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Otmar Venjakob

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On Kato's local ϵ-isomorphism conjecture for rank-one Iwasawa modules

Otmar Venjakob

This paper contains a complete proof of Fukaya and Kato's ϵ-isomorphism conjecture for invertible Λ-modules (the case of $V = V_0(r)$, where V_0 is unramified of dimension 1). Our results rely heavily on Kato's proof, in an unpublished set of lecture notes, of (commutative) ϵ-isomorphisms for one-dimensional representations of $G_{\mathbb{Q}_p}$, but apart from fixing some sign ambiguities in Kato's notes, we use the theory of (ϕ, Γ)-modules instead of syntomic cohomology. Also, for the convenience of the reader we give a slight modification or rather reformulation of it in the language of Fukuya and Kato and extend it to the (slightly noncommutative) semiglobal setting. Finally we discuss some direct applications concerning the Iwasawa theory of CM elliptic curves, in particular the local Iwasawa Main Conjecture for CM elliptic curves E over the extension of $\mathbb{Q}_p$ which trivialises the p-power division points $E(p)$ of E. In this sense the paper is complimentary to our work with Bouganis (*Asian J. Math.* **14**:3 (2010), 385–416) on noncommutative Main Conjectures for CM elliptic curves.

1. Introduction

The significance of local ϵ-factors à la Deligne and Tate, or more generally that of the (conjectural) ϵ-isomorphism suggested in [Fukaya and Kato 2006, §3] is at least twofold. First, they are important ingredients to obtain a precise functional equation for L-functions or more generally for (conjectural) ζ-isomorphisms [loc. cit., §2] of motives in the context of equivariant or noncommutative Tamagawa number conjectures (see, e.g., Theorem 4.1). Secondly, they are essential in interpolation formulae of (actual) p-adic L-functions and for the relation between ζ-isomorphisms and (conjectural, not necessarily commutative) p-adic L-functions as discussed in [loc. cit., §4]. Of course the two occurrences are closely related; for a survey on these ideas see also [Venjakob 2007].

Our motivation for writing this article stems from Theorem 8.4 of [Burns and Venjakob 2011] (see Theorem 4.2), which describes under what conditions the

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validity of a (noncommutative) Iwasawa Main Conjecture for a critical (ordinary at p) motive M over some p-adic Lie extension F_∞ of $\mathbb{Q}$ implies parts of the equivariant Tamagawa number conjecture (ETNC) by Burns and Flach for M with respect to a finite Galois extension $F \subseteq F_\infty$ of $\mathbb{Q}$. Due to the second above mentioned meaning it requires among others the existence of an ϵ-isomorphism

$$\epsilon_{p,\mathbb{Z}_p[G(F/\mathbb{Q})]}(\hat{\mathbb{T}}_F) : \mathbf{1}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \to \mathbf{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(R\Gamma(\mathbb{Q}_p, \hat{\mathbb{T}}_F))\mathbf{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\hat{\mathbb{T}}_F) \qquad (1)$$

in the sense of [Fukaya and Kato 2006, Conjecture 3.4.3], where the Iwasawa module $\hat{\mathbb{T}}_F$ is related to the ordinary condition of M; e.g., for an (ordinary) elliptic curve E it arises from the formal group part of the usual Tate module of E. Unfortunately, very little is known about the existence of such ϵ-isomorphisms in general. To the knowledge of the author it is not even contained in the literature for $\hat{\mathbb{T}}_F$ attached to a CM-elliptic curve E and the trivialising extension $F_\infty := F(E(p))$, where $E(p)$ denotes the group of p-power division points of E. In principle a rough sketch of a proof is contained in [Kato 1993b], which unfortunately has never been published. Moreover there were still some sign ambiguities which we fix in this paper; in particular, it turns out that one has to take $-\mathscr{L}_{K,\epsilon^{-1}}$, that is, -1 times the classical Coleman map (6), in the construction of the epsilon isomorphism (17).

Benois and Berger [2008] have proved the conjecture $C_{EP}(L/K, V)$ for arbitrary crystalline representations V of G_K, where K is an unramified extension of $\mathbb{Q}_p$ and L a finite subextension of $K_\infty = K(\mu(p))$ over K. Although they mention in their introduction that "Les mêmes arguments, avec un peu plus de calculs, permettent de démontrer la conjecture $C_{EP}(L/K, V)$ pour toute extension L/K contenue dans $\mathbb{Q}_p^{ab}$. Cette petite généralisation est importante pour la version équivariante des conjectures de Bloch et Kato", they leave it as an "exercise" to the reader. In the special case $V = \mathbb{Q}_p(r)$, $r \in \mathbb{Z}$, Burns and Flach [2006] proved a local ETNC using global ingredients in a semilocal setting, while in the above example we need it for $V = \mathbb{Q}_p(\eta)(r)$, where η denotes an unramified character. Also we would like to stress that the existence of the ϵ-isomorphisms à la Fukaya and Kato is a slightly finer statement than the $C_{EP}(L/K, V)$-conjecture or the result of Burns and Flach, because the former one states that a certain family of certain precisely defined units of integral group algebras of finite groups in a certain tower can be interpolated by a unit in the corresponding Iwasawa algebra while in the latter ones "only" a family of lattices is "interpolated" by one over the Iwasawa algebra.

The aim of this article, which also might hopefully serve as a survey into the subject, is to provide detailed and complete arguments for the existence of the ϵ-isomorphism

$$\epsilon_\Lambda(\mathbb{T}(T)) : \mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}(T)))_{\widetilde{\Lambda}}\mathbf{d}_\Lambda(\mathbb{T}(T))_{\widetilde{\Lambda}},$$

where $\Lambda = \Lambda(G)$ is the Iwasawa algebra of $G = G(K_\infty/\mathbb{Q}_p)$ for any (possibly

infinite) unramified extension K of $\mathbb{Q}_p$, $T = \mathbb{Z}_p(\eta)(r)$ and $R\Gamma(\mathbb{Q}_p, \mathbb{T}(T))$ denotes the complex calculating local Galois cohomology of $\mathbb{T}(T)$, the usual Iwasawa theoretic deformation of T (see (28)). Furthermore, for an associative ring R with one, $\mathbf{d}_R$ denotes the determinant functor with $\mathbf{1}_R = \mathbf{d}_R(0)$ (see Appendix B) while $\widetilde{\Lambda}$ is defined in (2). We are mainly interested in the case where $G \cong \mathbb{Z}_p^2 \times \Delta$ for a finite group Δ — such extensions arise for example by adjoining the p-power division points of a CM elliptic curve to the base field as above. This corresponds to a (generalised) conjecture $C_{IW}(K_\infty/\mathbb{Q}_p)$ (in the notation of Benois and Berger) originally due to Perrin-Riou. It is the first example of an ϵ-isomorphism associated with a two-dimensional p-adic Lie group extension. Following Kato's approach we construct a *universal* ϵ-isomorphism $\epsilon_\Lambda(\mathbb{T}(\mathbb{Z}_p(1)))$, from which all the others arise by suitable twists and descent. But while Kato constructs it first over cyclotomic $\mathbb{Z}_p$-extensions and then takes limits, here we construct it directly over $(\mathbb{Z}_p^2 \times \Delta)$-extensions (and then take limits). To show that they satisfy the right interpolation property with respect to Artin (Dirichlet) characters of G, we use the theory of (ϕ, Γ)-modules and the explicit formulae in [Berger 2003], instead of the much more involved syntomic cohomology and Kato's reciprocity laws for formal groups. In contrast to Kato's unpublished preprint, in which he uses the language of étale sheaves and cohomology, we prefer Galois cohomology as used also in [Fukaya and Kato 2006]. In order to work out in detail Kato's reduction argument [1993b] to the case of trivial η we have to show a certain twist compatibility of Perrin-Riou's exponential map/Coleman map for T versus $\mathbb{Z}_p(r)$ over a trivialising extension K_∞ for η, see Lemma A.4. Going over to semilocal settings we obtain the first ϵ-isomorphism over a (slightly) noncommutative ring. In a forthcoming paper [Loeffler et al. 2013], using the techniques of [Benois and Berger 2008] and [Loeffler and Zerbes 2011], we are going to extend these results to the case of arbitrary crystalline representations for the same tower of local fields as above. Of course it would be most desirable to extend the existence of ϵ-isomorphism also to nonabelian local extensions, but this seems to require completely new ideas and to be out of reach at present (see [Izychev 2012] for some examples). Some evidence in that direction has been provided by Fukaya (unpublished).

Combined with Yasuda's work [2009] concerning ϵ-isomorphisms for $l \neq p$, we also obtain in principle a purely local proof of the Burns–Flach result for $V = \mathbb{Q}_p(r)$.

2. Kato's proof for one-dimensional representations

Let p be a prime and let K be any unramified (possibly infinite) Galois extension of $\mathbb{Q}_p$. We set $K_n := K(\mu_{p^n})$ for $0 \leq n \leq \infty$ and

$$\Gamma = G(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times.$$

Recall that the maximal unramified extension $\mathbb{Q}_p^{ur}$ and the maximal abelian extension $\mathbb{Q}_p^{ab}$ of $\mathbb{Q}_p$ are given as $\mathbb{Q}_p(\mu(p'))$ and $\mathbb{Q}_p(\mu) = \mathbb{Q}_p^{ur}(\mu(p))$, where $\mu(p)$ and $\mu(p')$ denote the p-primary and prime-to-p part of μ, the group of all roots of unity, respectively. In particular, we have the canonical decomposition

$$G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p) = G(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \times G(\mathbb{Q}_{p,\infty}/\mathbb{Q}_p) = \hat{\mathbb{Z}} \times \mathbb{Z}_p^\times,$$

under which by definition τ_p corresponds to $(\phi, 1)$ (and by abuse of notation also to its image in G below), where $\phi := \mathrm{Frob}_p$ denotes the arithmetic Frobenius $x \mapsto x^p$. We put

$$H := H_K := G(K/\mathbb{Q}_p) = \overline{\langle \phi \rangle}$$

and

$$G := G(K_\infty/\mathbb{Q}_p) \cong H \times \Gamma.$$

Assume that G is a p-adic Lie group, that is, H is the product of a finite abelian group of order prime to p with a (not necessarily strict) quotient of $\mathbb{Z}_p$. By

$$\Lambda := \Lambda(G) := \mathbb{Z}_p[\![G]\!]$$

we denote as usual the Iwasawa algebra of G. Also we write $\widehat{\mathbb{Z}_p^{ur}}$ for the ring of Witt vectors $W(\overline{\mathbb{F}_p})$ with its natural action by ϕ and we set

$$\widetilde{\Lambda} = \Lambda \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}_p^{ur}} = \widehat{\mathbb{Z}_p^{ur}}[\![G]\!]. \tag{2}$$

By

$$\mathbb{T}_{un} := \Lambda^\sharp(1)$$

we denote the free Λ-module of rank one with the Galois action

$$\chi_{un} : G_{\mathbb{Q}_p} \to \Lambda^\times, \quad \sigma \mapsto [\mathbb{T}_{un}, \sigma] := \bar{\sigma}^{-1} \kappa(\sigma),$$

where $\bar{\ } : G_{\mathbb{Q}_p} \twoheadrightarrow G$ is the natural projection map and $\kappa : G_{\mathbb{Q}_p} \twoheadrightarrow \mathbb{Z}_p^\times$ is the p-cyclotomic character. Furthermore, we write

$$\mathbb{U}(K_\infty) := \varprojlim_{L,i} \mathcal{O}_L^\times / p^i$$

for the Λ-module of local units, where L and i run through the finite subextensions of $K_\infty/\mathbb{Q}_p$ and the natural numbers, respectively, and the transition maps are induced by the norm. Finally we fix once and for all a $\mathbb{Z}_p$-basis $\epsilon = (\epsilon_n)_n$ of $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$.

We set

$$\Lambda_a = \{x \in \widetilde{\Lambda} \mid (1 \otimes \phi)(x) = (a \otimes 1) \cdot x\} \quad \text{for } a \in \Lambda^\times = K_1(\Lambda).$$

Proposition 2.1. *For $a = [\mathbb{T}_{un}, \tau_p]^{-1} = \tau_p$ there is a canonical isomorphism*

$$\Lambda_a \cong \begin{cases} \mathbb{O}_K[\![\Gamma]\!] & \text{if } H \text{ is finite}, \\ \varprojlim_{\mathbb{Q}_p \subseteq K' \subseteq K \text{finite}, \mathrm{Tr}} \mathbb{O}_{K'}[\![\Gamma]\!] & \text{if } H \text{ is infinite}, \end{cases}$$

as Λ-modules. All modules are free of rank one.

Proof. We first assume $H = \langle \tau_p \rangle$ to be finite of order d and replace Γ by a finite quotient without changing the notation. Then any element $x \in \widetilde{\Lambda} = \widehat{\mathbb{Z}_p^{ur}}[\Gamma][H]$ can be uniquely written as $\sum_{i=0}^{d-1} a_i \tau_p^i$ with $a_i \in \widehat{\mathbb{Z}_p^{ur}}[\Gamma]$ and ϕ acts coefficientwise on the latter elements. The calculation

$$\begin{aligned}(1 \otimes \phi)(x) - (\tau_p \otimes 1)x &= \sum_{i=0}^{d-1} \phi(a_i)\tau_p^i - \sum_{i=0}^{d-1} a_i \tau_p^{i+1} \\ &= \sum_{i=0}^{d-1} \big(\phi(a_i) - a_{i-1}\big)\tau_p^i\end{aligned}$$

with $a_{-1} := a_{d-1}$ shows that x belongs to Λ_a if and only if $\phi^d(a_i) = a_i$ and $\phi^{-i}(a_0) = a_i$ for all i. As $\widehat{\mathbb{Z}_p^{ur}}^{\phi^d = 1} = \mathbb{O}_K$, the canonical map

$$\Lambda_a \cong \mathbb{O}_K[\Gamma], \quad \sum a_i \tau_p^i \mapsto a_0,$$

is an isomorphism of Λ-modules, the inverse of which is

$$x \mapsto \sum_{h \in H} h \otimes h^{-1}(x)$$

and which is obviously functorial in Γ, whence the same result follows for the original (infinite) Γ.

Now, for a surjection $\pi : H'' \twoheadrightarrow H'$ it is easy to check that the trace $\mathrm{Tr}_{K''/K'} : \mathbb{O}_{K''} \to \mathbb{O}_{K'}$ induces a commutative diagram

$$\begin{array}{ccc} \Lambda''_{a''} & \xrightarrow{\cong} & \mathbb{O}_{K''}[\![\Gamma]\!] \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{K''/K'}} \\ \Lambda'_{a'} & \xrightarrow{\cong} & \mathbb{O}_{K'}[\![\Gamma]\!], \end{array} \tag{3}$$

whence the first claim follows. From the normal basis theorem for finite fields we obtain (noncanonical) isomorphisms

$$\mathbb{O}_{K'} \cong \mathbb{Z}_p[H_{K'}],$$

which are compatible with trace and natural projection maps. Indeed, the sets $S_{K'} := \{a \in \mathbb{O}_{K'} \mid \mathbb{Z}_p[H_{K'}]a = \mathbb{O}_{K'}\} \cong \mathbb{Z}_p[H_{K'}]^\times$ are compact, since $1 + \mathrm{Jac}(\mathbb{Z}_p[H_{K'}])$

for the Jacobson radical $\mathrm{Jac}(\mathbb{Z}_p[H_{K'}])$ is open in $\mathbb{Z}_p[H_{K'}]^\times$, and thus $\varprojlim_{K'} S_{K'}$ is nonempty. Hence the trace maps induce (noncanonical) isomorphisms

$$\varprojlim_{K'} \mathbb{O}_{K'} \cong \mathbb{Z}_p[\![H]\!] \quad \text{and} \quad \varprojlim_{K'} \mathbb{O}_{K'}[\![\Gamma]\!] \cong \mathbb{Z}_p[\![G]\!]. \qquad \square$$

We now review Coleman's exact sequence [1979; 1983], which is one crucial ingredient in the construction of the ϵ-isomorphism.

Assume first that $K/\mathbb{Q}_p$ *is finite*. Then $\mathbb{U}(K_\infty) := \varprojlim_{n,i} \mathbb{O}_{K_n}^\times/p^i$ with $K_n := K(\mu_{p^n})$, and the sequence

$$0 \longrightarrow \mathbb{Z}_p(1) \xrightarrow{\iota} \mathbb{U}(K_\infty) \xrightarrow{\mathrm{Col}} \mathbb{O}_K[\![\Gamma]\!] \xrightarrow{\pi} \mathbb{Z}_p(1) \longrightarrow 0 \tag{4}$$

of Λ-modules is exact, where the maps are defined as follows:

- $\iota(\epsilon) = \epsilon$.
- $\mathrm{Col}(u) := \mathrm{Col}_\epsilon(u)$ is defined by the rule

$$\mathscr{L}(g_u) := \left(1 - \frac{\varphi}{p}\right)\log(g_u) = \frac{1}{p}\log\frac{g_u^p}{\varphi(g_u)} = \mathrm{Col}(u)\cdot(X+1) \tag{5}$$

  in $\mathbb{O}_K[\![X]\!]$, with $g_u := g_{u,\epsilon} \in \mathbb{O}_K[\![X]\!]$ the Coleman power series satisfying $g^{\phi^{-n}}(\epsilon_n - 1) = u_n$ for all n. Here ϕ is acting coefficientwise on $g_u = g_u(X)$, while $\varphi : \mathbb{O}_K[\![X]\!] \to \mathbb{O}_K[\![X]\!]$ is induced by $X \mapsto (X+1)^p - 1$ and the action of ϕ on the coefficients. Furthermore, the $\mathbb{O}_K$-linear action of $\mathbb{O}_K[\![\Gamma]\!]$ on $\mathbb{O}_K[\![X]\!]$ is induced by $\gamma \cdot X = (1+X)^{\kappa(\gamma)} - 1$.
- π is the composite of $\mathbb{O}_K[\![\Gamma]\!] \to \mathbb{O}_K$, $\gamma \mapsto \kappa(\gamma)$, followed by the trace $\mathrm{Tr}_{K/\mathbb{Q}_p} : \mathbb{O}_K \to \mathbb{Z}_p$ (and strictly speaking followed by $\mathbb{Z}_p \to \mathbb{Z}_p(1)$, $c \mapsto c\epsilon$).

Using Proposition 2.1 and the isomorphism

$$\Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} \cong \mathbb{T}_{un} \otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}}, \quad a \mapsto (1\otimes\epsilon)\otimes a,$$

we thus obtain an exact sequence of Λ-modules

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow \mathbb{U}(K_\infty) \xrightarrow{\mathscr{L}_{K,\epsilon}} \mathbb{T}_{un}(K_\infty) \otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} \longrightarrow \mathbb{Z}_p(1) \longrightarrow 0. \tag{6}$$

In the end we actually shall need the analogous exact sequence

$$0 \to \mathbb{Z}_p(1) \to \mathbb{U}(K_\infty) \xrightarrow{-\mathscr{L}_{K,-\epsilon}} \mathbb{T}_{un}(K_\infty) \otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} \to \mathbb{Z}_p(1) \to 0, \tag{7}$$

where we replace ϵ by $-\epsilon$ everywhere in the construction and where we multiply (only) the middle map by -1. Note that the maps involving $\mathbb{Z}_p(1)$ do not change compared with (6).

To deal with the case where $K/\mathbb{Q}_p$ *is infinite*, that is, $p^\infty \mid [K:\mathbb{Q}_p]$, consider finite intermediate extensions $\mathbb{Q}_p \subseteq L \subseteq L' \subseteq K$. We claim that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \mathbb{U}(L_\infty) & \xrightarrow{\mathscr{L}_{L',\epsilon}} & \mathbb{T}_{un}(L'_\infty)\otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} & \longrightarrow & \mathbb{Z}_p(1) \longrightarrow 0 \\ & & \Big\downarrow{\scriptstyle N_{L'_\infty/L_\infty}=[L':L]\cdot} & & \Big\downarrow{\scriptstyle N_{L'_\infty/L_\infty}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{L'/L}} & & \Big\downarrow{\scriptstyle =} \\ 0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \mathbb{U}(L_\infty) & \xrightarrow{\mathscr{L}_{L,\epsilon}} & \mathbb{T}_{un}(L_\infty)\otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} & \longrightarrow & \mathbb{Z}_p(1) \longrightarrow 0 \end{array} \tag{8}$$

commutes, where the norm maps $N_{L'_\infty/L_\infty} = N_{L'/L}$ are induced by $N_{L'_n/L_n}$ for all n, which on $\mathbb{Z}_p(1)$ amounts to multiplication by $[L':L]$ while $N_{L'_\infty/L_\infty}: \mathbb{U}(L'_\infty) \to \mathbb{U}(L_\infty)$ is nothing else than the projection on the corresponding inverse (sub)system. Recalling (3) this is equivalent to the commutativity of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \mathbb{U}(L'_\infty) & \xrightarrow{\mathrm{Col}_{L',\epsilon}} & \mathcal{O}_{L'}[\![\Gamma]\!] & \longrightarrow & \mathbb{Z}_p(1) \longrightarrow 0 \\ & & \Big\downarrow{\scriptstyle N_{L'_\infty/L_\infty}=[L':L]\cdot} & & \Big\downarrow{\scriptstyle N_{L'_\infty/L_\infty}} & & \Big\downarrow{\scriptstyle \mathrm{Tr}_{L'/L}} & & \Big\downarrow{\scriptstyle =} \\ 0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & \mathbb{U}(L_\infty) & \xrightarrow{\mathrm{Col}_{L,\epsilon}} & \mathcal{O}_{L}[\![\Gamma]\!] & \longrightarrow & \mathbb{Z}_p(1) \longrightarrow 0 \end{array} \tag{9}$$

where $\mathrm{Tr}_{L'/L}: \mathcal{O}_{L'}[\![\Gamma]\!] \to \mathcal{O}_L[\![\Gamma]\!]$ is induced by the trace on the coefficients. While the left and right square obviously commute, we sketch how to check this for the middle one.

By the uniqueness of the Coleman power series we have

$$N_{L'/L}(g_{u'}) = g_{N_{L'/L}(u')} \quad \text{for } u' \in \mathbb{U}(L'_\infty),$$

where $N_{L'/L}: \mathcal{O}_{L'}[\![X]\!] \to \mathcal{O}_L[\![X]\!]$ is defined by $f(X) \mapsto \prod_{\sigma\in G(L'/L)} f^\sigma(X)$, where σ acts coefficientwise on f (see the proof of Lemma 2 in [Yager 1982] for a similar argument). Next, one has

$$\mathscr{L}(N_{L'/L}(g)) = \mathrm{Tr}_{L'/L}\mathscr{L}(g)$$

for $g \in \mathcal{O}_{L'}[\![X]\!]^\times$, since $N_{L'/L}$ and ϕ commute. So far we have seen that

$$\mathrm{Tr}_{L'/L}\mathscr{L}(g_{u'}) = \mathscr{L}(g_{N_{L'/L}(u')}),$$

which implies the claim

$$\mathrm{Tr}_{L'/L}(\mathrm{Col}(u')) = \mathrm{Col}(g_{N_{L'/L}(u')})$$

using the defining equation (5) and the compatibility of $\mathrm{Tr}_{L'/L}$ with the Mahler transform $\mathfrak{M}: \mathcal{O}_K[\![\Gamma]\!] \to \mathcal{O}_K[\![X]\!]$, $\lambda \mapsto \lambda\cdot(1+X)$.

Taking inverse limits of (8) we obtain the exact sequence

$$0 \to \mathbb{U}(K_\infty) \xrightarrow{\mathscr{L}_{K,\epsilon}} \mathbb{T}_{un}(K_\infty)\otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} \longrightarrow \mathbb{Z}_p(1) \to 0. \tag{10}$$

Similarly, starting with (7) we obtain the exact sequence

$$0 \to \mathbb{U}(K_\infty) \xrightarrow{-\mathscr{L}_{K,-\epsilon}} \mathbb{T}_{un}(K_\infty) \otimes_\Lambda \Lambda_{[\mathbb{T}_{un},\tau_p]^{-1}} \longrightarrow \mathbb{Z}_p(1) \to 0. \tag{11}$$

Galois cohomology. The complex $R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))$ of continuous cochains has only nontrivial cohomology groups for $i = 1, 2$:

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) = \varprojlim_{\mathbb{Q}_p \subseteq L \subseteq K_\infty \text{finite}} \mathrm{H}^1(L, \mathbb{Z}_p(1)) = \varprojlim_L (L^\times)^{\wedge p} \tag{12}$$

by Kummer theory and

$$\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) = \varprojlim_{\mathbb{Q}_p \subseteq L \subseteq K_\infty \text{finite}} \mathrm{H}^2(L, \mathbb{Z}_p(1)) = \mathbb{Z}_p \tag{13}$$

by local Tate duality; here the sign of the trace map $\mathrm{tr} : \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \cong \mathbb{Z}_p$ is normalised according to [Kato 1993a, Chapter II, §1.4] as follows: If $\theta \in \mathrm{H}^1(\mathbb{Q}_p, \Lambda)$ denotes the character $G_{\mathbb{Q}_p} \xrightarrow{w} \hat{\mathbb{Z}} \xrightarrow{\text{canon}} \Lambda$, where w is the map which sends Frob_p to 1 and the inertia subgroup to 0, then we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{Q}_p^\times & \xrightarrow{v} & \mathbb{Z} & \xrightarrow{\text{canon}} & \mathbb{Z}_p \\ \downarrow{\scriptstyle \delta} & & & & \uparrow{\scriptstyle \cong\ \mathrm{tr}} \\ \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) & & \xrightarrow{-\bigcup\theta} & & \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Z}_p(1)), \end{array} \tag{14}$$

where v denotes the normalised valuation map and δ is the Kummer map. The first isomorphism (12) induces

- a canonical exact sequence

$$0 \to \mathbb{U}(K_\infty) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \xrightarrow{-\hat{v}} \mathbb{Z}_p \to 0, \tag{15}$$

  if $K/\mathbb{Q}_p$ is finite, $\hat{v}$ being induced from the valuation maps $v_L : L^\times \to \mathbb{Z}$ (the sign before $\hat{v}$ will become evident by the descent calculation (54));

- an isomorphism

$$\mathbb{U}(K_\infty) \cong \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)), \tag{16}$$

  if $p^\infty \mid [K : \mathbb{Q}_p]$.

Determinants. Now we assume that $K/\mathbb{Q}_p$ is *infinite*. Then

$$G \cong G' \times \Delta,$$

where Δ is a finite abelian group of order d prime to p and $G' \cong \mathbb{Z}_p^2$. Thus

$$\Lambda(G) = \mathbb{Z}_p[\Delta][\![\mathbb{Z}_p^2]\!]$$

is a product of regular, hence Cohen–Macaulay, rings. Set

$$\mathcal{O} := \mathbb{Z}_p[\mu_d].$$

Then

$$\Lambda(G) \subseteq \Lambda_{\mathbb{O}}(G) = \prod_{\chi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_p}(\Delta)} \Lambda_{\mathbb{O}}(G')e_\chi,$$

where e_χ denotes the idempotent corresponding to χ, while $\mathrm{Irr}_{\overline{\mathbb{Q}}_p}(\Delta)$ denotes the set of $\overline{\mathbb{Q}}_p$-rational characters of Δ. Since regular rings are normal (or by Wedderburn theory) it follows that there is a product decomposition into local regular integral domains

$$\Lambda(G) = \prod_{\chi \in \mathrm{Irr}_{\mathbb{Q}_p}(\Delta)} \Lambda_{\mathbb{O}_\chi}(G')e_\chi,$$

where now $\mathrm{Irr}_{\mathbb{Q}_p}(\Delta)$ denotes the set of $\mathbb{Q}_p$-rational characters and $\mathbb{O}_\chi$ is the ring of integers of $K_\chi := \mathrm{End}_{\mathbb{Z}_p[\Delta]}(\chi)$.

For the various rings R showing up like $\Lambda(G)$ for different G, we fix compatible determinant functors $\mathbf{d}_R : \mathrm{D}^{\mathrm{p}}(R) \to \mathcal{P}_R$ from the category of perfect complexes of R-modules (consisting of (bounded) complexes of finitely generated R-modules quasi-isomorphic to strictly perfect complexes, that is, bounded complexes of finitely generated projective R-modules) into the Picard category $\mathcal{P}_R$ with unit object $\mathbf{1}_R = \mathbf{d}_R(0)$, see Appendix B) for the yoga of determinants used in this article.

Lemma 2.2. *For all $r \in \mathbb{Z}$ there exists a canonical isomorphism*

$$\mathbf{1}_\Lambda \xrightarrow{\mathrm{can}_{\mathbb{Z}_p(r)}} \mathbf{d}_\Lambda(\mathbb{Z}_p(r)).$$

Remark 2.3. The proof will show that the same result holds for $G \cong \mathbb{Z}_p^k \times \Delta$, $k \geq 2$ and any $\Lambda(G)$-module M of Krull codimension at least 2.

Proof. Since

$$\mathrm{Ext}^i_{\Lambda(G)}(\mathbb{Z}_p(r), \Lambda(G)) \cong \mathrm{Ext}^i_{\Lambda(G')}(\mathbb{Z}_p(r), \Lambda(G')) = 0$$

for $i \neq k$ (≥ 2) we see that the codimension of $\mathbb{Z}_p(r)$ equals $k + 1 - 1 = k \geq 2$. Setting $M = \mathbb{Z}_p(r)$ we first show that the class $[M]$ in $G_0(\Lambda) = K_0(\Lambda)$ vanishes; i.e., there exists an isomorphism $c_0 : \mathbf{1} \cong \mathbf{d}(M)$ by the definition of $\mathcal{P}_R$ in [Fukaya and Kato 2006]. Since

$$K_0(\Lambda) = \bigoplus_\chi K_0(\Lambda_{\mathbb{O}_\chi}(G')) \cong \bigoplus_\chi \mathbb{Z},$$

where the last map is given by the rank, the claim follows because the $e_\chi M$ are torsion $\Lambda_{\mathbb{O}_\chi}(G')$-modules. By the knowledge of the codimension we have $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \subset \Lambda$ of height at most 1. In particular, we obtain canonical isomorphisms

$$c_{\mathfrak{p}} : \mathbf{1}_{\Lambda_{\mathfrak{p}}} \cong \mathbf{d}_{\Lambda_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Since $\mathrm{Mor}(\mathbf{1}_{\Lambda_\mathfrak{p}}, \mathbf{d}_{\Lambda_\mathfrak{p}}(M_\mathfrak{p}))$ is a (nonempty) $K_1(\Lambda_\mathfrak{p})$-torsor, there exists for each $\mathfrak{p}$ a unique $\lambda_\mathfrak{p} \in \Lambda_\mathfrak{p}^\times = K_1(\Lambda_\mathfrak{p})$ such that

$$c_\mathfrak{p} = (c_0)_\mathfrak{p} \cdot \lambda_\mathfrak{p},$$

where $(c_0)_\mathfrak{p} = \Lambda_\mathfrak{p} \otimes_\Lambda c_0$. Now let $\mathfrak{q} = \mathfrak{q}_\chi$ be a prime of height zero corresponding to $\chi \in \mathrm{Irr}_{\mathbb{Q}_p}(\Delta)$. Then

$$\begin{aligned} c_\mathfrak{q} &= \Lambda_\mathfrak{q} \otimes_{\Lambda_\mathfrak{p}} c_\mathfrak{p} \\ &= \Lambda_\mathfrak{q} \otimes_\Lambda c_0 \cdot \lambda_\mathfrak{p} = (c_0)_\mathfrak{q} \lambda_\mathfrak{p} \end{aligned}$$

for all prime ideals $\mathfrak{p} \supset \mathfrak{q}$ of height one, whence

$$\lambda_\mathfrak{p} = \lambda_\mathfrak{q}.$$

Thus

$$\lambda_\mathfrak{q} \in \bigcap_{\mathfrak{p} \supset \mathfrak{q}, \mathrm{ht}(\mathfrak{p})=1} \Lambda_\mathfrak{p}^\times = \Lambda_{\mathbb{O}_\chi}(G')^\times$$

($\Lambda_{\mathbb{O}_\chi}(G')$ being regular, that is, $\bigcap_{\mathfrak{p} \supset \mathfrak{q}, \mathrm{ht}(\mathfrak{p})=1} \Lambda_\mathfrak{p} = \Lambda_{\mathbb{O}_\chi}(G')$) and

$$\mathrm{can}_M := (c_0 \cdot \lambda_{\mathfrak{q}_\chi})_\chi : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(M)$$

is unique and independent of the choice of c_0. Here we used the canonical decomposition $K_1(\Lambda(G)) \cong \bigoplus_\chi K_1(\Lambda_{\mathbb{O}_\chi}(G'))$. □

Now we can finally define the ϵ-isomorphism for the pair $(\Lambda(G), \mathbb{T}_{un})$:

$$\epsilon_\Lambda(\mathbb{T}_{un}) := \epsilon_{\Lambda,\epsilon}(\mathbb{T}_{un}) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}))\mathbf{d}_\Lambda(\mathbb{T}_{un} \otimes_\Lambda \Lambda_{\tau_p}). \tag{17}$$

Since Λ is regular we obtain, by property (B.h) in the Appendix,

$$\begin{aligned} \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}))^{-1} &\cong \mathbf{d}_\Lambda(\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}))\mathbf{d}_\Lambda(\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}))^{-1} \\ &\cong \mathbf{d}_\Lambda(\mathbb{U}(K_\infty))\mathbf{d}_\Lambda(\mathbb{Z}_p)^{-1} \\ &\cong \mathbf{d}_\Lambda(\mathbb{T}_{un} \otimes_\Lambda \Lambda_{\tau_p})\mathbf{d}_\Lambda(\mathbb{Z}_p(1))^{-1}\mathbf{d}_\Lambda(\mathbb{Z}_p)^{-1} \\ &\cong \mathbf{d}_\Lambda(\mathbb{T}_{un} \otimes_\Lambda \Lambda_{\tau_p}); \end{aligned}$$

here we have used (13) and (16) for the second isomorphism, regularity and the sequence (11) with its map $-\mathscr{L}_{K,\epsilon^{-1}}$ (sic!) for the third, and the identifications $\mathrm{can}_{\mathbb{Z}_p(1)}$ and $\mathrm{can}_{\mathbb{Z}_p}$ in the last step. This induces (17).

In the spirit of Fukaya and Kato, this can be reformulated in a way that also covers noncommutative rings Λ later. For any $a \in K_1(\widetilde{\Lambda})$ define

$$K_1(\Lambda)_a := \{x \in K_1(\widetilde{\Lambda}) \mid (1 \otimes \phi)_*(x) = a \cdot x\},$$

which is nonempty by [Fukaya and Kato 2006, Proposition 3.4.5]. If Λ is the Iwasawa algebra of an abelian p-adic Lie group, that is, $K_1(\widetilde{\Lambda}) = \widetilde{\Lambda}^\times$, this implies

in particular that $\Lambda_a \cap \widetilde{\Lambda}^\times = K_1(\Lambda)_a \neq \varnothing$, whence we obtain an isomorphism of $\widetilde{\Lambda}$-modules

$$\Lambda_a \otimes_\Lambda \widetilde{\Lambda} \cong \widetilde{\Lambda}, \quad x \otimes y \mapsto x \cdot y. \tag{18}$$

Thus, one immediately sees that the map

$$\mathbb{U}(K_\infty) \to \mathbb{T}_{un} \otimes_\Lambda \Lambda_{\tau_p} \subseteq \mathbb{T}_{un} \otimes_\Lambda \widetilde{\Lambda}$$

extends to an exact sequence of $\widetilde{\Lambda}$-modules

$$0 \to \mathbb{U}(K_\infty) \otimes_\Lambda \widetilde{\Lambda} \longrightarrow \mathbb{T}_{un} \otimes_\Lambda \widetilde{\Lambda} \longrightarrow \widehat{\mathbb{Z}_p^{ur}}(1) \to 0, \tag{19}$$

which in fact is canonically isomorphic to the base change of (10) from Λ- to $\widetilde{\Lambda}$-modules. Therefore base changing (17) by $\widetilde{\Lambda} \otimes_\Lambda -$ and using (18) (tensored with $\mathbb{T}_{un}(K_\infty)$) we obtain

$$\epsilon'_\Lambda(\mathbb{T}_{un}) := \epsilon'_{\Lambda,\epsilon}(\mathbb{T}_{un}) : \mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}))_{\widetilde{\Lambda}} \mathbf{d}_\Lambda(\mathbb{T}_{un})_{\widetilde{\Lambda}}, \tag{20}$$

which actually arises as base change from some

$$\epsilon_0 : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))\mathbf{d}_\Lambda(\mathbb{T}_{un}(K_\infty))$$

plus a twisting by an element $\delta \in K_1(\Lambda)_{\tau_p}$, that is,

$$\epsilon'_\Lambda(\mathbb{T}_{un}) \in \mathrm{Mor}(\mathbf{1}_\Lambda, \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))\mathbf{d}_\Lambda(\mathbb{T}_{un}(K_\infty))) \times^{K_1(\Lambda)} K_1(\Lambda)_{\tau_p}.$$

Indeed, fixing an isomorphism $\psi : \Lambda \cong \Lambda_{\tau_p}$ (see Proposition 2.1) sending 1 to δ, (18) implies that $\delta \in K_1(\Lambda)_{\tau_p}$ and the claim follows from the commutative diagram

$$\begin{array}{ccc} \mathbb{T}_{un} \otimes_\Lambda \widetilde{\Lambda} & \xrightarrow{\mathbb{T}_{un} \otimes \delta^{-1}} & \mathbb{T}_{un} \otimes_\Lambda \widetilde{\Lambda} \\ \uparrow & & \uparrow \\ \mathbb{T}_{un} \otimes_\Lambda \Lambda_{\tau_p} & \xrightarrow{\mathbb{T}_{un} \otimes \psi^{-1}} & \mathbb{T}_{un} \otimes_\Lambda \Lambda \end{array}$$

($\epsilon'_\Lambda(\mathbb{T}_{un})$ equals δ times the base change of $\epsilon_0 := (\mathbb{T}_{un} \otimes \psi^{-1}) \circ \epsilon_\Lambda(\mathbb{T}_{un})$).

Twisting. We recall the following definition from [Fukaya and Kato 2006, §1.4]:

Definition 2.4. A ring R is *of type* 1 if there exists a two-sided ideal I of R such that R/I^n is finite of order a power of p for any $n \geq 1$ and such that $R \cong \varprojlim_n R/I^n$.

A ring R is *of type* 2 if it is the matrix algebra $M_n(L)$ of some finite extension L over $\mathbb{Q}_p$, for some $n \geq 1$.

By Lemma 1.4.4 in the same work, R is of type 1 if and only if the defining condition above holds for the Jacobson ideal $J = J(R)$. Such rings are always semilocal and R/J is a finite product of matrix algebras over finite fields.

Now let R be a commutative ring of type 1 and let $\mathbb{T} = \mathbb{T}_\chi$ be a free R-module of rank one with Galois action given by

$$\chi = \chi_{\mathbb{T}} : G_{\mathbb{Q}_p} \to R^\times$$

which factors through G. By $\tilde{\chi}_{\mathbb{T}}$ we denote the induced ring homomorphism $\Lambda(G) \to R$. Furthermore let $Y = Y_\chi$ be the $(R, \Lambda(G))$-bimodule which is R as R-module and where $\Lambda(G)$ is acting via

$$\chi_Y := \tilde{\chi}_{\mathbb{T}}^{-1} \chi_{cyc} : \Lambda(G) \to R$$

(from the right), where

$$\chi_{cyc} : \Lambda(G) \to \mathbb{Z}_p \to R$$

is induced by the cyclotomic character and the unique ring homomorphism $\mathbb{Z}_p \to R$. Then the map

$$Y \otimes_{\Lambda(G)} \mathbb{T}_{un} \xrightarrow{\cong} \mathbb{T}, \qquad y \otimes t \mapsto y \cdot \chi_Y(t),$$

is an isomorphism of R-modules which is Galois equivariant, where the Galois action on the tensor product is given by $\sigma(y \otimes t) = y \otimes \sigma(t)$ for $\sigma \in G_{\mathbb{Q}_p}$.

Let $\widetilde{R}$ and R_a be defined in the same way as for Λ. Then, using the isomorphisms

$$Y \otimes_\Lambda \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un})) \cong \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, Y \otimes_\Lambda \mathbb{T}_{un})) \cong \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, \mathbb{T}))$$

by [Fukaya and Kato 2006, 1.6.5] and

$$R \otimes_\Lambda \Lambda_a \cong R_{\chi(a)},$$

where $\chi : \Lambda \to R$ denotes a continuous ring homomorphism, we may define the following ϵ-isomorphisms:

Definition 2.5. In the above situation we set

$$\epsilon_R(\mathbb{T}) := \epsilon_{R,\epsilon}(\mathbb{T}) := Y \otimes_\Lambda \epsilon_{\Lambda,\epsilon}(\mathbb{T}_{un}) : \mathbf{1}_R \to \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, \mathbb{T}))\mathbf{d}_R(\mathbb{T} \otimes_R R_{\chi(\tau_p)})$$

and

$$\epsilon'_R(\mathbb{T}) := \epsilon'_{R,\epsilon}(\mathbb{T}) := Y \otimes_\Lambda \epsilon'_{\Lambda,\epsilon}(\mathbb{T}_{un}) : \mathbf{1}_{\widetilde{R}} \to \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, \mathbb{T}))_{\widetilde{R}}\mathbf{d}_R(\mathbb{T})_{\widetilde{R}}.$$

By definition we have an important twist invariance property: if R and R' are commutative rings of type 1 or 2 and Y' is any (R', R)-bimodule that is projective as an R'-module and satisfies $Y' \otimes_R \mathbb{T} \cong \mathbb{T}'$, we have

$$Y' \otimes_R \epsilon_R(\mathbb{T}) = \epsilon_{R'}(\mathbb{T}') \quad \text{and} \quad Y' \otimes_R \epsilon'_R(\mathbb{T}) = \epsilon'_{R'}(\mathbb{T}'). \tag{21}$$

Indeed, to this end the definition extends to all pairs $(R, \mathbb{T})$, where R is a (not necessarily commutative) ring of type 1 or 2 and $\mathbb{T}$ stands for a projective R-module such that there exists a (R, Λ)-bimodule Y which is projective as R-module and

such that $\mathbb{T} \cong Y \otimes_\Lambda \mathbb{T}_{un}$. In this context we denote by $[\mathbb{T}, \sigma]$, $\sigma \in G_{\mathbb{Q}_p}$, the element in $K_1(R)$ induced by the action of $G_{\mathbb{Q}_p}$ on $\mathbb{T}$; note that this induces a homomorphism $[\mathbb{T}, -] : G(\mathbb{Q}_p^{ab}) \to K_1(R)$.

Example 2.6. Let $\psi : G_F \to \mathbb{Z}_p^\times$ be a Grössencharacter of an imaginary quadratic field F such that p is split in F and assume that its restriction to G_{F_v}, v a place above p, factors through G. We write $\mathbb{T}_\psi$ for the free rank-one $\Lambda(G)$-module with Galois action given by $\sigma(\lambda) = \lambda \bar{\sigma}^{-1} \psi(\sigma)$. Then we also write $\epsilon_\Lambda(\psi)$ for $\epsilon_\Lambda(\mathbb{T}_\psi)$.

The ϵ-conjecture. We fix $K/\mathbb{Q}_p$ infinite and recall that $G = G(K_\infty/\mathbb{Q}_p)$ as well as $\Lambda = \Lambda(G)$ and $\Lambda_{\mathbb{O}} = \Lambda_{\mathbb{O}}(G)$ for $\mathbb{O} = \mathbb{O}_L$ the ring of integers of some finite extension L of $\mathbb{Q}_p$. If $\chi : G \to \mathbb{O}_L^\times$ denotes any continuous character such that the representation

$$V_\chi := L(\chi),$$

whose underlying vector space is just L and whose $G_{\mathbb{Q}_p}$-action is given by χ, is de Rham, hence potentially semistable by [Serre 1968] (in this classical case) or by [Berger 2002] (in general) then we have

$$L \otimes_{\mathbb{O}_L} \epsilon'_{\mathbb{O}_L}(\mathbb{T}_\chi) = \epsilon'_L(V_\chi)$$

by definition. The ϵ-isomorphism conjecture (Conjecture 3.4.3 of [Fukaya and Kato 2006]) states that

$$\epsilon'_L(V_\chi) = \Gamma_L(V_\chi) \cdot \epsilon_{L,\epsilon,dR}(V_\chi) \cdot \theta_L(V_\chi), \tag{22}$$

where, for any de Rham p-adic representation V of G, the notation used is as follows:

(a) $\Gamma_L(V) := \prod_{\mathbb{Z}} \Gamma^*(j)^{-h(-j)}$ with $h(j) = \dim_L gr^j D_{dR}(V)$ and

$$\Gamma^*(j) = \begin{cases} (-1)^j (-j)!^{-1} & \text{for } j \le 0, \\ \Gamma(j) & \text{for } j > 0, \end{cases}$$

denotes the leading coefficient of the Γ-function.

(b) The map

$$\epsilon_{dR}(V) := \epsilon_{L,\epsilon,dR}(V) : \mathbf{1}_{\widetilde{L}} \to \mathbf{d}_{\widetilde{L}}(V) \mathbf{d}_{\widetilde{L}}(D_{dR}(V))^{-1},$$

with $\widetilde{L} := \widehat{\mathbb{Q}_p^{ur}} \otimes_{\mathbb{Q}_p} L$, is defined in [Fukaya and Kato 2006, Proposition 3.3.5]. We shall recall its definition after the proof of Lemma A.5.

(c) $\theta_L(V)$ is defined as follows: Firstly, $R\Gamma_f(\mathbb{Q}_p, V)$ is defined as a certain subcomplex of the local cohomology complex $R\Gamma(\mathbb{Q}_p, V)$, concentrated in degrees 0 and 1, whose image in the derived category is isomorphic to

$$R\Gamma_f(\mathbb{Q}_p, V) \cong \left[D_{\mathrm{cris}}(V) \xrightarrow{(1-\varphi_p, 1)} D_{\mathrm{cris}}(V) \oplus D_{dR}(V)/D^0_{dR}(V) \right]. \tag{23}$$

Here φ_p denotes the usual Frobenius homomorphism and the induced map $t(V) := D_{dR}(V)/D^0_{dR}(V) \to H^1_f(\mathbb{Q}_p, V)$ is the exponential map $\exp_{BK}(V)$ of Bloch–Kato, where we write $\mathrm{H}^n_f(\mathbb{Q}_p, V)$ for the cohomology of $R\Gamma_f(\mathbb{Q}_p, V)$. Now

$$\theta_L(V) : \mathbf{1}_L \to \mathbf{d}_L(R\Gamma(\mathbb{Q}_p, V)) \cdot \mathbf{d}_L(D_{dR}(V)) \tag{24}$$

is by definition induced from $\eta_p(V) \cdot \overline{(\eta_p(V^*(1))^*)}$ (see Remark B.1 for the notation) — with

$$\eta_p(V) : \mathbf{1}_L \to \mathbf{d}_L(R\Gamma_f(\mathbb{Q}_p, V))\mathbf{d}_L(t(V)) \tag{25}$$

arising by trivialising $D_{\mathrm{cris}}(V)$ in (23) by the identity — followed by an isomorphism induced by local Tate duality

$$R\Gamma_f(\mathbb{Q}_l, V) \cong \big(R\Gamma(\mathbb{Q}_l, V^*(1))/R\Gamma_f(\mathbb{Q}_l, V^*(1))\big)^*[-2] \tag{26}$$

and using $D^0_{dR}(V) = t(V^*(1))^*$.

More explicitly, $\theta_L(V)$ is obtained from applying the determinant functor to the following exact sequence:

$$0 \to \mathrm{H}^0(\mathbb{Q}_p, V) \to D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(V) \oplus t(V) \xrightarrow{\exp_{BK}(V)} \mathrm{H}^1(\mathbb{Q}_p, V)$$
$$\xrightarrow{\exp_{BK}(V^*(1))^*} D_{\mathrm{cris}}(V^*(1))^* \oplus t(V^*(1))^* \to D_{\mathrm{cris}}(V^*(1))^* \to \mathrm{H}^2(\mathbb{Q}_p, V) \to 0,$$

which arises from joining the defining sequences of $\exp_{BK}(V)$ with the dual sequence for $\exp_{BK}(V^*(1))$ by local duality (26).

Remark 2.7. (a) The ϵ-conjecture may analogously be formulated using $\epsilon_R(\mathbb{T})$ instead of $\epsilon'_R(\mathbb{T})$. In the following we will amply switch between the two versions.

(b) Since by definition of $\epsilon_{\mathbb{O}_L}(\mathbb{T}_\chi)$ we have

$$L \otimes_{\mathbb{O}_L} \epsilon'_{\mathbb{O}_L}(\mathbb{T}_\chi) = L \otimes_{\mathbb{O}_L} \big(Y_\chi \otimes_\Lambda \epsilon'_\Lambda(\mathbb{T}_{un})\big) = (L \otimes_{\mathbb{O}_L} Y_\chi) \otimes_\Lambda \epsilon'_\Lambda(\mathbb{T}_{un}),$$

proving (22) amounts to showing that

$$L \otimes_\Lambda \epsilon_\Lambda(\mathbb{T}_{un}) = \epsilon_L(V_\chi), \tag{27}$$

where Λ acts on L via $\chi^{-1}\chi_{cyc} : \Lambda(G) \to \mathbb{O}_L \subseteq L$. Once we have shown (27) for all possible χ as above, it follows immediately by twisting that for example $\epsilon_\Lambda(\mathbb{T}_{K_\infty}(T))$ for $T = \mathbb{Z}_p(\eta)(r)$ as below satisfies the descent property

$$V_\rho \otimes_\Lambda \epsilon_\Lambda(\mathbb{T}_{K_\infty}(T)) = \epsilon_L(V(\rho^*))$$

with $V(\rho^*) := V \otimes_{\mathbb{Q}_p} V_{\rho^*}$ for all one-dimensional representations V_ρ arising from some continuous $\rho : G \to \mathbb{O}_L^\times$ and its contragredient representation V_{ρ^*}.

Note that by [Serre 1968] any V_χ as above is of the form

$$W = L(\eta\rho)(r) = Lt_{\rho\eta,r},$$

where r is some integer, $\eta : G \to \mathbb{O}_L^\times$ is an unramified character and $\rho : G \twoheadrightarrow G(K'_m/\mathbb{Q}_p) \to \mathbb{O}_L^\times$ denotes an Artin character for some finite subextension K' of $K/\mathbb{Q}_p$ and with $m = a(\rho)$ chosen minimal, that is, $p^{a(\rho)}$ is the p-part of the conductor of ρ.

In the following we fix η and r and we set $T := \mathbb{Z}_p(\eta)(r)$, $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and

$$\mathbb{T}_{K_\infty} = \mathbb{T}_{K_\infty}(T) = \Lambda^\sharp \otimes_{\mathbb{Z}_p} T, \tag{28}$$

the free Λ-module on which $\sigma \in G_{\mathbb{Q}_p}$ acts as $\bar\sigma^{-1}\eta\kappa^r(\sigma)$.

Now we are going to make the map (24) explicit. First we describe the local cohomology groups:

$$\mathrm{H}^0(\mathbb{Q}_p, W) = \begin{cases} L & \text{if } r = 0 \text{ and } \rho\eta = \mathbb{1}, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

By local Tate duality we have

$$\mathrm{H}^2(\mathbb{Q}_p, W) \cong \mathrm{H}^0(\mathbb{Q}_p, W^*(1))^* = \begin{cases} L & \text{if } r = 1 \text{ and } \rho\eta = \mathbb{1}, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

From the local Euler–Poincaré characteristic formula one immediately obtains

$$\dim_L \mathrm{H}^1(\mathbb{Q}_p, W) = \dim_L \mathrm{H}^1(\mathbb{Q}_p, W^*(1)) = \begin{cases} 2 & \text{if } r = 0 \text{ or } 1 \text{ and } \rho\eta = \mathbb{1}, \\ 1 & \text{otherwise.} \end{cases} \tag{31}$$

Following the same reasoning used for Lemma 1.3.1 of [Benois and Nguyen Quang Do 2002], one sees that

$$\begin{aligned} \mathrm{H}^1_f(\mathbb{Q}_p, W) &\cong \big(\mathrm{H}^1(\mathbb{Q}_p, W^*(1))/\mathrm{H}^1_f(\mathbb{Q}_p, W^*(1))\big)^* \\ &= \begin{cases} \mathrm{H}^1(\mathbb{Q}_p, W) & \text{if } r \geq 2, \text{ or } r = 1 \text{ and } \rho\eta \neq \mathbb{1}, \\ \mathrm{im}\left(\mathbb{U}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))\right) & \text{if } r = 1 \text{ and } \rho\eta = \mathbb{1}, \\ \mathrm{H}^1(\mathbb{F}_p, \mathbb{Q}_p) & \text{if } r = 0 \text{ and } \rho\eta = \mathbb{1}, \\ 0, & r \leq -1, \text{ or } r = 0 \text{ and } \rho\eta \neq \mathbb{1}, \end{cases} \end{aligned}$$

where the map in the second line is the Kummer map. Hence we call the cases where $r = 0$ or 1 and $\rho\eta = \mathbb{1}$ *exceptional* and all the others *generic*.

For the tangent space we have by (61)

$$t(W) = \begin{cases} D_{dR}(W) = L & \text{if } r > 0, \\ 0 & \text{if } r \leq 0, \end{cases} \tag{32}$$

$$t(W^*(1)) = \begin{cases} 0 & \text{if } r > 0, \\ D_{dR}(W^*(1)) = L & \text{if } r \leq 0, \end{cases} \tag{33}$$

and

$$D_{\mathrm{cris}}(W) = \begin{cases} 0 & \text{if } a(\rho) \neq 0, \\ Le_{\rho\eta,r} & \text{otherwise,} \end{cases} \tag{34}$$

with Frobenius action given as $\phi(e_{\rho\eta,r}) = p^{-r}\rho\eta(\tau_p^{-1})e_{\rho\eta,r}$.

The case $r \geq 1$. In this case we have $\Gamma_L(W) = \Gamma(r)^{-1} = \dfrac{1}{(r-1)!}$ and $\mathrm{H}^0(\mathbb{Q}_p, W) = 0$, whence

$$1 - \phi : D_{\mathrm{cris}}(W) \to D_{\mathrm{cris}}(W) \tag{35}$$

and

$$\exp(W) : D_{dR}(W) \cong H^1_f(\mathbb{Q}_p, W) \tag{36}$$

are bijections. Combined with the exact sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, W^*(1))^* &\xrightarrow{\exp(W^*(1))^*} D_{\mathrm{cris}}(W^*(1))^* \\ &\xrightarrow{1-\phi^*} D_{\mathrm{cris}}(W^*(1))^* \longrightarrow \mathrm{H}^2(\mathbb{Q}_p, W) \longrightarrow 0 \end{aligned} \tag{37}$$

and

$$0 \to \mathrm{H}^1_f(\mathbb{Q}_p, W) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, W) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, W^*(1))^* \to 0,$$

they induce the following isomorphism corresponding to $\theta_L(W)^{-1}$:

$$\mathbf{d}_L(D_{dR}(W)) \to \mathbf{d}_L(R\Gamma(\mathbb{Q}_p, W))^{-1}.$$

In the *generic* case this decomposes as

$$\mathbf{d}_L(\exp(W)) : \mathbf{d}_L(D_{dR}(W)) \to \mathbf{d}_L(\mathrm{H}^1(\mathbb{Q}_p, W)) = \mathbf{d}_L(R\Gamma(\mathbb{Q}_p, W))^{-1}$$

times

$$\frac{\det(1-\phi^* \mid D_{\mathrm{cris}}(W^*(1))^*)}{\det(1-\phi \mid D_{\mathrm{cris}}(W))} : \mathbf{1}_L \to \mathbf{1}_L,$$

which equals

$$\frac{\det(1-\phi \mid D_{\mathrm{cris}}(W^*(1)))}{\det(1-\phi \mid D_{\mathrm{cris}}(W))} = \begin{cases} \dfrac{1 - p^{r-1}\rho\eta(\tau_p)}{1 - p^{-r}\rho\eta(\tau_p^{-1})} & \text{if } a(\rho) = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{38}$$

Now let $r = 1$ and $\rho\eta = \mathbb{1}$, that is, we consider the *exceptional* case $W = \mathbb{Q}_p(1)$. As now $\det(1-\phi \mid D_{\mathrm{cris}}(W^*(1))) = 0$ and the two occurrences of $D_{\mathrm{cris}}(W^*(1))^*$ in (37) are identified via the identity, the map $\theta_L(W)^{-1}$ is also induced by (35), (36) together with the (second) exact sequence in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_p^\times \otimes \mathbb{Q}_p & \longrightarrow & \widehat{\mathbb{Q}_p^\times} \otimes \mathbb{Q}_p & \xrightarrow{-\hat{v}\otimes\mathbb{Q}_p} & \mathbb{Q}_p & \longrightarrow & 0 \\ & & \cong\downarrow\delta & & \cong\downarrow\delta & & \cong\uparrow\mathrm{Tr} & & \\ 0 & \longrightarrow & \mathrm{H}^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1)) & \longrightarrow & \mathrm{H}^1(\mathbb{Q}_p,\mathbb{Q}_p(1)) & \dashrightarrow & \mathrm{H}^2(\mathbb{Q}_p,\mathbb{Q}_p(1)) & \longrightarrow & 0 \end{array} \tag{39}$$

where the first two vertical maps δ are induced by Kummer theory, v denotes the normalised valuation map and the dotted arrow is defined by commutativity; that is, $\theta_L(W)^{-1}$ arises from

$$\mathbf{d}_{\mathbb{Q}_p}(D_{dR}(\mathbb{Q}_p(1))) \xrightarrow{\exp_{\mathbb{Q}_p(1)}} \mathbf{d}_{\mathbb{Q}_p}(H^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1))) \cong \mathbf{d}_{\mathbb{Q}_p}(R\Gamma(\mathbb{Q}_p,W))^{-1} \tag{40}$$

times

$$\det(1-\phi \mid D_{\mathrm{cris}}(\mathbb{Q}_p(1))) = (1-p^{-1}). \tag{41}$$

Combining (39), (40) and (41) this can rephrased as follows:

Proposition 2.8. *The map $\theta(\mathbb{Q}_p(1))$ is just induced by the single exact sequence*

$$0 \to t(\mathbb{Q}_p(1)) \cong \mathbb{Q}_p \xrightarrow{(1-p^{-1})^{-1}\exp_{\mathbb{Q}_p(1)}} \mathrm{H}^1(\mathbb{Q}_p,\mathbb{Q}_p(1)) \xrightarrow{-\hat{v}\otimes\mathbb{Q}_p} \mathrm{H}^2(\mathbb{Q}_p,\mathbb{Q}_p(1)) \to 0.$$

Proof. Since $t(\mathbb{Q}_p)=0$, it follows directly from its definition as a connecting homomorphism that

$$\exp_{\mathbb{Q}_p} : \mathbb{Q}_p = D_{\mathrm{cris}}(\mathbb{Q}_p) \to \mathrm{H}^1_f(\mathbb{Q}_p,\mathbb{Q}_p) \subseteq \mathrm{H}^1(\mathbb{Q}_p,\mathbb{Q}_p) \tag{42}$$

sends $\alpha \in \mathbb{Q}_p$ to the character $\chi_\alpha : G_{\mathbb{Q}_p} \to \mathbb{Q}_p$, $g \mapsto (g-1)c$, where $c \in \widehat{\mathbb{Q}_p^{nr}}$ satisfies $(1-\varphi)c = \alpha$, that is, $\chi_\alpha(\phi) = -\alpha$. As noted in [Benois and Nguyen Quang Do 2002, Lemma 1.3.1], we thus may identify $\mathrm{H}^1_f(\mathbb{Q}_p,\mathbb{Q}_p) = \mathrm{H}^1(\mathbb{F}_p,\mathbb{Q}_p)$. Identifying the copies of $D_{\mathrm{cris}}(\mathbb{Q}_p)$ (in the dual of (37)) gives rise to a map

$$\psi : \mathbb{Q}_p = \mathrm{H}^0(\mathbb{Q}_p,\mathbb{Q}_p) \to \mathrm{H}^1_f(\mathbb{Q}_p,\mathbb{Q}_p), \quad \alpha \mapsto \chi_\alpha.$$

By local Tate duality

$$\begin{array}{ccccc} \mathrm{H}^1(\mathbb{Q}_p,\mathbb{Q}_p(1))/\mathrm{H}^1_f(\mathbb{Q}_p,\mathbb{Q}_p(1)) & \times & \mathrm{H}^1_f(\mathbb{Q}_p,\mathbb{Q}_p) & \to & \mathrm{H}^2(\mathbb{Q}_p,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p \\ \psi^*\downarrow & & \uparrow\psi & & \| \\ \mathrm{H}^2(\mathbb{Q}_p,\mathbb{Q}_p(1)) & \times & \mathrm{H}^0(\mathbb{Q}_p,\mathbb{Q}_p) & \to & \mathrm{H}^2(\mathbb{Q}_p,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p \end{array}$$

we obtain for the dual map ψ^* using the normalisation (14)

$$\mathrm{tr}(\psi^*(\delta(p)) = \mathrm{tr}(\delta(p)\cup\chi_1) = \chi_1(\phi) = -1.$$

The dotted arrow in (39) being ψ^*, this diagram commutes as claimed. □

The case $r \le 0$. This case is dual to the previous one, replacing W by $W^*(1)$.

The descent. Let K be infinite. In order to describe the descent of $\mathscr{L}_{K,\epsilon^{-1}}$ in (10) we set

$$\mathscr{L}_{\mathbb{T}} := \mathscr{L}_{\mathbb{T},\epsilon^{-1}} := Y \otimes_{\Lambda} \mathscr{L}_{K,\epsilon^{-1}} \tag{43}$$

if the projective left Λ'-module Y (with commuting right Λ-module structure) satisfies $Y \otimes_{\Lambda} \mathbb{T}_{un} \cong \mathbb{T}$ as Λ'-modules. Since $\mathscr{L}_{K,\epsilon^{-1}}$ is the crucial ingredient in the definition of $\epsilon'_{\Lambda}(\mathbb{T})$, the following descent diagram will be important:

For fixed ρ as before we choose $K' \subseteq K$ and $n \geq \max\{1, a(\rho)\}$ such that ρ factorises over $G_n := G(K'_n/\mathbb{Q}_p)$. Setting $\Lambda' := \mathbb{Q}_p[G_n]$ and $V' := \mathbb{Q}_p[G_n]^{\sharp} \otimes \mathbb{Q}_p(\eta)(r)$ we first note that

$$\mathrm{H}^i(\mathbb{Q}_p, V') \cong \mathrm{H}^i(K'_n, \mathbb{Q}_p(\eta)(r))$$

by Shapiro's lemma. Also, let Y' be the (Λ', Λ)-bimodule such that $Y' \otimes_{\Lambda} \mathbb{T}_{un} \cong V'$. We write $e_{\chi} := (1/\#G_n) \sum_{g \in G_n} \chi(g^{-1})g$ for the usual idempotent, which induces a canonical decomposition $\Lambda' \cong \prod L_{\chi}$ into a product of finite extensions L_{χ} of $\mathbb{Q}_p$. In particular, for $L = L_{\rho}$ we have $W \cong e_{\rho^{-1}} V' = L_{\rho}(\rho\eta)(r)$.

Then, for $r \geq 1$ and with $\Gamma(V') := \bigoplus_{\chi} \Gamma(e_{\chi} V')$, we have a commutative diagram

$$\begin{array}{ccccc}
Y' \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) & & \xrightarrow{-\mathscr{L}_{V'} = -Y' \otimes_{\Lambda} \mathscr{L}_{K,\epsilon^{-1}}} & & Y' \otimes_{\Lambda} \mathbb{T}_{un} \otimes_{\Lambda} \Lambda_{[\mathbb{T}_{un}, \tau_p^{-1}]} \\
\downarrow{\scriptstyle \mathrm{pr}_n} & & & & \downarrow{\scriptstyle \cong\ \mathrm{pr}_n} \\
\mathrm{H}^1(K'_n, \mathbb{Q}_p(\eta)(r)) & \xleftarrow{\Gamma(V')^{-1} \exp_{V'}} & D_{dR}(V') & \longleftarrow & V' \otimes_{\Lambda'} (\Lambda')_{[V', \tau_p^{-1}]}
\end{array}$$

of Λ'-modules as will be explained in the Appendix, Proposition A.6.

Applying the exact functor $V_{\rho^*} \otimes_{\Lambda'} -$ leads to the final commutative descent diagram — at least for $W \neq \mathbb{Q}_p(1)$ —

$$\begin{array}{ccccc}
Y'' \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) & & \xrightarrow{-\mathscr{L}_W = -Y'' \otimes_{\Lambda} \mathscr{L}_{K,\epsilon^{-1}}} & & Y'' \otimes_{\Lambda} \mathbb{T}_{un} \otimes_{\Lambda} \Lambda_{[\mathbb{T}_{un}, \tau_p^{-1}]} \\
\downarrow{\scriptstyle \mathrm{pr}_n} & & & & \downarrow{\scriptstyle \cong\ \mathrm{pr}_n} \\
\mathrm{H}^1(\mathbb{Q}_p, W) & \xleftarrow{\Gamma(W)^{-1} \exp_W} & D_{dR}(W) & \xleftarrow{\frac{\det(1-\varphi|D_{\mathrm{cris}}(W^*(1)))}{\det(1-\varphi|D_{\mathrm{cris}}(W))} \cdot \epsilon_{L,\epsilon,dR}(W)^{-1}} & W \otimes_L L_{[W, \tau_p^{-1}]},
\end{array} \tag{44}$$

where $Y'' := V_{\rho^*} \otimes_{\Lambda'} Y' = V_{\rho^*} \otimes_{\Lambda} Y$ is a (L, Λ)-bimodule. For $W = \mathbb{Q}_p(1)$ the Euler factor in the denominator and the map pr_0 become zero, so we shall instead apply a direct descent calculation in Lemma 2.9 using semisimplicity and a Bockstein homomorphism.

For the descent we need

- the long Tor-exact sequence by applying $Y'' \otimes_{\Lambda(G)} -$ to the defining sequence (10) for $-\mathscr{L}_{K,\epsilon^{-1}}$;

- the convergent cohomological spectral sequence

$$E_2^{i,j} := \mathrm{Tor}_{-i}^{\Lambda}(Y'', H^j(\mathbb{Q}_p, \mathbb{T}_{un})) \Rightarrow \mathrm{H}^{i+j}(\mathbb{Q}_p, W), \tag{45}$$

  which is induced from the isomorphism

$$Y'' \otimes^{\mathbb{L}} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}) \cong R\Gamma(\mathbb{Q}_p, Y'' \otimes_{\Lambda} \mathbb{T}_{un}),$$

  proved in [Fukaya and Kato 2006] and using $W \cong Y'' \otimes \mathbb{T}_{un}$;

- and the fact that the determinant functor is compatible with both these ingredients [Venjakob 2012].

For $\mathbb{T} = \mathbb{T}(T) := \Lambda^{\sharp} \otimes_{\mathbb{Z}_p} T \cong Y \otimes_{\Lambda} \mathbb{T}_{un}$, we have

$$\mathrm{H}^i(\mathbb{Q}_p, \mathbb{T}) \cong \begin{cases} T & \text{if } i = 0 \text{ and } r = 0, \eta = \mathbb{1}, \\ \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}) \neq 0 & \text{if } i = 1, \\ T(-1) & \text{if } i = 2 \text{ and } r = 1, \eta = \mathbb{1}, \\ 0 & \text{otherwise.} \end{cases} \tag{46}$$

Hence we obtain for $r \geq 1$ the following exact sequence of terms in lower degree:

$$\begin{aligned} 0 \to \mathrm{Tor}_1^{\Lambda}(Y'', \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})) \to \mathrm{H}^0(\mathbb{Q}_p, W) \to \mathrm{Tor}_2^{\Lambda}(Y'', \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un})) \\ \to Y'' \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \to \mathrm{H}^1(\mathbb{Q}_p, W) \to \mathrm{Tor}_1^{\Lambda}(Y'', \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un})) \to 0, \end{aligned} \tag{47}$$

and we also obtain

$$\mathrm{Tor}_2^{\Lambda}(Y'', \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})) = 0 \quad \text{and} \quad Y'' \otimes_{\Lambda} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}) \cong \mathrm{H}^2(\mathbb{Q}_p, W).$$

Since $Y'' \otimes_{\Lambda}^{\mathbb{L}} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}) \cong V_{\rho^*} \otimes_{\Lambda}^{\mathbb{L}} \big(Y \otimes_{\Lambda} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un})\big) \cong V_{\rho^*} \otimes_{\Lambda}^{\mathbb{L}} R\Gamma(\mathbb{Q}_p, \mathbb{T})$, the preceding sequence is canonically isomorphic to

$$\begin{aligned} 0 \to \mathrm{Tor}_1^{\Lambda}(V_{\rho^*}, \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T})) \to \mathrm{H}^0(\mathbb{Q}_p, W) \to \mathrm{Tor}_2^{\Lambda}(V_{\rho^*}, \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T})) \\ \to V_{\rho^*} \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}) \to \mathrm{H}^1(\mathbb{Q}_p, W) \to \mathrm{Tor}_1^{\Lambda}(V_{\rho^*}, \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T})) \to 0, \end{aligned} \tag{48}$$

and we get

$$\mathrm{Tor}_2^{\Lambda}(V_{\rho^*}, \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T})) = 0 \quad \text{and} \quad V_{\rho^*} \otimes_{\Lambda} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}) \cong \mathrm{H}^2(\mathbb{Q}_p, W).$$

In the *generic case* the spectral sequence boils down to the isomorphism

$$Y'' \otimes_{\Lambda} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \cong \mathrm{H}^1(\mathbb{Q}_p, W). \tag{49}$$

Considering the support of $\mathbb{Z}_p(1)$, one easily sees that $\mathrm{Tor}_i^{\Lambda}(Y'', \mathbb{Z}_p(1)) = 0$ for all $i \geq 0$. Hence the long exact Tor-sequence associated with (10) combined with

(16) degenerates to

$$Y'' \otimes_\Lambda \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \xrightarrow[\cong]{-\mathscr{L}_W} W \otimes_L L_{[W,\tau_p]^{-1}}, \tag{50}$$

while for all $i \geq 0$

$$\mathrm{Tor}_i^\Lambda(Y'', \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un})) = \mathrm{Tor}_i^\Lambda(Y'', \mathbb{Z}_p) = 0. \tag{51}$$

Thus the conjectured equation (22) holds by (44), (49), (50) and the definition (17) with $-\mathscr{L}_{K,\epsilon^{-1}}$.

For the *exceptional case* $W = \mathbb{Z}_p(1)$ we set $R = \Lambda(\Gamma)_\mathfrak{p}$, where $\mathfrak{p}$ denotes the augmentation ideal of $\Lambda(\Gamma)$ and recall that R is a discrete valuation ring with uniformising element $\pi := 1 - \gamma_0$, where γ_0 is a fixed element in Γ sent to 1 under

$$\Gamma \xrightarrow{\kappa} \mathbb{Z}_p^\times \xrightarrow{\log_p} \mathbb{Z}_p,$$

and residue field $R/\pi = \mathbb{Q}_p$. The commutative diagram of homomorphisms of rings

$$\begin{array}{ccccc} \Lambda = \Lambda(G) & \longrightarrow & \Lambda(\Gamma) & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p & \longrightarrow & \mathbb{Q}_p \end{array}$$

induces with $Y'' := R/\pi$ the isomorphism

$$\begin{aligned} R\Gamma(\mathbb{Q}_p, \mathbb{Q}_p(1)) &\cong Y'' \otimes^{\mathbb{L}}_\Lambda R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \\ &\cong \mathbb{Q}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} \left(\Lambda(\Gamma) \otimes^{\mathbb{L}}_\Lambda R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))\right) \\ &\cong \mathbb{Q}_p \otimes^{\mathbb{L}}_R R \otimes_{\Lambda(\Gamma)} \otimes^{\mathbb{L}}_\Lambda R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty})) \\ &\cong \mathbb{Q}_p \otimes^{\mathbb{L}}_R R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_\mathfrak{p}. \end{aligned} \tag{52}$$

In particular, the descent calculation factorises over the cyclotomic level; that is,

$$\epsilon'_{\mathbb{Q}_p}(\mathbb{Q}_p(1)) = R/\pi \otimes_R \epsilon'_R(R \otimes_{\Lambda(\Gamma)} \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))$$

is induced by $\epsilon'_R(R \otimes_{\Lambda(\Gamma)} \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))$, which in turn is induced by the localisation at $\mathfrak{p}$ of the exact sequences (6) and (15) for $K = \mathbb{Q}_p$, which are respectively

$$\mathbb{U}(\mathbb{Q}_{p,\infty})_\mathfrak{p} \xrightarrow[\cong]{-\mathscr{L}_{\mathbb{T}_{un}(\mathbb{Q}_{p,\infty})_\mathfrak{p}}} \mathbb{T}_{un}(\mathbb{Q}_{p,\infty})_\mathfrak{p} \otimes_R R_{[\mathbb{T}_{un},\tau_p]^{-1}} \tag{53}$$

(this arises as the long exact Tor-sequence from (10)) and

$$0 \to \mathbb{U}(\mathbb{Q}_{p,\infty})_\mathfrak{p} \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_\mathfrak{p} \xrightarrow{-\hat{v}} \mathbb{Q}_p \cong \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_\mathfrak{p} \to 0. \tag{54}$$

This last sequence arises from an analogue of the spectral sequence (45) above — which gives with $\mathscr{H} = G(K_\infty/\mathbb{Q}_{p,\infty})$ an exact sequence

$$0 \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathscr{H}} \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty})) \to \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))^{\mathscr{H}} \to 0,$$

and

$$\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathscr{H}} = \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))$$

— combined with (13) and an identification of $\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))^{\mathscr{H}} = \mathbb{Z}_p$ with $\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathscr{H}} = \mathbb{Z}_p$ induced by the base change of $\mathrm{can}_{\mathbb{Z}_p}$. Indeed, it is easy to check that the long exact $\mathscr{H}$-homology ($= \mathrm{Tor}_i^{\Lambda}(\Lambda(\Gamma), -)$) sequence associated with (10) recovers (6), in particular $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathscr{H}} \cong \mathbb{U}(K_\infty)_H \cong \mathbb{U}(\mathbb{Q}_{p,\infty})$. Moreover, the composite

$$\begin{aligned}\tilde{\beta} : \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty})) \to \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))^{\mathscr{H}} &= \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathscr{H}} \\ &= \mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))\end{aligned}$$

is via restriction and taking $G(K_\infty^{\mathscr{H}'}/\mathbb{Q}_{p,\infty})$-invariants by construction induced by the Bockstein homomorphism β associated to the exact triangle in the derived category

$$R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \xrightarrow{1-h_0} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty)) \longrightarrow R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty^{\mathscr{H}'})),$$

where $\mathscr{H}'$ is the maximal pro-p quotient of $\mathscr{H}$ and h_0 is the image of ϕ. By [Flach 2004, Lemma 5.9] (and the argument following directly afterwards using the projection formula for the cup product) it follows that $\tilde{\beta}$ is given by the cup product $\theta \cup -$, where

$$\theta : G_{\mathbb{Q}_{p,\infty}} \twoheadrightarrow \mathscr{H}' \cong \mathbb{Z}_p$$

is the unique character such that h_0 is sent to 1 under the second isomorphism. Using our above convention of the trace map (14) one finds according to [Kato 1993a, Chapter II, §1.4.2] that the above composite equals $-\hat{v}$. Indeed

$$\mathrm{tr}(\tilde{\beta}(\delta(p))) = \mathrm{tr}(\theta \cup \delta(p)) = -\theta(\phi) = -\theta(h_0) = -1.$$

Now consider the element

$$u := (1 - \epsilon_n^{-1})_n \in \varprojlim_n (\mathbb{Q}_p(\mu_{p^n})^\times)^\wedge \cong \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))$$

and its image $u_{\mathfrak{p}}$ in $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathfrak{p}}$.

Lemma 2.9. $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathfrak{p}} \cong Ru_{\mathfrak{p}}$ *is a free* R*-module of rank one and* $\mathscr{L}_{\mathbb{T}_{un}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}}}$ *induces modulo* π *a canonical isomorphism*

$$t(\mathbb{Q}_p(1)) \xleftarrow{-\mathscr{L}_{\mathbb{Q}_p(1)}} \mathbb{U}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}}/\pi \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathfrak{p}}/\pi \qquad (55)$$

which sends $(1-p^{-1})e \in \mathbb{Q}_p e = t(\mathbb{Q}_p(1))$ *to* $\bar{u}$, *the image of* $u_{\mathfrak{p}}$ (*but which is of course not induced by the map* $\mathbb{U}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}} \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathfrak{p}}$ *as the latter map becomes trivial modulo* π!).

Proof. The natural inclusion $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}}/\pi \subseteq \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ maps $\bar{u}$ to the image of p under $(\mathbb{Q}_p^\times)^\wedge \otimes \mathbb{Q}_p \cong \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$, the isomorphism of Kummer theory, because p is the image of the elements $1-\epsilon_n^{-1}$ under the norm maps. In particular, $\bar{u}$ is nonzero. By (15) the element u^{γ_0-1} belongs to $\mathbb{U}(\mathbb{Q}_{p,\infty})$. In order to calculate the image of the class $\overline{u^{\gamma_0-1}}$ of $u_{\mathfrak{p}}^{\gamma_0-1}$ modulo π under $-\mathscr{L}_{\mathbb{Q}_p(1)}$ we note that

$$g(X) = g_{u^{\gamma_0-1},-\epsilon}(X) = \frac{(1+X)^{\kappa(\gamma_0)}-1}{X} \equiv \kappa(\gamma_0) \bmod (X),$$

whence we obtain from setting $X=0$ in $-(5)$ (i.e., Equation (5) multiplied by -1) that

$$-(1-p^{-1}) = -(1-p^{-1})\log(\kappa(\gamma_0)) = -\mathrm{Col}_{-\epsilon}(u^{\gamma_0-1})\cdot 1$$

equals the image of $\overline{u^{\gamma_0-1}}$ in $\mathbb{Q}_p = R/\pi \otimes_{\Lambda(\Gamma)} \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}) \otimes_{\Lambda(\Gamma)} \Lambda(\Gamma)_{[\mathbb{T}_{un},\tau_p]^{-1}}$ under $-\mathscr{L}_{\mathbb{Q}_p(1)}$. In particular, $\overline{u^{\gamma_0-1}}$ is a basis of $\mathbb{U}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}}/\pi$, which is mapped to zero in $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}}/\pi$, whence the long exact Tor-sequence associated with (54) induces the isomorphisms

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}}/\pi \xrightarrow{-v} \mathbb{Q}_p, \qquad \bar{u} \mapsto -1$$

(since $v(p)=1$) and

$$\mathbb{Q}_p \longrightarrow \mathbb{U}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}}/\pi, \qquad 1 \mapsto \overline{u^{\gamma_0-1}},$$

where the latter formula follows from the snake lemma. By the first isomorphism and Nakayama's lemma the first statement is proven and therefore

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}}[\pi] = \mathbb{U}(\mathbb{Q}_{p,\infty})_{\mathfrak{p}}[\pi] = 0.$$

The second claim follows now from the composition of these isomorphisms. □

Finally, the exact triangle in the derived category of R-modules

$$R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}} \xrightarrow{1-\gamma_0} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}} \longrightarrow \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}(\mathbb{Q}_{p,\infty}))_{\mathfrak{p}} \longrightarrow$$

combined with (52) induces the Bockstein map $\beta = \theta\cup$ sitting in the canonical exact sequence (depending on γ_0)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathfrak{p}}/\pi & \longrightarrow & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) & \xrightarrow{\beta} & \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) & \longrightarrow & 0, \\ & & & & \cong\downarrow & & \downarrow\cong & & \\ & & & & (\mathbb{Q}_p^\times)^\wedge \otimes \mathbb{Q}_p & \xrightarrow{\log_p} & \mathbb{Q}_p & & \end{array} \tag{56}$$

where θ denotes the composite $G_{\mathbb{Q}_p} \xrightarrow{\kappa} \mathbb{Z}_p^\times \xrightarrow{\log_p} \mathbb{Z}_p$ considered as an element of $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Z}_p)$ (see [Flach 2004, Lemma 5.7-9; Burns and Venjakob 2006, §3.1; Burns and Flach 2006, §5.3] and, for the commutativity of the square, [Kato 1993a, Ch. II, 1.4.5]). The last zero on the upper line comes from $\mathrm{H}^3(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))_{\mathfrak{p}}[\pi] = 0$. Combining with (55) it follows that $\epsilon'_{\mathbb{Q}_p}(\mathbb{Q}_p(1))$ is induced from the exact sequence

$$0 \to t(\mathbb{Q}_p(1)) \cong \mathbb{Q}_p \xrightarrow{(1-p^{-1})^{-1}\mathrm{im}(p)} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \xrightarrow{\beta=\log_p} \mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \to 0,$$

which does *not* coincide at all with the sequence of Proposition 2.8 (not even up to sign). Nevertheless they induce the same map on determinants: both induce a map

$$\mathbf{d}_{\mathbb{Q}_p}(\mathrm{H}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))) \to \mathbf{d}_{\mathbb{Q}_p}(\mathrm{H}^2(\mathbb{Q}_p, \mathbb{Q}_p(1))) \otimes \mathbf{d}_{\mathbb{Q}_p}(t(\mathbb{Q}_p(1)) \cong \mathbf{d}_{\mathbb{Q}_p}(\mathbb{Q}_p) \otimes \mathbf{d}_{\mathbb{Q}_p}(\mathbb{Q}_p) \quad (57)$$

sending $(1-p^{-1})^{-1}\exp(1) \wedge -\mathrm{im}(p) = (1-p^{-1})^{-1}\mathrm{im}(p) \wedge \exp(1)$ to $1 \wedge 1$. This completes the proof in the exceptional case.

For $r \leq 0$ one has symmetric calculations — at least in the generic case — using a descent diagram analogous to (44), except that the left map on the bottom is now induced by the dual Bloch–Kato exponential map $\Gamma(V)\exp^*_{V^*(1)}$ as indicated in (67) (left to the reader). The exceptional case can be dealt with by using the duality principle (generalised reciprocity law) as follows:

Let $\mathbb{T}$ be a free R-module of rank one with compatible $G_{\mathbb{Q}_p}$-action as above. Then

$$\mathbb{T}^* := \mathrm{Hom}_R(\mathbb{T}, R)$$

is a free R°-module of rank one — for the action $h \mapsto h(-)r$, r in the opposite ring R° of R — with compatible $G_{\mathbb{Q}_p}$-action given by $h \mapsto h \circ \sigma^{-1}$. Recall that in Iwasawa theory we have the canonical involution $\iota : \Lambda^\circ \to \Lambda$, induced by $g \mapsto g^{-1}$, which allows us to consider (left) Λ°-modules again as (left) Λ-modules; for example, one has $\mathbb{T}^*(T)^\iota \cong \mathbb{T}(T^*)$ as $(\Lambda, G_{\mathbb{Q}_p})$-modules, where $M^\iota := \Lambda \otimes_{\iota,\Lambda^\circ} M$ denotes the Λ-module with underlying abelian group M, but on which $g \in G$ acts as g^{-1} for any Λ°-module M.

Given $\epsilon'_{R^\circ,-\epsilon}(\mathbb{T}^*(1))$ we may apply the dualising functor $-^*$ (compare (B.j) in Appendix B) to obtain an isomorphism

$$\epsilon'_{R^\circ,-\epsilon}(\mathbb{T}^*(1))^* : (\mathbf{d}_{R^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)))_{\widetilde{R^\circ}})^*(\mathbf{d}_{R^\circ}(\mathbb{T}^*(1))_{\widetilde{R^\circ}})^* \to \mathbf{1}_{\widetilde{R^\circ}},$$

while the local Tate-duality isomorphism [Fukaya and Kato 2006, §1.6.12]

$$\psi(\mathbb{T}) : R\Gamma(\mathbb{Q}_p, \mathbb{T}) \cong R\mathrm{Hom}_{R^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), R^\circ)[-2]$$

induces an isomorphism

$$\overline{\mathbf{d}_R(\psi(\mathbb{T}))_{\widetilde{R}}}^{-1} : \big((\mathbf{d}_{R^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)))_{\widetilde{R^\circ}})^*\big)^{-1}$$
$$\cong \mathbf{d}_R(R\mathrm{Hom}_{R^\circ}(R\Gamma(\mathbb{Q}_p, \mathbb{T}^*(1)), R^\circ))^{-1}_{\widetilde{R}} \to \mathbf{d}_R(R\Gamma(\mathbb{Q}_p, \mathbb{T}))^{-1}_{\widetilde{R}}, \quad (58)$$

in the notation of Remark B.1. Consider the product

$$\epsilon'_{R,\epsilon}(\mathbb{T}) \cdot \epsilon'_{R^\circ,-\epsilon}(\mathbb{T}^*(1))^* \cdot \overline{\mathbf{d}_R(\psi(\mathbb{T}))_{\widetilde{R}}}^{-1} : \mathbf{d}_R(\mathbb{T}(-1))_{\widetilde{R}} \cong \mathbf{d}_R(\mathbb{T}^*(1)^*)_{\widetilde{R}} \to \mathbf{d}_R(\mathbb{T})_{\widetilde{R}}$$

and the isomorphism $\mathbb{T}(-1) \xrightarrow{\cdot\epsilon} \mathbb{T}$ that sends $t \otimes \epsilon^{\otimes -1}$ to t.

Proposition 2.10 (duality). *Let $\mathbb{T}$ be as above and such that $\mathbb{T} \cong Y \otimes_\Lambda \mathbb{T}_{un}$ for some (R, Λ)-bimodule Y that is projective as R-module. Then*

$$\epsilon'_{R,\epsilon}(\mathbb{T}) \cdot \epsilon'_{R^\circ,-\epsilon}(\mathbb{T}^*(1))^* \cdot \overline{\mathbf{d}_R(\psi(\mathbb{T}))_{\widetilde{R}}}^{-1} = \mathbf{d}_R\big(\ \mathbb{T}(-1) \xrightarrow{\cdot\epsilon} \mathbb{T}\ \big)_{\widetilde{R}}.$$

Proof. The statement is stable under applying $Y' \otimes_R -$ for some (R', R)-bimodule Y', which is projective as an R'-module by the functoriality of local Tate duality and the lemma below. This reduces the proof to the case $(R, \mathbb{T}) = (\Lambda, \mathbb{T}(T))$, where $T = \mathbb{Z}_p(r)(\eta)$ is generic. Since the morphisms between $\mathbf{d}_R(\mathbb{T}(-1))_{\widetilde{R}}$ and $\mathbf{d}_R(\mathbb{T})_{\widetilde{R}}$ form a $K_1(\widetilde{\Lambda})$-torsor and the kernel

$$SK_1(\widetilde{\Lambda}) := \ker\left(K_1(\widetilde{\Lambda}) \to \prod_{\rho \in \mathrm{Irr}\, G} K_1(\widetilde{L}_\rho)\right) = 1$$

is trivial (because G is abelian), it suffices to check the statement for all $(L, V(\rho))$, which is nothing else than the content of [Fukaya and Kato 2006, Proposition 3.3.8]. Here $\mathrm{Irr}\, G$ denotes the set of $\overline{\mathbb{Q}}_p$-valued irreducible representations of G with finite image. □

Lemma 2.11. *Let Y be a (R', R)-bimodule such that $Y \otimes_R \mathbb{T} \cong \mathbb{T}'$ as $(R', G_{\mathbb{Q}_p})$-module and let $Y^* = \mathrm{Hom}_{R'}(Y, R')$ the induced (R'°, R°)-bimodule. Then there is a natural equivalence of functors*

$$Y \otimes_R \mathrm{Hom}_{R^\circ}(-, R^\circ) \cong \mathrm{Hom}_{R'^\circ}(Y^* \otimes_{R^\circ} -, R'^\circ)$$

on $P(R^\circ)$, and a natural isomorphism $Y^* \otimes_{R^\circ} \mathbb{T}^* \cong (\mathbb{T}')^*$ of $(R'^\circ, G_{\mathbb{Q}_p})$-modules.

Proof. This is easily checked using the adjointness of Hom and $\otimes$. □

Proposition 2.12 (Change of ϵ). *Let $c \in \mathbb{Z}_p^\times$ and let σ_c be the unique element of the inertia subgroup of $G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$ such that $\sigma_c(\epsilon) = c\epsilon$ (in the $\mathbb{Z}_p$-module $\mathbb{Z}_p(1)$, whence written additively). Then*

$$\epsilon'_{R,c\epsilon}(\mathbb{T}) = [\mathbb{T}, \sigma_c]\epsilon'_{R,\epsilon}(\mathbb{T}).$$

Proof. As in the proof of Proposition 2.10 this is easily reduced to the pairs $(L, V(\rho))$, for which the statement follows from the functorial properties of ϵ-constants [Fukaya and Kato 2006, §3.2.2(2)]. □

Altogether we have proved this:

Theorem 2.13 (Kato, ϵ-isomorphisms). *Let $\mathbb{T}$ be such that $\mathbb{T} \cong Y \otimes_\Lambda \mathbb{T}_{un}$ as $(R, G_{\mathbb{Q}_p})$-modules for some (R, Λ)-bimodule Y which is projective as R-module, where $\Lambda = \Lambda(G)$ with $G = G(L/\mathbb{Q}_p)$ for any $L \subseteq \mathbb{Q}_p^{ab}$. Then a unique epsilon isomorphism $\epsilon'_R(\mathbb{T})$ exists satisfying the twist invariance property (21), the descent property (22), the "change of ϵ" relation (Proposition 2.12) and the duality relation (Proposition 2.10). In particular $\epsilon'_\Lambda(\mathbb{T})$ exists for all pairs $(\Lambda, \mathbb{T})$ with $\mathbb{T} \cong \Lambda$ one-dimensional (free) as a Λ-module.*

Proof. For G a two-dimensional p-adic Lie group this has been shown explicitly above. The general case follows by taking limits. □

We will indicate shortly how this result implies the validity of a local Main Conjecture in this context. Here again we restrict to the universal case $\mathbb{T}_{un}$, but we point out that similar statements hold for general $\mathbb{T}$ as in the above theorem by the twisting principle; in particular it applies to $\mathbb{T}_E$ for the local representation given by a CM elliptic curve as in Example 3.1 below.

We place ourselves in the situation described at the bottom of page 2376; in particular, G is a two-dimensional p-adic Lie group. Denote by

$$S := \{\lambda \in \Lambda \mid \Lambda/\Lambda\lambda \text{ is finitely generated over } \Lambda(G(K_\infty/\mathbb{Q}_{p,\infty}))\}$$

the canonical Ore set of Λ (see [Coates et al. 2005]) and by $\widetilde{S}$ the canonical Ore set of $\widetilde{\Lambda}$. Fix an element u of $\mathbb{U}(K_\infty) = \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))$ such that the map $\Lambda \to \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}(K_\infty))$ taking 1 to u becomes an isomorphism after base change to $\widetilde{\Lambda}_{\widetilde{S}}$ (such "generators" exist according to (19) and Proposition 2.1). Then, with $\mathbb{L} := -\mathscr{L}_{K,\epsilon^{-1}}$, the map

$$\epsilon'_\Lambda(\mathbb{T}_{un}) : \mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(R\Gamma(\mathbb{Q}_p, \mathbb{T}_{un}))_{\widetilde{\Lambda}} \mathbf{d}_\Lambda(\mathbb{T}_{un})_{\widetilde{\Lambda}}$$

induces a map

$$\mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})/\Lambda u)^{-1}_{\widetilde{\Lambda}} \mathbf{d}_\Lambda(\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}))_{\widetilde{\Lambda}} \mathbf{d}_\Lambda(\mathbb{T}_{un}/\mathbb{L}(u))_{\widetilde{\Lambda}} \tag{59}$$

whose base change followed by the canonical trivialisations

$$\mathbf{1}_{\widetilde{\Lambda}_{\widetilde{S}}} \to \mathbf{d}_\Lambda(\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})/\Lambda u)^{-1}_{\widetilde{\Lambda}_{\widetilde{S}}} \mathbf{d}_\Lambda(\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un}))_{\widetilde{\Lambda}_{\widetilde{S}}} \mathbf{d}_\Lambda(\mathbb{T}_{un}/\Lambda\mathbb{L}(u))_{\widetilde{\Lambda}_{\widetilde{S}}}$$
$$\cong \mathbf{d}_{\widetilde{\Lambda}_{\widetilde{S}}}(\widehat{\mathbb{Z}_p^{ur}}) \mathbf{d}_{\widetilde{\Lambda}_{\widetilde{S}}}(\widehat{\mathbb{Z}_p^{ur}}(1)) \to \mathbf{1}_{\widetilde{\Lambda}_{\widetilde{S}}}$$

(here all arguments on the right are $\tilde{S}$-torsion modules!) equals the identity in

$\mathrm{Aut}(\mathbf{1}_{\widetilde{\Lambda}_{\widetilde{S}}}) = K_1(\widetilde{\Lambda}_{\widetilde{S}})$ by Lemma 2.2. Let $\mathscr{E}_u$ be the element in $K_1(\widetilde{\Lambda}_{\widetilde{S}})$ such that

$$\mathbb{L}(u) = \mathscr{E}_u^{-1} \cdot (1 \otimes \epsilon).$$

Consider the connecting homomorphism ∂ in the exact localisation sequence

$$K_1(\widetilde{\Lambda}) \longrightarrow K_1(\widetilde{\Lambda}_{\widetilde{S}}) \xrightarrow{\partial} K_0(\widetilde{S}\text{-tor}) \longrightarrow 0,$$

where $\widetilde{S}$-tor denotes the category of finitely generated $\widetilde{\Lambda}$-modules which are $\widetilde{S}$-torsion. Then we obviously have

$$\partial(\mathscr{E}_u) = -[\mathbb{T}_{un}/\Lambda\mathbb{L}(u)] = [\mathrm{H}^2(\mathbb{Q}_p, \mathbb{T}_{un})] - [\mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un})/\Lambda u]$$

in $K_0(\widetilde{S}\text{-tor})$. Moreover one can evaluate $\mathscr{E}_u$ at Artin characters ρ of G as in [Coates et al. 2005] and derive an interpolation property for $\mathscr{E}(\rho)$ from Theorem 2.13 by the techniques of [Fukaya and Kato 2006, Lemma 4.3.10]; this is carried out in [Schmitt $\geq$ 2013]. These two properties build the local Main Conjecture as suggested by Fukaya and Kato in a much more general, not necessarily commutative setting. Kato (unpublished) has shown that $\widetilde{\Lambda}_{\widetilde{S}} \otimes_{\Lambda} \mathbb{U}(K_\infty) \cong \widetilde{\Lambda}_{\widetilde{S}}$ does hold in vast generality for p-adic Lie extensions.

3. The semilocal case

Let $F_\infty/\mathbb{Q}$ be a p-adic Lie extension with Galois group G and v be any place of F_∞ above p such that $G_v = G(F_{\infty,v}/\mathbb{Q}_p)$ is the decomposition group at v. For any free $\mathbb{Z}_p$-module T of finite rank with continuous Galois action by $G_{\mathbb{Q}}$ we define the free $\Lambda(G)$-module

$$\mathbb{T} := \mathbb{T}(T)_{F_\infty} := \Lambda(G)^\natural \otimes_{\mathbb{Z}_p} T$$

with the usual diagonal $G_{\mathbb{Q}}$-action. Similarly, we define the free $\Lambda(G_v)$-module

$$\mathbb{T}^{\mathrm{loc}} := \Lambda(G_v)^\natural \otimes_{\mathbb{Z}_p} T$$

with the usual diagonal $G_{\mathbb{Q}_p}$-action. Then we have the canonical isomorphism of $(\Lambda(G), G_{\mathbb{Q}_p})$-bimodules

$$\mathbb{T} \cong \Lambda(G) \otimes_{\Lambda(G_v)} \mathbb{T}^{\mathrm{loc}}.$$

Thus we might define

$$\epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T}) := \Lambda(G) \otimes_{\Lambda(G_v)} \epsilon'_{\Lambda(G_v)}(\mathbb{T}^{\mathrm{loc}}) : \mathbf{1}_{\widetilde{\Lambda(G)}} \to \mathbf{d}_{\Lambda(G)}(R\Gamma(\mathbb{Q}_p, \mathbb{T}))_{\widetilde{\Lambda(G)}} \mathbf{d}_{\Lambda(G)}(\mathbb{T})_{\widetilde{\Lambda(G)}}.$$

Now let $\rho : G \to GL_n(\mathbb{O}_L)$ be a continuous map and ρ_v its restriction to G_v, where L is a finite extension of $\mathbb{Q}_p$. By abuse of notation we shall denote the

induced ring homomorphisms $\Lambda(G) \to M_n(\mathbb{O}_L)$ and $\Lambda(G_v) \to M_n(\mathbb{O}_L)$ by the same letters. Since we have a canonical isomorphism

$$L^n \otimes_{\rho,\Lambda(G)} \mathbb{T} \cong L^n \otimes_{\rho_v,\Lambda(G_v)} \mathbb{T}^{\mathrm{loc}}$$

of $(L, G_{\mathbb{Q}_p})$-bimodules, we obtain

$$\begin{aligned} &L^n \otimes_{\rho,\Lambda(G)} \epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T}) \\ &\qquad = L^n \otimes_{\rho_v,\Lambda(G_v)} \epsilon'_{\Lambda(G_v)}(\mathbb{T}^{\mathrm{loc}}) : \mathbf{1}_{\widetilde{L}} \to \mathbf{d}_L(R\Gamma(\mathbb{Q}_p, V(\rho^*)))_{\widetilde{L}} \mathbf{d}_L(V(\rho^*))_{\widetilde{L}}, \end{aligned}$$

where $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Example 3.1. Let E be a elliptic curve defined over $\mathbb{Q}$ with CM by the ring of integers of an imaginary quadratic extension $K \subseteq F_\infty$ of $\mathbb{Q}$ and let ψ denote the Grössencharacter associated to E. Then $T_E \cong \mathrm{Ind}_{\mathbb{Q}}^K T_\psi$, which is isomorphic to $T_\psi \oplus T_{\psi^c}$ as representation of G_K. Here T_ψ equals $\mathbb{Z}_p$ on which $G_{\mathbb{Q}}$ acts via ψ, while ψ^c is the conjugate of ψ by complex multiplication $c \in G(K/\mathbb{Q})$.

Assuming $K_v = \mathbb{Q}_p$ and setting $\mathbb{T}_E := \mathbb{T}$, $\mathbb{T}_E^{\mathrm{loc}} := \mathbb{T}^{\mathrm{loc}}$ for $T = T_E$ as well as $\mathbb{T}_\psi := \Lambda(G)^\natural \otimes_{\mathbb{Z}_p} T_\psi$, $\mathbb{T}_\psi^{\mathrm{loc}} := \Lambda(G_v)^\natural \otimes_{\mathbb{Z}_p} T_\psi$ we obtain

$$\mathbb{T}_E \cong \mathbb{T}_\psi \oplus \mathbb{T}_{\psi^c}$$

as $(\Lambda(G), G_K)$-modules and hence

$$\begin{aligned} \epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T}_E) &= \epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T}_\psi)\epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{T}_{\psi^c}) \\ &= \Lambda(G) \otimes_{\Lambda(G_v)} \left(\epsilon_{\Lambda(G_v)}(\mathbb{Q}_p, \mathbb{T}_\psi^{\mathrm{loc}})\epsilon_{\Lambda(G_v)}(\mathbb{Q}_p, \mathbb{T}_{\psi^c}^{\mathrm{loc}})\right). \end{aligned}$$

If F is a number field and F_∞ a p-adic Lie extension of F again with Galois group G, then, for a place $\mathfrak{p}$ above p and a projective $\Lambda(G)$-module $\mathbb{T}$ with continuous $G_{F_\mathfrak{p}}$-action, we define a corresponding ϵ-isomorphism

$$\epsilon_{\Lambda(G)}(F_\mathfrak{p}, \mathbb{T}) : \mathbf{1}_{\widetilde{\Lambda(G)}} \to \mathbf{d}_{\Lambda(G)}(R\Gamma(F_\mathfrak{p}, \mathbb{T}))_{\widetilde{\Lambda(G)}} \mathbf{d}_{\Lambda(G)}(\mathbb{T})^{[F_\mathfrak{p}:\mathbb{Q}_p]}_{\widetilde{\Lambda(G)}}$$

to be induced from

$$\begin{aligned} &\epsilon_{\Lambda(G)}(\mathbb{Q}_p, \mathbb{Z}[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[G_{F_\mathfrak{p}}]} \mathbb{T}) \\ &: \mathbf{1}_{\widetilde{\Lambda(G)}} \to \mathbf{d}_{\Lambda(G)}(R\Gamma(\mathbb{Q}_p, \mathbb{Z}[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[G_{F_\mathfrak{p}}]} \mathbb{T}))_{\widetilde{\Lambda(G)}} \mathbf{d}_{\Lambda(G)}(\mathbb{Z}[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[G_{F_\mathfrak{p}}]} \mathbb{T})_{\widetilde{\Lambda(G)}}. \end{aligned}$$

Finally we put

$$\epsilon_\Lambda(F \otimes_{\mathbb{Q}} \mathbb{Q}_p, \mathbb{T}) = \epsilon_\Lambda\left(\mathbb{Q}_p, \bigoplus_{\mathfrak{p}|p} \mathbb{Z}[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Z}[G_{F_\mathfrak{p}}]} \mathbb{T}\right) = \prod_{\mathfrak{p}|p} \epsilon_\Lambda(F_\mathfrak{p}, \mathbb{T}),$$

where $\mathfrak{p}$ runs through the places of F above p.

4. Global functional equation

In this section we would like to explain the applications addressed in the introduction. In the same setting as in Example 3.1 we assume that p is a prime of good ordinary reduction for the CM elliptic curve E and we set $F_\infty = \mathbb{Q}(E(p))$, as well as $G = G(F_\infty/\mathbb{Q})$ and $\Lambda := \Lambda(G)$. We write $M = h^1(E)(1)$ for the motive attached to E and set $\epsilon_{p,\Lambda}(M) = \epsilon_\Lambda(\mathbb{Q}_p, \mathbb{T}_E)$. Using [Yasuda 2009] one obtains similarly ϵ-isomorphisms over $\mathbb{Q}_l$, $l \neq p$, which we call analogously $\epsilon_{l,\Lambda}(M)$. Finally, one can define $\epsilon_{\infty,\Lambda}(M)$ also at the place at infinity; this is done in [Fukaya and Kato 2006, §3.5, Conjecture] and, with a slightly different normalisation, at the end of [Venjakob 2007, §5]. We choose the latter normalisation. Let S be the finite set of places of $\mathbb{Q}$ consisting of p, ∞ as well as the places of bad reduction of M.

Now, according to the conjectures of [Fukaya and Kato 2006] there exists a ζ-isomorphism

$$\zeta_\Lambda(M) := \zeta_\Lambda(\mathbb{T}_E) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(R\Gamma_c(U, \mathbb{T}_E))^{-1}$$

which is the global analogue of the ϵ-isomorphism concerning special L-values (at motivic points in the sense of [Flach 2009]) instead of ϵ- and Γ-factors; here $R\Gamma_c(U, \mathbb{T}_E)$ denotes the perfect complex calculating étale cohomology with compact support of $\mathbb{T}_E$ with respect to $U = \mathrm{Spec}(\mathbb{Z}) \setminus S$. Good evidence for the existence of $\zeta_\Lambda(M)$ is given in (loc. cit.) although Flach concentrates on the commutative case, that is, he considers $\Lambda(G(F_\infty/K))$ instead of $\Lambda(G)$; from this the noncommutative version probably follows by similar techniques as in [Bouganis and Venjakob 2010], but as a detailed discussion would lead us too far away from the topic of this article, we just assume the existence here for simplicity. Then we obtain the following:

Theorem 4.1. *There is the functional equation*

$$\zeta_\Lambda(M) = (\overline{\zeta_\Lambda(M)^*})^{-1} \cdot \prod_{v \in S} \epsilon_{v,\Lambda}(M).$$

This result is motivated by [Fukaya and Kato 2006, Conjecture 3.5.5]; for more details see [Venjakob 2007, Theorem 5.11], and compare with [Burns and Flach 2001, §5]. Observe that we used the self-duality $M = M^*(1)$ of M here.

Finally we want to address the application towards the descent result with Burns mentioned in the introduction. If ω denotes the Neron differential of E, we obtain the usual real and complex periods $\Omega_\pm = \int_{\gamma^\pm} \omega$ by integrating along paths $\gamma^\pm$ which generate $H_1(E(\mathbb{C}), \mathbb{Z})^\pm$. We set $R = \{q \text{ prime} : |j(E)|_q > 1\} \cup \{p\}$ and let u, w be the roots of the characteristic polynomial of the action of Frobenius on the Tate module T_E of E, which is

$$1 - a_p T + pT^2 = (1 - uT)(1 - wT), \quad u \in \mathbb{Z}_p^\times.$$

Further let $p^{\mathfrak{f}_p(\rho)}$ be the p-part of the conductor of an Artin representation ρ, while $P_p(\rho, T) = \det(1 - \mathrm{Frob}_p^{-1} T | V_\rho^{I_p})$ describes the Euler factor of ρ at p. We also set $d^{\pm}(\rho) = \dim_{\mathbb{C}} V_\rho^{\pm}$ and denote by ρ^* the contragredient representation of ρ. By $e_p(\rho)$ we denote the local ϵ-factor of ρ at p. In the notation of [Tate 1979] this is $e_p(\rho, \psi(-x), dx_1)$, where ψ is the additive character of $\mathbb{Q}_p$ defined by $x \to \exp(2\pi i x)$ and dx_1 is the Haar measure that gives volume 1 to $\mathbb{Z}_p$. Moreover, we write $R_\infty(\rho^*)$ and $R_p(\rho^*)$ for the complex and p-adic regulators of E twisted by ρ^*. Finally, in order to express special values of complex L-functions in the p-adic world, we fix embeddings of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$.

In [Bouganis and Venjakob 2010, Theorem 2.14] we have shown that as a consequence of the work of Rubin and Yager there exists $\mathscr{L}_E \in K_1(\Lambda_{\mathbb{Z}_p}(G)_S)$ satisfying the interpolation property

$$\mathscr{L}_E(\rho) = \frac{L_R(E, \rho^*, 1)}{\Omega_+^{d^+(\rho)} \Omega_-^{d^-(\rho)}} e_p(\rho) \frac{P_p(\rho, u^{-1})}{P_p(\rho^*, w^{-1})} u^{-\mathfrak{f}_p(\rho)}$$

for all Artin representations ρ of G. Moreover the (slightly noncommutative) Iwasawa Main Conjecture (see [Coates et al. 2005] or Conjecture 1.4 in [loc. cit.]) is true provided that the $\mathfrak{M}_H(G)$ conjecture (see [Coates et al. 2005] or Conjecture 1.2 in [loc. cit.]) holds; for CM elliptic curves this conjecture is equivalent to the vanishing of the cyclotomic μ-invariant of E. In [Burns and Venjakob 2011, Conjecture 7.4/9 and Proposition 7.8] a refined Main Conjecture was formulated requiring the following p-adic BSD-type formula:

At each Artin representation ρ of G (with coefficients in L) the leading term $\mathscr{L}_E^*(\rho)$ of $\mathscr{L}_E$ (as defined in [Burns and Venjakob 2006]) equals

$$(-1)^{r(E)(\rho^*)} \frac{L_R^*(E, \rho^*) R_p(\rho^*)}{\Omega_+^{d_+(\rho)} \Omega_-^{d_-(\rho)} R_\infty(\rho^*)} e_p(\rho) \frac{P_p(\rho, u^{-1})}{P_p(\rho^*, w^{-1})} u^{-\mathfrak{f}_p(\rho)}, \tag{60}$$

where $L_R^*(E, \rho^*)$ is the leading coefficient at $s = 1$ of the L-function $L_R(E, \rho^*, s)$ obtained from the Hasse–Weil L-function of E twisted by ρ^* by removing the Euler factors at R. Here the number $r(E)(\rho^*)$ is defined in [Burns and Venjakob 2011, (51)] (with $M = h^1(E)(1)$) and equals $\dim_{\mathbb{C}_p}(e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)})))$ if the Tate–Shafarevich group $Ш(E/F_\infty^{\ker(\rho)})$ is finite.

We write $X(E/F_\infty)$ for the Pontryagin dual of the (p-primary) Selmer group of E over F_∞.

Theorem 4.2. *Let F be a number field contained in F_∞ and assume that*

(i) *the $\mathfrak{M}_H(G)$ conjecture holds,*

(ii) *$\mathscr{L}_E$ satisfies the refined interpolation property (60), and*

(iii) $X(E/F_\infty)$ *is semisimple at all* ρ *in* $\operatorname{Irr} G_{F/\mathbb{Q}}$ (in the sense of [Burns and Venjakob 2006, Definition 3.11]).

The p-part of the equivariant Tamagawa number conjecture for $(E, \mathbb{Z}[G(F/\mathbb{Q})])$ *is true in this situation. If, moreover, the Tate–Shafarevich group* $Ш(E/F)$ *of* E *over* F *is finite, this implies the p-part of a Birch–Swinnerton-Dyer type formula* (see, for example, [Venjakob 2007, §3.1]).

For more details on the "p-part" of the ETNC and the proof of this result, which uses the existence of (1) as shown in this paper, see [Burns and Venjakob 2011, Theorem 8.4]. Note that due to our semisimplicity assumption combined with Remark 7.6 and Proposition 7.8 of [loc. cit.], formula (60) coincides with that of [loc. cit., Conjecture 7.4]. Also Assumption (W) of Theorem 8.4 is valid for weight reasons. Finally we note that by [Burns and Venjakob 2006, Lemma 3.13, 6.7] $X(E/F_\infty)$ is semisimple at ρ if and only if the p-adic height pairing

$$h_p(V_p(E)\otimes\rho^*) : H^1_f(\mathbb{Q}, V_p(E)\otimes\rho^*) \times H^1_f(\mathbb{Q}, V_p(E)\otimes\rho) \to L$$

from [Nekovář 2006, §11] (see also [Schneider 1982] or [Perrin-Riou 1992]) is nondegenerate, where $V_p(E) = \mathbb{Q}_p \otimes T_E$ is the usual p-adic representation attached to E. As far as we are aware, the only theoretical evidence for nondegeneracy is a result in [Bertrand 1982] that for an elliptic curve with complex multiplication, the height of a point of infinite order is nonzero. Computationally, however, a lot of work has been done recently by Stein and Wuthrich [Wuthrich 2004].

Appendix A: p-adic Hodge theory and (φ, Γ)-modules

As before in the local situation K denotes a (finite) unramified extension of $\mathbb{Q}_p$. Let $\eta : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ (here $\mathbb{Z}_p^\times$ can also be replaced by $\mathbb{O}_L^\times$, but for simplicity of notation we won't do that in this exposition) be an unramified character and let T_0 be the free $\mathbb{Z}_p$-module with basis $t_{\eta,0}$ such that $\sigma \in G_{\mathbb{Q}_p}$ acts via $\sigma t_{\eta,0} = \eta(\sigma)t_{\eta,0}$. More generally, for $r \in \mathbb{Z}$, we consider the $G_{\mathbb{Q}_p}$-module

$$T := T_0(r),$$

which is free as a $\mathbb{Z}_p$-module with basis $t_{\eta,r} := t_{\eta,0} \otimes \epsilon^{\otimes r}$, where $\epsilon = (\epsilon_n)_n$ denotes a fixed generator of $\mathbb{Z}_p(1)$, that is, $\epsilon_n^p = \epsilon_{n-1}$ for all $n \geq 1$, $\epsilon_0 = 1$ and $\epsilon_1 \neq 1$. Thus we have $\sigma(t_{\eta,r}) = \eta(\sigma)\kappa^r(\sigma)t_{\eta,r}$, where $\kappa : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ denotes the p-cyclotomic character. Setting $V := \mathbb{Q}_p \otimes T = V_0(r)$ we obtain for its de Rham filtration

$$D^i_{dR}(V) = \begin{cases} D_{dR}(V) \cong K e_{\eta,r} & \text{if } i \leq -r, \\ 0 & \text{otherwise,} \end{cases} \tag{61}$$

where $e_{\eta,r} := at^{-r} \otimes t_{\eta,r}$ with a unique $a = a_\eta \in \widehat{\mathbb{Z}_p^{ur}}^\times$, such that $\tau_p(a) = \eta^{-1}(\tau_p)a$, see [Serre 1968, Theorem 1, p. III-31]. Here as usual $t = \log[\epsilon] \in B_{\mathrm{cris}} \subseteq B_{dR}$

denotes the p-adic period analogous to $2\pi i$. Furthermore we have

$$D_{\mathrm{cris}}(V) = K e_{\eta,r}$$

with

$$\varphi(e_{\eta,r}) = p^{-r}\eta^{-1}(\tau_p)e_{\eta,r}.$$

If η is trivial, we also write t_r and e_r for $t_{\eta,r}$ and $e_{\eta,r}$, respectively.

Now consider the $\mathbb{O}_K$-lattices

$$M_0 := \mathbb{O}_K e_{\eta,0} = (\widehat{\mathbb{Z}_p^{ur}} \otimes_{\mathbb{Z}_p} T_0)^{G_K} \subseteq D_{\mathrm{cris}}(V_0)$$

and

$$M := (t^{-r} \otimes \epsilon^{\otimes r})M_0 = \mathbb{O}_K e_{\eta,r} \subseteq D_{\mathrm{cris}}(V).$$

Using the variable $X = [\epsilon] - 1$ we have $t = \log(1+X)$ and on the rings

$$\mathbb{O}_K[\![X]\!] \subseteq B^+_{\mathrm{rig},K} := \left\{ f(X) = \sum_{k\geq 0} a_k X^k \;\middle|\; a_k \in K,\ f(X) \text{ converges on } \{x \in \mathbb{C}_p \mid |x|_p < 1\} \right\}$$

we have the following operations: φ is induced by the usual action of ϕ on the coefficients and by $\varphi(X) := (1+X)^p - 1$, while $\gamma \in \Gamma$ acts trivially on coefficients and by $\gamma(X) = (1+X)^{\kappa(\gamma)} - 1$; letting $H_K = G(K/\mathbb{Q}_p)$ act just on the coefficients we obtain a $\Lambda(G)$-module structure on $\mathbb{O}_K[\![X]\!]$. Moreover, φ has a left inverse operator ψ uniquely determined via $\varphi \circ \psi(f) = (1/p)\sum_{\zeta^p=1} f(\zeta(1+X)-1)$. The differential operator $D := (1+X)\,d/dX$ satisfies

$$D\varphi f = p\varphi Df \quad \text{and} \quad D\gamma f = \kappa(\gamma)\gamma Df. \tag{62}$$

It is well-known [Perrin-Riou 1994, Lemma 1.1.6] that D induces an isomorphism of $\mathbb{O}_K[\![X]\!]^{\psi=0}$. Furthermore, setting $\Delta_i f := D^i f(0)$ for $f \in \mathbb{O}_K[\![X]\!]^{\psi=0}$, we have an exact sequence [loc. cit., §2.2.7, (2.1)]

$$\begin{aligned} 0 \longrightarrow t^r \otimes D_{\mathrm{cris}}(V)^{\varphi=p^{-r}} \longrightarrow (B^+_{\mathrm{rig},K} \otimes_K D_{\mathrm{cris}}(V))^{\psi=1} \xrightarrow{1-\varphi} \\ (B^+_{\mathrm{rig},K})^{\psi=0} \otimes_K D_{\mathrm{cris}}(V) \xrightarrow{\Delta_r} (D_{\mathrm{cris}}(V)/(1-p^r\varphi))(r) \longrightarrow 0, \end{aligned} \tag{63}$$

where φ (and ψ) acts diagonally on $B^+_{\mathrm{rig},K} \otimes_K D_{\mathrm{cris}}(V)$, while D operates just on the first tensor factor. We set

$$\mathcal{D}_M := \mathbb{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathbb{O}_K} M,$$

and denote by

$$D(T) = (\mathbb{A} \otimes_{\mathbb{Z}_p} T)^{G_{K_\infty}}$$

the (φ, Γ)-module attached to T, where the definition of the ring $\mathbb{A}$ together with its φ- and Γ-action can be found for example in [Berger 2003]. Here we only recall

that $\mathbb{A}_K^+ \cong \mathcal{O}_K[\![X]\!]$ and $\mathbb{A}_K \cong (\mathcal{O}_K[\![X]\!][1/X])^{\wedge p\text{-adic}}$ is the p-adic completion of the Laurent series ring.

Remark A.1. (i) Let η be nontrivial. From [Berger 2003, Theorem A.3] and its proof one sees immediately that for the Wach module $N(T_0)$, which according to Proposition A.1 of [loc. cit.] equals $\mathcal{O}_K[\![X]\!] \otimes_{\mathcal{O}_K} M_0$, the natural inclusion $N(T_0) \hookrightarrow \mathbb{A}_K \otimes_{\mathbb{A}_K^+} N(T_0)$ induces an isomorphism

$$(\mathcal{O}_K[\![X]\!] \otimes_{\mathcal{O}_K} M_0)^{\psi=1} \xrightarrow{\cong} N(T_0)^{\psi=1} \xrightarrow{\cong} \left(\mathbb{A}_K \otimes_{\mathbb{A}_K^+} N(T_0)\right)^{\psi=1} \cong D(T_0)^{\psi=1}.$$

(ii) If η is trivial, one has similarly $N(\mathbb{Z}_p) = \mathbb{A}_K^+ = \mathcal{O}_K[\![X]\!]$ by the same Proposition A.1, whence $N(\mathbb{Z}_p(1)) = X^{-1}\mathbb{A}_K^+ \otimes t_1 = X^{-1}\mathcal{O}_K[\![X]\!] \otimes t_1$ by the usual twist behaviour of Wach modules. We obtain

$$\begin{aligned} D(\mathbb{Z}_p(1))^{\psi=1} \cong N(\mathbb{Z}_p(1))^{\psi=1} &= (X^{-1}\mathcal{O}_K[\![X]\!] \otimes t_1)^{\psi=1} \\ &= \mathbb{Z}_p X^{-1} \otimes t_1 \oplus (\mathcal{O}_K[\![X]\!] \otimes t_1)^{\psi=1}, \end{aligned}$$

but $N(\mathbb{Z}_p)^{\psi=1} \ncong D(\mathbb{Z}_p)^{\psi=1}$ according to Proposition A.3 of [loc. cit.].

We define $\tilde{D}(\mathbb{Z}_p(r))^{\psi=1} = (\mathcal{O}_K[\![X]\!] \otimes t_r)^{\psi=1}$ and $\tilde{D}(T)^{\psi=1} = D(T)^{\psi=1}$ for nontrivial η and obtain a canonical isomorphism

$$(\mathcal{O}_K[\![X]\!] \otimes_{\mathcal{O}_K} M)^{\psi=p^r} \cong \tilde{D}(T)^{\psi=1} \tag{64}$$

induced by multiplication with t^r:

$$f(X) \otimes (oat^{-r} \otimes t_{\eta,r}) \mapsto f(X)oa \otimes t_{\eta,r},$$

where $o \in \mathcal{O}_K$ and a is as before.

Setting $\mathbb{T}_{K_\infty} := \mathbb{T}_{K_\infty}(T) := \Lambda(G(K_\infty/\mathbb{Q}_p))^\sharp \otimes_{\mathbb{Z}_p} T$ we recall that there is a canonical isomorphism due to Fontaine

$$D(T)^{\psi=1} \cong \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}), \tag{65}$$

which for example is called $\{h^1_{K_n,V}\}_n$ in [Berger 2003] and its inverse $\mathrm{Log}^*_{T^*(1)}$ in [Cherbonnier and Colmez 1999, Remark II.1.4].

I am very grateful to Denis Benois for parts of the proof of the following proposition, which has been stated in [Perrin-Riou 1994, Proposition 4.1.3] in a slightly different form, but without proof.[1]

Proposition A.2. (i) *There is a canonical exact sequence of $\mathcal{O}_K$-modules*

$$0 \to 1 \otimes M^{\varphi=p^{-r}} \to (\mathcal{O}_K[\![X]\!] \otimes_{\mathcal{O}_K} M)^{\psi=p^r} \to \mathcal{D}_M \xrightarrow{\Delta_{M,r}} M/(1-p^r\varphi)M \to 0,$$

where the map in the middle is induced by $1-\varphi$ up to twisting (see the first diagram

[1] As twisting with the cyclotomic character starting from $\mathbb{Q}_p(1)$ only recovers the representations $V = \mathbb{Q}_p(r)$, the general case where V_0 is nontrivial is not covered in that reference.

in the proof below).

(ii) *Assume that η is nontrivial. Then, using the isomorphisms (64) and (65) we obtain the following commutative diagram of $\Lambda(G)$-modules, in which the maps $\mathscr{C}(\mathbb{T}_{K_\infty})$ $(= (D^{-r} \otimes t^{-r})(1-\varphi))$ and $\mathscr{L}_0(\mathbb{T}_{K_\infty})$ are defined by the property that the rows become isomorphic to the exact sequence in* (i):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D(T)^{\varphi=1} & \longrightarrow & D(T)^{\psi=1} & \xrightarrow{\mathscr{C}(\mathbb{T}_{K_\infty})} & \mathscr{D}_M & \xrightarrow{\Delta_{M,r}} & M/(1-p^r\varphi)M \to 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow= & & \downarrow= \\
0 & \to & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty})_{tors} & \to & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}) & \xrightarrow{\mathscr{L}_0(\mathbb{T}_{K_\infty})} & \mathscr{D}_M & \xrightarrow{\Delta_{M,r}} & M/(1-p^r\varphi)M \to 0.
\end{array}$$

(iii) *The sequence (15) can be interpreted in terms of (φ, Γ)-modules by the commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{U}(K_\infty) & \xrightarrow{\delta} & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{un}) \cong \varprojlim_n \widehat{K_n^\times} & \xrightarrow{-\hat{v}} & \mathbb{Z}_p & \longrightarrow & 0 \\
 & & \downarrow\cong \mathrm{Dlog}\, g_- & & \downarrow\cong & & \| & & \\
0 & \longrightarrow & \tilde{D}(\mathbb{Z}_p(1))^{\psi=1} & \longrightarrow & D(\mathbb{Z}_p(1))^{\psi=1} & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 0,
\end{array}$$

where δ denotes the Kummer map and $\hat{v}$ is induced from the normalised valuation map. Furthermore, we obtain again a commutative diagram of $\Lambda(G)$-modules, in which the maps $\mathscr{C}(\mathbb{T}_{K_\infty})$ $(= (D^{-r} \otimes t^{-r})(1-\varphi))$ and $\mathscr{L}_0(\mathbb{T}_{K_\infty})$ are defined by the property that the rows become isomorphic to the exact sequence in (i):

$$\begin{array}{ccccccccc}
0 & \to & \tilde{D}(\mathbb{Z}_p(r))^{\varphi=1} & \to & \tilde{D}(\mathbb{Z}_p(r))^{\psi=1} & \xrightarrow{\mathscr{C}(\mathbb{T}_{K_\infty})} & \mathscr{D}_M & \xrightarrow{\Delta_{M,r}} & M/(1-p^r\varphi)M \to 0 \\
 & & \uparrow\cong & & \cong\uparrow \mathrm{Dlog}\, g_- & & \| & & \uparrow\cong \\
0 & \longrightarrow & \mathbb{Z}_p(r) & \longrightarrow & \mathbb{U}(K_\infty)(r-1) & \xrightarrow{\mathscr{L}_0(\mathbb{T}_{K_\infty})} & \mathscr{D}_M & \xrightarrow{\Delta_{M,r}} & \mathbb{Z}_p(r) \longrightarrow 0.
\end{array}$$

Hence, using the map $\mathfrak{M} \otimes e_1 : \mathbb{O}_K[\![\Gamma]\!] \xrightarrow{\cong} \mathscr{D}_M$, $\lambda \mapsto \lambda \cdot (1+X) \otimes e_1$, *where $\mathfrak{M}$ denotes the Mahler (or p-adic Mellin) transform* (see [Coates and Sujatha 2006, Theorem 3.3.3]), *the lower sequence can be canonically identified with Coleman's exact sequence (4): $\mathscr{L}_0(\mathbb{T}_{K_\infty}) = (\mathfrak{M} \otimes e_1) \circ \mathrm{Col}_\epsilon$.*

Proof. The exactness in (i) for M_0 can be checked as follows. Let $f(X) \otimes e_{\eta,0}$ be in ${\mathscr{D}_{M_0}}^{\Delta_{M_0,0}=0}$, that is, $f(0)e_{\eta,0} = (1-\varphi)b$ for some $b \in M_0$. Hence

$$(f(X) - f(0)) \otimes e_{\eta,0} = Xg(X) \otimes e_{\eta,0}$$

for some $g \in \mathbb{O}_K[\![X]\!]$ and

$$F' := (1-\varphi)^{-1}(Xg(X) \otimes e_{\eta,0}) := \sum_{i \geq 0} \varphi^i (Xg(X) \otimes e_{\eta,0}) \in \mathbb{O}_K[\![X]\!] \otimes M_0$$

is a well-defined element. Setting $F := F' + b$ we have $(1-\varphi)F = f(X) \otimes e_{\eta,0}$ as desired. Now exactness follows from (63). The general case follows from the following commutative "twist diagram" of $\mathbb{O}_K$-modules:

$$\begin{array}{ccccccccc}
0 & \to & 1 \otimes M^{\varphi=p^{-r}} & \to & (\mathbb{O}_K[\![X]\!] \otimes_{\mathbb{O}_K} M)^{\psi=p^r} & \longrightarrow & \mathcal{D}_M & \xrightarrow{\Delta_{M,r}} & M/(1-p^r\varphi)M \to 0 \\
 & & \downarrow{\scriptstyle 1\otimes(t^r\otimes\epsilon^{\otimes -r})}\cong & & \downarrow{\scriptstyle 1\otimes(t^r\otimes\epsilon^{\otimes -r})}\cong & & \downarrow{\scriptstyle D^r\otimes(t^r\otimes\epsilon^{\otimes -r})}\cong & & \downarrow{\scriptstyle t^r\otimes\epsilon^{\otimes -r}}\cong \\
0 & \to & 1 \otimes M_0^{\varphi=1} & \to & (\mathbb{O}_K[\![X]\!] \otimes_{\mathbb{O}_K} M_0)^{\psi=1} & \xrightarrow{1-\varphi} & \mathcal{D}_{M_0} & \xrightarrow{\Delta_{M_0,0}} & M_0/(1-p^r\varphi)M_0 \to 0.
\end{array}$$

Item (ii) is clear from the fact that $D(T) = \mathbb{A}_K \cdot a \otimes t_{\eta,r}$, which can either be calculated directly or deduced from the above remark. The statement about the torsion (first vertical isomorphism) follows from [Colmez 2004, Theorem 5.3.15].

For (iii) first note that by [Cherbonnier and Colmez 1999, Proposition V.3.2(iii)] we have a commutative diagram

$$\begin{array}{ccccc}
\mathbb{U}(K_\infty) & \xrightarrow{\Upsilon} & D(\mathbb{Z}_p(1))^{\psi=1} & = & D(\mathbb{Z}_p)^{\psi=1}(1) \\
 & \searrow^{\delta} & \downarrow{\scriptstyle \mathrm{Log}^*_{\mathbb{Q}_p}} & & \\
 & & \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(\mathbb{Z}_p(1))), & &
\end{array}$$

where Υ maps u to $(Dg_u/g_u) \otimes t_1 = \mathrm{Dlog}\, g_u \otimes t_1$. The statements concerning the first diagram follow easily, see also [Colmez 2004, §7.2]. The second diagram follows as above. By construction the composite

$$\mathbb{U}(K_\infty) \to D(\mathbb{Z}_p(1))^{\psi=1} \to \mathcal{D}_M$$

maps $u = (u_n)_n$ to

$$\begin{aligned}
\left(D^{-1}(1-\varphi)\,\mathrm{Dlog}\, g_u\right) \otimes e_1 &= \left((1-p^{-1}\varphi)\log g_u\right) \otimes e_1 \\
&= (1-\varphi)\,(\log g_u \otimes e_1) \\
&= \mathcal{L}(g_u) \otimes e_1 \\
&= \mathrm{Col}(u) \cdot (1+X) \otimes e_1,
\end{aligned}$$

where $\mathcal{L}$ was defined in (5). This implies the last statement. □

Now let K be again a finite extension of degree d_K over $\mathbb{Q}_p$. For a uniform treatment we define

$$\tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) := \begin{cases} \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) & \text{if } \eta \neq \mathbb{1}, \\ \mathbb{U}(K_\infty)(r-1) & \text{if } T = \mathbb{Z}_p(r). \end{cases}$$

Now set

$$\mathcal{H}_M := \{F \in B^+_{\mathrm{rig},K} \otimes_{\mathbb{O}_K} M \mid (1-\varphi)f \in \mathcal{D}_M\}.$$

Using [Berger 2003, Theorem II.11] and the commutativity of the diagram

$$\begin{array}{ccc} \mathscr{H}_M & \xrightarrow{1-\varphi} & \mathscr{D}_M^{\Delta_{M,r}=0} \\ \downarrow{\scriptstyle D^r\otimes(t^r\otimes\epsilon^{\otimes -r})} & & \downarrow{\scriptstyle D^r\otimes(t^r\otimes\epsilon^{\otimes -r})} \\ (\mathcal{O}_K[\![X]\!]\otimes_{\mathcal{O}_K} M_0)^{\psi=1} & \xrightarrow{1-\varphi} & \mathscr{D}_{M_0}^{\Delta_{M_0,0}=0} \end{array}$$

we see that the map $\mathscr{L}_0(\mathbb{T}_{K_\infty})$ coincides with the "inverse" of Perrin-Riou's [1999] large exponential map

$$\Omega_{T,r} : \mathscr{D}_M^{\Delta_{M,r}=0} \to D(T)^{\psi=1}/T^{H_K} \quad (\cong \mathrm{H}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty})/T^{H_K}),$$

(which is $(-1)^{r-1}$ times the one in [Perrin-Riou 1994]). This map sends f to $(D^r \otimes t^r)F$, where $F \in \mathscr{H}_M$ satisfies $(1-\varphi)F = f$. Here "$D^r \otimes t^r$" denotes the composite

$$\mathscr{H}_M \xrightarrow{D^r\otimes(t^r\otimes\epsilon^{\otimes -r})} (\mathcal{O}_K[\![X]\!]\otimes_{\mathcal{O}_K} M_0)^{\psi=1} \xrightarrow{1\otimes(t^{-r}\otimes\epsilon^{\otimes r})} (\mathcal{O}_K[\![X]\!]\otimes_{\mathcal{O}_K} M)^{\psi=p^r} \xrightarrow{t^r} D(T)^{\psi=1}$$

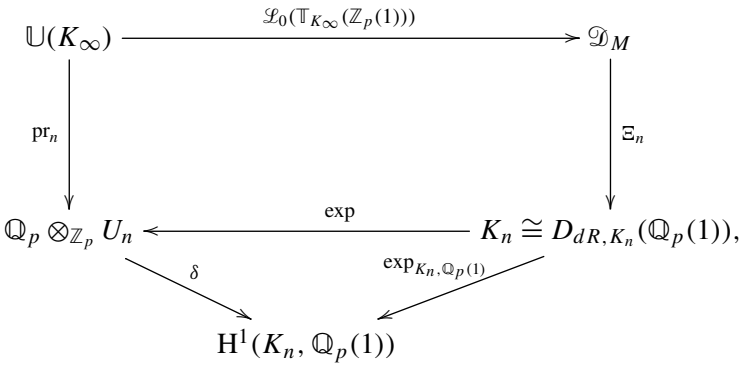
and corresponds to the operator $\nabla_{r-1}\circ\cdots\circ\nabla_0$ in [Berger 2003] for $r \geq 1$. In particular, by Theorem II.10/13 of the same reference we obtain the following descent diagram for $r, n \geq 1$, where the maps $\Xi_{M,n} = \Xi^{\epsilon}_{M,n}$ are recalled in (71):

$$\begin{array}{ccc} \tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) & \xrightarrow{\mathscr{L}_0(\mathbb{T}_{K_\infty}(T))} & \mathscr{D}_M \\ \downarrow{\scriptstyle \mathrm{pr}_n} & & \downarrow{\scriptstyle \Xi_{M,n}} \\ \mathrm{H}^1(K_n, V) & \xleftarrow{(-1)^{r-1}(r-1)!\exp_{K_n,V}} & K_n \cong D_{dR,K_n}(V), \end{array} \tag{66}$$

while for $r \leq 0$

$$\begin{array}{ccc} \tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) & \xrightarrow{\mathscr{L}_0(\mathbb{T}_{K_\infty}(T))} & \mathscr{D}_M \\ \downarrow{\scriptstyle \mathrm{pr}_n} & & \downarrow{\scriptstyle \Xi_{M,n}} \\ \mathrm{H}^1(K_n, V) & \xrightarrow{(-r)!\exp^*_{K_n,V^*(1)}} & K_n \cong D_{dR,K_n}(V). \end{array} \tag{67}$$

Remark A.3. In particular, for $T = \mathbb{Z}_p(1)$ we have the following commutative descent diagram for $n \geq 1$:



where exp denotes the usual p-adic exponential (series), while Ξ_n maps the element $((1-p^{-1}\varphi)\log g_u)\otimes e_1$ to $\log g_u^{\phi^{-n}}(\epsilon_n - 1) = \log u_n$.

In order to arrive at a morphism

$$\mathscr{L}(\mathbb{T}_{K_\infty}(T)) : \tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) \to \mathbb{T}_{K_\infty}(T) \otimes_\Lambda \Lambda_{[\mathbb{T}(T),\tau_p]^{-1}},$$

where $[\mathbb{T}, \tau_p]^{-1} = \tau_p \eta^{-1}(\tau_p)$, generalising $\mathscr{L}_{K,\epsilon}$ in (6), we compose $\mathscr{L}_0(\mathbb{T}_{K_\infty}(T))$ with the following canonical isomorphisms:

$$\mathscr{D}_M = \mathbb{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathbb{O}_K} M \xleftarrow[\cong]{\Psi_M} \mathbb{O}_K[\![\Gamma]\!] \otimes_{\mathbb{O}_K} M \xrightarrow[\cong]{\Theta_M} \mathbb{T}_{K_\infty} \otimes_\Lambda \Lambda_{[\mathbb{T},\tau_p]^{-1}}, \quad (68)$$

where the left one, $\Psi_M(\lambda \otimes m) = \lambda \cdot (1+X) \otimes m$, is induced by $\mathfrak{M}$, while the right one is given by

$$\begin{aligned}\Theta_M(\lambda \otimes (at^{-r} \otimes t_{\eta,r})) &= (1 \otimes t_{\eta,r}) \otimes \left(\sum_{i=0}^{d_K-1} \tau_p^i \otimes \eta^{-i}(\tau_p)\phi^{-i}(\lambda a) \right) \\ &= (1 \otimes t_{\eta,r}) \otimes \left(\sum_i \tau_p^i \otimes \phi^{-i}(\lambda) a \right).\end{aligned}$$

Similarly to the original Coleman map Col in (4), the homomorphisms $\mathscr{C}(\mathbb{T}_{K_\infty})$, $\mathscr{L}_0(\mathbb{T}_{K_\infty})$ and $\mathscr{L}(\mathbb{T}_{K_\infty})$ are norm compatible when enlarging K within $\mathbb{Q}_p^{ur}$. Thus, by taking inverse limits we may and do define them also for infinite unramified extensions K of $\mathbb{Q}_p$. Then we have the following twist and descent properties:

Lemma A.4. *Let $K' \subseteq K$ be (possibly infinite) unramified extensions of $\mathbb{Q}_p$ and Y a $(\Lambda(G(K'_\infty/\mathbb{Q}_p)), \Lambda(G(K_\infty/\mathbb{Q}_p))$-module such that $Y \otimes_{\Lambda(G(K_\infty/\mathbb{Q}_p))} \mathbb{T}_{K_\infty}(T) \cong \mathbb{T}_{K'_\infty}(T')$ as $\Lambda(G(K'_\infty/\mathbb{Q}_p))$-modules with compatible $G_{\mathbb{Q}_p}$-action. Then*

$$Y \otimes_{\Lambda(G(K_\infty/\mathbb{Q}_p))} \mathscr{L}_0(\mathbb{T}_{K_\infty}(T)) = \mathscr{L}_0(\mathbb{T}_{K'_\infty}(T'))$$

and

$$Y \otimes_{\Lambda(G(K_\infty/\mathbb{Q}_p))} \mathscr{L}(\mathbb{T}_{K_\infty}(T)) = \mathscr{L}(\mathbb{T}_{K'_\infty}(T')).$$

In particular, $\mathscr{L}(\mathbb{T}_{K'_\infty}(T)) = \mathscr{L}_{\mathbb{T}_{K'_\infty}(T),\epsilon}$ in (43).

Proof. The proof can be divided into a twist statement, where $K' = K$ and $T' \cong T \otimes_{\mathbb{Z}_p} T''$, such that $G_{\mathbb{Q}_p}$ acts diagonally on the tensor product and T' is a rank-one $\mathbb{Z}_p$-representation of G, and a descent statement. One first proves the twist statement for T''/p^n, n fix, and all finite subextensions K' of K, such that $G(K/K')$ acts trivially on T''/p^n. Afterwards one takes limits over K' obtaining the twist statement for T''/p^n. Then, taking the projective limit with respect to n (see [Berger 2004] for the correct behaviour of (φ, Γ)-modules under such limits) one shows the full twist statement (compare with the well-known twisting for H^i_{IW}). The descent statement then follows easily from the norm compatibility and the fact that the twisted analogue of the exact sequence (10)

$$0 \to \tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K_\infty}(T)) \xrightarrow{\mathscr{L}(\mathbb{T}_{K_\infty})} \mathbb{T}_{K_\infty}(T) \otimes_\Lambda \Lambda_{[\mathbb{T}(T),\tau_p]^{-1}} \longrightarrow T \to 0$$

recovers (for finite extension K' of $\mathbb{Q}_p$) the exact sequence

$$0 \to T^{G(K/K')} \to \tilde{\mathrm{H}}^1(\mathbb{Q}_p, \mathbb{T}_{K'_\infty}(T)) \xrightarrow{\mathscr{L}(\mathbb{T}_{K'_\infty})} \mathbb{T}_{K'_\infty}(T) \otimes_\Lambda \Lambda_{[\mathbb{T}(T),\tau_p]^{-1}} \to T \to 0$$

by taking $G(K/K')$-coinvariants. We explain the unramified twist in more detail (the cyclotomic twist being well known): Assume that η factorises over $G(K/\mathbb{Q}_p)$, that is, $a = a_\eta \in \mathbb{O}_K^\times$, and let $N := \mathbb{O}_K e_r \subseteq D_{\mathrm{cris}}(\mathbb{Q}_p(r))$ be the lattice associated to $\mathbb{Q}_p(r)$. Then we have the following commutative diagram of Λ-modules:

$$\begin{array}{ccccc}
\big(\mathbb{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathbb{O}_K} N\big) \otimes_{\mathbb{Z}_p} T_0 & \xleftarrow{\Psi_N \otimes T_0} & (\mathbb{O}_K[\![\Gamma]\!] \otimes N) \otimes_{\mathbb{Z}_p} T_0 & \xrightarrow{\Theta_N \otimes T_0} & \Lambda \otimes_{\Lambda,f} \mathbb{T}(\mathbb{Z}_p(r)) \otimes \Lambda_{\tau_p} \\
\downarrow{\scriptstyle a^{-1}\otimes a \otimes 1} & & \downarrow{\scriptstyle a^{-1}\otimes a\otimes 1} & & \downarrow{\scriptstyle \vartheta \otimes \tilde{f}} \\
\mathbb{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathbb{O}_K} M & \xleftarrow{\Psi_M} & \mathbb{O}_K[\![\Gamma]\!] \otimes M & \xrightarrow{\Theta_M} & \mathbb{T}(T) \otimes_\Lambda \Lambda_{\tau_p \eta(\tau_p)^{-1}},
\end{array}$$

where in the top line the Λ-action is induced by the diagonal G-action and via left multiplication on Λ, respectively,

$$\Theta_N \otimes T_0(\lambda \otimes (t^{-r} \otimes t_r) \otimes t_{\eta,0}) = 1 \otimes 1 \otimes t_r \otimes \sum_i \tau_p^i \otimes \phi^{-i}(\lambda)$$

and $\tilde{f} := f \otimes 1$ on $\Lambda \widehat{\otimes} \widehat{\mathbb{Z}_p^{ur}}$ is induced by $f : \Lambda \to \Lambda$, $g \mapsto \eta(g)^{-1} g$, while

$$\vartheta : \Lambda \otimes_{\Lambda,f} \mathbb{T}(\mathbb{Z}_p(r)) \to \mathbb{T}(T), \quad a \otimes (b \otimes t_r) \mapsto a f(b) \otimes t_{\eta,r}.$$

Here $\Lambda \otimes_{\Lambda,f} -$ indicates that the tensor product is formed with respect to f. Also we have the commutative diagram

$$\begin{array}{ccc}
D(\mathbb{Z}_p(r))^{\psi=1} \otimes T_0 & \xrightarrow{\mathscr{C}(\mathbb{T}_{K_\infty}(\mathbb{Z}_p(r)))} & \big(\mathbb{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathbb{O}_K} N\big) \otimes_{\mathbb{Z}_p} T_0 \\
\cong \downarrow & & \downarrow a^{-1} \\
D(T)^{\psi=1} & \xrightarrow{\mathscr{C}(\mathbb{T}_{K_\infty}(T))} & \mathbb{O}_K[\![X]\!]^{\psi=0} \otimes_{\mathbb{O}_K} M.
\end{array}$$

□

As on page 2386 we set $\Lambda' = \mathbb{Q}_p[G_n]$.

Lemma A.5. *There are natural isomorphisms*

(i) $\Sigma_{M,n} : K_n' \otimes M = K_n'(at^{-r} \otimes t_{r,\eta}) \cong D_{dR}(V')$ *of* Λ'*-modules*;

(ii) $1 \otimes \Sigma_{M,n} : V_{\rho^*} \otimes_{\Lambda'} K_n' \otimes M \cong V_{\rho^*} \otimes_{\Lambda'} D_{dR}(V') \cong D_{dR}(W)$ *of* L*-vector spaces.*

Proof. The canonical isomorphism (which makes explicit the general formula $(\mathrm{Ind}_G^H(B \otimes V)) \cong (B \otimes \mathrm{Ind} V)$)

$$\mathbb{Q}_p[G_{\mathbb{Q}_p}] \otimes_{\mathbb{Q}_p[G_{K_n'}]} \left(B_{dR} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)(r)\right) \cong B_{dR} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[G_n]^{\sharp} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)(r),$$

which maps $g \otimes a \otimes b$ to $ga \otimes \bar{g}^{-1} \otimes gb$ with $g \in G_{\mathbb{Q}_p}$ induces the isomorphism (via the general isomorphism $N^H \cong (\mathrm{Ind}_G^H N)^G$, $n \mapsto \sum_{\bar{g} \in G/H} g \otimes n$)

$$K_n' \cdot (at^{-r} \otimes t_{r,\eta}) = \left(B_{dR} \otimes \mathbb{Q}_p(\eta)(r)\right)^{G_{K_n'}} \cong D_{dR}(V'),$$

which maps $x \cdot at^{-r} \otimes t_{r,\eta}$ to

$$\sum_{g \in G_n} g(xat^{-r}) \otimes g^{-1} \otimes gt_{r,\eta} = \sum_{g \in G_n} g(x)at^{-r} \otimes g^{-1} \otimes t_{r,\eta}. \tag{69}$$

Putting $e_{\eta,r} := at^{-r} \otimes t_{r,\eta}$ we similarly obtain the isomorphism in (ii) sending $l \otimes x \otimes e_{\eta,r}$ to

$$\sum_{g \in G_n} g(x)at^{-r} \otimes \rho(g)l \otimes t_{r,\eta},$$

where this element is regarded in $B_{dR} \otimes_{\mathbb{Q}_p} W = B_{dR} \otimes_{\mathbb{Q}_p} L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)(r)$. Alternatively we can read it in $(B_{dR} \otimes_{\mathbb{Q}_p} L) \otimes_L W$ as

$$\#G_n at^{-r} e_{\rho^*}(x)l \otimes t_{\rho\eta,r}. \qquad \square$$

Any embedding $\sigma : L_\rho \to \overline{\mathbb{Q}}_p$ induces a map $A_\rho := \mathbb{Q}_p^{nr} \otimes_{\mathbb{Q}_p} L_\rho \to \overline{\mathbb{Q}}_p$ taking $x \otimes y$ to $x\sigma(y)$; we still call this map σ.

Consider the Weil group $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, which fits into a short exact sequence

$$1 \to I \longrightarrow W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \xrightarrow{v} \mathbb{Z} \to 0,$$

and let D be the linearised $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-module associated to $D_{pst}(W) = A_\rho e_{\eta,r}(\rho)$, that is, $g \in W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts as $g_{\mathrm{old}}\varphi^{-v(g)}$ or explicitly via the character

$$\chi_D(g) := \rho(g)\eta(\tau_p)^{v(g)}p^{rv(g)}.$$

For an embedding σ we write $\bar{D}_\sigma := \overline{\mathbb{Q}}_p \otimes_{A_\rho,\sigma} D \cong \overline{\mathbb{Q}}_p e_{\eta,r}(\rho^\sigma)$, where σ acts coefficientwise on ρ. If $n \geq 0$ is minimal with the property that $G(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{nr}(\mu(p^n)))$

acts trivially on $\bar{D}_\sigma$, then by properties (3) and (7) in [Fukaya and Kato 2006, §3.2.2][2] we obtain for the epsilon constant attached to $\bar{D}_\sigma$ (see loc. cit.)

$$\epsilon(\bar{D}_\sigma, -\psi) = 1$$

if $n = 0$, while for $n \geq 1$

$$\begin{aligned}\epsilon(\bar{D}_\sigma, -\psi) &= \epsilon(\bar{D}_\sigma^*(1), \psi)^{-1} \\ &= \left((\rho^\sigma \eta(\tau_p) p^{r-1})^n \sum_{\gamma \in \Gamma_n} \rho^\sigma(\gamma)\gamma \cdot \epsilon_n\right)^{-1} \\ &= \left((\rho^\sigma \eta(\tau_p) p^{r-1})^n \tau(\rho^\sigma, \epsilon_n)\right)^{-1}.\end{aligned}$$

Here $\Gamma_n := G(K_n/K)$, $\psi : \mathbb{Q}_p \to \bar{\mathbb{Q}}_p^{\times}$ corresponds to the compatible system $(\epsilon_n)_n$, that is $\psi(1/p^n) = \epsilon_n$, and $\bar{D}_\sigma^*(1)$ denotes the linearised Kummer dual of $\bar{D}_\sigma$, that is,

$$\chi_{\bar{D}_\sigma^*(1)}(g) = \rho^\sigma(g)^{-1} \eta(\tau_p)^{-v(g)} p^{-(r-1)v(g)},$$

while

$$\tau(\rho^\sigma, \epsilon_n) := \sum_{\gamma \in \Gamma_n} \rho^\sigma(\gamma)\gamma \cdot \epsilon_n = \#\Gamma_n e_{\rho^*}^{\Gamma_n} \epsilon_n$$

denotes the usual Gauss sum. Furthermore

$$\epsilon_L(D, -\psi) = \left(\epsilon(\bar{D}_\sigma, -\psi)\right)_\sigma \in \prod_\sigma \mathbb{Q}_p^{\times} \cong (\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} L)^{\times} \subseteq (B_{dR} \otimes_{\mathbb{Q}_p} L)^{\times}$$

is the ϵ-element as defined in [Fukaya and Kato 2006, §3.3.4]. We may assume that L contains $\mathbb{Q}_p(\mu_{p^n})$; then $\epsilon_L(D, -\psi)$ can be identified with

$$1 \otimes (\rho \eta(\tau_p) p^{r-1})^{-n} \tau(\rho, \epsilon_n)^{-1}.$$

Hence the comparison isomorphism renormalised by $\epsilon_L(D, -\psi)$

$$\epsilon_{L,-\epsilon,dR}(W)^{-1} : W \otimes L_{[W,\tau_p^{-1}]} \to D_{dR}(W) \subseteq B_{dR} \otimes_{\mathbb{Q}_p} L \otimes_L W$$

is explicitly given as

$$x \otimes l \mapsto \epsilon_L(D, -\psi)^{-1}(-t)^r l \otimes x = (-1)^r (\rho \eta(\tau_p) p^{r-1})^n \tau(\rho, \epsilon_n) t^r l \otimes x, \quad (70)$$

[2] Apparently, the formula in §3.2.2 (7) of [Fukaya and Kato 2006] is not compatible with Deligne as claimed: Deligne identifies $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ via class field theory with $\mathbb{Q}_p^{\times}$ by sending the *geometric* Frobenius automorphism to p, which induces, by a standard calculation applied to Definition (3.4.3.2) for epsilon constants of quasicharacters of $\mathbb{Q}_p^{\times}$ in [Deligne 1973] (see for example [Hida 1993, §8.5 between (4a) and (4b)]), the formula $\epsilon(V_\chi, \psi) = \chi(\tau_p)^{-n} \sum_{\sigma \in \Gamma_n} \chi(\sigma)^{-1} \sigma \epsilon_n$, while in [Fukaya and Kato 2006] the factor is just $\chi(\tau_p)^n$. Here $\chi : W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to E^{\times}$ is a character which gives the action on the E-vector space V_χ.

where $\epsilon_L(D,-\psi)^{-1}(-t)^r l$ is considered as an element of $B_{dR}\otimes_{\mathbb{Q}_p} L$.

In order to deduce the descent diagram (44) from (66), for $n\geq 1$, we have to add a commutative diagram of the form

$$\begin{array}{ccc} \mathscr{D}_M & \longrightarrow & \mathbb{T}_{K_\infty}\otimes_\Lambda \Lambda_{[\mathbb{T},\tau_p]^{-1}} \\ {\scriptstyle \Xi_{M,n}}\downarrow & & \downarrow{\scriptstyle Y\otimes_\Lambda -} \\ K_n\otimes M \cong D_{dR,K_n}(\mathbb{Q}_p(\eta)(r))/D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1} & \longleftarrow & V'\otimes_{\Lambda'}(\Lambda')_{[V',\tau_p^{-1}]}, \end{array}$$

where

$$\Xi_{M,n}(f)=\Xi^{\epsilon}_{M,n}(f)=p^{-n}(\phi\otimes\varphi)^{-n}\,(F)\,(\epsilon_n-1)=p^{-n}(\varphi\otimes\varphi)^{-n}(F)(0), \quad (71)$$

with $F\in\mathscr{H}_M$ such that $(1-\varphi)F=f=\tilde{f}\otimes e_{\eta,r}$ (recall that φ acts as $\varphi\otimes\varphi$ here) on ${\mathscr{D}_M}^{\Delta=0}$; and more generally we have, mod $D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1}$ (recalling that $D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1}=0$ in the generic case),

$$\begin{aligned}\Xi_{M,n}(f)&=p^{-n}\left(\sum_{k=1}^{n}(\phi\otimes\varphi)^{-k}\,(f(\epsilon_k-1))+(1-\phi\otimes\varphi)^{-1}(f(0))\right)\\ &=p^{-n}\left(\sum_{k=1}^{n}p^{kr}\eta(\tau_p)^k\tilde{f}^{\phi^{-k}}(\epsilon_k-1)+(1-p^{-r}\eta(\tau_p)^{-1}\phi)^{-1}\tilde{f}(0)\right)\otimes e_{\eta,r}\end{aligned}$$

(see [Benois and Berger 2008, Lemma 4.9], where $f(0)$ is considered in $D_{\mathrm{cris}}(V)$ and hence the last summand above equals $(1-\varphi)^{-1}f(0)$ there by the ϕ-linearity of φ). Here, for any $H(X)=\tilde{H}(X)\otimes e\in B^+_{\mathrm{rig},K}\otimes_{\mathbb{O}_K}M$ we consider $H(\epsilon_k-1)=\tilde{H}(\epsilon_k-1)\otimes e$, $k\leq n$, as an element in $K_n\otimes_{\mathbb{O}_K}M$, on which $\phi\otimes\varphi$ acts naturally.

First we note that for $n\geq 1$ we have a commutative diagram

$$\begin{array}{ccccc} \mathscr{D}_M & \xleftarrow{\Psi_M} & \mathbb{O}_K[\![\Gamma]\!]\otimes M & \xrightarrow{\Theta_M} & \mathbb{T}_{K_\infty}\otimes_\Lambda \Lambda_{[\mathbb{T},\tau_p]^{-1}} \\ \downarrow{\scriptstyle \Xi_{M,n}} & & \downarrow{\scriptstyle \mathrm{pr}_n\otimes id} & & \downarrow{\scriptstyle Y\otimes_\Lambda -} \\ K_n\otimes_{\mathbb{O}_K} M/D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1} & \xleftarrow{\Psi_{M,n}} & K[\Gamma_n]\otimes M & \longrightarrow & V'\otimes_{\Lambda'}(\Lambda')_{[V',\tau_p^{-1}]}, \end{array} \quad (72)$$

where

$$\begin{aligned}&\Psi_{M,n}(\mu\otimes e_{\eta,r})\\ &\quad=\Psi^{\epsilon}_{M,n}(\mu\otimes e_{\eta,r})\\ &\quad=p^{-n}\left(\sum_{k=1}^{n}\epsilon_k^{\phi^{-k}(\mu)}\otimes\varphi^{-k}(e_{\eta,r})+(1-\phi\otimes\varphi)^{-1}(1^\mu\otimes e_{\eta,r})\right)\\ &\quad=\left(\sum_{k=1}^{n}p^{kr-n}\eta(\tau_p)^k\epsilon_k^{\phi^{-k}(\mu)}+p^{-n}(1-p^{-r}\eta(\tau_p)^{-1}\phi)^{-1}(1^\mu)\right)\otimes e_{\eta,r} \quad (73)\end{aligned}$$

modulo $D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)(r))^{\varphi=1}$. Here ϕ acts coefficientwise on $K[\Gamma_n]$ and 1^μ is the same as the image of μ under the augmentation map $\mathcal{O}_K[\Gamma_n] \to \mathcal{O}_K$.

Proposition A.6. (i) *For $n \geq \max\{1, a(\rho)\}$ and $W \neq \mathbb{Q}_p(1)$, the following diagram is commutative*:

$$\begin{array}{ccccc}
 & & V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} K[\Gamma_n] \otimes M & & \\
 & \swarrow^{1\otimes\Psi_{M,n}} & & \searrow^{1\otimes\Theta_{M,n}} & \\
V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} K_n \otimes M & & & & V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} V' \otimes_{\mathbb{Q}_p[G_n]} \Lambda'_{[V',\tau_p^{-1}]} \\
\downarrow{\scriptstyle 1\otimes\Sigma_{M,n}} & & & & \downarrow{\scriptstyle \cong} \\
V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} D_{dR}(V') & & & & \\
\cong D_{dR}(W) & \xleftarrow{\Phi_W} & D_{dR}(W) & \xleftarrow{(-1)^r \epsilon_{L,-\epsilon,dR}(W)^{-1}} & W \otimes_L L_{[W,\tau_p^{-1}]},
\end{array}$$

where

$$\Phi_W := \begin{cases} \mathrm{id}_{D_{dR}(W)} & \text{if } a(\rho) \neq 0, \\ \dfrac{\det(1-\varphi \mid D_{\mathrm{cris}}(W^*(1)))}{\det(1-\varphi \mid D_{\mathrm{cris}}(W))} & \text{otherwise.} \end{cases}$$

(ii) *For $W \neq \mathbb{Q}_p(1)$ the diagram (44) commutes.*

Proof. Let b denote a normal basis of $\mathcal{O}_K$, that is, $\mathcal{O}_K = \mathbb{Z}_p[\overline{H}]b$ with $\overline{H} = G(K/\mathbb{Q}_p)$, which can be lifted from the residue field, K being unramified, and $e := e_{\eta,r}$. Then $1 \otimes b \otimes e = 1 \otimes e_{\rho^*} b \otimes e$ is a basis of $V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} K[\Gamma_n] \otimes M$ as L-vector space (in general $\rho(g)$ does not lie in K, but using $V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} K[\Gamma_n] \cong V_{\rho^*} \otimes_{L[G_n]} L[G_n] \otimes_{\mathbb{Q}_p[G_n]} K[\Gamma_n] \cong V_{\rho^*} \otimes_{L[G_n]} (L \otimes_{\mathbb{Q}_p} K[\Gamma_n])$ one can make sense of it). We calculate (going clockwise in the above diagram)

$$\begin{aligned}
&1 \otimes \Theta_{M,n}(1 \otimes b \otimes e) \\
&\quad = 1 \otimes (1 \otimes t_{\eta,r}) \otimes \sum_{i=0}^{d_K-1} \tau_p^i \otimes \phi^{-i}(b)a \qquad (\subseteq V_{\rho^*} \otimes_{\mathbb{Q}_p[G_n]} V' \otimes_{\mathbb{Q}_p[G_n]} \Lambda'_{[V',\tau_p^{-1}]}) \\
&\quad = t_{\rho\eta,r} \otimes \sum_{i=0}^{d_K-1} \rho(\tau_p)^{-i} \rho^*(\phi^{-i}(b))a \qquad (\subseteq W \otimes_L L_{[W,\tau_p^{-1}]}) \\
&\quad = t_{\rho\eta,r} \otimes \sum_{i=0}^{d_K-1} \rho(\tau_p)^{-i} \phi^{-i}(b)a \\
&\quad = t_{\rho\eta,r} \otimes \varsigma(\rho, b)a,
\end{aligned}$$

with

$$\varsigma(\rho, b) := \sum_{i=0}^{d_K-1} \rho(\tau_p)^{-i} \phi^{-i}(b) = d_K e_{\rho^*}^{\overline{H}} b$$

a Gauss-like sum, where $e_{\rho^*}^{\overline{H}} = \frac{1}{\#\overline{H}} \sum_{h \in H} \rho(h)h$. The image of this element under $(-1)^r \epsilon_{L,-\epsilon,dR}(W)$ is

$$(-1)^r \epsilon_L(D, -\psi)^{-1}(-t)^{-r} \varsigma(\rho, b) a \otimes t_{\rho\eta,r} = p^{mr-m}(\rho\eta)(\tau_p^m)\tau(\rho, \epsilon_m)\varsigma(\rho, b) a t^{-r} \otimes t_{\rho\eta,r} \quad (74)$$

in $D_{dR}(W)$, where we used (70) with $m = a(\rho)$.

Now we determine the image of $1 \otimes b \otimes e = 1 \otimes e_{\rho^*} b \otimes e$ anticlockwise. First note that the idempotent e_{ρ^*} decomposes as $e_{\rho^*}^{\Gamma_n} \cdot e_{\rho^*}^{\overline{H}}$.

Hence, for $n \geq a(\rho) \geq 1$, where $p^{a(\rho)}$ denotes the conductor of ρ restricted to Γ_n, we have

$$\begin{aligned}(1 \otimes \Psi_{M,n})(1 \otimes b \otimes e) &= 1 \otimes e_{\rho^*} \Psi_{M,n}(b \otimes e) \\ &= 1 \otimes p^{nr-n} \eta(\tau_p)^n \phi^{-n}(e_{\rho^*}^{\overline{H}} b) e_{\rho^*}^{\Gamma_n} \cdot \epsilon_n \otimes e \\ &= 1 \otimes p^{nr-n} \eta(\tau_p)^n \rho^*(\tau_p^{-n}) e_{\rho^*}^{\overline{H}} b e_{\rho^*}^{\Gamma_n} \cdot \epsilon_n \otimes e \\ &= 1 \otimes \frac{p^{nr-n}}{\#G_n} (\rho\eta)(\tau_p^n)\varsigma(\rho, b)\tau(\rho, \epsilon_n) \otimes e,\end{aligned}$$

where we have used the explicit formula (73) and the following fact about Gauss sums, valid for $k \leq n$ (see for example [Burns and Flach 2006, Lemma 5.2]):

$$e_\rho^{\Gamma_n}(\epsilon_k) = \begin{cases} e_\rho^{\Gamma_n}(\epsilon_k) & \text{if } a(\rho) = k, \\ (1-p)^{-1} & \text{if } a(\rho) = 0 \text{ and } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now from the end of the proof of Lemma A.5 we see that $\Sigma_{M,n}$ sends this element, which already "lies in the right eigenspace" to

$$a t^{-r} p^{nr-n} (\rho\eta)(\tau_p^n)\tau(\rho, \epsilon_n)\varsigma(\rho, b) \otimes t_{\rho\eta,r} = p^{nr-n}(\rho\eta)(\tau_p^n)\tau(\rho, \epsilon_n)\varsigma(\rho, b) a t^{-r} \otimes t_{\rho\eta,r},$$

that is, to the same element as in (74), whence the result follows if $a(\rho) \neq 0$.

Now assume that $a(\rho) = 0$, that is, $\rho \mid \Gamma_n$, the restriction to Γ_n, is trivial. Setting $n = 1$ we then have

$$\begin{aligned}&(1 \otimes \Psi_{M,1})(1 \otimes b \otimes e) \\ &\quad = 1 \otimes \Psi_{M,1}(e_{\rho^*} b \otimes e) \\ &\quad = 1 \otimes \left(p^{r-1}\eta(\tau_p)\epsilon_1^{\phi^{-1}(e_{\rho^*} b)} + p^{-1}(1 - p^{-r}\eta(\tau_p)^{-1}\phi)^{-1}(e_{\rho^*}^{\overline{H}} b)\right) \otimes e \\ &\quad = 1 \otimes \left(p^{r-1}\eta(\tau_p)\phi^{-1}(e_{\rho^*}^{\overline{H}} b) e_{\rho^*}^{\Gamma_1} \cdot \epsilon_1 + p^{-1}(1 - p^{-r}\rho\eta(\tau_p)^{-1})^{-1}(e_{\rho^*}^{\overline{H}} b)\right) \otimes e \\ &\quad = 1 \otimes \left(p^{r-1}\rho\eta(\tau_p)(1-p)^{-1} + p^{-1}(1 - p^{-r}\rho\eta(\tau_p)^{-1})^{-1}\right) \frac{\varsigma(\rho, b)}{d_K} \otimes e\end{aligned}$$

$$= 1 \otimes \left(\frac{1 - p^{r-1}\rho\eta(\tau_p)}{1 - p^{-r}\rho\eta(\tau_p^{-1})} \right) \frac{\varsigma(\rho, b)}{d_K(p-1)} \otimes e,$$

which is sent under $\Sigma_{M,1}$ to

$$\left(\frac{\det(1-\varphi \mid D_{\mathrm{cris}}(W^*(1)))}{\det(1-\varphi \mid D_{\mathrm{cris}}(W))} \right) \varsigma(\rho, b) a t^{-r} \otimes t_{\rho\eta,r},$$

while (74) becomes just

$$\varsigma(\rho, b) a t^{-r} \otimes t_{\rho\eta,r}.$$

Upon replacing ϵ by $-\epsilon = \epsilon^{-1}$ (we have used both the additive and multiplicative notation!) the second statement follows from (66), (72) and the diagram in part (i) of the proposition, combined with the isomorphism (68) and Lemma A.4. □

Appendix B: Determinant functors

In this appendix we recall some details of the formalism of determinant functors introduced in [Fukaya and Kato 2006] (see also [Venjakob 2007]).

We fix an associative unital noetherian ring R. We write $B(R)$ for the category of bounded complexes of (left) R-modules, $C(R)$ for the category of bounded complexes of finitely generated (left) R-modules, $P(R)$ for the category of finitely generated projective (left) R-modules and $C^{\mathrm{p}}(R)$ for the category of bounded (cohomological) complexes of finitely generated projective (left) R-modules. By $D^{\mathrm{p}}(R)$ we denote the category of perfect complexes as a full triangulated subcategory of the derived category $D^{\mathrm{b}}(R)$ of $B(R)$. We write $(C^{\mathrm{p}}(R), \mathrm{quasi})$ for the subcategory of quasi-isomorphisms of $C^{\mathrm{p}}(R)$ and $(D^{\mathrm{p}}(R), \mathrm{isom})$ for the subcategory of isomorphisms of $D^{\mathrm{p}}(R)$.

For each complex $C = (C^\bullet, d_C^\bullet)$ and each integer r we define the r-fold shift $C[r]$ of C by setting $C[r]^i = C^{i+r}$ and $d^i_{C[r]} = (-1)^r d_C^{i+r}$ for each integer i.

We first recall that there exists a Picard category $\mathscr{C}_R$ and a determinant functor $\mathbf{d}_R : (C^{\mathrm{p}}(R), \mathrm{quasi}) \to \mathscr{C}_R$ with the following properties (for objects C, C' and C'' of $\mathrm{C}^{\mathrm{p}}(R)$):

(B.a) $\mathscr{C}_R$ has an associative and commutative product structure $(M, N) \mapsto M \cdot N$ (which we often write more simply as MN) with canonical unit object $\mathbf{1}_R = \mathbf{d}_R(0)$. If P is any object of $P(R)$, then in $\mathscr{C}_R$ the object $\mathbf{d}_R(P)$ has a canonical inverse $\mathbf{d}_R(P)^{-1}$. Every object of $\mathscr{C}_R$ is of the form $\mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)^{-1}$ for suitable objects P and Q of $P(R)$.

(B.b) All morphisms in $\mathscr{C}_R$ are isomorphisms and elements of the form $\mathbf{d}_R(P)$ and $\mathbf{d}_R(Q)$ are isomorphic in $\mathscr{C}_R$ if and only if P and Q correspond to the same element of the Grothendieck group $K_0(R)$. There is a natural identification $\mathrm{Aut}_{\mathscr{C}_R}(\mathbf{1}_R) \cong K_1(R)$ and if $\mathrm{Mor}_{\mathscr{C}_R}(M, N)$ is nonempty then it is a $K_1(R)$-torsor, where each

element α of $K_1(R) \cong \mathrm{Aut}_{\mathscr{C}_R}(\mathbf{1}_R)$ acts on $\phi \in \mathrm{Mor}_{\mathscr{C}_R}(M, N)$ to give

$$\alpha\phi : M = \mathbf{1}_R \cdot M \xrightarrow{\alpha\cdot\phi} \mathbf{1}_R \cdot N = N.$$

(B.c) $\mathbf{d}_R$ preserves the product structure: specifically, for each P and Q in $P(R)$ one has $\mathbf{d}_R(P \oplus Q) = \mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)$.

(B.d) If $C' \to C \to C''$ is a short exact sequence of complexes, there is a canonical isomorphism $\mathbf{d}_R(C) \cong \mathbf{d}_R(C')\mathbf{d}_R(C'')$ in $\mathscr{C}_R$ (which we usually take to be an identification).

(B.e) If C is acyclic, the quasi-isomorphism $0 \to C$ induces a canonical isomorphism $\mathbf{1}_R \to \mathbf{d}_R(C)$.

(B.f) For any integer r one has $\mathbf{d}_R(C[r]) = \mathbf{d}_R(C)^{(-1)^r}$.

(B.g) The functor $\mathbf{d}_R$ factorises over the image of $C^{\mathrm{p}}(R)$ in $D^{\mathrm{p}}(R)$ and extends (uniquely up to unique isomorphisms) to $(D^{\mathrm{p}}(R), \mathrm{isom})$. Moreover, if R is regular, also property (B.d) extends to all distinguished triangles.

(B.h) For each C in $D^{\mathrm{b}}(R)$ we write $\mathrm{H}(C)$ for the complex which has $\mathrm{H}(C)^i = H^i(C)$ in each degree i and in which all differentials are 0. If $\mathrm{H}(C)$ belongs to $D^{\mathrm{p}}(R)$ (in which case one says that C is *cohomologically perfect*), then C belongs to $D^{\mathrm{p}}(R)$ and there are canonical isomorphisms

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(\mathrm{H}(C)) \cong \prod_{i\in\mathbb{Z}} \mathbf{d}_R(H^i(C))^{(-1)^i}.$$

(For an explicit description of the first isomorphism see [Knudsen and Mumford 1976, §3] or [Breuning and Burns 2005, Remark 3.2].)

(B.i) If R' is another (associative unital noetherian) ring and Y an (R', R)-bimodule that is both finitely generated and projective as an R'-module then the functor $Y \otimes_R - : P(R) \to P(R')$ extends to a commutative diagram

$$\begin{array}{ccc} (\mathrm{D}^p(R), \mathrm{isom}) & \xrightarrow{\mathbf{d}_R} & \mathscr{C}_R \\ {\scriptstyle Y\otimes^{\mathbb{L}}_R -}\downarrow & & \downarrow{\scriptstyle Y\otimes_R -} \\ (\mathrm{D}^p(R'), \mathrm{isom}) & \xrightarrow{\mathbf{d}_{R'}} & \mathscr{C}_{R'}. \end{array}$$

In particular, if $R \to R'$ is a ring homomorphism and C is in $\mathrm{D}^{\mathrm{p}}(R)$ then we often simply write $\mathbf{d}_R(C)_{R'}$ in place of $R' \otimes_R \mathbf{d}_R(C)$.

(B.j) Let R° be the opposite ring of R. The functor $\mathrm{Hom}_R(-, R)$ induces an antiequivalence between $\mathscr{C}_R$ and $\mathscr{C}_{R^\circ}$, with quasi-inverse induced by $\mathrm{Hom}_{R^\circ}(-, R^\circ)$;

both functors will be denoted by $-^*$. This extends to give a diagram

$$\begin{array}{ccc} (\mathrm{D}^p(R), \mathrm{isom}) & \xrightarrow{\mathbf{d}_R} & \mathscr{C}_R \\ \downarrow{\scriptstyle \mathrm{RHom}_R(-,R)} & & \downarrow{\scriptstyle -^*} \\ (\mathrm{D}^p(R^\circ), \mathrm{isom}) & \xrightarrow{\mathbf{d}_{R^\circ}} & \mathscr{C}_{R^\circ} \end{array}$$

which commutes (up to unique isomorphism); similarly we have such a commutative diagram for $\mathrm{RHom}_{R^\circ}(-, R^\circ)$.

For the handling of the determinant functor the following considerations are important in practice:

Remark B.1. (i) For objects $A, B \in \mathscr{C}_R$ we often identify a morphism $f : A \to B$ with the induced morphism

$$\mathbf{1}_R = A \cdot A^{-1} \xrightarrow{f \cdot \mathrm{id}_{A^{-1}}} B \cdot A^{-1}.$$

Then for morphisms $f : A \to B$ and $g : B \to C$ in $\mathscr{C}_R$, the composition $g \circ f : A \to C$ is identified with the product $g \cdot f : \mathbf{1}_R \to C \cdot A^{-1}$ of $g : \mathbf{1}_R \to C \cdot B^{-1}$ and $f : \mathbf{1}_R \to B \cdot A^{-1}$. Also, by this identification a map $f : A \to A$ corresponds uniquely to an element in $K_1(R) = \mathrm{Aut}_{\mathscr{C}_R}(\mathbf{1}_R)$. Furthermore, for a map $f : A \to B$ in $\mathscr{C}_R$, we write $\bar{f} : B \to A$ for its inverse with respect to composition, while $f^{-1} =: \overline{\mathrm{id}_{B^{-1}} \cdot f \cdot \mathrm{id}_{A^{-1}}} : A^{-1} \to B^{-1}$ for its inverse with respect to the multiplication in $\mathscr{C}_R$, that is $f \cdot f^{-1} = \mathrm{id}_{\mathbf{1}_R}$. Obviously, for a map $f : A \to A$ both inverses $\bar{f}$ and f^{-1} coincide if all maps are considered as elements of $K_1(R)$ as above.

Convention B.2. If $f : \mathbf{1} \to A$ is a morphism and B an object in $\mathscr{C}_R$, we write $\bullet f : B \to B \cdot A$ for the morphism $\mathrm{id}_B \cdot f$. In particular, any morphism $f : B \to A$ can be written as $\bullet(\mathrm{id}_{B^{-1}} \cdot f) : B \to A$.

(ii) The determinant of the complex $C = [P_0 \xrightarrow{\phi} P_1]$ (in degrees 0 and 1) with $P_0 = P_1 = P$ is by definition $\mathbf{d}_R(C) = \mathbf{1}_R$; it is defined even if ϕ is not an isomorphism (in contrast to $\mathbf{d}_R(\phi)$). But if ϕ happens to be an isomorphism, i.e., if C is acyclic, then by (B.e) there is also a canonical map $\mathrm{acyc} : \mathbf{1}_R \to \mathbf{d}_R(C)$, which is none other than

$$\mathbf{1}_R = \mathbf{d}_R(P_1)\mathbf{d}_R(P_1)^{-1} \xrightarrow{\mathbf{d}(\phi)^{-1} \cdot \mathrm{id}_{\mathbf{d}(P_1)^{-1}}} \mathbf{d}_R(P_0)\mathbf{d}_R(P_1)^{-1} = \mathbf{d}_R(C)$$

(and which depends on ϕ, in contrast with the first identification). Hence, the composite

$$\mathbf{1}_R \xrightarrow{\mathrm{acyc}} \mathbf{d}_R(C) \overset{\mathrm{def}}{=\!=} \mathbf{1}_R$$

corresponds to $\mathbf{d}_R(\phi)^{-1} \in K_1(R)$ according to the first remark. In order to distinguish the above identifications between $\mathbf{1}_R$ and $\mathbf{d}_R(C)$ we also say that C is

trivialised by the identity when we refer to $\mathbf{d}_R(C) = \mathbf{1}_R$ (or its inverse with respect to composition). For $\phi = \mathrm{id}_P$ both identifications obviously agree.

We end this section by considering the example where $R = K$ is a field and V a finite-dimensional vector space over K. Then, according to [Fukaya and Kato 2006, 1.2.4], $\mathbf{d}_K(V)$ can be identified with the highest exterior product $\bigwedge^{\mathrm{top}} V$ of V and for an automorphism $\phi : V \to V$ the determinant $\mathbf{d}_K(\phi) \in K^\times = K_1(K)$ can be identified with the usual determinant $\det_K(\phi)$. In particular, we identify $\mathbf{d}_K = K$ with canonical basis 1. Then a map $\psi : \mathbf{1}_K \to \mathbf{1}_K$ corresponds uniquely to the value $\psi(1) \in K^\times$.

Remark B.3. Note that every *finite* $\mathbb{Z}_p$-module A possesses a free resolution C; that is, $\mathbf{d}_{\mathbb{Z}_p}(A) \cong \mathbf{d}_{\mathbb{Z}_p}(C)^{-1} = \mathbf{1}_{\mathbb{Z}_p}$. Then modulo $\mathbb{Z}_p^\times$ the composite

$$\mathbf{1}_{\mathbb{Q}_p} \xrightarrow{\text{acyc}} \mathbf{d}_{\mathbb{Z}_p}(C)_{\mathbb{Q}_p} \overset{\text{def}}{=\!=} \mathbf{1}_{\mathbb{Q}_p}$$

corresponds to the cardinality $|A|^{-1} \in \mathbb{Q}_p^\times$.

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venjakob@mathi.uni-heidelberg.de *Mathematisches Institut, Universität Heidelberg, Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany*

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