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The system of representations of the Weil–Deligne group associated to an abelian variety

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Fix a number field $F \subset \mathbb{C}$, an abelian variety A/F and let G_A be the Mumford–Tate group of $A_{/\mathbb{C}}$. After replacing F by finite extension one can assume that, for every prime number ℓ, the action of the absolute Galois group $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ on the étale cohomology group $\mathrm{H}^1_{\mathrm{\acute{e}t}}(A_{\bar{F}}, \mathbb{Q}_\ell)$ factors through a morphism $\rho_\ell \colon \Gamma_F \to G_A(\mathbb{Q}_\ell)$. Let v be a valuation of F and write Γ_{F_v} for the absolute Galois group of the completion F_v. For every ℓ with $v(\ell) = 0$, the restriction of ρ_ℓ to Γ_{F_v} defines a representation ${}'W_{F_v} \to G_{A/\mathbb{Q}_l}$ of the Weil–Deligne group.

It is conjectured that, for every ℓ, this representation of ${}'W_{F_v}$ is defined over $\mathbb{Q}$ as a representation with values in G_A and that the system above, for variable ℓ, forms a compatible system of representations of ${}'W_{F_v}$ with values in G_A. A somewhat weaker version of this conjecture is proved for the valuations of F, where A has semistable reduction and for which $\rho_\ell(\mathrm{Fr}_v)$ is neat.

Introduction

Let F_v be a finite extension of the field $\mathbb{Q}_p$ (for some prime number p) and let X be a proper and smooth variety over F_v. The Galois group $\Gamma_{F_v} = \mathrm{Gal}(\bar{F}_v/F_v)$ acts on the étale cohomology groups $\mathrm{H}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ for each prime number ℓ and each i. It is a major problem in arithmetic geometry to determine to what extent the properties of these representations are independent of ℓ. To obtain such independence results, one has to consider the restrictions of the representations above to the Weil group W_{F_v} of F_v. This is the subgroup formed by the elements of Γ_{F_v}, which induce an integral power of the Frobenius automorphism on the residue field of F_v.

In what follows it will always be assumed that $\ell \neq p$; the case where $\ell = p$ will be analysed in a later paper. Let us first assume that X has good reduction at v, that is, that X extends to a proper and smooth scheme over the ring of integers of F_v. This assumption implies that the inertia subgroup of Γ_{F_v} acts trivially on the

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étale cohomology groups of X. Moreover, it follows from Deligne's work [1974a] on the Weil conjectures that the character of the representation of W_{F_v} on each $\mathrm{H}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ has values in $\mathbb{Q}$ and is independent of $\ell \neq p$. In view of the triviality of the action of inertia, this amounts to a statement on the action of the subgroup of Γ_{F_v} generated by a Frobenius element. We will summarise this statement by saying that the representations of W_{F_v} on the $\mathrm{H}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ (for fixed i and variable ℓ) are *defined over* $\mathbb{Q}$ and that they form a *compatible system* of representations of W_{F_v}.

If X only has potentially good reduction, the inertia group no longer acts trivially but its action on a given étale cohomology group of X factors through a finite quotient. It is still conjectured that the system of $\mathrm{H}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ (as always for fixed i and variable ℓ) forms a compatible system of representations of W_{F_v} which are defined over $\mathbb{Q}$, see for example [Serre 1994, 12.13?].

Serre actually states his conjecture in more generality because it applies to motives instead of varieties. The category of (pure) motives can be seen as an intermediate between varieties and their cohomology. It is a $\mathbb{Q}$-linear tannakian category, that is, an abelian category in which the morphisms between two objects form a $\mathbb{Q}$-vector space and which is equipped with "tensor products". All reasonable cohomology functors on the category of varieties factor through the category of motives so that any motive has cohomology groups in the same way as varieties do. We also refer to the cohomology groups of a motive M as the *realisations* of M. There are different constructions of motives, depending on how the morphisms are defined, but in any case the category of motives has more morphisms and hence also more objects than the category of varieties.

In this paper we will consider the category of motives for absolute Hodge cycles developed in [Deligne and Milne 1982] where the morphisms are defined by absolute Hodge classes. This theory is particularly efficient for dealing with abelian varieties because the motivic Galois group of an abelian variety A coincides with its Mumford–Tate group, defined by the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(A_{/\mathbb{C}}, \mathbb{Q})$.

The motivic version of the conjecture can be stated in terms of motivic Galois groups in the following way. In any good category of motives it is possible to associate a motivic Galois group to any subcategory. This group is a linear proalgebraic group over $\mathbb{Q}$. Its defining property is the fact that any category of motives is equivalent to the category of representations of its motivic Galois group. The motivic Galois group G_M of any object M is defined as the group associated to the tannakian category generated by M and the Tate motive. The group G_M is related to the étale cohomology of M by the fact that, for each prime number ℓ, the ℓ-adic Galois representation associated to M factors through $G_M(\mathbb{Q}_\ell)$. This implies that the corresponding representations of the Weil group factor through $G_{M/\mathbb{Q}_\ell}$. Serre's ℓ-independence conjecture for objects in the $\otimes$-category generated by M is then equivalent to the statement that the representations $W_{F_v} \to G_{M/\mathbb{Q}_\ell}$

of the Weil group form a compatible system *defined over* $\mathbb{Q}$*, with values in* G_M; see 2.3 for the precise definition.

For abelian varieties of CM type, the conjecture follows from the theory of complex multiplication developed by Shimura and Taniyama [1961], which yields a considerably more precise result. Indeed, let $F \subset \mathbb{C}$ be a number field, A/F an abelian variety of CM type and G_A the motivic Galois group of A. Serre [1968] constructs a commutative algebraic group $S_{F,\mathfrak{m}}$ and a canonical system of representations $\varphi_\ell\colon \Gamma_F \to S_{F,\mathfrak{m}}(\mathbb{Q}_\ell)$. By the theory of complex multiplication, the system of ℓ-adic representations associated to A is the image of the system (φ_ℓ) by a $\mathbb{Q}$-rational morphism $S_{F,\mathfrak{m}} \to G_A$. Moreover, Deligne [1982] gives an explicit description of the motivic Galois group of the category of abelian varieties potentially of CM type in terms of the Taniyama group.

More generally, if $F \subset \mathbb{C}$ is a number field, A/F an abelian variety and v a valuation of F where A has good reduction, the ℓ-independence conjecture was proved in [Noot 2009], by a method and under additional hypotheses similar to those of the present paper. The case where A has ordinary reduction at v has been treated in [Noot 1995] by a completely different method. In the latter case, a stronger statement can be proved and it turns out that there is an element in $G_A(\mathbb{Q})$ that is conjugate to the ℓ-adic image of Frobenius for all $\ell \neq p$. As noted in the introduction of [Noot 1995], this is not the case in general. The reader may consult the introduction to [Noot 2009] for a more detailed discussion.

One may ask if these results can be generalised without any assumptions on the reduction of X. Before discussing the properties of the Galois representations provided by the étale cohomology groups of a variety X, consider *any* ℓ-adic representation ρ_ℓ of Γ_{F_v} for $\ell \neq p$. By a theorem of Grothendieck (see [Serre and Tate 1968, Appendix; Deligne 1973, §2, §8]), the action of a sufficiently small open subgroup of the inertia group can be described using a single endomorphism N_ℓ, the monodromy operator. The restriction of ρ_ℓ to the Weil group W_{F_v} can then be encoded by giving N_ℓ together with a representation ρ'_ℓ of W_{F_v}, which is trivial on an open subgroup of the inertia group. We will refer to such a pair (ρ'_ℓ, N_ℓ) as a *representation of the Weil–Deligne group* $'W_{F_v}$ *of* F_v. Where the p-adic étale cohomology is concerned, Fontaine's theory associates a representation of the Weil–Deligne group to any semistable p-adic representation as well, but we will not go into the details here.

It is conjectured that, applying the construction above to the system of Galois representations provided by the $\mathrm{H}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$, for fixed i and variable ℓ, one obtains a compatible system of representations of $'W_{F_v}$ which are defined over $\mathbb{Q}$; see for example [Fontaine 1994, 2.4.3, conjecture C_{WD}] for a statement encompassing the p-adic representation. The conjecture on the ℓ-independence of the representation of the Weil–Deligne group hinges on the monodromy-weight conjecture; see

[Illusie 1994, §3]. This elusive conjecture is somewhat more accessible under the hypothesis that X has semistable reduction, a hypothesis which implies in particular that the representations ρ'_ℓ are trivial on the inertia subgroup of W_{F_v}. The action of inertia on $\mathrm{H}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ is then determined by the monodromy operator N_ℓ. Even if X has semistable reduction however, the monodromy-weight conjecture has so far only been proved under far more restrictive hypotheses. Apart from the cases where X is a curve or an abelian variety, the main achievement is due to [Rapoport and Zink 1982], which treats the case where X has dimension 2. We refer to [Ochiai 1999; Ito 2004] and the work of Scholze for some recent and very recent progress on the problems. It should finally be pointed out that the discussion above has an analogue in equal characteristics, which has proved much more accessible; see for example [Deligne 1980].

This paper aims to study the motivic version of Fontaine's C_{WD} conjecture. Under some additional hypotheses, described below, we will prove the compatibility conjecture for the system $'W_{F_v} \to G_{A/\mathbb{Q}_\ell}$ associated to an abelian variety A defined over a number field $F \subset \mathbb{C}$. Here v is a fixed valuation of F and ℓ runs through the set of primes with $v(\ell) = 0$.

The hypotheses are twofold. First of all, we need to assume that the number field F is sufficiently big. We do not only need to ensure that A has semistable reduction, but also that the Mumford–Tate group G_A is connected and even that the Frobenius element at the given place of F is *weakly neat*; see Definition 3.5. Secondly, in certain cases we will need to work in a group that is slightly larger than the Mumford–Tate group; see Section 3.3. The Mumford–Tate group coincides with the identity component of this larger group. Enlarging the group obviously weakens the notion of conjugacy. The precise result is Theorem 3.6 and all definitions used in the statement are given in Section 3.

The strategy of the proof is inspired by the previous paper [Noot 2009], which treats the good reduction case. The idea is first to prove the statement for *tractable* abelian varieties (called *accomodantes* [ibid.]). In Section 4, we recall the notion of a tractable abelian variety as well as some related constructions stemming from [ibid.]. Tractable abelian varieties have many endomorphisms and the theorem is proved by combining more or less classical results concerning abelian varieties and 1-motives with the, equally classical, work of Springer and Steinberg on conjugacy classes in linear algebraic groups. The necessary results on representations of the Weil–Deligne group associated to a 1-motive follow from the theory sketched in [Fontaine 1994] by adding the action of an endomorphism algebra throughout. We make extensive use of [Raynaud 1994], which allows the reduction to the case of strict 1-motives. This is discussed, together with the relevant prerequisites, in Sections 1 and 2. Using the results of these sections, proof of the main theorem for a tractable abelian variety is given in Sections 5 and 6.

Once we have proved the main theorem for tractable abelian varieties, the general case can be deduced using the theory, developed in [Noot 2006], of lifting Galois representations along isogenies between the Mumford–Tate groups. This final step of the proof is carried out in Section 7. In order to construct the liftings of the Galois representations, one needs to extend the base field. At first, this leads to a proof of the main theorem over an uncontrollable extension of the base field. The results of Sections 1 and 2 are used again to deduce the theorem over the original base field. The condition that the Frobenius element is weakly neat is essential in this step.

In his thesis, Laskar [2011] generalises the results of this paper, as well as those of [Noot 2009], to a larger class of varieties. He proves the main theorem of [ibid.] for the absolute Hodge motive of any variety X with good reduction belonging to the tannakian category generated by the motives of abelian varieties. Under somewhat more restrictive conditions, Laskar also generalises the results of the present paper, treating the case of curves, $K3$ surfaces and a Fermat hypersurfaces with semistable reduction.

Another direction for generalisation is the case of 1-motives. In this context, an analogue of the theory of Mumford–Tate liftings remains to be developed. The analogue of our results in the case where the base field is a function field in characteristic p seems inaccessible with the techniques used in this paper. Indeed, it would be necessary to develop a substitute for the theory of absolute Hodge motives and, most importantly, the concept of Mumford–Tate liftings.

1. 1-motives with L-action

In this section, we indicate how the theory of 1-motives developed in [Raynaud 1994; Deligne 1974b, §10] works out for 1-motives with a given endomorphism field. Most of the statements are easy generalisations of those given in [Fontaine 1994] for the case where $L = \mathbb{Q}$. The results of this section will be applied to the study of the monodromy of an abelian variety with a given endomorphism algebra. Where these preliminary results are concerned, very little additional effort is required to deal with the more general case of 1-motives.

We also review the construction of the Weil–Deligne group of a local field and recall how to associate a representation of the Weil–Deligne group to a local Galois representation.

1.1. ***Generalities on 1-motives.*** In all of this section, F is a finite extension of $\mathbb{Q}_p$ and ℓ, ℓ' are prime numbers.

Let M be a 1-motive over F. By definition M is a complex $u\colon Y \to G$ where Y is a K-group scheme which is locally isomorphic, in the étale topology, to $\mathbb{Z}^r$ and G is a semiabelian variety over K. In this complex, Y is placed in degree -1 and

G in degree 0. The semiabelian variety G is an extension of an abelian variety A of dimension g by a torus T of dimension $r^\star$. Let $d = d(M) = r + r^\star + 2g$.

One defines the ℓ-adic realisations of M following [Deligne 1974b, 10.1], taking projective limits. This differs by a trivial manipulation from Raynaud's definition [1994, 3.1] where an inductive limit is used. To be precise, for an integer n, we define $\mathrm{Tors}_n(M)$ as the H^{-1} of the complex

$$\begin{aligned} &Y \to Y \oplus G(\bar{F}) \to G(\bar{F}),\\ &x \mapsto (-nx, -u(x)),\\ &\qquad\qquad\quad (x, y) \mapsto u(x) - ny, \end{aligned}$$

situated in degrees -2, -1, 0. Here and in what follows, $\bar{F}$ is an algebraic closure of F. For any ℓ we put

$$T_\ell(M) = \varprojlim \mathrm{Tors}_{\ell^n}(M) \quad \text{and} \quad V_\ell(M) = T_\ell(M) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Thus, $T_\ell(M)$ is a free $\mathbb{Z}_\ell$-module of rank d and $V_\ell(M)$ is a $\mathbb{Q}_\ell$-vector space of dimension d. For each ℓ, the absolute Galois group $\Gamma_F = \mathrm{Aut}_F(\bar{F}) = \mathrm{Gal}(\bar{F}/F)$ acts naturally on V_ℓ.

We fix a number field $L \subset \mathrm{End}^0(M) = \mathrm{End}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. The endomorphism ring $\mathrm{End}(M)$ can be interpreted either as the ring of endomorphisms of the complex $Y \to G$ or as the ring of endomorphisms of its image in the derived category $D^b(\mathrm{fppf})$; see [Raynaud 1994, 2.3]. The first interpretation of $\mathrm{End}(M)$ shows that L acts on $Y \otimes \mathbb{Q}$ and that L embeds into $\mathrm{End}^0(G)$. It follows (for example) from [Milne 1986, 3.9] that any morphism of the torus T to an abelian variety is trivial. This implies that we have an embedding $L \subset \mathrm{End}^0(T)$ so L also acts on $Y^\star \otimes \mathbb{Q}$ where $Y^\star = \mathrm{Hom}(T, \mathbb{G}_m)$. Finally, by passing to the quotient $A = G/T$, we obtain an embedding $L \subset \mathrm{End}^0(A)$.

By functoriality, L acts on $V_\ell = V_\ell(M)$, making it into an $L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module. To ease notation, we will write $L_\ell = L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ from now on. The weight filtration of M (see [Raynaud 1994, 2.2]) induces an increasing L_ℓ-linear filtration of V_ℓ such that the nonzero components of the associated graded are

$$\begin{aligned} \mathrm{Gr}_{-2}(V_\ell) &\cong (Y^\star)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(1),\\ \mathrm{Gr}_{-1}(V_\ell) &\cong V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,\\ \mathrm{Gr}_0(V_\ell) &\cong Y \otimes_{\mathbb{Z}} \mathbb{Q}_\ell. \end{aligned}$$

The action of Γ_F respects the weight filtration and commutes with the L_ℓ-action so Γ_F acts L_ℓ-linearly on $\mathrm{Gr}_\bullet(V_\ell)$. The isomorphisms above are L_ℓ-linear and Γ_F-equivariant. One has $L_\ell \cong \bigoplus_\lambda L_\lambda$, where λ runs through the primes of L lying over ℓ. This decomposition gives rise to corresponding decompositions of each L_ℓ-module occurring above as a direct sum of L_λ-modules of the same rank.

The next aim is to establish a common $\mathbb{Q}$-structure V on the modules V_ℓ, endowed with L-action and weight filtration. We first describe the associated graded. For $V^0 = Y \otimes_{\mathbb{Z}} \mathbb{Q}$, there is a system of canonical L_ℓ-linear isomorphisms $\mathrm{Gr}_0(V_\ell) \cong V^0 \otimes \mathbb{Q}_\ell$. Next, put $V^{-2} = (Y^\star \otimes_{\mathbb{Z}} \mathbb{Q})^\vee$ and fix isomorphisms $\mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell$ for each ℓ. This gives rise to a system of isomorphisms $\mathrm{Gr}_{-2}(V_\ell) \cong V^{-2} \otimes \mathbb{Q}_\ell$, depending only on the identifications $\mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell$.

We finally consider Gr_{-1}. By the theorem of the primitive element, $L = \mathbb{Q}(\alpha)$, so it follows from [Mumford 1970, §19, Theorem 4] that there exists an L-vector space V^{-1} endowed with L_ℓ-linear isomorphisms

$$\mathrm{Gr}_{-1}(V_\ell) \cong V^{-1} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

for every ℓ; see [Noot 2006, proof of 6.13]. As a vector space is determined up to isomorphism by its dimension, V^{-1} is unique up to L-linear isomorphisms and each isomorphism $\mathrm{Gr}_{-1}(V_\ell) \cong V^{-1} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is unique up to L_ℓ-linear automorphisms of $\mathrm{Gr}_{-1}(V_\ell)$.

Define

$$V = V^{-2} \oplus V^{-1} \oplus V^0,$$

endowed with the natural L-action and the increasing filtration defined by the grading. As L_ℓ is a semisimple algebra, V_ℓ is L_ℓ-isomorphic to its associated graded, so the preceding discussion gives rise to a noncanonical L_ℓ-linear isomorphism $V_\ell \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ for every ℓ. We proved the following lemma.

Lemma 1.2. *There exists an L-vector space V endowed with an increasing filtration and, for each ℓ, an L_ℓ-linear isomorphism $V_\ell \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ compatible with the filtrations. In particular, each V_ℓ is a free L_ℓ-module whose rank is independent of ℓ and the weight filtration is a filtration by free L_ℓ-submodules of ranks independent of ℓ.*

1.3. ***The group H.*** We fix an L-vector space V together with a system of isomorphisms as in the lemma. Let $H = \mathrm{Res}_{L/\mathbb{Q}} \mathrm{GL}_{/L}(V)$ be the linear algebraic group over $\mathbb{Q}$ of L-linear automorphisms of V and let $\mathfrak{h}$ be its Lie algebra. This means that $\mathfrak{h} = \mathfrak{gl}_L(V)$ is the $\mathbb{Q}$-Lie algebra of L-linear endomorphisms of V. Obviously, H acts on $\mathfrak{h}$ through the adjoint representation. By means of the identifications above, $H_{/\mathbb{Q}_\ell}$ and $\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ act on V_ℓ. This identifies $H_{/\mathbb{Q}_\ell}$ with the group of L_ℓ-linear automorphisms of V_ℓ and $\mathfrak{h} \otimes \mathbb{Q}_\ell$ with its Lie algebra. These identifications are determined up to inner automorphisms for the groups and up to the adjoint action of $H_{/\mathbb{Q}_\ell}$ where the Lie algebras are concerned. Finally note that $L = \mathrm{End}_H(V)$ and that $L_\ell = \mathrm{End}_{H_{/\mathbb{Q}_\ell}}(V_\ell)$.

1.4. ***The monodromy operator.*** From now on we assume that the 1-motive M is *strict* in the sense of [Raynaud 1994, Définition 4.2.3], which means that the semiabelian variety G has potentially good reduction. In this case, [ibid., 4.3] defines the *geometric monodromy*, a canonical additive map $\mu\colon Y \otimes Y^\star \to \mathbb{Q}$. Giving μ is equivalent to giving the induced map

$$N\colon V^0 = Y \otimes \mathbb{Q} \to V^{-2} = (Y^\star \otimes \mathbb{Q})^\vee. \tag{1.4$*$}$$

By functoriality this map is L-linear, so we can interpret N as an element of the Lie algebra $\mathfrak{h}^{\mathrm{ss}} \subset \mathrm{End}(V)$. As an endomorphism of V, it is nilpotent of echelon 2. In what follows, the notation N will be reserved for this element of $\mathfrak{h}^{\mathrm{ss}}$.

The map N defines a morphism $\mathrm{Gr}_0(V_\ell) \to \mathrm{Gr}_{-2}(V_\ell)(-1)$ and thus a morphism $N_\ell\colon V_\ell \to V_\ell(-1)$. As N_ℓ is L_ℓ-linear, $N_\ell \in \mathfrak{h}^{\mathrm{ss}} \otimes \mathbb{Q}_\ell(-1)$ for each ℓ. Recall that the identification $V_\ell \cong V \otimes \mathbb{Q}_\ell$ depends on the identification $\mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell$ fixed in Section 1.1. Using the same identification, we identify $\mathfrak{h}^{\mathrm{ss}} \otimes \mathbb{Q}_\ell(-1)$ with $\mathfrak{h}^{\mathrm{ss}} \otimes \mathbb{Q}_\ell$ and under these isomorphisms the images of $N \otimes 1$ and N_ℓ in $\mathfrak{h}^{\mathrm{ss}} \otimes \mathbb{Q}_\ell$ coincide.

The discussion above only depends on the isomorphisms $\mathrm{Gr}_i(V_\ell) \cong V^i \otimes \mathbb{Q}_\ell$ for $i = -2, 0$, which in turn depend only on the choice of an identification $\mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell$. In what follows we may thus change the splitting of the weight filtration and the identification $\mathrm{Gr}_{-1}(V_\ell) \cong V^{-1} \otimes \mathbb{Q}_\ell$ without affecting the properties above. We have established the following proposition.

Proposition 1.5. *Let notation and assumptions be as above, in particular the motive M is assumed to be strict and $N_\ell\colon V_\ell \to V_\ell(-1)$ is the ℓ-adic monodromy operator. For each ℓ, fix an identification of $H_{/\mathbb{Q}_\ell}$ with the group of L_ℓ-linear automorphisms of V_ℓ and an isomorphism $\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(1)$. Using these identifications we consider N_ℓ as an element of $\mathfrak{h} \otimes \mathbb{Q}_\ell$.*

- *For every algebraically closed field $\Omega \supset \mathbb{Q}_\ell$ and every $\sigma \in \mathrm{Aut}(\Omega)$, the image of N_ℓ in $(\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \Omega = \mathfrak{h} \otimes \Omega$ is conjugate to $\sigma(N_\ell)$ under the adjoint action of $H(\Omega)$.*
- *If Ω is an algebraically closed field containing both $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$ then the images in $\mathfrak{h} \otimes \Omega$ of N_ℓ and $N_{\ell'}$ are $H(\Omega)$-conjugate.*

1.6. ***The action of inertia.*** The Γ_F-action on each V_ℓ is L_ℓ-linear so the realisations of M give rise to a system of representations

$$\rho_\ell\colon \Gamma_F \to H(\mathbb{Q}_\ell),$$

using the identifications from Section 1.1 and the group H from Section 1.3.

Following [Deligne 1973, §2] for the basic notation, we discuss the action of the inertia group $I_F \subset \Gamma_F$. Let v be the valuation of F with value group $\mathbb{Z}$ and let k be the (finite) residue field. We will write $\bar{v}$ for the valuation on $\bar{F}$ extending v and write $\bar{k}$ for the residue field of $\bar{F}$.

Let

$$\widehat{\mathbb{Z}}_{\neq p} = \lim_{p \nmid n} \mathbb{Z}/n\mathbb{Z} = \prod_{\ell \neq p} \mathbb{Z}_\ell$$

be the p-primary part of $\widehat{\mathbb{Z}}$ and let $\mathbb{A}_{\neq p} = \widehat{\mathbb{Z}}_{\neq p} \otimes_{\mathbb{Z}} \mathbb{Q}$. The prime-to-$p$ part of $\mathbb{Q}/\mathbb{Z}$ is $\mathbb{A}_{\neq p}/\widehat{\mathbb{Z}}_{\neq p} = (\mathbb{Q}/\mathbb{Z})_{\neq p}$. For every n with $p \nmid n$, we identify $(\frac{1}{n}\mathbb{Z}/\mathbb{Z})(1)$ with the group of n-th roots of unity in $\bar{k}$. This identifies $(\mathbb{Q}/\mathbb{Z})_{\neq p}(1)$ with the multiplicative group $\bar{k}^\times$. There is a natural morphism $t\colon I_F \to \widehat{\mathbb{Z}}_{\neq p}(1)$ such that $\sigma(x)x^{-1} = [t(\sigma)\bar{v}(x)]$ for all $\sigma \in I_F$ and $x \in \bar{F}^\times$. Here $[t(\sigma)\bar{v}(x)]$ is the image in $\bar{k}^\times \cong (\mathbb{Q}/\mathbb{Z})_{\neq p}(1)$ of $t(\sigma)\bar{v}(x) \in \mathbb{A}_{\neq p}(1)$. For $\ell \neq p$, we write $t_\ell\colon I_F \to \mathbb{Z}_\ell(1)$ for the composite of t with the projection $\widehat{\mathbb{Z}}_{\neq p}(1) \to \mathbb{Z}_\ell(1)$.

In the case where Y and G have good reduction, it follows from [Raynaud 1994, Proposition 4.6.1] that if $\ell \neq p$ then for each $\sigma \in I_F$ one has

$$\rho_\ell(\sigma) = \exp(N_\ell \otimes t_\ell(\sigma)), \tag{1.6$*$}$$

where $N_\ell \in \mathfrak{h}^{\mathrm{ss}} \otimes \mathbb{Q}_\ell(-1)$ is the ℓ-adic monodromy operator defined in Section 1.4. For an arbitrary strict 1-motive, the equality above holds for all σ in a sufficiently small open subgroup of I_F. We finally note that (1.6$*$) characterises the operator N_ℓ as a map $N_\ell\colon V_\ell \to V_\ell(-1)$. This will play an important role in Section 2.2, in particular in the formula (2.2$*$).

1.7. ***Characteristic polynomials.*** Write $q = |k|$ and let φ be the arithmetic Frobenius automorphism $\varphi\colon x \mapsto x^q$ of $\bar{k}$ over k. The *Weil group* of F is the subgroup of Γ_F consisting of the elements ψ inducing an integral power of $\varphi^{\alpha(\psi)}$ of φ. The map $\alpha\colon W_F \to \mathbb{Z}$ thus defined is a group homomorphism and its kernel is the inertia group $I_F \subset \Gamma_F$. We endow the Weil group with the topology determined by the condition that $I_F \subset W_F$ is an open subgroup carrying the topology inherited from its topology as a Galois group.

For a 1-motive M/F with L-action as before, $k = -2, -1$ or 0 and $\psi \in W_F$, let

$$P^{(k)}_{L_\ell,\psi} \in L_\ell[T]$$

be the characteristic polynomial of $\rho_\ell(\psi)$ acting as an L_ℓ-linear endomorphism on the free L_ℓ-module $\mathrm{Gr}_k(V_\ell)$. Let $P_{L_\ell,\psi}$ be the characteristic polynomial of $\rho_\ell(\psi)$ acting L_ℓ-linearly on V_ℓ. Obviously, one has

$$P_{L_\ell,\psi} = \prod_{k=-2}^{0} P^{(k)}_{L_\ell,\psi}.$$

Proposition 1.8. *Let notation and hypotheses be as above*; *in particular M is assumed to be strict. Let ℓ run though the primes different from p. Then for each $k = -2, -1, 0$ and any $\psi \in W_F$, we have $P^{(k)}_{L_\ell,\psi} \in L[T]$. For fixed k and ψ, the* *polynomial $P^{(k)}_{L_\ell,\psi}$ is independent of ℓ and all its complex roots have absolute value $q^{-\alpha(\psi)k/2}$. The polynomial $P_{L_\ell,\psi}$ belongs to $L[T]$ and is independent of ℓ.*

Proof. It is sufficient to prove the statements concerning the $P^{(k)}_{L_\ell,\psi}$. For $k=-2,0$, these follow from the Γ_F-equivariant isomorphisms $\mathrm{Gr}_{-2}(V_\ell) \cong (Y^\star)^\vee \otimes \mathbb{Q}_\ell(1)$ and $\mathrm{Gr}_0(V_\ell) \cong Y \otimes \mathbb{Q}_\ell$ and the fact that Γ_F acts on Y and on $Y^\star$ through finite quotients.

For $k=-1$ we have $\mathrm{Gr}_{-1}(V_\ell) \cong V_\ell A$, where A is an abelian variety with $L \subset \mathrm{End}^0(A)$. The statement about the absolute values of the roots of $P^{(-1)}_{L_\ell,\psi}$ therefore follows from the corollary to Theorem 3 in [Serre and Tate 1968]; see also [Raynaud 1994, 4.7.4]. Under the assumption that A has good reduction and that ψ is a Frobenius element, a proof of the claims that $P^{(-1)}_{L_\ell,\psi} \in L[T]$ and that this polynomial is independent of ℓ is sketched in [Noot 2009, 2.1]. Taking into account [Serre and Tate 1968, Theorem 2], the argument remains valid when A only has potentially good reduction and $\psi \in I_F$. For the case where ψ reduces to a nontrivial power of the Frobenius element, one replaces the use of [Serre and Tate 1968, Theorem 2] by the corollary to Theorem 3 in the same paper. □

1.9. ***Remark.*** The action of any $\psi \in W_F$ on $\mathrm{Gr}_k(V_\ell)$ is semisimple for any k. For $k=-2,0$ this results from the fact that Γ_F acts on $\mathrm{Gr}_0(V_\ell)$ and on $\mathrm{Gr}_{-2}(V_\ell)(-1)$ through a finite quotient. For $k=-1$ it follows from the fact that $\mathrm{Gr}_{-1}(V_\ell) \cong V_\ell A$, where A is an abelian variety over F with potentially good reduction. Combining this statement with the Proposition 1.8, it follows that each $\psi \in W_F$ with $\alpha(\psi) \neq 0$ acts semisimply on V_ℓ.

1.10. ***Frobenius weights.*** As before, M is a strict 1-motive over F with L-action. We fix an arithmetic Frobenius element $\Phi \in \Gamma_F$, that is, a lifting of the Frobenius automorphism φ of $\bar{k}$; see Section 1.7. The operator $N_\ell \colon V_\ell \to V_\ell(-1)$ defined in Section 1.4 is Γ_F-equivariant, which implies that $\mathrm{Ad}(\rho_\ell(\Phi))(N_\ell) = qN_\ell$.

As noted in Section 1.9, the image $\rho_\ell(\Phi)$ is semisimple and by Proposition 1.8 its eigenvalues are algebraic integers and, for any eigenvalue, all complex absolute values coincide and are equal to q, $q^{1/2}$ or 1. For $k=-2,-1,0$, let $V^k_\ell \otimes \overline{\mathbb{Q}}_\ell$ be the sum of the eigenspaces associated to the eigenvalues with absolute value $q^{-k/2}$. This defines a splitting $V_\ell = V^{-2}_\ell \oplus V^{-1}_\ell \oplus V^0_\ell$ of the weight filtration. The *Frobenius weight cocharacter* $w_\ell \colon \mathbb{G}_{m/\mathbb{Q}_\ell} \to H_{/\mathbb{Q}_\ell}$ is the morphism making $\mathbb{G}_{m/\mathbb{Q}_\ell}$ act on V^k_ℓ through the $(k+1)$-st power map. The reader should take note of the shift in filtration, which is introduced to simplify matters later on. Through the adjoint representation, $w_\ell(t)$ acts on the line in $\mathfrak{h}^{\mathrm{ss}} \otimes \mathbb{Q}_\ell(-1)$ generated by N_ℓ as multiplication by t^{-2}.

If M/F is any, not necessarily strict, 1-motive with L-action, then the complex absolute values of any eigenvalue of $\rho_\ell(\Phi)$ are still equal to q, $q^{1/2}$ or 1. This follows from the existence [Raynaud 1994, 4.2.2] of a strict 1-motive M' with L-action endowed with a system of canonical isomorphisms $V_\ell(M) \cong V_\ell(M')$. The Frobenius weight cocharacter w_ℓ of M can therefore be defined exactly as before. It corresponds to the cocharacter associated to M' by transport via the isomorphism $V_\ell(M) \cong V_\ell(M')$ above. In general, w_ℓ does not split the weight filtration.

Finally note that, for M strict, the identification $V_\ell \cong V \otimes \mathbb{Q}_\ell$ can be modified, without affecting its previously established properties, to ensure that the grading on V_ℓ defined by the Frobenius weights corresponds to the grading on V.

2. The representations of the Weil–Deligne group associated to a 1-motive

2.1. ***The Weil–Deligne group.*** In addition to the conventions in Section 1.1, we will from now on assume that $\ell, \ell' \neq p$. As in Section 1.7, W_F is the Weil group of F. We briefly summarise some of the notions introduced in [Deligne 1973, §8]; see also [Fontaine 1994].

Letting $\psi \in W_F$ operate on the additive group $\mathbb{G}_{a/\mathbb{Q}}$ as multiplication by $q^{\alpha(\psi)}$, one defines an action of the constant topological group scheme W_F on $\mathbb{G}_{a/\mathbb{Q}}$. The Weil–Deligne group of F is the semidirect product

$$'W_F = W_F \ltimes \mathbb{G}_a$$

defined by this action, viewed as a group scheme over $\mathbb{Q}$.

Fix an identification $\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(1)$ as in Section 1.1, an arithmetic Frobenius element $\Phi \in W_F$ as in Section 1.10 and consider the map t_ℓ from Section 1.6 as a morphism

$$I_F \to \mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell = \mathbb{G}_a(\mathbb{Q}_\ell).$$

We define a system of ℓ-adic representations of W_F with values in $'W_F(\mathbb{Q}_\ell)$ by

$$\psi \mapsto \big(\psi, t_\ell(\Phi^{-\alpha(\psi)}\psi)\big) \in (W_F \ltimes \mathbb{G}_a)\,(\mathbb{Q}_\ell).$$

For a field E of characteristic 0 and a linear algebraic group $G_{/E}$ over E, giving an algebraic representation $('W_F)_{/E} \to G_{/E}$ is equivalent to giving a pair (ρ', N) where $\rho' \colon W_F \to G_{/E}(E)$ is a linear representation that is trivial on some open subgroup of I_F and $N \in \mathrm{Lie}(G_{/E})$ is a nilpotent element satisfying the condition that

$$\mathrm{Ad}(\rho'(\psi))N = q^{\alpha(\psi)}N \tag{2.1$*$}$$

for all $\psi \in W_F$. The representation of $('W_F)_{/E}$ corresponding to the pair (ρ', N) is given by $(\psi, x) \mapsto \rho'(\psi)\exp(Nx)$.

2.2. ***$'W_F$ and ℓ-adic Galois representations.*** Let $H_{/\mathbb{Q}_\ell}$ be a $\mathbb{Q}_\ell$-linear algebraic group and $\rho_\ell \colon W_F \to H_{/\mathbb{Q}_\ell}(\mathbb{Q}_\ell)$ a continuous representation. By Grothendieck's

ℓ-adic monodromy theorem (see [Deligne 1973, 8.2]), there exists a nilpotent element $N'_\ell \in \mathfrak{h}^{\mathrm{ss}}(-1) = \mathrm{Lie}(H_{/\mathbb{Q}_\ell})^{\mathrm{ss}}(-1)$ such that

$$\rho_\ell(\psi) = \exp(N'_\ell t_\ell(\psi)) \tag{2.2$*$}$$

for all ψ in a sufficiently small open subgroup of I_F; see (1.6$*$). One can therefore associate to ρ_ℓ a representation (ρ'_ℓ, N'_ℓ) of $'W_F$ with values in $H_{/\mathbb{Q}_\ell}$ as follows.

Using the identification $\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(1)$ to interpret N'_ℓ as an element of $\mathfrak{h}^{\mathrm{ss}}$, one defines

$$\rho'_\ell(\psi) = \rho_\ell(\psi)\exp\bigl(-N'_\ell t_\ell(\Phi^{-\alpha(\psi)}\psi)\bigr).$$

Composing the corresponding algebraic representation of $'W_F$ with the natural representation $W_F \to {'W_F}(\mathbb{Q}_\ell)$ defined above, one recovers ρ_ℓ.

According to [Deligne 1973, 8.11], the geometric conjugacy class of (ρ'_ℓ, N'_ℓ) is independent of the choices of Φ and of the identification $\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(1)$ made in this construction.

2.3. ***Compatible systems of representations of $'W_F$.*** Let H be a reductive algebraic group over $\mathbb{Q}$. For a fixed ℓ, we say that a representation $'W_{F/\mathbb{Q}_\ell} \to H_{\mathbb{Q}_\ell}$ is *defined over* $\mathbb{Q}$ (*as a representation with values in* H) if for every algebraically closed field $\Omega \supset \mathbb{Q}_\ell$, the base extension $'W_{F/\Omega} \to H_{/\Omega}$ is conjugate under $H(\Omega)$ to all its images under $\mathrm{Aut}_{\mathbb{Q}}(\Omega)$. In terms of the pair (ρ'_ℓ, N'_ℓ), let

$$\rho'_\ell \otimes_{\mathbb{Q}_\ell} \Omega \colon W_F \to H_{/\Omega}(\Omega)$$

be the extension of scalars and let $N'_\ell \otimes_{\mathbb{Q}_\ell} 1 \in (\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \Omega = \mathfrak{h} \otimes \Omega$ be the image of N'_ℓ. Then the condition above is equivalent to the condition that for every $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$ there is an element $g \in H(\Omega)$ such that

$$\sigma(\rho'_\ell \otimes_{\mathbb{Q}_\ell} \Omega) = g(\rho'_\ell \otimes_{\mathbb{Q}_\ell} \Omega)g^{-1} \quad \text{and} \quad \sigma(N'_\ell \otimes_{\mathbb{Q}_\ell} 1) = \mathrm{Ad}(g)(N'_\ell \otimes_{\mathbb{Q}_\ell} 1). \tag{2.3$*$}$$

We say that a family of representations $'W_{F/\mathbb{Q}_\ell} \to H_{/\mathbb{Q}_\ell}$ is a *compatible system of representations of* $'W_F$ (*with values in* H) if for every pair (ℓ, ℓ') and every algebraically closed field Ω containing $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$, the base extensions to Ω of the ℓ-adic and ℓ'-adic representations of $'W_F$ are $H(\Omega)$-conjugate. In terms of the pairs (ρ'_ℓ, N'_ℓ) and $(\rho'_{\ell'}, N'_{\ell'})$, this means that there is a $g \in H(\Omega)$ such that

$$\rho'_\ell \otimes_{\mathbb{Q}_\ell} \Omega = g(\rho'_{\ell'} \otimes_{\mathbb{Q}_{\ell'}} \Omega)g^{-1} \quad \text{and} \quad N'_\ell \otimes_{\mathbb{Q}_\ell} 1 = \mathrm{Ad}(g)(N'_{\ell'} \otimes_{\mathbb{Q}_{\ell'}} 1) \in \mathfrak{h} \otimes \Omega. \tag{2.3$\dagger$}$$

The action of $H(\Omega)$ by conjugation factors through $H(\Omega) \to H^{\mathrm{ad}}(\Omega)$ so $H(\Omega)$-conjugacy may be replaced by $H^{\mathrm{ad}}(\Omega)$-conjugacy everywhere.

2.4. ***Application to* 1*-motives.*** We apply the discussion above to the system of ℓ-adic representations $V_\ell(M)$ associated to a 1-motive M with L action. Let M be as in Section 1.1 and, as was the case from Section 1.4 onward, continue to assume M to be strict. The numbers $r = \mathrm{rank}(Y)$, $r^\star = \dim(T)$ and $g = \dim(A)$ are as in Section 1.1 and we fix an L-vector space V and a system of identifications $V_\ell \cong V \otimes \mathbb{Q}_\ell$ as in Lemma 1.2. Let the algebraic group $H = \mathrm{Res}_{L/\mathbb{Q}} \mathrm{GL}_{/L}(V)$ be as in Section 1.3. For every ℓ, the group $H_{/\mathbb{Q}_\ell}$ identifies with the group of $L_\ell = L \otimes \mathbb{Q}_\ell$-linear endomorphisms of V_ℓ. The action of Γ_F on $V_\ell(M)$ is L_ℓ-linear, so it provides us with an ℓ-adic representation of ${}'W_F$ with values in $H_{/\mathbb{Q}_\ell}$, that is, a system of pairs (ρ'_ℓ, N'_ℓ), where $N'_\ell \in \mathfrak{h} \otimes \mathbb{Q}_\ell = \mathrm{Lie}(H) \otimes \mathbb{Q}_\ell$ and $\rho'_\ell \colon W_F \to H_{/\mathbb{Q}_\ell}$.

Lemma 2.5. *Let M/F be any 1-motive with L action. The Frobenius weight cocharacter w_ℓ commutes with the representation ρ'_ℓ.*

Proof. By construction, $\rho_\ell(\Phi) = \rho'_\ell(\Phi)$ and the same equality holds for all powers of Φ. In the construction of the Frobenius weight cocharacter, one may replace the Frobenius element Φ, and hence q, by any strictly positive power without modifying w_ℓ. This means that it is sufficient to prove that there is a strictly positive power Φ^n such that $\rho'_\ell(\Phi^n)$ lies in the centre of the image of ρ'_ℓ. This is obvious since ρ'_ℓ factors through an extension of the group generated by Φ by a finite quotient of I_F. □

Proposition 2.6. *Assume that we are in the situation of Section 2.4, so in particular M is strict. Each ℓ-adic representation*

$$ {}'W_{F/\mathbb{Q}_\ell} \to H_{/\mathbb{Q}_\ell} $$

is defined over $\mathbb{Q}$ and these representations form a compatible system of representations of ${}'W_F$ with values in H.

Proof. We first show that each representation is defined over $\mathbb{Q}$.

The operator N'_ℓ is determined by the fact that it satisfies (2.2$*$) for all ψ in a sufficiently small open subgroup of the inertia group I_F. The equality (1.6$*$) implies that the monodromy operator N_ℓ has the same property so we conclude that $N'_\ell = N_\ell$.

It follows from Proposition 1.5 that for every $\Omega \supset \mathbb{Q}_\ell$ and $\sigma \in \mathrm{Aut}(\Omega)$ as in Section 2.3, $N_\ell \in \mathfrak{h} \otimes \mathbb{Q}_\ell$ is $H(\Omega)$-conjugate to $\sigma(N_\ell)$. Let $g \in H(\Omega)$ be such that $\sigma(N_\ell) = \mathrm{Ad}(g)(N_\ell)$. It is sufficient to show that $\rho'_\ell \otimes \Omega$ and $g^{-1}\sigma(\rho'_\ell \otimes \Omega)g$ are conjugate under the stabiliser of N_ℓ in $H(\Omega)$. By elementary representation theory (see [Deligne 1973, Proposition 8.9]), it suffices to show that the representation ρ'_ℓ is semisimple and that, for every $\psi \in W_F$, the L_ℓ-linear characteristic polynomials of $\rho'_\ell(\psi)$ and of $g^{-1}\sigma(\rho'_\ell \otimes \Omega)g$ acting on each $\mathrm{Gr}_i^{\mathrm{mon}}(V_\ell)$ coincide. In fact, it is sufficient to prove that the traces coincide. Here $\mathrm{Gr}_i^{\mathrm{mon}}(V_\ell)$ is the associated graded of V_ℓ for the monodromy filtration defined by N_ℓ.

We first treat the semisimplicity. The restriction $\rho'_\ell|_{I_F}$ is semisimple because it factors through a finite quotient of I_F. As the action of W_F on each $\mathrm{Gr}_i(V_\ell)$ is semisimple, the semisimplicity of ρ'_ℓ results from Section 1.9 applied to any

$\psi \in W_F$ with $\alpha(\psi) \neq 0$ and the fact that W_F is an extension of the group generated by the Frobenius element to Φ by the inertia group I_F.

To establish the putative equality of the characteristic polynomials we will prove that the L_ℓ-linear characteristic polynomials of the $\rho'_\ell(\psi)$ acting on the $\mathrm{Gr}_i^{\mathrm{mon}}(V_\ell)$ lie in $L[T]$. By Proposition 1.8 the corresponding statement is true for the action of W_F on the $\mathrm{Gr}_i(V_\ell)$, the associated graded for the weight filtration. We finish the argument by passing to the graded for the monodromy filtration.

This is accomplished by considering the filtration

$$W_{-2}^{\mathrm{mon}} V_\ell = \mathrm{im}(N_\ell) \subset W_{-2} V_\ell \subset W_{-1} V_\ell \subset W_{-1}^{\mathrm{mon}} V_\ell = \ker(N_\ell) \subset W_0 V_\ell = V_\ell.$$

The isomorphism $\mathrm{Gr}_0(V_\ell) \cong V^0 \otimes \mathbb{Q}_\ell$ is Γ_F-equivariant and the action of Γ_F on $V^0 \otimes \mathbb{Q}_\ell$ comes from its L-linear action on V^0. Similarly, the action of Γ_F on $\mathrm{Gr}_{-2}(V_\ell) \cong V^{-2} \otimes \mathbb{Q}_\ell(1)$ comes from its L-linear action on V^{-2} and the cyclotomic action on $\mathbb{Q}_\ell(1)$. Finally, N_ℓ comes from the L-linear map $N\colon V^0 \to V^{-2}$ so by (2.1$*$), the kernel of N in V^0 and the image of N in V^{-2} are W_F-invariant L-linear subspaces. The representation induced by ρ'_ℓ on each of the spaces

$$\begin{aligned} &\mathrm{Gr}_{-2}^{\mathrm{mon}} V_\ell = \mathrm{im}(N_\ell) \subset \mathrm{Gr}_{-2} V_\ell, && W_{-2} V_\ell / \mathrm{im}(N_\ell) \cong \mathrm{Gr}_{-2} V_\ell / \mathrm{Gr}_{-2}^{\mathrm{mon}} V_\ell, \\ &\ker(N_\ell)/W_{-1} V_\ell \subset \mathrm{Gr}_0 V_\ell, \quad \text{and} && \mathrm{Gr}_0^{\mathrm{mon}} V_\ell = V_\ell / \ker(N_\ell), \end{aligned}$$

therefore, is a base extension of a representation of W_F on an L-vector space. This proves the claim for $\mathrm{Gr}_{-2}^{\mathrm{mon}} V_\ell$ and $\mathrm{Gr}_0^{\mathrm{mon}} V_\ell$. For $\mathrm{Gr}_{-1}^{\mathrm{mon}} V_\ell$ the claim follows similarly by considering the graded for the filtration

$$W_{-2} V_\ell / \mathrm{im}(N_\ell) \subset W_{-1} V_\ell / \mathrm{im}(N_\ell) \subset \ker(N_\ell)/\mathrm{im}(N_\ell) = \mathrm{Gr}_{-1}^{\mathrm{mon}} V_\ell.$$

The fact that the representations ${}'W_{F/\mathbb{Q}_\ell} \to H_{/\mathbb{Q}_\ell}$ form a compatible system is proved by an analogous argument. One now has to prove that, for $i = -2, -1, 0$ and for each $\psi \in W_F$, the L_ℓ-linear characteristic polynomials of $\rho'_\ell(\psi)$ acting on $\mathrm{Gr}_i^{\mathrm{mon}}(V_\ell)$ are independent of ℓ. Again by Proposition 1.8, this is true for the characteristic polynomials on the $\mathrm{Gr}_i(V_\ell)$. The ℓ independence of the characteristic polynomials on the $\mathrm{Gr}_i^{\mathrm{mon}}(V_\ell)$ follows from this by considering the combined filtration and adapting the argument above. □

Corollary 2.7. *Let M be any 1-motive over F with L-action. Then for each ℓ, the ℓ-adic realisation $V_\ell(M)$ is a free L_ℓ-module. For $\ell \neq p$, the representations ${}'W_{F/\mathbb{Q}_\ell} \to H_{/\mathbb{Q}_\ell}$ are defined over $\mathbb{Q}$ and form a compatible system of representations of ${}'W_F$ with values in H.*

Proof. By Lemma 1.2, each $V_\ell(M)$ is a free L_ℓ-module. This implies that $H_{/\mathbb{Q}_\ell}$ identifies with the group of L_ℓ-linear endomorphisms of $V_\ell(M)$ and that any two such identifications differ by an inner automorphism of $H_{/\mathbb{Q}_\ell}$. It is therefore sufficient to prove the second statement for one system of such identifications.

By [Raynaud 1994, 4.2.2] there are a strict 1-motive M' over F and a system of canonical isomorphisms $V_\ell(M) \cong V_\ell(M')$ (for every ℓ). Using these identifications and the remark above, the corollary follows from Proposition 2.6. □

Corollary 2.8. *With the notation and hypotheses of Corollary 2.7,* $\ker(\rho'_\ell) \subset W_F$ *is independent of* ℓ.

3. Application to abelian varieties and statement of the main theorem

We turn our attention to an abelian variety A over a number field $F \subset \mathbb{C}$. If F is sufficiently big, each ℓ-adic representation associated to A factors through $\rho_\ell \colon \Gamma_F \to G_A(\mathbb{Q}_\ell)$, where G_A is the Mumford–Tate group of $A_{/\mathbb{C}}$. For a fixed valuation v of F, the construction sketched in Section 2.2 gives rise to a system of ℓ-adic representations ${}'W_{F_v/\mathbb{Q}_\ell} \to G_{A/\mathbb{Q}_\ell}$ of the Weil–Deligne group of F_v. It is hoped that these representations are defined over $\mathbb{Q}$ and that they form a compatible system of representations with values in G_A.

The statement of the main theorem is somewhat weaker; loosely speaking, it states that, after a finite extension of F, the representations of ${}'W_{F_v}$ form a compatible system when G_A is replaced by a larger group of which G_A is the identity component. As the construction will show, only certain factors of G^{der} of type D are affected by this modification. In order to formulate the precise statement we need a number of constructions from the previous paper [Noot 2009].

3.1. ***Notation.*** From now on, $F \subset \mathbb{C}$ is a number field and A/F an abelian variety. Let $\bar{F}$ be the algebraic closure of F in $\mathbb{C}$ and $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group. We fix a valuation $\bar{v}$ of $\bar{F}$ and let v be its restriction to F. Let p be the residue characteristic of v and F_v the completion of F at v. It is a finite extension of $\mathbb{Q}_p$. Let $\ell, \ell' \neq p$ prime numbers.

3.2. ***Abelian varieties.*** Betti cohomology defines a fibre functor $\mathrm{H}_{\mathrm{B}} = \mathrm{H}^1_{\mathrm{B}}$ on the category of absolute Hodge motives generated by the motive of A and the Tate motive $\mathbb{Q}(1)$. The Mumford–Tate G_A of A is the group of $\otimes$-automorphisms of this fibre functor; see [Noot 2009, 1.2] for a more detailed explanation. We will assume throughout that G_A is connected, this condition holds after replacing F by a finite extension and it implies that G_A is the smallest linear algebraic $\mathbb{Q}$-group such that the Hodge structure on $\mathrm{H}^1_{\mathrm{B}}(A(\mathbb{C}), \mathbb{Q})$ is defined by a morphism $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m \to G_{A/\mathbb{R}}$; see [ibid., 1.2]. Let $\mathfrak{g}_A$ be the Lie algebra of G_A.

For every ℓ, the fibre functor $\mathrm{H}_{\text{ét},\ell}$ defined by the ℓ-adic étale cohomology is canonically isomorphic to $\mathrm{H}_{\mathrm{B}} \otimes \mathbb{Q}_\ell$. The representation of Γ_F on the ℓ-adic étale cohomology makes Γ_F act on the functor $\mathrm{H}_{\text{ét},\ell}$ and this gives rise to a morphism

$$\rho_\ell \colon \Gamma_F \to G_A(\mathbb{Q}_\ell).$$

Let Φ_v an arithmetic Frobenius element, belonging to the decomposition group $\Gamma_{F_v} \cong D_{\bar{v}} \subset \Gamma_F$. This gives rise to local data as in Section 2.1. We consider the restriction of the ρ_ℓ to the Weil group $W_v = W_{F_v}$. As explained in Section 2.2, it defines a representation of the Weil–Deligne group ${}'W_v = {}'W_{F_v}$, that is, a pair (ρ'_ℓ, N'_ℓ) with $\rho'_\ell \colon W_v \to G_A(\mathbb{Q}_\ell)$ and $N'_\ell \in \mathfrak{g}_A^{\mathrm{ss}} \otimes \mathbb{Q}_\ell$. By Corollary 2.8, there is an open subgroup J of the inertia group $I_{\bar{v}}$ such that each ρ'_ℓ is trivial on J.

To apply the results above on 1-motives, it is convenient to work with Tate modules instead of étale cohomology groups. The ℓ-adic Galois representation $V_\ell A = T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is dual to $\mathrm{H}^1_{\text{ét}}(A_{/\bar{F}}, \mathbb{Q}_\ell)$. Identification of the fibre functor defined by V_ℓ to the dual of the one defined by $\mathrm{H}_{\text{ét},\ell}$ endows V_ℓ with the structure of a representation of $G_{A/\mathbb{Q}_\ell}$. The action of Γ_F on $V_\ell A$ is given by the same morphism $\rho_\ell \colon \Gamma_F \to G_A(\mathbb{Q}_\ell)$ as before. The corresponding representation (ρ'_ℓ, N'_ℓ) of the Weil–Deligne group ${}'W_{F_v}$ is also unchanged.

3.3. ***The group $G^{\natural\,\mathrm{ad}}$.*** In [Noot 2009, 1.5] one finds the construction of a group $\mathrm{Aut}'(G)_{/\overline{\mathbb{Q}}}$ of automorphisms of $G_{A/\overline{\mathbb{Q}}}$. In this paper we will write $G^{\natural\,\mathrm{ad}}$ for this "natural extension" of G^{ad}. We briefly sketch its construction.

The derived group $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$ is the almost direct product of almost simple subgroups $G_i \subset G_{A/\overline{\mathbb{Q}}}$, for i in some index set I. Let $J \subset I$ be the set of indices i such that $G_i \cong \mathrm{SO}(2k_i)_{/\overline{\mathbb{Q}}}$ for some $k_i \geq 4$ and for each $i \in J$ put $G'_i = \mathrm{O}(2k_i) \supset G_i$. We define

$$G^{\natural\,\mathrm{ad}} = \prod_{i\in J} G_i'^{\,\mathrm{ad}} \times \prod_{i\in I\setminus J} G_i^{\mathrm{ad}} \supset G^{\mathrm{ad}}_{A/\overline{\mathbb{Q}}}.$$

As this group operates trivially on the centre of $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$, we can define an action of $G^{\natural\,\mathrm{ad}}$ on $G_{A/\overline{\mathbb{Q}}}$ extending the adjoint action on $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$ and with $G^{\natural\,\mathrm{ad}}$ acting trivially on the centre of $G_{A/\overline{\mathbb{Q}}}$. Through the adjoint representation, the group $G^{\natural\,\mathrm{ad}}$ also acts on the Lie algebra $\mathfrak{g} \otimes \overline{\mathbb{Q}}$.

3.4. ***Compatible systems revisited.*** We introduce a variant of the notion, introduced in Section 2.3, of a compatible system defined over $\mathbb{Q}$ of representations of ${}'W_{F_v}$ with values in G_A. This time we allow conjugation by the group $G^{\natural\,\mathrm{ad}}$ so the condition is weaker than G-conjugacy if $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$ has factors of the form $\mathrm{SO}(2k)$.

Let v, ${}'W_v$ and Φ_v be as in Sections 3.1 and 3.2. For a fixed ℓ, let (ρ'_ℓ, N'_ℓ) define a representation of ${}'W_{v/\mathbb{Q}_\ell}$ with values in $G_{A/\mathbb{Q}_\ell}$. We say that this representation is *defined over $\mathbb{Q}$ modulo $G_A^{\natural\,\mathrm{ad}}$* if, for every algebraically closed field $\Omega \supset \mathbb{Q}_\ell$ and every $\sigma \in \mathrm{Aut}_{\mathbb{Q}}(\Omega)$, there is a $g \in G_A^{\natural\,\mathrm{ad}}(\Omega)$ such that

$$\sigma(\rho'_\ell \otimes_{\mathbb{Q}_\ell} \Omega) = g(\rho'_\ell \otimes_{\mathbb{Q}_\ell} \Omega)g^{-1} \quad \text{and} \quad \sigma(N'_\ell \otimes_{\mathbb{Q}_\ell} 1) = \mathrm{Ad}(g)(N'_\ell \otimes_{\mathbb{Q}_\ell} 1).$$

We say that a system of representations (ρ'_ℓ, N'_ℓ) of ${}'W_{v/\mathbb{Q}_\ell}$ is a *compatible system of representations of ${}'W_v$ modulo $G_A^{\natural\,\mathrm{ad}}$* if for every pair (ℓ, ℓ') and every algebraically closed field $\Omega \supset \mathbb{Q}_\ell, \mathbb{Q}_{\ell'}$, there is a $g \in G_A^{\natural\,\mathrm{ad}}(\Omega)$ such that

$$\rho'_\ell \otimes_{\mathbb{Q}_\ell} \Omega = g(\rho'_{\ell'} \otimes_{\mathbb{Q}_{\ell'}} \Omega)g^{-1} \quad \text{and} \quad N'_\ell \otimes_{\mathbb{Q}_\ell} 1 = \mathrm{Ad}(g)(N'_{\ell'} \otimes_{\mathbb{Q}_{\ell'}} 1) \in \mathfrak{g} \otimes \Omega.$$

Definition 3.5. Let Ω be an algebraically closed field, G a linear algebraic group over a subfield of Ω and V a representation of G. A semisimple $g \in G(\Omega)$ is *neat* if the Zariski closure of the subgroup of $G(\Omega)$ generated by g is connected. A semisimple element $g \in G(\Omega)$ is *weakly neat* (*with respect to* V) if 1 is the only root of unity among the quotients $\lambda\mu^{-1}$ of eigenvalues λ and μ of g.

For weak neatness, we will suppress the reference to V if it is clear which representation is being considered. For elements of the Mumford–Tate group of an abelian variety, we always consider the representation defined by the Tate module. A neat element is weakly neat with respect to any representation.

Theorem 3.6. *Assume that A has semistable reduction at v and that, for some ℓ, the image $\rho'_\ell(\Phi_v)$ is weakly neat. Then the representations (ρ'_ℓ, N'_ℓ) of $'W_v$ corresponding to A are defined over $\mathbb{Q}$ modulo the action $G_A^{\natural\,\mathrm{ad}}$. For $\ell \neq p$, these representations form a compatible system of representations of $'W_v$ modulo the action of $G_A^{\natural\,\mathrm{ad}}$.*

3.7. ***Remarks.***

3.7.1. The condition that A has semistable reduction at v implies that ρ'_ℓ is trivial on $I_{\bar{v}}$. In particular, the condition that $\rho'_\ell(\Phi_v)$ is weakly neat does not depend on the choice of the Frobenius element Φ_v. Also note that $\rho'_\ell(\Phi_v) = \rho_\ell(\Phi_v)$ so that the condition can also be checked on $\rho_\ell(\Phi_v)$. By the main theorem, or in a more elementary fashion by Proposition 1.8, the condition that $\rho'_\ell(\Phi_v)$ is weakly neat is independent of ℓ.

3.7.2. In general, A only has potentially semistable reduction at v. In this case, $\rho'_\ell|_{I_{\bar{v}}}$ has finite image so, for $\sigma \in I_{\bar{v}}$, all eigenvalues of $\rho'_\ell(\sigma)$ are roots of unity. The elements of $\rho'_\ell(I_{\bar{v}})$ are not neat and the methods of this paper do not seem to permit one to prove that the ρ'_ℓ form a compatible system in this case. As the monodromy operators are unchanged by a finite base extension, one may reduce to the case of stable reduction to prove that the N'_ℓ do form a compatible system defined over $\mathbb{Q}$.

3.7.3. To check the condition of weak neatness, one has to determine the characteristic polynomial of $\rho_\ell(\Phi_v)$ for at least one ℓ, which is not always feasible in practice. However, the condition always holds for a power of Φ_v, that is, after replacing F by a finite extension F' and v by a valuation of F' lying over v. Also note that if, for some $\ell \neq p$, the elements of $\rho_\ell(\Gamma_F)$ are congruent to 1 modulo ℓ (or congruent to 1 modulo 4 if $\ell = 2$), then $\rho_\ell(\Phi_v)$ is necessarily weakly neat.

On the other hand, it is easy to construct abelian varieties, even with good reduction, that do not satisfy the condition of weak neatness. For this, one may choose a p^a-Weil number α having two conjugates differing by a nontrivial root of unity. There exists an abelian variety A_0 over a finite field of characteristic p such that the characteristic polynomial of Frobenius is a power of the minimum polynomial of α. Any lifting of A_0 over a number field provides a counterexample to the neatness condition of the theorem.

3.7.4. If the abelian variety A has good reduction at v then the monodromy N'_ℓ is trivial and the theorem reduces to the main result, [Noot 2009, Théorème 1.8].

3.7.5. We finally refer to [ibid., Remark 1.9(4)] for a note on the density of the set of places v of good reduction where $\rho_\ell(\Phi_v)$ is weakly neat. Density statements of this type are not useful in the present context as the number of places where A does not have good reduction is finite.

3.8. ***G_A, monodromy and Frobenius weights.*** As pointed out in Section 3.2, the system (ρ_ℓ) is determined by the Galois representations on the Tate modules of A. From now on, we systematically adopt this point of view.

In the proof of Corollary 2.7, we applied [Raynaud 1994, 4.2] to the motive M in order to reduce to a strict motive M'. Applying the same argument to the abelian variety $A_v = A_{/F_v}$, one again obtains a strict 1-motive M'/F_v endowed with a system of canonical Γ_{F_v}-equivariant isomorphisms

$$V_\ell(A_v) = T_\ell(A_v) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong V_\ell(M').$$

Let $M' = [Y \to G]$, let $Y^\star$ be the character group of the toric part of G and write r and $r^\star$ for the ranks of Y and $Y^\star$. Let g be the dimension of the quotient of G by its maximal torus.

In Section 1.10 we defined the Frobenius weight cocharacter of a local Galois representation associated to a 1-motive. Applying this construction to the restrictions $\rho_\ell|_{D_v}$ we obtain the Frobenius weight cocharacter

$$w_\ell \colon \mathbb{G}_m \to \mathrm{GL}(V_\ell(A)).$$

Lemma 3.9. *Under the conditions above we have $r = r^\star$ and the monodromy operator $N \colon Y \otimes \mathbb{Q} \to Y^\star \otimes \mathbb{Q}$ associated to M' is an isomorphism. For each ℓ, the map N'_ℓ defines an isomorphism from the t-eigenspace of w_ℓ acting on $V_\ell(A)$ onto the t^{-1}-eigenspace.*

Proof. The arguments used in [Raynaud 1994, 4.2] show that $Y = \Lambda$, where Λ is the $\mathbb{Z}$-module in the diagram (**) of [loc. cit.], so r is equal to the rank of Λ. This reference also implies that $r^\star$ is equal to the rank of Λ, so $r = r^\star$. Still by [ibid., 4.2], the intersection of Λ with the rigid analytic generic fibre of G is trivial. With the notation of [ibid., 4.3], this means that for any $y \in Y$, there exists $y^\star \in Y^\star$ with

$\mu_o(y \otimes y^\star) > 0$. It follows that N induces an isomorphism $Y \otimes \mathbb{Q} \cong Y^\star \otimes \mathbb{Q}$. All this is classical; see for example [Grothendieck et al. 1972, Exposé IX, Théorème 10.4].

The monodromy filtration on $V_\ell(M')$ coincides with the weight filtration so the last statement immediately follows from the previous ones. □

Lemma 3.10. *The Frobenius weight cocharacter w_ℓ factors through $G_{A/\mathbb{Q}_\ell}$. In fact, this cocharacter factors through a torus $T_v \subset G_{A/\mathbb{Q}_\ell}$ containing $\rho_\ell(\Phi_v)$.*

Proof. The Mumford–Tate group G_A contains the group $\mathbb{G}_{m/\mathbb{Q}}$ of scalar multiplications of $\mathrm{H}^1_{\mathrm{B}}(A(\mathbb{C}), \mathbb{Q})$. Let $w'_\ell = t \cdot w_\ell$, let $T'_v \subset G_{A/\mathbb{Q}_\ell}$ be the identity component of the Zariski closure of the subgroup of $G_{A/\mathbb{Q}_\ell}$ generated by $\rho_\ell(\Phi_v)$ and put $T_v = \mathbb{G}_m T'_v$. We will prove that w_ℓ factors through T_v by showing that w'_ℓ factors through T'_v.

The last statement follows from the argument used in [Serre 2000], §4 of the first letter. The proof comes down to the fact that the eigenvalues of $w'_\ell(t)$ satisfy all the multiplicative relations satisfied by the archimedean absolute values of the eigenvalues of $\rho_\ell(\Phi_v)$. □

4. Generalities on tractable abelian varieties

4.1. ***Tractable abelian varieties.*** The notion of "variété abélienne accommodante" was introduced in [Noot 2009, 2.3]; in this paper we call such a variety a *tractable* abelian variety. Let us recall the relevant ideas.

First of all, we define the notion of an admissible representation of a reductive group. Heuristically, the admissible representations are the representations encountered when studying Shimura data of abelian type that admit an embedding into the Siegel Shimura datum. We refer to [Deligne 1979, 1.3] for this classification. To be precise, let K be a field of characteristic 0 and let $\Omega \supset K$ be an algebraically closed extension. Assume that G^s is a linear algebraic group over K such that $G^s_{/\Omega}$ is almost simple of type A, B, C or D. Let V^s be a faithful K-linear representation of G^s. We say that V^s is an *admissible* representation of G^s in the following cases:

- $G^s_{/\Omega}$ is of type A_n and $V^s \otimes_K \Omega$ is a multiple of the direct sum of the representations of highest weights ϖ_1 and ϖ_n if $n \geq 2$ and a multiple of the representation of highest weight ϖ_1 if $n = 1$.
- $G^s_{/\Omega}$ is of type B_n and $V^s \otimes_K \Omega$ is a multiple of the representation of highest weight ϖ_n.
- $G^s_{/\Omega}$ is of type C_n and $V^s \otimes_K \Omega$ is a multiple of the representation of highest weight ϖ_1.
- $G^s_{/\Omega}$ is of type D_n and $V^s \otimes_K \Omega$ is a multiple of the representation of highest weight ϖ_1.

- $G^s_{/\Omega}$ is of type D_n and $V^s \otimes_K \Omega$ is a multiple of the direct sum of the representations of highest weights ϖ_{n-1} and ϖ_n.

In the first three cases, we will say that the pair (G^s, V^s) is of type A_n, B_n or C_n, in the last two cases we say that (G^s, V^s) is of type $D_n^{\mathbb{H}}$ or of type $D_n^{\mathbb{R}}$, respectively.

Returning to abelian varieties, we let A be an abelian variety over $\mathbb{C}$ and write $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbb{C}), \mathbb{Q})$. We say that A is *strictly tractable* if

- there exists a totally real number field K and an almost simple linear algebraic group G^s over K such that $G_A^{\mathrm{der}} = \mathrm{Res}_{K/\mathbb{Q}} G^s$;
- as a representation of G_A^{der}, the cohomology group V is the restriction of scalars of an admissible representation V^s of G^s;
- if (G^s, V^s) is of type $D_n^{\mathbb{R}}$ then every character space in $V \otimes \overline{\mathbb{Q}}$ for the action of the centre of $G_{A/\overline{\mathbb{Q}}}$ is an admissible representation of a factor of $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$; and
- the conditions above do not hold for any proper abelian subvariety of A.

The type of a strictly tractable abelian variety is the type of the pair (G^s, V^s).

We will say that A is *tractable* if A is isogenous to a product $\prod_{i=1}^m A_i$ of strictly tractable abelian varieties A_i and $G_A^{\mathrm{der}} \cong \prod_{i=1}^m G_{A_i}^{\mathrm{der}}$. If $F \subset \mathbb{C}$ is a subfield, an abelian variety A/F is (strictly) tractable if $A_{/\mathbb{C}}$ is and if G_A is connected.

4.2. *Remark.* The concept of tractability is an auxiliary notion used in the proof of the main theorem. It does not seem to be of independent interest, though it is conceivable that the method of the present paper can be used in other contexts.

Heuristically, the fact that an abelian variety A is strictly tractable means that the representation of G_A^{der} on $V = \mathrm{H}^1_{\mathrm{B}}(A(\mathbb{C}), \mathbb{Q})$ decomposes over $\overline{\mathbb{Q}}$ as a direct sum of irreducible representations of the almost simple factors of $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$. In particular, any abelian variety $A/\mathbb{C}$ of dimension g with $G_A^{\mathrm{der}} = \mathrm{Sp}_{2g}$ is tractable. This means that, in the moduli space of g-dimensional abelian varieties, the points corresponding to nontractable varieties belong to a countable union of closed subvarieties and thus the general abelian variety is tractable.

If the simple factors of $V \otimes \overline{\mathbb{Q}}$, as a representation of $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$, are tensor products of irreducible representations of the almost simple factors $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$, then A is not tractable. The simplest such example was given by Mumford [1969]. The generic members of the families constructed there are not tractable.

More generally, any simple abelian variety $A/\mathbb{C}$, with $L = \mathrm{End}^0(A)$, whose Mumford–Tate group coincides with the group of L-linear symplectic similitudes of $\mathrm{H}^1_{\mathrm{B}}(A(\mathbb{C}), \mathbb{Q})$ is tractable. The converse is not true, as can be seen for example in the case where the Mumford–Tate group is of type D_n. Nevertheless, being tractable still means that the endomorphism algebra is big compared to the Mumford–Tate group. This is the key to the proof of the main theorem for tractable abelian varieties.

By definition, abelian varieties of CM type are not tractable. As pointed out in the introduction, the system of ℓ-adic representations associated to an abelian variety of CM type is described by the theory of complex multiplication and the main theorem is true in the CM case.

4.3. ***The algebra $L \subset$ End$^0(A)$.*** In the proof of the main Theorem 3.6, we will adapt the ideas used in [Noot 2009]. We will in particular make use of the algebra $L \subset \mathrm{End}(A) \otimes \mathbb{Q}$ constructed in the beginning of the proof of [ibid., Théorème 2.4]. For this construction, first decompose $V \otimes \overline{\mathbb{Q}} = \bigoplus_{i=1}^{n} V_i$, where the V_i are the isotypic components of the representation of $G_{A/\overline{\mathbb{Q}}}$ on $V \otimes \overline{\mathbb{Q}}$. In other words, each V_i is a multiple of an irreducible representation of $G_{A/\overline{\mathbb{Q}}}$ and $\mathrm{Hom}_{G_{A/\overline{\mathbb{Q}}}}(V_i, V_j) = 0$ for $i \neq j$. This decomposition defines a subalgebra $\overline{\mathbb{Q}}^n \subset \mathrm{End}_{G_{A/\overline{\mathbb{Q}}}}(V \otimes \overline{\mathbb{Q}})$, with the i-th factor $\overline{\mathbb{Q}}$ acting on the factor V_i by scalar multiplication. Taking $\Gamma_{\mathbb{Q}}$-invariants, this inclusion descends to

$$L \subset \mathrm{End}_{G_{A/\overline{\mathbb{Q}}}}(V \otimes \overline{\mathbb{Q}}) = \mathrm{End}^0(A_{/\mathbb{C}}) = \mathrm{End}^0(A),$$

where L is a finite, semisimple, commutative $\mathbb{Q}$-algebra. The last equality follows from the fact that G_A is connected and is justified in [Noot 2009, proof of 2.4].

There is a canonical isomorphism $L \otimes \overline{\mathbb{Q}} \cong \prod_{\iota :\, L \to \overline{\mathbb{Q}}} \overline{\mathbb{Q}}$; in fact L is defined as the algebra of $\Gamma_{\mathbb{Q}}$-invariants in the product on the right hand side. The direct factors of $V \otimes \overline{\mathbb{Q}}$ are indexed by the morphisms $\iota \colon L \to \overline{\mathbb{Q}}$, with the ι-factor of $\prod_\iota \overline{\mathbb{Q}}$ acting on V_ι by scalar multiplications and acting trivially on the other V_κ. The decomposition of $L \otimes \overline{\mathbb{Q}}$ thus gives rise to a decomposition $V \otimes \overline{\mathbb{Q}} = \prod_\iota V_\iota$. There is a similar decomposition $V \otimes \Omega = \prod_\iota V_\iota$ for any algebraically closed field Ω of characteristic 0, with the product taken over all morphisms $\iota \colon L \to \Omega$.

Lemma 4.4. *Assume that $A/\mathbb{C}$ is a strictly tractable abelian variety. Unless A is of type $D_n^{\mathbb{R}}$, the algebra above $L \subset \mathrm{End}(A) \otimes \mathbb{Q}$ is a field. If $A/\mathbb{C}$ of type $D_n^{\mathbb{R}}$ then the algebra L is either a field or it is isomorphic to $L' \times L'$ for a field L'.*

Proof. As L is a semisimple algebra, it decomposes as a product of fields. This decomposition gives rise to a corresponding decomposition of A and unless the pair (G^s, V^s) associated to A is of type A_n or $D_n^{\mathbb{R}}$, each factor still satisfies the first three conditions of the definition of a strictly tractable abelian variety. If there is more than one factor, this violates the minimality condition.

If the pair (G^s, V^s) associated to A is of type A_n, then the argument used in [Noot 2006, 5.1] shows that the complex conjugation acts on the Dynkin diagram of G^s by the main involution. This implies that a direct factor of $V \otimes \overline{\mathbb{Q}}$, which is a representation of highest weight ϖ_1 of some factor of $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$ belongs to the same $\Gamma_{\overline{\mathbb{Q}}}$-orbit as the representation of highest weight ϖ_n of the same factor. It follows that for any decomposition of A as above, each factor still satisfies the first three conditions of the definition. The minimality condition again implies that there is only one factor.

If (G^s, V^s) is of type $D_n^{\mathbb{R}}$, then a direct factor L' of L may define a factor A' of A such that the associated pair (G^s, V^s) is a half spin representation. In that case, A has a factor A'' for which (G^s, V'^s) is the other half spin representation. By minimality, one must have $A \sim A' \oplus A''$. In this case, the set of vertices ϖ_{n-1} and the set of vertices ϖ_n of the Dynkin diagram of G_A^{der} form two separate orbits for the $\Gamma_{\mathbb{Q}}$ action. Since these orbits are isomorphic as $\Gamma_{\mathbb{Q}}$-sets, it follows that $L = L' \oplus L'$. □

4.5. ***The group H.*** For the rest of this section we will assume, in addition to the hypotheses of Section 3.1, that A/F is tractable. Let $G_A^{\natural\,\mathrm{ad}}$ be the linear algebraic group over $\overline{\mathbb{Q}}$ introduced in Section 3.3.

To prove the theorem for A, we will use the results on 1-motives obtained in Section 2 so it is convenient to study the Galois representations defined by the Tate-modules; see Section 3.2. Let $V = \mathrm{H}_1(A(\mathbb{C}), \mathbb{Q})$ and for each prime number ℓ put $V_\ell = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, where $T_\ell(A)$ is the ℓ-adic Tate module of A. As in Section 3.2, there is a natural representation of G_A on V and there are canonical isomorphisms $V_\ell \cong V \otimes \mathbb{Q}_\ell$. The action of Γ_F on the V_ℓ is given by the representations $\rho_\ell \colon \Gamma_F \to G_A(\mathbb{Q}_\ell)$.

We closely follow the proof of [Noot 2009, Théorème 2.4]. Let the endomorphism algebra $L \subset \mathrm{End}^0(A)$ be as in Section 4.3. It is a product of number fields L_i for $i = 1, \ldots, s$. This decomposition gives rise to a decomposition up to isogeny $A \sim \prod_{i=1}^s A_i$ and to a corresponding decomposition $V = \bigoplus_{i=1}^s V_i$, where $V_i = \mathrm{H}_1(A_i(\mathbb{C}), \mathbb{Q})$. For each i one has $L_i \subset \mathrm{End}^0(A_i)$ and this action endows V_i with the structure of L_i-vector space. It follows from Lemma 4.4 that each factor A_i is either strictly tractable or that it is of type $D_n^{\mathbb{R}}$. If A_i is not strictly tractable, it follows from the definition and again from Lemma 4.4 that there is another factor A_j such that $L_i \cong L_j$ and the product $A_i \times A_j$ is strictly tractable. In this case we change definitions and put $A_i = A_i \times A_j$ and $L_i = L_i \times L_j$. We suppress the factors A_j and L_j and modify the value of s accordingly. After these modifications, we have decompositions $L = \prod_{i=1}^s L_i$ and $A \sim \prod_{i=1}^s A_i$, where all factors A_i are strictly tractable and each L_i is either a field of a product $L'_i \times L'_i$ of fields.

Let $d_i = \dim_{L_i} V_i$ and let

$$H = \prod_{i=1}^{s} \mathrm{Res}_{L_i/\mathbb{Q}}\ \mathrm{GL}_{/L_i}(V_i) = \prod_{i=1}^{s} H_i \cong \prod_{i=1}^{s} \mathrm{Res}_{L_i/\mathbb{Q}}\ \mathrm{GL}_{d_i/L_i}$$

be the centraliser of L in $\mathrm{GL}(V)$. In the case where $L_i \cong L'_i \times L'_i$, the factor H_i is

$$\left(\mathrm{Res}_{L'_i/\mathbb{Q}}\, \mathrm{GL}_{d_i/L'_i}\right)^2.$$

It is a linear algebraic group over $\mathbb{Q}$. The action of G_A on V commutes with the action of $\operatorname{End}^0(A)$, so $G_A \subset H$. The decomposition $H = \prod_{i=1}^s H_i$ corresponds to the decompositions of L, of A and of V. In particular, H_i is the only factor of H acting nontrivially on V_i. Writing G_{A_i} for the Mumford–Tate group of A_i one has $G_{A_i} \subset H_i$.

Let Ω be an algebraically closed extension of $\mathbb{Q}$. For each $\mathbb{Q}$-algebra homomorphism $\iota\colon L \to \Omega$ there is a unique index $i = i(\iota)$ such that ι factors through $L \to L_i \to \Omega$, where $L \to L_i$ is the projection and $L_i \to \Omega$ a ring homomorphism. This final map is an embedding if L_i is a field and an embedding of one of the factors of L_i if A_i is of type $D_n^{\mathbb{R}}$ and L_i is a product $L_i' \times L_i'$. Let $d_\iota = d_i$. Extending the base field to Ω one obtains

$$H_{/\Omega} \cong \prod_{\iota\colon L\to\Omega} \mathrm{GL}_{d_\iota/\Omega}\,.$$

The group $G_{A/\Omega}$ embeds into this product and for $\iota\colon L \to \Omega$ we let G_ι be its image in the factor $\mathrm{GL}_{d_\iota/\Omega}$ corresponding to ι.

This gives rise to similar decompositions of the Lie algebras. For $\mathfrak{h} = \operatorname{Lie}(H)$ we have $\mathfrak{h} \otimes_{\mathbb{Q}} \Omega = \bigoplus \mathfrak{h}_\iota = \bigoplus \operatorname{End}(V_\iota) \cong \bigoplus \mathfrak{gl}_{d_\iota/\Omega}$. The inclusion $G_A \subset H$ induces

$$\mathfrak{g}_A \otimes \Omega = \operatorname{Lie}(G_A) \otimes \Omega \hookrightarrow \bigoplus_{\iota\colon L\to\Omega} \mathfrak{g}_\iota \subset \bigoplus_\iota \operatorname{End}(V_\iota) = \mathfrak{h} \otimes \Omega, \tag{4.5$*$}$$

where $\mathfrak{g}_\iota \subset \mathfrak{h}_\iota = \operatorname{End}(V_\iota \otimes \Omega)$ is the Lie algebra of G_ι. For both the group $H_{/\Omega} \cong \prod \mathrm{GL}_{d_\iota/\Omega}$ and the Lie algebra $\mathfrak{h} \otimes_{\mathbb{Q}} \Omega \cong \prod \mathfrak{gl}_{d_\iota/\Omega}$, the obvious action of $\sigma \in \operatorname{Aut}(\Omega)$ on the left hand side translates on the right hand side to

$$\sigma\colon\ (x_\iota)_{\iota\colon L\to\Omega} \mapsto (\sigma(x_{\sigma^{-1}\iota}))_{\iota\colon L\to\Omega}\,. \tag{4.5†}$$

4.6. ***Monodromy and Frobenius weights.*** The ℓ-adic realisations of A decompose in the same way as the Betti realisation. Define $V_{i,\ell} = T_\ell(A_i) \otimes \mathbb{Q}_\ell$ for every ℓ. The decomposition of A gives rise to Γ_{F_v}-equivariant $L_\ell = L \otimes \mathbb{Q}_\ell$-linear decompositions $V_\ell = \bigoplus_{i=1}^s V_{i,\ell}$ and we have canonical L_i-linear isomorphisms $V_{i,\ell} \cong V_i \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

Assume that A has semistable reduction at v and let further notation be as in Section 3.1. Let q be the order of the residue field of F_v. To study the monodromy we follow Section 2.2, so we fix identifications $\mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell$ and define the monodromy operators N_ℓ' by (2.2$*$). One has $N_\ell' \in \mathfrak{g}_A^{\mathrm{ss}} \otimes \mathbb{Q}_\ell$ by Section 3.2. Similarly, for each A_i we have the monodromy operator $N_{i,\ell}' \in \mathfrak{g}_{A_i}^{\mathrm{ss}} \otimes \mathbb{Q}_\ell$. Under the embedding $\mathfrak{g}_A \subset \bigoplus_{i=1}^s \mathfrak{g}_{A_i}$ the element N_ℓ' maps to $(N_{i,\ell}')_{i=1,\dots,s}$.

For A and each one of the A_i, let the Frobenius weight cocharacters

$$w_\ell\colon \mathbb{G}_{m/\overline{\mathbb{Q}}_\ell} \to G_{A/\overline{\mathbb{Q}}_\ell} \quad \text{and} \quad w_{i,\ell}\colon \mathbb{G}_{m/\overline{\mathbb{Q}}_\ell} \to G_{A_i/\overline{\mathbb{Q}}_\ell}$$

be as in Section 3.8. These cocharacters factor through the Mumford–Tate groups by Lemma 3.10. As in the case of the monodromy operators, the cocharacter

$$(w_{i,\ell})_{i=1,\dots,s}\colon \mathbb{G}_{m/\overline{\mathbb{Q}}_\ell} \to \prod_{i=1,\dots,s} G_{A_i/\overline{\mathbb{Q}}_\ell}$$

is the composite of w_ℓ with the inclusion $G_A \subset \prod G_{A_i}$. It follows from Lemma 3.9 that w_ℓ and the $w_{i,\ell}$ split the monodromy filtrations on V_ℓ and on the $V_{i,\ell}$, respectively.

5. Strictly tractable abelian varieties of types A_n, C_n and $D_n^{\mathbb{H}}$

5.1. We keep the notation of Section 3.1. In this section, we will assume that A is a strictly tractable abelian variety and that its Mumford–Tate group G_A is of type A_n, C_n or $D_n^{\mathbb{H}}$. According to Lemma 4.4, the algebra L then is a field. In particular, where the group H introduced in Section 4.5 is concerned, we have $d_\iota = d_1 = d$ for all $\iota\colon L \to \Omega$. As in Section 4.6, consider the monodromy operator N'_ℓ and the weight cocharacter w_ℓ. It follows from Corollary 2.7 that the H-conjugacy class of (N'_ℓ, w_ℓ) is defined over $\mathbb{Q}$ and that it is independent of ℓ, in accordance with the terminology developed in Section 2.3.

For any algebraically closed extension Ω of $\mathbb{Q}$, the Lie algebra $\mathfrak{g}^{\mathrm{ss}} \otimes \Omega$ is a direct sum of simple Lie algebras. The groups $G_A^{\mathrm{ad}}(\Omega)$ and $G_A^{\natural\,\mathrm{ad}}(\Omega)$ act on each direct factor of this Lie algebra factor through a unique simple factor. Fix a factor $\mathfrak{g}_\iota^{\mathrm{ss}}$ of $\mathfrak{g}^{\mathrm{ss}} \otimes \Omega$ and consider the corresponding factors G_ι^{der} of $G^{\mathrm{der}}_{A/\Omega}$ and $G_\iota^{\natural\,\mathrm{ad}}$ of $G^{\natural\,\mathrm{ad}}_{A/\Omega}$. Under the sequence of embeddings (4.5$*$), $\mathfrak{g}_\iota$ embeds into a simple factor $\mathfrak{h}_\iota \cong \mathfrak{gl}_d$ of $\mathfrak{h} \otimes \Omega$. If G_A is of type A_n, then $\mathfrak{g}_\iota^{\mathrm{ss}}$ and G_ι^{der} act on the direct factor V_ι of $V \otimes \Omega$ either as a multiple of the standard representation or as its dual. If G_A is of type C_n or $D_n^{\mathbb{H}}$ in the classification, $\mathfrak{g}_\iota^{\mathrm{ss}}$ and G_ι^{der} act on V_ι as a multiple of the symplectic or the orthogonal representation respectively.

Let $\Omega \supset \mathbb{Q}_\ell$ be an algebraically closed field and let $\sigma \in \mathrm{Aut}(\Omega)$. As we saw above, (N'_ℓ, w_ℓ) and $\sigma(N'_\ell, w_\ell) \in \mathfrak{h} \otimes \Omega$ are conjugate under the adjoint action of $H(\Omega)$. Writing $N'_\ell = (N'_{\ell,\iota})_\iota \in \prod \mathfrak{h}_\iota$ as in (4.5$*$) and $w_\ell = (w_{\ell,\iota})_\iota$ it follows from the formula (4.5†) that the projections $(N'_{\ell,\iota}, w_{\ell,\iota})$ and $\sigma(N'_{\ell,\sigma^{-1}\iota}, w_{\ell,\sigma^{-1}\iota})$ of these pairs are $H_\iota(\Omega)$-conjugate.

In the case where G_A is of type A_n, it trivially follows that $N'_{\ell,\iota}$ and $\sigma(N'_{\ell,\sigma^{-1}\iota})$ are conjugate under the action of $G_\iota^{\mathrm{ad}}(\Omega) = G_\iota^{\natural\,\mathrm{ad}}(\Omega)$ on $\mathfrak{g}_\iota$. In the cases where G_A is of type C_n or D_n, it follows from [Springer and Steinberg 1970, IV §2], in particular from 2.14, that $N'_{\ell,\iota}$ and $\sigma(N'_{\ell,\sigma^{-1}\iota})$ are conjugate under the action of $G_\iota^{\natural\,\mathrm{ad}}(\Omega)$ on $\mathfrak{g}_\iota$. See also [Humphreys 1995, 7.11] for a summary of the results concerning the nilpotent conjugacy classes in the classical Lie algebras.

Similarly, if Ω is an algebraically closed field containing $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$ then the images of N'_ℓ and $N'_{\ell'}$ in $\mathfrak{h}^{\mathrm{ss}} \otimes \Omega$ are conjugate under $H(\Omega)$. The argument above implies that, for each ι, the operators $N'_{\ell,\iota}$ and $N'_{\ell',\iota}$ are conjugate under the action of $G_\iota^{\natural\,\mathrm{ad}}(\Omega)$ on $\mathfrak{g}_\iota$.

Next consider the weight cocharacter w_ℓ. For each ι, let $w_{\ell,\iota}$ be the projection of w_ℓ to G_ι. Recall that by Lemma 3.9, the monodromy operator $N'_{\ell,\iota}$ induces an isomorphism of the t-eigenspace of $w_{\ell,\iota}$ in V_ι onto the t^{-1}-eigenspace. Going through the arguments of [Springer and Steinberg 1970, IV], with $N'_{\ell,\iota}$ playing the role of X, one deduces that there exists a basis of V_ι satisfying the conditions of [ibid., IV, 2.19(b)] such that $w_{\ell,\iota}$ is the *inverse* of the cocharacter λ defined in [ibid., IV, 2.22]. In particular, $w_{\ell,\iota}$ factors through the derived group G_ι^{der}. This fact can also quite easily be shown directly.

If Ω is an algebraically closed field and $\sigma \in \mathrm{Aut}(\Omega)$ then, for $X = \sigma(N'_{\ell,\sigma^{-1}\iota})$, there is a basis of V_ι as in [ibid., IV, 2.19(b)] such that $\sigma(w_{\ell,\sigma^{-1}\iota})$ coincides with $\sigma(\lambda^{-1})$. We know that $N'_{\ell,\iota}$ and $\sigma(N'_{\ell,\sigma^{-1}\iota})$ are conjugate under $G_\iota^{\natural\,\mathrm{ad}}(\Omega)$. Moreover, any two bases of V that satisfy the conditions of [ibid., IV, 2.19(b)] are conjugate under the centraliser $Z_{\ell,\iota}^{\natural\,\mathrm{ad}}$ of $N'_{\ell,\iota}$ in $G_\iota^{\natural\,\mathrm{ad}}$. It follows that $(N'_{\ell,\iota}, w_{\ell,\iota})$ and $\sigma(N'_{\ell,\sigma^{-1}\iota}, w_{\ell,\sigma^{-1}\iota})$ are $G_\iota^{\natural\,\mathrm{ad}}(\Omega)$-conjugate.

If ℓ and ℓ' are two prime numbers and if Ω is an algebraically closed field containing both $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$, then the same argument, applied to $(N'_{\ell,\iota}, w_{\ell,\iota})$ and $(N'_{\ell',\iota}, w_{\ell',\iota})$, proves that these two pairs are $G_\iota^{\natural\,\mathrm{ad}}(\Omega)$-conjugate.

Proposition 5.2. *Let notation and assumptions be as above. In particular, A/F is a strictly tractable abelian variety of type A_n, B_n or $D_n^{\mathbb{H}}$.*

- *For every ℓ, every algebraically closed field $\Omega \supset \mathbb{Q}_\ell$ and every $\sigma \in \mathrm{Aut}(\Omega)$, the image of (N'_ℓ, w_ℓ) in $\mathfrak{g} \otimes \Omega \times X(G_{A/\Omega})$ is conjugate to $\sigma(N'_\ell, w_\ell)$ under the adjoint action of $G_A^{\natural\,\mathrm{ad}}(\Omega)$.*
- *If Ω is an algebraically closed field containing both $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$, then the images in $\mathfrak{g} \otimes \Omega \times X(G_{A/\Omega})$ of (N'_ℓ, w_ℓ) and $(N'_{\ell'}, w_{\ell'})$ are $G_A^{\natural\,\mathrm{ad}}(\Omega)$-conjugate.*

Proof. As A is tractable, the group $G^{\mathrm{der}}_{A/\Omega}$ is the product of its almost simple factors. If G_A is of type C_n or $D_n^{\mathbb{H}}$, then $G^{\mathrm{der}}_{A/\Omega}$ is the product of the G_ι^{der}. In the case where G_A is of type A_n, we saw in the proof of Lemma 4.4 that the complex conjugation acts nontrivially on each component of the Dynkin diagram. If $n \geq 2$, it follows from [Noot 2006, 5.1] that L is a CM field and hence that the complex conjugation defines an nontrivial involution $\iota \mapsto \iota'$ on the set of embeddings $L \to \Omega$. In this case, $G^{\mathrm{der}}_{A/\Omega}$ is a product of groups $\Delta_{\{\iota,\iota'\}}$, where each $\Delta_{\{\iota,\iota'\}} \subset G_\iota^{\mathrm{der}} \times G_{\iota'}^{\mathrm{der}}$ is the graph of an isomorphism $G_\iota^{\mathrm{der}} \cong G_{\iota'}^{\mathrm{der}}$. Identifying $\Delta_{\{\iota,\iota'\}}$ with G_ι^{der} through the projection on the first factor, the representation of $\Delta_{\{\iota,\iota'\}}$ on V_ι is a multiple of the representation with highest weight ϖ_1 and $V_{\iota'}$ is its dual, a multiple of the representation with highest weight ϖ_n. The case where $n = 1$ is left to the reader.

The $N'_{\ell,\iota}$ belong to $\mathfrak{g}_\iota^{\mathrm{ss}}$ and, as we pointed out in Section 5.1, the $w_{\ell,\iota}$ factor through G_ι^{der}. The proposition follows from the fact, proved in Section 5.1, that

$(N'_{\ell,\iota}, w_{\ell,\iota})$ and $\sigma(N'_{\ell,\iota}, w_{\ell,\iota})$ are $G_\iota^{\natural\,\mathrm{ad}}(\Omega)$-conjugate for each ι, combined with the formula (4.5†). □

5.3. ***The centraliser of (N'_ℓ, w_ℓ) in $G^\natural_{A/\overline{\mathbb{Q}}_\ell}$.*** We return to the construction 3.3 of the group $G_A^{\natural\,\mathrm{ad}}$. In the case considered here, this group is the adjoint group of a $\overline{\mathbb{Q}}$-group $G_A^\natural$ containing the derived Mumford–Tate group $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$. In fact, one has to take $G_A^\natural = G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$ if G_A is of type A_n or C_n and in this case we put $G_\iota^\natural = G_\iota^{\mathrm{der}}$ for each embedding $\iota\colon L \to \overline{\mathbb{Q}}$. If G_A is of type $D_n^{\mathbb{H}}$, then

$$G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}} = \prod G_\iota^{\mathrm{der}},$$

where each $G_\iota^{\mathrm{der}} \cong \mathrm{SO}_{2n}$. We put $G_\iota^\natural = \mathrm{O}_{2n}$ and $G_A^\natural = \prod G_\iota^\natural$. In all cases, it is clear from the construction Section 3.3 that $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}} \subset G_A^\natural$ is the identity component and that the group $G_A^{\natural\,\mathrm{ad}}$ defined in that construction is indeed the adjoint group of $G_A^\natural$. Working with this group $G_A^\natural$, we may apply [Springer and Steinberg 1970, IV].

In what follows, the centralisers $C_\ell^\natural \subset G^\natural_{A/\overline{\mathbb{Q}}_\ell}$ and $C_\ell \subset G_{A/\overline{\mathbb{Q}}_\ell}$ of the pair (N'_ℓ, w_ℓ) will play an important role. The group C_ℓ is the subgroup of $G_{A/\overline{\mathbb{Q}}_\ell}$ generated by $C_\ell^\natural \cap G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}_\ell}$ and the centre of $G_{A/\overline{\mathbb{Q}}_\ell}$. The embedding

$$G^\natural_{A/\overline{\mathbb{Q}}_\ell} \to \prod_\iota G^\natural_{\iota/\overline{\mathbb{Q}}_\ell}$$

gives rise to a similar embedding $C^\natural_{\ell/\overline{\mathbb{Q}}_\ell} \to \prod_\iota C^\natural_{\ell,\iota}$, where each $C^\natural_{\ell,\iota}$ is the centraliser of $(N'_{\ell,\iota}, w_{\ell,\iota})$ in $G^\natural{}_\iota$. We first determine these groups $C^\natural{}_{\ell,\iota}$.

By Lemma 3.9, N'_ℓ induces an isomorphism from the t-eigenspace of w_ℓ onto the t^{-1}-eigenspace of w_ℓ. As we saw in Section 5.1, this implies that, taking $G = G_\iota^\natural$ and $X = N'_{\ell,\iota}$ in [Springer and Steinberg 1970, IV §2], the proper choice of a basis of V_ι ensures that the cocharacter $w_{\ell,\iota}$ is the inverse of the cocharacter λ defined in that paper, IV 2.22. The group $C^\natural_{\ell,\iota}$ is therefore equal to the group C of [ibid., IV 2.23(iii)]. Note that, contrary to what is affirmed in that statement, this group is not necessarily connected. Indeed, there may be two connected components; see [ibid., 2.25 and 2.26].

To give an explicit description of $C^\natural_{\ell,\iota}$, let the 1-motive M' and the dimensions $r = r^\star$ and g be as in Section 3.8. In this case, $r = r^\star$ is equal to the dimension of both the t and the t^{-1}-eigenspaces of w_ℓ. Since $N'_{\ell,\iota}$ is nilpotent of echelon at most 2, it is easily deduced from [ibid., IV 1.8, 2.25] that the group $C^\natural_{\ell,\iota}$ is isomorphic to the $\overline{\mathbb{Q}}_\ell$-group

- $\mathrm{SL}_r \times \mathrm{SL}_{2g}$ if G_A is of type A_n,
- $\mathrm{O}_r \times \mathrm{Sp}_{2g}$ if G_A is of type C_n or
- $\mathrm{Sp}_r \times \mathrm{O}_{2g}$ if G_A is of type D_n.

Each factor $C^{\natural}_{\ell,\iota}$ is therefore a product $C^{\natural}_{\ell,\iota,0} \times C^{\natural}_{\ell,\iota,-1}$ and this decomposition is determined by the cocharacter $w_{\ell,\iota}$. In fact, for each integer k let V_{ℓ}^{k-1} and $V_{\ell,\iota}^{k-1}$ be the t^k-eigenspaces of w_ℓ and $w_{\ell,\iota}$ in $V_\ell = V \otimes \mathbb{Q}_\ell$ and $V_\iota \otimes \overline{\mathbb{Q}}_\ell$, respectively. This seemingly confusing notation is consistent with Section 1.10. As $C^{\natural}_{\ell,\iota}$ commutes with $w_{\ell,\iota}$, it respects the grading $V_{\ell,\iota} = \bigoplus_{k=-2,-1,0} V^k_{\ell,\iota}$ and it follows from [ibid., IV 1.8, 2.25] that for $k = 0, -1$, the group $C^{\natural}_{\ell,\iota,k}$ is the image of $C^{\natural}_{\ell}$ in $\mathrm{GL}(V^k_{\ell,\iota})$. For a group of type A_n or C_n, this embedding is given by the standard or symplectic representation, respectively, and for a group of type B_n or D_n it is defined by the orthogonal representation. The monodromy operator $N_{\ell,\iota}$ defines an isomorphism of the representations of $C^{\natural}_{\ell,\iota,0}$ on $V^0_{\ell,\iota}$ and on $V^{-2}_{\ell,\iota}$.

The decomposition above can be defined on the level of the group C_ℓ by taking $C_{\ell,k}$ equal to the image of C_ℓ in $\mathrm{GL}(V^k_\ell)$, for $k = 0, -1$. Finally, let $C_{\ell,\iota,k}$ be the image of $C_{\ell/\overline{\mathbb{Q}}_\ell}$ in $\mathrm{GL}(V^k_{\ell,\iota})$, so that each $C^{\mathrm{der}}_{\ell,\iota,k} \subset C^{\natural}_{\ell,\iota,k}$ is the identity component, with equality for all factors other than those isomorphic to an SO_{2n}.

5.4. ***The representations ρ'_ℓ.*** We now turn to the representations ρ'_ℓ of the Weil group $W_v = W_{F_v}$. For general $\psi \in W_v$, the image $\rho'_\ell(\psi)$ does not belong to $C_\ell(\mathbb{Q}_\ell)$ so in order to apply the arguments of [Noot 2009, §2], we replace C_ℓ by the group $\widetilde{C}_\ell \subset G_{A/\overline{\mathbb{Q}}_\ell}$ generated by C_ℓ and the image of w_ℓ. For $\psi \in W_v$ one has

$$\mathrm{Ad}\left(\rho'_\ell(\psi)\right)(N'_\ell) = q^{\alpha(\psi)} N'_\ell = \mathrm{Ad}\left(w_\ell(q^{\alpha(\psi)/2})\right)(N'_\ell).$$

On the other had, it follows from Lemma 2.5 that $\rho'_\ell(\psi)$ and w_ℓ commute. This implies that $\rho'_\ell(\psi) w_\ell(q^{-\alpha(\psi)/2})$ lies in $C_\ell(\overline{\mathbb{Q}}_\ell)$ and hence that $\rho'_\ell(\psi) \in \widetilde{C}_\ell(\mathbb{Q}_\ell)$. The action of $\widetilde{C}_\ell$ on V_ℓ respects the grading $V_\ell = \bigoplus_{k=-2,-1,0} V^k_\ell$ so, for $k = 0, -1$, it makes sense to define $\widetilde{C}_{\ell,k}$ as the image of $\widetilde{C}_\ell$ in $\mathrm{GL}(V^k_{\ell,\iota})$. The adjoint action of $C^{\natural}_\ell$ on C_ℓ extends to an action of $C^{\natural}_\ell$ on $\widetilde{C}_\ell$, with the former group acting trivially on the image of w_ℓ.

Recall that, according to Lemma 4.4, the algebra L is a field in the cases considered here. We write $\mathbb{A}^d_{(L)} = \mathrm{Res}_{L/\mathbb{Q}} \mathbb{A}^d_{/L}$. The discussion above shows that taking the L_ℓ-linear characteristic polynomials of the elements of $\widetilde{C}_\ell$ acting on the w_ℓ-eigenspace V^{-1}_ℓ, one defines a map

$$P'_L \colon \widetilde{C}_\ell \to \widetilde{C}_{\ell,-1} \to \mathbb{A}^{2g}_{(L)/\overline{\mathbb{Q}}_\ell}. \tag{5.4$*$}$$

We will also write P'_L for the map $\widetilde{C}_{\ell,-1} \to \mathbb{A}^{2g}_{(L)/\overline{\mathbb{Q}}_\ell}$. As in [Noot 2009], the maps P'_L factor through the quotients of $\widetilde{C}_\ell$ and $\widetilde{C}_{\ell,-1}$ by the adjoint $C^{\natural}_\ell$-action.

It follows from Proposition 1.8 and from Section 3.8 that for any $\psi \in W_v$, the characteristic polynomial of $\rho'_\ell(\psi)$ acting as an L_ℓ-linear automorphism on V^{-1}_ℓ has coefficients in L and is independent of ℓ. This proves the following lemma.

Lemma 5.5. *Under the hypotheses above, the image of $\rho'_\ell(\psi) \in \widetilde{C}_\ell(\mathbb{Q}_\ell)$ under the map P'_L defined in* (5.4$*$) *lies in $\mathbb{A}^{2g}_{(L)}(\mathbb{Q})$ and is independent of ℓ.*

5.6. *Remark.* The statement of the lemma also holds for the image of $\rho'_\ell(\psi)$ under the map $P''_L\colon \widetilde{C}_\ell \to \mathbb{A}^r(L)$ defined by taking the characteristic polynomial on V_ℓ^0. This observation is of little interest for weakly neat elements.

Proposition 5.7. *The main theorem, Theorem 3.6, holds if A is strictly tractable and its Mumford–Tate group G_A is of type A_n, C_n or $D_n^{\mathbb{H}}$.*

Proof. As the assumptions of Theorem 3.6 are now in force, A has semistable reduction at v and the image $\rho'_\ell(\Phi_v)$ of Frobenius is weakly neat. This implies that the restriction of ρ'_ℓ to $I_{\bar{b}}$ is trivial and that $\rho'_\ell(\Phi_v)$ acts on V_ℓ^0 as multiplication by $\varepsilon = \pm 1$ and on V_ℓ^{-2} as multiplication by $q\varepsilon$.

As in Section 5.1, let $\Omega \supset \mathbb{Q}_\ell$ be an algebraically closed field and let $\sigma \in \mathrm{Aut}(\Omega)$. We have to show that the pairs $(N'_\ell, \rho'_\ell(\Phi_v))$ and $(\sigma(N'_\ell), \sigma(\rho'_\ell(\Phi_v)))$ are conjugate under the action of $G_A^\natural(\Omega)$. By Proposition 5.2, there is a $g \in G_A^\natural(\Omega)$ such that

$$(N'_\ell, w_\ell) = \mathrm{Ad}(g)(\sigma(N'_\ell), \sigma(w_\ell)).$$

This implies that $\rho'_\ell(\Phi_v)$ and $g\sigma(\rho'_\ell(\Phi_v))g^{-1}$ belong to $\widetilde{C}_\ell(\Omega)$. As $C_\ell^\natural$ centralises (N'_ℓ, w_ℓ), it is enough to show that $\rho'_\ell(\Phi_v)$ and $g\sigma(\rho'_\ell(\Phi_v))g^{-1}$ are conjugate under $C_\ell^\natural(\Omega)$. By Section 1.9 and the proof of Proposition 2.6, the element $\rho'_\ell(\Phi_v)$ is semisimple. Moreover

$$P'_L(g\sigma(\rho'_\ell(\Phi_v))g^{-1}) = \sigma(P'_L(\rho'_\ell(\Phi_v))) = P'_L(\rho'_\ell(\Phi_v)),$$

where the former equality is elementary and the latter one follows from Lemma 5.5. The projections of these elements to $\widetilde{C}_{\ell,-1}(\Omega)$ are semisimple and weakly neat and their projections to $C_{\ell,0}(\Omega)$ lie in the centre of this group. The required statement therefore follows from Lemma 5.8.

For the ℓ-independence, let ℓ and ℓ' be two prime numbers and let Ω be an algebraically closed field containing $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$. It follows from Proposition 5.2 that there exists $g \in G_A^\natural(\Omega)$ such that $(N'_\ell, w_\ell) = \mathrm{Ad}(g)(N'_{\ell'}, w_{\ell'})$. Exactly as before we combine the Lemmas 5.5 and 5.8 to show that $\rho'_\ell(\Phi_v)$ and $\rho'_{\ell'}(\Phi_v)$ are conjugate under $C_\ell^\natural(\Omega)$. □

Lemma 5.8. *Let $\widetilde{C}_\ell$, $C_\ell^\natural$ and P'_L be as above and let $g_1, g_2 \in \widetilde{C}_\ell(\Omega)$ be semisimple elements whose projections to $C_{\ell,-1}^\natural(\Omega)$ are weakly neat and whose projections to $\widetilde{C}_{\ell,0}(\Omega)$ act on V_ℓ^0 by the same scalar multiplication. If $P'_L(g_1) = P'_L(g_2)$ then g_1 and g_2 are conjugate under $C_\ell^\natural(\Omega)$.*

Proof. This essentially results from [Noot 2009, Lemmas 2.5 and 2.6]. First note that the variety $\mathrm{Conj}'(\widetilde{C}_\ell)$ considered in [ibid., Lemma 2.5] is the variety of semisimple $C_\ell^\natural$-conjugacy classes in $\widetilde{C}_\ell$. Similarly, for each ι and k, the variety $\mathrm{Conj}'(\widetilde{C}_{\ell,\iota,k})$ is the variety of semisimple $C_{\ell,\iota,k}^\natural$-conjugacy classes in $\widetilde{C}_{\ell,\iota,k}$. It follows that $\mathrm{Conj}'(\widetilde{C}_\ell)(\Omega)$ and $\mathrm{Conj}'(\widetilde{C}_{\ell,\iota,k})(\Omega)$ are the sets of semisimple $C_\ell^\natural(\Omega)$ and $C_{\ell,\iota,k}^\natural(\Omega)$-conjugacy classes in $\widetilde{C}_\ell(\Omega)$ and in $\widetilde{C}_{\ell,\iota,k}(\Omega)$, respectively.

It thus follows from [ibid., Lemma 2.5] and its proof that the map

$$\widetilde{C}_\ell(\Omega) \to \widetilde{C}_{\ell,0}(\Omega) \times \widetilde{C}_{\ell,-1}(\Omega) \to \prod_\iota \widetilde{C}_{\ell,\iota,0}(\Omega) \times \prod_\iota \widetilde{C}_{\ell,\iota,-1}(\Omega)$$

induces an injection on weakly neat $C_\ell^\natural(\Omega)$-conjugacy classes. As the images of g_1 and g_2 in $\widetilde{C}_{\ell,0}(\Omega)$ are equal and since these images are $C_\ell^\natural(\Omega)$-invariant, it is sufficient to show that the images of g_1 and g_2 in each $\mathrm{Conj}'(\widetilde{C}_{\ell,\iota,-1})(\Omega)$ coincide.

To this end, note that each map

$$P'_L\colon\ \mathrm{Conj}'(\widetilde{C}_{\ell,\iota,-1})(\Omega) \to \mathbb{A}^{2g}(\Omega)$$

is injective. This is the statement of [ibid., Lemma 2.6] for the group $C^{\mathrm{der}}_{\ell,\iota,-1}$, with Ω instead of $\overline{\mathbb{Q}}$. Since $C^{\mathrm{der}}_{\ell,\iota,-1}$ is of type A_n, C_n or $D_n^{\mathbb{H}}$, it follows from the remark in the beginning of the proof of [ibid., Lemma 2.6] that the lemma in question is valid in this setting. □

6. Strictly tractable abelian varieties of types B_n and $D_n^{\mathbb{R}}$

6.1. ***The monodromy in a Mumford–Tate group of type B_n.*** With the notation of Section 3.1, we turn to the case where A/F is a strictly tractable abelian variety with Mumford–Tate group G_A of type B_n. We will adapt the arguments of Section 5 to this case. The endomorphism algebra L and the group H are defined as in Sections 4.3 and 4.5. As in the previous cases, it follows from Lemma 4.4 that L is a field. Moreover we have $G_A^{\natural\,\mathrm{ad}} = G_A^{\mathrm{ad}}$. For each prime number ℓ, the monodromy operator N'_ℓ and the Frobenius weight cocharacter w_ℓ are defined as before. Each w_ℓ acts on $V_\ell = V \otimes \mathbb{Q}_\ell$ with at most three eigenvalues. It presents a single weight if and only if A has good reduction, which is the case if and only if $N'_\ell = 0$.

Recall the decomposition $L \otimes \overline{\mathbb{Q}} = \bigoplus_\iota \overline{\mathbb{Q}}$ from Section 4.3, where the direct sum is indexed by the maps $\iota\colon L \to \overline{\mathbb{Q}}$. As in Sections 4.3 and 4.5, we consider the resulting decompositions $V \otimes \overline{\mathbb{Q}} = \bigoplus_\iota V_\iota$ and $\mathfrak{h} \otimes \overline{\mathbb{Q}} = \bigoplus_\iota \mathfrak{h}_\iota$ as well as the embedding $\mathfrak{g} \otimes \overline{\mathbb{Q}} \hookrightarrow \bigoplus_\iota \mathfrak{g}_\iota$. We write G_ι for the image of $G_{A/\overline{\mathbb{Q}}}$ in $\mathrm{GL}(V_\iota)$. The derived group $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$ identifies with the product $\prod_\iota G_\iota^{\mathrm{der}}$. Each G_ι^{der} is a spin group of type B_n and its representation on V_ι is a multiple of the irreducible representation V_ι^{irr} of highest weight ϖ_n and hence of dimension $d = 2^n$.

As before, let $(N'_{\ell,\iota})_\iota$ be the image of N'_ℓ in $\bigoplus_\iota \mathfrak{g}_\iota \otimes \overline{\mathbb{Q}}_\ell$. Of course, N'_ℓ lies in $\mathfrak{g}^{\mathrm{ss}} \otimes \mathbb{Q}_\ell$ and $N'_{\ell,\iota} \in \mathfrak{g}_\iota^{\mathrm{ss}} \otimes \overline{\mathbb{Q}}_\ell$. The operator N'_ℓ is L_ℓ-linear so it belongs to $\mathfrak{h} \otimes \mathbb{Q}_\ell$ and it follows from the Corollary 2.7 that its $H(\overline{\mathbb{Q}}_\ell)$-orbit is defined over $\mathbb{Q}$. This implies the rank of the projection $N'_{\ell,\iota}$ is independent of ι. If $N'_{\ell,\iota} = 0$ for some ι, then all $N'_{\ell,\iota}$ are trivial and then the abelian variety A has potentially good reduction. This is the situation treated in [Noot 2009]. In what follows, we will assume that this is not the case. We investigate the possible ranks of the $N'_{\ell,\iota}$ and the corresponding forms of the cocharacters $w_{\ell,\iota}$.

Let T_ι be a maximal torus of G_ι and let $\widetilde{T}_\iota = \mathbb{G}_m^d$ be a maximal torus of $\mathrm{GL}(V_\iota^{\mathrm{irr}})$ containing it. According to [ibid., 2.6.4] we have $d = 2^n$ and we can assume that T_ι is the image of the application $\mathbb{G}_m^{n+1} \to \widetilde{T}_\iota$ given by

$$(\lambda_0, \lambda_1, \ldots, \lambda_n) \mapsto (\lambda_0 \lambda_1^{\varepsilon_1} \cdots \lambda_n^{\varepsilon_n})_{(\varepsilon_1,\ldots,\varepsilon_n)=(\pm 1,\ldots,\pm 1)},$$

where the factors of $\widetilde{T}_\iota$ are indexed by n-tuples of signs $(\varepsilon_1, \ldots, \varepsilon_n)$.

For each ι, the Frobenius weight cocharacter w_ℓ defines a cocharacter $w_{\ell,\iota}$ of $G_{\iota/\overline{\mathbb{Q}}_\ell}$. The monodromy operator $N'_{\ell,\iota}$ defines an isomorphism between the t and t^{-1} eigenspaces of $w_{\ell,\iota}$ acting on V_ι, so these eigenspaces have the same dimension and it follows that $w_{\ell,\iota}$ factors through $G^{\mathrm{der}}_{\iota/\overline{\mathbb{Q}}_\ell}$. Up to conjugation by an element of $G_\iota^{\mathrm{der}}(\overline{\mathbb{Q}}_\ell)$, we can assume that $w_{\ell,\iota}$ factors through $T_{\iota/\overline{\mathbb{Q}}_\ell}$. It then lifts to a quasicocharacter $\tilde{w}_{\ell,\iota}$ with values in $\mathbb{G}_m^{n+1}$. The filtration on V_ι^{irr} defined by $w_{\ell,\iota}$ has at most three weights and it follows that $\tilde{w}_{\ell,\iota}$ projects nontrivially to at most two factors of $\mathbb{G}_m^n$. Moreover, if it projects nontrivially to two factors, then the two projections must coincide.

The filtration by Frobenius weights coincides with the monodromy filtration and it follows that

- if $\tilde{w}_{\ell,\iota}$ is trivial, then $N'_{\ell,\iota} = 0$;
- if $\tilde{w}_{\ell,\iota}$ projects nontrivially to exactly one factor of $\mathbb{G}_m^n$, then $N'_{\ell,\iota}$ is of rank 2^{n-1} (as an endomorphism of V_ι^{irr}); and
- if $\tilde{w}_{\ell,\iota}$ projects nontrivially to exactly two factors of $\mathbb{G}_m^n$, then $N'_{\ell,\iota}$ is of rank 2^{n-2}.

The first possibility is excluded by the hypothesis that $N'_\ell \neq 0$.

As for the other cases, we consider the adjoint group G_ι^{ad} of G_ι^{der}, which is isomorphic to the special orthogonal group $\mathrm{SO}_{2n+1/\overline{\mathbb{Q}}_\ell}$. Let W_ι be the orthogonal representation of this group, that is, the representation with highest weight ϖ_1, and let $w_{\ell,\iota}^{\mathrm{ad}}$ be the projection of the cocharacter $w_{\ell,\iota}$ to G_ι^{ad}.

If we are in the second case then $w_{\ell,\iota}^{\mathrm{ad}}$ acts on W_ι with eigenvalues t, 1 and t^{-1} and the eigenspaces for t and for t^{-1} are 1-dimensional. The relation

$$\mathrm{Ad}(w_{\ell,\iota}(t))(N'_{\ell,\iota}) = t^2 N'_{\ell,\iota}$$

implies that the same relation holds with $w_{\ell,\iota}^{\mathrm{ad}}$ instead of $w_{\ell,\iota}$. It follows that the Jordan normal form of the image of $N'_{\ell,\iota}$ in the orthogonal representation of $\mathfrak{g}_\iota^{\mathrm{ss}}$ has one block of size 2 and that all other blocks are of size 1. This is impossible according to [Springer and Steinberg 1970, IV 2.14; Humphreys 1995, 7.11] so the second possibility is excluded.

We study of the conjugacy class of (N'_ℓ, w_ℓ) in the third, and only possible, case. The argument above shows that the image of $N'_{\ell,\iota}$ in the orthogonal representation

W_ι of $\mathfrak{g}_\iota^{\mathrm{ss}}$ has two blocks of size 2 and that all other blocks are of size 1. Considering the orthogonal representation W_ι, we also see that if we take $X = N'_{\ell,\iota}$ in [Springer and Steinberg 1970, IV 2.19(b)], then we can assume that $w^{\mathrm{ad}}_{\ell,\iota}$ is the inverse of the cocharacter λ of [ibid., IV 2.22]. This remains true after passing to any algebraically closed field $\Omega \supset \overline{\mathbb{Q}}_\ell$ and also after replacing the data (N'_ℓ, w_ℓ) by $\sigma(N'_\ell, w_\ell)$, where σ is an automorphism of Ω. Applying [ibid., IV, §2] in the same way as in Section 5.1, we prove that for any such Ω and σ, the pairs $(N'_{\ell,\iota}, w^{\mathrm{ad}}_{\ell,\iota})$ and $\sigma(N'_{\ell,\iota}, w^{\mathrm{ad}}_{\ell,\iota})$ are conjugate under the $G_\iota^{\mathrm{der}}(\Omega)$-action. It follows that this is also the case for $(N'_{\ell,\iota}, w_{\ell,\iota})$ and $\sigma(N'_{\ell,\iota}, w_{\ell,\iota})$.

One shows by the same argument that if ℓ' is a second prime number and if Ω is an algebraically closed field containing $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$, then (N'_ℓ, w_ℓ) and $(N'_{\ell'}, w_{\ell'})$ are $G_A(\Omega)$-conjugate. This proves Proposition 5.2 in the case where A/F is a strictly tractable abelian variety of type B_n.

6.2. ***The Frobenius elements in Mumford–Tate groups of type B_n.*** To prove the conjugacy of the Frobenius elements, we adapt the argument used from Section 5.3 to Lemma 5.8. On the one hand, notation is simplified because $G_A^{\natural\,\mathrm{ad}} = G_A^{\mathrm{ad}}$, but on the other hand, they are complicated by the fact that we need to consider the orthogonal groups G_ι^{ad} in order to apply [Springer and Steinberg 1970]. As in Section 5.3, consider the centraliser $C_\ell \subset G_{A/\mathbb{Q}_\ell}$ of (N'_ℓ, w_ℓ) and note that $\rho'_\ell(\Phi_v) \in \widetilde{C}_\ell(\mathbb{Q}_\ell)$, where $\widetilde{C}_\ell$ is the subgroup of $G_{A/\mathbb{Q}_\ell}$ generated by C_ℓ and the image of w_ℓ. For any fixed embedding $\iota\colon L \to \overline{\mathbb{Q}}$, let G_ι be the image of $G_{A/\overline{\mathbb{Q}}_\ell}$ in $\mathrm{GL}(V_\iota)$. The centraliser $C_{\ell,\iota} \subset G_{\iota/\overline{\mathbb{Q}}_\ell}$ of $(N'_{\ell,\iota}, w_{\ell,\iota})$ can be described by projecting it to the adjoint group.

The action of G_ι on itself by conjugation factors through the adjoint group $G_\iota^{\mathrm{ad}} \cong \mathrm{SO}_{2n+1}$. Recall that W_ι is the orthogonal representation of this group. In view of [ibid., IV §2] and the dimension count carried out in Section 6.1, the centraliser $C^\flat_{\ell,\iota} \subset G^{\mathrm{ad}}_{\iota/\overline{\mathbb{Q}}_\ell}$ of the pair $(N_{\ell,\iota}, w^{\mathrm{ad}}_{\ell,\iota})$ satisfies

$$C^\flat_{\ell,\iota} \cong \mathrm{SL}_{2/\overline{\mathbb{Q}}_\ell} \times \mathrm{SO}_{2n-3/\overline{\mathbb{Q}}_\ell}. \tag{6.2$*$}$$

Consider the tori $T_\iota \subset \widetilde{T}_\iota$ defined in Section 6.1. Up to conjugation, we can assume that $w_{\ell,\iota}$ factors through T_ι and that its lift $\tilde{w}_{\ell,\iota}$ along $\mathbb{G}_m^{n+1} \to \widetilde{T}_\iota$ is given by

$$\tilde{w}_{\ell,\iota}\colon \mathbb{G}_m \to \mathbb{G}_m^{n+1}, \quad t \mapsto (1, t^{1/2}, t^{1/2}, 1, \dots, 1).$$

The image $\Delta'' \subset T_\iota \subset \widetilde{T}_\iota$ of the map $t \mapsto (1, t^{1/2}, t^{-1/2}, 1, \dots, 1)$ then projects to a maximal torus of the factor SL_2 of $C^\flat_{\ell,\iota} \subset G^{\mathrm{ad}}_{\iota/\overline{\mathbb{Q}}_\ell}$. The product $\Delta' = \mathbb{G}_m^{n-2}$ of the last $n-2$ factors projects to a maximal torus of the factor SO_{2n-3}.

The group $C_{\ell,\iota} \subset G_\iota$ is the inverse image of $C^\flat_{\ell,\iota}$. Its derived group therefore admits an isogeny

$$C''_{\ell,\iota} \times C'_{\ell,\iota} = \mathrm{SL}_{2/\overline{\mathbb{Q}}_\ell} \times C'_{\ell,\iota} \to C^{\mathrm{der}}_{\ell,\iota},$$

where $C'_{\ell,\iota}$ is a spin group of type B_{n-2}.

The map from Δ'' to G_ι^{der} factors through $C''_{\ell,\iota} \times 1 = \mathrm{SL}_2$ in the product above and the image of Δ'' in $C''_{\ell,\iota}$ is a maximal torus. Similarly, Δ' maps through $1 \times C'_\iota$ and defines a maximal torus in C'_ι. Recall from Section 6.1 that V_ι is a multiple of the spin representation V_ι^{irr} of G_ι. Considering the characters occurring in the representation of $\Delta'' \times \Delta'$ on V_ι^{irr}, one concludes that, as a representation of $\mathrm{SL}_{2/\overline{\mathbb{Q}}_\ell} \times C'_\iota$, the space V_ι^{irr} is a tensor product $V''_\iota \otimes V'_\iota$. Here V''_ι is a multiple of the direct sum of the standard representation and two copies of the trivial representation of SL_2 and V'_ι is the spin representation of C'_ι. As representations of $C_{\ell,\iota}$, the t- and t^{-1}-eigenspaces $V^0_{\ell,\iota}$ and $V^{-2}_{\ell,\iota} \subset V_{\ell,\iota}$ of $w_{\ell,\iota}$ are both isomorphic to a multiple of V'_ι, so the representation of $C''_{\ell,\iota} \times C'_{\ell,\iota}$ on $V^0_{\ell,\iota}$ identifies $C'_{\ell,\iota}$ with its image in $\mathrm{GL}(V^0_{\ell,\iota})$.

These observations imply that the isogeny above is in fact an isomorphism $C''_{\ell,\iota} \times C'_{\ell,\iota} = \mathrm{SL}_{2/\overline{\mathbb{Q}}_\ell} \times C'_{\ell,\iota} \cong C^{\mathrm{der}}_{\ell,\iota}$ and hence

$$C^{\mathrm{der}}_{\ell/\overline{\mathbb{Q}}_\ell} \cong \prod_{\iota\colon L \to \overline{\mathbb{Q}}_\ell} (C''_{\ell,\iota} \times C'_{\ell,\iota}).$$

It also follows that C^{der}_ℓ itself decomposes as a product $C''_\ell \times C'_\ell$ of algebraic groups over $\mathbb{Q}_\ell$. The group C_ℓ is generated by C^{der}_ℓ and the centre of $G_{A/\mathbb{Q}_\ell}$ and $\widetilde{C}_\ell$ is generated by C_ℓ and the image of w_ℓ. We will show that $\rho'_\ell(\Phi_v)$ lies in $\widetilde{C}''_\ell \subset \widetilde{C}_\ell$, the subgroup generated by C''_ℓ and the centre of $\widetilde{C}_\ell$.

Indeed, as in Section 5.4, we consider the action of $\rho'_\ell(\Phi_v)$ on the different w_ℓ-eigenspaces in V_ℓ. It was pointed out in the beginning of the proof of Proposition 5.7 that, since $\rho'_\ell(\Phi_v)$ is weakly neat, it acts on V^0_ℓ as multiplication by $\varepsilon = \pm 1$ and on V^{-2}_ℓ as multiplication by εq. This means that $\rho'_\ell(\Phi_v) \in \widetilde{C}''_\ell(\mathbb{Q}_\ell)$, as claimed.

The group $C''_{\ell,\iota}$, which is isomorphic to SL_2, acts trivially on $V^0_{\ell,\iota}$ and on $V^{-2}_{\ell,\iota}$ and $V^{-1}_{\ell,\iota}$ is a multiple of the standard representation. The centre of C_ℓ acts on each V_ι through a fixed character. Through w_ℓ, the group $\mathbb{G}_m$ acts on V^{-2}_ℓ, on V^{-1}_ℓ and on V^0_ℓ as multiplication by t^{-1}, by 1 and by t, respectively. For the group $\widetilde{C}''_\ell$ defined above, this discussion implies that the map

$$\widetilde{C}''_\ell \to \mathrm{GL}(V^{-2}_\ell) \times \mathrm{GL}(V^{-1}_\ell) \times \mathrm{GL}(V^0_\ell)$$

is injective. As in the proof of Lemma 5.5, the image of $\rho'_\ell(\Phi_v)$ under the map P'_L, defined in (5.4$*$) by taking the L-linear characteristic polynomial on V^{-1}_ℓ, lies in $\mathbb{A}^{2g}_{(L)}(\mathbb{Q})$ and is independent of ℓ. We already know that the images of $\rho'_\ell(\Phi_v)$ in $\mathrm{GL}(V^{-2}_\ell)$ and in $\mathrm{GL}(V^0_\ell)$ are rational scalars, independent of ℓ. In the case of an abelian variety of type B_n, the main theorem now follows using a variant of Lemma 5.8, again using [Noot 2009, 2.5, 2.6]. Note that, as before, the statement of [ibid., 2.6] is valid for any Ω, instead of just $\mathbb{Q}$, because the group $\widetilde{C}''_\ell$ is of type A_1.

6.3. ***Abelian varieties of type $D_n^{\mathbb{R}}$.*** The case where the abelian variety A is strictly tractable with Mumford–Tate group of type $D_n^{\mathbb{R}}$ can be treated by analogous arguments. We will just indicate the points where the discussion of Sections 6.1 and 6.2 needs to be modified.

First of all, the quotient of G_A one has to consider in order apply [Springer and Steinberg 1970] is not the adjoint group, but the intermediate quotient of $G_{A/\overline{\mathbb{Q}}_\ell}$ for which the simple factors $G_\iota^\flat$ are groups of the form SO_{2n}. Also, A is not necessarily simple in this case. If it is not, then the endomorphism algebra L is of the form $L = L' \times L'$, where L' is a number field; see Lemma 4.4.

We now follow the proof of [Noot 2009, Théorème 2.4] for this type. For each $\iota\colon L \to \overline{\mathbb{Q}}$, let V_ι be the direct factor of $V \otimes \overline{\mathbb{Q}}$ on which L acts through ι. The group $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$ acts on V_ι through a single direct factor G_ι^{der}, but for $n \geq 4$ this factor does not act faithfully on V_ι. In fact V_ι is a multiple of a semispin representation V_ι^{irr} of G_ι, with highest weight ϖ_{n-1} say. For $\iota = \iota^+$, there is a $\iota^-\colon L \to \overline{\mathbb{Q}}$ such that V_{ι^-} is a multiple of the other semispin representation $V_{\iota^-}^{\mathrm{irr}}$ of G_ι, with highest weight ϖ_n. The representation of $G_{A/\overline{\mathbb{Q}}}$ on $V_{\iota^+} \oplus V_{\iota^-}$ restricts to a faithful representation of G_ι^{der}. We redefine G_ι as the image of $G^{\mathrm{der}}_{A/\overline{\mathbb{Q}}}$ in $\mathrm{GL}(V_{\iota^+}) \times \mathrm{GL}(V_{\iota^-})$.

If L is a field, then it is a CM field and the involution $\iota^+ \mapsto \iota^-$ on the set of maps $L \to \overline{\mathbb{Q}}$ defined by this construction is given by the composition with the complex conjugation on L. If $L = L' \times L'$ is a product of two fields, then ι^- is the composite of ι^+ with the involution exchanging the factors. Using the spin representation $V_{\iota^+}^{\mathrm{irr}} \oplus V_{\iota^-}^{\mathrm{irr}}$ instead of V_ι^{irr}, the arguments of Section 6.1 and hence the proof of Proposition 5.2 carry over to this case.

Where the discussion of Section 6.2 is concerned, the analogue of (6.2$*$) states that the centraliser $C_{\ell,\iota}^\flat$ of $(N_{\ell,\iota}, w_{\ell,\iota}^{\mathrm{ad}})$ in $G_\iota^\flat$ is given by

$$C_{\ell,\iota}^\flat \cong \mathrm{SL}_{2,\overline{\mathbb{Q}}_\ell} \times \mathrm{SO}_{2n-4}\,.$$

Once again, $C_{\ell,\iota} \subset G_\iota$ is the inverse image of $C_{\ell,\iota}^\flat$ and there is an isogeny

$$C''_{\ell,\iota} \times C'_{\ell,\iota} \cong \mathrm{SL}_{2/\mathbb{Q}_\ell} \times C'_{\ell,\iota} \to C^{\mathrm{der}}_{\ell,\iota}.$$

Here the group $C'_{\ell,\iota}$ is a spin group of type D_{n-2}. Similarly to the previous case, one shows that $V_{\iota^+}^{\mathrm{irr}} \oplus V_{\iota^-}^{\mathrm{irr}}$ is of the form $V_\iota'' \otimes V_\iota'$, where V_ι'' is a multiple of the direct sum of the standard representation and two copies of the trivial representation of SL_2 and V_ι' is the spin representation of C_ι'. We prove once again that

$$C^{\mathrm{der}}_{\ell,\iota} = C''_{\ell,\iota} \times C'_{\ell,\iota}$$

and we define $\widetilde{C}_\ell$ and $\widetilde{C}''_\ell$ as before. As in Section 6.2 one has $\rho'_\ell(\Phi_v) \in \widetilde{C}''_\ell(\mathbb{Q}_\ell)$. For each pair ι^+, ι^- as above, the image of C''_ℓ in $C''_{\ell,\iota^+} \times C''_{\ell,\iota^-}$ is the graph of an isomorphism. Each C''_ℓ is again a group of type A_1, acting on the 1-eigenspace

$V_{\iota^+}^{-1} \oplus V_{\iota^-}^{-1}$ for w_ℓ as a multiple of the standard representation. The argument can be completed as in Section 6.2.

7. The proof of the main theorem, Theorem 3.6

Proposition 7.1. *Assume that we are in the situation of Theorem 3.6 and that the variety A is tractable. Then there is a finite extension F' of F such that Theorem 3.6 holds for $A_{/F'}$.*

Proof. The Proposition 5.7 and the results of Section 6 prove the proposition in the case where A is strictly tractable.

If A is tractable then there exists a finite extension $F' \supset F$, strictly tractable abelian varieties $A_1, \dots A_m/F'$ and an isogeny $A_{/F'} \sim \prod_{i=1}^m A_i$ such that the inclusion $f\colon G_A \to \prod_{i=1}^m G_{A_i}$ induces an isomorphism $G_A^{\mathrm{der}} \cong \prod_{i=1}^m G_{A_i}^{\mathrm{der}}$. In that case there are isomorphisms

$$\mathfrak{g}_A^{\mathrm{ss}} \cong \bigoplus_{i=1}^m \mathfrak{g}_{A_i}^{\mathrm{ss}} \quad \text{and} \quad G_A^\natural \cong \prod_{i=1}^m G_{A_i}^\natural .$$

For the induced map

$$f_\ell\colon G_A(\mathbb{Q}_\ell) \to \prod_{i=1}^m G_{A_i}(\mathbb{Q}_\ell),$$

one has $f_\ell \circ \rho_{A,\ell} = (\rho_{A_i,\ell})_{i=1,\dots,m}$, so the tangent map to f_ℓ sends the monodromy operator $N'_\ell \in \mathfrak{g}_A^{\mathrm{ss}} \otimes \mathbb{Q}_\ell$ to the m-tuple in $\bigoplus_{i=1}^m \mathfrak{g}_{A_i}^{\mathrm{ss}} \otimes \mathbb{Q}_\ell$ of the monodromy operators associated to the A_i. This obviously implies that $f_\ell \circ \rho'_{A,\ell} = (\rho'_{A_i,\ell})_{i=1,\dots,m}$. The statement for $A_{/F'}$ therefore results immediately from the corresponding statements for the A_i. □

7.2. ***Preliminaries to the proof of Theorem 3.6.*** We use the method of the proof of [Noot 2009, Théorème 1.8] in Section 3 of that paper. After fixing the notation we will indicate an omission, pointed out by Abhijit Laskar, in [Noot 2009] and explain how to complete the argument.

In what follows, the notation and the hypotheses of Section 3.1 and of Theorem 3.6 are in force, so A has semistable reduction at v and the image $\rho_\ell(\Phi_v)$ of the arithmetic Frobenius is weakly neat. However, since the argument involves auxiliary abelian varieties, we write $(\rho'_{A,\ell}, N'_{A,\ell})$ for the representation of $'W_v$ associated to $A_{/F_v}$.

By [Noot 2006, §2 and Corollary 3.2] of that paper, there is a tractable abelian variety $B_{/\bar{F}}$ such that $B_{/\mathbb{C}}$ provides a weak Mumford–Tate lift of $A_{/\mathbb{C}}$. Following [Noot 2009, §3], this implies that there exists an abelian variety of CM-type $C_{/\bar{F}}$ such that $A_{/\bar{F}}$ belongs to the category of absolute Hodge motives generated by $B_{/\bar{F}}$ and $C_{/\bar{F}}$. This fact determines a morphism of Mumford–Tate groups $\pi\colon G_{B\times C} \to G_A$

but, contrary to what is stated in [ibid., §3], not a morphism $G_B \times G_C \to G_A$. In fact, $G_{B\times C}$ is a closed subgroup of the product $G_B \times G_C$ and the inclusion identifies the derived groups. This means that the diagram considered in the proof of [ibid., Théorème 1.8] has to be replaced by the diagram

$$\begin{tikzcd} \Gamma_{F'} \arrow[r, "\rho_{B\times C,\ell}"] \arrow[rr, bend left, "(\rho_{B,\ell},\rho_{B,\ell})"] \arrow[dr, "\rho_{A,\ell}"'] & G_{B\times C}(\mathbb{Q}_\ell) \arrow[r, hook] \arrow[d, "\pi"] & G_B(\mathbb{Q}_\ell) \times G_C(\mathbb{Q}_\ell) \\ & G_A(\mathbb{Q}_\ell), & \end{tikzcd} \tag{7.2$*$}$$

which commutes for a sufficiently large finite extension F' of F. None of the above depends on v but this will not play any role in what follows.

7.3. ***Addendum to the proof of*** **[Noot 2009, Théorème 1.8].** Recall that the statement of the theorem in question is essentially the special case of the main theorem of this paper where A has good reduction. In [Noot 2009] it is formulated in terms of the variety of geometric conjugacy classes of the Mumford–Tate group. We have to prove that there exists a conjugacy class $\mathrm{Cl}_A \mathrm{Fr}_v \in \mathrm{Conj}'(G_A)(\mathbb{Q})$ containing the image of $\rho_{A,\ell}(\Phi_v^{-1})$ of any ℓ with $v(\ell) = 0$. Here $\mathrm{Conj}'(G_A)_{/\overline{\mathbb{Q}}}$ is the quotient of $G_{A/\overline{\mathbb{Q}}}$ by the adjoint action of $G_A^{\natural\,\mathrm{ad}}$. We refer to [ibid., 1.5] for the construction of a natural model $\mathrm{Conj}'(G_A)$ over $\mathbb{Q}$. Assume for the moment that [ibid., Theorem 1.8] holds for $B \times C/F'$, where F' is a sufficiently big finite extension of F and v' an extension of the valuation v to F'. We then obtain a conjugacy class $\mathrm{Cl}_{B\times C} \mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_{B\times C})(\mathbb{Q})$ and its image $\mathrm{Cl}_A \mathrm{Fr}_{v'} \in \mathrm{Conj}'(G_A)(\mathbb{Q})$ fulfils the statement [ibid., Theorem 1.8] for $A_{/F'}$. The proof of Theorem 1.8 there then applies and it follows that the theorem also holds for A/F.

It remains to construct $\mathrm{Cl}_{B\times C} \mathrm{Fr}_{v'}$. As G_C is a torus, $\mathrm{Conj}'(G_{B\times C})_{/\overline{\mathbb{Q}}}$ and $\mathrm{Conj}'(G_B \times G_C)_{/\overline{\mathbb{Q}}}$ are the quotients of $G_{B\times C/\overline{\mathbb{Q}}}$ and of $G_{B/\overline{\mathbb{Q}}} \times G_{C/\overline{\mathbb{Q}}}$ for the adjoint action by the same group, denoted $\mathrm{Aut}'(G_B)$ in [Noot 2009, 1.5] and $G_B^{\natural\,\mathrm{ad}}$ in Section 3.3 of this paper. If $T_{B\times C} \subset T_B \times G_C$ denote maximal tori of $G_{B\times C}$ and of $G_B \times G_C$, then $\mathrm{Conj}'(G_{B\times C})$ and $\mathrm{Conj}'(G_B \times G_C)$ are also quotients of these tori by the finite group $\widetilde{W}$ of [ibid., 1.6]. The group $\widetilde{W}$ is an extension of a finite group of outer automorphisms by the Weyl group of G_B^{der}. We claim that the closed immersion $T_{B\times C} \subset T_B \times G_C$ induces a closed immersion on the quotients for the $\widetilde{W}$-action.

To justify the claim, assume that $R \to S$ is a surjective morphism of $\mathbb{Q}$-algebras with $\widetilde{W}$ action. Let $b \in S^{\widetilde{W}}$ and assume that $a \in R$ maps to b. The average of the elements of the $\widetilde{W}$-orbit of a then is an element of $R^{\widetilde{W}}$ mapping to b. It follows that $R^{\widetilde{W}} \to S^{\widetilde{W}}$ is also surjective.

As B is tractable, Thórème 2.4 of [Noot 2009] provides a conjugacy class

$$(\mathrm{Cl}_B \operatorname{Fr}_{v'}, \mathrm{Cl}_C \operatorname{Fr}_{v'}) \in \operatorname{Conj}'(G_B \times G_C)(\mathbb{Q})$$

containing $(\rho_{B,\ell}(\Phi_{v'}^{-1}), \rho_{C,\ell}(\Phi_{v'}^{-1}))$ for any $\ell \neq p$. As $(\rho_{B,\ell}, \rho_{C,\ell})$ factors through $G_{B\times C}$ for all ℓ, it follows that $(\mathrm{Cl}_B \operatorname{Fr}_{v'}, \mathrm{Cl}_C \operatorname{Fr}_{v'}) \in \operatorname{Conj}'(G_{B\times C})(\mathbb{Q})$. It is obviously the class $\mathrm{Cl}_{B\times C} \operatorname{Fr}_{v'}$ we had to construct.

Proof of Theorem 3.6. We take up the thread of the proof of Theorem 3.6 by considering the diagram (7.2∗). In this diagram, the map $G_{B\times C} \hookrightarrow G_B \times G_C$ induces an isomorphism on the derived groups and it follows that $G^{\natural\,\mathrm{ad}}_{B\times C} = G^{\natural\,\mathrm{ad}}_B$ and that both the subgroup $G_{B\times C/\overline{\mathbb{Q}}} \subset (G_B \times G_C)_{/\overline{\mathbb{Q}}}$ and the Lie subalgebra $\mathfrak{g}_{B\times C} \otimes \overline{\mathbb{Q}} \subset (\mathfrak{g}_B \oplus \mathfrak{g}_C) \otimes \overline{\mathbb{Q}}$ are stable under the adjoint action of $G^{\natural\,\mathrm{ad}}_B$. Taking F' big enough and fixing an extension v' of the valuation, we can assume, by Proposition 7.1, that the conclusion of the main theorem holds for B. By [Noot 2009, Corollaire 2.2] we can also assume that it is valid for C. This implies that the theorem is true for $(B \times C)_{/F'}$.

Consider the statement of Theorem 3.6 for the representation of ${}'W_{F'_{v'}}$ associated to $A_{F'}$. The monodromy operators are unaffected by passing from A to $A_{F'}$, whereas Φ_v, and hence the $\rho'_{A,\ell}(\Phi_v)$, are replaced by their f-th powers, where f is the residue degree of the extension $F'_{v'}/F_v$. This exponent is independent of ℓ.

The variety C has potentially good reduction at v', so for every prime number ℓ, the monodromy operator $N'_{A,\ell} \in \mathfrak{g}_A \otimes \mathbb{Q}_\ell$ is the image of

$$(N'_{B,\ell}, 0) \in \mathfrak{g}_{B\times C} \otimes \mathbb{Q}_\ell \subset (\mathfrak{g}_B \oplus \mathfrak{g}_C) \otimes \mathbb{Q}_\ell$$

under the tangent map to π. Here $N'_{B,\ell}$ is the monodromy operator associated to $B_{/F_v}$. We have made use of the fact, expressed by the diagram (7.2∗), that the product of the ℓ-adic Galois representations associated to B and C factors through $G_{B\times C}(\mathbb{Q}_\ell)$.

Similarly,

$$\rho'_{A,\ell}(\Phi_{v'}) = \pi(\rho'_{B,\ell}(\Phi_{v'}), \rho'_{C,\ell}(\Phi_{v'})),$$

which makes sense since $(\rho'_{B,\ell}(\Phi_{v'}), \rho'_{C,\ell}(\Phi_{v'})) \in G_{B\times C}(\mathbb{Q}_\ell) \subset (G_B \times G_C)(\mathbb{Q}_\ell)$. As the theorem holds for $(B \times C)_{/F'}$, it follows that the theorem is true for $A_{/F'}$.

Now return to the original field F. Let $\Omega \supset \mathbb{Q}_\ell$ be an algebraically closed field and $\sigma \in \operatorname{Aut}(\Omega)$. By what we just proved, the images of $(N'_{A,\ell}, \rho'_\ell(\Phi_v^f))$ and $\sigma(N'_{A,\ell}, \rho'_\ell(\Phi_v^f))$ in $\mathfrak{g}_A \otimes \Omega \times G_A(\Omega)$ are conjugate by an element $g \in G^\natural_A(\Omega)$. Thus $N'_{A,\ell} = \operatorname{Ad}(g)(\sigma(N'_{A,\ell}))$ and we will show that $\rho'_\ell(\Phi_v) = g\sigma(\rho'_\ell(\Phi_v))g^{-1}$ as well. Indeed, applying [Raynaud 1994, 4.2] as in Section 3.8, we obtain a strict 1-motive M'/F_v and a system of Γ_{F_v}-equivariant isomorphisms $V_\ell(A_{/F_v}) \cong V_\ell(M')$. By Proposition 1.8, the characteristic polynomials of $\rho'_\ell(\Phi_v)$ and $\sigma(\rho'_\ell(\Phi_v))$ acting on $V_\ell(A)$ coincide. This common polynomial is also the characteristic polynomial of

$g\sigma(\rho'_\ell(\Phi_v))g^{-1}$. As we already know that $\rho'_\ell(\Phi_v^f) = g\sigma(\rho'_\ell(\Phi_v^f))g^{-1}$, the equality $\rho'_\ell(\Phi_v) = g\sigma(\rho'_\ell(\Phi_v))g^{-1}$ follows from Lemma 7.4 below.

Similarly, let Ω be an algebraically closed field containing $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$. We know that the images of the pairs $(N'_{A,\ell}, \rho'_\ell(\Phi_v^f))$ and $(N'_{A,\ell'}, \rho'_{\ell'}(\Phi_v^f))$ are conjugate by some $g \in G_A^\natural(\Omega)$. Again by Proposition 1.8, the characteristic polynomials of $\rho'_\ell(\Phi_v)$ and $\rho'_{\ell'}(\Phi_v))$ coincide so Lemma 7.4 implies that $\rho'_\ell(\Phi_v) = \rho'_{\ell'}(\Phi_v))$. This proves the theorem for A. □

Lemma 7.4. *Assume that Ω is an algebraically closed field, $d > 0$ is an integer and that $x, y \in \mathrm{GL}_d(\Omega)$ are two semisimple and weakly neat elements. Assume that $x^f = y^f$ for some integer f and that x and y have the same characteristic polynomial. Then $x = y$.*

Proof. This is a variant of [Noot 2009, Proposition 3.2].

For any semisimple element $z \in \mathrm{GL}_d(\Omega)$, let $T_z \subset \mathrm{GL}_d$ be the torus acting by scalar multiplication on each eigenspace of z. Up to conjugation, z is a point of the diagonal torus $\mathbb{G}_m^d \subset \mathrm{GL}_d$ and, writing $t_1, \dots, t_d$ for the coordinates on $\mathbb{G}_m^d$ and $z = (z_1, \dots, z_d)$, one then has

$$T_z = \{(t_1, \dots, t_d) \in \mathbb{G}_m^d \mid t_i = t_j \text{ if } z_i = z_j\}. \tag{7.4$*$}$$

Note that for every positive integer n one has $T_{z^n} \subset T_z$ and that this inclusion is an equality if z is weakly neat.

With this notation we prove the lemma. As x and y are weakly neat and satisfy $x^f = y^f$, we get $T_x = T_{x^f} = T_{y^f} = T_y$. This implies in particular that $y \in T_x(\Omega)$. We can assume that x lies in the diagonal torus $\mathbb{G}_m^d \subset \mathrm{GL}_d$ and we write $x = (x_1, \dots, x_d) \in \mathbb{G}_m^d(\Omega)$. The fact that x and y have the same characteristic polynomial implies that there is a permutation $\sigma \in \mathrm{S}_d$ of the factors of the product $\mathbb{G}_m^d$ such that $y = \sigma(x)$. We have $x^f = y^f = \sigma(x^f)$ and, considering the equations (7.4$*$) for T_{x^f}, it follows that $\sigma|_{T_{x^f}} = \mathrm{id}$. Since $T_{x^f} = T_x$ we conclude that $x = \sigma(x) = y$ as claimed. □

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Fourier–Jacobi coefficients of Eisenstein series on unitary groups

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This paper studies the Fourier–Jacobi expansions of Eisenstein series on U(3, 1). I relate the Fourier–Jacobi coefficients of the Eisenstein series with special values of L-functions. This relationship can be applied to verify the existence of certain Eisenstein series on U(3, 1) that do not vanish modulo p. This is a crucial step towards one divisibility of the main conjecture for $\mathrm{GL}_2 \times K^\times$ using the method of Eisenstein congruences.

1. Introduction

Eisenstein congruence and Iwasawa main conjecture. Eisenstein series have been intensively used in constructions of p-adic L-functions and in the Iwasawa main conjectures. Ribet [1976] used the congruences between Eisenstein series and cusp forms to prove the converse of the Herbrand theorem. This idea was extended to congruences between p-adic families of modular forms, which was successfully used, first by Mazur and Wiles [1984] to prove the main conjecture for real abelian fields, then by Wiles [1990] for all totally real fields. Subsequently, Skinner and Urban used this technique to study the main conjectures for the motives attached to modular forms; see [Urban 2001; Urban 2006; Skinner and Urban 2012].

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In the ongoing joint project with Hsieh, we want to apply Eisenstein congruences to the following main conjecture for elliptic curves. Take an imaginary quadratic field K in which p splits. Fix an embedding $i_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and an isomorphism $i : \mathbb{C} \cong \mathbb{C}_p$. Let

$$\rho_E : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}(H^1_{\mathrm{et}}(E, \mathbb{Q}_p))$$

be the p-adic Galois representation associated to an elliptic curve E over $\mathbb{Q}$ with good ordinary reduction at p. Let $\eta : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ be a Hecke character with the infinity type $(k, 0)$, where k is an integer and $k > 1$. Put $\mathbb{O} = \mathbb{Z}_p[\mathrm{Im}\,\eta]$, the ring of values of η. Let $K_{\max}$ be the unique $\mathbb{Z}_p^2$-extension of K, and let $\Gamma = \mathrm{Gal}(K_{\max}/K)$. Define $\Lambda = \mathbb{O}[[\Gamma]]$ the Iwasawa algebra of Γ. Given Ψ a Λ-valued character of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ that interpolates η, there exists a unique element $L_p(\rho_E \otimes \Psi)$ in Λ interpolating $((2\pi i)^{2k}/\Omega_K^{2k})L_K(0, E \otimes \eta)$, where Ω_K is the CM-period associated to K. (About the precise normalization of $L_K(E \otimes \eta, 0)$, one may check [Hsieh 2011b, Definition 1]. Especially, since the weight of the CM form associated to η is greater than 2, the period appearing in the denominator of the normalization of $L_K(E \otimes \eta, 0)$ is not the period of E, but the CM period attached to η and K.) Let $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ be the two primes in K above p. Let K_S be the maximal unramified extension of K outside S, where

$$S = \{\mathfrak{p}, \bar{\mathfrak{p}}\} \cup \{v \text{ finite} \mid \eta_v \text{ is ramified, or } E \text{ has a bad reduction at } v\} \cup \{\infty\}.$$

Use M^* to denote the Pontryagin dual of a Λ-module M. Define the nonprimitive Λ-adic Selmer group to $\rho_E \otimes \Psi$ by

$$\mathrm{Sel}_K(\rho_E \otimes \Psi) = \ker\{H^1(\mathrm{Gal}(K_S/K), T \otimes \Lambda^*) \to H^1(I_{\bar{\mathfrak{p}}}, T_{\bar{\mathfrak{p}}} \otimes \Lambda^*)\},$$

where $T = H^1_{\mathrm{et}}(E, \mathbb{Z}_p)$, $I_{\bar{\mathfrak{p}}}$ is the inertia group at $\bar{\mathfrak{p}}$; see [Hsieh 2011b]. The following conjecture is formulated in [Greenberg 1994]:

Conjecture 1.1. $\mathrm{Sel}_K(\rho_E \otimes \Psi)$ *is cotorsion over* Λ, *and for any height* 1 *prime* P,

$$\mathrm{ord}_P\, L_p(\rho_E \otimes \Psi) = l_P(\mathrm{Sel}_K(\rho_E \otimes \Psi)),$$

where $l_P(\mathrm{Sel}_K(\rho_E \otimes \Psi)) = \mathrm{length}_{\Lambda_P}(\mathrm{Sel}_K(\rho_E \otimes \Psi)^* \otimes_\Lambda \Lambda_P)$.

Remark 1.2. Though they have similar formulations, the conjecture above is different from the main conjecture of elliptic curves studied in [Skinner and Urban 2012], because the specializations of Ψ are different. Our Ψ interpolates Hecke characters over K with the infinity type $(k, 0)$ for an integer $k > 1$; the infinity types in [ibid.] are different. In addition, we consider different $L_p(\rho_E \otimes \Psi)$ and $\mathrm{Sel}_K(\rho_E \otimes \Psi)$.

Nonvanishing modulo p *of Eisenstein series.* Both for Ribet's original argument and for various cases of Iwasawa main conjectures, we crucially need to guarantee the Eisenstein series (or the p-adic Eisenstein series) used in the proofs does not vanish modulo p, for the naive reason that this is sufficient to deduce that the congruent cusp form is nontrivial. A more technical reason is that it is necessary to show that the constant term of this Eisenstein series divides the Eisenstein ideal, which is the first step towards one divisibility of the main conjecture.

For main conjectures of different cuspidal representations, Eisenstein series on different reductive groups are constructed so that the p-adic L-functions in the main conjectures interpolate the constant terms of the Eisenstein series. There is no consistent way to argue nonvanishing modulo p of an Eisenstein series on a general reductive group. This question has been one of the obstacles of Iwasawa theory for L-functions of higher degrees. Urban [2006] argued that an Eisenstein–Klingen series on GSp(4) is nontrivial modulo p in the following way: Find an algebraic linear combination of its Fourier coefficients (this is essentially a period integral of the Eisenstein series) that turns out to be a special L-value, and use Vatsal's result [2003] that this L-value does not vanish modulo p. A similar argument is used as well by Skinner and Urban [2012] to show an Eisenstein series on U(2, 2) does not vanish modulo p.

To solve the main conjecture in our ongoing project, we need to construct an Eisenstein series on U(3, 1). To attack the question about nonvanishing modulo p of this Eisenstein series, the method is a bit different from other cases. Because U(3, 1) is nonquasisplit, the Eisenstein series has a Fourier–Jacobi expansion where the coefficients are theta functions.

Results on Fourier–Jacobi coefficients. From now on, let K be an imaginary quadratic extension of a totally real field F. Assume the degree of F over $\mathbb{Q}$ is r. Let P be the minimal parabolic subgroup of U(3, 1) with $\mathrm{U}(2) \times K^\times$ as the Levi part. Let Π be an automorphic representation of $\mathrm{U}(2)(F) \setminus U(2)(\mathbb{A}_F)$, which corresponds to a holomorphic weight 2 cuspidal eigenform on $\mathrm{GL}_2(F)$. Take a Hecke character η of $K^\times$ such that $\eta_\infty(z_\infty) = |z_\infty|^k / {z_\infty}^k$. The desirable Eisenstein series $E_k(\,\cdot\,, \Pi, \eta)$ of weight k on U(3, 1) is defined by pulling back a Siegel–Eisenstein series on U(3, 3). This pullback idea was due to Shimura and was applied in the constructions of Eisenstein series in many cases; see [Urban 2006; Skinner and Urban 2012; Hsieh 2011b].

Let $\mathcal{M}_k(U(3, 1))$ be the space of weight k automorphic forms on U(3, 1). A linear functional associated to a Bruhat–Schwartz function ϕ on $\mathbb{A}^2$ (where $\mathbb{A}$ is the adèle of F) can be defined by

$$l_\phi : \mathcal{M}_k(\mathrm{U}(3, 1)) \to \mathbb{C}, \quad \mathbf{F} \mapsto \frac{\langle \mathbf{F}_\psi, \theta_\phi \rangle \cdot \theta_\phi^\Pi(1)}{\langle \theta_\phi, \theta_\phi \rangle}, \tag{1}$$

where θ_ϕ^Π is the theta lifting from the cuspidal representation Π on U(2) to U(1) defined by θ_ϕ, and $\mathbf{F}_\psi$ is the Fourier–Jacobi coefficient of $\mathbf{F}$ attached to the additive character ψ. When defining the theta lifting, a Hecke character χ will be introduced. So in the following Theorem 1.3, χ is implicit in the left side of the equation, but appears in the right. For the precise definition of the Fourier–Jacobi coefficient of $\mathbf{F}$, one can refer to (25), and change $E(\,\cdot\,, f, s)$ in that equation to $\mathbf{F}$. Let $T_{\mathbb{A}}(\psi)$ be the space of adelic theta functions on $N(F)U(2)(F) \setminus N(\mathbb{A})U(2)(\mathbb{A})$ with a well-defined inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. Then $\mathbf{F}_\psi$ and θ_ϕ are both elements inside.

In the following theorem, S denotes the set of ramified places (the ramified places include places where any data used in the computation is ramified, for example, the characters η and χ are ramified, the field extension K/F is ramified, the representation Π is ramified, and so on). We use L^S to denote the partial L-function omitting the local L-factors at primes in S.

Theorem 1.3. *For $E_k(\,\cdot\,, \Pi, \eta)$,*

$$l_\phi(E_k(\,\cdot\,, \Pi, \eta)) \\ = C \cdot \frac{(2\pi i)^{(2k-4)r} L^S(\eta\chi^{-1}, \frac{1}{2}(k-2)) L^S(\eta^{-1}\chi, \frac{1}{2}(k-4)) L^S(\Pi, \chi, 1)}{\Omega_K^{2k}},$$

where C is a nonzero constant which can be explicitly calculated, $r = [F : \mathbb{Q}]$ and Ω_K is the CM-period associated to K.

The nature of $l_\phi(E_k(\,\cdot\,, \Pi, \eta))$ depends on the normalization of $E_k(\,\cdot\,, \Pi, \eta)$. However, when we choose $E_k(\,\cdot\,, \Pi, \eta)$ to be rational, we can then show that $l_\phi(E_k(\,\cdot\,, \Pi, \eta)) \in \overline{\mathbb{Q}}$. So C must be an algebraic number.

In the paper, the value of $l_\phi(E_k(\,\cdot\,, \Pi, \eta))$ can simply be obtained by detailed computations of $\langle E_k(\,\cdot\,, \Pi, \eta), \theta_\phi\rangle$, from which one can prove Theorem 4.12. The proof uses computations of Fourier–Jacobi coefficients of the Siegel–Eisenstein series on U(3, 3), whose definition can be found in (21); pullback formulas; the theta liftings for unitary groups; and the Siegel–Weil formula. By unfolding integrals step by step, the questions are translated to computations of Rallis inner product type. Then by studying the integral structure of the space of theta functions $T_{\mathbb{A}}(\psi)$, we can prove:

Proposition 1.4. *For the p-integral holomorphic Eisenstein series E_k, $l_\phi(E_k)$ is a p-integer.*

Conjecture 1.5. $E_k(\,\cdot\,, \Pi, \eta)$ *does not vanish* modulo p.

By the nonvanishing modulo p of Hecke L-values (see [Hida 2004a]), we see that this conjecture is reduced to the question "Does there exist a Hecke character χ such that $L^S(\Pi, \chi, 1)$ does not vanish modulo p?" There have been many results of this flavor by Vatsal, Hida, Sun and Brakočević. I will consider this question in the near future.

The method introduced in this paper for calculating Fourier–Jacobi coefficients of Eisenstein series on U(3, 1) can be generalized well to other nonquasisplit unitary groups. I will briefly explain the generalizations in Remark 5.7 at the end of this paper. For example, this method gives an alternative way to calculate the Fourier–Jacobi coefficients of the Eisenstein series on U(2, 1) other than the one given in [Murase and Sugano 2002], and recovers the proof of nonvanishing modulo p of this Eisenstein series (compare to the discussion in [Mainardi 2004], which uses results in [Murase and Sugano 2002]).

The generalized computational results for Fourier–Jacobi coefficients of certain Eisenstein series on arbitrary unitary groups might inspire a new argument on the nonvanishing modulo p of the Eisenstein series $E_k(\,\cdot\,, \Pi, \eta)$ on U(3, 1). Briefly, one can construct an Eisenstein series on U(4, 2) of the type explained in Remark 5.7. Thus the linear functional on this Eisenstein series will be the product of central special L-values, whose p-adic properties are much easier to study. Especially, we can first obtain nonvanishing modulo p of this Eisenstein series. Then by studying the relations between Fourier–Jacobi coefficients of $E_k(\,\cdot\,, \Pi, \eta)$ on U(3, 1) and this Eisenstein series on U(4, 2), we can furthermore argue the nonvanishing modulo p of $E_k(\,\cdot\,, \Pi, \eta)$. I hope to address this question soon in another paper.

Structure of the paper and some notation. Section 2 is about the theory of Eisenstein series on unitary groups. Two types of Eisenstein series are defined: one is on U(3, 1) (U(I_ζ)), and the other is the Siegel type on U(3, 3). Section 2C discusses the pullback formula. Theorem 2.6 gives the precise form, which is actually an adelic counterpart of "pullback of Eisenstein series" in [Shimura 1997].

Section 3 recalls the theory of Weil representations and theta functions. First, Section 3A introduces the Schrödinger representation ρ_ψ of the Heisenberg group and the Weil representation ω_ψ of the metaplectic group. For this paper, we mainly need the Weil representation restricted to the dual reductive pair of unitary groups. The needed results are summarized in Section 3B, which also explains the relation between $\omega_\psi|_{U(2)}$ and $\omega_\psi|_{U(2,2)}$. Section 3C briefly recalls the theta lifting that appears in (1) and the Siegel–Weil formula, which is used to calculate the theta lifting.

Section 4 is about computations of Fourier–Jacobi coefficients of the two types of Eisenstein series used in the paper. The very important result that helps give a nice expression of the Fourier–Jacobi coefficients of $E_k(\,\cdot\,, \Pi, \eta)$ is summarized in Theorem 4.9.

Section 5, gives a strategic answer to the question of how to apply the results in Section 4 to show $E_k(\,\cdot\,, \Pi, \eta))$ on U(3, 1) does not vanish modulo p.

Unfortunately, the Siegel–Weil formula for the reductive pair (U(2), U(1)) that I use in this paper is not to be found in the literature — I give a proof in Appendix A. Appendix B gives a very brief discussion about integral theta functions, which is used in Section 5.

Let G be an algebraic group defined over the totally real number field F. In this paper, we use $G(\mathbb{A})$ and $G(F)$ to denote the groups of adelic points and F-points of G, respectively. We use $[G]$ to denote the quotient $G(F)\backslash G(\mathbb{A})$. Given an arbitrary number field L (for example, $\mathbb{Q}$, F, K), for each place v, we choose the additive Haar measure on L_v so that, if $L_v \simeq \mathbb{R}$, the measure is the usual Lebesgue measure, if $L_v \simeq \mathbb{C}$, the measure is $2\,dx\,dy$ ($z = x + iy \in \mathbb{C}$), and if v is a finite place, the measure gives the volume of the integer ring of L_v to be $D_{L_v}^{-1/2}$ (D_{L_v} is the absolute discriminant of L_v). The product of these local measures gives a measure on $\mathbb{A}_L$, and thus induces a measure on $L\backslash\mathbb{A}_L$. At each finite place v, the multiplicative measure on $L_v^\times$ is taken such that $\mathrm{vol}(\mathbb{O}_{L_v}^\times) = 1$, from which we obtain the measure on $\mathbb{A}_L^\times$. Notice that this multiplicative measure and the measure chosen in [Skinner and Urban 2012, 8.2.1] differ by a constant. The Haar measures on local and adelic points of algebraic groups should be clear from the definitions above and from the context.

2. Eisenstein series on U(3, 1)

2A. ***Unitary groups.*** Let K be a totally imaginary quadratic extension of a totally real number field F. Consider an n-dimensional K-vector space V with an ϵ-Hermitian form σ for $\epsilon = \pm 1$. Without loss of generality, from now on we fix $\epsilon = -1$. The unitary group associated to (V, σ) can be defined as follows:

$$\mathrm{U}(\sigma) = \{g \in \mathrm{GL}(V) \mid \sigma(xg, yg) = (x, y) \text{ for all } x, y \in V\}.$$

To obtain a good matrix representation of $\mathrm{U}(\sigma)$, let us fix a Witt decomposition of V, with $V = J + Z + J'$, where J and J' are maximal totally σ-isotropic subspaces, and Z is anisotropic or empty, so that $\dim J = \dim J' = r$ and $\dim Z = t = n - 2r$. Under a suitable basis of V consistent with the Witt decomposition, σ can be expressed by the matrix

$$I_\zeta = \begin{pmatrix} 0 & 0 & I_r \\ 0 & \zeta & 0 \\ -I_r & 0 & 0 \end{pmatrix}$$

with $\zeta = -\zeta^* \in \mathrm{GL}_t(K)$. (From now on, we use x^* to denote $\bar{x}^t$ for a matrix x.) Then:

$$U(\sigma) = U(I_\zeta) = \{g \in \mathrm{GL}_n(K) \mid g I_\zeta g^* = I_\zeta\}.$$

Accordingly, the adelic unitary group $\mathrm{U}(I_\zeta)(\mathbb{A})$ and local groups $\mathrm{U}(I_\zeta)_v$ can be defined. Given a totally σ-isotropic subspace J (which may not be maximal), put $P_J = \{g \in U(I_\zeta) \mid Jg = J\}$. Then P_J is the parabolic subgroup of $\mathrm{U}(I_\zeta)$ associated to J.

The Hermitian domain associated to $\mathrm{U}(I_\zeta)$ is defined as

$$\mathfrak{Z} = \mathfrak{Z}(r,\zeta) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}_r^{r+t} \,\middle|\, x \in \mathbb{C}_r^r,\ y \in \mathbb{C}_r^t,\ i(x^* - x) > iy^*\zeta^{-1}y \right\} \tag{2}$$

Let

$$g_\infty = \begin{pmatrix} a & b & c \\ d & e & f \\ h & l & p \end{pmatrix} \in U(I_\zeta)_\infty,$$

with $a, p \in \mathbb{C}_r^r$ and $e \in \mathbb{C}_t^t$. The action of g_∞ on $\mathfrak{Z}(r,\zeta)$ is

$$g_\infty \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (ax+by+c)(hx+ly+p)^{-1} \\ (dx+ey+f)(hx+ly+p)^{-1} \end{pmatrix}.$$

The automorphy factor is

$$j\left(g_\infty, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \det(hx+ly+p). \tag{3}$$

Proposition 2.1. (1) *Pick the origin* $\boldsymbol{i}$ *of* $\mathfrak{Z}$ *and put* $C = \{g_\infty \in U(I_\zeta)_\infty \mid g_\infty \boldsymbol{i} = \boldsymbol{i}\}$. *Then* C *is a maximal compact subgroup of* $\mathrm{U}(I_\zeta)_\infty$.

(2) *Let* $P = P_J$, *where* J *is a maximal totally isotropic subspace of* V. *Then* $\mathrm{U}(I_\zeta)_\infty = P_\infty C$.

Assume $\dim V = 4$ so that J and J' are two isotropic lines, and Z is a 2-dimensional anisotropic space; then $\mathrm{U}(I_\zeta) \subset \mathrm{GL}_4(K)$ is a degree 4 unitary group. From now on, whenever we write $\mathrm{U}(I_\zeta)$, I mean particularly this unitary group. To $\mathrm{U}(I_\zeta)$, there is only one nontrivial parabolic subgroup up to conjugation that is P_J. We can simply denote it by P. It consists of such elements

$$p = \begin{pmatrix} a & * & * \\ & u & * \\ & & (a^*)^{-1} \end{pmatrix},$$

where $a \in K^\times$ and

$$u \in U(\zeta) = \{u \mid u\zeta u^* = \zeta\}. \tag{4}$$

2B. ***Eisenstein series.*** Following the notation above, P is the parabolic subgroup of $\mathrm{U}(I_\zeta)$. It has the Levi decomposition $P = M \cdot N$, where M is the Levi part which is isomorphic to $\mathrm{U}(\zeta) \times \mathbb{G}_m/K$, and N is the unipotent radical, consisting of elements like

$$\begin{pmatrix} 1 & * & * \\ & I_2 & * \\ & & 1 \end{pmatrix}.$$

Given a cuspidal representation Π of $\mathrm{U}(\zeta)$ and a Hecke character η of K, we get a cuspidal representation $\Pi \otimes \eta$ on the Levi part M. Then one has the induced representation

$$I_P^{U(I_\zeta)}(\Pi \otimes \eta, s) := \mathrm{Ind}_{P(\mathbb{A})}^{U(I_\zeta)(\mathbb{A})} \delta_P{}^{\frac{1}{2}+s} \cdot \Pi \otimes \eta,$$

where δ_P is the modulus character on P. If we denote by V_Π the representation space of Π, then the representation space for $I_P^{U(I_\zeta)}(\Pi \otimes \eta, s)$ is the set of smooth functions $\tilde{f}_s : U(I_\zeta)(\mathbb{A}) \to V_\Pi$ such that

(1) $\tilde{f}_s(pg) = \delta_P(p)^{\frac{1}{2}+s} \cdot \Pi \otimes \eta(p)(\tilde{f}_s(g))$, $p \in P(\mathbb{A})$,

(2) $\tilde{f}_s$ is right $\mathbf{K}$-finite with $\mathbf{K}$ a maximal open compact subgroup of $\mathrm{U}(I_\zeta)(\mathbb{A})$.

The action of $\mathrm{U}(I_\zeta)(\mathbb{A})$ on $I_P^{U(3,1)}(\Pi \otimes \eta, s)$ is by right translation.

If we further assume that Π is an irreducible submodule of $\mathscr{A}(M)$, where $\mathscr{A}(M)$ denotes the set of automorphic forms on M, we can realize $I_P^{U(I_\zeta)}(\Pi \otimes \eta, s)$ as $\mathbb{C}$-valued functions rather than functions valued in V_Π. For $\tilde{f}_s \in I_P^{U(I_\zeta)}(\Pi \otimes \eta, s)$, let $f_s(g) = (\tilde{f}_s(g))(1)$. Then $f_s(g)$ satisfies $f_s(nmg) = \tilde{f}_s(g)(m)$. Define the Eisenstein series

$$E(g, f, s) = \sum_{\gamma \in P(F)\backslash U(I_\zeta)(F)} f_s(\gamma g). \tag{5}$$

It can be proved that the sum converges absolutely when $\mathrm{Re}(s) \gg 0$, and can be continued to a meromorphic function on $\mathbb{C}$.

Aside from the Eisenstein series on $\mathrm{U}(I_\zeta)$, we also want to define an Eisenstein series on $\mathrm{U}(3, 3)$, which we are going to use in next section. Following definitions of unitary groups in the previous section,

$$U(3,3) = \left\{ g \in \mathrm{GL}_6(K) \;\middle|\; g \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} g^* = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} \right\}. \tag{6}$$

The Siegel parabolic subgroup $\mathscr{P}$ of $\mathrm{U}(3, 3)$ is the one that fixes the maximal totally isotropic space of dimension 3 in a 6-dimensional Hermitian vector space. The Levi part of $\mathscr{P}$ is isomorphic to $\mathrm{GL}_3(K)$. Take the one-dimensional representation $\eta(\det \cdot)$ on the Levi part. Then it induces the representation on $\mathrm{U}(3, 3)$ as

$$I_{\mathscr{P}}^{U(3,3)}(\eta, s) := \mathrm{Ind}_{\mathscr{P}(\mathbb{A})}^{U(3,3)(\mathbb{A})} \delta_{\mathscr{P}}^{\frac{1}{2}+s} \cdot \eta(\det \cdot).$$

Pick a section $\mathbb{f}_s$ in the induced representation. The Siegel–Eisenstein series is defined by

$$\mathbb{E}(g, \mathbb{f}, s) = \sum_{\gamma \in \mathcal{P}(F) \backslash U(3,3)(F)} \mathbb{f}_s(\gamma g) \quad \text{for } g \in U(3,3)(\mathbb{A}). \tag{7}$$

It satisfies analytic properties similar to $E(\cdot, f, s)$.

2C. *Pullback formulas.* In this section, the Eisenstein series $E(g, f, s)$ on $\mathrm{U}(I_\zeta)$ is constructed using the pullback of a Siegel Eisenstein series on U(3, 3).

The unitary group $\mathrm{U}(\zeta)$ is closely related to the quaternion algebra, about which, let me quote two lemmas from [Shimura 1997].

Lemma 2.2. *V is a 2-dimensional K-vector space with a nondegenerate Hermitian pairing described by ζ. Then V is anisotropic if and only if* $\det \zeta$ *is represented by* -1 *in* $K^\times/N_{K/F}(K^\times)$.

Lemma 2.3. *Let (V, ζ) be anisotropic.*

(1) $D := \{\alpha \in \mathrm{End}(V) \mid \sigma(\alpha^\iota x, y) = \sigma(x, \alpha y)$ *for all* $x, y \in V\}$, *where ι is the main involution of* $\mathrm{End}(V)$ *such that* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\iota = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. *Then D is a definite quaternion algebra over F.*

(2) $\mathrm{U}(\zeta) = \{s\alpha \mid s \in K^\times, \alpha \in D^\times, ss^\rho \det\alpha = 1\}$.

So, locally:

(1) When K_v splits, namely $K_v = F_v \times F_v$, we have $\mathrm{U}(\zeta)_v \simeq \mathrm{GL}_2(F_v)$.

(2) When K_v is a field and $d(\sigma_v)$ is represented by 1, we have $\mathrm{U}(\zeta)_v = U(1,1)_v$.

(3) When K_v is a field and $d(\sigma_v)$ is represented by -1 (including archimedean places), we have $\mathrm{U}(\zeta)_v \subset K_v^\times \cdot D_v^\times$, and D_v is a quaternion algebra. In this case, $\mathrm{U}(\zeta)_v$ is compact.

The following theorem about the Jacquet–Langlands correspondence relates automorphic forms on $D^\times$ to automorphic forms on $\mathrm{GL}_2(F)$.

Theorem 2.4 [Gelbart 1975]. *S is the set of places of F where D is ramified. To each irreducible unitary admissible representation $\pi = \otimes_v \pi_v$ of $D_{\mathbb{A}}^\times$, let π' denote the representation of* $\mathrm{GL}_2(\mathbb{A})$ *whose v-th component is equivalent to π_v if $v \notin S$, and special or supercuspidal if $v \in S$.*

(1) *The map $\pi \to \pi'$ restricted to the collection of* (*greater than one-dimensional*) *cusp forms on $D^\times$ is one-to-one onto the collection of all* (*equivalence classes of*) *cusp forms on* $\mathrm{GL}_2(F)$ *such that π'_v is square-integrable for each $v \in S$.*

(2) *If we require further that $\pi \subset L^2(D^\times(F) \setminus D^\times(\mathbb{A}))$, then it implies that π'_v is one-dimensional for $v \in S$.*

For an irreducible representation π on the definite quaternion algebra D, π_∞ as a finite-dimensional representation of $D_\infty^\times$ is equivalent to $|\det|^r \otimes \rho_n$, where ρ_n is the n-th symmetric tensor product. It corresponds to the discrete series

$$\sigma(\mu_1, \mu_2) \text{ of } \mathrm{GL}_2(\mathbb{R}) \quad \text{with } \mu_1(x) = |x|^{r+n+\frac{1}{2}} \quad \text{and} \quad \mu_2(x) = |x|^{r-\frac{1}{2}} \operatorname{sgn}(x)^n.$$

Globally, π corresponds to a cuspidal newform f that is also a Hecke eigenform of weight $k = n + 2$.

Take a representation Π of $\mathrm{U}(\zeta)$ in this way: First pick an irreducible representation $\lambda \cdot \pi$ on $\mathbb{A}_K^\times \cdot D_\mathbb{A}^\times$ satisfying $\lambda|_{\mathbb{A}^\times} = \chi_\pi$, where χ_π is the central character of π. Then $\Pi = (\lambda \cdot \pi)|_{U(\zeta)(\mathbb{A})}$ gives an irreducible representation on $\mathrm{U}(\zeta)$. For our later application, the cusp form f that π corresponds to comes from an elliptic curve. So f has weight $k = 2$. Then Π_∞ must be one-dimensional. In this case, the pullback formula will have a simple form. So in this paper, we always assume Π satisfies this condition.

Define an embedding

$$\mathfrak{e} : U(I_\zeta) \times U(\zeta) \to U(3,3), \quad (g, u) \mapsto \mathfrak{e}(g, u) = A_\zeta \begin{pmatrix} g & \\ & u \end{pmatrix} A_\zeta^{-1}, \tag{8}$$

where

$$A_\zeta = \begin{pmatrix} 1 & & & \\ & \zeta^{-1} & & -\zeta^{-1} \\ & & 1 & \\ & \frac{1}{2} & & \frac{1}{2} \end{pmatrix}.$$

Pay attention that $\mathrm{U}(\zeta)$ has no nontrivial parabolic subgroup. If $\mathcal{P}$ is the Siegel parabolic subgroup of $\mathrm{U}(3,3)$ and P is the only nontrivial parabolic subgroup of $\mathrm{U}(I_\zeta)$, we have the following lemma:

Lemma 2.5 [Shimura 1997]. $\mathcal{P}(F) \setminus U(3,3)(F) = \mathfrak{e}(P(F) \setminus U(I_\zeta)(F), U(\zeta)(F))$.

Take $\mathbb{f}_s \in I_{\mathcal{P}}^{U(3,3)}(\eta, s)$. Define the Eisenstein series $\mathbb{E}(g, \mathbb{f}, s)$ on $\mathrm{U}(3,3)$ by (7).

Theorem 2.6. *Take $\beta \in V_\Pi \subset L^2(U(\zeta)(F) \setminus U(\zeta)(\mathbb{A}))$. Then the integral*

$$\int_{U(\zeta)(F) \setminus U(\zeta)(\mathbb{A})} \mathbb{E}(\mathfrak{e}(g, u), \mathbb{f}, s) \beta(u) \eta^{-1}(\det u)\, du$$

gives an Eisenstein series $E(g, f, s)$ on $\mathrm{U}(I_\zeta)$ associated to $f_s \in I_P^{U(I_\zeta)}(\Pi \otimes \eta, s)$, and

$$f_s(g) = \int_{U(\zeta)(\mathbb{A})} \mathbb{f}_s(\mathfrak{e}(g, u)) \beta(u) \eta^{-1}(\det u)\, du.$$

Here we understand $\beta(u)$ as $(\Pi(u)\beta)(1)$.

Proof. General results about the pullback formula at infinity are essentially discussed in [Shimura 1997]. Here I give a proof using the adelic language. Using Lemma 2.5,

$$\begin{aligned}\mathbb{E}(\mathfrak{e}(g,u),\mathbb{f},s) &= \sum_{\gamma\in\mathcal{P}(F)\backslash U(3,3)(F)} \mathbb{f}_s(\gamma\,\mathfrak{e}(g,u))\\ &= \sum_{\gamma_1\in P(F)\backslash U(I_\zeta)(F)}\ \sum_{\gamma_2\in U(\zeta)(F)} \mathbb{f}_s(\mathfrak{e}(\gamma_1 g,\gamma_2 u)),\end{aligned}$$

unfold the integral:

$$\begin{aligned}&\int_{U(\zeta)(F)\backslash U(\zeta)(\mathbb{A})} \mathbb{E}(\mathfrak{e}(g,u),\mathbb{f},s)\beta(u)\eta^{-1}(\det u)du\\ &\qquad= \int_{U(\zeta)(F)\backslash U(\zeta)(\mathbb{A})}\ \sum_{\gamma_1\in P(F)\backslash U(I_\zeta)(F)}\ \sum_{\gamma_2\in U(\zeta)(F)} \mathbb{f}_s(\mathfrak{e}(\gamma_1 g,\gamma_2 u))du\\ &\qquad= \int_{U(\zeta)(\mathbb{A})}\ \sum_{\gamma_1\in P(F)\backslash U(I_\zeta)(F)} \mathbb{f}_s(\mathfrak{e}(\gamma_1 g,u))\beta(u)\eta^{-1}(\det u)du\\ &\qquad= \sum_{\gamma_1\in P(F)\backslash U(3,1)(F)} \int_{U(\zeta)(\mathbb{A})} \mathbb{f}_s(\mathfrak{e}(\gamma_1 g,u))\beta(u)\eta^{-1}(\det u)du. \qquad (*)\end{aligned}$$

In the last step, we suppose s is in a proper range so that there are no convergence problems. Then we can change the order of the integral and the summation. Formally, $(*)$ looks like the Eisenstein series $E(g,f,s)$ on $\mathrm{U}(I_\zeta)$ defined by (5) with

$$f_s = \int_{U(\zeta)(\mathbb{A})} \mathbb{f}_s(\mathfrak{e}(g,u))\beta(u)\eta^{-1}(\det u)\,du.$$

We are left to show that $f_s \in I_P^{U(I_\zeta)}(\Pi\otimes\eta,s)$. Take

$$p = \begin{pmatrix} a & * & * \\ & v & * \\ & & (a^*)^{-1}\end{pmatrix} \in P(\mathbb{A}),$$

and also note that

$$\mathfrak{e}\left(\begin{pmatrix} a & * & * \\ & v & * \\ & & (a^*)^{-1}\end{pmatrix}, v\right) = \begin{pmatrix} a & * & * & * \\ & \zeta^{-1}v\zeta & * & * \\ & & (a^*)^{-1} & * \\ & & & v\end{pmatrix} \in \mathcal{P}(\mathbb{A})$$

by the embedding formula (8). So

$$\begin{aligned}
f_s(pg) &= \int_{U(\zeta)(\mathbb{A})} \mathbb{f}_s(\mathfrak{e}(pg,u))\beta(u)\eta^{-1}(\det u)\,du \\
&= \int_{U(\zeta)(\mathbb{A})} \mathbb{f}_s(\mathfrak{e}(pg,vu))\beta(vu)\eta^{-1}(\det vu)\,du \\
&= \int_{U(\zeta)(\mathbb{A})} \mathbb{f}_s\left(\mathfrak{e}\left(\begin{pmatrix} a & * & * \\ & v & * \\ & & (a^*)^{-1}\end{pmatrix}, v\right)\cdot \mathfrak{e}(g,u)\right)\beta(vu)\eta^{-1}(\det vu)\,du \\
&= \int_{U(\zeta)(\mathbb{A})} \eta(a\det v)|a\det v|^{\frac{1}{2}+s}\mathbb{f}_s(\mathfrak{e}(g,u))\beta(vu)\eta^{-1}(\det vu)\,du \\
&= \Pi\otimes\eta(p)f_s(g)|a|^{\frac{1}{2}+s}.
\end{aligned}$$

This means that $f_s \in I_P^{U(I_\zeta)}(\Pi\otimes\eta, s)$. □

3. Theta functions

3A. *Weil representations.* Suppose V is a finite-dimensional vector space over a field $\mathbb{F}$. When $\mathbb{F}$ is a nonarchimedean local field, use $S(V)$ to denote the space of Bruhat–Schwartz functions (locally constant compactly supported functions) on V. If $F=\mathbb{R}$ or $\mathbb{C}$, we first take $S(V)$ to be $L^2(V)$ (but later, we may add holomorphic conditions when needed).

Following the notation in previous sections, F is a totally real number field. Let W be a finite-dimensional symplectic vector space over F_v with a nondegenerate alternating form $\langle\cdot,\cdot\rangle$. The Heisenberg group $H(W)$ associated to W is a nontrivial central extension of W by F_v and is defined to be a group of pairs $\{(w,t)\mid w\in W, t\in F_v\}$ with the law of multiplication

$$(w_1,t_1)(w_2,t_2)=\bigl(w_1+w_2,t_1+t_2+\tfrac{1}{2}\langle w_1,w_2\rangle\bigr).$$

Fix an additive character ψ of F_v and a complete polarization of W as $W=X\oplus Y$ where X and Y are maximal totally isotropic subspaces of W. Define the Schrödinger representation ρ_ψ of $H(W)$ on $S(X)$ as follows:

$$\begin{aligned}
\rho_\psi(x)f(z)&=f(x+z) && \text{for } x\in X,\\
\rho_\psi(y)f(z)&=\psi(\langle z,y\rangle)f(z) && \text{for } y\in Y,\\
\rho_\psi(t)f(z)&=\psi(t)f(z) && \text{for } t\in F_v.
\end{aligned}$$

Theorem 3.1 (Stone, von Neumann). *$H(W)$ has a unique irreducible smooth representation on which F_v operates via the character ψ.*

One may have seen other constructions of smooth irreducible representations of $H(W)$ on which the center acts by the character ψ. However, by Theorem 3.1, they are isomorphic to one another. In this paper, I only use $(\rho_\psi, S(X))$ defined above, which is so-called the Schrödinger model.

The symplectic group $\mathrm{Sp}(W)$ has an action on $H(W)$ as $g\cdot(w,t) = (gw,t)$. By the uniqueness theorem of Stone and von Neumann, there is an operator $\omega_\psi(g)$ on $S(X)$ that it is unique up to scalar and satisfies

$$\rho_\psi(gw,t)\omega_\psi(g) = \omega_\psi(g)\rho_\psi(w,t) \quad \text{for all } (w,t)\in H(W). \tag{10}$$

Define the metaplectic group

$$\widetilde{\mathrm{Sp}}_\psi(W) = \{(g,\omega_\psi(g)) \text{ such that (10) holds}\},$$

which fits in the following exact sequence:

$$1 \to \mathbb{C}^\times \to \widetilde{\mathrm{Sp}}_\psi(W) \xrightarrow{\text{proj}} \mathrm{Sp}(W) \to 1.$$

The Weil representation of $\widetilde{\mathrm{Sp}}_\psi(W)$ is the one obtained by projecting to the second factor $(g,\omega_\psi(g)) \to \omega_\psi(g)$. Under the Schrödinger model $(\rho_\psi, S(X))$, the Weil representation can be explicitly written down:

$$\begin{aligned}
\omega_\psi\begin{pmatrix} A & \\ & (A^t)^{-1}\end{pmatrix}\phi(x) &= |\det A|^{\frac{1}{2}}\phi(xA),\\
\omega_\psi\begin{pmatrix} I & B\\ & I\end{pmatrix}\phi(x) &= \psi\Big(\frac{xBx^t}{2}\Big)\phi(x),\\
\omega_\psi\begin{pmatrix} & I\\ -I & \end{pmatrix}\phi(x) &= \gamma\hat{\phi}(x),
\end{aligned}$$

where $\phi\in S(X)$, $\hat{\phi}$ is the Fourier transform

$$\hat{\phi}(x) = \int_{F_v^n}\phi(y)\psi\left(\sum_{i=1}^n x_i y_i\right)dy,$$

the Haar measure dy is chosen so that $\hat{\hat{\phi}} = \phi(-x)$, and γ is an 8-th root of unity determined by ψ.

The discussion above in the local case can mostly be generalized to the global case. The global Schrödinger representation ρ_ψ of $H(W)(\mathbb{A})$ on $S(X(\mathbb{A}))$ can be defined, where

$$S(X(\mathbb{A})) = \{\otimes_v\phi_v \mid \phi_\infty\in L^2(X_\infty),\ \phi_v\in S(X_v),\ \text{and for a.a. } v,\ \hat{\phi}_v = \phi_v\}.$$

Also we have the global Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ on $S(X(\mathbb{A}))$, and for each place v of F, one has the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{C}^\times \arrow[r] \arrow[d] & \widetilde{\mathrm{Sp}}(W)(F_v) \arrow[r, "\mathrm{proj}"] & \mathrm{Sp}(W)(F_v) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{C}^\times \arrow[r] & \widetilde{\mathrm{Sp}}(W)(\mathbb{A}) \arrow[r, "\mathrm{proj}"] & \mathrm{Sp}(W)(\mathbb{A}) \arrow[r] & 1
\end{tikzcd}$$

3B. ***Dual reductive pair.***

Definition. A dual reductive pair is a pair of subgroups (G, G') of the symplectic group $\mathrm{Sp}(W)$ such that

(1) G is the centralizer of G' in $\mathrm{Sp}(W)$ and vice versa, and

(2) the actions of G and G' are completely reducible on W. (An action is called completely reducible if every invariant subspace has an invariant complement.)

Obviously this definition can be applied locally or globally.

In this paper, the types of reductive pairs used are the following: K is again an imaginary quadratic extension of F, $(V_1, (\cdot\,,\cdot)_1)$ is a skew-Hermitian space over K, and $(V_2, (\cdot\,,\cdot)_2)$ is a Hermitian space over K. Take $W = V_1 \otimes V_2$, and on W define an alternating form $\frac{1}{2}\,\mathrm{tr}_{K/F}(\cdot\,,\cdot)_1 \otimes \overline{(\cdot\,,\cdot)}_2$. Then the unitary groups $\mathrm{U}(V_1)$ and $\mathrm{U}(V_2)$ form a dual reductive pair in $\mathrm{Sp}(W)$. We have the embedding

$$e : U(V_1) \times U(V_2) \to \mathrm{Sp}(W), \quad (g_1, g_2) \cdot (v_1 \otimes v_2) \mapsto= v_1 g_1 \otimes g_2^{-1} v_2. \tag{11}$$

In this paper, we use e to denote the embeddings of unitary groups into symplectic groups, and use $\mathfrak{e}$ to denote the embeddings of the same type of groups, such as the embedding of one unitary group into another unitary group, or the embedding of one symplectic group into another symplectic group.

Splittings. For a fixed Weil representation $g \to \omega(g)$ (for example, the Weil representation ω_ψ defined through the Schrödinger model), we can define $c(g_1, g_2)$ so that

$$\omega(g_1)\omega(g_2) = c(g_1, g_2)\omega(g_1 g_2).$$

Then $c : \mathrm{Sp}(W) \times \mathrm{Sp}(W) \to \mathbb{C}^\times$ is a 2-cocycle whose class is in the cohomology group $H^2(\mathrm{Sp}(W), \mathbb{C}^\times)$. When the additive character ψ of F and one maximal isotropic subspace of W are fixed, c is determined using the Leray invariant. The following proposition claims that under certain condition, c could be a coboundary. Returning to the dual reductive pair $(U(V_1), U(V_2)))$ of $\mathrm{Sp}(W)$, we have this:

Proposition 3.2 [Harris et al. 1996]. $\widetilde{\mathrm{Sp}}(W)$ *splits over* $\mathrm{U}(V_i)$ *compatibly with respect to rational points for* $i = 1, 2$. *In particular*, *there is a splitting homomorphism*

$s_i : U(V_i)(\mathbb{A}) \to \widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ *such that we have the commutative diagram*

$$\begin{tikzcd} & \widetilde{\mathrm{Sp}}(W) \arrow[d, "\mathrm{proj}"] \\ U(V_i) \arrow[ur, "s_i"] \arrow[r, "e_i"] & \mathrm{Sp}(W), \end{tikzcd}$$

where e_i *is the embedding of* $\mathrm{U}(V_i)$ *into* $\mathrm{Sp}(W)$, *which is the restriction of* e *defined in* (11). *Further*, $s_i(U(V_i)(F)) \subset \widetilde{\mathrm{Sp}}(W)(F)$.

If $\dim_K V_2 = m$, we choose a character χ_{V_2} of $K^\times$ such that $\chi_{V_2}|_{F^\times} = \epsilon_{K/F}^m$, where $\epsilon_{K/F}$ is the quadratic character associated to the extension K/F. This choice determines a lifting $s_1 : U(V_1) \to \widetilde{\mathrm{Sp}}(W)$, which can be explicitly formulated. Notice that two choices χ and χ' of χ_{V_2} differ by a character μ of $K^\times$, namely $\chi = \mu\chi'$ and $\mu|_{F^\times} = 1$. Then μ defines a character μ' on K^1 by $\mu'(x/\bar{x}) = \mu(x)$, with $x \in K^\times$, and we have:

Lemma 3.3 [Harris et al. 1996]. $s_{1,\chi} = (\mu' \circ \det) \cdot s_{1,\chi'}$

Doubling method. From the symplectic space $W = V_1 \otimes V_2$, we can create a new symplectic space $\mathbb{W}$, which is essentially two copies of W, in this way: Take $W^- = V_1^- \otimes V_2$. As a vector space, V_1^- is the same as V_1, but the skew-Hermitian form defined on it is $-(\cdot\,,\cdot)_1$. To $\mathbb{W} = W \oplus W^-$, we have one dual reductive paier $(U(V_1 \oplus V_1^-), U(V_2))$. We have the following commutative diagram:

$$\begin{tikzcd} U(V_1 \oplus V_1^-) \arrow[r, "e_1"] & \mathrm{Sp}(W \oplus W^-) = \mathrm{Sp}(\mathbb{W}) \\ U(V_1) \times U(V_1) \arrow[u, "\mathfrak{e}"] \arrow[r, "e_1 \times e_1"] & \mathrm{Sp}(W) \times \mathrm{Sp}(W) \arrow[u, "\mathfrak{e}"] \\ U(V_1) \arrow[u, "\triangle"] \arrow[r, "e_1"] & \mathrm{Sp}(W) \arrow[u, "\triangle"] \end{tikzcd}$$

By Proposition 3.2, $\widetilde{\mathrm{Sp}}(\mathbb{W})$ splits over each of the unitary groups in the diagram above. We want to determine the compatibility among the splittings. The Weil representation of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ determines an isomorphism

$$\widetilde{\mathrm{Sp}}(\mathbb{W}) \simeq \mathrm{Sp}(\mathbb{W}) \times \mathbb{C}^\times.$$

Group multiplications on the right hand side are described by the cocycle $c(g_1, g_2)$:

$$(g_1, c_1)(g_2, c_2) = (g_1 g_2, c(g_1, g_2) c_1 c_2).$$

The inverse image of $\mathfrak{e}(\mathrm{Sp}(W)\times 1)$ in $\widetilde{\mathrm{Sp}}(W\oplus W^-)$ is isomorphic to $\widetilde{\mathrm{Sp}}(W)$. We choose a lift $\tilde{\mathfrak{e}}$ of $\mathfrak{e}$ so that

$$\tilde{\mathfrak{e}}:\widetilde{\mathrm{Sp}}(W)\times\widetilde{\mathrm{Sp}}(W)\to\widetilde{\mathrm{Sp}}(W\oplus W^-)$$

restricted to the $\mathbb{C}^\times$-component is

$$\mathbb{C}^\times\times\mathbb{C}^\times\to\mathbb{C}^\times,\quad (c_1,c_2)\mapsto c_1\bar{c}_2.$$

Since $(U(V_1\oplus V_1^-),U(V_2))$ is a dual reductive pair of $\mathrm{Sp}(\mathbb{W})$, and $\dim_K V_2=m$, choose χ such that $\chi|_{\mathbb{A}^\times}=\epsilon_{K/F}^m$. We can obtain an explicit homomorphism

$$s_\chi:U(V_1\oplus V_1^-)\to\widetilde{\mathrm{Sp}}(\mathbb{W}),$$

so that

$$\begin{tikzcd}
U(V_1\oplus V_1^-) \arrow[r, "s_\chi"] & \widetilde{\mathrm{Sp}}(\mathbb{W}) \\
U(V_1)\times U(V_1) \arrow[u, "\mathfrak{e}"] \arrow[r, "s_{1,\chi}\times s_{1,\chi,-}"] & \widetilde{\mathrm{Sp}}(W)\times\widetilde{\mathrm{Sp}}(W) \arrow[u, "\tilde{\mathfrak{e}}"']
\end{tikzcd}$$

Lemma 3.4 [Harris et al. 1996]. *In the commutative diagram above*,

$$s_\chi|_{U(V_1)\times 1}=s_{1,\chi}\quad\textit{and}\quad s_{1,\chi,-}=(\chi^{-1}\circ\det)\cdot s_{1,\chi}.$$

Weil representations on dual reductive pairs. Let $W=X\oplus Y$ be the complete polarization. Then $\mathbb{W}$ naturally has the polarization $\mathbb{W}=(X\oplus X)\oplus(Y\oplus Y)$. Now take the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ constructed from the standard Schrödinger model associated to this polarization. Fix a pair of characters χ_n and χ_m of $K^\times$ with $\chi_n|_{\mathbb{A}^\times}=\epsilon_{K/F}^n$ and $\chi_m|_{\mathbb{A}^\times}=\epsilon_{K/F}^m$, where $n=\dim_K V_1$ and $m=\dim_K V_2$. These characters determine the splitting homomorphisms

$$\begin{aligned}
s_{\chi_m}&:U(V_1\oplus V_1^-)\to\widetilde{\mathrm{Sp}}(\mathbb{W}),\\
s_{1,\chi_m}&:U(V_1)\to\widetilde{\mathrm{Sp}}(W),\\
s_{2,\chi_m}&:U(V_2)\to\widetilde{\mathrm{Sp}}(W).
\end{aligned}$$

Define the representation of $\mathrm{U}(V_1\oplus V_1^-)$ to be $\omega_{\chi_m}=\omega_\psi\circ s_{\chi_m}$; the representations ω_{1,χ_m} and ω_{2,χ_n} of $\mathrm{U}(V_1)$ and $\mathrm{U}(V_2)$ can be defined similarly.

There is another polarization of $\mathbb{W}$ that we are also interested in. The skew-Hermitian space $V_1\oplus V_1^-$ has the decomposition $V_1\oplus V_1^-=V_d\oplus V^d$ where $V_d=\{(x,x)\mid x\in V_1\}$ and $V^d=\{(x,-x)\mid x\in V_1\}$. They are maximal isotropic subspaces. Thus $\mathbb{W}=V_d\otimes V_2\oplus V^d\otimes V_2=\mathbb{W}_d\oplus\mathbb{W}^d$ is a complete polarization. We abuse the notation and still use ω_ψ to denote the Weil representation defined from this polarization, since one can easily distinguish representations attached to two polarizations from the context. First, let us write down the Weil representation

ω_{χ_m} of $\mathrm{U}(V_1 \oplus V_1^-)$ on $S(\mathbb{W}_d)$, which we are going to use in later calculations. By the Witt decomposition $V_1 \oplus V_1^- = V_d \oplus V^d$, $\begin{pmatrix} A & \\ & (A^*)^{-1} \end{pmatrix} \in U(V_1 \oplus V_1^-)$ for $A \in \mathrm{GL}(V_d)$, and $\begin{pmatrix} 1 & B \\ & 1 \end{pmatrix} \in U(V_1 \oplus V_1^-)$ if $B = B^*$. Given $\phi(x) \in S(V_d \otimes V_2)$, we have

$$\begin{aligned} \omega_{\chi_m} \begin{pmatrix} A & \\ & (A^*)^{-1} \end{pmatrix} \phi(x) &= \chi_m(\det A)|\det A|_K^{m/2} \phi(A^t x), \\ \omega_{\chi_m} \begin{pmatrix} 1 & B \\ & 1 \end{pmatrix} \phi(x) &= \psi(x^* B x)\phi(x), \\ \omega_{\chi_m} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \phi(x) &= \hat{\phi}(x). \end{aligned} \tag{12}$$

Proposition 3.5. (1) *Under the homomorphism* $\tilde{\mathfrak{e}} : \widetilde{\mathrm{Sp}}(W) \times \widetilde{\mathrm{Sp}}(W) \to \widetilde{\mathrm{Sp}}(\mathbb{W})$, *we have* $\omega_\psi \circ \tilde{\mathfrak{e}} = \omega_\psi \otimes \check{\omega}_\psi$, *where* $\check{\omega}_\psi$ *is the contragredient of* ω_ψ.

(2) *As the representation of* $\mathrm{U}(V_1) \times U(V_1)$, *we have* $\omega_{\chi_m} \circ \mathfrak{e} \simeq \omega_{1,\chi_m} \otimes (\chi_m \circ \check{\omega}_{1,\chi_m})$.

(3) *The representation* $\omega_{\chi_m} \circ \mathfrak{e} \circ \triangle$ *of* $\mathrm{U}(V_1)$ *is isomorphic to the twist by* χ_m *of the linear action of* $\mathrm{U}(V_1)$ *on* $S(\mathbb{W}_d)$, *that is, for* $\phi \in S(\mathbb{W}_d)$ *and* $x \in \mathbb{W}_d$,

$$\omega_{\chi_m}(\mathfrak{e}(g, g))\phi(x) = \chi_m(\det g) \cdot \phi(xg).$$

Using two polarizations of $\mathbb{W}$, two Weil representations of $\mathrm{U}(V_1 \oplus V_1^-)$ are defined above. An isometry between the two representation spaces $S((X \oplus X)(\mathbb{A}))$ and $S(\mathbb{W}_d(\mathbb{A}))$ can be given so that it intertwines the two representations on the spaces. Let

$$\delta_\psi : S((X \oplus X)(\mathbb{A})) \to S(\mathbb{W}_d(\mathbb{A}))$$

be the intertwining isometry. Identify $\mathbb{W}_d$ with W via the map $(w, -w) \to w$ and write $w \in W$ as $w = (x, y)$ with respect to the polarization $W = X \oplus Y$. Then for $\phi \in S((X \oplus X)(\mathbb{A}))$, one has

$$\delta_\psi(\phi)(w) = \int_{X(\mathbb{A})} \psi(2\langle u, y\rangle)\phi(u + x, u - x)\, du, \tag{13}$$

where $\langle\, \cdot\,,\, \cdot\, \rangle$ is the alternating pair on W. The map δ_ψ intertwines the action of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ on those two spaces. In particular, if $\phi = \phi_1 \otimes \bar{\phi}_2$ for $\phi_1, \phi_2 \in S(X(\mathbb{A}))$,

$$\delta_\psi(\phi_1 \otimes \bar{\phi}_2)(0) = (\phi_1, \phi_2),$$

where $(\,\cdot\,,\,\cdot\,)$ is the Hermitian inner product in the Hilbert space $L^2(X(\mathbb{A}))$, and

$$\omega_{\chi_m}(i(g_1, g_2))\delta_\psi(\phi_1 \otimes \bar{\phi}_2) = \delta_\psi(\omega_{1,\chi_m}(g_1)\phi_1 \otimes \chi_m(\det g_2)\check{\omega}_{1,\chi_m}(g_2)\bar{\phi}_2).$$

3C. ***Theta functions and applications.*** In this section, I will recall some facts about theta functions that are used in calculations of Fourier–Jacobi coefficients of Eisenstein series. More discussions about arithmetic properties of theta functions will be left to Appendix B.

Introduction. Recall W has a complete polarization $W = X \oplus Y$. On $S(X(\mathbb{A}))$, there is a distribution θ defined by

$$\theta(\phi) = \sum_{l \in X(F)} \phi(l).$$

It can be proved that for each $\phi \in S(X(\mathbb{A}))$, the sum on the right converges absolutely. Let the Jacobi group $J(W)$ be the semidirect product of $H(W)$ and $\mathrm{Sp}(W)$. Put $\widetilde{J}(W)(\mathbb{A}) = H(W)(\mathbb{A}) \cdot \widetilde{\mathrm{Sp}}(W)(\mathbb{A})$. Use $\tilde{g} \cdot \phi$ to represent the Weil representation of $\tilde{g} \in \widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ on ϕ. So for each $\phi \in S(X(\mathbb{A}))$, the theta function θ_ϕ on $\widetilde{J}(W)(\mathbb{A})$ is defined as

$$\theta_\phi((w, t)\tilde{g}) = \sum_{l \in X(F)} \rho_\psi(w, t)(\tilde{g} \cdot \phi)(l).$$

It is $J(W)(F)$-invariant.

When χ_m is chosen, the splitting $s_{1,\chi_m} : U(V_1) \to \widetilde{\mathrm{Sp}}(W)$ is fixed as stated in Proposition 3.2. Let

$$\omega(g_1, g_2)\phi(x) = \omega_{1,\chi_m}(g_1)\phi(g_2^{-1}x) \tag{14}$$

for $g_1 \in U(V_1)$ and $g_2 \in U(V_2)$. Thus θ_ϕ can also be regarded as a function on the dual reductive pair $(U(V_1), U(V_2))$, and

$$\theta_\phi(g_1, g_2) = \theta(\omega(g_1, g_2)\phi). \tag{15}$$

Theta liftings. From now on, assume that $\dim_K V_1 = 2,\ \dim_K V_2 = 1$ and assume $(V_1, (\cdot\,,\cdot)_1)$ is an anisotropic skew-Hermitian space. This is exactly the situation that one will see in next sections.

Let Π be a cuspidal representation contained in $L^2(U(V_1)(F) \setminus U(V_1)(\mathbb{A}))$. For convenience, given a reductive group G over F, we use $[G]$ to denote the quotient $G(F) \setminus G(\mathbb{A})$. For each smooth $\beta \in V_\Pi$, where V_Π is the representation space of Π, the function

$$\theta_\phi^\beta(g_2) = \int_{[U(V_1)]} \theta_\phi(g_1, g_2)\beta(g_1)\, dg_1$$

is well-defined, where $\theta_\phi(g_1, g_2)$ is given in (15). Actually, θ_ϕ^β determines a slowly increasing function on $[U(V_2)]$. It is expected that $\theta_\phi^\beta(g_2)$ is nonzero and generates an irreducible automorphic representation on $\mathrm{U}(V_2)$. So θ_ϕ^β defines a theta lifting from $\mathrm{U}(V_1)$ to $\mathrm{U}(V_2)$.

By (11), $e(g_1, g_2)\cdot(v_1\otimes v_2) = v_1g_1\otimes g_2^{-1}v_2$. So

$$\theta_\phi^\beta(g_2) = \chi_\Pi(g_2)\int_{[U(V_1)]}\theta_\phi(g_1,1)\beta(g_1)\,dg_1, \tag{16}$$

where χ_Π is the central character of Π. Whether θ_ϕ^β is zero or not only depends on the scalar $\int_{[U(V_1)]}\theta_\phi(g_1,1)\beta(g_1)\,dg_1$. In fact, $|\int_{[U(V_1)]}\theta_\phi(g_1,1)\beta(g_1)\,dg_1|^2$ can be transformed into the Rallis inner product using the Siegel–Weil formula. I am going to introduce the Siegel–Weil formula in the following and show its application in the calculation of $|\int_{[U(V_1)]}\theta_\phi(g_1,1)\beta(g_1)dg_1|^2$.

The Siegel–Weil formula. The Siegel–Weil formula on unitary groups relates the value of a Siegel–Eisenstein series to the integral of a theta function. We temporarily return to the general case. Consider the reductive pair $(U(n,n),U(V))$ with $\dim_K V = m$. Use $S(V^n(\mathbb{A}))$ to denote the space of the Weil representation ω_ψ. Here we take $S(V_\infty^n)$ to be the subspace of $L^2(V_\infty^n)$ called the Fock space, that is,

$$S(V_\infty^n) = \left\{\phi : V_\infty^n \to \mathbb{C} \;\middle|\; \phi(v_1,\dots,v_n)\psi\left(i\sum_{i=1}^n |v_i|^2\right) \text{ is antiholomorphic as a function of } v_i \text{ for } v_i\in V_\infty\right\}.$$

Fix χ_m and χ_{2n} such that the Weil representation on the reductive pair can be defined. For $\phi\in S(V^n(\mathbb{A}))$, $g\in U(n,n)(\mathbb{A})$ and $u\in U(V)(\mathbb{A})$, define such a theta integral:

$$I(g,\phi) = \int_{[U(V)]}\theta_\phi(g,u)\,du.$$

On the other hand, let $\mathcal{P}'$ be the Siegel parabolic subgroup of $\mathrm{U}(n,n)$ and $\mathcal{K}'$ be the maximal open compact subgroup. So $\mathrm{U}(n,n)(\mathbb{A}) = \mathcal{P}'(\mathbb{A})\mathcal{K}'(\mathbb{A})$. For $g = pk$ with $p = \begin{pmatrix} A & * \\ & (A^*)^{-1}\end{pmatrix}\in\mathcal{P}'(\mathbb{A})$ and $k\in\mathcal{K}'(\mathbb{A})$, put $|a(g)|_K = |\det A|_K$. Given $\phi\in S(V^n(\mathbb{A}))$, let

$$f_{\phi,s}(g) = |a(g)|_K^{s-s_0}\omega(g,1)\phi(0), \tag{17}$$

where $s_0 = (m-n)/2$. Then $f_{\phi,s}\in I_{\mathcal{P}'}^{U(n,n)}(\chi_m,s)$. The Siegel–Eisenstein series can be defined as

$$E(g,f_{\phi,s}) = \sum_{\gamma\in\mathcal{P}'(F)\backslash U(n,n)(F)} f_{\phi,s}(\gamma g).$$

It has been proved in many cases that $E(g,f_{\phi,s_0}) = I(g,\phi)$ if $E(g,f_{\phi,s})$ is holomorphic at $s=s_0$ and $I(g,\phi)$ is absolutely convergent. The case we are interested in is when $n=2$ and $m=1$. By a result of Weil [1965], $I(g,\phi)$ is absolutely convergent if V is anisotropic. So it is automatically satisfied if $\dim_K V = 1$.

Theorem 3.6. *When $n = 2$ and $m = 1$, $E(g, f_{\phi,s})$ is holomorphic at $s = -\frac{1}{2}$ for all $\phi \in S(V^n(\mathbb{A}))$, and*

$$E(g, f_{\phi,s})|_{s=-\frac{1}{2}} = 2I(g, \phi).$$

For the reductive pair $(\mathrm{Sp}_n, \mathrm{O}(V))$ when V is an anisotropic F-vector space, the Siegel–Weil formula has been proved by Kudla and Rallis [1988]. The proof of Theorem 3.6 is very similar to theirs, and is found in Appendix A. Let us return to the theta lifting question. With the Siegel–Weil formula, we are ready to calculate $|\int_{[U(V_1)]} \theta_\phi(g, 1)\beta(g)\, dg|^2$. According to (16), we have

$$\left|\int_{[U(V_1)]} \theta_\phi(g, 1)\beta(g)\, dg\right|^2 = (\theta_\phi^\beta, \theta_\phi^\beta), \tag{18}$$

where $(\theta_\phi^\beta, \theta_\phi^\beta) = \int_{[U(1)]} \theta_\phi^\beta(u)\overline{\theta_\phi^\beta}(u)\, du$.

Proposition 3.7. *We have*

$$(\theta_\phi^\beta, \theta_\phi^\beta) = \tfrac{1}{2} \int_{U(V_1)(\mathbb{A})} f_{\delta_\psi(\phi\otimes\bar{\phi}),s}(\mathfrak{e}(g, 1))\langle \Pi(g)\beta, \beta\rangle\, dg|_{s=-\frac{1}{2}},$$

where $\langle \Pi(g)\beta, \beta\rangle = \int_{[U(V_1)]} \beta(g'g)\overline{\beta(g')}\, dg'$ is the matrix coefficient of the representation Π.

Proof. First,

$$\begin{aligned}(\theta_\phi^\beta, \theta_\phi^\beta) &= \int_{[U(1)]}\int_{[U(V_1)]} \theta_\phi(g, u)\beta(g) \int_{[U(V_1)]} \overline{\theta_\phi(g', u)\beta(g')}\, dg\, dg'\, du \\ &= \int_{[U(V_1)\times U(V_1)]} \beta(g)\overline{\beta(g')}\Big(\int_{[U(1)]} \theta_\phi(g, u)\overline{\theta_\phi(g', u)}\, du\Big)\, dg\, dg'.\end{aligned}$$

Notice that

$$\theta_\phi(g, u)\overline{\theta_\phi(g', u)} = \chi_1^{-1}(\det g')\theta_{\delta_\psi(\phi\otimes\bar{\phi})}(\mathfrak{e}(g, g'), u),$$

where δ_ψ is defined by (13), and $\mathfrak{e}(g, g') \in U(V_1 \oplus V_1^-)(\mathbb{A}) \simeq U(2, 2)(\mathbb{A})$. So $\theta_{\delta_\psi(\phi\times\bar{\phi})}$ is a theta function of the reductive pair $(U(2, 2), U(1))$. Applying Theorem 3.6, we have

$$\begin{aligned}\int_{[U(1)]} \theta_\phi(g, u)\overline{\theta_\phi(g', u)}\, du &= \chi_1^{-1}(\det g') \int_{[U(1)]} \theta_{\delta_\psi(\phi\otimes\bar{\phi})}(\mathfrak{e}(g, g'), u)\, du \\ &= \tfrac{1}{2}\chi_1^{-1}(\det g')E(\mathfrak{e}(g, g'), f_{\delta_\psi(\phi\otimes\bar{\phi}),s})|_{s=-\frac{1}{2}}.\end{aligned}$$

Unfolding the Eisenstein series, we have

$$\int_{[U(V_1)\times U(V_1)]} \beta(g)\overline{\beta(g')}\chi_1^{-1}(\det g')E(\mathfrak{e}(g,g'), f_{\delta_\psi(\phi\times\bar{\phi}),s})\,dg\,dg' \\ = \int_{U(V_1(\mathbb{A}))} f_{\delta_\psi(\phi\times\bar{\phi}),s}(i(g,1))\langle\Pi(g)\beta,\beta\rangle\,dg. \quad \square$$

4. Fourier–Jacobi coefficients of Eisenstein series

In Sections 2A and 2B, we first define the unitary group $\mathrm{U}(I_\zeta)$ for

$$I_\zeta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \zeta & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

with $\zeta \in \mathrm{GL}_2(K)$ such that $\zeta = -\zeta^*$ and $\det\zeta \notin N_{K/F}(K^\times)$, and then define an Eisenstein series $E(g, f, s)$ on $\mathrm{U}(I_\zeta)$. In this section, we are going to use the pullback formula introduced in Section 2C (refer to Theorem 2.6) to calculate the Fourier–Jacobi coefficients of $E(g, f, s)$.

4A. ***Fourier–Jacobi coefficients of Siegel–Eisenstein series on* U(3, 3).** Recall that

$$\mathrm{U}(3,3) = \left\{ g \in \mathrm{GL}_6(K) \;\middle|\; g\begin{pmatrix} & I_3 \\ -I_3 & \end{pmatrix} g^* = \begin{pmatrix} & I_3 \\ -I_3 & \end{pmatrix} \right\}.$$

In Section 2B, we define a Siegel–Eisenstein series as follows: Take $\mathbb{f} \in I_{\mathcal{P}}^{U(3,3)}(\eta, s)$ and define

$$\mathbb{E}(g, \mathbb{f}, s) = \sum_{\gamma\in\mathcal{P}(F)\backslash U(3,3)(F)} \mathbb{f}_s(\gamma g).$$

In this section, we will define the Fourier–Jacobi coefficients of this Eisenstein series, and show that if the holomorphic section $\mathbb{f}$ is chosen properly, the Fourier–Jacobi coefficient is a product of a theta function and a Siegel–Eisenstein series on $\mathrm{U}(2,2)$.

Let $H = \{(x, y, t) \mid x, y \in K^2, t \in F\} \subset U(3,3)$, where

$$(x,y,t) = \begin{pmatrix} 1 & x & t+\frac{1}{2}(xy^*-yx^*) & y \\ & I_2 & y^* & 0_2 \\ & & 1 & \\ & & -x^* & I_2 \end{pmatrix}.$$

Notice that

$$(x_1,y_1,t_1)\cdot(x_2,y_2,t_2) = \left(x_1+x_2,\, y_1+y_2,\, t_1+t_2+\tfrac{1}{2}(x_1y_2^*+y_2x_1^*)-\tfrac{1}{2}(x_2y_1^*-y_1x_2^*)\right).$$

So if we take $\mathbb{W}_d = \{(x, 0, 0)\}$ and $\mathbb{W}^d = \{(0, y, 0)\}$, then $\mathbb{W} = \mathbb{W}_d \oplus \mathbb{W}^d$ is a symplectic space with the alternating pair

$$\langle (x_1, y_1), (x_2, y_2)\rangle = x_1 y_2^* + y_2 x_1^* - x_2 y_1^* - y_1 x_2^*,$$

and $\mathbb{W}_d$ and $\mathbb{W}^d$ are two isotropic subspaces. By the definition of Heisenberg groups in Section 3A, H is a Heisenberg group associated to $\mathbb{W}$, and

$$T = \{(0, 0, t), t \in F\}$$

is the center of H. There is another subgroup of U(3, 3):

$$U(\mathbb{W}_d \oplus \mathbb{W}^d) = \left\{ \begin{pmatrix} 1 & & & \\ & A & & B \\ & & 1 & \\ & C & & D \end{pmatrix} \,\middle|\, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2, 2) \right\} \simeq U(2, 2). \tag{19}$$

The action of $\mathrm{U}(W_d \oplus \mathbb{W}^d)$ on H is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot (x, y, t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} (x, y, t) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (xA + yC, xB + yD, t).$$

Denote by $\psi_{\mathbb{Q}}$ the additive character of $\mathbb{Q} \setminus \mathbb{A}_{\mathbb{Q}}$ with

$$\psi_{\mathbb{Q}}(x_\infty) = \exp(2\pi \sqrt{-1} x_\infty) \quad \text{for } x_\infty \in \mathbb{R}.$$

For $m \in F$, define an additive character ψ_m (later in this paper, we may simply denote it by ψ) of $F \setminus \mathbb{A}$ by

$$\psi_m(x) = \psi_{\mathbb{A}}(\mathrm{Tr}_{F/\mathbb{Q}}(mx)) \quad \text{for } x \in \mathbb{A}. \tag{20}$$

Define the Fourier–Jacobi coefficient of the Eisenstein series $\mathbb{E}(\,\cdot\,, \mathbb{f}, s)$ associated to ψ as follows:

$$\mathbb{E}_\psi(g, \mathbb{f}, s) = \int_{[T]} \mathbb{E}(tg, \mathbb{f}, s) \psi(-t)\, dt \quad \text{for } g \in U(3, 3)(\mathbb{A}). \tag{21}$$

So $\mathbb{E}$ admits the Fourier–Jacobi expansion

$$\mathbb{E}(g, \mathbb{f}, s) = \sum_{m \in F} \mathbb{E}_{\psi_m}(g, \mathbb{f}, s).$$

We can regard $\mathbb{E}_\psi$ as a function on the semidirect product of H and U(2, 2), denoted by J_H. Thus $\mathbb{E}_\psi(hg, \mathbb{f}, s) \in C^\infty_\psi([J_H])$, where the subindex ψ means that the left action of T on the functions is given by ψ.

Lemma 4.1. (1) *Let*

$$\xi_0 = \begin{pmatrix} 0 & & 1 & \\ & I_2 & & \\ -1 & & 0 & \\ & & & I_2 \end{pmatrix}.$$

Then $\mathrm{U}(3, 3) = \mathcal{P} J_H \sqcup \mathcal{P} \xi_0 J_H$, *where* $\mathcal{P}$ *is the Siegel parabolic subgroup of* U(3, 3).

(2) $\mathcal{P}\backslash\mathcal{P}\xi_0 J_H = \xi_0\cdot\mathbb{W}^d\backslash H\cdot\mathcal{P}'\backslash U(2,2) = w_3\cdot\mathbb{W}_d\backslash H\cdot w_2\cdot\mathcal{P}'\backslash U(2,2)$, *where* $\mathcal{P}'$ *is the Siegel parabolic subgroup of* $\mathrm{U}(2,2)$ *and* $w_n = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \in U(n,n)$.

Applying this lemma to $\mathbb{E}_\psi(hg,\mathbb{f},s)$, we have

$$\begin{aligned}\mathbb{E}_\psi(hg,\mathbb{f},s) &= \int_{[T]}\psi(-t)\sum_{\gamma\in\mathcal{P}(F)\backslash U(3,3)(F)}\mathbb{f}_s(\gamma thg)\,dt\\ &= \int_{[T]}\psi(-t)\Bigg(\sum_{\gamma\in\mathcal{P}(F)\backslash\mathcal{P}(F)J_H(F)}+\sum_{\gamma\in\mathcal{P}\backslash\mathcal{P}(F)\xi_0 J_H(F)}\mathbb{f}_s(\gamma thg)\Bigg)\,dt\\ &= \int_{[T]}\psi(-t)\sum_{\gamma\in w_3\cdot\mathbb{W}_d(F)\backslash H(F)\cdot w_2\cdot\mathcal{P}'(F)\backslash U(2,2)(F)}\mathbb{f}_s(\gamma thg)\,dt,\end{aligned}\tag{22}$$

because $\int_{[T]}\psi(-t)\sum_{\gamma\in\mathcal{P}(F)\backslash\mathcal{P}(F)J_H(F)}\mathbb{f}_s(\gamma thg)\,dt = 0$.

Pick a character χ of $K^\times$ such that $\chi|_{\mathbb{A}^\times} = \epsilon_{K/F}$ and $\chi_\infty(z) = |z|/z$ for $z\in K_\infty^\times$. We defined a Weil representation ω_χ of $\mathrm{U}(2,2)$ on $S(\mathbb{W}_d)$ by (12). Here what we actually use is a bit different from the conventional one. The difference is that instead of taking $\omega_\chi(w_2)\phi(x) = \hat{\phi}(x)$ we take

$$\omega_\chi\left(\begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}\right)\phi(x) = \hat{\phi}(x).$$

Fix such an ω_χ; we are going to prove that if $\mathbb{f}_s$ is chosen properly, then

$$\int_{T(\mathbb{A})}\mathbb{f}_s(w_3 t(0,y,0)w_2 g')\psi(-t)\,dt = c_{\mathbb{f},\psi}\,\omega_\chi(g')\phi_{\mathbb{f}}(y)R_s(g'),\tag{23}$$

where $R_s(g')\in I_{\mathcal{P}'}^{U(2,2)}(\eta\chi^{-1},s)$, $c_{\mathbb{f},\psi}$ is a nonzero constant and $\phi_{\mathbb{f}}\in S(\mathbb{W}_d(\mathbb{A}))$. Since the integral (23) can be decomposed into the product of local integrals, we can do the calculations place by place. The results are stated in Theorem 4.9 at the end of this section. Take a subgroup $\mathcal{K}(\mathbb{A}) = \prod\mathcal{K}_v$ of $\mathrm{U}(3,3)(\mathbb{A})$, such that $\mathcal{K}_\infty$ is a maximal compact subgroup of $\mathrm{U}(3,3)_\infty$, and $\mathcal{K}_v$ is a maximal open compact subgroup of $\mathrm{U}(3,3)_v$. Let $\mathbb{f} = \otimes'\mathbb{f}_v$, with $\mathbb{f}_\infty$ being right $\mathcal{K}_\infty$-finite and $\mathbb{f}_v$ being spherical for almost all finite places v. By spherical, I mean that $\mathbb{f}_v$ is $\mathcal{K}_v$-invariant. In the following, the computations are purely local, so we omit the subindex v.

Case: $\mathbb{f}_s$ *is spherical and* χ *is unramified.* In this case, we make these assumptions:

Assumption 4.2. (1). Assume $\mathbb{f}$ is normalized, so $\mathbb{f}(1) = 1$.

(2). Assume $\mathbb{W}_d\cap\mathcal{K} = \mathcal{O}_K^2$. The dual of $\mathbb{W}_d\cap\mathcal{K}$ with respect to $\psi(xy^*+yx^*)$ is defined as

$$(\mathbb{W}_d\cap\mathcal{K})^\vee = \{(0,y,0)\in\mathbb{W}^d \mid \psi(xy^*+yx^*) = 1 \text{ for all } (x,0,0)\in\mathbb{W}_d\cap\mathcal{K}\}.$$

We assume $(\mathbb{W}_d\cap\mathcal{K})^\vee$ is also $\mathcal{O}_K^2$ and $(\mathbb{W}_d\cap\mathcal{K})^\vee = \mathbb{W}^d\cap\mathcal{K}$.

Lemma 4.3. *$\int_T \mathbb{f}_s(w_3(0,y,t)w_2)\psi(-t)\,dt = \Phi_{\mathbb{O}_K^2}(y)$, where $\Phi_{\mathbb{O}_K^2}$ is the characteristic function supported on $\mathbb{O}_K^2$.*

Proof. Take

$$\boldsymbol{x} = \begin{pmatrix} 1 & & & \\ -x^* & I_2 & & \\ & & 1 & x \\ & & & I_2 \end{pmatrix} \in \mathcal{P}$$

and notice that $\boldsymbol{x} w_3 = w_3(x,0,0)$. So,

$$\boldsymbol{x} w_3 t(0,y,0)w_2 = w_3(0,y,t+xy^*+yx^*)(x,0,0)w_2.$$

Then

$$\begin{aligned}
\int_T &\mathbb{f}_s(w_3(0,y,t)w_2)\psi(-t)\,dt \\
&= \int_T \mathbb{f}_s(\boldsymbol{x} w_3(0,y,t)w_2)\psi(-t)\,dt \\
&= \int_T \mathbb{f}_s(w_3(0,y,t+xy^*+yx^*)(x,0,0)w_2)\psi(-t)\,dt \\
&= \psi(xy^*+yx^*)\int_T \mathbb{f}_s(w_3(0,y,t)(x,0,0)w_2)\psi(-t)\,dt.
\end{aligned}$$

For $(0,y,0) \in \mathbb{W}^d \cap \mathcal{K}$,

$$\int_T \mathbb{f}_s(w_3(0,y,t)w_2)\psi(-t)\,dt = \int_T \mathbb{f}_s(w_3 t)\psi(-t)\,dt = 1.$$

If $(0,y,0) \notin \mathbb{W}^d \cap \mathcal{K}$, there must exist $(x,0,0) \in \mathbb{W}_d \cap \mathcal{K}$, such that $\psi(xy^*+yx^*) \neq 1$. So,

$$\begin{aligned}
\int_T &\mathbb{f}_s(w_3(0,y,t)w_2)\psi(-t)\,dt \\
&= \int_T \mathbb{f}_s\left(\begin{pmatrix} 1 & & & \\ -x^* & I_2 & & \\ & & 1 & x \\ & & & I_2 \end{pmatrix} w_3(0,y,t)w_2\right)\psi(-t)\,dt \\
&= \psi(xy^*+yx^*)\int_T \mathbb{f}_s(w_3(0,y,t)(x,0,0)w_2)\psi(-t)\,dt \\
&= \psi(xy^*+yx^*)\int_T \mathbb{f}_s(w_3(0,y,t))\psi(-t)\,dt.
\end{aligned}$$

Then $\int_T \mathbb{f}_s(w_3(0,y,t)w_2)\psi(-t)dt = 0$, if $(0,y,0) \notin \mathbb{W}^d \cap \mathcal{K}$. □

We have the Iwasawa decomposition $\mathrm{U}(2,2) = \mathcal{P}'\mathcal{K}'$, where $\mathcal{P}'$ is the Siegel parabolic subgroup of $\mathrm{U}(2,2)$, $\mathcal{K}'$ is the maximal open compact subgroup and

$\mathcal{K}' = \mathcal{K} \cap U(2,2)$. Take $g' = p'k' = \left(\begin{smallmatrix} a & b \\ & (a^*)^{-1} \end{smallmatrix}\right) k' \in U(2,2)$, where $k' \in \mathcal{K}'$. Then

$$\begin{aligned}
\int_T & \mathbb{f}_s\left(w_3(0,y,t)w_2\begin{pmatrix} a & b \\ & (a^*)^{-1}\end{pmatrix}k'\right)\psi(-t)\,dt \\
&= \int_T \mathbb{f}_s\left(w_3(0,y,t)\begin{pmatrix} (a^*)^{-1} & \\ -b & a\end{pmatrix}\right)\psi(-t)\,dt \\
&= \int_T \mathbb{f}_s\left(\begin{pmatrix} a & b \\ & (a^*)^{-1}\end{pmatrix}w_3(-yb, ya, t)\right)\psi(-t)\,dt \\
&= \eta(\det a)|\det a|^{s+\frac{3}{2}}\int_T \mathbb{f}_s(w_3(0, ya, t+yba^*y^*))\psi(-t)\,dt \\
&= \eta(\det a)|\det a|^{s+\frac{3}{2}}\psi(yba^*y^*)\Phi_{\mathbb{O}_K^2}(ya) \\
&= \chi(\det a)|\det a|^{\frac{1}{2}}\Phi_{\mathbb{O}_K^2}(ya)\psi(yba^*y^*)\eta\chi^{-1}(\det a)|\det a|^{s+1} \\
&= \omega_\chi(p')\Phi_{\mathbb{O}_K^2}(y)R_s(g').
\end{aligned}$$

Notice that $\omega_\chi(k')\Phi_{\mathbb{O}_K^2} = \Phi_{\mathbb{O}_K^2}$. So

$$\int_T \mathbb{f}_s(w_3(0,y,t)w_2g')\psi(-t)\,dt = \omega_\chi(g')\Phi_{\mathbb{O}_K^2}(y)R_s(g')$$

with normalized spherical $R_s \in I_{\mathcal{P}'}^{U(2,2)}(\eta\chi^{-1}, s)$.

Archimedean places. Let $\mathrm{U}(n) = \{u \in \mathrm{GL}_n(\mathbb{C}) \mid uu^* = I_n\}$ be the unitary group of degree n at an archimedean place. Take such an embedding:

$$\begin{aligned}
&\mathfrak{e} : U(3)\times U(3) \hookrightarrow U(3,3), \\
&\qquad (u,v) \mapsto \mathfrak{e}(u,v) = \begin{pmatrix} -\frac{i}{2}I_3 & \frac{i}{2}I_3 \\ \frac{1}{2}I_3 & \frac{1}{2}I_3 \end{pmatrix}\begin{pmatrix} u & \\ & v\end{pmatrix}\begin{pmatrix} iI_3 & I_3 \\ -iI_3 & I_3\end{pmatrix}.
\end{aligned}$$

The Hermitian domain of a unitary group is defined in (2). Choose an initial point in the Hermitian domain of U(3, 3) to be $\boldsymbol{i} = iI_3$, where we fix $i = \sqrt{-1}$ once and for all. From Proposition 2.1, one choice of a maximal compact subgroup at an archimedean place for U(3, 3) is $\mathcal{K} = \{g \in U(3,3) \mid g\boldsymbol{i} = \boldsymbol{i}\}$. Notice that the image of the embedding $\mathfrak{e}$ defined above is exactly $\mathcal{K}$. So $\mathcal{K} \overset{\mathfrak{e}}{\simeq} U(3)\times U(3)$. Recall that U(2, 2), which is isomorphic to $\mathrm{U}(\mathbb{W}_d \oplus \mathbb{W}^d)$, is naturally embedded into U(3, 3) (refer to (19)). So we can choose compatibly the maximal compact subgroup $\mathcal{K}'$ of U(2, 2) by $\mathcal{K}' = \mathcal{K} \cap U(2,2)$, and obviously $\mathcal{K}' \simeq U(2)\times U(2)$. The initial point $\boldsymbol{i}' = iI_2$ of the Hermitian domain of U(2, 2) is invariant under the action of elements in $\mathcal{K}'$.

At archimedean places, take a weight $(0,k)$ section $\mathbb{f}_k$ such that

$$\mathbb{f}_k(g) = j(g, \boldsymbol{i})^{-k},$$

where j is the automorphy factor defined in (3). Assume η satisfies $\eta(z) = \bar{z}^{l'}/z^{l}$; then $l' + l = k$, and $\mathbb{f}_k \in I(\eta, l - 3/2)$. Recall that $[F : \mathbb{Q}] = r$. Then for $m \in F$, let $m_\infty = (m_1, m_2, \ldots, m_r) \in F_\infty \simeq \mathbb{R}^r$. Given $z = (z_1, z_2, \ldots, z_r) \in F_\infty$, $\psi(z) = \exp(2\pi i m_1 z_1)\exp(2\pi i m_2 z_2)\cdots\exp(2\pi i m_r z_r)$.

Proposition 4.4. *Take $\mathbb{f}_k$ and ψ as above. We have*

$$\int_T \psi(-t)\mathbb{f}_k(w_3(0, y, t)w_2 g')\,dt = c_\psi \omega_\chi(g')\phi_{\mathbb{f}_k}(y)R_{k-1}(g'), \tag{24}$$

where $g' \in U(2,2)$, $\phi_{\mathbb{f}_k}(y) = \psi(iyy^*)$, χ satisfies $\chi(z) = |z|/z$ for $z \in K_\infty^\times$, and

$$c_\psi = \frac{(-2\pi i)^{kr}(m_1 m_2 \cdots m_r)^{k-1} e^{-2\pi(m_1+m_2+\cdots+m_r)}}{[(k-1)!]^r}.$$

The function R_{k-1} satisfies $R_{k-1}(g') = j(g', \boldsymbol{i}')^{-k+1}$. As an induced representation on $\mathrm{U}(2,2)$, *it is in $I(\eta\chi^{-1}, l - \frac{3}{2})$ and is a weight $(0, k-1)$ section.*

Proof. For simplicity, we prove this proposition under the assumption that $r = 1$, from which the general result should be easily derived. So regard m as a real number first.

Step 1. Let $g' = I_4$. We have

$$\begin{aligned}\int_{\mathbb{R}} \psi(-t)\mathbb{f}_k(w_3(0, y, t)w_2)\,dt &= (-1)^k \int_{\mathbb{R}} \psi(-t)(t + i + iyy^*)^{-k}\,dt \\ &= (-1)^k \psi(iyy^* + i) \int_{\mathbb{R}+i+iyy^*} \psi(-t)t^{-k}\,dt \\ &= \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!}\,\psi(iyy^*).\end{aligned}$$

This calculation hints that we should take $\phi_{\mathbb{f}_k}$ to be $\psi(iyy^*)$.

Step 2. Replace g' by

$$p'g' = \begin{pmatrix} a & \\ & (a^*)^{-1} \end{pmatrix} \begin{pmatrix} I_2 & b \\ & I_2 \end{pmatrix} g'.$$

If we assume that

$$\int_{\mathbb{R}} \psi(-t)\mathbb{f}_k(w_3(0, y, t)w_2 g')\,dt = \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!}\,\psi(iyy^*)R_{k-1}(g'),$$

then

$$\begin{aligned}
\int_{\mathbb{R}} & \psi(-t)\mathbb{f}_k(w_3(0, y, t)w_2 p' g')\, dt \\
&= (\overline{\det a})^k \psi(yaba^* y^*) \int_{\mathbb{R}} \psi(-t)\mathbb{f}_k(w_3(0, ya, t)w_2 g')\, dt \\
&= \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!} (\overline{\det a})^k \psi(yaba^* y^*)\psi(iyaa^* y^*) R_{k-1}(g') \\
&= \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!} \omega_\chi(p'g')\phi_{\mathbb{f}_k}(y) R_{k-1}(p'g').
\end{aligned}$$

Step 3. Take $g' = w_2 n' = w_2\left(\begin{smallmatrix} I_2 & b \\ & I_2 \end{smallmatrix}\right)$. Note that we take

$$\omega_\chi\left(\begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}\right)\phi(x) = \hat{\phi}(x),$$

so one easily checks that at archimedean places, $\omega_\chi(w_2)\phi(x) = -\hat{\phi}(x)$. Then,

$$\begin{aligned}
\int_{\mathbb{R}} & \psi(-t)\mathbb{f}_k(w_3(0, y, t)w_2 w_2 n'\, dt \\
&= (-1)^k \int_{\mathbb{R}} \psi(-t) \det\begin{pmatrix} t+i & -y \\ -y^* & b+i \end{pmatrix}^{-k} dt \\
&= (-1)^k \int_{\mathbb{R}} \psi(-t) \det(b+i)^{-k}(t+i-y(b+i)^{-1}y^*)^{-k}\, dt \\
&= \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!} \det(b+i)^{-k}\psi(-y(b+i)^{-1}y^*) \\
&= \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!} \omega_\chi(w_2 n')\phi_{\mathbb{f}_k}(y) R_{k-1}(w_2 n').
\end{aligned}$$

Here we use that the Fourier transform of $\psi(y(i+b)y^*)$ is $\det(1-ib)^{-1}\psi(y(-i-b)^{-1}y^*)$.

Step 4. Let $g' = w_2 n_1' w_2 n_2'$, where $n_1' = \left(\begin{smallmatrix} I_2 & b_1 \\ & I_2 \end{smallmatrix}\right)$ and $n_2' = \left(\begin{smallmatrix} I_2 & b_2 \\ & I_2 \end{smallmatrix}\right)$. Both b_1 and b_2 are 2×2 Hermitian matrices. We have a decomposition

$$\begin{aligned}
w_2 n_1' w_2 n_2' &= w_2 \begin{pmatrix} I_2 & b_1 \\ & I_2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{b_2^2+1}} & \frac{-b_2}{\sqrt{b_2^2+1}} \\ & \sqrt{b_2^2+1} \end{pmatrix} \mathfrak{e}\left(\frac{-b_2+i}{\sqrt{b_2^2+1}}, \frac{-b_2-i}{\sqrt{b_2^2+1}}\right) \\
&= \begin{pmatrix} \sqrt{b_2^2+1} & \\ & 1/\sqrt{b_2^2+1} \end{pmatrix} w_2 \begin{pmatrix} I_2 & -b_2 + \sqrt{b_2^2+1}\, b_1 \sqrt{b_2^2+1} \\ & I_2 \end{pmatrix} \\
&\qquad\qquad \cdot \mathfrak{e}\left(\frac{-b_2+i}{\sqrt{b_2^2+1}}, \frac{-b_2-i}{\sqrt{b_2^2+1}}\right).
\end{aligned}$$

From Steps 2 and 3, one can see that when $g' \in \mathcal{P}' w_2 \mathcal{N}' \subset U(2,2)$, where $\mathcal{N}'$ is the unipotent radical of $\mathcal{P}'$, the equality (24) holds. Then let us prove that when taking $g' = w_2 n_1' w_2 n_2'$, the equality still holds. We have

$$\mathbb{f}_k(w_3(0,y,t)w_2w_2n_1w_2n_2) = \mathbb{f}_k(w_3(0,y,t)w_2p'w_2n')\det\left(\frac{b_2+i}{\sqrt{b_2^2+1}}\right)^{-k},$$

where

$$p' = \begin{pmatrix} \sqrt{b_2^2+1} & \\ & 1/\sqrt{b_2^2+1} \end{pmatrix} \quad \text{and} \quad n' = \begin{pmatrix} I_2 & -b_2+\sqrt{b_2^2+1}\,b_1\sqrt{b_2^2+1} \\ & I_2 \end{pmatrix}.$$

So

$$\begin{aligned}
&\int_{\mathbb{R}} \psi(-t)\mathbb{f}_k(w_3(0,y,t)w_2w_2n_1w_2n_2)\,dt \\
&\quad = \det\left(\frac{b_2+i}{\sqrt{b_2^2+1}}\right)^{-k} \int_{\mathbb{R}} \psi(-t)\mathbb{f}_k(w_3(0,y,t)w_2p'w_2n')\,dt \\
&\quad = \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!} \det\left(\frac{b_2+i}{\sqrt{b_2^2+1}}\right)^{-k} \omega_\chi(p'w_2n')\phi_{\mathbb{f}_k}(y) R_{k-1}(p'w_2n') \\
&\quad = \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!} \det\left(\frac{b_2+i}{\sqrt{b_2^2+1}}\right)^{-1} \omega_\chi(p'w_2n')\phi_{\mathbb{f}_k}(y) R_{k-1}(w_2n_1w_2n_2).
\end{aligned}$$

Note that

$$\omega_\chi(p'w_2n')\phi_{\mathbb{f}_k}(y) = -\frac{1}{\det\left(\sqrt{b_2^2+1}\right)} \frac{1}{\det\left(\frac{b_2 i+1}{b_2^2+1} - b_1 i\right)} \psi\left(y\left(\frac{b_2-i}{b_2^2+1} - b_1\right)^{-1} y^*\right).$$

Also,

$$\begin{aligned}
&\omega_\chi(w_2n_1'w_2n_2')\phi_{\mathbb{f}_k}(y) \\
&\qquad = \frac{1}{\det(1-b_2 i)} \frac{1}{\det((1-b_2 i)^{-1} - b_1 i)} \psi(y((b_2+i)^{-1} - b_1)^{-1} y^*).
\end{aligned}$$

By comparison,

$$\int_{\mathbb{R}} \psi(-t)\mathbb{f}_k(w_3(0,y,t)w_2g')\,dt = \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!} \omega_\chi(g')\phi_{\mathbb{f}_k}(y) R_{k-1}(g')$$

for $g' = w_2 n_1' w_2 n_2'$.

Step 5. It is known that elements in U(2, 2) can be generated by w_2 and $p' \in \mathcal{P}'$, where $\mathcal{P}'$ is the Siegel parabolic subgroup of U(2, 2). The following lemma shows how these elements generate U(2, 2).

Lemma 4.5. $\mathrm{U}(2,2) = \mathcal{P}' \cup \mathcal{P}' w_2 \mathcal{N}' \cup \mathcal{P}' w_2 \mathcal{N}' w_2 \mathcal{N}'$.

The proof is straightforward, so let me skip it. In Steps 1 to 4, we have verified that

$$\int_{\mathbb{R}} \psi(-t)\mathbb{f}_k(w_3(0, y, t)w_2 g')\,dt = \frac{(-2\pi i)^k m^{k-1} e^{-2\pi m}}{(k-1)!}\,\omega_\chi(g')\phi_{\mathbb{f}_k}(y)R_{k-1}(g')$$

for g' in each of the subsets of U(2, 2).

For each archimedean place, the computations are exactly as above as long as we change m to the corresponding m_i. Putting them together, we can prove the proposition. □

$\mathbb{f}_s$ is not spherical. Denote the set of places where $\mathbb{f}_s$ is not the spherical element in the induced representation space by S. When $\mathbb{f}_s$ is not spherical, we make such a choice: $\mathbb{f}_s \in I(\eta, s)$ and $\mathbb{f}_s$ is supported in the big cell $\mathcal{P}w_3\mathcal{P}$ such that $\mathbb{f}_s(w_3 n(b)) = \Phi(b)$, where $n(b) = \begin{pmatrix} I_3 & b \\ & I_3 \end{pmatrix}$, and b is a 3×3 Hermitian matrix. Let $b = \begin{pmatrix} t & y \\ y^* & b' \end{pmatrix}$, where $t \in F$ and b' is a 2×2 Hermitian matrix. Assume $\Phi(b) = \phi'(t)\phi(y)\Phi'(b')$ such that ϕ, ϕ', Φ' are all Bruhat–Schwartz functions. Further we make these assumptions concerning $\mathbb{f}_s$ and the additive character ψ:

Assumption 4.6. (1) The set S includes all the places where η or χ is ramified. In another words, when χ or η is ramified, we should take $\mathbb{f}_s$ supported on the big cell, and as long as such an $\mathbb{f}_s$ satisfies the following two assumptions, Theorem 4.9 can be derived. However, for the sake of later computations in Remark 4.14, we need to further assume (28).

(2) For $b' \in \operatorname{supp}\Phi'$ and $y \in \operatorname{supp}\phi$, we have $\psi(yb'y^*) = 1$.

(3) When $t \in \operatorname{supp}\phi'$, we have $\psi(t) = 1$.

Remark 4.7. The assumptions above are in general quite weak. In practice, we may first determine Φ' and ϕ. Because they are both compactly supported, as a set of F, $\{yb'y^* \mid y \in \operatorname{supp}\Phi', b' \in \operatorname{supp}\phi\}$ must be compact. So there exists an additive character ψ (in fact, infinitely many such characters) that is constant of value 1 on this set. Moreover, we can even choose a universal ψ that is independent of the conductors of η and χ. Then based on Assumption 4.6(3), one can determine the function ϕ'. Then in this way, we can determine Φ, and in turn $\mathbb{f}_s$.

Lemma 4.8. *Suppose R_s is in $I_{\mathcal{P}'}^{U(2,2)}(\eta\chi^{-1}, s)$ and is supported on the big cell $\mathcal{P}'w_2\mathcal{P}'$, such that*

$$R_s\left(w_2\begin{pmatrix} I_2 & b' \\ & I_2 \end{pmatrix}\right) = \Phi'(b').$$

Let $\phi_{\mathbb{f}_s}(y) = \hat{\phi}(-y)$. Then

$$\int_T \psi(-t)\mathbb{f}_s(w_3(0, y, t)w_2 g')\,dt = \left(\int_T \psi(-t)\phi'(t)\,dt\right)\omega_\chi(g')\phi_{\mathbb{f}_s}(y)R_s(g').$$

Proof. In order that $w_3(0, y, t)w_2 g' \in \mathcal{P} w_3 \mathcal{P}$, it must be that g' is in $\mathcal{P}' w_2 \mathcal{P}'$, which implies that R_s is supported on $\mathcal{P}' w_2 \mathcal{P}'$. Let $g' = w_2 n'(b') = w_2\left(\begin{smallmatrix} I_2 & b' \\ & I_2 \end{smallmatrix}\right)$ for $n'(b') \in \mathcal{N}'$; then

$$\begin{aligned}
\int_T \psi(-t)\mathbb{f}_k(w_3(0, y, t)w_2 w_2 n'(b'))\,dt \\
&= \int_T \psi(-t)\Phi\left(\begin{pmatrix} t & y \\ y^* & b' \end{pmatrix}\right) dt \\
&= \left(\int_T \psi(-t)\phi'(t)\,dt\right)\phi(y)\Phi'(b') \\
&= \left(\int_T \psi(-t)\phi'(t)\,dt\right)\omega_\chi(w_2 n'(b'))\phi_{\mathbb{f}_s}(y)R_s(w_2 n'(b')). \quad \square
\end{aligned}$$

Let me summarize the local computations of the three cases above in the following theorem. Let S be a finite set of local places, such that $\mathbb{f}_{k,v}$ is not spherical if and only if $v \in S$.

Theorem 4.9. *For a Hecke character η of K and $\eta_\infty(z) = \bar{z}^{k-l}/z^l$, take*

$$\mathbb{f}_k \in I_{\mathcal{P}(\mathbb{A})}^{U(3,3)(\mathbb{A})}\left(\eta, l - \tfrac{3}{2}\right).$$

Assume that $\mathbb{f}_{k,\infty}(g) = j(g, \boldsymbol{i})^{-k}$, and $\mathbb{f}_{k,v}$ is supported on the big cell when $v \in S$. Choose a Hecke character χ of K so that $\chi_\infty(z) = |z|/z$. Fix an additive character ψ of F as in (20). *When $\mathbb{f}_k$, χ and ψ satisfy Assumptions 4.2 and 4.6, we have*

$$\mathbb{E}_\psi(hg') = c_{\mathbb{f}_k,\psi}\theta_{\phi_{\mathbb{f}_k}}(hw_2^{-1}g')\mathcal{E}(g', R_{k-1}) \quad \textit{for } h \in H(\mathbb{A}),\ g' \in U(2,2)(\mathbb{A}),$$

where

$$c_{\mathbb{f}_k,\psi} = c_\psi \prod_{v \in S} \operatorname{vol}(\operatorname{supp}\phi'_v),$$

c_ψ is given in Proposition 4.4, $\phi_{\mathbb{f}_k} \in S(\mathbb{W}^d(\mathbb{A}))$, and $R_{k-1} \in I_{\mathcal{P}'(\mathbb{A})}^{U(2,2)(\mathbb{A})}(\eta\chi^{-1}, l - \frac{3}{2})$. Specifically, $R_{k-1,\infty}(g') = j(g', \boldsymbol{i}')^{-k+1}$ and $R_{k-1,v}$ is normalized and spherical when $v \notin S$. Otherwise, $R_{k-1,v}$ is supported on the big cell. The Siegel–Eisenstein series $\mathcal{E}$ is associated to R_{k-1}.

Proof. Recall (22):

$$\mathbb{E}_\psi(hg', \mathbb{f}_k) = \int_{[T]} \psi(-t') \sum_{\substack{\gamma \in w_3 \cdot \mathbb{W}_d(F)\backslash H(F)\cdot w_2 \\ \cdot \mathcal{P}'(F)\backslash U(2,2)(F)}} \mathbb{f}_k(\gamma t' h g')\,dt'.$$

Let $h = (x, y, t) \in H(\mathbb{A})$; then

$$\begin{aligned}
\mathbb{f}_k(w_3 h w_2 g')) &= \mathbb{f}_k\big(w_3(x, 0, 0)(0, y, t - \tfrac{1}{2}(xy^* + yx^*))w_2 g'\big) \\
&= \mathbb{f}_k\big(w_3(0, y, t - \tfrac{1}{2}(xy^* + yx^*))w_2 g'\big).
\end{aligned}$$

So we get

$$\begin{aligned}
&\mathbb{E}_\psi(hg', \mathbb{f}_k)\\
&\quad= \int_{T(\mathbb{A})} \psi(-t') \sum_{\substack{y_0\in\mathbb{W}^d(F)\\ g_0\in\mathcal{P}'(F)\backslash U(2,2)(F)}} \mathbb{f}_k(w_3(0, y_0, t')w_2 g_0 hg')\, dt'\\
&\quad= \int_{T(\mathbb{A})} \psi(-t') \sum_{\substack{y_0\in\mathbb{W}^d(F)\\ g_0\in\mathcal{P}'(F)\backslash U(2,2)(F)}} \mathbb{f}_k(w_3(0, y_0, t')((x, y)g_0^{-1}w_2^{-1}, t)w_2 g_0 g')\, dt'\\
&\quad= c_\psi \prod_{v\in S} \int_{T_v} \psi_v(-t')\mathbb{f}_{k,v}(w_3(0, 0, t'))\, dt'\\
&\qquad\qquad\cdot \sum_{\substack{y_0\in\mathbb{W}^d(F)\\ g_0\in\mathcal{P}'(F)\backslash U(2,2)(F)}} \omega_\chi(((x, y)g_0^{-1}w_2^{-1}, t)g_0 g)\, \phi_{\mathbb{f}_k}(y_0)R_{k-1}(g_0 g').
\end{aligned}$$

Notice that $\int_{T_v} \psi_v(-t')\mathbb{f}_{k,v}(w_3(0, 0, t'))\, dt' = \mathrm{vol}(\mathrm{supp}\, \phi'_v)$ for $v \in S$. So

$$\begin{aligned}
\mathbb{E}_\psi(hg', \mathbb{f}_k) &= c_{\mathbb{f}_k,\psi}\theta_{\phi_{\mathbb{f}_k}}(((x, y)w_2^{-1}, t)g')\mathscr{E}(g', R_{k-1})\\
&= c_{\mathbb{f}_k,\psi}\theta_{\phi_{\mathbb{f}_k}}(hw_2^{-1}g')\mathscr{E}(g', R_{k-1}). \qquad \square
\end{aligned}$$

4B. ***Fourier–Jacobi coefficients of Eisenstein series on* U(3, 1).** In this section, I will define the Fourier–Jacobi coefficients of the Eisenstein series $E(g, f, s)$ on $\mathrm{U}(I_\zeta)$. If this $E(g, f, s)$ is from the pullback of a Siegel–Eisenstein series $\mathbb{E}(\,\cdot\,, \mathbb{f}, s)$, then by applying the pullback formula (Theorem 2.6) and results about the Fourier–Jacobi coefficients of $\mathbb{E}(\,\cdot\,, \mathbb{f}, s)$ (Theorem 4.9), we will get formulas for the Fourier–Jacobi coefficients of $E(g, f, s)$.

Definitions. Let P be the only nontrivial parabolic subgroup P of $\mathrm{U}(I_\zeta)$; then the unipotent radical of P is

$$N = \left\{ \begin{pmatrix} 1 & x & t + x\zeta x^*/2\\ & I_2 & \zeta x^*\\ & & 1 \end{pmatrix} \;\middle|\; t \in F \right\}.$$

From another point of view, N can be regarded as a Heisenberg group attached to a 4-dimensional symplectic space W of F. Using conventional notation, denote

$$\begin{pmatrix} 1 & x & t + x\zeta x^*/2\\ & I_2 & \zeta x^*\\ & & 1 \end{pmatrix} \quad \text{by } (x, t);$$

then

$$(x_1, t_1)(x_2, t_2) = (x_1 + x_2, t_1 + t_2 + \langle x_1, x_2\rangle/2),$$

where $\langle \cdot, \cdot \rangle$ represents the alternating pair on W, and

$$\langle x_1, x_2 \rangle = x_1 \zeta x_2^* - x_2 \zeta x_1^*.$$

The degree two unitary group $\mathrm{U}(\zeta) = \{u \mid u\zeta u^* = \zeta\}$ is a subgroup of $\mathrm{U}(I_\zeta)$, and has an action on N by $u \cdot (x, t) = u^{-1}(x, t)u = (xu, t)$.

Given the additive character ψ as in (20), define the Fourier–Jacobi coefficient of $E(g, f, s)$ as

$$E_\psi(nu, f, s) = \int_{[T]} \psi(-t) E(tnu, f, s)\, dt \tag{25}$$

for $u \in U(\zeta)(\mathbb{A})$ and $n \in N(\mathbb{A})$, where T is the center of N. So $E_\psi \in C_\psi^\infty([NU(\zeta)])$.

Pullback of $\mathbb{E}_\psi$. Recall in (8), we define an embedding $\mathfrak{e} : U(I_\zeta) \times U(\zeta) \hookrightarrow U(3, 3)$. See that

$$\mathfrak{e}|_{U(\zeta)\times U(\zeta)} : U(\zeta) \times U(\zeta) \hookrightarrow U(2, 2) = U(\mathbb{W}_d \oplus \mathbb{W}^d) \subset U(3, 3).$$

Also for $(x, t) \in N$, we have $\mathfrak{e}((x, t), I_2) = (x\zeta/2, x, t)$. This implies $\mathfrak{e}(N, I_2) \subset H$. Specifically, the center of N is mapped to the center of H under $\mathfrak{e}$, which explains why I use T to denote both centers of H and N.

From Theorem 2.6, we know that

$$E(g, f, s) = \int_{[U(\zeta)]} \mathbb{E}(\mathfrak{e}(g, u), \mathbb{f}, s)\beta(u)\eta^{-1}(\det u)\, du \tag{26}$$

for $f_s(g) = \int_{U(\zeta)(\mathbb{A})} \mathbb{f}_s(\mathfrak{e}(g, u))\beta(u)\eta^{-1}(\det u)\, du \in I_P^{U(I_\zeta)}(\Pi \otimes \eta, s)$, where $\beta \in V_\Pi$. So it is reasonable to infer that E_ψ is also the pullback of $\mathbb{E}_\psi$. The following proposition is easy to verify.

Proposition 4.10. *If $E(g, f, s)$ is defined by* (26), *then*

$$E_\psi(nu, f, s) = \int_{[U(\zeta)]} \mathbb{E}_\psi(\mathfrak{e}(nu, u'), \mathbb{f}, s)\beta(u')\eta^{-1}(\det u')\, du'.$$

Let us study the relation between $N \cdot U(\zeta)$ and $H \cdot U(2, 2)$ induced by the embedding $\mathfrak{e}$. N is the Heisenberg group associated to the symplectic group W, and $\mathrm{U}(W) = U(\zeta)$, while $H \subset U(3, 3)$ is the Heisenberg group associated to $\mathbb{W} = \mathbb{W}_d + \mathbb{W}^d$, and $\mathrm{U}(\mathbb{W}) = U(2, 2)$. Notice that

$$W \overset{\mathfrak{e}}{\hookrightarrow} \mathbb{W}, \quad x \mapsto \Big(\frac{x\zeta}{z}, x\Big).$$

At the same time, we can define another embedding of W^-:

$$W^- \hookrightarrow \mathbb{W}, \quad x \mapsto \Big(-\frac{x\zeta}{z}, x\Big).$$

So, the two embeddings combine to give

$$\begin{aligned} W+W^- &\xrightarrow{\sim} \mathbb{W}=\mathbb{W}_d+\mathbb{W}^d,\\ (x,-x)&\mapsto (x\zeta,0),\\ (x,x)&\mapsto (0,x). \end{aligned}$$

Since W has the polarization $W=X+Y$, the Weil representation of $\mathrm{U}(2,2)$ can be realized on $S(X+X)$ or on $S(\mathbb{W}_d)$. Recall that we define an intertwining isometry operator δ_ψ between the two representations by (13) . It can be applied here with a little revision. Take $\phi\in S(X+X)$, and let $\delta_\psi{}'(\phi)(w)=\delta_\psi(\phi)(w\zeta^{-1})$ be the corresponding function in $S(\mathbb{W}_d)$. If we take $\phi=\phi_1\otimes\bar{\phi}_2$ for $\phi_1,\phi_2\in S(X)$, then

$$\omega_\chi(\mathfrak{e}(u_1,u_2))\delta'_\psi(\phi_1\otimes\bar{\phi}_2)=\delta'_\psi(\omega_\chi(u_1)\phi_1\otimes\chi(\det u_2)\check{\omega}_\chi(u_2)\bar{\phi}_2).$$

It is straightforward to verify that

$$\theta_{\delta'_\psi(\phi_1\otimes\bar{\phi}_2)}(\mathfrak{e}(nu_1,u_2))=\chi(\det u_2)\theta_{\phi_1}(nu_1)\bar{\theta}_{\phi_2}(u_2),$$

where $\bar{\theta}_{\phi_2}(u_2)=\sum_{x\in X(F)}\check{\omega}_\chi(u_2)\overline{\phi_2}(x)$. Then Proposition 4.10 and Theorem 4.9 give this:

Corollary 4.11. *Assume that $\phi_{\mathbb{f}_k}=\delta'_\psi(\phi_1\otimes\bar{\phi}_2)$, and $E(g,f_k)$ is defined as the pullback of $\mathbb{E}(\,\cdot\,,\mathbb{f}_k)$, where $\mathbb{f}_k$ satisfies the conditions of Theorem 4.9. Then*

$$E_\psi(nu,f_k)=c_{\mathbb{f}_k,\psi}\theta_{\phi_1}(nu)\int_{[U(\zeta)]}\beta(u')\chi\eta^{-1}(\det u')\mathscr{E}(\mathfrak{e}(u,u'),R_{k-1})\bar{\theta}_{\phi_2}(u')\,du'.$$

Inner product of E_ψ with θ_φ. As we mentioned, both E_ψ and θ_φ for $\varphi\in S(X)$ are functions in $C^\infty_\psi([NU(\zeta)])$. Let $L^2_\psi([NU(\zeta)])$ be the completion of $C^\infty_\psi([NU(\zeta)])$ with respect to the inner product

$$\langle\theta,\theta'\rangle=\int_{[NU(\zeta)]}\theta(r)\overline{\theta'}(r)\,dr.$$

Treat E_ψ and θ_φ as elements in $L^2_\psi([NU(\zeta)])$. Let us calculate $\langle E_\psi,\theta_\varphi\rangle$.

Theorem 4.12. *In the setting above, we have*

$$\langle E_\psi(nu,f_k),\theta_\varphi(nu)\rangle = c_{\mathbb{f}_k,\psi}(\phi_1,\varphi)\int_{U(\zeta)(\mathbb{A})}R_{k-1}(\mathfrak{e}(u,1))\,du\int_{[U(\zeta)]}\beta(u)\bar{\theta}_{\phi_2}(u)\,du,$$

$$\prod_{v\notin S}\int_{U(\zeta)_v}R_{k-1,v}(\mathfrak{e}(u,1))\,du=\frac{L^S(\eta\chi^{-1},l-\frac{1}{2})L^S(\eta\chi^{-1},l-\frac{3}{2})}{L^S(\eta_F\epsilon_{K/F},2l-1)L^S(\eta_F,2l-2)},$$

where S is the set of places introduced in Theorem 4.9, and L^S is the partial L-function skipping the factors at $v\in S$.

(I will discuss the case $v \in S$ in Remark 4.14.)

Proof. We prove the theorem in three steps:

Step 1. $\langle E_\psi(nu, f_k), \theta_\varphi(nu)\rangle = c_{\mathbb{f}_k,\psi}(\phi_1, \varphi) I(E_\psi)$, where

$$I(E_\psi) = \int_{[U(\zeta)\times U(\zeta)]} \beta(u')\chi\eta^{-1}(\det u')\mathscr{E}(\mathfrak{e}(u, u'), R_{k-1})\, \bar{\theta}_{\phi_2}(u')\, du\, du'.$$

Step 2. $I(E_\psi) = \int_{U(\zeta)(\mathbb{A})} R_{k-1}(\mathfrak{e}(u, 1))\, du \cdot \int_{[U(\zeta)]} \beta(u)\bar{\theta}_{\phi_2}(u)\, du.$

Step 3. We show

$$\int_{U(\zeta)_v} R_{k-1,v}(\mathfrak{e}(u, 1))\, du = \frac{L_v(\eta\chi^{-1}, l - \frac{1}{2}) L_v(\eta\chi^{-1}, l - \frac{3}{2})}{L_v(\eta_F \epsilon_{K/F}, 2l - 1) L_v(\eta_F, 2l - 2)}$$

for a finite place $v \notin S$.

The equation in Step 1 is straightforward from Corollary 4.11, because first for $\theta_{\phi_1}, \theta_\varphi \in L^2_\psi([NU(\zeta)])$, we have $\langle\theta_{\phi_1}, \theta_\varphi\rangle = (\phi_1, \varphi)$; then

$$\begin{aligned}
&\langle E_\psi(nu, f_k), \theta_\varphi(nu)\rangle \\
&\quad = \int_{[NU(\zeta)]} c_{\mathbb{f}_k,\psi}\Big(\int_{[U(\zeta)]} \beta(u')\chi\eta^{-1}(\det u') \\
&\qquad\qquad \mathscr{E}(\mathfrak{e}(u, u'))\bar{\theta}_{\phi_2}(u')\, du'\Big)\theta_{\phi_1}(nu)\bar{\theta}_\varphi(nu)\, dn\, du \\
&\quad = c_{\mathbb{f}_k,\psi}(\phi_1, \varphi) \int_{[U(\zeta)\times U(\zeta)]} \beta(u')\chi\eta^{-1}(\det u')\mathscr{E}(\mathfrak{e}(u, u'), R_{k-1})\bar{\theta}_{\phi_2}(u')\, du\, du' \\
&\quad = c_{\mathbb{f}_k,\psi}(\phi_1, \varphi) I(E_\psi).
\end{aligned}$$

To get the expression of $I(E_\psi)$ in Step 2, let us unfold $\mathscr{E}(\cdot, R_{k-1})$:

$$\begin{aligned}
\mathscr{E}(\mathfrak{e}(u, u'), R_{k-1}) &= \sum_{\gamma\in\mathscr{P}'(F)\backslash U(2,2)(F)} R_{k-1}(\gamma\,\mathfrak{e}(u, u')) \\
&= \sum_{\gamma=\mathfrak{e}(u_1,u_2)\in\Delta(U(\zeta))(F)\backslash U(\zeta)(F)\times U(\zeta)(F)} R_{k-1}(\gamma\,\mathfrak{e}(u, u')),
\end{aligned}$$

where $\Delta(U(\zeta))$ is the image of the diagonal embedding of $\mathrm{U}(\zeta)$ to $\mathrm{U}(\zeta)\times U(\zeta)$. Then,

$$\begin{aligned}
&I(E_\psi) \\
&\quad = \int_{\Delta(U(\zeta))(F)\backslash U(\zeta)(\mathbb{A})\times U(\zeta)(\mathbb{A})} \beta(u')\chi\eta^{-1}(\det u')\bar{\theta}_{\phi_2}(u')R_{k-1}(\mathfrak{e}(u, u'))\, du\, du' \\
&\quad \overset{u=u'v}{=} \int_{U(\zeta)(\mathbb{A})\times[U(\zeta)]} \beta(u')\chi\eta^{-1}(\det u')\bar{\theta}_{\phi_2}(u')R_{k-1}(\mathfrak{e}(u'v, u'))\, dv\, du' \\
&\quad = \int_{U(\zeta)(\mathbb{A})} R_{k-1}(\mathfrak{e}(u, 1))\, du \cdot \int_{[U(\zeta)]} \beta(u)\bar{\theta}_{\phi_2}(u)\, du.
\end{aligned}$$

Now we are left with the calculation of $\int_{U(\zeta)(\mathbb{A})} R_{k-1}(\mathfrak{e}(u,1))\,du$. It can be written as the product of local integrals:

$$\int_{U(\zeta)(\mathbb{A})} R_{k-1}(\mathfrak{e}(u,1))\,du = \prod_v \int_{U(\zeta)(F_v)} R_{k-1,v}(\mathfrak{e}(u,1))\,du.$$

In the following computations, we drop the subscript v if this does not cause confusion.

First, $v=\infty$. Now $R_{k-1,\infty}$ is defined by the automorphy factor, namely $R_{k-1,\infty}(g') = j(g',\boldsymbol{i}')^{-k+1}$, and $\mathrm{U}(\zeta)(F_\infty)$ is isomorphic to copies of the compact group $\mathrm{U}(2)(\mathbb{R})$. By Proposition 4.4, $R_{k-1}(\mathfrak{e}(u,1))=1$. So,

$$\int_{U(\zeta)(F_\infty)} R_{k-1,\infty}(\mathfrak{e}(u,1))\,du = \mathrm{vol}(U(\zeta)(F_\infty)).$$

Second, v splits in F. Then $\mathrm{U}(\zeta)(F_v)\simeq \mathrm{GL}_2(F_v)$. Assume $R_{k-1,v}$ is normalized and spherical. Now, $K=F+F$, $\eta(a,b)=\eta_1(a)\eta_2(b)$ for $(a,b)\in K$, and similarly for χ. Since $\chi|_{\mathbb{A}^\times}=\epsilon_{K/F}$, we have $\chi_1\chi_2=1$. Let us take the Godement section representation of R_{k-1}:

$$\begin{aligned}&L(\eta_F,2l-1)L(\eta_F,2l-2)R_{k-1}(g)\\&\quad=\eta_1\chi_1^{-1}(\det g)|\det g|^{l-\frac12}\int_{\mathrm{GL}_2(F)}\Phi_{M_{2\times4}(\mathbb{O}_F)}((0,X)g)\eta_1\eta_2(\det X)|\det X|^{2l-1}\,dX,\end{aligned}$$

where $\Phi_{M_{2\times4}(\mathbb{O}_F)}$ is the characteristic function of $M_{2\times4}(\mathbb{O}_F)$. Then

$$\begin{aligned}&L(\eta_F,2l-1)L(\eta_F,2l-2)R_{k-1}(\mathfrak{e}(u,1))\\&\qquad=\eta_1\chi_1^{-1}(\det u)|\det u|^{l-\frac12}\int_{\mathrm{GL}_2(F)}\Phi_{M_{2\times4}(\mathbb{O}_F)}(X,Xu)\eta_1\eta_2(\det X)|\det X|^{2l-1}\,dX.\end{aligned}$$

So,

$$\begin{aligned}&L(\eta_F,2l-1)L(\eta_F,2l-2)\int_{U(\zeta)}R_{k-1}(\mathfrak{e}(u,1))\,du\\&\quad= L(\eta_F,2l-1)L(\eta_F,2l-2)\int_{\mathrm{GL}_2(F)}R_{k-1}(\mathfrak{e}(u,1))\,du\\&\quad=\int_{\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)}\eta_1\chi_1^{-1}(\det u)|\det u|^{l-\frac12}\\&\qquad\qquad\qquad\Phi_{M_{2\times4}(\mathbb{O}_F)}(X,Xu)\eta_1\eta_2(\det X)|\det X|^{2l-1}\,dX\,du\\&\overset{Y=Xu}{=}\int_{\mathrm{GL}_2}\eta_1\chi_1^{-1}(\det Y)|\det Y|^{l-\frac12}\Phi_{\mathbb{O}_F^2}(Y)\,dY\\&\qquad\qquad\cdot\int_{\mathrm{GL}_2}\eta_2\chi_1(\det X)|\det X|^{l-\frac12}\Phi_{\mathbb{O}_F^2}(X)\,dX\end{aligned}$$

which is equal to

$$L\big(\eta\chi^{-1}, l-\tfrac{1}{2}\big)L\big(\eta\chi^{-1}, l-\tfrac{3}{2}\big).$$

Then we have

$$\int_{\mathrm{GL}_2(F)} R_{k-1}(\mathfrak{e}(u,1))\,du = \frac{L(\eta\chi^{-1}, l-\frac{1}{2})L(\eta\chi^{-1}, l-\frac{3}{2})}{L(\eta_F, 2l-1)L(\eta_F, 2l-2)}.$$

First, $R_{k-1,v}$ is normalized and spherical, and $\mathrm{U}(\zeta)(F_v)$ is quasisplit over F_v. So $\mathrm{U}(\zeta) \simeq U(1,1)$. The embedding $\mathfrak{e}$ of $\mathrm{U}(1,1)\times U(1,1)$ to $\mathrm{U}(2,2)$ is inherited from the global definition of $\mathfrak{e}|_{U(\zeta)\times U(\zeta)}$.

Take the local Iwasawa decomposition $\mathrm{U}(1,1) = PK = NMK$, where N, M and K are the unipotent radical, Levi part and maximal open compact subgroup, respectively. Then $K = U(1,1)(\mathbb{O}_F)$, and $\mathfrak{e}(K,K) \subset \mathcal{K}'$, which is the maximal compact subgroup of $\mathrm{U}(2,2)$. Let $u = nmk$; then $du = \delta_P^{-1}(m)\,dn\,dm\,dk$, where δ_P is the modular character on P. If $m = \begin{pmatrix} \bar{\alpha}^{-1} & \\ & \alpha \end{pmatrix}$, then $\delta_P(m)^{-1} = |\alpha|_K$. Let

$$w_{13} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix},$$

then

$$\begin{aligned}\int_{U(\zeta)} R_{k-1}(\mathfrak{e}(u,1))\,du &= \int_{NM} R_{k-1}(\mathfrak{e}(nm,1))\delta_P^{-1}(m)\,dn\,dm \\ &= \int_M \int_F R_{k-1}\left(w_{13}\begin{pmatrix} 1 & & x & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} w_{13}^{-1}\mathfrak{e}(m,1)\right)\delta_P^{-1}(m)\,dx\,dm.\end{aligned}$$

Consider such a function on $\mathrm{U}(2,2)$:

$$R'_{k-1}(g) = \int_F R_{k-1}\left(w_{13}\begin{pmatrix} 1 & & x & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} g\right)dx.$$

We have

$$\int_{U(\zeta)} R_{k-1}(\mathfrak{e}(u,1))\,du = \int_M R'_{k-1}(w_{13}^{-1}\mathfrak{e}(m,1))\delta_P^{-1}(m)\,dm. \tag{27}$$

Recall that $\mathrm{U}(2,2) = U(\mathbb{W}) = U(\mathbb{W}_d + \mathbb{W}^d)$. If we have

$$\mathbb{W}_d + \mathbb{W}^d = Ke_1 + Ke_2 + Kf_1 + Kf_2,$$

then a parabolic subgroup P' fixing both $\mathbb{W}_d$ and Ke_2 can be defined. An element $p' \in P'$ looks like

$$\begin{pmatrix} a & & & d \\ b & c & e & f \\ & & a' & b' \\ & & & c' \end{pmatrix}.$$

Lemma 4.13. *R'_{k-1} is in the space of the induced representations from P', so that*

$$R'_{k-1}(p'gk') = \eta\chi^{-1}(a'c)|a'|_K^{l-\frac{3}{2}}|c|_K^{l-\frac{1}{2}} R'_{k-1}(g)$$

for $p' \in P'$, $k' \in \mathcal{K}'$ and $g \in U(2,2)$.

This lemma can be proved by direct calculations, which I will skip here. Applying the lemma, we see R'_{k-1} has a Godement section representation as follows:

$$\begin{aligned}
&\frac{L(\eta_F\epsilon_{K/F}, 2l-1)}{L(\eta_F\epsilon_{K/F}, 2l-2)} R'_{k-1}(g) \\
&\qquad = c\int_{\mathrm{GL}_2(K)} \Phi_{M_{2\times 4}(\mathbb{O}_K)}((0,X)g)\eta^{-1}\chi(\det X)|\det X|_K^{-l+\frac{3}{2}}\, dX\cdot \\
&\qquad\qquad\qquad \cdot\int_{K^\times} \Phi_{\mathbb{O}_K^4}((0,0,0,Z)g)\eta\chi^{-1}(|Z|_K)|Z|_K^{2l-2}\, dZ,
\end{aligned}$$

where the normalizing constant c satisfies

$$\begin{aligned}
1 = c\int_{\mathrm{GL}_2(K)} &\Phi_{M_{2\times 2}(\mathbb{O}_K)}(X)\eta^{-1}\chi(\det X)\,|\det X|_K^{-l+\frac{3}{2}}\, dX \\
&\cdot\int_{K^\times} \Phi_{\mathbb{O}_K}(Z)\eta\chi^{-1}(|Z|_K)|Z|_K^{2l-2}\, dZ,
\end{aligned}$$

$$R'_{k-1}(1) = \int_F R_{k-1}\left(w_{13}\begin{pmatrix} 1 & & x & \\ & 1 & & \\ & & 1 & \\ & & & 1\end{pmatrix}\right) dx = \frac{L(\eta_F\epsilon_{K/F}, 2l-2)}{L(\eta_F\epsilon_{K/F}, 2l-1)}.$$

Let $g = w_{13}^{-1}\mathfrak{e}(m,1) = w_{13}^{-1}\mathfrak{e}\big(\big(\begin{smallmatrix}\bar{\alpha}^{-1} & \\ & \alpha\end{smallmatrix}\big), 1\big)$. It can be verified that

$$\begin{aligned}
&R'_{k-1}(w_{13}^{-1}\mathfrak{e}(m,1)) \\
&= \frac{\eta\chi^{-1}(\alpha)|\alpha|_K^{l-\frac{3}{2}}}{L(\eta_F\epsilon_{K/F}, 2l-1)L(\eta_F, 2l-2)} \int_{K^\times} \Phi_{\mathbb{O}_K}(Z)\Phi_{\mathbb{O}_K}(Z\alpha)\eta\chi^{-1}(|Z|_K)|Z|_K^{2l-2}\, dZ.
\end{aligned}$$

Substituting the expression of $R'_{k-1}(w_{13}\mathfrak{e}(m,1))$ in (27), we have

$$\begin{aligned}
&\int_{U(\zeta)} R_{k-1}(\mathfrak{e}(u,1))\, du \\
&\qquad = \int_M R'_{k-1}(w_{13}^{-1}\mathfrak{e}(m,1))\delta_P^{-1}(m)\, dm \\
&\qquad = \frac{1}{L(\eta_F\epsilon_{K/F}, 2l-1)L(\eta_F, 2l-2)} \\
&\qquad\quad \cdot\int_{K^\times\times K^\times} \eta\chi^{-1}(\alpha)|\alpha|_K^{l-\frac{1}{2}}\Phi_{\mathbb{O}_K}(Z)\Phi_{\mathbb{O}_K}(Z\alpha)\eta\chi^{-1}(|Z|_K)|Z|_K^{2l-2}\, dZ\, d\alpha \\
&\qquad = \frac{L\big(\eta\chi^{-1}, l-\frac{1}{2}\big)L\big(\eta\chi^{-1}, l-\frac{3}{2}\big)}{L(\eta_F\epsilon_{K/F}, 2l-1)L(\eta_F, 2l-2)}. \qquad \square
\end{aligned}$$

Remark 4.14. In Theorem 4.12, $\int_{U(\zeta)(F_v)} R_{k-1,v}(\mathfrak{e}(u,1))\,du$ was not explicitly calculated when $v \in S$, because we need more assumptions and things become more technical. Let me put it here. When $v \in S$, $R_{k-1,v}$ is supported in the big cell associated to a characteristic function Φ' on $\mathrm{Her}_2(F_v)$. (One can refer to the section about nonspherical $\mathbb{f}_k$, especially Lemma 4.8.)

Skip the subindex v now. Fix an integral ideal $\mathfrak{c}$ of $\mathbb{O}_K$ so that $\eta\chi^{-1}(1+\mathfrak{c}) = 1$. Then pick a totally imaginary element $\delta \in \mathbb{O}_K$ satisfying $1+\delta \in \mathbb{O}_K^\times$. Assume that

$$\operatorname{supp}\Phi' = \mathrm{Her}_2(F) \cap \ \delta\zeta^{-1}(2I_2 + 2\mathfrak{c}\,\mathrm{GL}_2(\mathbb{O}_K)). \tag{28}$$

Define $D_1(\mathfrak{c})$, a subset of $\mathrm{U}(\zeta)$, by

$$D_1(\mathfrak{c}) = \left\{ u \in U(\zeta) \;\middle|\; u = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ a, d \in -\frac{1+\delta}{1-\delta} + \delta\mathfrak{c},\ b, c \in \delta\mathfrak{c} \right\}.$$

Lemma 4.15. *Let $u \in U(\zeta)$. Then*

$$\mathfrak{e}(u,1) \in \operatorname{supp} R_{k-1} \Longleftrightarrow u \in D_1(\mathfrak{c}).$$

Proof. First notice that for a matrix in $\mathrm{U}(2,2)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{supp} R_{k-1} \Longleftrightarrow c^{-1}d \in \operatorname{supp}\Phi'.$$

By the definition of the embedding $\mathfrak{e}$ in (8),

$$\mathfrak{e}(u,1) = \begin{pmatrix} (\zeta^{-1}(u+I_2)\zeta)/2 & \zeta^{-1}(u-I_2) \\ ((u-I_2)\zeta)/4 & (u+I_2)/2 \end{pmatrix}.$$

So

$$\begin{aligned} \mathfrak{e}(u,1) \in \operatorname{supp} R_{k-1} &\Longleftrightarrow 2\zeta^{-1}(u-I_2)^{-1}(u+I_2) \in \operatorname{supp}\Phi' \\ &\Longleftrightarrow u \in D_1(\mathfrak{c}). \end{aligned}$$

□

Then we get

$$R_{k-1}(\mathfrak{e}(u,1)) = \begin{cases} \eta\chi^{-1}(\delta) & \text{if } u \in D_1(\mathfrak{c}), \\ 0 & \text{otherwise.} \end{cases}$$

The integral

$$\int_{U(\zeta)} R_{k-1}(\mathfrak{e}(u,1))\,du = \eta\chi^{-1}(\delta)\,\mathrm{vol}(D_1(\mathfrak{c})).$$

As we see from Theorem 4.12, $\langle E_\psi, \theta_\varphi\rangle$ equals the product of an explicit constant and the integral $\int_{[U(\zeta)]} \beta(u)\bar{\theta}_{\phi_2}(u)\,du$. Using the discussion of Section 3C, I will show that this integral can be interpreted by the theta lifting from $\mathrm{U}(\zeta)$ to $\mathrm{U}(1)$.

Recall that N is the unipotent radical of the parabolic subgroup P of $\mathrm{U}(I_\zeta)$. We take N as the Heisenberg group associated to the 4-dimensional symplectic space W over F. On W, the alternating pairing is defined as $\langle x_1, x_2\rangle = x_1\zeta x_2^* - x_2\zeta x_1^*$, while, as a K-vector space, $W = V_1 \otimes V_2$, where V_1 and V_2 are Hermitian vector spaces with dimensions 2 and 1 over K, respectively. The skew-Hermitian form on V_1 is $(x, y)_1 = x\zeta y^*$. The Hermitian form on V_2 is defined as $(x, y)_2 = x\bar{y}$. So, we immediately have the reductive pair $(U(V_1), U(V_2))$ in $\mathrm{Sp}(W)$ with $\mathrm{U}(V_1) \simeq U(\zeta)$, and $\mathrm{U}(V_2) \simeq U(1)$.

Consider the theta lifting

$$\theta^{\beta}_{\phi_2}(u) = \int_{U(\zeta)} \theta_{\phi_2}(g, u)\beta(g)\, dg$$

from $\mathrm{U}(\zeta)$ to $\mathrm{U}(1)$. In Section 3C, I give the definition of a theta lifting and explain the way to compute $|\theta^{\beta}_{\phi_2}(1)|^2$ using the Siegel–Weil formula. If we assume that the representation Π on $\mathrm{U}(\zeta)$ is self-dual, then $V_{\Pi} \simeq V_{\check{\Pi}}$, and

$$\Big|\int_{[U(\zeta)]} \beta(u)\bar{\theta}_{\phi_2}(u)\, du\Big|^2 = \big|\theta^{\beta}_{\phi_2}(1)\big|^2$$

Proposition 3.7 and (18) together imply:

Corollary 4.16. $|\theta^{\beta}_{\phi_2}(1)|^2 = \frac{1}{2}\displaystyle\int_{U(\zeta)(\mathbb{A})} f_{\delta_{\psi}(\phi_2\otimes\bar{\phi}_2),s}(\mathfrak{e}(g, 1))\langle\Pi(g)\beta, \beta\rangle\, dg|_{s=-\frac{1}{2}}$.

Remark 4.17. The integral in the corollary above should be nonzero if ϕ_2 is chosen properly, because the theta lifting of Π should define a nonzero representation space of $\mathrm{U}(1)$. This is a special case in [Li 1992]. The nonzero result is crucial for our application.

Remark 4.18. If $\phi_{2,v}$ is a standard characteristic function, then $\delta_{\psi}(\phi_2 \otimes \bar{\phi}_2)$ is also a standard characteristic function, and $f_{\delta_{\psi}(\phi_2\otimes\bar{\phi}_2),s,v}$ is normalized spherical.

Proposition 4.19. *If Π_v is an unramified representation with a spherical vector β, $\mathfrak{f}_{s,v}$ is the unique $\mathrm{U}(2, 2)(\mathbb{O}_{F_v})$-invariant section in $I^{U(2,2)}_{\mathcal{P}'}(\chi, s)_v$ and $\mathfrak{f}_{s,v}(1) = 1$, then*

$$\int_{U(\zeta)_v} \mathfrak{f}_{s,v}(\mathfrak{e}(u, 1))\langle\Pi(u)\beta, \beta\rangle_v\, du = \frac{L\big(\Pi_v, \chi_v, s + \frac{1}{2}\big)}{L_v(\epsilon_{K/F}, 2s + 2)L_v(1, 2s + 1),}$$

where $L_v(1, \cdot\,)$ is the Zeta function of the local field F_v.

Remark 4.20. This type of integral was considered by Piatetski-Shapiro and Rallis in many cases. Similar calculations have been done in [Li 1992]. In the $(U(1), U(1))$ case, Yang [1997] had explicit formulas.

Proof. Let me calculate the integral above in the case when $\mathrm{U}(\zeta)_v \simeq \mathrm{GL}_2(F_v)$. The computations at other types of unramified places are skipped here. Now we omit the subindex v.

Denote the matrix coefficient of Π by w_Π. Let $\chi|_{F\times F} = (\chi_1, \chi_2)$. Using the Godement section of $\mathfrak{f}_s$, we have

$$\begin{aligned}
&\frac{1}{L(\epsilon_{K/F}, 2s+2)L(\epsilon_{K/F}, 2s+1)} \int_{U(\zeta)} \mathfrak{f}_s(\mathfrak{e}(u,1))\langle \Pi(u)\beta, \beta\rangle \, du \\
&\quad = \int_{\mathrm{GL}_2(F)} \chi_1(\det u)|\det u|^{s+1} \\
&\qquad \cdot \int_{\mathrm{GL}_2(F)} \Phi_{M_{2\times 4}(\mathbb{O}_F)}((0, Z)\mathfrak{e}(u,1))\epsilon_{K/F}(\det Z)|\det Z|^{2s+2} w_\Pi(u)\, du\, dZ \\
&\quad = \int_{\mathrm{GL}_2(F)\times \mathrm{GL}_2(F)} \Phi_{M_{2\times 4}(\mathbb{O}_F)}(Z, Zu)\chi_1(\det u)|\det u|^{s+1} \\
&\qquad\qquad \cdot \epsilon_{K/F}(\det Z)|\det Z|^{2s+2} w_\Pi(u)\, du\, dZ \\
&\quad \overset{Y=Zu}{=} \int_{\mathrm{GL}_2(F)\times \mathrm{GL}_2(F)} \Phi_{M_{2\times 4}(\mathbb{O}_F)}(Y, Z)\chi_1(\det Y)|\det Y|^{s+1} \\
&\qquad\qquad \cdot \chi_2(\det Z)|\det Z|^{s+1} w_\Pi(Z^{-1}Y)\, dY\, dZ. \qquad (29)
\end{aligned}$$

Let $\mathrm{GL}_2(F) = BK$, where B consists of upper triangular matrices and $K = \mathrm{GL}_2(\mathbb{O}_F)$. The matrix coefficient w_Π is a zonal spherical function and satisfies

$$\int_K w_\Pi(XkY)\, dk = w_\Pi(X) w_\Pi(Y) \quad \text{for } X, Y \in \mathrm{GL}_2(F).$$

Lemma 4.21. *The expression in* (29) *equals the product of*

$$\int_{\mathrm{GL}_2(F)} \Phi_{M_2(\mathbb{O}_F)}(Y)\chi_1(Y)|\det Y|^{s+1} w_\Pi(Y)\, dY$$

and

$$\int_{\mathrm{GL}_2(F)} \Phi_{M_2(\mathbb{O}_F)}(X)\chi_2(X)|\det X|^{s+1} w_\Pi(X)\, dX.$$

Proof. First, it is obvious that

$$\begin{aligned}
(29) = \int_{\mathrm{GL}_2(F)\times \mathrm{GL}_2(F)\times K} &\Phi_{M_{2\times 4}(\mathbb{O}_F)}(kY, Z)\chi_1(\det kY)|\det kY|^{s+1} \\
&\cdot \chi_2(\det Z)|\det Z|^{s+1} w_\Pi(Z^{-1}Y)\, dY\, dZ\, dk. \qquad (30)
\end{aligned}$$

Then, substitute kY by Y':

$$
\begin{aligned}
(30) &= \int_{\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)} \Phi_{M_{2\times 4}(\mathbb{O}_F)}(Y', Z)\chi_1(\det Y')|\det Y'|^{s+1}\chi_2(\det Z)|\det Z|^{s+1} \\
&\qquad\qquad \cdot \int_K w_\Pi(Z^{-1}k^{-1}Y')\,dk\,dY'\,dZ \\
&= \int_{\mathrm{GL}_2(F)} \Phi_{M_2(\mathbb{O}_F)}(Y)\chi_1(Y)|\det Y|^{s+1} w_\Pi(Y)\,dY \\
&\qquad\qquad \cdot \int_{\mathrm{GL}_2(F)} \Phi_{M_2(\mathbb{O}_F)}(Z)\chi_2(Z)|\det Z|^{s+1} w_\Pi(Z^{-1})\,dZ.
\end{aligned}
$$

So this proves Lemma 4.21. □

Let

$$Z(\Phi_{M_2(\mathbb{O}_F)}, \chi_1 \otimes w_\Pi, s+1) = \int_{\mathrm{GL}_2(F)} \Phi_{M_2(\mathbb{O}_F)}(Y)\chi_1(Y)|\det Y|^{s+1} w_\Pi(Y)\,dY.$$

Zeta integrals such as this are discussed in [Godement and Jacquet 1972].

Lemma 4.22. *If* $\Pi = \pi(\mu_1, \mu_2)$, *then*

$$Z(\Phi_{M_2(\mathbb{O}_F)}, \chi_1 \otimes w_\Pi, s+1) = L\big(\chi_1\mu_1, s+\tfrac{1}{2}\big)L\big(\chi_2\mu_2, s+\tfrac{1}{2}\big).$$

Proof. Let $Y = pk$, for $p = \left(\begin{smallmatrix} a_1 & b \\ & a_2 \end{smallmatrix}\right) \in B$ and $k \in K$. Then $dY = (1/|a_1|)\,db\,da_1\,da_2$. $w_\Pi(Y) = w_\Pi(p) = \mu_1(a_1)\mu_2(a_2)(|a_1|/|a_2|)^{\frac{1}{2}}$. So,

$$
\begin{aligned}
Z(\Phi_{M_2(\mathbb{O}_F)}, \chi_1 \otimes w_\Pi, s+1) &= \int_{\mathbb{O}_F\times\mathbb{O}_F} \chi_1\mu_1(a_1)|a_1|^{s+\frac{1}{2}}\chi_1\mu_2(a_2)|a_2|^{s+\frac{1}{2}}\,da_1\,da_2 \\
&= L\big(\chi_1\mu_1, s+\tfrac{1}{2}\big)L\big(\chi_2\mu_2, s+\tfrac{1}{2}\big).
\end{aligned}
$$

□

By Lemmas 4.21 and 4.22, we have

$$\int_{U(\zeta)} \mathfrak{f}_s(\mathfrak{e}(u,1))\langle \Pi(u)\beta, \beta\rangle\,du = \frac{L\big(\Pi, \chi, s+\frac{1}{2}\big)}{L(\epsilon_{K/F}, 2s+2)L(\epsilon_{K/F}, 2s+1)}.$$

And notice that in this case, $\epsilon_{K/F}(\varpi) = 1$ for the prime element ϖ in F. This proves Proposition 4.19 when $\mathrm{U}(\zeta) \simeq \mathrm{GL}_2(F)$. □

Next, let me explain the local integrals of $|\theta^{\beta}_{\phi_2}(1)|^2$ at archimedean places and ramified finite places. At archimedean places, $f_{\delta_\psi(\phi_2\otimes\bar{\phi}_2),s,\infty}$ is $\mathrm{U}(\zeta)_\infty \times U(\zeta)_\infty$-invariant, and Π_∞ is one-dimensional. Apparently,

$$\int_{U(\zeta)_\infty} f_{\delta_\psi(\phi_2\otimes\bar{\phi}_2),s,\infty}(\mathfrak{e}(g,1))\langle \Pi(g)\beta, \beta\rangle_\infty\,dg = \mathrm{vol}(U(\zeta)_\infty) \neq 0.$$

As for finite places, first we are always allowed to take the standard characteristic function $\phi_{2,v}$ when Π_v is unramified. This means if Π_v is unramified, we can guarantee $f_{\delta_\psi(\phi_2\otimes\bar{\phi}_2),s,v}$ is spherical.

If Π_v is ramified at a finite place v, let $D_v \subset U(\zeta)_v$ be a compact open subgroup fixing Π_v. Especially, when $\Pi_v = I(\mu_1, \mu_2)$ is a ramified principal series associated to the local characters μ_1 and μ_2, we assume that μ_1 is ramified and μ_2 is unramified. Then $D_v = \left\{ g \in U(\zeta)_v \;\middle|\; g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), a \in 1 + \mathfrak{m}, b, d \in \mathcal{O}_{K_v}, c \in \mathfrak{m} \right\}$, where $\mathfrak{m}$ is the conductor of μ_1. We can choose $\phi_{2,v}$ so that $f_{\delta_\psi(\phi_2\otimes\bar{\phi}_2),s,v}(\mathfrak{e}(g, 1)) = \Phi_{D_v}(g)$, where Φ_{D_v} is the characteristic function with the support in D_v. For instance, if v splits, $\mathrm{U}(\zeta)_v \simeq \mathrm{GL}_2(F_v)$. Then μ_1 and μ_2 are both characters of F_v. We can take ϕ_2 such that

$$\phi_2(x) = \begin{cases} 1 & \text{if } x \in \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) + \mathfrak{m}\mathcal{O}_{F_v}^2, \\ 0 & \text{otherwise.} \end{cases}$$

In this way,

$$\int_{U(\zeta)_v} f_{\delta_\psi(\phi_2\otimes\bar{\phi}_2),s,v}(\mathfrak{e}(g, 1)) \langle \Pi(g)\beta, \beta \rangle_v \, dg = \mathrm{vol}(D_v) \neq 0.$$

5. Applications to Eisenstein series nonvanishing modulo *p*

Section 4 calculated the Fourier–Jacobi coefficients of two Eisenstein series. One is the holomorphic Siegel–Eisenstein series $\mathbb{E}_k(\,\cdot\,, \eta)$ on U(3, 3). The other is the holomorphic Eisenstein series $E_k(\,\cdot\,, \Pi, \eta)$ on $\mathrm{U}(I_\zeta)$ that is from the pullback of $\mathbb{E}_k(\,\cdot\,, \eta)$. From now on, we fix a prime p, so that every prime in F above p is unramified for both Eisenstein series. In this section, we will briefly discuss how to apply the computations of Section 4 to look for an Eisenstein series on U(3, 1) that does not vanish modulo p.

Remark 5.1. (1) Recall that $\mathbb{E}_k(\,\cdot\,, \eta)$ is defined from the section $\mathbb{f}_k$ that is spherical outside S (for the definition of S, one may refer to Theorem 4.9). In the application, S is usually taken to be the set of places where the given data (for instance the number fields F and K, the characters η and χ, the representation Π and so on) are ramified. Notice that we can assume that all the data at p are unramified. So $p \notin S$. In other words, $\mathbb{f}_k$ is spherical at p.

(2) $\mathbb{E}_k(\,\cdot\,, \eta)$ can be normalized to become a p-integral Eisenstein series $\mathbb{E}^{\mathrm{int}}$. About this claim, one can refer to (3.3.5.3) and Remark (3.3.5.5) in [Harris et al. 2006]. The normalizing factor is given in (3.3.5.1). In fact, the situation here is simpler than [ibid.], because $\mathbb{E}_k(\,\cdot\,, \eta)$ is unramified at p.

(3) The pullback of $\mathbb{E}^{\text{int}}$ denoted by $\mathbb{E}^{\text{int}}_{\text{pb}}$ is an Eisenstein series on $\mathrm{U}(I_\zeta)$. It only differs by a constant with $E(\,\cdot\,, \Pi, \eta)$. Define

$$E^{\text{int}} = \frac{\mathbb{E}^{\text{int}}_{\text{pb}}}{\Omega_K^k}, \tag{31}$$

where Ω_K is the CM-period of K and it is well-defined up to $\overline{\mathbb{Z}}_p^\times$.

Lemma 5.2. *E^{int} is a p-integral holomorphic Eisenstein series.*

About Ω_K and this lemma, one can refer to [Hsieh 2011b, Section 7.2].

As mentioned in the introduction, one of the motivations of this paper is to provide a possible way to argue nonvanishing modulo p of the Eisenstein series on the unitary group U(3, 1) used in the Iwasawa theory through the calculation of its Fourier–Jacobi coefficients. For the discussion on this topic, let me assume that $F = \mathbb{Q}$ and that the imaginary quadratic extension $K/\mathbb{Q}$ splits at p.

Following the idea of Skinner and Urban [Skinner and Urban 2012; Urban 2006] to show one divisibility of the main conjecture for $\mathrm{GL}_2 \times K^\times$ by the method of Eisenstein congruence on U(3, 1), a Hida family of holomorphic Eisenstein series $\mathscr{E}^{\text{ord}}$ on U(3, 1) is constructed so that its constant terms at all cusps are divisible by the p-adic L-function of $\mathrm{GL}_2 \times K^\times$. Suppose $\mathscr{E}^{\text{ord}}$ is defined over a two-variable Iwasawa algebra Λ (refer to Conjecture 1.1), and denote by $\mathfrak{m}_\Lambda$ the maximal ideal of Λ. It is required that $\mathscr{E}^{\text{ord}} \not\equiv 0 \pmod{\mathfrak{m}_\Lambda}$. Since $\mathscr{E}^{\text{ord}}$ is obtained by interpolating a p-ordinary holomorphic Eisenstein series E^{ord}, we have:

Lemma 5.3. *If $E^{\text{ord}} \not\equiv 0 \pmod{\mathfrak{m}_p}$, then $\mathscr{E}^{\text{ord}} \not\equiv 0 \pmod{\mathfrak{m}_\Lambda}$, where $\mathfrak{m}_p$ is the maximal ideal of $\overline{\mathbb{Z}}_{(p)}$ induced by $i : \mathbb{C} \to \mathbb{C}_p$.*

So it is enough to show E^{ord} does not vanish modulo $\mathfrak{m}_p$. For the strict definition and construction of E^{ord}, see [Hsieh 2011b]. Although E^{ord} and E^{int} are both p-integral holomorphic Eisenstein series, they are not the same, because we assume E^{int} is unramified at p, but E^{ord} is ramified at p. So in order to apply the computation of Fourier–Jacobi coefficients of E^{int} in Section 4, two points will be addressed in this section. The first is to relate E^{ord} used in the proof of the main conjecture to E^{int}. The second is to give a strategy of showing nonvanishing modulo p of E^{int}.

At p, $\mathrm{U}(\zeta)(\mathbb{Q}_p) \simeq \mathrm{GL}_2(\mathbb{Q}_p)$. In [Hsieh 2011b], the representation Π of $\mathrm{U}(\zeta)$ is chosen so that the local representation Π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ is ordinary; then Π_p must be of the type $\pi(\mu_1, \mu_2)$. Assume that the characters μ_i of $\mathbb{Q}_p$ are unramified. Then an Eisenstein series E_0 is defined with the data of Π and η, so that E^{ord} is exactly the ordinary projection of E_0 ($E^{\text{ord}} = eE_0$, where e is the map of ordinary projection). The only difference between E_0 considered in [ibid.] and E^{int} in (31) is that local sections at p are different. A special section at p is taken to make sure the corresponding E_0 leads to the ordinary p-adic Eisenstein series with the optimal constant terms at cusps. But, for E^{int}, the local section at p is spherical.

Define normalized actions of U_p and T_p operators ($\parallel U_p$ and $\parallel T_p$) to a modular form on $\mathrm{U}(I_\zeta)$; refer to [Hsieh 2011a; 2011b].

Lemma 5.4. *Let E be a p-adic modular form of weight $(0, k)$, $k > 2$. Suppose that E is unramified at p. Then we have*

$$E \parallel U_p(\alpha_i) \equiv E \parallel T_p(\alpha_i) \pmod p,$$
$$\text{for } \alpha_i = \begin{pmatrix} I_{6-i} & 0 \\ 0 & pI_i \end{pmatrix} \in \mathrm{GL}_6(\mathbb{Q}_p) \simeq U(I_\zeta)(\mathbb{Q}_p),\ i = 1, 2, \ldots, 6.$$

The analogous result for modular forms on $\mathrm{U}(2, 1)$ was due to Hida's observation. Hsieh summarized it in [2011a, Lemma 7.3], whose proof is essentially the same as that of Lemma 5.4. I want to emphasize that the formulas of the normalized Hecke operators at p used in the proof are especially for holomorphic forms. So, we can safely apply this lemma to E^{int} and E^{ord}.

Lemma 5.5. $E^{\mathrm{ord}} = eE_0 = C \cdot eE^{\mathrm{int}}$, *where C is a p-adic unit.*

Roughly speaking, the proof is as follows. Suppose the local sections at p for the Eisenstein series E^{ord} and E^{int} are, respectively, f_p^{ord} and f_p^{int}. There is a unique normalized ordinary local section at p, denoted by $f_p^{\mathrm{ord},N}$ (for the uniqueness, see [Hida 2004b; Hsieh 2011b, Remark 6.3]). Then

$$f_p^{\mathrm{ord}} = C_1 f_p^{\mathrm{ord},N} \quad \text{and} \quad ef_p^{\mathrm{int}} = C_2 f_p^{\mathrm{ord},N}.$$

Moreover it can be shown that $C = C_1 C_2^{-1}$ is a p-adic unit. Combining the two lemmas above, we get

$$E^{\mathrm{ord}} \not\equiv 0 \pmod{\mathfrak{m}_p} \Longleftrightarrow E^{\mathrm{int}} \not\equiv 0 \pmod{\mathfrak{m}_p}.$$

We are left to show that $E^{\mathrm{int}} \not\equiv 0 \pmod{\mathfrak{m}_p}$.

From the calculation of Section 4, we can construct a linear functional on the space of holomorphic modular forms on $\mathrm{U}(3, 1)$, such that

$$l_\theta(E^{\mathrm{int}}) = \frac{\langle E_\psi^{\mathrm{int}}, \theta \rangle}{\langle \theta, \theta \rangle} \cdot \theta^\beta(1).$$

And notice that E_ψ^{int} and θ are both in the space of holomorphic theta functions, and $\langle \cdot, \cdot \rangle$ is defined to be the inner product in this space. When θ is θ_φ, we have $l_{\theta_\varphi}(E^{\mathrm{int}}) \sim L^{\mathrm{alg}}(\eta\chi^{-1}, l - \frac{1}{2}) L^{\mathrm{alg}}(\eta\chi^{-1}, l - \frac{3}{2}) L^{\mathrm{alg}}(\Pi, \chi, 1)$. Notice that χ is from the Weil representation and we can vary χ with only the restrictions that $\chi_\infty(z) = |z|/z$ and $\chi|_{\mathbb{A}^\times} = \epsilon_{K/\mathbb{Q}}$. $L^{\mathrm{alg}}(\eta\chi^{-1}, l - \frac{1}{2})$ and $L^{\mathrm{alg}}(\eta\chi^{-1}, l - \frac{3}{2})$ are both p-units for almost all χ. These facts are due to Hida [2004a]. The question of whether $L^{\mathrm{alg}}(\Pi, \chi, 1)$ is a p-unit or not for infinitely many χ remains an open problem. Many results are known about nonvanishing modulo p of special values of L-functions (refer to [Hida 2004a; Vatsal 2003] for results on GL_1 and GL_2 L-functions). These facts suggest $L^{\mathrm{alg}}(\Pi, \chi, 1)$ may share the same property. Recent work of M. Brakočević on anticyclotomic p-adic L-functions of central critical Rankin–Selberg L-value may help prove the conjecture. I will address this question in my next paper.

From Appendix B, we see by choosing proper φ that θ_φ is a p-integral theta function, and can serve as one element of the basis that spans the space of p-integral holomorphic theta functions $T(m, L, U_f)$. So l_{θ_φ} maps a p-integral modular form in $\mathrm{U}(3, 1)$ to a p-adic integer.

From the discussion above, we have this:

Conjecture 5.6. $E^{\mathrm{int}} \not\equiv 0 \pmod{\mathfrak{m}_\Lambda}$.

Remark 5.7. As mentioned in the introduction, the computations explained in this paper can be generalized to an arbitrary nonquasisplit unitary group $\mathrm{U}(m, n)$ for $m > n$. Let $P = MN$ be the minimal parabolic subgroup of $\mathrm{U}(m, n)$, and $M = \mathrm{GL}_n(K) \times U(m-n)$. Given $\Phi \in I_P^{U(m,n)}(\Pi \otimes \eta(\det \cdot), s)$, define the Eisenstein series $E(\Phi)$ and consider the Fourier–Jacobi coefficients $E_\psi(\Phi)$ as a theta function on the Jacobi group $\mathrm{U}(m-n) \cdot N$. Because $E(\Phi)$ can be written as the pullback of a Siegel–Eisenstein series $\mathbb{E}(\eta)$ on $\mathrm{U}(m, m)$, we can study $E_\psi(\Phi)$ using certain Fourier–Jacobi coefficients of $\mathbb{E}(\eta)$. So $E_\psi(\Phi)$ will have similar expressions as in Corollary 4.11. Again using the inner product of the space of theta functions on $\mathrm{U}(m-n) \cdot N$, we can define a linear functional on $E(\Phi)$ that leads to special L-values.

Appendix A: Proof of the Siegel–Weil formula for $(U(2, 2), U(1))$

The Siegel–Weil formula for the dual reductive pair $(U(2, 2), U(1))$ is formulated in Theorem 3.6. Let us first fix necessary notation and then give the proof.

Consider the dual reductive pair $(U(2, 2), U(V))$, where V is a Hermitian vector space of dimension 1 over K. Now $\mathrm{U}(2, 2) \times U(V)$ acts on $S(V^2(\mathbb{A}))$ via the Weil representation ω_χ determined by an additive character ψ of $\mathbb{A}$ and a character χ of $\mathbb{A}_K^\times / K^\times$ such that $\chi|_{\mathbb{A}^\times} = \epsilon_{K/F}$. For $\phi \in S(V^2(\mathbb{A}))$, we have $f_{\phi,s} \in I_{\mathcal{P}'}^{U(2,2)}(\chi, s)$, where $\mathcal{P}'$ is the Siegel parabolic subgroup of $\mathrm{U}(2, 2)$ (refer to (17)). Theorem 3.6 states that the Eisenstein series $E(g, f_{\phi,s})$ on $\mathrm{U}(2, 2)$ is holomorphic at $s = -\frac{1}{2}$ and

$$E(g, f_{\phi,s})|_{s=-\frac{1}{2}} = 2 \int_{[U(V)]} \theta_\phi(g, u)\, du.$$

I give the proof of this theorem following the idea of Kudla and Rallis [1988], who proved the Siegel–Weil formula for $(\mathrm{Sp}_n, \mathrm{O}(V))$ when V is anisotropic. First, we prove that $E(g, f_{\phi,s})$ is holomorphic at $s = -\frac{1}{2}$ by studying the analytic properties of the constant term. Denote $\int_{[U(V)]}\theta_\phi(g,u)\,du$ by $I(g,\phi)$. Notice that $I(g,\phi)$ is orthogonal to all cusp forms on $\mathrm{U}(2,2)(\mathbb{A})$ and is moreover concentrated on the Borel subgroup B in the sense that the constant term of $I(g,\phi)$ along any parabolic subgroup P strictly containing B is orthogonal to all cusp forms on the Levi factor of P. Since the same is true for $E(g, f_{\phi,s})$, it suffices to show that the constant term of $E(g, f_{\phi,s})$ along the Siegel parabolic subgroup at $s=-\frac{1}{2}$ is equal to that of $I(g,\phi)$.

Denote the constant term of $E(g, f_{\phi,s})$ with respect to $\mathcal{P}'$ by $E_{\mathcal{P}'}(g, f_{\phi,s})$. Then we have this:

Lemma A.1. *For any $\mathcal{K}'$-finite section f, $E(g, f_s)$ and $E_{\mathcal{P}'}(g, f_s)$ have the same set of poles.*

If ϕ is $\mathcal{K}'$-finite, so is f_ϕ. We can apply the lemma above to find the poles of $E(g, f_{\phi,s})$. There are three terms in $E_{\mathcal{P}'}(g, f_{\phi,s})$, that is

$$E_{\mathcal{P}'}(g, f_{\phi,s}) = E^0_{\mathcal{P}'}(g, f_{\phi,s}) + E^1_{\mathcal{P}'}(g, f_{\phi,s}) + E^2_{\mathcal{P}'}(g, f_{\phi,s}), \tag{32}$$

corresponding to the Bruhat decomposition

$$\mathrm{U}(2,2)(F) = \mathcal{P}' \sqcup \mathcal{P}' w_{13}\mathcal{P}' \sqcup \mathcal{P}' w_2 \mathcal{P}',$$

where

$$w_{13} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}.$$

We have $E^0_{\mathcal{P}'}(g, f_{\phi,s}) = f_{\phi,s}(g)$. So $E^0_{\mathcal{P}'}(g, f_{\phi,s})|_{s=-\frac{1}{2}} = \omega_\chi(g)\phi(0)$. The third term has

$$E^2_{\mathcal{P}'}(g, f_{\phi,s}) = \int_{\mathcal{N}'(\mathbb{A})} f_{\phi,s}(w_2 ng)\,dn = M(s) f_{\phi,s}(g),$$

where $M(s)$ is the intertwining operator. The second term has

$$E^1_{\mathcal{P}'}(g, f_{\phi,s}) = \sum_{\gamma\in B_1\backslash \mathrm{GL}_2(K)} f^1_{\phi,s}(\gamma g), \tag{33}$$

where B_1 is the Borel subgroup of $\mathrm{GL}_2(K)$ and

$$f^1_{\phi,s}(\gamma g) = \int_{\mathcal{N}''(\mathbb{A})} f_{\phi,s}(w_{13} ng)\,dn,$$

where $\mathcal{N}''$ is a subgroup of $\mathcal{N}'$ such that for $n\in\mathcal{N}''$,

$$n = \begin{pmatrix} 1 & & x & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Since the Levi part of $\mathcal{P}'(F)$ is isomorphic to $\mathrm{GL}_2(K)$, we see $E^1_{\mathcal{P}'}(g, f_{\phi,s})|_{\mathrm{GL}_2(\mathbb{A}_K)}$ is an Eisenstein series on $\mathrm{GL}_2(\mathbb{A}_K)$ associated to the section

$$f^1_{\phi,s} \in I^{\mathrm{GL}_2}_{B^1}\big(\chi|\cdot|_K^{-s-\frac{1}{2}}, \chi|\cdot|_K^{s+\frac{3}{2}}\big),$$

that is,

$$f^1_{\phi,s}\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g\right) = \chi(a)|a|_K^{-s-\frac{1}{2}}\chi(b)|b|_K^{s+\frac{3}{2}}\left|\frac{a}{b}\right|_K^{\frac{1}{2}} f^1_{\phi,s}(g). \tag{34}$$

Let us consider the analytic property of $M(s)f_{\phi,s}(g)$. It is well known that $M(s)$ is well-defined for s if $\mathrm{Re}\, s$ is big enough and it has the meromorphic continuation to the complex plane. Let S be a finite set of places of F such that $f_{\phi,s,v} = f_{0,s,v}$ is spherical for all $v \notin S$. Then we have

$$M(s)f_{\phi,s} = \frac{a(s)}{b(s)} \prod_{v\in S} \frac{b(s)_v}{a(s)_v} M(s)_v f_{\phi,s,v} \prod_{v\notin S} f_{0,-s,v},$$

where

$$\frac{a(s)}{b(s)} = \frac{\xi(2s-1,1)\xi(2s,\epsilon_{K/F})}{\xi(2s+2,1)\xi(2s+1,\epsilon_{K/F})};$$

specifically,

$$\frac{a(s)_v}{b(s)_v} = \frac{L_v(2s-1,1)L_v(2s,\epsilon_{K/F})}{L_v(2s+2,1)L_v(2s+1,\epsilon_{K/F})}$$

for $v \neq \infty$, and

$$\frac{a(s)_\infty}{b(s)_\infty} = c(s)\,\frac{\Gamma(2s-1)\Gamma(2s+1)}{\Gamma(2s+2)\Gamma(2s+2)}, \tag{35}$$

where $c(s)$ is some exponential, which will not affect the analytic property of $M(s)f_{\phi,s}$.

Theorem A.2. $M(s)f_{\phi,s}|_{s=-\frac{1}{2}} = 0$

Proof. Notice that

$$\frac{a(s)}{b(s)}\Big|_{s=-\frac{1}{2}} = 0.$$

So we only need to show that

$$\frac{b(s)_v}{a(s)_v} M(s)_v f_{\phi,s,v}$$

for $v \in S$ at $s = -\frac{1}{2}$ is holomorphic.

When $v \in S$ and v is a finite place, we always have this:

Lemma A.3. $\dfrac{1}{a(s)_v} M(s)_v f_{s,v}$ *is holomorphic at* $s = -\frac{1}{2}$ *for any* f_s.

The general form of the lemma is stated in [Kudla and Rallis 1988]. If $v \in S$ is an inert place, $b(s)_v$ is obviously holomorphic at $s = -\frac{1}{2}$. So

$$\frac{b(s)_v}{a(s)_v} M(s)_v f_{\phi,s,v}$$

is holomorphic at $s = -\frac{1}{2}$.

If $v \in S$ is a splitting place, pay attention that now $b(s)_v$ has a simple pole. We need a refinement of Lemma A.3 as follows:

Lemma A.4. *If v is a splitting place,*

$$\frac{1}{a(s)_v} M(s)_v f_{\phi,s,v}$$

vanishes at $s = -\frac{1}{2}$.

I have to emphasize that this lemma is only right for the Siegel–Weil section. Then in this case, we still have

$$\frac{b(s)_v}{a(s)_v} M(s)_v f_{\phi,s,v}$$

is holomorphic at $s = -\frac{1}{2}$. I omit the proof of Lemma A.4. It is a direct corollary of [Kudla and Sweet 1997, Theorem 1.3].

For $v = \infty$, for convenience, we take $\phi^0_\infty(x) = \psi(ixx^*)$ as it is what we use in the paper. For general functions in $S(V^2)_\infty$ in the space of the Fock representation, the holomorphic result still holds. We have that ϕ^0_∞ is an eigenfunction under the action of an element k in the maximal open compact subgroup of $\mathrm{U}(2,2)_\infty$. Then,

Lemma A.5. $M(s)_\infty f_{\phi^0_\infty,s} = c \dfrac{\Gamma(2s)\Gamma(2s-1)}{\Gamma(s+\frac{3}{2})\Gamma(s+\frac{1}{2})^2\Gamma(s-\frac{1}{2})} f_{\phi^0_\infty,-s}.$

From (35) and Lemma A.5, we can see that

$$\frac{b(s)_\infty}{a(s)_\infty} M(s)_\infty f_{\phi^0_\infty,s}$$

is holomorphic at $s = -\frac{1}{2}$. Combining the discussions above at each place, we prove Theorem A.2. □

Now let us consider (33), which is the second term of the constant term of $E(g, f_{\phi,s})$. As we see, restricted to the Levi part of $\mathrm{U}(2,2)$, it is an Eisenstein series associated to the (34). In [Kudla and Rallis 1988, Proposition 6.4], the Eisenstein series like this was discussed and the holomorphic property was confirmed. The idea is to obtain a relation of the form

$$E^1_{\mathcal{P}'}(g, f_{\phi,s}) = \sum_j \alpha_j(s) E^1(g, F_j(s)), \tag{36}$$

where $\alpha_j(s)$ is a meromorphic function holomorphic at $s=-\frac{1}{2}$, and $E^1(g, F_j(s))$ is an Eisenstein series on $\mathrm{GL}_2(\mathbb{A}_K)$ and

$$F_j(g,s)=\mu_1(\det g)|\det g|_K^{\frac{1}{2}}\int_{\mathbb{A}_K^\times}\phi_j((0,x)g)\mu_1\mu_2^{-1}(\det g)|\det g|_K\,dx$$

for $\phi_j\in S(M_{12}(\mathbb{A}_K))$, and $\mu_1=\chi|\cdot|_K^{-s-\frac{1}{2}}$, $\mu_2=\chi|\cdot|_K^{s+\frac{3}{2}}$. The analytic property of this type of Eisenstein series is explicitly worked out. In our case, each $E^1(\cdot\,,F_j(s))$ is holomorphic at $s=-\frac{1}{2}$. Furthermore, because the sum in (36) is a finite sum, $E^1_{\mathcal{P}'}(g, f_{\phi,s})$ is also holomorphic at $s=-\frac{1}{2}$.

Theorem A.6. *$E(g, f_{\phi,s})$ is holomorphic at $s=-\frac{1}{2}$.*

It is not hard to verify that $I(g,\phi)$ as an automorphic form on $\mathrm{U}(2,2)$ is orthogonal to all cusp forms on $\mathrm{U}(2,2)$ and is concentrated on the Borel subgroup B. Since the constant term of $I(g,\phi)$ with respect to the Siegel parabolic subgroup $\mathcal{P}'$ is $\omega_\chi(g)\phi(0)$, the only thing remaining to prove the Siegel–Weil formula is to confirm the constant term of $E(g, f_{\phi,s})$ at $s=-\frac{1}{2}$ is $2\omega_\chi(g)\phi(0)$.

Let us first calculate the constant term $E_{\mathcal{P}'_1}(g, f_{\phi,s})$ of $E(g, f_{\phi,s})$ with respect to the parabolic subgroup $\mathcal{P}'_1$ whose Levi factor is isomorphic to $K^\times\times U(1,1)$. Let

$$i: U(1,1)\to U(2,2),\qquad \left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)\mapsto\begin{pmatrix}1&&&\\&a&&b\\&&1&\\&c&&d\end{pmatrix}.$$

Then:

Proposition A.7. *For $g\in U(1,1)$,*

$$E_{\mathcal{P}'_1}(g, f_{\phi,s})=E^0_{\mathcal{P}'_1}\big(g, s+\tfrac{1}{2}, i^*f_{\phi,s}\big)+E^1_{\mathcal{P}'_1}\big(g,\tfrac{1}{2}-s, i^*M(s)f_{\phi,s}\big),\tag{37}$$

where $E^0_{\mathcal{P}'_1}$ and $E^1_{\mathcal{P}'_1}$ are both Eisenstein series on $\mathrm{U}(1,1)$.

Notice that $E^0_{\mathcal{P}'_1}$ is an Eisenstein series associated to the Siegel–Weil section $i^*f_{\phi,s}$. Applying the result about the Siegel–Weil formula for the dual reductive pair $(U(1,1),U(1))$ [Ichino 2004], we see that the constant term of

$$E^0_{\mathcal{P}'_1}(g, s+\tfrac{1}{2}, i^*f_{\phi,s})$$

with respect to the Siegel parabolic subgroup of $\mathrm{U}(1,1)$ has two terms and they are both equal to $\omega_\chi(i(g))\phi(0)$ for $g\in U(1,1)$ when $s=-\frac{1}{2}$.

First, $E^1_{\mathcal{P}'_1}\big(g,\frac{1}{2}-s, i^*M(s)f_{\phi,s}\big)$ is holomorphic at $s=-\frac{1}{2}$ by arguments similar to those in [Kudla and Rallis 1988, Section 8]. Then applying Theorem A.2, we have this Eisenstein series is zero at $s=-\frac{1}{2}$.

It seems that too many constant terms are involved. The relation can be described by the following diamond graph used in [ibid.]:

$$\begin{tikzcd} & U(2,2) \arrow[dl] \arrow[dr] & \\ \mathrm{GL}_1 \times U(1,1) \arrow[dr] & & \mathrm{GL}_2 \arrow[dl] \\ & \mathrm{GL}_1 \times \mathrm{GL}_1 & \end{tikzcd}$$

From the line

$$\mathrm{U}(2,2) \to \mathrm{GL}_1 \times U(1,1) \to \mathrm{GL}_1 \times \mathrm{GL}_1,$$

we get $E^{00}_{\mathcal{P}'_1}$, $E^{01}_{\mathcal{P}'_1}$, $E^{10}_{\mathcal{P}'_1}$, and $E^{11}_{\mathcal{P}'_1}$. They are constant terms of $E^{0}_{\mathcal{P}'_1}$ and $E^{1}_{\mathcal{P}'_1}$. From the line

$$\mathrm{U}(2,2) \to \mathrm{GL}_2 \to \mathrm{GL}_1 \times \mathrm{GL}_1,$$

we get $E^{0}_{\mathcal{P}'}$, $E^{10}_{\mathcal{P}'}$, $E^{11}_{\mathcal{P}'}$, and $E^{2}_{\mathcal{P}'}$ (refer to (32)), where $E^{10}_{\mathcal{P}'}$ and $E^{11}_{\mathcal{P}'}$ are the constant terms of $E^{1}_{\mathcal{P}'}$. Restricting all these to $\mathrm{GL}_1 \times \mathrm{GL}_1$, we have the following match up relation:

$$\begin{array}{cccc} E^{00}_{\mathcal{P}'_1} & E^{01}_{\mathcal{P}'_1} & E^{10}_{\mathcal{P}'_1} & E^{11}_{\mathcal{P}'_1} \\ E^{0}_{\mathcal{P}'} & E^{10}_{\mathcal{P}'} & E^{11}_{\mathcal{P}'} & E^{2}_{\mathcal{P}'} \end{array}$$

The top terms match up with the bottom terms. Because $E^{1}_{\mathcal{P}'_1}$ is zero at $s = -\frac{1}{2}$, then we have that $E^{11}_{\mathcal{P}'}$ is zero at $s = -\frac{1}{2}$.

Proposition A.8. $\quad E^{1}_{\mathcal{P}'}(g, f_{\phi,s})|_{s=-\frac{1}{2}} = \omega_\chi(g)\phi(0).$

Thus, we obtain that $E_{\mathcal{P}'}(g, f_{\phi,s})|_{s=-\frac{1}{2}} = 2\omega_\chi(g)\phi(0)$. Then Theorem 3.6 is proved.

Appendix B: Integral theta functions

First recall classical definitions of theta functions. For applications, let us restrict to the 2-dimensional case here. Most of the results can be generalized to arbitrary dimensions without difficulty. Fix the embeddings

$$\overline{\mathbb{Q}} \xrightarrow{i_\infty} \mathbb{C} \quad \text{and} \quad \overline{\mathbb{Q}} \xrightarrow{i_p} \mathbb{C}_p.$$

Let V be a 2-dimensional Hermitian vector space over K. Choose an $\mathbb{O}_K$-lattice L so that it fixes an abelian variety $\mathcal{A}_L$ with complex multiplication defined over a number field M, where $K \subset M \subset \overline{\mathbb{Q}}$. Under the embedding i_∞, L can be regarded as a $\mathbb{Z}$-module of rank 4, and so a lattice of $\mathbb{C}^2$. Then there exists an analytic parametrization over $\mathbb{C}$ such that

$$\mathcal{A}_L \otimes_{i_\infty} \mathbb{C} \simeq \mathbb{C}^2/L.$$

For a Riemann form H on $\mathbb{C}^2/L$, and a map $\epsilon : L \to U$, where U is the unit circle of $\mathbb{C}$, define an analytic line bundle $\mathfrak{L}^{\mathrm{an}}_{H,\epsilon} \simeq \mathbb{C} \times \mathbb{C}^2/L$ with the action of L given by

$$l \cdot (w, x) = \Big(w + l, \epsilon(l) e\Big(\frac{1}{2i} H\Big(l, w + \frac{l}{2}\Big)\Big) x\Big) \quad \text{for } l \in L, (w, x) \in \mathbb{C}^2 \times \mathbb{C},$$

where $e(x) = e^{2\pi i x}$. Then the space of global sections $\Gamma(\mathbb{C}^2/L, \mathfrak{L}^{\mathrm{an}}_{H,\epsilon})$ can be identified with the space of holomorphic theta functions $T(H, \epsilon, L)$ such that for $f \in T(H, \epsilon, L)$,

$$f(w + l) = f(w)\epsilon(l) e\Big(\frac{1}{2i} H\Big(l, w + \frac{l}{2}\Big)\Big) \quad \text{for } w \in \mathbb{C}^2, l \in L.$$

To study arithmetic theta functions and furthermore integral theta functions inside $T(H, \epsilon, L)$, let us consider the line bundle $\mathfrak{L}_{H,\epsilon}$ on $\mathscr{A}_L$ defined over M, and give $\mathfrak{L}_{H,\epsilon}$ a rigidification at the origin. At ∞, fix an isomorphism

$$\mathfrak{L}_{H,\epsilon} \otimes_{i_\infty} \mathbb{C} \simeq \mathfrak{L}^{\mathrm{an}}_{H,\epsilon},$$

such that it is consistent with the analytic parametrization of $\mathscr{A}_L$, and carries the rigidification of $\mathfrak{L}_{H,\epsilon}$ into the canonical one of $\mathfrak{L}^{\mathrm{an}}_{H,\epsilon}$. For a prime p, when $\mathscr{A}_L \otimes_{i_p} \mathbb{C}_p$ has a good reduction, we can assume it is defined over the ring of integers $\mathbb{O} = \mathbb{O}(\mathbb{C}_p)$. Then we require that the rigidification of $\mathfrak{L}_{H,\epsilon}$ satisfies that the p-integral elements of the stalk of $\mathfrak{L}_{H,\epsilon}$ over the origin correspond to the p-integral points on the affine line.

Within the context above, we see that $i_\infty(\Gamma(\mathscr{A}_L, \mathfrak{L}_{H,\epsilon}))$ is an $i_\infty(M)$-vector space of theta functions inside $\Gamma(\mathbb{C}^2/L, \mathfrak{L}^{\mathrm{an}}_{H,\epsilon}) = T(H, \epsilon, L)$. Let us denote this space by $T^{\mathrm{ar}}(H, \epsilon, L)$; each element inside is an arithmetic theta function. Inside the $\mathbb{C}_p$-vector space $\Gamma(\mathscr{A}_L \otimes_{i_p} \mathbb{C}_p, \mathfrak{L}_{H,\epsilon} \otimes_{i_p} \mathbb{C}_p)$, we have the $\mathbb{O}$-module of integral sections. It is in turn the $i_\infty(i_p^{-1}(\mathbb{O}) \cap M)$-module of p-integral theta functions in $i_\infty(\Gamma(\mathscr{A}_L, \mathfrak{L}_{H,\epsilon})) = T^{\mathrm{ar}}(H, \epsilon, L)$. We give it a notation $T^{\mathrm{int}}(H, \epsilon, L)$. Thus we have

$$T^{\mathrm{int}}(H, \epsilon, L) \subset T^{\mathrm{ar}}(H, \epsilon, L) \subset T(H, \epsilon, L).$$

For $f \in T(H, \epsilon, L)$, define

$$f_*(w) = e\Big(\frac{i}{4} H(w, w)\Big) f(w).$$

The following lemma gives simple characterizations of arithmetic theta functions and integral theta functions.

Lemma B.1. *$T^{\mathrm{ar}}(H, \epsilon, L)$ consists of all functions $f \in T(H, \epsilon, L)$ such that $f_*(w)$ is an algebraic number in $\mathbb{C}$ for any $w \in KL$. The module $T^{\mathrm{int}}(H, \epsilon, L)$ of p-integral theta functions consists of $f \in T(H, \epsilon, L)$ such that $i_p \cdot i_\infty^{-1}(f_*(w))$ is integral in $\mathbb{C}_p$ for $w \in KL$.*

For the proof, see the proof [Finis 2006, Lemma 3.1]. Even though, there, only the case of one-dimensional Hermitian space was considered, the idea can be easily generalized here.

Shimura [1976] studied the arithmetic properties of theta functions with complex multiplication, and gave a basis for $T^{\mathrm{ar}}(H, \epsilon, L)$. Let me mainly discuss the case when V/K is a 2-dimensional anisotropic Hermitian space. Also, V can be regarded as a symplectic vector space with the alternating form

$$\langle v_1, v_2\rangle = v_1\zeta v_2^* - v_2\zeta v_1^*,$$

with $v_1\zeta v_2^*$ the anti-Hermitian pairing on V. There exist $w_1, w_2 \in M_2(K)$, such that $z = w_2^{-1}w_1$ is a point in the Siegel upper half plane $\mathfrak{H}_2$, and

$$L = \left\{ (a\ b)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \,\middle|\, a \in \mathbb{Z}^2, b \in \mathbb{Z}^2\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \text{ for a fixed } \alpha \in \mathbb{Z} \right\}.$$

Then the Riemann form on V can be written as

$$H(uw_2, vw_2) = 2im \cdot u(z-\bar{z})^{-1}v^* \quad \text{for} \quad u, v \in K^2, \tag{38}$$

and ϵ is of the form

$$\epsilon(aw_1 + bw_2) = e\Big(\frac{m}{2}\cdot ab^t + br^t + as^t\Big) \quad \text{for } a \in \mathbb{Z}^2,\ b \in \mathbb{Z}^2\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix},$$

with a choice of r and s in $\mathbb{Q}^2$. For $u \in \mathbb{C}^2$, $z \in \mathfrak{H}_2$, $r, s \in \mathbb{R}^2$, and a positive integer μ, define

$$\theta_{r,s}(u, z) = e\big(\tfrac{1}{2}u(z-\bar{z})^{-1}u^t\big)\sum_{x\in\mathbb{Z}^2} e\big(\tfrac{1}{2}(x+r)z(x+r)^t + (x+r)(u+s)^t\big).$$

Put $f_{m,r,s}(u) = \theta_{r,s}(muw_2^{-1}, mw_2^{-1}w_1)$.

Theorem B.2. (1) *For $r, s \in \mathbb{Q}^2$, we have $f_{m,r,s}(u) \in T^{\mathrm{ar}}(H, \epsilon, L)$ for H defined in* (38) *and ϵ defined by*

$$\epsilon(aw_1 + bw_2) = e\Big(\tfrac{1}{2}m\cdot ab^t + mbr^t - as^t\Big).$$

Moreover, the functions

$$f_{m,r+j,s} \quad \textit{for } j \in m^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\mathbb{Z}^2/\mathbb{Z}^2$$

give a basis of $T^{\mathrm{ar}}(H, \epsilon, L)$ over the field of algebraic numbers in $\mathbb{C}$.

(2) *Let p be an unramified prime number for $T(H, \epsilon, L)$. When z is diagonal, $f_{m,r,s}$ is a p-integral element in $T^{\mathrm{int}}(H, \epsilon, L)$. Thus $T^{\mathrm{int}}(H, \epsilon, L)$ is spanned by the functions $f_{m,r+j,s}$ over $i_\infty(i_p^{-1}(\mathbb{O}))$.*

Proof. See [Shimura 1976, Proposition 2.5] for the proof of (1). For (2), though it is possible to remove the condition that z is diagonal, I will skip the discussion of the more general results because the proof would be more technical. When $z = \left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right)$ is diagonal,

$$\theta_{r,s}(u, z) = \prod_{i=1}^{2} \theta_{r_i,s_i}(u_i, z_i),$$

where

$$\theta_{r_i,s_i}(u_i, z_i) = e\big(\tfrac{1}{2}u_i(z_i - \bar{z}_i)^{-1}u_i\big) \sum_{x\in\mathbb{Z}} e\big(\tfrac{1}{2}(x+r_i)z_i(x+r_i) + (x+r_i)(u_i+s_i)\big),$$

for $r = (r_1, r_2)$ and so on. Notice that these $\theta_{r_i,s_i}(u_i, z_i)$ are essentially theta functions in the one-dimensional case. The p-integrality of them is confirmed by [Finis 2006, Lemma 3.3]. Thus, the p-integrality of the $f_{m,r+j,s}$ follows. □

Remark B.3. When L corresponds to a diagonal element in $\mathfrak{H}_2$, $\mathcal{A}_L$ is isomorphic to the product of two elliptic curves. In this case, the $i_\infty(i_p^{-1}(\mathbb{O}))$-basis of $T^{\mathrm{int}}(H, \epsilon, L)$ is given in Theorem B.2(2), and the number of elements inside is exactly equal to the dimension of $T^{\mathrm{ar}}(H, \epsilon, L)$. In general, when $\mathcal{A}_L$ is an abelian variety with CM, it is always isogenous to $E_1 \times E_2$, where E_1 and E_2 are elliptic curves with CM. Such isogeny induces the isomorphism between modules of integral theta functions of $\mathcal{A}_L$ and $E_1 \times E_2$ with appropriate choices of Riemann forms accordingly.

Then let me briefly explain the relation between classical theta functions and adelic theta functions. One can find the discussion of one-dimensional case in [Finis 2006]. Define $T_{\mathbb{A}}(m, L, U_f)$ to be the space of all smooth functions

$$\Theta : N(\mathbb{Q})U(\zeta)(\mathbb{Q})\diagdown N(\mathbb{A})U(\zeta)(\mathbb{A})\diagup U(\zeta)_\infty U_f N(L)_f \to \mathbb{C},$$

where U_f is a certain open compact subgroup of $\mathrm{U}(\zeta)$ at finite places such that the level is prime to p, and

$$N(L)_f = \big\{(w, t) \;\big|\; w \in \hat{L},\, t + \tfrac{1}{2}w\zeta w^* \in \mu(L)\hat{\mathbb{O}}_K\big\},$$

where $\mu(L)$ is the ideal generated by $w\zeta w^*$ for all $w \in L$. The function Θ satisfies

$$\Theta((0, t)r) = e(mt)\Theta(r) \quad \text{for } r \in N(\mathbb{A})U(\zeta)(\mathbb{A}).$$

Because $\mathrm{U}(\zeta)$ is anisotropic, $\mathrm{U}(\zeta)(\mathbb{Q})\diagdown U(\zeta)(\mathbb{A})\diagup U(\zeta)_\infty U_f$ consists of a finite set of points $\{u_1, \dots, u_s\} \subset U(\zeta)(\mathbb{A}_f)$. So we have

$$T_{\mathbb{A}}(m, L, U_f) = \textstyle\bigoplus_{i=1}^{s} T(m, u_i L), \quad \Theta \mapsto (\Theta_i)$$

such that $\Theta_i(n) = \Theta(nu_i)$for $n \in N(\mathbb{A})$. Then one may check for each i,

$$\theta_i(w_\infty) = e\Big(-m\frac{w\zeta w^*}{2}\Big)\Theta_i((w_\infty, 0))$$

is a classical theta function in $T(H, \epsilon, u_i L)$, where H and ϵ are defined according to the lattice $u_i L$, and

$$\langle \Theta, \Theta' \rangle = \sum_i \langle \theta_i, \theta_i' \rangle \quad \text{for } \Theta, \Theta' \in T_{\mathbb{A}}(m, L, U_f),$$

where

$$\langle \Theta, \Theta' \rangle = \int_{N(\mathbb{Q})U(\zeta)(\mathbb{Q})\diagdown N(\mathbb{A})U(\zeta)(\mathbb{A})} \Theta(r)\overline{\Theta'(r)}\, dr$$

and

$$\langle \theta_i, \theta_i' \rangle = \frac{1}{\mu(L)} \int_{\mathbb{C}/L} \theta_i(w)\overline{\theta_i'(w)}\, dw.$$

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The phase limit set of a variety

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A coamoeba is the image of a subvariety of a complex torus under the argument map to the real torus. We describe the structure of the boundary of the coamoeba of a variety, which we relate to its logarithmic limit set. Detailed examples of lines in three-dimensional space illustrate and motivate these results.

1. Introduction

A coamoeba is the image of a subvariety of a complex torus under the argument map to the real torus. Coamoebae are cousins to amoebae, which are images of subvarieties under the coordinatewise logarithm map $z \mapsto \log|z|$. Amoebae were introduced by Gelfand, Kapranov, and Zelevinsky [1994] and have subsequently been widely studied [Kenyon et al. 2006; Mikhalkin 2000; Passare and Rullgård 2004; Purbhoo 2008]. Coamoebae were introduced by Passare in a talk in 2004, and they appear to have many beautiful and interesting properties. For example, coamoebae of $\mathscr{A}$-discriminants in dimension 2 are unions of two nonconvex polyhedra [Nilsson and Passare 2010], and a hypersurface coamoeba has an associated arrangement of codimension-1 tori contained in its closure [Nisse 2009].

Bergman [1971] introduced the logarithmic limit set $\mathscr{L}^\infty(X)$ of a subvariety X of the torus as the set of limiting directions of points in its amoeba. Bieri and Groves [1984] showed that $\mathscr{L}^\infty(X)$ is a rational polyhedral complex in the sphere. Logarithmic limit sets are now called tropical algebraic varieties [Speyer and Sturmfels 2004]. For a hypersurface $\mathscr{V}(f)$, logarithmic limit set $\mathscr{L}^\infty(\mathscr{V}(f))$ consists of the directions of nonmaximal cones in the outer normal fan of the Newton polytope of f. We introduce a similar object for coamoebae and establish a structure theorem for coamoebae similar to those of Bergman and of Bieri and Groves for amoebae.

Let co$\mathscr{A}(X)$ be the coamoeba of a subvariety X of $(\mathbb{C}^*)^n$ with ideal I. The *phase limit set* $\mathscr{P}^\infty(X)$ of X is the set of accumulation points of arguments of sequences in X with unbounded logarithm. For $w \in \mathbb{R}^n$, the initial variety $\mathrm{in}_w X \subset (\mathbb{C}^*)^n$ is the variety of the initial ideal of I. The fundamental theorem of tropical geometry

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asserts that $\mathrm{in}_w X \neq \varnothing$ exactly when the direction of $-w$ lies in $\mathscr{L}^\infty(X)$. We establish its analog for coamoebae.

Theorem 1. *The closure of* $\mathrm{co}\mathscr{A}$ *is* $\mathrm{co}\mathscr{A}(X) \cup \mathscr{P}^\infty(X)$, *and*

$$\mathscr{P}^\infty(X) = \bigcup_{w \neq 0} \mathrm{co}\mathscr{A}(\mathrm{in}_w X).$$

Johansson [2013] used different methods to prove this when X is a complete intersection.

The cone over the logarithmic limit set admits the structure of a rational polyhedral fan Σ in which all weights w in the relative interior of a cone $\sigma \in \Sigma$ give the same initial scheme $\mathrm{in}_w X$. Thus, the union in Theorem 1 is finite and indexed by the images of these cones σ in the logarithmic limit set of X. The logarithmic limit set or tropical algebraic variety is a combinatorial shadow of X encoding many properties of X. While the coamoeba of X is typically not purely combinatorial (see the examples of lines in $(\mathbb{C}^*)^3$ in Section 3), the phase limit set does provide a combinatorial skeleton and that we believe will be useful in the further study of coamoebae.

We give definitions and background in Section 2 and detailed examples of lines in three-dimensional space in Section 3. These examples are reminiscent of the concrete descriptions of amoebae of lines in [Theobald 2002]. We prove Theorem 1 in Section 4.

2. Coamoebae, tropical varieties, and initial ideals

As a real algebraic group, the set $\mathbb{T} := \mathbb{C}^*$ of invertible complex numbers is isomorphic to $\mathbb{R} \times \mathbb{U}$ under the map $(r, \theta) \mapsto e^{r+\sqrt{-1}\theta}$. Here, $\mathbb{U}$ is the set of complex numbers of norm 1 that may be identified with $\mathbb{R}/2\pi\mathbb{Z}$. The inverse map is $z \mapsto (\log|z|, \arg(z))$.

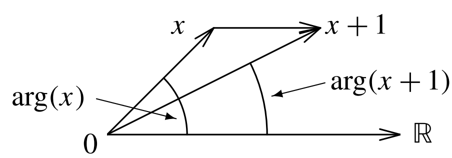
Let M be a free abelian group of finite rank and $N = \mathrm{Hom}(M, \mathbb{Z})$ its dual group. We use $\langle \cdot, \cdot \rangle$ for the pairing between M and N. The group ring $\mathbb{C}[M]$ is the ring of Laurent polynomials with exponents in M. It is the coordinate ring of a torus $\mathbb{T}_N$ that is identified with $N \otimes_{\mathbb{Z}} \mathbb{T} = \mathrm{Hom}(M, \mathbb{T})$, the set of group homomorphisms $M \to \mathbb{T}$. There are natural maps $\mathrm{Log}\colon \mathbb{T}_N \to \mathbb{R}_N = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $\arg\colon \mathbb{T}_N \to \mathbb{U}_N = N \otimes_{\mathbb{Z}} \mathbb{U}$, which are induced by the maps $\mathbb{C}^* \ni z \mapsto \log|z|$ and $z \mapsto \arg(z) \in \mathbb{U}$. Maps $N \to N'$ of free abelian groups induce corresponding maps $\mathbb{T}_N \to \mathbb{T}_{N'}$ of tori and also of $\mathbb{R}_N$ and $\mathbb{U}_N$. If n is the rank of N, we may identify N with $\mathbb{Z}^n$, which identifies $\mathbb{T}_N$ with $\mathbb{T}^n$, $\mathbb{U}_N$ with $\mathbb{U}^n$, and $\mathbb{R}_N$ with $\mathbb{R}^n$.

The *amoeba* $\mathscr{A}(X)$ of a subvariety $X \subset \mathbb{T}_N$ is its image under $\mathrm{Log}\colon \mathbb{T}_N \to \mathbb{R}_N$, and the *coamoeba* $\mathrm{co}\mathscr{A}(X)$ of X is the image of X under $\arg\colon \mathbb{T}_N \to \mathbb{U}_N$. An amoeba has a geometric-combinatorial structure at infinity encoded by the logarithmic limit set [Bergman 1971; Bieri and Groves 1984]. Coamoebae similarly have phase limit sets that have a related combinatorial structure that we define and study in Section 4.

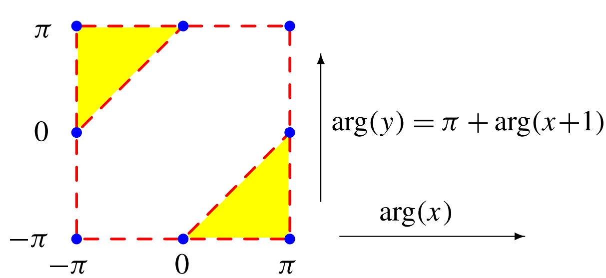
If we identify $\mathbb{C}^*$ with $\mathbb{R}^2 \setminus \{(0,0)\}$, then the map $\arg\colon \mathbb{C}^* \to \mathbb{U}$ given by $(a,b) \mapsto (a,b)/\sqrt{a^2+b^2}$ is a real algebraic map. Thus, coamoebae, as they are the image of a real algebraic subset of the real algebraic variety $\mathbb{T}_N$ under the real algebraic map arg, are semialgebraic subsets of $\mathbb{U}_N$ [Basu et al. 2006]. It would be very interesting to study them as semialgebraic sets; in particular, what are the equations and inequalities satisfied by a coamoeba? When X is a Grassmannian, such a description would generalize Richter-Gebert's five-point condition for phirotopes from rank 2 to arbitrary rank [Below et al. 2003].

Similarly, we may replace the map $\mathbb{C}^* \ni z \mapsto \log|z| \in \mathbb{R}$ in the definition of amoebae by the map $\mathbb{C}^* \ni z \mapsto |z| \in \mathbb{R}^+ := \{r \in \mathbb{R} \mid r > 0\}$ to obtain the *algebraic amoeba* of X, which is a subset of $\mathbb{R}^+_N$. The algebraic amoeba is a semialgebraic subset of $\mathbb{R}^+_N$, and we also ask for its description as a semialgebraic set.

Example 2. Let $\ell \subset \mathbb{T}^2$ be defined by $x+y+1=0$. The coamoeba $co\mathscr{A}(\ell)$ is the set of points of $\mathbb{U}^2$ of the form $(\arg(x), \pi+\arg(x+1))$ for $x \in \mathbb{C}\setminus\{0,-1\}$. If x is real, then these points are $(\pm\pi, 0)$, $(\pm\pi, \pm\pi)$, and $(0, \pm\pi)$ if x lies in the intervals $(-\infty,-1)$, $(-1,0)$, and $(0,\infty)$, respectively. For other values, consider the picture below in the complex plane:



For $\arg(x) \notin \{0,\pi\}$ fixed, $\pi+\arg(x+1)$ can take on any value strictly between $\pi+\arg(x)$ (for w near ∞) and 0 (for x near 0), and thus, $co\mathscr{A}(\ell)$ consists of the three points $(\pi,0)$, (π,π), and $(0,\pi)$ and the interiors of the two triangles displayed below in the fundamental domain $[-\pi,\pi]^2 \subset \mathbb{R}^2$ of $\mathbb{U}^2$. This should be understood modulo 2π so that $\pi = -\pi$.



(1)

The coamoeba is the complement of the region

$$\big\{(\alpha,\beta) \in [-\pi,\pi]^2 \;\big|\; |\alpha-\beta| \le \pi = \arg(-1)\big\}$$

together with the three images of real points $(\pm\pi, 0)$, $(\pm\pi, \pm\pi)$, and $(0, \pm\pi)$.

Given a general line $ax + by + c = 0$ with $a, b, c \in \mathbb{C}^*$, we may replace x by cx'/a and y by cy'/b to obtain the line $x' + y' + 1 = 0$ with coamoeba (1). This transformation rotates the coamoeba (1) by $\arg(a/c)$ horizontally and $\arg(b/c)$ vertically.

Let $f \in \mathbb{C}[M]$ be a polynomial with support $\mathscr{A} \subset M$

$$f := \sum_{\boldsymbol{m}\in\mathscr{A}} c_{\boldsymbol{m}} \cdot \xi^{\boldsymbol{m}}, \quad \text{where } c_{\boldsymbol{m}} \in \mathbb{C}^*, \tag{2}$$

and we write $\xi^{\boldsymbol{m}}$ for the element of $\mathbb{C}[M]$ corresponding to $\boldsymbol{m} \in M$. Given $w \in \mathbb{R}_N$, let $w(f)$ be the minimum of $\langle \boldsymbol{m}, w\rangle$ for $\boldsymbol{m} \in \mathscr{A}$. Then the initial form $\mathrm{in}_w f$ of f with respect to $w \in \mathbb{R}_N$ is the polynomial $\mathrm{in}_w f \in \mathbb{C}[M]$ defined by

$$\mathrm{in}_w f := \sum_{\langle \boldsymbol{m}, w\rangle = w(f)} c_{\boldsymbol{m}} \cdot \xi^{\boldsymbol{m}}.$$

Given an ideal $I \subset \mathbb{C}[M]$ and $w \in \mathbb{R}_N$, the *initial ideal* with respect to w is

$$\mathrm{in}_w I := \langle \mathrm{in}_w f \mid f \in I\rangle \subset \mathbb{C}[M].$$

Lastly, when I is the ideal of a subvariety X, the *initial scheme* $\mathrm{in}_w X \subset \mathbb{T}_N$ is defined by the initial ideal $\mathrm{in}_w I$.

The sphere $\mathbb{S}_N := (\mathbb{R}_N \setminus \{0\})/\mathbb{R}^+$ is the set of directions in $\mathbb{R}_N$. Let $\pi : \mathbb{R}_N \setminus \{0\} \to \mathbb{S}_N$ be the projection. The *logarithmic limit set* $\mathscr{L}^\infty(X)$ of a subvariety X of $\mathbb{T}_N$ is the set of accumulation points in $\mathbb{S}_N$ of sequences $\{\pi(\mathrm{Log}(x_n))\}$, where $\{x_n\} \subset X$ is an unbounded set. A sequence $\{x_n\} \subset \mathbb{T}_N$ is unbounded if its sequence of logarithms $\{\mathrm{Log}(x_n)\}$ is unbounded.

A *rational polyhedral cone* $\sigma \subset \mathbb{R}_N$ is the set of points $w \in \mathbb{R}_N$ that satisfy finitely many inequalities and equations of the form

$$\langle \boldsymbol{m}, w\rangle \geq 0 \quad \text{and} \quad \langle \boldsymbol{m}', w\rangle = 0,$$

where $\boldsymbol{m}, \boldsymbol{m}' \in M$. The *dimension* of σ is the dimension of its linear span, and *faces* of σ are proper subsets of σ obtained by replacing some inequalities by equations. The relative interior of σ consists of its points not lying in any face. Also, σ is determined by $\sigma \cap N$, which is a finitely generated subsemigroup of N.

A *rational polyhedral fan* Σ is a collection of rational polyhedral cones in $\mathbb{R}_N$ in which every two cones of Σ meet along a common face.

Theorem 3. *The cone in $\mathbb{R}_N$ over the negative $-\mathscr{L}^\infty(X)$ of the logarithmic limit set of X is the set of $w \in \mathbb{R}_N$ such that $\mathrm{in}_w X \neq \varnothing$. Equivalently, it is the set of $w \in \mathbb{R}_N$ such that for every $f \in \mathbb{C}[M]$ lying in the ideal I of X, $\mathrm{in}_w f$ is not a monomial. This cone over $-\mathscr{L}^\infty(X)$ admits the structure of a rational polyhedral* *fan Σ with the property that if u and w lie in the relative interior of a cone σ of Σ, then* $\mathrm{in}_u I = \mathrm{in}_w I$.

It is important to take $-\mathscr{L}^\infty(X)$. This is correct as we use the tropical convention of minimum, which is forced by our use of toric varieties to prove Theorem 1 in Section 4.2.

We write $\mathrm{in}_\sigma I$ for the initial ideal defined by points in the relative interior of a cone σ of Σ. The fan structure Σ is not canonical for it depends upon an identification $M \xrightarrow{\sim} \mathbb{Z}^n$. Moreover, it may be the case that $\sigma \neq \tau$ but $\mathrm{in}_\sigma I = \mathrm{in}_\tau I$.

Bergman [1971] defined the logarithmic limit set of a subvariety of the torus $\mathbb{T}_N$, and Bieri and Groves [1984] showed it was a finite union of convex polyhedral cones. The connection to initial ideals was made more explicit through work of Kapranov [2006], and the form above is adapted from Speyer and Sturmfels [2004]. The logarithmic limit set of X is now called the tropical algebraic variety of X, and this latter work led to the field of tropical geometry.

3. Lines in space

We consider coamoebae of lines in three-dimensional space. We will work in the torus $\mathbb{TP}^3$ of $\mathbb{P}^3$, which is the quotient of $\mathbb{T}^4$ by the diagonal torus $\Delta_\mathbb{T}$ and similarly in $\mathbb{UP}^3$, the quotient of $\mathbb{U}^4$ by the diagonal $\Delta_\mathbb{U} := \{(\theta,\theta,\theta,\theta) \mid \theta \in \mathbb{U}\}$. By *coordinate lines and planes* in $\mathbb{UP}^3$, we mean the images in $\mathbb{UP}^3$ of lines and planes in $\mathbb{U}^4$ parallel to some coordinate plane.

Let ℓ be a line in $\mathbb{P}^3$ not lying in any coordinate plane, so ℓ has a parametrization

$$\phi\colon \mathbb{P}^1 \ni [s:t] \mapsto [\ell_0(s,t) : \ell_1(s,t) : \ell_2(s,t) : \ell_3(s,t)], \tag{3}$$

where ℓ_0, ℓ_1, ℓ_2, and ℓ_3 are nonzero linear forms that do not all vanish at the same point. For $i = 0, \ldots, 3$, let $\zeta_i \in \mathbb{P}^1$ be the zero of ℓ_i. The configuration of these zeroes determine the coamoeba of $\ell \cap \mathbb{TP}^3$, which we will simply write as $co\mathscr{A}(\ell)$.

Suppose that two zeroes coincide; say $\zeta_3 = \zeta_2$. Then $\ell_3 = a\ell_2$ for some $a \in \mathbb{C}^*$, and so ℓ lies in the translated subtorus $z_3 = az_2$, and its coamoeba $co\mathscr{A}(\ell)$ lies in the coordinate subspace of $\mathbb{U}_3$ defined by $\theta_3 = \arg(a) + \theta_2$. In fact, $co\mathscr{A}(\ell)$ is pulled back from the coamoeba of the projection of ℓ to the $\theta_3 = 0$ plane. It follows that if there are only two distinct roots among $\zeta_0, \ldots, \zeta_3$, then $co\mathscr{A}(\ell)$ is a coordinate line of $\mathbb{U}_3$. If three of the roots are distinct, then (up to a translation) the projection of the coamoeba $co\mathscr{A}(\ell)$ to the $\theta_3 = 0$ plane looks like (1) so that $co\mathscr{A}(\ell)$ consists of two triangles lying in a coordinate plane.

For each $i = 0, \ldots, 3$, define a function depending upon a point $[s:t] \in \mathbb{P}^1$ and $\theta \in \mathbb{U}$ by

$$\varphi_i(s,t;\theta) := \begin{cases} \theta & \text{if } \ell_i(s,t) = 0, \\ \arg(\ell_i(s,t)) & \text{otherwise.} \end{cases}$$

For each $i = 0, \ldots, 3$, let h_i be the image in $\mathbb{UP}^3$ of $\mathbb{U}$ under the map

$$\theta \mapsto [\varphi_0(\zeta_i, \theta), \varphi_1(\zeta_i, \theta), \varphi_2(\zeta_i, \theta), \varphi_3(\zeta_i, \theta)].$$

Lemma 4. *For each $i = 0, \ldots, 3$, h_i is a coordinate line in $\mathbb{UP}^3$ that consists of accumulation points of* $\mathrm{co}\mathscr{A}(\ell)$.

This follows from Theorem 1. For the main idea, note that $\arg \circ \phi(\zeta_i + \epsilon e^{\theta\sqrt{-1}})$ for $\theta \in \mathbb{U}$ is a curve in $\mathbb{UP}^3$ whose Hausdorff distance to the line h_i approaches 0 as $\epsilon \to 0$. The *phase limit set* of ℓ is the union of these four lines.

Lemma 5. *Suppose that the zeroes ζ_0, ζ_1, and ζ_2 are distinct. Then*

$$\mathbb{P}^1 \setminus \{\zeta_0, \zeta_1, \zeta_2\} \ni x \mapsto \arg(\ell_0(x), \ell_1(x), \ell_2(x)) \in \mathbb{U}^3/\Delta_{\mathbb{U}} = \mathbb{UP}^2$$

is constant along each arc of the circle in $\mathbb{P}^1$ through ζ_0, ζ_1, and ζ_2.

Proof. After changing coordinates in $\mathbb{P}^1$ and translating in $\mathbb{UP}^2$ (rotating coordinates), we may assume that these roots are ∞, 0, and -1, and so the circle becomes the real line. Choosing affine coordinates, we may assume that $\ell_0 = 1$, $\ell_1 = x$, and $\ell_2 = x + 1$ so that we are in the situation of Example 2. Then the statement of the lemma is the computation there for x real in which we obtained the coordinate points $(\pi, 0)$, (π, π), and $(0, \pi)$. $\square$

Lemma 6. *The phase limit lines h_0, h_1, h_2, and h_3 are disjoint if and only if the roots $\zeta_0, \ldots, \zeta_3$ do not all lie on a circle.*

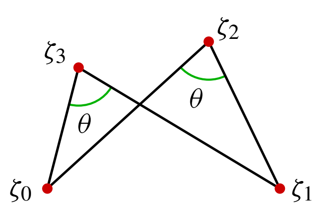
Proof. Suppose that two of the limit lines meet, say h_0 and h_1. Without loss of generality, we suppose that we have chosen coordinates on $\mathbb{U}^4$ and $\mathbb{P}^1$ so that $\zeta_i \in \mathbb{C}$ and $\ell_i(x) = x - \zeta_i$ for $i = 0, \ldots, 3$. Then there are points $\alpha, \beta, \theta \in \mathbb{U}$ such that

$$\begin{aligned}&(\varphi_0(\zeta_0, \alpha), \varphi_1(\zeta_0, \alpha), \varphi_2(\zeta_0, \alpha), \varphi_3(\zeta_0, \alpha))\\ &\qquad = (\varphi_0(\zeta_1, \beta), \varphi_1(\zeta_1, \beta), \varphi_2(\zeta_1, \beta), \varphi_3(\zeta_1, \beta)) + (\theta, \theta, \theta, \theta).\end{aligned}$$

Comparing the last two components, we obtain

$$\arg(\zeta_0 - \zeta_2) = \arg(\zeta_1 - \zeta_2) + \theta \quad \text{and} \quad \arg(\zeta_0 - \zeta_3) = \arg(\zeta_1 - \zeta_3) + \theta,$$

and so the zeroes $\zeta_0, \ldots, \zeta_3$ have the configuration below:



But then $\zeta_0, \dots, \zeta_3$ are cocircular. Conversely, if $\zeta_0, \dots, \zeta_3$ lie on a circle C, then by Lemma 5, the lines h_i and h_j meet only if ζ_i and ζ_j are the endpoints of an arc of $C \setminus \{\zeta_0, \dots, \zeta_3\}$. $\square$

Lemma 7. *If the roots $\zeta_0, \dots, \zeta_3$ do not all lie on a circle, then the map*

$$\arg \circ \phi \colon \mathbb{P}^1 \setminus \{\zeta_0, \zeta_1, \zeta_2, \zeta_3\} \to \mathbb{UP}^3$$

is an immersion.

Proof. Let $x \in \mathbb{P}^1 \setminus \{\zeta_0, \zeta_1, \zeta_2, \zeta_3\}$, which we consider to be a real two-dimensional manifold. After possibly reordering the roots, the circle C_1 containing x, ζ_0, and ζ_1 meets the circle C_2 containing x, ζ_2, and ζ_3 transversally at x. Under the derivative of the map $\arg \circ \phi$, tangent vectors at x to C_1 and C_2 are taken to nonzero vectors $(0, 0, u_1, v_1)$ and $(u_2, v_2, 0, 0)$ in the tangent space to $\mathbb{U}^4$. Furthermore, as the four roots do not all lie on a circle, we cannot have both $u_1 = v_1$ and $u_2 = v_2$, and so this derivative has full rank two at x as a map from $\mathbb{P}^1 \setminus \{\zeta_0, \zeta_1, \zeta_2, \zeta_3\} \to \mathbb{UP}^3$, which proves the lemma. $\square$

By these lemmas, there is a fundamental difference between the coamoebae of lines when the roots of the linear forms ℓ_i are cocircular and when they are not. We examine each case in detail. First, choose coordinates so that $\zeta_0 = \infty$. After dehomogenizing and separately rescaling each affine coordinate (e.g., identifying $\mathbb{UP}^3$ with $\mathbb{U}^3$ and applying phase shifts to each coordinate $\theta_1, \theta_2, \theta_3$ of $\mathbb{U}^3$), we may assume that the map (3) parametrizing ℓ is

$$\phi \colon \mathbb{C} \ni x \mapsto (x - \zeta_1, x - \zeta_2, x - \zeta_3) \in \mathbb{C}^3. \tag{4}$$

Suppose first that the four roots are cocircular. As $z_0 = \infty$, the other three lie on a real line in $\mathbb{C}$, which we may assume is $\mathbb{R}$. That is, if the four roots are cocircular, then up to coordinate change, we may assume that the line ℓ is real and the affine parametrization (4) is also real. For this reason, we will call such lines ℓ *real lines*. We first study the boundary of $co\mathscr{A}(\ell)$. Suppose that x lies on a contour C in the upper half plane as in Figure 1 that contains semicircles of radius ϵ centered at each root and a semicircle of radius $1/\epsilon$ centered at 0 but otherwise lies along the real

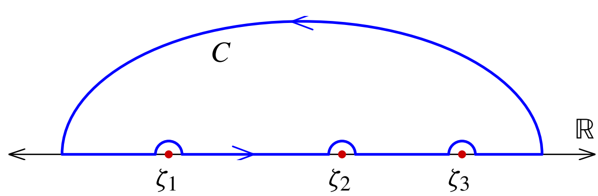


Figure 1. Contour in upper half plane.

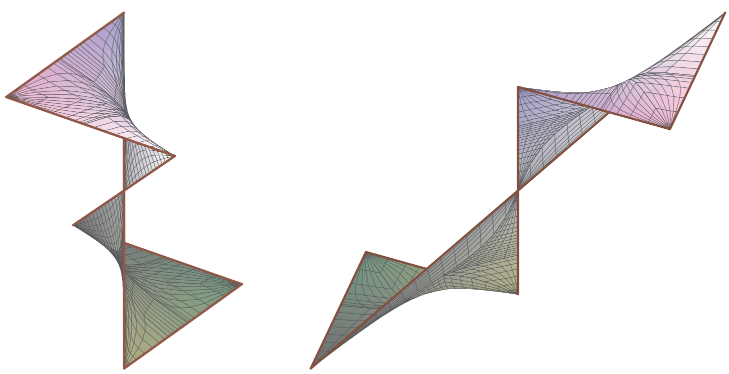
axis for ϵ, a sufficiently small positive number. Then $\arg(\phi(w)) \in \mathbb{U}^3$ is constant on the four segments of C lying along $\mathbb{R}$ with respective values

$$(\pi, \pi, \pi), \quad (0, \pi, \pi), \quad (0, 0, \pi), \quad \text{and} \quad (0, 0, 0), \tag{5}$$

moving from left to right. On the semicircles around ζ_1, ζ_2, and ζ_3, two of the coordinates are essentially constant (but not quite equal to either 0 or π!) while the third decreases from π to 0. Finally, on the large semicircle, the three coordinates are nearly equal and increase from $(0, 0, 0)$ to (π, π, π). The image $\arg(\phi(C))$ can be made as close as we please to the quadrilateral in $\mathbb{U}^3$ connecting the points of (5) in cyclic order when ϵ is sufficiently small. Thus, the image of the upper half plane under the map $\arg \circ \phi$ is a relatively open membrane in $\mathbb{U}^3$ that spans the quadrilateral. It lies within the convex hull of this quadrilateral, which is computed using the affine structure induced from $\mathbb{R}^3$ by the quotient $\mathbb{U}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$.

For this, observe that its projection in any of the four coordinate directions parallel to its edges is one of the triangles of the coamoeba of the projected line in $\mathbb{CP}^2$ of Example 2, and the convex hull of the quadrilateral is the intersection of the four preimages of these triangles.

Because ℓ is real, the image of the lower half plane is isomorphic to the image of the upper half plane under the map $(\theta_1, \theta_2, \theta_3) \mapsto (-\theta_1, -\theta_2, -\theta_3)$, and so the coamoeba is symmetric in the origin of $\mathbb{U}^3$ and consists of two quadrilateral patches that meet at their vertices. Here are two views of the coamoeba of the line where the roots are ∞, $-1/2$, 0, and $3/2$:



Now suppose that the roots $\zeta_0, \ldots, \zeta_3$ do not all lie on a circle. By Lemma 6, the four phase limit lines $h_1, \ldots, h_3$ are disjoint, and the map from ℓ to the coamoeba is an immersion. Figure 2 shows two views of the coamoeba in a fundamental domain of $\mathbb{UP}^3$ when the roots are ∞, 1, ζ, and ζ^2, where ζ is a primitive third root of infinity. This and other pictures of coamoebae of lines are animated on the website [Nisse and Sottile 2010].

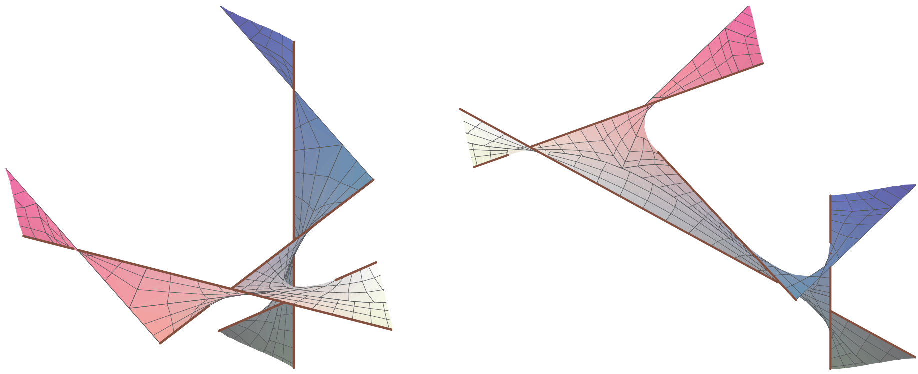


Figure 2. Two views of the coamoeba of a symmetric line.

The projection of this coamoeba along a coordinate direction (parallel to one of the phase limit lines h_i) gives a coamoeba of a line in $\mathbb{TP}^2$ as we saw in Example 2. The line h_i is mapped to the interior of one triangle, and the vertices of the triangles are the images of line segments lying on the coamoeba. These three line segments come from the three arcs of the circle through the three roots other than ζ_i, the root corresponding to h_i.

Proposition 8. *The interior of the coamoeba of a general line in $\mathbb{TP}^3$ contains twelve line segments in triples parallel to each of the four coordinate directions.*

The symmetric coamoeba we show in Figure 2 has six additional line segments, two each coming from the three longitudinal circles through a third root of unity and 0 and ∞. Two such segments are visible as pinch points in the leftmost view in Figure 2. We ask, What is the maximal number of line segments on a coamoeba of a line in $\mathbb{TP}^3$?

4. Structure of the phase limit set

The *phase limit set* $\mathscr{P}^\infty(X)$ of a complex subvariety $X \subset \mathbb{T}_N$ is the set of all accumulation points of sequences $\{\arg(x_n) \mid n \in \mathbb{N}\} \subset \mathbb{U}_N$, where $\{x_n \mid n \in \mathbb{N}\} \subset X$ is an unbounded sequence. For $w \in N$, $\operatorname{in}_w X \subset \mathbb{T}_N$ is the (possibly empty) initial scheme of X, whose ideal is the initial ideal $\operatorname{in}_w I$, where I is the ideal of X. Our main result, Theorem 1, is that the phase limit set of X is the union of the coamoebae of all its initial schemes.

Remark 9. The union of Theorem 1 is finite. By Theorem 3, $\operatorname{in}_w X$ is nonempty only when w lies in the cone over the logarithmic set $\mathscr{L}^\infty(X)$, which can be given the structure of a finite union of rational polyhedral cones such that any two points in the relative interior of the same cone σ have the same initial scheme. If we write $\operatorname{in}_\sigma X$ for the initial scheme corresponding to a cone σ, the torus $\mathbb{T}_{\langle\sigma\rangle} \simeq (\mathbb{C}^*)^{\dim\sigma}$

acts on $\mathrm{in}_\sigma X$ by translation (e.g., see Corollary 13). (Here, $\langle\sigma\rangle \subset N$ is the span of $\sigma \cap N$, a free abelian group of rank $\dim\sigma$.) This implies $\mathrm{co}\mathscr{A}(\mathrm{in}_\sigma X)$ is a union of orbits of $\mathrm{co}\mathscr{A}(\mathbb{T}_{\langle\sigma\rangle}) = \mathbb{U}_{\langle\sigma\rangle}$ and thus that $\dim(\mathrm{co}\mathscr{A}(\mathrm{in}_\sigma X)) \leq 2\dim(X) - \dim(\sigma)$.

This discussion implies the following proposition:

Proposition 10. *Let $X \subset \mathbb{T}_N$ be a subvariety, and suppose that $\mathbb{T}_X \subset \mathbb{T}_N$ is the largest subtorus acting on X. Then* $\dim \mathrm{co}\mathscr{A}(X) \leq \min\{\dim \mathbb{T}_N, 2\dim X - \dim \mathbb{T}_X\}$.

We prove Theorem 1 in the next two subsections.

4.1. ***Coamoebae of initial schemes.*** We review the standard dictionary relating initial ideals to toric degenerations in the context of subvarieties of $\mathbb{T}_N$ [Gelfand et al. 1994, Chapter 6]. Let $X \subset \mathbb{T}_N$ be a subvariety with ideal $I \subset \mathbb{C}[M]$. We study $\mathrm{in}_w I$ and the initial schemes $\mathrm{in}_w X = \mathscr{V}(\mathrm{in}_w I) \subset \mathbb{T}_N$ for $w \in N$. Since $\mathrm{in}_0 I = I$ so that $\mathrm{in}_0 X = X$, we may assume that $w \neq 0$. As N is the lattice of one-parameter subgroups of $\mathbb{T}_N$, w corresponds to a one-parameter subgroup written as $\mathbb{C}^* \ni t \mapsto t^w \in \mathbb{T}_N$. Define $\mathscr{X} \subset \mathbb{C} \times \mathbb{T}_N$ by

$$\mathscr{X} := \{(t, x) \in \mathbb{C}^* \times \mathbb{T}_N \mid t^w \cdot x \in X\}. \tag{6}$$

The fiber of $\mathscr{X}$ over a point $t \in \mathbb{C}^*$ is $t^{-w}X$. Let $\overline{\mathscr{X}}$ be the closure of $\mathscr{X}$ in $\mathbb{C} \times \mathbb{T}_N$, and set X_0 to be the fiber of $\overline{\mathscr{X}}$ over $0 \in \mathbb{C}$.

Proposition 11. $X_0 = \mathrm{in}_w X$.

Proof. We first describe the ideal $\mathscr{I}$ of $\mathscr{X}$. For $\boldsymbol{m} \in M$, the element $\xi^{\boldsymbol{m}} \in \mathbb{C}[M]$ takes the value $t^{\langle \boldsymbol{m}, w\rangle} \in \mathbb{C}^*$ on the element $t^w \in \mathbb{T}_N$, and so if $x \in \mathbb{T}_N$, then $\xi^{\boldsymbol{m}}$ takes the value $t^{\langle \boldsymbol{m}, w\rangle}\xi^{\boldsymbol{m}}(x) = t^{\langle \boldsymbol{m}, w\rangle}\boldsymbol{m}(x)$ on $t^w x$. Given a polynomial $f \in \mathbb{C}[M]$ of the form

$$f := \sum_{\boldsymbol{m}\in\mathscr{A}} c_{\boldsymbol{m}}\xi^{\boldsymbol{m}} \quad \text{for } c_{\boldsymbol{m}} \in \mathbb{C}^*,$$

define the polynomial $f(t) \in \mathbb{C}[t, t^{-1}][M]$ by

$$f(t) := \sum_{\boldsymbol{m}\in\mathscr{A}} c_{\boldsymbol{m}} t^{\langle \boldsymbol{m}, w\rangle}\xi^{\boldsymbol{m}}. \tag{7}$$

Then $f(t)(x) = f(t^w x)$, so $\mathscr{I}$ is generated by the polynomials $f(t)$ of (7) for $f \in I$. A general element of $\mathscr{I}$ is a linear combination of translates $t^a f(t)$ of such polynomials for $a \in \mathbb{Z}$.

If we set $w(f)$ to be the minimal exponent of t occurring in $f(t)$, then

$$\mathrm{in}_w f = \sum_{\langle \boldsymbol{m}, w\rangle = w(f)} c_{\boldsymbol{m}}\xi^{\boldsymbol{m}},$$

and

$$t^{-w(f)} f(t) = \mathrm{in}_w f + \sum_{\langle \boldsymbol{m}, w\rangle > w(f)} t^{\langle \boldsymbol{m}, w\rangle - w(f)} c_{\boldsymbol{m}}\xi^{\boldsymbol{m}}.$$

This shows that $\mathcal{I} \cap \mathbb{C}[t][M]$ is generated by polynomials $t^{-w(f)} f(t)$, where $f \in I$. Since $\mathrm{in}_w f \in \mathbb{C}[M]$ and the remaining terms are divisible by t, we see that the ideal of X_0 is generated by $\{\mathrm{in}_w f \mid f \in \mathcal{I}\}$, which completes the proof. $\square$

We use Proposition 11 to prove one inclusion of Theorem 1, namely that

$$\mathcal{P}^\infty(X) \supset \bigcup_{w \in N \setminus \{0\}} \mathrm{co}\mathcal{A}(\mathrm{in}_w X). \tag{8}$$

Fix $0 \neq w \in N$, and let $\mathcal{X}$, $\overline{\mathcal{X}}$, and $X_0 = \mathrm{in}_w X$ be as in Proposition 11, and let $x_0 \in X_0$. We show that $\arg(x_0) \in \mathcal{P}^\infty(X)$. Since $(0, x_0) \in \overline{\mathcal{X}}$, there is an irreducible curve $C \subset \mathcal{X}$ with $(0, x_0) \in \overline{C}$. The projection of $C \subset \mathbb{C}^* \times \mathbb{T}_N$ to $\mathbb{C}^*$ is dominant, so there exists a sequence $\{(t_n, x_n) \mid n \in \mathbb{N}\} \subset C$ that converges to $(0, x_0)$ with each t_n real and positive. Then $\arg(x_0)$ is the limit of the sequence $\{\arg(x_n)\}$.

For each $n \in \mathbb{N}$, set $y_n := t_n^w \cdot x_n \in X$. Since t_n is positive and real, every component of t_n^w is positive and real, and so $\arg(y_n) = \arg(x_n)$. Thus, $\arg(x_0)$ is the limit of the sequence $\{\arg(y_n)\}$. Since x_n converges to x_0 and t_n converges to 0, the sequence $\{y_n\} \subset X$ is unbounded, which implies that $\arg(x_0)$ lies in the phase limit set of X. This proves (8).

4.2. ***Coamoebae and tropical compactifications.*** We finish the proof of Theorem 1 by establishing the other inclusion,

$$\mathcal{P}^\infty(X) \subset \bigcup_{w \in N \setminus \{0\}} \mathrm{co}\mathcal{A}(\mathrm{in}_w X).$$

Suppose that $\{x_n \mid n \in \mathbb{N}\} \subset X$ is an unbounded sequence. To study the accumulation points of the sequence $\{\arg(x_n) \mid n \in \mathbb{N}\}$, we use a compactification of X that is adapted to its inclusion in $\mathbb{T}_N$. Suitable compactifications are the tropical compactifications of Tevelev [2007] for in these the boundary of X is composed of initial schemes $\mathrm{in}_w X$ of X in a manner we describe below.

By Theorem 3, the cone over the logarithmic limit set $\mathcal{L}^\infty(X)$ of X is the support of a rational polyhedral fan Σ whose cones σ have the property that all initial ideals $\mathrm{in}_w I$ coincide for w in the relative interior of σ.

Recall the construction of the toric variety Y_Σ associated with a fan Σ [Fulton 1993; Gelfand et al. 1994, Chapter 6]. For $\sigma \in \Sigma$, set

$$\sigma^\vee := \{\boldsymbol{m} \in M \mid \langle \boldsymbol{m}, w\rangle \geq 0 \text{ for all } w \in \sigma\},$$
$$\sigma^\perp := \{\boldsymbol{m} \in M \mid \langle \boldsymbol{m}, w\rangle = 0 \text{ for all } w \in \sigma\}.$$

Set $V_\sigma := \mathrm{spec}\, \mathbb{C}[\sigma^\vee]$ and $\mathcal{O}_\sigma := \mathrm{spec}\, \mathbb{C}[\sigma^\perp]$, which is naturally isomorphic to $\mathbb{T}_N/\mathbb{T}_{\langle\sigma\rangle}$, where $\langle\sigma\rangle \subset N$ is the subgroup generated by $\sigma \cap N$. The map $\boldsymbol{m} \mapsto \boldsymbol{m} \otimes \boldsymbol{m}$ determines a comodule map $\mathbb{C}[\sigma^\vee] \to \mathbb{C}[\sigma^\vee] \otimes \mathbb{C}[M]$, which induces the action of the torus $\mathbb{T}_N$ on V_σ. Its orbits correspond to faces of the cone σ, and the smallest orbit $\mathbb{O}_\sigma$ corresponds to σ itself. The inclusion $\sigma^\perp \subset \sigma^\vee$ is split by the semigroup map

$$\sigma^\vee \ni \boldsymbol{m} \mapsto \begin{cases} \boldsymbol{m} & \text{if } \boldsymbol{m} \in \sigma^\perp, \\ 0 & \text{if } \boldsymbol{m} \notin \sigma^\perp, \end{cases} \tag{9}$$

which induces a map $\mathbb{C}[M] \twoheadrightarrow \mathbb{C}[\sigma^\perp]$, and thus, we have the $\mathbb{T}_N$-equivariant split inclusion

$$\mathbb{O}_\sigma \hookrightarrow V_\sigma \xrightarrow{\pi_\sigma} \mathbb{O}_\sigma. \tag{10}$$

On orbits $\mathbb{O}_\tau$ in V_σ, the map π_σ is simply the quotient by $\mathbb{T}_{\langle\sigma\rangle}$.

If $\sigma, \tau \in \Sigma$ with $\sigma \subset \tau$, then $\sigma^\vee \supset \tau^\vee$, so $\mathbb{C}[\sigma^\vee] \supset \mathbb{C}[\tau^\vee]$, and so $V_\sigma \subset V_\tau$. Since the quotient fields of $\mathbb{C}[\sigma^\vee]$ and $\mathbb{C}[M]$ coincide, these are inclusions of open sets, and these varieties V_σ for $\sigma \in \Sigma$ glue together along these natural inclusions to give the toric variety Y_Σ. The torus $\mathbb{T}_N$ acts on Y_Σ with an orbit $\mathbb{O}_\sigma$ for each cone σ of Σ.

Since $V_0 = \mathbb{T}_N$, Y_Σ contains $\mathbb{T}_N$ as a dense subset, and thus X is a (nonclosed) subvariety. Let $\overline{X}$ be the closure of X in Y_Σ. As the fan Σ is supported on the cone over $\mathscr{L}^\infty(X)$, $\overline{X}$ will be a tropical compactification of X, and $\overline{X}$ is complete [Tevelev 2007, Proposition 2.3]. To understand the points of $\overline{X} \setminus X$, we study the intersection $\overline{X} \cap V_\sigma$, which is defined by $I \cap \mathbb{C}[\sigma^\vee]$, as well as the intersection $\overline{X} \cap \mathbb{O}_\sigma$, which is defined in $\mathbb{C}[\sigma^\perp]$ by the image $I(\sigma)$ of $I \cap \mathbb{C}[\sigma^\vee]$ under the map $\mathbb{C}[\sigma^\vee] \twoheadrightarrow \mathbb{C}[\sigma^\perp]$ induced by (10).

Lemma 12. *The initial ideal* $\mathrm{in}_\sigma I \subset \mathbb{C}[M]$ *of* I *is generated by* $I(\sigma)$ *under the inclusion* $\mathbb{C}[\sigma^\perp] \hookrightarrow \mathbb{C}[M]$.

Proof. Let $f \in I$. Since σ is a cone in Σ, we have that $\mathrm{in}_\sigma f = \mathrm{in}_w f$ for all w in the relative interior of σ. Thus, for $w \in \sigma$, the function $\boldsymbol{m} \mapsto \langle \boldsymbol{m}, w\rangle$ on exponents of monomials of f is minimized on (a superset of) the support of $\mathrm{in}_\sigma f$, and if w lies in the relative interior of σ, then the minimizing set is the support of $\mathrm{in}_\sigma f$. Multiplying f if necessary by $\xi^{-\boldsymbol{m}}$, where $\boldsymbol{m}$ is some monomial of $\mathrm{in}_\sigma f$, we may assume that for every $w \in \sigma$, the linear function $\boldsymbol{m} \mapsto \langle \boldsymbol{m}, w\rangle$ is nonnegative on the support of f so that $f \in \mathbb{C}[\sigma^\vee]$, and the function is zero on the support of $\mathrm{in}_\sigma f$. Furthermore, if w lies in the relative interior of σ, then it vanishes exactly on the support of $\mathrm{in}_\sigma f$. Thus, $\mathrm{in}_\sigma f \in \mathbb{C}[\sigma^\perp]$, which completes the proof. □

Since $\mathbb{O}_\sigma = \mathbb{T}_N/\mathbb{T}_{\langle\sigma\rangle}$, Lemma 12 has the following geometric counterpart:

Corollary 13. *By translation with* $\mathrm{in}_\sigma X/\mathbb{T}_{\langle\sigma\rangle} = \overline{X} \cap \mathbb{O}_\sigma$, $\mathbb{T}_{\langle\sigma\rangle}$ *acts* (*freely*) *on* $\mathrm{in}_\sigma X$.

Proof of Theorem 1. Let $\theta \in \mathscr{P}^\infty(X)$ be a point in the phase limit set of X. Then there exists an unbounded sequence $\{x_n \mid n \in \mathbb{N}\} \subset X$ with

$$\lim_{n\to\infty} \arg(x_n) = \theta.$$

Since $\overline{X}$ is compact, the sequence $\{x_n \mid n \in \mathbb{N}\}$ has an accumulation point x in $\overline{X}$. As the sequence is unbounded, $x \notin \mathbb{O}_0$, and so $x \in \overline{X} \setminus X$. Thus, x is a point of $\overline{X} \cap \mathbb{O}_\sigma$ for some cone $\sigma \neq 0$ of Σ. Replacing $\{x_n\}$ by a subsequence, we may assume that $\lim_n x_n = x$.

Because the map π_σ of (10) is continuous and is the identity on $\mathbb{O}_\sigma$, we have that $\{\pi_\sigma(x_n)\}$ converges to $\pi_\sigma(x) = x$, and thus,

$$\pi_\sigma(\theta) = \pi_\sigma\big(\lim_{n\to\infty} \arg(x_n)\big) = \arg\big(\lim_{n\to\infty} \pi_\sigma(x_n)\big) = \arg(x) \in \operatorname{co}\mathscr{A}(\overline{X} \cap \mathbb{O}_\sigma). \quad (11)$$

Corollary 13 implies that $\operatorname{co}\mathscr{A}(\overline{X} \cap \mathbb{O}_\sigma) = \operatorname{co}\mathscr{A}(\operatorname{in}_\sigma X)/\mathbb{U}_\sigma$ as $\mathbb{U}_{\langle\sigma\rangle} = \arg(\mathbb{T}_{\langle\sigma\rangle})$. Recall that on $\mathbb{O}_0$, π_σ is the quotient by $\mathbb{T}_{\langle\sigma\rangle}$. Thus, we conclude from (11) that $\theta \in \operatorname{co}\mathscr{A}(\operatorname{in}_\sigma X)$, which completes the proof of Theorem 1 as $\operatorname{in}_\sigma X = \operatorname{in}_w X$ for any w in the relative interior of σ. □

Example 14. In [Nisse 2009], the closure of a hypersurface coamoeba $\operatorname{co}\mathscr{A}(\mathscr{V}(f))$ for $f \in \mathbb{C}[M]$ was shown to contain a finite collection of *codual hyperplanes*. These are translates of codimension-1 subtori $\mathbb{U}_\sigma$ for σ a cone in the normal fan of the Newton polytope of f corresponding to an edge. By Theorem 1, these translated tori are that part of the phase limit set of X corresponding to the cones σ dual to the edges, specifically $\operatorname{co}\mathscr{A}(\operatorname{in}_\sigma X)$. Since σ has dimension $n-1$, the torus $\mathbb{T}_\sigma$ acts with finitely many orbits on $\operatorname{in}_\sigma X$, which is therefore a union of finitely many translates of $\mathbb{T}_\sigma$. Thus, $\operatorname{co}\mathscr{A}(\operatorname{in}_\sigma X)$ is a union of finitely many translates of $\mathbb{U}_\sigma$.

The logarithmic limit set $\mathscr{L}^\infty(C)$ of a curve $C \subset \mathbb{T}_N$ is a finite collection of points in $\mathbb{S}_N$. Each point gives a ray in the cone over $\mathscr{L}^\infty(C)$, and the components of $\mathscr{P}^\infty(C)$ corresponding to a ray σ are finitely many translations of the dimension-1 subtorus $\mathbb{U}_\sigma$ of $\mathbb{U}_N$, which we referred to as lines in Section 3. These were the lines lying in the boundaries of the coamoebae $\operatorname{co}\mathscr{A}(\ell)$ of the lines ℓ in $\mathbb{T}^2$ and $\mathbb{T}^3$.

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Base change behavior of the relative canonical sheaf related to higher dimensional moduli

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We show that the compatibility of the relative canonical sheaf with base change fails generally in families of normal varieties. Furthermore, it always fails if the general fiber of a family of pure dimension n is Cohen–Macaulay and the special fiber contains a strictly S_{n-1} point. In particular, in moduli spaces with functorial relative canonical sheaves Cohen–Macaulay schemes can not degenerate to S_{n-1} schemes. Another, less immediate consequence is that the canonical sheaf of an S_{n-1}, G_2 scheme of pure dimension n is not S_3.

1. Introduction

The canonical sheaf plays a crucial role in the classification of varieties of characteristic zero. Global sections of its powers define the canonical map, which is birational onto its image for varieties of general type with mild singularities. The image is called the canonical model, and it is a unique representative of the birational equivalence class of the original variety. In particular, the canonical model can be used to construct a moduli space that classifies varieties of general type up to birational equivalence. This moduli space $\overline{\mathfrak{M}}_h$ of stable schemes is the higher dimensional generalization of the intensely investigated space $\overline{\mathfrak{M}}_g$ of stable curves. In order to build $\overline{\mathfrak{M}}_h$, it is important to understand when the canonical sheaf behaves functorially in families, that is, when it is compatible with base change.

More precisely, to obtain a compact moduli space, in $\overline{\mathfrak{M}}_h$, not only canonical models are allowed, but also their generalizations, the semi-log canonical models [Kollár 2010, Definition 15]. By definition these are projective schemes with semi-log canonical singularities [Hacon and Kovács 2010, Definition 3.13.5] and ample canonical bundles. The first naive definition of the moduli functor of stable schemes

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with Hilbert function h is then as follows. Here $h : \mathbb{Z} \to \mathbb{Z}$ is an arbitrary function.

$$\overline{\mathfrak{M}}_h(B) = \left\{ f : X \to B \;\middle|\; \begin{array}{l} f \text{ is flat, proper, } X_{\bar{b}} \text{ is a semi-} \\ \text{log canonical model } (\forall b \in B), \\ h(m) = \chi(\omega_{X_b}^{[m]}) \ (\forall m \in \mathbb{Z}, b \in B) \end{array} \right\} \Big/ \cong \text{ over } B \qquad (1.0.a)$$

As usual, the naive definition works only in the naive cases but not in general. More precisely, (1.0.a) is insufficient to prove the existence of a projective coarse moduli space or a proper Deligne–Mumford stack structure on $\overline{\mathfrak{M}}_h$; see [Kollár 2008; 2010]. In general, (1.0.a) has to be complemented with

$$\omega_{X/B}^{[m]}\big|_{X_b} \cong \omega_{X_b}^{[m]} \quad \text{for every integer } m \text{ and } b \in B. \qquad (1.0.b)$$

Usually (1.0.b) is referred to as *Kollár's condition* (for instance, in [Hassett and Kovács 2004, page 238]). Note also that (1.0.b) is not necessary for reduced B, but it does add important extra restrictions when B is nonreduced.

Currently, it is not understood in every aspect why and how deeply this condition is needed. For example it is not known if in characteristic zero it is equivalent or not to the other possible choice, called *Viehweg's condition* (see [Viehweg 1995, Assumption 8.30; Hassett and Kovács 2004, page 238]):

$$\text{There is an integer } m \text{ such that } \omega_{X/B}^{[m]} \text{ is a line bundle.} \qquad (1.0.c)$$

The starting point of this article is the $m = 1$ case of (1.0.b), that is, the compatibility of the relative canonical sheaf with base change. We will try to understand how restrictive this condition is on flat families. The results will also yield statements about how Serre's S_n condition behaves in families and for the canonical sheaves of single schemes.

Recently it has been proven in [Kollár and Kovács 2010, Theorem 7.9.3] that the relative canonical sheaf of flat families of projective schemes (over $\mathbb{C}$) with Du Bois fibers is compatible with base change. According to [Kollár and Kovács 2010, Theorem 1.4] this pertains to families with semi-log canonical fibers as well. Furthermore, compatibility holds whenever the fibers are Cohen–Macaulay [Conrad 2000, Theorem 3.6.1].

It is important to note at this point that the $m = 1$ case of (1.0.b) behaves differently than the rest. For $m > 1$ there are examples of families of normal surfaces for which (1.0.b) does not hold; see [Hacon and Kovács 2010, Section 14.A]. However, since normal surfaces are Cohen–Macaulay, condition (1.0.b) with $m = 1$ holds for every flat family of normal surfaces. Hence, any incompatibility can be observed only in higher dimensions. Partly due to this fact, there has been a common misbelief, sometimes even stated in articles, that the relative canonical sheaf is compatible with base change for flat families of normal varieties. The question if this compatibility holds indeed was asked about the same time independently by János Kollár and the author.

Question 1.1 (Kollár). Is $\omega_{X/B}|_{X_b} \cong \omega_{X_b}$ for every flat family $X \to B$ of normal varieties?

Here we construct examples showing that the answer is no. That is, there are flat families of normal varieties over smooth curves such that the relative canonical sheaves are not compatible with base change. The examples also show that the known results are optimal in many senses. That is, the fibers of the given families can be chosen to be S_j for any $n > j \geq 2$ and their relative canonical sheaves to be $\mathbb{Q}$-line bundles. The precise statement is as follows.

Theorem 1.2. *For each $n \geq 3$ and $n > j \geq 2$ there is a flat family $\mathcal{X} \to B$ of S_j (but not S_{j+1}) normal varieties of dimension n over some open set $B \subseteq \mathbb{P}^1$, with $\omega_{\mathcal{X}/B}$ a $\mathbb{Q}$-line bundle, such that*

$$\omega_{\mathcal{X}/B}|_{\mathcal{X}_0} \ncong \omega_{\mathcal{X}_0}, \tag{1.2.a}$$

(Here $\mathcal{X}_0$ is the central fiber of $\mathcal{X}$.)

Moreover, the general fiber of $\mathcal{X}$ can be chosen to be smooth and the central fiber to have only one singular point.

When $j = n-1$ and the general fiber is Cohen–Macaulay, somewhat surprisingly, the incompatibility of (1.2.a) always holds. Furthermore, one can allow S_{n-1} points also in the general fibers provided the relative S_{n-1} locus has a component in the central fiber. The precise statement is as follows. (See Section 2 for the assumptions of the article, for instance, scheme is always separated and of finite type over $k = \bar{k}$, etc.)

Theorem 1.3. *If $f : \mathcal{X} \to B$ is a flat family of schemes of pure dimension n over a smooth curve, such that a component of the locus*

$$\overline{\{x \in \mathcal{X} \mid x \text{ is closed},\ \operatorname{depth} \mathcal{O}_{\mathcal{X}_{f(x)},x} = n-1\}} \tag{1.3.a}$$

is contained in the special fiber $\mathcal{X}_0$, then the restriction homomorphism $\omega_{\mathcal{X}/B}|_{\mathcal{X}_0} \to \omega_{\mathcal{X}_0}$ is not an isomorphism.

In particular, the contrapositive of Theorem 1.3 when the general fiber is Cohen–Macaulay yields the following corollary.

Corollary 1.4. *If $f : \mathcal{X} \to B$ is a flat family of schemes of pure dimension n such that $\omega_{\mathcal{X}/B}$ is compatible with base change and the general fiber of f is Cohen–Macaulay, then the central fiber of f cannot have a closed point x such that* $\operatorname{depth} \mathcal{O}_{\mathcal{X}_{f(x)},x} = n-1$.

Corollary 1.4 has many geometric consequences with respect to building moduli spaces with functorial relative canonical sheaves. For example, cone singularities over abelian surfaces cannot be smoothed over irreducible bases. It also generalizes some aspects of theorems by Kollár and Kovács [2010, Theorem 7.12] and Hassett [2001, Theorem 1.1] stating that if all fibers are Du Bois schemes or log canonical surfaces and the general fiber is S_k or Cohen–Macaulay, respectively, then so is the central fiber.

Interestingly, the nonexistence of a depth $n-1$ point is the strongest implication of the compatibility of the relative canonical sheaf with base change.

Proposition 1.5. *Corollary 1.4 is sharp in the sense that $n-1$ cannot be replaced by i for any $i < n-1$.*

Summarizing, Corollary 1.4 and Proposition 1.5 state that in moduli spaces satisfying Kollár's condition, S_{n-1} schemes do not appear in the irreducible components containing Cohen–Macaulay schemes. However, S_j schemes can possibly show up for some $j < n-1$.

If a scheme X is Cohen–Macaulay, which by definition means that $\mathcal{O}_X$ is Cohen–Macaulay, then ω_X is Cohen–Macaulay as well [Kollár and Mori 1998, Corollary 5.70]. One would expect that if $\mathcal{O}_X$ is only S_{n-1}, then typically ω_X is also S_{n-1} or at least it can be S_{n-1}. Surprisingly the truth is quite the opposite. The following application of Theorem 1.3 states that in certain cases an S_{n-1} scheme cannot have even an S_3 canonical sheaf.

Theorem 1.6. *If X is an S_3, G_2 scheme of pure dimension n, which has a closed point with depth $n-1$, then ω_X is not S_3.*

The most immediate consequences of Theorem 1.6 deal with compatibility of restriction to subvarieties. For example, one can show that on a cone X over a Calabi–Yau threefold Y with $h^2(\mathcal{O}_Y) \neq 0$, for an effective, normal Cartier divisor D,

$$\omega_X(D)|_D \cong \omega_D \iff D \text{ does not pass through the vertex.}$$

Or more generally, for an S_{n-1}, normal variety X and an effective, normal Cartier divisor D,

$$\omega_X(D)|_D \cong \omega_D \iff D \text{ does not pass through any closed point with depth } n-1.$$

Theorem 1.6 can also be related to log canonical centers. If (X, D) is a log canonical pair, $D \sim_{\mathbb{Q}} -K_X$ and ω_X is not S_3 at $x \in X$, then x is a log canonical center of the pair (X, D) [Kollár 2011, Theorem 3]. Hence by Theorem 1.6, if X is S_{n-1} and (X, D) log canonical such that $D \sim_{\mathbb{Q}} -K_X$, then (X, D) has a log-canonical center at all closed points with depth $n-1$. This statement is of course obvious if we know that the depth $n-1$ closed points are already log-canonical centers of X. However, that is not always the case. For example, let X be the cone, with high enough polarization, over the product Y of a K3 surface with the projective line and let D be the cone over an anti canonical divisor of Y. Then, (X, D) is log-canonical, X is S_{n-1} and the cone point is the only closed point with depth $n-1$; see Lemma 4.3. Still, the vertex is not a log-canonical center of X, because K_X is not $\mathbb{Q}$-Cartier.

Theorem 1.6 raises the following question as well.

Question 1.7. Is it true that if X is a pure n-dimensional scheme such that $\mathcal{O}_X$ is S_l, but not S_{l+1}, and ω_X is S_j, but not S_{j+1}, for some $j, l < n$, then $j + l \le n + 1$?

Remark 1.8. By the methods of Section 4, the answer to Question 1.7 is positive if X is a cone over a smooth projective variety.

There are a couple of intuitive reasons for the failure of compatibility in (1.2.a). First, compatibility holds for the relative dualizing complex if the base is smooth by Proposition 3.3.(1). Hence $\omega_{\mathcal{X}/B}$ is a nonfunctorial component, the $-n$-th cohomology sheaf, of the functorial object $\omega^\bullet_{\mathcal{X}/B}$. For example, by the proof of Theorem 1.3, if the general fiber is Cohen–Macaulay and the central fiber is S_{n-1}, the restriction homomorphism fits into an exact sequence as follows, with a nonzero term on the right.

$$0 \to \omega_{\mathcal{X}/B}|_{X_b} \to \omega_{\mathcal{X}_b} \to \mathcal{T}or^1(h^{-(n-1)}(\omega^\bullet_{\mathcal{X}/B}), \mathcal{O}_{\mathcal{X}_0}) \to 0 \tag{1.8.a}$$

This shows in a precise way how the functoriality might be destroyed by passing to the lowest cohomology sheaf of $\omega^\bullet_{\mathcal{X}/B}$.

Another explanation for the incompatibility (1.2.a) is that $\mathcal{X}_0$ is too singular. Using stable reduction one may find a replacement for $\mathcal{X}_0$ with the mildest possible singularities. The reduction steps consist of blow-ups, finite surjective normalized base changes and contractions on the total space of the family. The output is a family, the relative canonical sheaf of which is compatible with base change by [Kollár and Kovács 2010]. At the end of the article, we also present the stable reduction of our construction using a straightforward ad hoc method. The algorithmic, and lengthy, method can be found in the preprint version of the article.

In Section 3, we start with a short background overview on the base-change properties of relative dualizing complexes and relative canonical sheaves. The proofs of the main theorems can be found in Section 6 and Section 7. Some of these results are based on the existence of projective cones with appropriately chosen singularities. In Section 4 we give a cohomological characterization of when certain sheaves on a cone are S_d. Then in Section 5 we use this characterization to give the desired examples of projective cones. In Section 8 we compute the stable limit of our construction.

2. Notation and assumptions

Unless otherwise stated, *scheme* means a separated scheme of finite type over a fixed field k of characteristic zero and every morphism is separated. A *variety* is an integral scheme. A *projective* or *quasiprojective* scheme means a projective or quasiprojective scheme over k. A *curve* is a quasiprojective, integral scheme of dimension one. If Y is a subscheme of X, then $\mathcal{I}_{Y,X}$ is the ideal sheaf of Y in X. If $\mathcal{I}_{Y,X}$ is a line bundle (that is, a locally free sheaf of rank one), then we define $\mathcal{O}_X(-Y) := \mathcal{I}_{Y,X}$ and $\mathcal{O}_X(Y) := \mathcal{O}_X(-Y)^{-1}$. Notice that $\mathcal{I}_{Y,X}$ being a line bundle is equivalent to Y being defined around every point P by a single nonzero divisor element of $\mathcal{O}_{X,P}$.

A hypersurface of a quasiprojective scheme $X \subseteq \mathbb{P}^N$ is a subscheme $H \subseteq X$ defined by a section of $\mathcal{O}_X(d)$ for some $d > 0$. If H and H' are hypersurfaces of a quasiprojective scheme $X \subseteq \mathbb{P}^N$, defined by f_0 and $f_\infty \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$, respectively, then the *pencil* generated by H and H' is the subscheme $\mathcal{H} \subseteq X \times \mathbb{P}^1$ defined by the section $f_0 t_0 + f_\infty t_1$ of $H^0(X \times \mathbb{P}^1, \mathcal{O}(d, 1))$. Here t_0 and t_1 are the usual parameters of $\mathbb{P}^1$, and f_0 and f_∞ are viewed as elements of $H^0(X, \mathcal{O}_X(d))$ via the natural homomorphism $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(X, \mathcal{O}_X(d))$.

For a complex $\mathcal{C}^\bullet$ of sheaves, $h^i(\mathcal{C}^\bullet)$ is the i-th cohomology sheaf of $\mathcal{C}$. For a morphism $f : X \to Y$, $\omega^\bullet_{X/Y} := f^!\mathcal{O}_Y$, where $f^!$ is the functor obtained in [Hartshorne 1966, Corollary VII.3.4.a]. If f has equidimensional fibers of dimension n, then $\omega_{X/Y} := h^{-n}(\omega^\bullet_{X/Y})$. Every complex and morphism of complexes is considered in the derived category $D(qc/\cdot)$ of quasicoherent sheaves up to the equivalences defined there. If Z is a closed subscheme of X, where $\iota : Z \to X$ is the embedding morphism, then the map $\mathcal{R}\iota_* \cong \iota_*$ identifies $D(qc/Z)$ with a full subcategory of $D(qc/X)$. We use this identification at multiple places, equating $\mathcal{C}^\bullet$ and $\mathcal{R}\iota_*\mathcal{C}^\bullet$ for every $\mathcal{C}^\bullet \in D(qc/Z)$. If Z is a closed subscheme of a scheme X, then $(\,\cdot\,)|^L_Z$ denotes the derived restriction functor, which is naturally isomorphic to $\cdot \otimes^L \mathcal{O}_Z$ via the identification mentioned above. A *line bundle* is a locally free sheaf of rank one.

If $X \to B$ is a morphism of schemes, then X_b is the scheme theoretic fiber of X over B. If a sheaf $\mathcal{F}$ on X is given, then $\mathcal{F}_b := \mathcal{F}|_{X_b}$. The dimension $\dim_X P$ of a point $P \in X$ is the dimension of its closure in X. The acronym *slc* stands for semi-log canonical [Hacon and Kovács 2010, Definition 3.13.5].

The *depth* of a coherent sheaf $\mathcal{F}$ at a point $x \in X$ is by definition the depth of $\mathcal{F}_x$ with respect to the maximal ideal $m_{X,x}$ at x and is denoted by $\operatorname{depth} \mathcal{F}_x$. The depth of a scheme X at x is $\operatorname{depth} \mathcal{O}_{X,x}$. A coherent sheaf $\mathcal{F}$ is S_d on X if for every $x \in X$,

$$\operatorname{depth} \mathcal{F}_x \geq \min\{d, \dim \mathcal{O}_{X,x}\}. \tag{2.0.b}$$

Note that there is an ambiguity in the literature about the definition of S_d sheaves. Many sources replace $\mathcal{O}_{X,x}$ in (2.0.b) by $\mathcal{F}_x$, thus gaining a weaker notion. Since every sheaf of this article has full support, or equivalently every sheaf is considered over its support, the two definitions are equivalent for all cases considered here. Hence, we decided to include the stronger notion, but the reader should feel free to think about the other one as well. For a morphism $f : X \to B$, $\mathcal{F}$ is *relative* S_d if $\mathcal{F}|_{X_b}$ is S_d for all $b \in B$. The word (relative) *Cohen–Macaulay* is a synonym for (relative) $S_{\dim X}$.

A scheme X is G_r for some $r \geq 0$ if it is Gorenstein in codimension r. A point $P \in X$ is an *associated point* of a coherent sheaf $\mathcal{F}$ if $m_{X,P}$ is the annihilator of some element of $\mathcal{F}_P$. An *associated component* of a coherent sheaf is the closure of an associated point. One can show that if $Q \in X$, $\mathcal{F}_Q \neq 0$ and $\mathcal{P}$ is the set of prime ideals of $\mathcal{O}_{X,Q}$ corresponding to generalizations of Q that are associated points of $\mathcal{F}$, then

$$\bigcup_{P \in \mathcal{P}} P = \{x \in \mathcal{O}_{X,Q} \mid \text{there exists } 0 \neq m \in \mathcal{F}_Q \text{ with } xm = 0\}$$

Consequently, if s is a section of a line bundle, then it does not vanish on any associated component of X (that is, of $\mathcal{O}_X$) if and only if s_P is not a zero divisor for every $P \in X$. That is, if H is the subscheme of X cut out by s, then $\mathcal{I}_{H,X}$ is a line bundle if and only if s does not vanish on any associated component of X.

For an S_2, G_1 scheme and an arbitrary coherent sheaf $\mathcal{F}$, the n-th *reflexive power* is

$$\mathcal{F}^{[n]} := \begin{cases} (\mathcal{F}^{\otimes n})^{**} & \text{if } n \geq 0, \\ (\mathcal{F}^{\otimes(-n)})^{*} & \text{if } n < 0. \end{cases}$$

That is, it is the reflexive hull of the n-th tensor power. A coherent sheaf $\mathcal{F}$ is a $\mathbb{Q}$-line bundle if $\mathcal{F}^{[n]}$ is a line bundle for some $n > 0$. Note that if $f : X \to B$ is a family with ω_{X_b} a $\mathbb{Q}$-line bundle for all $b \in B$, then $\omega_{X/B}$ is not necessarily a $\mathbb{Q}$-line bundle [Hacon and Kovács 2010, Section 14.A]. However, if the X_b are S_2, G_1 schemes and $\omega_{X/B}$ a $\mathbb{Q}$-line bundle then ω_{X_b} is a $\mathbb{Q}$-line bundle for all $b \in B$; see [Hassett and Kovács 2004, Lemma 2.6].

3. Background on base change for dualizing complexes

This section contains a general overview on the base change properties of relative dualizing complexes and relative canonical sheaves. For experts, some of the statements might be well known, but they are included here for completeness and easier reference. Readers more interested in geometric arguments and willing to accept the statements of this section without proof should feel free to skip to the next section.

Recall that the relative dualizing complex $\omega^{\bullet}_{X/B}$ of a quasiprojective family $f : X \to B$ is defined as $f^{!}\mathcal{O}_B$. Here $f^{!}$ is the functor constructed in [Hartshorne 1966, Corollary VII.3.4.a]. The following technical point should be noted here.

Remark 3.1. There is also another definition of $f^{!}$ in [Neeman 1996] as the right adjoint of Rf_*. The two definitions coincide for proper morphisms by [Hartshorne 1966, Theorem VII.3.3; Neeman 1996, Section 6], but not in general. For example, if X is smooth affine variety over $B = \operatorname{Spec} k$ and f is the structure map, then Hartshorne's definition of $f^{!}\mathcal{O}_{\operatorname{Spec} k}$ lives in cohomological degree $-\dim X$ while Neeman's is in cohomological degree zero. See [Lipman and Hashimoto 2009, Part I, Exercise 4.2.3.d] for more details on the differences (Neeman's $f^{!}$ is denoted $f^{\times}$ there). We use Hartshorne's definition in this article.

The dualizing complex of a single scheme Y is $\omega^{\bullet}_Y := \omega^{\bullet}_{Y/\operatorname{Spec} k}$. The following fact is needed in the proof of Proposition 3.3(11). It follows from the invariance of the length of maximal regular sequences [Bruns and Herzog 1993, Theorem 1.2.5].

Fact 3.2. *Let P be a point of a subscheme H of a scheme X such that $(\mathcal{I}_{H,X})_P$ is a line bundle, d is an integer, and $\mathcal{F}$ is a coherent S_1 sheaf with full support (that is, $\operatorname{supp}\mathcal{F} = X$) on X.*

(1) $\operatorname{depth}\mathcal{F}_P \ge d \iff \operatorname{depth}(\mathcal{F}|_H)_P \ge d-1$ (*here $\mathcal{F}|_H$ is regarded as a sheaf on H, not on X*),

(2) $\operatorname{depth}\mathcal{F}_P \ge \min\{d, \dim \mathcal{O}_{X,P}\} \iff \operatorname{depth}(\mathcal{F}|_H)_P \ge \min\{d-1, \dim \mathcal{O}_{H,P}\}$.

Proposition 3.3. *Suppose we have a flat family $f : \mathcal{X} \to B$ of schemes of pure dimension n over a smooth base, a point $0 \in B$ and a single quasiprojective scheme X of pure dimension n.*

(1) *There is an isomorphism*

$$\omega^{\bullet}_{\mathcal{X}/B}|^{L}_{\mathcal{X}_0} \cong \omega^{\bullet}_{\mathcal{X}_0}. \tag{3.3.a}$$

(2) *Fixing any isomorphism in* (3.3.a) *yields natural homomorphism*

$$\omega_{\mathcal{X}/B}|_{\mathcal{X}_0} \to \omega_{\mathcal{X}_0}. \tag{3.3.b}$$

(3) *If B is of pure dimension d with $\mathcal{O}_B \cong \omega_B$, then $\omega^{\bullet}_{\mathcal{X}/B} \cong \omega^{\bullet}_{\mathcal{X}}[-d]$.*

(4) *If $V \subseteq X$ is any open set, then $\omega^{\bullet}_V \cong \omega^{\bullet}_X|_V$.*

(5) *If $U \subseteq \mathcal{X}$ is any open set, then $\omega^{\bullet}_{U/B} \cong \omega^{\bullet}_{\mathcal{X}/B}|_U$.*

(6) *If $P \in X$ is a point, then $\operatorname{depth}_P \mathcal{O}_X = d$ if and only if $h^i(\omega^{\bullet}_X)_P$ is zero for $i > -d - \dim_X P$ and nonzero for $i = -d - \dim_X P$.*

(7) *If $P \in \mathcal{X}$ is a point, then $\operatorname{depth}_P \mathcal{O}_{\mathcal{X}_{f(P)}} = d$ if and only if $h^i(\omega^{\bullet}_{\mathcal{X}/B})_P$ is zero for $i > -d - \dim_{\mathcal{X}_{f(P)}} P$ and nonzero for $i = -d - \dim_{\mathcal{X}_{f(P)}} P$.*

(8) *ω_X is S_2.*

(9) *$\omega_{\mathcal{X}/B}$ is S_2.*

(10) *If the fibers of f are Cohen–Macaulay then $\omega_{\mathcal{X}/B} \cong \omega^{\bullet}_{\mathcal{X}/B}$ and consequently* (3.3.b) *is an isomorphism.*

(11) *If $\mathcal{X}_0$ is S_2 and G_1, then* (3.3.b) *is isomorphism if and only if*

$$\operatorname{depth} \omega_{\mathcal{X}/B,P} \geq \min\{3, \dim \mathcal{O}_{\mathcal{X},P}\} \quad \textit{for every } P \in \mathcal{X}_0. \tag{3.3.c}$$

Furthermore if (3.3.c) *is not satisfied then not only is* (3.3.b) *not an isomorphism, but $\omega_{\mathcal{X}/B}|_{\mathcal{X}_0} \ncong \omega_{\mathcal{X}_0}$.*

Proof. First, we prove point (1). It will be an ad hoc proof, since we have not found the exact statement in the literature. The statements we found are either only for flat base change morphisms [Hartshorne 1966, Corollary VII.3.4.a] or for proper f [Lipman and Hashimoto 2009, Part I, Corollary 4.4.3]. Note that, however, it might seem that point (1) follows from base change for proper f, to the best knowledge of the author, it is not clear whether one can compactify a flat morphism to a flat morphism.

First, by [Hartshorne 1966, Corollary VII.3.4.a], $\omega^{\bullet}_{\mathcal{X}/B}$ is compatible with flat base change. So, since $\operatorname{Spec} \widehat{\mathcal{O}_{B,0}}$ is flat over B, we may assume that B is the spectrum of a complete local ring of a smooth scheme and 0 is the unique closed point. In particular, then $B \cong \operatorname{Spec} k[\![x_1, \dots, x_m]\!]$. Hence, by induction on m, it is enough to prove that

$$\omega^{\bullet}_{\mathcal{X}/B}|_Y \cong \omega^{\bullet}_{Y/C}, \tag{3.3.d}$$

where $C := \operatorname{Spec} k[\![x_1, \dots, x_{m-1}]\!]$ and $Y := \mathcal{X} \times_B C$. To prove (3.3.d), first consider the usual exact triangle

$$\mathcal{O}_{\mathcal{X}} \xrightarrow{\mu} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_Y \xrightarrow{+1}, \tag{3.3.e}$$

where μ is multiplication by x_m. Tensoring (3.3.e) by $\omega^{\bullet}_{\mathcal{X}/B}$ yields

$$\omega^{\bullet}_{\mathcal{X}/B} \xrightarrow{\mu \otimes \mathrm{id}_{\omega^{\bullet}_{\mathcal{X}/B}}} \omega^{\bullet}_{\mathcal{X}/B} \to \omega^{\bullet}_{\mathcal{X}/B}|^L_Y \xrightarrow{+1} . \tag{3.3.f}$$

On the other hand, applying $\mathcal{R}\mathcal{H}om(\,\cdot\,, \omega^{\bullet}_{\mathcal{X}/B})$ and a rotation to (3.3.e) yields

$$\omega^{\bullet}_{\mathcal{X}/B} \xrightarrow{\mu \otimes \mathrm{id}_{\omega^{\bullet}_{\mathcal{X}/B}}} \omega^{\bullet}_{\mathcal{X}/B} \to \mathcal{R}\mathcal{H}om(\mathcal{O}_Y, \omega^{\bullet}_{\mathcal{X}/B})[1] \xrightarrow{+1} . \tag{3.3.g}$$

So, (3.3.f) and (3.3.g) together imply that

$$\mathcal{R}\mathcal{H}om(\mathcal{O}_Y, \omega^{\bullet}_{\mathcal{X}/B})[1] \cong \omega^{\bullet}_{\mathcal{X}/B}|_Y. \tag{3.3.h}$$

Denote by ι and g the maps $Y \to X$ and $Y \to C$, respectively. The following stream of isomorphisms finishes then the proof of point (1).

$$\begin{aligned}\omega^\bullet_{\mathcal{X}/B}|_Y &\cong \underbrace{\mathcal{R}\mathcal{H}om_X(\mathcal{R}\iota_*\mathcal{O}_Y, \omega^\bullet_{\mathcal{X}/B})[1]}_{\text{by (3.3.h)}} \cong \underbrace{\mathcal{R}\mathcal{H}om_Y(\mathcal{O}_Y, \iota^!\omega^\bullet_{\mathcal{X}/B})[1]}_{\text{by Grothendieck duality}} \\ &\cong \iota^!\omega^\bullet_{\mathcal{X}/B}[1]! \cong \underbrace{\iota^! f^! \mathcal{O}_B[1]}_{\text{definition of } \omega^\bullet_{\mathcal{X}/B}} \cong \underbrace{i^! f^! \omega^\bullet_B[-(m-1)]}_{\omega^\bullet_B[-m]\cong\mathcal{O}_B} \\ &\cong \underbrace{\omega^\bullet_Y[-(m-1)]}_{\omega^\bullet_Y \cong (f\circ\iota)^!\omega^\bullet_B} \cong \underbrace{g^!\omega^\bullet_C[-(m-1)]}_{\omega^\bullet_Y\cong g^!\omega^\bullet_C} \underbrace{\cong g^!\mathcal{O}_C}_{\omega^\bullet_C[-(m-1)]\cong\mathcal{O}_C} \cong \omega^\bullet_{Y/C}.\end{aligned}$$

To prove point (2), notice that since $\omega_{\mathcal{X}/B} := h^{-n}(\omega^\bullet_{\mathcal{X}/B})$ is the lowest cohomology sheaf of $\omega^\bullet_{\mathcal{X}/B}$, there is a homomorphism

$$\omega_{\mathcal{X}/B}[n] \to \omega^\bullet_{\mathcal{X}/B}. \tag{3.3.i}$$

Applying $(\,\cdot\,)|^L_{\mathcal{X}_0}$ to (3.3.i) and then composing with the isomorphism given by (3.3.a) yields a homomorphism

$$\omega_{\mathcal{X}/B}[n]|^L_{\mathcal{X}_0} \to \omega^\bullet_{\mathcal{X}_0}. \tag{3.3.j}$$

Finally taking $-n$-th cohomology sheaves of (3.3.j) yields the restriction homomorphism of (3.3.b).

Point (3) is shown by the following line of isomorphisms:

$$\omega^\bullet_{\mathcal{X}/B} = f^!\mathcal{O}_B \cong f^!\omega_B \cong f^!\omega^\bullet_B[-d] \cong \omega^\bullet_{\mathcal{X}}[-d].$$

To prove point (4), consider the following commutative diagram.

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "\nu"'] & X \arrow[d, "\mu"] \\ & \operatorname{Spec} k \end{tikzcd}$$

Since j is smooth of relative dimension 0, using the notation of [Hartshorne 1966], we have $j^! \cong j^\# \cong j^*$, and then

$$\omega^\bullet_V = \nu^!\mathcal{O}_{\operatorname{Spec} k} \cong j^!\mu^!\mathcal{O}_{\operatorname{Spec} k} \cong j^!\omega^\bullet_X \cong j^*\omega^\bullet_X = \omega^\bullet_X|_V.$$

Point (5) follows from points (4) and (3).

Point (6) is proved in [Kovács 2011, Proposition 3.2] (by taking $\mathscr{F} := \mathscr{O}_X$). To prove point (7), let $b := f(P)$ and consider the following Cartesian square.

$$\begin{tikzcd} \mathscr{X} \arrow[d, "f"] & \mathscr{X}' \arrow[l, "\lambda'"'] \arrow[d, "f'"] \\ B & \operatorname{Spec} \mathscr{O}_{B,b} \arrow[l, "\lambda"] \end{tikzcd}$$

By flat base change,

$$(\lambda')^* \omega^\bullet_{\mathscr{X}/B} \cong \omega^\bullet_{\mathscr{X}'/\operatorname{Spec} \mathscr{O}_{B,b}}. \tag{3.3.k}$$

That is,

$$h^i(\omega^\bullet_{\mathscr{X}/B})_P \cong \underbrace{h^i(\omega^\bullet_{\mathscr{X}'/\operatorname{Spec}\mathscr{O}_{B,b}})_P}_{\text{by (3.3.k)}} \cong \underbrace{h^i(\omega^\bullet_{\mathscr{X}'}[-\dim \mathscr{O}_{B,b}])_P}_{\text{by point (3)}} \cong h^{i-\dim \mathscr{O}_{B,b}}(\omega^\bullet_{\mathscr{X}'})_P.$$

Hence,

$$h^i(\omega^\bullet_{\mathscr{X}/B})_P \ \text{ is } \begin{cases} 0 & \text{if } i > -d - \dim_{\mathscr{X}_b} P, \\ \neq 0 & \text{if } i = -d - \dim_{\mathscr{X}_b} P. \end{cases}$$

$$\Updownarrow$$

$$h^i(\omega^\bullet_{\mathscr{X}'})_P \ \text{ is } \begin{cases} 0 & \text{if } i > -d - \dim_{\mathscr{X}_b} P - \dim \mathscr{O}_{B,b}, \\ \neq 0 & \text{if } i = -d - \dim_{\mathscr{X}_b} P - \dim \mathscr{O}_{B,b}. \end{cases}$$

$$\Updownarrow$$

$$\operatorname{depth}_P \mathscr{O}_{\mathscr{X}'} = d + \dim \mathscr{O}_{B,b}$$

$$\Updownarrow \quad \text{(by Fact 3.2)}$$

$$(\operatorname{depth}_P \mathscr{O}_{(\mathscr{X}')_{f(P)}} =) \operatorname{depth}_P \mathscr{O}_{\mathscr{X}_{f(P)}} = d.$$

To prove point (8), by point (4) we may assume that X is affine. Using point (4) again we may also assume that it is projective. Then [Kollár and Mori 1998, Corollary 5.69] concludes the proof of point (8). Point (9) is a consequence of point (8) and point (3). Point (10) is shown in [Conrad 2000, Theorem 3.5.1].

To prove point (11), notice that by point (8), $\omega_{\mathscr{X}_0}$ is S_2. Also since $\mathscr{X}_0$ is G_1, using point (10), the homomorphism $\omega_{\mathscr{X}/B}|_{\mathscr{X}_0} \to \omega_{\mathscr{X}_0}$ is isomorphism in codimension one. Then by [Hartshorne 1994, Theorem 1.9 and Theorem 1.12], using that $\mathscr{X}_0$ is S_2 and G_1, $\omega_{\mathscr{X}/B}|_{\mathscr{X}_0} \to \omega_{\mathscr{X}_0}$ is an isomorphism if and only if $\omega_{\mathscr{X}/B}|_{\mathscr{X}_0}$ is S_2. Finally, by Fact 3.2(2), this is equivalent to (3.3.c).

Notice that if (3.3.c) is not satisfied, then $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0}$ is not S_2 over $\mathcal{H}_0$. Hence in this case not only can (3.3.b) not be isomorphism, but any isomorphism between $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0}$ and $\omega_{\mathcal{H}_0}$ is impossible. $\square$

Remark 3.4. A priori, saying that (3.3.b) is an isomorphism is a stronger statement than that $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0}$ is isomorphic to $\omega_{\mathcal{H}_0}$. However, if B is smooth, $\mathcal{H}_0$ is projective, S_2 and G_1, they are equivalent by the following argument. In this case $\omega_{\mathcal{H}_0}$ is S_2 and is a line bundle over the Gorenstein locus U. Assume that $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0} \cong \omega_{\mathcal{H}_0}$ via an arbitrary isomorphism α. Then $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0}$ is also S_2 and a line bundle over U. Since both are S_2, homomorphisms $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0} \to \omega_{\mathcal{H}_0}$ are determined in codimension one, e.g., over U. Furthermore, any two isomorphisms over U between any two line bundles differ by multiplication with an element of $H^0(U, \mathcal{O}_{\mathcal{H}_0})$, where $H^0(U, \mathcal{O}_{\mathcal{H}_0}) \cong k^*$, by $\mathcal{H}_0$ being S_2 and projective. Since the restriction of the natural morphism $\beta : \omega_{\mathcal{H}/B}|_{\mathcal{H}_0} \to \omega_{\mathcal{H}_0}$ over U is an isomorphism, α differs from β over U by a multiplication with an element of k^*. However, then the same is true over entire X, by the codimension one determination. Hence β is also an isomorphism.

Finally, we conclude with a statement about restriction behavior of relative dualizing complexes and relative canonical sheaves to hypersurfaces. For that we also need a lemma about flatness of hypersurfaces.

Lemma 3.5. *If $f : \mathcal{X} \to B$ is a flat morphism onto a smooth curve and $\mathcal{H} \subseteq \mathcal{X}$ is a subscheme for which $\mathcal{I}_{\mathcal{H},\mathcal{X}}$ is a line bundle, then the following are equivalent:*

(1) *$\mathcal{I}_{\mathcal{H}_b,\mathcal{X}_b}$ is a line bundle for every $b \in B$.*

(2) *$\mathcal{H}$ is flat over B.*

In particular, if $f : \mathcal{X} \to B$ is flat with fibers of pure dimension n and $\mathcal{H} \subseteq \mathcal{X}$ is also flat with $\mathcal{I}_{\mathcal{H},\mathcal{X}}$ a line bundle, then fibers of $\mathcal{H}$ are of pure dimension $n-1$.

Proof. We prove only the equivalence statement, since the addendum follows from the ideals $\mathcal{I}_{\mathcal{H}_b,\mathcal{X}_b}$ being line bundles.

The statement is local on $\mathcal{H}$. So, fix $P \in \mathcal{H}$ and let $Q := f(P)$. By [Hartshorne 1977, Proposition 9.1A.a], $\mathcal{H}$ and $\mathcal{X}$ are flat over B at P if and only if the respective homomorphisms $\mathcal{O}_{\mathcal{H},P} \to \mathcal{O}_{\mathcal{H},P}$ and $\mathcal{O}_{\mathcal{X},P} \to \mathcal{O}_{\mathcal{X},P}$ induced by multiplication with some power of the local parameter t of $\mathcal{O}_{B,Q}$ are injective. Furthermore, by induction this is equivalent to the injectivity of multiplication with the first power t.

The assumptions of the lemma state that $(\mathcal{I}_{\mathcal{H},\mathcal{X}})_P \subseteq \mathcal{O}_{\mathcal{X},P}$ is generated by a nonzero divisor element s. Hence there is a commutative diagram with exact rows and columns as follows.

$$\begin{tikzcd}
 & & 0 & 0 & & \\
0 \arrow[r] & \ker(\cdot s) \arrow[r] & \mathcal{O}_{\mathcal{X}_Q,P} \arrow[r, "\cdot s"] \arrow[u] & \mathcal{O}_{\mathcal{X}_Q,P} \arrow[u] & & \\
 & 0 \arrow[r] & \mathcal{O}_{\mathcal{X},P} \arrow[r, "\cdot s"] \arrow[u] & \mathcal{O}_{\mathcal{X},P} \arrow[r, "\cdot s"] \arrow[u] & \mathcal{O}_{\mathcal{H},P} \arrow[r] & 0 \\
 & 0 \arrow[r] & \mathcal{O}_{\mathcal{X},P} \arrow[r, "\cdot s"] \arrow[u, "\cdot t"] & \mathcal{O}_{\mathcal{X},P} \arrow[r, "\cdot s"] \arrow[u, "\cdot t"] & \mathcal{O}_{\mathcal{H},P} \arrow[r] \arrow[u, "\cdot t"] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] & \ker(\cdot t) \arrow[u] & \\
 & & & & 0 \arrow[u] &
\end{tikzcd}$$

By the snake lemma applied vertically, $\ker(\cdot t) = \ker(\cdot s)$. In particular, $\ker(\cdot t) = 0$ if and only if $\ker(\cdot s) = 0$. The former is equivalent to flatness of $\mathcal{H} \to B$ at P while the latter is equivalent to $\mathcal{I}_{\mathcal{H}_Q,\mathcal{X}_Q}$ being a line bundle at P. ☐

Proposition 3.6. *If $\mathcal{X} \to B$ is a flat family of pure n-dimensional schemes, and $\mathcal{H} \subseteq \mathcal{X}$ a flat subscheme such that $\mathcal{I}_{\mathcal{H},\mathcal{X}}$ is a line bundle, then*

(1) *there is an isomorphism*

$$\omega^{\bullet}_{\mathcal{X}/B}(\mathcal{H})|^{L}_{\mathcal{H}}[-1] \cong \omega^{\bullet}_{\mathcal{H}/B}, \tag{3.6.a}$$

(2) *there is a homomorphism*

$$\omega_{\mathcal{X}/B}(\mathcal{H})|_{\mathcal{H}} \to \omega_{\mathcal{H}/B}, \tag{3.6.b}$$

which is isomorphism over the relative Cohen–Macaulay locus of $\mathcal{H} \to B$.

Proof. Notice first that by Lemma 3.5, $\mathcal{H}$ has equidimensional fibers and hence $\omega_{\mathcal{H}/B}$ is defined indeed. To prove point (1), consider the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}(\mathcal{H}) \to \mathcal{O}_{\mathcal{H}}(\mathcal{H}) \to 0. \tag{3.6.c}$$

Applying $(\,\cdot\,) \otimes^{L} \omega^{\bullet}_{\mathcal{X}/B}$ to (3.6.c) and then translating yields the exact triangle

$$\omega^{\bullet}_{\mathcal{X}/B}(\mathcal{H})|^{L}_{\mathcal{H}}[-1] \to \omega^{\bullet}_{\mathcal{X}/B} \to \omega^{\bullet}_{\mathcal{X}/B}(\mathcal{H}) \xrightarrow{+1} . \tag{3.6.d}$$

On the other hand if $\iota : \mathcal{H} \to \mathcal{X}$ is the embedding morphism, then

$$\omega^\bullet_{\mathcal{H}/B} \cong \iota^! \omega^\bullet_{\mathcal{X}/B} = R\mathcal{H}om_{\mathcal{H}}(\mathcal{O}_{\mathcal{H}}, \iota^! \omega^\bullet_{\mathcal{X}/B}) \cong \underbrace{R\mathcal{H}om_{\mathcal{X}}(\mathcal{O}_{\mathcal{H}}, \omega^\bullet_{\mathcal{X}/B})}_{\text{by Grothendieck duality}}.$$

Now, applying $R\mathcal{H}om_{\mathcal{X}}(\,\cdot\,, \omega^\bullet_{\mathcal{X}/B})$ to the twist of (3.6.c) by $\mathcal{O}_{\mathcal{X}}(-\mathcal{H})$ yields the exact triangle

$$\omega^\bullet_{\mathcal{H}/B} \cong R\mathcal{H}om_{\mathcal{X}}(\mathcal{O}_{\mathcal{H}}, \omega^\bullet_{\mathcal{X}/B}) \to \omega^\bullet_{\mathcal{X}/B} \to \omega^\bullet_{\mathcal{X}/B}(\mathcal{H}) \xrightarrow{+1}. \tag{3.6.e}$$

Putting together (3.6.d) and (3.6.e) finishes the proof of point (1).

To prove (2), take the natural map $\omega_{\mathcal{X}/B}[n] \to \omega^\bullet_{\mathcal{X}/B}$, twist it with $\mathcal{O}_{\mathcal{X}}(\mathcal{H})$ and then restrict to $\mathcal{H}$. This yields the commutative diagram

$$\begin{tikzcd} \omega_{\mathcal{X}/B}[n-1](\mathcal{H})|^L_{\mathcal{H}} \arrow[r] \arrow[rr, bend left] & \omega^\bullet_{\mathcal{X}/B}(\mathcal{H})[-1]|^L_{\mathcal{H}} \arrow[r, "\underbrace{\cong}_{\text{by point (1)}}"'] & \omega^\bullet_{\mathcal{H}/B} \end{tikzcd} \tag{3.6.f}$$

Applying then $h^{-(n-1)}(\,\cdot\,)$ to the long composition arrow of (3.6.f) yields the homomorphism (3.6.b).

Let P be a point of $\mathcal{H}$ that is relatively Cohen–Macaulay over B, and let b be the image of P in B. By the openness of the relative Cohen–Macaulay locus, there is a neighborhood U of P where $X \to B$ is relatively Cohen–Macaulay. In particular, then $\omega_{\mathcal{X}/B}[n-1] \to \omega^\bullet_{\mathcal{X}/B}[-1]$ is an isomorphism over U by Proposition 3.3(10) and hence so is the first arrow of (3.6.f). This proves that (3.6.b) is an isomorphism in a neighborhood of P, which finishes the proof of point (2) as well. □

Remark 3.7. The homomorphisms constructed in Propositions 3.3 and 3.6, for example the isomorphisms (3.3.a) and (3.6.a), are not canonical in any sense.

4. Serre's condition on projective cones

In this section we consider sheaves on projective cones that are isomorphic to pullbacks from the base outside the vertex. Lemma 4.3 gives a cohomological description of when such sheaves are S_d. Before that we also need a short lemma, Lemma 4.2, about how the property S_d pulls back in flat relatively Cohen–Macaulay families.

We cite the following fact separately here, because it is used at many places throughout the article, including the aforementioned Lemma 4.2.

Fact 4.1 [Grothendieck 1965, Proposition 6.3.1]. *Let X and Y be two noetherian schemes, $f : X \to Y$ a flat morphism, $P \in X$ arbitrary and $\mathcal{F}$ a coherent Y module. In this situation,*

$$\operatorname{depth}_{\mathcal{O}_{X,P}} (f^*\mathcal{F})_P = \operatorname{depth}_{\mathcal{O}_{Y,f(P)}} \mathcal{F}_{f(P)} + \operatorname{depth}_{\mathcal{O}_{X_{f(P)},P}} \mathcal{O}_{X_{f(P)},P}.$$

Lemma 4.2. *If $\mathscr{G}$ is a full-dimensional coherent S_d sheaf on the scheme X, and $f : \mathscr{X} \to X$ is a flat, relatively Cohen–Macaulay family, then $\mathscr{F} := f^*\mathscr{G}$ is S_d as well.*

Proof. For every $x \in X$,

$$
\begin{aligned}
\operatorname{depth} \mathscr{F}_x &= \underbrace{\operatorname{depth} \mathscr{G}_{f(x)} + \operatorname{depth} \mathcal{O}_{\mathscr{X}_{f(x)},x}}_{\text{Fact 4.1}} = \underbrace{\operatorname{depth} \mathscr{G}_{f(x)} + \dim \mathcal{O}_{\mathscr{X}_{f(x)},x}}_{\mathscr{X}_{f(x)} \text{ is Cohen–Macaulay}} \\
&\geq \underbrace{\min\{d, \dim \mathcal{O}_{X,f(x)}\} + \dim \mathcal{O}_{\mathscr{X}_{f(x)},x}}_{\mathscr{G} \text{ is } S_d} \\
&\geq \min\{d, \dim \mathcal{O}_{X,f(x)} + \dim \mathcal{O}_{\mathscr{X}_{f(x)},x}\} \\
&= \underbrace{\min\{d, \dim \mathcal{O}_{\mathscr{X},x}\}}_{\substack{\dim \mathcal{O}_{X,f(x)} + \dim \mathcal{O}_{\mathscr{X}_{f(x)},x} = \dim \mathcal{O}_{\mathscr{X},x} \\ \text{by [Matsumura 1989, Theorem 15.1.ii].}}} \qquad \square
\end{aligned}
$$

Lemma 4.3. *Assume that we are in the following situation:*

- *Y is a projective scheme,*
- *X is the projectivized cone over Y,*
- *P is the vertex of X and $V := X \setminus P$,*
- *d is an integer such that $2 \leq d \leq \dim X$, and*
- *$\mathscr{F}$ is a coherent sheaf on X, such that $\mathscr{F}|_V = \pi^*\mathscr{G}$ for some S_d coherent sheaf $\mathscr{G}$ on Y, where $\pi : V \to Y$ is the natural projection.*

Then the following conditions are equivalent:

(1) $\operatorname{depth} \mathscr{F}_P \geq d$.

(2) $\operatorname{depth} \mathscr{F}_P \geq \min\{d, \dim \mathcal{O}_{X,P}\}$.

(3) *$\mathscr{F}$ is S_d.*

(4) *$\mathscr{F}$ is S_2 and $H^i(Y, \mathscr{G}(n)) = 0$ for all $0 < i < d - 1$ and $n \in \mathbb{Z}$.*

Proof. Since $\mathscr{G}$ is S_d, $\mathscr{F}$ is S_d everywhere except at the vertex P by Lemma 4.2. Hence, using the assumption $d \leq \dim X$,

$$
\begin{gathered}
\mathscr{F} \text{ is } S_d, \\
\Updownarrow \\
\operatorname{depth} \mathscr{F}_P \geq \min\{d, \dim \mathcal{O}_{X,P}\} \\
\Updownarrow \\
\operatorname{depth} \mathscr{F}_P \geq d \\
\Updownarrow \\
H^i_P(Z, \mathscr{F}) = 0 \text{ for all } i < d \text{ and for the affine cone } Z,
\end{gathered}
$$

where the latter equivalence follows from [Hartshorne 1977, Exercises III.3.4.b and III.2.5]. So, we are left to show that the condition $H^i_P(Z, \mathscr{F}) = 0$ for all $i < d$ is equivalent to point (4). Define $U := Z \setminus P$. Then there is a long exact sequence

$$\cdots \to H^i_P(Z, \mathscr{F}) \to H^i(Z, \mathscr{F}) \to H^i(U, \mathscr{F}) \to \cdots .$$

Since Z is affine $H^i(Z, \mathscr{F}) = 0$ for all $i > 0$. Hence

$$H^i(U, \mathscr{F}) \cong H^{i+1}_P(Z, \mathscr{F}) \text{ for all } i > 0. \tag{4.3.a}$$

So, since $H^0_P(Z, \mathscr{F}) = H^1_P(Z, \mathscr{F}) = 0$ is assumed in point (4), it is enough to show that for all $0 < i < d - 1$,

$$H^i(U, \mathscr{F}) \cong \bigoplus_{n \in \mathbb{Z}} H^i(Y, \mathscr{G}(n)). \tag{4.3.b}$$

In fact we will prove this for all i. First, notice that $U \cong \operatorname{Spec}_Y(\bigoplus_{n\in\mathbb{Z}} \mathscr{O}_Y(n))$ and the natural projection $\operatorname{Spec}_Y(\bigoplus_{n\in\mathbb{Z}} \mathscr{O}_Y(n)) \to Y$ can be identified with $\pi|_U$ via this isomorphism. Hence $(\pi|_U)_*\mathscr{O}_U \cong \bigoplus_{n\in\mathbb{Z}} \mathscr{O}_Y(n)$ and $R^i(\pi|_U)_*\mathscr{O}_U = 0$ for $i > 0$. So:

$$\begin{aligned} H^i(U, \mathscr{F}) \cong H^i(Y, (\pi|_U)_*\mathscr{F}|_U) \cong H^i(Y, (\pi|_U)_*(\pi|_U)^*\mathscr{G}) \cong \\ \cong H^i(Y, \bigoplus_{n\in\mathbb{Z}} \mathscr{G}(n)) \cong \bigoplus_{n\in\mathbb{Z}} H^i(Y, \mathscr{G}(n)) \end{aligned}$$

as claimed in (4.3.b). □

5. Construction of varieties with prescribed singularities

In this section, normal S_j (but not S_{j+1}) varieties of dimension $n \geq 3$ with S_l (but not S_{l+1}), $\mathbb{Q}$-line bundle canonical sheaves are constructed for certain values of j and l. They are going to be used in Section 6 and in Section 7 to build families with prescribed base change behavior for the relative canonical sheaves. First we need some lemmas.

Lemma 5.1. *If H is a general, high enough degree hypersurface in a projective variety X, then $H^i(H, \mathscr{O}_H) \cong H^i(X, \mathscr{O}_X)$ for every i such that $0 < i < \dim H$.*

Proof. We start with the usual exact sequence

$$0 \to \mathscr{O}_X(-H) \to \mathscr{O}_X \to \mathscr{O}_H \to 0. \tag{5.1.a}$$

Since $\deg H \gg 0$,

$$H^i(X, \mathscr{O}_X(-H)) = 0 \text{ whenever } i < \dim X. \tag{5.1.b}$$

Taking the cohomology long exact sequence of (5.1.a) and using (5.1.b) finishes the proof. □

Iterated use of Lemma 5.1 yields the following:

Lemma 5.2. *If H is a general, high enough degree complete intersection (that is, it is the intersection of hypersurfaces, all of which are high enough degree) in a smooth projective variety X, then $H^i(H, \mathcal{O}_H) \cong H^i(X, \mathcal{O}_X)$ for every i such that* $0 < i < \dim H$.

Finally, iterated use of the adjunction formula yields the following:

Lemma 5.3. *If H is a complete intersection in a smooth projective variety X, then $\omega_H \cong \omega_X(m)|_H$ for some $m > 0$ (here $\mathcal{O}_X(1)$ is the very ample line bundle given by the projective embedding of X).*

Proposition 5.4. *For each $n \geq 2$ and $2 \leq d, l \leq n$ such that $l \leq d$ and $d + l \leq n + 2$ there is an $(n+1$-dimensional projective variety X_{n+1} for which*

- *X_{n+1} is the projective cone over a smooth projective variety Y_n with vertex P,*
- *X_{n+1} is S_d and* $\operatorname{depth} \mathcal{O}_{X_{n+1},P} = d$,
- *$\omega_{X_{n+1}}$ is S_l and* $\operatorname{depth} \omega_{X_{n+1},P} = l$, *and*
- *$\omega_{X_{n+1}}$ is a $\mathbb{Q}$-line bundle.*

Proof. Take first two Calabi–Yau hypersurfaces Z and W of dimensions $d-1$ and $n+1-l$, respectively. Let $Y := Y_n$ be a general high enough degree complete intersection of codimension $d-l$ in $Z \times W$. Notice that $d - l \geq 0$ by assumption. Finally, let X_{n+1} be the projective cone over Y polarized by $\mathcal{O}_Y(1) := \mathcal{O}_{Z\times W}(p)|_Y$ for some $p \gg 0$ (after fixing Y). Here $\mathcal{O}_{Z\times W}(1)$ is the very ample line bundle on $Z \times W$ coming from its projective embedding.

The Künneth isomorphism yields

$$H^q(Z \times W, \mathcal{O}_{Z\times W}) \cong \bigoplus_{r=0}^{q} H^r(Z, \mathcal{O}_Z) \otimes H^{q-r}(W, \mathcal{O}_W).$$

Since Z and W are Calabi–Yau hypersurfaces of dimension $d-1$ and $n+1-l$, respectively, the following holds for their cohomology table:

$$\begin{aligned} H^q(Z, \mathcal{O}_Z) \neq 0 &\iff q = 0 \text{ or } d-1, \\ H^s(W, \mathcal{O}_W) \neq 0 &\iff s = 0 \text{ or } n+1-l. \end{aligned}$$

Hence

$$H^q(Z \times E, \mathcal{O}_{Z\times E}) \neq 0 \iff q = 0, d-1, n+1-l \text{ or } n-l+d.$$

Using Lemma 5.2 yields, for $0 < q < n$,

$$H^q(Y, \mathcal{O}_Y) \neq 0 \iff q = d-1 \text{ or } n+1-l. \tag{5.4.a}$$

Since $p \gg 0$, also,

$$H^q(Y, \mathcal{O}_Y(r)) = 0 \quad \text{for every } r \text{ and } 0 < q < n.$$

Then by Lemma 4.3 using that $d-1 \leq n+1-l$ by assumption, X_{n+1} is S_d and $\operatorname{depth} \mathcal{O}_{X_{n+1},P} = d$ (X_{n+1} is S_2 at the vertex, because $p \gg 0$ and hence Y is projectively normal).

Serre duality implies that

$$H^q(Y, \omega_Y) \cong (H^{n-q}(Y, \mathcal{O}_Y))^*.$$

So, by (5.4.a), for $0 < q < n$,

$$H^q(Y, \omega_Y) \neq 0 \iff q = l-1 \text{ or } n+1-d.$$

Since X_{n+1} is an affine bundle over Y, $\omega_{X_{n+1}}$ is isomorphic to the pullback of ω_Y outside of the vertex. Then by Lemma 4.3 using that $l-1 \leq n+1-d$, $\omega_{X_{n+1}}$ is S_l and $\operatorname{depth} \omega_{X_{n+1},P} = l$ ($\omega_{X_{n+1}}$ is always S_2 by Proposition 3.3(8)).

We have left to show that the $\omega_{X_{n+1}}$ are $\mathbb{Q}$-Cartier. By Lemma 5.3,

$$\omega_Y^{\otimes p} \cong (\omega_{Z \times E}(m)|_Y)^{\otimes p} \cong (\mathcal{O}_{Z \times E}(m)|_Y)^{\otimes p} \cong \mathcal{O}_Y(m).$$

That is, $\omega_Y^{\otimes p}$ is an integer multiple of the polarization of Y used at the construction of X_{n+1}. Hence, [Hacon and Kovács 2010, Exercise 3.5] concludes the proof. □

6. Construction of families without the base change property

In this section we present the proof of Theorem 1.2. The following lemma contains the key argument. It is also used in the proofs of Proposition 1.5 and Theorem 1.6.

Lemma 6.1. *Let $f : \mathcal{H} \to B = \mathbb{P}^1$ be a flat pencil of hypersurfaces of a quasi-projective, equidimensional scheme X, such that $\mathcal{I}_{\mathcal{H}, X \times B}$ is a line bundle and $\mathcal{H}$ and the closed fibers of f are S_2 and G_1.*

(1) *If ω_X is S_3, the restriction map $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0} \to \omega_{\mathcal{H}_0}$ is an isomorphism.*

(2) *If $\operatorname{depth} \omega_{X,P} \not\geq \min\{3, \dim \mathcal{O}_{X,P}\}$ for some $P \in X$, such that $P \in \mathcal{H}_0$, but $P \notin \mathcal{H}_\infty$, then $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0} \not\cong \omega_{\mathcal{H}_0}$.*

Proof. Notice that by flatness of $\mathcal{H}$ and by Lemma 3.5, it does make sense to talk about $\omega_{\mathcal{H}/B}$. Define $\mathcal{X} := X \times B$. Then $\mathcal{H}$ is a hypersurface of $\mathcal{X}$. By Proposition 3.6.(2) there is a homomorphism $\omega_{\mathcal{X}/B}(\mathcal{H})|_{\mathcal{H}} \to \omega_{\mathcal{H}/B}$, which is an isomorphism in codimension one, over the Gorenstein locus of $\mathcal{H}$. Fix this homomorphism for the course of the proof.

Now, we show point (1). If ω_X is S_3, then so is $\omega_{\mathcal{X}/B} \cong p_1^* \omega_X$ by Lemma 4.2. Hence, by Fact 3.2.(2), $\omega_{\mathcal{X}/B}(\mathcal{H})|_{\mathcal{H}}$ is S_2. Then, since $\omega_{\mathcal{H}/B}$ is S_2 by Proposition 3.3(9), $\omega_{\mathcal{X}/B}(\mathcal{H})|_{\mathcal{H}} \to \omega_{\mathcal{H}/B}$ is an isomorphism everywhere by [Hartshorne 1994, Theorems 1.9 and 1.12]. However, for every $P \in \mathcal{X}_0$,

$$\operatorname{depth} \omega_{\mathcal{X}/B,P} = \underbrace{\operatorname{depth} \omega_{X,p_1(P)} + 1}_{\text{Fact 4.1, applied to } \omega_{\mathcal{X}/B} \cong p_1^* \omega_X} \geq \underbrace{\min\{3, \dim \mathcal{O}_{X,p_1(P)}\} + 1}_{\omega_X \text{ is } S_3} = \min\{4, \dim \mathcal{O}_{\mathcal{X},P}\}. \quad (6.1.a)$$

But then, for every $P \in \mathcal{H}_0$,

$$\operatorname{depth} \omega_{\mathcal{H}/B,P} = \underbrace{\operatorname{depth}(\omega_{\mathcal{X}/B}(\mathcal{H})|_{\mathcal{H}})_P}_{\omega_{\mathcal{H}/B} \cong \omega_{\mathcal{X}/B}(\mathcal{H})|_{\mathcal{H}}} \geq \underbrace{\min\{3, \dim \mathcal{O}_{\mathcal{H},P}\}}_{\text{Fact 3.2.(2) and (6.1.a)}},$$

which implies point (1) by Proposition 3.3(11).

To prove point (2), denote by U the open set $p_1^{-1}(X \setminus (\mathcal{H}_0 \cap \mathcal{H}_\infty)) \subseteq \mathcal{X}$. This is the set of points, the first coordinates of which are not contained in every element of the pencil $\mathcal{H} \to B$. By Proposition 3.3(4) and 3.3(5), we may replace $\mathcal{X}$ by U, or with other words, X by $X \setminus (\mathcal{H}_0 \cap \mathcal{H}_\infty)$. In particular, then $\mathcal{H}_0 \cap \mathcal{H}_\infty = \varnothing$ and P is an arbitrary point of $\mathcal{H}_0$, such that

$$\operatorname{depth} \omega_{X,P} \not\geq \min\{3, \dim \mathcal{O}_{X,P}\}. \quad (6.1.b)$$

Then all fibers of the projection $p_1|_{\mathcal{H}} : \mathcal{H} \to X$ have dimension zero. So, for every $Q \in \mathcal{H}$,

$$\operatorname{depth} \omega_{\mathcal{X}/B,Q} = \underbrace{\operatorname{depth} \omega_{X,p_1(Q)} + 1}_{\text{Fact 4.1, applied to } \omega_{\mathcal{X}/B} \cong p_1^* \omega_X} \geq \underbrace{\min\{2, \dim \mathcal{O}_{X,p_1(Q)}\} + 1}_{\text{Proposition 3.3.(8)}} = \min\{3, \dim \mathcal{O}_{\mathcal{X},Q}\}.$$

Then, repeating the argument of the previous paragraph $\omega_{\mathcal{X}/B}(\mathcal{H})|_{\mathcal{H}} \cong \omega_{\mathcal{H}/B}$. Also, at the fixed $P \in \mathcal{H}_0$, the following computation estimates the depth more precisely.

$$\operatorname{depth} \omega_{\mathcal{X}/B,P} = \underbrace{\operatorname{depth} \omega_{X,P} + 1}_{\text{Fact 4.1, applied to } \omega_{\mathcal{X}/B} \cong p_1^* \omega_X} \not\geq \underbrace{\min\{3, \dim \mathcal{O}_{X,P}\} + 1}_{(6.1.b)} = \min\{4, \dim \mathcal{O}_{\mathcal{X},P}\} \quad (6.1.c)$$

However, then

$$\operatorname{depth} \omega_{\mathcal{H}/B,P} = \underbrace{\operatorname{depth}(\omega_{\mathcal{X}/B}(\mathcal{H})|_{\mathcal{H}})_P}_{\omega_{\mathcal{H}/B} \cong \omega_{\mathcal{X}/B}(\mathcal{H})|_{\mathcal{H}}} \not\geq \underbrace{\min\{3, \dim \mathcal{O}_{\mathcal{H},P}\}}_{\text{by Fact 3.2.(2)}},$$

which concludes the proof by Proposition 3.3(11). □

Remark 6.2. The condition of $\mathscr{I}_{\mathscr{H},\mathscr{X}}$ being a line bundle in Lemma 6.1 might look superfluous at first sight, since $\mathscr{H}$ is a hypersurface in $\mathscr{X}$. However, according to Section 2, the latter only means that $\mathscr{H}$ is the zero locus of some special section of a line bundle. That is, $\mathscr{H}$ or $\mathscr{H}_b$ for some $b \in B$ could contain an entire irreducible component of $\mathscr{X}$ or $\mathscr{X}_b$, respectively. Then Proposition 3.6 would not apply. Such situations should definitely be avoided.

The following is the main construction to which Lemma 6.1 is applied in this section.

Construction 6.3. Consider a projective cone X over a variety Y. Let P be the vertex of X. Take two hypersurfaces in X. The first one H is a projective cone over a degree d generic hypersurface D of Y. The second one $\widetilde{H}$ is a general degree d hypersurface of X. Denote by $\mathscr{H} \to B$ the pencil generated by H and $\widetilde{H}$ (for which $H = \mathscr{H}_0$ and $\widetilde{H} = \mathscr{H}_\infty$). Throughout the paper we allow ourselves to replace this family by its restriction to a small enough open neighborhood of $0 \in B$. Furthermore, when we compute stable reduction in Section 8, we will assume that $d \gg 0$.

Lemma 6.4. *In the situation of Construction 6.3, if X is S_3 and Y is R_1, then*

(1) *$\mathscr{H}$ and the closed fibers of f are normal varieties,*

(2) *$\mathscr{I}_{\mathscr{H},X\times B}$ is a line bundle,*

(3) *f is flat.*

Proof. We use the notation $\mathscr{X} := X \times B$. Since Y is a variety (that is, integral), so are D, X, $\mathscr{X}$, $\mathscr{H}_0$ and $\mathscr{H}_\infty$. By the definition of a pencil, $\mathscr{H}$ is defined by a single nonzero equation locally on $\mathscr{X}$. So, since $\mathscr{X}$ is integral, point (2) follows. Similarly, for every $b \in B$, $\mathscr{H}_b$ is defined locally by a single nonzero equation locally. Hence by integrality of X, $\mathscr{I}_{\mathscr{H}_b,\mathscr{X}_b}$ is also a line bundle for every $b \in B$. Thus, Lemma 3.5 yields point (3).

To prove point (1), note that $\mathscr{X}$ is S_3 by Lemma 4.2 and by the assumption of the lemma. So, by Fact 3.2, $\mathscr{H}$ and the closed fibers of $\mathscr{H}$ are S_2. (Remember, in Construction 6.3 we allowed ourselves to shrink B around $0 \in B$). Since D is general and Y is R_1, D is R_1 as well by Bertini's theorem; see [Harris 1992, Theorem 17.16]. Therefore, so is H. Then, by possibly shrinking B, each closed fiber of $\mathscr{H}$ is R_1. Thus all closed fibers of $\mathscr{H}$, and $\mathscr{H}$ itself, are normal. □

Theorem 6.5. *In the situation of Construction 6.3, if* $\dim X \geq 3$, *X is S_3, Y is R_1, and* $\operatorname{depth} \omega_{X,P} = 2$, *then*

$$\omega_{\mathscr{H}/B}|_{\mathscr{H}_0} \ncong \omega_{\mathscr{H}_0}. \tag{6.5.a}$$

In addition:

(1) *If ω_X is a $\mathbb{Q}$-line bundle, then $\omega_{\mathscr{X}/B}$ is a $\mathbb{Q}$-line bundle. In particular then $\omega_{\mathscr{X}_b}$ is a $\mathbb{Q}$-line bundle for all $b \in B$.*

(2) *If X is S_d and* $\operatorname{depth} \mathscr{O}_{X,P} = d$, *then $\mathscr{X}_b$ is S_{d-1} for all $b \in B$, and*

$$\operatorname{depth} \mathscr{O}_{\mathscr{X}_0,P} = d-1.$$

Proof. By Lemma 6.4, we may apply Lemma 6.1(2) to obtain the main statement of the theorem.

To prove addendum (1), note that the normality of $\mathscr{X}$ and $\mathscr{X}_b$ for every $b \in B$, [Hartshorne 1994, Theorem 1.12] and Proposition 3.6 imply that

$$\omega_{\mathscr{X}_b}^{[n]} \cong (\omega_{\mathscr{X}_b}(\mathscr{X}_b)|_{\mathscr{X}_b})^{[n]} \text{ for any } b \in B, \text{ and } \omega_{\mathscr{X}/B}^{[n]} \cong (\omega_{\mathscr{X}/B}(\mathscr{X})|_{\mathscr{X}})^{[n]} \qquad (6.5.b)$$

for all $n \in \mathbb{Z}$. Hence if ω_X is a $\mathbb{Q}$-line bundle, then (6.5.b) implies that so is $\omega_{\mathscr{X}/B}$ and $\omega_{\mathscr{X}_b}$ for all $b \in B$. To prove (2) we use Fact 3.2 once again. □

Theorem 1.2 now follows as a corollary:

Proof of Theorem 1.2. It follows by combining Proposition 5.4 (setting $d = j+1$ and $l = 2$), Construction 6.3 and Theorem 6.5. □

7. Degenerations and Serre's condition

We turn to proving the statements relating Serre's condition S_d to degenerations of flat families. The first half of the section is devoted to Theorem 1.3.

Remark 7.1. By the restriction homomorphism $\omega_{\mathscr{X}/B} \to \omega_{\mathscr{X}_0}$ we mean any homomorphism obtained as in Proposition 3.3(2).

Theorem 1.3 might look technical, but it applies for example to the special case, when the general fiber is Cohen–Macaulay and the central fiber contains at least one closed point with depth $n-1$.

Proof of Corollary 1.4 and Proposition 1.5. Fix a $2 \leq i < n-1$. Consider the projective cone X given by Proposition 5.4, setting $d = i+1$ and $l = 3$. Use then Construction 6.3 for X. By Lemma 6.4, this yields a flat family $f : \mathscr{X} \to B$ of normal varieties for which Lemma 6.1(1) applies. That is, the restriction homomorphisms $\omega_{\mathscr{X}/B}|_{\mathscr{X}_b} \to \omega_{\mathscr{X}_b}$ are isomorphisms for every $b \in B$. Finally, since X is Cohen–Macaulay outside of P and $\operatorname{depth} \mathscr{O}_{X,P} = i+1$, by Fact 3.2, $\mathscr{X}_b$ is Cohen–Macaulay outside of P, where $\operatorname{depth} \mathscr{O}_{\mathscr{X}_0,P} = i$. □

We also need the following lemma in the proof of Theorem 1.3.

Lemma 7.2. *If $f : \mathscr{X} \to B$ is a flat morphism of schemes onto a smooth curve, $\mathscr{F}$ is a coherent $\mathscr{O}_{\mathscr{X}}$-module on $\mathscr{X}$, and $P \in \mathscr{X}_0$, then*

(1) *$\mathcal{T}or^1_{\mathscr{X}}(\mathscr{F}, \mathscr{O}_{\mathscr{X}_0})_P \neq 0$ if and only if $\mathscr{F}$ has an associated component W such that $P \in W \subseteq \mathscr{X}_0$, and*

(2) $\mathcal{T}or^i_{\mathcal{H}}(\mathcal{F}, \mathcal{O}_{\mathcal{H}_0}) = 0$ *for* $i > 1$.

Proof. By restricting B, we may assume that $\mathcal{I}_{0,B} \cong \mathcal{O}_B$. Denote by s a generator of $\mathcal{I}_{0,B}$ and consider the exact sequence

$$0 \to \mathcal{O}_{\mathcal{H}} \xrightarrow{\cdot s} \mathcal{O}_{\mathcal{H}} \to \mathcal{O}_{\mathcal{H}_0} \to 0. \tag{7.2.a}$$

Then the long exact sequence of $\mathcal{T}or^{\bullet}_{\mathcal{H}}(\mathcal{F}, \cdot)$ applied to (7.2.a) yields

$$\mathcal{T}or^1_{\mathcal{H}}(\mathcal{F}, \mathcal{O}_{\mathcal{H}}) = 0 \to \mathcal{T}or^1_{\mathcal{H}}(\mathcal{F}, \mathcal{O}_{\mathcal{H}_0}) \to \mathcal{F} \xrightarrow{\cdot s} \mathcal{F}.$$

Hence $\mathcal{T}or^1_{\mathcal{H}}(\mathcal{F}, \mathcal{O}_{\mathcal{H}_0})_P \neq 0$ if and only if s annihilates something in $\mathcal{F}_P$, if and only if $\mathcal{F}$ has an associated component W such that $P \in W \subseteq \mathcal{H}_0$.

Another part of the long exact sequence of $\mathcal{T}or^{\bullet}_{\mathcal{H}}(\mathcal{F}, \cdot)$ applied to (7.2.a) yields the following for $i > 1$:

$$\mathcal{T}or^i_{\mathcal{H}}(\mathcal{F}, \mathcal{O}_{\mathcal{H}}) = 0 \to \mathcal{T}or^i_{\mathcal{H}}(\mathcal{F}, \mathcal{O}_{\mathcal{H}_0}) \to \mathcal{T}or^{i-1}_{\mathcal{H}}(\mathcal{F}, \mathcal{O}_{\mathcal{H}}) = 0.$$

Hence, $\mathcal{T}or^i_{\mathcal{H}}(\mathcal{F}, \mathcal{O}_{\mathcal{H}_0}) = 0$ indeed if $i > 1$. □

Proof of Theorem 1.3. Fix a closed point $x \in \mathcal{H}_0$ with $\operatorname{depth} \mathcal{O}_{\mathcal{H}_0,x} = n-1$, contained in a component $W \subseteq \mathcal{H}_0$ of the locus (1.3.a). The locus (1.3.a) is $\operatorname{supp}(h^{-(n-1)}(\omega^{\bullet}_{\mathcal{H}/B}))$ by Proposition 3.3(7); hence W is also an associated component of $h^{-(n-1)}(\omega^{\bullet}_{X/B})$. Consider an open neighborhood of x, where every closed point has depth at least $n-1$. Replacing $\mathcal{H}$ by this neighborhood, all assumptions of the theorem stay valid, and moreover we may assume that every closed point of $\mathcal{H}$ has depth at least $n-1$. In particular, then

$$h^i(\omega^{\bullet}_{\mathcal{H}/B}) \neq 0 \iff i = -n \text{ or } -(n-1). \tag{7.2.b}$$

Define $\mathcal{E} := h^{-(n-1)}(\omega^{\bullet}_{\mathcal{H}/B})$. By (7.2.b), there is an exact triangle

$$\omega_{\mathcal{H}/B}[n] \to \omega^{\bullet}_{\mathcal{H}/B} \to \mathcal{E}[n-1] \xrightarrow{+1}. \tag{7.2.c}$$

Applying $\cdot \otimes_L \mathcal{O}_{\mathcal{H}_0}$ to (7.2.c) and then considering the long exact sequence of cohomology sheaves yields

$$\begin{aligned} h^{-n-1}(\mathcal{E}[n-1] \otimes_L \mathcal{O}_{\mathcal{H}_0}) \to h^{-n}(\omega_{\mathcal{H}/B}[n] \otimes_L \mathcal{O}_{\mathcal{H}_0}) \to h^{-n}(\omega^{\bullet}_{\mathcal{H}/B} \otimes_L \mathcal{O}_{\mathcal{H}_0}) \\ \to h^{-n}(\mathcal{E}[n-1] \otimes_L \mathcal{O}_{\mathcal{H}_0}) \to h^{-n+1}(\omega_{\mathcal{H}/B}[n] \otimes_L \mathcal{O}_{\mathcal{H}_0}), \end{aligned} \tag{7.2.d}$$

where

- $h^{-n-1}(\mathcal{E}[n-1] \otimes_L \mathcal{O}_{\mathcal{H}_0}) \cong \mathcal{T}or^2_{\mathcal{H}}(\mathcal{E}, \mathcal{O}_{\mathcal{H}_0}) = 0$ by Lemma 7.2,
- $h^{-n}(\omega_{\mathcal{H}/B}[n] \otimes_L \mathcal{O}_{\mathcal{H}_0}) \cong \omega_{\mathcal{H}/B}|_{\mathcal{H}_0}$,
- $h^{-n}(\omega^{\bullet}_{\mathcal{H}/B} \otimes_L \mathcal{O}_{\mathcal{H}_0}) \cong h^{-n}(\omega^{\bullet}_{\mathcal{H}_0}) \cong \omega_{\mathcal{H}_0}$ by Proposition 3.3(1),
- $h^{-n}(\mathcal{E}[n-1] \otimes_L \mathcal{O}_{\mathcal{H}_0}) \cong \mathcal{T}or^1_{\mathcal{H}}(\mathcal{E}, \mathcal{O}_{\mathcal{H}_0})$ and

- $h^{-n+1}(\omega_{\mathcal{H}/B}[n]\otimes_L \mathcal{O}_{\mathcal{H}_0}) \cong h^1(\omega_{\mathcal{H}/B}\otimes_L \mathcal{O}_{\mathcal{H}_0}) = 0$ since $\cdot \otimes_L \mathcal{O}_{\mathcal{H}_0}$ is a left derived functor, so $\omega_{\mathcal{H}/B} \otimes_L \mathcal{O}_{\mathcal{H}_0}$ is supported in negative cohomological degrees.

Therefore, (7.2.d) is isomorphic to the exact sequence

$$0 \to \omega_{\mathcal{H}/B}|_{\mathcal{H}_0} \to \omega_{\mathcal{H}_0} \to \mathcal{T}or^1_{\mathcal{H}}(\mathcal{E}, \mathcal{O}_{\mathcal{H}_0}) \to 0.$$

Since $\mathcal{E}$ has an associated component through x contained in $\mathcal{H}_0$, we know that $\mathcal{T}or^1_{\mathcal{H}}(\mathcal{E}, \mathcal{O}_{\mathcal{H}_0})_x \neq 0$ by Lemma 7.2, which concludes our proof. □

Having finished the proof of Theorem 1.3, the rest of the section is devoted its consequence, Theorem 1.6. See Section 1 for its motivation.

Proof of Theorem 1.6. Since the statement of the theorem is local, we may assume X is affine and hence quasiprojective. Restricting to a sufficiently small neighborhood of a point with depth $n-1$, all assumptions of the theorem stay valid and we may assume that all closed points of X have depth at least $n-1$. We use the notation $\mathcal{X} := X \times B$. Let $X = \bigcup_{i=1}^r X_i$ be the decomposition into irreducible components.

Consider a pencil $f : \mathcal{H} \to B = \mathbb{P}^1$ of hypersurfaces of X such that

(1) $\mathcal{H}_0$ contains the entire non-Gorenstein locus,

(2) $\varnothing \neq \mathcal{H}_0 \cap X_j \neq X_j$ for every $1 \leq j \leq r$,

(3) $\mathcal{H}_\infty$ is a general hypersurface.

In particular then,

$$(\mathcal{H}_0 \setminus \mathcal{H}_\infty) \cap X_j \neq \varnothing \quad \text{for every } 1 \leq j \leq r. \tag{7.2.e}$$

By definition of the pencil, if $P \in \mathcal{H}_0 \setminus \mathcal{H}_\infty$, then $P \notin \mathcal{H}_b$ for any $b \neq 0$. Hence assumption (2) and (7.2.e) imply that for all $b \in B$, there is a point of X_j not contained in $\mathcal{H}_b$. Note now, that since X is S_1, by Lemma 4.2, so is $\mathcal{X}$. In particular, then all associated points of X and $\mathcal{X}$ are general points of components. So, since none of the $\mathcal{H}_b$ contains any of the X_j, $\mathcal{I}_{\mathcal{H},\mathcal{X}}$ and $\mathcal{I}_{\mathcal{H}_b,\mathcal{X}_b}$ for every $b \in B$ have nonzero divisor local generators and hence are line bundles. Then $\mathcal{H}$ is flat over B by Lemma 3.5.

Define the loci

$$Z := \overline{\{x \in X \mid x \text{ is closed, } \operatorname{depth} \mathcal{O}_{X,x} = n-1\}},$$
$$W := \overline{\{x \in X \mid x \text{ is closed, } \operatorname{depth} \mathcal{O}_{\mathcal{H}_{f(x)},x} = n-2\}}.$$

By construction and by Fact 3.2, $W_0 = Z$ and $W = (p^{-1}Z)_{\text{red}}$, where $p : \mathcal{H} \to X$ is the natural projection. Let Z' be an irreducible component of Z of the highest dimension. By the choice of $\mathcal{H}_0$ and $\mathcal{H}_\infty$, we have $Z' \subseteq \mathcal{H}_0$, and $Z' \not\subseteq \mathcal{H}_\infty$. Furthermore, $\mathcal{H}_\infty$ does not contain any of the irreducible components of Z. Hence, the general fiber of the map $W \to B$ will have dimension at most $\dim Z' - 1$. So,

W has dimension $\dim Z'$. Hence, $Z' \subseteq W_0$ is an irreducible component of W. In particular, by Theorem 1.3, the restriction morphism $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0} \to \omega_{\mathcal{H}_0}$ is not an isomorphism.

On the other hand assume that ω_X is S_3. Since X is G_2, $\mathcal{H}$ and $\mathcal{H}_b$ are G_1 for every $b \in B$. In fact, $\mathcal{H}$, and $\mathcal{H}_b$ for a general $b \in B$, are G_2 also, but for $\mathcal{H}_0$ only G_1 can be guaranteed. Also, X is S_3 by assumption and $\mathcal{X}$ is S_3 because of Lemma 4.2. Then $\mathcal{H}$ and $\mathcal{H}_b$ are S_2 for every $b \in B$ by Fact 3.2. That is, we may apply Lemma 6.1(1), which states that the restriction homomorphism $\omega_{\mathcal{H}/B}|_{\mathcal{H}_0} \to \omega_{\mathcal{H}_0}$ is an isomorphism. This is a contradiction; hence ω_X cannot be S_3. □

8. Stable reduction

In Construction 6.3, although the general fiber of $\mathcal{H} \to B$ has mild, that is, log canonical, singularities, $\mathcal{H}_0$ is very singular. The failure of base change for $\omega_{\mathcal{H}/B}$ implies that by [Kollár and Kovács 2010, Theorem 7.9] $\mathcal{H}_0$ is not Du Bois. By [ibid., Theorem 1.4], it is also not log canonical. In this section, we compute the stable limit of $\mathcal{H} \to B$. It is the limit at 0 of some stable family $\mathcal{H}' \to \widetilde{B}$. This family has two important properties. First, $\mathcal{H} \times_B \widetilde{B}|_{\widetilde{B}\setminus\{0\}} \cong \mathcal{H}'|_{\widetilde{B}\setminus\{0\}}$ for a finite cover $\phi : (\widetilde{B}, 0) \to (B, 0)$ totally ramified at 0. Second, $(\mathcal{H}')_0$ is log canonical, and hence by [ibid., Theorems 1.4 and 7.9], $\omega_{\mathcal{H}'/\widetilde{B}}$ commutes with base change. So, $(\mathcal{H}')_0$ is the "right" limit of $\mathcal{H}$, and the incompatibility of Theorem 1.2 can be thought of as a consequence of using the wrong limit in Construction 6.3.

Proposition 8.1. *Assuming that Y is smooth, the stable limit of Construction 6.3 is the d-fold cyclic cover of Y ramified exactly over D, with eigen-line bundles $\mathcal{O}_Y(-i)$ for $i = 0, \ldots, d-1$.*

Proof. First, shrink B if necessary so that $\infty \notin B$ and that every fiber apart from $\mathcal{H}_0$ is log canonical. This is possible because the general fiber of $\mathcal{H}$ is smooth by Bertini's theorem. Also, since we assumed that $d \gg 0$, the family $\mathcal{H} \to B$ has canonically polarized fibers and hence is stable over $B^* := B \setminus \{0\}$. Define $\mathcal{X} := X \times B$.

The closed embedding $Y \subseteq \mathbb{P}^{N-1}$ induces a natural closed embedding $X \subseteq \mathbb{P}^N$. This yields very ample line bundles $\mathcal{O}_{\mathbb{P}^N}(1)$ and $\mathcal{O}_X(1)$. Then, $\mathcal{H}$ is the zero locus of a section $f_0 + t f_\infty$ of $\mathcal{O}_{\mathcal{X}}(d) := p_1^* \mathcal{O}_X(d)$ for some $f_0, f_\infty \in H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$, as explained in Section 2.

Choose a basis $z_0, \ldots, z_N$ of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$, such that $z_0, \ldots, z_{N-1}$ form a basis of $H^0(\mathbb{P}^{N-1}, \mathcal{O}_{\mathbb{P}^{N-1}}(1))$. Then f_0 and f_∞ correspond to degree d homogeneous polynomials in variables $z_0, \ldots, z_{N-1}$ and $z_0, \ldots, z_N$, respectively. Also, the fact that $P \notin \mathcal{H}_\infty$ implies that the coefficient of z_N^d in f_∞ is nonzero, say c.

Let $\phi : (\widetilde{B}, 0) \to (B, 0)$ be the degree d cyclic cover branched only at 0, where it is totally ramified, and let s be a local parameter of $\widetilde{B}$ at 0, such that $s^d = t$.

Consider the subscheme $\mathcal{H}' \subseteq \mathcal{X} \times_B \widetilde{B} =: \mathcal{X}_\phi$ defined by

$$f_0(z_0, \dots, z_{N-1}) + s^d f_\infty\big(z_0, \dots, z_{N-1}, \tfrac{1}{s} z_N\big) \in H^0(\mathcal{X}_\phi, \mathcal{O}_{\mathcal{X}_\phi}(d)),$$

where $\mathcal{O}_{\mathcal{X}_\phi}(d)$ is the pullback of $\mathcal{O}_{\mathcal{X}}(d)$ to $\mathcal{X}_\phi$.

By the uniqueness of stable limit, $\mathcal{H}'$ is a stable reduction of $\mathcal{H}$ (that is, a stable family isomorphic generically to the pullback of $\mathcal{H}$), if

(1) $(\mathcal{H}')_0$ is a canonically polarized manifold, and

(2) $\mathcal{H}'|_{\widetilde{B}^*} \cong \mathcal{H}_\phi|_{\widetilde{B}^*}$, where $\mathcal{H}_\phi := \mathcal{H} \times_B \widetilde{B}$ and $\widetilde{B}^* := \widetilde{B} \setminus \{0\}$.

To prove point (1), notice that $(\mathcal{H}')_0$ is defined by the zero locus of s on $\mathcal{H}'$ or equivalently by the zero locus of the following section of $\mathcal{O}_X(d)$ on X:

$$f_0(z_0, \dots, z_{N-1}) + c z_N^d.$$

Hence it is the cyclic cover of Y of degree d branched along D with eigensheaves $\mathcal{O}_Y(-i)$ for $0 \le i \le d-1$. So first, it is smooth by [Kollár and Mori 1998, Lemma 2.51]. Second, since $(\mathcal{H}')_0$ is contained in the smooth part of X, we have $\omega_{(\mathcal{H}')_0} \cong \omega_X(d)|_{(\mathcal{H}')_0}$ by Proposition 3.6 and it is a line bundle. So, since $d \gg 0$, $(\mathcal{H}')_0$ is a canonically polarized manifold indeed.

To prove point (2), notice that the equation of $\mathcal{H}_\phi$ in $\mathcal{X}_\phi$ is

$$f_0(z_0, \dots, z_{N-1}) + s^d f_\infty(z_0, \dots, z_{N-1}, z_N) \in H^0(\mathcal{X}_\phi, \mathcal{O}_{\mathcal{X}_\phi}(d)).$$

Hence, $\mathcal{H}_\phi|_{\widetilde{B}^*} \cong \mathcal{H}'|_{\widetilde{B}^*}$ via the isomorphism induced by the following automorphism of $\mathbb{P}^N \times \widetilde{B}^*$:

$$(z_0, \dots, z_{N+1}, z_N) \mapsto (z_0, \dots, z_{N-1}, s z_N).$$

We proved both points (1) and (2). Consequently, $\mathcal{H}'$ is a stable reduction of $\mathcal{H}$ indeed. Through the course of the proof of point (1), we also proved that $(\mathcal{H}')_0$ is indeed the cyclic cover described in the statement of the proposition. □

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Two ways to degenerate the Jacobian are the same

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We provide sufficient conditions for the line bundle locus in a family of compact moduli spaces of pure sheaves to be isomorphic to the Néron model. The result applies to moduli spaces constructed by Eduardo Esteves and Carlos Simpson, extending results of Busonero, Caporaso, Melo, Oda, Seshadri, and Viviani.

1. Introduction

1.1. ***Background.*** This paper relates two different approaches to extending families of Jacobian varieties. Recall that if X_0 is a smooth projective curve of genus g, then the associated Jacobian variety is a g-dimensional smooth projective variety J_0 that can be described in two different ways: as the universal abelian variety that contains X_0 (the Albanese variety), and as the moduli space of degree 0 line bundles on X_0 (the Picard variety). If $X_U \to U$ is a family of smooth, projective curves, then the Jacobians of the fibers fit together to form a family $J_U \to U$. In this paper, U will be an open subset of a smooth curve B (or, more generally, a Dedekind scheme), and we will be interested in extending J_U to a family over B. Corresponding to the two different ways of describing the Jacobian (Albanese vs. Picard) are two different approaches to extending the family $J_U \to U$.

Viewing the Jacobian as the Albanese variety, it is natural to try to extend $J_U \to U$ by extending it to a family of group varieties over B. Néron [1964] showed that this can be done in a canonical way. He worked with an arbitrary family of abelian varieties $A_U \to U$ and proved that there is a unique B-smooth group scheme $\mathrm{N} := \mathrm{N}(A_U) \to B$ extending $A_U \to U$ which is universal with respect to a natural mapping property. This scheme is called the Néron model. Arithmetic geometry has seen the use of the Néron model in a number of important results, e.g., [Mazur 1972; 1977; Mazur and Wiles 1984; Gross 1990]. The Néron model of a Jacobian variety plays a particularly prominent role, and an alternative description

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of this scheme in terms of the relative Picard functor was given by Raynaud [1970]. We primarily work with Raynaud's description, which is recalled in Section 2.

An alternative approach, suggested by viewing the Jacobian as the Picard variety, is to extend $J_U \to U$ as a family of moduli spaces of sheaves. This approach was first proposed by Mayer and Mumford [1964]. Typically, one first extends $X_U \to U$ to a family of curves $X \to B$ and then extends J_U to a family $\bar{J} \to B$ with the property that the fiber over a point $b \in B$ is a moduli space of sheaves on X_b parametrizing certain line bundles, together with their degenerations. In this paper, we show that the line bundle locus J in $\bar{J}$ is canonically isomorphic to the Néron model for some schemes $\bar{J}$ constructed in the literature.

To state this more precisely, we need to specify which schemes $\bar{J}$ we consider. The problem of constructing such a family of moduli spaces has been studied by many mathematicians, and they have constructed a number of different compactifications; see for example [Ishida 1978; D'Souza 1979; Oda and Seshadri 1979; Altman and Kleiman 1980; Caporaso 1994; Simpson 1994; Pandharipande 1996; Jarvis 2000; Esteves 2001]. Many of the difficulties to performing such a construction arise from the fact that, when X_b is reducible, the degree 0 line bundles on a fiber X_b do not form a bounded family.

For simplicity, assume the residue field $k(b)$ is algebraically closed and X_b is reduced with components labeled $X_1, \dots, X_n$. Given a line bundle $\mathcal{M}$ of degree 0 on X_b, the sequence $(\deg(\mathcal{M}|_{X_1}), \dots, \deg(\mathcal{M}|_{X_n}))$ is called the multidegree of $\mathcal{M}$. This sequence must sum to 0, but may otherwise be arbitrary, which implies unboundedness. A bounded family can be obtained by fixing the multidegree, and typically the scheme $\bar{J}$ is defined so that it parametrizes (possibly coarsely) line bundles (and their degenerations) that satisfy a numerical condition on the multidegree. This paper focuses on constructions given by Simpson [1994] and by Esteves [2001], which we now describe in more detail.

For the Simpson moduli space, the numerical condition imposed on line bundles is slope semistabilty with respect to an auxiliary ample line bundle. This condition arises from the method of construction: the moduli space is constructed using geometric invariant theory (GIT), and slope stability corresponds to GIT stability. In general, the Simpson moduli space is a coarse moduli space in the sense that nonisomorphic sheaves may correspond to the same point of the space, but it contains an open subscheme (the stable locus) that is a fine moduli space, and we will work exclusively with this locus. Families of moduli spaces of semistable sheaves have been constructed for arbitrary families of projective schemes, but we will only be concerned with the case of families of curves.

The moduli spaces of Esteves parametrize sheaves that are σ-quasistable. Like slope stability, σ-quasistabillity is a numerical conditions on the multidegree, but it is defined in terms of an auxiliary vector bundle $\mathcal{E}$ and a section σ, rather than an ample line bundle. The moduli spaces are constructed for families over an arbitrary locally noetherian base, but strong conditions are required of the fibers: They must be geometrically reduced. The space is constructed as a closed subspace of a (nonnoetherian, nonseparated) algebraic space that was constructed in [Altman and Kleiman 1980]. For nodal curves, Melo and Viviani [2012] describe the relation between the Esteves moduli spaces and the Simpson moduli spaces. However, here we treat these moduli spaces separately.

For a discussion of the relation between these moduli spaces and other moduli spaces constructed in the literature, the reader is directed to [Alexeev 2004; Casalaina-Martin et al. 2011, Section 2]. The reader familiar with the work of Caporaso is warned of one potential point of confusion. In [Caporaso 1994], the compactified Jacobian associated to a stable curve X parametrizes pairs (Y, L) consisting of a line bundle L on a nodal curve Y stably equivalent to X that satisfies certain conditions. The line bundle locus J that we study corresponds to the locus parametrizing pairs (Y, L) with $X = Y$.

1.2. ***Main result.*** The main result of this paper relates the line bundle locus in a proper family of moduli spaces of sheaves to the Néron model of the Jacobian:

Theorem 1. *Fix a Dedekind scheme B. Let $f : X \to B$ be a family of geometrically reduced curves with regular total space X and smooth generic fiber X_η. Let $J \subset \bar{J}$ the locus of line bundles in one of the following moduli spaces*:

- *the Esteves compactified Jacobian $\bar{J}^{\sigma}_{\mathscr{E}}$;*
- *the Simpson compactified Jacobian $\bar{J}^{0}_{\mathscr{L}}$ associated to an f-ample line bundle $\mathscr{L}$ such that slope semistability coincides with slope stability.*

Then J is the Néron model of its generic fiber.

Theorem 1 is the combination of Corollaries 4.2 and 4.5, which themselves are consequences of Theorem 3.9. Theorem 3.9 is quite general, and we expect that it applies to many other fine moduli spaces of sheaves (but *not* coarse ones). In particular, Theorem 3.9 applies to families of curves with possibly nonreduced fibers, though then general results asserting the existence of a suitable moduli space are unknown (but see Section 4.3 for some simple examples).

The arithmetically inclined reader should note Theorem 1 and the results later in this paper do not place any hypotheses on the base Dedekind scheme B. In particular, we do not assume that the residue fields are perfect. This surprised the author initially as there is a body of work (e.g., [Liu et al. 2004; Raynaud 1970]) showing that various pathologies can arise when $k(b)$ fails to be perfect.

Theorem 1 has interesting consequences for both the Néron model and the compactified Jacobian. One consequence of the theorem is that Néron models of Jacobians can often be constructed over high-dimensional bases. The Néron model of an abelian variety is only defined over a (regular) 1-dimensional base B, but no such dimensional hypotheses are needed to apply the existence results from [Simpson 1994; Esteves 2001]. At the end of Section 4.3, we examine a family $J \to \mathbb{P}^2$ over the plane with the property that a dense, open subset of $\mathbb{P}^2$ is covered by lines C such that the restriction J_C of J is the Néron model of its generic fiber. Surprisingly, while the Néron models fit into a 2-dimensional family, their group structure does not.

Theorem 1 also has interesting consequences for the moduli spaces of Esteves and Simpson. Indeed, if $f : X \to B$ is a family of curves satisfying the hypotheses of the theorem, then both the Esteves Jacobians $J^{\sigma}_{\mathscr{E}}$ and the Simpson Jacobians $J^{0}_{\mathscr{L}}$ (for $\mathscr{L}$ as in the hypothesis) are independent of the particular polarizations, and every such Simpson Jacobian is isomorphic to every Esteves Jacobian. This is not immediate from the definitions. At the end of Section 4.1, we discuss this fact in greater detail and pose a related question.

1.3. ***Past results.*** Certain cases of Theorem 1 were already known. In his (unpublished) thesis, Simone Busonero [2008] established Theorem 1 for certain Esteves Jacobians. A different proof using similar techniques that extends the result to the Simpson moduli spaces is due to Melo and Viviani [2012, Theorem 3.1]. They prove Theorem 1 when the fibers of f are nodal and X is regular. We do not discuss the Caporaso universal compactified Jacobian here, but the relation between that scheme and the Néron model was described by Caporaso [2008a; 2008b; 2012, especially Theorem 2.9]. Earlier still, Oda and Seshadri related their ϕ-semistable compactified Jacobians, also not discussed here, to Néron models [Oda and Seshadri 1979, Corollary 14.4]. In each of those papers, an important step in the proof is a combinatorial argument establishing that, for example, the natural map from the set of σ-quasistable multidegrees to the degree class group is a bijection.

The proof given here does not use any combinatorics, and the idea can be described succinctly. Consider the special case where $B := \operatorname{Spec}(\mathbb{C}[\![t]\!])$, which is a strict henselian discrete valuation ring with algebraically closed residue field. There is a natural map $J \to \mathrm{N}$ to the Néron model coming from the universal property of N, and an application of Zariski's main theorem shows that this morphism is an open immersion. Thus the only issue is set-theoretic surjectivity. Because B is henselian, every point on the special fiber of N is the specialization of a section, so surjectivity is equivalent to the surjectivity of the map $J(\mathbb{C}[\![t]\!]) \to J(\operatorname{Frac} \mathbb{C}[\![t]\!])$ that sends a section to its restriction to the generic fiber. A given point $p \in J(\operatorname{Frac} \mathbb{C}[\![t]\!])$ may be extended to a section $\sigma \in \bar{J}(\mathbb{C}[\![t]\!])$ of $\bar{J}$ by the valuative criteria. As $\bar{J}$ is a fine moduli space, σ corresponds to a family of rank 1, torsion-free sheaves, which in fact must be a family of line bundles because X is factorial. We may conclude that $\sigma \in J(\mathbb{C}[\![t]\!])$, yielding the result.

1.4. ***Questions.*** It would be interesting to know when a Simpson Jacobian $J^0_{\mathscr{L}}$ satisfying the hypotheses of Theorem 1 exists; that is, given a family $f : X \to B$, does there exist an ample line bundle $\mathscr{L}$ such that every $\mathscr{L}$-slope semistable sheaf of degree 0 is stable? We briefly survey the literature on this question at the end of Section 4.2.

More generally, given a family $f : X \to B$, it would be desirable to have a description of the maximal subfunctors of the degree 0 relative Picard functor P^0 representable by a separated B-scheme. We discuss this question in Section 4.3, where we analyze the simple case of genus 1 curves.

1.5. ***Organization.*** We end this introduction with a few technical remarks about the paper. The moduli spaces of sheaves that we consider are moduli spaces of pure sheaves. On a curve, a coherent sheaf is pure if and only if it is Cohen–Macaulay. This condition is also equivalent to the condition of being torsion-free in the sense of elementary algebra when the curve is integral, and the term "torsion-free" is sometimes used in place of "pure".

The term "family of curves" only to refers to families with geometrically irreducible generic fibers. This is done to avoid notational complications concerning multidegrees. Families of curves are required to be proper, but not projective. A family of curves over a Dedekind scheme can fail to be projective (e.g., [Altman and Kleiman 1980, 8.10]), but projectivity is automatic if the local rings of the total space are factorial, which is the main case of interest. (See Proposition 4.1.)

We prove the main results for families over a base scheme S that is the spectrum of a strict henselian discrete valuation ring rather than a more general Dedekind scheme. Doing so lets us make sectionwise arguments because a smooth family of a henselian base admits many sections. Furthermore, this is not a real restriction: Results over a general Dedekind base can be deduced by passing to the strict henselization.

The body of the paper is organized as follows. In Section 2, we review Rayanud's construction of the maximal separated quotient. We then relate this scheme to a general moduli space of line bundles satisfying some axioms in Section 3. Finally, we describe some schemes that satisfy these axioms in the final section, Section 4.

Conventions

1.1. The symbol X_T denotes the fiber product $X \times_S T$.

1.2. The letter S denotes the spectrum of a strict henselian discrete valuation ring with special point 0 and generic point η.

1.3. A curve over a field k is a proper k-scheme $f_0 : X_0 \to \mathrm{Spec}(k)$ that is geometrically connected and of pure dimension 1.

1.4. If B is a scheme, then a family of curves over B is a proper, flat morphism $f : X \to B$ whose fibers are curves and whose geometric generic fibers are irreducible.

1.5. If $f : Y \to B$ is a finitely presented morphism, then we write $Y^{\mathrm{sm}} \subset Y$ for the smooth locus of f.

1.6. A coherent module I_0 on a noetherian scheme X_0 is rank 1 if the localization of I_0 at x is isomorphic to $\mathcal{O}_{X_0,x}$ for every generic point x.

1.7. A coherent module I_0 on a noetherian scheme X_0 is pure if the dimension of $\mathrm{Supp}(I_0)$ equals the dimension of $\mathrm{Supp}(J_0)$ for every nonzero subsheaf J_0 of I_0.

1.8. If $X_0 \to \mathrm{Spec}(k)$ is proper, then the degree of a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}$ is defined by $\deg(\mathcal{F}) := \chi(\mathcal{F}) - \chi(\mathcal{O}_X)$.

2. Raynaud's maximal separated quotient

We begin by reviewing Raynaud's construction of the Néron model of a Jacobian and, more generally, the maximal separated quotient of the relative Picard functor [Raynaud 1970]. Much of this material is also treated in [Bosch et al. 1990, Chapter 9].

Let S be a strict henselian discrete valuation ring with generic point η and special point 0. Given a family of curves $f : X \to S$, the *relative Picard functor* P of f is defined to be the étale sheaf $\mathrm{P} : S\text{-Sch} \to \mathrm{Grp}$ associated to the functor

$$T \mapsto \mathrm{Pic}(X_T). \tag{2-1}$$

Here $\mathrm{Pic}(X_T)$ is the set of isomorphism classes of line bundles on X_T. Rayanud actually defines P to be the associated fppf sheaf, but then observes that this is the same as the associated étale sheaf ([Raynaud 1970, 1.2]; see also [Kleiman 2005, Remark 9.2.11]). The fibers of P are representable by group schemes locally of finite type, and P itself is representable by an algebraic space if and only if f is cohomologically flat [Raynaud 1970, Theorem 5.2]. Regardless of its representability properties, P is locally finitely presented and formally S-smooth.

Inside of P, we may consider the functor $\mathrm{E} : S\text{-Sch} \to \mathrm{Grp}$ that is defined to be the scheme-theoretic closure of the identity section. This is the fppf subsheaf of P generated by the elements $g \in \mathrm{P}(T)$, where $T \to S$ is flat and $g_\eta \in \mathrm{P}(T_\eta)$ is the identity element. When P is a scheme, this coincides with the usual notion of closure. The representability properties of E are similar to those of P: The fibers of E are group schemes locally of finite type, and E is representable by an algebraic space precisely when f is cohomologically flat [Raynaud 1970, Proposition 5.2]. When representable, $\mathrm{E} \to S$ is an étale S-group space; in general, the generic fiber of E is the trivial group scheme, and the special fiber is a group scheme of dimension equal to $h^0(\mathcal{O}_{X_0}) - h^0(\mathcal{O}_{X_\eta})$.

When E is not the trivial S-group scheme, P does not satisfy the valuative criteria of separatedness. We can, however, form the *maximal separated quotient* $\mathrm{Q} : S\text{-Sch} \to \mathrm{Grp}$ of P. By definition, this is the fppf quotient sheaf $\mathrm{Q} := \mathrm{P}/\mathrm{E}$. The maximal separated quotient Q is always representable by a scheme that is S-smooth, separated, and locally of finite type [Raynaud 1970, Theorem 4.1.1, Proposition 8.0.1]. Rather than working directly with Q, we shall primarily work with the slightly smaller subfunctor $\mathrm{Q}^{\tau} : S\text{-Sch} \to \mathrm{Grp}$, which we now define.

Suppose generally that B is a scheme and $G : S\text{-Sch} \to \mathrm{Grp}$ is a B-group functor whose fibers are representable by group schemes locally of finite type. For every point $b \in B$, we may form the identity component $G_b^{\mathrm{o}} \subset G_b$ and the component group G_b/G_b^{o}. The subgroup functor $G^{\tau} \subset G$ is defined to the subfunctor whose T-valued points are elements $g \in G(T)$ with the property that, for every $t \in T$ mapping to $b \in B$, the element $g_t \in G_b(k(t))$ maps to a torsion element of $G_b/G_b^{\mathrm{o}}(k(t))$. If we instead require that g_t maps to the identity element, then we obtain the subgroup functor $G^{\mathrm{o}} \subset G$. Let us examine these constructions when B equals S and G equals P or Q.

The functors P^{o} and P^{τ} coincide, and this common functor is the étale sheaf associated to the assignment sending T to the set of isomorphism classes of line bundles on X_T that fiberwise have multidegree 0. From this description, it is easy to see that $\mathrm{P}^{\mathrm{o}} = \mathrm{P}^{\tau} \subset \mathrm{P}$ is an open subfunctor. Another open subfunctor of P is the subfunctor parametrizing line bundles on X_T with fiberwise degree 0, which we denote by P^0. It is typographically difficult to distinguish between P^0 and P^{o}, but we will not make use of P^{o} in this paper, so this should not cause confusion.

The functors Q^{o} and Q^{τ} are different in general. They are, however, both open subfunctors of Q [Grothendieck 1966b, Theorem 1.1(i.i), Corollary 1.7]. In particular, they are both representable by smooth and separated S-group schemes that are locally of finite type. In fact, both schemes are of finite type over S as their fibers are easily seen to have a finite number of connected components. The condition that $\mathrm{Q}^{\tau} \subset \mathrm{Q}$ is a closed subscheme is important, but slightly subtle. A characterization of this condition is given by [Raynaud 1970, Proposition 8.1.2(iii)]; one sufficient (but not necessary) condition for $\mathrm{Q}^{\tau} \subset \mathrm{Q}$ to be closed is that the local rings of X are factorial.

The factoriality condition is also almost sufficient to ensure that Q^{τ} is the Néron model of its generic fiber. Suppose that the generic fiber of f is smooth, so the generic fiber of $\mathrm{Q}^{\tau} \to S$ is an abelian variety, and thus it makes sense to speak of the Néron model $\mathrm{N} := \mathrm{N}(\mathrm{Q}^{\tau}_{\eta})$. By the universal property, there is a unique morphism $\mathrm{Q}^{\tau} \to \mathrm{N}$ that is the identity on the generic fiber. Theorem 8.1.4 of [Raynaud 1970] states that if the local rings of X are factorial, then $\mathrm{Q}^{\tau} \to \mathrm{N}$ is an isomorphism in the cases that $k(0)$ is perfect and that a certain invariant δ is coprime to the residual characteristic.

The proof uses the characterization of the Néron model in terms of the weak Néron mapping property. Recall that a S-scheme $Y \to S$ is said to be a weak Néron model of its generic fiber if the natural map $Y(S) \to Y(\eta)$ is bijective. If $G \to S$ is a finite type S-group scheme whose generic fiber is an abelian variety, then G is the Néron model of its generic fiber if and only if it satisfies the weak Néron mapping property [Bosch et al. 1990, Section 7.1, Theorem 1].

3. The main theorem

Here we derive the main results for families over a strict henselian discrete valuation ring S with generic point η and special point 0. Specifically, we provide sufficient conditions for the maximal separated quotient Q^{τ} of the Picard functor to be the Néron model and we relate Q^{τ} to a fine moduli space of line bundles that satisfies certain axioms. These moduli spaces are, by definition, subfunctors of a (large) functor that we now define.

Definition 3.1. If T is a S-scheme, then we define $\mathrm{Sheaf}(X_T)$ to be the set of isomorphism classes of $\mathcal{O}_T$-flat, finitely presented $\mathcal{O}_{X_T}$-modules $\mathcal{I}$ on X_T that are fiberwise pure, rank 1, and of degree 0.

The functor $\mathrm{Sh} = \mathrm{Sh}_{X/S} : S\text{-Sch} \to \mathrm{Sets}$ is defined to be the étale sheaf associated to the functor

$$T \mapsto \mathrm{Sheaf}(X_T). \tag{3-1}$$

There is a tautological transformation $\mathrm{P}^0 \to \mathrm{Sh}$ that realizes P^0 as a subfunctor of Sh.

Lemma 3.2. *The subfunctor* $\mathrm{P}^0 \subset \mathrm{Sh}$ *is open.*

Proof. Given a S-scheme T and a morphism $g : T \to \mathrm{Sh}$, we must show that $T \times_{\mathrm{Sh}} \mathrm{P}^0$ is representable by a scheme and that $T \times_{\mathrm{Sh}} \mathrm{P}^0 \to T$ is an open immersion. Thus, let g be given.

By definition, there exists an étale surjection $p : T' \to T$ and a sheaf $\mathcal{I}' \in \mathrm{Sheaf}(X_{T'})$ that represents $g \circ p : T' \to \mathrm{Sh}$. Consider the subset $U' \subset T'$ of points $t \in T'$ with the property that the restriction of $\mathcal{I}$ to the fiber X_t is a line bundle. This locus is open by [Altman and Kleiman 1980, Lemma 5.12(a)], and one may easily show that U' represents $T' \times_{\mathrm{Sh}} \mathrm{P}^0$. A descent argument establishes the analogous property for the image U of U' under $T' \to T$. This completes the proof. □

A remark about topologies: We work with the étale sheaf associated to (3-1), but one could instead work with the associated fppf sheaf. When f is projective, it is a theorem of Altman and Kleiman [1980, Theorem 7.4] that the subfunctor of Sh parametrizing simple sheaves can be represented by a quasiseparated, locally finitely presented S-algebraic space, and hence is an fppf sheaf. We do not know if Sh is an fppf sheaf in general. Here Sh is just used as a tool for keeping track of representable functors, and certainly any representable subfunctor of Sh is an fppf sheaf.

One reason for working with the étale topology instead of the fppf topology is that it makes the following fact easy to prove.

Fact 3.3. The natural map $\mathrm{Sheaf}(X) \to \mathrm{Sh}(S)$ is surjective.

Proof. Let $g \in \mathrm{Sh}(S)$ be given. By definition, there is an étale morphism $S' \to S$ and an element $\mathscr{I}' \in \mathrm{Sheaf}(X_{S'})$ that maps to $g_{S'} \in \mathrm{Sh}(S')$. But S is strict henselian, so $S' \to S$ may be taken to be an isomorphism $S \to S$ [Grothendieck 1967, Proposition 18.8.1(c)], in which case the result is obvious. □

The following two facts about separably closed fields are standard, but they will be used so frequently that it is convenient to record them.

Fact 3.4. If $k(0)$ is a separably closed field and $f_0 : Y_0 \to \mathrm{Spec}(k(0))$ is smooth of relative dimension n, then the closed points of Y_0 with residue field $k(0)$ are dense.

Proof. This is [Bosch et al. 1990, Corollary 13]. The scheme Y_0 can be covered by affine opens U_0 that admit an étale morphism $p : U_0 \to \mathbb{A}^n_{k(0)}$. Certainly, the closed points with residue field $k(0)$ are dense in the image of p. If $v_0 \in \mathbb{A}^n_{k(0)}$ is one such point, then $p^{-1}(v_0)$ is a finite, étale $k(0)$-algebra, and hence a disjoint union of closed points defined over $k(0)$. Density follows. □

Fact 3.4 is typically used in conjunction with the following fact to assert that a smooth morphism has many sections.

Fact 3.5. Let $Y \to S$ be a smooth morphism over strict henselian discrete valuation ring. Then $Y(S) \to Y(k(0))$ is surjective.

Proof. This is [Grothendieck 1967, Corollary 17.17.3], or [Bosch et al. 1990, Proposition 14]. If U and X' are as in the statement of the former, then we must have $U = S$ and $X' \to U$ may be taken to be an isomorphism (again, by [Grothendieck 1967, Proposition 18.8.1(c)]). □

We now prove the main results of the paper.

Proposition 3.6. *Let $f : X \to S$ be a family of curves and $J \subset \mathrm{P}^0$ a subfunctor such that the generic fibers $J_\eta = \mathrm{P}^0_\eta$ coincide. Assume J is represented by a smooth, finitely presented S-scheme.*

If J is S-separated, then $J \to \mathrm{Q}$ is an open immersion. Furthermore, the image is contained in Q^τ provided $\mathrm{Q}^\tau \subset \mathrm{Q}$ is closed (e.g., the local rings of X are factorial).

Proof. This is an application of Zariski's main theorem. We begin by showing that the induced map $J \to \mathrm{Q}$ is injective on closed points. It is enough to verify this after extending base S so that $k(0)$ is algebraically closed. Thus, we will temporarily assume $k := k(0)$ is algebraically closed and work with k-valued points instead of closed points. Given $q \in \mathrm{Q}(k)$, there is nothing to show when the fiber over q is empty. If nonempty, pick $p \in J(k)$ mapping to q. We may invoke Fact 3.5 to assert that there exists a section $\sigma \in J(S)$ with $\sigma(0) = p$.

The fiber of $\mathrm{P} \to \mathrm{Q}$ over q is the set of elements of the form $p+e$ with $e \in \mathrm{E}(k)$ or, equivalently, the elements of $(\sigma + \mathrm{E})(k)$ [Raynaud 1970, Corollary 4.1.2]. Restricting to J, we see that the fiber of q under $J \to \mathrm{Q}$ is the set of k-valued points of $(\sigma + \mathrm{E}) \cap J$. But $(\sigma + \mathrm{E}) \cap J$ is the scheme-theoretic closure of σ in J (by [Grothendieck 1965, 2.8.5]), which is just the image of σ by separatedness. In particular, the preimage of q under $J \to \mathrm{Q}$ must be the singleton set $\{p\}$. This proves that the map is injective on closed points. We now return to the case where S is a henselian discrete valuation ring (so $k(0)$ is no longer assumed to be algebraically closed).

It follows that the set-theoretic fibers of $J \to \mathrm{Q}$ are finite sets. Indeed, if $Z \subset J$ is the locus of points $x \in J$ with the property that x lies in a positive dimensional fiber, then Z is closed by Chevalley's theorem [Grothendieck 1965, 13.1.3]. Furthermore, Z is contained in the special fiber J_0 and contains no closed points. This is only possible if Z is the empty scheme. In other words, the set-theoretic fibers of $J \to \mathrm{Q}$ are 0-dimensional, and hence finite (by [Grothendieck 1964, 14.1.9]).

It follows immediately from Zariski's main theorem [Grothendieck 1961, 4.4.9] that $J \to \mathrm{Q}$ is an open immersion. This proves the first part of the theorem. To complete the proof, observe that flatness implies that the generic fiber of J_η is dense in J [Grothendieck 1965, 2.8.5]. In particular, J is contained in the closure of J_η in Q. The generic fiber of J coincides with the generic fiber of Q^τ, so the closure of this common scheme is contained in Q^τ when $\mathrm{Q}^\tau \subset \mathrm{Q}$ is closed. This completes the proof. □

Remark 3.7. In Proposition 3.6, we do *not* assume that $J \subset \mathrm{P}^0$ is an open subfunctor, but this condition holds in most cases of interest. When open, J is automatically formally smooth and locally of finite presentation. Thus, the key hypothesis in the proposition is that J is represented by a S-separated scheme. A similar remark holds for Theorem 3.9; there the key hypotheses are that $\bar{J}$ satisfies the valuative criteria of properness and that J is representable. Indeed, we do not even need to assume that $\bar{J}$ is representable.

Under stronger assumptions, we can actually show that the natural map $J \to \mathrm{Q}^\tau$ is an isomorphism. The essential point is to prove that J satisfies the weak Néron mapping property. When J can be embedded in a S-proper moduli space $\bar{J}$, this property holds provided that the local rings of X are factorial. The content of this claim is that a line bundle on the generic fiber can specialize only to a line bundle on the special fiber. By localizing, the claim is equivalent to the following lemma, which is based on a proof from [Altman and Kleiman 1979, p. 27 after Step XII].

Lemma 3.8. *Suppose that (R, π) is a discrete valuation ring and $R \to \mathbb{O}$ a local, flat algebra extension with $\mathbb{O}$ noetherian. Let M be a R-flat, finite $\mathbb{O}$-module with the property that $M[\pi^{-1}]$ is free of rank 1 and $M/\pi M$ is a rank 1, pure module. If $\mathbb{O}$ is factorial, then M is free of rank 1.*

Proof. We can certainly assume $\mathbb{O}$ is not the zero ring. To ease notation, we write $\overline{M} := M/\pi M$ and $\overline{\mathbb{O}} := \mathbb{O}/\pi\mathbb{O}$. It is enough to prove that M is isomorphic to a height 1 ideal. Indeed, such an ideal is principal by the factoriality assumption.

We argue by first showing that M is isomorphic to an ideal of $\mathbb{O}$. Let $\bar{\mathfrak{p}}_1, \dots, \bar{\mathfrak{p}}_n$ be the minimal primes of $\overline{\mathbb{O}}$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ the corresponding primes of $\mathbb{O}$. We have assumed that the stalk $\overline{M} \otimes k(\bar{\mathfrak{p}}_i)$ is 1-dimensional. This stalk coincides with the stalk $M \otimes k(\mathfrak{p}_i)$, so we can conclude that the localization $M_{\mathfrak{p}_i}$ is free of rank 1 for $i = 1, \dots, n$.

We can also conclude that the same holds for the localizations of the dual module $M^\vee := \operatorname{Hom}(M, \mathbb{O})$. An application of the prime avoidance lemma shows that there exists an element $\phi \in M^\vee$ that maps to a generator of $M^\vee_{\mathfrak{p}_i}$ for all i. We will show that $\phi : M \to \mathbb{O}$ realizes M as a R-flat family of ideals (i.e., ϕ is injective with R-flat cokernel).

It is enough to show that the reduction $\bar{\phi} : \overline{M} \to \overline{\mathbb{O}}$ is injective. An element of the kernel of this map is also in the kernel of the composition

$$\overline{M} \to \bigoplus \overline{M}_{\bar{\mathfrak{p}}_i} \to \bigoplus \overline{\mathbb{O}}_{\bar{\mathfrak{p}}_i}.$$

The kernel of the leftmost map is a submodule whose support does not contain any of the primes $\bar{\mathfrak{p}}_i$, and thus must be zero by pureness. Furthermore, the rightmost map is an isomorphism by construction. This proves injectivity.

Consider the ideal $I[\pi^{-1}]$ given by the image of $\phi[\pi^{-1}] : M[\pi^{-1}] \to \mathbb{O}[\pi^{-1}]$. This is a principal ideal, and hence is either the unit ideal or an ideal of height at most 1 (Hauptidealsatz!). By flatness, the same is true of the image I of ϕ. In fact, I cannot be a height zero ideal: The only such prime is the zero ideal, which does not satisfy the hypotheses. Thus, I is either the unit ideal or a height 1 ideal. In either case, I must be principal, and the proof is complete. □

We record the factorial condition as a hypothesis.

Hypothesis 1. We say a family of curves $f : X \to B$ over a Dedekind scheme satisfies Hypothesis 1 if the generic fiber X_η is smooth and the local rings of X_S are factorial for every strict henselization $S \to B$.

Hypothesis 1 is satisfied when X is regular and X_η is smooth. We now prove the main theorem of this paper.

Theorem 3.9. *Let $f : X \to S$ be a family of curves and $\bar{J}$ a subfunctor of* Sh *such that the generic fibers $\bar{J}_\eta = \mathrm{Sh}_\eta$ coincide. Assume the line bundle locus $J \subset \bar{J}$ is represented by a smooth and finitely presented S-scheme.*

If $\bar{J}$ satisfies the valuative criteria of properness and f satisfies Hypothesis 1, then Q^τ is the Néron model and

$$J \subset \mathrm{Q}^\tau = \mathrm{N}$$

is an open subscheme that contains all the $k(0)$-valued points of Q^τ. Furthermore,

$$J = \mathrm{Q}^\tau = \mathrm{N}$$

provided one of the following conditions hold:

(1) *$k(0)$ is algebraically closed*;

(2) *J is stabilized by the identity component Q^{o}.*

Proof. By Proposition 3.6, the natural map $J \to \mathrm{Q}$ is an open immersion with image contained in Q^τ. Using this fact, we can prove that Q^τ is the Néron model of its generic fiber. Indeed, it is enough to prove that Q^τ satisfies the weak Néron mapping property. The open subscheme $J \subset \mathrm{Q}^\tau$, in fact, already satisfies this property. Let $\sigma_\eta \in \mathrm{Q}^\tau(\eta) = J(\eta)$ be given. By properness, we can extend σ_η to a section $\sigma \in \bar{J}(S)$, and this element can be represented by a family $\mathscr{I}$ of pure, rank 1 sheaves (by Fact 3.3). But every such family is a family of line bundles (Lemma 3.8), and hence σ lies in $J(S) \subset \bar{J}(S)$. In other words, J satisfies the weak Néron mapping property.

The weak Néron mapping property of J also implies that the image of J contains all the $k(0)$-valued points of Q^τ. Indeed, every $k(0)$-valued point of Q^τ is the specialization of a section by Fact 3.5. If $k(0)$ is algebraically closed, then we have shown that J contains every $k(0)$-valued point of Q^τ, hence every closed point. Thus, $J = \mathrm{Q}^\tau$, and there is nothing more to show.

Let us now turn our attention to the case where $k(0)$ is only separably closed, but J is stabilized by Q^{o}. Our goal is to show $J = \mathrm{Q}^\tau$, and to show this, we pass to the special fiber $J_0 \to \mathrm{Q}^\tau_0$ and argue with points. Let x be a $\bar{k}(0)$-valued point of Q^τ, where $\bar{k}(0)$ is the algebraic closure of the residue field. By density (Fact 3.4), there exists a $\bar{k}(0)$-valued point y in the image of $J_0 \to \mathrm{Q}^\tau_0$ that lies in the same connected component as x. We have $x = y + (x - y)$, which expresses x as the sum of a point of $\mathrm{Q}^{\mathrm{o}}_0$ and a point of J_0. The point x must lie in J_0 by assumption. This shows that the image of J contains all of Q^τ, completing the proof. $\square$

Remark 3.10. The hypothesis that J is stabilized by the identity component Q^{o} is perhaps unexpected, but it is often satisfied in practice. The moduli space $\bar{J}$ is typically constructed by imposing numerical conditions on the multidegree of a sheaf, and the multidegree is invariant under the action of Q^{o} (because the action is given by tensoring with a multidegree 0 line bundle).

In the next section, we will show that certain moduli spaces constructed in the literature satisfy the hypotheses of Theorem 3.9. There are, however, families of curves $f : X \to S$ with factorial local rings $\mathcal{O}_{X,x}$ such that there does not exist a S-scheme $\bar{J}$ satisfying the conditions of the theorem. Indeed, the family $f : X \to S$ in [Raynaud 1970, Example 9.2.3] is a family of genus 1 curves such that the local rings of X are factorial (even regular), but the natural map $\mathrm{Q}^{\tau} \to \mathrm{N}$ is not an isomorphism. In particular, no $\bar{J}$ satisfying the hypotheses of Theorem 3.9 can exist.

4. Applications

Here we apply Theorem 3.9 to some families of moduli spaces from the literature and then deduce consequences. The two moduli spaces that we are interested in are the Esteves moduli space of quasistable sheaves (Section 4.1) and the Simpson moduli space of slope stable sheaves (Section 4.2). In Section 4.3, we discuss the special case of families of genus 1 curves, where suitable moduli spaces can be constructed explicitly.

The moduli spaces we study are associated to a relatively projective family of curves. We are primarily interested in families over a Dedekind scheme with locally factorial total space, in which case projectivity is automatic. This fact is a consequence of the generalized Chevalley Conjecture when the Dedekind scheme is defined over a field, but we do not know a reference. For completeness, we prove:

Proposition 4.1. *Let $f : X \to B$ be a family of curves over a Dedekind scheme. If the local rings of X are factorial, then f is projective.*

Proof. This proof was explained to the author by Steven Kleiman. Fix a closed point $b_0 \in B$. Given any component $F \subset X_{b_0}$, I claim that we can find a line bundle $\mathcal{L}$ on X that has nonnegative degree on every component of every fiber and strictly positive degree on F.

Pick a closed point $x \in F$ and an open affine neighborhood $U \subset X$ of that point. By the prime avoidance lemma, we can find a regular function $r \in H^0(U, \mathcal{O}_X)$ that does not vanish on any component of X_{b_0} but does vanish at x. Pick a component D of the closure of $\{r = 0\} \subset U$ in X. Then D is a Cartier divisor (by the factoriality assumption) that does not contain any component of any fiber X_b (by construction).

Furthermore, D has nontrivial intersection with F. The associated line bundle $\mathcal{L} := \mathcal{O}_X(D)$ has the desired positivity property.

Now construct one such line bundle for every irreducible component F of X_{b_0} and define $\mathcal{M}$ to be their tensor product. The line bundle $\mathcal{M}$ is nef on every fiber and ample on X_{b_0}. Ampleness is an open condition, so $\mathcal{M}$ is in fact ample on all but finitely many fibers of f. After repeating the construction for each such fiber and forming the tensor product, we have constructed a f-relatively ample line bundle on X. This completes the proof. $\square$

We now turn our attention to the moduli spaces.

4.1. *Esteves Jacobians.* We first discuss the Esteves moduli space of quasistable sheaves. This moduli space fits very naturally into the framework of the previous section.

Suppose B be a locally noetherian scheme and $f : X \to B$ a projective family of curves whose fibers are geometrically reduced. Quasistability is defined in terms of a section $\sigma : B \to X^{\text{sm}}$ and a vector bundle $\mathcal{E}$ on X with fiberwise integral slope $\deg(\mathcal{E}_b)/\operatorname{rank}(\mathcal{E}_b)$, which we assume is constant as a function of $b \in B$. Given σ and $\mathcal{E}$, σ-quasistability is a numerical condition on the multidegree of a rank 1, torsion-free sheaf of degree

$$d(\mathcal{E}) = d := -\chi(\mathcal{O}_{X_b}) - \deg(\mathcal{E}_b)/\operatorname{rank}(\mathcal{E}_b).$$

For the definitions (which we will not use), we direct the reader to [Esteves 2001, p. 3051] (for a single sheaf) and [ibid., p. 3054] (for a family). The basic existence theorem is [ibid., Theorem A on p. 3047], which is proved in [ibid., Section 4]). It states that if $\operatorname{Sheaf}^{\sigma}_{\mathcal{E}} : S\text{-Sch} \to \text{Sets}$ is the functor defined by setting $\operatorname{Sheaf}^{\sigma}_{\mathcal{E}}(T)$ equal to the set of isomorphism classes of $\mathcal{O}_T$-flat, finitely presented $\mathcal{O}_{X_T}$-modules that are fiberwise σ-quasistable, then there is a B-proper algebraic space $\bar{J}^{\sigma}_{\mathcal{E}} \to B$ of finite type that represents the étale sheaf associated to $\operatorname{Sheaf}^{\sigma}_{\mathcal{E}}$.

Strictly speaking, our definition differs from the one given in [ibid.] in two ways. First, Esteves does not work with isomorphism classes of sheaves but rather with equivalence classes under the relation given by identifying two sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ on X_T when $\mathcal{I}_1$ is isomorphic to $\mathcal{I}_2 \otimes f^*(\mathcal{L})$ for some line bundle $\mathcal{L}$ on T. Zariski locally on T, the sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ are isomorphic, and it follows that the étale sheaf associated to $\operatorname{Sheaf}^{\sigma}_{\mathcal{E}}$ is canonically isomorphic to the sheaf considered by Esteves. Second, Esteves only defines his functor as a functor from locally noetherian schemes to sets. However, the functor $\operatorname{Sheaf}^{\sigma}_{\mathcal{E}}$ and its associated étale sheaf are easily seen to be locally finitely presented. It follows that $\bar{J}^{\sigma}_{\mathcal{E}}$ represents the étale sheaf associated to $\operatorname{Sheaf}^{\sigma}_{\mathcal{E}}$, rather than just the restriction of this sheaf to locally noetherian schemes.

If f satisfies stronger conditions, then the space $\bar{J}^{\sigma}_{\mathscr{E}}$ is actually a scheme. This is the content of [Esteves 2001, Theorem B, p. 3048], proved on [ibid., p. 3086]. The theorem states that if there exist sections $\sigma_1, \dots, \sigma_n : B \to X^{\text{sm}}$ of f with the property that every irreducible component of a fiber X_b is geometrically integral and contains one of the points $\sigma_1(b), \dots, \sigma_n(b)$, then $\bar{J}^{\sigma}_{\mathscr{E}}$ is a scheme.

In the special case where $B = S$ is a strict henselian discrete valuation ring with generic point η and special point 0, the hypotheses of Theorem B are automatically satisfied. Indeed, the locus of $k(0)$-valued points is dense in the smooth locus X_0^{sm} (Fact 3.4), which in turn is dense in X_0 as X_0 is geometrically reduced. We may conclude that the irreducible components of X_0 are geometrically integral. Finally, every $k(0)$-valued point of X_0 extends to a section $\sigma : S \to X$ (Fact 3.5), so the hypotheses of Theorem B are certainly satisfied.

We call $\bar{J}^{\sigma}_{\mathscr{E}}$ the *Esteves compactified Jacobian*. Inside of the Esteves compactified Jacobian, we can consider the open subscheme parametrizing line bundles. This scheme is called the *Esteves Jacobian* and denoted by $J^{\sigma}_{\mathscr{E}}$. While the scheme $\bar{J}^{\sigma}_{\mathscr{E}}$ parametrizes sheaves, it is not naturally a subfunctor of Sh because it does not parametrize degree 0 sheaves. We can, however, define a natural transformation $\bar{J}^{\sigma}_{\mathscr{E}} \to \text{Sh}$ by the rule

$$\mathscr{I} \mapsto \mathscr{I}(-d \cdot \sigma)$$

Both Proposition 3.6 and Theorem 3.9 apply to $\bar{J}^{\sigma}_{\mathscr{E}}$.

Corollary 4.2. *Fix a Dedekind scheme B. Let $f : X \to B$ be a projective family of geometrically reduced curves. Let $\sigma : B \to X^{sm}$ be a section and $\mathscr{E}$ a vector bundle on X with fiberwise integral slope.*

Then the natural map $J^{\sigma}_{\mathscr{E}} \to \text{Q}$ is an open immersion.

Assume further that f satisfies Hypothesis 1. Then $J^{\sigma}_{\mathscr{E}} = \text{Q}^{\tau}$, and this scheme is the Néron model.

Proof. By localizing, we can assume that $B = S$ is a strict henselian discrete valuation ring, in which case we are reduced to proving that the hypotheses of Proposition 3.6 and Theorem 3.9 hold. The scheme $J^{\sigma}_{\mathscr{E}}$ is easily seen to be formally S-smooth. Indeed, σ-quasistability is a condition on fibers, so the formal smoothness of P^0 implies the formal smoothness of $J^{\sigma}_{\mathscr{E}}$. The remaining hypotheses of Proposition 3.6 are explicitly assumed, so we can deduce the first part of the theorem. To complete the proof, it is enough to show that $J^{\sigma}_{\mathscr{E}}$ is stabilized by Q^{o}. But the action of Q^{o} on $J^{\sigma}_{\mathscr{E}}$ is given by the tensor product against a multidegree 0 line bundle, so this action preserves multidegree and hence σ-quasistability. □

Corollary 4.2 implies that $J^{\sigma}_{\mathscr{E}}$ is a scheme with (unique) B-group scheme structure that extends the group scheme structure of the generic fiber. It is not immediate from the definition that $J^{\sigma}_{\mathscr{E}}$ admits such structure, and Example 4.9 shows that the group structure is special to the case of families over a 1-dimensional base. The result also implies uniqueness results for the Esteves Jacobian; if $\sigma' : B \to X^{\mathrm{sm}}$ is a second section and $\mathscr{E}'$ a second vector bundle on X, then $J^{\sigma'}_{\mathscr{E}'}$ is canonically isomorphic to $J^{\sigma}_{\mathscr{E}}$. In the next section, we will define the Simpson stable Jacobian $J^{0}_{\mathscr{L}}(X)$, and this scheme is also isomorphic to $J^{\sigma}_{\mathscr{E}}$ provided every slope semistable sheaf is stable. It would be interesting to know if these isomorphisms extend to the compactifications. Important results along these lines can be found in [Melo and Viviani 2012; Esteves 2009], but many basic question remain unanswered. Currently, there is no example of a curve $X_0 \to \mathrm{Spec}(k)$ such that two Esteves compactified Jacobians associated to X_0 are nonisomorphic.

4.2. ***Simpson Jacobians.*** The hypotheses to Proposition 3.6 and Theorem 3.9 are satisfied by certain moduli spaces of stable sheaves, which we call Simpson Jacobians. Here we recall Simpson's construction, along with later work of Langer and Maruyama, and then apply results from Section 3. We restrict our attention to families of reduced curves (but see Remark 4.4, and the discussion preceding Example 4.9).

We work over a scheme B that is finitely generated over a universally Japanese ring R (e.g., $R=\mathbb{C}, \mathbb{F}_p, \mathbb{Z}, \dots$). Let $f : X \to B$ a family of curves with f-relatively ample line bundle $\mathscr{L}$, and assume the Euler–Poincaré characteristics $\chi(\mathscr{O}_{X_b})$ and $\chi(\mathscr{L}_b)$ are constant as functions of the base B. Set P_d equal to the polynomial

$$P_d(t) := \deg(\mathscr{L}_b)\cdot t + d + \chi, \tag{4-1}$$

where χ is the Euler–Poincaré characteristic of a fiber of f and $\deg(\mathscr{L}_b)$ is the degree of the restriction of $\mathscr{L}$ to a fiber. This is the Hilbert polynomial of a degree d line bundle.

Given this data, Simpson constructed an associated moduli space in the case that $R=\mathbb{C}$. The Simpson moduli space $\mathrm{M}(\mathscr{O}_X, P_d)$ parametrizes slope semistable sheaves with Hilbert polynomial P_d. (See [Simpson 1994, pp. 54–56] for the definition of semistability). To be precise, define $\mathrm{M}^{\sharp}(\mathscr{O}_X, P_d)$ to be the functor whose T-valued points are isomorphism classes of $\mathscr{O}_T$-flat, finitely presented $\mathscr{O}_{X_T}$-modules whose fibers are $\mathscr{L}$-slope semistable sheaves with Hilbert polynomial P_d. The main existence result [Simpson 1994, Theorem 1.21] asserts that there is a projective scheme $\mathrm{M}(\mathscr{O}_X, P_d)$ that corepresents $\mathrm{M}^{\sharp}(\mathscr{O}_X, P_d)$. Inside of $\mathrm{M}(\mathscr{O}_X, P_d)$, we may consider the open subscheme $\mathrm{M}^{\mathrm{st}}(\mathscr{O}_X, P_d)$ parametrizing $\mathscr{L}$-slope stable sheaves. The stable locus is a fine moduli space: Its $\mathbb{C}$-valued points are in natural bijection with the isomorphism classes of $\mathscr{L}$-slope stable sheaves with Hilbert polynomial P_d, and étale locally on $\mathrm{M}^{\mathrm{st}}(\mathscr{O}_X, P_d)$, the product $X \times_B \mathrm{M}^{\mathrm{st}}(\mathscr{O}_X, P_d)$ admits a universal family of sheaves. The reader may check that these conditions are equivalent to the condition that $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ represents the étale sheaf associated the functor parametrizing stable sheaves. While Simpson only considers the case $R = \mathbb{C}$, later work of Langer [2004a, Theorem 4.1; 2004b, Theorem 0.2] and Maruyama [1996] extends these results to the case where R is an arbitrary universally Japanese ring.

Let us now specialize to the case where B is a Dedekind scheme. When f has reducible fibers, $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ may contain points corresponding to sheaves that are not rank 1; see [López-Martín 2005, Example 2.2]. This is potentially a major source of confusion: The term "rank" is used in a different way in [Simpson 1994], and the sheaves parametrized by $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ are rank 1 in Simpson's sense but not necessary in the sense used here.

We avoid these sheaves. Define the *Simpson stable Jacobian* $J^d_{\mathcal{L}}$ of degree d to be the locus of stable line bundles in $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ (which is an open subscheme by [Altman and Kleiman 1980, Lemma 5.12(a)]). We define the *Simpson stable compactified Jacobian* $\bar{J}^d_{\mathcal{L}}$ to be the subset of the stable locus $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ that corresponds to pure, rank 1 sheaves. (Warning: The compactified Jacobian is a B-proper scheme when every semistable pure sheaf with Hilbert polynomial P_d is stable but not in general!)

When the fibers of $X \to B$ are *geometrically reduced*, a minor modification of the proof of [Pandharipande 1996, Lemma 8.1.1] shows that the subset $\bar{J}^d_{\mathcal{L}} \subset \mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ is closed and open, and hence has a natural scheme structure:

Lemma 4.3. *Assume the fibers of $f : X \to B$ are geometrically reduced. Then the subset $\bar{J}^d_{\mathcal{L}}$ is closed and open in* $\mathrm{M}^{st}(\mathcal{O}_X, P_d)$.

Proof. The main point to prove is that a 1-parameter family of line bundles cannot specialize to a pure sheaf that fails to have rank 1, and this is shown by examining the leading term of the Hilbert polynomial. To begin, we may cover $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ by étale morphisms $\mathrm{M} \to \mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ with the property that a universal family $\mathcal{I}_{\mathrm{uni.}}$ on $\mathrm{M} \times_B X$ exists. It is enough to verify the claim after passing from $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ to an arbitrary such scheme, and so for the remainder of the proof we work with M in place of $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$. We will also abuse notation by denoting the pullback of $\bar{J}^d_{\mathcal{L}}$ under $\mathrm{M} \to \mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ by the same symbol $\bar{J}^d_{\mathcal{L}}$.

We first need to check that $\bar{J}^d_{\mathcal{L}} \subset \mathrm{M}$ is constructible, so that we can make use of the valuative criteria. Given $m \in M$ mapping to $b \in B$, the condition that the fiber I_m is rank 1 is just the condition that the restriction of I_m to X^{sm}_b is a line bundle. Constructibility thus follows from [Grothendieck 1966a, 9.4.7] applied to $\mathrm{M} \times_B X^{\mathrm{sm}} \to \mathrm{M}$.

To finish, it is enough to prove that $\bar{J}^d_{\mathcal{L}}$ is closed under specialization and generalization. Thus, we pass from M to a discrete valuation ring T mapping to M. If

$\mathcal{I}$ is the sheaf on X_T given by the pullback of the universal family, then we need to show that the generic fiber of I_η is rank 1 if and only if the special fiber I_0 is.

To prove this, we turn our attention to the Hilbert polynomial P_d of a fiber of I. This polynomial is defined so that the leading term is $\deg(\mathcal{L}_b)$, and we can express this number in terms of components of a fiber of $X_T \to T$ as follows. If x is generic point of the special fiber X_0, then we define $\deg_x(\mathcal{L}_0)$ to be the degree of the restriction of $\mathcal{L}_0$ to the irreducible component corresponding to x. (Give the component the reduced subscheme structure.) For any generic point y of X_η, we define $\deg_y(\mathcal{L}_\eta)$ in the analogous manner. If $x_1, \dots, x_n$ are all the generic points of X_0 and $y_1, \dots, y_m$ all the generic points of X_η, then we have

$$\begin{aligned}\deg(\mathcal{L}_b) &= \deg_{x_1}(\mathcal{L}_0) + \cdots + \deg_{x_n}(\mathcal{L}_0)\\ &= \deg_{y_1}(\mathcal{L}_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\end{aligned}$$

by, say, [Altman and Kleiman 1979, 2.5.1]. The terms $\deg_{x_i}(\mathcal{L}_0)$ and $\deg_{y_j}(\mathcal{L}_\eta)$ in the equation above are each strictly positive as $\mathcal{L}$ is relatively ample.

We can also express $\deg(\mathcal{L}_b)$ in terms of the generic rank of a fiber of I. Using [Altman and Kleiman 1979, 2.5.1] again, we have

$$\begin{aligned}\deg(\mathcal{L}_b) &= \deg_{x_1}(\mathcal{L}_0)\cdot \ell_{x_1}(I_0) + \cdots + \deg_{x_n}(\mathcal{L}_0)\cdot \ell_{x_n}(I_0)\\ &= \deg_{y_1}(\mathcal{L}_\eta)\cdot \ell_{y_1}(I_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\cdot \ell_{y_m}(I_\eta).\end{aligned}$$

Here $\ell_{x_i}(I_0)$ denotes the length of the localization of I_0 at x_i and similarly for $\ell_{y_j}(I_\eta)$. The fibers of $X_T \to T$ are reduced, so such a length is equal to the minimal number of generators. In particular, these numbers are upper semicontinuous. In other words, if y_j specializes to x_i, then we have $\ell_{y_j}(I_\eta) \le \ell_{x_i}(I_0)$ (by Nakayama's lemma).

The desired result now follows. Suppose first that I_0 is rank 1. Then we have $\ell_{y_i}(I_\eta) \le 1$ for all i by semicontinuity. If some inequality was strict, say $\ell_{y_1}(I_\eta) = 0$, then we would have

$$\begin{aligned}\deg(\mathcal{L}_b) &= \deg_{y_1}(\mathcal{L}_\eta) + \deg_{y_2}(\mathcal{L}_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\\ &> \qquad\qquad \deg_{y_2}(\mathcal{L}_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\\ &\ge \deg_{y_1}(\mathcal{L}_\eta)\cdot \ell_{y_1}(I_\eta) + \cdots + \deg_{y_m}(\mathcal{L}_\eta)\cdot \ell_{y_m}(I_\eta)\\ &= \deg(\mathcal{L}_b).\end{aligned}$$

This is absurd! Thus, we must have $\ell_{y_i}(I_\eta) = 1$ for all y_i and I_η is rank 1. Similar reasoning shows that if I_η is rank 1, then I_0 is rank 1. □

Remark 4.4. The hypothesis that the fibers of f are geometrically reduced is necessary. Indeed, the moduli space $\mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ was described in [Chen and Kass 2011] in the case that X is a nonreduced curve whose reduced subscheme X_{red}

is smooth and whose nilradical $\mathcal{N}$ is square-zero (i.e., X is a ribbon). Using that description it is easy to produce examples where $\bar{J}^d_{\mathcal{L}} \subset \mathrm{M}^{\mathrm{st}}(\mathcal{O}_X, P_d)$ is not closed (e.g., take d equal to 0, X to have even genus, and X_{red} to have genus 1). The points of the complement in the closure correspond to stable rank 2 vector bundles on X_{red}.

We now apply Proposition 3.6 and Theorem 3.9 to the Simpson Jacobians.

Corollary 4.5. *Fix a Dedekind scheme B that is finitely generated over a universally Japanese ring. Let $f : X \to B$ be a family of geometrically reduced curves. Let $\mathcal{L}$ be f-relatively ample line bundle.*

Then the natural map $J^0_{\mathcal{L}}(X) \to \mathrm{Q}^{\tau}$ is an open immersion. Assume further that both of the following conditions hold:

- *Every $\mathcal{L}$-slope semistable rank* 1*, torsion free sheaf of degree* 0 *is $\mathcal{L}$-slope stable.*
- *f satisfies Hypothesis 1.*

Then $J^0_{\mathcal{L}}(X) = \mathrm{Q}^{\tau}$, and this scheme is the Néron model.

Proof. The local existence of a universal family [Simpson 1994, Theorem 2.1(4)] implies that there is a natural transformation $\bar{J}_{\mathcal{L}}(X) \to \mathrm{Sh}$ with the property that $J_{\mathcal{L}}(X)$ is the preimage of $\mathrm{P}^0 \subset \mathrm{Sh}$. Furthermore, the slope stability condition is a fiberwise condition, so a modification of the argument given in Corollary 4.2 completes the proof. □

Remark 4.6. A minor generalization of Corollary 4.5 can be obtained by allowing for moduli spaces of degree d lines bundles, with $d \neq 0$. If we are given a line bundle $\mathcal{M}$ on X with fiberwise degree d, then there is an associated map $J^d_{\mathcal{L}}(X) \to \mathrm{Q}$ that extends the map on the generic fiber given by tensoring with $\mathcal{M}^{-1}$. With only notational changes the previous corollary generalizes to a statement about this map.

Corollary 4.5 is, of course, only of interest when there exists an $\mathcal{L}$ such that $\mathcal{L}$-slope stability coincides with $\mathcal{L}$-slope semistability. Thus, we ask, When does such an $\mathcal{L}$ exist? A comprehensive discussion of this question would require a digression on stability conditions, so we limit ourselves to reviewing known results about a single curve X_0 over an algebraically closed field. When X_0 is integral, the stability condition is vacuous, so every ample $\mathcal{L}_0$ has the desired property. If X_0 is reducible of genus $g \neq 1$ with only nodes as singularities, then Melo and Viviani have proven the existence of a suitable $\mathcal{L}_0$ [2012, Proposition 6.4]. Stability conditions on reduced, genus 1 curves were analyzed by López-Martín [2005]. She exhibits curves X_0 with the property that there is no $\mathcal{L}_0$ such that every $\mathcal{L}_0$-slope semistable, pure, rank 1 sheaf degree 0 is stable, but a suitable $\mathcal{L}_0$ always exists if one considers sheaves of fixed degree $d \neq 0$. Finally, stability conditions for a ribbon were analyzed in [Chen and Kass 2011]. On a ribbon, the stability condition is independent of $\mathscr{L}_0$, and for rational ribbons, slope stability coincides with slope semistability precisely when the genus g is even. It would be desirable to have a general result asserting (non)existence of a suitable $\mathscr{L}_0$.

4.3. *Genus* 1 *curves.* The Néron model of the Jacobian of a genus 1 curve can be quite complicated (see for example [Liu et al. 2004]), but these complications do not arise if the family admits a section. Suppose B is a Dedekind scheme and $f : X \to B$ is a family of curves such that the total space X is regular and the generic fiber X_η is smooth. If $\sigma : B \to X^{\text{sm}}$ is a section contained in the smooth locus, then there is a canonical identification of the smooth locus X^{sm} with the Néron model N of the Jacobian of X_η. Here we examine how this fact fits into the preceding framework.

Definition 4.7. Let $f : X \to B$ be a family of genus 1 curves over a Dedekind scheme and $\sigma : B \to X^{\text{sm}}$ a section contained in the smooth locus. We define a sheaf $\mathscr{I}_{\text{uni.}}$ on $X \times_B X$ by the formula

$$\mathscr{I}_{\text{uni.}} := \mathscr{I}_\Delta(\pi_1^*(\sigma) + \pi_2^*(\sigma)). \tag{4-2}$$

Here $\mathscr{I}_\Delta$ is the ideal sheaf of the diagonal, and $\pi_1, \pi_2 : X \times_B X \to X$ are the projection maps.

The sheaf $\mathscr{I}_{\text{uni.}}$ determines a transformation $X \to \text{Sh}$ that realizes X as a moduli space of sheaves over itself. Proposition 3.6 and Theorem 3.9 apply to this moduli space.

Corollary 4.8. *Fix a Dedekind scheme B. Let $f : X \to B$ be a family of genus* 1 *curves. Let $\sigma : B \to X^{\text{sm}}$ be a section.*

Then the natural map $X^{\text{sm}} \to \text{Q}$ is an open immersion.

Assume further that f satisfies Hypothesis 1. Then $X^{\text{sm}} = \text{Q}^\tau$, and this scheme is the Néron model.

Let us consider the special case where B is a discrete valuation ring, X is a minimal regular surface, and the residue field $k(0)$ is algebraically closed. The possibilities for the special fiber X_0 are given by the Kodaria–Néron classification ([Kodaira 1960; Néron 1964]; see [Silverman 1994, pp. 353–354] for a recent exposition). The reduced curves appearing in the classification are the reduction types I_n, II, III, and IV. In these cases, one may show that the induced morphism $X \to \text{Sh}$ identifies X with the Esteves compactified Jacobian $\bar{J}^{\sigma}_{\mathscr{E}}$.

In every remaining case (reduction type I_n^*, II^*, III^*, or IV^*) the morphism $X \to \text{Sh}$ is not a special case of the fine moduli spaces discussed in the previous two sections. Indeed, the special fiber X_0 is nonreduced, so the Esteves Jacobian of X is not defined. In Section 4.2, we reviewed Simpson's moduli space $\text{M}^{\text{st}}(\mathscr{O}_X, P_d)$ of stable sheaves, but the image of $X \to \mathrm{Sh}$ cannot be described as a closed subscheme of that space. The reason is that slope stable sheaves are simple, but some fibers of $\mathcal{I}_{\text{uni.}}$ are not simple. Specifically, if $p_0 \in X_0$ lies on the intersection of two components, then the fiber of $\mathcal{I}_{\text{uni}}$ of p_0 fails to be simple. This can be seen as follows. This fiber is the sheaf $\mathcal{I}_{p_0}(+\sigma(0))$, where $\mathcal{I}_{p_0}$ is the ideal of p_0. If $\nu : X'_0 \to X_0$ is the blow-up of X_0 at p_0, then one may show that $H^0(X'_0, \mathcal{O}_{X'_0})$ is canonically isomorphic to the endomorphism ring of $\mathcal{I}_{p_0}(+\sigma(0))$. An inspection of the Kodaria–Néron table shows that X'_0 is disconnected, so $H^0(X'_0, \mathcal{O}_{X'_0})$ does not equal $k(0)$ and $\mathcal{I}_{p_0}(+\sigma(0))$ is not simple.

Corollary 4.8 provides a partial answer to a question posed in the introduction: What are the maximal subfunctors J of P^0 represented by a separated B-scheme? When X is, say, regular, a strong result one could hope for is that there is always a subfunctor $\bar{J}$ of Sh satisfying the hypotheses of Theorem 3.9. The line bundle locus $J \subset \bar{J}$ in such a functor has the property that $J \to \mathrm{Q}^{\tau}$ is an isomorphism, and hence J is maximal. Corollary 4.8 shows that such a $\bar{J}$ exists when $f : X \to S$ is a family of genus 1 curves that f admits a section. Similarly, the Esteves compactified Jacobian represents a suitable subfunctor when f has geometrically reduced fibers and admits a section. In general, however, the hope is too optimistic: Raynaud's family, mentioned at the end of Section 3, has that property that no such $\bar{J}$ can exist.

The question of describing maximal subfunctors J is most interesting when f has nonreduced fibers. The slope stable line bundles form a subfunctor $J \subset \mathrm{P}^0$ represented by a S-separated scheme, but our discussion of genus 1 families together with Remark 4.4 suggest that we should consider other methods for constructing a suitable J when f has nonreduced fibers.

In a different direction, one nice property of the moduli spaces described by Corollary 4.8 is that their geometry is very simple. We use these spaces to provide an example showing that a family $J \to B$ of Esteves Jacobians over a regular 2-dimensional base may not have group scheme structure.

Example 4.9 (Néron models in 2-dimensional families). We will construct a 2-dimensional family $f : X \to B$ of plane cubics and an associated Esteves Jacobian $J \to B$ with the property that the group law on the locus $J_U \to U$ parametrizing nonsingular cubics does not extend over all of B. Furthermore, the family is constructed so that a dense open subset of B is covered by nonsingular curves C with the property that the restriction X_C of X to C is regular, so J_C is the Néron model of its generic fiber (and in particular admits group scheme structure that extends the group scheme structure over $C \cap U$). Thus, the Néron models fit into a 2-dimensional family, but their group scheme structure does not.

The idea is as follows. The family we construct has a reducible element $X_{b_0} \to b_0$ with the property that, for every nonsingular curve $C \subset B$ passing through b_0

such that X_C is regular, the restriction of the Esteves Jacobian J_C is the Néron model of its generic fiber. The fiber J_{b_0} inherits a group law from this Néron model, and we show explicitly that this group law depends on the particular choice of C. But, if the group law on J_U extended to J, then all the different group laws on J_{b_0} coming from the different curves C would be the restriction of one common group law on J, which is absurd. We now construct the family.

We work over an algebraically closed field k. The family $X \to B$ will be a net of plane cubics. Let $X_0 \subset \mathbb{P}^2_k$ be a reducible plane cubics that is the union of a smooth quadric Q_0 and a line L_0 that meet in two distinct points. (See Figure 1.) Fix two general points $p_1, p_2 \in L_0(k)$ on the line and one general point $q_1 \in Q_0(k)$ on the quadric. Say that $F \in H^0(\mathbb{P}^2_k, \mathbb{O}(3))$ is an equation for X_0 and $G, H \in H^0(\mathbb{P}^2_k, \mathbb{O}(3))$ are two general cubic equations that vanish on all of the points p_1, p_2, q_1. We will work with the net $V := \langle F, G, H\rangle \subset H^0(\mathbb{P}^2_k, \mathbb{O}(3))$ and the associated family of curves

$$\begin{aligned} X := \{(p, [r, s, t]) : r \cdot F(p) + s \cdot G(p) + t \cdot H(p) = 0\} \\ \subset \mathbb{P}^2_k \times \mathbb{P}^2_k. \end{aligned} \tag{4-3}$$

There are two obvious morphisms $e, f : X \to \mathbb{P}^2_k$ given by the two projections. If we set $B := \mathbb{P}^2_k$ equal to the plane, then the second morphism $f : X \to B$ realizes X as a family of genus 1 curves with $X_0 = f^{-1}(b_0)$, where $b_0 := [1, 0, 0]$. Corresponding to the points $p_1, p_2, q_1 \in X_0(k)$ are three section $\sigma_1, \sigma_2, \tau_1 : B \to X^{\mathrm{sm}}$, which lie in the smooth locus by the generality assumption.

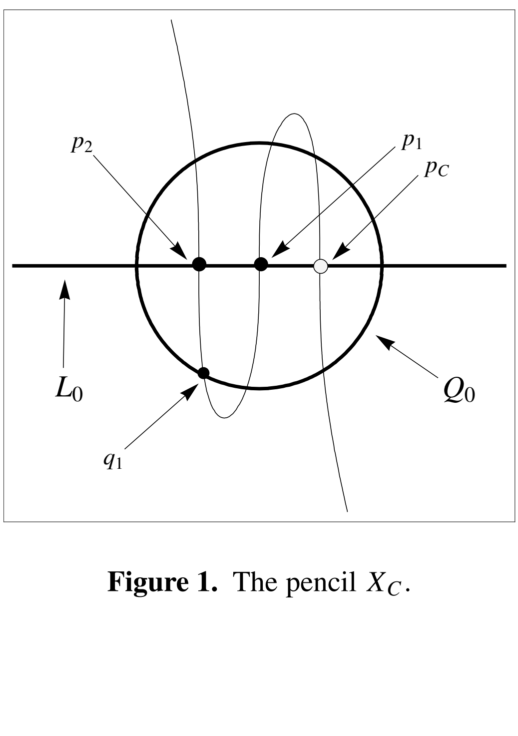

Figure 1. The pencil X_C.

Another application of the generality assumption shows that the fibers of f are reduced, so we can form the Esteves Jacobian $J := J_{\mathscr{E}}^{\sigma_1}$, where $\mathscr{E} = \mathbb{O}_X$. The quasistability condition on a line bundle $\mathscr{L}_0$ on X_0 is the condition that the bidegree $(\deg(\mathscr{L}_{L_0}), \deg(\mathscr{L}_{Q_0}))$ equals $(0, 0)$ or $(1, -1)$. Now we assume $J \to B$ is a group scheme and derive a contradiction.

Suppose that we are given a general line $C \subset B$ in the plane that contains b_0. Such a line corresponds to a 2-dimensional linear subspace of the form $W := \langle F, G_C \rangle \subset V$ for some $G_C \in V$. Invoking generality again, the base locus

$$\{p \in \mathbb{P}^2_k : F(p) = G_C(p) = 0\} \tag{4-4}$$

consists of 9 distinct points. The first projection map $e : X \to \mathbb{P}^2_k$ realizes X_C as the blow-up of the plane at these points, so X_C is regular, and thus J_C is the Néron model of its generic fiber. We now study the group of sections of $J_C \to C$.

The base locus (4-4) includes the points p_1, p_2, q_1. In addition to the points p_1, p_2, a unique third point of the base locus must lie on the line L_0. Let us label that point p_C and write $\sigma_C : C \to X_C$ for the corresponding section.

Now consider the following line bundles on X_C:

$$\begin{aligned} \mathscr{L}_1 &:= \mathbb{O}(\sigma_1 - \tau_1), \qquad & \mathscr{L}_C &:= \mathbb{O}(\sigma_C - \tau_1), \\ \mathscr{L}_2 &:= \mathbb{O}(\sigma_2 - \tau_1), \qquad & \mathscr{M} &:= \mathbb{O}(1) \otimes \mathbb{O}(-3 \cdot \tau_1). \end{aligned}$$

These lines bundles are all σ_1-quasistable. If we let $g_1, g_2, g_C, h \in J_C(C)$ respectively correspond to $\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_C, \mathscr{M}$, then I claim we have

$$g_1 + g_2 + g_C = h. \tag{4-5}$$

Indeed, it is enough to verify the claim after passing to the generic fiber of $J_C \to C$, where the equation is just the statement that the points p_1, p_2, p_C all lie on a line (the line L_0). Now suppose that $J \to \mathbb{P}^2_k$ admits a group law extending the group law of the generic fiber. Then the specialization of (4-5) to J_{b_0} holds for all C simultaneously. In particular, the isomorphism class of the line bundle $\mathbb{O}_{X_{b_0}}(p_C - q_1)$ is independent of the particular line $C \subset \mathbb{P}^2_k$ chosen. But this is absurd: For distinct general lines C_1, C_2, the points p_{C_1} and p_{C_2} (and hence the associated line bundles) are distinct! This completes our discussion of this example.

This example is particularly interesting in light of [Oda and Seshadri 1979]. The authors of that paper consider the case of a family of nodal curves $f : X \to B$ over a suitable Dedekind scheme with the property that X is regular. Let J_η be the Jacobian of the generic fiber. Given a closed point $0 \in B$, they prove that the special fiber N_0 of the Néron model of J_η depends only on the curve X_0 and not the particular family f [ibid., Corollary 14.4]. This result must be interpreted with care: In our example, the group law depends on a particular choice of family, but any two such group laws define isomorphic group schemes.

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Arithmetic motivic Poincaré series of toric varieties

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The *arithmetic motivic Poincaré series* of a variety V defined over a field of characteristic zero is an invariant of singularities that was introduced by Denef and Loeser by analogy with the Serre–Oesterlé series in arithmetic geometry. They proved that this motivic series has a rational form that specializes to the Serre–Oesterlé series when V is defined over the integers. This invariant, which is known explicitly for a few classes of singularities, remains quite mysterious. In this paper, we study this motivic series when V is an affine toric variety. We obtain a formula for the rational form of this series in terms of the Newton polyhedra of the *ideals of sums of combinations* associated to the minimal system of generators of the semigroup of the toric variety. In particular, we explicitly deduce a finite set of candidate poles for this invariant.

Introduction

Let S denote an irreducible and reduced algebraic variety defined over a field k of characteristic zero. The set $H(S)$ of formal arcs of the form $\operatorname{Spec} k[[t]] \to S$ can be given the structure of scheme over k (not necessarily of finite type). If $0 \in S$, we denote by $H(S)_0$ the subscheme of the arc space consisting of arcs in $H(S)$ with origin at 0. The set $H_m(S)$ of m-jets of S of the form $\operatorname{Spec} k[t]/(t^{m+1}) \to S$ has the structure of algebraic variety over k. By a theorem of Greenberg, the image of the space of arcs $H(S)$ by the natural morphism of schemes $j_m : H(S) \to H_m(S)$ that maps any arc to its m-jet is a constructible subset of $H_m(S)$.

It follows that $j_m(H(S))$ defines a class $[j_m(H(S))]$ in the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ and also a class $\chi_c([H_m(S)]) \in K_0(\mathrm{CHMot}_k)$ in the Grothendieck ring of Chow motives, where $\chi_c : K_0(\mathrm{Var}_k) \to K_0(\mathrm{CHMot}_k)$ is the unique ring homomorphism that maps the class of a smooth projective variety to its Chow motive (see [Gillet and Soulé 1996; Guillén and Navarro Aznar 2002]). We denote

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by $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ the image of $K_0(\mathrm{Var}_k)$ by the homomorphism χ_c. We use the same symbol $\mathbf{L}$ to denote the class $[\mathbf{A}_k^1] \in K_0(\mathrm{Var}_k)$ and the class $\chi_c([\mathbf{A}_k^1]) \in K_0^{\mathrm{mot}}(\mathrm{Var}_k)$.

Denef and Loeser [DL 2004] have defined various notions of motivic Poincaré series, motivated by some generating series in arithmetic geometry. Assume for simplicity that the variety S is defined over the integers. We denote by p a prime number and by $\mathbb{Z}_p$ the p-adic integers. For every positive integer m, the symbol $N_{p,m}(S)$ denotes the number of rational points of S over $\mathbb{Z}/p^{m+1}\mathbb{Z}$ that can be lifted to rational points of S over $\mathbb{Z}_p$ by the projection induced by the natural map $\mathbb{Z}_p \to \mathbb{Z}/p^{m+1}\mathbb{Z}$. The *Serre–Oesterlé* series of S at the prime p is

$$P_p^S(T) = \sum_{m \geq 0} N_{p,m}(S) T^m \in \mathbb{Z}[[T]].$$

The definition of the *geometric motivic Poincaré series*,

$$P_{\mathrm{geom}}^S(T) = \sum_{m \geq 0} \chi_c([j_m(H)]) T^m \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}[[T]],$$

is inspired by that of the Serre–Oesterlé series. However, there is no specialization of the series $P_{\mathrm{geom}}^S(T)$ into $P_p^S(T)$ in general [DL 2004].

Denef and Loeser studied the "motivic nature" of the series $P_p^S(T)$, passing through the Grothendieck ring $K_0(\mathrm{Field}_k)$ of ring formulas over k. By Greenberg's theorem, for every m there exists a formula ψ_m over k, such that for any field extension $k \subset K$, the m-jets over k that can be lifted to arcs defined over K correspond to the tuples satisfying ψ_m in K. It follows that ψ_m defines an element $[\psi_m] \in K_0(\mathrm{Field}_k)$. Then, Denef and Loeser defined a ring homomorphism $\chi_f : K_0(\mathrm{Field}_k) \to K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}$. This homomorphism can be seen as a generalization of χ_c, since the image by χ_f of the class of the ring formula defining a variety V coincides with the class $\chi_c([V])$ in $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}$. The *arithmetic motivic Poincaré series* of S is defined as

$$P_{\mathrm{ar}}^S(T) = \sum_{m \geq 0} \chi_f([\psi_m]) T^m \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}[[T]].$$

Denef [1984] proved the rationality of the series $P_p^S(T)$ using quantifier elimination results. Denef and Loeser [1999; 2001] proved the rationality of the series $P_{\mathrm{geom}}^S(T)$ and $P_{\mathrm{ar}}^S(T)$ by using quantifier eliminations theorems, various forms of motivic integration, and the existence of resolution of singularities.

If V is a variety defined over the integers and p is a prime number, the symbol $N_p(V)$ denotes the number of rational points of V over the field of p elements. Denef and Loeser proved that the result of applying the operator N_p to the motivic arithmetic series $P_{\mathrm{ar}}^S(T)$ provides the Serre–Oesterlé series $P_p^S(T)$ for almost all primes p.

If we fix the origin of the arcs at a fixed point $0 \in S$, we obtain local versions of the series $P_{\mathrm{ar}}^{(S,0)}(T)$ and $P_{\mathrm{geom}}^{(S,0)}(T)$, which are also rational; see [DL 1999; 2001]. The rationality proofs therein are qualitative in nature; in particular, there is no conjecture on the significance of the terms appearing in the denominator of the rational form of the series $P_{\mathrm{ar}}^{(S,0)}(T)$ or in $P_{\mathrm{geom}}^{(S,0)}(T)$.

The rational form of the series $P_{\mathrm{ar}}^{(S,0)}(T)$ is known explicitly for a few classes of singularities. If $(S, 0)$ is an analytically irreducible germ of plane curve, the information provided by the series $P_{\mathrm{ar}}^{(S,0)}(T)$ is equivalent to the data of the Puiseux pairs [DL 2001]. Nicaise [2005a] proved the equality of the geometric and arithmetic motivic Poincaré series in the case of varieties that admit a very special resolution of singularities, in particular for normal toric surfaces; see also [Nicaise 2005b; Lejeune-Jalabert and Reguera 2003]. He gave a criterion for the equality $P_{\mathrm{ar}}^{(S,0)}(T) = P_{\mathrm{geom}}^{(S,0)}(T)$ for various classes of singularities and also an example of a normal toric threefold $(S_0, 0)$ such that the series $P_{\mathrm{ar}}^{(S_0,0)}(T)$ and $P_{\mathrm{geom}}^{(S_0,0)}(T)$ are different. Some features of the motivic arithmetic series are studied for quasi-ordinary singularities in [Rond 2009].

In this paper, we describe the arithmetic motivic Poincaré series of an affine toric variety $Z^{\Lambda} = \operatorname{Spec} k[\Lambda]$, in terms of the semigroup Λ. We assume that Λ is a semigroup of finite type of a rank d lattice M (lattice of characters), which generates M as a group, and such that the cone $\mathbb{R}_{\geq 0}\Lambda$ contains no lines. In this situation, there is a unique minimal system of generators $e_1, \dots, e_n$ of the semigroup Λ. The monomial ideal $(X^{e_i})_{i=1,\dots,n} \subset k[\Lambda]$ is maximal, and defines the distinguished point $0 \in Z^{\Lambda}$. In this paper we consider other monomial ideals as the *logarithmic jacobian ideals* $\mathcal{J}_l$, generated by monomials of the form X^u for u in the set

$$\{e_{i_1} + \cdots + e_{i_l} \mid e_{i_1} \wedge \cdots \wedge e_{i_l} \neq 0\}$$

for $l = 1, \dots, d$ (see [Cobo Pablos and González Pérez 2012]), and the *ideals of sums of combinations* $\mathcal{C}_j$, defined by monomials X^w, with w in the set

$$\left\{ e_{i_1} + \cdots + e_{i_j} \;\middle|\; \{i_1, \dots, i_j\} \in \binom{\{1, \dots, n\}}{j} \right\},$$

where $\binom{\{1,\dots,n\}}{j}$ denotes the set of combinations of j elements of $\{1, \dots, n\}$ for $j = 1, \dots, n$.

We study the motivic arithmetic series $P_{\mathrm{ar}}^{(Z^{\Lambda},0)}(T)$ by extending the approach we used in [CoGP 2010; 2012] to describe the geometric motivic Poincaré series of toric and quasi-ordinary singularities.

For convenience, we explain the methods and results first when the variety Z^{Λ} is normal. The set $j_m(H(Z^{\Lambda})_0)$ of m-jets of arcs through $(Z^{\Lambda}, 0)$ is constructible; it is a finite disjoint union of locally closed subsets of the form $j_m(H_\nu^*)$ [Cobo Pablos and González Pérez 2012]. Here H_ν^* denotes the set of arcs through $(Z^{\Lambda}, 0)$ that have generic point in the torus and a given order $\nu \in M^*$. The set H_ν^* is an orbit of the natural action of the arc space of the torus on the arc space of the toric variety Z^Λ [Ishii 2004; 2005].

We describe the class, denoted by $\chi_f([j_m(H_\nu^*)]_f)$, of the formula defining the locally closed subset $j_m(H_\nu^*)$ in terms of the Newton polyhedra of the logarithmic jacobian ideals and the degree of a certain Galois cover. This Galois cover reflects the relation between the initial coefficients of the arcs in H_ν^* and the initial coefficients of the m-jets in $j_m(H_\nu^*)$; see Section 5.

A key point in the description of the rational form of the series $P_{\text{ar}}^{(Z^\Lambda,0)}(T)$ is that using the ideals $\mathscr{C}_j$, we can refine a finite partition of the set of possible pairs $\{(\nu, m)\}$, which was defined in [Cobo Pablos and González Pérez 2012] to describe the sum of $P_{\text{geom}}^{(Z^\Lambda,0)}(T)$. If (ν, m) and (ν', m') belong to the same subset of this refinement, then the degrees of the Galois covers associated to $j_m(H_\nu^*)$ and $j_{m'}(H_{\nu'}^*)$ coincide (see Sections 6 and 7). Using these partitions, we decompose the series $P_{\text{ar}}^{(Z^\Lambda,0)}(T)$ as a sum of a finite number of contributions. The main result is a formula for the rational form of $P_{\text{ar}}^{(Z^\Lambda,0)}(T)$ (see Theorem 11.4 and Corollary 10.4). The proofs pass by the results on the generating function of the projection of the set of integral points in the interior of a rational polyhedral cone; see [Cobo Pablos and González Pérez 2012]. The denominator of $P_{\text{ar}}^{(Z^\Lambda,0)}(T)$ is a finite product of terms of the form $1-\mathbf{L}^a T^b$ with $a \geq 0$ and $b > 0$, which are determined explicitly in terms of the ideals of sums of combinations $\mathscr{C}_j$. The integers a and b can be described in terms of the orders of vanishing of the ideals $\mathscr{C}_j$ and $\mathscr{J}_l$ at the codimension-one orbits of various toric modifications given by the Newton polyhedra of the ideals $\mathscr{C}_j$ (see Remark 10.8). In the normal toric case, we obtain a formula for $P_{\text{ar}}^{Z^\Lambda}(T)$ in terms of arithmetic motivic series at the distinguished points of the orbits.

In the nonnormal case, we obtain in a similar way a formula for the rational form of $P_{\text{ar}}^{(Z^\Lambda,0)}(T)$ and the factors of its denominator. The main difference is that we have to consider contributions of jets of arcs with generic point in the various orbits of Z^Γ. We deduce a formula for the difference $P_{\text{geom}}^{(Z^\Lambda,0)}(T) - P_{\text{ar}}^{(Z^\Lambda,0)}(T)$ and we give a criterion for the equality of these two series that generalizes the one given by Nicaise [2005b] (see Proposition 10.5 and Corollary 10.6).

The paper is organized as follows. In Sections 1 and 2 we introduce the Grothendieck rings, the arc and jet spaces, and the motivic Poincaré series. The notations on toric varieties, their monomial ideals, and their arcs are introduced in Sections 3 and 4. The computation of the class $\chi_f([j_m(H_\nu^*)]_f)$ is given in Section 5. Sections 6 and 7 deal with the partitions associated to sequences of monomial ideals. The main results are stated and proved in Sections 8, 9 and 10. In the case of normal toric varieties, some features of the computation can be simplified (see Section 11). We discuss some examples in Section 12.

1. Grothendieck rings of varieties and of ring formulas

The *Grothendieck ring* $K_0(\mathrm{Var}_k)$ *of k-varieties* is the free abelian group of isomorphism classes $[X]$ of k-varieties X modulo the relations $[X] = [X'] + [X \setminus X']$ if X' is closed in X, and where the product is defined by $[X][X'] = [X \times X']$. We denote by $\mathbf{L} := [\mathbf{A}_k^1]$ the class of the affine line. If C is a constructible subset of some variety X, that is, a disjoint union of finitely many locally closed subvarieties A_i of X, then $[C] \in K_0(\mathrm{Var}_k)$ is well-defined as $[C] := \sum_i [A_i]$ independently of the representation. Bittner [2004] proved, using the weak factorization theorem, that the ring $K_0(\mathrm{Var}_k)$ is generated by classes of smooth projective k-varieties, modulo relations of the form $[W] - [E] = [X] - [Y]$, where $Y \subset X$ is a closed subvariety, and W is the blowing up of X along Y with exceptional divisor E.

We refer to [Scholl 1994; Gillet and Soulé 1996; Guillén and Navarro Aznar 2002] for the definition of the category of *Chow motives*. Roughly speaking, its definition involves replacing the category of smooth projective algebraic varieties over k by a category with basically the same objects, and whose morphisms are suitable correspondences modulo rational equivalence. There exists a unique ring homomorphism

$$\chi_c : K_0(\mathrm{Var}_k) \to K_0(\mathrm{CHMot}_k) \tag{1}$$

that maps the class of a smooth projective variety over k to its Chow motive, where $K_0(\mathrm{CHMot}_k)$ denotes the *Grothendieck ring of the category of Chow motives* over k (with coefficients in $\mathbb{Q}$). This fundamental theorem, which is due to Gillet and Soulé [1996] and Guillén and Navarro Aznar [2002], can be seen also in terms of Bittner's result. We refer to [Gillet and Soulé 1996; Guillén and Navarro Aznar 2002; Bittner 2004] for precise definitions and proofs and to [Scholl 1994] for a survey on the notion of motives. We denote by $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ the image of $K_0(\mathrm{Var}_k)$ in $K_0(\mathrm{CHMot}_k)$ under this homomorphism.

Notice that the image of $\mathbf{L}$ in $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$, which we denote with the same symbol, is not a zero divisor in $K_0^{\mathrm{mot}}(\mathrm{Var}_k)$ since it is a unit in $K_0(\mathrm{CHMot}_k)$. However, it seems that it is not known if $\mathbf{L}$ is a zero divisor in $K_0(\mathrm{Var}_k)$.

A *ring formula* ψ over k is a first-order formula in the language of k-algebras and free variables $x_1, \dots, x_n$, that is, the formula ψ is built from boolean combinations ("and", "or", "not") of polynomial equations over k and existential and universal quantifiers. The *Grothendieck ring* $K_0(\mathrm{Field}_k)$ *of ring formulas over k* is generated by symbols $[\psi]$, where ψ is a ring formula over k, subject to the relations

$$[\psi_1 \vee \psi_2] = [\psi_1] + [\psi_2] - [\psi_1 \wedge \psi_2]$$

if ψ_1 and ψ_2 have the same free variables, and $[\psi_1] = [\psi_2]$ if there exists a ring formula Ψ over k such that when interpreted in any field, $K \supseteq k$ provides the graph of a bijection between the tuples of elements of K satisfying ψ_1 and those satisfying

ψ_2. The ring multiplication is induced by the conjunction of formulas in disjoint sets of variables [DL 2001]. Denef and Loeser defined a ring homomorphism

$$\chi_f : K_0(\mathrm{Field}_k) \to K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}. \tag{2}$$

They proved that this homomorphism is characterized by two conditions. The first one is that for any ring formula ψ that is a conjunction of polynomial equations over k, the element $\chi_f([\psi])$ is equal to the class $\chi_c([V])$ in $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}$ of the variety V defined by ψ. The second condition, which is more technical, expresses that certain relations should hold in terms of unramified Galois coverings over k. We refer to [DL 2001; 2004] for the precise statement. In the simplest case it implies the following:

Example 1.1 [DL 2004, Example 6.4.3]. If $n \geq 1$ is a fixed integer, k is a field containing all n-th roots of unity, and ψ is the ring formula

$$\psi : \text{there exists y such that } x = y^n \text{ and } x \neq 0,$$

then we have that $\chi_f([\psi]) = (1/n)(\mathbf{L} - 1)$.

Lemma 1.2. *Let ψ be the ring formula whose interpretation in any field $K \supseteq k$ provides the set of K-rational points of T that lift to K-rational points of T' by a Galois covering $T' \to T$ of degree n of d-dimensional algebraic k-tori. If the field k contains all the n-th roots of unity, then we have that $\chi_f([\psi]) = (1/n)(\mathbf{L} - 1)^d$.*

Proof. The morphism $T' \to T$ induces a finite index inclusion of the corresponding character group $M \subseteq M'$, and hence a map of k-algebras $k[M] \hookrightarrow k[M']$. By the classification theorem of finitely generated abelian groups applied to M'/M, there exists a basis $\{v_1, \dots, v_d\}$ of M' and unique integers $b_1 \mid b_2 \mid \cdots \mid b_d$, where $\mid$ denotes division, such that $\{b_1 v_1, \dots, b_d v_d\}$ is a basis of M and $n = b_1 \dots b_d$. It follows that the map of coordinate rings $K[M] \hookrightarrow K[M']$ expresses in coordinates as $K[z_1^{\pm b_1}, \dots, z_d^{\pm b_d}] \hookrightarrow K[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$. We deduce that the ring formula ψ is the conjunction of formulas ψ_i : there exists y_i such that $x_i = y_i^{b_i}$ and $x_i \neq 0$ for $i = 1, \dots, d$, where the variables $x_1, \dots, x_d$ are independent. Then we get that

$$\chi_f([\psi]) = \frac{1}{b_1 \dots b_d}(\mathbf{L} - 1)^d. \qquad \square$$

Remark 1.3. Denef and Loeser defined the map χ_f by factoring it through the Grothendieck ring $K_0(\mathrm{PFF}_k)$ of ring formulas for the category of *pseudofinite fields* containing k. See [DL 2001; 2004; 2002].

2. Arcs, jets and motivic Poincaré series

We start this section by recalling the definition of the space of arcs of a variety S. We assume for simplicity that S is an affine irreducible and reduced algebraic variety defined over a field k of characteristic zero.

For any integer $m \geq 0$, the functor from the category of k-algebras to the category of sets, sending a k-algebra R to the set of $R[t]/(t^{m+1})$-rational points of S, is representable by a k-scheme $H_m(S)$ of finite type over k, called the m-jet scheme of S. The natural maps induced by truncation $j_m^{m+1} : H_{m+1}(S) \to H_m(S)$ are affine, and hence the projective limit $H(S) := \varprojlim H_m(S)$ is a k-scheme, not necessarily of finite type, called the *arc space* of S.

In what follows, we consider the schemes $H_m(S)$ and $H(S)$ with their reduced structure. We have natural morphisms $j_m : H(S) \to H_m(S)$. By an *arc* we mean a k-rational point of $H(S)$, that is, a morphism $\operatorname{Spec} k[[t]] \to S$. By an m-jet we mean a k-rational point of $H_m(S)$, that is, a morphism $\operatorname{Spec} k[t]/(t^{m+1}) \to S$. The origin of the arc (respectively of the m-jet) is the image of the closed point 0 of $\operatorname{Spec} k[[t]]$ (respectively of $\operatorname{Spec} k[t]/(t^{m+1})$).

If $Z \subset S$ is a closed subvariety, then $H(S)_Z := j_0^{-1}(Z)$ (respectively $H_m(S)_Z := (j_0^m)^{-1}(Z)$) denotes the subscheme of $H(S)$ (respectively of $H_m(S)$) formed by arcs (respectively m-jets) in S with origin in Z.

By a theorem of Greenberg [1966], $j_m(H(S))$ is a constructible subset of the k-variety $H_m(S)$ for any integer $m \geq 0$. We can then consider the class

$$[j_m(H(S))] \in K_0(\mathrm{Var}_k).$$

Greenberg's result implies also that there is a ring formula ψ_m over k, such that for any field K containing k, the k-rational points of $H_m(S)$ that can be lifted to K-rational points of $H(S)$ correspond to the tuples satisfying ψ_m in K. If ψ'_m is another ring formula over k with the same property, then $[\psi_m] = [\psi'_m]$ in $K_0(\mathrm{Field}_k)$. The same applies for $j_m(H(S)_Z)$ if $Z \subset S$ is a closed subvariety.

Notation 2.1. We denote the class $[\psi_m]$ by $[j_m(H(S))]_f$ to avoid confusion with the class $[j_m(H(S))] \in K_0(\mathrm{Var}_k)$.

Definition 2.2 [DL 1999; 2001].

(1) The *geometric motivic Poincaré series* of (S, Z) is

$$P_{\mathrm{geom}}^{(S,Z)}(T) := \sum_{m \geq 0} \chi_c([j_m(H(S)_Z)])T^m \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}[[T]].$$

(2) The *arithmetic motivic Poincaré series* of (S, Z) is

$$P_{\mathrm{ar}}^{(S,Z)}(T) := \sum_{m \geq 0} \chi_f([j_m(H(S)_Z)]_f)T^m \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}[[T]].$$

Remark 2.3. In order to have the geometric and arithmetic setting in the same ring, we have slightly modified the original definition of the geometric motivic Poincaré series, since $\sum_{m\geq 0}[j_m(H(S)_Z)]T^m \in K_0(\mathrm{Var}_k)[[T]]$; see [DL 1999]. This does not affect the rationality results below.

Denef and Loeser proved that these series have a rational form:

Theorem 2.4 [DL 1999, Theorem 1.1; 2001, Theorem 9.2.1]. *The series $P_{\mathrm{geom}}^{(S,Z)}(T)$ and $P_{\mathrm{ar}}^{(S,Z)}(T)$ belong to the subring of $K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}[[T]]$ generated by*

$$K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}[T]$$

and the series $(1 - \mathbf{L}^a T^b)^{-1}$, with $a \in \mathbb{Z}$ and $b > 0$.

The arithmetic motivic Poincaré series has interesting properties of specialization to classical arithmetic series. Let p be a prime number. The operators N_p and $N_{p,m}$ are applied to a variety V defined over the integers by $N_p(V) := \#V(\mathbb{Z}/p\mathbb{Z})$ and $N_{p,m}(V) := \#\{\pi_m(V(\mathbb{Z}_p))\}$, where $\mathbb{Z}_p$ denotes the p-adic integers,

$$\pi_m(V(\mathbb{Z}_p)) \subset V(\mathbb{Z}/p^{m+1}\mathbb{Z})$$

is the image of $V(\mathbb{Z}_p))$ by the natural projection induced by $\mathbb{Z}_p \to \mathbb{Z}/p^{m+1}\mathbb{Z}$, and # denotes the cardinality. Suppose that the variety S is defined over the integers. The *Serre–Oesterlé* series $P_p^S(T) := \sum_{m\geq 0} N_{p,m}T^m \in \mathbb{Z}[[T]]$ of S at the prime p is a rational function of T [Denef 1984]. Denef and Loeser proved that for $p \gg 0$, the series $P_p^S(T)$ is obtained from $P_{\mathrm{ar}}^S(T)$ by applying to each coefficient the operator N_p [DL 2001; 2002; 2004].

Remark 2.5. These results hold in a more general setting, in particular when S is not affine as assumed here [DL 1999; 2001]. The proof of the rationality of $P_p^S(T)$ involves the use of quantifier elimination results and p-adic integration [Denef 1984]. The proof of the rationality of $P_{\mathrm{ar}}^S(T)$ requires also quantifier elimination results and arithmetic motivic integration [DL 2001; 2004; 2002].

3. Affine toric varieties and monomial ideals

In this section we introduce the basic notions and notations from toric geometry; see [Ewald 1996; Oda 1988; Fulton 1993; Gel'fand et al. 1994] for proofs.

If $N \cong \mathbb{Z}^d$ is a lattice, we denote by $N_{\mathbb{R}} := N \otimes \mathbb{R}$ the vector space spanned by N over the field $\mathbb{R}$, and by $N_{\mathbb{Q}} := N \otimes \mathbb{Q}$ the vector space spanned by N over $\mathbb{Q}$. In what follows, a *cone* in $N_{\mathbb{R}}$ means a *rational convex polyhedral cone*: the set of nonnegative linear combinations of vectors $a_1, \dots, a_r \in N$. The cone τ is *strictly convex* if it contains no line through the origin, in which case we denote by 0 the 0-dimensional face of τ; the cone τ is *simplicial* if the primitive vectors of the

1-dimensional faces are linearly independent over $\mathbb{R}$. We denote by $\mathring{\tau}$ or by $\operatorname{int}(\tau)$ the relative interior of the cone τ.

We denote by M the dual lattice. The *dual* cone $\tau^\vee \subset M_{\mathbb{R}}$ of τ is the set $\{w \in M_{\mathbb{R}} \mid \langle w, u\rangle \geq 0 \text{ for all } u \in \tau\}$. The *orthogonal* cone $\tau^\perp$ has the condition $\langle w, u\rangle = 0$ instead of $\langle w, u\rangle \geq 0$.

A *fan* Σ is a family of strictly convex cones in $N_{\mathbb{R}}$ such that any face of such a cone is in the family and the intersection of any two of them is a face of each. The relation $\theta \leq \tau$ denotes that θ is a face of τ. By $\theta < \tau$, we mean $\theta \neq \tau$ is a face of τ. The *support* of the fan Σ is the set $|\Sigma| := \bigcup_{\tau\in\Sigma} \tau \subset N_{\mathbb{R}}$. The *k-skeleton* of the fan Σ is $\Sigma^{(k)} = \{\tau \in \Sigma \mid \dim\tau = k\}$. We say that a fan Σ' is a *subdivision* of the fan Σ if both fans have the same support and if every cone of Σ' is contained in a cone of Σ. If Σ_i for $i = 1, \dots, n$ are fans with the same support, their intersection $\bigcap_{i=1}^n \Sigma_i := \{\bigcap_{i=1}^n \tau_i \mid \tau_i \in \Sigma_i\}$ is also a fan.

Notation 3.1. In this paper, Λ is a subsemigroup of finite type of a lattice M, which generates M as a group and such that the cone $\sigma^\vee = \mathbb{R}_{\geq 0}\Lambda$ is strictly convex and of dimension d. We denote by N the dual lattice of M and by $\sigma \subset N_{\mathbb{R}}$ the dual cone of $\sigma^\vee$. We denote by Z^Λ the *affine toric variety* $Z^\Lambda = \operatorname{Spec} k[\Lambda]$, where $k[\Lambda] = \{\sum_{\text{finite}} a_\lambda X^\lambda \mid a_\lambda \in k\}$ denotes the semigroup algebra of the semigroup Λ with coefficients in the field k. The semigroup Λ has a unique minimal set of generators $e_1, \dots, e_n$; see the proof of [Ewald 1996, Lemma V.3.5, page 155]. We have an *embedding* of $Z^\Lambda \subset \mathbf{A}_k^n$ given by $x_i := X^{e_i}$ for $i = 1, \dots, n$.

If $\Lambda = \sigma^\vee \cap M$, then the variety Z^Λ, which we denote also by $Z_{\sigma,N}$ or by Z_σ when the lattice is clear from the context, is normal. If $\Lambda \neq \sigma^\vee \cap M$, the inclusion of semigroups $\Lambda \to \bar{\Lambda} := \sigma^\vee \cap M$ defines a toric modification $Z^{\bar{\Lambda}} \to Z^\Lambda$, which is the *normalization map*.

The torus $T_N := Z^M$ is an open dense subset of Z^Λ that acts on Z^Λ, and the action extends the action of the torus on itself by multiplication. The *origin* 0 of the affine toric variety Z^Λ is the 0-dimensional orbit, defined by the maximal ideal $(X^\lambda)_{0\neq\lambda\in\Lambda}$ of $k[\Lambda]$. There is a one-to-one inclusion reversing correspondence between the faces of σ and the orbit closures of the torus action on Z^Λ. If $\theta \leq \sigma$, we denote by $\operatorname{orb}_\theta^\Lambda$ the orbit corresponding to the face θ of σ. The orbit closures are of the form $Z^{\Lambda\cap\theta^\perp}$ for $\theta \leq \sigma$.

Notation 3.2. The *Newton polyhedron* of a monomial ideal corresponding to a nonempty set of lattice vectors $\mathcal{I} \subset \Lambda$ is defined as the convex hull of the Minkowski sum of sets $\mathcal{I} + \sigma^\vee$. We denote this polyhedron by $\mathcal{N}(\mathcal{I})$. Notice that the vertices of $\mathcal{N}(\mathcal{I})$ are elements of $\mathcal{I}$. We denote by $\operatorname{ord}_{\mathcal{I}}$ the *support function* of the polyhedron $\mathcal{N}(\mathcal{I})$, which is defined by $\operatorname{ord}_{\mathcal{I}} : \sigma \to \mathbb{R}$, $\nu \mapsto \inf_{\omega\in\mathcal{N}(\mathcal{I})}\langle \nu, \omega\rangle$. A vector $\nu \in \sigma$ defines the face $\mathcal{F}_\nu := \{\omega \in \mathcal{N}(\mathcal{I}) \mid \langle \nu, \omega\rangle = \operatorname{ord}_{\mathcal{I}}(\nu)\}$ of the polyhedron $\mathcal{N}(\mathcal{I})$. All faces of $\mathcal{N}(\mathcal{I})$ are of this form, and the compact faces are defined by vectors $\nu \in \mathring{\sigma}$.

The *fan* $\Sigma(\mathcal{I})$ *of the polyhedron* $\mathcal{N}(\mathcal{I})$ is the set of cones

$$\sigma(\mathcal{F}) := \{\nu \in \sigma \mid \langle \nu, \omega \rangle = \mathrm{ord}_{\mathcal{I}}(\nu) \text{ for all } \omega \in \mathcal{F}\}$$

for $\mathcal{F}$ running through the faces of $\mathcal{N}(\mathcal{I})$. The fan $\Sigma(\mathcal{I})$ is supported on σ. By definition, it is easy to check that if $\tau \in \Sigma(\mathcal{I})$ and if $\nu, \nu' \in \mathring{\tau}$, then the faces of the polyhedron $\mathcal{N}(\mathcal{I})$ defined by ν and ν' coincide, that is, $\mathcal{F}_\nu = \mathcal{F}_{\nu'}$; we denote this face also by $\mathcal{F}_\tau$.

The affine varieties Z_τ corresponding to cones τ in a fan Σ glue up to define a *toric variety* Z_Σ. A fan Σ subdividing the cone σ defines a *toric modification* $\pi_\Sigma : Z_\Sigma \to Z_\sigma$.

If $\mathcal{I} \subset \Lambda$ defines a monomial ideal, the composite $Z_{\Sigma(\mathcal{I})} \xrightarrow{\pi_{\Sigma(\mathcal{I})}} Z_\sigma \longrightarrow Z^\Lambda$ is equal to the *normalized blowing up* of Z^Λ centered at $\mathcal{I}$; see for instance [Lejeune-Jalabert and Reguera 2003].

Definition 3.3. For $1 \le j \le n$, the *j-th ideal of sums of combinations* of Z^Λ is the monomial ideal $\mathcal{C}_j$ of $k[\Lambda]$ generated by X^α, where α runs through

$$\left\{ e_{i_1} + \cdots + e_{i_j} \;\middle|\; \{i_1, \dots, i_j\} \in \binom{\{1, \dots, n\}}{j} \right\}, \tag{3}$$

where $\binom{\{1,\dots,n\}}{j}$ denotes the set of combinations of j elements of $\{1, \dots, n\}$ for $j = 1, \dots, n$. We denote by Θ_j the fan of the polyhedron $\mathcal{N}(\mathcal{C}_j)$, and by $\mathrm{ord}_{\mathcal{C}_j}$ its support function; see Notation 3.2. The maps

$$\varphi_1 := \mathrm{ord}_{\mathcal{C}_1} \quad \text{and} \quad \varphi_j := \mathrm{ord}_{\mathcal{C}_j} - \mathrm{ord}_{\mathcal{C}_{j-1}} \quad \text{for } j = 2, \dots, n,$$

are piecewise linear functions defined on the cone σ. If $\nu \in \sigma$, we write $\varphi_0(\nu) := 0$ and $\varphi_{n+1}(\nu) := +\infty$ for convenience.

Definition 3.4. For $1 \le l \le d$, the *l-th logarithmic jacobian ideal* of Z^Λ is the monomial ideal $\mathcal{J}_l$ of $k[\Lambda]$ generated by X^α, where α runs through

$$\{e_{i_1} + \cdots + e_{i_l} \mid e_{i_1} \wedge \cdots \wedge e_{i_l} \ne 0 \text{ for } 1 \le i_1, \dots, i_l \le n\}. \tag{4}$$

We denote by Σ_l the fan of the polyhedron $\mathcal{N}(\mathcal{J}_l)$, and by $\mathrm{ord}_{\mathcal{J}_l}$ its support function; see Notation 3.2. The maps

$$\phi_1 := \mathrm{ord}_{\mathcal{J}_1} \quad \text{and} \quad \phi_l := \mathrm{ord}_{\mathcal{J}_l} - \mathrm{ord}_{\mathcal{J}_{l-1}} \quad \text{for } l = 2, \dots, d,$$

are piecewise linear functions defined on the cone σ. If $\nu \in \sigma$, we write $\phi_0(\nu) := 0$ and $\phi_{d+1}(\nu) := +\infty$ for convenience.

We also use the notations $\mathcal{J}_l$ and $\mathcal{C}_j$ for the sets (4) and (3), respectively.

Lemma 3.5. *If $\nu \in \mathring{\sigma}$ and if $(p_1, \dots, p_n)$ is a permutation of $(1, \dots, n)$ such that*

$$\langle \nu, e_{p_1} \rangle \le \cdots \le \langle \nu, e_{p_n} \rangle,$$

then $\mathrm{ord}_{\mathscr{C}_j}(\nu) = \langle \nu, \sum_{r=1}^{j} e_{p_r} \rangle$ *and* $\varphi_j(\nu) = \langle \nu, e_{p_j} \rangle$ *for* $1 \leq j \leq n$. *Moreover,*

$$0 = \varphi_0(\nu) \leq \varphi_1(\nu) \leq \cdots \leq \varphi_n(\nu) \quad \text{and} \quad 0 = \phi_0(\nu) \leq \phi_1(\nu) \leq \cdots \leq \phi_d(\nu).$$

Proof. The first assertion follows by induction on $j \in \{1, \dots, n\}$.

See [Cobo Pablos and González Pérez 2012, Lemma 5.3] for the second sequence of inequalities. □

Proposition 3.6. *The Newton polyhedra of the ideals* $\mathscr{C}_j$ *for* $j = 1, \dots, n$ *determine and are determined by the minimal system of generators of the semigroup* Λ.

Proof. The Newton polyhedron $\mathcal{N}(\mathscr{C}_j)$ determines and is determined by its support function $\mathrm{ord}_{\mathscr{C}_j}$ for $j = 1, \dots, n$. By Lemma 3.5 and the definitions if θ is a d-dimensional cone of the fan $\bigcap_{r=1}^{n} \Theta_r$, there exists a permutation $i_1, \dots, i_n$ of $1, \dots, n$ such that $\varphi_j(\nu) = \langle \nu, e_{i_j} \rangle$ for $j = 1, \dots, n$ and all $\nu \in \mathring{\theta}$. Thus, the functions φ_j for $j = 1, \dots, n$ or equivalently, $\mathrm{ord}_{\mathscr{C}_j}$ for $j = 1, \dots, n$ determine the vectors $e_1, \dots, e_n$. □

4. Arcs and jets on a toric singularity

Let Λ be a semigroup as in Notation 3.1. If R is a k-algebra, an R-rational point of Z^Λ is a homomorphism of semigroups $(\Lambda, +) \to (R, \cdot)$, where $(R, \cdot)$ denotes the semigroup R for the multiplication. In particular, the closed points are obtained for $R = k$. An arc h on the affine toric variety Z^Λ is given by a semigroup homomorphism $(\Lambda, +) \to (k[[t]], \cdot)$. An arc in the torus T_N is defined by a semigroup homomorphism $\Lambda \to k[[t]]^*$, where $k[[t]]^*$ denotes the group of units of the ring $k[[t]]$.

Notation 4.1. We denote the set of arcs $H(Z^\Lambda)_0$ of Z^Λ with origin at the distinguished point 0 of Z^Λ simply by H_Λ, and by H_Λ^* the set consisting of those arcs of H_Λ with generic point in the torus T_N.

Notice that $h \in H_\Lambda^*$ if and only if the formal power series $X^u \circ h \in k[[t]]$ is nonzero for all $u \in \Lambda$. Any arc $h \in H_\Lambda^*$ defines two group homomorphisms $\nu_h : M \to \mathbb{Z}$ and $\omega_h : M \to k[[t]]^*$ by $X^m \circ h = t^{\nu_h(m)} \omega_h(m)$. If $m \in \Lambda$, then $\nu_h(m) > 0$, and hence ν_h belongs to $\mathring{\sigma} \cap N$. Notice that ω_h defines an arc in the torus, that is, $\omega_h \in H(T_N)$.

Remark 4.2. The space of arcs in the torus acts on the arc space of a toric variety; [Ishii 2004; 2005].

Lemma 4.3 [Ishii 2004, Theorem 4.1; 2005, Lemma 5.6; Lejeune-Jalabert and Reguera 2003, Proposition 3.3]. *The map* $\mathring{\sigma} \cap N \times H(T_N) \to H_\Lambda^*$, *which applies a pair* (ν, ω) *to the arc* h *defined by* $X^u \circ h = t^{\langle \nu, u \rangle} \omega(u)$ *for* $u \in \Lambda$ *is a bijection. The sets* $H_{\Lambda,\nu}^* := \{h \in H_\Lambda^* \mid \nu_h = \nu\}$ *for* $\nu \in \mathring{\sigma} \cap N$ *are orbits for the action of* $H(T_N)$ *on* H_Λ^*, *and we have* $H_\Lambda^* = \bigsqcup_{\nu \in \mathring{\sigma} \cap N} H_{\Lambda,\nu}^*$.

Remark 4.4. We often denote the set H^*_Λ and orbit $H^*_{\Lambda,\nu}$ by H^* and H^*_ν, respectively, if Λ is clear from the context.

An arc $h \in H_\Lambda$ has its generic point η contained in exactly one orbit of the torus action on Z^Λ. If $h(\eta) \in \mathrm{orb}^\Lambda_\theta$ for some $\theta \leq \sigma$, then h factors through the orbit closure $Z^{\Lambda\cap\theta^\perp}$ and $h \in H^*_{\Lambda\cap\theta^\perp}$, that is, h is an arc through $(Z^{\Lambda\cap\theta^\perp}, 0)$ with generic point in the torus $\mathrm{orb}^\Lambda_\theta$. We can apply Lemma 4.3 to describe the set $H^*_{\Lambda\cap\theta^\perp}$, just replacing the semigroup Λ by $\Lambda\cap\theta^\perp$; see [Cobo Pablos and González Pérez 2012]. In particular, if $\theta = 0$, then $h \in H^*_\Lambda$; if $\theta = \sigma$, then $\Lambda\cap\theta^\perp = 0$ and h is the constant arc at the distinguished point $0 \in Z^\Lambda$. We have a partition $H_\Lambda = \bigsqcup_{\theta\leq\sigma} H^*_{\Lambda\cap\theta^\perp}$.

5. The image of the class of the formula defining $j_s(H^*_\nu)$

Definition 5.1. We associate to $(\nu, s) \in (\mathring{\sigma} \cap N) \times \mathbb{Z}_{>0}$ the sets

$$M^s_\nu := \mathrm{span}_\mathbb{Z}\{e_i \mid \langle \nu, e_i\rangle \leq s, i = 1, \dots, n\},$$
$$\ell^s_\nu := \mathrm{span}_\mathbb{Q}\{e_i \mid \langle \nu, e_i\rangle \leq s, i = 1, \dots, n\}.$$

We denote by $l(\nu, s)$ the dimension of the $\mathbb{Q}$-vector space ℓ^s_ν. The integer $l(\nu, s)$ is also the rank of the lattice M^s_ν. We denote by $q(\nu, s)$ the index of the lattice extension $M^s_\nu \subset \ell^s_\nu \cap M$.

Proposition 5.2. *If $(\nu, s) \in \mathring{\sigma} \times \mathbb{Z}_{>0}$, with $l(\nu, s) > 0$, and if the field k contains all the $q(\nu, s)$-th roots of unity, then we have*

$$\chi_f([j_s(H^*_\nu)]_f) = \frac{1}{q(\nu, s)}(\mathbf{L}-1)^{l(\nu,s)} \times \mathbf{L}^{sl(\nu,s)-\mathrm{ord}_{\mathcal{J}_{l(\nu,s)}}(\nu)}.$$

If $l(\nu, s) = 0$, then we have $\chi_f([j_s(H^*_\nu)]_f) = 1$.

Proof. If $h \in H^*_\nu$, the equality $\mathrm{ord}_t(X^{e_i} \circ h) = \langle \nu, e_i\rangle$ holds for $1 \leq i \leq n$. By Definition 5.1, those vectors e_i such that $j_s(X^{e_i} \circ h) \neq 0$ span the $\mathbb{Q}$-vector space ℓ^s_ν, since $\langle \nu, e_i\rangle \leq s$. If $l(\nu, s) = 0$, this vector space is empty, the jet space $j_s(H^*_\nu)$ consists of the constant 0-jet, and the conclusion follows easily from the definitions.

Suppose then that $l := l(\nu, s) > 0$. If $h \in H^*_\nu$, then it is given by n series of the form

$$X^{e_i} \circ h = t^{\langle \nu, e_i\rangle} c(e_i)\Big(1 + \sum_{m\geq 1} u_m(e_i)t^m\Big) \quad \text{for } i = 1, \dots, n.$$

The s-jet $j_s(X^{e_i} \circ h)$ is different from zero if and only if $\langle \nu, e_i\rangle \leq s$.

By [Cobo Pablos and González Pérez 2012, Lemma 5.7], there exist integers $1 \leq k_1, \dots, k_l \leq n$ such that $\phi_i(\nu) = \langle \nu, e_{k_i}\rangle \leq s$ for $i = 1, \dots, l$,

$$\ell^s_\nu = \mathrm{span}_\mathbb{Q}\{e_{k_1}, \dots, e_{k_l}\}, \quad \text{and} \quad \mathrm{ord}_{\mathcal{J}_l}(\nu) = \sum_{i=1}^{l} \langle \nu, e_{k_i}\rangle.$$

By [Cobo Pablos and González Pérez 2012, Section 6], if h is the universal family of arcs parametrizing H_ν^*, the terms $\{u_m(e_{k_i}) \mid i = 1, \dots, l, m \geq 1\}$ are algebraically independent over $\mathbb{Q}$ and the terms $\{c(e_i)^{\pm 1} \mid i = 1, \dots, n\}$ generate a k-algebra isomorphic to $k[M]$ by the isomorphism that maps $c(e_i) \mapsto X^{e_i}$.

By the proof of [Cobo Pablos and González Pérez 2012, Theorem 7.1], a formula defining $j_s(H_\nu^*)$ is the conjunction of two formulas, ψ_1 and ψ_2, with independent sets of variables. The first formula, ψ_1, is a finite sequence of polynomial equalities with rational coefficients expressing the terms $u_r(e_i)$ appearing in $j_s(X^{e_i} \circ h)$, for $1 \leq r \leq s - \langle \nu, e_i \rangle$, in terms of the variables

$$\{u_r(e_{k_i}) \mid 1 \leq i \leq l,\ 1 \leq r \leq s - \langle \nu, e_{k_i} \rangle\}.$$

We deduce that $\chi_f([\psi_1]) = \mathbf{L}^{sl - \mathrm{ord}_{\mathcal{J}_l}(\nu)}$. The second formula comes from the effect on the initial coefficients $c(e_i)$ for $e_i \in \ell_\nu^s$ of the operation taking the s-jet of an arc. This operation is described by taking the image by the map

$$\Psi : T' := \operatorname{Spec} k[c(e_i)^{\pm 1}]_{e_i \in \ell_\nu^s} \to T := \operatorname{Spec} k[c(e_i)^{\pm 1}]_{\langle \nu, e_i \rangle \leq s}$$

of the point determined by $h \in H_\nu^*$. The map Ψ is the unramified covering of l-dimensional algebraic tori determined by the inclusion $M_\nu^s \subset \ell_\nu^s \cap M$ of index $q(\nu, s)$ of rank $l(\nu, s)$ lattices. Thus the second formula is equivalent to ψ_2: there exists $y \in T'$ such that $\Psi(y) = x$ for $x \in T$, and hence by Lemma 1.2 we get that $\chi_f([\psi_2]) = (1/q(\nu, s))(\mathbf{L} - 1)^l$. □

6. Sequences of convex piecewise linear functions and fans

Let $\sigma \subset N_\mathbb{R}$ be a rational convex polyhedral cone of dimension $d = \dim N_\mathbb{R}$. Consider a sequence of piecewise linear continuous functions

$$h_p : \sigma \to \mathbb{R} \quad \text{for } 1 \leq p \leq m,$$

such that $h_p(\sigma \cap N) \subset \mathbb{Z}$, and

$$0 \leq h_1(\nu) \leq \cdots \leq h_m(\nu) \quad \text{for all } \nu \in \sigma. \tag{5}$$

By convenience we set $h_0(\nu) = 0$ and $h_{m+1}(\nu) = +\infty$. We denote by Ξ_0 the fan consisting on the faces of σ and by Ξ_p the coarser fan such that the restriction of h_p to η is linear for any cone $\eta \in \Xi_p$ for $1 \leq p \leq m$. In addition we assume that for any cone $\eta \in \Xi_{p-1}$ the restriction $h_{p|\eta}$ is *upper convex*, that is, $h_p(\nu) + h_p(\nu') \leq h_p(\nu + \nu')$ for all $\nu, \nu' \in \eta$.

Notation 6.1. For $0 \leq p \leq m$ and for $\eta \in \bigcap_{r=1}^p \Xi_r$ we set

$$\eta(\underline{h}, p) := \{(\nu, s) \in N_\mathbb{R} \times \mathbb{R}_{\geq 0} \mid \nu \in \mathring{\sigma} \cap \mathring{\eta},\ h_p(\nu) \leq s < h_{p+1}(\nu)\}.$$

Lemma 6.2. *The closure $\bar{\eta}(\underline{h}, p)$ of the set $\eta(\underline{h}, p)$ is a convex polyhedral cone that is rational for the lattice $N \times \mathbb{Z}$.*

Proof. If $\eta \in \bigcap_{r=0}^{p} \Xi_r$, then the restriction $h_{j|\eta} \colon \eta \to \mathbb{R}$ is linear if $j = p$, and upper convex if $j = p+1$. It follows that $\bar{\eta}(\underline{h}, p)$ is a convex polyhedral cone, rational for the lattice $N \times \mathbb{Z}$, since h_p and h_{p+1} take integral values on N. □

Notation 6.3. For $0 \le p \le m$ and $\eta \in \Xi_p$ we define the following sets:

(i) $A(\underline{h}, p) := \{(\nu, s) \in N \times \mathbb{Z} \mid \nu \in \mathring{\sigma},\ h_p(\nu) \le s < h_{p+1}(\nu)\}$.

(ii) $A(\underline{h}, p, \eta) := \{(\nu, s) \in N \times \mathbb{Z} \mid \nu \in \mathring{\sigma} \cap \mathring{\eta},\ h_p(\nu) \le s < h_{p+1}(\nu)\}$.

Remark 6.4. We have partitions

$$(\mathring{\sigma} \cap N) \times \mathbb{Z}_{\ge 0} = \bigsqcup_{p=0}^{m} A(\underline{h}, p) \quad \text{and} \quad A(\underline{h}, p) = \bigsqcup_{\eta \in \bigcap_{r=0}^{p} \Xi_r} A(\underline{h}, p, \eta).$$

7. Refinements of partitions

We apply the procedure of Section 6 to both sequences $\underline{\phi} = (\phi_1, \dots, \phi_d)$ and $\underline{\varphi} = (\varphi_1, \dots, \varphi_n)$ (see Lemma 3.5).

Remark 7.1. The sequence of fans associated to $\underline{\phi}$ (respectively $\underline{\varphi}$) is $\bigcap_{r=0}^{i} \Sigma_r$ for $i = 0, \dots, d$ (respectively $\bigcap_{r=0}^{i} \Theta_r$ for $i = 0, \dots, n$), where for convenience we denote by Σ_0 or by Θ_0 the fan consisting of the faces of the cone σ.

Lemma 7.2. *If $A(\underline{\varphi}, j, \theta) \neq \varnothing$ for some $1 \le j \le n$ and $\theta \in \bigcap_{r=1}^{j} \Theta_r$ (using Notation 6.3), then the restriction of the functions $(\mathring{\sigma} \cap N) \times \mathbb{Z}_{>0} \to \mathbb{Z}_{\ge 0}$ given by*

$$(\nu, s) \mapsto l(\nu, s) \quad \textit{and} \quad (\nu, s) \mapsto q(\nu, s)$$

to the set $A(\underline{\varphi}, j, \theta)$ are constant functions. We denote their values on the set $A(\underline{\varphi}, j, \theta)$ by $l(j, \theta)$ and $q(j, \theta)$, respectively.

Proof. If $\nu \in \mathring{\theta}$ for $\theta \in \bigcap_{r=1}^{j} \Theta_r$, then there exists a unique cone $\theta_r \in \Theta_r$ such that $\nu \in \mathring{\theta}_r$ for $r = 1, \dots, j$. We denote by $\mathcal{F}_{r,\theta}$ the face of the polyhedron $\mathcal{N}(\mathscr{C}_r)$ defined by any such vector $\nu \in \mathring{\theta}$ for $r = 1, \dots, j$ (see Notation 3.2).

Suppose that $\nu, \nu' \in \mathring{\theta}$ and $(p_1, \dots, p_n)$ and $(p'_1, \dots, p'_n)$ are two permutations of $(1, \dots, n)$ such that the inequalities

$$\langle \nu, e_{p_1} \rangle \le \cdots \le \langle \nu, e_{p_n} \rangle \quad \text{and} \quad \langle \nu', e_{p'_1} \rangle \le \cdots \le \langle \nu', e_{p'_n} \rangle \tag{6}$$

hold. We prove first that

$$\langle \nu, e_{p'_1} \rangle \le \cdots \le \langle \nu, e_{p'_j} \rangle. \tag{7}$$

By definition, for any $1 \le r \le j$, we have that $\mathrm{ord}_{\mathscr{C}_r}(\nu) = \langle \nu, u_r\rangle$ for any $u_r \in \mathscr{F}_{r,\theta}$. We get from Lemma 3.5 that the vectors $u_r := e_{p_1} + \cdots + e_{p_r}$ and $u'_r := e_{p'_1} + \cdots + e_{p'_r}$ belong to $\mathscr{F}_{r,\theta}$ for $1 \le r \le j$. This implies (7).

If $(\nu, s) \in A(\underline{\varphi}, j, \theta)$, then by Lemma 3.5, we obtain $\varphi_j(\nu) = \langle \nu, e_{p_j}\rangle \le s < \varphi_{j+1}(\nu)$. We deduce that if (ν, s) and (ν', s') belong to $A(\underline{\varphi}, j, \theta)$, then

$$\{e_i \mid 1 \le i \le n,\ \langle \nu, e_i\rangle \le s\} = \{e_i \mid 1 \le i \le n,\ \langle \nu', e_i\rangle \le s'\} = \{e_{p_1}, \dots, e_{p_j}\}. \tag{8}$$

Since (8) spans the lattice M^s_ν and the vector space ℓ^s_ν, the sublattices $\ell^s_\nu \cap M$ and M^s_ν are independent of the choice of (ν, s) in $A(\underline{\varphi}, j, \theta)$. This implies the constancy of the functions l and q on $A(\underline{\varphi}, j, \theta)$. $\square$

Remark 7.3. If $1 \le l \le d$ and if $\tau \in \bigcap_{r=1}^{l} \Sigma_r$, we denoted in [Cobo Pablos and González Pérez 2012] the sets $A(\underline{\phi}, l)$ and $A(\underline{\phi}, l, \tau)$ by A_l and $A_{l,\tau}$, respectively. The map $l(\nu, s)$ is also constant on the sets of the form $A(\underline{\phi}, l, \tau)$ for $\tau \in \bigcap_{i=0}^{l} \Sigma_i$; see [Cobo Pablos and González Pérez 2012, Lemma 5.7].

By Remark 6.4, we have two partitions

$$\begin{aligned}(\mathring{\sigma} \cap N) \times \mathbb{Z}_{>0} &= \bigsqcup_{j=0}^{n} \ \bigsqcup_{\theta \in \bigcap_{r=0}^{j} \Theta_r} A(\underline{\varphi}, j, \theta),\\ (\mathring{\sigma} \cap N) \times \mathbb{Z}_{>0} &= \bigsqcup_{l=0}^{d} \ \bigsqcup_{\eta \in \bigcap_{i=0}^{l} \Sigma_i} A(\underline{\phi}, l, \eta)\end{aligned} \tag{9}$$

associated to the sequences $\underline{\varphi}$ and $\underline{\phi}$.

Proposition 7.4. *If $\theta(\underline{\varphi}, j) \neq \varnothing$ for some $1 \le j \le n$ and $\theta \in \bigcap_{r=1}^{j} \Theta_r$, then there exists a unique cone $\tau \in \bigcap_{r=1}^{l(j,\theta)} \Sigma_r$ such that $\theta \subset \tau$ and*

$$A(\underline{\varphi}, j, \theta) \subset A(\underline{\phi}, l(j, \theta), \tau). \tag{10}$$

Proof. Given (ν, s) and (ν', s') in $A(\underline{\varphi}, j, \theta) \subset (\mathring{\sigma} \cap N) \times \mathbb{Z}_{>0}$, we deduce from (9) that there exist cones $\tau \in \bigcap_{i=0}^{l} \Sigma_i$ and $\tau' \in \bigcap_{i=0}^{l'} \Sigma_i$ for integers $0 \le l, l' \le d$ such that $(\nu, s) \in A(\underline{\phi}, l, \tau)$ and $(\nu', s') \in A(\underline{\phi}, l', \tau')$. By [Cobo Pablos and González Pérez 2012, Lemma 5.7], we have $l = l(\nu, s)$ and $l' = l(\nu', s')$, and then $l = l'$ by (8). Then $l = l(j, \theta)$ by definition in Lemma 7.2.

Let $(p_1, \dots, p_n)$ and $(p'_1, \dots, p'_n)$ be two permutations of $(1, \dots, n)$ such that (6) holds. Then we can apply the method given in [Cobo Pablos and González Pérez 2012, Proposition 5.1] to determine the value of $\mathrm{ord}_{\mathscr{J}_i}(\nu)$ for $1 \le i \le l(\nu, s)$. Moreover, it is enough to apply this on the set (8) instead of $\{e_1, \dots, e_n\}$. We deduce from (7) that ν and ν' define the same face of $\mathcal{N}(\mathscr{J}_i)$ for $1 \le i \le l(\nu, s)$. This is equivalent to the equality $\tau = \tau'$. We have proven (10), and as a consequence, the inclusion $\theta \subset \tau$ holds. $\square$

Definition 7.5 [Cobo Pablos and González Pérez 2012, Definition 8.1 and Remark 8.6]. We consider the equivalence relation $\sim$ defined on the set $(\mathring{\sigma} \cap N) \times \mathbb{Z}_{>0}$ by

$$(\nu, s) \sim (\nu', s') \quad \Longleftrightarrow \quad s = s',\ \ell_\nu^s = \ell_{\nu'}^s \text{ and } \nu_{|\ell_\nu^s} = \nu'_{|\ell_{\nu'}^s}.$$

Lemma 7.6. *The set $A(\underline{\varphi}, j, \theta)$ is a union of equivalence classes by the relation $\sim$ of Definition 7.5 for $1 \le j \le n$ and $\theta \in \bigcap_{r=1}^{j} \Theta_r$. Moreover, we have*

$$A(\underline{\phi}, l, \tau)/\!\sim \;= \bigsqcup_{\theta \in \bigcap_{r=1}^{j} \Theta_r,\, l(j,\theta)=l}^{\theta \subset \tau} A(\underline{\varphi}, j, \theta)/\!\sim. \tag{11}$$

Proof. By (9) and Proposition 7.4, it follows that

$$A(\underline{\phi}, l, \tau) = \bigsqcup_{\theta \in \bigcap_{r=1}^{j} \Theta_r,\, l(j,\theta)=l}^{\theta \subset \tau} A(\underline{\varphi}, j, \theta).$$

If (ν, s) belongs to $A(\underline{\varphi}, j, \theta)$ and $(\nu, s) \sim (\nu', s)$, then (8) holds. The vectors ν and ν' define the same face of $\mathcal{N}(\mathcal{C}_r)$ for $1 \le r \le j$, and therefore $\nu' \in \operatorname{int} \theta$. Since $\varphi_j(\nu') \le s < \varphi_{j+1}(\nu')$, we conclude that $(\nu', s) \in A(\underline{\varphi}, j, \theta)$. $\square$

8. The structure of the series $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$

We consider the auxiliary Poincaré series

$$P_{\mathrm{ar}}(\Lambda) := \sum_{s \ge 0} \chi_f\Big(\Big[j_s(H_\Lambda) \setminus \bigcup_{0 \ne \theta \le \sigma} j_s(H_{\Lambda \cap \theta^\perp})\Big]_f\Big) T^s \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}[\![T]\!]. \tag{12}$$

Notice that the Poincaré series $P_{\mathrm{ar}}(\Lambda)$ measures the class of the formula defining the set of jets of arcs with origin in 0 that are not jets of arcs factoring through proper orbit closures of the toric variety Z^Λ.

Proposition 8.1. *We have* $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T) = \sum_{\theta \le \sigma} P_{\mathrm{ar}}(\Lambda \cap \theta^\perp)$.

It follows from Proposition 8.1 that in order to describe the motivic series $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$, it is enough to describe the form of the auxiliary series $P_{\mathrm{ar}}(\Lambda)$ for any semigroup Λ.

Remark 8.2. In the normal case, the equality $j_m(H_\Lambda) = j_m(H_\Lambda^*)$ holds for all $m \ge 0$ [Nicaise 2005b], but this property fails in general.

Definition 8.3 [Cobo Pablos and González Pérez 2012, Definition 8.9]. If $1 \le l \le d$, $\tau \in \bigcap_{i=1}^{l} \Sigma_i$, and $\nu \in \mathring{\tau}$, then ν defines a face $\mathcal{F}_{l,\nu}$ of the polyhedron $\mathcal{N}(\mathcal{J}_l)$. Since the face $\mathcal{F}_{l,\nu}$ is independent of the choice of $\nu \in \mathring{\tau}$, we denote it by $\mathcal{F}_{l,\tau}$. If $1 \le l \le d$, the set $\mathcal{D}_l$ is the subset of cones $\tau \in \bigcap_{i=1}^{l} \Sigma_i$ such that the face $\mathcal{F}_{l,\tau}$ of $\mathcal{N}(\mathcal{J}_l)$ is contained in the interior of $\sigma^\vee$. We denote by $|\mathcal{D}_l|$ the set $\bigcup_{\tau \in \mathcal{D}_l} \tau$.

Proposition 8.4. *Let us fix an integer $s_0 \geq 1$. The set $j_{s_0}(H^*_\Lambda) \setminus \bigcup_{0 \neq \theta \leq \sigma} j_{s_0}(H_{\Lambda \cap \theta^\perp})$ expresses as a finite disjoint union of locally closed subsets, as follows*:

$$j_{s_0}(H^*_\Lambda) \setminus \bigcup_{0 \neq \theta \leq \sigma} j_{s_0}(H_{\Lambda \cap \theta^\perp}) = \bigsqcup_{j=1}^{n} \bigsqcup_{\theta \in \bigcap_{r=1}^{j} \Theta_r}^{\theta \subset |\mathcal{D}_{l(j,\theta)}|} \bigsqcup_{[(\nu, s_0)] \in A(\underline{\varphi}, j, \theta)/\sim} j_{s_0}(H^*_{\Lambda,\nu}). \qquad (13)$$

Proof. This partition follows from the partition in [Cobo Pablos and González Pérez 2012, Proposition 8.11] by using formula (11) (see Remark 7.3). □

If $s_0 \geq 1$, the coefficient of T^{s_0} in the auxiliary series $P(\Lambda)$ is obtained by applying the map χ_f to the class of the formula defining (13). Then we determine this class by using Proposition 5.2.

We introduce the following auxiliary series for $\theta \in \bigcap_{r=1}^{j} \Theta_r$:

$$P_{\underline{\varphi},j,\theta}(\Lambda) := (\mathbf{L}-1)^{l(j,\theta)} \sum_{s \geq 1} \sum_{[(\nu,s)] \in A(\underline{\varphi},j,\theta)/\sim} \mathbf{L}^{l(j,\theta)s - \mathrm{ord}_{\mathcal{J}_{l(j,\theta)}}(\nu)} T^s. \qquad (14)$$

We deduce the next proposition from Propositions 8.4 and 5.2 and formula (14).

Proposition 8.5. $$P_{\mathrm{ar}}(\Lambda) = \sum_{j=1}^{n} \sum_{\theta \in \bigcap_{r=1}^{j} \Theta_r}^{\theta \subset |\mathcal{D}_{l(j,\theta)}|} \frac{1}{q(j,\theta)} P_{\underline{\varphi},j,\theta}(\Lambda).$$

9. The rational form of some generating series

In this section, we fix an integer $1 \leq j \leq n$ and a cone $\theta \in \bigcap_{r=1}^{j} \Theta_r$ such that $A(\underline{\varphi}, j, \theta) \neq \varnothing$. For simplicity, we denote by l the integer $l(j,\theta)$ defined in Lemma 7.2 and by τ the unique element of the fan $\bigcap_{r=1}^{l} \Sigma_r$ such that (10) holds.

Since $\theta \subset \tau \subset \bigcap_{r=1}^{l} \Sigma_r$, the restriction of ϕ_r to θ is a linear function of the form

$$(\phi_r)_{|\theta}(\nu) = \langle \nu, e_{i_r} \rangle \quad \text{for } r = 1, \ldots, l,$$

where $\{i_1, \ldots, i_l\} \subset \{1, \ldots, n\}$.

Consider the lattice homomorphisms

$$\mu : N \times \mathbb{Z} \to \mathbb{Z}^{l+1}, \quad (\nu, s) \mapsto (\langle \nu, e_{i_1} \rangle, \ldots, \langle \nu, e_{i_l} \rangle, s)$$

and

$$\pi = (\pi_1, \pi_2) : \mathbb{Z}^{l+1} \to \mathbb{Z}^2, \quad (a_1, \ldots, a_{l+1}) \mapsto (l a_{l+1} - a_1 - \cdots - a_l, a_{l+1}).$$

We set $\xi = \pi \circ \mu$.

Remark 9.1. The homomorphisms π, μ, and ξ were considered in [Cobo Pablos and González Pérez 2012]. Since θ is contained in τ, the kernels of μ and ξ intersect the cone θ only at the origin. Similarly, by formula (10), the inclusion $\xi(A(\underline{\varphi}, j, \theta)) \subset \mathbb{Z}^2_{\geq 0} \setminus \{(0,0)\}$ holds. See [Cobo Pablos and González Pérez 2012, Section 9].

If $j \neq n$, the *lower boundary* of the cone θ is the set

$$\partial_- \theta(\underline{\varphi}, j) := \{(\nu, s) \mid \nu \in \theta,\ s = \varphi_j(\nu)\}.$$

Notice that $\partial_-\theta(\underline{\varphi}, j)$ is a cone since $\theta \in \bigcap_{r=1}^{j} \Theta_j$, and then the function φ_j is linear on θ. The *upper boundary* is the set

$$\partial_+ \theta(\underline{\varphi}, j) := \{(\nu, s) \mid \nu \in \theta,\ s = \varphi_{j+1}(\nu) \neq \varphi_j(\nu)\}.$$

If $j = n$, then $l = d$ and $\varphi_{n+1}(\nu) = +\infty$, and the upper boundary is the union of cones spanned by $(0,1) \in N_{\mathbb{R}} \times \mathbb{R}$ and the proper faces of the cone $\partial_-\theta(\underline{\varphi}, j)$. The edges of the cone $\theta(\underline{\varphi}, j)$ are edges of $\partial_-\theta(\underline{\varphi}, j) \cup \partial_+\theta(\underline{\varphi}, j)$.

Notation 9.2. If $\rho \subset \tau$ is a one-dimensional cone rational for the lattice N, we denote by ν_ρ the primitive integral vector on ρ, that is, the generator of the semigroup $\rho \cap N$.

Remark 9.3. The primitive integral vectors for the lattice $N \times \mathbb{Z}$ on the edges of the cone θ are

$$(\nu_\rho, \varphi_j(\nu_\rho)) \quad \text{for } \rho \leq \theta,\ \dim \rho = 1$$

together with

$$\begin{cases} (0,1) & \text{if } j = n, \\ (\nu_\rho, \varphi_{j+1}(\nu_\rho)) & \text{if } j \neq n \end{cases}$$

for $\rho \in \Theta_{j+1}$, $\rho \subset \theta$, $\dim \rho = 1$, and $\varphi_j(\nu) \neq \varphi_{j+1}(\nu)$. Then notice that

$$\xi(\nu, s) = \begin{cases} (l\varphi_j(\nu_\rho) - \operatorname{ord}_{\mathcal{J}_l}(\nu_\rho), \varphi_j(\nu_\rho)) & \text{if } (\nu, s) = (\nu_\rho, \varphi_j(\nu_\rho)), \\ (l\varphi_{j+1}(\nu_\rho) - \operatorname{ord}_{\mathcal{J}_l}(\nu_\rho), \varphi_{j+1}(\nu_\rho)) & \text{if } (\nu, s) = (\nu_\rho, \varphi_{j+1}(\nu_\rho)), \\ (d, 1) & \text{if } (\nu, s) = (0,1). \end{cases} \tag{15}$$

Definition 9.4. Suppose that $A(\underline{\varphi}, j, \theta) \neq \varnothing$. We denote by $B_{\underline{\varphi}, j, \theta}(\Lambda)$ the finite set

$$\bigl\{(l\varphi_j(\nu_\rho) - \operatorname{ord}_{\mathcal{J}_l}(\nu_\rho), \varphi_j(\nu_\rho)) \bigm| \rho \leq \theta,\ \dim \rho = 1\bigr\}$$
$$\cup \begin{cases} \bigl\{(l\varphi_{j+1}(\nu_\rho) - \operatorname{ord}_{\mathcal{J}_l}(\nu_\rho), \varphi_{j+1}(\nu_\rho)) \bigm| \rho \in \Theta_{j+1}^{(1)},\ \rho \subset \theta\bigr\} & \text{if } j \neq n, \\ \{(d,1)\} & \text{if } j = n. \end{cases}$$

Definition 9.5. If $A \subset \mathbb{Z}^{l+1}$ is a set, we denote by $F_A(x) := \sum_{a \in A} x^a$ the *generating function* of A; see [Cobo Pablos and González Pérez 2012, Section 12].

Proposition 9.6. *We have the following equality*:

$$P_{\underline{\varphi},j,\theta}(\Lambda) = (\mathbf{L}-1)^{l(j,\theta)} \sum_{a\in\mu(A(\underline{\varphi},j,\theta))} \mathbf{L}^{\pi_1(a)}T^{\pi_2(a)} \in \mathbb{Z}[\mathbf{L}][\![T]\!]. \tag{16}$$

There exists a polynomial $R_{\underline{\varphi},j,\theta} \in \mathbb{Z}[\mathbf{L}, T]$ *such that* $P_{\underline{\varphi},j,\theta}(\Lambda)$ *has the rational form*

$$P_{\underline{\varphi},j,\theta}(\Lambda) = R_{\underline{\varphi},j,\theta} \prod_{(a,b)\in B_{\underline{\varphi},j,\theta}(\Lambda)} (1-\mathbf{L}^a T^b)^{-1}.$$

Proof. The map μ defines a bijection

$$A(\underline{\varphi}, j, \theta)/_\sim \to \mu(A(\underline{\varphi}, j, \theta)), \quad [(\nu, s)] \mapsto \mu(\nu, s);$$

see [Cobo Pablos and González Pérez 2012, Lemma 9.3] and Lemma 7.6. Then the equality (16) follows from the definitions.

We denote by $\pi_* : k[\![\mathbb{Z}^{l+1}]\!] \to k[\![\mathbf{L}, T]\!]$ the monomial transformation defined by $\pi_*(x^a) := \mathbf{L}^{\pi_1(a)}T^{\pi_2(a)}$ for $a \in \mathbb{Z}^{l+1}$. Then we get that

$$P_{\underline{\varphi},j,\theta}(\Lambda) = (\mathbf{L}-1)^{l(j,\theta)}\pi_*(F_{\mu(A_{\underline{\varphi},j,\theta})}(x)).$$

We apply [Cobo Pablos and González Pérez 2012, Theorem 12.4]. We obtain that the denominator of a rational form of $F_{\mu(A_{\underline{\varphi},j,\theta})}(x)$ consists of products of terms $1 - x^{\mu(b)}$ for b running through the primitive integral vectors in the edges of the closure of the cone $\theta(\underline{\varphi}, j)$. Then the result follows by Remark 9.3 and Definition 9.4. □

10. Main results

Definition 10.1. (i) For a semigroup Λ generating a rank $d \geq 1$ lattice, we define a finite subset $B_{\text{ar}}(\Lambda)$ of $\mathbb{Z}^2_{\geq 0}$ as (see Definition 9.4):

$$B_{\text{ar}}(\Lambda) := \bigcup_{\theta\in\bigcap_{r=1}^{j}\Theta_r,\ \theta\subset|\mathfrak{D}_{l(j,\theta)}|}^{1\leq j\leq n} B_{\underline{\varphi},j,\theta}(\Lambda). \tag{17}$$

If $0 < \eta < \sigma$, then $\Lambda \cap \eta^\perp$ is a semigroup generating lattice of rank $d - \dim\eta$. We use formula (17) and Definition 9.4 to define in this case a finite subset $B_{\text{ar}}(\Lambda \cap \eta^\perp)$ of $\mathbb{Z}^2_{\geq 0}$. We set

$$B_{\text{ar}}(\Lambda \cap \sigma^\perp) := \{(0, 1)\} \quad \text{and} \quad B_{\text{ar},\Lambda} := \bigcup_{\tau\leq\sigma} B_{\text{ar}}(\Lambda \cap \sigma^\perp).$$

(ii) We define the integer

$$q(\Lambda) := \operatorname{lcm}\big\{q(j,\theta) \bigm| \theta \in \textstyle\bigcap_{r=1}^{j}\Theta_r,\ \theta \subseteq |D_{l(j,\theta)}|,\ 1 \leq j \leq n\big\}. \tag{18}$$

If $0 < \eta < \sigma$, then $q(\Lambda \cap \eta^{\perp})$ is the number obtained by replacing Λ by $\Lambda \cap \eta^{\perp}$ in (18). We set $q(\Lambda \cap \sigma^{\perp}) := 1$. We define also the integer

$$q_{\Lambda} := \operatorname{lcm}\{q(\Lambda \cap \eta^{\perp}) \mid \eta \leq \sigma\}.$$

Theorem 10.2. *Suppose that the field k contains all $q(\Lambda)$-th roots of unity. Then there exists a polynomial $Q_{\mathrm{ar}}(\Lambda) \in \mathbb{Z}[\mathbf{L}, T]$ such that*

$$P_{\mathrm{ar}}(\Lambda) = \frac{1}{q(\Lambda)} Q_{\mathrm{ar}}(\Lambda) \prod_{(a,b)\in B_{\mathrm{ar}}(\Lambda)} (1 - \mathbf{L}^{a} T^{b})^{-1}.$$

Proof. This follows from Propositions 8.5 and 9.6. □

Notation 10.3. If $\eta < \sigma$, then the polynomial $Q_{\mathrm{ar}}(\Lambda \cap \eta^{\perp})$ is obtained from Theorem 10.2 by replacing Λ by the semigroup $\Lambda \cap \eta^{\perp}$. We set $Q_{\mathrm{ar}}(\Lambda \cap \sigma^{\perp}) := 1$.

Corollary 10.4. *If the field k contains all q_{Λ}-th roots of unity, then there exists a polynomial $Q_{\mathrm{ar},\Lambda} \in \mathbb{Z}[\mathbf{L}, T]$ such that*

$$P_{\mathrm{ar}}^{(Z^{\Lambda},0)}(T) = \frac{1}{q_{\Lambda}} Q_{\mathrm{ar},\Lambda} \prod_{(a,b)\in B_{\mathrm{ar},\Lambda}} (1 - \mathbf{L}^{a} T^{b})^{-1}.$$

Moreover, we have the equality

$$P_{\mathrm{ar}}^{(Z^{\Lambda},0)}(T) = \sum_{\eta\leq\sigma} \frac{1}{q(\Lambda\cap\eta^{\perp})} Q_{\mathrm{ar}}(\Lambda \cap \eta^{\perp}) \prod_{(a,b)\in B_{\mathrm{ar}}(\Lambda\cap\eta^{\perp})} (1 - \mathbf{L}^{a} T^{b})^{-1}. \tag{19}$$

Proof. The result follows by Theorem 10.2 and Proposition 8.1. □

We can now compare the series $P_{\mathrm{geom}}^{(Z,0)}(T)$ and $P_{\mathrm{ar}}^{(Z,0)}(T)$ (see Definition 2.2). In [Cobo Pablos and González Pérez 2012] we introduced the series

$$P_{\mathrm{geom}}(\Lambda) := \sum_{s\geq 0} \chi_c\Big(\Big[j_s(H_{\Lambda}^{*}) \setminus \bigcup_{0\neq\theta\leq\sigma} j_s(H_{\Lambda\cap\theta^{\perp}}) \Big]\Big) T^{s} \in K_0^{\mathrm{mot}}(\mathrm{Var}_k) \otimes \mathbb{Q}[[T]], \tag{20}$$

and we proved that

$$P_{\mathrm{geom}}^{(Z^{\Lambda},0)}(T) = \sum_{\theta\leq\sigma} P_{\mathrm{geom}}(\Lambda \cap \theta^{\perp}).$$

Proposition 10.5. *If the field k contains all $q(\Lambda)$-th roots of unity, then*

$$P_{\mathrm{ar}}(\Lambda) - P_{\mathrm{geom}}(\Lambda) = \sum_{j=1}^{n} \sum_{\theta\in\bigcap_{r=1}^{j}\Theta_r}^{\theta\subset|\mathcal{D}_{l(j,\theta)}|} \Big(1 - \frac{1}{q(j,\theta)}\Big) R_{\underline{\varphi},j,\theta} \prod_{(a,b)\in B_{\underline{\varphi},j,\theta}(\Lambda)} (1 - \mathbf{L}^{a} T^{b})^{-1}.$$

Proof. This follows from Proposition 9.6, formula (20), Theorem 10.2, and the results in [Cobo Pablos and González Pérez 2012] for $P_{\mathrm{geom}}(\Lambda)$. □

Corollary 10.6. *If for every integer $1 \leq l \leq d$ and any vertex v of the Newton polyhedra $\mathcal{N}(\mathcal{J}_l)$, there exists a subset $I_v \subset \{1, \ldots, n\}$ of l elements such that*

$$v = \sum_{i \in I_v} e_i$$

and the vectors e_i for $i \in I_v$ form part of a basis of M, then the series $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$ and $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T)$ coincide.

Proof. This condition implies that $q(\nu, s) = 1$ for every $(\nu, s) \in (\mathring{\sigma} \cap N) \times \mathbb{Z}_{>0}$. By Proposition 10.5, we get that $P_{\mathrm{ar}}(\Lambda) = P_{\mathrm{geom}}(\Lambda)$. Now for any face $\eta \leq \sigma$, the vertices of the Newton polyhedra of the logarithmic jacobian ideals of $\Lambda \cap \eta^\perp$ are also vertices of the logarithmic jacobian ideals of Λ. The hypothesis implies that $\Lambda \cap \eta^\perp$ spans the lattice $M \cap \eta^\perp$ and then also that $P_{\mathrm{ar}}(\Lambda \cap \eta^\perp) = P_{\mathrm{geom}}(\Lambda \cap \eta^\perp)$. □

Remark 10.7. Corollary 10.6 is a generalization of the Nicaise condition [2005b, Theorem 1] in the case of normal toric varieties.

Remark 10.8. The coordinates of the vectors in the set $B_{\underline{\varphi}, j, \theta}(\Lambda)$ can be described geometrically in terms of the ideals $\mathcal{C}_j$ and $\mathcal{J}_l$ for $l = l(j, \theta)$. Let $\pi_j : Z_j \to Z^\Lambda$ be the composite of the normalization of Z^Λ, with the toric modification defined by the subdivision $\bigcap_{r=1}^{j} \Theta_r$ of σ. The modification π_j is the minimal toric modification that factors through the normalized blowing up with center $\mathcal{C}_r$ for $r = 1, \ldots, j$. If ρ is an edge of θ, the orbit closure E_ρ of the orbit associated to ρ on Z_j has codimension one. We denote by v_ρ the divisorial valuation defined by E_ρ. It satisfies $v_\rho(X^m) = \langle \nu_\rho, m \rangle$ for $m \in M$. The pullback $\pi_j^*(\mathcal{J})$ of a monomial ideal $\mathcal{J}$ of Z^Λ is a sheaf of monomial ideals on Z_j. The ideals $\pi_j^*(\mathcal{C}_r)$ for $r = 1, \ldots, j$ are locally principal on Z_j. Then we get the following identities:

$$\begin{aligned}
\varphi_j(\nu_\rho) &= v_\rho(\pi_j^*(\mathcal{C}_j)) - v_\rho(\pi_j^*(\mathcal{C}_{j-1})), \\
\varphi_{j+1}(\nu_\rho) &= v_\rho(\pi_j^*(\mathcal{C}_{j+1})) - v_\rho(\pi_j^*(\mathcal{C}_j)), \\
\mathrm{ord}_{\mathcal{J}_l}(\nu_\rho) &= v_\rho(\pi_j^*(\mathcal{J}_l)).
\end{aligned}$$

Compare this with the geometrical description of the set of candidate poles of $P_{\mathrm{geom}}^{(Z^\Lambda,0)}(T)$; see [Cobo Pablos and González Pérez 2012].

11. The normal case

In the normal case, when the semigroup Λ is saturated, that is, $\Lambda = \sigma^\vee \cap M$, we describe the motivic arithmetic series in a simpler way by using $j^s(H_\Lambda) = j_s(H_\Lambda^*)$; see [Nicaise 2005b].

Notation 11.1. (i) $\mathcal{A} = \bigsqcup_{l=1}^{d} \bigsqcup_{\tau \in \bigcap_{r=1}^{l} \Sigma_r} A(\underline{\phi}, l, \tau)/\sim$.

(ii) For $s_0 \geq 0$ we set $\mathcal{A}_{s_0} = \{[(\nu, s)] \in \mathcal{A} \mid s = s_0\}$.

Remark 11.2. The set $\mathscr{A}_s$ is finite; see [Cobo Pablos and González Pérez 2012, Remark 8.2]. By (9) and Lemma 7.6, we deduce that

$$\mathscr{A} = \bigsqcup_{j=1}^{n} \bigsqcup_{\theta \in \bigcap_{r=1}^{j} \Theta_r} A(\underline{\varphi}, j, \theta)/\sim.$$

Proposition 11.3. *Let us fix an integer $s_0 \geq 1$. The set $j_{s_0}(H^*)$ expresses as a finite disjoint union of locally closed subsets as $j_{s_0}(H^*) = \bigsqcup_{[(\nu,s)] \in \mathscr{A}_{s_0}} j_{s_0}(H^*_\nu)$. We deduce that $\chi_f([j_s(H^*)]_f) = \sum_{[(\nu,s)] \in \mathscr{A}_s} \chi_f([j_s(H^*_\nu)]_f)$.*

Proof. The first claim follows by the method of [Cobo Pablos and González Pérez 2012, Proposition 8.11]. The second is a consequence of the first and Proposition 5.2. □

Theorem 11.4. *If Z^Λ is normal, then we have*

$$P_{\mathrm{ar}}^{(Z^\Lambda,0)} = \sum_{j=1}^{n} \sum_{\theta \in \bigcap_{r=1}^{j} \Theta_r} \frac{1}{q(j,\theta)} R_{\underline{\varphi},j,\theta} \prod_{(a,b) \in B_{\underline{\varphi},j,\theta}(\Lambda)} (1 - \mathbf{L}^a T^b)^{-1}.$$

Proof. It is a consequence of Propositions 11.3 and 9.6 and Remark 11.2. □

Corollary 11.5. *Suppose that the affine toric variety Z^Λ is normal. If $\theta \leq \sigma$, we denote by $\sigma_\theta^\vee$ the image of the cone $\sigma^\vee$ in $(M_\theta)_\mathbb{R}$, where $M_\theta := M/\theta^\perp \cap M$, and by $\Lambda(\theta)$ the saturated semigroup $\Lambda(\theta) := (\sigma_\theta^\vee \cap M_\theta) \times \mathbb{Z}_{\geq 0}^{\operatorname{codim}\theta}$. With this notation, we have*

$$P_{\mathrm{ar}}^{Z^\Lambda}(T) = \sum_{\theta \leq \sigma} (\mathbf{L} - 1)^{\operatorname{codim}\theta} P_{\mathrm{ar}}^{(Z^{\Lambda(\theta)},0)}(T).$$

Proof. The proof follows by the arguments of [Cobo Pablos and González Pérez 2012, Corollary 4.11]. □

12. Examples

12a. *The case of monomial curves.* Let $\Lambda \subset \mathbb{Z}_{\geq 0}$ be a semigroup with minimal system of generators $e_1 < e_2 < \cdots < e_n$ such that $\gcd\{e_1, \ldots, e_n\} = 1$. In this case, we have $\mathring{\sigma} \cap N = \mathbb{Z}_{>0}$. If $q_i := \gcd\{e_1, \ldots, e_i\}$, then we obtain

$$P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T) = \frac{1}{1-T} + \frac{\mathbf{L}-1}{1-\mathbf{L}T}\left(\frac{1}{q_1}\frac{T^{e_1}}{1-T^{e_1}} + \sum_{i=2}^{n} \frac{q_{i-1}-q_i}{q_{i-1}q_i} \frac{\mathbf{L}^{e_i-e_1}T^{e_i}}{1-\mathbf{L}^{e_i-e_1}T^{e_i}}\right). \quad (21)$$

This follows from the results of this paper, taking the following observations into account:

- We have the equality $j_s(H) = j_s(H^*)$.
- If $\nu, \nu' \in \mathbb{Z}_{>0}$ satisfy $j_s(H^*_\nu)$ and $j_s(H^*_{\nu'}) \neq \{0\}$, then the equality $j_s(H^*_\nu) = j_s(H^*_{\nu'})$ implies $\nu = \nu'$.

- If $\nu \in \mathbb{Z}_{>0}$ satisfies $\nu e_i \leq s < \nu e_{i+1}$, then $q(\nu, s) = q_i$.

Then, setting $e_{d+1} := \infty$, we get the following equality, which implies (21):

$$P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T) = \frac{1}{1-T} + \sum_{i=1}^{n} \sum_{\nu=1}^{\infty} \sum_{s=\nu e_i}^{\nu e_{i+1}-1} (\mathbf{L}-1)\frac{1}{q_i}\mathbf{L}^{s-\nu e_1} T^s.$$

Remark 12.1. The inequalities $q_1 \geq q_2 \geq \cdots \geq q_n = 1$ are not always strict. For instance, if Λ is generated by $e_1 = 8$, $e_2 = 18$, $e_3 = 20$ and $e_4 = 21$, then we get $q_1 = 8$, $q_2 = q_3 = 2$ and $q_4 = 1$. It follows from (21) that the term $1 - \mathbf{L}^{12}T^{20}$ is not a factor of the denominator of the series $P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T)$. If Λ' is the semigroup generated by e_1, e_2 and e_4, then we obtain from (21) that

$$P_{\mathrm{ar}}^{(Z^\Lambda,0)}(T) = P_{\mathrm{ar}}^{(Z^{\Lambda'},0)}(T),$$

while the semigroups Λ and Λ' are not isomorphic. In contrast with this behavior, the motivic series $P_{\mathrm{ar}}^{(C,0)}(T)$ of a plane branch $(C, 0)$ determines the semigroup of the branch $(C, 0)$; see [DL 2001].

12b. ***An example of non-normal toric surface.*** Consider the semigroup Λ generated by the vectors $e_1 = (5, 0)$, $e_2 = (0, 2)$, $e_3 = (0, 3)$ and $e_4 = (6, 2)$. The cone σ is $\mathbb{R}^2_{\geq 0}$ and the lattice M is equal to $\mathbb{Z}^2$. We have the semigroups $\Lambda \cap \eta_1^\perp = (5, 0)\mathbb{Z}_{>0}$ and $\Lambda \cap \eta_2^\perp = (0, 2)\mathbb{Z}_{>0} + (0, 3)\mathbb{Z}_{>0}$, where η_1 and η_2 are the one-dimensional faces of σ. By the case of monomial curves, we get

$$P_{\mathrm{ar}}(\Lambda\cap\eta_1^\perp) = \frac{\mathbf{L}-1}{1-\mathbf{L}T}\frac{T}{1-T} \quad \text{and} \quad P_{\mathrm{ar}}(\Lambda\cap\eta_2^\perp) = \frac{\mathbf{L}-1}{2(1-\mathbf{L}T)}\left(\frac{T^2}{1-T^2} + \frac{\mathbf{L}T^3}{1-\mathbf{L}T^3}\right).$$

Figure 1 shows the subdivisions associated with the ideals $\mathscr{C}_r$ for $r = 1, 2, 3$.

In the following table, we give the different values of $q(j, \theta)$ and $l(j, \theta)$ for θ in the subdivisions of Figure 1 and j such that $A(\underline{\varphi}, j, \theta) \neq \varnothing$. We exclude from

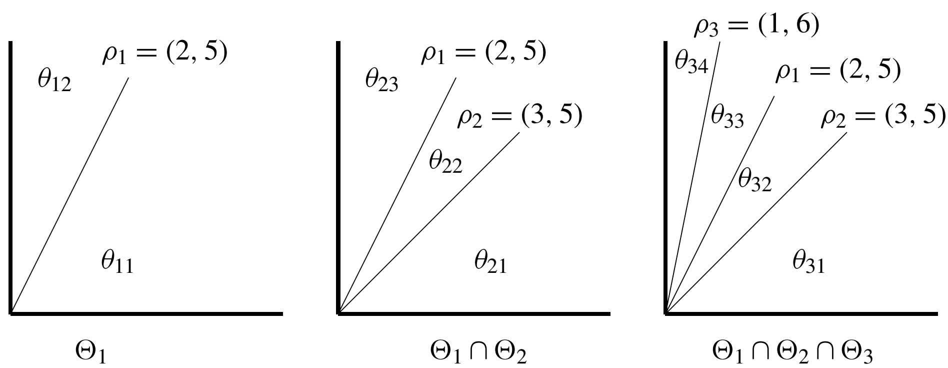

Figure 1. The subdivisions Θ_1, $\Theta_1 \cap \Theta_2$ and $\Theta_1 \cap \Theta_2 \cap \Theta_3$.

this table the cones in $\theta \in \bigcap_{r=1}^{4} \Theta_r$ for $j = 4$, since in this case $q(4, \theta) = 1$ and $l(4, \theta) = 2$.

$q(1, \theta_{11}) = 2$ $l(1, \theta_{11}) = 1$	$q(1, \theta_{12}) = 5$ $l(1, \theta_{12}) = 1$			
$q(2, \theta_{21}) = 1$ $l(2, \theta_{21}) = 1$	$q(2, \theta_{22}) = 10$ $l(2, \theta_{22}) = 2$	$q(2, \theta_{23}) = 10$ $l(2, \theta_{23}) = 2$	$q(2, \rho_1) = 10$ $l(2, \rho_1) = 2$	
$q(3, \theta_{31}) = 5$ $l(3, \theta_{31}) = 2$	$q(3, \theta_{32}) = 5$ $l(3, \theta_{32}) = 5$	$q(3, \theta_{33}) = 5$ $l(3, \theta_{33}) = 2$	$q(3, \theta_{34}) = 2$ $l(3, \theta_{34}) = 2$	$q(3, \rho_1) = 5$ $q(3, \rho_2) = 5$ $l(3, \rho_1) = 2$ $l(3, \rho_2) = 2$

Notice that $A(\underline{\varphi}, 1, \rho_1) = A(\underline{\varphi}, 2, \rho_2) = A(\underline{\varphi}, 3, \rho_3) = \varnothing$. In the following table, we have filled in the cases corresponding to the pairs $(a, b) \in B_{\text{ar}}(\Lambda)$, $(a, b) \neq (2, 1)$ in terms of the rays appearing in the subdivisions of Figure 1:

$(a, b) \in B_{\text{ar}}(\Lambda)$	$\nu_{\rho_1} = (2, 5)$	$\nu_{\rho_2} = (3, 5)$	$\nu_{\rho_3} = (1, 6)$	$\nu_{\sigma^\vee \cap \eta_1^\perp} = (1, 0)$	$\nu_{\sigma^\vee \cap \eta_2^\perp} = (0, 1)$
$(2\varphi_2 - \text{ord}_{\mathcal{J}_2}, \varphi_2)$	$(0, 10)$	$(5, 15)$			$(2, 2)$
$(2\varphi_3 - \text{ord}_{\mathcal{J}_2}, \varphi_3)$	$(10, 15)$	$(5, 15)$	$(19, 18)$	$(5, 5)$	$(2, 2)$
$(2\varphi_4 - \text{ord}_{\mathcal{J}_2}, \varphi_4)$	$(24, 22)$	$(31, 28)$	$(19, 18)$	$(7, 6)$	$(4, 3)$

It follows that $B_{\text{ar},\Lambda} = B_{\text{ar}}(\Lambda) \cup \{(1, 3), (0, 2), (1, 1), (0, 1)\}$. We have computed the sum of the series $P_{\text{ar}}^{(Z^\Lambda, 0)}(T)$ with the methods of [Cobo Pablos and González Pérez 2012]. We have obtained an irredundant representation of the form

$$P_{\text{ar}}^{(Z^\Lambda, 0)}(T) = R(\mathbf{L}, T) \prod_{(a,b) \in B} (1 - \mathbf{L}^a T^b)^{-1}$$

with $R(\mathbf{L}, T) \in \mathbb{Q}[\mathbf{L}, T]$, and where $B = B_{\text{ar},\Lambda} \setminus \{(24, 22), (31, 28)\}$.

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Maximal ideals and representations of twisted forms of algebras

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Given a central simple algebra $\mathfrak{g}$ and a Galois extension of base rings S/R, we show that the maximal ideals of twisted S/R-forms of the algebra of currents $\mathfrak{g}(R)$ are in natural bijection with the maximal ideals of R. When $\mathfrak{g}$ is a Lie algebra, we use this to give a complete classification of the finite-dimensional simple modules over twisted forms of $\mathfrak{g}(R)$.

1. Introduction

Let S/R be a (finite) Galois extension of commutative, associative, and unital algebras over a field k, and let $\mathfrak{g}$ be a finite-dimensional central simple k-algebra. Let $\mathscr{L}$ be an S/R-form of $\mathfrak{g} \otimes_k R$, that is, an R-algebra $\mathscr{L}$ such that

$$\mathscr{L} \otimes_R S \simeq \mathfrak{g} \otimes_k S \tag{1.1}$$

as algebras over S.

In this paper we accomplish two tasks:

(1) We establish a natural correspondence between the maximal ideals of $\mathscr{L}$ and those of the base ring R.

(2) If $\mathfrak{g}$ is a Lie algebra, k is algebraically closed of characteristic 0, and R is of finite type, we describe all the finite-dimensional irreducible modules of $\mathscr{L}$ and classify them up to isomorphism.

In what follows, we will denote $\mathfrak{g} \otimes_k S$ as $\mathfrak{g}(S)$. Recall that if Γ is the Galois group of S/R, then there is a natural correspondence between the set of isomorphism classes of S/R-forms of $\mathfrak{g}(R) = \mathfrak{g} \otimes_k R$ and the pointed set of nonabelian Galois cohomology $\mathrm{H}^1(\Gamma, \mathrm{Aut}_{S\text{-alg}}\, \mathfrak{g}(S))$. See [Knus and Ojanguren 1974], for example.

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For example, consider the multiloop algebra $\mathscr{L}(\mathfrak{g}, \sigma)$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra over an algebraically closed field k of characteristic 0, and σ is an N-tuple of commuting automorphisms

$$\sigma_1, \dots, \sigma_N : \mathfrak{g} \to \mathfrak{g}$$

of finite orders $m_1, \dots, m_N$, respectively. This is a $\mathbb{Z}^N$-graded Lie subalgebra of the Lie algebra $\mathfrak{g}(S)$, where $S = k[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$:

$$\mathscr{L}(\mathfrak{g}; \sigma) = \bigoplus_{j \in \mathbb{Z}^N} \mathfrak{g}_j \otimes t_1^{j_1} t_2^{j_2} \cdots t_N^{j_N},$$

where $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \sigma_i(x) = \xi_i^{j_i} x \text{ for all } i\}$, for fixed primitive m_i-th roots of unity $\xi_i \in k$. Then $\mathscr{L}(\mathfrak{g}, \sigma)$ is an S/R-form of $\mathfrak{g}(R)$, where $R = k\big[t_1^{\pm m_1}, \dots, t_N^{\pm m_N}\big]$. The Galois group Γ of S/R is $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_N}$, and the corresponding (constant) 1-cocycle in $\mathrm{H}^1(\Gamma, \mathrm{Aut}_{S\text{-alg}}\, \mathfrak{g}(S))$ is the group homomorphism taking a fixed generator α_i of $\mathbb{Z}_{m_i}$ to $\sigma_i^{-1} \otimes 1$. Such algebras play an important role in affine Kac–Moody, toroidal, and extended affine Lie theory.[1]

We open the paper with a detailed investigation of the maximal ideals of twisted forms $\mathscr{L}$.[2] Given any ideal $\mathscr{I}$ of the R-algebra $\mathscr{L}$, we show that there is a unique Γ-stable ideal $J(\mathscr{I}) \subseteq S$ for which $\mathscr{I} \otimes_R S$ maps to $\mathfrak{g} \otimes_k J(\mathscr{I})$ under the isomorphism $\mathscr{L} \otimes_R S \to \mathfrak{g} \otimes_k S$. As all maximal ideals $\mathscr{I}$ of the k-algebra $\mathscr{L}$ are R-stable, this produces a bijection $\psi : \mathscr{I} \mapsto J(\mathscr{I}) \cap R$ between maximal ideals of the k-algebra $\mathscr{L}$ and the set $\mathrm{Max}(R)$ of maximal ideals of R. Explicitly, $\psi^{-1} : I \mapsto I\mathscr{L}$ for maximal ideals $I \subseteq R$.

To have access to the attractive results of classical representation theory, we then assume that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra and R is of finite type over an algebraically closed field k of characteristic 0. The classification of finite-dimensional simple $\mathscr{L}$-modules V proceeds by observing that the kernel of the representation $\phi : \mathscr{L} \to \mathrm{End}_k(V)$ is an intersection of a finite collection of distinct maximal ideals $\mathscr{I}_1, \dots, \mathscr{I}_n \subseteq \mathscr{L}$. Given any maximal ideals $M_1, \dots, M_n \in \mathrm{Max}(S)$ lying over the maximal ideals $\psi(\mathscr{I}_1), \dots, \psi(\mathscr{I}_n) \in \mathrm{Max}(R)$, respectively, we obtain evaluation maps

$$\mathrm{ev}_M : \mathscr{L} \hookrightarrow \mathfrak{g} \otimes_k S \to (\mathfrak{g} \otimes_k S/M_1) \oplus \cdots \oplus (\mathfrak{g} \otimes_k S/M_n) \simeq \mathfrak{g}^{\oplus n}.$$

We then use properties of forms to show that ev_M is surjective and descends to an isomorphism $\mathrm{ev}_M : \mathscr{L}/\ker\phi \xrightarrow{\sim} \mathfrak{g}^{\oplus n}$. The finite-dimensional simple $\mathscr{L}$-modules V

[1] For simplicity of notation, we use integral powers of the variables t_i, though fractional exponents are sometimes used to work with the absolute Galois group of the base ring R or with twisted modules for vertex algebras.

[2] Throughout this paper, all ideals are assumed to be two-sided unless there is an explicit mention to the contrary.

are thus pullbacks of tensor products of $\mathfrak{g}$-modules along ev_M:

$$V \simeq V(\lambda, M) = V_{\lambda_1}(M_1) \otimes_k \cdots \otimes_k V_{\lambda_n}(M_n),$$

for some nonzero dominant integral highest weights $\lambda_1, \ldots, \lambda_n$ of $\mathfrak{g}$ (relative to a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$) and maximal ideals $M_1, \ldots, M_n \in \mathrm{Max}(S)$, where $V_{\lambda_i}(M_i)$ is the simple $\mathfrak{g}$-module of highest weight λ_i, viewed as an $\mathscr{L}$-module via the composition of maps

$$\mathscr{L} \xrightarrow{\mathrm{ev}_{M_i}} \mathfrak{g} \otimes_k S/M_i \simeq \mathfrak{g} \to \mathrm{End}(V_{\lambda_i}).$$

Two such representations $V(\lambda, M) = V_{\lambda_1}(M_1) \otimes_k \cdots \otimes_k V_{\lambda_m}(M_m)$ and $V(\mu, N) = V_{\mu_1}(N_1) \otimes_k \cdots \otimes_k V_{\mu_n}(N_n)$ are isomorphic $(\mathscr{L}/\ker\phi)$-modules, and thus isomorphic $\mathscr{L}$-modules, if and only if their highest weights are equal, relative to the induced triangular decomposition

$$\mathscr{L}/\ker\phi = \mathrm{ev}_M^{-1}(\mathfrak{n}_-^{\oplus n}) \oplus \mathrm{ev}_M^{-1}(\mathfrak{h}^{\oplus n}) \oplus \mathrm{ev}_M^{-1}(\mathfrak{n}_+^{\oplus n}).$$

The cohomological interpretation of forms leads to an action of the group Γ on $P_+ \times \mathrm{Max}(S)$, for which $V(\lambda, M) \simeq V(\mu, N)$ if and only if $m = n$ and

$$(\lambda_i, M_i) = {}^{\gamma_i}(\mu_i, N_i)$$

for some $\gamma_1, \ldots, \gamma_n \in \Gamma$. This classification (Proposition 3.7) is then described in terms of Γ-invariant functions from the maximal spectrum $\mathrm{Max}(S)$ to the set P_+ of dominant integral weights. This gives a constructive description (Theorem 3.9) of the moduli space of finite-dimensional simple $\mathscr{L}$-modules in terms of finitely supported Γ-invariant functions $\mathrm{Max}(S) \to P_+$.

One of our main motivations in the present paper was to generalize and provide more intuitive proofs of previous work on (twisted) loop and multiloop algebras. See [Lau 2010; Senesi 2010] for a summary of past work on this problem. However, the interpretation of isomorphism classes as spaces of Γ-equivariant maps used in past work does not generalize to our context of twisted forms. Instead, the Γ-equivariant functions had to be reinterpreted as Γ-*invariant* functions $\mathrm{Max}(S) \to P_+$. This turned out to be the correct perspective to include cases where there is no natural action of Γ on the space $P_+^\times$ of nonzero dominant integral weights. More significantly, with new proofs, we have eliminated all dependence on the $\mathbb{Z}^N$-grading of $\mathscr{L}(\mathfrak{g}, \sigma)$, a point that was crucial in the arguments of [Lau 2010]. This lets us apply our work to nongraded contexts, including a classification of modules for the mysterious Margaux algebras explained in Section 4.

Perhaps the most striking feature of the present work is its nearly complete independence from the particular S/R-form under consideration. The maximal ideals of *any* S/R-form $\mathscr{L}$ of $\mathfrak{g}(R)$ are in bijection with $\mathrm{Max}(R)$, and the finite-dimensional simple $\mathscr{L}$-modules are evaluation modules enumerated by finitely supported Γ-invariant maps $\mathrm{Max}(S) \to P_+$. Indeed, the only place where the Galois cocycle (and hence the isomorphism class) of the S/R-form plays an explicit role is in the isomorphism criterion for $\mathscr{L}$-modules (Proposition 3.7). But in many interesting examples, even this condition vanishes, as we illustrate in Section 4.

Notation. Throughout this paper, k will denote a field. We let $k^\times = k \setminus \{0\}$ and denote the set of nonnegative integers by $\mathbb{Z}_+$. The category of finitely generated unital commutative associative k-algebras will be denoted by k-alg, and we will write $\mathrm{Max}(S)$ for the maximal spectrum of each $S \in k$-alg.

2. Twisted forms and their maximal ideals

In this section, k will denote an arbitrary field and S/R will be a finite Galois extension in k-alg with Galois group Γ. Let $\mathfrak{g}$ be a finite-dimensional central simple algebra over k, and let $R \in k$-alg. We may view $\mathfrak{g}(R) \cong \mathfrak{g} \otimes_k R$ as an algebra over R by base change, the multiplication given by $(x \otimes r)(y \otimes s) = xy \otimes rs$ (for each $x, y \in \mathfrak{g}$ and $r, s \in R$). As before, $\mathscr{L}$ will denote an S/R-form of $\mathfrak{g}(R)$. Any such $\mathscr{L}$ is obviously an algebra over k by restriction of scalars, and we may thus speak of *k-ideals* and *R-ideals* of $\mathscr{L}$, namely the ideals of $\mathscr{L}$ viewed as an algebra over k and over R, respectively.[3] The goal of this section is to classify the maximal k-ideals of $\mathscr{L}$.

Since Galois extensions are faithfully flat, we have the following general facts. See [Matsumura 1989, Theorem 7.5], for instance.

Lemma 2.1. *Let I be an ideal of R, and let M be an R-module.*

(1) *The canonical map*

$$M \to M \otimes_R S, \quad x \mapsto x \otimes 1$$

is injective. In particular, R can be identified with a k-subalgebra of S.

(2) *After viewing R inside of S via* (1), *IS is an ideal of S and $R \cap IS = I$.*

Up to coboundary, we can associate a Galois 1-cocycle

$$u = (u_\gamma)_{\gamma\in\Gamma} \in Z^1(\Gamma, \mathrm{Aut}_{S\text{-alg}}(\mathfrak{g}(S)))$$

to $\mathscr{L}$, such that $\mathscr{L} \simeq \mathscr{L}_u = \{z \in \mathfrak{g} \otimes_k S \mid u_\gamma{}^\gamma z = z \text{ for all } \gamma \in \Gamma\}$. We therefore can (and henceforth will) view $\mathscr{L}$ as an R-subalgebra of $\mathfrak{g}(S) = \mathfrak{g} \otimes_k S$. Note that the S-algebra isomorphism

$$\mathscr{L} \otimes_R S \simeq \mathfrak{g}(R) \otimes_R S = \mathfrak{g}(S)$$

[3] We remind the reader that the word *ideal* means two-sided ideal.

may be realized as the multiplication map

$$\mu : \mathcal{L} \otimes_R S \to \mathfrak{g}(S), \quad \Big(\sum_i x_i \otimes s_i\Big) \otimes s \mapsto \sum_i x_i \otimes s_i s \tag{2.2}$$

for all $\sum_i x_i \otimes s_i \in \mathcal{L}$ and $s \in S$. This will allow us to associate an ideal of S to every R-ideal of $\mathcal{L}$.

Lemma 2.3. *Let $\mathcal{I}$ be an R-ideal of $\mathcal{L}$. Then $\mathcal{I} \otimes_R S$ is an S-ideal of $\mathcal{L} \otimes_R S$, and there is a unique ideal $J = J(\mathcal{I}) \subseteq S$ such that $\mathfrak{g} \otimes_k J = \mu(\mathcal{I} \otimes_R S)$.*

Proof. Fix a k-basis $\{x_1, \ldots, x_m\}$ of $\mathfrak{g}$. Let $J = J(\mathcal{I})$ be the set of all $s \in S$ for which there exists $\sum_{i=1}^m x_i \otimes s_i \in \mu(\mathcal{I} \otimes_R S)$ such that $s = s_i$ for some i. By the definition of J, it is clear that $\mu(\mathcal{I} \otimes_R S) \subseteq \mathfrak{g} \otimes_k J$. Moreover, since $\mathfrak{g} \otimes 1 \subseteq \mathfrak{g} \otimes_k S$ is a finite-dimensional central simple k-algebra, it follows from the Jacobson density theorem that $x_i \otimes s \in \mu(\mathcal{I} \otimes_R S)$ for all $s \in J$ and for all $i \leq m$. Thus $\mathfrak{g} \otimes_k J \subseteq \mu(\mathcal{I} \otimes_R S)$. The uniqueness of J is clear since the tensor product $\mathfrak{g} \otimes_k J$ is being taken over a field k. □

Proposition 2.4. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be R-ideals of $\mathcal{L}$. Then $J(\mathcal{I}_1) \subseteq J(\mathcal{I}_2)$ if and only if $\mathcal{I}_1 \subseteq \mathcal{I}_2$. In particular, the map $J : \{R\text{-ideals of } \mathcal{L}\} \to \{\text{ideals of } S\}$ is injective.*

Proof. Let $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2$. The restriction of the multiplication map

$$\mu : \mathcal{L} \otimes_R S \to \mathfrak{g}(S)$$

to $\mathcal{I} \otimes_R S$ gives an isomorphism $\mu_{\mathcal{I}} : \mathcal{I} \otimes_R S \to \mathfrak{g} \otimes_k J(\mathcal{I})$ with $J(\mathcal{I}) = J(\mathcal{I}_1) + J(\mathcal{I}_2)$. By flatness of S/R,

$$(\mathcal{I}/\mathcal{I}_2) \otimes_R S \simeq \frac{\mathcal{I} \otimes_R S}{\mathcal{I}_2 \otimes_R S}$$

as S-modules. The injection $\mu_{\mathcal{I}}$ restricts to an isomorphism

$$\mathcal{I}_2 \otimes_R S \to \mathfrak{g} \otimes_k J(\mathcal{I}_2),$$

so we see that

$$\frac{\mathcal{I} \otimes_R S}{\mathcal{I}_2 \otimes_R S} \simeq \frac{\mathfrak{g} \otimes_k J(\mathcal{I})}{\mathfrak{g} \otimes_k J(\mathcal{I}_2)} = \mathfrak{g} \otimes_k (J(\mathcal{I})/J(\mathcal{I}_2)).$$

Thus $(\mathcal{I}/\mathcal{I}_2) \otimes_R S = 0$ if and only if $\mathfrak{g} \otimes_k (J(\mathcal{I})/J(\mathcal{I}_2)) = 0$; then by faithful flatness, $\mathcal{I}/\mathcal{I}_2 = 0$ if and only if $J(\mathcal{I})/J(\mathcal{I}_2) = 0$. That is, $\mathcal{I}_1 \subseteq \mathcal{I}_2$ if and only if $J(\mathcal{I}_1) \subseteq J(\mathcal{I}_2)$. □

Proposition 2.5. *Let $\mathcal{I} \subseteq \mathcal{L}$ be an R-ideal. Then $J(\mathcal{I})$ is stable under the action of the Galois group $\Gamma = \mathcal{G}al(S/R)$.*

Proof. As in the proof of Lemma 2.3, we fix a k-basis $\beta = \{x_1, \dots, x_m\}$ of $\mathfrak{g}$. From the definition of $J = J(\mathcal{I})$, it is easy to see that J is the ideal of S generated by the set $E_\beta(\mathcal{I})$ of those elements $s \in S$ for which there is an element $\sum_i x_i \otimes s_i \in \mathcal{I}$ for which $s_i = s$ for some i. It is thus enough to show ${}^\gamma s \in J$ for all $\gamma \in \Gamma$ and $s \in E_\beta(\mathcal{I})$.

Let $u \in Z^1(\Gamma, \mathrm{Aut}_{S\text{-alg}}(\mathfrak{g}(S)))$ be a cocycle corresponding to the S/R-form $\mathcal{L}$. Fix $\gamma \in \Gamma$, and write $u_\gamma(x_i \otimes 1) = \sum_{j=1}^m x_j \otimes a_{ij}$. Since u_γ is an automorphism of $\mathfrak{g}(S)$, the matrix $A = (a_{ij})$ is invertible in $M_m(S)$. Let $z = \sum x_i \otimes s_i \in \mathcal{I}$. It suffices to show that ${}^\gamma s_i \in J$ for $i = 1, \dots, m$. We have

$$\begin{aligned}\sum x_i \otimes s_i &= \mu(z \otimes 1) = \mu(u_\gamma {}^\gamma z \otimes 1) = \mu\Big(\sum_i u_\gamma(x_i \otimes {}^\gamma s_i) \otimes 1\Big)\\ &= \mu\Big(\sum_i {}^\gamma s_i u_\gamma(x_i \otimes 1) \otimes 1\Big) = \mu\Big(\sum_i u_\gamma(x_i \otimes 1) \otimes {}^\gamma s_i\Big)\\ &= \mu\Big(\sum_{i,j} x_j \otimes a_{ij} \otimes {}^\gamma s_i\Big) = \sum_j x_j \otimes \Big(\sum_i a_{ij} {}^\gamma s_i\Big).\end{aligned}$$

In matrix form, we see that

$$\begin{pmatrix} {}^\gamma s_1 \\ \vdots \\ {}^\gamma s_m \end{pmatrix} = (A^t)^{-1} \begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix}.$$

By definition, $s_i \in E_\beta(\mathcal{I}) \subseteq J$ for all i, and $(A^t)^{-1} \in M_m(S)$. Hence ${}^\gamma s_i \in J$ for all i. $\square$

Lemma 2.6. *Let I be an ideal of R. Then $I\mathcal{L}$ is an ideal of $\mathcal{L}$, and $J(I\mathcal{L}) = IS$.*

Proof. It is obvious that $I\mathcal{L}$ is an ideal of $\mathcal{L}$. As S-modules (in fact, as S-algebras),

$$I\mathcal{L} \otimes_R S = \mathcal{L} \otimes_R IS \simeq \mathcal{L} \otimes_R S \otimes_S IS \simeq \mathfrak{g} \otimes_k S \otimes_S IS \simeq \mathfrak{g} \otimes_k IS,$$

so $J(I\mathcal{L}) = IS$. $\square$

We now turn to the classification of maximal k-ideals $\mathcal{I}$ of the S/R-form $\mathcal{L}$.

Lemma 2.7. *The sets of maximal k-ideals and maximal R-ideals of $\mathcal{L}$ coincide.*

Proof. Let $\mathcal{I}$ be a maximal k-ideal of $\mathcal{L}$. We claim that $\mathcal{I}$ is stable under the action of R. For any $r \in R$, the space $r\mathcal{I}$ is clearly a k-ideal of $\mathcal{L}$, and if $r\mathcal{I} \not\subseteq \mathcal{I}$, then $\mathcal{I} + r\mathcal{I} = \mathcal{L}$ by the maximality of $\mathcal{I}$. The algebra $\mathcal{L}$ is perfect by descent considerations, as has already been noted in [Gille and Pianzola 2007], for instance. Thus

$$\mathcal{L} = \mathcal{L}\mathcal{L} = (\mathcal{I} + r\mathcal{I})\mathcal{L} = \mathcal{I}\mathcal{L} + \mathcal{I}(r\mathcal{L}) \subseteq \mathcal{I}\mathcal{L} \subseteq \mathcal{I},$$

since $\mathscr{L}$ is an R-algebra. But this contradicts the proper inclusion $\mathscr{I} \subsetneq \mathscr{L}$, so $r\mathscr{I} \subseteq \mathscr{I}$ as claimed. From this, it follows that every maximal k-ideal of $\mathscr{L}$ is also a maximal R-ideal of $\mathscr{L}$ and conversely. □

Lemma 2.8. *Let M be a maximal ideal of R.*

(1) *There exist prime ideals of S lying over M, and any such ideal is maximal. The group Γ acts transitively on the set of such maximal ideals. In particular, this set is finite.*

(2) *$MS = \bigcap_i M_i$, where the intersection is taken over the (finite) set of maximal ideals of S lying over M.*

Proof. (1) This is well-known, but we recall the main ideas for completeness. From basic properties of Galois extensions, we know that $R = S^{\Gamma}$, and hence S/R is integral. From this it follows that the set of prime ideals of S lying over M is not empty, that any such ideal is maximal, and that the action of Γ on this set is transitive. (See [Bourbaki 1964, §2.1 Proposition 1 and §2.2 Théorème 2].)

(2) Any maximal ideal $\mathfrak{m}$ of S containing MS will lie over M, since the intersection $\mathfrak{m} \cap R$ is a proper ideal of R containing $MS \cap R$, which is equal to the maximal ideal M by Lemma 2.1(2). Thus $\mathfrak{m} = M_i$ for some i, and $\bigcap_i M_i$ is the radical of MS. Since S/R is flat,

$$S/MS \simeq (R \otimes_R S)/(M \otimes_R S) \simeq (R/M) \otimes_R S.$$

Let $L = R/M$, a field extension of k. Since the extension S is Galois over R, general facts about base change guarantee that the extension $(R/M) \otimes_R S$ is Galois over $(R/M) \otimes_R R \simeq L$. (See [Milne 1980, §I.5], for instance.) That is, S/MS is a Galois extension of L. Galois extensions are finite étale and the only such extensions of L are products $L_1 \times \cdots \times L_m$, where the L_i are finite separable field extensions of L. We see from this that S/MS has trivial Jacobson radical. Hence MS is a radical ideal of S, and $MS = \bigcap_i M_i$. □

Theorem 2.9. *The map $\psi : I \mapsto I\mathscr{L}$ defines a bijection between the set of maximal ideals of R and the set of maximal ideals of $\mathscr{L}$.*

Proof. Let $\mathscr{I}$ be a maximal ideal of $\mathscr{L}$, and let $J = J(\mathscr{I}) \subseteq S$ be the ideal corresponding to $\mathscr{I}$. Let $P \subseteq S$ be a maximal ideal containing J, and let $M = P \cap R$. Since S/R is integral, M is a maximal ideal of R [Bourbaki 1964, §2.1 Proposition 1].

As explained in Lemma 2.8(1), the Galois group Γ acts transitively on the finite set $M_1, \dots, M_N$ of maximal ideals S lying over M. Since J is Γ-stable (Proposition 2.5) and contained in a maximal ideal P lying over M, we see that $J \subseteq \bigcap_{i=1}^N M_i$. By Lemma 2.8(2), $MS = \bigcap_{i=1}^N M_i$. Hence $J \subseteq MS$.

Note that $M\mathscr{L}$ is an ideal of $\mathscr{L}$ whose corresponding ideal is MS, by Lemma 2.6. By Proposition 2.4, $\mathscr{I} \subseteq M\mathscr{L}$. Since $MS = \bigcap_{i=1}^N M_i$ is a proper ideal of S, Lemma 2.3 guarantees that $M\mathscr{L}$ is a proper ideal of $\mathscr{L}$. Hence $\mathscr{I} = M\mathscr{L}$ by the maximality of $\mathscr{I}$, so the image of the map ψ includes all maximal ideals of $\mathscr{L}$.

Let I_1 and I_2 be maximal ideals of R. If $I_1\mathscr{L} = I_2\mathscr{L}$, then $I_1 S = I_2 S$ by Proposition 2.4 and Lemma 2.6. Now Lemma 2.1(2) yields that $I_1 = I_2$, hence that ψ is injective. It remains only to check that $I\mathscr{L} \subseteq \mathscr{L}$ is maximal whenever $I \subseteq R$ is maximal. Suppose that $I \subseteq R$ is a maximal ideal, and let $\mathscr{I} \subseteq \mathscr{L}$ be a maximal ideal containing $I\mathscr{L}$. We have already shown that there is a maximal ideal $M \subseteq R$ for which $\mathscr{I} = M\mathscr{L}$. By Lemma 2.1(2) and Lemma 2.6,

$$M = MS \cap R = J(M\mathscr{L}) \cap R = J(\mathscr{I}) \cap R.$$

By Proposition 2.4, $J(I\mathscr{L}) \subseteq J(\mathscr{I})$, so

$$I = IS \cap R = J(I\mathscr{L}) \cap R \subseteq J(\mathscr{I}) \cap R = M.$$

By the maximality of I, we see that $I = M$. Hence $I\mathscr{L} = M\mathscr{L} = \mathscr{I}$ is a maximal ideal of $\mathscr{L}$. □

As an application, we recover the following well-known fact; see [Knus and Ojanguren 1974, Corollary III.5.2].

Corollary 2.10. *Let $\mathscr{A}$ be an Azumaya algebra over R. Every* (*two-sided*) *maximal ideal of $\mathscr{A}$ is of the form $I\mathscr{A}$ for some maximal ideal I of R.*

3. Classification of simple modules

We maintain the notation of the previous section but now assume that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over an algebraically closed field k of characteristic zero. The base ring R will be of finite type in k-alg, and all modules (representations) will be of finite dimension over k. Unless explicitly indicated otherwise, $\otimes$ will denote a tensor product $\otimes_k$ taken over the base field k.

Let $\mathscr{L} \subset \mathfrak{g} \otimes S$ be an S/R-form of $\mathfrak{g}(R)$ as before, and let $\phi : \mathscr{L} \to \mathrm{End}_k(V)$ be a finite-dimensional irreducible representation of $\mathscr{L}$. We fix a cocycle $u \in Z^1(\Gamma, \mathrm{Aut}_{S\text{-Lie}}(\mathfrak{g}(S)))$ so that $\mathscr{L} = \mathscr{L}_u$.

3a. ***Evaluation maps and simple modules.*** Since $\mathscr{L}$ is perfect, $\mathscr{L}/\ker\phi$ is a finite-dimensional semisimple Lie algebra over k [Lau 2010, Proposition 2.1]. Hence there is an isomorphism

$$f : \mathscr{L}/\ker\phi \to \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$$

for some collection of finite-dimensional simple k-Lie algebras $\mathfrak{g}_1, \dots, \mathfrak{g}_n$. Let $\pi : \mathscr{L} \to \mathscr{L}/\ker\phi$ be the natural projection. Then

$$\mathscr{L}/\ker\phi \simeq \mathscr{L}/\mathscr{M}_1 \oplus \cdots \oplus \mathscr{L}/\mathscr{M}_n,$$

where $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are pairwise distinct maximal ideals of $\mathcal{L}$ whose intersection is $\ker \phi$. More precisely, we can take

$$\mathcal{M}_i = \pi^{-1} \circ f^{-1}(\mathfrak{g}_1 \oplus \cdots \oplus \widehat{\mathfrak{g}_i} \oplus \cdots \oplus \mathfrak{g}_n)$$

for $i = 1, \ldots, n$, where $\widehat{\mathfrak{g}_i}$ indicates that the i-th summand is omitted. To classify the simple modules of $\mathcal{L}$, it thus suffices to consider quotients of $\mathcal{L}$ by maximal ideals.[4]

Let $\mathcal{I} \subseteq \mathcal{L}$ be a maximal ideal. By Theorem 2.9, $\mathcal{I} = I\mathcal{L}$ for some maximal ideal $I \subseteq R$. Let $P \subseteq S$ be a maximal ideal lying over I, and let

$$\epsilon : S \to S/P \simeq k \tag{3.1}$$

be the natural evaluation map.[5] Then the composition

$$\mathrm{ev}_P : \mathcal{L} \hookrightarrow \mathfrak{g} \otimes S \xrightarrow{1 \otimes \epsilon} \mathfrak{g} \otimes k \simeq \mathfrak{g} \tag{3.2}$$

is a homomorphism of k-Lie algebras.

Proposition 3.3. *The map* $\mathrm{ev}_P : \mathcal{L} \to \mathfrak{g}$ *is surjective and has kernel* $\mathcal{I} = (P \cap R)\mathcal{L}$.

Proof. The multiplication map $\mu : \mathcal{L} \otimes_R S \to \mathfrak{g}(S)$ is an isomorphism (2.2), so given any element $x \in \mathfrak{g}$, there exist elements $z_i \in \mathcal{L}$ and $t_i \in S$ such that

$$\mu\Big(\sum_i z_i \otimes t_i\Big) = x \otimes 1.$$

That is, if $z_i = \sum_j x_j \otimes s_{ij}$ for some k-basis $\{x_j\}$ of $\mathfrak{g}$ and $s_{ij} \in S$, then

$$\sum_{i,j} x_j \otimes s_{ij} t_i = x \otimes 1.$$

Applying the map $1 \otimes \epsilon$ introduced in (3.1), we get $\sum_{i,j} x_j \otimes \epsilon(s_{ij})\epsilon(t_i) = x \otimes 1$. But $\mathcal{L}$ is closed under multiplication by elements of k, so $\sum_i \epsilon(t_i) z_i \in \mathcal{L}$, and

$$\mathrm{ev}_P\Big(\sum_i \epsilon(t_i) z_i\Big) = \sum_{i,j} x_j \epsilon(s_{ij})\epsilon(t_i) = x.$$

Hence ev_P is surjective.

Let $z = \sum_i x_i \otimes s_i \in \mathcal{L}$ and $r \in I$. Then $\epsilon(r) = 0$, since $I = P \cap R \subseteq P = \ker \epsilon$. Hence

$$\mathrm{ev}_P(rz) = \sum x_i \epsilon(r s_i) = \sum x_i \epsilon(r)\epsilon(s_i) = 0,$$

[4] Recall that there is no difference in the concept of maximal ideal if we view $\mathcal{L}$ as an R- or k-Lie algebra.

[5] S is of finite type over R and R is assumed to be of finite type over k. Thus S is of finite type over k and therefore $S/P \simeq k$ by the Nullstellensatz.

so $I\mathcal{L} \subseteq \ker \mathrm{ev}_P$. Since $\mathcal{I} = I\mathcal{L}$ is a maximal ideal and ev_P is nonzero, the kernel of ev_P is precisely $\mathcal{I}$. □

We have now shown that $\mathcal{L}/\ker\phi$ is isomorphic to a direct sum of finitely many copies of $\mathfrak{g}$. Explicitly, $\ker\phi$ is the intersection of a (finite) family of distinct maximal ideals $\mathcal{M}_1, \dots, \mathcal{M}_n$ in $\mathcal{L}$. Let $I_1, \dots, I_n$ be the (distinct) maximal ideals of R given by Theorem 2.9. For any collection $\underline{M}$ of maximal ideals $M_1, \dots, M_n$ of S lying over $I_1, \dots, I_n$, respectively, the map

$$\mathrm{ev}_{\underline{M}} = (\mathrm{ev}_{M_1}, \dots, \mathrm{ev}_{M_n}) : \mathcal{L} \to \mathfrak{g} \oplus \cdots \oplus \mathfrak{g},$$
$$z \mapsto (\mathrm{ev}_{M_1}(z), \dots, \mathrm{ev}_{M_n}(z))$$

descends to an isomorphism $\mathrm{ev}_{\underline{M}} : \mathcal{L}/\ker\phi \to \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$.

Since the irreducible representations of $\mathfrak{g}^{\oplus n} = \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ are precisely the tensor products

$$\rho = (\rho_1, \dots, \rho_n) : \mathfrak{g} \oplus \cdots \oplus \mathfrak{g} \to \mathrm{End}_k(V_1 \otimes \cdots \otimes V_n),$$
$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} \mathrm{id} \otimes \cdots \otimes \rho_i(x_i) \otimes \cdots \otimes \mathrm{id}$$

of simple $\mathfrak{g}$-modules (ρ_i, V_i), we now have a complete list of the simple $\mathcal{L}$-modules.

Theorem 3.4. *Let $\phi : \mathcal{L} \to End_k(V)$ be a finite-dimensional irreducible representation of $\mathcal{L}$. Then there exists a finite collection $\underline{P} = (P_1, \dots, P_n)$ of maximal ideals of S with $P_i \cap R \neq P_j \cap R$ for $i \neq j$, and a simple $\mathfrak{g}^{\oplus n}$-module $(\rho, V_1 \otimes \cdots \otimes V_n)$ such that $V \simeq V_1 \otimes \cdots \otimes V_n$ and $\phi = \rho \circ \mathrm{ev}_{\underline{P}}$.*

Remark 3.5. The converse of Theorem 3.4 is obvious. Given a collection of maximal ideals $P_1, \dots, P_n$ of S for which the ideals $P_i \cap R$ of R are pairwise distinct, the Chinese remainder theorem gives an isomorphism

$$\mathcal{L}/\mathcal{M}_1 \oplus \cdots \oplus \mathcal{L}/\mathcal{M}_n \simeq \mathcal{L}/\cap_i \mathcal{M}_i,$$

where $\mathcal{M}_i = (P_i \cap R)\mathcal{L}$. (This uses the fact that the $P_i \cap R$ are maximal, as shown in the proof of Theorem 2.9.) Thus the map

$$\mathcal{L} \to \mathcal{L}/\mathcal{M}_1 \oplus \cdots \oplus \mathcal{L}/\mathcal{M}_n \simeq \mathfrak{g}^{\oplus n}$$

is surjective, so the pullback of any simple $\mathfrak{g}^{\oplus n}$-module $V = V_1 \otimes \cdots \otimes V_n$ will be a simple $\mathcal{L}$-module.

3b. ***Isomorphism classes of simple modules.*** Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and an épinglage of $(\mathfrak{g}, \mathfrak{h})$; see [Bourbaki 1975, VIII, §4.1]. Given a maximal ideal $M \in \mathrm{Max}(S)$ and a finite-dimensional representation $\rho : \mathfrak{g} \to \mathrm{End}_k(W)$, we write

$W(M)$ for the vector space W, viewed as an $\mathcal{L}$-module with action given by the composition of maps

$$\mathcal{L} \hookrightarrow \mathfrak{g} \otimes S \xrightarrow{\mathrm{ev}_M} \mathfrak{g} \xrightarrow{\rho} \mathrm{End}_k(W),$$

where ev_M is the quotient map

$$\begin{aligned} \mathrm{ev}_M : \mathfrak{g} \otimes S &\to (\mathfrak{g} \otimes S)/(\mathfrak{g} \otimes M) = \mathfrak{g} \otimes (S/M) \simeq \mathfrak{g}, \\ x \otimes s &\mapsto (x \otimes s)(M) = s(M)x \end{aligned}$$

for all $x \in \mathfrak{g}$ and $s \in S$. For each automorphism $\alpha \in \mathrm{Aut}_{S\text{-Lie}}(\mathfrak{g}(S))$ and $M \in \mathrm{Max}(S)$, we write $\alpha(M) \in \mathrm{Aut}(\mathfrak{g})$ for the automorphism defined by

$$(\alpha(M))(x) = (\alpha(x \otimes 1))(M) = \mathrm{ev}_M(\alpha(x \otimes 1))$$

for each $x \in \mathfrak{g}$. It is straightforward to verify that the map

$$\mathrm{Aut}_{S\text{-Lie}}(\mathfrak{g}(S)) \to \mathrm{Aut}(\mathfrak{g}), \quad \alpha \mapsto \alpha(M)$$

is a group homomorphism for each $M \in \mathrm{Max}(S)$. We write $\mathrm{Out}\,\alpha(M)$ and $\mathrm{Int}\,\alpha(M)$ for the outer and inner parts, respectively, of the automorphism

$$\alpha(M) = \mathrm{Int}\,\alpha(M) \circ \mathrm{Out}\,\alpha(M).$$

See [Bourbaki 1975, VIII, §5.3 corollaire 1] for details.

By Theorem 3.4, the (finite-dimensional) simple $\mathcal{L}$-modules are those of the form $V(\lambda, M) = V_{\lambda_1}(M_1) \otimes \cdots \otimes V_{\lambda_n}(M_n)$, where each λ_i is in the set $P_+^\times$ of nonzero dominant integral weights, V_{λ_i} is the simple $\mathfrak{g}$-module of highest weight λ_i, and $M = (M_1, \ldots, M_n)$ is an n-tuple of maximal ideals of S lying over distinct (closed) points of $\mathrm{Spec}(R)$.

Lemma 3.6. *Suppose that the $\mathcal{L}$-modules $V(\lambda, M) = V_{\lambda_1}(M_1) \otimes \cdots \otimes V_{\lambda_m}(M_m)$ and $V(\mu, N) = V_{\mu_1}(N_1) \otimes \cdots \otimes V_{\mu_n}(N_n)$ are isomorphic for certain $\lambda_1, \ldots, \lambda_m$, $\mu_1, \ldots, \mu_n \in P_+^\times$ and $M_1, \ldots, M_m, N_1, \ldots, N_n \in \mathrm{Max}(S)$. Then $m = n$, and up to reordering, $M_i \cap R = N_i \cap R$ for all i.*

Proof. Let $\phi_{\lambda,M} : \mathcal{L} \to \mathrm{End}_k(V(\lambda, M))$ and $\phi_{\mu,N} : \mathcal{L} \to \mathrm{End}_k(V(\mu, N))$ be the homomorphisms determining the module actions. Since $V(\lambda, M) \simeq V(\mu, N)$, their kernels are equal, so

$$\bigcap_{i=1}^{m} (M_i \cap R)\mathcal{L} = \ker \phi_{\lambda,M} = \ker \phi_{\mu,N} = \bigcap_{j=1}^{n} (N_j \cap R)\mathcal{L}.$$

By Lemma 2.1(2) and Lemma 2.6,

$$\bigcap_{i=1}^{m}(M_i \cap R) = \left(\bigcap_{i=1}^{m}(M_i \cap R)S\right) \cap R = J\left(\bigcap_{i=1}^{m}(M_i \cap R)\mathscr{L}\right) \cap R$$
$$= J\left(\bigcap_{j=1}^{n}(N_j \cap R)\mathscr{L}\right) \cap R = \bigcap_{j=1}^{n}(N_j \cap R).$$

For $I \subseteq R$, let $\operatorname{Var} I$ be the set of $\mathfrak{m} \in \operatorname{Spec} R$ with $I \subseteq \mathfrak{m}$. Then

$$\bigcup_{i=1}^{m}\{M_i \cap R\} = \bigcup_{i=1}^{m}\operatorname{Var}(M_i \cap R) = \operatorname{Var}\left(\bigcap_{i=1}^{m}(M_i \cap R)\right)$$
$$= \operatorname{Var}\left(\bigcap_{j=1}^{n}(N_j \cap R)\right) = \bigcup_{j=1}^{n}\{N_j \cap R\}.$$

Thus $m = n$, and after reordering, $M_i \cap R = N_i \cap R$ for all i. □

Recall that u_γ is the image of $\gamma \in \Gamma = \mathscr{G}al(S/R)$ under the Galois cocycle $u : \Gamma \to \operatorname{Aut}_{S\text{-Lie}}(\mathfrak{g}(S))$. The group Γ acts on the set of pairs $(\lambda, M) \in P_+^\times \times \operatorname{Max}(S)$ by ${}^\gamma(\mu, N) = (\mu \circ \operatorname{Out} u_\gamma{}^{-1}({}^\gamma N), {}^\gamma N)$.

Proposition 3.7. *Suppose*

$$V(\lambda, M) = V_{\lambda_1}(M_1) \otimes \cdots \otimes V_{\lambda_n}(M_n) \quad \textit{and} \quad V(\mu, N) = V_{\mu_1}(N_1) \otimes \cdots \otimes V_{\mu_n}(N_n)$$

are irreducible $\mathscr{L}$-modules with $\lambda, \mu \in (P_+^\times)^n$ and $M_i \cap R = N_i \cap R$ for all i. Then $V(\lambda, M) \simeq V(\mu, N)$ if and only if there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that

$$(\lambda_i, M_i) = {}^{\gamma_i}(\mu_i, N_i) \quad \textit{for } i = 1, \ldots, n.$$

Proof. Let $\phi_{\lambda,M} : \mathscr{L} \to \operatorname{End}_k(V(\lambda, M))$ and $\phi_{\mu,N} : \mathscr{L} \to \operatorname{End}_k(V(\mu, N))$ be the homomorphisms defining the module actions. Since each λ_i is nonzero, the kernel of the action of $\mathfrak{g}^{\oplus n}$ on $V(\lambda, M)$ is trivial, and the evaluation maps ev_{M_i} induce an automorphism

$$\operatorname{ev}_M = \operatorname{ev}_{M_1} \oplus \cdots \oplus \operatorname{ev}_{M_n} : \mathscr{L}/\ker \phi_{\lambda,M} \xrightarrow{\sim} \mathfrak{g}^{\oplus n}.$$

Similarly, $\operatorname{ev}_N : \mathscr{L}/\ker \phi_{\mu,N} \to \mathfrak{g}^{\oplus n}$ is a Lie algebra isomorphism.

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$ relative to the épinglage of $(\mathfrak{g}, \mathfrak{h})$. We pull back the corresponding triangular decomposition of $\mathfrak{g}^{\oplus n}$ to obtain the triangular decomposition

$$\mathscr{L}/\ker \phi_{\lambda,M} = \operatorname{ev}_M^{-1}(\mathfrak{n}_-^{\oplus n}) \oplus \operatorname{ev}_M^{-1}(\mathfrak{h}^{\oplus n}) \oplus \operatorname{ev}_M^{-1}(\mathfrak{n}_+^{\oplus n}). \tag{3.8}$$

The representations $V(\lambda, M)$ and $V(\mu, N)$ will be isomorphic precisely when they have the same highest weights relative to the decomposition (3.8).

The Galois group $\Gamma = \mathcal{G}al(S/R)$ acts transitively on the fibers of the pullback map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ over maximal ideals of R. Choose $\gamma_i \in \Gamma$ so that $M_i = {}^{\gamma_i}N_i$ for all i.

Let $\mathfrak{g}^i = 0 \oplus \cdots \oplus \mathfrak{g} \oplus \cdots \oplus 0$ be the i-th component of $\mathfrak{g}^{\oplus n}$. Note that

$$\operatorname{ev}_M^{-1}(\mathfrak{g}^i) = \bigcap_{r \neq i} \ker \operatorname{ev}_{M_r} = \bigcap_{r \neq i} (M_r \cap R)\mathcal{L} = \bigcap_{r \neq i} (N_r \cap R)\mathcal{L} = \bigcap_{r \neq i} \ker \operatorname{ev}_{N_r}.$$

Therefore, $\operatorname{ev}_{N_j} \circ \operatorname{ev}_M^{-1}(\mathfrak{g}^i) = 0$ for all $i \neq j$, and

$$\operatorname{ev}_N \circ \operatorname{ev}_M^{-1}(x^i) = \iota_i \circ \operatorname{ev}_{N_i} \circ \operatorname{ev}_M^{-1}(x^i) = \iota_i \circ \operatorname{ev}_{N_i} \circ \operatorname{ev}_{M_i}^{-1}(x)$$

for all $x^i \in \mathfrak{g}^i$, where ι_i is the inclusion of $\mathfrak{g}$ as the i-th component of $\mathfrak{g}^{\oplus n}$:

$$\iota_i : \mathfrak{g} \hookrightarrow 0 \oplus \cdots \oplus \mathfrak{g} \oplus \cdots \oplus 0 \subseteq \mathfrak{g}^{\oplus n}.$$

Relative to the decomposition (3.8), the highest weight of $V(\lambda, M)$ is thus $\sum_{i=1}^n \lambda_i \circ \operatorname{ev}_{M_i}$ and the highest weight of $V(\mu, N)$ is $\sum_{i=1}^n \nu_i \circ \operatorname{ev}_{N_i}$, where $\nu_i \in (\operatorname{ev}_{N_i} \circ \operatorname{ev}_{M_i}^{-1}(\mathfrak{h}))^*$ is the highest weight of V_{μ_i}, relative to the new triangular decomposition

$$\mathfrak{g} = \operatorname{ev}_{N_i} \circ \operatorname{ev}_{M_i}^{-1}(\mathfrak{n}_-) \oplus \operatorname{ev}_{N_i} \circ \operatorname{ev}_{M_i}^{-1}(\mathfrak{h}) \oplus \operatorname{ev}_{N_i} \circ \operatorname{ev}_{M_i}^{-1}(\mathfrak{n}_+).$$

By [Lau 2010, Lemma 5.2], $\nu_i = \mu_i \circ \tau_i^{-1}$, where $\tau_i = \operatorname{Int}(\operatorname{ev}_{N_i} \circ \operatorname{ev}_{M_i}^{-1})$. That is, $V(\lambda, M) \simeq V(\mu, N)$ if and only if

$$\sum_{i=1}^n \lambda_i \circ \operatorname{ev}_{M_i} = \sum_{i=1}^n \mu_i \circ \tau_i^{-1} \circ \operatorname{ev}_{N_i}$$

on $\operatorname{ev}_M^{-1}(\mathfrak{h}^{\oplus n})$. For the i-th component $\mathfrak{h}^i = 0 \oplus \cdots \oplus \mathfrak{h} \oplus \cdots \oplus 0$, we have

$$\operatorname{ev}_M^{-1}(\mathfrak{h}^i) \subseteq \operatorname{ev}_M^{-1}(\mathfrak{g}^i) = \bigcap_{j \neq i} (M_j \cap R)\mathcal{L},$$

so $\lambda_j \circ \operatorname{ev}_{M_j}(\operatorname{ev}_M^{-1}(\mathfrak{h}^i)) = 0$ for $i \neq j$. Therefore, $V(\lambda, M) \simeq V(\mu, N)$ if and only if $\lambda_i \circ \operatorname{ev}_{M_i} = \mu_i \circ \tau_i^{-1} \circ \operatorname{ev}_{N_i}$ for all i; that is, if and only if $\lambda_i = \mu_i \circ \operatorname{Out}(\operatorname{ev}_{N_i} \circ \operatorname{ev}_{M_i}^{-1})$.

We now simplify the expression for the automorphism $\operatorname{ev}_{N_i} \circ \operatorname{ev}_{M_i}^{-1} : \mathfrak{g} \to \mathfrak{g}$. For $x \in \mathfrak{g}$, write $\operatorname{ev}_{M_i}^{-1}(x) = \sum_j x_j \otimes s_j + \ker \operatorname{ev}_{M_i} \in \mathcal{L}/\ker \operatorname{ev}_{M_i} = \mathcal{L}/\ker \operatorname{ev}_{N_i}$, where $x_j \in \mathfrak{g}$ and $s_j \in S$ for all j. Then $\operatorname{ev}_{N_i} \circ \operatorname{ev}_{M_i}^{-1}(x) = \sum_j s_j(N_i) x_j$. By definition,

$$s_j(N_i) + N_i = s_j + N_i \in S/N_i,$$

and $s_j(N_i) \in k \subseteq R$ is clearly fixed by $\gamma_i \in \Gamma$. Hence

$$s_j(N_i) + {}^{\gamma_i}N_i = {}^{\gamma_i}s_j + {}^{\gamma_i}N_i \in S/{}^{\gamma_i}N_i = S/M_i,$$

and $s_j(N_i) = {}^{\gamma_i}s_j(M_i)$. Therefore,

$$\mathrm{ev}_{N_i} \circ \mathrm{ev}_{M_i}^{-1}(x) = \sum_j {}^{\gamma_i}s_j(M_i)x_j.$$

Moreover, $\sum_j x_j \otimes s_j \in \mathscr{L} = \{z \in \mathfrak{g} \otimes S \mid u_\gamma\, {}^\gamma z = z \text{ for all } \gamma \in \Gamma\}$, so

$$\begin{aligned}\mathrm{ev}_{N_i} \circ \mathrm{ev}_{M_i}^{-1}(x) &= {}^{\gamma_i}\Big(\sum_j x_j \otimes s_j\Big)(M_i) = (u_{\gamma_i})^{-1}\Big(\sum_j x_j \otimes s_j\Big)(M_i)\\ &= u_{\gamma_i}^{-1}(M_i)\sum_j s_j(M_i)x_j = u_{\gamma_i}^{-1}(M_i)(x),\end{aligned}$$

and $\mathrm{ev}_{N_i} \circ \mathrm{ev}_{M_i}^{-1} = u_{\gamma_i}^{-1}(M_i)$. Hence $V(\lambda, M) \simeq V(\mu, N)$ if and only if there exist $\gamma_1, \dots, \gamma_n \in \Gamma$ such that ${}^{\gamma_i}(\mu_i, N_i) = (\lambda_i, M_i)$ for all i. □

We identify the $\mathscr{L}$-module $V(\lambda, M) = V_{\lambda_1}(M_1) \otimes \cdots \otimes V_{\lambda_n}(M_n)$ with the map

$$\chi_{[\lambda,M]} : \mathrm{Max}(S) \to P_+,$$

where $\chi_{[\lambda,M]} = \sum_{\gamma\in\Gamma} \sum_{i=1}^n \chi_{{}^\gamma(\lambda_i,M_i)}$ and

$$\chi_{(\mu_i,N_i)} : \mathrm{Max}(S) \to P_+, \quad I \mapsto \begin{cases} \mu_i & \text{if } I = N_i,\\ 0 & \text{otherwise.}\end{cases}$$

The Galois group Γ acts on the set $\mathscr{F}$ of finitely supported functions $\mathrm{Max}(S) \to P_+$, by identifying each function f with the set of ordered pairs $\{(f(M), M) \mid M \in \mathrm{Max}(S)\}$ and defining ${}^\gamma f = \{{}^\gamma(f(M), M) \mid M \in \mathrm{Max}(S)\}$. The function $\chi_{[\gamma,M]}$ is then Γ-invariant, and the set $\mathscr{F}^\Gamma$ of Γ-invariant functions in $\mathscr{F}$ is in bijection with the set $\mathscr{C}$ of isomorphism classes $[V]$ of (finite-dimensional) simple $\mathscr{L}$-modules V:

Theorem 3.9. *The map $\psi : [V(\lambda, M)] \mapsto \chi_{[\lambda,M]}$ is a well-defined natural bijection between $\mathscr{C}$ and $\mathscr{F}^\Gamma$.*

Proof. By Theorem 3.4, Lemma 3.6, and Proposition 3.7, two simple $\mathscr{L}$-modules W_1 and W_2 are isomorphic if and only if there exist $n \geq 0$, ordered pairs

$$(M, \lambda), (N, \mu) \in (\mathrm{Max}(S))^n \times (P_+^\times)^n$$

with $M_i \cap R = N_i \cap R \neq N_j \cap R = M_j \cap R$ for $i \neq j$, and $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $W_1 \simeq V(\lambda, M)$, $W_2 \simeq V(\mu, N)$, and $(M_i, \lambda_i) = {}^{\gamma_i}(N_i, \mu_i)$ for all i. Thus $V(\lambda, M) \simeq V(\mu, N)$ if and only if $\chi_{[\lambda,M]} = \chi_{[\mu,N]}$. In particular, the map $\psi : \mathscr{C} \to \mathscr{F}^\Gamma$ is well-defined and injective. It is also surjective, as the support of any $f \in \mathscr{F}^\Gamma$ decomposes into a disjoint union of Γ-orbits. Therefore,

$$f = \sum_{\gamma\in\Gamma} \sum_{i=1}^m \chi_{{}^\gamma(\lambda_i,M_i)}$$

for some collection of orbit representatives $M_1, \dots, M_m \in \mathrm{Max}(S)$. $\square$

4. Applications

In this section, k will denote an algebraically closed field of characteristic zero.

4a. ***Multiloop algebras.*** Multiloop algebras are multivariable generalizations of the loop algebras in affine Kac–Moody theory. The study of these algebras and their extensions includes a substantial body of work on (twisted and untwisted) multiloop, toroidal, and extended affine Lie algebras. The representation theory of multiloop algebras has also been adapted to include generalized current algebras and equivariant map algebras [Chari et al. 2010; Neher et al. 2012]. When R and S are Laurent polynomial rings, the intersection of the class of algebras with the class of twisted forms discussed in the present paper includes multiloop algebras (Section 4a), but not Margaux algebras (Section 4b), for instance.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over k, with commuting automorphisms $\sigma_1, \dots, \sigma_N : \mathfrak{g} \to \mathfrak{g}$ of finite orders $m_1, \dots, m_N$, respectively. Fix a primitive m_j-th root of unity $\xi_j \in k$ for each j, and let $R = k[t_1^{\pm m_1}, \dots, t_N^{\pm m_N}] \subseteq S = k[t_1^{\pm 1}, \dots, t_N^{\pm 1}]$.

The (twisted) multiloop algebra $\mathscr{L} = \mathscr{L}(\mathfrak{g}, \sigma)$ is a $\mathbb{Z}^N$-graded subalgebra of $\mathfrak{g}(S) = \mathfrak{g} \otimes S$:

$$\mathscr{L}(\mathfrak{g}, \sigma) = \bigoplus_{j \in \mathbb{Z}^N} \mathfrak{g}_j \otimes t^j,$$

where $j = (j_1, \dots, j_N)$, $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \sigma_i(x) = \xi_i^{j_i} x \text{ for } i = 1, \dots, N\}$, and $t^j = t_1^{j_1} t_2^{j_2} \cdots t_N^{j_N}$. It is easy to see that $\mathscr{L}$ is a Lie algebra over R and an S/R-form of $\mathfrak{g}(R)$.

Specializing our main theorems to the case of multiloop algebras, we recover the results of [Lau 2010]. Maximal ideals $M_i = M_{a_i} = (t_1 - a_{i1}, \dots, t_N - a_{iN})$ of S correspond to points $a_i = (a_{i1}, \dots, a_{iN})$ on the algebraic n-torus $(k^\times)^N = k^\times \times \cdots \times k^\times$. Note that $M_i \cap R$ is the ideal (of R) of polynomials vanishing at a_i. Thus $M_i \cap R \in \mathrm{Max}\ R$ is generated by $\{t_1^{m_1} - a_{i1}^{m_1}, \dots, t_N^{m_N} - a_{iN}^{m_N}\}$. Therefore, $M_i \cap R = M_j \cap R$ if and only if $m(a_i) = m(a_j)$, where we write $m(a_\ell) = (a_{\ell 1}^{m_1}, \dots, a_{\ell N}^{m_N})$ for all $a_\ell \in (k^\times)^N$.

The Galois group $\Gamma = \mathscr{G}al(S/R)$ is $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_N}$, where each $\mathbb{Z}_{m_i}$ is generated by an element

$$\alpha_i : t_j \mapsto \begin{cases} \xi_i t_i & \text{if } i = j, \\ t_j & \text{otherwise.} \end{cases}$$

The 1-cocycle $u : \Gamma \to \mathrm{Aut}_{S\text{-Lie}}(\mathfrak{g}(S))$ corresponding to $\mathscr{L}$ is given by

$$u_\gamma = \sigma_1^{-r_1} \cdots \sigma_N^{-r_N} \otimes 1,$$

for each $\gamma = (\alpha_1^{r_1}, \dots, \alpha_N^{r_N}) \in \Gamma$. Then $u_\gamma(M) = \sigma_1^{-r_1} \cdots \sigma_N^{-r_N}$ for all $M \in \mathrm{Max}(S)$. The fact that

$$u_\gamma : \mathrm{Max}(S) \to \mathrm{Aut}\,\mathfrak{g}, \quad M \mapsto u_\gamma(M)$$

is constant means that the action of Γ on $P_+^\times \times \mathrm{Max}(S)$ splits into separate actions of Γ on $\mathrm{Max}(S)$ and on $P_+^\times$ by

$$\psi : \Gamma \times P_+^\times \to P_+^\times, \quad (\gamma, \lambda) \mapsto \lambda \circ \mathrm{Out}\,\sigma_1^{-r_1} \cdots \sigma_N^{-r_N}.$$

In this language, Γ acts on $P_+^\times \times \mathrm{Max}(S)$ as ${}^\gamma(\lambda, M) = (\psi(\gamma^{-1}, \lambda), {}^\gamma M)$. The Γ-invariant functions $\chi_{[\lambda,M]} : \mathrm{Max}(S) \to P_+$ become Γ-equivariant functions under the new action ψ on $P_+^\times$. We thus recover the following theorem [Lau 2010, Corollary 4.4, Theorem 4.5, and Corollary 5.10]:

Theorem 4.1. (1) *The finite-dimensional simple modules of $\mathcal{L}(\mathfrak{g}; \sigma)$ are those of the form $V(\lambda, a) = V_{\lambda_1}(M_{a_1}) \otimes \cdots \otimes V_{\lambda_n}(M_{a_n})$ for $n \geq 0$, $a_i \in (k^\times)^N$, and $m(a_i) \neq m(a_j)$ whenever $i \neq j$.*

(2) *The isomorphism classes of finite-dimensional simple $\mathcal{L}(\mathfrak{g}; \sigma)$-modules are in bijection with the finitely supported Γ-equivariant maps $(k^\times)^N \to P_+$.*

4b. ***Azumaya and Margaux algebras.*** Fix Laurent polynomial rings

$$R = k\big[t_1^{\pm 2}, t_2^{\pm 2}\big] \quad \text{and} \quad S = k\big[t_1^{\pm 1}, t_2^{\pm 1}\big].$$

Let $A = A(1, 2)$ be the standard Azumaya algebra, the unital associative R-algebra generated by $\{T_1^{\pm 1}, T_2^{\pm 1}\}$ with relations $T_2 T_1 = -T_1 T_2$ and $T_i^2 = t_i^2$ for $i = 1, 2$. Then A is an S/R-form of the associative algebra $M_2(R)$ of 2×2 matrices over R, as can be readily verified using one of the well-known representations of the quaternions as matrices in $M_2(\mathbb{C})$.

Since $\mathbf{PGL}_2$ is the automorphism group (scheme) of both $M_2(k)$ and $\mathfrak{sl}_2(k)$, there is a natural correspondence between S/R-forms of $M_2(R)$ and $\mathfrak{sl}_2(R)$. Namely, given any S/R-form B of the matrix algebra $M_2(R)$, view B as a Lie algebra $\mathrm{Lie}\,B$ with bracket $[a, b] = ab - ba$. Its derived subalgebra $(\mathrm{Lie}\,B)' = \mathrm{Span}\{[a, b] \mid a, b \in B\}$ is then an S/R-form of $\mathfrak{sl}_2(R)$.

Applying this construction to $\mathcal{L}_1 = (\mathrm{Lie}\,A)'$ and computing explicitly, it follows that $\mathcal{L}_1 \simeq \mathcal{L}(\mathfrak{sl}_2(k), \sigma_1, \sigma_2)$ where σ_1 and σ_2 are conjugation by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, respectively [Gille and Pianzola 2007]. Therefore, we obtain the representations of $\mathcal{L}_1$ as in the previous section.

Surprisingly, not every twisted form of $\mathfrak{g}(k[t_1^{\pm 1}, t_2^{\pm 1}])$ is a multiloop algebra. This can be seen using loop torsors. The only known S/R-forms of $\mathfrak{g}(R)$ that are not isomorphic to multiloop algebras are called *Margaux algebras*. The simplest of these can be constructed concretely as follows. See [Gille and Pianzola 2007] for details.

Let A, R, and S be as in Section 4a. The right A-module

$$M = \{(\lambda, \mu) \in A \oplus A \mid (1+T_1)\lambda = (1+T_2)\mu\}$$

is projective but not free. This can be used to show that its endomorphism ring $\mathcal{M} = \mathrm{End}_A(M)$, while also an S/R-form of $M_2(R)$, is not isomorphic to A as an A-algebra. It follows that $\mathcal{L}_1$ and $\mathcal{L}_2 = (\mathrm{Lie}\,\mathcal{M})'$ are nonisomorphic S/R-forms of $\mathfrak{sl}_2(R)$. By the classification of involutions in $\mathrm{PGL}_2(k)$ and a study of loop torsors, it can be shown that $\mathcal{L}_2$ is *not* a (twisted) multiloop algebra.

By Theorems 3.4 and 3.9, the irreducible representations of $\mathcal{L}_2$ are the tensor products $V(\lambda, M) = V_{\lambda_1}(M_1) \otimes \cdots \otimes V_{\lambda_n}(M_n)$, where $\lambda_1, \dots, \lambda_n \in \mathbb{Z}_+ \setminus \{0\}$ are highest weights of $\mathfrak{sl}_2(k)$ and $M_i = \langle t_1 - a_{i1}, t_2 - a_{i2}\rangle$ are maximal ideals of $S = k[t_1^{\pm 1}, t_2^{\pm 1}]$ corresponding to points in distinct fibers over $\mathrm{Spec}\, R$. That is, $(a_{i1}^2, a_{i2}^2) \neq (a_{j1}^2, a_{j2}^2)$ for $i \neq j$.

Two such representations

$$V(\lambda, M) = V_{\lambda_1}(M_1) \otimes \cdots \otimes V_{\lambda_m}(M_m) \quad \text{and} \quad V(\mu, N) = V_{\mu_1}(N_1) \otimes \cdots \otimes V_{\mu_n}(N_n)$$

are isomorphic precisely when the corresponding $\mathcal{G}al(S/R)$-invariant functions $\chi_{[\lambda, M]}$ and $\chi_{[\mu, N]}$ are equal. But the action

$${}^{\gamma}(\lambda_i, M_i) = (\lambda_i \circ \mathrm{Out}\, {u_\gamma}^{-1}({}^{\gamma}M_i), {}^{\gamma}M_i)$$

is simply an action on $\mathrm{Max}(S)$,

$${}^{\gamma}(\lambda_i, M_i) = (\lambda_i, {}^{\gamma}M_i),$$

since ${u_\gamma}^{-1}({}^{\gamma}M) \in \mathrm{Aut}\, \mathfrak{sl}_2(k)$, and every automorphism of $\mathfrak{sl}_2(k)$ is inner! Thus $V(\lambda, M) \simeq V(\mu, N)$ if and only if (after reordering the tensor factors) $m = n$, $\lambda_i = \mu_i$, and the $a_i, b_i \in k^\times \times k^\times$ corresponding to M_i and N_i satisfy $a_{ij} = \pm b_{ij}$ for all i and j.

As for any Galois extension S/R, the isomorphism classes of the (finite-dimensional) simple modules of any S/R-form of $\mathfrak{sl}_2(R)$ are given by restrictions of the same evaluation modules of $\mathfrak{sl}_2(S)$. In particular, the irreducible $\mathcal{L}_1$- and $\mathcal{L}_2$-modules come from the same $\mathfrak{sl}_2(S)$-modules.

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Higher Chow groups of varieties with group action

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We give explicit descriptions of the higher Chow groups of toric bundles and flag bundles over schemes. We derive several consequences of these descriptions for the equivariant and ordinary higher Chow groups of schemes with group action.

We prove a decomposition theorem for the equivariant higher Chow groups of a smooth scheme with action of a diagonalizable group. This theorem is applied to compute the equivariant and ordinary higher Chow groups of smooth toric varieties. The results of this paper play fundamental roles in the proof of the Riemann–Roch theorems for equivariant higher K-theory.

1. Introduction

A *scheme* in this paper will mean a separated and reduced scheme of finite type over a perfect field k, which admits an ample line bundle. This base field k will be fixed throughout this paper. A *linear algebraic group* G over k will mean a smooth and affine group scheme over k. By a closed subgroup H of an algebraic group G, we shall mean a morphism $H \to G$ of algebraic groups over k that is a closed immersion of k-schemes. In particular, a closed subgroup of a linear algebraic group will be of the same type and hence smooth. Recall from [Borel 1991, Proposition 1.10] that a linear algebraic group over k is a closed subgroup of a general linear group, defined over k.

Recall that an action of a linear algebraic group G on a k-scheme X is said to be *linear* if X admits a G-equivariant ample line bundle, a condition that is always satisfied if X is normal (see [Sumihiro 1975, Theorem 2.5] for G connected and [Thomason 1988, 5.7] for G general). All G-actions in this paper will be assumed to be linear. Let $\mathcal{V}_k$ denote the category of quasiprojective k-schemes and let $\mathcal{V}^S_k$ denote the full subcategory of smooth k-schemes. We shall denote the category of quasiprojective G-schemes with G-equivariant maps by $\mathcal{V}_G$, and the full subcategory of smooth G-schemes will be denoted by $\mathcal{V}^S_G$.

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The G-equivariant higher Chow groups $\mathrm{CH}_*^G(X, i)$ of $X \in \mathcal{V}_G$ were defined by Edidin and Graham [1998] in terms of the ordinary higher Chow groups (motivic Borel–Moore homology) of the quotient space $X \times^G U$. Here, U is a G-invariant open subscheme of a finite-dimensional representation V of G such that it acts freely on U and $V \setminus U$ is of sufficiently high codimension. This definition of Edidin and Graham is based on an earlier construction of Totaro [1999], who invented the idea above to define the Chow groups of the classifying spaces of linear algebraic groups.

In this paper, we develop further the Edidin–Graham theory of equivariant higher Chow groups and establish many important properties of this theory. We also prove some decomposition theorems for the equivariant higher Chow groups of smooth schemes with torus action. These results turn out to have many applications.

Brion [1997] proved many results about the equivariant Chow groups of the form $\mathrm{CH}_*^G(X, 0)$. Many of the structural results in this paper can be described as the generalization of the results of [Brion 1997] to the case of equivariant *higher* Chow groups. In Section 2, we recall the definition of equivariant higher Chow groups from [Edidin and Graham 1998] and prove its basic properties, which are all well known for the ordinary higher Chow groups; see [Bloch 1986]. As a consequence, one finds that the equivariant higher Chow groups form a Borel–Moore oriented bigraded homology theory in the category of schemes with the action of a given linear algebraic group. Other important results about these groups such as the Morita isomorphism are proven in Section 3. We also prove a structure theorem (see Theorem 3.5) for the equivariant higher Chow groups of schemes with action of tori.

Section 4 contains the proof of the self-intersection formula for the higher Chow groups. This formula plays a very important role in the proofs of the main results of this paper. In Section 5, we construct Demazure operators on equivariant higher Chow groups and give some consequences of these operators.

In Section 6, we prove the Leray–Hirsch theorem for the higher Chow groups. As a consequence of this theorem, we compute the higher Chow groups of toric bundles in Section 7. In Section 8, we turn to the description of the higher Chow groups of principal bundles and flag bundles over schemes. We give several applications of these descriptions in the study of equivariant higher Chow groups.

In Sections 9 and 10, we prove a decomposition theorem (see Theorem 10.3) for the equivariant higher Chow groups of smooth schemes with action of a diagonalizable group G. This result describes the equivariant higher Chow group of a G-scheme in terms of the equivariant higher Chow groups of the loci where the stabilizers have a fixed dimension. This result is an analogue of a similar result of Vezzosi and Vistoli [2003] in equivariant K-theory and has many important applications in the study of equivariant and ordinary higher Chow groups of smooth schemes.

Theorem 10.3 is the basic step in the proof of the equivariant Riemann–Roch theorem in [Krishna 2009b]. This theorem presents an explicit relation between the equivariant K-theory and the equivariant higher Chow groups. Like in the ordinary case, this Riemann–Roch is a fundamental result in equivariant algebraic geometry. This theorem was in fact one of the main motivations for the author to work on this paper. We expect Theorem 10.3 to have many more applications in the computation of equivariant and ordinary higher Chow groups. In Section 11, we apply this theorem to compute the equivariant and ordinary higher Chow groups of smooth toric varieties. We shall follow the following convention while studying the equivariant and ordinary higher Chow groups with the rational coefficients.

Convention. In this paper, all the results and statements up to Section 7 do not make any assumption on the coefficient ring of the higher Chow groups. On the other hand, all the results and statements from Section 8 onwards assume rational coefficients. In order to simplify the notation, the following convention will be followed.

From Section 8 onwards, an abelian group A will actually mean its extension $A \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular, all higher Chow groups and other cohomology groups will be considered with the rational coefficients. For $\mathbb{Q}$-vector spaces A and B, the tensor product $A \otimes_{\mathbb{Q}} B$ will be simply written as $A \otimes B$. We shall however, indicate the appropriate coefficients in the statements of the all results.

2. Equivariant higher Chow groups

In this section, we recall the definition of the equivariant higher Chow groups from [Edidin and Graham 1998] and review their main functorial properties. It turns out in particular that the equivariant higher Chow groups have all the properties of an oriented bigraded Borel–Moore homology theory.

Let G be a linear algebraic group and let X be a scheme over k with a G-action. We shall denote the dimension of the underlying group G usually by the letter g. All representations of G in this paper will be finite-dimensional. The definition of equivariant higher Chow groups of X needs one to consider certain kind of mixed spaces which in general may not be schemes even if the original spaces are schemes. The following well-known (see [Edidin and Graham 1998, Proposition 23]) lemma shows that this problem does not occur in our context and all the mixed spaces in this paper are schemes with ample line bundles.

Lemma 2.1. *Let H be a linear algebraic group acting freely and linearly on a k-scheme U such that the quotient U/H exists as a quasiprojective scheme. Let X be a k-scheme with a linear action of H. Then the mixed quotient $X \times^H U$ exists for* *the diagonal action of H on $X \times U$ and is quasiprojective. Moreover, this quotient is smooth if both U and X are so. In particular, if H is a closed subgroup of a linear algebraic group G and X is a k-scheme with a linear action of H, then the quotient $G \times^H X$ is a quasiprojective scheme.*

Proof. It is already shown in [Edidin and Graham 1998, Proposition 23] using [Mumford et al. 1994, Proposition 7.1] that the quotient $X \times^H U$ is a scheme. Moreover, as U/H is quasiprojective, [Mumford et al. 1994, Proposition 7.1] in fact shows that $X \times^H U$ is also quasiprojective. The similar conclusion about $G \times^H X$ follows from the first case by taking $U = G$ and by observing that G/H is a smooth quasiprojective scheme; see [Borel 1991, Theorem 6.8]. The assertion about the smoothness is clear since $X \times U \to X \times^H U$ is a principal H-bundle. □

2a. ***Good pairs and equivariant higher Chow groups.*** For any integer $j \geq 0$, let V be an l-dimensional representation of G and let U be a G-invariant open subset of V such that the codimension of the complement $V \setminus U$ in V is sufficiently larger than j, and G acts freely on U such that the quotient U/G is a quasiprojective scheme. Such a pair (V, U) will be called a *good* pair for the G-action corresponding to j. It is easy to see that a good pair always exists; see [Edidin and Graham 1998, Lemma 9].

For an equidimensional G-scheme X, let X_G denote the quotient $X \times^G U$ of the product $X \times U$ by the diagonal action of G, which is free. We define the equivariant higher Chow group $\mathrm{CH}^j_G(X, i)$ as the homology group $H_i(\mathcal{Z}^j(X_G, \bullet))$, where $\mathcal{Z}^j(X_G, \bullet)$ is the Bloch cycle complex of the scheme X_G. It is known [Edidin and Graham 1998, Section 2] that this definition of $\mathrm{CH}^j_G(X, i)$ is independent of the choice of a good pair (V, U) for the G-action up to unique isomorphisms. One should also observe that $\mathrm{CH}^j_G(X, i)$ may be nonzero for infinitely many values of j, a crucial change from the case of nonequivariant (ordinary) higher Chow groups.

If X is of dimension d, which is not necessarily equidimensional, one defines the equivariant higher Chow groups as

$$\mathrm{CH}^G_j(X, i) := H_i(\mathcal{Z}_{j+l-g}(X_G, \bullet)), \tag{2-1}$$

where (V, U) is an l-dimensional good pair for the G-action corresponding to $d - j$, and $\mathcal{Z}_p(X_G, \bullet)$ is the homological cycle complex of Bloch such that $\mathcal{Z}_p(X_G, i)$ is the group of admissible algebraic cycles on $X_G \times \Delta^i$ of dimension $p + i$. We write

$$\mathrm{CH}^G_*(X, i) = \bigoplus_{-\infty < j \leq d} \mathrm{CH}^G_j(X, i) \quad \text{and} \quad \mathrm{CH}^G_*(X) = \bigoplus_{i \geq 0} \mathrm{CH}^G_*(X, i). \tag{2-2}$$

It is easy to see that $\mathrm{CH}^G_j(X, i) = \mathrm{CH}^{d-j}_G(X, i)$ if X is equidimensional of dimension d. For most of this paper, we shall use the cohomological indexing for the equivariant higher Chow groups while dealing with smooth schemes. In particular, $\mathrm{CH}^*_G(X)$ will denote the sum $\bigoplus_{i\geq 0}\mathrm{CH}^*_G(X,i)$.

For a commutative ring R, the equivariant higher Chow groups $\mathrm{CH}^j_G(X,i;R)$ are defined as the homology groups of the complex $\mathcal{Z}^j(X_G,\bullet)\otimes_{\mathbb{Z}} R$. The symbol $\mathrm{CH}^*_G(X;R)$ will denote the direct sum of all $\mathrm{CH}^j_G(X,i;R)$. We shall denote the rings $\mathrm{CH}^*_G(k,0)$ and $\mathrm{CH}^*_G(k,0;R)$ by $S(G)$ and $S(G;R)$, respectively.

2b. ***Equivariant operational Chow groups.*** For $X\in\mathcal{V}_G$, we define

$$\mathrm{OPCH}^j_G(X,i)=\varinjlim \mathrm{CH}^j_G(Y,i), \tag{2-3}$$

where the limit is taken over the category of arrows $X\to Y$ in $\mathcal{V}_G$ with $Y\in\mathcal{V}^S_G$. Notice that the natural map $\mathrm{OPCH}^j_G(X,i)\to\mathrm{CH}^j_G(X,i)$ is an isomorphism if X is smooth. We shall write the sum $\bigoplus_{i,j\geq 0}\mathrm{OPCH}^j_G(X,i)$ as $\mathrm{OPCH}^*_G(X)$.

It follows from [Bloch 1986, Proposition 5.5, Corollary 5.6] that $\mathrm{OPCH}^*_G(X)$ has a ring structure and $\mathrm{OPCH}^*_G(X,0)$ is a subring of $\mathrm{OPCH}^*_G(X)$. Moreover, $X\mapsto\mathrm{OPCH}^*_G(X)$ is a contravariant functor on $\mathcal{V}_G$ which acts on the higher Chow groups of X. In particular, $\mathrm{OPCH}^1_G(X,0)\xrightarrow{\simeq}\mathrm{Pic}_G(X)$ acts on $\mathrm{CH}^G_*(X,i)$. This action is same as the action of the Chern classes of equivariant line bundles on the homology theory $\mathrm{CH}^G_*(X,i)$.

2c. ***Main properties of equivariant higher Chow groups.*** The following result summarizes most of the essential properties of the equivariant higher Chow groups that will be used in this paper.

Proposition 2.2. *The equivariant higher Chow groups as defined above satisfy the following properties.*

(1) *Functoriality*: *Covariance for proper maps, contravariance for flat maps and their compatibility. That is, for a fiber diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

in $\mathcal{V}_G$ with f proper and g flat, one has

$$g^*\circ f_* = f'_*\circ g'^* : \mathrm{CH}^G_*(X,i)\to\mathrm{CH}^G_*(Y',i).$$

Moreover, if $f:X\to Y$ is a morphism in $\mathcal{V}_G$ with Y in $\mathcal{V}^S_G$, then there is a pull-back map $f^*:\mathrm{CH}^G_*(Y,i)\to\mathrm{CH}^G_*(X,i)$.

(2) *Homotopy*: *If $f:X\to Y$ is an equivariant vector bundle, then*

$$f^*:\mathrm{CH}^G_*(Y,i)\xrightarrow{\simeq}\mathrm{CH}^G_*(X,i).$$

(3) *Exterior product*: *There is a natural product map*

$$\mathrm{CH}^G_*(X, i) \otimes \mathrm{CH}^G_*(Y, i') \to \mathrm{CH}^G_*(X \times Y, i + i').$$

Moreover, *if* $f : X \to Y$ *is such that* $Y \in \mathcal{V}^S_G$, *then there is a pull-back via the graph map* $\Gamma_f : X \to X \times Y$, *which makes* $\mathrm{CH}^*_G(Y)$ *a bigraded ring and* $\mathrm{CH}^G_*(X)$ *a module over this ring. In particular*, $\mathrm{CH}^G_*(X, i)$ *an* $S(G)$*-module for* $X \in \mathcal{V}_G$ *and* $i \geq 0$.

(4) *Localization*: *If* $Y \subset X$ *is a* G*-invariant closed subscheme with complement* U, *then there is a long exact localization sequence of* $S(G)$*-modules*

$$\cdots \to \mathrm{CH}^G_*(Y, i) \to \mathrm{CH}^G_*(X, i) \to \mathrm{CH}^G_*(U, i) \to \mathrm{CH}^G_*(Y, i-1) \to \cdots .$$

This sequence is compatible with the push-forward and flat pull-back maps of higher Chow groups.

(5) *Chern classes*: *For any* G*-equivariant vector bundle of rank* r, *there are equivariant Chern classes* $c^G_l(E) : \mathrm{CH}^G_j(X, i) \to \mathrm{CH}^G_{j-l}(X, i)$ *for* $0 \leq l \leq r$, *having the same functoriality properties as in the nonequivariant case and* $c^G_0(E) = 1$.

(6) *Projection formula*: *For a proper map* $f : X \to Y$ *in* $\mathcal{V}_G$ *and for* $x \in \mathrm{CH}^G_*(X)$, $y \in \mathrm{OPCH}^*_G(Y)$, *one has* $f_*(f^*(y) \cdot x) = y \cdot f_*(x)$. *Here*, *the action of* $\mathrm{OPCH}^*_G(Y)$ *on* $\mathrm{CH}^G_*(X)$ *is given by* (3) *above.*

(7) *Free action*: *If* G *acts freely on* X *with quotient* Y, *then there is a canonical isomorphism* $\mathrm{CH}^G_*(X, i) \xrightarrow{\simeq} \mathrm{CH}_*(Y, i)$.

Proof. Since the equivariant higher Chow groups of X are defined in terms of the higher Chow groups of X_G, the proposition (except possibly the last property) can be easily deduced from the similar results for the nonequivariant higher Chow groups as in [Bloch 1986] and the techniques of [Edidin and Graham 1998]. We therefore skip the proof. To see that the maps in the localization sequence are $S(G)$-linear, it suffices to know that for a good pair (V, U), the long exact sequence

$$\cdots \to \mathrm{CH}_*(Y_G, i) \to \mathrm{CH}_*(X_G, i) \to \mathrm{CH}_*(U_G, i) \to \mathrm{CH}_*(Y_G, i-1) \to \cdots$$

is a sequence of $\mathrm{CH}_*(U/G)$-modules. But this is a well-known fact as U/G is smooth and the above is a sequence of higher Chow groups of schemes over it; see [Bloch 1986, Exercise 5.8(ii)].

To prove (6), we need to show that if $Y \xrightarrow{g} Z$ is a G-equivariant map with $Z \in \mathcal{V}^S_G$, then the map $\mathrm{CH}^G_*(X) \xrightarrow{f_*} \mathrm{CH}^G_*(Y)$ is $\mathrm{CH}^*_G(Z)$-linear. Since the push-forward and the pull-back maps of equivariant Chow groups are nothing but the maps of ordinary higher Chow groups of suitable mixed quotients, it suffices to prove the statement above for the push-forward map of the higher Chow groups corresponding to the maps of mixed quotients $X_G \to Y_G \to Z_G$. Since Z_G is smooth, this nonequivariant version is well-known [Bloch 1986, §5.5, Exercise 5.8].

For the last property, fix $j \le d$ and choose a good pair (V, U) of dimension l for the G-action corresponding to $d - j \ge 0$. Since G acts freely on X, it acts likewise also on $X \times V$ with quotient, say X_V. Then X_G is an open subset of X_V and $X_V \to Y$ is a vector bundle, which implies that the map $\mathrm{CH}_j(Y, i) \to \mathrm{CH}_{j+l}(X_V, i)$ is an isomorphism by the homotopy invariance. On the other hand, the restriction map $\mathrm{CH}_{j+l}(X_V, i) \to \mathrm{CH}_{j+l}(X_G, i) = \mathrm{CH}^G_{j-g}(X, i)$ is an isomorphism by the property (4) as $d - j$ is sufficiently small. □

Remark 2.3. The reader should be warned that the various isomorphisms between the (equivariant) higher Chow groups in the proposition above are true only up to some obvious shift in the dimension of cycles, which we have chosen not to write.

We next recall from [Edidin and Graham 1998] that the Chern classes $c_l^G(E)$ of an equivariant vector bundle E, as described in Proposition 2.2 above, live in the operational Chow groups $\mathrm{OPCH}^l(X_G)$. If X is in $\mathcal{V}_G^S$ however, this operational Chow group is isomorphic to the equivariant Chow group $\mathrm{CH}^l_G(X, 0)$ and the action of $c_l^G(E)$ on $\mathrm{CH}^*_G(X)$ then coincides with the intersection product in the ring $\mathrm{CH}^*_G(X)$.

Finally, we recall from [ibid.] that if $H \subset G$ is a closed subgroup and if (V, U) is a good pair, then for $X \in \mathcal{V}_G$, the natural map of quotients $X \times^H U \to X \times^G U$ is an étale locally trivial G/H-fibration and hence there is a natural restriction map

$$r^G_{H,X} : \mathrm{CH}^G_*(X, i) \to \mathrm{CH}^H_*(X, i). \tag{2-4}$$

Taking $H = \{1\}$, one obtains the *forgetful* map

$$r^G_X : \mathrm{CH}^G_*(X, i) \to \mathrm{CH}_*(X, i). \tag{2-5}$$

Moreover, as $r^G_{H,X}$ is the pull-back under a flat (in fact, a smooth) map, it commutes (see Proposition 2.2) with the pull-back for any flat map, and with the push-forward for any proper map in $\mathcal{V}_G$. We remark here that although the definition of $r^G_{H,X}$ uses a good pair (V, U) for any given $j \le \dim(X)$, it is easy to check from the homotopy invariance that $r^G_{H,X}$ is independent of the choice of the good pair (V, U).

3. Morita isomorphisms

In this section, we prove some Morita-type isomorphisms that address the question of comparison between the equivariant higher Chow groups for the action of two different algebraic groups. We also prove a structure theorem for these equivariant higher Chow groups under the trivial action of split tori. These results are analogues of the similar results of Thomason in equivariant K-theory; see [Thomason 1986, Lemma 5.6; 1988, Section 1].

Proposition 3.1 (Morita isomorphism). *Let H be a normal subgroup of a linear algebraic group G and let $F = G/H$. Let $f : X \to Y$ be a G-equivariant morphism of G-varieties that is an H-torsor for the restricted action. Then the map induced on the equivariant higher Chow groups*

$$\mathrm{CH}^F_*(Y, i) \xrightarrow{f^*} \mathrm{CH}^G_*(X, i).$$

is an isomorphism.

Proof. We first observe from [Springer 1998, Corollary 12.2.2] that F is also a linear algebraic group over the given ground field k. Now, since f is an H-torsor, it is clear that G acts on Y via F. Fix $j \le \dim(X)$ and choose a good pair (V, U) of dimension l for the F-action corresponding to $\dim(Y) - j$. Then V is also a representation of G in which U is G-invariant. In particular, G acts on $X \times U$ via the diagonal action, which is easily seen to be free since H acts freely on X and F acts freely on U. By the same reason, we see that $X \times U \to Y \times U$, which is a principal H-bundle, is G-equivariant. This in turn implies that the map $(X \times U)/G \to Y_F$ is an isomorphism and hence we get

$$\mathrm{CH}^F_j(Y, i) \cong \mathrm{CH}_{j+l-g+h}(Y_F, i) \underset{f^*}{\xrightarrow{\simeq}} \mathrm{CH}_{j+l-g+h}(X \times^G U, i), \tag{3-1}$$

where $\dim(H) = h$. On the other hand, we have

$$\mathrm{CH}^G_{j+h}(X, i) \cong \mathrm{CH}^G_{j+h+l}(X \times V, i) \cong \mathrm{CH}^G_{j+h+l}(X \times U, i) \cong \mathrm{CH}_{j+h+l-g}(X \times^G U, i),$$

where the first isomorphism is due to the homotopy invariance, the second follows from the localization property (see Proposition 2.2(4)) as j is sufficiently small, and the third isomorphism follows from Proposition 2.2(7). The proof of the proposition now follows by combining this with (3-1). □

Corollary 3.2 (see [Edidin and Graham 2000]). *Let $H \subset G$ be a closed subgroup and let $X \in \mathcal{V}_H$. Then for any $i \ge 0$, there is a natural isomorphism*

$$\mathrm{CH}^G_*(G \times^H X, i) \xrightarrow{\simeq} \mathrm{CH}^H_*(X, i). \tag{3-2}$$

Proof. Define an action of $H \times G$ on $G \times X$ by

$$(h, g) \cdot (g', x) = (gg'h^{-1}, hx) \tag{3-3}$$

and an action of $H \times G$ on X by $(h, g) \cdot x = hx$. Then the projection map $G \times X \xrightarrow{p} X$ is $(H \times G)$-equivariant and is a G-torsor. Hence by Proposition 3.1, the natural map

$$\mathrm{CH}^H_*(X, i) \xrightarrow{p^*} \mathrm{CH}^{H \times G}_*(G \times X, i)$$

is an isomorphism. On the other hand, the projection map $G \times X \to G \times^H X$ is $(H \times G)$-equivariant and is an H-torsor. Hence we get an isomorphism

$$\mathrm{CH}_*^G(G \times^H X, i) \xrightarrow{\simeq} \mathrm{CH}_*^{H \times G}(G \times X, i).$$

The corollary follows by combining these two isomorphisms. $\square$

Proposition 3.3. *Let G be a connected reductive group over k. Let B be a Borel subgroup of G containing a maximal torus T over k. Then for any $i \geq 0$, the restriction map*

$$\mathrm{CH}_*^B(X, i) \xrightarrow{r_{T,X}^B} \mathrm{CH}_*^T(X, i) \tag{3-4}$$

is an isomorphism for any $X \in \mathcal{V}_B$.

Proof. By Corollary 3.2, we only need to show that

$$\mathrm{CH}_*^B(X, i) \xrightarrow{\simeq} \mathrm{CH}_*^B(B \times^T X, i). \tag{3-5}$$

By [M. Demazure 1970, XXII, 5.9.5], there exists a characteristic filtration

$$B^u = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_n = \{1\}$$

of the unipotent radical B^u of B such that U_{j-1}/U_j is a vector group, each U_j is normal in B and $TU_j = T \ltimes U_j$. Moreover, this filtration also implies that for each j, the natural map $B/TU_j \to B/TU_{j-1}$ is a torsor under the vector bundle $U_{j-1}/U_j \times B/TU_{j-1}$ on B/TU_{j-1}. Hence, the homotopy invariance gives an isomorphism

$$\mathrm{CH}_*^B(B/TU_{j-1} \times X, i) \xrightarrow{\simeq} \mathrm{CH}_*^B(B/TU_j \times X, i).$$

Composing these isomorphisms successively for $j = 1, \ldots, n$, we get

$$\mathrm{CH}_*^B(X, i) \xrightarrow{\simeq} \mathrm{CH}_*^B(B/T \times X, i).$$

The canonical isomorphism of B-varieties $B \times^T X \cong B/T \times X$ (see Corollary 3.2) now proves (3-5) and hence (3-4). $\square$

Recall that a linear algebraic group G over k of dimension g is *diagonalizable* if $G_{\bar{k}} \cong H \times (\mathbb{G}_m)^g$, where H is a finite abelian group. The group G is called split diagonalizable, if such an isomorphism is defined over k. A connected reductive group G over k is said to be *split* if it contains a split maximal torus T over k such that G is given by a root system relative to T. Every connected and reductive group containing a split maximal torus is split; see [M. Demazure 1970, Chapter XXII, Proposition 2.1].

Recall from [Springer 1998, Lemma 14.1.1] that every solvable group G over k has a filtration $\{e\} = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$ by closed normal k-subgroups such that each quotient group G_j/G_{j-1} is either diagonalizable or an elementary unipotent group; see [Springer 1998, §3.4]. The group G is called *split* over k if each G_j/G_{j-1} is either split diagonalizable or $\mathbb{G}_a$. It is known [Springer 1998, Corollary 14.3.10] that every unipotent group over a perfect field is split.

Proposition 3.4. *Let H be a possibly nonreductive group over k. Assume that H has a Levi decomposition $H = L \ltimes H^u$ such that H^u is split over k (for example, if k is of characteristic zero). Then for each $i \geq 0$, the map*

$$\mathrm{CH}^H_*(X, i) \xrightarrow{r^H_{L,X}} \mathrm{CH}^L_*(X, i) \tag{3-6}$$

is an isomorphism.

Proof. Since the unipotent radical of H is split over k, the proof is exactly same as in the proof of (3-4), where we just have to replace B and T by H and L, respectively. □

3a. ***A structure theorem for*** $\mathbf{CH}^T_*(X)$**.** We end this section with the following structure theorem for the equivariant higher Chow groups of a scheme with the action of a split diagonalizable group on which certain subgroup acts trivially. This theorem is the initial step in the proof of its far reaching generalization in Theorem 10.3.

Theorem 3.5. *Let T be a split diagonalizable group and let $X \in \mathcal{V}_T$. Let H be a connected closed subgroup of T that acts trivially on X. Then there is a natural isomorphism*

$$\mathrm{CH}^{T/H}_*(X) \otimes_{\mathbb{Z}} S(H) \xrightarrow{i^T_{H,X}} \mathrm{CH}^T_*(X).$$

This is a bigraded ring isomorphism if X is smooth.

Proof. Put $T' = T/H$. Since H is a split torus, we can choose a decomposition (not necessarily canonical) $T = H \times T'$. Fix an integer $j \leq \dim(X)$ and let (V, U) and (V', U') be good pairs for the actions of H and T', respectively, corresponding to $\dim(X) - j$ as in [Edidin and Graham 1998, Example 3.1]. Thus U is a product of punctured affine spaces and $U/H = (\mathbb{P}^n)^r$ for some $n \gg 0$, where $r = \mathrm{rank}(H)$. Then (V_T, U_T), with $V_T = V \times V'$ and $U_T = U \times U'$, is a good pair for the action of T. We now have

$$X_T = (X \times U \times U')/(H \times T') = (X \times U') \times^{T'} U/H = X_{T'} \times (\mathbb{P}^n)^r,$$

where the second equality holds since H acts trivially on $X \times U'$ and the third equality holds because T' acts trivially on U. It follows from the projective bundle formula (see also Lemma 6.2) for the ordinary higher Chow groups that the map

$$\mathrm{CH}_*(X_{T'}) \otimes_{\mathbb{Z}} \mathrm{CH}_*((\mathbb{P}^n)^r, 0) \to \mathrm{CH}_*(X_T) \tag{3-7}$$

is an isomorphism. We conclude the proof by noting that $\mathrm{CH}_p(X_{T'}) \cong \mathrm{CH}_p^{T'}(X)$ and $\mathrm{CH}_p(X_T) \cong \mathrm{CH}_p^{T}(X)$ for all $p \le j$. If X is smooth, the assertion about the ring isomorphism of $i^T_{H,X}$ now follows because (3-7) is known to be a bigraded ring isomorphism in that case. $\square$

4. Self-intersection and projection formulas

Our aim in this section is to prove the following two results for the ordinary and equivariant higher Chow groups. The first result is the self-intersection formula for the higher Chow groups of smooth schemes. The analogue of this formula for the higher K-theory was proven by Thomason [1993, Theorem 3.1]. Surprisingly, this formula for the higher Chow groups has remained unnoticed. Its equivariant version will play a very crucial role in the decomposition Theorem 10.3 for the equivariant higher Chow groups of smooth schemes with an action of a diagonalizable group.

The second result of this section is a version of projection formula for the higher Chow groups of singular schemes. Such a formula for the smooth schemes was proven by Bloch [1986]. We shall need this version of the projection formula in our construction of Demazure operators on the equivariant higher Chow groups.

4a. ***Self-intersection formula.*** We shall use the method of *deformation to the normal cone* as the main technical tool to prove the self-intersection formula. Since this technique will be used several times in this paper, we briefly recall the construction from [Fulton 1984, Chapter 5] for our, as well as reader's, convenience. Let X be a smooth scheme over k and let $f : Y \hookrightarrow X$ be a smooth closed subscheme of codimension $d \ge 1$. Let $\widetilde{M}$ be the blow-up of $X \times \mathbb{P}^1$ along $Y \times \infty$. Then $Bl_Y(X)$ is a closed subscheme of $\widetilde{M}$ and one denotes its complement by M. There is a natural map $\pi : M \to \mathbb{P}^1$ such that $\pi^{-1}(\mathbb{A}^1) \cong X \times \mathbb{A}^1$ with π the projection map and $\pi^{-1}(\infty) \cong X'$, where X' is the total space of the normal bundle $N_{Y/X}$ of Y in X. One also gets the following diagram, where all the squares and the triangles commute.

$$\begin{tikzcd}
Y \arrow[rr, bend left, "i_0"] \arrow[dr, "u'"'] \arrow[dd, "f"'] & & Y \times \mathbb{P}^1 \arrow[ll, "p_Y"] \arrow[dd, "F"] & Y \arrow[l, "i_\infty"'] \arrow[dd, "f'"] \\
& Y \times \mathbb{A}^1 \arrow[ur, "j'"'] \arrow[dd, "F'"] & & \\
X \arrow[rr, bend left, "h"] \arrow[dr, "u"'] & & M & X' \arrow[l, "i"] \\
& X \times \mathbb{A}^1. \arrow[ur, "j"'] & &
\end{tikzcd} \tag{4-1}$$

In this diagram, all the vertical arrows are the closed embeddings, i_0 and i_∞ are the obvious inclusions of Y in $Y \times \mathbb{P}^1$ along the specified points, i and j are inclusions of the inverse images of ∞ and $\mathbb{A}^1$, respectively, under the map π, u and f' are zero section embeddings and p_Y is the projection map. In particular, one has $p_Y \circ i_0 = p_Y \circ i_\infty = \mathrm{id}_Y$.

In case X is a G-scheme and Y is G-invariant, then by letting G act trivially on $\mathbb{P}^1$ and diagonally on $X \times \mathbb{P}^1$, one gets a natural action of G on M, and all the spaces in the diagram above become G-spaces and all the morphisms become G-equivariant. This observation will be used later in this paper.

We shall need the following result about the higher Chow groups, which is an easy consequence of Bloch's moving lemma.

Lemma 4.1. *Let*

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "f"'] & X \end{tikzcd}$$

be a fiber diagram of closed immersions of schemes such that X and Y are smooth and Y and Z intersect properly in X. Then one has for each $i \geq 0$,

$$f^* \circ g_* = {g'}_* \circ {f'}^* : \mathrm{CH}_*(Y, i) \to \mathrm{CH}_*(Z, i).$$

Proof. Since X and Y are smooth, we can assume them to be equidimensional. Let

$$\mathfrak{Z}^p_{ZW}(Y, \bullet) \xhookrightarrow{i_Y} \mathfrak{Z}^p(Y, \bullet)$$

be the subcomplex that is generated by cycles on $Y \times \Delta^\bullet$ which intersect all faces of $Z \times \Delta^\bullet$ and $W \times \Delta^\bullet$ properly. Similarly, let

$$\mathfrak{Z}^p_{Z}(X, \bullet) \xhookrightarrow{i_X} \mathfrak{Z}^p(X, \bullet)$$

be the subcomplex generated by the cycles on $X \times \Delta^\bullet$ that intersect all faces of $Z \times \Delta^\bullet$ properly. Then i_X and i_Y are quasi-isomorphisms by the moving lemma; see [Krishna and Levine 2008, Theorem 1.10]. However, if $V \in \mathfrak{Z}^p_{ZW}(Y, \bullet)$ is an irreducible cycle in $Y \times \Delta^n$, then the conclusion of the lemma is checked easily. $\square$

Corollary 4.2. *Let G be a linear algebraic group and let*

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "f"'] & X \end{tikzcd}$$

be a fiber diagram of closed immersions of smooth G-schemes such that Y and Z intersect properly in X. Then one has $f^* \circ g_* = {g'}_* \circ {f'}^* : \mathrm{CH}^*_G(Y, i) \to \mathrm{CH}^*_G(Z, i)$.

Proof. By choosing a good pair (V, U) for the G-action and then considering the appropriate mixed quotients, we can reduce to proving the corollary for the nonequivariant higher Chow groups. But this is shown in Lemma 4.1. □

Lemma 4.3. *Consider the diagram* (4-1) *and let $y \in \mathrm{CH}^*(Y, m)$. Then there exists $z \in \mathrm{CH}^*(M, m)$ such that $f_*(y) = h^*(z)$ and ${f'}_*(y) = i^*(z)$.*

Proof. Put $\tilde{y} = p_Y^*(y)$ and $z = F_*(\tilde{y})$. Then

$$\begin{aligned}
f_*(y) &= f_*((p_Y \circ i_0)^*(x)) = f_* \circ i_0^* \circ p_Y^*(y) = f_* \circ i_0^*(\tilde{y}) = f_* \circ {u'}^* \circ {j'}^*(\tilde{y}) \\
&= u^* \circ {F'}_*({j'}^*(\tilde{y})) && \text{(by Lemma 4.1)} \\
&= u^* \circ j^* \circ F_*(\tilde{y}) && \text{(since } j \text{ is an open immersion)} \\
&= h^* \circ F_*(\tilde{y}) = h^*(z).
\end{aligned}$$

Similarly,

$$\begin{aligned}
{f'}_*(y) &= {f'}_*((p_Y \circ i_\infty)^*(x)) = {f'}_* \circ i_\infty^* \circ p_Y^*(y) = {f'}_* \circ i_\infty^*(\tilde{y}) \\
&= i^* \circ F_*(\tilde{y}) && \text{(by Lemma 4.1)} \\
&= i^*(z).
\end{aligned}$$

□

Theorem 4.4 (self-intersection formula). *Let $Y \overset{f}{\hookrightarrow} X$ be a closed immersion of smooth varieties of codimension $d \geq 0$, and let $N_{Y/X}$ be the normal bundle of Y in X. Then for every $y \in \mathrm{CH}^*(Y, i)$, one has $f^* \circ f_*(y) = c_d(N_{Y/X}) \cdot y$.*

Proof. There is nothing to prove when $d = 0$ and so we assume $d \geq 1$. We first consider the case when $X \xrightarrow{p} Y$ is a vector bundle of rank d and f is the zero section embedding so that $p \circ f = \mathrm{id}_Y$. In that case, we have

$$\begin{aligned}
f^* \circ f_*(y) &= f^* \circ f_*(f^* \circ p^*(y)) = f^*(f_*(1) \cdot p^*(y)) \quad \text{(by Proposition 2.2(6))} \\
&= f^*(f_*(1)) \cdot (f^* \circ p^*(y)) = f^*(f_*(1)) \cdot y = c_d(N_{Y/X}) \cdot y,
\end{aligned}$$

where the last equality follows from the self-intersection formula for Fulton's Chow groups; see [Fulton 1984, Corollary 6.3]. This proves the theorem in the case of zero section embedding.

Now let $Y \hookrightarrow X$ be as in the theorem. We consider the deformation to the normal cone diagram (4-1) and choose $z \in \mathrm{CH}^*(M, i)$ as in Lemma 4.3. Then we have

$$\begin{aligned}
f^* \circ f_*(y) &= f^* \circ h^*(z) = {i_0}^* \circ F^*(z) = i_\infty^* \circ F^*(z) = {f'}^* \circ i^*(z) \\
&= {f'}^* \circ {f'}_*(y) && \text{(by Lemma 4.3)} \\
&= c_d(N_{Y/X'}) \cdot y \quad \text{(by the case of vector bundle above)} \\
&= c_d(N_{Y/X}) \cdot y.
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 4.5. *Let G be a linear algebraic group over k and let $Y \xhookrightarrow{f} X$ be a closed immersion of codimension $d \geq 0$ in $\mathcal{V}^S_G$. Then for every $i \geq 0$ and $y \in \mathrm{CH}^*_G(Y, i)$, one has $f^* \circ f_*(y) = c^G_d(N_{Y/X}) \cdot y$.*

Proof. There is nothing to prove if $d = 0$ and so we assume $d \geq 1$. Fix $i, j \geq 0$ and choose a good pair (V, U) for $n \gg j + d$. We can then identify $\mathrm{CH}^p_G(X, i)$ with $\mathrm{CH}^p(X_G, i)$ (and same for Y) for $p \leq n$. We can also identify $c^G_d(E)$ with $c_d(E_G)$ for any equivariant vector bundle E on Y; see [Edidin and Graham 1998, Section 2.4]. Now, the proof of the corollary would follow straightaway from Theorem 4.4, once we show that $(N_{Y/X})_G$ is the normal bundle of Y_G in X_G. But this follows immediately from the elementary fact that if G acts freely on a smooth scheme Z and W is a smooth closed and G-invariant subscheme of Z with normal bundle N, then G acts freely on N, and moreover, N/G is the normal bundle of W/G in Z/G. We leave the proof of this fact to the reader. □

4b. ***A projection formula for singular schemes.*** Recall from Section 2b (see also [Bloch 1986, Corollary 5.6]) that the operational Chow groups $X \mapsto \mathrm{OPCH}^*(X)$ form a ring-valued contravariant functor on $\mathcal{V}_k$ that acts on the higher Chow groups. The action of $\mathrm{OPCH}^1(X, 0) \xrightarrow{\simeq} \mathrm{Pic}(X)$ on $\mathrm{CH}_*(X, j)$ coincides with the action of the Chern classes of line bundles.

Proposition 4.6. *Let $X \in \mathcal{V}_k$ and let $f : Y = \mathbb{P}(E) \to X$ be the projective bundle associated to a vector bundle E of rank $n+1$ on X and let*

$$\xi = c_1(\mathcal{O}_Y(1)) \in \mathrm{OPCH}^1(Y, 0)$$

be the first Chern class of the relative tautological line bundle on Y. Then for any $x \in \mathrm{CH}_*(X, j)$, one has

$$f_*(\xi^i \cdot f^*(x)) = \begin{cases} 0 & \text{if } i < n, \\ x & \text{if } i = n. \end{cases}$$

Proof. If X is smooth, the proposition is an easy consequence of the projection formula [Bloch 1986, Exercise 5.8], as this formula implies that

$$f_*(\xi^i \cdot f^*(x)) = f_*(\xi^i) \cdot x.$$

Moreover, it follows from [Fulton 1984, Proposition 3.1] that $f_*(\xi^i) = 1$ if $i = n$ and zero otherwise. The case of singular schemes is the hard part of the theorem because we cannot directly apply the projection formula of [Bloch 1986]. We obtain a proof by an indirect approach of reduction to the smooth case and by unravelling the action of Chern classes on the higher Chow cycles on singular schemes.

By [Fulton 1984, Lemma 18.2], we can find a closed embedding $\iota : X \to X'$ and a vector bundle E' of rank $n+1$ on X' such that $E \cong \iota^*(E')$ and X' is smooth. We set $Y' = \mathbb{P}(E')$, $\xi' = c_1(\mathbb{O}_{Y'}(1))$ and consider the Cartesian diagram

$$\begin{tikzcd} Y \arrow[r, "\iota'"] \arrow[d, "f"'] & Y' \arrow[d, "f'"] \\ X \arrow[r, "\iota"'] & X'. \end{tikzcd} \tag{4-2}$$

Recall from the construction of the action of ξ'^i on $\mathrm{CH}_*(Y, j)$ in [Bloch 1986] that for any irreducible cycle $[V]$ on $Y \times \Delta^j$, the support of $\xi^i \cdot [V] = \xi'^i \cdot [V]$ is $\mathrm{supp}(V \cap \alpha)$, where α is cycle on Y' representing ξ'^i and such that each component of α intersects V properly. This is achieved by using the moving lemma on Y', a smooth scheme. Since ξ' reduces the dimension of a cycle on $Y \times \Delta^j$ by exactly one, we see that $\dim(\xi^i \cdot f^*([W])) = \dim(W) + n - i$ and the support of $f_*(\xi^i \cdot f^*([W]))$ is contained in W, whenever W is an irreducible admissible cycle on $X \times \Delta^j$. We conclude from the definition of the push-forward map that $f_*(\xi^i \cdot f^*([W]))$ must be zero if $n - i > 0$. Since any admissible cycle on $X \times \Delta^j$ is a sum of irreducible admissible cycles, this proves the first case.

We prove the case of $i = n$ by induction on n. If E is of rank one, then f is an isomorphism and $\mathbb{O}_Y(1)$ is trivial and hence $c_0(\mathbb{O}_Y(1)) = 1$. So we assume that $n \geq 1$. We let $\iota : X \hookrightarrow X'$ be a closed embedding into a smooth scheme as in (4-2).

By [Panin 2003, Lemma 3.24], there is a morphism $\phi' : T' \to X'$, which is a composite of projective and affine bundles on X' such that $\phi'^*(E') = F' \oplus L'$, where L' is a line bundle on T'. Moreover, if $\phi : T \to X$ is the restriction of ϕ' on X, then the pull-back map $\phi^* : \mathrm{CH}_*(X, j) \to \mathrm{CH}_*(T, j)$ is a split injection and the same holds for ϕ'^*. Notice here that X is a closed subscheme of X' and hence $\mathrm{CH}_*(X) \xrightarrow{\sim} \mathrm{CH}^X_*(X')$ in the notation of [ibid., Definition 2.1]. We denote the restrictions of F' and L' on X by F and L respectively.

Consider the Cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "\psi"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ T \arrow[r, "\phi"'] & X \end{tikzcd} \tag{4-3}$$

and suppose the given assertion holds for the projective bundle g. We then have

$$\begin{aligned} \phi^*(f_*(\xi^n \cdot f^*(x))) &= g_*(\psi^*(\xi^n \cdot f^*(x))) = g_*((\psi^*(\xi))^n \cdot \psi^* \circ f^*(x)) \\ &= g_*((\psi^*(\xi))^n \cdot g^* \circ \phi^*(x)) = \phi^*(x), \end{aligned}$$

where the first equality holds by Proposition 2.2 and the last equality holds by our assumption. Since ϕ^* is injective, we see that the conclusion holds for f as well. Thus we have reduced the problem to the case when $E' = F' \oplus L'$ and $E = F \oplus L$.

If we set $\widetilde{E} = E \otimes L^{-1}$ and $\widetilde{Y} = \mathbb{P}(\widetilde{E})$, there is a commutative diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[rr, "h"] \arrow[dr, "\tilde{f}"'] & & \mathbb{P}(E) \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

such that h is an isomorphism and $\mathbb{O}_{\widetilde{Y}}(1) = h^*(\mathbb{O}_Y(1)) \otimes \tilde{f}^*(L^{-1})$. Set $\eta = c_1(L)$ in $\mathrm{OPCH}^1(X, 0)$ and $\tilde{\eta} = \tilde{f}^*(\eta)$ and $\tilde{\xi} = c_1(\mathbb{O}_{\widetilde{Y}}(1))$ in $\mathrm{OPCH}^1(\widetilde{Y}, 0)$.

Suppose that our assertion holds for the projective bundle $\tilde{f}$. In this case, we get

$$f_*(\xi^n \cdot f^*(x)) = \tilde{f}_*((\tilde{\eta} + \tilde{\xi})^n \cdot \tilde{f}^*(x)) = \tilde{f}_*((\tilde{\xi})^n \cdot \tilde{f}^*(x)) = x,$$

where the last equality holds by our assertion about $\tilde{f}$. The second equality holds because of the fact that

$$\begin{aligned} \tilde{f}_*\big((\eta + \tilde{\xi})^n \cdot \tilde{f}^*(x)\big) &= \sum_{i=0}^{n} \binom{n}{i} \tilde{f}_*((\tilde{\eta})^i (\tilde{\xi})^{n-i} \cdot \tilde{f}^*(x)) \\ &= \sum_{i=0}^{n} \binom{n}{i} \eta^i \cdot \tilde{f}_*((\tilde{\xi})^{n-i} \cdot \tilde{f}^*(x)) \end{aligned}$$

(see [Bloch 1986, Exercise 5.8]) and that the term $\tilde{f}_*((\tilde{\xi})^{n-i} \cdot \tilde{f}^*(x))$ vanishes for $0 \le n - i < n$ by the first assertion of proposition. Hence, we are further reduced to the case when $E' = F' \oplus \mathbb{O}_{X'}$ and $E = F \oplus \mathbb{O}_X$.

Let $Z = \mathbb{P}(F)$ and let p and q be the closed and the open inclusions of $Z \hookrightarrow Y$ and $F \hookrightarrow Y$, respectively. Let $g : Z \hookrightarrow Y \to X$ be the composite map and set $\zeta = c_1(\mathbb{O}_Z(1))$. We observe that Z is a Cartier divisor on Y such that $\mathbb{O}_Y(1) = \mathscr{L}(Z)$. In particular, the pull-back p^* is defined and we have $\xi \cdot f^*(a) = p_*(p^* \circ f^*(a))$. Since F is of rank n, the desired assertion holds for $g : Z \to X$ by induction. That is, $g_*(\zeta^{n-1} \cdot g^*(x)) = x$. We now have

$$\begin{aligned} f_*(\xi^n \cdot f^*(x)) &= f_*(\xi^{n-1} p_*(p^* \circ f^*(x))) \\ &= f_*(p_*(\zeta^{n-1} \cdot p^* \circ f^*(x))) = g_*(\zeta^{n-1} \cdot g^*(x)) = x. \end{aligned}$$

This proves the desired assertion and the proof of the proposition is complete. $\square$

5. Demazure operators on equivariant higher Chow groups

In this section, we introduce Demazure (divided difference) operators on the equivariant higher Chow groups of schemes. Such operators were constructed by Brion

[1997] on the equivariant Chow groups $\mathrm{CH}^G_*(X, 0)$ and by Holm and Sjamaar [2008] on the equivariant singular cohomology $H^*_G(X)$. We extend these operators to the higher Chow groups and discuss some consequences.

Let G be a connected reductive group with a split maximal torus T of rank n. Let $\Phi = \Phi(G, T) \subset \mathscr{X}(T)$ be the root system of G with respect to T, where $\mathscr{X}(T)$ is the character group of T. For any $\alpha \in \Phi$, let P_α be the minimal parabolic subgroup of G corresponding to α and let $B = TU_\alpha U$ and $B' = TU_{-\alpha}U$ be the opposite Borel subgroups of G containing T in P_α. Let $W_\alpha = \{s_\alpha, s_{-\alpha}\}$ denote the Weyl group of P_α, where s_α is the reflection in $\mathscr{X}(T)$ given by

$$s_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha \quad \text{for } \lambda \in \mathscr{X}(T).$$

Let X be a k-scheme with a free G-action and let B and B' act on $X \times P_\alpha$ by $b \cdot (x, g) = (x, bg)$ and $b' \cdot (x, g) = (b'x, gb'^{-1})$. It is easy to check that the two actions are free and they commute with each other. Hence we get a free action of $B \times B'$ on $X \times G$ by $(b, b') \cdot (x, g) = (b'x, bgb'^{-1})$. One checks that B acts freely on $X' = X \times^{B'} P_\alpha$. P_α acts freely on X' by acting trivially on its X-factor and by left multiplication on the P_α-factor. In particular, we have a B-equivariant map $X' \to X$ given by $[x, g] \mapsto x$, which yields the projection map on quotients $f_1 : X'/B \to X/B \cong X'/P_\alpha$. One also has the P_α-equivariant map $X' \to X$ given by $[x, g] \mapsto [gx]$, which yields the map $f_2 : X'/B \to X/B$ on quotients by the action of B. It is also easy to check that the data above yield the commutative diagram

$$\begin{tikzcd} X'/B \arrow[r, "f_2"] \arrow[d, "f_1"] & X/B \arrow[d, "p_X"] \arrow[l, bend right, dotted, "\sigma_2"'] \\ X/B \arrow[r, "p_X"'] \arrow[u, bend left, dotted, "\sigma_1"] & X/P_\alpha, \end{tikzcd} \tag{5-1}$$

which is Cartesian. The section σ_2 of f_2 is defined by $\sigma_2([x]) = [x, 1]$ and the section σ_1 of f_1 is defined by $\sigma_1([x]) = [x, n_\alpha]$, where n_α is a representative of s_α in $N_{P_\alpha}(T)$. Since f_2 is induced by a P_α-equivariant map, we see that it is W_α-equivariant with respect to the natural action of W_α on X'/B and X/B. In particular, we get

$$\begin{aligned} &f_2^* \circ s_\alpha = s_\alpha \circ f_2^*, \\ &f_2^*(ax) = af_2^*(x) \quad \text{for all } a \in S \text{ and } x \in \mathrm{CH}_*(X/B). \end{aligned} \tag{5-2}$$

Note that the all the maps (except the sections) in (5-1) are $P_\alpha/B \cong \mathbb{P}^1$-bundles and hence they are all smooth and projective. Let $\mathscr{O}_{X/B}(1)$ denote the universal quotient line bundle on X/B relative to p_X.

Let D denote the image of σ_1 and let $\mathscr{L}(D)$ denote the associated line bundle on X'/B. Let $\mathscr{L}_\lambda$ and $\mathscr{L}'_\lambda$ denote the line bundles $X \times^B L_\lambda$ and $X' \times^B L_\lambda$ on X/B and X'/B respectively, where L_λ is the B-equivariant line bundle on $\mathrm{Spec}(k)$ corresponding to the character λ of T. It follows from [Demazure 1974, Lemme 2, Proposition 2] that

$$\begin{aligned} \mathscr{L}'_\alpha &= f_1^*(\mathscr{L}_\alpha) \otimes (\mathscr{L}(D))^{\otimes 2}, \\ s_\alpha(\mathscr{L}(D)) &= (\mathscr{L}(D) \otimes f_1^*(\mathscr{L}_\alpha))^{\otimes(-1)}. \end{aligned} \tag{5-3}$$

5a. ***Demazure operators on*** $\mathbf{CH}_*(X)$**.** Let G be a connected reductive group with split maximal torus T as above and let X be a G-scheme of dimension d. Let $j \le d$ be an integer and let (V, U) be a good pair for the G-action corresponding to $d - j$. The smooth and projective morphism $p_{X \times U} : X_B \to X_{P_\alpha}$ yields the maps

$$p^*_{X\times U}: \mathrm{CH}^{P_\alpha}_j(X, i) \to \mathrm{CH}^B_j(X, i) \text{ and } p_{X\times U *}: \mathrm{CH}^B_j(X, i) \to \mathrm{CH}^{P_\alpha}_{j+1}(X, i). \tag{5-4}$$

For the rest of this text, the ring S will denote the equivariant Chow ring

$$S(T) = \mathrm{CH}^*_T(k, 0).$$

Lemma 5.1. *The maps $p^*_{X\times U}$ and $p_{X\times U *}$ do not depend on the choice of the good pair (V, U).*

Proof. We prove the lemma for the push-forward map and a very similar proof works also for the pull-back map; see [Edidin and Graham 1998, Section 1].

Let g and b denote the dimensions of P_α and B, respectively. Let (V, U) and (V', U') be good pairs of dimensions l and l', respectively, corresponding to $d - j$. We let $\mathscr{V} = V \oplus V'$ and $\mathscr{U} = (U \oplus V') \cup (V \oplus U')$. Let G act diagonally on $\mathscr{V}$. Then it is easy to see that the dimension of the complement of the open subset $X \times^B (U \oplus V')$ in $X \times^B \mathscr{U}$ is sufficiently smaller than $l + l' - b + j$. Similarly, the dimension of the complement of the open subset $X \times^{P_\alpha} (U \oplus V')$ in $X \times^{p_\alpha} \mathscr{U}$ is sufficiently smaller than $l + l' - g + j \le l + l' - b + j$. It follows from the localization sequence for the higher Chow groups and Lemma 5.2 that the there is a commutative diagram

$$\begin{tikzcd} \mathrm{CH}_{l+l'+j-b}(X \times^B \mathscr{U}, i) \arrow[r] \arrow[d] & \mathrm{CH}_{l+l'+j-b}(X \times^B (U \oplus V'), i) \arrow[d] \\ \mathrm{CH}_{l+l'+j-b}(X \times^{P_\alpha} \mathscr{U}, i) \arrow[r] & \mathrm{CH}_{l+l'+j-b}(X \times^{P_\alpha} (U \oplus V'), i) \end{tikzcd}$$

where the vertical maps are the push-forward maps and the horizontal maps are isomorphisms.

On the other hand, the maps

$$X \times^G (U \oplus V') \to X \times^G U \quad \text{and} \quad X \times^B (U \oplus V') \to X \times^B U$$

are vector bundles of rank l' and hence we get another commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_{l+j-b}(X \times^B U, i) \arrow[r] \arrow[d] & \mathrm{CH}_{l+l'+j-b}(X \times^B (U \oplus V'), i) \arrow[d] \\
\mathrm{CH}_{l+j-b}(X \times^{P_\alpha} U, i) \arrow[r] & \mathrm{CH}_{l+l'+j-b}(X \times^{P_\alpha} (U \oplus V'), i),
\end{tikzcd}$$

where the vertical arrows are the push-forward maps and the horizontal arrows are isomorphisms by the homotopy invariance.

Combining the two isomorphisms above, we get the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}_{j+l+l'-b}(X \times^B \mathcal{U}, i) \arrow[r] \arrow[d] & \mathrm{CH}_{l+j-b}(X \times^B U, i) \arrow[d] \\
\mathrm{CH}_{j+1+l+l'-g}(X \times^{P_\alpha} \mathcal{U}, i) \arrow[r] & \mathrm{CH}_{l+l'+j+1-g}(X \times^{P_\alpha} (U \oplus V'), i),
\end{tikzcd}$$

where the horizontal maps are isomorphisms. By repeating the same argument with U', we get the diagram above with U replaced by U' and V' replaced by V on the right column. This proves the lemma. $\square$

Lemma 5.2. *Let $p : X \to Y$ be a morphism in $\mathcal{V}_{P_\alpha}$ such that P_α acts freely on X and Y. Then the diagram of quotients*

$$\begin{tikzcd}
X/B \arrow[r] \arrow[d] & Y/B \arrow[d] \\
X/P_\alpha \arrow[r] & Y/P_\alpha
\end{tikzcd}$$

is Cartesian such that the vertical maps are smooth and projective.

Proof. This is an easy exercise. $\square$

Proposition 5.3. *For any $X \in \mathcal{V}_G$, one has the restriction and the push-forward maps*

$$r^{P_\alpha}_{B,X} : \mathrm{CH}^{P_\alpha}_*(X, i) \to \mathrm{CH}^B_*(X, i) \quad \textit{and} \quad p^{P_\alpha}_{B,X} : \mathrm{CH}^B_*(X, i) \to \mathrm{CH}^{P_\alpha}_{*+1}(X, i).$$

These maps are contravariant with respect to the flat maps and covariant with respect to the proper morphisms of schemes in $\mathcal{V}_G$.

Proof. Let $j \le d$ be an integer and let (V, U) be a good pair for the G-action corresponding to $d - j$. We define $r^{P_\alpha}_{B,X}$ and $p^{P_\alpha}_{B,X}$ to be $p^*_{X\times U}$ and $p_{X\times U *}$ respectively. It follows from Lemma 5.1 that these maps are well-defined.

The functoriality properties of $r^{P_\alpha}_{B,X}$ is already known; see [Edidin and Graham 1998, Section 1]. To prove these properties for $p^{P_\alpha}_{B,X}$, it suffices to prove the same for $p_{X\times U*}$. But this follows easily from Lemma 5.2 and the similar properties of the ordinary higher Chow groups. $\square$

The following result generalizes [Brion 1997, Theorem 6.3] to equivariant higher Chow groups.

Theorem 5.4. *Let α be a simple root of G. For any $X \in \mathcal{V}_G$ and $i \geq 0$, there is a unique operator δ_α^X on $\mathrm{CH}_*^T(X, i)$ such that for all $u \in S$ and $v \in \mathrm{CH}_*^T(X, i)$, we have*

(1) $\alpha\delta_\alpha^X(v) = v - s_\alpha(v)$ *and*

(2) $\delta_\alpha^X(uv) = u\delta_\alpha^X(v) + \delta_\alpha^k(u)s_\alpha(v)$ *if X is smooth.*

Moreover, δ_α^X commutes with the G-equivariant flat pull-back and proper push-forward maps between T-equivariant higher Chow groups.

Proof. Let B and B' be the opposite Borel subgroups of P_α containing T. Using Proposition 3.3, we can replace T by B to define δ_α^X. We let

$$\delta_\alpha^X := r_{B,X}^{P_\alpha} \circ p_{B,X}^{P_\alpha} : \mathrm{CH}_j^B(X, i) \to \mathrm{CH}_{j+1}^B(X, i). \tag{5-5}$$

The co- and contravariant functoriality of δ_α^X follows from Proposition 5.3. The uniqueness of δ_α^X follows from [Brion 1997, Theorem 6.3] since this definition of δ_α^X coincides with the one defined for $\mathrm{CH}_*^B(X, 0)$ by Brion. We only need to show the first and the second assertions.

Let $j \leq \dim(X)$ and (V, U) be a good pair the G-action corresponding to $d - j$. There is a G-equivariant projection $X \times U \to X$ such that

$$\mathrm{CH}_j^G(X, i) \xrightarrow{\sim} \mathrm{CH}_j^G(X \times U, i)$$

and δ_α^X on $\mathrm{CH}_j^G(X, i)$ coincides with the operator $\delta_\alpha^{X\times U}$ on $\mathrm{CH}_j^G(X \times U, i)$ by its construction. Hence we can assume that G acts freely on X.

We now consider the diagram (5-1). Since f_2 is a $\mathbb{P}^1$-bundle, the map f_2^* is split injective by Proposition 4.6. Hence it suffices to show that the two assertions of the theorem hold after applying f_2^*.

On the other hand, $f_2^*(s_\alpha(v)) = s_\alpha(f_2^*(v))$ and $f_2^*(\alpha\delta_\alpha^X(v)) = \alpha f_2^*(\delta_\alpha^X(v))$ by (5-2). Since f_2 is induced by a G-equivariant map $X' \to X$, we also have $f_2^*(\delta_\alpha^X(v)) = \delta_\alpha^{X'}(f_2^*(v))$ by the functoriality of δ_α. Thus we need to show for $u \in S$ and $v' \in \mathrm{CH}_*(X'/B, i)$ that

$$\alpha\delta_\alpha^{X'}(v') = v' - s_\alpha(v') \quad \text{and} \quad \delta_\alpha^{X'}(uv') = u\delta_\alpha^{X'}(v') + \delta_\alpha^k(u)s_\alpha(v') \quad \text{if } X \in \mathcal{V}_G. \tag{5-6}$$

Let $c : \mathrm{CH}_T^*(k, 0) \cong \mathrm{OPCH}_T^*(k, 0) \to \mathrm{OPCH}^*(X/B, 0)$ be the ring homomorphism on the operational Chow groups induced by the map on the Picard groups $L_\alpha \mapsto \mathcal{L}_\alpha$. We denote the corresponding map $\mathrm{OPCH}_T^*(k, 0) \to \mathrm{OPCH}^*(X'/B, 0)$ by c'. We set $\xi = c_1(\mathcal{L}(D))$ and $\zeta = c_1(\mathcal{O}_{X'/B}(1))$, where $\mathcal{O}_{X'/B}(1)$ is the universal quotient line bundle associated to the $\mathbb{P}^1$-bundle f_1. We shall write δ_α^k simply as δ_α in what follows.

Since $\mathcal{L}(D)$ and $\mathbb{O}_{X'/B}(1)$ have same degree on every fiber of f_1, there is a line bundle $\mathcal{M}$ on X/B such that $\mathbb{O}_{X'/B}(1) \cong \mathcal{L}(D) \otimes f_1^*(\mathcal{M})$. In particular, there exists $\eta \in \mathrm{OPCH}^1(X/B, 0)$ such that $\zeta = \xi + f_1^*(\eta)$. Since f_1^* commutes with the action of $\mathrm{OPCH}^*(X/B, 0)$ on the higher Chow groups, it follows from Proposition 4.6 that we can write $v' \in \mathrm{CH}_*(X'/B, i)$ as

$$v' = f_1^*(a) + \xi f_1^*(b), \quad \text{with } a, b \in \mathrm{CH}_*(X/B, i). \tag{5-7}$$

Furthermore, it also follows that for any $b \in \mathrm{CH}_*(X/B, i)$,

$$f_{1*}(\xi f_1^*(b)) = f_{1*}(\zeta f_1^*(b)) - f_{1*}(f_1^*(\eta b)) = b + 0 = b. \tag{5-8}$$

Since s_α keeps the elements of the form $f^*(a)$ invariant, we get

$$\begin{aligned} s_\alpha(v') &= f_1^*(a) + s_\alpha(\xi) f_1^*(b) \\ &= f_1^*(a) - [\xi + f_1^*(c(\alpha))] f_1^*(b), \end{aligned} \tag{5-9}$$

where the second equality follows from (5-3).

On the other hand, we have seen in (5-1) that f_1 is same as the quotient map $X'/B \to X'/P_\alpha$ and hence $\delta_\alpha^{X'} = f_1^* f_{1*}$. This yields

$$\begin{aligned} v' - c'(\alpha)\delta_\alpha^{X'}(v') &= f_1^*(a) + \xi f_1^*(b) - c'(\alpha) f_1^*[f_{1*} f_1^*(a) + f_{1*}(\xi f_1^*(b))] \\ &=^\dagger f_1^*(a) + \xi f_1^*(b) - c'(\alpha) f_1^*(b) \\ &=^\ddagger f_1^*(a) + \xi f_1^*(b) - [f_1^*(c(\alpha)) + 2\xi] f_1^*(b) \\ &= f_1^*(a) - [\xi + f_1^*(c(\alpha))] f_1^*(b), \end{aligned} \tag{5-10}$$

where $=^\dagger$ follows from Proposition 4.6 and (5-8), and $=^\ddagger$ follows from (5-3). The first equality of (5-6) follows at once by comparing (5-9) and (5-10).

To prove the second equality of (5-6) for $u \in S$ and $v' \in \mathrm{CH}_*(X'/B, i)$ with X smooth, we can assume using (5-7) that v' is either $f_1^*(a)$ or $\xi f_1^*(a)$. We now have

$$\begin{aligned} \delta_\alpha^X(u f_1^*(a)) &= f_1^* \circ f_{1*}(u f_1^*(a)) =^1 f_1^*(f_{1*}(u) \cdot a) = f_1^* f_{1*}(u) \cdot f_1^*(a) \\ &=^2 \delta_\alpha^k(u) \cdot f_1^*(a) = \delta_\alpha^k(u) \cdot s_\alpha(f_1^*(a)) + u \delta_\alpha^X(f_1^*(a)). \end{aligned}$$

The equality $=^1$ holds by the projection formula for smooth schemes (see [Bloch 1986, Exercise 5.8]) and $=^2$ holds by [Brion 1997, Theorem 6.3]. The last equality holds because $f_1^*(a)$ is invariant under s_α and $f_1^* f_{1*}(a)$ vanishes, again by the projection formula. The required formula for $v' = \xi f_1^*(a)$ is proved exactly in the similar way using the observation that $f_{1*}(\xi) = 1$ and that the equality $=^2$ holds even if we replace u by ξu. $\square$

Proposition 5.5. *For $X \in \mathcal{V}_G$, let $r^G_{T,X} : \mathrm{CH}^G_*(X, i) \to \mathrm{CH}^T_*(X, i)$ be the restriction map.*

(1) $\delta_\alpha \circ r^G_{T,X} = 0$.

(2) *If X is smooth, then δ_α is $\mathrm{CH}^*_G(X)$-linear.*

(3) $\delta^2_\alpha = 0$.

(4) $s_\alpha \delta_\alpha = \delta_\alpha = -\delta_{-\alpha},\ \delta_\alpha s_\alpha = -\delta_\alpha$.

Proof. Since $r^G_{T,X} = r^{P_\alpha}_{T,X} \circ r^G_{P_\alpha,X}$, we can replace G by P_α. It suffices then to show that

$$p^{P_\alpha}_{B,X} \circ r^{P_\alpha}_{B,X} = 0.$$

But this follows immediately from Proposition 4.6. The second point follows from the observation that f_1^* and f_{1*} in (5-1) are $\mathrm{CH}^*(X'/G)$-linear. The third point follows directly from the first and the fourth point is an immediate consequence of the other assertions of the proposition. □

5b. *Ring of Demazure operators.* Let $\{\alpha_1, \dots, \alpha_m\}$ be the set of all simple roots of G. For any sequence $I = \{i_1, \dots, i_l\}$ of integers in the interval $[1, m]$, we define the operator δ^X_I on $\mathrm{CH}^T_*(X, i)$ by

$$\delta^X_I = \delta_{\alpha_{i_l}} \circ \cdots \circ \delta_{\alpha_{i_1}}. \tag{5-11}$$

Following the notation of [Brion 1997, §6.4], we let $\mathbf{D}$ denote the subring of $\mathrm{End}_{\mathbb{Z}}(S)$ generated by the elements δ^k_α and the endomorphisms given by the multiplication in S. It is clear from the definition of $\mathbf{D}$ and Theorem 5.4 that $\mathbf{D}$ contains the twisted group algebra $S[W]$ and there are inclusions of rings $S \subsetneq S[W] \subsetneq \mathbf{D}$. It is known that $\mathbf{D}$ is a free S-module with basis $\{\partial_w\}_{w \in W}$, where ∂_w is same as δ^k_I above whenever $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_l}}$. As an immediate consequence of Theorem 5.4, Proposition 5.5 and (5-11), we get:

Corollary 5.6. *For any $X \in \mathcal{V}_G$ and $i \geq 0$, there is a unique $\mathbf{D}$-module structure on $\mathrm{CH}^T_*(X, i)$, which extends the action of $S[W]$. Moreover, the flat pull-back and proper push-forward maps between the T-equivariant higher Chow groups are $\mathbf{D}$-linear. For $X \in \mathcal{V}^S_G$, the $\mathbf{D}$-module structure on $\mathrm{CH}^*_T(X)$ commutes with its $\mathrm{CH}^*_G(X)$-module structure.*

Let $I(\mathbf{D})$ be the subset of $\mathbf{D}$ consisting of those operators δ such that $\delta(1) = 0$. It is easy to check that $I(\mathbf{D})$ is a left ideal of $\mathbf{D}$ generated by $\{\partial_w\}_{w \neq 1}$. For any $X \in \mathcal{V}_G$, let

$$(\mathrm{CH}^T_*(X, i))^{I(\mathbf{D})} = \big\{x \in \mathrm{CH}^T_*(X, i) \bigm| \delta(x) = 0\ \forall \delta \in I(\mathbf{D})\big\}. \tag{5-12}$$

Since the Weyl group is generated by simple reflections, it follows from Theorem 5.4 and Proposition 5.5 that

$$r^G_{T,X}(\mathrm{CH}^G_*(X,i)) \subseteq (\mathrm{CH}^T_*(X,i))^{I(D)} \subseteq (\mathrm{CH}^T_*(X,i))^W. \tag{5-13}$$

Recall that the torsion index t_G of the group G is the order of the cokernel of the map $S_N \to \mathrm{CH}^N(G/B)$, where $N = \dim(G/B) = |\Phi^+|$. Let R denote the localized ring $\mathbb{Z}[t_G^{-1}]$.

Theorem 5.7. *Let $X \in \mathcal{V}^S_G$ and $i \geq 0$ be such that $\mathrm{CH}^*_T(X,i)$ is torsion-free. Then the map $\mathrm{CH}^G_*(X,i) \to (\mathrm{CH}^T_*(X,i))^W$ is an isomorphism over R.*

Proof. Let B be a Borel subgroup of G containing T. Let (V,U) be a good pair for the G-action and consider the Cartesian diagram

$$\begin{tikzcd} X_B \arrow[r, "q_B"] \arrow[d, "p_X"'] & U/B \arrow[d, "p"] \\ X_G \arrow[r, "q_G"'] & U/G. \end{tikzcd} \tag{5-14}$$

By the definition of t_G, it follows that there is $a \in \mathrm{CH}^N(U/B, 0)$ such that $p_*(a) = t_G \in \mathrm{CH}^0(U/G, 0)$. Using the projection formula, we get for any $x \in \mathrm{CH}^*(X_G, i)$,

$$p_{X*}(q_B^*(a)p_X^*(x)) = p_{X*}(q_B^*(a))x = q_G^*(p_*(a))x = t_G x.$$

In particular, $r^G_{T,X}$ is split injective over R.

To show the surjectivity, we see from the above that for any $x \in \mathrm{CH}^*(X_G, i)$,

$$p_X^* f p_X^*(x) = p_X^*(t_G x) = t_G p_X^*(x),$$

where $f(y) = p_{X*}(ay)$. This in particular implies that $p_X^* f(y) = t_G y$ for all y in the image of p_X^*. It follows from Corollary 8.7 that

$$p_X^* f(y) = t_G y \quad \text{for all } y \in (\mathrm{CH}^*(X_B, i)_{\mathbb{Q}})^W.$$

Since $\mathrm{CH}^*_T(X,i)$ is torsion-free, we must have

$$p_X^* f(y) = t_G y \quad \text{for all } y \in (\mathrm{CH}^*(X_B, i))^W.$$

Hence the map $r^G_{T,X}$ is surjective onto the W-invariants over R. □

Remark 5.8. The proof of the theorem above in fact shows that the map $r^G_{T,X}$ is split injective over R for any $X \in \mathcal{V}^S_G$.

Corollary 5.9. *Let X be a smooth projective scheme with a G-action such that the fixed point locus X^T for the T-action is isolated. Then the map*

$$\mathrm{CH}^G_*(X, 0) \to (\mathrm{CH}^T_*(X, 0))^W$$

is an isomorphism over R.

Proof. This is an immediate consequence of [Krishna 2009a, Theorem 4.2] and Theorem 5.7. □

6. The Leray–Hirsch Theorem

In algebraic topology, the Leray–Hirsch theorem is a very important tool for describing the cohomology of the total space of a fiber bundle. Since the arguments in this theorem are mostly topological, one cannot always expect such results for the cohomology theories of algebraic varieties. A version of the Leray–Hirsch theorem was proven for the Chow groups of the total space of a Zariski-locally trivial fibration in [Ellingsrud and Strømme 1989, Lemma 2.8; Edidin and Graham 1997, Lemma 6]. In this section, we prove the general form of the Leray–Hirsch theorem for the higher Chow groups of schemes. We shall give several important applications of this theorem in the next few sections.

6a. ***A Künneth formula.*** In [Fulton 1984, Example 1.9.1], a k-scheme L is called *cellular* if it has a filtration $\varnothing = L_{n+1} \subsetneq L_n \subsetneq \cdots \subsetneq L_1 \subsetneq L_0 = L$ by closed subschemes such that each $L_i \setminus L_{i+1}$ is an affine space $\mathbb{A}^{r_i}_k$.[1] It follows from the Bruhat decomposition that schemes of the type G/B are cellular, where B is a Borel subgroup of a split reductive group G.

Lemma 6.1. *Let L be a cellular scheme with the cellular decomposition*

$$\varnothing = L_{n+1} \subsetneq L_n \subsetneq \cdots \subsetneq L_1 \subsetneq L_0 = L$$

and let $U_i = L \setminus L_i$ for $0 \le i \le n+1$. Then for any $0 \le i \le n$ and $p \ge 0$, the sequence

$$0 \to \mathrm{CH}_*(U_{i+1} \setminus U_i, p) \to \mathrm{CH}_*(U_{i+1}, p) \to \mathrm{CH}_*(U_i, p) \to 0$$

is exact.

Proof. The proof is very similar to the arguments of [Kahn 1999, Lemma 3.3] using an induction on the number of cells. □

Lemma 6.2. *Let L be a cellular scheme and let X be a any k-scheme. Then the exterior product map*

$$\mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathrm{CH}_*(L, 0) \to \mathrm{CH}_*(X \times L) \tag{6-1}$$

[1] Some authors allow $L_i \setminus L_{i+1}$ to be a disjoint union of affine spaces over k. But both definitions are equivalent.

is an isomorphism. In particular, the natural map $\mathrm{CH}^*(k)\otimes_{\mathbb{Z}}\mathrm{CH}_*(L,0)\to\mathrm{CH}_*(L)$ *is an isomorphism.*

Proof. Consider the cellular decomposition of L as in Lemma 6.1. Then each $U_i = L\setminus L_i$ is also a cellular scheme. It suffices to show by induction that (6-1) holds when L is any of these U_i. There is nothing to prove for $i=0$ and the case $i=1$ follows by the homotopy invariance since U_1 is an affine space. In general, we have the short exact sequence

$$0\to\mathrm{CH}_*(U_{i+1}\setminus U_i,0)\to\mathrm{CH}_*(U_{i+1},0)\to\mathrm{CH}_*(U_i,0)\to 0 \tag{6-2}$$

by applying Lemma 6.1 with $p=0$. Since each $U_{i+1}\setminus U_i$ is an affine space, it also follows from Lemma 6.1 and by induction on the number of affine cells that each $\mathrm{CH}_*(U_i,0)$ is a free abelian group of finite rank. Tensoring this with $\mathrm{CH}_*(X)$ over $\mathrm{CH}^*(k,0)=\mathbb{Z}$, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & {} \arrow[d] \\
\mathrm{CH}_*(X)\otimes\mathrm{CH}_*(U_{i+1}\setminus U_i,0) \arrow[r] \arrow[d] & \mathrm{CH}_*(X\times(U_{i+1}\setminus U_i)) \arrow[d, "i_*"] \\
\mathrm{CH}_*(X)\otimes\mathrm{CH}_*(U_{i+1},0) \arrow[r] \arrow[d] & \mathrm{CH}_*(X\times U_{i+1}) \arrow[d, "j^*"] \\
\mathrm{CH}_*(X)\otimes\mathrm{CH}_*(U_i,0) \arrow[r] \arrow[d] & \mathrm{CH}_*(X\times U_i), \arrow[d] \\
0 & {}
\end{tikzcd}$$

where the left column is exact by the freeness of each $\mathrm{CH}_*(U_i,0)$ and the right column is the localization exact sequence. The top horizontal arcolumn is an isomorphism by the homotopy invariance and the bottom horizontal arcolumn is an isomorphism by the induction. In particular, j^* is surjective in all indices. We conclude that i_* is injective in all indices and the middle horizontal arcolumn is also an isomorphism. □

6b. ***Leray–Hirsch with integral coefficients.*** Let F be a cellular scheme over k. For any field extension $k\hookrightarrow l$, the scheme F_l is also cellular, for which the cellular decomposition and the affine cells are the base extensions of the cellular decomposition and affine cells of F. It follows from Lemma 6.1 that the map $\mathrm{CH}_*(F,0)\to\mathrm{CH}_*(F_l,0)$ is an isomorphism. The following is the integral version of the Leray–Hirsch theorem for the Zariski-locally trivial fibrations.

Theorem 6.3. *Let B be a smooth k-scheme and let $F \xrightarrow{i} E \xrightarrow{p} B$ be a Zariski-locally trivial fibration such that the fiber F is a smooth cellular scheme. Assume that there are elements $\{e_1, \dots, e_r\}$ in $\mathrm{CH}^*(E, 0)$ such that*

$$\{f_1 = i^*(e_1), \dots, f_r = i^*(e_r)\}$$

forms a $\mathbb{Z}$-basis of $\mathrm{CH}^*(F_y, 0)$ for each fiber F_y of the fibration. Then the map

$$\Phi : \mathrm{CH}^*(F, 0) \otimes_{\mathbb{Z}} \mathrm{CH}^*(B) \to \mathrm{CH}^*(E), \qquad \sum_{1 \le i \le r} f_i \otimes b_i \mapsto \sum_{1 \le i \le r} p^*(b_i) e_i$$

is an isomorphism. In particular, $\mathrm{CH}^*(E)$ is a free $\mathrm{CH}^*(B)$-module with basis $\{e_1, \dots, e_r\}$.

Proof. Since k is perfect, we can find a filtration

$$\varnothing = B_{n+1} \subsetneq B_n \subsetneq \cdots \subsetneq B_1 \subsetneq B_0 = B$$

of B by closed subschemes such that for each $0 \le i \le n$, the scheme $B_i \setminus B_{i+1}$ is smooth and the given fibration is trivial over it. We set $U_i = B \setminus B_i$ and $V_i = U_i \setminus U_{i-1} = B_{i-1} \setminus B_i$. Observe then that each of the U_i and V_i is smooth. Set $E_i = p^{-1}(U_i)$ and $W_i = p^{-1}(V_i)$. We prove by induction on i that the map $\mathrm{CH}^*(F, 0) \otimes_{\mathbb{Z}} \mathrm{CH}^*(U_i) \to \mathrm{CH}^*(E_i)$ is an isomorphism, which will prove the theorem. Since $U_0 = \varnothing$ and $E_1 = U_1 \times F$, the desired isomorphism for $i = 1$ follows from Lemma 6.2. We now consider the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^*(U_i) \otimes \mathrm{CH}^*(F,0) \arrow[r] \arrow[d] & \mathrm{CH}^*(E_i) \arrow[d] \\
\mathrm{CH}^*(V_{i+1}) \otimes \mathrm{CH}^*(F,0) \arrow[r] \arrow[d] & \mathrm{CH}^*(W_{i+1}) \arrow[d] \\
\mathrm{CH}^*(U_{i+1}) \otimes \mathrm{CH}^*(F,0) \arrow[r] \arrow[d] & \mathrm{CH}^*(E_{i+1}) \arrow[d] \\
\mathrm{CH}^*(U_i) \otimes \mathrm{CH}^*(F,0) \arrow[r] \arrow[d] & \mathrm{CH}^*(E_i) \arrow[d] \\
\mathrm{CH}^*(V_{i+1}) \otimes \mathrm{CH}^*(F,0) \arrow[r] & \mathrm{CH}^*(W_{i+1}).
\end{tikzcd} \tag{6-3}$$

The left column is obtained by tensoring the long exact localization sequence for higher Chow groups with $\mathrm{CH}^*(F, 0)$ over $\mathbb{Z}$, and the right column is just the localization exact sequence. Since $\mathrm{CH}^*(F, 0)$ is a free abelian group, the left column is also exact.

It is easily checked that the second and the third squares commute using the commutativity property of the push-forward and pull-back maps of higher Chow groups in a fiber diagram. We show that the other squares also commute. It is enough to show that the first square commutes as the fourth one is same as the first.

Let δ denote the connecting homomorphism in a long exact localization sequence for the higher Chow groups.

Before we show the required commutativity, let us recall that if (X, Y) is pair of k-schemes where $i : Y \hookrightarrow X$ is a closed subscheme with complement $j : U \hookrightarrow X$, then the localization exact sequence is the long exact homology sequence associated to the short exact sequence of cycle complexes

$$0 \to \mathcal{Z}_n(Y, \bullet) \xrightarrow{i_*} \mathcal{Z}_n(X, \bullet) \xrightarrow{j^*} \frac{\mathcal{Z}_n(X, \bullet)}{\mathcal{Z}_n(Y, \bullet)} \to 0,$$

where the natural map

$$\frac{\mathcal{Z}_n(X, \bullet)}{\mathcal{Z}_n(Y, \bullet)} \to \mathcal{Z}_n(U, \bullet)$$

is a quasi-isomorphism. So we identify the last term with $\mathcal{Z}_n(U, \bullet)$. The formalism of the homological algebra now shows that the connecting homomorphism $\delta : \mathrm{CH}_n(U, i) \to \mathrm{CH}_n(Y, i-1)$ is obtained as one obtains the connecting homomorphism in the snake lemma. In particular, this is same as the differential map $\partial : \mathcal{Z}_n(X, i) \to \mathcal{Z}_n(X, i-1)$, evaluated on the homology groups. The Leibniz rule for this differential now implies that the connecting homomorphism δ also satisfies the Leibniz rule; see [Panin 2003, §2.4].

If we now start with an element $b \otimes i^*(e_j) \in \mathrm{CH}^*(U_i) \otimes \mathrm{CH}^*(F, 0)$ and map this vertically, we get $\delta b \otimes i^*(e_j)$, which maps horizontally down to $p^*(\delta b) \cdot e_j$. On the other hand, if we first map horizontally, we get $p^*(b) \cdot e_j$ which maps vertically to $\delta(p^*(b) \cdot e_j)$. Using the Leibniz rule above, this last term is same as $\delta p^*(b) \cdot e_j = p^*(\delta b) \cdot e_j$ since $\delta e_j = 0$. We have shown that the diagram above commutes.

The first and the fourth horizontal arrows in (6-3) are isomorphisms by induction. The second and the fifth horizontal arrows are isomorphisms by Lemma 6.2. Hence the middle horizontal arrow is also an isomorphism by the 5-lemma. □

6c. ***Leray–Hirsch with rational coefficients.*** We need the following step to prove the rational version of the Leray–Hirsch theorem for the étale locally trivial fibrations of smooth schemes.

Let F be a smooth cellular scheme over k. We have seen before that for any field extension $k \hookrightarrow l$, the natural map $\mathrm{CH}^*(F, 0) \to \mathrm{CH}^*(F_l, 0)$ is an isomorphism. Moreover, each of these is a free abelian group with the basis vectors given by the closures of the affine cells in the cellular decomposition. We fix this basis $\{f_1, \dots, f_r\}$ of $\mathrm{CH}^*(F_l, 0)$ in what follows. For a complete flag variety G/B, this set is same as the set of Schubert cycles $\{\zeta_w\}_{w \in W}$.

Lemma 6.4. *Let B be a smooth k-scheme and let $F \xrightarrow{i} E \xrightarrow{p} B$ be an étale locally trivial fibration such that the fiber F is a smooth cellular scheme. Assume that this fibration becomes trivial after a finite étale cover of B of degree d. Assume furthermore that there are elements $\{e_1, \dots, e_r\}$ in $\mathrm{CH}^*(E, 0)$ such that $\{f_1 = i^*(e_1), \dots, f_r = i^*(e_r)\}$ is the basis of $\mathrm{CH}^*(F_y, 0)$ for each geometric fiber F_y of the fibration. Then the map*

$$\Phi : \mathrm{CH}^*(F, 0) \otimes_{\mathbb{Z}} \mathrm{CH}^*(B) \to \mathrm{CH}^*(E), \qquad \sum_{1 \le i \le r} f_i \otimes b_i \mapsto \sum_{1 \le i \le r} p^*(b_i) e_i$$

is an isomorphism over $\mathbb{Z}[d^{-1}]$. In particular, $\mathrm{CH}^*(E)[d^{-1}]$ is a free $\mathrm{CH}^*(B)[d^{-1}]$-module with basis $\{e_1, \dots, e_r\}$.

Proof. Let $B' \xrightarrow{q} B$ be a finite étale cover such that $E' = E \times_B B' \xrightarrow{p'} B'$ is a trivial fibration and let $q' : E' \to E$ be the other projection. It follows from our assumption and the isomorphism of $\mathrm{CH}^*(F, 0)$ under the field extensions that the set $\{e'_i = q'^*(e_i)\}$ restricts to the basis $\{f_i\}$ of $\mathrm{CH}^*(F_y, 0)$ for every fiber F_y of the fibration p'.

Setting $\Phi'(b' \otimes f_j) = p'^*(b')e'_j$ and using the fact that $q'^* \circ p^* = p'^* \circ q^*$, $p^* \circ q_* = q'_* \circ p'^*$ and $q_* \circ q^* = d = q'_* \circ q'^*$ (see [Bloch 1986, Exercise 5.8(i)]), one checks that the diagram

$$\begin{array}{ccccc}
\mathrm{CH}^*(B) & & \mathrm{CH}^*(B') & & \mathrm{CH}^*(B) \\
\otimes & \xrightarrow{q^* \otimes 1} & \otimes & \xrightarrow{q_* \otimes 1} & \otimes \\
\mathrm{CH}^*(F, 0) & & \mathrm{CH}^*(F, 0) & & \mathrm{CH}^*(F, 0) \\
\Phi \big\downarrow & & \Phi' \big\downarrow & & \big\downarrow \Phi \\
\mathrm{CH}^*(E) & \xrightarrow[q'^*]{} & \mathrm{CH}^*(E') & \xrightarrow[q'_*]{} & \mathrm{CH}^*(E)
\end{array}$$

commutes. The middle vertical arrow is an isomorphism by Lemma 6.2. A diagram chase shows that Φ is an isomorphism over $\mathbb{Z}[d^{-1}]$. □

Theorem 6.5. *Let B be a smooth k-scheme and let $F \xrightarrow{i} E \xrightarrow{p} B$ be an étale locally trivial fibration such that the fiber F is a smooth cellular scheme. Assume that there are elements $\{e_1, \dots, e_r\}$ in $\mathrm{CH}^*(E, 0)$ such that $\{f_1 = i^*(e_1), \dots, f_r = i^*(e_r)\}$ is the basis of $\mathrm{CH}^*(F_y, 0)$ for each geometric fiber F_y of the fibration. Then the map*

$$\Phi : \mathrm{CH}^*(F, 0) \otimes_{\mathbb{Q}} \mathrm{CH}^*(B) \to \mathrm{CH}^*(E), \qquad \sum_{1 \le i \le r} f_i \otimes b_i \mapsto \sum_{1 \le i \le r} p^*(b_i) e_i$$

is an isomorphism over the rationals. In particular, $\mathrm{CH}^*(E)$ is a free $\mathrm{CH}^*(B)$-module with basis $\{e_1, \dots, e_r\}$ over the rationals.

Proof. We assume all abelian groups to be tensored with $\mathbb{Q}$ in this proof. Since k is perfect and since every étale cover is generically finite, we can find a filtration

$$\varnothing = B_{n+1} \subsetneq B_n \subsetneq \cdots \subsetneq B_1 \subsetneq B_0 = B$$

of B by closed subschemes such that for each $0 \le i \le n$, the scheme $V_i = B_{i-1} \setminus B_i$ is smooth and there is a finite étale cover $V_i' \to V_i$ such that the given fibration is trivial over V_i'. We set $U_i = B \setminus B_i$ as before, which implies that $V_i = U_i \setminus U_{i-1}$. Observe then that each of the U_i and V_i is smooth. Set $E_i = p^{-1}(U_i)$ and $W_i = p^{-1}(V_i)$. We prove by induction on i that the map

$$\mathrm{CH}^*(F, 0) \otimes_{\mathbb{Q}} \mathrm{CH}^*(U_i) \to \mathrm{CH}^*(E_i)$$

is an isomorphism, which will imply the proposition. Since $U_0 = \varnothing$ and since the map $E_1 \to U_1$ is a smooth fibration which becomes trivial over the finite étale cover $V_1' \to V_1 = U_1$, the desired isomorphism for $i = 1$ follows from Lemma 6.4. We now consider the diagram

$$\begin{tikzcd}
\begin{matrix}\mathrm{CH}^*(U_i)\\ \otimes \\ \mathrm{CH}^*(F,0)\end{matrix} \arrow[r] \arrow[d] & \begin{matrix}\mathrm{CH}^*(V_{i+1})\\ \otimes \\ \mathrm{CH}^*(F,0)\end{matrix} \arrow[r] \arrow[d] & \begin{matrix}\mathrm{CH}^*(U_{i+1})\\ \otimes \\ \mathrm{CH}^*(F,0)\end{matrix} \arrow[r] \arrow[d] & \begin{matrix}\mathrm{CH}^*(U_i)\\ \otimes \\ \mathrm{CH}^*(F,0)\end{matrix} \arrow[r] \arrow[d] & \begin{matrix}\mathrm{CH}^*(V_{i+1})\\ \otimes \\ \mathrm{CH}^*(F,0)\end{matrix} \arrow[d] \\
\mathrm{CH}^*(E_i) \arrow[r] & \mathrm{CH}^*(W_{i+1}) \arrow[r] & \mathrm{CH}^*(E_{i+1}) \arrow[r] & \mathrm{CH}^*(E_i) \arrow[r] & \mathrm{CH}^*(W_{i+1}).
\end{tikzcd}$$

The top row is obtained by tensoring the long exact localization sequence for higher Chow groups with $\mathrm{CH}^*(F, 0)$ over $\mathbb{Q}$ and hence is exact. The bottom row is just the localization exact sequence.

One checks as in the proof of Theorem 6.3 that the diagram above is commutative. The first and the fourth vertical arrows are isomorphisms by induction. The second and the fifth vertical arrows are isomorphisms by Lemma 6.4. Hence the middle vertical arrow is also an isomorphism by 5-lemma. $\square$

7. Higher Chow groups of toric bundles and applications

As an application of Theorem 6.3, we describe the ordinary higher Chow groups of toric bundles with integral coefficients. Let T be a split torus of rank n over k and let $M = \mathrm{Hom}(\mathbb{G}_m, T)$ be the group of its one-parameter subgroups. Let $X = X(\Delta)$ be a smooth projective toric variety associated to a fan Δ in $M_{\mathbb{R}}$ (see Section 11). Let B be a smooth k-scheme and let $p : E \to B$ be a principal T-bundle. Setting $E(X) = E \times^T X$, we see that

$$\pi : E(X) \to B, \qquad \pi((e, x)) = p(e)$$

is a Zariski-locally trivial smooth fibration with all fibers isomorphic to X. Since X is projective, it follows that π is a projective morphism.

We fix an ordering $\{\sigma_1, \dots, \sigma_m\}$ of $\Delta_{\max}$ and let $\tau_i \subset \sigma_i$ be the cone that is the intersection of σ_i with all those σ_j such that $j \geq i$ and that intersect σ_i in dimension $n-1$. Let $\tau'_i \subset \sigma_i$ be the cone such that $\tau_i \cap \tau'_i = \{0\}$ and $\dim(\tau_i) + \dim(\tau'_i) = n$ for $1 \leq i \leq m$. It is easy to see that τ'_i is the intersection of σ_i with all those σ_j such that $j \leq i$ and that intersect σ_i in dimension $n-1$. Since X is smooth and projective, it is well-known that we can choose the ordering above of $\Delta_{\max}$ such that

$$\tau_i \subset \sigma_j \quad \text{implies} \quad i \leq j \quad \text{and} \quad \tau'_i \subset \sigma_j \quad \text{implies} \quad j \leq i. \tag{7-1}$$

Let $\Delta_1 = \{\rho_1, \dots, \rho_d\}$ be the set of one-dimensional cones in Δ and let $\{v_1, \dots, v_d\}$ be the associated primitive elements of M. We choose $\{\rho_1, \dots, \rho_n\}$ to be a set of one-dimensional faces of σ_m such that $\{v_1, \dots, v_n\}$ is a basis of M. Let $\{\chi_1, \dots, \chi_n\}$ be the dual basis of $M^\vee$.

Definition 7.1. Let A be a commutative ring with unit and let $\{r_1, \dots, r_n\}$ be a subset of A. Let I_A denote the ideal of the polynomial algebra $A[t_1, \dots, t_d]$ generated by the elements

$$t_{j_1} \cdots t_{j_l} \quad \text{for } 1 \leq j_p \leq d \tag{7-2}$$

such that $\rho_{j_1}, \dots, \rho_{j_l}$ do not span a cone of Δ. Let $\bar{I}_\Delta$ denote the ideal of $A[t_1, \dots, t_d]$ generated by I_Δ and the relations

$$s_i := \Bigl(\sum_{j=1}^{d} \langle \chi_i, v_j \rangle t_j\Bigr) - r_i \quad \text{for } 1 \leq i \leq n. \tag{7-3}$$

We define the A-algebras $R_{\mathrm{eq}}(A, \Delta)$ and $R(A, \Delta)$ to be quotients of $A[t_1, \dots, t_d]$ by the ideals I_Δ and $\bar{I}_\Delta$, respectively.

The ring $R_{\mathrm{eq}}(A, \Delta)$ is also known in the literature as the *Stanley–Reisner* algebra over A associated to the fan Δ; see [Sankaran and Uma 2003, Definition 2.1]. Notice that any character χ of T acts on $R_{\mathrm{eq}}(A, \Delta)$ through the multiplication by the element $\sum_{j=1}^{d} \langle \chi, v_j \rangle\, t_j$. This makes $R_{\mathrm{eq}}(A, \Delta)$ into an $S = S(T)$-algebra.

Any T-equivariant line bundle $L \to X$ uniquely defines a line bundle

$$E(L) = E \times^T L$$

on $E(X)$. Every $\rho \in \Delta_1$ defines a unique T-equivariant line bundle L_ρ on X with a T-equivariant section $s_\rho : X \to L_\rho$ that is transverse to the zero section and whose zero locus is the orbit closure $V_\rho = \overline{O_\rho}$. For any $\sigma \in \Delta$, let u_σ denote the fundamental class of the T-invariant cycle $[V_\sigma]$ in $\mathrm{CH}^*(X, 0)$ and let y_σ denote the cycle $[E(V_\sigma)]$ in $\mathrm{CH}^*(E(X), 0)$. Notice that $\pi_\sigma : E(V_\sigma) \to B$ is a smooth projective toric subbundle of $\pi : E(X) \to B$ with fiber V_σ.

Suppose that $\rho_{j_1}, \dots, \rho_{j_l}$ do not span a cone in Δ. Then $s = (s_{j_1}, \dots, s_{j_l})$ is a nowhere vanishing section of $L_{\rho_{j_1}} \oplus \cdots \oplus L_{\rho_{j_l}}$ and hence

$$c_1(E(L_{\rho_{j_1}})) \cdots c_1(E(L_{\rho_{j_l}})) = 0 \quad \text{in } \mathrm{CH}^*(E(X)).$$

In particular, we get

$$y_{\rho_{j_1}} \cdots y_{\rho_{j_l}} = 0 \quad \text{in } \mathrm{CH}^*(E(X)). \tag{7-4}$$

We now consider the commutative diagram

$$\begin{tikzcd}
X_l \arrow[r, "\iota"] \arrow[d, "\pi_l"'] & E(X) \arrow[d, "\pi"'] & E \times X \arrow[l, "\bar{p}"'] \arrow[r, "p_X"] \arrow[d, "p_E"] & X \arrow[d, "\pi_X"] \\
\mathrm{Spec}(l) \arrow[r] & B & E \arrow[l, "p"] \arrow[r, "\pi_E"'] & \mathrm{Spec}(k),
\end{tikzcd} \tag{7-5}$$

where $\mathrm{Spec}(l)$ is any point of B. It is clear that all squares are Cartesian and all the maps in the right square are T-equivariant. Let L_χ denote the T-equivariant line bundle on $\mathrm{Spec}(k)$ associated to a character χ of T. Since p and $\bar{p}$ are principal T-bundles, we see that there is a unique line bundle ζ on B such that

$$\pi_E^*(L_\chi) = p^*(\zeta) \quad \text{and} \quad p_X^* \circ \pi_X^*(L_\chi) = \bar{p}^* \circ \pi^*(\zeta). \tag{7-6}$$

Using the identity

$$\pi_X^*(c_1^T(L_\chi)) = \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle \, u_\rho \quad \text{in } \mathrm{CH}_T^*(X)$$

and the isomorphisms $\mathrm{CH}^*(B) \cong \mathrm{CH}_T^*(E)$, $\mathrm{CH}^*(E(X)) \cong \mathrm{CH}_T^*(E \times X)$, we see that $\pi^*(c_1(\zeta)) = \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle y_\rho$. Let $\zeta_i \in \mathrm{Pic}(B)$ be such that $\pi_E^*(L_{\chi_i}) = p^*(\zeta_i)$, where $\{\chi_1, \dots, \chi_n\}$ is a chosen basis of $M^\vee$ as above. Setting $r_i = c_1(\zeta_i) \in \mathrm{CH}^*(B)$, we conclude that

$$\pi^*(r_i) = \sum_{j=1}^{d} \langle \chi_i, v_j \rangle y_{\rho_j} \quad \text{in } \mathrm{CH}^*(E(X)), \quad 1 \le i \le n. \tag{7-7}$$

We define a homomorphism of $\mathrm{CH}^*(B)$-algebras

$$\mathrm{CH}^*(B)[t_1, \dots, t_d] \to \mathrm{CH}^*(E(X))$$

by the assignment $t_i \mapsto y_{\rho_i}$ for $1 \le i \le d$. It follows from (7-4) and (7-7) that this homomorphism descends to a $\mathrm{CH}^*(B)$-algebra homomorphism

$$\psi : R(\mathrm{CH}^*(B), \Delta) \to \mathrm{CH}^*(E(X)). \tag{7-8}$$

The following result describes the higher Chow groups of the projective toric bundle $\pi : E(X) \to B$ and generalizes [Sankaran and Uma 2003, Theorem 1.2(iii)] to higher Chow groups.

Theorem 7.2. *The homomorphism ψ is an isomorphism.*

Proof. To prove this theorem, we first observe that for any $\sigma \in \Delta$ and any point $\mathrm{Spec}(l) \to B$, one has $\iota^*(E(V_\sigma)) = (V_\sigma)_l$. Also, it is well-known [Sankaran and Uma 2003, Lemma 3.1] that $\{\iota^*(y_{\tau_1}), \dots, \iota^*(y_{\tau_m})\}$ forms a $\mathbb{Z}$-basis of $\mathrm{CH}^*(X_l, 0)$.

Since $y_{\tau_i} = y_{\rho_{i_1}} \cdots y_{\rho_{i_p}}$ for every $1 \le i \le m$, where $\{\rho_{i_1}, \dots, \rho_{i_p}\}$ is the set of edges of τ_i, it follows from the proof of Theorem 6.3 that $\mathrm{CH}^*(E(X))$ is generated by $\{y_{\rho_1}, \dots, y_{\rho_d}\}$ as a $\mathrm{CH}^*(B)$-algebra. In particular, the map ψ is surjective.

To prove injectivity, let $x(\sigma)$ denote the monomial $t_{i_1} \cdots t_{i_p}$ in $R(\mathrm{CH}^*(B), \Delta)$ such that $\{\rho_{i_1}, \dots, \rho_{i_p}\}$ is the set of edges of $\sigma \in \Delta$. Then by [Sankaran and Uma 2003, Lemma 2.1(ii)], the set $\{x(\tau_1), \dots, x(\tau_m)\}$ spans $R(\mathrm{CH}^*(B), \Delta)$ as a $\mathrm{CH}^*(B)$-module. Since $\psi(x(\tau_i)) = y_{\tau_i}$ for $1 \le i \le m$ and since $\mathrm{CH}^*(E(X))$ is a free $\mathrm{CH}^*(B)$-module with basis $\{y_{\tau_1}, \dots, y_{\tau_m}\}$ by Theorem 6.3, we conclude that ψ must be injective. ☐

7a. ***Equivariant and ordinary higher Chow groups of smooth projective toric varieties.*** As a consequence of Theorem 7.2, we derive some explicit formulas for the equivariant and ordinary higher Chow groups of smooth projective toric varieties with integral coefficients. We shall later show in Section 11 that such formulas hold for all smooth toric varieties with rational coefficients. Recall that if $X = X(\Delta)$ is a toric variety, then for every $\sigma \in \Delta$, the orbit closure V_σ is a T-invariant closed toric subvariety of X and hence uniquely defines a class $y_\sigma = [V_\sigma]$ in $\mathrm{CH}^*_T(X, 0)$; see [Edidin and Graham 1998, Section 2.2]. This is called the fundamental equivariant class of V_σ.

We consider $A = \mathrm{CH}^*(k) \otimes_{\mathbb{Z}} S \cong \mathrm{CH}^*(k)[t_1, \dots, t_n]$ as a graded $\mathrm{CH}^*(k)$-algebra whose degree zero part is $\mathrm{CH}^*(k)$. We have seen above that $R_{\mathrm{eq}}(\mathrm{CH}^*(k), \Delta)$ has an action of S that makes it a graded A-algebra. Moreover, $R(\mathrm{CH}^*(k), \Delta)$ is just the quotient $R_{\mathrm{eq}}(\mathrm{CH}^*(k), \Delta) \otimes_S \mathbb{Z} \cong R_{\mathrm{eq}}(\mathrm{CH}^*(k), \Delta) \otimes_A \mathrm{CH}^*(k)$.

Corollary 7.3. *Let $X = X(\Delta)$ be a smooth projective toric variety as above. Then the assignment $t_i \mapsto y_{\rho_i}$ induces $\mathrm{CH}^*(k)$-algebra isomorphisms*

$$\Psi_X : R_{\mathrm{eq}}(\mathrm{CH}^*(k), \Delta) \xrightarrow{\simeq} \mathrm{CH}^*_T(X), \tag{7-9}$$

$$\overline{\Psi}_X : R(\mathrm{CH}^*(k), \Delta)) \xrightarrow{\simeq} \mathrm{CH}^*(X). \tag{7-10}$$

Proof. The second isomorphism is just a special case of Theorem 7.2 when $B = \mathrm{Spec}(k)$.

To prove the isomorphism of (7-9), we first observe that Ψ_X is a graded A-linear homomorphism; see Section 11. Let M denote the kernel of this map. It follows from [Krishna 2009a, Theorem 4.2] that $\mathrm{CH}^*_T(X)$ is a free A-module with basis $\{y_{\tau_i}\}$. It follows from this that Ψ_X is surjective and we get an exact sequence of graded A-modules

$$0 \to M \to R_{\mathrm{eq}}(\mathrm{CH}^*(k), \Delta) \xrightarrow{\Psi_X} \mathrm{CH}^*_T(X) \to 0.$$

The freeness of $\mathrm{CH}^*_T(X)$ as an A-module ensures that this sequence remains short exact after tensoring with $\mathrm{CH}^*(k)$ via the augmentation $A \twoheadrightarrow \mathrm{CH}^*(k)$.

It follows from [Krishna 2009a, Theorem 1.1] that

$$\mathrm{CH}^*_T(X) \otimes_A \mathrm{CH}^*(k) \xrightarrow{\sim} \mathrm{CH}^*(X).$$

In particular, Ψ_X becomes an isomorphism after tensoring with $\mathrm{CH}^*(k)$. We conclude that $M \otimes_A \mathrm{CH}^*(k) = 0$. Since M is a nonnegatively graded A-module, it must be zero. □

8. Higher Chow groups of flag bundles and applications

We remind the readers of our convention that all the higher Chow groups for the rest of this text will be considered with rational coefficients. We shall however, indicate the coefficients in the statement of all results.

In this section, we describe a formula for the higher Chow groups of complete flag bundles with rational coefficients. Such a formula for general flag bundles is an immediate consequence of the case of complete flag bundles. We also give some applications of this formula to the theory of equivariant higher Chow groups.

8a. ***Complete flag bundles.*** Let G be a connected reductive group over k and let B be a Borel subgroup of G containing a split maximal torus T. Let X be a k-scheme and let $p : E \to X$ be a principal G-bundle and let $\pi : E/B \to X$ be the associated complete flag bundle. Vistoli [1989] described the classical Chow groups $\mathrm{CH}_*(E, 0)$ and $\mathrm{CH}_*(E/B, 0)$ in terms of the Chow groups of X. In this section we generalize Vistoli's results to the case of higher Chow groups. The proof below is completely different from Vistoli's; it is much shorter and relies more on the equivariant techniques.

The restriction map $r^G_{T,X}$ induces for every $i \geq 0$, a natural map of $S(T)$-modules

$$\mathrm{CH}^G_*(E, i) \otimes_{S(G)} S(T) \to \mathrm{CH}^T_*(E, i), \quad w \otimes \alpha \mapsto \alpha \cdot r^G_{T,E}(w).$$

Since G acts freely on E, one identifies $\mathrm{CH}^G_*(E, i)$ with $\mathrm{CH}_*(X, i)$ by Proposition 2.2. The group $\mathrm{CH}^B_*(E, i)$ is canonically identified with $\mathrm{CH}_*(E/B, i)$ for the same reason. The map above then translates into a natural map of $S(T)$-modules

$$\lambda_X : \mathrm{CH}_*(X, i) \otimes_{S(G)} S(T) \to \mathrm{CH}_*(E/B, i). \tag{8-1}$$

Taking the direct sum over $\{\mathrm{CH}_*(X, i)\}_{i\geq 0}$, we get a natural map of $S(T)$-modules

$$\lambda_X : \mathrm{CH}_*(X) \otimes_{S(G)} S(T) \to \mathrm{CH}_*(E/B). \tag{8-2}$$

This map is a ring homomorphism if X is smooth. One can easily check that λ_X commutes with the flat pull-back and the proper push-forward maps between the higher Chow groups of the base schemes of the bundle. We wish to show that λ_X is an isomorphism. We begin with the following special case.

Lemma 8.1. *Let X be a smooth (not necessarily connected) scheme over k and let $f : X' \to X$ be a finite étale morphism such that the principal bundle $p : E \to X$ is trivialized over X'. Then the map λ_X is an isomorphism with rational coefficients. In particular, λ_X is an isomorphism with rational coefficients if X is zero-dimensional.*

Proof. Since X is a disjoint union of connected smooth schemes, it is enough to prove the lemma when X is smooth and connected. If $E/B = G/B \to \mathrm{Spec}(k)$ is the flag variety, then we have

$$\begin{aligned}
\mathrm{CH}_*(k) \otimes_{S(G)} S(T) &\cong \mathrm{CH}_*(k) \otimes_{\mathrm{CH}^*(k,0)} (\mathrm{CH}^*(k, 0) \otimes_{S(G)} S(T)) \\
&\cong^{\dagger} \mathrm{CH}_*(k) \otimes_{\mathrm{CH}^*(k,0)} \mathrm{CH}_*(G/B, 0) \\
&\cong \mathrm{CH}_*(G/B) \qquad\qquad \text{(by Lemma 6.2)},
\end{aligned}$$

where $\cong^{\dagger}$ follows from [Demazure 1973, théorème 2].

If $G/B \times X \xrightarrow{\pi} X$ is the trivial bundle, then we get

$$\begin{aligned}
\mathrm{CH}_*(X) \otimes_{S(G)} S(T) &\cong \mathrm{CH}_*(X) \otimes_{\mathrm{CH}^*(k,0)} (\mathrm{CH}^*(k, 0) \otimes_{S(G)} S(T)) \\
&\cong \mathrm{CH}_*(X) \otimes_{\mathrm{CH}^*(k,0)} \mathrm{CH}_*(G/B, 0) \\
&\cong \mathrm{CH}_*(G/B \times X) \qquad\qquad \text{(by Lemma 6.2)}.
\end{aligned}$$

In general, we consider the diagram

$$\begin{tikzcd}
\mathrm{CH}_*(X) \otimes S(T) \arrow[r, "f^*\otimes \mathrm{id}"] \arrow[d, "\lambda_X"'] & \mathrm{CH}_*(X') \otimes S(T) \arrow[r, "f_*\otimes \mathrm{id}"] \arrow[d, "\lambda_{X'}"'] & \mathrm{CH}_*(X) \otimes S(T) \arrow[d, "\lambda_X"] \\
\mathrm{CH}_*(E/B) \arrow[r, "\bar{f}^*"'] & \mathrm{CH}_*(E_{X'}/B) \arrow[r, "\bar{f}_*"'] & \mathrm{CH}_*(E/B),
\end{tikzcd}$$

where the tensor product in the top row is over $S(G)$. We have just shown that $\lambda_{X'}$ is an isomorphism. It follows from the projection formula (see [Bloch 1986, Exercise 5.8]) that the composite horizontal maps in both rows are multiplication by $[k(X') : k(X)]$. Hence, λ_X must be an isomorphism too. □

Theorem 8.2. *For any k-scheme X, the map λ_X is an isomorphism of $S(T)$-modules with rational coefficients. This is a ring isomorphism if X is smooth.*

Proof. We only need to prove the first assertion, for which we use induction on the dimension of X. The zero-dimensional case follows from Lemma 8.1. In general, we can find an étale cover of X over which the bundle $p : E \to X$ becomes trivial. Since any such cover is generically finite and since the base field k is perfect, we can find a dense open subset $j : U \hookrightarrow X$ and a finite étale cover $f : U' \to U$ such that U is a disjoint union of connected smooth schemes and the given bundle is trivial over U'. Let $\iota : Z \hookrightarrow X$ be the complement of U with its reduced induced closed subscheme structure.

We now consider the diagram

$$\begin{array}{ccccccccc}
\mathrm{CH}_*(U) & & \mathrm{CH}_*(Z) & & \mathrm{CH}_*(X) & & \mathrm{CH}_*(U) & & \mathrm{CH}_*(Z) \\
\otimes & \xrightarrow{\partial} & \otimes & \xrightarrow{\iota_*} & \otimes & \xrightarrow{j^*} & \otimes & \xrightarrow{\partial} & \otimes \\
S(T) & & S(T) & & S(T) & & S(T) & & S(T) \\
\downarrow{\scriptstyle \lambda_U} & & \downarrow{\scriptstyle \lambda_Z} & & \downarrow{\scriptstyle \lambda_X} & & \downarrow{\scriptstyle \lambda_U} & & \downarrow{\scriptstyle \lambda_Z} \\
\mathrm{CH}^*(E_U/B) & \to & \mathrm{CH}^*(E_Z/B) & \to & \mathrm{CH}^*(E/B) & \to & \mathrm{CH}^*(E_U/B) & \to & \mathrm{CH}^*(E_Z/B)
\end{array}$$

of localization exact sequences, where the tensor product in the top row is over the ring $S(G)$. In particular, this row is exact by the flatness of $S(T)$ over $S(G)$. The second and the third squares commute by the compatibility of λ_X with the push-forward and the pull-back maps as remarked above.

To see that the first square commutes, let us consider an element

$$a \otimes b \in \mathrm{CH}_*(U) \otimes S(T).$$

If we map this horizontally, we get $\partial(a) \otimes b$, which is mapped vertically down to $b \cdot \pi_Z^* \circ \partial(a)$. Since the localization sequence of higher Chow groups is compatible with respect to the flat pull-back, this last term is same as $b \cdot \partial \circ \pi_U^*(a)$. On the other hand, mapping $a \otimes b$ vertically down gives $b \cdot \pi_U^*(a)$ and if we map this horizontally, we get $\partial(b \cdot \pi_U^*(a))$. Since the horizontal maps in the bottom row are $S(T)$-linear (see Proposition 2.2), we conclude that the first (hence the fourth) square commutes.

The first and the fourth vertical arrows in the diagram above are isomorphisms by Lemma 8.1. Since U is a dense open in X, the dimension of Z is strictly smaller than that of X. Hence the second and the fifth vertical arrows are isomorphisms by induction on the dimension and Lemma 8.1. We conclude from the 5-lemma that λ_X is an isomorphism. □

8b. *Principal bundles and flag bundles.* The following extension of Theorem 8.2 to all flag bundles is a direct generalization of the projective bundle formula for higher Chow groups.

Corollary 8.3. *Let $p : E \to X$ be a principal G-bundle over a k-scheme X and let $\pi : E/P \to X$ denote the flag bundle associated to a parabolic subgroup P. Then* *the natural map of $S(P)$-modules*

$$\lambda_X : \mathrm{CH}_*(X;\mathbb{Q}) \underset{S(G;\mathbb{Q})}{\otimes} S(P;\mathbb{Q}) \to \mathrm{CH}_*(E/P;\mathbb{Q}), \quad w\otimes\alpha \mapsto \alpha\cdot r^G_{P,E}(w).$$

is an isomorphism. This is a ring isomorphism if X is smooth.

Proof. We only need to prove the first assertion. Let $B \subseteq P$ be a Borel subgroup of G containing a split maximal torus T. Let $\Phi(G,T)$ be the root system of G with respect to T such that B corresponds to the base Δ of $\Phi(G,T)$ and P corresponds to a subset $I\subset\Delta$. Let $P = M \ltimes N$ be the Levi decomposition and let $B_M = B\cap M$ be the Borel subgroup of M containing T. Let $W_P \subset W$ be the Weyl group of P with respect to T. It follows from Propositions 3.3, 3.4 and Corollary 8.7 that the natural map

$$\mathrm{CH}_*(E/P) \to (\mathrm{CH}_*(E/B))^{W_P} \tag{8-3}$$

is an isomorphism.

On the other hand, following the proof of Corollary 8.7, we get

$$\begin{aligned}(\mathrm{CH}_*(X)\otimes_{S(G)} S(T))^{W_P} &\cong \mathrm{CH}_*(X)\otimes_{S(G)} (S(T))^{W_P}\\ &\cong \mathrm{CH}_*(X)\otimes_{S(G)} S(M)\\ &\cong \mathrm{CH}_*(X)\otimes_{S(G)} S(P),\end{aligned}\tag{8-4}$$

where the last isomorphism follows again from Proposition 3.4. The corollary now follows from Theorem 8.2 by combining (8-3) and (8-4). □

The following result generalizes [Vistoli 1989, Corollary 3.2] to higher Chow groups.

Corollary 8.4. *Let G be connected and split reductive group over k and let $p : E \to X$ be a principal G-bundle over a k-scheme X. Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} = \mathrm{Tor}_p^{S(G;\mathbb{Q})}(\mathbb{Q}, \mathrm{CH}_*(X,q;\mathbb{Q})) \Rightarrow \mathrm{CH}_*(E, p+q;\mathbb{Q}).$$

The edge homomorphism yields an isomorphism

$$\mathrm{CH}_*(X,0;\mathbb{Q}) \underset{S(G;\mathbb{Q})}{\otimes} \mathbb{Q} \xrightarrow{\sim} \mathrm{CH}_*(E,0;\mathbb{Q}).$$

Proof. This is an immediate consequence of Theorem 8.2, [Krishna 2009a, Theorem 1.1] and flatness of $S(T)$ over $S(G)$. □

8c. ***A change of groups isomorphism.*** The following theorem is an analogue of a similar result in equivariant K-theory by Merkurjev [2005, Proposition 8]. However, this result for the equivariant higher Chow groups has an advantage over Merkurjev's theorem in that it holds for the action of any split reductive group (though with rational coefficients) whereas [ibid., Proposition 8] is known only for the groups whose derived subgroups are simply connected, for example, GL_n. The special case $\mathrm{CH}^G_*(X, 0)$ of the result below was proven by Brion [1997, Theorem 6.7].

Theorem 8.5. *Let G be a connected reductive group and let T be a split maximal torus of G. Then for any $X \in \mathcal{V}_G$, the natural map of $S(T)$-modules*

$$\lambda_X : \mathrm{CH}^G_*(X; \mathbb{Q}) \underset{S(G;\mathbb{Q})}{\otimes} S(T; \mathbb{Q}) \to \mathrm{CH}^T_*(X; \mathbb{Q}) \tag{8-5}$$

is an isomorphism. This is a ring isomorphism if X is smooth.

Proof. We only need to show the first assertion. If (V, U) is a good pair for the G-action, then $\mathrm{CH}^G_*(X, i)$ and $\mathrm{CH}^T_*(X, i)$ in suitable degrees are the same as $\mathrm{CH}_*(X_G, i)$ and $\mathrm{CH}_*(X_B, i)$, respectively, where B is a Borel subgroup of G containing T. Hence, it suffices to show that for any k-scheme Z with a free action of G with quotients G/B and Z/G, the natural map

$$\lambda_{Z/G} : \mathrm{CH}_*(Z/G) \otimes_{S(G)} S(T) \to \mathrm{CH}_*(Z/B)$$

is an isomorphism. But this follows immediately by applying Theorem 8.2 to the principal bundle $Z \to Z/G$. □

Remark 8.6. The first remark is that Theorem 8.5 holds if G is any connected linear algebraic group (not necessarily reductive) if the base field is of characteristic zero. This is an immediate consequence of Proposition 3.4. The second remark is that the theorem above can also be proven as a simple consequence of the Leray–Hirsch theorem 6.5. The case of smooth schemes is a direct consequence of Theorem 6.5 and the general case can be proven using noetherian induction and the localization sequence. We leave it an exercise to fill in the details.

8d. ***Some consequences of Theorem 8.5.*** Recall that if G is a connected reductive group with a split maximal torus T, then the normalizer N of T in G and all its connected components are defined over k and the quotient N/T is the Weyl group W of the corresponding root system. In particular, $W \subset G/T$. If G acts on a variety X and if (V, U) is a good pair for the G-action, then $X \times^T U \to X \times^G U$ is an étale-locally trivial smooth fibration with fiber G/T. In particular, W acts on each $\mathrm{CH}^T_j(X, i)$ and the map $\mathrm{CH}^G_j(X, i) \to \mathrm{CH}^T_j(X, i)$ factors through the W-invariants. We get the following consequence of Theorem 8.5.

Corollary 8.7 (see Theorem 5.7). *Let G be a connected reductive group and let T be a split maximal torus of G with the Weyl group W. Then for any $X \in \mathcal{V}_G$, the restriction map $r^G_{T,X}$ induces an isomorphism*

$$\mathrm{CH}^G_*(X; \mathbb{Q}) \xrightarrow{\sim} (\mathrm{CH}^T_*(X; \mathbb{Q}))^W.$$

Proof. Since W is a finite group, the trivial $\mathbb{Q}[W]$-module $\mathbb{Q}$ is a projective $\mathbb{Q}[W]$-module of finite rank. In particular, it follows from Theorem 8.5 that

$$\begin{aligned}(\mathrm{CH}^*_T(X))^W &= \mathrm{Hom}_{\mathbb{Q}[W]}(\mathbb{Q}, \mathrm{CH}^G_*(X) \otimes_{S(G)} S(T)) \\ &= \mathrm{CH}^G_*(X) \otimes_{S(G)} \mathrm{Hom}_{\mathbb{Q}[W]}(\mathbb{Q}, S(T)) \\ &= \mathrm{CH}^G_*(X) \otimes_{S(G)} (S(T))^W \\ &=^{\dagger} \mathrm{CH}^G_*(X) \otimes_{S(G)} S(G) \\ &= \mathrm{CH}^G_*(X),\end{aligned}$$

where $=^{\dagger}$ holds by [Edidin and Graham 1998, Proposition 6]. □

As an important consequence of the result above, we get the following analogue of a similar result of Thomason [1988, Theorem 1.13] in equivariant K-theory.

Corollary 8.8 (see Remark 5.8). *Let G be a connected and reductive group over k and let T be a split maximal torus in G. Then the restriction map*

$$\mathrm{CH}^G_*(X; \mathbb{Q}) \xrightarrow{r^G_{T,X}} \mathrm{CH}^T_*(X; \mathbb{Q}) \tag{8-6}$$

is a split monomorphism. Moreover, this splitting is natural for morphisms in $\mathcal{V}_G$. In particular, if H is any closed subgroup of G, then there is a split injective map

$$\mathrm{CH}^H_*(X; \mathbb{Q}) \xrightarrow{r^G_{T,X}} \mathrm{CH}^T_*(G \times^H X; \mathbb{Q}). \tag{8-7}$$

Proof. The first statement follows directly from Corollary 8.7, where the splitting is given by the trace map into the subgroup of W-invariants. The last statement follows from (8-6) and Corollary 3.2. □

Remark 8.9. Let $\varrho \in S(T) \cong \mathrm{CH}^*_G(G/B, 0)$ be such that the forgetful map takes ϱ to the class of the zero-dimensional Schubert cycle in $\mathrm{CH}^*(G/B, 0)$. For a flag bundle $\pi : E/B \to X$ over a scheme X, if we define $\psi_X : \mathrm{CH}_*(X) \to \mathrm{CH}_*(X)$ by $\psi_X(\alpha) = \pi_*(\varrho \cdot \pi^*(\alpha))$, then it can be shown that ψ_X is an isomorphism. In particular, π^* is split injective. This gives another (and more conceptual) proof of Corollary 8.8. In fact, this proof shows that $r^G_{T,X}$ is split injective with integer coefficients if G is *special*.

Let G be a connected reductive group and let B be a Borel subgroup of G containing a split maximal torus T. It follows from [Demazure 1973, théorème 2] that the forgetful map $S(T) \to \mathrm{CH}^*(G/B, 0)$ is surjective. Moreover, if $\{\varrho_w\}_{w \in W}$ are polynomials in $S(T)$ which map to the classes of Schubert cycles $\{\zeta_w\}_{w \in W}$ in $\mathrm{CH}^*(G/B, 0)$, then $S(T)$ is a free $S(G)$-module with basis $\{\varrho_w\}_{w \in W}$. The following is a direct generalization of Demazure's theorem to the case of all smooth schemes and all higher Chow groups. This also strengthens Corollary 8.7 for smooth schemes.

Corollary 8.10. *Let $X \in \mathcal{V}_G^S$ and let $X \xrightarrow{p_X} \mathrm{Spec}(k)$ be the structure map. Set $\varrho_{w,X} = p_X^*(\varrho_w)$. Then $\mathrm{CH}_T^*(X; \mathbb{Q})$ is a free $\mathrm{CH}_G^*(X; \mathbb{Q})$-module with basis $\{\varrho_{w,X}\}_{w \in W}$.*

Proof. It follows from the construction that the map

$$\lambda_X : \mathrm{CH}_G^*(X) \otimes_{S(G)} S(T) \to \mathrm{CH}_T^*(X)$$

takes $1 \otimes \varrho_w$ onto $\varrho_{w,X}$. The corollary is now an immediate consequence of Theorem 8.5. □

9. Cohomological rigidity and specializations

Let G be a split diagonalizable group over k acting on a smooth scheme X. Recall [Springer 1998, 13.2.5] that all the diagonalizable subgroups of G are defined and split over k. The equivariant K-theory of X for the G-action was studied by Vezzosi and Vistoli [2003]. Their main result (Theorem 1) is to reconstruct the equivariant K-theory ring of X in terms of the equivariant K-theory of the loci where the stabilizers have constant dimension. In the next two sections, we use the ideas of Vezzosi and Vistoli to prove an analogous decomposition theorem (see Theorem 10.3) for the equivariant higher Chow groups of X for the G-action. As mentioned in the introduction, this theorem and its compatibility with the corresponding result for the equivariant K-theory play fundamental roles in the proof of the equivariant Riemann–Roch theorems in [Krishna 2009b]. This theorem is very useful in computing the equivariant and ordinary higher Chow groups of smooth schemes with torus action. Some applications of this kind are given in Section 11.

This section is concerned with the study of the notion of cohomological rigidity and the construction of certain specialization maps in equivariant higher Chow groups. In this and the next section, the group G will denote a split diagonalizable group and the all schemes will be assumed to be smooth with G-action. We have seen (Proposition 2.2) that for such a scheme X, $\mathrm{CH}_G^*(X)$ is a bigraded ring, which is an algebra over the ring $\mathrm{CH}_G^*(k)$.

9a. *Cohomological rigidity.*

Definition 9.1. Let $Y \subset X$ be a smooth and G-invariant closed subscheme of codimension $d \geq 0$ and let $N_{Y/X}$ denote the normal bundle of Y in X. We say that Y is *cohomologically rigid* inside X if $c_d^G(N_{Y/X})$ is a not a zero-divisor in the ring $\mathrm{CH}_G^*(Y)$.

As one observes, this definition has reasonable meaning only in the equivariant setting, since every element of positive degree in the nonequivariant Chow ring is nilpotent. The importance of cohomological rigidity for the equivariant higher

Chow groups comes from the following analogue of the K-theory splitting theorem (Proposition 4.3) of [Vezzosi and Vistoli 2003].

Proposition 9.2. *Let Y be a smooth and G-invariant closed subscheme of X of codimension $d \geq 0$. Assume that Y is cohomologically rigid inside X, and put $U = X \setminus Y$. Let $i : Y \hookrightarrow X$ and $j : U \hookrightarrow X$ be the inclusion maps.*

(i) *The localization sequence*

$$0 \to \mathrm{CH}^*_G(Y; \mathbb{Q}) \xrightarrow{i_*} \mathrm{CH}^*_G(X; \mathbb{Q}) \xrightarrow{j^*} \mathrm{CH}^*_G(U; \mathbb{Q}) \to 0$$

is exact.

(ii) *The restriction ring homomorphisms*

$$\mathrm{CH}^*_G(X; \mathbb{Q}) \xrightarrow{(i^*, j^*)} \mathrm{CH}^*_G(Y; \mathbb{Q}) \times \mathrm{CH}^*_G(U; \mathbb{Q})$$

give an isomorphism of rings

$$\mathrm{CH}^*_G(X; \mathbb{Q}) \xrightarrow{\simeq} \mathrm{CH}^*_G(Y; \mathbb{Q}) \underset{\widetilde{\mathrm{CH}^*_G(Y; \mathbb{Q})}}{\times} \mathrm{CH}^*_G(U; \mathbb{Q}),$$

where $\widetilde{\mathrm{CH}^*_G(Y; \mathbb{Q})} = \mathrm{CH}^*_G(Y; \mathbb{Q})/(c^G_d(N_{Y/X}))$, and the maps

$$\mathrm{CH}^*_G(Y; \mathbb{Q}) \to \widetilde{\mathrm{CH}^*_G(Y; \mathbb{Q})}, \qquad \mathrm{CH}^*_G(U; \mathbb{Q}) \to \widetilde{\mathrm{CH}^*_G(Y; \mathbb{Q})}$$

are, respectively, the natural surjection and the map

$$\mathrm{CH}^*_G(U; \mathbb{Q}) = \frac{\mathrm{CH}^*_G(X; \mathbb{Q})}{i_*(\mathrm{CH}^*_G(Y; \mathbb{Q}))} \xrightarrow{i^*} \frac{\mathrm{CH}^*_G(Y; \mathbb{Q})}{c^G_d(N_{Y/X})} = \widetilde{\mathrm{CH}^*_G(Y; \mathbb{Q})},$$

which is well-defined by Corollary 4.5.

Proof. Part (i) follows directly from Corollary 4.5 and the definition of cohomological rigidity. Since i^* and j^* are ring homomorphisms, the proof of the second part follows directly from the first part and [Vezzosi and Vistoli 2003, Lemma 4.4]. □

To apply the result above in our context, we need to find some sufficient conditions for checking the cohomological rigidity in specific examples. We begin with the following elementary result.

Lemma 9.3. *Let A be a ring which is a $\mathbb{Q}$-algebra. Then an element of the form t^d, where $t = \sum_{j=1}^{r} a_j t_{i_j} \in A[t_1, \ldots, t_n]$, is not a zero-divisor for any $d \geq 0$ whenever $a_j \in \mathbb{Q}$ for all j and $a_j \neq 0$ for some j.*

Proof. Since all $a_j \in \mathbb{Q}$ and some $a_j \neq 0$, we see that t^d is a nonzero element of $\mathbb{Q}[t_1, \ldots, t_n]$ and hence a nonzero divisor in this ring. Since tensoring with A over $\mathbb{Q}$ is exact, we see that the multiplication by t^d is injective in $A[t_1, \ldots, t_n]$ too. □

Proposition 9.4. *Let G be a split diagonalizable group acting on a smooth scheme X and let E be a G-equivariant vector bundle of rank d on X. Assume that there is a subtorus $T \subset G$ of positive rank which acts trivially on X, such that in the eigenspace decomposition of E with respect to T, the submodule corresponding to the trivial character is zero. Then $c^G_d(E)$ is not a zero-divisor in $\mathrm{CH}^*_G(X; \mathbb{Q})$.*

Proof. By [Thomason 1986, Lemma 5.6], E has a unique direct sum decomposition

$$E = \bigoplus_{i=1}^{r} E_{\chi_i} \otimes L_{\chi_i},$$

where we choose a splitting $G = D \times T$, E_{χ_i} are D-bundles and χ_i are characters of T with associated line bundles $L_{\chi_i} \in \mathrm{Pic}_T(k)$. This decomposition is via the functor

$$\mathrm{Bun}^D_X \times \mathrm{Rep}(T) \to \mathrm{Bun}^G_X, \quad (F, \rho) \mapsto p_1^*(F) \otimes p_2^*(\rho),$$

where $p_1 : D \times T \to D$ and $p_2 : D \times T \to T$ are the projections.

Since $\mathrm{rank}(E) = d$, the Whitney sum formula yields $c^G_d(E) = \prod_{i=1}^r c^G_{d_i}(E_{\chi_i} \otimes L_{\chi_i})$, where $d_i = \mathrm{rank}(E_{\chi_i})$. We can thus assume that $E = E_\chi \otimes L_\chi$, where χ is not a trivial character by our assumption. In particular, we can write

$$c^T_1(L_\chi) = t = \sum_{i=1}^{p} n_i t_i \in \mathbb{Q}[t_1, \dots, t_n] \tag{9-1}$$

with $n_i \neq 0$ for some i. By neglecting those i for which the coefficients n_i are zero, we can assume that $n_i \neq 0$ for all i. Now we have

$$\begin{aligned} c^G_d(E) = c^G_d(p_1^*(E_\chi) \otimes p_2^*(L_\chi)) &=^\dagger \sum_{i=0}^{d} c^G_{d-i}(p_1^*(E_\chi)) \cdot (c^G_1(p_2^*(L_\chi)))^i \\ &= \sum_{i=0}^{d} p_1^*(c^D_{d-i}(E_\chi)) \cdot p_2^*((c^T_1(L_\chi))^i) = \sum_{i=0}^{d} \alpha_i t^i, \end{aligned}$$

where $\alpha_i \in \mathrm{CH}^*_D(X)$ and $c^G_d(E) \in \mathrm{CH}^*_G(X) \cong \mathrm{CH}^*_D(X) \otimes S(T)$ by Theorem 3.5 and $=^\dagger$ holds by [Fulton 1984, Remark 3.2.3]. Furthermore, $\alpha_d = p_1^*(c^D_0(E_\chi)) = 1$. Thus we get $c^G_d(E) = t^d + \alpha_{d-1} t^{d-1} + \dots + \alpha_1 t + \alpha_0 = g(t)$.

We need to show that $g(t)$ is not a zero divisor in $\mathrm{CH}^*_D(X)[t_1, \dots, t_n]$. So suppose $f(\underline{t})$ is a nonzero polynomial such that $g(t) f(\underline{t}) = 0$, and let $f'(\underline{t})$ be the homogeneous part of $f(\underline{t})$ of largest degree which is not zero. By comparing the homogeneous parts, it is easy to see that $g(t) f(\underline{t}) = 0$ only if $t^d f'(\underline{t}) = 0$. But this is a contradiction since t satisfies the condition of Lemma 9.3 by (9-1), and hence is not a zero-divisor. □

Let G be a split diagonalizable group as above and let $X \in \mathcal{V}_G^S$. Following the notation of [Vezzosi and Vistoli 2003], for any $s \geq 0$, we let $X_{\leq s} \subset X$ be the open subset of points whose stabilizers have dimension at most s. We shall often write $X_{\leq s-1}$ also as $X_{<s}$. Let $X_s = X_{\leq s} \setminus X_{<s}$ denote the locally closed subset of X, where the stabilizers have dimension exactly s. We think of X_s as a subspace of X with the reduced induced structure. It is clear that $X_{\leq s}$ and X_s are G-invariant subspaces of X. Let N_s denote the normal bundle of X_s in $X_{\leq s}$, and let N_s^0 denote the complement of the 0-section in N_s. Then G clearly acts on N_s. The following result describes some very useful properties of these subspaces.

Proposition 9.5. *Let $s \geq 0$ be an integer.*

(i) *There exists a finite number of s-dimensional subtori $T_1, \dots, T_r$ in G such that X_s is the disjoint union of the fixed point spaces $X_{\leq s}^{T_j}$.*

(ii) *X_s is smooth locally closed subscheme of X.*

(iii) $N_s^0 = (N_s)_{<s}$.

Proof. Since the base field k is perfect, this is a special case of [Vezzosi and Vistoli 2003, Proposition 2.2], which holds for regular G-schemes over any connected and separated Noetherian base scheme. □

Remark 9.6. We mention here that although the proposition above has been stated for the smooth schemes, part (i) of the proposition holds also when X is not necessarily smooth, since the proof given in [loc. cit.] only uses Thomason's generic étale slice theorem, which holds very generally.

Corollary 9.7. *For $s \geq 1$, X_s is cohomologically rigid inside $X_{\leq s}$.*

Proof. Let d_s be the codimension of X_s in $X_{\leq s}$. We need to show that $c_{d_s}^G(N_s)$ is not a zero-divisor in $\mathrm{CH}_G^*(X_s)$. By Proposition 9.4, it suffices to show that there exists a subtorus T in G of positive rank that acts trivially on X_s, such that in the eigenspace decomposition of N_s with respect to T, the submodule corresponding to the trivial character is zero. But this follows directly from parts (i) and (iii) of Proposition 9.5 and the fact that $s \geq 1$; see [Vezzosi and Vistoli 2003, Proposition 4.6]. □

9b. ***Specialization maps.*** Let G and X be as above and let n be the dimension of G. As seen above, there is a filtration of X by G-invariant open subsets

$$\varnothing = X_{\leq -1} \subset X_{\leq 0} \subset \cdots \subset X_{\leq n} = X.$$

In particular, G acts on $X_{\leq 0}$ with finite stabilizers, and the toral component of G acts trivially on X_n. We fix $1 \leq s \leq n$ and let $f_s : X_s \hookrightarrow X_{\leq s}$ and $g_s : X_{<s} \hookrightarrow X_{\leq s}$ denote the closed and the open embeddings, respectively. Let $\pi : M_s \to \mathbb{P}^1$ be the deformation to the normal cone for the embedding f_s as in Section 4. We have already observed there that for the trivial action of G on $\mathbb{P}^1$, M_s has a

natural G-action. Moreover, the deformation diagram (4-1) is a diagram of smooth G-spaces. For $0 \le t \le s$, we shall often denote the open subspace $(M_s)_{\le t}$ of M_s by $M_{s,\le t}$. The terms like $M_{s,t}$ and $M_{s,<t}$ (and also for N_s) will have similar meaning in what follows. Since G acts trivially on $\mathbb{P}^1$, it acts on M_s fiberwise with $N_s = \pi^{-1}(\infty)$ and

$$M_{s,\le t} \cap \pi^{-1}(\mathbb{A}^1) = X_{\le t} \times \mathbb{A}^1, \qquad M_{s,t} \cap \pi^{-1}(\mathbb{A}^1) = X_t \times \mathbb{A}^1. \tag{9-2}$$

Let $i_{s,\le t} : N_{s,\le t} \hookrightarrow M_{s,\le t}$ and $j_{s,\le t} : X_{\le t} \times \mathbb{A}^1 \hookrightarrow M_{s,\le t}$ denote the obvious closed and open embeddings. We define $i_{s,t}$ and $j_{s,t}$ similarly. Let $\eta_{s,t} : N_{s,t} \hookrightarrow N_{s,\le t}$ and $\delta_{s,t} : M_{s,t} \hookrightarrow M_{s,\le t}$ denote the other closed embeddings. One has a commutative diagram

$$\begin{tikzcd}
X_{\le t} \arrow[r, "g_{\le t}"] \arrow[d, "f_{s,\le t}"'] & X_{\le t} \times \mathbb{A}^1 \arrow[r, "j_{s,\le t}"] \arrow[d] & M_{s,\le t} \arrow[d] & & \\
X_{\le s} \arrow[r, "g_{\le s}"] & X_{\le s} \times \mathbb{A}^1 \arrow[r, "j_{\le s}"] & M_s \arrow[r] & X_{\le s} \times \mathbb{P}^1 \arrow[r] & X_{\le s},
\end{tikzcd} \tag{9-3}$$

where $g_{\le t}$ is the 0-section embedding, and the composite of all the maps in the bottom row is identity. This gives us the diagram

$$\begin{tikzcd}
\mathrm{CH}^*_G(N_{s,t}) \arrow[r, "i_{s,t*}"] \arrow[d, "\eta_{s,t*}"'] & \mathrm{CH}^*_G(M_{s,t}) \arrow[r, "j^*_{s,t}"] \arrow[d, "\delta_{s,t*}"] & \mathrm{CH}^*_G(X_t \times \mathbb{A}^1) \arrow[r, "g^*_t"] \arrow[d, "f_{t*}"] & \mathrm{CH}^*_G(X_t) \arrow[d, "f_{t*}"] \\
\mathrm{CH}^*_G(N_{s,\le t}) \arrow[r, "i_{s,\le t*}"] & \mathrm{CH}^*_G(M_{s,\le t}) \arrow[r, "j^*_{s,\le t}"] & \mathrm{CH}^*_G(X_{\le t} \times \mathbb{A}^1) \arrow[r, "g^*_{\le t}"] & \mathrm{CH}^*_G(X_{\le t})
\end{tikzcd}$$

of equivariant higher Chow groups, where the left square commutes by the covariance of the push-forward map, the middle commutes by Proposition 2.2(1) and the right commutes by Corollary 4.2. Since the last horizontal maps in both rows are natural isomorphisms by the homotopy invariance, we shall often identify the last two terms in both rows and use $j^*_{s,\le t}$ and $(j_{s,\le t} \circ g_{\le t})^*$ interchangeably.

Theorem 9.8. *The maps $j^*_{s,\le t}$ and $j^*_{s,t}$ are surjective and there are ring homomorphisms*

$$\overline{\mathrm{Sp}}^{\le t}_{X,s} : \mathrm{CH}^*_G(X_{\le t}; \mathbb{Q}) \to \mathrm{CH}^*_G(N_{s,\le t}; \mathbb{Q}) \quad \textit{and}$$
$$\overline{\mathrm{Sp}}^{t}_{X,s} : \mathrm{CH}^*_G(X_t; \mathbb{Q}) \to \mathrm{CH}^*_G(N_{s,t}; \mathbb{Q})$$

such that $i^*_{s,\le t} = \overline{\mathrm{Sp}}^{\le t}_{X,s} \circ j^*_{s,\le t}$ and $i^*_{s,t} = \overline{\mathrm{Sp}}^{t}_{X,s} \circ j^*_{s,t}$. Moreover, both the squares in the following diagram commute:

$$\begin{tikzcd}
\mathrm{CH}^*_G(X_{\le t}; \mathbb{Q}) \arrow[r, "f^*_t"] \arrow[d, "\overline{\mathrm{Sp}}^{\le t}_{X,s}"'] & \mathrm{CH}^*_G(X_t; \mathbb{Q}) \arrow[r, "f_{t*}"] \arrow[d, "\overline{\mathrm{Sp}}^{t}_{X,s}"] & \mathrm{CH}^*_G(X_{\le t}; \mathbb{Q}) \arrow[d, "\overline{\mathrm{Sp}}^{\le t}_{X,s}"] \\
\mathrm{CH}^*_G(N_{s,\le t}; \mathbb{Q}) \arrow[r, "\eta^*_{s,t}"'] & \mathrm{CH}^*_G(N_{s,t}; \mathbb{Q}) \arrow[r, "\eta_{s,t*}"'] & \mathrm{CH}^*_G(N_{s,\le t}; \mathbb{Q})
\end{tikzcd} \tag{9-4}$$

Proof. Using the results obtained so far in this section, one can define the specialization maps along the lines of the construction of such maps for K-theory in [Vezzosi and Vistoli 2003, Theorem 3.2; 2005]. However, it is not at all clear from the construction of the specialization maps in [Vezzosi and Vistoli 2005] that these maps have good functorial properties, and, in particular, if they are ring homomorphisms. Moreover, it is not clear if these maps will have the compatibility properties with the Chern character and Riemann–Roch maps (see [Krishna 2009b]) from the equivariant K-groups to higher Chow groups.

We give here a more direct and functorial construction of the specialization maps, which works both for the K-theory as well as the higher Chow groups, and the proof of various compatibilities of these maps then becomes essentially obvious. We give here the construction of these maps for the higher Chow groups. The same construction works also for the K-theory without any change.

First of all, using Corollary 9.7 and Proposition 9.2, we see that for $1 \le s \le n$ and $0 \le t \le s$, the map $\mathrm{CH}^*_G(X_{\le s}) \to \mathrm{CH}^*_G(X_{\le t})$ is surjective. We now consider the commutative diagram

$$\begin{tikzcd} \mathrm{CH}^*_G(M_s) \arrow[r, "j^*_{\le s}"] \arrow[d] & \mathrm{CH}^*_G(X_{\le s}) \arrow[d] \\ \mathrm{CH}^*_G(M_{s,\le t}) \arrow[r, "j^*_{s,\le t}"] & \mathrm{CH}^*_G(X_{\le t}). \end{tikzcd}$$

Since the composite map in the bottom row of (9-3) is identity, we see by the homotopy invariance that $j^*_{\le s}$ is surjective. Thus $j^*_{s,\le t}$ is also surjective. Applying this surjectivity for $j^*_{s,\le t}$ and $j^*_{s,\le t-1}$, we obtain the commutative diagram of Figure 1, which is such that the second and the third rows are exact. All the columns are exact by Corollary 9.7 and Proposition 9.2. We conclude that the localization sequence of the top row is also exact. This proves the surjectivity part of the theorem.

Next, we apply the self-intersection formula (Corollary 4.5) to the inclusions $i_{s,\le t}$ and $i_{s,t}$ to see that the composites $i^*_{s,\le t} \circ i_{s,\le t *}$ and $i^*_{s,t} \circ i_{s,t *}$ are multiplication by the first Chern class c_1^G of the corresponding normal bundles. But these normal bundles are the inverse images of a line bundle on $\mathbb{P}^1$. It follows that these normal bundles are trivial, because the restriction of any line bundle on $\mathbb{P}^1$ to $\infty \in \mathbb{P}^1$ and hence on the fiber over ∞ is clearly trivial. We conclude that the composites $i^*_{s,\le t} \circ i_{s,\le t *}$ and $i^*_{s,t} \circ i_{s,t *}$ are zero.

The diagram above now automatically defines the specializations $\overline{\mathrm{Sp}}^{\le t}_{X,s}$ and $\overline{\mathrm{Sp}}^{t}_{X,s}$ and gives the desired factorization of $i^*_{s,\le t}$ and $i^*_{s,t}$. Since $i^*_{s,t}$ and $j^*_{s,t}$ are ring homomorphisms, and since the latter is surjective as shown in 1, we deduce that $\overline{\mathrm{Sp}}^{t}_{X,s}$ is also a ring homomorphism. The map $\overline{\mathrm{Sp}}^{\le t}_{X,s}$ is a ring homomorphism for the same reason.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{CH}^*_G(N_{s,t}) \arrow[r, "i_{s,t*}"] \arrow[d, "\eta_{s,t*}"'] & \mathrm{CH}^*_G(M_{s,t}) \arrow[r, "j^*_{s,t}"] \arrow[d, "\delta_{s,t*}"] & \mathrm{CH}^*_G(X_t) \arrow[r] \arrow[d, "f_{t*}"] & 0 \\
0 \arrow[r] & \mathrm{CH}^*_G(N_{s,\le t}) \arrow[r, "i_{s,\le t*}"] \arrow[d] & \mathrm{CH}^*_G(M_{s,\le t}) \arrow[r, "j^*_{s,\le t}"] \arrow[d] & \mathrm{CH}^*_G(X_{\le t}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{CH}^*_G(N_{s,\le t-1}) \arrow[r, "i_{s,\le t-1*}"] \arrow[d] & \mathrm{CH}^*_G(M_{s,\le t-1}) \arrow[r, "j^*_{s,\le t-1}"] \arrow[d] & \mathrm{CH}^*_G(X_{\le t-1}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Figure 1

We are now left with the proof of the commutativity of (9-4). To prove that the right square commutes, we consider the following diagram.

$$
\begin{tikzcd}
\mathrm{CH}^*_G(M_{s,t}) \arrow[rr, "\delta_{s,t*}"] \arrow[dd, "i^*_{s,t}"'] \arrow[dr, "j^*_{s,t}"] & & \mathrm{CH}^*_G(M_{s,\le t}) \arrow[dd, "i^*_{s,\le t}" near start] \arrow[dr, "j^*_{s,\le t}"] & \\
 & \mathrm{CH}^*_G(X_t) \arrow[rr, "f_{t*}" near end] \arrow[dl, "\overline{\mathrm{Sp}}^t_{X,s}"] & & \mathrm{CH}^*_G(X_{\le t}) \arrow[dl, "\overline{\mathrm{Sp}}^{\le t}_{X,s}"] \\
\mathrm{CH}^*_G(N_{s,t}) \arrow[rr, "\eta_{s,t*}"] & & \mathrm{CH}^*_G(N_{s,\le t}) &
\end{tikzcd}
\tag{9-5}
$$

It is easy to check that $N_{s,\le t}$ and $M_{s,t}$ are Tor-independent over $M_{s,\le t}$ (see [Vezzosi and Vistoli 2005, Lemma 1]) and hence the back face of the diagram above commutes by Corollary 4.2. The upper face commutes by diagram 1. Since $j^*_{s,t}$ is surjective, a diagram chase shows that the lower face also commutes, which is what we needed to prove.

Finally, since we have shown that η_{s,t_*} is injective, and the right square commutes, it now suffices to show that the composite square in (9-4) commutes in order to show that the left square commutes.

By the projection formula, the composite maps $f_{t*} \circ f^*_t$ and $\eta_{s,t*} \circ \eta^*_{s,t}$ are multiplication by $f_{t*}(1)$ and $\eta_{s,t*}(1)$ respectively. Since

$$\overline{\mathrm{Sp}}^{\le t}_{X,s} \quad \text{and} \quad \overline{\mathrm{Sp}}^{t}_{X,s}$$

are ring homomorphisms, it suffices to show that

$$\overline{\mathrm{Sp}}^{\le t}_{X,s}(f_{t*} \circ j^*_{s,t}(1)) = \overline{\mathrm{Sp}}^{\le t}_{X,s}(f_{t*}(1)) = \eta_{s,t*}(1).$$

But this follows directly from the commutativity of the right square. □

10. Decomposition theorem for equivariant higher Chow groups

We use the specialization maps to prove the main decomposition theorem for the equivariant higher Chow groups of $X \in \mathcal{V}^S_G$, where G is a split diagonalizable group. We continue with the notation of the previous section.

Proposition 10.1. *The restriction maps*

$$\mathrm{CH}^*_G(X_{\le s}; \mathbb{Q}) \xrightarrow{(f^*_s, g^*_s)} \mathrm{CH}^*_G(X_s; \mathbb{Q}) \times \mathrm{CH}^*_G(X_{<s}; \mathbb{Q})$$

define an isomorphism of rings

$$\mathrm{CH}^*_G(X_{\le s}; \mathbb{Q}) \xrightarrow{\simeq} \mathrm{CH}^*_G(X_s; \mathbb{Q}) \underset{\mathrm{CH}^*_G(N^0_s; \mathbb{Q})}{\times} \mathrm{CH}^*_G(X_{<s}; \mathbb{Q}),$$

where $\mathrm{CH}^*_G(X_s; \mathbb{Q}) \xrightarrow{\eta^*_{s,\le s-1}} \mathrm{CH}^*_G(N^0_s; \mathbb{Q})$ *is the pull-back*

$$\mathrm{CH}^*_G(X_s; \mathbb{Q}) \xrightarrow{\simeq} \mathrm{CH}^*_G(N_s; \mathbb{Q}) \to \mathrm{CH}^*_G(N^0_s; \mathbb{Q})$$

and

$$\mathrm{CH}^*_G(X_{<s}; \mathbb{Q}) \xrightarrow{\overline{\mathrm{Sp}}^{\le s-1}_{X,s}} \mathrm{CH}^*_G(N_{s,\le s-1}; \mathbb{Q}) = \mathrm{CH}^*_G(N^0_s; \mathbb{Q})$$

is the specialization map of Theorem 9.8.

Proof. We only need to identify the pull-back and the specialization maps with the appropriate maps of Proposition 9.2. In the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{CH}^*_G(X_s) \arrow[r, "f_{s,\infty *}"] \arrow[dr, "c^G_{d_s}"'] & \mathrm{CH}^*_G(N_s) \arrow[r, "\eta^*_{s,\le s-1}"] \arrow[d, "f^*_{s,\infty}"] & \mathrm{CH}^*_G(N^0_s) \arrow[r] & 0 \\ & & \mathrm{CH}^*_G(X_s), & & \end{tikzcd}$$

where $f_{s,\infty} : X_s \to N_s$ is the 0-section embedding, the top sequence is exact, and the lower triangle commutes by Corollary 4.5. Since $f^*_{s,\infty}$ is an isomorphism, this immediately identifies the pull-back map of the proposition with the quotient map

$$\mathrm{CH}^*_G(X_s) \to \frac{\mathrm{CH}^*_G(X_s)}{(c^G_{d_s}(N_s))}.$$

Next, we consider the diagram

$$
\begin{tikzcd}
\mathrm{CH}^*_G(X_{\le s}) \arrow[r, "\overline{\mathrm{Sp}}^{\le s}_{X,s}"] \arrow[d, "f^*_{s,\le s-1}"'] & \mathrm{CH}^*_G(N_s) \arrow[r, "f^*_{s,\infty}"] \arrow[d] & \mathrm{CH}^*_G(X_s) \arrow[dl, "\eta^*_{s,\le s-1}"] \\
\mathrm{CH}^*_G(X_{<s}) \arrow[r, "\overline{\mathrm{Sp}}^{\le s-1}_{X,s}"'] & \mathrm{CH}^*_G(N^0_s).
\end{tikzcd}
\tag{10-1}
$$

Since the left vertical arrow in the diagram above is surjective, we only need to show that

$$\overline{\mathrm{Sp}}^{\le s-1}_{X,s} \circ f^*_{s,\le s-1} = \eta^*_{s,\le s-1} \circ f^*_s$$

in order to identify $\overline{\mathrm{Sp}}^{\le s-1}_{X,s}$ with the map j^* of Proposition 9.2. It is clear from the diagram 1 and the definition of the specialization maps that the left square in the diagram (10-1) commutes. We have just shown above that the right side triangle also commutes. This reduces us to showing that

$$f^*_{s,\infty} \circ \overline{\mathrm{Sp}}^{\le s}_{X,s} = f^*_s. \tag{10-2}$$

If $X_s \times \mathbb{P}^1 \xrightarrow{F_s} M_s$ denotes the embedding (see (4-1)), then for $x \in \mathrm{CH}^*_G(X_{\le s})$, we can write $x = j^*_{\le s}(y)$ by Theorem 9.8. Then

$$
\begin{aligned}
f^*_{s,\infty} \circ \overline{\mathrm{Sp}}^{\le s}_{X,s} \circ j^*_{\le s}(y) &= f^*_{s,\infty} \circ i^*_{s,\le s}(y) =^{\dagger} g^*_{\infty,\le s} \circ F^*_s(y) \\
&= g^*_{0,\le s} \circ F^*_s(y) = f^*_s \circ j^*_{\le s}(y) = f^*_s(x),
\end{aligned}
$$

where $=^{\dagger}$ follows from Corollary 4.2. This proves (10-2) and the proposition. $\square$

We need the following algebraic result before we prove the main theorem. Let A be a $\mathbb{Q}$-algebra (not necessarily commutative) and let $Z(A)$ denote the center of A. For any linear form $f(\underline{t}) = \sum_{i=1}^n a_i t_i$ in $A[t_1, \dots, t_n]$ such that $a_i \in \mathbb{Q}$ for each i, let $c(f)$ denote the vector $(a_1, \dots, a_n) \in \mathbb{Q}^n$ consisting of the coefficients of the form f.

Lemma 10.2. *Let A be as above and let $S = \{f_1, \dots, f_s\}$ be a set of linear forms in $A[t_1, \dots, t_n]$ such that the vectors $\{c(f_1), \dots, c(f_s)\}$ are pairwise nonproportional in $\mathbb{Q}^n$. Let*

$$\gamma_j = \sum_{i=0}^{d_j} m^j_i f^i_j$$

such that $m^j_{d_j} \in \mathbb{Q}^*$ for $1 \le j \le s$, and $m^j_{j'} \in Z(A)$ for all j, j'. Then one has

$$(\gamma_1 \cdots \gamma_s) = \bigcap_{j=1}^s (\gamma_j)$$

as ideals in $A[t_1, \dots, t_n]$.

Proof. Using a simple induction, it suffices to show that for $j \neq j'$, the relation $\gamma_j \mid q\gamma_{j'}$ implies that $\gamma_j \mid q$. So we can assume $S = \{f_1, f_2\}$. Since $c(f_1)$ and $c(f_2)$ are nonproportional, we can extend the set $\{c(f_1), c(f_2)\}$ to a basis B of $\mathbb{Q}^n$. Applying the linear automorphism of $A[t_1, \dots, t_n]$ given by the invertible matrix B, we can assume that $f_j = t_j$ for $j = 1, 2$. Now the proof follows along the same lines as the proof of [Vezzosi and Vistoli 2003, Lemma 4.9]. $\square$

Theorem 10.3. *Let G be a split diagonalizable group of dimension n and let $X \in \mathcal{V}_G^S$. The ring homomorphism*

$$\mathrm{CH}^*_G(X; \mathbb{Q}) \to \prod_{s=0}^{n} \mathrm{CH}^*_G(X_s; \mathbb{Q})$$

is injective. Moreover, its image consists of the n-tuples (α_s) in the product with the property that for each $s = 1, \dots, n$, the pull-back of $\alpha_s \in \mathrm{CH}^*_G(X_s; \mathbb{Q})$ in $\mathrm{CH}^*_G(N_{s,s-1}; \mathbb{Q})$ is the same as $\overline{\mathrm{Sp}}^{s-1}_{X,s}(\alpha_{s-1}) \in \mathrm{CH}^*_G(N_{s,s-1}; \mathbb{Q})$. In other words, there is a ring isomorphism

$$\mathrm{CH}^*_G(X; \mathbb{Q}) \xrightarrow{\simeq} \mathrm{CH}^*_G(X_n; \mathbb{Q}) \underset{\mathrm{CH}^*_G(N_{n,n-1};\mathbb{Q})}{\times} \cdots \underset{\mathrm{CH}^*_G(N_{1,0};\mathbb{Q})}{\times} \mathrm{CH}^*_G(X_0; \mathbb{Q}).$$

Proof. We prove by induction on the largest integer s such that $X_s \neq \varnothing$.

If $s = 0$, there is nothing to prove. If $s > 0$, we have by induction

$$\mathrm{CH}^*_G(X_{<s}) \xrightarrow{\cong} \mathrm{CH}^*_G(X_{s-1}) \underset{\mathrm{CH}^*_G(N_{s-1,s-2})}{\times} \cdots \underset{\mathrm{CH}^*_G(N_{1,0})}{\times} \mathrm{CH}^*_G(X_0). \tag{10-3}$$

Using (10-3) and Proposition 10.1, it suffices to show that if $\alpha_s \in \mathrm{CH}^*_G(X_s)$ and if $\alpha_{<s} \in \mathrm{CH}^*_G(X_{<s})$ with the restriction $\alpha_{s-1} \in \mathrm{CH}^*_G(X_{s-1})$ are such that $\alpha_s \mapsto \alpha_s^0 \in \mathrm{CH}^*_G(N_s^0)$ and $\alpha_s \mapsto \alpha_{s,s-1} \in \mathrm{CH}^*_G(N_{s,s-1})$, then

$$\overline{\mathrm{Sp}}^{\leq s-1}_{X,s}(\alpha_{<s}) = \alpha_s^0 \quad \text{if and only if} \quad \overline{\mathrm{Sp}}^{s-1}_{X,s}(\alpha_{s-1}) = \alpha_{s,s-1}.$$

Using the commutativity of the left square in Theorem 9.8, this is reduced to showing that the restriction map

$$\mathrm{CH}^*_G(N_s^0) \to \mathrm{CH}^*_G(N_{s,s-1}) \tag{10-4}$$

is injective.

To prove this, we first use Proposition 9.5 to assume that the toral component T of the isotropy groups of the points of X_s is fixed, and choose a splitting $G = D \times T$.

Now, following the proof of the analogous result for K-theory [Vezzosi and Vistoli 2003, Theorem 4.5], we can write

$$N_s = E = \bigoplus_{i=1}^{q} E_i \quad \text{and} \quad N_{s,s-1} = \coprod_i E_i^0,$$

where each E_i is of the form $\bigoplus E_{m_j} \otimes \chi_i^{m_j}$ such that for $i \neq i'$, χ_i and $\chi_{i'}$ are nonproportional characters of T, and E_i^0 is embedded in E by setting all the other components equal to zero. Let $d_i = \operatorname{rank}(E_i)$.

Now we see from Proposition 10.1 that

$$\operatorname{Ker}(\mathrm{CH}^*_G(X_s) \to \mathrm{CH}^*_G(N_{s,s-1})) = \bigcap_i (c^G_{d_i}(E_i)),$$

$$\operatorname{Ker}(\mathrm{CH}^*_G(X_s) \to \mathrm{CH}^*_G(N^0_s)) = (c^G_{d_s}(N_s)) \quad \text{with } d_s = \sum d_i .$$

Put $\gamma_i = c^G_{d_i}(E_i)$ and $\gamma = c^G_{d_s}(N_s)$. Since the map $\mathrm{CH}^*_G(X_s) \to \mathrm{CH}^*_G(N^0_s)$ is surjective, showing the injectivity of the map in (10-4) is equivalent to showing that

$$(\gamma) = \Big(\prod_i \gamma_i\Big) = \bigcap_i (\gamma_i) \tag{10-5}$$

in $\mathrm{CH}^*_D(X_s)[t_1, \dots, t_s]$.

However, we have seen in the proof of Proposition 9.4 that each γ_i is of the form

$$\gamma_i = u_i^{d_i} + \alpha^i_{d_i-1} u_i^{d_i-1} + \cdots + \alpha^i_1 u_i + \alpha^i_0,$$

where $\alpha^i_j \in \mathrm{CH}^*_D(X_s, 0) \subseteq Z(\mathrm{CH}^*_D(X_s))$ and $u_i = c^T_1(L_{\chi_i}) = \sum_{j=1}^s b^i_j t_j \neq 0$ in $\mathbb{Q}[t_1, \dots, t_s]$. Moreover, the pairwise nonproportionality of χ_i implies the same for the vectors $\{c(u_1), \dots, c(u_q)\}$ in $\mathbb{Q}^s$. We now apply Lemma 10.2 to conclude the proof of (10-5) and hence the theorem. □

11. Equivariant higher Chow groups of toric varieties

In this section, we apply our decomposition theorem to give explicit descriptions of the equivariant higher Chow groups of smooth toric varieties. An analogous description of the equivariant cohomology of such varieties was earlier given by Bifet, De Concini and Procesi in [Bifet et al. 1990] and such a description of the equivariant K-theory was given by Vezzosi and Vistoli [2003]. Brion [1997, Theorem 5.4] had proven similar results for the classical equivariant Chow groups of toric varieties. As a consequence of our descriptions of the equivariant higher Chow groups, we shall obtain formulas for the ordinary higher Chow groups (or motivic cohomology) of smooth toric varieties.

Let T be a split torus of rank n over k. Let $M = \operatorname{Hom}(\mathbb{G}_m, T)$ be the lattice of the one-parameter subgroups of T and let $M^\vee$ be the character lattice of T. Let Δ be a fan in $M_{\mathbb{R}}$ and let Δ_1 and $\Delta_{\max}$ denote the subsets of the one-dimensional cones and the maximal cones in Δ, respectively.

Let $X = X(\Delta)$ be the smooth toric variety associated to the fan Δ. The smoothness of X is equivalent to the condition that every positive dimensional cone of Δ is generated by it edges such that the primitive vectors along these edges form a subset of a basis of M. In this case, there is an one-to-one correspondence between the T-orbits in X and the cones in Δ. For every cone $\sigma \in \Delta$, the corresponding orbit O_σ is isomorphic to the torus T/T_σ, where T_σ is associated to the sublattice M_σ of M generated by $\sigma \cap M$. Under this isomorphism, the origin (identity point) of T/T_σ corresponds to the distinguished k-rational point x_σ of O_σ. In particular, for every $0 \le s \le n$, X_s is of the form

$$X_s = \coprod_{\dim(\sigma)=s} O_\sigma \cong \coprod_{\dim(\sigma)=s} T/T_\sigma. \tag{11-1}$$

We shall write $\tau \le \sigma$ if τ is a face of σ as cones in Δ. The orbit closure V_σ of O_σ is the toric variety associated to the fan $*(\sigma) = \{\tau \in \Delta \mid \sigma \le \tau\}$, called the star of σ. Moreover, it is clear from the characterization of the smoothness of toric varieties that V_σ is also smooth and is the disjoint union of all orbits O_τ such that σ is a face of τ. In particular, O_σ is closed in X if and only if $\sigma \in \Delta_{\max}$. The following is our first description of the equivariant higher Chow groups of smooth toric varieties.

Theorem 11.1. *Let $X = X(\Delta)$ be a smooth toric variety associated to a fan Δ in $M_{\mathbb{R}}$. There is an injective homomorphism of S-algebras*

$$\Phi_X : \mathrm{CH}^*_T(X; \mathbb{Q}) \to \prod_{\sigma \in \Delta_{\max}} \mathrm{CH}^*(k; \mathbb{Q}) \otimes S(T_\sigma; \mathbb{Q}).$$

An element

$$(a_\sigma) \in \prod_{\sigma \in \Delta_{\max}} \mathrm{CH}^*(k; \mathbb{Q}) \otimes S(T_\sigma; \mathbb{Q})$$

is in the image of this homomorphism if and only if for any two maximal cones σ_1 and σ_2, the restrictions of a_{σ_1} and a_{σ_2} to $\mathrm{CH}^*(k; \mathbb{Q}) \otimes S(T_{\sigma_1 \cap \sigma_2}; \mathbb{Q})$ coincide.

Proof. We only need to appropriately identify the various terms and the maps in the statement of Theorem 10.3. We follow the notation of Section 9 and Section 10. It follows from [Vezzosi and Vistoli 2003, Lemma 6.1] that for every $s \ge 1$, there is a canonical isomorphism

$$N_{s,s-1} = \coprod_{\substack{\sigma \in \Delta \\ \dim(\sigma)=s}} \coprod_{\tau \in \partial\sigma} O_\tau. \tag{11-2}$$

Furthermore, for each s-dimensional cone σ and $\tau \in \partial\sigma$, the composition of the map

$$\mathrm{Sp}^{s-1}_{X,s} : \mathrm{CH}^*_T(X_{s-1}) = \prod_{\substack{\tau \in \Delta \\ \dim(\tau)=s-1}} \mathrm{CH}^*_T(O_\tau) \to \mathrm{CH}^*_T(N_{s,s-1}) = \prod_{\substack{\sigma \in \Delta \\ \dim(\sigma)=s}} \prod_{\tau \in \partial\sigma} \mathrm{CH}^*_T(O_\tau)$$

with the projection

$$\mathrm{Pr}_{\sigma,\tau} : \prod_{\substack{\sigma\in\Delta \\ \dim(\sigma)=s}} \prod_{\tau\in\partial\sigma} \mathrm{CH}^*_T(O_\tau) \to \mathrm{CH}^*_T(O_\tau)$$

is the projection map

$$\prod_{\substack{\tau\in\Delta \\ \dim(\tau)=s-1}} \mathrm{CH}^*_T(O_\tau) \to \mathrm{CH}^*_T(O_\tau).$$

The identification above of the specialization maps in [Vezzosi and Vistoli 2003, Lemma 6.1] was shown for the equivariant K-theory, but the same holds in the present case as well without any modification in view of the construction of these specializations for higher Chow groups in Section 9.

It also follows from Corollary 3.2 and Theorem 3.5 that

$$\mathrm{CH}^*_T(O_\sigma) \cong \mathrm{CH}^*_T(T/T_\sigma) \cong \mathrm{CH}^*_{T_\sigma}(k) \cong \mathrm{CH}^*(k) \otimes S(T_\sigma). \tag{11-3}$$

We conclude now from (11-1) and Theorem 10.3 that $\mathrm{CH}^*_T(X)$ is a subring of $\prod_{\sigma\in\Delta} \mathrm{CH}^*(k) \otimes S(T_\sigma)$, consisting of elements (a_σ) with the property that the restriction of $a_\sigma \in \mathrm{CH}^*(k) \otimes S(T_\sigma)$ to $\mathrm{CH}^*(k) \otimes S(T_\tau)$ coincides with a_τ whenever $\tau \leq \sigma$. The theorem now follows from the fact that every cone in Δ is contained in a maximal cone in Δ. □

As an immediate consequence of Theorem 11.1, we obtain the following localization theorem for the equivariant higher Chow groups of smooth projective toric varieties. This was earlier proven for $\mathrm{CH}^*_T(X, 0)$ by Brion [1997, Theorem 3.4].

Corollary 11.2. *Let $X = X(\Delta)$ be a smooth projective toric variety and $\{x_1, \ldots, x_r\}$ be the fixed point locus of X. Then the map*

$$\mathrm{CH}^*_T(X; \mathbb{Q}) \to \mathrm{CH}^*_T(X^T; \mathbb{Q}) \xrightarrow{\simeq} (\mathrm{CH}^*(k; \mathbb{Q})[t_1, \ldots, t_n])^r$$

is injective and its image is the set of all n-tuples $(f_1, \ldots, f_n)$ such that $f_i \equiv f_j \pmod{\chi}$ whenever x_i and x_j lie on a T-invariant smooth irreducible curve on which T acts through its character χ.

Proof. This follows directly from Theorem 11.1 once we observe that the fixed points of X for the torus action are same as the T-orbits corresponding to the maximal cones in Δ that are n-dimensional. Moreover, the orbit closures corresponding to the codimension one cones in Δ are the smooth T-invariant curves. □

11a. ***Stanley–Reisner presentation.*** Using Theorem 11.1, we now give another explicit presentation of the equivariant higher Chow groups of smooth toric varieties. This presentation is analogous to the Stanley–Reisner presentation of the equivariant cohomology in [Bifet et al. 1990, Theorem 8] and equivariant K-theory in [Vezzosi and Vistoli 2003, Theorem 6.1]. This presentation has the advantage that it can often be used to describe the ordinary higher Chow groups of smooth toric varieties.

Let T be a split torus of rank n and let M denote the lattice of the one-parameter subgroups of T. Let $X = X(\Delta)$ be a smooth toric variety associated to a fan Δ in $M_{\mathbb{R}}$. For $r \geq 1$, let Δ_r denote the set of r-dimensional cones in Δ. For $\sigma \in \Delta_{\max}$, let M_σ denote the sublattice of one-parameter subgroups of T_σ so that $\widehat{T_\sigma} = M_\sigma^\vee$ as an abelian group. For any $\rho \in \Delta_1$, let v_ρ denote the generator of the monoid $\rho \cap M$. Note that if $\{\rho_1, \dots, \rho_s\}$ is the set of one-dimensional faces of $\sigma \in \Delta_{\max}$, then the smoothness of X implies that $\{v_{\rho_1}, \dots, v_{\rho_s}\}$ is a basis of M_σ. Let $\{v_{\rho_1}^\vee, \dots, v_{\rho_s}^\vee\}$ denote the dual basis of $M_\sigma^\vee$.

We recall that for $\sigma \in \Delta$, there is a canonical isomorphism of abelian groups $\widehat{T_\sigma} \hookrightarrow S(T_\sigma)_1$ given by $\chi \mapsto c_1^{T_\sigma}(L_\chi)$. For each $\rho \in \Delta_1$, we define an element $u_\rho = (u_\rho^\sigma) \in \prod_{\sigma \in \Delta_{\max}} \prod S(T_\sigma)$ such that

$$u_\rho^\sigma = \begin{cases} v_\rho^\vee & \text{if } \rho \leq \sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{11-4}$$

Then u_ρ has the property that for all $\sigma_1, \sigma_2 \in \Delta_{\max}$, the restrictions of $u_\rho^{\sigma_1} \in \widehat{T_{\sigma_1}}$ and $u_\rho^{\sigma_2} \in \widehat{T_{\sigma_2}}$ in $\widehat{T_{\sigma_1 \cap \sigma_2}}$ coincide.

We have the obvious inclusion

$$\prod_{\sigma \in \Delta_{\max}} S(T_\sigma) \subseteq \prod_{\sigma \in \Delta_{\max}} \mathrm{CH}^*(k) \otimes S(T_\sigma) \tag{11-5}$$

and using the description of $\mathrm{CH}_T^*(X)$ in Theorem 11.1 and the description of u_ρ above, we can consider these u_ρ as elements of the ring $\mathrm{CH}_T^*(X)$. In other words, we get a bigraded $\mathrm{CH}^*(k)$-algebra homomorphism

$$\mathrm{CH}^*(k)[t_\rho] \to \mathrm{CH}_T^*(X), \quad t_\rho \mapsto u_\rho,$$

where $\mathrm{CH}^*(k)[t_\rho]$ is the polynomial algebra $\mathrm{CH}^*(k)[t_\rho \mid \rho \in \Delta_1]$.

If S is a subset of Δ_1 that is not contained in any maximal cone of Δ, then for any given $\sigma \in \Delta_{\max}$, there is one $\rho \in S$ such that $\rho \not\leq \sigma$. This implies in particular that $u_\rho^\sigma = 0$. We conclude from this that the elements u_ρ satisfy the relation

$$\prod_{\rho \in S} u_\rho = 0 \quad \text{in } \mathrm{CH}_T^*(X) \tag{11-6}$$

whenever $S \subseteq \Delta_1$ is such that it is not contained in any maximal cone of Δ. We shall denote the collection of all such subsets of Δ_1 by Δ_1^0. We conclude that if I_Δ denotes the graded ideal of $\mathrm{CH}^*(k)[t_\rho]$ generated by the set of monomials $\left\{\prod_{\rho \in S} t_\rho \mid S \in \Delta_1^0\right\}$, then there is a $\mathrm{CH}^*(k)$-algebra homomorphism

$$\Psi_X : \frac{\mathrm{CH}^*(k)[t_\rho]}{I_\Delta} \to \mathrm{CH}_T^*(X), \quad t_\rho \mapsto u_\rho. \tag{11-7}$$

Note also that any character $\chi \in M^\vee$ defines multiplication by the element $t_\chi = \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle t_\rho$ in $\mathrm{CH}^*(k)[t_\rho]$, and this makes the term on the left hand side of (11-7) an S-algebra and Ψ_X is also an S-algebra homomorphism. Furthermore, it is easy to check from the definition of u_ρ that it is the fundamental class (see [Edidin and Graham 1998, Section 2]) of the T-equivariant Chow cycle

$$[V_\rho \to X] \in \mathrm{CH}^1_T(X, 0) \subsetneq \mathrm{CH}^*_T(X),$$

where V_σ is the orbit closure in X associated to a cone $\sigma \in \Delta$.

Theorem 11.3. *For a smooth toric variety $X = X(\Delta)$ associated to a fan Δ in $M_\mathbb{R}$, the homomorphism Ψ_X is an isomorphism with rational coefficients.*

Proof. We prove the theorem by induction on the number of maximal cones in Δ. Suppose $\Delta_{\max} = \{\sigma\}$ is a singleton set. In that case, σ is the only maximal cone and $X = U_\sigma$ is a T-equivariant vector bundle over O_σ such that the inclusion

$$O_\sigma \overset{i_\sigma}{\hookrightarrow} X$$

is the zero-section embedding. Hence, we conclude from (11-3) that there are isomorphisms

$$\mathrm{CH}^*_T(X) \underset{i^*_\sigma}{\xrightarrow{\sim}} \mathrm{CH}^*_T(O_\sigma) \cong \mathrm{CH}^*(k) \otimes S(T_\sigma) = \mathrm{CH}^*(k)[t_1, \dots, t_s],$$

where s is the dimension of σ. It is also clear in this case that the ideal I_Δ in (11-7) is zero. Hence, we have isomorphisms

$$\mathrm{CH}^*(k)[t_1, \dots, t_s] \underset{\Psi_X}{\xrightarrow{\sim}} \mathrm{CH}^*_T(X) \underset{\Phi_X}{\xrightarrow{\sim}} \mathrm{CH}^*(k) \otimes S(T_\sigma).$$

We consider now the general case. We assume that $|\Delta_{\max}| \geq 2$ and choose a maximal cone σ of dimension $s \geq 1$ in Δ. Let $X' = X'(\Delta')$ be the toric variety associated to the fan $\Delta' = \Delta \setminus \{\sigma\}$. Note that O_σ is a closed T-orbit in X and X' is the complement of O_σ in X. Let $U_\sigma \subset X$ be the principal open set associated to the fan consisting of all faces of σ and let U' be the complement of O_σ in U_σ. Then U' is nothing but the complement of the zero-section in the T-equivariant vector bundle $U_\sigma \to O_\sigma$. Let $i_\sigma : O_\sigma \hookrightarrow X$ and $j_\sigma : X' \hookrightarrow X$ denote the T-invariant closed and open embeddings respectively. Let $S_\sigma = \{\rho_1, \dots, \rho_s\}$ be the set of one-dimensional faces of σ and set

$$x_\sigma = \prod_{j=1}^{s} t_{\rho_j} \in \frac{\mathrm{CH}^*(k)[t_\rho]}{I_\Delta} \quad \text{and} \quad y_\sigma = \prod_{j=1}^{s} u_{\rho_j} \in \mathrm{CH}^*_T(X).$$

Since $N_{O_\sigma/X} = N_{O_\sigma/U_\sigma}$ and since the latter is of the form $\bigoplus_{j=1}^s L_{\chi_j}$, where $\{\chi_1, \dots, \chi_s\}$ is a basis of $\widehat{T_\sigma}$, it follows from the definition of the elements u_ρ (see

Proposition 9.4) that

$$c_s^T(N_{O_\sigma/X}) = y_\sigma \in \mathrm{CH}_T^*(X). \tag{11-8}$$

We consider the diagram

$$\begin{tikzcd}
\mathrm{CH}^*(k)[t_{\rho_1}, \dots, t_{\rho_s}] \arrow[r, "\cong"] \arrow[d, "x_\sigma"'] & \mathrm{CH}_T^*(O_\sigma) \arrow[r, "\cong"] \arrow[d, "i_{\sigma *}"] & \mathrm{CH}^*(k) \otimes S(T_\sigma) \arrow[d, "y_\sigma"] \\
\dfrac{\mathrm{CH}^*(k)[t_\rho]}{I_\Delta} \arrow[r, "\Psi_X"] & \mathrm{CH}_T^*(X) \arrow[r, "\Phi_X"] & \prod_{\tau \in \Delta_{\max}} \mathrm{CH}^*(k) \otimes S(T_\tau),
\end{tikzcd} \tag{11-9}$$

where the horizontal maps on the top are the obvious isomorphisms taking t_{ρ_j} to u_{ρ_j}. The left and the right vertical maps are the multiplication by the indicated elements in the target rings. We claim that all the vertical arrows are injective and the left square in this diagram commutes.

To prove the claim, notice that the composite outer square clearly commutes by the definition of x_σ and y_σ and the map Ψ_X. Since Φ_X is injective by Theorem 11.1, we only need to show that the right square commutes and the right vertical arrow is injective to prove the claim.

We first observe that the right vertical arrow is the multiplication by y_σ on the factor $\mathrm{CH}^*(k) \otimes S(T_\sigma)$ and is zero on the other factors of $\prod_{\tau \in \Delta_{\max}} \mathrm{CH}^*(k) \otimes S(T_\tau)$. Thus the required injectivity is equivalent to showing that the multiplication by y_σ is injective in $\mathrm{CH}^*(k) \otimes S(T_\sigma)$. We can thus assume that $X = U_\sigma$ and then $\mathrm{CH}_T^*(X) \cong \mathrm{CH}^*(k)[t_1, \dots, t_s]$. In this case, y is just the element $t_1 \cdots t_s$ and hence is a nonzero divisor in $\mathrm{CH}^*(k)[t_1, \dots, t_s]$.

To show the commutativity of the right square, we observe from the proof of Theorem 11.1 that Φ_X is simply the product of the pull-back maps

$$i_\tau^* : \mathrm{CH}_T^*(X) \to \mathrm{CH}_T^*(O_\tau) \quad \text{for } \tau \in \Delta_{\max}.$$

Hence the composite $\Phi_X \circ i_{\sigma *}$ is $i_\sigma^* \circ i_{\sigma *}$ on the factor $\mathrm{CH}_T^*(O_\sigma)$ and zero on the other factors of $\prod_{\tau \in \Delta_{\max}} \mathrm{CH}_T^*(O_\tau)$. Since we have just seen that the composite

$$\mathrm{CH}_T^*(O_\sigma) \xrightarrow{y_\sigma} \prod_{\tau \in \Delta_{\max}} \mathrm{CH}_T^*(O_\tau)$$

is of similar type, we are reduced to showing that the triangle

$$\begin{tikzcd}
\mathrm{CH}_T^*(O_\sigma) \arrow[d, "i_{\sigma *}"'] \arrow[dr, "y_\sigma"] & \\
\mathrm{CH}_T^*(X) \arrow[r, "i_\sigma^*"'] & \mathrm{CH}_T^*(O_\sigma)
\end{tikzcd}$$

commutes. But this follows immediately from Corollary 4.5 and (11-8). This proves the claim.

To complete the proof of the theorem, we now consider the diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^*(k)[t_{\rho_1}, \ldots, t_{\rho_s}] \arrow[r, "x_\sigma"] \arrow[d, "\cong"'] & \dfrac{\mathrm{CH}^*(k)[t_\rho]}{I_\Delta} \arrow[r, "\overline{j}^*_\sigma"] \arrow[d, "\Psi_X"] & \dfrac{\mathrm{CH}^*(k)[t_\rho]}{(I_\Delta, x_\sigma)} \arrow[r] \arrow[d, "\overline{\Psi}_{X'}"] & 0 \\
0 \arrow[r] & \mathrm{CH}^*_T(O_\sigma) \arrow[r, "i_{\sigma *}"'] & \mathrm{CH}^*_T(X) \arrow[r, "j^*_\sigma"'] & \mathrm{CH}^*_T(X') \arrow[r] & 0,
\end{tikzcd}
\tag{11-10}
$$

where $\overline{j}^*_\sigma$ is the natural quotient map by the ideal (x_σ) in $\mathrm{CH}^*(k)[t_\rho]/I_\Delta$. Note that the image of the first map in the top row is the ideal (x_σ) because the product of x_σ with any t_ρ for $\rho \notin \{\rho_1, \ldots, \rho_s\}$ is zero.

The left square in this diagram commutes and the first maps in both the rows are injective by the claim above. The bottom row is exact by Proposition 2.2. Since σ is not a cone of Δ', the element x_σ is zero in $\mathrm{CH}^*(k)[t_\rho, \rho \in {\Delta'}_1]/I_{\Delta'}$ and hence the map $j^*_\sigma \circ \Psi_X$ has a factorization:

$$
\frac{\mathrm{CH}^*(k)[t_\rho]}{I_\Delta} \twoheadrightarrow \frac{\mathrm{CH}^*(k)[t_\rho]}{(I_\Delta, x_\sigma)} \to \frac{\mathrm{CH}^*(k)[t_\rho,\ \rho \in {\Delta'}_1]}{I_{\Delta'}} \xrightarrow{\Psi_{X'}} \mathrm{CH}^*_T(X'),
$$

where the middle arrow is the natural map of the Stanley–Reisner rings induced by the inclusion of the fans $\Delta' \subset \Delta$. Letting $\overline{\Psi}_{X'}$ denote the composite

$$
\frac{\mathrm{CH}^*(k)[t_\rho]}{(I_\Delta, x_\sigma)} \to \frac{\mathrm{CH}^*(k)[t_\rho,\ \rho \in {\Delta'}_1]}{I_{\Delta'}} \xrightarrow{\Psi_{X'}} \mathrm{CH}^*_T(X'),
$$

we see that the right square in the diagram (11-10) also commutes.

If all the cones of Δ are at most one-dimensional, then $x_\sigma = t_\rho$, where $\rho = \sigma$ and it is obvious that $\mathrm{CH}^*(k)[t_\rho]/(I_\Delta, x_\sigma)$ is the Stanley–Reisner ring associated to the fan Δ'. If Δ has a cone of dimension at least two, we can assume that σ is of dimension at least two. In that case, we have ${\Delta'}_1 = \Delta_1$ and the natural inclusion $\Delta^0_1 \subseteq {\Delta'}^0_1$ gives the equality ${\Delta'}^0_1 = \Delta^0_1 \coprod \{S_\sigma\}$. In particular, we have

$$
\frac{\mathrm{CH}^*(k)[t_\rho, \rho \in \Delta_1]}{(I_\Delta, x_\sigma)} \xrightarrow{\sim} \frac{\mathrm{CH}^*(k)[t_\rho, \rho \in {\Delta'}_1]}{I_{\Delta'}}.
$$

On the other hand, Δ' is a fan with smaller number of maximal cones than in Δ and $X' = X'(\Delta')$. Hence the map

$$
\frac{\mathrm{CH}^*(k)[t_\rho, \rho \in {\Delta'}_1]}{I_{\Delta'}} \xrightarrow{\Psi_{X'}} \mathrm{CH}^*_T(X')
$$

is an isomorphism by induction. We conclude that the map $\overline{\Psi}_{X'}$ in the diagram (11-10) is an isomorphism. A diagram chase in (11-10) now shows that Ψ_X is also an isomorphism. □

As an important application of Theorem 11.3, we obtain the following presentation of the ordinary higher Chow groups (motivic cohomology groups) of smooth toric varieties. An explicit description of $\mathrm{CH}^*(X, 0)$ for a smooth projective toric variety X was given in [Fulton 1993, Proposition 5.2]. The following result extends this to all smooth toric varieties, not necessarily projective. In fact, such a description extends to all higher Chow groups of smooth projective toric varieties. In particular, we obtain another proof of Corollary 7.3 with rational coefficients. Recall that for every $\sigma \in \Delta$, the orbit closure $V_\sigma = \overline{O_\sigma}$ in X is a T-invariant Weil divisor and defines a unique element $[V_\sigma] \in \mathrm{CH}^1(X, 0)$.

Corollary 11.4. *Let $X = X(\Delta)$ be a smooth projective toric variety. Then the assignment $t_\rho \mapsto [V_\sigma]$ defines a $\mathrm{CH}^*(k; \mathbb{Q})$-algebra isomorphism*

$$\overline{\Psi}_X : \frac{\mathrm{CH}^*(k; \mathbb{Q})[t_\rho]}{\left(I_\Delta, \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle\, t_\rho\right)} \to \mathrm{CH}^*(X; \mathbb{Q}), \tag{11-11}$$

where χ runs over $M^\vee$.

If X is not necessarily projective, the map

$$\frac{\mathbb{Q}[t_\rho]}{\left(I_\Delta, \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle t_\rho\right)} \to \mathrm{CH}^*(X, 0; \mathbb{Q})$$

is a ring isomorphism.

Proof. We have already seen before that every character $\chi \in M^\vee$ acts on $\mathrm{CH}^*(k)[t_\rho]$ by multiplication with the element $\sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle\, t_\rho$ which makes the left hand side of (11-11) an S-algebra. The corollary now follows directly from Theorem 11.3 and [Krishna 2009a, Theorem 1.3]. The second isomorphism follows in the same way from Theorem 11.3 and [Brion 1997, Corollary 2.3]. □

Corollary 11.5. *Let $X = X(\Delta)$ be a smooth toric variety. Then there are canonical ring isomorphisms*

$$\mathrm{CH}^*(k; \mathbb{Q}) \otimes \mathrm{CH}^*_T(X, 0; \mathbb{Q}) \xrightarrow{\simeq} \mathrm{CH}^*_T(X; \mathbb{Q}),$$
$$\mathrm{CH}^*(k; \mathbb{Q}) \otimes \mathrm{CH}^*(X, 0; \mathbb{Q}) \xrightarrow{\simeq} \mathrm{CH}^*(X; \mathbb{Q}).$$

Proof. It follows from (11-7) that $\mathrm{CH}^*(k) \otimes (\mathbb{Q}[t_\rho]/I_\Delta) \xrightarrow{\simeq} \mathrm{CH}^*(k)[t_\rho]/I_\Delta$. The first part of the corollary now follows directly from Theorem 11.3. The second part follows from the first and [Krishna 2009a, Theorem 1.1], which says that there is a convergent spectral sequence

$$\mathrm{Tor}^S_p(\mathbb{Q}, \mathrm{CH}^*_T(X, q)) \Rightarrow \mathrm{CH}^*(X, p+q).$$ □

Remark 11.6. All the results in this section about the (equivariant) higher Chow groups of smooth toric varieties have been stated over the rationals. However, an attentive reader can check that these results (and the proofs) for the subrings $\mathrm{CH}^*_T(X, 0)$ and $\mathrm{CH}^*(X, 0)$ hold true with the integral coefficients. The basic reason is that $\mathrm{CH}^*_T(k, 0)$ and $\mathrm{CH}^*(k, 0)$ are torsion-free abelian groups. But this is false for the higher Chow groups of k.

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