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**Higher Chow groups of varieties
with group action**

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We give explicit descriptions of the higher Chow groups of toric bundles and flag bundles over schemes. We derive several consequences of these descriptions for the equivariant and ordinary higher Chow groups of schemes with group action.

We prove a decomposition theorem for the equivariant higher Chow groups of a smooth scheme with action of a diagonalizable group. This theorem is applied to compute the equivariant and ordinary higher Chow groups of smooth toric varieties. The results of this paper play fundamental roles in the proof of the Riemann–Roch theorems for equivariant higher K -theory.

1. Introduction

A *scheme* in this paper will mean a separated and reduced scheme of finite type over a perfect field k , which admits an ample line bundle. This base field k will be fixed throughout this paper. A *linear algebraic group* G over k will mean a smooth and affine group scheme over k . By a closed subgroup H of an algebraic group G , we shall mean a morphism $H \rightarrow G$ of algebraic groups over k that is a closed immersion of k -schemes. In particular, a closed subgroup of a linear algebraic group will be of the same type and hence smooth. Recall from [Borel 1991, Proposition 1.10] that a linear algebraic group over k is a closed subgroup of a general linear group, defined over k .

Recall that an action of a linear algebraic group G on a k -scheme X is said to be *linear* if X admits a G -equivariant ample line bundle, a condition that is always satisfied if X is normal (see [Sumihiro 1975, Theorem 2.5] for G connected and [Thomason 1988, 5.7] for G general). All G -actions in this paper will be assumed to be linear. Let \mathcal{V}_k denote the category of quasiprojective k -schemes and let \mathcal{V}_k^S denote the full subcategory of smooth k -schemes. We shall denote the category of quasiprojective G -schemes with G -equivariant maps by \mathcal{V}_G , and the full subcategory of smooth G -schemes will be denoted by \mathcal{V}_G^S .

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The G -equivariant higher Chow groups $\mathrm{CH}_*^G(X, i)$ of $X \in \mathcal{V}_G$ were defined by Edidin and Graham [1998] in terms of the ordinary higher Chow groups (motivic Borel–Moore homology) of the quotient space $X \times^G U$. Here, U is a G -invariant open subscheme of a finite-dimensional representation V of G such that it acts freely on U and $V \setminus U$ is of sufficiently high codimension. This definition of Edidin and Graham is based on an earlier construction of Totaro [1999], who invented the idea above to define the Chow groups of the classifying spaces of linear algebraic groups.

In this paper, we develop further the Edidin–Graham theory of equivariant higher Chow groups and establish many important properties of this theory. We also prove some decomposition theorems for the equivariant higher Chow groups of smooth schemes with torus action. These results turn out to have many applications.

Brion [1997] proved many results about the equivariant Chow groups of the form $\mathrm{CH}_*^G(X, 0)$. Many of the structural results in this paper can be described as the generalization of the results of [Brion 1997] to the case of equivariant *higher* Chow groups. In Section 2, we recall the definition of equivariant higher Chow groups from [Edidin and Graham 1998] and prove its basic properties, which are all well known for the ordinary higher Chow groups; see [Bloch 1986]. As a consequence, one finds that the equivariant higher Chow groups form a Borel–Moore oriented bigraded homology theory in the category of schemes with the action of a given linear algebraic group. Other important results about these groups such as the Morita isomorphism are proven in Section 3. We also prove a structure theorem (see Theorem 3.5) for the equivariant higher Chow groups of schemes with action of tori.

Section 4 contains the proof of the self-intersection formula for the higher Chow groups. This formula plays a very important role in the proofs of the main results of this paper. In Section 5, we construct Demazure operators on equivariant higher Chow groups and give some consequences of these operators.

In Section 6, we prove the Leray–Hirsch theorem for the higher Chow groups. As a consequence of this theorem, we compute the higher Chow groups of toric bundles in Section 7. In Section 8, we turn to the description of the higher Chow groups of principal bundles and flag bundles over schemes. We give several applications of these descriptions in the study of equivariant higher Chow groups.

In Sections 9 and 10, we prove a decomposition theorem (see Theorem 10.3) for the equivariant higher Chow groups of smooth schemes with action of a diagonalizable group G . This result describes the equivariant higher Chow group of a G -scheme in terms of the equivariant higher Chow groups of the loci where the stabilizers have a fixed dimension. This result is an analogue of a similar result of Vezzosi and Vistoli [2003] in equivariant K -theory and has many important

applications in the study of equivariant and ordinary higher Chow groups of smooth schemes.

Theorem 10.3 is the basic step in the proof of the equivariant Riemann–Roch theorem in [Krishna 2009b]. This theorem presents an explicit relation between the equivariant K -theory and the equivariant higher Chow groups. Like in the ordinary case, this Riemann–Roch is a fundamental result in equivariant algebraic geometry. This theorem was in fact one of the main motivations for the author to work on this paper. We expect **Theorem 10.3** to have many more applications in the computation of equivariant and ordinary higher Chow groups. In **Section 11**, we apply this theorem to compute the equivariant and ordinary higher Chow groups of smooth toric varieties. We shall follow the following convention while studying the equivariant and ordinary higher Chow groups with the rational coefficients.

Convention. In this paper, all the results and statements up to **Section 7** do not make any assumption on the coefficient ring of the higher Chow groups. On the other hand, all the results and statements from **Section 8** onwards assume rational coefficients. In order to simplify the notation, the following convention will be followed.

From **Section 8** onwards, an abelian group A will actually mean its extension $A \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular, all higher Chow groups and other cohomology groups will be considered with the rational coefficients. For \mathbb{Q} -vector spaces A and B , the tensor product $A \otimes_{\mathbb{Q}} B$ will be simply written as $A \otimes B$. We shall however, indicate the appropriate coefficients in the statements of the all results.

2. Equivariant higher Chow groups

In this section, we recall the definition of the equivariant higher Chow groups from [Edidin and Graham 1998] and review their main functorial properties. It turns out in particular that the equivariant higher Chow groups have all the properties of an oriented bigraded Borel–Moore homology theory.

Let G be a linear algebraic group and let X be a scheme over k with a G -action. We shall denote the dimension of the underlying group G usually by the letter g . All representations of G in this paper will be finite-dimensional. The definition of equivariant higher Chow groups of X needs one to consider certain kind of mixed spaces which in general may not be schemes even if the original spaces are schemes. The following well-known (see [Edidin and Graham 1998, Proposition 23]) lemma shows that this problem does not occur in our context and all the mixed spaces in this paper are schemes with ample line bundles.

Lemma 2.1. *Let H be a linear algebraic group acting freely and linearly on a k -scheme U such that the quotient U/H exists as a quasiprojective scheme. Let X be a k -scheme with a linear action of H . Then the mixed quotient $X \times^H U$ exists for*

the diagonal action of H on $X \times U$ and is quasiprojective. Moreover, this quotient is smooth if both U and X are so. In particular, if H is a closed subgroup of a linear algebraic group G and X is a k -scheme with a linear action of H , then the quotient $G \times^H X$ is a quasiprojective scheme.

Proof. It is already shown in [Edidin and Graham 1998, Proposition 23] using [Mumford et al. 1994, Proposition 7.1] that the quotient $X \times^H U$ is a scheme. Moreover, as U/H is quasiprojective, [Mumford et al. 1994, Proposition 7.1] in fact shows that $X \times^H U$ is also quasiprojective. The similar conclusion about $G \times^H X$ follows from the first case by taking $U = G$ and by observing that G/H is a smooth quasiprojective scheme; see [Borel 1991, Theorem 6.8]. The assertion about the smoothness is clear since $X \times U \rightarrow X \times^H U$ is a principal H -bundle. \square

2a. Good pairs and equivariant higher Chow groups. For any integer $j \geq 0$, let V be an l -dimensional representation of G and let U be a G -invariant open subset of V such that the codimension of the complement $V \setminus U$ in V is sufficiently larger than j , and G acts freely on U such that the quotient U/G is a quasiprojective scheme. Such a pair (V, U) will be called a *good pair* for the G -action corresponding to j . It is easy to see that a good pair always exists; see [Edidin and Graham 1998, Lemma 9].

For an equidimensional G -scheme X , let X_G denote the quotient $X \times^G U$ of the product $X \times U$ by the diagonal action of G , which is free. We define the equivariant higher Chow group $\mathrm{CH}_G^j(X, i)$ as the homology group $H_i(\mathcal{Z}^j(X_G, \bullet))$, where $\mathcal{Z}^j(X_G, \bullet)$ is the Bloch cycle complex of the scheme X_G . It is known [Edidin and Graham 1998, Section 2] that this definition of $\mathrm{CH}_G^j(X, i)$ is independent of the choice of a good pair (V, U) for the G -action up to unique isomorphisms. One should also observe that $\mathrm{CH}_G^j(X, i)$ may be nonzero for infinitely many values of j , a crucial change from the case of nonequivariant (ordinary) higher Chow groups.

If X is of dimension d , which is not necessarily equidimensional, one defines the equivariant higher Chow groups as

$$\mathrm{CH}_j^G(X, i) := H_i(\mathcal{Z}_{j+l-g}(X_G, \bullet)), \quad (2-1)$$

where (V, U) is an l -dimensional good pair for the G -action corresponding to $d - j$, and $\mathcal{Z}_p(X_G, \bullet)$ is the homological cycle complex of Bloch such that $\mathcal{Z}_p(X_G, i)$ is the group of admissible algebraic cycles on $X_G \times \Delta^i$ of dimension $p + i$. We write

$$\mathrm{CH}_*^G(X, i) = \bigoplus_{-\infty < j \leq d} \mathrm{CH}_j^G(X, i) \quad \text{and} \quad \mathrm{CH}_*^G(X) = \bigoplus_{i \geq 0} \mathrm{CH}_*^G(X, i). \quad (2-2)$$

It is easy to see that $\mathrm{CH}_j^G(X, i) = \mathrm{CH}_G^{d-j}(X, i)$ if X is equidimensional of dimension d . For most of this paper, we shall use the cohomological indexing for the equivariant

higher Chow groups while dealing with smooth schemes. In particular, $\mathrm{CH}_G^*(X)$ will denote the sum $\bigoplus_{i \geq 0} \mathrm{CH}_G^*(X, i)$.

For a commutative ring R , the equivariant higher Chow groups $\mathrm{CH}_G^j(X, i; R)$ are defined as the homology groups of the complex $\mathcal{L}^j(X_G, \bullet) \otimes_{\mathbb{Z}} R$. The symbol $\mathrm{CH}_G^*(X; R)$ will denote the direct sum of all $\mathrm{CH}_G^j(X, i; R)$. We shall denote the rings $\mathrm{CH}_G^*(k, 0)$ and $\mathrm{CH}_G^*(k, 0; R)$ by $S(G)$ and $S(G; R)$, respectively.

2b. Equivariant operational Chow groups. For $X \in \mathcal{V}_G$, we define

$$\mathrm{OPCH}_G^j(X, i) = \varinjlim \mathrm{CH}_G^j(Y, i), \quad (2-3)$$

where the limit is taken over the category of arrows $X \rightarrow Y$ in \mathcal{V}_G with $Y \in \mathcal{V}_G^S$. Notice that the natural map $\mathrm{OPCH}_G^j(X, i) \rightarrow \mathrm{CH}_G^j(X, i)$ is an isomorphism if X is smooth. We shall write the sum $\bigoplus_{i, j \geq 0} \mathrm{OPCH}_G^j(X, i)$ as $\mathrm{OPCH}_G^*(X)$.

It follows from [Bloch 1986, Proposition 5.5, Corollary 5.6] that $\mathrm{OPCH}_G^*(X)$ has a ring structure and $\mathrm{OPCH}_G^*(X, 0)$ is a subring of $\mathrm{OPCH}_G^*(X)$. Moreover, $X \mapsto \mathrm{OPCH}_G^*(X)$ is a contravariant functor on \mathcal{V}_G which acts on the higher Chow groups of X . In particular, $\mathrm{OPCH}_G^1(X, 0) \xrightarrow{\sim} \mathrm{Pic}_G(X)$ acts on $\mathrm{CH}_*^G(X, i)$. This action is same as the action of the Chern classes of equivariant line bundles on the homology theory $\mathrm{CH}_*^G(X, i)$.

2c. Main properties of equivariant higher Chow groups. The following result summarizes most of the essential properties of the equivariant higher Chow groups that will be used in this paper.

Proposition 2.2. *The equivariant higher Chow groups as defined above satisfy the following properties.*

- (1) *Functoriality: Covariance for proper maps, contravariance for flat maps and their compatibility. That is, for a fiber diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

in \mathcal{V}_G with f proper and g flat, one has

$$g^* \circ f_* = f'_* \circ g'^* : \mathrm{CH}_*^G(X, i) \rightarrow \mathrm{CH}_*^G(Y', i).$$

Moreover, if $f : X \rightarrow Y$ is a morphism in \mathcal{V}_G with Y in \mathcal{V}_G^S , then there is a pull-back map $f^* : \mathrm{CH}_*^G(Y, i) \rightarrow \mathrm{CH}_*^G(X, i)$.

- (2) *Homotopy: If $f : X \rightarrow Y$ is an equivariant vector bundle, then*

$$f^* : \mathrm{CH}_*^G(Y, i) \xrightarrow{\sim} \mathrm{CH}_*^G(X, i).$$

(3) *Exterior product: There is a natural product map*

$$\mathrm{CH}_*^G(X, i) \otimes \mathrm{CH}_*^G(Y, i') \rightarrow \mathrm{CH}_*^G(X \times Y, i + i').$$

Moreover, if $f : X \rightarrow Y$ is such that $Y \in \mathcal{V}_G^S$, then there is a pull-back via the graph map $\Gamma_f : X \rightarrow X \times Y$, which makes $\mathrm{CH}_G^*(Y)$ a bigraded ring and $\mathrm{CH}_*^G(X)$ a module over this ring. In particular, $\mathrm{CH}_*^G(X, i)$ an $S(G)$ -module for $X \in \mathcal{V}_G$ and $i \geq 0$.

(4) *Localization: If $Y \subset X$ is a G -invariant closed subscheme with complement U , then there is a long exact localization sequence of $S(G)$ -modules*

$$\cdots \rightarrow \mathrm{CH}_*^G(Y, i) \rightarrow \mathrm{CH}_*^G(X, i) \rightarrow \mathrm{CH}_*^G(U, i) \rightarrow \mathrm{CH}_*^G(Y, i - 1) \rightarrow \cdots .$$

This sequence is compatible with the push-forward and flat pull-back maps of higher Chow groups.

(5) *Chern classes: For any G -equivariant vector bundle of rank r , there are equivariant Chern classes $c_l^G(E) : \mathrm{CH}_j^G(X, i) \rightarrow \mathrm{CH}_{j-l}^G(X, i)$ for $0 \leq l \leq r$, having the same functoriality properties as in the nonequivariant case and $c_0^G(E) = 1$.*

(6) *Projection formula: For a proper map $f : X \rightarrow Y$ in \mathcal{V}_G and for $x \in \mathrm{CH}_*^G(X)$, $y \in \mathrm{OPCH}_G^*(Y)$, one has $f_*(f^*(y) \cdot x) = y \cdot f_*(x)$. Here, the action of $\mathrm{OPCH}_G^*(Y)$ on $\mathrm{CH}_*^G(X)$ is given by (3) above.*

(7) *Free action: If G acts freely on X with quotient Y , then there is a canonical isomorphism $\mathrm{CH}_*^G(X, i) \xrightarrow{\sim} \mathrm{CH}_*(Y, i)$.*

Proof. Since the equivariant higher Chow groups of X are defined in terms of the higher Chow groups of X_G , the proposition (except possibly the last property) can be easily deduced from the similar results for the nonequivariant higher Chow groups as in [Bloch 1986] and the techniques of [Edidin and Graham 1998]. We therefore skip the proof. To see that the maps in the localization sequence are $S(G)$ -linear, it suffices to know that for a good pair (V, U) , the long exact sequence

$$\cdots \rightarrow \mathrm{CH}_*(Y_G, i) \rightarrow \mathrm{CH}_*(X_G, i) \rightarrow \mathrm{CH}_*(U_G, i) \rightarrow \mathrm{CH}_*(Y_G, i - 1) \rightarrow \cdots$$

is a sequence of $\mathrm{CH}_*(U/G)$ -modules. But this is a well-known fact as U/G is smooth and the above is a sequence of higher Chow groups of schemes over it; see [Bloch 1986, Exercise 5.8(ii)].

To prove (6), we need to show that if $Y \xrightarrow{g} Z$ is a G -equivariant map with $Z \in \mathcal{V}_G^S$, then the map $\mathrm{CH}_*^G(X) \xrightarrow{f_*} \mathrm{CH}_*^G(Y)$ is $\mathrm{CH}_G^*(Z)$ -linear. Since the push-forward and the pull-back maps of equivariant Chow groups are nothing but the maps of ordinary higher Chow groups of suitable mixed quotients, it suffices to prove the statement above for the push-forward map of the higher Chow groups corresponding to the

maps of mixed quotients $X_G \rightarrow Y_G \rightarrow Z_G$. Since Z_G is smooth, this nonequivariant version is well-known [Bloch 1986, §5.5, Exercise 5.8].

For the last property, fix $j \leq d$ and choose a good pair (V, U) of dimension l for the G -action corresponding to $d - j \geq 0$. Since G acts freely on X , it acts likewise also on $X \times V$ with quotient, say X_V . Then X_G is an open subset of X_V and $X_V \rightarrow Y$ is a vector bundle, which implies that the map $\mathrm{CH}_j(Y, i) \rightarrow \mathrm{CH}_{j+l}(X_V, i)$ is an isomorphism by the homotopy invariance. On the other hand, the restriction map $\mathrm{CH}_{j+l}(X_V, i) \rightarrow \mathrm{CH}_{j+l}(X_G, i) = \mathrm{CH}_{j-g}^G(X, i)$ is an isomorphism by the property (4) as $d - j$ is sufficiently small. \square

Remark 2.3. The reader should be warned that the various isomorphisms between the (equivariant) higher Chow groups in the proposition above are true only up to some obvious shift in the dimension of cycles, which we have chosen not to write.

We next recall from [Edidin and Graham 1998] that the Chern classes $c_l^G(E)$ of an equivariant vector bundle E , as described in Proposition 2.2 above, live in the operational Chow groups $\mathrm{OPCH}^l(X_G)$. If X is in \mathcal{V}_G^S however, this operational Chow group is isomorphic to the equivariant Chow group $\mathrm{CH}_G^l(X, 0)$ and the action of $c_l^G(E)$ on $\mathrm{CH}_G^*(X)$ then coincides with the intersection product in the ring $\mathrm{CH}_G^*(X)$.

Finally, we recall from [ibid.] that if $H \subset G$ is a closed subgroup and if (V, U) is a good pair, then for $X \in \mathcal{V}_G$, the natural map of quotients $X \times^H U \rightarrow X \times^G U$ is an étale locally trivial G/H -fibration and hence there is a natural restriction map

$$r_{H,X}^G : \mathrm{CH}_*^G(X, i) \rightarrow \mathrm{CH}_*^H(X, i). \quad (2-4)$$

Taking $H = \{1\}$, one obtains the *forgetful* map

$$r_X^G : \mathrm{CH}_*^G(X, i) \rightarrow \mathrm{CH}_*(X, i). \quad (2-5)$$

Moreover, as $r_{H,X}^G$ is the pull-back under a flat (in fact, a smooth) map, it commutes (see Proposition 2.2) with the pull-back for any flat map, and with the push-forward for any proper map in \mathcal{V}_G . We remark here that although the definition of $r_{H,X}^G$ uses a good pair (V, U) for any given $j \leq \dim(X)$, it is easy to check from the homotopy invariance that $r_{H,X}^G$ is independent of the choice of the good pair (V, U) .

3. Morita isomorphisms

In this section, we prove some Morita-type isomorphisms that address the question of comparison between the equivariant higher Chow groups for the action of two different algebraic groups. We also prove a structure theorem for these equivariant higher Chow groups under the trivial action of split tori. These results are analogues of the similar results of Thomason in equivariant K -theory; see [Thomason 1986, Lemma 5.6; 1988, Section 1].

Proposition 3.1 (Morita isomorphism). *Let H be a normal subgroup of a linear algebraic group G and let $F = G/H$. Let $f : X \rightarrow Y$ be a G -equivariant morphism of G -varieties that is an H -torsor for the restricted action. Then the map induced on the equivariant higher Chow groups*

$$\mathrm{CH}_*^F(Y, i) \xrightarrow{f^*} \mathrm{CH}_*^G(X, i).$$

is an isomorphism.

Proof. We first observe from [Springer 1998, Corollary 12.2.2] that F is also a linear algebraic group over the given ground field k . Now, since f is an H -torsor, it is clear that G acts on Y via F . Fix $j \leq \dim(X)$ and choose a good pair (V, U) of dimension l for the F -action corresponding to $\dim(Y) - j$. Then V is also a representation of G in which U is G -invariant. In particular, G acts on $X \times U$ via the diagonal action, which is easily seen to be free since H acts freely on X and F acts freely on U . By the same reason, we see that $X \times U \rightarrow Y \times U$, which is a principal H -bundle, is G -equivariant. This in turn implies that the map $(X \times U)/G \rightarrow Y_F$ is an isomorphism and hence we get

$$\mathrm{CH}_j^F(Y, i) \cong \mathrm{CH}_{j+l-g+h}(Y_F, i) \xrightarrow{f^*} \mathrm{CH}_{j+l-g+h}(X \times^G U, i), \tag{3-1}$$

where $\dim(H) = h$. On the other hand, we have

$$\mathrm{CH}_{j+h}^G(X, i) \cong \mathrm{CH}_{j+h+l}^G(X \times V, i) \cong \mathrm{CH}_{j+h+l}^G(X \times U, i) \cong \mathrm{CH}_{j+h+l-g}(X \times^G U, i),$$

where the first isomorphism is due to the homotopy invariance, the second follows from the localization property (see Proposition 2.2(4)) as j is sufficiently small, and the third isomorphism follows from Proposition 2.2(7). The proof of the proposition now follows by combining this with (3-1). □

Corollary 3.2 (see [Edidin and Graham 2000]). *Let $H \subset G$ be a closed subgroup and let $X \in \mathcal{V}_H$. Then for any $i \geq 0$, there is a natural isomorphism*

$$\mathrm{CH}_*^G(G \times^H X, i) \xrightarrow{\sim} \mathrm{CH}_*^H(X, i). \tag{3-2}$$

Proof. Define an action of $H \times G$ on $G \times X$ by

$$(h, g) \cdot (g', x) = (gg'h^{-1}, hx) \tag{3-3}$$

and an action of $H \times G$ on X by $(h, g) \cdot x = hx$. Then the projection map $G \times X \xrightarrow{p} X$ is $(H \times G)$ -equivariant and is a G -torsor. Hence by Proposition 3.1, the natural map

$$\mathrm{CH}_*^H(X, i) \xrightarrow{p^*} \mathrm{CH}_*^{H \times G}(G \times X, i)$$

is an isomorphism. On the other hand, the projection map $G \times X \rightarrow G \times^H X$ is $(H \times G)$ -equivariant and is an H -torsor. Hence we get an isomorphism

$$\mathrm{CH}_*^G(G \times^H X, i) \xrightarrow{\sim} \mathrm{CH}_*^{H \times G}(G \times X, i).$$

The corollary follows by combining these two isomorphisms. □

Proposition 3.3. *Let G be a connected reductive group over k . Let B be a Borel subgroup of G containing a maximal torus T over k . Then for any $i \geq 0$, the restriction map*

$$\mathrm{CH}_*^B(X, i) \xrightarrow{r_{T,X}^B} \mathrm{CH}_*^T(X, i) \tag{3-4}$$

is an isomorphism for any $X \in \mathcal{V}_B$.

Proof. By [Corollary 3.2](#), we only need to show that

$$\mathrm{CH}_*^B(X, i) \xrightarrow{\sim} \mathrm{CH}_*^B(B \times^T X, i). \tag{3-5}$$

By [\[M. Demazure 1970, XXII, 5.9.5\]](#), there exists a characteristic filtration

$$B^u = U_0 \supseteq U_1 \supseteq \dots \supseteq U_n = \{1\}$$

of the unipotent radical B^u of B such that U_{j-1}/U_j is a vector group, each U_j is normal in B and $TU_j = T \times U_j$. Moreover, this filtration also implies that for each j , the natural map $B/TU_j \rightarrow B/TU_{j-1}$ is a torsor under the vector bundle $U_{j-1}/U_j \times B/TU_{j-1}$ on B/TU_{j-1} . Hence, the homotopy invariance gives an isomorphism

$$\mathrm{CH}_*^B(B/TU_{j-1} \times X, i) \xrightarrow{\sim} \mathrm{CH}_*^B(B/TU_j \times X, i).$$

Composing these isomorphisms successively for $j = 1, \dots, n$, we get

$$\mathrm{CH}_*^B(X, i) \xrightarrow{\sim} \mathrm{CH}_*^B(B/T \times X, i).$$

The canonical isomorphism of B -varieties $B \times^T X \cong B/T \times X$ (see [Corollary 3.2](#)) now proves [\(3-5\)](#) and hence [\(3-4\)](#). □

Recall that a linear algebraic group G over k of dimension g is *diagonalizable* if $G_{\bar{k}} \cong H \times (\mathbb{G}_m)^g$, where H is a finite abelian group. The group G is called split diagonalizable, if such an isomorphism is defined over k . A connected reductive group G over k is said to be *split* if it contains a split maximal torus T over k such that G is given by a root system relative to T . Every connected and reductive group containing a split maximal torus is split; see [\[M. Demazure 1970, Chapter XXII, Proposition 2.1\]](#).

Recall from [\[Springer 1998, Lemma 14.1.1\]](#) that every solvable group G over k has a filtration $\{e\} = G_0 \subseteq G_1 \subseteq \dots \subseteq G_n = G$ by closed normal k -subgroups such that each quotient group G_j/G_{j-1} is either diagonalizable or an elementary

unipotent group; see [Springer 1998, §3.4]. The group G is called *split* over k if each G_j/G_{j-1} is either split diagonalizable or \mathbb{G}_a . It is known [Springer 1998, Corollary 14.3.10] that every unipotent group over a perfect field is split.

Proposition 3.4. *Let H be a possibly nonreductive group over k . Assume that H has a Levi decomposition $H = L \ltimes H^u$ such that H^u is split over k (for example, if k is of characteristic zero). Then for each $i \geq 0$, the map*

$$\mathrm{CH}_*^H(X, i) \xrightarrow{r_{L,X}^H} \mathrm{CH}_*^L(X, i) \tag{3-6}$$

is an isomorphism.

Proof. Since the unipotent radical of H is split over k , the proof is exactly same as in the proof of (3-4), where we just have to replace B and T by H and L , respectively. □

3a. A structure theorem for $\mathrm{CH}_*^T(X)$. We end this section with the following structure theorem for the equivariant higher Chow groups of a scheme with the action of a split diagonalizable group on which certain subgroup acts trivially. This theorem is the initial step in the proof of its far reaching generalization in Theorem 10.3.

Theorem 3.5. *Let T be a split diagonalizable group and let $X \in \mathcal{V}_T$. Let H be a connected closed subgroup of T that acts trivially on X . Then there is a natural isomorphism*

$$\mathrm{CH}_*^{T/H}(X) \otimes_{\mathbb{Z}} \mathcal{S}(H) \xrightarrow{i_{H,X}^T} \mathrm{CH}_*^T(X).$$

This is a bigraded ring isomorphism if X is smooth.

Proof. Put $T' = T/H$. Since H is a split torus, we can choose a decomposition (not necessarily canonical) $T = H \times T'$. Fix an integer $j \leq \dim(X)$ and let (V, U) and (V', U') be good pairs for the actions of H and T' , respectively, corresponding to $\dim(X) - j$ as in [Edidin and Graham 1998, Example 3.1]. Thus U is a product of punctured affine spaces and $U/H = (\mathbb{P}^n)^r$ for some $n \gg 0$, where $r = \mathrm{rank}(H)$. Then (V_T, U_T) , with $V_T = V \times V'$ and $U_T = U \times U'$, is a good pair for the action of T . We now have

$$X_T = (X \times U \times U') / (H \times T') = (X \times U') \times^{T'} U/H = X_{T'} \times (\mathbb{P}^n)^r,$$

where the second equality holds since H acts trivially on $X \times U'$ and the third equality holds because T' acts trivially on U . It follows from the projective bundle formula (see also Lemma 6.2) for the ordinary higher Chow groups that the map

$$\mathrm{CH}_*(X_{T'}) \otimes_{\mathbb{Z}} \mathrm{CH}_*((\mathbb{P}^n)^r, 0) \rightarrow \mathrm{CH}_*(X_T) \tag{3-7}$$

is an isomorphism. We conclude the proof by noting that $\text{CH}_p(X_{T'}) \cong \text{CH}_p^{T'}(X)$ and $\text{CH}_p(X_T) \cong \text{CH}_p^T(X)$ for all $p \leq j$. If X is smooth, the assertion about the ring isomorphism of $i_{H,X}^T$ now follows because (3-7) is known to be a bigraded ring isomorphism in that case. \square

4. Self-intersection and projection formulas

Our aim in this section is to prove the following two results for the ordinary and equivariant higher Chow groups. The first result is the self-intersection formula for the higher Chow groups of smooth schemes. The analogue of this formula for the higher K -theory was proven by Thomason [1993, Theorem 3.1]. Surprisingly, this formula for the higher Chow groups has remained unnoticed. Its equivariant version will play a very crucial role in the decomposition Theorem 10.3 for the equivariant higher Chow groups of smooth schemes with an action of a diagonalizable group.

The second result of this section is a version of projection formula for the higher Chow groups of singular schemes. Such a formula for the smooth schemes was proven by Bloch [1986]. We shall need this version of the projection formula in our construction of Demazure operators on the equivariant higher Chow groups.

4a. Self-intersection formula. We shall use the method of *deformation to the normal cone* as the main technical tool to prove the self-intersection formula. Since this technique will be used several times in this paper, we briefly recall the construction from [Fulton 1984, Chapter 5] for our, as well as reader’s, convenience. Let X be a smooth scheme over k and let $f : Y \hookrightarrow X$ be a smooth closed subscheme of codimension $d \geq 1$. Let \tilde{M} be the blow-up of $X \times \mathbb{P}^1$ along $Y \times \infty$. Then $Bl_Y(X)$ is a closed subscheme of \tilde{M} and one denotes its complement by M . There is a natural map $\pi : M \rightarrow \mathbb{P}^1$ such that $\pi^{-1}(\mathbb{A}^1) \cong X \times \mathbb{A}^1$ with π the projection map and $\pi^{-1}(\infty) \cong X'$, where X' is the total space of the normal bundle $N_{Y/X}$ of Y in X . One also gets the following diagram, where all the squares and the triangles commute.

$$\begin{array}{ccccc}
 & & & & i_0 \\
 & & & & \curvearrowright \\
 Y & & & Y \times \mathbb{P}^1 & \xleftarrow{i_\infty} Y \\
 & \searrow^{u'} & & \nearrow^{j'} & \\
 & & Y \times \mathbb{A}^1 & & \\
 & \searrow^{f'} & & \nearrow^{F'} & \\
 X & & & M & \xleftarrow{i} X' \\
 & \searrow^u & & \nearrow^j & \\
 & & X \times \mathbb{A}^1 & & \\
 & & & & h \\
 & & & & \curvearrowleft \\
 & & & & F'
 \end{array}
 \tag{4-1}$$

In this diagram, all the vertical arrows are the closed embeddings, i_0 and i_∞ are the obvious inclusions of Y in $Y \times \mathbb{P}^1$ along the specified points, i and j are inclusions of the inverse images of ∞ and \mathbb{A}^1 , respectively, under the map π , u and f' are zero section embeddings and p_Y is the projection map. In particular, one has $p_Y \circ i_0 = p_Y \circ i_\infty = \text{id}_Y$.

In case X is a G -scheme and Y is G -invariant, then by letting G act trivially on \mathbb{P}^1 and diagonally on $X \times \mathbb{P}^1$, one gets a natural action of G on M , and all the spaces in the diagram above become G -spaces and all the morphisms become G -equivariant. This observation will be used later in this paper.

We shall need the following result about the higher Chow groups, which is an easy consequence of Bloch’s moving lemma.

Lemma 4.1. *Let*

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ g' \downarrow & & \downarrow g \\ Z & \xrightarrow{f} & X \end{array}$$

be a fiber diagram of closed immersions of schemes such that X and Y are smooth and Y and Z intersect properly in X . Then one has for each $i \geq 0$,

$$f^* \circ g_* = g'_* \circ f'^* : \text{CH}_*(Y, i) \rightarrow \text{CH}_*(Z, i).$$

Proof. Since X and Y are smooth, we can assume them to be equidimensional. Let

$$\mathcal{Z}_{ZW}^p(Y, \bullet) \xrightarrow{i_Y} \mathcal{Z}^p(Y, \bullet)$$

be the subcomplex that is generated by cycles on $Y \times \Delta^\bullet$ which intersect all faces of $Z \times \Delta^\bullet$ and $W \times \Delta^\bullet$ properly. Similarly, let

$$\mathcal{Z}_Z^p(X, \bullet) \xrightarrow{i_X} \mathcal{Z}^p(X, \bullet)$$

be the subcomplex generated by the cycles on $X \times \Delta^\bullet$ that intersect all faces of $Z \times \Delta^\bullet$ properly. Then i_X and i_Y are quasi-isomorphisms by the moving lemma; see [Krishna and Levine 2008, Theorem 1.10]. However, if $V \in \mathcal{Z}_{ZW}^p(Y, \bullet)$ is an irreducible cycle in $Y \times \Delta^n$, then the conclusion of the lemma is checked easily. \square

Corollary 4.2. *Let G be a linear algebraic group and let*

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ g' \downarrow & & \downarrow g \\ Z & \xrightarrow{f} & X \end{array}$$

be a fiber diagram of closed immersions of smooth G -schemes such that Y and Z intersect properly in X . Then one has $f^* \circ g_* = g'_* \circ f'^* : \mathrm{CH}_G^*(Y, i) \rightarrow \mathrm{CH}_G^*(Z, i)$.

Proof. By choosing a good pair (V, U) for the G -action and then considering the appropriate mixed quotients, we can reduce to proving the corollary for the nonequivariant higher Chow groups. But this is shown in [Lemma 4.1](#). \square

Lemma 4.3. *Consider the diagram (4-1) and let $y \in \mathrm{CH}^*(Y, m)$. Then there exists $z \in \mathrm{CH}^*(M, m)$ such that $f_*(y) = h^*(z)$ and $f'_*(y) = i^*(z)$.*

Proof. Put $\tilde{y} = p_Y^*(y)$ and $z = F_*(\tilde{y})$. Then

$$\begin{aligned} f_*(y) &= f_*((p_Y \circ i_0)^*(x)) = f_* \circ i_0^* \circ p_Y^*(y) = f_* \circ i_0^*(\tilde{y}) = f_* \circ u'^* \circ j'^*(\tilde{y}) \\ &= u'^* \circ F'_*(j'^*(\tilde{y})) && \text{(by Lemma 4.1)} \\ &= u'^* \circ j^* \circ F_*(\tilde{y}) && \text{(since } j \text{ is an open immersion)} \\ &= h^* \circ F_*(\tilde{y}) = h^*(z). \end{aligned}$$

Similarly,

$$\begin{aligned} f'_*(y) &= f'_*((p_Y \circ i_\infty)^*(x)) = f'_* \circ i_\infty^* \circ p_Y^*(y) = f'_* \circ i_\infty^*(\tilde{y}) \\ &= i^* \circ F_*(\tilde{y}) && \text{(by Lemma 4.1)} \\ &= i^*(z). \end{aligned} \quad \square$$

Theorem 4.4 (self-intersection formula). *Let $Y \xrightarrow{f} X$ be a closed immersion of smooth varieties of codimension $d \geq 0$, and let $N_{Y/X}$ be the normal bundle of Y in X . Then for every $y \in \mathrm{CH}^*(Y, i)$, one has $f^* \circ f_*(y) = c_d(N_{Y/X}) \cdot y$.*

Proof. There is nothing to prove when $d = 0$ and so we assume $d \geq 1$. We first consider the case when $X \xrightarrow{p} Y$ is a vector bundle of rank d and f is the zero section embedding so that $p \circ f = \mathrm{id}_Y$. In that case, we have

$$\begin{aligned} f^* \circ f_*(y) &= f^* \circ f_*(f^* \circ p^*(y)) = f^*(f_*(1) \cdot p^*(y)) \quad \text{(by Proposition 2.2(6))} \\ &= f^*(f_*(1)) \cdot (f^* \circ p^*(y)) = f^*(f_*(1)) \cdot y = c_d(N_{Y/X}) \cdot y, \end{aligned}$$

where the last equality follows from the self-intersection formula for Fulton's Chow groups; see [\[Fulton 1984, Corollary 6.3\]](#). This proves the theorem in the case of zero section embedding.

Now let $Y \hookrightarrow X$ be as in the theorem. We consider the deformation to the normal cone diagram (4-1) and choose $z \in \mathrm{CH}^*(M, i)$ as in [Lemma 4.3](#). Then we have

$$\begin{aligned} f^* \circ f_*(y) &= f^* \circ h^*(z) = i_0^* \circ F^*(z) = i_\infty^* \circ F^*(z) = f'^* \circ i^*(z) \\ &= f'^* \circ f'_*(y) && \text{(by Lemma 4.3)} \\ &= c_d(N_{Y/X}) \cdot y \quad \text{(by the case of vector bundle above)} \\ &= c_d(N_{Y/X}) \cdot y. \end{aligned}$$

This completes the proof of the theorem. □

Corollary 4.5. *Let G be a linear algebraic group over k and let $Y \xrightarrow{f} X$ be a closed immersion of codimension $d \geq 0$ in \mathcal{V}_G^S . Then for every $i \geq 0$ and $y \in \text{CH}_G^*(Y, i)$, one has $f^* \circ f_*(y) = c_d^G(N_{Y/X}) \cdot y$.*

Proof. There is nothing to prove if $d = 0$ and so we assume $d \geq 1$. Fix $i, j \geq 0$ and choose a good pair (V, U) for $n \gg j + d$. We can then identify $\text{CH}_G^p(X, i)$ with $\text{CH}^p(X_G, i)$ (and same for Y) for $p \leq n$. We can also identify $c_d^G(E)$ with $c_d(E_G)$ for any equivariant vector bundle E on Y ; see [Edidin and Graham 1998, Section 2.4]. Now, the proof of the corollary would follow straightaway from Theorem 4.4, once we show that $(N_{Y/X})_G$ is the normal bundle of Y_G in X_G . But this follows immediately from the elementary fact that if G acts freely on a smooth scheme Z and W is a smooth closed and G -invariant subscheme of Z with normal bundle N , then G acts freely on N , and moreover, N/G is the normal bundle of W/G in Z/G . We leave the proof of this fact to the reader. □

4b. A projection formula for singular schemes. Recall from Section 2b (see also [Bloch 1986, Corollary 5.6]) that the operational Chow groups $X \mapsto \text{OPCH}^*(X)$ form a ring-valued contravariant functor on \mathcal{V}_k that acts on the higher Chow groups. The action of $\text{OPCH}^1(X, 0) \xrightarrow{\sim} \text{Pic}(X)$ on $\text{CH}_*(X, j)$ coincides with the action of the Chern classes of line bundles.

Proposition 4.6. *Let $X \in \mathcal{V}_k$ and let $f : Y = \mathbb{P}(E) \rightarrow X$ be the projective bundle associated to a vector bundle E of rank $n + 1$ on X and let*

$$\xi = c_1(\mathcal{O}_Y(1)) \in \text{OPCH}^1(Y, 0)$$

be the first Chern class of the relative tautological line bundle on Y . Then for any $x \in \text{CH}_(X, j)$, one has*

$$f_*(\xi^i \cdot f^*(x)) = \begin{cases} 0 & \text{if } i < n, \\ x & \text{if } i = n. \end{cases}$$

Proof. If X is smooth, the proposition is an easy consequence of the projection formula [Bloch 1986, Exercise 5.8], as this formula implies that

$$f_*(\xi^i \cdot f^*(x)) = f_*(\xi^i) \cdot x.$$

Moreover, it follows from [Fulton 1984, Proposition 3.1] that $f_*(\xi^i) = 1$ if $i = n$ and zero otherwise. The case of singular schemes is the hard part of the theorem because we cannot directly apply the projection formula of [Bloch 1986]. We obtain a proof by an indirect approach of reduction to the smooth case and by unravelling the action of Chern classes on the higher Chow cycles on singular schemes.

By [Fulton 1984, Lemma 18.2], we can find a closed embedding $\iota : X \rightarrow X'$ and a vector bundle E' of rank $n + 1$ on X' such that $E \cong \iota^*(E')$ and X' is smooth. We set $Y' = \mathbb{P}(E')$, $\xi' = c_1(\mathcal{O}_{Y'}(1))$ and consider the Cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\iota'} & Y' \\ f \downarrow & & \downarrow f' \\ X & \xrightarrow{\iota} & X'. \end{array} \quad (4-2)$$

Recall from the construction of the action of ξ'^i on $\text{CH}_*(Y, j)$ in [Bloch 1986] that for any irreducible cycle $[V]$ on $Y \times \Delta^j$, the support of $\xi'^i \cdot [V] = \xi'^i \cdot [V]$ is $\text{supp}(V \cap \alpha)$, where α is cycle on Y' representing ξ'^i and such that each component of α intersects V properly. This is achieved by using the moving lemma on Y' , a smooth scheme. Since ξ' reduces the dimension of a cycle on $Y \times \Delta^j$ by exactly one, we see that $\dim(\xi'^i \cdot f^*([W])) = \dim(W) + n - i$ and the support of $f_*(\xi'^i \cdot f^*([W]))$ is contained in W , whenever W is an irreducible admissible cycle on $X \times \Delta^j$. We conclude from the definition of the push-forward map that $f_*(\xi'^i \cdot f^*([W]))$ must be zero if $n - i > 0$. Since any admissible cycle on $X \times \Delta^j$ is a sum of irreducible admissible cycles, this proves the first case.

We prove the case of $i = n$ by induction on n . If E is of rank one, then f is an isomorphism and $\mathcal{O}_Y(1)$ is trivial and hence $c_0(\mathcal{O}_Y(1)) = 1$. So we assume that $n \geq 1$. We let $\iota : X \hookrightarrow X'$ be a closed embedding into a smooth scheme as in (4-2).

By [Panin 2003, Lemma 3.24], there is a morphism $\phi' : T' \rightarrow X'$, which is a composite of projective and affine bundles on X' such that $\phi'^*(E') = F' \oplus L'$, where L' is a line bundle on T' . Moreover, if $\phi : T \rightarrow X$ is the restriction of ϕ' on X , then the pull-back map $\phi^* : \text{CH}_*(X, j) \rightarrow \text{CH}_*(T, j)$ is a split injection and the same holds for ϕ'^* . Notice here that X is a closed subscheme of X' and hence $\text{CH}_*(X) \xrightarrow{\sim} \text{CH}_*^X(X')$ in the notation of [ibid., Definition 2.1]. We denote the restrictions of F' and L' on X by F and L respectively.

Consider the Cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{\psi} & Y \\ g \downarrow & & \downarrow f \\ T & \xrightarrow{\phi} & X \end{array} \quad (4-3)$$

and suppose the given assertion holds for the projective bundle g . We then have

$$\begin{aligned} \phi^*(f_*(\xi'^n \cdot f^*(x))) &= g_*(\psi^*(\xi'^n \cdot f^*(x))) = g_*((\psi^*(\xi'))^n \cdot \psi^* \circ f^*(x)) \\ &= g_*((\psi^*(\xi))^n \cdot g^* \circ \phi^*(x)) = \phi^*(x), \end{aligned}$$

where the first equality holds by [Proposition 2.2](#) and the last equality holds by our assumption. Since ϕ^* is injective, we see that the conclusion holds for f as well. Thus we have reduced the problem to the case when $E' = F' \oplus L'$ and $E = F \oplus L$.

If we set $\tilde{E} = E \otimes L^{-1}$ and $\tilde{Y} = \mathbb{P}(\tilde{E})$, there is a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{h} & \mathbb{P}(E) \\ & \searrow \tilde{f} & \swarrow f \\ & & X \end{array}$$

such that h is an isomorphism and $\mathcal{O}_{\tilde{Y}}(1) = h^*(\mathcal{O}_Y(1)) \otimes \tilde{f}^*(L^{-1})$. Set $\eta = c_1(L)$ in $\text{OPCH}^1(X, 0)$ and $\tilde{\eta} = \tilde{f}^*(\eta)$ and $\tilde{\xi} = c_1(\mathcal{O}_{\tilde{Y}}(1))$ in $\text{OPCH}^1(\tilde{Y}, 0)$.

Suppose that our assertion holds for the projective bundle \tilde{f} . In this case, we get

$$f_*(\xi^n \cdot f^*(x)) = \tilde{f}_*((\tilde{\eta} + \tilde{\xi})^n \cdot \tilde{f}^*(x)) = \tilde{f}_*((\tilde{\xi})^n \cdot \tilde{f}^*(x)) = x,$$

where the last equality holds by our assertion about \tilde{f} . The second equality holds because of the fact that

$$\begin{aligned} \tilde{f}_*((\eta + \tilde{\xi})^n \cdot \tilde{f}^*(x)) &= \sum_{i=0}^n \binom{n}{i} \tilde{f}_*((\tilde{\eta})^i (\tilde{\xi})^{n-i} \cdot \tilde{f}^*(x)) \\ &= \sum_{i=0}^n \binom{n}{i} \eta^i \cdot \tilde{f}_*((\tilde{\xi})^{n-i} \cdot \tilde{f}^*(x)) \end{aligned}$$

(see [Bloch 1986](#), Exercise 5.8) and that the term $\tilde{f}_*((\tilde{\xi})^{n-i} \cdot \tilde{f}^*(x))$ vanishes for $0 \leq n - i < n$ by the first assertion of proposition. Hence, we are further reduced to the case when $E' = F' \oplus \mathcal{O}_{X'}$ and $E = F \oplus \mathcal{O}_X$.

Let $Z = \mathbb{P}(F)$ and let p and q be the closed and the open inclusions of $Z \hookrightarrow Y$ and $F \hookrightarrow Y$, respectively. Let $g : Z \hookrightarrow Y \rightarrow X$ be the composite map and set $\zeta = c_1(\mathcal{O}_Z(1))$. We observe that Z is a Cartier divisor on Y such that $\mathcal{O}_Y(1) = \mathcal{L}(Z)$. In particular, the pull-back p^* is defined and we have $\xi \cdot f^*(a) = p_*(p^* \circ f^*(a))$. Since F is of rank n , the desired assertion holds for $g : Z \rightarrow X$ by induction. That is, $g_*(\zeta^{n-1} \cdot g^*(x)) = x$. We now have

$$\begin{aligned} f_*(\xi^n \cdot f^*(x)) &= f_*(\xi^{n-1} p_*(p^* \circ f^*(x))) \\ &= f_*(p_*(\zeta^{n-1} \cdot p^* \circ f^*(x))) = g_*(\zeta^{n-1} \cdot g^*(x)) = x. \end{aligned}$$

This proves the desired assertion and the proof of the proposition is complete. \square

5. Demazure operators on equivariant higher Chow groups

In this section, we introduce Demazure (divided difference) operators on the equivariant higher Chow groups of schemes. Such operators were constructed by Brion

[1997] on the equivariant Chow groups $\text{CH}_*^G(X, 0)$ and by Holm and Sjamaar [2008] on the equivariant singular cohomology $H_G^*(X)$. We extend these operators to the higher Chow groups and discuss some consequences.

Let G be a connected reductive group with a split maximal torus T of rank n . Let $\Phi = \Phi(G, T) \subset \mathcal{X}(T)$ be the root system of G with respect to T , where $\mathcal{X}(T)$ is the character group of T . For any $\alpha \in \Phi$, let P_α be the minimal parabolic subgroup of G corresponding to α and let $B = TU_\alpha U$ and $B' = TU_{-\alpha}U$ be the opposite Borel subgroups of G containing T in P_α . Let $W_\alpha = \{s_\alpha, s_{-\alpha}\}$ denote the Weyl group of P_α , where s_α is the reflection in $\mathcal{X}(T)$ given by

$$s_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha \quad \text{for } \lambda \in \mathcal{X}(T).$$

Let X be a k -scheme with a free G -action and let B and B' act on $X \times P_\alpha$ by $b \cdot (x, g) = (x, bg)$ and $b' \cdot (x, g) = (b'x, gb'^{-1})$. It is easy to check that the two actions are free and they commute with each other. Hence we get a free action of $B \times B'$ on $X \times G$ by $(b, b') \cdot (x, g) = (b'x, bgb'^{-1})$. One checks that B acts freely on $X' = X \times^{B'} P_\alpha$. P_α acts freely on X' by acting trivially on its X -factor and by left multiplication on the P_α -factor. In particular, we have a B -equivariant map $X' \rightarrow X$ given by $[x, g] \mapsto x$, which yields the projection map on quotients $f_1 : X'/B \rightarrow X/B \cong X'/P_\alpha$. One also has the P_α -equivariant map $X' \rightarrow X$ given by $[x, g] \mapsto [gx]$, which yields the map $f_2 : X'/B \rightarrow X/B$ on quotients by the action of B . It is also easy to check that the data above yield the commutative diagram

$$\begin{array}{ccc}
 & & \sigma_2 \\
 & \swarrow & \searrow \\
 X'/B & \xrightarrow{f_2} & X/B \\
 \downarrow f_1 & & \downarrow p_X \\
 X/B & \xrightarrow{p_X} & X/P_\alpha,
 \end{array}
 \tag{5-1}$$

which is Cartesian. The section σ_2 of f_2 is defined by $\sigma_2([x]) = [x, 1]$ and the section σ_1 of f_1 is defined by $\sigma_1([x]) = [x, n_\alpha]$, where n_α is a representative of s_α in $N_{P_\alpha}(T)$. Since f_2 is induced by a P_α -equivariant map, we see that it is W_α -equivariant with respect to the natural action of W_α on X'/B and X/B . In particular, we get

$$\begin{aligned}
 f_2^* \circ s_\alpha &= s_\alpha \circ f_2^*, \\
 f_2^*(ax) &= af_2^*(x) \quad \text{for all } a \in S \text{ and } x \in \text{CH}_*(X/B).
 \end{aligned}
 \tag{5-2}$$

Note that the all the maps (except the sections) in (5-1) are $P_\alpha/B \cong \mathbb{P}^1$ -bundles and hence they are all smooth and projective. Let $\mathcal{O}_{X/B}(1)$ denote the universal quotient line bundle on X/B relative to p_X .

Let D denote the image of σ_1 and let $\mathcal{L}(D)$ denote the associated line bundle on X'/B . Let \mathcal{L}_λ and \mathcal{L}'_λ denote the line bundles $X \times^B L_\lambda$ and $X' \times^B L_\lambda$ on X/B and X'/B respectively, where L_λ is the B -equivariant line bundle on $\text{Spec}(k)$ corresponding to the character λ of T . It follows from [Demazure 1974, Lemme 2, Proposition 2] that

$$\begin{aligned} \mathcal{L}'_\alpha &= f_1^*(\mathcal{L}_\alpha) \otimes (\mathcal{L}(D))^{\otimes 2}, \\ s_\alpha(\mathcal{L}(D)) &= (\mathcal{L}(D) \otimes f_1^*(\mathcal{L}_\alpha))^{\otimes (-1)}. \end{aligned} \tag{5-3}$$

5a. Demazure operators on $\text{CH}_*(X)$. Let G be a connected reductive group with split maximal torus T as above and let X be a G -scheme of dimension d . Let $j \leq d$ be an integer and let (V, U) be a good pair for the G -action corresponding to $d - j$. The smooth and projective morphism $p_{X \times U} : X_B \rightarrow X_{P_\alpha}$ yields the maps

$$p_{X \times U}^* : \text{CH}_j^{P_\alpha}(X, i) \rightarrow \text{CH}_j^B(X, i) \quad \text{and} \quad p_{X \times U*} : \text{CH}_j^B(X, i) \rightarrow \text{CH}_{j+1}^{P_\alpha}(X, i). \tag{5-4}$$

For the rest of this text, the ring S will denote the equivariant Chow ring

$$S(T) = \text{CH}_T^*(k, 0).$$

Lemma 5.1. *The maps $p_{X \times U}^*$ and $p_{X \times U*}$ do not depend on the choice of the good pair (V, U) .*

Proof. We prove the lemma for the push-forward map and a very similar proof works also for the pull-back map; see [Edidin and Graham 1998, Section 1].

Let g and b denote the dimensions of P_α and B , respectively. Let (V, U) and (V', U') be good pairs of dimensions l and l' , respectively, corresponding to $d - j$. We let $\mathcal{V} = V \oplus V'$ and $\mathcal{U} = (U \oplus V') \cup (V \oplus U')$. Let G act diagonally on \mathcal{V} . Then it is easy to see that the dimension of the complement of the open subset $X \times^B (U \oplus V')$ in $X \times^B \mathcal{U}$ is sufficiently smaller than $l + l' - b + j$. Similarly, the dimension of the complement of the open subset $X \times^{P_\alpha} (U \oplus V')$ in $X \times^{P_\alpha} \mathcal{U}$ is sufficiently smaller than $l + l' - g + j \leq l + l' - b + j$. It follows from the localization sequence for the higher Chow groups and Lemma 5.2 that there is a commutative diagram

$$\begin{array}{ccc} \text{CH}_{l+l'+j-b}(X \times^B \mathcal{U}, i) & \longrightarrow & \text{CH}_{l+l'+j-b}(X \times^B (U \oplus V'), i) \\ \downarrow & & \downarrow \\ \text{CH}_{l+l'+j-b}(X \times^{P_\alpha} \mathcal{U}, i) & \longrightarrow & \text{CH}_{l+l'+j-b}(X \times^{P_\alpha} (U \oplus V'), i) \end{array}$$

where the vertical maps are the push-forward maps and the horizontal maps are isomorphisms.

On the other hand, the maps

$$X \times^G (U \oplus V') \rightarrow X \times^G U \quad \text{and} \quad X \times^B (U \oplus V') \rightarrow X \times^B U$$

are vector bundles of rank l' and hence we get another commutative diagram

$$\begin{CD} \mathrm{CH}_{l+j-b}(X \times^B U, i) @>>> \mathrm{CH}_{l+l'+j-b}(X \times^B (U \oplus V'), i) \\ @VVV @VVV \\ \mathrm{CH}_{l+j-b}(X \times^{P_\alpha} U, i) @>>> \mathrm{CH}_{l+l'+j-b}(X \times^{P_\alpha} (U \oplus V'), i), \end{CD}$$

where the vertical arrows are the push-forward maps and the horizontal arrows are isomorphisms by the homotopy invariance.

Combining the two isomorphisms above, we get the commutative diagram

$$\begin{CD} \mathrm{CH}_{j+l+l'-b}(X \times^B \mathcal{U}, i) @>>> \mathrm{CH}_{l+j-b}(X \times^B U, i) \\ @VVV @VVV \\ \mathrm{CH}_{j+l+l'-g}(X \times^{P_\alpha} \mathcal{U}, i) @>>> \mathrm{CH}_{l+l'+j+1-g}(X \times^{P_\alpha} (U \oplus V'), i), \end{CD}$$

where the horizontal maps are isomorphisms. By repeating the same argument with U' , we get the diagram above with U replaced by U' and V' replaced by V on the right column. This proves the lemma. \square

Lemma 5.2. *Let $p : X \rightarrow Y$ be a morphism in \mathcal{V}_{P_α} such that P_α acts freely on X and Y . Then the diagram of quotients*

$$\begin{CD} X/B @>>> Y/B \\ @VVV @VVV \\ X/P_\alpha @>>> Y/P_\alpha \end{CD}$$

is Cartesian such that the vertical maps are smooth and projective.

Proof. This is an easy exercise. \square

Proposition 5.3. *For any $X \in \mathcal{V}_G$, one has the restriction and the push-forward maps*

$$r_{B,X}^{P_\alpha} : \mathrm{CH}_*^{P_\alpha}(X, i) \rightarrow \mathrm{CH}_*^B(X, i) \quad \text{and} \quad p_{B,X}^{P_\alpha} : \mathrm{CH}_*^B(X, i) \rightarrow \mathrm{CH}_{*+1}^{P_\alpha}(X, i).$$

These maps are contravariant with respect to the flat maps and covariant with respect to the proper morphisms of schemes in \mathcal{V}_G .

Proof. Let $j \leq d$ be an integer and let (V, U) be a good pair for the G -action corresponding to $d - j$. We define $r_{B,X}^{P_\alpha}$ and $p_{B,X}^{P_\alpha}$ to be $p_{X \times U}^*$ and $p_{X \times U*}$ respectively. It follows from Lemma 5.1 that these maps are well-defined.

The functoriality properties of $r_{B,X}^{P_\alpha}$ is already known; see [Edidin and Graham 1998, Section 1]. To prove these properties for $p_{B,X}^{P_\alpha}$, it suffices to prove the same

for $p_{X \times U*}$. But this follows easily from [Lemma 5.2](#) and the similar properties of the ordinary higher Chow groups. \square

The following result generalizes [\[Brion 1997, Theorem 6.3\]](#) to equivariant higher Chow groups.

Theorem 5.4. *Let α be a simple root of G . For any $X \in \mathcal{V}_G$ and $i \geq 0$, there is a unique operator δ_α^X on $\text{CH}_*^T(X, i)$ such that for all $u \in S$ and $v \in \text{CH}_*^T(X, i)$, we have*

- (1) $\alpha \delta_\alpha^X(v) = v - s_\alpha(v)$ and
- (2) $\delta_\alpha^X(uv) = u \delta_\alpha^X(v) + \delta_\alpha^k(u) s_\alpha(v)$ if X is smooth.

Moreover, δ_α^X commutes with the G -equivariant flat pull-back and proper push-forward maps between T -equivariant higher Chow groups.

Proof. Let B and B' be the opposite Borel subgroups of P_α containing T . Using [Proposition 3.3](#), we can replace T by B to define δ_α^X . We let

$$\delta_\alpha^X := r_{B,X}^{P_\alpha} \circ p_{B,X}^{P_\alpha} : \text{CH}_j^B(X, i) \rightarrow \text{CH}_{j+1}^B(X, i). \tag{5-5}$$

The co- and contravariant functoriality of δ_α^X follows from [Proposition 5.3](#). The uniqueness of δ_α^X follows from [\[Brion 1997, Theorem 6.3\]](#) since this definition of δ_α^X coincides with the one defined for $\text{CH}_*^B(X, 0)$ by Brion. We only need to show the first and the second assertions.

Let $j \leq \dim(X)$ and (V, U) be a good pair the G -action corresponding to $d - j$. There is a G -equivariant projection $X \times U \rightarrow X$ such that

$$\text{CH}_j^G(X, i) \xrightarrow{\sim} \text{CH}_j^G(X \times U, i)$$

and δ_α^X on $\text{CH}_j^G(X, i)$ coincides with the operator $\delta_\alpha^{X \times U}$ on $\text{CH}_j^G(X \times U, i)$ by its construction. Hence we can assume that G acts freely on X .

We now consider the diagram (5-1). Since f_2 is a \mathbb{P}^1 -bundle, the map f_2^* is split injective by [Proposition 4.6](#). Hence it suffices to show that the two assertions of the theorem hold after applying f_2^* .

On the other hand, $f_2^*(s_\alpha(v)) = s_\alpha(f_2^*(v))$ and $f_2^*(\alpha \delta_\alpha^X(v)) = \alpha f_2^*(\delta_\alpha^X(v))$ by (5-2). Since f_2 is induced by a G -equivariant map $X' \rightarrow X$, we also have $f_2^*(\delta_\alpha^X(v)) = \delta_\alpha^{X'}(f_2^*(v))$ by the functoriality of δ_α . Thus we need to show for $u \in S$ and $v' \in \text{CH}_*(X'/B, i)$ that

$$\alpha \delta_\alpha^{X'}(v') = v' - s_\alpha(v') \quad \text{and} \quad \delta_\alpha^{X'}(uv') = u \delta_\alpha^{X'}(v') + \delta_\alpha^k(u) s_\alpha(v') \quad \text{if } X \in \mathcal{V}_G. \tag{5-6}$$

Let $c : \text{CH}_T^*(k, 0) \cong \text{OPCH}_T^*(k, 0) \rightarrow \text{OPCH}^*(X/B, 0)$ be the ring homomorphism on the operational Chow groups induced by the map on the Picard groups $L_\alpha \mapsto \mathcal{L}_\alpha$. We denote the corresponding map $\text{OPCH}_T^*(k, 0) \rightarrow \text{OPCH}^*(X'/B, 0)$ by c' . We set $\xi = c_1(\mathcal{L}(D))$ and $\zeta = c_1(\mathbb{O}_{X'/B}(1))$, where $\mathbb{O}_{X'/B}(1)$ is the universal

quotient line bundle associated to the \mathbb{P}^1 -bundle f_1 . We shall write δ_α^k simply as δ_α in what follows.

Since $\mathcal{L}(D)$ and $\mathcal{O}_{X'/B}(1)$ have same degree on every fiber of f_1 , there is a line bundle \mathcal{M} on X/B such that $\mathcal{O}_{X'/B}(1) \cong \mathcal{L}(D) \otimes f_1^*(\mathcal{M})$. In particular, there exists $\eta \in \text{OPCH}^1(X/B, 0)$ such that $\zeta = \xi + f_1^*(\eta)$. Since f_1^* commutes with the action of $\text{OPCH}^*(X/B, 0)$ on the higher Chow groups, it follows from [Proposition 4.6](#) that we can write $v' \in \text{CH}_*(X'/B, i)$ as

$$v' = f_1^*(a) + \xi f_1^*(b), \quad \text{with } a, b \in \text{CH}_*(X/B, i). \quad (5-7)$$

Furthermore, it also follows that for any $b \in \text{CH}_*(X/B, i)$,

$$f_{1*}(\xi f_1^*(b)) = f_{1*}(\zeta f_1^*(b)) - f_{1*}(f_1^*(\eta b)) = b + 0 = b. \quad (5-8)$$

Since s_α keeps the elements of the form $f^*(a)$ invariant, we get

$$\begin{aligned} s_\alpha(v') &= f_1^*(a) + s_\alpha(\xi) f_1^*(b) \\ &= f_1^*(a) - [\xi + f_1^*(c(\alpha))] f_1^*(b), \end{aligned} \quad (5-9)$$

where the second equality follows from [\(5-3\)](#).

On the other hand, we have seen in [\(5-1\)](#) that f_1 is same as the quotient map $X'/B \rightarrow X'/P_\alpha$ and hence $\delta_\alpha^{X'} = f_1^* f_{1*}$. This yields

$$\begin{aligned} v' - c'(\alpha) \delta_\alpha^{X'}(v') &= f_1^*(a) + \xi f_1^*(b) - c'(\alpha) f_1^*[f_{1*} f_1^*(a) + f_{1*}(\xi f_1^*(b))] \\ &=^\dagger f_1^*(a) + \xi f_1^*(b) - c'(\alpha) f_1^*(b) \\ &=^\ddagger f_1^*(a) + \xi f_1^*(b) - [f_1^*(c(\alpha)) + 2\xi] f_1^*(b) \\ &= f_1^*(a) - [\xi + f_1^*(c(\alpha))] f_1^*(b), \end{aligned} \quad (5-10)$$

where $=^\dagger$ follows from [Proposition 4.6](#) and [\(5-8\)](#), and $=^\ddagger$ follows from [\(5-3\)](#). The first equality of [\(5-6\)](#) follows at once by comparing [\(5-9\)](#) and [\(5-10\)](#).

To prove the second equality of [\(5-6\)](#) for $u \in S$ and $v' \in \text{CH}_*(X'/B, i)$ with X smooth, we can assume using [\(5-7\)](#) that v' is either $f_1^*(a)$ or $\xi f_1^*(a)$. We now have

$$\begin{aligned} \delta_\alpha^X(uf_1^*(a)) &= f_1^* \circ f_{1*}(uf_1^*(a)) =^1 f_1^*(f_{1*}(u) \cdot a) = f_1^* f_{1*}(u) \cdot f_1^*(a) \\ &=^2 \delta_\alpha^k(u) \cdot f_1^*(a) = \delta_\alpha^k(u) \cdot s_\alpha(f_1^*(a)) + u \delta_\alpha^X(f_1^*(a)). \end{aligned}$$

The equality $=^1$ holds by the projection formula for smooth schemes (see [\[Bloch 1986, Exercise 5.8\]](#)) and $=^2$ holds by [\[Brion 1997, Theorem 6.3\]](#). The last equality holds because $f_1^*(a)$ is invariant under s_α and $f_1^* f_{1*}(a)$ vanishes, again by the projection formula. The required formula for $v' = \xi f_1^*(a)$ is proved exactly in the similar way using the observation that $f_{1*}(\xi) = 1$ and that the equality $=^2$ holds even if we replace u by ξu . \square

Proposition 5.5. *For $X \in \mathcal{V}_G$, let $r_{T,X}^G : \mathrm{CH}_*^G(X, i) \rightarrow \mathrm{CH}_*^T(X, i)$ be the restriction map.*

- (1) $\delta_\alpha \circ r_{T,X}^G = 0$.
- (2) *If X is smooth, then δ_α is $\mathrm{CH}_G^*(X)$ -linear.*
- (3) $\delta_\alpha^2 = 0$.
- (4) $s_\alpha \delta_\alpha = \delta_\alpha = -\delta_{-\alpha}$, $\delta_\alpha s_\alpha = -\delta_\alpha$.

Proof. Since $r_{T,X}^G = r_{T,X}^{P_\alpha} \circ r_{P_\alpha,X}^G$, we can replace G by P_α . It suffices then to show that

$$p_{B,X}^{P_\alpha} \circ r_{B,X}^{P_\alpha} = 0.$$

But this follows immediately from [Proposition 4.6](#). The second point follows from the observation that f_1^* and f_{1*} in [\(5-1\)](#) are $\mathrm{CH}^*(X'/G)$ -linear. The third point follows directly from the first and the fourth point is an immediate consequence of the other assertions of the proposition. □

5b. Ring of Demazure operators. Let $\{\alpha_1, \dots, \alpha_m\}$ be the set of all simple roots of G . For any sequence $I = \{i_1, \dots, i_l\}$ of integers in the interval $[1, m]$, we define the operator δ_I^X on $\mathrm{CH}_*^T(X, i)$ by

$$\delta_I^X = \delta_{\alpha_{i_1}} \circ \dots \circ \delta_{\alpha_{i_l}}. \tag{5-11}$$

Following the notation of [\[Brion 1997, §6.4\]](#), we let \mathbf{D} denote the subring of $\mathrm{End}_{\mathbb{Z}}(S)$ generated by the elements δ_α^k and the endomorphisms given by the multiplication in S . It is clear from the definition of \mathbf{D} and [Theorem 5.4](#) that \mathbf{D} contains the twisted group algebra $S[W]$ and there are inclusions of rings $S \subseteq S[W] \subseteq \mathbf{D}$. It is known that \mathbf{D} is a free S -module with basis $\{\partial_w\}_{w \in W}$, where ∂_w is same as δ_I^k above whenever $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_l}}$. As an immediate consequence of [Theorem 5.4](#), [Proposition 5.5](#) and [\(5-11\)](#), we get:

Corollary 5.6. *For any $X \in \mathcal{V}_G$ and $i \geq 0$, there is a unique \mathbf{D} -module structure on $\mathrm{CH}_*^T(X, i)$, which extends the action of $S[W]$. Moreover, the flat pull-back and proper push-forward maps between the T -equivariant higher Chow groups are \mathbf{D} -linear. For $X \in \mathcal{V}_G^S$, the \mathbf{D} -module structure on $\mathrm{CH}_T^*(X)$ commutes with its $\mathrm{CH}_G^*(X)$ -module structure.*

Let $I(\mathbf{D})$ be the subset of \mathbf{D} consisting of those operators δ such that $\delta(1) = 0$. It is easy to check that $I(\mathbf{D})$ is a left ideal of \mathbf{D} generated by $\{\partial_w\}_{w \neq 1}$. For any $X \in \mathcal{V}_G$, let

$$(\mathrm{CH}_*^T(X, i))^{I(\mathbf{D})} = \{x \in \mathrm{CH}_*^T(X, i) \mid \delta(x) = 0 \ \forall \delta \in I(\mathbf{D})\}. \tag{5-12}$$

Since the Weyl group is generated by simple reflections, it follows from [Theorem 5.4](#) and [Proposition 5.5](#) that

$$r_{T,X}^G(\mathrm{CH}_*^G(X, i)) \subseteq (\mathrm{CH}_*^T(X, i))^{I(D)} \subseteq (\mathrm{CH}_*^T(X, i))^W. \tag{5-13}$$

Recall that the torsion index t_G of the group G is the order of the cokernel of the map $S_N \rightarrow \mathrm{CH}^N(G/B)$, where $N = \dim(G/B) = |\Phi^+|$. Let R denote the localized ring $\mathbb{Z}[t_G^{-1}]$.

Theorem 5.7. *Let $X \in \mathcal{V}_G^S$ and $i \geq 0$ be such that $\mathrm{CH}_T^*(X, i)$ is torsion-free. Then the map $\mathrm{CH}_*^G(X, i) \rightarrow (\mathrm{CH}_*^T(X, i))^W$ is an isomorphism over R .*

Proof. Let B be a Borel subgroup of G containing T . Let (V, U) be a good pair for the G -action and consider the Cartesian diagram

$$\begin{array}{ccc} X_B & \xrightarrow{q_B} & U/B \\ p_X \downarrow & & \downarrow p \\ X_G & \xrightarrow{q_G} & U/G. \end{array} \tag{5-14}$$

By the definition of t_G , it follows that there is $a \in \mathrm{CH}^N(U/B, 0)$ such that $p_*(a) = t_G \in \mathrm{CH}^0(U/G, 0)$. Using the projection formula, we get for any $x \in \mathrm{CH}^*(X_G, i)$,

$$p_{X*}(q_B^*(a)p_X^*(x)) = p_{X*}(q_B^*(a))x = q_G^*(p_*(a))x = t_G x.$$

In particular, $r_{T,X}^G$ is split injective over R .

To show the surjectivity, we see from the above that for any $x \in \mathrm{CH}^*(X_G, i)$,

$$p_X^* f p_X^*(x) = p_X^*(t_G x) = t_G p_X^*(x),$$

where $f(y) = p_{X*}(ay)$. This in particular implies that $p_X^* f(y) = t_G y$ for all y in the image of p_X^* . It follows from [Corollary 8.7](#) that

$$p_X^* f(y) = t_G y \quad \text{for all } y \in (\mathrm{CH}^*(X_B, i)_{\mathbb{Q}})^W.$$

Since $\mathrm{CH}_T^*(X, i)$ is torsion-free, we must have

$$p_X^* f(y) = t_G y \quad \text{for all } y \in (\mathrm{CH}^*(X_B, i))^W.$$

Hence the map $r_{T,X}^G$ is surjective onto the W -invariants over R . □

Remark 5.8. The proof of the theorem above in fact shows that the map $r_{T,X}^G$ is split injective over R for any $X \in \mathcal{V}_G^S$.

Corollary 5.9. *Let X be a smooth projective scheme with a G -action such that the fixed point locus X^T for the T -action is isolated. Then the map*

$$\mathrm{CH}_*^G(X, 0) \rightarrow (\mathrm{CH}_*^T(X, 0))^W$$

is an isomorphism over R .

Proof. This is an immediate consequence of [Krishna 2009a, Theorem 4.2] and Theorem 5.7. □

6. The Leray–Hirsch Theorem

In algebraic topology, the Leray–Hirsch theorem is a very important tool for describing the cohomology of the total space of a fiber bundle. Since the arguments in this theorem are mostly topological, one cannot always expect such results for the cohomology theories of algebraic varieties. A version of the Leray–Hirsch theorem was proven for the Chow groups of the total space of a Zariski-locally trivial fibration in [Ellingsrud and Strømme 1989, Lemma 2.8; Edidin and Graham 1997, Lemma 6]. In this section, we prove the general form of the Leray–Hirsch theorem for the higher Chow groups of schemes. We shall give several important applications of this theorem in the next few sections.

6a. A Künneth formula. In [Fulton 1984, Example 1.9.1], a k -scheme L is called *cellular* if it has a filtration $\emptyset = L_{n+1} \subsetneq L_n \subsetneq \cdots \subsetneq L_1 \subsetneq L_0 = L$ by closed subschemes such that each $L_i \setminus L_{i+1}$ is an affine space $\mathbb{A}_k^{r_i}$.¹ It follows from the Bruhat decomposition that schemes of the type G/B are cellular, where B is a Borel subgroup of a split reductive group G .

Lemma 6.1. *Let L be a cellular scheme with the cellular decomposition*

$$\emptyset = L_{n+1} \subsetneq L_n \subsetneq \cdots \subsetneq L_1 \subsetneq L_0 = L$$

and let $U_i = L \setminus L_i$ for $0 \leq i \leq n + 1$. Then for any $0 \leq i \leq n$ and $p \geq 0$, the sequence

$$0 \rightarrow \mathrm{CH}_*(U_{i+1} \setminus U_i, p) \rightarrow \mathrm{CH}_*(U_{i+1}, p) \rightarrow \mathrm{CH}_*(U_i, p) \rightarrow 0$$

is exact.

Proof. The proof is very similar to the arguments of [Kahn 1999, Lemma 3.3] using an induction on the number of cells. □

Lemma 6.2. *Let L be a cellular scheme and let X be a any k -scheme. Then the exterior product map*

$$\mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathrm{CH}_*(L, 0) \rightarrow \mathrm{CH}_*(X \times L) \tag{6-1}$$

¹Some authors allow $L_i \setminus L_{i+1}$ to be a disjoint union of affine spaces over k . But both definitions are equivalent.

is an isomorphism. In particular, the natural map $\text{CH}^*(k) \otimes_{\mathbb{Z}} \text{CH}_*(L, 0) \rightarrow \text{CH}_*(L)$ is an isomorphism.

Proof. Consider the cellular decomposition of L as in Lemma 6.1. Then each $U_i = L \setminus L_i$ is also a cellular scheme. It suffices to show by induction that (6-1) holds when L is any of these U_i . There is nothing to prove for $i = 0$ and the case $i = 1$ follows by the homotopy invariance since U_1 is an affine space. In general, we have the short exact sequence

$$0 \rightarrow \text{CH}_*(U_{i+1} \setminus U_i, 0) \rightarrow \text{CH}_*(U_{i+1}, 0) \rightarrow \text{CH}_*(U_i, 0) \rightarrow 0 \quad (6-2)$$

by applying Lemma 6.1 with $p = 0$. Since each $U_{i+1} \setminus U_i$ is an affine space, it also follows from Lemma 6.1 and by induction on the number of affine cells that each $\text{CH}_*(U_i, 0)$ is a free abelian group of finite rank. Tensoring this with $\text{CH}_*(X)$ over $\text{CH}^*(k, 0) = \mathbb{Z}$, we get a commutative diagram

$$\begin{array}{ccc}
 0 & & 0 \\
 \downarrow & & \downarrow \\
 \text{CH}_*(X) \otimes \text{CH}_*(U_{i+1} \setminus U_i, 0) & \longrightarrow & \text{CH}_*(X \times (U_{i+1} \setminus U_i)) \\
 \downarrow & & \downarrow i_* \\
 \text{CH}_*(X) \otimes \text{CH}_*(U_{i+1}, 0) & \longrightarrow & \text{CH}_*(X \times U_{i+1}) \\
 \downarrow & & \downarrow j_* \\
 \text{CH}_*(X) \otimes \text{CH}_*(U_i, 0) & \longrightarrow & \text{CH}_*(X \times U_i), \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

where the left column is exact by the freeness of each $\text{CH}_*(U_i, 0)$ and the right column is the localization exact sequence. The top horizontal arcolumn is an isomorphism by the homotopy invariance and the bottom horizontal arcolumn is an isomorphism by the induction. In particular, j_* is surjective in all indices. We conclude that i_* is injective in all indices and the middle horizontal arcolumn is also an isomorphism. □

6b. Leray–Hirsch with integral coefficients. Let F be a cellular scheme over k . For any field extension $k \hookrightarrow l$, the scheme F_l is also cellular, for which the cellular decomposition and the affine cells are the base extensions of the cellular decomposition and affine cells of F . It follows from Lemma 6.1 that the map $\text{CH}_*(F, 0) \rightarrow \text{CH}_*(F_l, 0)$ is an isomorphism. The following is the integral version of the Leray–Hirsch theorem for the Zariski-locally trivial fibrations.

Theorem 6.3. *Let B be a smooth k -scheme and let $F \xrightarrow{i} E \xrightarrow{p} B$ be a Zariski-locally trivial fibration such that the fiber F is a smooth cellular scheme. Assume that there are elements $\{e_1, \dots, e_r\}$ in $\text{CH}^*(E, 0)$ such that*

$$\{f_1 = i^*(e_1), \dots, f_r = i^*(e_r)\}$$

forms a \mathbb{Z} -basis of $\text{CH}^(F_y, 0)$ for each fiber F_y of the fibration. Then the map*

$$\Phi : \text{CH}^*(F, 0) \otimes_{\mathbb{Z}} \text{CH}^*(B) \rightarrow \text{CH}^*(E), \quad \sum_{1 \leq i \leq r} f_i \otimes b_i \mapsto \sum_{1 \leq i \leq r} p^*(b_i)e_i$$

is an isomorphism. In particular, $\text{CH}^(E)$ is a free $\text{CH}^*(B)$ -module with basis $\{e_1, \dots, e_r\}$.*

Proof. Since k is perfect, we can find a filtration

$$\emptyset = B_{n+1} \subsetneq B_n \subsetneq \dots \subsetneq B_1 \subsetneq B_0 = B$$

of B by closed subschemes such that for each $0 \leq i \leq n$, the scheme $B_i \setminus B_{i+1}$ is smooth and the given fibration is trivial over it. We set $U_i = B \setminus B_i$ and $V_i = U_i \setminus U_{i-1} = B_{i-1} \setminus B_i$. Observe then that each of the U_i and V_i is smooth. Set $E_i = p^{-1}(U_i)$ and $W_i = p^{-1}(V_i)$. We prove by induction on i that the map $\text{CH}^*(F, 0) \otimes_{\mathbb{Z}} \text{CH}^*(U_i) \rightarrow \text{CH}^*(E_i)$ is an isomorphism, which will prove the theorem. Since $U_0 = \emptyset$ and $E_1 = U_1 \times F$, the desired isomorphism for $i = 1$ follows from [Lemma 6.2](#). We now consider the commutative diagram

$$\begin{array}{ccc}
 \text{CH}^*(U_i) \otimes \text{CH}^*(F, 0) & \longrightarrow & \text{CH}^*(E_i) \\
 \downarrow & & \downarrow \\
 \text{CH}^*(V_{i+1}) \otimes \text{CH}^*(F, 0) & \longrightarrow & \text{CH}^*(W_{i+1}) \\
 \downarrow & & \downarrow \\
 \text{CH}^*(U_{i+1}) \otimes \text{CH}^*(F, 0) & \longrightarrow & \text{CH}^*(E_{i+1}) \\
 \downarrow & & \downarrow \\
 \text{CH}^*(U_i) \otimes \text{CH}^*(F, 0) & \longrightarrow & \text{CH}^*(E_i) \\
 \downarrow & & \downarrow \\
 \text{CH}^*(V_{i+1}) \otimes \text{CH}^*(F, 0) & \longrightarrow & \text{CH}^*(W_{i+1}).
 \end{array} \tag{6-3}$$

The left column is obtained by tensoring the long exact localization sequence for higher Chow groups with $\text{CH}^*(F, 0)$ over \mathbb{Z} , and the right column is just the localization exact sequence. Since $\text{CH}^*(F, 0)$ is a free abelian group, the left column is also exact.

It is easily checked that the second and the third squares commute using the commutativity property of the push-forward and pull-back maps of higher Chow groups in a fiber diagram. We show that the other squares also commute. It is enough to show that the first square commutes as the fourth one is same as the first.

Let δ denote the connecting homomorphism in a long exact localization sequence for the higher Chow groups.

Before we show the required commutativity, let us recall that if (X, Y) is pair of k -schemes where $i : Y \hookrightarrow X$ is a closed subscheme with complement $j : U \hookrightarrow X$, then the localization exact sequence is the long exact homology sequence associated to the short exact sequence of cycle complexes

$$0 \rightarrow \mathcal{L}_n(Y, \bullet) \xrightarrow{i_*} \mathcal{L}_n(X, \bullet) \xrightarrow{j^*} \frac{\mathcal{L}_n(X, \bullet)}{\mathcal{L}_n(Y, \bullet)} \rightarrow 0,$$

where the natural map

$$\frac{\mathcal{L}_n(X, \bullet)}{\mathcal{L}_n(Y, \bullet)} \rightarrow \mathcal{L}_n(U, \bullet)$$

is a quasi-isomorphism. So we identify the last term with $\mathcal{L}_n(U, \bullet)$. The formalism of the homological algebra now shows that the connecting homomorphism $\delta : \text{CH}_n(U, i) \rightarrow \text{CH}_n(Y, i - 1)$ is obtained as one obtains the connecting homomorphism in the snake lemma. In particular, this is same as the differential map $\partial : \mathcal{L}_n(X, i) \rightarrow \mathcal{L}_n(X, i - 1)$, evaluated on the homology groups. The Leibniz rule for this differential now implies that the connecting homomorphism δ also satisfies the Leibniz rule; see [Panin 2003, §2.4].

If we now start with an element $b \otimes i^*(e_j) \in \text{CH}^*(U_i) \otimes \text{CH}^*(F, 0)$ and map this vertically, we get $\delta b \otimes i^*(e_j)$, which maps horizontally down to $p^*(\delta b) \cdot e_j$. On the other hand, if we first map horizontally, we get $p^*(b) \cdot e_j$ which maps vertically to $\delta(p^*(b) \cdot e_j)$. Using the Leibniz rule above, this last term is same as $\delta p^*(b) \cdot e_j = p^*(\delta b) \cdot e_j$ since $\delta e_j = 0$. We have shown that the diagram above commutes.

The first and the fourth horizontal arrows in (6-3) are isomorphisms by induction. The second and the fifth horizontal arrows are isomorphisms by Lemma 6.2. Hence the middle horizontal arrow is also an isomorphism by the 5-lemma. \square

6c. Leray–Hirsch with rational coefficients. We need the following step to prove the rational version of the Leray–Hirsch theorem for the étale locally trivial fibrations of smooth schemes.

Let F be a smooth cellular scheme over k . We have seen before that for any field extension $k \hookrightarrow l$, the natural map $\text{CH}^*(F, 0) \rightarrow \text{CH}^*(F_l, 0)$ is an isomorphism. Moreover, each of these is a free abelian group with the basis vectors given by the closures of the affine cells in the cellular decomposition. We fix this basis $\{f_1, \dots, f_r\}$ of $\text{CH}^*(F_l, 0)$ in what follows. For a complete flag variety G/B , this set is same as the set of Schubert cycles $\{\zeta_w\}_{w \in W}$.

Lemma 6.4. *Let B be a smooth k -scheme and let $F \xrightarrow{i} E \xrightarrow{p} B$ be an étale locally trivial fibration such that the fiber F is a smooth cellular scheme. Assume that this fibration becomes trivial after a finite étale cover of B of degree d . Assume furthermore that there are elements $\{e_1, \dots, e_r\}$ in $\text{CH}^*(E, 0)$ such that $\{f_1 = i^*(e_1), \dots, f_r = i^*(e_r)\}$ is the basis of $\text{CH}^*(F_y, 0)$ for each geometric fiber F_y of the fibration. Then the map*

$$\Phi : \text{CH}^*(F, 0) \otimes_{\mathbb{Z}} \text{CH}^*(B) \rightarrow \text{CH}^*(E), \quad \sum_{1 \leq i \leq r} f_i \otimes b_i \mapsto \sum_{1 \leq i \leq r} p^*(b_i)e_i$$

is an isomorphism over $\mathbb{Z}[d^{-1}]$. In particular, $\text{CH}^*(E)[d^{-1}]$ is a free $\text{CH}^*(B)[d^{-1}]$ -module with basis $\{e_1, \dots, e_r\}$.

Proof. Let $B' \xrightarrow{q} B$ be a finite étale cover such that $E' = E \times_B B' \xrightarrow{p'} B'$ is a trivial fibration and let $q' : E' \rightarrow E$ be the other projection. It follows from our assumption and the isomorphism of $\text{CH}^*(F, 0)$ under the field extensions that the set $\{e'_i = q'^*(e_i)\}$ restricts to the basis $\{f_i\}$ of $\text{CH}^*(F_y, 0)$ for every fiber F_y of the fibration p' .

Setting $\Phi'(b' \otimes f_j) = p'^*(b')e'_j$ and using the fact that $q'^* \circ p^* = p'^* \circ q^*$, $p^* \circ q_* = q'_* \circ p'^*$ and $q_* \circ q'^* = d = q'_* \circ q^*$ (see [Bloch 1986, Exercise 5.8(i)]), one checks that the diagram

$$\begin{array}{ccccc} \text{CH}^*(B) & \xrightarrow{q^* \otimes 1} & \text{CH}^*(B') & \xrightarrow{q_* \otimes 1} & \text{CH}^*(B) \\ \otimes & & \otimes & & \otimes \\ \text{CH}^*(F, 0) & & \text{CH}^*(F, 0) & & \text{CH}^*(F, 0) \\ \Phi \downarrow & & \Phi' \downarrow & & \downarrow \Phi \\ \text{CH}^*(E) & \xrightarrow{q'^*} & \text{CH}^*(E') & \xrightarrow{q'_*} & \text{CH}^*(E) \end{array}$$

commutes. The middle vertical arrow is an isomorphism by Lemma 6.2. A diagram chase shows that Φ is an isomorphism over $\mathbb{Z}[d^{-1}]$. □

Theorem 6.5. *Let B be a smooth k -scheme and let $F \xrightarrow{i} E \xrightarrow{p} B$ be an étale locally trivial fibration such that the fiber F is a smooth cellular scheme. Assume that there are elements $\{e_1, \dots, e_r\}$ in $\text{CH}^*(E, 0)$ such that $\{f_1 = i^*(e_1), \dots, f_r = i^*(e_r)\}$ is the basis of $\text{CH}^*(F_y, 0)$ for each geometric fiber F_y of the fibration. Then the map*

$$\Phi : \text{CH}^*(F, 0) \otimes_{\mathbb{Q}} \text{CH}^*(B) \rightarrow \text{CH}^*(E), \quad \sum_{1 \leq i \leq r} f_i \otimes b_i \mapsto \sum_{1 \leq i \leq r} p^*(b_i)e_i$$

is an isomorphism over the rationals. In particular, $\text{CH}^*(E)$ is a free $\text{CH}^*(B)$ -module with basis $\{e_1, \dots, e_r\}$ over the rationals.

Proof. We assume all abelian groups to be tensored with \mathbb{Q} in this proof. Since k is perfect and since every étale cover is generically finite, we can find a filtration

$$\emptyset = B_{n+1} \subsetneq B_n \subsetneq \cdots \subsetneq B_1 \subsetneq B_0 = B$$

of B by closed subschemes such that for each $0 \leq i \leq n$, the scheme $V_i = B_{i-1} \setminus B_i$ is smooth and there is a finite étale cover $V'_i \rightarrow V_i$ such that the given fibration is trivial over V'_i . We set $U_i = B \setminus B_i$ as before, which implies that $V_i = U_i \setminus U_{i-1}$. Observe then that each of the U_i and V_i is smooth. Set $E_i = p^{-1}(U_i)$ and $W_i = p^{-1}(V_i)$. We prove by induction on i that the map

$$\mathrm{CH}^*(F, 0) \otimes_{\mathbb{Q}} \mathrm{CH}^*(U_i) \rightarrow \mathrm{CH}^*(E_i)$$

is an isomorphism, which will imply the proposition. Since $U_0 = \emptyset$ and since the map $E_1 \rightarrow U_1$ is a smooth fibration which becomes trivial over the finite étale cover $V'_1 \rightarrow V_1 = U_1$, the desired isomorphism for $i = 1$ follows from [Lemma 6.4](#). We now consider the diagram

$$\begin{array}{ccccccccc} \mathrm{CH}^*(U_i) & & \mathrm{CH}^*(V_{i+1}) & & \mathrm{CH}^*(U_{i+1}) & & \mathrm{CH}^*(U_i) & & \mathrm{CH}^*(V_{i+1}) \\ \otimes & \longrightarrow & \otimes & \longrightarrow & \otimes & \longrightarrow & \otimes & \longrightarrow & \otimes \\ \mathrm{CH}^*(F, 0) & & \mathrm{CH}^*(F, 0) & & \mathrm{CH}^*(F, 0) & & \mathrm{CH}^*(F, 0) & & \mathrm{CH}^*(F, 0) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{CH}^*(E_i) & \longrightarrow & \mathrm{CH}^*(W_{i+1}) & \longrightarrow & \mathrm{CH}^*(E_{i+1}) & \longrightarrow & \mathrm{CH}^*(E_i) & \longrightarrow & \mathrm{CH}^*(W_{i+1}). \end{array}$$

The top row is obtained by tensoring the long exact localization sequence for higher Chow groups with $\mathrm{CH}^*(F, 0)$ over \mathbb{Q} and hence is exact. The bottom row is just the localization exact sequence.

One checks as in the proof of [Theorem 6.3](#) that the diagram above is commutative. The first and the fourth vertical arrows are isomorphisms by induction. The second and the fifth vertical arrows are isomorphisms by [Lemma 6.4](#). Hence the middle vertical arrow is also an isomorphism by 5-lemma. \square

7. Higher Chow groups of toric bundles and applications

As an application of [Theorem 6.3](#), we describe the ordinary higher Chow groups of toric bundles with integral coefficients. Let T be a split torus of rank n over k and let $M = \mathrm{Hom}(\mathbb{G}_m, T)$ be the group of its one-parameter subgroups. Let $X = X(\Delta)$ be a smooth projective toric variety associated to a fan Δ in $M_{\mathbb{R}}$ (see [Section 11](#)). Let B be a smooth k -scheme and let $p : E \rightarrow B$ be a principal T -bundle. Setting $E(X) = E \times^T X$, we see that

$$\pi : E(X) \rightarrow B, \quad \pi((e, x)) = p(e)$$

is a Zariski-locally trivial smooth fibration with all fibers isomorphic to X . Since X is projective, it follows that π is a projective morphism.

We fix an ordering $\{\sigma_1, \dots, \sigma_m\}$ of Δ_{\max} and let $\tau_i \subset \sigma_i$ be the cone that is the intersection of σ_i with all those σ_j such that $j \geq i$ and that intersect σ_i in dimension $n - 1$. Let $\tau'_i \subset \sigma_i$ be the cone such that $\tau_i \cap \tau'_i = \{0\}$ and $\dim(\tau_i) + \dim(\tau'_i) = n$ for $1 \leq i \leq m$. It is easy to see that τ'_i is the intersection of σ_i with all those σ_j such that $j \leq i$ and that intersect σ_i in dimension $n - 1$. Since X is smooth and projective, it is well-known that we can choose the ordering above of Δ_{\max} such that

$$\tau_i \subset \sigma_j \text{ implies } i \leq j \text{ and } \tau'_i \subset \sigma_j \text{ implies } j \leq i. \tag{7-1}$$

Let $\Delta_1 = \{\rho_1, \dots, \rho_d\}$ be the set of one-dimensional cones in Δ and let $\{v_1, \dots, v_d\}$ be the associated primitive elements of M . We choose $\{\rho_1, \dots, \rho_n\}$ to be a set of one-dimensional faces of σ_m such that $\{v_1, \dots, v_n\}$ is a basis of M . Let $\{\chi_1, \dots, \chi_n\}$ be the dual basis of M^\vee .

Definition 7.1. Let A be a commutative ring with unit and let $\{r_1, \dots, r_n\}$ be a subset of A . Let I_A denote the ideal of the polynomial algebra $A[t_1, \dots, t_d]$ generated by the elements

$$t_{j_1} \cdots t_{j_l} \text{ for } 1 \leq j_p \leq d \tag{7-2}$$

such that $\rho_{j_1}, \dots, \rho_{j_l}$ do not span a cone of Δ . Let \bar{I}_Δ denote the ideal of $A[t_1, \dots, t_d]$ generated by I_Δ and the relations

$$s_i := \left(\sum_{j=1}^d \langle \chi_i, v_j \rangle t_j \right) - r_i \text{ for } 1 \leq i \leq n. \tag{7-3}$$

We define the A -algebras $R_{\text{eq}}(A, \Delta)$ and $R(A, \Delta)$ to be quotients of $A[t_1, \dots, t_d]$ by the ideals I_Δ and \bar{I}_Δ , respectively.

The ring $R_{\text{eq}}(A, \Delta)$ is also known in the literature as the *Stanley–Reisner algebra* over A associated to the fan Δ ; see [Sankaran and Uma 2003, Definition 2.1]. Notice that any character χ of T acts on $R_{\text{eq}}(A, \Delta)$ through the multiplication by the element $\sum_{j=1}^d \langle \chi, v_j \rangle t_j$. This makes $R_{\text{eq}}(A, \Delta)$ into an $S = S(T)$ -algebra.

Any T -equivariant line bundle $L \rightarrow X$ uniquely defines a line bundle

$$E(L) = E \times^T L$$

on $E(X)$. Every $\rho \in \Delta_1$ defines a unique T -equivariant line bundle L_ρ on X with a T -equivariant section $s_\rho : X \rightarrow L_\rho$ that is transverse to the zero section and whose zero locus is the orbit closure $V_\rho = \overline{O_\rho}$. For any $\sigma \in \Delta$, let u_σ denote the fundamental class of the T -invariant cycle $[V_\sigma]$ in $\text{CH}^*(X, 0)$ and let y_σ denote the cycle $[E(V_\sigma)]$ in $\text{CH}^*(E(X), 0)$. Notice that $\pi_\sigma : E(V_\sigma) \rightarrow B$ is a smooth projective toric subbundle of $\pi : E(X) \rightarrow B$ with fiber V_σ .

Suppose that $\rho_{j_1}, \dots, \rho_{j_l}$ do not span a cone in Δ . Then $s = (s_{j_1}, \dots, s_{j_l})$ is a nowhere vanishing section of $L_{\rho_{j_1}} \oplus \dots \oplus L_{\rho_{j_l}}$ and hence

$$c_1(E(L_{\rho_{j_1}})) \cdots c_1(E(L_{\rho_{j_l}})) = 0 \quad \text{in } \text{CH}^*(E(X)).$$

In particular, we get

$$y_{\rho_{j_1}} \cdots y_{\rho_{j_l}} = 0 \quad \text{in } \text{CH}^*(E(X)). \quad (7-4)$$

We now consider the commutative diagram

$$\begin{array}{ccccccc} X_l & \xrightarrow{\iota} & E(X) & \xleftarrow{\bar{p}} & E \times X & \xrightarrow{p_X} & X \\ \pi_l \downarrow & & \pi \downarrow & & \downarrow p_E & & \downarrow \pi_X \\ \text{Spec}(l) & \longrightarrow & B & \xleftarrow{p} & E & \xrightarrow{\pi_E} & \text{Spec}(k), \end{array} \quad (7-5)$$

where $\text{Spec}(l)$ is any point of B . It is clear that all squares are Cartesian and all the maps in the right square are T -equivariant. Let L_χ denote the T -equivariant line bundle on $\text{Spec}(k)$ associated to a character χ of T . Since p and \bar{p} are principal T -bundles, we see that there is a unique line bundle ζ on B such that

$$\pi_E^*(L_\chi) = p^*(\zeta) \quad \text{and} \quad p_X^* \circ \pi_X^*(L_\chi) = \bar{p}^* \circ \pi^*(\zeta). \quad (7-6)$$

Using the identity

$$\pi_X^*(c_1^T(L_\chi)) = \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle u_\rho \quad \text{in } \text{CH}_T^*(X)$$

and the isomorphisms $\text{CH}^*(B) \cong \text{CH}_T^*(E)$, $\text{CH}^*(E(X)) \cong \text{CH}_T^*(E \times X)$, we see that $\pi^*(c_1(\zeta)) = \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle y_\rho$. Let $\zeta_i \in \text{Pic}(B)$ be such that $\pi_E^*(L_{\chi_i}) = p^*(\zeta_i)$, where $\{\chi_1, \dots, \chi_n\}$ is a chosen basis of M^\vee as above. Setting $r_i = c_1(\zeta_i) \in \text{CH}^*(B)$, we conclude that

$$\pi^*(r_i) = \sum_{j=1}^d \langle \chi_i, v_j \rangle y_{\rho_j} \quad \text{in } \text{CH}^*(E(X)), \quad 1 \leq i \leq n. \quad (7-7)$$

We define a homomorphism of $\text{CH}^*(B)$ -algebras

$$\text{CH}^*(B)[t_1, \dots, t_d] \rightarrow \text{CH}^*(E(X))$$

by the assignment $t_i \mapsto y_{\rho_i}$ for $1 \leq i \leq d$. It follows from (7-4) and (7-7) that this homomorphism descends to a $\text{CH}^*(B)$ -algebra homomorphism

$$\psi : R(\text{CH}^*(B), \Delta) \rightarrow \text{CH}^*(E(X)). \quad (7-8)$$

The following result describes the higher Chow groups of the projective toric bundle $\pi : E(X) \rightarrow B$ and generalizes [Sankaran and Uma 2003, Theorem 1.2(iii)] to higher Chow groups.

Theorem 7.2. *The homomorphism ψ is an isomorphism.*

Proof. To prove this theorem, we first observe that for any $\sigma \in \Delta$ and any point $\text{Spec}(l) \rightarrow B$, one has $l^*(E(V_\sigma)) = (V_\sigma)_l$. Also, it is well-known [Sankaran and Uma 2003, Lemma 3.1] that $\{l^*(y_{\tau_1}), \dots, l^*(y_{\tau_m})\}$ forms a \mathbb{Z} -basis of $\text{CH}^*(X_l, 0)$.

Since $y_{\tau_i} = y_{\rho_{i_1}} \cdots y_{\rho_{i_p}}$ for every $1 \leq i \leq m$, where $\{\rho_{i_1}, \dots, \rho_{i_p}\}$ is the set of edges of τ_i , it follows from the proof of Theorem 6.3 that $\text{CH}^*(E(X))$ is generated by $\{y_{\rho_1}, \dots, y_{\rho_d}\}$ as a $\text{CH}^*(B)$ -algebra. In particular, the map ψ is surjective.

To prove injectivity, let $x(\sigma)$ denote the monomial $t_{i_1} \cdots t_{i_p}$ in $R(\text{CH}^*(B), \Delta)$ such that $\{\rho_{i_1}, \dots, \rho_{i_p}\}$ is the set of edges of $\sigma \in \Delta$. Then by [Sankaran and Uma 2003, Lemma 2.1(ii)], the set $\{x(\tau_1), \dots, x(\tau_m)\}$ spans $R(\text{CH}^*(B), \Delta)$ as a $\text{CH}^*(B)$ -module. Since $\psi(x(\tau_i)) = y_{\tau_i}$ for $1 \leq i \leq m$ and since $\text{CH}^*(E(X))$ is a free $\text{CH}^*(B)$ -module with basis $\{y_{\tau_1}, \dots, y_{\tau_m}\}$ by Theorem 6.3, we conclude that ψ must be injective. \square

7a. Equivariant and ordinary higher Chow groups of smooth projective toric varieties. As a consequence of Theorem 7.2, we derive some explicit formulas for the equivariant and ordinary higher Chow groups of smooth projective toric varieties with integral coefficients. We shall later show in Section 11 that such formulas hold for all smooth toric varieties with rational coefficients. Recall that if $X = X(\Delta)$ is a toric variety, then for every $\sigma \in \Delta$, the orbit closure V_σ is a T -invariant closed toric subvariety of X and hence uniquely defines a class $y_\sigma = [V_\sigma]$ in $\text{CH}_T^*(X, 0)$; see [Edidin and Graham 1998, Section 2.2]. This is called the fundamental equivariant class of V_σ .

We consider $A = \text{CH}^*(k) \otimes_{\mathbb{Z}} S \cong \text{CH}^*(k)[t_1, \dots, t_n]$ as a graded $\text{CH}^*(k)$ -algebra whose degree zero part is $\text{CH}^*(k)$. We have seen above that $R_{\text{eq}}(\text{CH}^*(k), \Delta)$ has an action of S that makes it a graded A -algebra. Moreover, $R(\text{CH}^*(k), \Delta)$ is just the quotient $R_{\text{eq}}(\text{CH}^*(k), \Delta) \otimes_{\mathbb{Z}} \mathbb{Z} \cong R_{\text{eq}}(\text{CH}^*(k), \Delta) \otimes_A \text{CH}^*(k)$.

Corollary 7.3. *Let $X = X(\Delta)$ be a smooth projective toric variety as above. Then the assignment $t_i \mapsto y_{\rho_i}$ induces $\text{CH}^*(k)$ -algebra isomorphisms*

$$\Psi_X : R_{\text{eq}}(\text{CH}^*(k), \Delta) \xrightarrow{\sim} \text{CH}_T^*(X), \quad (7-9)$$

$$\overline{\Psi}_X : R(\text{CH}^*(k), \Delta) \xrightarrow{\sim} \text{CH}^*(X). \quad (7-10)$$

Proof. The second isomorphism is just a special case of Theorem 7.2 when $B = \text{Spec}(k)$.

To prove the isomorphism of (7-9), we first observe that Ψ_X is a graded A -linear homomorphism; see Section 11. Let M denote the kernel of this map. It follows

from [Krishna 2009a, Theorem 4.2] that $\mathrm{CH}_T^*(X)$ is a free A -module with basis $\{y_{\tau_i}\}$. It follows from this that Ψ_X is surjective and we get an exact sequence of graded A -modules

$$0 \rightarrow M \rightarrow R_{\mathrm{eq}}(\mathrm{CH}^*(k), \Delta) \xrightarrow{\Psi_X} \mathrm{CH}_T^*(X) \rightarrow 0.$$

The freeness of $\mathrm{CH}_T^*(X)$ as an A -module ensures that this sequence remains short exact after tensoring with $\mathrm{CH}^*(k)$ via the augmentation $A \rightarrow \mathrm{CH}^*(k)$.

It follows from [Krishna 2009a, Theorem 1.1] that

$$\mathrm{CH}_T^*(X) \otimes_A \mathrm{CH}^*(k) \xrightarrow{\sim} \mathrm{CH}^*(X).$$

In particular, Ψ_X becomes an isomorphism after tensoring with $\mathrm{CH}^*(k)$. We conclude that $M \otimes_A \mathrm{CH}^*(k) = 0$. Since M is a nonnegatively graded A -module, it must be zero. \square

8. Higher Chow groups of flag bundles and applications

We remind the readers of our convention that all the higher Chow groups for the rest of this text will be considered with rational coefficients. We shall however, indicate the coefficients in the statement of all results.

In this section, we describe a formula for the higher Chow groups of complete flag bundles with rational coefficients. Such a formula for general flag bundles is an immediate consequence of the case of complete flag bundles. We also give some applications of this formula to the theory of equivariant higher Chow groups.

8a. Complete flag bundles. Let G be a connected reductive group over k and let B be a Borel subgroup of G containing a split maximal torus T . Let X be a k -scheme and let $p : E \rightarrow X$ be a principal G -bundle and let $\pi : E/B \rightarrow X$ be the associated complete flag bundle. Vistoli [1989] described the classical Chow groups $\mathrm{CH}_*(E, 0)$ and $\mathrm{CH}_*(E/B, 0)$ in terms of the Chow groups of X . In this section we generalize Vistoli's results to the case of higher Chow groups. The proof below is completely different from Vistoli's; it is much shorter and relies more on the equivariant techniques.

The restriction map $r_{T,X}^G$ induces for every $i \geq 0$, a natural map of $S(T)$ -modules

$$\mathrm{CH}_*^G(E, i) \otimes_{S(G)} S(T) \rightarrow \mathrm{CH}_*^T(E, i), \quad w \otimes \alpha \mapsto \alpha \cdot r_{T,E}^G(w).$$

Since G acts freely on E , one identifies $\mathrm{CH}_*^G(E, i)$ with $\mathrm{CH}_*(X, i)$ by Proposition 2.2. The group $\mathrm{CH}_*^B(E, i)$ is canonically identified with $\mathrm{CH}_*(E/B, i)$ for the same reason. The map above then translates into a natural map of $S(T)$ -modules

$$\lambda_X : \mathrm{CH}_*(X, i) \otimes_{S(G)} S(T) \rightarrow \mathrm{CH}_*(E/B, i). \quad (8-1)$$

Taking the direct sum over $\{\text{CH}_*(X, i)\}_{i \geq 0}$, we get a natural map of $S(T)$ -modules

$$\lambda_X : \text{CH}_*(X) \otimes_{S(G)} S(T) \rightarrow \text{CH}_*(E/B). \tag{8-2}$$

This map is a ring homomorphism if X is smooth. One can easily check that λ_X commutes with the flat pull-back and the proper push-forward maps between the higher Chow groups of the base schemes of the bundle. We wish to show that λ_X is an isomorphism. We begin with the following special case.

Lemma 8.1. *Let X be a smooth (not necessarily connected) scheme over k and let $f : X' \rightarrow X$ be a finite étale morphism such that the principal bundle $p : E \rightarrow X$ is trivialized over X' . Then the map λ_X is an isomorphism with rational coefficients. In particular, λ_X is an isomorphism with rational coefficients if X is zero-dimensional.*

Proof. Since X is a disjoint union of connected smooth schemes, it is enough to prove the lemma when X is smooth and connected. If $E/B = G/B \rightarrow \text{Spec}(k)$ is the flag variety, then we have

$$\begin{aligned} \text{CH}_*(k) \otimes_{S(G)} S(T) &\cong \text{CH}_*(k) \otimes_{\text{CH}^*(k,0)} (\text{CH}^*(k, 0) \otimes_{S(G)} S(T)) \\ &\cong^\dagger \text{CH}_*(k) \otimes_{\text{CH}^*(k,0)} \text{CH}_*(G/B, 0) \\ &\cong \text{CH}_*(G/B) \quad (\text{by Lemma 6.2}), \end{aligned}$$

where \cong^\dagger follows from [Demazure 1973, théorème 2].

If $G/B \times X \xrightarrow{\pi} X$ is the trivial bundle, then we get

$$\begin{aligned} \text{CH}_*(X) \otimes_{S(G)} S(T) &\cong \text{CH}_*(X) \otimes_{\text{CH}^*(k,0)} (\text{CH}^*(k, 0) \otimes_{S(G)} S(T)) \\ &\cong \text{CH}_*(X) \otimes_{\text{CH}^*(k,0)} \text{CH}_*(G/B, 0) \\ &\cong \text{CH}_*(G/B \times X) \quad (\text{by Lemma 6.2}). \end{aligned}$$

In general, we consider the diagram

$$\begin{array}{ccccc} \text{CH}_*(X) \otimes S(T) & \xrightarrow{f^* \otimes \text{id}} & \text{CH}_*(X') \otimes S(T) & \xrightarrow{f_* \otimes \text{id}} & \text{CH}_*(X) \otimes S(T) \\ \lambda_X \downarrow & & \lambda_{X'} \downarrow & & \downarrow \lambda_X \\ \text{CH}_*(E/B) & \xrightarrow{\bar{f}^*} & \text{CH}_*(E_{X'}/B) & \xrightarrow{\bar{f}_*} & \text{CH}_*(E/B), \end{array}$$

where the tensor product in the top row is over $S(G)$. We have just shown that $\lambda_{X'}$ is an isomorphism. It follows from the projection formula (see [Bloch 1986, Exercise 5.8]) that the composite horizontal maps in both rows are multiplication by $[k(X') : k(X)]$. Hence, λ_X must be an isomorphism too. \square

Theorem 8.2. *For any k -scheme X , the map λ_X is an isomorphism of $S(T)$ -modules with rational coefficients. This is a ring isomorphism if X is smooth.*

Proof. We only need to prove the first assertion, for which we use induction on the dimension of X . The zero-dimensional case follows from [Lemma 8.1](#). In general, we can find an étale cover of X over which the bundle $p : E \rightarrow X$ becomes trivial. Since any such cover is generically finite and since the base field k is perfect, we can find a dense open subset $j : U \hookrightarrow X$ and a finite étale cover $f : U' \rightarrow U$ such that U is a disjoint union of connected smooth schemes and the given bundle is trivial over U' . Let $\iota : Z \hookrightarrow X$ be the complement of U with its reduced induced closed subscheme structure.

We now consider the diagram

$$\begin{array}{ccccccccc}
 \mathrm{CH}_*(U) & & \mathrm{CH}_*(Z) & & \mathrm{CH}_*(X) & & \mathrm{CH}_*(U) & & \mathrm{CH}_*(Z) \\
 \otimes & \xrightarrow{\partial} & \otimes & \xrightarrow{\iota_*} & \otimes & \xrightarrow{j^*} & \otimes & \xrightarrow{\partial} & \otimes \\
 S(T) & & S(T) & & S(T) & & S(T) & & S(T) \\
 \lambda_U \downarrow & & \downarrow \lambda_Z & & \downarrow \lambda_X & & \downarrow \lambda_U & & \downarrow \lambda_Z \\
 \mathrm{CH}^*(E_U/B) & \simeq & \mathrm{CH}^*(E_Z/B) & \simeq & \mathrm{CH}^*(E/B) & \simeq & \mathrm{CH}^*(E_U/B) & \simeq & \mathrm{CH}^*(E_Z/B)
 \end{array}$$

of localization exact sequences, where the tensor product in the top row is over the ring $S(G)$. In particular, this row is exact by the flatness of $S(T)$ over $S(G)$. The second and the third squares commute by the compatibility of λ_X with the push-forward and the pull-back maps as remarked above.

To see that the first square commutes, let us consider an element

$$a \otimes b \in \mathrm{CH}_*(U) \otimes S(T).$$

If we map this horizontally, we get $\partial(a) \otimes b$, which is mapped vertically down to $b \cdot \pi_Z^* \circ \partial(a)$. Since the localization sequence of higher Chow groups is compatible with respect to the flat pull-back, this last term is same as $b \cdot \partial \circ \pi_U^*(a)$. On the other hand, mapping $a \otimes b$ vertically down gives $b \cdot \pi_U^*(a)$ and if we map this horizontally, we get $\partial(b \cdot \pi_U^*(a))$. Since the horizontal maps in the bottom row are $S(T)$ -linear (see [Proposition 2.2](#)), we conclude that the first (hence the fourth) square commutes.

The first and the fourth vertical arrows in the diagram above are isomorphisms by [Lemma 8.1](#). Since U is a dense open in X , the dimension of Z is strictly smaller than that of X . Hence the second and the fifth vertical arrows are isomorphisms by induction on the dimension and [Lemma 8.1](#). We conclude from the 5-lemma that λ_X is an isomorphism. □

8b. Principal bundles and flag bundles. The following extension of [Theorem 8.2](#) to all flag bundles is a direct generalization of the projective bundle formula for higher Chow groups.

Corollary 8.3. *Let $p : E \rightarrow X$ be a principal G -bundle over a k -scheme X and let $\pi : E/P \rightarrow X$ denote the flag bundle associated to a parabolic subgroup P . Then*

the natural map of $S(P)$ -modules

$$\lambda_X : \mathrm{CH}_*(X; \mathbb{Q}) \otimes_{S(G; \mathbb{Q})} S(P; \mathbb{Q}) \rightarrow \mathrm{CH}_*(E/P; \mathbb{Q}), \quad w \otimes \alpha \mapsto \alpha \cdot r_{P,E}^G(w).$$

is an isomorphism. This is a ring isomorphism if X is smooth.

Proof. We only need to prove the first assertion. Let $B \subseteq P$ be a Borel subgroup of G containing a split maximal torus T . Let $\Phi(G, T)$ be the root system of G with respect to T such that B corresponds to the base Δ of $\Phi(G, T)$ and P corresponds to a subset $I \subset \Delta$. Let $P = M \ltimes N$ be the Levi decomposition and let $B_M = B \cap M$ be the Borel subgroup of M containing T . Let $W_P \subset W$ be the Weyl group of P with respect to T . It follows from Propositions 3.3, 3.4 and Corollary 8.7 that the natural map

$$\mathrm{CH}_*(E/P) \rightarrow (\mathrm{CH}_*(E/B))^{W_P} \tag{8-3}$$

is an isomorphism.

On the other hand, following the proof of Corollary 8.7, we get

$$\begin{aligned} (\mathrm{CH}_*(X) \otimes_{S(G)} S(T))^{W_P} &\cong \mathrm{CH}_*(X) \otimes_{S(G)} (S(T))^{W_P} \\ &\cong \mathrm{CH}_*(X) \otimes_{S(G)} S(M) \\ &\cong \mathrm{CH}_*(X) \otimes_{S(G)} S(P), \end{aligned} \tag{8-4}$$

where the last isomorphism follows again from Proposition 3.4. The corollary now follows from Theorem 8.2 by combining (8-3) and (8-4). \square

The following result generalizes [Vistoli 1989, Corollary 3.2] to higher Chow groups.

Corollary 8.4. *Let G be connected and split reductive group over k and let $p : E \rightarrow X$ be a principal G -bundle over a k -scheme X . Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} = \mathrm{Tor}_p^{S(G; \mathbb{Q})}(\mathbb{Q}, \mathrm{CH}_*(X, q; \mathbb{Q})) \Rightarrow \mathrm{CH}_*(E, p + q; \mathbb{Q}).$$

The edge homomorphism yields an isomorphism

$$\mathrm{CH}_*(X, 0; \mathbb{Q}) \otimes_{S(G; \mathbb{Q})} \mathbb{Q} \xrightarrow{\sim} \mathrm{CH}_*(E, 0; \mathbb{Q}).$$

Proof. This is an immediate consequence of Theorem 8.2, [Krishna 2009a, Theorem 1.1] and flatness of $S(T)$ over $S(G)$. \square

8c. A change of groups isomorphism. The following theorem is an analogue of a similar result in equivariant K -theory by Merkurjev [2005, Proposition 8]. However, this result for the equivariant higher Chow groups has an advantage over Merkurjev’s theorem in that it holds for the action of any split reductive group (though with

rational coefficients) whereas [*ibid.*, Proposition 8] is known only for the groups whose derived subgroups are simply connected, for example, GL_n . The special case $\mathrm{CH}_*^G(X, 0)$ of the result below was proven by Brion [1997, Theorem 6.7].

Theorem 8.5. *Let G be a connected reductive group and let T be a split maximal torus of G . Then for any $X \in \mathcal{V}_G$, the natural map of $S(T)$ -modules*

$$\lambda_X : \mathrm{CH}_*^G(X; \mathbb{Q}) \otimes_{S(G; \mathbb{Q})} S(T; \mathbb{Q}) \rightarrow \mathrm{CH}_*^T(X; \mathbb{Q}) \quad (8-5)$$

is an isomorphism. This is a ring isomorphism if X is smooth.

Proof. We only need to show the first assertion. If (V, U) is a good pair for the G -action, then $\mathrm{CH}_*^G(X, i)$ and $\mathrm{CH}_*^T(X, i)$ in suitable degrees are the same as $\mathrm{CH}_*(X_G, i)$ and $\mathrm{CH}_*(X_B, i)$, respectively, where B is a Borel subgroup of G containing T . Hence, it suffices to show that for any k -scheme Z with a free action of G with quotients G/B and Z/G , the natural map

$$\lambda_{Z/G} : \mathrm{CH}_*(Z/G) \otimes_{S(G)} S(T) \rightarrow \mathrm{CH}_*(Z/B)$$

is an isomorphism. But this follows immediately by applying [Theorem 8.2](#) to the principal bundle $Z \rightarrow Z/G$. \square

Remark 8.6. The first remark is that [Theorem 8.5](#) holds if G is any connected linear algebraic group (not necessarily reductive) if the base field is of characteristic zero. This is an immediate consequence of [Proposition 3.4](#). The second remark is that the theorem above can also be proven as a simple consequence of the Leray–Hirsch theorem [6.5](#). The case of smooth schemes is a direct consequence of [Theorem 6.5](#) and the general case can be proven using noetherian induction and the localization sequence. We leave it an exercise to fill in the details.

8d. Some consequences of [Theorem 8.5](#). Recall that if G is a connected reductive group with a split maximal torus T , then the normalizer N of T in G and all its connected components are defined over k and the quotient N/T is the Weyl group W of the corresponding root system. In particular, $W \subset G/T$. If G acts on a variety X and if (V, U) is a good pair for the G -action, then $X \times^T U \rightarrow X \times^G U$ is an étale-locally trivial smooth fibration with fiber G/T . In particular, W acts on each $\mathrm{CH}_j^T(X, i)$ and the map $\mathrm{CH}_j^G(X, i) \rightarrow \mathrm{CH}_j^T(X, i)$ factors through the W -invariants. We get the following consequence of [Theorem 8.5](#).

Corollary 8.7 (see [Theorem 5.7](#)). *Let G be a connected reductive group and let T be a split maximal torus of G with the Weyl group W . Then for any $X \in \mathcal{V}_G$, the restriction map $r_{T, X}^G$ induces an isomorphism*

$$\mathrm{CH}_*^G(X; \mathbb{Q}) \xrightarrow{\sim} (\mathrm{CH}_*^T(X; \mathbb{Q}))^W.$$

Proof. Since W is a finite group, the trivial $\mathbb{Q}[W]$ -module \mathbb{Q} is a projective $\mathbb{Q}[W]$ -module of finite rank. In particular, it follows from [Theorem 8.5](#) that

$$\begin{aligned} (\mathrm{CH}_T^*(X))^W &= \mathrm{Hom}_{\mathbb{Q}[W]}(\mathbb{Q}, \mathrm{CH}_*^G(X) \otimes_{S(G)} S(T)) \\ &= \mathrm{CH}_*^G(X) \otimes_{S(G)} \mathrm{Hom}_{\mathbb{Q}[W]}(\mathbb{Q}, S(T)) \\ &= \mathrm{CH}_*^G(X) \otimes_{S(G)} (S(T))^W \\ &= {}^\dagger \mathrm{CH}_*^G(X) \otimes_{S(G)} S(G) \\ &= \mathrm{CH}_*^G(X), \end{aligned}$$

where $= {}^\dagger$ holds by [\[Edidin and Graham 1998, Proposition 6\]](#). □

As an important consequence of the result above, we get the following analogue of a similar result of Thomason [\[1988, Theorem 1.13\]](#) in equivariant K -theory.

Corollary 8.8 (see [Remark 5.8](#)). *Let G be a connected and reductive group over k and let T be a split maximal torus in G . Then the restriction map*

$$\mathrm{CH}_*^G(X; \mathbb{Q}) \xrightarrow{r_{T,X}^G} \mathrm{CH}_*^T(X; \mathbb{Q}) \tag{8-6}$$

is a split monomorphism. Moreover, this splitting is natural for morphisms in \mathcal{V}_G . In particular, if H is any closed subgroup of G , then there is a split injective map

$$\mathrm{CH}_*^H(X; \mathbb{Q}) \xrightarrow{r_{T,X}^G} \mathrm{CH}_*^T(G \times^H X; \mathbb{Q}). \tag{8-7}$$

Proof. The first statement follows directly from [Corollary 8.7](#), where the splitting is given by the trace map into the subgroup of W -invariants. The last statement follows from [\(8-6\)](#) and [Corollary 3.2](#). □

Remark 8.9. Let $\varrho \in S(T) \cong \mathrm{CH}_G^*(G/B, 0)$ be such that the forgetful map takes ϱ to the class of the zero-dimensional Schubert cycle in $\mathrm{CH}^*(G/B, 0)$. For a flag bundle $\pi : E/B \rightarrow X$ over a scheme X , if we define $\psi_X : \mathrm{CH}_*(X) \rightarrow \mathrm{CH}_*(X)$ by $\psi_X(\alpha) = \pi_*(\varrho \cdot \pi^*(\alpha))$, then it can be shown that ψ_X is an isomorphism. In particular, π^* is split injective. This gives another (and more conceptual) proof of [Corollary 8.8](#). In fact, this proof shows that $r_{T,X}^G$ is split injective with integer coefficients if G is *special*.

Let G be a connected reductive group and let B be a Borel subgroup of G containing a split maximal torus T . It follows from [\[Demazure 1973, théorème 2\]](#) that the forgetful map $S(T) \rightarrow \mathrm{CH}^*(G/B, 0)$ is surjective. Moreover, if $\{\varrho_w\}_{w \in W}$ are polynomials in $S(T)$ which map to the classes of Schubert cycles $\{\zeta_w\}_{w \in W}$ in $\mathrm{CH}^*(G/B, 0)$, then $S(T)$ is a free $S(G)$ -module with basis $\{\varrho_w\}_{w \in W}$. The following is a direct generalization of Demazure’s theorem to the case of all smooth schemes and all higher Chow groups. This also strengthens [Corollary 8.7](#) for smooth schemes.

Corollary 8.10. *Let $X \in \mathcal{V}_G^S$ and let $X \xrightarrow{p_X} \text{Spec}(k)$ be the structure map. Set $\varrho_{w,X} = p_X^*(\varrho_w)$. Then $\text{CH}_T^*(X; \mathbb{Q})$ is a free $\text{CH}_G^*(X; \mathbb{Q})$ -module with basis $\{\varrho_{w,X}\}_{w \in W}$.*

Proof. It follows from the construction that the map

$$\lambda_X : \text{CH}_G^*(X) \otimes_{S(G)} S(T) \rightarrow \text{CH}_T^*(X)$$

takes $1 \otimes \varrho_w$ onto $\varrho_{w,X}$. The corollary is now an immediate consequence of [Theorem 8.5](#). \square

9. Cohomological rigidity and specializations

Let G be a split diagonalizable group over k acting on a smooth scheme X . Recall [\[Springer 1998, 13.2.5\]](#) that all the diagonalizable subgroups of G are defined and split over k . The equivariant K -theory of X for the G -action was studied by Vezzosi and Vistoli [\[2003\]](#). Their main result (Theorem 1) is to reconstruct the equivariant K -theory ring of X in terms of the equivariant K -theory of the loci where the stabilizers have constant dimension. In the next two sections, we use the ideas of Vezzosi and Vistoli to prove an analogous decomposition theorem (see [Theorem 10.3](#)) for the equivariant higher Chow groups of X for the G -action. As mentioned in the introduction, this theorem and its compatibility with the corresponding result for the equivariant K -theory play fundamental roles in the proof of the equivariant Riemann–Roch theorems in [\[Krishna 2009b\]](#). This theorem is very useful in computing the equivariant and ordinary higher Chow groups of smooth schemes with torus action. Some applications of this kind are given in [Section 11](#).

This section is concerned with the study of the notion of cohomological rigidity and the construction of certain specialization maps in equivariant higher Chow groups. In this and the next section, the group G will denote a split diagonalizable group and the all schemes will be assumed to be smooth with G -action. We have seen ([Proposition 2.2](#)) that for such a scheme X , $\text{CH}_G^*(X)$ is a bigraded ring, which is an algebra over the ring $\text{CH}_G^*(k)$.

9a. Cohomological rigidity.

Definition 9.1. Let $Y \subset X$ be a smooth and G -invariant closed subscheme of codimension $d \geq 0$ and let $N_{Y/X}$ denote the normal bundle of Y in X . We say that Y is *cohomologically rigid* inside X if $c_d^G(N_{Y/X})$ is a not a zero-divisor in the ring $\text{CH}_G^*(Y)$.

As one observes, this definition has reasonable meaning only in the equivariant setting, since every element of positive degree in the nonequivariant Chow ring is nilpotent. The importance of cohomological rigidity for the equivariant higher

Chow groups comes from the following analogue of the K -theory splitting theorem (Proposition 4.3) of [Vezzosi and Vistoli 2003].

Proposition 9.2. *Let Y be a smooth and G -invariant closed subscheme of X of codimension $d \geq 0$. Assume that Y is cohomologically rigid inside X , and put $U = X \setminus Y$. Let $i : Y \hookrightarrow X$ and $j : U \hookrightarrow X$ be the inclusion maps.*

(i) *The localization sequence*

$$0 \rightarrow \mathrm{CH}_G^*(Y; \mathbb{Q}) \xrightarrow{i_*} \mathrm{CH}_G^*(X; \mathbb{Q}) \xrightarrow{j^*} \mathrm{CH}_G^*(U; \mathbb{Q}) \rightarrow 0$$

is exact.

(ii) *The restriction ring homomorphisms*

$$\mathrm{CH}_G^*(X; \mathbb{Q}) \xrightarrow{(i^*, j^*)} \mathrm{CH}_G^*(Y; \mathbb{Q}) \times \mathrm{CH}_G^*(U; \mathbb{Q})$$

give an isomorphism of rings

$$\mathrm{CH}_G^*(X; \mathbb{Q}) \xrightarrow{\sim} \mathrm{CH}_G^*(Y; \mathbb{Q}) \times_{\widetilde{\mathrm{CH}}_G^*(Y; \mathbb{Q})} \mathrm{CH}_G^*(U; \mathbb{Q}),$$

where $\widetilde{\mathrm{CH}}_G^(Y; \mathbb{Q}) = \mathrm{CH}_G^*(Y; \mathbb{Q}) / (c_d^G(N_{Y/X}))$, and the maps*

$$\mathrm{CH}_G^*(Y; \mathbb{Q}) \rightarrow \widetilde{\mathrm{CH}}_G^*(Y; \mathbb{Q}), \quad \mathrm{CH}_G^*(U; \mathbb{Q}) \rightarrow \widetilde{\mathrm{CH}}_G^*(Y; \mathbb{Q})$$

are, respectively, the natural surjection and the map

$$\mathrm{CH}_G^*(U; \mathbb{Q}) = \frac{\mathrm{CH}_G^*(X; \mathbb{Q})}{i_*(\mathrm{CH}_G^*(Y; \mathbb{Q}))} \xrightarrow{i^*} \frac{\mathrm{CH}_G^*(Y; \mathbb{Q})}{c_d^G(N_{Y/X})} = \widetilde{\mathrm{CH}}_G^*(Y; \mathbb{Q}),$$

which is well-defined by Corollary 4.5.

Proof. Part (i) follows directly from Corollary 4.5 and the definition of cohomological rigidity. Since i^* and j^* are ring homomorphisms, the proof of the second part follows directly from the first part and [Vezzosi and Vistoli 2003, Lemma 4.4]. \square

To apply the result above in our context, we need to find some sufficient conditions for checking the cohomological rigidity in specific examples. We begin with the following elementary result.

Lemma 9.3. *Let A be a ring which is a \mathbb{Q} -algebra. Then an element of the form t^d , where $t = \sum_{j=1}^r a_j t_j \in A[t_1, \dots, t_n]$, is not a zero-divisor for any $d \geq 0$ whenever $a_j \in \mathbb{Q}$ for all j and $a_j \neq 0$ for some j .*

Proof. Since all $a_j \in \mathbb{Q}$ and some $a_j \neq 0$, we see that t^d is a nonzero element of $\mathbb{Q}[t_1, \dots, t_n]$ and hence a nonzero divisor in this ring. Since tensoring with A over \mathbb{Q} is exact, we see that the multiplication by t^d is injective in $A[t_1, \dots, t_n]$ too. \square

Proposition 9.4. *Let G be a split diagonalizable group acting on a smooth scheme X and let E be a G -equivariant vector bundle of rank d on X . Assume that there is a subtorus $T \subset G$ of positive rank which acts trivially on X , such that in the eigenspace decomposition of E with respect to T , the submodule corresponding to the trivial character is zero. Then $c_d^G(E)$ is not a zero-divisor in $\text{CH}_G^*(X; \mathbb{Q})$.*

Proof. By [Thomason 1986, Lemma 5.6], E has a unique direct sum decomposition

$$E = \bigoplus_{i=1}^r E_{\chi_i} \otimes L_{\chi_i},$$

where we choose a splitting $G = D \times T$, E_{χ_i} are D -bundles and χ_i are characters of T with associated line bundles $L_{\chi_i} \in \text{Pic}_T(k)$. This decomposition is via the functor

$$\text{Bun}_X^D \times \text{Rep}(T) \rightarrow \text{Bun}_X^G, \quad (F, \rho) \mapsto p_1^*(F) \otimes p_2^*(\rho),$$

where $p_1 : D \times T \rightarrow D$ and $p_2 : D \times T \rightarrow T$ are the projections.

Since $\text{rank}(E) = d$, the Whitney sum formula yields $c_d^G(E) = \prod_{i=1}^r c_{d_i}^G(E_{\chi_i} \otimes L_{\chi_i})$, where $d_i = \text{rank}(E_{\chi_i})$. We can thus assume that $E = E_\chi \otimes L_\chi$, where χ is not a trivial character by our assumption. In particular, we can write

$$c_1^T(L_\chi) = t = \sum_{i=1}^p n_i t_i \in \mathbb{Q}[t_1, \dots, t_n] \tag{9-1}$$

with $n_i \neq 0$ for some i . By neglecting those i for which the coefficients n_i are zero, we can assume that $n_i \neq 0$ for all i . Now we have

$$\begin{aligned} c_d^G(E) &= c_d^G(p_1^*(E_\chi) \otimes p_2^*(L_\chi)) = \dagger \sum_{i=0}^d c_{d-i}^G(p_1^*(E_\chi)) \cdot (c_1^G(p_2^*(L_\chi)))^i \\ &= \sum_{i=0}^d p_1^*(c_{d-i}^D(E_\chi)) \cdot p_2^*((c_1^T(L_\chi))^i) = \sum_{i=0}^d \alpha_i t^i, \end{aligned}$$

where $\alpha_i \in \text{CH}_D^*(X)$ and $c_d^G(E) \in \text{CH}_G^*(X) \cong \text{CH}_D^*(X) \otimes S(T)$ by Theorem 3.5 and $=\dagger$ holds by [Fulton 1984, Remark 3.2.3]. Furthermore, $\alpha_d = p_1^*(c_0^D(E_\chi)) = 1$. Thus we get $c_d^G(E) = t^d + \alpha_{d-1}t^{d-1} + \dots + \alpha_1 t + \alpha_0 = g(t)$.

We need to show that $g(t)$ is not a zero divisor in $\text{CH}_D^*(X)[t_1, \dots, t_n]$. So suppose $f(\underline{t})$ is a nonzero polynomial such that $g(t)f(\underline{t}) = 0$, and let $f'(\underline{t})$ be the homogeneous part of $f(\underline{t})$ of largest degree which is not zero. By comparing the homogeneous parts, it is easy to see that $g(t)f(\underline{t}) = 0$ only if $t^d f'(\underline{t}) = 0$. But this is a contradiction since t satisfies the condition of Lemma 9.3 by (9-1), and hence is not a zero-divisor. □

Let G be a split diagonalizable group as above and let $X \in \mathcal{V}_G^S$. Following the notation of [Vezzosi and Vistoli 2003], for any $s \geq 0$, we let $X_{\leq s} \subset X$ be the open subset of points whose stabilizers have dimension at most s . We shall often write $X_{\leq s-1}$ also as $X_{< s}$. Let $X_s = X_{\leq s} \setminus X_{< s}$ denote the locally closed subset of X , where the stabilizers have dimension exactly s . We think of X_s as a subspace of X with the reduced induced structure. It is clear that $X_{\leq s}$ and X_s are G -invariant subspaces of X . Let N_s denote the normal bundle of X_s in $X_{\leq s}$, and let N_s^0 denote the complement of the 0-section in N_s . Then G clearly acts on N_s . The following result describes some very useful properties of these subspaces.

Proposition 9.5. *Let $s \geq 0$ be an integer.*

- (i) *There exists a finite number of s -dimensional subtori T_1, \dots, T_r in G such that X_s is the disjoint union of the fixed point spaces $X_{\leq s}^{T_j}$.*
- (ii) *X_s is smooth locally closed subscheme of X .*
- (iii) *$N_s^0 = (N_s)_{< s}$.*

Proof. Since the base field k is perfect, this is a special case of [Vezzosi and Vistoli 2003, Proposition 2.2], which holds for regular G -schemes over any connected and separated Noetherian base scheme. \square

Remark 9.6. We mention here that although the proposition above has been stated for the smooth schemes, part (i) of the proposition holds also when X is not necessarily smooth, since the proof given in [loc. cit.] only uses Thomason's generic étale slice theorem, which holds very generally.

Corollary 9.7. *For $s \geq 1$, X_s is cohomologically rigid inside $X_{\leq s}$.*

Proof. Let d_s be the codimension of X_s in $X_{\leq s}$. We need to show that $c_{d_s}^G(N_s)$ is not a zero-divisor in $\mathrm{CH}_G^*(X_s)$. By Proposition 9.4, it suffices to show that there exists a subtorus T in G of positive rank that acts trivially on X_s , such that in the eigenspace decomposition of N_s with respect to T , the submodule corresponding to the trivial character is zero. But this follows directly from parts (i) and (iii) of Proposition 9.5 and the fact that $s \geq 1$; see [Vezzosi and Vistoli 2003, Proposition 4.6]. \square

9b. Specialization maps. Let G and X be as above and let n be the dimension of G . As seen above, there is a filtration of X by G -invariant open subsets

$$\emptyset = X_{\leq -1} \subset X_{\leq 0} \subset \cdots \subset X_{\leq n} = X.$$

In particular, G acts on $X_{\leq 0}$ with finite stabilizers, and the toral component of G acts trivially on X_n . We fix $1 \leq s \leq n$ and let $f_s : X_s \hookrightarrow X_{\leq s}$ and $g_s : X_{< s} \hookrightarrow X_{\leq s}$ denote the closed and the open embeddings, respectively. Let $\pi : M_s \rightarrow \mathbb{P}^1$ be the deformation to the normal cone for the embedding f_s as in Section 4. We have already observed there that for the trivial action of G on \mathbb{P}^1 , M_s has a

natural G -action. Moreover, the deformation diagram (4-1) is a diagram of smooth G -spaces. For $0 \leq t \leq s$, we shall often denote the open subspace $(M_s)_{\leq t}$ of M_s by $M_{s,\leq t}$. The terms like $M_{s,t}$ and $M_{s,<t}$ (and also for N_s) will have similar meaning in what follows. Since G acts trivially on \mathbb{P}^1 , it acts on M_s fiberwise with $N_s = \pi^{-1}(\infty)$ and

$$M_{s,\leq t} \cap \pi^{-1}(\mathbb{A}^1) = X_{\leq t} \times \mathbb{A}^1, \quad M_{s,t} \cap \pi^{-1}(\mathbb{A}^1) = X_t \times \mathbb{A}^1. \quad (9-2)$$

Let $i_{s,\leq t} : N_{s,\leq t} \hookrightarrow M_{s,\leq t}$ and $j_{s,\leq t} : X_{\leq t} \times \mathbb{A}^1 \hookrightarrow M_{s,\leq t}$ denote the obvious closed and open embeddings. We define $i_{s,t}$ and $j_{s,t}$ similarly. Let $\eta_{s,t} : N_{s,t} \hookrightarrow N_{s,\leq t}$ and $\delta_{s,t} : M_{s,t} \hookrightarrow M_{s,\leq t}$ denote the other closed embeddings. One has a commutative diagram

$$\begin{array}{ccccccc} X_{\leq t} & \xrightarrow{g_{\leq t}} & X_{\leq t} \times \mathbb{A}^1 & \xrightarrow{j_{s,\leq t}} & M_{s,\leq t} & & \\ f_{s,\leq t} \downarrow & & \downarrow & & \downarrow & & \\ X_{\leq s} & \xrightarrow{g_{\leq s}} & X_{\leq s} \times \mathbb{A}^1 & \xrightarrow{j_{s,s}} & M_s & \longrightarrow & X_{\leq s} \times \mathbb{P}^1 \longrightarrow X_{\leq s}, \end{array} \quad (9-3)$$

where $g_{\leq t}$ is the 0-section embedding, and the composite of all the maps in the bottom row is identity. This gives us the diagram

$$\begin{array}{ccccccc} \mathrm{CH}_G^*(N_{s,t}) & \xrightarrow{i_{s,t*}} & \mathrm{CH}_G^*(M_{s,t}) & \xrightarrow{j_{s,t}^*} & \mathrm{CH}_G^*(X_t \times \mathbb{A}^1) & \xrightarrow{g_t^*} & \mathrm{CH}_G^*(X_t) \\ \eta_{s,t*} \downarrow & & \downarrow \delta_{s,t*} & & \downarrow f_{t*} & & \downarrow f_{t*} \\ \mathrm{CH}_G^*(N_{s,\leq t}) & \xrightarrow{i_{s,\leq t*}} & \mathrm{CH}_G^*(M_{s,\leq t}) & \xrightarrow{j_{s,\leq t}^*} & \mathrm{CH}_G^*(X_{\leq t} \times \mathbb{A}^1) & \xrightarrow{g_{\leq t}^*} & \mathrm{CH}_G^*(X_{\leq t}) \end{array}$$

of equivariant higher Chow groups, where the left square commutes by the covariance of the push-forward map, the middle commutes by Proposition 2.2(1) and the right commutes by Corollary 4.2. Since the last horizontal maps in both rows are natural isomorphisms by the homotopy invariance, we shall often identify the last two terms in both rows and use $j_{s,\leq t}^*$ and $(j_{s,\leq t} \circ g_{\leq t})^*$ interchangeably.

Theorem 9.8. *The maps $j_{s,\leq t}^*$ and $j_{s,t}^*$ are surjective and there are ring homomorphisms*

$$\begin{aligned} \overline{\mathrm{Sp}}_{X,s}^{\leq t} : \mathrm{CH}_G^*(X_{\leq t}; \mathbb{Q}) &\rightarrow \mathrm{CH}_G^*(N_{s,\leq t}; \mathbb{Q}) \quad \text{and} \\ \overline{\mathrm{Sp}}_{X,s}^t : \mathrm{CH}_G^*(X_t; \mathbb{Q}) &\rightarrow \mathrm{CH}_G^*(N_{s,t}; \mathbb{Q}) \end{aligned}$$

such that $i_{s,\leq t}^* = \overline{\mathrm{Sp}}_{X,s}^{\leq t} \circ j_{s,\leq t}^*$ and $i_{s,t}^* = \overline{\mathrm{Sp}}_{X,s}^t \circ j_{s,t}^*$. Moreover, both the squares in the following diagram commute:

$$\begin{array}{ccccc} \mathrm{CH}_G^*(X_{\leq t}; \mathbb{Q}) & \xrightarrow{f_t^*} & \mathrm{CH}_G^*(X_t; \mathbb{Q}) & \xrightarrow{f_{t*}} & \mathrm{CH}_G^*(X_{\leq t}; \mathbb{Q}) \\ \overline{\mathrm{Sp}}_{X,s}^{\leq t} \downarrow & & \downarrow \overline{\mathrm{Sp}}_{X,s}^t & & \downarrow \overline{\mathrm{Sp}}_{X,s}^{\leq t} \\ \mathrm{CH}_G^*(N_{s,\leq t}; \mathbb{Q}) & \xrightarrow{\eta_{s,t}^*} & \mathrm{CH}_G^*(N_{s,t}; \mathbb{Q}) & \xrightarrow{\eta_{s,t*}} & \mathrm{CH}_G^*(N_{s,\leq t}; \mathbb{Q}) \end{array} \quad (9-4)$$

Proof. Using the results obtained so far in this section, one can define the specialization maps along the lines of the construction of such maps for K -theory in [Vezzosi and Vistoli 2003, Theorem 3.2; 2005]. However, it is not at all clear from the construction of the specialization maps in [Vezzosi and Vistoli 2005] that these maps have good functorial properties, and, in particular, if they are ring homomorphisms. Moreover, it is not clear if these maps will have the compatibility properties with the Chern character and Riemann–Roch maps (see [Krishna 2009b]) from the equivariant K -groups to higher Chow groups.

We give here a more direct and functorial construction of the specialization maps, which works both for the K -theory as well as the higher Chow groups, and the proof of various compatibilities of these maps then becomes essentially obvious. We give here the construction of these maps for the higher Chow groups. The same construction works also for the K -theory without any change.

First of all, using Corollary 9.7 and Proposition 9.2, we see that for $1 \leq s \leq n$ and $0 \leq t \leq s$, the map $\text{CH}_G^*(X_{\leq s}) \rightarrow \text{CH}_G^*(X_{\leq t})$ is surjective. We now consider the commutative diagram

$$\begin{CD} \text{CH}_G^*(M_s) @>j_{\leq s}^*>> \text{CH}_G^*(X_{\leq s}) \\ @VVV @VVV \\ \text{CH}_G^*(M_{s,\leq t}) @>j_{s,\leq t}^*>> \text{CH}_G^*(X_{\leq t}). \end{CD}$$

Since the composite map in the bottom row of (9-3) is identity, we see by the homotopy invariance that $j_{\leq s}^*$ is surjective. Thus $j_{s,\leq t}^*$ is also surjective. Applying this surjectivity for $j_{s,\leq t}^*$ and $j_{s,\leq t-1}^*$, we obtain the commutative diagram of Figure 1, which is such that the second and the third rows are exact. All the columns are exact by Corollary 9.7 and Proposition 9.2. We conclude that the localization sequence of the top row is also exact. This proves the surjectivity part of the theorem.

Next, we apply the self-intersection formula (Corollary 4.5) to the inclusions $i_{s,\leq t}$ and $i_{s,t}$ to see that the composites $i_{s,\leq t}^* \circ i_{s,\leq t*}$ and $i_{s,t}^* \circ i_{s,t*}$ are multiplication by the first Chern class c_1^G of the corresponding normal bundles. But these normal bundles are the inverse images of a line bundle on \mathbb{P}^1 . It follows that these normal bundles are trivial, because the restriction of any line bundle on \mathbb{P}^1 to $\infty \in \mathbb{P}^1$ and hence on the fiber over ∞ is clearly trivial. We conclude that the composites $i_{s,\leq t}^* \circ i_{s,\leq t*}$ and $i_{s,t}^* \circ i_{s,t*}$ are zero.

The diagram above now automatically defines the specializations $\overline{\text{Sp}}_{X,s}^{\leq t}$ and $\overline{\text{Sp}}_{X,s}^t$ and gives the desired factorization of $i_{s,\leq t}^*$ and $i_{s,t}^*$. Since $i_{s,t}^*$ and $j_{s,t}^*$ are ring homomorphisms, and since the latter is surjective as shown in 1, we deduce that $\overline{\text{Sp}}_{X,s}^t$ is also a ring homomorphism. The map $\overline{\text{Sp}}_{X,s}^{\leq t}$ is a ring homomorphism for the same reason.

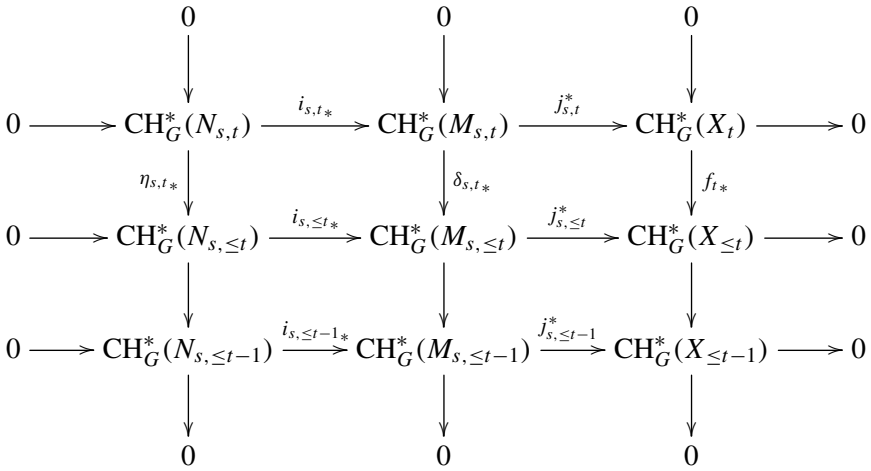
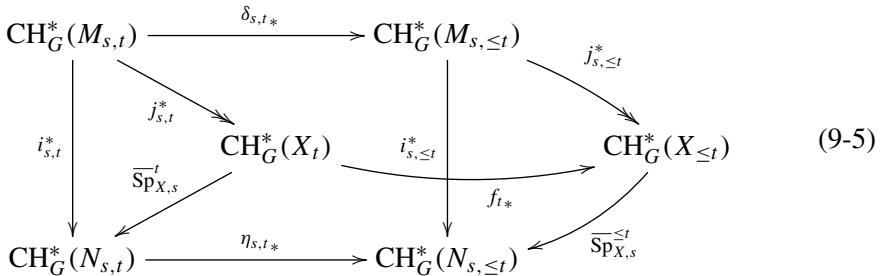


Figure 1

We are now left with the proof of the commutativity of (9-4). To prove that the right square commutes, we consider the following diagram.



It is easy to check that $N_{s,\leq t}$ and $M_{s,t}$ are Tor-independent over $M_{s,\leq t}$ (see [Vezzosi and Vistoli 2005, Lemma 1]) and hence the back face of the diagram above commutes by Corollary 4.2. The upper face commutes by diagram 1. Since $j_{s,t}^*$ is surjective, a diagram chase shows that the lower face also commutes, which is what we needed to prove.

Finally, since we have shown that $\eta_{s,t*}$ is injective, and the right square commutes, it now suffices to show that the composite square in (9-4) commutes in order to show that the left square commutes.

By the projection formula, the composite maps $f_{t*} \circ f_t^*$ and $\eta_{s,t*} \circ \eta_{s,t}^*$ are multiplication by $f_{t*}(1)$ and $\eta_{s,t*}(1)$ respectively. Since

$$\overline{\mathrm{Sp}}_{X,s}^{\leq t} \quad \text{and} \quad \overline{\mathrm{Sp}}_{X,s}^t$$

are ring homomorphisms, it suffices to show that

$$\overline{\text{Sp}}_{X,s}^{\leq t}(f_{t*} \circ j_{s,t}^*(1)) = \overline{\text{Sp}}_{X,s}^{\leq t}(f_{t*}(1)) = \eta_{s,t*}(1).$$

But this follows directly from the commutativity of the right square. □

10. Decomposition theorem for equivariant higher Chow groups

We use the specialization maps to prove the main decomposition theorem for the equivariant higher Chow groups of $X \in \mathcal{V}_G^S$, where G is a split diagonalizable group. We continue with the notation of the previous section.

Proposition 10.1. *The restriction maps*

$$\text{CH}_G^*(X_{\leq s}; \mathbb{Q}) \xrightarrow{(f_s^*, g_s^*)} \text{CH}_G^*(X_s; \mathbb{Q}) \times \text{CH}_G^*(X_{< s}; \mathbb{Q})$$

define an isomorphism of rings

$$\text{CH}_G^*(X_{\leq s}; \mathbb{Q}) \xrightarrow{\sim} \text{CH}_G^*(X_s; \mathbb{Q}) \times_{\text{CH}_G^*(N_s^0; \mathbb{Q})} \text{CH}_G^*(X_{< s}; \mathbb{Q}),$$

where $\text{CH}_G^*(X_s; \mathbb{Q}) \xrightarrow{\eta_{s, \leq s-1}^*} \text{CH}_G^*(N_s^0; \mathbb{Q})$ is the pull-back

$$\text{CH}_G^*(X_s; \mathbb{Q}) \xrightarrow{\sim} \text{CH}_G^*(N_s; \mathbb{Q}) \rightarrow \text{CH}_G^*(N_s^0; \mathbb{Q})$$

and

$$\text{CH}_G^*(X_{< s}; \mathbb{Q}) \xrightarrow{\overline{\text{Sp}}_{X,s}^{\leq s-1}} \text{CH}_G^*(N_{s, \leq s-1}; \mathbb{Q}) = \text{CH}_G^*(N_s^0; \mathbb{Q})$$

is the specialization map of [Theorem 9.8](#).

Proof. We only need to identify the pull-back and the specialization maps with the appropriate maps of [Proposition 9.2](#). In the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{CH}_G^*(X_s) & \xrightarrow{f_{s,\infty}^*} & \text{CH}_G^*(N_s) & \xrightarrow{\eta_{s, \leq s-1}^*} & \text{CH}_G^*(N_s^0) \longrightarrow 0 \\ & & \searrow c_{d_s}^G & & \downarrow f_{s,\infty}^* & & \\ & & & & \text{CH}_G^*(X_s), & & \end{array}$$

where $f_{s,\infty} : X_s \rightarrow N_s$ is the 0-section embedding, the top sequence is exact, and the lower triangle commutes by [Corollary 4.5](#). Since $f_{s,\infty}^*$ is an isomorphism, this immediately identifies the pull-back map of the proposition with the quotient map

$$\text{CH}_G^*(X_s) \rightarrow \frac{\text{CH}_G^*(X_s)}{(c_{d_s}^G(N_s))}.$$

Next, we consider the diagram

$$\begin{array}{ccccc}
 \mathrm{CH}_G^*(X_{\leq s}) & \xrightarrow{\overline{\mathrm{Sp}}_{X,s}^{\leq s}} & \mathrm{CH}_G^*(N_s) & \xrightarrow{f_{s,\infty}^*} & \mathrm{CH}_G^*(X_s) \\
 \downarrow f_{s,\leq s-1}^* & & \downarrow & \swarrow \eta_{s,\leq s-1}^* & \\
 \mathrm{CH}_G^*(X_{< s}) & \xrightarrow{\overline{\mathrm{Sp}}_{X,s}^{\leq s-1}} & \mathrm{CH}_G^*(N_s^0) & &
 \end{array} \tag{10-1}$$

Since the left vertical arrow in the diagram above is surjective, we only need to show that

$$\overline{\mathrm{Sp}}_{X,s}^{\leq s-1} \circ f_{s,\leq s-1}^* = \eta_{s,\leq s-1}^* \circ f_s^*$$

in order to identify $\overline{\mathrm{Sp}}_{X,s}^{\leq s-1}$ with the map j^* of [Proposition 9.2](#). It is clear from the diagram [1](#) and the definition of the specialization maps that the left square in the diagram [\(10-1\)](#) commutes. We have just shown above that the right side triangle also commutes. This reduces us to showing that

$$f_{s,\infty}^* \circ \overline{\mathrm{Sp}}_{X,s}^{\leq s} = f_s^*. \tag{10-2}$$

If $X_s \times \mathbb{P}^1 \xrightarrow{F_s} M_s$ denotes the embedding (see [\(4-1\)](#)), then for $x \in \mathrm{CH}_G^*(X_{\leq s})$, we can write $x = j_{\leq s}^*(y)$ by [Theorem 9.8](#). Then

$$\begin{aligned}
 f_{s,\infty}^* \circ \overline{\mathrm{Sp}}_{X,s}^{\leq s} \circ j_{\leq s}^*(y) &= f_{s,\infty}^* \circ t_{s,\leq s}^*(y) = {}^\dagger g_{\infty,\leq s}^* \circ F_s^*(y) \\
 &= g_{0,\leq s}^* \circ F_s^*(y) = f_s^* \circ j_{\leq s}^*(y) = f_s^*(x),
 \end{aligned}$$

where $= {}^\dagger$ follows from [Corollary 4.2](#). This proves [\(10-2\)](#) and the proposition. \square

We need the following algebraic result before we prove the main theorem. Let A be a \mathbb{Q} -algebra (not necessarily commutative) and let $Z(A)$ denote the center of A . For any linear form $f(t) = \sum_{i=1}^n a_i t_i$ in $A[t_1, \dots, t_n]$ such that $a_i \in \mathbb{Q}$ for each i , let $c(f)$ denote the vector $(a_1, \dots, a_n) \in \mathbb{Q}^n$ consisting of the coefficients of the form f .

Lemma 10.2. *Let A be as above and let $S = \{f_1, \dots, f_s\}$ be a set of linear forms in $A[t_1, \dots, t_n]$ such that the vectors $\{c(f_1), \dots, c(f_s)\}$ are pairwise nonproportional in \mathbb{Q}^n . Let*

$$\gamma_j = \sum_{i=0}^{d_j} m_i^j f_j^i$$

such that $m_{d_j}^j \in \mathbb{Q}^*$ for $1 \leq j \leq s$, and $m_{j'}^j \in Z(A)$ for all j, j' . Then one has

$$(\gamma_1 \cdots \gamma_s) = \bigcap_{j=1}^s (\gamma_j)$$

as ideals in $A[t_1, \dots, t_n]$.

Proof. Using a simple induction, it suffices to show that for $j \neq j'$, the relation $\gamma_j | q \gamma_{j'}$ implies that $\gamma_j | q$. So we can assume $S = \{f_1, f_2\}$. Since $c(f_1)$ and $c(f_2)$ are nonproportional, we can extend the set $\{c(f_1), c(f_2)\}$ to a basis B of \mathbb{Q}^n . Applying the linear automorphism of $A[t_1, \dots, t_n]$ given by the invertible matrix B , we can assume that $f_j = t_j$ for $j = 1, 2$. Now the proof follows along the same lines as the proof of [Vezzosi and Vistoli 2003, Lemma 4.9]. \square

Theorem 10.3. *Let G be a split diagonalizable group of dimension n and let $X \in \mathcal{V}_G^S$. The ring homomorphism*

$$\mathrm{CH}_G^*(X; \mathbb{Q}) \rightarrow \prod_{s=0}^n \mathrm{CH}_G^*(X_s; \mathbb{Q})$$

is injective. Moreover, its image consists of the n -tuples (α_s) in the product with the property that for each $s = 1, \dots, n$, the pull-back of $\alpha_s \in \mathrm{CH}_G^(X_s; \mathbb{Q})$ in $\mathrm{CH}_G^*(N_{s,s-1}; \mathbb{Q})$ is the same as $\overline{\mathrm{Sp}}_{X,s}^{s-1}(\alpha_{s-1}) \in \mathrm{CH}_G^*(N_{s,s-1}; \mathbb{Q})$. In other words, there is a ring isomorphism*

$$\mathrm{CH}_G^*(X; \mathbb{Q}) \xrightarrow{\cong} \mathrm{CH}_G^*(X_n; \mathbb{Q}) \times_{\mathrm{CH}_G^*(N_{n,n-1}; \mathbb{Q})} \cdots \times_{\mathrm{CH}_G^*(N_{1,0}; \mathbb{Q})} \mathrm{CH}_G^*(X_0; \mathbb{Q}).$$

Proof. We prove by induction on the largest integer s such that $X_s \neq \emptyset$.

If $s = 0$, there is nothing to prove. If $s > 0$, we have by induction

$$\mathrm{CH}_G^*(X_{<s}) \xrightarrow{\cong} \mathrm{CH}_G^*(X_{s-1}) \times_{\mathrm{CH}_G^*(N_{s-1,s-2})} \cdots \times_{\mathrm{CH}_G^*(N_{1,0})} \mathrm{CH}_G^*(X_0). \tag{10-3}$$

Using (10-3) and Proposition 10.1, it suffices to show that if $\alpha_s \in \mathrm{CH}_G^*(X_s)$ and if $\alpha_{<s} \in \mathrm{CH}_G^*(X_{<s})$ with the restriction $\alpha_{s-1} \in \mathrm{CH}_G^*(X_{s-1})$ are such that $\alpha_s \mapsto \alpha_s^0 \in \mathrm{CH}_G^*(N_s^0)$ and $\alpha_s \mapsto \alpha_{s,s-1} \in \mathrm{CH}_G^*(N_{s,s-1})$, then

$$\overline{\mathrm{Sp}}_{X,s}^{\leq s-1}(\alpha_{<s}) = \alpha_s^0 \quad \text{if and only if} \quad \overline{\mathrm{Sp}}_{X,s}^{s-1}(\alpha_{s-1}) = \alpha_{s,s-1}.$$

Using the commutativity of the left square in Theorem 9.8, this is reduced to showing that the restriction map

$$\mathrm{CH}_G^*(N_s^0) \rightarrow \mathrm{CH}_G^*(N_{s,s-1}) \tag{10-4}$$

is injective.

To prove this, we first use Proposition 9.5 to assume that the toral component T of the isotropy groups of the points of X_s is fixed, and choose a splitting $G = D \times T$.

Now, following the proof of the analogous result for K -theory [Vezzosi and Vistoli 2003, Theorem 4.5], we can write

$$N_s = E = \bigoplus_{i=1}^q E_i \quad \text{and} \quad N_{s,s-1} = \prod_i E_i^0,$$

where each E_i is of the form $\bigoplus E_{m_j} \otimes \chi_i^{m_j}$ such that for $i \neq i'$, χ_i and $\chi_{i'}$ are nonproportional characters of T , and E_i^0 is embedded in E by setting all the other components equal to zero. Let $d_i = \text{rank}(E_i)$.

Now we see from [Proposition 10.1](#) that

$$\text{Ker}(\text{CH}_G^*(X_s) \rightarrow \text{CH}_G^*(N_{s,s-1})) = \bigcap_i (c_{d_i}^G(E_i)),$$

$$\text{Ker}(\text{CH}_G^*(X_s) \rightarrow \text{CH}_G^*(N_s^0)) = (c_{d_s}^G(N_s)) \quad \text{with } d_s = \sum d_i.$$

Put $\gamma_i = c_{d_i}^G(E_i)$ and $\gamma = c_{d_s}^G(N_s)$. Since the map $\text{CH}_G^*(X_s) \rightarrow \text{CH}_G^*(N_s^0)$ is surjective, showing the injectivity of the map in [\(10-4\)](#) is equivalent to showing that

$$(\gamma) = \left(\prod_i \gamma_i \right) = \bigcap_i (\gamma_i) \tag{10-5}$$

in $\text{CH}_D^*(X_s)[t_1, \dots, t_s]$.

However, we have seen in the proof of [Proposition 9.4](#) that each γ_i is of the form

$$\gamma_i = u_i^{d_i} + \alpha_{d_i-1}^i u_i^{d_i-1} + \dots + \alpha_1^i u_i + \alpha_0^i,$$

where $\alpha_j^i \in \text{CH}_D^*(X_s, 0) \subseteq Z(\text{CH}_D^*(X_s))$ and $u_i = c_1^T(L_{\chi_i}) = \sum_{j=1}^s b_j^i t_j \neq 0$ in $\mathbb{Q}[t_1, \dots, t_s]$. Moreover, the pairwise nonproportionality of χ_i implies the same for the vectors $\{c(u_1), \dots, c(u_q)\}$ in \mathbb{Q}^s . We now apply [Lemma 10.2](#) to conclude the proof of [\(10-5\)](#) and hence the theorem. \square

11. Equivariant higher Chow groups of toric varieties

In this section, we apply our decomposition theorem to give explicit descriptions of the equivariant higher Chow groups of smooth toric varieties. An analogous description of the equivariant cohomology of such varieties was earlier given by Bifet, De Concini and Procesi in [\[Bifet et al. 1990\]](#) and such a description of the equivariant K -theory was given by Vezzosi and Vistoli [\[2003\]](#). Brion [\[1997, Theorem 5.4\]](#) had proven similar results for the classical equivariant Chow groups of toric varieties. As a consequence of our descriptions of the equivariant higher Chow groups, we shall obtain formulas for the ordinary higher Chow groups (or motivic cohomology) of smooth toric varieties.

Let T be a split torus of rank n over k . Let $M = \text{Hom}(\mathbb{G}_m, T)$ be the lattice of the one-parameter subgroups of T and let M^\vee be the character lattice of T . Let Δ be a fan in $M_{\mathbb{R}}$ and let Δ_1 and Δ_{\max} denote the subsets of the one-dimensional cones and the maximal cones in Δ , respectively.

Let $X = X(\Delta)$ be the smooth toric variety associated to the fan Δ . The smoothness of X is equivalent to the condition that every positive dimensional cone of Δ is generated by its edges such that the primitive vectors along these edges form a

subset of a basis of M . In this case, there is an one-to-one correspondence between the T -orbits in X and the cones in Δ . For every cone $\sigma \in \Delta$, the corresponding orbit O_σ is isomorphic to the torus T/T_σ , where T_σ is associated to the sublattice M_σ of M generated by $\sigma \cap M$. Under this isomorphism, the origin (identity point) of T/T_σ corresponds to the distinguished k -rational point x_σ of O_σ . In particular, for every $0 \leq s \leq n$, X_s is of the form

$$X_s = \coprod_{\dim(\sigma)=s} O_\sigma \cong \coprod_{\dim(\sigma)=s} T/T_\sigma. \tag{11-1}$$

We shall write $\tau \leq \sigma$ if τ is a face of σ as cones in Δ . The orbit closure V_σ of O_σ is the toric variety associated to the fan $\ast(\sigma) = \{\tau \in \Delta \mid \sigma \leq \tau\}$, called the star of σ . Moreover, it is clear from the characterization of the smoothness of toric varieties that V_σ is also smooth and is the disjoint union of all orbits O_τ such that σ is a face of τ . In particular, O_σ is closed in X if and only if $\sigma \in \Delta_{\max}$. The following is our first description of the equivariant higher Chow groups of smooth toric varieties.

Theorem 11.1. *Let $X = X(\Delta)$ be a smooth toric variety associated to a fan Δ in $M_{\mathbb{R}}$. There is an injective homomorphism of S -algebras*

$$\Phi_X : \text{CH}_T^*(X; \mathbb{Q}) \rightarrow \prod_{\sigma \in \Delta_{\max}} \text{CH}^*(k; \mathbb{Q}) \otimes S(T_\sigma; \mathbb{Q}).$$

An element

$$(a_\sigma) \in \prod_{\sigma \in \Delta_{\max}} \text{CH}^*(k; \mathbb{Q}) \otimes S(T_\sigma; \mathbb{Q})$$

is in the image of this homomorphism if and only if for any two maximal cones σ_1 and σ_2 , the restrictions of a_{σ_1} and a_{σ_2} to $\text{CH}^*(k; \mathbb{Q}) \otimes S(T_{\sigma_1 \cap \sigma_2}; \mathbb{Q})$ coincide.

Proof. We only need to appropriately identify the various terms and the maps in the statement of [Theorem 10.3](#). We follow the notation of [Section 9](#) and [Section 10](#). It follows from [\[Vezzosi and Vistoli 2003, Lemma 6.1\]](#) that for every $s \geq 1$, there is a canonical isomorphism

$$N_{s,s-1} = \coprod_{\substack{\sigma \in \Delta \\ \dim(\sigma)=s}} \coprod_{\tau \in \partial\sigma} O_\tau. \tag{11-2}$$

Furthermore, for each s -dimensional cone σ and $\tau \in \partial\sigma$, the composition of the map

$$\text{Sp}_{X,s}^{s-1} : \text{CH}_T^*(X_{s-1}) = \prod_{\substack{\tau \in \Delta \\ \dim(\tau)=s-1}} \text{CH}_T^*(O_\tau) \rightarrow \text{CH}_T^*(N_{s,s-1}) = \prod_{\substack{\sigma \in \Delta \\ \dim(\sigma)=s}} \prod_{\tau \in \partial\sigma} \text{CH}_T^*(O_\tau)$$

with the projection

$$\text{Pr}_{\sigma, \tau} : \prod_{\substack{\sigma \in \Delta \\ \dim(\sigma) = s}} \prod_{\tau \in \partial\sigma} \text{CH}_T^*(O_\tau) \rightarrow \text{CH}_T^*(O_\sigma)$$

is the projection map

$$\prod_{\substack{\tau \in \Delta \\ \dim(\tau) = s-1}} \text{CH}_T^*(O_\tau) \rightarrow \text{CH}_T^*(O_\sigma).$$

The identification above of the specialization maps in [Vezzosi and Vistoli 2003, Lemma 6.1] was shown for the equivariant K -theory, but the same holds in the present case as well without any modification in view of the construction of these specializations for higher Chow groups in Section 9.

It also follows from Corollary 3.2 and Theorem 3.5 that

$$\text{CH}_T^*(O_\sigma) \cong \text{CH}_T^*(T/T_\sigma) \cong \text{CH}_T^*(k) \cong \text{CH}^*(k) \otimes S(T_\sigma). \quad (11-3)$$

We conclude now from (11-1) and Theorem 10.3 that $\text{CH}_T^*(X)$ is a subring of $\prod_{\sigma \in \Delta} \text{CH}^*(k) \otimes S(T_\sigma)$, consisting of elements (a_σ) with the property that the restriction of $a_\sigma \in \text{CH}^*(k) \otimes S(T_\sigma)$ to $\text{CH}^*(k) \otimes S(T_\tau)$ coincides with a_τ whenever $\tau \leq \sigma$. The theorem now follows from the fact that every cone in Δ is contained in a maximal cone in Δ . \square

As an immediate consequence of Theorem 11.1, we obtain the following localization theorem for the equivariant higher Chow groups of smooth projective toric varieties. This was earlier proven for $\text{CH}_T^*(X, 0)$ by Brion [1997, Theorem 3.4].

Corollary 11.2. *Let $X = X(\Delta)$ be a smooth projective toric variety and $\{x_1, \dots, x_r\}$ be the fixed point locus of X . Then the map*

$$\text{CH}_T^*(X; \mathbb{Q}) \rightarrow \text{CH}_T^*(X^T; \mathbb{Q}) \xrightarrow{\sim} (\text{CH}^*(k; \mathbb{Q})[t_1, \dots, t_n])^r$$

is injective and its image is the set of all n -tuples (f_1, \dots, f_n) such that $f_i \equiv f_j \pmod{\chi}$ whenever x_i and x_j lie on a T -invariant smooth irreducible curve on which T acts through its character χ .

Proof. This follows directly from Theorem 11.1 once we observe that the fixed points of X for the torus action are same as the T -orbits corresponding to the maximal cones in Δ that are n -dimensional. Moreover, the orbit closures corresponding to the codimension one cones in Δ are the smooth T -invariant curves. \square

11a. Stanley–Reisner presentation. Using Theorem 11.1, we now give another explicit presentation of the equivariant higher Chow groups of smooth toric varieties. This presentation is analogous to the Stanley–Reisner presentation of the equivariant cohomology in [Bifet et al. 1990, Theorem 8] and equivariant K -theory in [Vezzosi

and Vistoli 2003, Theorem 6.1]. This presentation has the advantage that it can often be used to describe the ordinary higher Chow groups of smooth toric varieties.

Let T be a split torus of rank n and let M denote the lattice of the one-parameter subgroups of T . Let $X = X(\Delta)$ be a smooth toric variety associated to a fan Δ in $M_{\mathbb{R}}$. For $r \geq 1$, let Δ_r denote the set of r -dimensional cones in Δ . For $\sigma \in \Delta_{\max}$, let M_σ denote the sublattice of one-parameter subgroups of T_σ so that $\widehat{T}_\sigma = M_\sigma^\vee$ as an abelian group. For any $\rho \in \Delta_1$, let v_ρ denote the generator of the monoid $\rho \cap M$. Note that if $\{\rho_1, \dots, \rho_s\}$ is the set of one-dimensional faces of $\sigma \in \Delta_{\max}$, then the smoothness of X implies that $\{v_{\rho_1}, \dots, v_{\rho_s}\}$ is a basis of M_σ . Let $\{v_{\rho_1}^\vee, \dots, v_{\rho_s}^\vee\}$ denote the dual basis of M_σ^\vee .

We recall that for $\sigma \in \Delta$, there is a canonical isomorphism of abelian groups $\widehat{T}_\sigma \hookrightarrow S(T_\sigma)_1$ given by $\chi \mapsto c_1^{T_\sigma}(L_\chi)$. For each $\rho \in \Delta_1$, we define an element $u_\rho = (u_\rho^\sigma) \in \prod_{\sigma \in \Delta_{\max}} \prod S(T_\sigma)$ such that

$$u_\rho^\sigma = \begin{cases} v_\rho^\vee & \text{if } \rho \leq \sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{11-4}$$

Then u_ρ has the property that for all $\sigma_1, \sigma_2 \in \Delta_{\max}$, the restrictions of $u_{\rho_1}^{\sigma_1} \in \widehat{T}_{\sigma_1}$ and $u_{\rho_2}^{\sigma_2} \in \widehat{T}_{\sigma_2}$ in $\widehat{T}_{\sigma_1 \cap \sigma_2}$ coincide.

We have the obvious inclusion

$$\prod_{\sigma \in \Delta_{\max}} S(T_\sigma) \subseteq \prod_{\sigma \in \Delta_{\max}} \text{CH}^*(k) \otimes S(T_\sigma) \tag{11-5}$$

and using the description of $\text{CH}_T^*(X)$ in Theorem 11.1 and the description of u_ρ above, we can consider these u_ρ as elements of the ring $\text{CH}_T^*(X)$. In other words, we get a bigraded $\text{CH}^*(k)$ -algebra homomorphism

$$\text{CH}^*(k)[t_\rho] \rightarrow \text{CH}_T^*(X), \quad t_\rho \mapsto u_\rho,$$

where $\text{CH}^*(k)[t_\rho]$ is the polynomial algebra $\text{CH}^*(k)[t_\rho \mid \rho \in \Delta_1]$.

If S is a subset of Δ_1 that is not contained in any maximal cone of Δ , then for any given $\sigma \in \Delta_{\max}$, there is one $\rho \in S$ such that $\rho \not\leq \sigma$. This implies in particular that $u_\rho^\sigma = 0$. We conclude from this that the elements u_ρ satisfy the relation

$$\prod_{\rho \in S} u_\rho = 0 \quad \text{in } \text{CH}_T^*(X) \tag{11-6}$$

whenever $S \subseteq \Delta_1$ is such that it is not contained in any maximal cone of Δ . We shall denote the collection of all such subsets of Δ_1 by Δ_1^0 . We conclude that if I_Δ denotes the graded ideal of $\text{CH}^*(k)[t_\rho]$ generated by the set of monomials

$\{\prod_{\rho \in S} t_\rho \mid S \in \Delta_1^0\}$, then there is a $\text{CH}^*(k)$ -algebra homomorphism

$$\Psi_X : \frac{\text{CH}^*(k)[t_\rho]}{I_\Delta} \rightarrow \text{CH}_T^*(X), \quad t_\rho \mapsto u_\rho. \quad (11-7)$$

Note also that any character $\chi \in M^\vee$ defines multiplication by the element $t_\chi = \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle t_\rho$ in $\text{CH}^*(k)[t_\rho]$, and this makes the term on the left hand side of (11-7) an S -algebra and Ψ_X is also an S -algebra homomorphism. Furthermore, it is easy to check from the definition of u_ρ that it is the fundamental class (see [Edidin and Graham 1998, Section 2]) of the T -equivariant Chow cycle

$$[V_\rho \rightarrow X] \in \text{CH}_T^1(X, 0) \subsetneq \text{CH}_T^*(X),$$

where V_σ is the orbit closure in X associated to a cone $\sigma \in \Delta$.

Theorem 11.3. *For a smooth toric variety $X = X(\Delta)$ associated to a fan Δ in $M_{\mathbb{R}}$, the homomorphism Ψ_X is an isomorphism with rational coefficients.*

Proof. We prove the theorem by induction on the number of maximal cones in Δ . Suppose $\Delta_{\max} = \{\sigma\}$ is a singleton set. In that case, σ is the only maximal cone and $X = U_\sigma$ is a T -equivariant vector bundle over O_σ such that the inclusion

$$O_\sigma \xrightarrow{i_\sigma} X$$

is the zero-section embedding. Hence, we conclude from (11-3) that there are isomorphisms

$$\text{CH}_T^*(X) \xrightarrow{i_\sigma^*} \text{CH}_T^*(O_\sigma) \cong \text{CH}^*(k) \otimes S(T_\sigma) = \text{CH}^*(k)[t_1, \dots, t_s],$$

where s is the dimension of σ . It is also clear in this case that the ideal I_Δ in (11-7) is zero. Hence, we have isomorphisms

$$\text{CH}^*(k)[t_1, \dots, t_s] \xrightarrow{\Psi_X} \text{CH}_T^*(X) \xrightarrow{\Phi_X} \text{CH}^*(k) \otimes S(T_\sigma).$$

We consider now the general case. We assume that $|\Delta_{\max}| \geq 2$ and choose a maximal cone σ of dimension $s \geq 1$ in Δ . Let $X' = X'(\Delta')$ be the toric variety associated to the fan $\Delta' = \Delta \setminus \{\sigma\}$. Note that O_σ is a closed T -orbit in X and X' is the complement of O_σ in X . Let $U_\sigma \subset X$ be the principal open set associated to the fan consisting of all faces of σ and let U' be the complement of O_σ in U_σ . Then U' is nothing but the complement of the zero-section in the T -equivariant vector bundle $U_\sigma \rightarrow O_\sigma$. Let $i_\sigma : O_\sigma \hookrightarrow X$ and $j_\sigma : X' \hookrightarrow X$ denote the T -invariant closed and open embeddings respectively. Let $S_\sigma = \{\rho_1, \dots, \rho_s\}$ be the set of one-dimensional faces of σ and set

$$x_\sigma = \prod_{j=1}^s t_{\rho_j} \in \frac{\text{CH}^*(k)[t_\rho]}{I_\Delta} \quad \text{and} \quad y_\sigma = \prod_{j=1}^s u_{\rho_j} \in \text{CH}_T^*(X).$$

Since $N_{O_\sigma/X} = N_{O_\sigma/U_\sigma}$ and since the latter is of the form $\bigoplus_{j=1}^s L_{\chi_j}$, where $\{\chi_1, \dots, \chi_s\}$ is a basis of \widehat{T}_σ , it follows from the definition of the elements u_ρ (see [Proposition 9.4](#)) that

$$c_s^T(N_{O_\sigma/X}) = y_\sigma \in \mathrm{CH}_T^*(X). \quad (11-8)$$

We consider the diagram

$$\begin{array}{ccccc} \mathrm{CH}^*(k)[t_{\rho_1}, \dots, t_{\rho_s}] & \xrightarrow{\cong} & \mathrm{CH}_T^*(O_\sigma) & \xrightarrow{\cong} & \mathrm{CH}^*(k) \otimes S(T_\sigma) \\ \downarrow x_\sigma & & \downarrow i_{\sigma*} & & \downarrow y_\sigma \\ \frac{\mathrm{CH}^*(k)[t_\rho]}{I_\Delta} & \xrightarrow{\Psi_X} & \mathrm{CH}_T^*(X) & \xrightarrow{\Phi_X} & \prod_{\tau \in \Delta_{\max}} \mathrm{CH}^*(k) \otimes S(T_\tau), \end{array} \quad (11-9)$$

where the horizontal maps on the top are the obvious isomorphisms taking t_{ρ_j} to u_{ρ_j} . The left and the right vertical maps are the multiplication by the indicated elements in the target rings. We claim that all the vertical arrows are injective and the left square in this diagram commutes.

To prove the claim, notice that the composite outer square clearly commutes by the definition of x_σ and y_σ and the map Ψ_X . Since Φ_X is injective by [Theorem 11.1](#), we only need to show that the right square commutes and the right vertical arrow is injective to prove the claim.

We first observe that the right vertical arrow is the multiplication by y_σ on the factor $\mathrm{CH}^*(k) \otimes S(T_\sigma)$ and is zero on the other factors of $\prod_{\tau \in \Delta_{\max}} \mathrm{CH}^*(k) \otimes S(T_\tau)$. Thus the required injectivity is equivalent to showing that the multiplication by y_σ is injective in $\mathrm{CH}^*(k) \otimes S(T_\sigma)$. We can thus assume that $X = U_\sigma$ and then $\mathrm{CH}_T^*(X) \cong \mathrm{CH}^*(k)[t_1, \dots, t_s]$. In this case, y is just the element $t_1 \cdots t_s$ and hence is a nonzero divisor in $\mathrm{CH}^*(k)[t_1, \dots, t_s]$.

To show the commutativity of the right square, we observe from the proof of [Theorem 11.1](#) that Φ_X is simply the product of the pull-back maps

$$i_\tau^* : \mathrm{CH}_T^*(X) \rightarrow \mathrm{CH}_T^*(O_\tau) \quad \text{for } \tau \in \Delta_{\max}.$$

Hence the composite $\Phi_X \circ i_{\sigma*}$ is $i_\sigma^* \circ i_{\sigma*}$ on the factor $\mathrm{CH}_T^*(O_\sigma)$ and zero on the other factors of $\prod_{\tau \in \Delta_{\max}} \mathrm{CH}_T^*(O_\tau)$. Since we have just seen that the composite

$$\mathrm{CH}_T^*(O_\sigma) \xrightarrow{y_\sigma} \prod_{\tau \in \Delta_{\max}} \mathrm{CH}_T^*(O_\tau)$$

is of similar type, we are reduced to showing that the triangle

$$\begin{array}{ccc} \mathrm{CH}_T^*(O_\sigma) & & \\ i_{\sigma*} \downarrow & \searrow y_\sigma & \\ \mathrm{CH}_T^*(X) & \xrightarrow{i_\sigma^*} & \mathrm{CH}_T^*(O_\sigma) \end{array}$$

commutes. But this follows immediately from [Corollary 4.5](#) and [\(11-8\)](#). This proves the claim.

To complete the proof of the theorem, we now consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{CH}^*(k)[t_{\rho_1}, \dots, t_{\rho_s}] & \xrightarrow{x_\sigma} & \frac{\mathrm{CH}^*(k)[t_\rho]}{I_\Delta} & \xrightarrow{\bar{j}_\sigma^*} & \frac{\mathrm{CH}^*(k)[t_\rho]}{(I_\Delta, x_\sigma)} \longrightarrow 0 \\ & & \cong \downarrow & & \downarrow \Psi_X & & \downarrow \bar{\Psi}_{X'} \\ 0 & \longrightarrow & \mathrm{CH}_T^*(O_\sigma) & \xrightarrow{i_{\sigma*}} & \mathrm{CH}_T^*(X) & \xrightarrow{j_\sigma^*} & \mathrm{CH}_T^*(X') \longrightarrow 0, \end{array} \quad (11-10)$$

where \bar{j}_σ^* is the natural quotient map by the ideal (x_σ) in $\mathrm{CH}^*(k)[t_\rho]/I_\Delta$. Note that the image of the first map in the top row is the ideal (x_σ) because the product of x_σ with any t_ρ for $\rho \notin \{\rho_1, \dots, \rho_s\}$ is zero.

The left square in this diagram commutes and the first maps in both the rows are injective by the claim above. The bottom row is exact by [Proposition 2.2](#). Since σ is not a cone of Δ' , the element x_σ is zero in $\mathrm{CH}^*(k)[t_\rho, \rho \in \Delta'_1]/I_{\Delta'}$ and hence the map $j_\sigma^* \circ \Psi_X$ has a factorization:

$$\frac{\mathrm{CH}^*(k)[t_\rho]}{I_\Delta} \xrightarrow{\sim} \frac{\mathrm{CH}^*(k)[t_\rho]}{(I_\Delta, x_\sigma)} \rightarrow \frac{\mathrm{CH}^*(k)[t_\rho, \rho \in \Delta'_1]}{I_{\Delta'}} \xrightarrow{\Psi_{X'}} \mathrm{CH}_T^*(X'),$$

where the middle arrow is the natural map of the Stanley–Reisner rings induced by the inclusion of the fans $\Delta' \subset \Delta$. Letting $\bar{\Psi}_{X'}$ denote the composite

$$\frac{\mathrm{CH}^*(k)[t_\rho]}{(I_\Delta, x_\sigma)} \rightarrow \frac{\mathrm{CH}^*(k)[t_\rho, \rho \in \Delta'_1]}{I_{\Delta'}} \xrightarrow{\Psi_{X'}} \mathrm{CH}_T^*(X'),$$

we see that the right square in the diagram [\(11-10\)](#) also commutes.

If all the cones of Δ are at most one-dimensional, then $x_\sigma = t_\rho$, where $\rho = \sigma$ and it is obvious that $\mathrm{CH}^*(k)[t_\rho]/(I_\Delta, x_\sigma)$ is the Stanley–Reisner ring associated to the fan Δ' . If Δ has a cone of dimension at least two, we can assume that σ is of dimension at least two. In that case, we have $\Delta'_1 = \Delta_1$ and the natural inclusion $\Delta_1^0 \subseteq \Delta_1^0$ gives the equality $\Delta_1^0 = \Delta_1^0 \coprod \{S_\sigma\}$. In particular, we have

$$\frac{\mathrm{CH}^*(k)[t_\rho, \rho \in \Delta_1]}{(I_\Delta, x_\sigma)} \xrightarrow{\sim} \frac{\mathrm{CH}^*(k)[t_\rho, \rho \in \Delta'_1]}{I_{\Delta'}}.$$

On the other hand, Δ' is a fan with smaller number of maximal cones than in Δ and $X' = X'(\Delta')$. Hence the map

$$\frac{\text{CH}^*(k)[t_\rho, \rho \in \Delta'_1]}{I_{\Delta'}} \xrightarrow{\Psi_{X'}} \text{CH}_T^*(X')$$

is an isomorphism by induction. We conclude that the map $\bar{\Psi}_{X'}$ in the diagram (11-10) is an isomorphism. A diagram chase in (11-10) now shows that Ψ_X is also an isomorphism. \square

As an important application of [Theorem 11.3](#), we obtain the following presentation of the ordinary higher Chow groups (motivic cohomology groups) of smooth toric varieties. An explicit description of $\text{CH}^*(X, 0)$ for a smooth projective toric variety X was given in [\[Fulton 1993, Proposition 5.2\]](#). The following result extends this to all smooth toric varieties, not necessarily projective. In fact, such a description extends to all higher Chow groups of smooth projective toric varieties. In particular, we obtain another proof of [Corollary 7.3](#) with rational coefficients. Recall that for every $\sigma \in \Delta$, the orbit closure $V_\sigma = \overline{O_\sigma}$ in X is a T -invariant Weil divisor and defines a unique element $[V_\sigma] \in \text{CH}^1(X, 0)$.

Corollary 11.4. *Let $X = X(\Delta)$ be a smooth projective toric variety. Then the assignment $t_\rho \mapsto [V_\sigma]$ defines a $\text{CH}^*(k; \mathbb{Q})$ -algebra isomorphism*

$$\bar{\Psi}_X : \frac{\text{CH}^*(k; \mathbb{Q})[t_\rho]}{(I_\Delta, \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle t_\rho)} \rightarrow \text{CH}^*(X; \mathbb{Q}), \tag{11-11}$$

where χ runs over M^\vee .

If X is not necessarily projective, the map

$$\frac{\mathbb{Q}[t_\rho]}{(I_\Delta, \sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle t_\rho)} \rightarrow \text{CH}^*(X, 0; \mathbb{Q})$$

is a ring isomorphism.

Proof. We have already seen before that every character $\chi \in M^\vee$ acts on $\text{CH}^*(k)[t_\rho]$ by multiplication with the element $\sum_{\rho \in \Delta_1} \langle \chi, v_\rho \rangle t_\rho$ which makes the left hand side of (11-11) an S -algebra. The corollary now follows directly from [Theorem 11.3](#) and [\[Krishna 2009a, Theorem 1.3\]](#). The second isomorphism follows in the same way from [Theorem 11.3](#) and [\[Brion 1997, Corollary 2.3\]](#). \square

Corollary 11.5. *Let $X = X(\Delta)$ be a smooth toric variety. Then there are canonical ring isomorphisms*

$$\begin{aligned} \text{CH}^*(k; \mathbb{Q}) \otimes \text{CH}_T^*(X, 0; \mathbb{Q}) &\xrightarrow{\sim} \text{CH}_T^*(X; \mathbb{Q}), \\ \text{CH}^*(k; \mathbb{Q}) \otimes \text{CH}^*(X, 0; \mathbb{Q}) &\xrightarrow{\sim} \text{CH}^*(X; \mathbb{Q}). \end{aligned}$$

Proof. It follows from (11-7) that $\mathrm{CH}^*(k) \otimes (\mathbb{Q}[t_\rho]/I_\Delta) \xrightarrow{\sim} \mathrm{CH}^*(k)[t_\rho]/I_\Delta$. The first part of the corollary now follows directly from [Theorem 11.3](#). The second part follows from the first and [[Krishna 2009a](#), Theorem 1.1], which says that there is a convergent spectral sequence

$$\mathrm{Tor}_p^S(\mathbb{Q}, \mathrm{CH}_T^*(X, q)) \Rightarrow \mathrm{CH}^*(X, p+q). \quad \square$$

Remark 11.6. All the results in this section about the (equivariant) higher Chow groups of smooth toric varieties have been stated over the rationals. However, an attentive reader can check that these results (and the proofs) for the subrings $\mathrm{CH}_T^*(X, 0)$ and $\mathrm{CH}^*(X, 0)$ hold true with the integral coefficients. The basic reason is that $\mathrm{CH}_T^*(k, 0)$ and $\mathrm{CH}^*(k, 0)$ are torsion-free abelian groups. But this is false for the higher Chow groups of k .

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Algebra & Number Theory

Volume 7 No. 2 2013

The system of representations of the Weil–Deligne group associated to an abelian variety	243
RUTGER NOOT	
Fourier–Jacobi coefficients of Eisenstein series on unitary groups	283
BEI ZHANG	
The phase limit set of a variety	339
MOUNIR NISSE and FRANK SOTTILE	
Base change behavior of the relative canonical sheaf related to higher dimensional moduli	353
ZSOLT PATAKFALVI	
Two ways to degenerate the Jacobian are the same	379
JESSE LEO KASS	
Arithmetic motivic Poincaré series of toric varieties	405
HELENA COBO PABLOS and PEDRO DANIEL GONZÁLEZ PÉREZ	
Maximal ideals and representations of twisted forms of algebras	431
MICHAEL LAU and ARTURO PIANZOLA	
Higher Chow groups of varieties with group action	449
AMALENDU KRISHNA	