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Bruno Chiarellotto, Alice Ciccioni and Nicola Mazzari



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Let $\mathcal{V} = \text{Spec}(R)$ and R be a complete discrete valuation ring of mixed characteristic $(0, p)$. For any flat R -scheme \mathcal{X} , we prove the compatibility of the de Rham fundamental class of the generic fiber and the rigid fundamental class of the special fiber. We use this result to construct a syntomic regulator map $\text{reg}_{\text{syn}} : CH^i(\mathcal{X}/\mathcal{V}, 2i - n) \rightarrow H_{\text{syn}}^n(\mathcal{X}, i)$ when \mathcal{X} is smooth over R with values in the syntomic cohomology defined by A. Besser. Motivated by the previous result, we also prove some of the Bloch–Ogus axioms for the syntomic cohomology theory but viewed as an absolute cohomology theory.

Introduction

Let $\mathcal{V} = \text{Spec}(R)$ with R a complete discrete valuation ring of mixed characteristic and perfect residue field. Given \mathcal{X} , an algebraic \mathcal{V} -scheme, one can consider the de Rham cohomology of its generic fiber \mathcal{X}_K and the rigid cohomology of its special fiber \mathcal{X}_k . These two cohomology groups are related by a canonical *cospecialization* map $\text{cosp} : H_{\text{rig},c}^n(\mathcal{X}_k) \rightarrow H_{\text{dR},c}^n(\mathcal{X}_K)$ (in general not an isomorphism) [Baldassarri et al. 2004, Section 6]. There is also the notion of rigid and de Rham cycle class. The starting result of this paper is the compatibility of these cycle classes with respect to the cospecialization map (see Theorem 1.4.1 for the precise statement).

In the case \mathcal{X} is smooth (possibly nonproper) over \mathcal{V} , we get the following corollary (see Corollary 1.5.1). Let $\text{sp}_{CH} : CH^*(\mathcal{X}_K) \otimes \mathbb{Q} \rightarrow CH^*(\mathcal{X}_k) \otimes \mathbb{Q}$ be the specialization of Chow rings constructed by Grothendieck in [Berthelot et al. 1971, Appendix]. Then the diagram

$$\begin{array}{ccc}
 CH^q(\mathcal{X}_K/K) \otimes \mathbb{Q} & \xrightarrow{\eta_{\text{dR}}} & H_{\text{dR}}^{2q}(\mathcal{X}_K) \\
 \text{sp}_{CH} \downarrow & & \downarrow \text{sp} \\
 CH^q(\mathcal{X}_k/k) \otimes \mathbb{Q} & \xrightarrow{\eta_{\text{rig}}} & H_{\text{rig}}^{2q}(\mathcal{X}_k/K)
 \end{array}$$

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is commutative, where η_{dR} and η_{rig} are the de Rham and cycle class maps, respectively, and sp is the Poincaré dual of $cosp$. In the proof, we use the main results of [Baldassarri et al. 2004; Bosch et al. 1995; Petrequin 2003].

This result can be viewed as a generalization of a theorem of Messing [1987, Theorem B3.1], in which he further assumes \mathcal{X} to be proper (not only smooth) over \mathcal{V} . In that case, rigid and crystalline coincide, and the map sp is an isomorphism [Berthelot 1997b].

This compatibility result is the motivation for an alternative construction of the regulator map (see Proposition 1.6.6)

$$\text{reg}_{\text{syn}} : CH^i(\mathcal{X}/\mathcal{V}, 2i - n) \rightarrow H_{\text{syn}}^n(\mathcal{X}, i)$$

with values in the syntomic cohomology group defined by Besser [2000] (for \mathcal{X} smooth over \mathcal{V}). For this proof, we use an argument of Bloch [1986] and the existence of a syntomic cycle class (see Proposition 1.6.2).

The aforementioned results motivated us to investigate further the properties of syntomic cohomology. We are not able to formulate even the basic Bloch–Ogus axioms using Besser’s framework. Thus, we have followed the interpretation of syntomic cohomology of Bannai [2002] as an absolute one. To this end, we define a triangulated category of p -adic Hodge complexes, pHD (see Definition 2.0.11). An object M of pHD can be represented by a diagram of the form $M_{\text{rig}} \rightarrow M_K \leftarrow M_{\text{dR}}$, where $M_?$ is a complex of K -vector spaces endowed with a Frobenius automorphism when $? = \text{rig}$ and with a filtration when $? = \text{dR}$. In pHD , there is a naturally defined tensor product, and \mathbb{k} denotes the unit object of pHD . The main difference with respect to [Bannai 2002] is that the maps in the diagram are not necessarily quasi-isomorphisms.

From [Besser 2000], we get (in Proposition 5.3.1) that there are functorial p -adic Hodge complexes $R\Gamma(\mathcal{X})$ satisfying

$$R\Gamma_{\text{rig}}(\mathcal{X}) \rightarrow R\Gamma_K(\mathcal{X}) \leftarrow R\Gamma_{\text{dR}}(\mathcal{X})$$

and inducing the specialization map in cohomology (that is, taking the cohomology of each element of the diagram). Meanwhile, we show how the constructions made by Besser may be obtained using the theory of generalized Godement resolution (also called the bar resolution). In particular, we use the results of [van der Put and Schneider 1995] in order to have enough points for rigid analytic spaces. Further, we consider the twisted version $R\Gamma(\mathcal{X})(i)$, which is given by the same complexes but with the Frobenius (respectively the filtration) twisted by i (see Remark 2.2.1).

We then prove (see Proposition 5.3.4) that the syntomic cohomology groups $H_{\text{syn}}^n(\mathcal{X}, i)$ of [Besser 2000] are isomorphic to the (absolute cohomology) groups $H_{\text{abs}}^n(\mathcal{X}, i) := \text{Hom}_{pHD}(\mathbb{k}, R\Gamma(\mathcal{X})(i)[n])$. This result generalizes that of [Bannai

2002] (which was given only for \mathcal{V} -schemes with good compactification) to any smooth algebraic \mathcal{V} -scheme.

For this absolute cohomology, we can prove some of the Bloch–Ogus axioms. In fact, we construct a p -adic Hodge complex $R\Gamma_c(\mathcal{X})(i)$ related to rigid and de Rham cohomology with compact support. Therefore, we can define an absolute cohomology with compact support functorial with respect to proper maps $H_{\text{abs},c}^n(\mathcal{X}, i) := \text{Hom}_{pHD}(\mathbb{K}, R\Gamma_c(\mathcal{X})(i)[n])$ and an absolute homology theory $H_n^{\text{abs}}(\mathcal{X}, i) := \text{Hom}_{pHD}(R\Gamma_c(\mathcal{X})(i)[n], \mathbb{K})$.

We wish to point out that the constructions above are essentially consequences of the work done by Besser and Bannai, but it seems hard to prove the following results without the formalism of Godement resolutions that we develop in Section 3. Let \mathcal{X} be a smooth scheme over \mathcal{V} . Then

- (i) there is a cup product pairing

$$H_{\text{abs}}^n(\mathcal{X}, i) \otimes H_{\text{abs},c}^m(\mathcal{X}, j) \rightarrow H_{\text{abs},c}^{n+m}(\mathcal{X}, i + j)$$

induced by the natural pairings defined on the cohomology of the generic and the special fiber (see Corollary 5.4.4),

- (ii) there is a Poincaré duality isomorphism (see Proposition 5.4.5) and
- (iii) there is a Gysin map; that is, given a proper morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of smooth algebraic \mathcal{V} -schemes of relative dimension d and e , respectively, then there is a canonical map

$$f_* : H_{\text{abs}}^n(\mathcal{X}, i) \rightarrow H_{\text{abs}}^{n+2c}(\mathcal{Y}, i + c),$$

where $c = e - d$ (see Corollary 5.4.7).

Notation. In this paper, R is a complete discrete valuation ring with fraction field K and residue field k with k perfect. We assume $\text{char}(K) = 0$ and $\text{char}(k) = p > 0$.

The ring of Witt vectors of k is denoted by R_0 , and K_0 is its field of fractions. The Frobenius of K_0 is denoted by σ . The category of bounded complexes of K -vector spaces is denoted by $C^b(K)$.

If V is a K -vector space, then V^\vee is the dual vector space.

We use X, Y, \dots for schemes over k or K ; $\mathcal{X}, \mathcal{Y}, \dots$ for K -analytic spaces; P, Q for formal K -schemes; P_K, Q_K for the associated Raynaud fibers and \mathcal{X}, \mathcal{Y} for (algebraic) \mathcal{V} -schemes, where $\mathcal{V} = \text{Spec}(R)$. Finally, $\widehat{\mathcal{X}}$ denotes the p -adic completion of a \mathcal{V} -scheme \mathcal{X} .

1. Cycle classes

1.1. Higher Chow groups. Following Bloch [1986], we recall the definition of the higher Chow groups $CH^*(X/K, *)$ of a K -scheme X . For any $n \geq 0$, let

$\Delta^n := \text{Spec}(\mathbb{Z}[t_0, \dots, t_n]/(\sum_i t_i - 1))$ with face maps $\partial_i(n) : \Delta^n \rightarrow \Delta^{n+1}$, which in coordinates are given by $\partial_i(n)(t_0, \dots, t_n) := (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$. Let X be a smooth K -scheme of relative dimension d (this hypothesis is not necessary, but we deal only with smooth schemes). Let $z^q(X/K, 0)$ be the free abelian group generated by the irreducible and closed subschemes of X of codimension q . Let $z^q(X/K, n)$ denote the free abelian group generated by the closed subschemes $W \subset \Delta_X^n := X \times \Delta^n$ such that $W \in z^q(\Delta_X^n/K, 0)$ and meets all faces properly; that is, if $F \subset \Delta_X^n$ is a face of codimension c , then the codimension of each irreducible component of the intersection $F \cap W$ is greater than or equal to $c + q$ on Δ_X^n .

Using the differential $\sum_i (-1)^i \partial_i^*(n) : z^q(X/K, n + 1) \rightarrow z^q(X/K, n)$, one obtains a complex of abelian groups $z^q(X/K, *)$. We set $\Gamma_X^i(q) := z^q(X/K, 2q - i)$ and

$$CH^q(X/K, 2q - i) := H^i(\Gamma_X(q)).$$

These groups are in fact isomorphic to the Voevodsky–Suslin motivic cohomology $H_{mot}^i(X/K, q)$ of the generic fiber X [Mazza et al. 2006, Theorem 19.1].

1.2. Relative cycles. Let \mathcal{X} be an algebraic and flat \mathcal{V} -scheme. By the theory of relative cycles [Suslin and Voevodsky 2000], one can define the group $z^q(\mathcal{X}/\mathcal{V}, 0)$ to be the free abelian group generated by *universally integral relative cycles* of codimension q (we can use the codimension because \mathcal{X} is assumed to be equidimensional over \mathcal{V}). By [Ivorra 2005, Part I, Lemma 1.2.6], $z^q(\mathcal{X}/\mathcal{V}, 0)$ is the free abelian group generated by the closed subschemes $\mathcal{W} \subset \mathcal{X}$ that are integral, of codimension q and flat over \mathcal{V} . Then we can define the group $z^q(\mathcal{X}/\mathcal{V}, n)$ as the free abelian group generated by the integral and flat \mathcal{V} -schemes $\mathcal{W} \in z^q(\Delta_{\mathcal{X}}^n, 0)$ meeting all faces properly and such that $\partial_i(n - 1)^*\mathcal{W}$ is flat over \mathcal{V} for all i . Thus, we can form a complex $z^q(\mathcal{X}/\mathcal{V}, *)$ with the same boundary maps as $z^q(X/K, *)$.

Definition 1.2.1. With the notation above, we define the higher Chow groups of \mathcal{X} over \mathcal{V} to be

$$CH^q(\mathcal{X}/\mathcal{V}, 2q - i) := H^i(\Gamma_{\mathcal{X}}(q)),$$

where $\Gamma_{\mathcal{X}/\mathcal{V}}^i(q) := z^q(\mathcal{X}/\mathcal{V}, 2q - i)$.

Remark 1.2.2. (i) Recall that by Lemma 5.1.1, any closed and flat subscheme of \mathcal{X} is completely determined by its generic fiber. Then $z^q(\mathcal{X}/\mathcal{V}, *)$ is a subcomplex of $z^q(\mathcal{X}_K/K, *)$ inducing a canonical map in (co)homology

$$\gamma : CH^q(\mathcal{X}/\mathcal{V}, 2q - i) := H^i(\Gamma_{\mathcal{X}/\mathcal{V}}(q)) \rightarrow CH^q(\mathcal{X}_K/K, 2q - i).$$

(ii) It follows easily that, for $i = 2q$, the map $\gamma : CH^q(\mathcal{X}/\mathcal{V}, 0) \rightarrow CH^q(\mathcal{X}_K/K, 0)$ is surjective by the snake lemma. In the general case, we don't know whether or not this map is injective or surjective.

1.3. De Rham and rigid fundamental/cycle classes. In the following, we refer to [Hartshorne 1975; Baldassarri et al. 2004; Petrequin 2003] for the definitions and the properties of the (algebraic) de Rham and the rigid cohomology theory. Let W be an integral scheme of dimension r over K (resp. over k). Then we can associate to W its de Rham (resp. rigid) fundamental class, which is an element of the dual of the top de Rham (resp. rigid) cohomology with compact support

$$[W]_{\text{dR}} = \text{tr}_{\text{dR}} \in H_{\text{dR},c}^{2r}(W)^\vee \quad (\text{resp. } [W]_{\text{rig}} = \text{tr}_{\text{rig}} \in H_{\text{rig},c}^{2r}(W)^\vee).$$

For the de Rham case, this class is first defined in [Hartshorne 1975, Section 7] as an element of the de Rham homology; by Poincaré duality [Baldassarri et al. 2004, Theorem 3.4], it corresponds to the trace map. The rigid case is treated in [Petrequin 2003, Section 2.1, Section 6].

Now let X be a K -scheme (resp. k -scheme) of dimension d and $w = \sum_i n_i W_i \in z^q(X/K, 0)$ (resp. $\in z^q(X/k, 0)$) be a dimension- r cycle on X . The cohomology with compact support is functorial with respect to proper maps; hence, there is a canonical map

$$f : \bigoplus_i H_{?,c}^{2r}(W_i)^\vee \rightarrow H_{?,c}^{2r}(|w|)^\vee,$$

where $|w| = \bigcup_i W_i$ is the support of w and $?$ is dR or rig according to the choice of the base field. With the notation above, we define the de Rham (resp. rigid) cycle class of w as

$$[w]_? := f \left(\sum_i [W_i]_? \right) \in H_{?,c}^{2r}(|w|)^\vee, \quad \text{where } ? = \text{dR, rig}.$$

Again by functoriality, this defines an element of $H_{?,c}^{2r}(X)^\vee$.

1.4. Compatibility. Let \mathcal{X} be a flat \mathcal{V} -scheme, and let $w \in z^q(\mathcal{X}/\mathcal{V}, 0)$ be a relative cycle. We can write $w = \sum_i n_i \mathcal{W}_i$, where \mathcal{W}_i is an integral flat \mathcal{V} -scheme closed in \mathcal{X} and of codimension q . Then w defines a cycle $w_K \in z^q(\mathcal{X}_K/K, 0)$ (resp. $w_k \in z^q(\mathcal{X}_k/k, 0)$); on the generic fiber, we get simply $w_K = \sum_i n_i (\mathcal{W}_i)_K$. However, on the special fiber, we must write the irreducible decomposition of each $(\mathcal{W}_i)_k^{\text{red}}$, say $W_{i,1} \cup \dots \cup W_{i,r_i}$, and then consider the multiplicities, that is,

$$w_k = \sum_i \left(n_i \sum_j m_{i,j} W_{i,j} \right),$$

where $m_{i,j} := \text{length}(\mathbb{O}_{\mathcal{W}_k, W_{i,j}})$.

We can consider the de Rham and the rigid cycle classes of w , that is,

$$[w_K]_{\text{dR}} \in H_{\text{dR},c}^{2r}(|w_K|)^\vee \quad \text{and} \quad [w_k]_{\text{rig}} \in H_{\text{rig},c}^{2r}(|w_k|)^\vee.$$

The rigid homology groups $H_{2r}^{\text{rig}}(|w_k|)$ are defined as the dual of $H_{\text{rig},c}^{2r}(|w_k|)$; see [Petrequin 2003, Section 2]. We will prove that these cycle classes are compatible under (co)specialization and induce a well-defined syntomic cohomology class.

Theorem 1.4.1. *Let \mathcal{X} be a flat \mathcal{V} -scheme of relative dimension d , and let $w \in z^q(\mathcal{X}/\mathcal{V}, 0)$ be a relative cycle of codimension q (and relative dimension $r := d - q$). Then $\text{cosp}([w_k]_{\text{rig}}) = [w_K]_{\text{dR}}$.*

Proof. First of all, consider the basic case: $w = \mathcal{W}$ is an integral closed subscheme of \mathcal{X} smooth over \mathcal{V} . By [Baldassarri et al. 2004, proof of Theorem 6.9], we have a commutative diagram

$$\begin{array}{ccc}
 H_{\text{rig},c}^{2r}(\mathcal{W}_k/K) & \xrightarrow{\text{cosp}} & H_{\text{dR},c}^{2r}(\mathcal{W}_K) \\
 \searrow [w_k]_{\text{rig}} & & \swarrow [w_K]_{\text{dR}} \\
 & K &
 \end{array}$$

where the rigid (resp. de Rham) trace map $[w_k]_{\text{rig}}$ (resp. $[w_K]_{\text{dR}}$) is an isomorphism of K -vector spaces.

Given a general relative cycle w , we can reduce to the basic case by arguing as follows. First by linearity and the functoriality of the specialization map, we can restrict to the case $w = \mathcal{W}$ with \mathcal{W} integral. Then the generic fiber \mathcal{W}_K is integral, and by [Grothendieck 1967, Proposition 17.15.12], there exists a closed K -subscheme T such that $\mathcal{W}_K \setminus T$ is smooth over K . Let \mathcal{T} be the flat extension of T (see Lemma 5.1.1); then \mathcal{T} is of codimension at least 1 in \mathcal{W} , and the complement $\mathcal{W} \setminus \mathcal{T}$ is a flat \mathcal{V} -scheme of relative dimension r . Consider the long exact sequence

$$\dots \rightarrow H_{\text{dR},c}^{2r-1}(T) \rightarrow H_{\text{dR},c}^{2r}(\mathcal{W}_K \setminus \mathcal{T}_K) \rightarrow H_{\text{dR},c}^{2r}(\mathcal{W}_K) \rightarrow H_{\text{dR},c}^{2r}(T) \dots$$

Note that here the first and last terms vanish for dimensional reasons. The same happens (*mutatis mutandis*) for the rigid cohomology of the special fiber.

Hence, from now on, we can assume that \mathcal{W} is integral and that its generic fiber \mathcal{W}_K is smooth. In this setting, we apply the reduced fiber theorem for schemes [Bosch et al. 1995, Theorem 2.1]; that is, there exist a finite field extension K'/K and a finite morphism $f : \mathcal{Y} \rightarrow \mathcal{W} \times_{\mathcal{V}} \mathcal{V}'$ of \mathcal{V}' -schemes such that

- (i) $f_{K'} : \mathcal{Y}_{K'} \rightarrow \mathcal{W}_{K'}$ is an isomorphism and
- (ii) \mathcal{Y}/\mathcal{V}' is flat and has reduced geometric fibers.

Recall that the cospecialization map commutes with finite field extension, and the same holds for both the rigid and de Rham trace maps. By [Petrequin 2003, proof of Proposition 6.4], the rigid fundamental/cycle class is preserved by finite morphisms, that is,

$$[y_{k'}]_{\text{rig}} \circ f_{k'}^* = [w_{k'}]_{\text{rig}} \quad \text{where } f_{k'}^* : H_{\text{rig},c}^{2r}(\mathcal{W}_{k'}) \rightarrow H_{\text{rig},c}^{2r}(\mathcal{Y}_{k'}).$$

From the discussion above, there is no loss of generality in assuming that \mathcal{W} has reduced geometric fibers and smooth generic fiber \mathcal{W}_K . Now let S be the singular locus of the special fiber \mathcal{W}_k . Again by [Grothendieck 1967, Proposition 17.15.12], the complement $\mathcal{W}_k \setminus S$ is an open and dense subscheme of \mathcal{W}_k , and it is smooth over k . The scheme S has codimension at least 1 in \mathcal{W}_k ; hence, $H_{\text{rig},c}^{2r}(\mathcal{W}_k \setminus S) \rightarrow H_{\text{rig},c}^{2r}(\mathcal{W}_k)$ is an isomorphism. From this, it follows that we can assume \mathcal{W} to be smooth over \mathcal{V} , where we know that the claim is true. \square

1.5. The smooth case. From now on, assume \mathcal{X} to be smooth over \mathcal{V} . By the compatibility of (co)specialization with Poincaré duality [Baldassarri et al. 2004, Theorem 6.9], the (de Rham or rigid) cycle class map defines an element $\eta_{\text{rig}}(w_k) \in H_{\text{rig},|w_k|}^{2q}(\mathcal{X}_k/K)$ (resp. $\eta_{\text{dR}}(w_K) \in H_{\text{dR},|w_K|}^{2q}(\mathcal{X}_K)$) compatible with respect to the specialization morphism

$$sp(\eta_{\text{dR}}(w_K)) = \eta_{\text{rig}}(w_k). \tag{1}$$

Before stating the next corollary, we need to introduce some further notation. Let X be a smooth scheme over K (resp. over k); then the de Rham (resp. rigid) cycle class map factors through the Chow groups, inducing a map

$$\eta_{\text{dR}} : CH^q(X/K) \rightarrow H_{\text{dR}}^{2q}(X) \quad (\text{resp. } \eta_{\text{rig}} : CH^q(X/k) \rightarrow H_{\text{rig}}^{2q}(X/K)),$$

where, by abuse of notation, $\eta_{\text{dR}}(W)$ (resp. $\eta_{\text{rig}}(W)$) is viewed as an element of $H_{\text{dR}}^{2q}(X)$ (resp. $H_{\text{rig}}^{2q}(X/K)$) via the canonical map $H_{\text{dR},W}^{2q}(X) \rightarrow H_{\text{dR}}^{2q}(X)$ (resp. $H_{\text{rig},W}^{2q}(X/K) \rightarrow H_{\text{rig}}^{2q}(X/K)$) for any W integral subscheme of codimension q [Hartshorne 1975, Proposition 7.8.1, page 60; Petrequin 2003, Corollary 7.6].

Also, we recall that in [Berthelot et al. 1971, Exp. X, Appendix], a specialization map for the classical Chow ring is constructed:

$$sp_{CH} : CH^*(\mathcal{X}_K/K) \otimes \mathbb{Q} \rightarrow CH^*(\mathcal{X}_k/k) \otimes \mathbb{Q}.$$

Explicitly, the map is given as follows. Let $W \subset \mathcal{X}_K$ be an integral scheme of codimension q , and let \mathcal{W} denote its Zariski closure in \mathcal{X} . Then the specialization of $sp_{CH}[W]$ is the class representing the subscheme \mathcal{W}_k .

Corollary 1.5.1. *With the notation above, the diagram*

$$\begin{array}{ccc} CH^q(\mathcal{X}_K/K) \otimes \mathbb{Q} & \xrightarrow{\eta_{\text{dR}}} & H_{\text{dR}}^{2q}(\mathcal{X}_K) \\ \downarrow sp_{CH} & & \downarrow sp \\ CH^q(\mathcal{X}_k/k) \otimes \mathbb{Q} & \xrightarrow{\eta_{\text{rig}}} & H_{\text{rig}}^{2q}(\mathcal{X}_k/K) \end{array}$$

commutes (we tensored each term by \mathbb{Q} to guarantee the existence of sp_{CH}).

1.6. Syntomic cohomology. For any smooth \mathcal{V} -scheme \mathcal{X} , Besser [2000, Definition 6.1] defined the (rigid) syntomic cohomology groups $H_{\text{syn}}^n(\mathcal{X}, i)$. We will be rather sketchy on the definitions because we will give another construction later. For such a cohomology, there is a long exact sequence

$$\begin{aligned} \longrightarrow H_{\text{syn}}^n(\mathcal{X}, i) &\longrightarrow H_{\text{rig}}^n(\mathcal{X}_k/K_0) \oplus F^i H_{\text{dR}}^n(\mathcal{X}_K) \\ &\xrightarrow{h} H_{\text{rig}}^n(\mathcal{X}_k/K_0) \oplus H_{\text{rig}}^n(\mathcal{X}_k/K) \xrightarrow{+} \end{aligned} \quad (2)$$

where $h(x_0, x_{\text{dR}}) = (\phi(x_0^\sigma) - p^i x_0, x_0 \otimes 1_K - sp(x_{\text{dR}}))$.

Roughly, these groups are defined as $H_{\text{syn}}^n(\mathcal{X}, i) := H^n(\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i))$, where

$\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i)$

$$:= \text{Cone}(\mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus \text{Fil}^i \mathbb{R}\Gamma_{\text{dR}}(\mathcal{X}) \rightarrow \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K))[-1]$$

is a complex of abelian groups functorial in \mathcal{X} and such that

$$H^n(\mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0)) = H_{\text{rig}}^n(\mathcal{X}_k/K_0) \quad \text{and} \quad H^n(\text{Fil}^i \mathbb{R}\Gamma_{\text{dR}}(\mathcal{X})) = F^i H_{\text{dR}}^n(\mathcal{X}_K).$$

The functoriality of $\mathbb{R}\Gamma_{\text{Bes}}(\cdot, i)$ allows us to give the following definition:

Definition 1.6.1. Let \mathcal{X} be a smooth \mathcal{V} -scheme. Let $\mathcal{Z} \subset \mathcal{X}$ be a closed subscheme of \mathcal{X} . We define the syntomic complex with support in \mathcal{Z} using the complexes defined by Besser as

$$\mathbb{R}\Gamma_{\text{Bes}, \mathcal{Z}}(\mathcal{X}, i) := \text{Cone}(\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i) \rightarrow \mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X} \setminus \mathcal{Z}, i))[-1].$$

This is a complex of abelian groups functorial with respect to cartesian squares. This fact will be used in the proof of [Proposition 1.6.6](#). The cohomology of this complex is the *syntomic cohomology with support in \mathcal{Z}* denoted by

$$H_{\text{syn}, \mathcal{Z}}^n(\mathcal{X}, i) := H^n(\mathbb{R}\Gamma_{\text{Bes}, \mathcal{Z}}(\mathcal{X}, i))$$

so that we get a long exact sequence

$$\cdots H_{\text{syn}, \mathcal{Z}}^n(\mathcal{X}, i) \rightarrow H_{\text{syn}}^n(\mathcal{X}, i) \rightarrow H_{\text{syn}}^n(\mathcal{X} \setminus \mathcal{Z}, i) \rightarrow \cdots$$

Proposition 1.6.2 (syntomic cycle class). *Let \mathcal{X} be a smooth \mathcal{V} -scheme, and let $w \in z^q(\mathcal{X}/\mathcal{V}, 0)$ be a relative cycle of codimension q . Then the canonical mapping*

$$\psi : H_{\text{syn}, |w|}^{2q}(\mathcal{X}, q) \rightarrow H_{\text{rig}, |w_k|}^{2q}(\mathcal{X}_k/K_0) \oplus F^q H_{\text{dR}, |w_K|}^{2q}(\mathcal{X}_K)$$

is injective, and there exists a unique element $\eta_{\text{syn}}(w) \in H_{\text{syn}, |w|}^{2q}(\mathcal{X}, q)$ such that $\psi(\eta_{\text{syn}}(w)) = (\eta_{\text{rig}}(w_k), \eta_{\text{dR}}(w_K))$.

Proof. By the definition of syntomic cohomology with support, there is a long exact sequence similar to (2):

$$\begin{aligned} \rightarrow H_{\text{rig},|w_k|}^{2q-1}(\mathcal{X}_k/K_0) \oplus H_{\text{rig},|w_k|}^{2q-1}(\mathcal{X}_k/K) &\rightarrow H_{\text{syn},|w|}^{2q}(X, q) \rightarrow \\ H_{\text{rig},|w_k|}^{2q}(\mathcal{X}_k/K_0) \oplus F^q H_{\text{dR},|w_k|}^{2q}(\mathcal{X}_K) &\rightarrow H_{\text{rig},|w_k|}^{2q}(\mathcal{X}_k/K_0) \oplus H_{\text{rig},|w_k|}^{2q}(\mathcal{X}_k/K) \rightarrow . \end{aligned}$$

The last term on the left vanishes because of weak purity in rigid cohomology [Berthelot 1997b, Corollary 5.7]. It follows that $H_{\text{syn},|w|}^{2q}(\mathcal{X}, q)$ consists of the pairs $(x, y) \in H_{\text{rig},|w_k|}^{2q}(\mathcal{X}_k/K_0) \oplus F^q H_{\text{dR},|w_k|}^{2q}(\mathcal{X}_K)$ such that $\phi(x^\sigma) = p^q x$ and $sp(y) = x \otimes 1_K$. By Hodge theory, we have $F^q H_{\text{dR},|w_k|}^{2q}(\mathcal{X}_K) = H_{\text{dR},|w_k|}^{2q}(\mathcal{X}_K)$. Moreover, the Frobenius acts on $[w_k]_{\text{rig}}$ as multiplication by p^q [Petrequin 2003, Proposition 7.13]. Hence, in view of (1), we can easily conclude the proof. \square

Lemma 1.6.3 (functoriality). *Let $f : \mathcal{X}' \rightarrow \mathcal{X}$ be a closed immersion of smooth \mathcal{V} -schemes, and let $w \in z^q(\mathcal{X}/\mathcal{V}, 0)$ be a relative cycle of codimension q . Assume that the preimage $f^{-1}w$ lies in $z^q(\mathcal{X}'/\mathcal{V}, 0)$; then $f^* \eta_{\text{syn}}(w) = \eta_{\text{syn}}(f^{-1}w)$.*

Proof. It is not restrictive to assume that $w = \mathcal{W}$ is an integral subscheme of \mathcal{X} flat over \mathcal{V} .

We first show that it is sufficient to prove that $f^* \eta_{\text{dR}}(w_K) = \eta_{\text{dR}}(f^{-1}w_K)$. In fact, the syntomic cycle class can be viewed as an element $(\eta_{\text{rig}}(w_k), \eta_{\text{dR}}(w_K))$ in the direct sum of rigid and de Rham cohomology. Then note that $sp(\eta_{\text{dR}}(w_K)) = \eta_{\text{rig}}(w_k) \otimes 1_K$ and that the specialization map is functorial.

To prove that $f^* \eta_{\text{dR}}(\mathcal{W}_K) = \eta_{\text{dR}}(f^{-1}\mathcal{W}_K)$, we first reduce by excision to the case where \mathcal{W}_K is smooth over K (just remove from \mathcal{W}_K , \mathcal{X}_K and \mathcal{X}'_K the singular points of \mathcal{W}_K). In the same way, we can further assume $f^{-1}\mathcal{W}_K$ to be smooth over K . Now we can use the same proof as [Petrequin 2003, Proposition 7.1] to conclude. \square

Lemma 1.6.4 (homotopy). *The (rigid) syntomic cohomology is homotopy invariant:*

$$H_{\text{syn}}^n(\mathcal{X} \times \mathbb{A}_{\mathcal{V}}^1, q) \cong H_{\text{syn}}^n(\mathcal{X}, q).$$

Proof. Just consider the long exact sequences of syntomic cohomology, and note that the de Rham cohomology (of smooth schemes) is homotopy invariant by [Hartshorne 1975, Proposition 7.9.1]. The same holds for rigid cohomology, for instance, using the Künneth formula [Berthelot 1997a]. \square

Lemma 1.6.5 (weak purity). *Let \mathcal{X} be a smooth \mathcal{V} -scheme. Let $\mathcal{X} \subset \mathcal{X}$ be a closed subscheme of \mathcal{X} of codimension q . Then*

$$H_{\text{syn},\mathcal{X}}^n(\mathcal{X}, i) = 0 \quad \text{for all } n < 2q.$$

Proof. This follows directly from the long exact sequence of syntomic cohomology and the weak purity in de Rham and rigid cohomology [Hartshorne 1975, Section 7.2; Petrequin 2003, Section 1]. \square

Proposition 1.6.6. *Let \mathcal{X} be a smooth \mathcal{V} -scheme. The syntomic cycle class map induces a group homomorphism $\text{reg}_{\text{syn}} : CH^i(\mathcal{X}/\mathcal{V}, 2i - n) \rightarrow H_{\text{syn}}^n(\mathcal{X}, i)$.*

Proof. The construction is analogous to that provided in [Bloch 1986]. Consider the cohomological double complex $\mathbb{R}\Gamma_{\text{Bes}}(\Delta_{\mathcal{X}}^{-n}, q)^m$ nonzero for $m \geq 0$ and $n \leq 0$; the differential in n is induced by $\partial_i^{(-n)}$ in the usual way. Similarly define the double complex

$$\mathbb{R}\Gamma_{\text{Bes, sup}}(\Delta_{\mathcal{X}}^{-n}, q)^m := \text{colim}_{w \in z^q(\mathcal{X}, -n)} \mathbb{R}\Gamma_{\text{Bes, } |w|}(\Delta_{\mathcal{X}}^{-n}, q)^m.$$

For technical reasons, we truncate these complexes (nontrivially):

$$A_{?}^{n,m} = \tau_{n \geq -N} \mathbb{R}\Gamma_{\text{Bes, ?}}(\Delta_{\mathcal{X}}^{-n}, q)^m, \quad \text{where } ? = \emptyset, \text{ sup},$$

for N even and $N \gg 0$.

Consider the spectral sequence

$$E_1^{n,m} := H^m(A^{\bullet, n}) \Rightarrow H^{n+m}(sA^{\bullet, *}),$$

where s denotes the associated simple complex of a double complex. By homotopy invariance (Lemma 1.6.4), $E_1^{n,m} := H_{\text{syn}}^m(\Delta_{\mathcal{X}}^{-n}, q)$ is isomorphic to $H_{\text{syn}}^m(\mathcal{X}, q)$ for $-N \leq n \leq 0$ and $m \geq 0$; otherwise, it is 0. Moreover, $d_1^{n,m} = 0$ except for n even, $-N \leq n < 0$ and $m \geq 0$, in which case $d_1^{n,m} = \text{id}$. This gives an isomorphism $H^i(sA^{n,m}) \cong H_{\text{syn}}^i(\mathcal{X}, q)$.

In the spectral sequence

$$E_{1, \text{sup}}^{n,m} := H^m(A_{\text{sup}}^{\bullet, n}) \Rightarrow H^{n+m}(sA_{\text{sup}}^{\bullet, *}),$$

we have

$$E_{1, \text{sup}}^{n,m} = \text{colim}_{w \in z^q(X, -n)} H_{\text{syn, } |w|}^m(\Delta_{\mathcal{X}}^{-n}, q) \quad \text{for } -N \leq n \leq 0 \text{ and } m \geq 0$$

and $E_{1, \text{sup}}^{n,m} = 0$ otherwise. Applying Lemma 1.6.3 to the face morphisms, it is easy to prove that the syntomic cycle class induces a natural map of complexes $\Gamma_{\mathcal{X}/\mathcal{V}}^{\bullet}(q) \rightarrow E_{1, \text{sup}}^{\bullet-2q, 2q}$, and hence, for all i , a map $CH^q(\mathcal{X}/\mathcal{V}, 2q - i) \rightarrow E_{2, \text{sup}}^{i-2q, 2q}$. The groups $E_{r, \text{sup}}^{n,m}$ are zero for $m < 2q$ and $r \geq 1$ due to weak purity. Hence, there are natural maps $E_{2, \text{sup}}^{i-2q, 2q} \rightarrow E_{\infty, \text{sup}}^{i-2q, 2q} \rightarrow H^i(sA_{\text{sup}}^{\bullet, *})$. By construction, there is a map $H^i(sA_{\text{sup}}^{\bullet, *}) \rightarrow H^i(sA^{\bullet, *})$. Composing all these maps, we obtain the expected map $CH^q(\mathcal{X}, 2q - i) \rightarrow H_{\text{syn}}^i(\mathcal{X}, q)$. \square

Corollary 1.6.7. *With the notation above, there is a commutative diagram*

$$\begin{CD}
 CH^q(\mathcal{X}/\mathcal{V}) \otimes \mathbb{Q} @>\gamma \otimes 1_{\mathbb{Q}}>> CH^q(\mathcal{X}_K/K) \otimes \mathbb{Q} @>sp_{CH}>> CH^q(\mathcal{X}_k/k) \otimes \mathbb{Q} \\
 @V{\text{reg}_{\text{syn}}}\!V\!V @V{\eta_{\text{dR}}}\!V\!V @V{\eta_{\text{rig}}}\!V\!V \\
 H_{\text{syn}}^{2q}(\mathcal{X}, q) @>\pi>> F^q H_{\text{dR}}^{2q}(\mathcal{X}_K) @>sp>> H_{\text{rig}}^{2q}(\mathcal{X}_k/K),
 \end{CD}$$

where π is the composition

$$H_{\text{syn}}^n(\mathcal{X}, i) \longrightarrow H_{\text{rig}}^n(\mathcal{X}_k/K_0) \oplus F^i H_{\text{dR}}^n(\mathcal{X}_K) \xrightarrow{\text{pr}_2} F^i H_{\text{dR}}^n(\mathcal{X}_K)$$

and γ is the map described in Remark 1.2.2.

Proof. Just note that in this case, reg_{syn} is the map induced by the syntomic cycle class in the usual way. □

Remark 1.6.8. Via Chern classes, Besser [2000, Theorem 7.5] obtained a regulator $c_i^{2i-n} : K_{2i-n}(X) \rightarrow H_{\text{syn}}^n(\mathcal{X}, i)$. At present, we cannot compare it with the regulator of Proposition 1.6.6. This is because we don't know how to relate K -theory with the higher Chow groups we have defined. Nevertheless, we expect that there exists a map $CH^i(\mathcal{X}/\mathcal{V}, 2i - n)_{\mathbb{Q}} \rightarrow K_{2i-n}(X)_{\mathbb{Q}}$. This issue will be treated in a future work.

2. p -adic Hodge complexes

Having defined a regulator map with values in the syntomic cohomology, it is tempting to check (some of) the Bloch–Ogus axioms for this theory. We address this problem by viewing the syntomic cohomology as an absolute cohomology theory.

Thus, in this section, we define a triangulated category of p -adic Hodge complexes similar to that of [Bannai 2002]. See also [Beilinson 1986; Huber 1995; Levine 1998, Chapter V, Section 2.3].

The syntomic cohomology will be computed by Hom groups in this category.

Definition 2.0.9 (see [Bannai 2002, Section 2]). Let $C_{\text{rig}}^b(K)$ be the category of pairs (M^\bullet, ϕ) , where

- (i) $M^\bullet = M_0^\bullet \otimes_{K_0} K$ and M_0^\bullet is a complex in $C^b(K_0)$;
- (ii) (Frobenius structure) if $(M_0^\bullet)^\sigma := M_0^\bullet \otimes_\sigma K_0$, then $\phi : (M_0^\bullet)^\sigma \rightarrow M_0^\bullet$ is a K_0 -linear morphism.

The morphisms in this category are morphisms in $C^b(K_0)$ compatible with respect to the Frobenius structure. In this way, we get an abelian category.

Let Filt_K be the category of K -vector spaces with a descending, exhaustive and separated filtration. We write $C_{\text{dR}}^b(K) = C^b(\text{Filt}_K)$, and we write the objects of this category as pairs (M^\bullet, F) , where

- (i) M^\bullet is a complex in $C^b(K)$ and
- (ii) (Hodge filtration) F is a (separated and exhaustive) filtration on M^\bullet .

Remark 2.0.10 (strictness). We review some technical facts about filtered categories. For a full discussion, see [Huber 1995, Sections 2 and 3].

The category Filt_K (and also $C_{\text{dR}}^b(K)$) is additive but not abelian. It is an exact category when one takes for short exact sequences those that are exact as sequences of K -vector spaces and are such that the morphisms are strict with respect to the filtrations; recall that a morphism $f : (M, F) \rightarrow (N, F)$ is *strict* if $f(F^i M) = F^i N \cap \text{Im}(f)$.

An object $(M^\bullet, F) \in C_{\text{dR}}^b(K)$ is a *strict complex* if its differentials are strict as morphism of filtered vector spaces. Strict complexes can be characterized also by the fact that the canonical spectral sequence $H^q(F^p) \Rightarrow H^{p+q}(M^\bullet)$ degenerates at E_1 .

One can define canonical truncation functors on $C_{\text{dR}}^b(K)$: For $M^\bullet \in C_{\text{dR}}^b(K)$, let

$$\tau_{\leq n}(M^\bullet, F)^i := \begin{cases} M^i & \text{if } i < n, \\ \text{Ker}(d^n) & \text{if } i = n, \\ 0 & \text{if } i > n, \end{cases} \quad \tau_{\geq n}(M^\bullet, F)^i := \begin{cases} 0 & \text{if } i < n-1, \\ \text{Coim}(d^n) & \text{if } i = n-1, \\ M^i & \text{if } i \geq n. \end{cases}$$

It is important to note that the naive cohomology object $\tau_{\leq n} \tau_{\geq n}(M^\bullet, F)$ of a strict complex (M^\bullet, F) agrees with the cohomology $H^n(M^\bullet)$ of the complex of K -vector spaces underlying (M^\bullet, F) [Huber 1995, Proposition 2.1.3 and Section 3].

Definition 2.0.11 (see [Bannai 2002, Definition 2.2]). From the discussion above, there is an exact functor $\Phi_{\text{rig}} : C_{\text{rig}}^b(K) \rightarrow C^b(K)$ (resp. $\Phi_{\text{dR}} : C_{\text{dR}}^b(K) \rightarrow C^b(K)$) induced by $(M^\bullet, \phi) \mapsto M^\bullet$ (resp. $(M^\bullet, F) \mapsto M^\bullet$). We define the category pHC of *p-adic Hodge complexes* whose objects are systems $M = (M_{\text{rig}}^\bullet, M_{\text{dR}}^\bullet, M_K^\bullet, c, s)$, where

- (i) $(M_{\text{rig}}^\bullet, \phi)$ is an object of $C_{\text{rig}}^b(K)$ and $H^*(M_{\text{rig}}^\bullet)$ is finitely generated over K ,
- (ii) $(M_{\text{dR}}^\bullet, F)$ is an object of $C_{\text{dR}}^b(K)$ and $H^*(M_{\text{dR}}^\bullet)$ is finitely generated over K and
- (iii) M_K^\bullet is an object of $C^b(K)$ and $c : M_{\text{rig}}^\bullet \rightarrow M_K^\bullet$ (resp. $s : M_{\text{dR}}^\bullet \rightarrow M_K^\bullet$) is a morphism in $C^b(K)$. Hence, c and s give a diagram in C_K^b

$$M_{\text{rig}}^\bullet \xrightarrow{c} M_K^\bullet \xleftarrow{s} M_{\text{dR}}^\bullet.$$

A morphism in pHC is given by a system $f := (f_{\text{rig}}, f_{\text{dR}}, f_K)$, where $f_\? : M_\?^\bullet \rightarrow N_\?^\bullet$ is a morphism in $C_{\text{rig}}^b(K)$, $C_{\text{dR}}^b(K)$ or $C^b(K)$ for $\? = \text{rig}, \text{dR}$ or K , respectively, and such that they are compatible with respect to the diagram in (iii) above.

2.1. Derived version. A homotopy in pHC is a system of homotopies h_i compatible with the comparison maps c and s . We define the category pHK to be the category pHC modulo morphisms homotopic to 0. We say that a morphism $f = (f_{\text{rig}}, f_{\text{dR}}, f_K)$ in pHC (or pHK) is a *quasi-isomorphism* if $f_?$ is a quasi-isomorphism for $? = \text{rig}$ (or K) and f_{dR} is a filtered quasi-isomorphism, that is, $\text{gr}_F(f_{\text{dR}})$ is a quasi-isomorphism. Finally, we say that $M \in pHC$ (or pHK) is *acyclic* if $M_? = 0$ is acyclic for any $? = \text{rig}, \text{dR}, K$.

Lemma 2.1.1. (i) *The category pHK is a triangulated category.*

(ii) *The localization of pHK with respect to the class of quasi-isomorphisms exists. This category, denoted pHD , is a triangulated category.*

(iii) *On the category pHD , it is possible to define a nondegenerate t -structure (resp. truncation functors) compatible with the standard t -structure (resp. truncation functors) defined on $C^b(K)$, $C_{\text{rig}}^b(K)$ and $C_{\text{dR}}^b(K)$.*

Proof. The proof is the same as [Bannai 2002, Proposition 2.6]. See also [Huber 1995, Section 2] for a survey on how to derive exact categories. \square

Remark 2.1.2. In the terminology of Huber [1995, Section 4], the category pHC is the *glued exact category* of C_{rig}^b and C_{dR}^b via C_K^b . The foregoing definition is inspired by Bannai [2002], who constructs a *rigid glued exact category* C_{MF}^b , that is, the comparison maps are all quasi-isomorphisms. For our purposes, we cannot impose this strong assumption. This is motivated by the fact that the (co)specialization is not an isomorphism for a general smooth \mathcal{V} -scheme.

2.2. Derived Hom. Let M^\bullet and M'^\bullet be two objects of pHC . Consider the diagram $\mathcal{H}(M^\bullet, M'^\bullet)$ (of complexes of abelian groups)

$$\begin{array}{ccccc}
 \text{Hom}_{K_0}^\bullet((M_0^\bullet)^\sigma, M_0'^\bullet) & & \text{Hom}_K^\bullet(M_{\text{rig}}^\bullet, M_K'^\bullet) & & \text{Hom}_K^\bullet(M_{\text{dR}}^\bullet, M_K'^\bullet) \\
 \uparrow h_0 & \nearrow h_1 & \uparrow h_2 & \nearrow h_3 & \uparrow h_4 \\
 \text{Hom}_{K_0}^\bullet(M_0^\bullet, M_0'^\bullet) & & \text{Hom}_K^\bullet(M_K^\bullet, M_K'^\bullet) & & \text{Hom}_K^{\bullet F}(M_{\text{dR}}^\bullet, M_{\text{dR}}'^\bullet)
 \end{array}$$

where $h_0(x_0) = x_0 \circ \phi - \phi' \circ x_0^\sigma$; $h_1(x_0) = c' \circ (x_0 \otimes \text{id}_K)$; $h_2(x_K) = x_K \circ c$; $h_3(x_K) = x_K \circ s$ and $h_4(x_{\text{dR}}) = s' \circ x_{\text{dR}}$; $\text{Hom}_K^{\bullet F}(M_{\text{dR}}^\bullet, M_{\text{dR}}'^\bullet)$ is the complex of morphisms compatible with respect to the filtrations. Then define the two complexes of abelian groups $\Gamma_0(M^\bullet, M'^\bullet) :=$ (direct sum of the bottom row) and $\Gamma_1(M^\bullet, M'^\bullet) :=$ (direct sum of the top row). Finally, consider the cone

$$\Gamma(M^\bullet, M'^\bullet) := \text{Cone}(\psi_{M^\bullet, M'^\bullet} : \Gamma_0(M^\bullet, M'^\bullet) \rightarrow \Gamma_1(M^\bullet, M'^\bullet))[-1],$$

where

$$\psi_{M^\bullet, M'^\bullet} : (x_0, x_K, x_{\text{dR}}) \mapsto (-h_0(x_0), h_1(x_0) - h_2(x_K), h_3(x_K) - h_4(x_{\text{dR}})).$$

- Remark 2.2.1.** (i) Let $\mathbb{K}(-n)$ be the Tate twisted p -adic Hodge complex: i.e., $\mathbb{K}(-n)_{\text{rig}}$ (resp. $\mathbb{K}(-n)_{\text{dR}}$, $\mathbb{K}(-n)$) is equal to K concentrated in degree 0; the Frobenius is $\phi(\lambda) := p^n \sigma(\lambda)$; the filtration is $F^i = K$ for $i \leq n$ and 0 otherwise.
- (ii) Given two p -adic Hodge complexes M^\bullet and M'^\bullet , we define their tensor product $M^\bullet \otimes M'^\bullet$ component-wise, that is, $(M_{\text{rig}}^\bullet \otimes M'_{\text{rig}}{}^\bullet, M_{\text{dR}}^\bullet \otimes M'_{\text{dR}}{}^\bullet, M_K^\bullet \otimes M'_K{}^\bullet, c \otimes c', s \otimes s')$. The complex $M^\bullet \otimes \mathbb{K}(n)$ is denoted by $M^\bullet(n)$.
- (iii) The complex $\Gamma(\mathbb{K}, M^\bullet(n))$ is quasi-isomorphic to

$$\text{Cone}(M_0^\bullet \oplus F^n M_{\text{dR}}^\bullet \xrightarrow{\eta} M_0^\bullet \oplus M_K^\bullet)[-1],$$

$$\eta(x_0, x_{\text{dR}}) = (p^{-n} \phi(x_0^\sigma) - x_0, c(x_0 \otimes \text{id}_K) - s(x_{\text{dR}})),$$

where $x_0 \in M_0^\bullet$ and $x_{\text{dR}} \in F^n M_{\text{dR}}^\bullet$.

If c is a quasi-isomorphism, letting sp denote the composition of

$$H^q(F^n M_{\text{dR}}^\bullet) \xrightarrow{s^*} H^q(M_K^\bullet) \xleftarrow{\cong} H^q(M_{\text{rig}}^\bullet),$$

we obtain a long exact sequence

$$\rightarrow H^q(\Gamma(\mathbb{K}, M^\bullet(n))) \rightarrow H^q(M_0^\bullet) \oplus H^q(F^n M_{\text{dR}}^\bullet) \xrightarrow{\eta'} H^q(M_0^\bullet) \oplus H^q(M_{\text{rig}}^\bullet) \rightarrow,$$

where $\eta'(x_0, x_{\text{dR}}) = (p^{-n} \phi(x_0^\sigma) - x_0, x_0 \otimes 1_K - sp(x_{\text{dR}}))$.

If s is a quasi-isomorphism, letting $cosp$ denote the composition of

$$H^q(M_{\text{rig}}^\bullet) \xrightarrow{c^*} H^q(M_K^\bullet) \xleftarrow{\cong} H^q(M_{\text{dR}}^\bullet),$$

we obtain a long exact sequence

$$\rightarrow H^q(\Gamma(\mathbb{K}, M^\bullet(n))) \rightarrow H^q(M_0^\bullet) \oplus H^q(F^n M_{\text{dR}}^\bullet) \xrightarrow{\eta''} H^q(M_0^\bullet) \oplus H^q(M_{\text{dR}}^\bullet) \rightarrow,$$

where $\eta''(x_0, x_{\text{dR}}) = (p^{-n} \phi(x_0^\sigma) - x_0, cosp(x_0 \otimes 1_K) - x_{\text{dR}})$.

Proposition 2.2.2 (extension formula). *With the notation above, there is a canonical morphism of abelian groups*

$$\text{Hom}_{p\text{HD}}(M^\bullet, M'^\bullet[n]) \cong H^n(\Gamma(M^\bullet, M'^\bullet)).$$

In particular, if $M^\bullet = M$ and $M'^\bullet = M'$ are concentrated in degree 0, then $H^n(\Gamma(M, M')) = 0$ for $n \geq 2$ and $n < 0$.

Proof. By the octahedron axiom, we have the following triangle in $D^b(\text{Ab})$:

$$\text{Ker } \psi_{M^\bullet, M'^\bullet} \rightarrow \Gamma(M^\bullet, M'^\bullet) \rightarrow \text{Coker } \psi_{M^\bullet, M'^\bullet}[-1] \xrightarrow{+}.$$

Its cohomological long exact sequence is

$$\xrightarrow{\partial} H^n(\text{Ker } \psi_{M^\bullet, M'^\bullet}) \rightarrow H^n(\Gamma(M, M')) \rightarrow H^n(\text{Coker } \psi_{M^\bullet, M'^\bullet}[-1]) \xrightarrow{\partial}.$$

Note that by construction, $H^n(\text{Ker } \psi_{M^\bullet, M'^\bullet}) = \text{Hom}_{pHK}(M^\bullet, M'^\bullet[n])$. Also, we have

$$\text{Hom}_{pHD}(M^\bullet, M'^\bullet[n]) = \text{colim}_I \text{Hom}_{pHK}(M^\bullet, M''^\bullet[n]), \quad I = \{\text{quis } g : M'^\bullet \rightarrow M''^\bullet\}.$$

Thus, the result is proved if we show that

- (i) $H^n(\Gamma(M^\bullet, M'^\bullet)) \cong H^n(\Gamma(M^\bullet, M''^\bullet))$ holds given any $g : M'^\bullet \rightarrow M''^\bullet$ quasi-isomorphism and
- (ii) $\text{colim}_I H^n(\text{Coker } \psi_{M^\bullet, M''^\bullet}[-1]) = 0$.

The first claim follows from the exactness of $\Gamma(M^\bullet, \cdot)$, and the second is proved in [Beilinson 1986, 1.7, 1.8] (and with more details in [Huber 1995, Lemma 4.2.8; Bannai 2002, Lemma 2.15]) with the assumption that all the gluing maps are quasi-isomorphisms, but this hypothesis is not necessary. □

Lemma 2.2.3 (tensor product). *Let M^\bullet, M'^\bullet and I^\bullet be p -adic Hodge complexes. For any $\alpha \in K$, there is a morphism of complexes*

$$\cup_\alpha : \Gamma(I^\bullet, M^\bullet) \otimes \Gamma(I^\bullet, M'^\bullet) \rightarrow \Gamma(I^\bullet, M^\bullet \otimes M'^\bullet).$$

All such \cup_α are equivalent up to homotopy.

Proof. See [Beilinson 1986, 1.11]. □

Remark 2.2.4 (enlarging the diagram). We recall some results from [Levine 1998, Chapter V, 2.3.3]. Let $M_1^\bullet \xrightarrow{f} M_2^\bullet \xleftarrow{g} M_3^\bullet$ (resp. $M_1^\bullet \xleftarrow{f} M_2^\bullet \xrightarrow{g} M_3^\bullet$) be a diagram of complexes in $C^b(K)$. Let $P^\bullet = \text{Cone}(f - g : M_1^\bullet \oplus M_3^\bullet \rightarrow M_2^\bullet)[-1]$ be the *quasipullback* complex (resp. $Q^\bullet = \text{Cone}((f, -g) : M_2^\bullet \rightarrow M_1^\bullet \oplus M_3^\bullet)$ be the *quasipushout*). Assume that f is a quasi-isomorphism. Then the diagrams

$$\begin{array}{ccc} P^\bullet & \xrightarrow{h} & M_3^\bullet \\ \downarrow & & \downarrow g \\ M_1^\bullet & \xrightarrow{f} & M_2^\bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} M_2^\bullet & \xrightarrow{f} & M_1^\bullet \\ g \downarrow & & \downarrow \\ M_3^\bullet & \xrightarrow{k} & P^\bullet \end{array}$$

are commutative up to homotopy and are such that h and k are quasi-isomorphisms.

Now let pHC' be a category of systems $(M_{\text{rig}}^\bullet, M_{\text{dR}}^\bullet, M_1^\bullet, M_2^\bullet, M_3^\bullet, c, s, f, g)$ similar to Definition 2.0.11 and such that there is a diagram

$$M_{\text{rig}}^\bullet \xrightarrow{c} M_1^\bullet \xleftarrow{f} M_2^\bullet \xrightarrow{g} M_3^\bullet \xleftarrow{s} M_{\text{dR}}^\bullet.$$

Then the quasipushout induces a functor from the category pHC' to the category pHC . This functor is compatible with tensor product after passing to the categories pHK' and pHK .

3. Godement resolution

Here we recall some facts about the *generalized* Godement resolution, also called the bar-resolution. We refer to [Ivorra 2005]; see also [Weibel 1994, Section 8.6].

3.1. General construction. Let $u : P \rightarrow X$ be a morphism of Grothendieck topologies so that P^\sim (resp. X^\sim) is the category of abelian sheaves on P (resp. X). Then we have a pair of adjoint functors (u^*, u_*) , where $u^* : X^\sim \rightarrow P^\sim$ and $u_* : P^\sim \rightarrow X^\sim$. For any object \mathcal{F} of X^\sim , we can define a cosimplicial object $B^*(\mathcal{F}) : \Delta \rightarrow X^\sim$ in the following way. First let $\eta : \text{id}_{X^\sim} \rightarrow u_*u^*$ and $\epsilon : u^*u_* \rightarrow \text{id}_{P^\sim}$ be the natural transformations induced by adjunction.

Endow $B^n(\mathcal{F}) := (u_*u^*)^{n+1}(\mathcal{F})$ with codegeneracy maps

$$\sigma_i^n := (u_*u^*)^i u_* \epsilon u^* (u_*u^*)^{n-1-i} : B^n(\mathcal{F}) \rightarrow B^{n-1}(\mathcal{F}) \quad \text{for } i = 0, \dots, n-1$$

and cofaces

$$\delta_i^{n-1} := (u_*u^*)^i \eta (u_*u^*)^{n-i} : B^{n-1}(\mathcal{F}) \rightarrow B^n(\mathcal{F}) \quad \text{for } i = 0, \dots, n.$$

Lemma 3.1.1. *With the notation above, let $sB^*(\mathcal{F})$ be the associated complex of objects of X^\sim . Then there is a canonical map $b_{\mathcal{F}} : \mathcal{F} \rightarrow sB^*(\mathcal{F})$ such that $u^*(b_{\mathcal{F}})$ is a quasi-isomorphism. Moreover, if u^* is exact and conservative, then $b_{\mathcal{F}}$ is a quasi-isomorphism.*

Proof. See [Ivorra 2005, Chapter III, Lemma 3.4.1]. □

Thus, for any sheaf $\mathcal{F} \in X^\sim$ (or complex of sheaves), we can define a functorial map $b_{\mathcal{F}} : \mathcal{F} \rightarrow sB^*(\mathcal{F})$ with $sB^n(\mathcal{F}) := (u_*u^*)^{n+1}\mathcal{F}$. We will denote this complex of sheaves $\text{Gd}_P(\mathcal{F})$. In the case u^* is exact and conservative, $\text{Gd}_P(\mathcal{F})$ is a canonical resolution of \mathcal{F} . If \mathcal{F}^\bullet is a complex of sheaves on X , $\text{Gd}_P(\mathcal{F}^\bullet)$ denotes the simple complex $s(\text{Gd}_P(\mathcal{F}^i)^j)$. Often, we will need to iterate this process, and we will write $\text{Gd}_P^2(\mathcal{F}) := \text{Gd}_P(\text{Gd}_P(\mathcal{F}))$.

Now suppose there is a commutative diagram of sites

$$\begin{array}{ccc} P & \xrightarrow{g} & Q \\ u \downarrow & & \downarrow v \\ X & \xrightarrow{f} & Y \end{array}$$

and a morphism of sheaves $a : \mathcal{G} \rightarrow f_*\mathcal{F}$, where \mathcal{F} (resp. \mathcal{G}) is a sheaf on X (resp. Y).

Lemma 3.1.2. *There is a canonical map $\text{Gd}_Q(\mathcal{G}) \rightarrow f_*\text{Gd}_P(\mathcal{F})$ compatible with $b_{\mathcal{F}}$ and $b_{\mathcal{G}}$.*

Proof. We need only show that there is a canonical map $v_*v^*\mathcal{G} \rightarrow f_*u_*u^*\mathcal{F}$ lifting a . First consider the composition $\mathcal{G} \rightarrow f_*\mathcal{F} \rightarrow f_*u_*u^*\mathcal{F}$. Then we get a

map $\mathcal{G} \rightarrow v_* g_* u^* \mathcal{F}$ because $v_* g_* = f_* u_*$. By adjunction, this gives $v^* \mathcal{G} \rightarrow g_* u^* \mathcal{F}$. Then we apply v_* and use $v_* g_* = f_* u_*$ to obtain the desired map. \square

Remark 3.1.3 (tensor product). The Godement resolution is compatible with tensor products; that is, for any pair of sheaves \mathcal{F} and \mathcal{G} on X , there is a canonical quasi-isomorphism $\text{Gd}_P(\mathcal{F}) \otimes \text{Gd}_P(\mathcal{G}) \rightarrow \text{Gd}_P(\mathcal{F} \otimes \mathcal{G})$. The same holds for complexes that are bounded below [Friedlander and Suslin 2002, Appendix A].

3.2. Points of sites/topoi.

Definition 3.2.1 (see [Artin et al. 1972a, Example IV, Section 6]). Let X be a site and $Sh(X)$ be the associated topos of sheaves of sets. A *point* of X is a morphism of topoi $\pi : Set \rightarrow Sh(X)$, that is, a pair of adjoint functors (π^*, π_*) such that π^* is left exact.

Example 3.2.2. Let X be a scheme. Then any point x of the topological space underlying X gives a point π_x of the Zariski site of X . We call them Zariski points.

Now let x be a geometric point of X ; then it induces a point π_x for the étale site of X . We call them étale points of X .

Let \mathcal{F} be a Zariski (resp. étale) sheaf X and P be the set of Zariski (resp. étale) points of X . Then the functor $\mathcal{F} \mapsto \bigsqcup_{\pi \in P} \mathcal{F}_\pi := \pi^* \mathcal{F}$ is exact and conservative. In other words, the Zariski (resp. étale) site of X has enough points.

Example 3.2.3 (points on rigid analytic spaces [van der Put and Schneider 1995]). Let \mathcal{X} be a rigid analytic space over K . We recall that a *filter* f on \mathcal{X} is a collection $(\mathcal{U}_\alpha)_\alpha$ of admissible open subsets of X satisfying

- (i) $\mathcal{X} \in f$ and $\emptyset \notin f$,
- (ii) if $\mathcal{U}_\alpha, \mathcal{U}_\beta \in f$, then $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \in f$ and
- (iii) if $\mathcal{U}_\alpha \in f$ and the admissible open \mathcal{V} contains \mathcal{U}_α , then $\mathcal{V} \in f$.

A *prime filter* on \mathcal{X} is a filter p satisfying moreover

- (iv) if $\mathcal{U} \in p$ and $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}'_i$ is an admissible covering of \mathcal{U} , then $\mathcal{U}'_{i_0} \in p$ for some $i_0 \in I$.

Let $P(\mathcal{X})$ be the set of all prime filters of \mathcal{X} . The filters on \mathcal{X} are ordered with respect to inclusion. We can give to $P(\mathcal{X})$ a topology and define a morphism of sites $\sigma : P(\mathcal{X}) \rightarrow \mathcal{X}$. Also, we let $Pt(\mathcal{X})$ denote the set of prime filters with the discrete topology. Let $i : Pt(\mathcal{X}) \rightarrow P(\mathcal{X})$ be the canonical inclusion and $\xi = \sigma \circ i$.

Remark 3.2.4. Let $p = (\mathcal{U}_\alpha)_\alpha$ be a prime filter on \mathcal{X} as above. Then p is a point of the site \mathcal{X} (see Definition 3.2.1). Using the construction of the continuous map σ of [van der Put and Schneider 1995], we get that the morphism of topoi $\pi : Set \rightarrow Sh(\mathcal{X})$,

associated to p , is defined in the following way. For any sheaf (of sets) \mathcal{F} on \mathcal{X} , let $\pi^*(\mathcal{F}) = \text{colim}_\alpha \mathcal{F}(\mathcal{U}_\alpha)$; for any set \mathcal{S} and \mathcal{V} admissible open in \mathcal{X} , let

$$\pi_*\mathcal{S}(\mathcal{V}) = \begin{cases} \mathcal{S} & \text{if } \mathcal{V} = \mathcal{U}_\alpha \text{ for some } \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where 0 denotes the final object in the category *Set*. In fact, with the notation above, we easily get the adjunction

$$\text{Hom}_{\text{Set}}(\pi^*(\mathcal{F}), \mathcal{S}) = \lim_\alpha \text{Hom}_{\text{Set}}(\mathcal{F}(\mathcal{U}_\alpha), \mathcal{S}) = \text{Hom}_{\text{Sh}(\mathcal{X})}(\mathcal{F}, \pi_*\mathcal{S}).$$

Lemma 3.2.5. *With the notation above, the functor $\xi^{-1} : \text{Sh}(\mathcal{X}) \rightarrow \text{Sh}(\text{Pt}(\mathcal{X}))$ is exact and conservative. In other words, for any $p \in \text{Pt}(\mathcal{X})$, $\mathcal{F} = 0$ if all $\mathcal{F}_p = 0$ and the functors $\text{Sh}(\mathcal{X}) \ni \mathcal{F} \mapsto \mathcal{F}_p$ are exact.*

Proof. See [van der Put and Schneider 1995, Section 4] after the proof of Theorem 1. □

4. Rigid and de Rham complexes

We begin this section by recalling the construction of the rigid complexes of [Besser 2000, Section 4]. Instead of the techniques of [Artin et al. 1972b, Exposé V.bis], we use the machinery of generalized Godement resolution as developed in Section 3. This alternative approach was also mentioned by Besser in the introduction of his paper. We then recall the construction of the de Rham complexes.

4.1. Rigid complexes. We define a *rigid triple* to be a system (X, \bar{X}, P) , where X is an algebraic k -scheme, $j : X \rightarrow \bar{X}$ is an open embedding into a proper k -scheme and $\bar{X} \rightarrow P$ is a closed embedding into a p -adic formal \mathcal{V} -scheme P that is smooth in a neighborhood of X .

Definition 4.1.1 [Besser 2000, 4.2, 4.4]. Let (X, \bar{X}, P) and (Y, \bar{Y}, Q) be two rigid triples, and let $f : X \rightarrow Y$ be a morphism of k -schemes. Let $\mathcal{U} \subset]\bar{X}[_P$ be a strict neighborhood of $]X[_P$ and $F : \mathcal{U} \rightarrow Q_K$ be a morphism of K -rigid spaces. We say that F is *compatible* with f if it induces the following commutative diagram:

$$\begin{array}{ccc}]X[_P & \xrightarrow{F} &]Y[_Q \\ \text{sp} \downarrow & & \downarrow \text{sp} \\ X & \xrightarrow{f} & Y \end{array}$$

We write $\text{Hom}_f(\mathcal{U}, Q_K)$ for the collection of morphisms compatible with f .

The collection of rigid triples forms a category RT with the set of pairs (f, F) written as $\text{Hom}((X, \bar{X}, P), (Y, \bar{Y}, Q))$, where $f : X \rightarrow Y$ is a k -morphism and $F \in \text{colim}_{\mathcal{U}} \text{Hom}_f(\mathcal{U}, Q)$.

Proposition 4.1.2. (i) *There is a functor*

$$(\text{Sch}/k)^\circ \rightarrow C(K_0), \quad X \mapsto R\Gamma_{\text{rig}}(X/K_0)$$

from the category of algebraic k -schemes with proper morphisms Sch/k , such that $H^i(R\Gamma_{\text{rig}}(X/K_0)) \cong H^i_{\text{rig}}(X/K_0)$. Moreover, a canonical σ -linear endomorphism of $R\Gamma_{\text{rig}}(X/K_0)$ exists inducing the Frobenius on cohomology.

(ii) *There are two functors $\text{RT} \rightarrow C(K)$*

$$\widetilde{R}\Gamma_{\text{rig}}(X)_{\bar{X},P} \quad \text{and} \quad R\Gamma_{\text{rig}}(X/K)_{\bar{X},P}$$

and functorial quasi-isomorphisms with respect to maps of rigid triples

$$R\Gamma_{\text{rig}}(X/K) \leftarrow \widetilde{R}\Gamma_{\text{rig}}(X)_{\bar{X},P} \rightarrow R\Gamma_{\text{rig}}(X)_{\bar{X},P}.$$

Proof. See [Besser 2000, 4.9, 4.21, 4.22]. □

Remark 4.1.3. The building block of the construction is the functor $R\Gamma_{\text{rig}}(X/K)_{\bar{X},P}$. That complex is constructed with a system of compatible resolution of the over-convergent de Rham complexes $j_X^\dagger \Omega_{\mathcal{U}}^\bullet$, where \mathcal{U} runs over all strict neighborhoods of the tube of X . Using Godement resolution, we can explicitly define

$$R\Gamma_{\text{rig}}(X/K)_{\bar{X},P} := \text{colim}_{\mathcal{U}} \Gamma(\mathcal{U}, \text{Gd}_{\text{an}} j_X^\dagger \text{Gd}_{\text{an}} \Omega_{\mathcal{U}}^\bullet),$$

where $\text{Gd}_{\text{an}} = \text{Gd}_{P_t(\mathcal{U})}$. This will be an essential ingredient for achieving the main results of the paper.

All the proofs of [Besser 2000, Section 4] work using this *Godement* complex. We recall that

$$\begin{aligned} R\Gamma_{\text{rig}}(X/K) &:= \text{colim}_{A \in \text{SET}_X^0} R\Gamma_{\text{rig}}(X/K)_{\bar{X}_A, P_A}, \\ \widetilde{R}\Gamma_{\text{rig}}(X/K) &:= \text{colim}_{A \in \text{SET}_{(X, \bar{X}, P)}^0} R\Gamma_{\text{rig}}(X/K)_{\bar{X}_A, P_A}, \end{aligned}$$

where SET_X^0 and $\text{SET}_{(X, \bar{X}, P)}^0$ are filtered categories of indexes.

With some modifications, we can provide a compact support version of the functors above. We just need to be careful in the choice of morphisms of rigid triples.

Definition 4.1.4. Let (X, \bar{X}, P) and (Y, \bar{Y}, Q) be two rigid triples, and let $f : X \rightarrow Y$ be a morphism of k -schemes. Let $F : \mathcal{U} \rightarrow \mathcal{Q}_K$ be compatible with f (as in

Definition 4.1.1. We say that F is *strict* if there is a commutative diagram

$$\begin{array}{ccccc}]X[_P & \longrightarrow & \mathcal{U} & \longleftarrow & \mathcal{U} \setminus]X[_P \\ \downarrow F & & \downarrow F & & \downarrow F \\]Y[_Q & \longrightarrow & \mathcal{V} & \longleftarrow & \mathcal{V} \setminus]Y[_Q, \end{array}$$

where \mathcal{V} is a strict neighborhood of $]Y[_Q$ in $\bar{Y}[_Q$.

It is easy to show that strict morphisms are composable. We let RT_c denote the category of *rigid triples with proper morphisms*, which is the (not full) subcategory of RT with the same objects and morphisms pairs (f, F) , where f is proper and F is a germ of a strict compatible morphism.

Lemma 4.1.5. (i) *Let (X, \bar{X}, P) be a rigid triple, and let \mathcal{U} be a strict neighborhood of $]X[_P$. Then*

$$H^i(\Gamma(\mathcal{U}, \text{Gd}_{\text{an}} \underline{\Gamma}_{]X[_P} \text{Gd}_{\text{an}} \Omega_{\mathcal{U}}^\bullet)) = H_{\text{rig},c}^i(X).$$

(ii) *Let (Y, \bar{Y}, Q) be another rigid triple, $f : X \rightarrow Y$ be a proper k -morphism and $F : \mathcal{U} \rightarrow Q_K$ be a morphism of K -analytic space compatible with f and strict. Then there is a canonical map*

$$F^* : \text{Gd}_{\text{an}} \underline{\Gamma}_{]Y[_Q} \text{Gd}_{\text{an}} \Omega_{Q_K}^\bullet \rightarrow F_* \text{Gd}_{\text{an}} \underline{\Gamma}_{]X[_P} \text{Gd}_{\text{an}} \Omega_{\mathcal{U}}^\bullet.$$

Proof. (i) It is sufficient to note that $\text{Gd}_{\text{an}}(\Omega_{\mathcal{U}}^\bullet)$ is a complex of flasque sheaves and that a flasque sheaf is acyclic for $\underline{\Gamma}_{]X[_P}$. Let F be a flasque sheaf on the rigid analytic space \mathcal{U} . By definition, $\underline{\Gamma}_{]X[_P} = \text{Ker}(a : F \rightarrow i_* i^* F)$. It is easy to check that $\mathbb{R}^q i_* i^* F = 0$ for $q \geq 1$. Hence, $\mathbb{R}\underline{\Gamma}_{]X[_P} F \cong \text{Cone}(a : F \rightarrow i_* i^* F)[-1]$. But by hypothesis, the map a is surjective, so $\text{Cone}(a : F \rightarrow i_* i^* F)[-1] \cong \text{Ker}(a) = \mathbb{R}^0 \underline{\Gamma}_{]X[_P}$.

(ii) First consider the canonical pullback of differential forms $F^* : \Omega_{Q_K} \rightarrow F_* \Omega_{\mathcal{U}}$. Then by [Lemma 3.1.2](#), we get a map $\text{Gd}_{P_t(Q_K)} \Omega_{Q_K}^\bullet \rightarrow F_* \text{Gd}_{P_t(\mathcal{U})} \Omega_{\mathcal{U}}$. Applying the functor $\underline{\Gamma}_{]X[_P}$ to the adjoint map, we get

$$\underline{\Gamma}_{]X[_P} F^{-1} \text{Gd}_{P_t(Q_K)} \Omega_{Q_K} \rightarrow \underline{\Gamma}_{]X[_P} \text{Gd}_{P_t(\mathcal{U})} \Omega_{\mathcal{U}}.$$

But the strictness of F implies that there is a canonical map $F^{-1} \underline{\Gamma}_{]Y[_Q} \rightarrow \underline{\Gamma}_{]X[_P} F^{-1}$ [[Le Stum 2007](#), proof of Proposition 5.2.17]. Hence, we have a map

$$F^{-1} \underline{\Gamma}_{]Y[_Q} \text{Gd}_{P_t(Q_K)} \Omega_{Q_K} \rightarrow \underline{\Gamma}_{]X[_P} \text{Gd}_{P_t(\mathcal{U})} \Omega_{\mathcal{U}}.$$

We can conclude the proof by taking the adjoint of this map and again applying [Lemma 3.1.2](#). \square

Proposition 4.1.6. (i) *There is a functor*

$$(\text{Sch}_c/k)^\circ \rightarrow C(K_0), \quad X \mapsto R\Gamma_{\text{rig},c}(X/K_0)$$

from the category of algebraic k -schemes with proper morphisms Sch_c/k , such that $H^i(R\Gamma_{\text{rig},c}(X/K_0)) \cong H^i_{\text{rig},c}(X/K_0)$. Moreover, there exists a canonical σ -linear endomorphism of $R\Gamma_{\text{rig},c}(X/K_0)$ inducing the Frobenius on cohomology.

(ii) There are two functors $\text{RT}_c \rightarrow C(K)$

$$\widetilde{R\Gamma}_{\text{rig},c}(X)_{\bar{X},P} \quad \text{and} \quad R\Gamma_{\text{rig},c}(X/K)_{\bar{X},P}$$

and functorial quasi-isomorphisms with respect to maps of rigid triples

$$R\Gamma_{\text{rig},c}(X/K) \leftarrow \widetilde{R\Gamma}_{\text{rig},c}(X)_{\bar{X},P} \rightarrow R\Gamma_{\text{rig},c}(X)_{\bar{X},P}.$$

Proof. In view of [Lemma 4.1.5](#), it suffices to mimic the construction given in [\[Besser 2000, 4.9, 4.21, 4.22\]](#) but using only proper morphisms of k -schemes and strict compatible maps. In this case, the functors used in the construction are

$$R\Gamma_{\text{rig},c}(X/K)_{\bar{X},P} := \text{colim}_{\mathcal{U}} \Gamma(\mathcal{U}, \text{Gd}_{\text{an}} \underline{\Gamma}_{X|P} \text{Gd}_{\text{an}} \Omega_{\mathcal{U}}^{\bullet}),$$

$$R\Gamma_{\text{rig},c}(X/K) := \text{colim}_{A \in \text{SET}_X^0} R\Gamma_{\text{rig},c}(X/K)_{\bar{X}_A, P_A},$$

$$\widetilde{R\Gamma}_{\text{rig},c}(X/K) := \text{colim}_{A \in \text{SET}_X^0(X, \bar{X}, P)} R\Gamma_{\text{rig},c}(X/K)_{\bar{X}_A, P_A}. \quad \square$$

4.2. de Rham complexes. Now we focus on de Rham complexes, and we deal with smooth K -algebraic schemes. Let X be a smooth algebraic K -scheme. The (algebraic) de Rham cohomology of X is the hypercohomology of its complex of Kähler differentials $H^i_{\text{dR}}(X/K) := H^i(X, \Omega_{X/K}^{\bullet})$ [\[Grothendieck 1966b\]](#). We can also define the de Rham cohomology with compact support [\[Baldassarri et al. 2004, Section 1\]](#) as the hypercohomology groups $H^i_{\text{dR},c}(X/K) := H^i(\bar{X}, \lim_n J^n \Omega_{\bar{X}/K}^{\bullet})$, where $X \rightarrow \bar{X}$ is a smooth compactification and J is the sheaf of ideals associated to the complement $\bar{X} \setminus X$ (this definition does not depend on the choice of \bar{X} [\[Baldassarri et al. 2004, Theorem 1.8\]](#)). In order to consider the Hodge filtration on the de Rham cohomology groups, we fix a normal crossings compactification $g : X \rightarrow Y$ and let $D := Y \setminus X$ be the complement divisor (this is possible by the Nagata compactification theorem and the Hironaka resolution theorem [\[Deligne 1971, Section 3.2.1\]](#)). We let $\Omega_Y^{\bullet}(D)$ denote the de Rham complex of Y with logarithmic poles along D (in the Zariski topology) [\[Jannsen 1990, 3.3\]](#). Let $I \subset \mathcal{O}_Y$ be the defining sheaf of ideals of D .

Proposition 4.2.1. *With the notation above,*

(i) *there is a canonical isomorphism*

$$H^i_{\text{dR}}(X) \cong H^i(Y, \Omega_Y^{\bullet}(D)) \quad (\text{resp. } H^i_{\text{dR},c}(X) \cong H^i(Y, I\Omega_Y^{\bullet}(D))),$$

(ii) *the spectral sequence*

$$E_1^{p,q} = H^q(Y, \Omega^p) \Rightarrow H^{p+q}(Y, \Omega^\bullet)$$

degenerates at 1 for $\Omega^\bullet = \Omega_Y^\bullet\langle D \rangle$ and $I\Omega_Y^\bullet\langle D \rangle$ and

(iii) *the filtration induced by this spectral sequence on $H_{\text{dR}}^i(X)$ (resp. $H_{\text{dR},c}^i(X)$) is independent of the choice of Y . Namely,*

$$F^j H_{\text{dR}}^i(X) := H^i(Y, \sigma^{\geq j} \Omega_Y^\bullet\langle D \rangle) \quad (\text{resp. } H_{\text{dR},c}^i(X) := H^i(Y, \sigma^{\geq j} I\Omega_Y^\bullet\langle D \rangle)),$$

where $\sigma^{\geq j}$ is the stupid filtration.

Proof. Using the argument of [Deligne 1971, 3.2.11], we get the independence of the choice of Y . The same holds for $H_{\text{dR},c}^i(X)$.

Since our base field K is of characteristic 0, we can find an embedding $\tau : K \rightarrow \mathbb{C}$. Then by [GAGA 1955–1956], we get these isomorphisms of filtered vector spaces:

$$\begin{aligned} H^i(Y, \Omega_Y^\bullet\langle D \rangle) \otimes_K \mathbb{C} &\cong H^i(Y_h, \Omega_{Y_h}^\bullet\langle D_h \rangle) \\ (\text{resp. } H^i(Y, I\Omega_Y^\bullet\langle D \rangle) \otimes_K \mathbb{C} &\cong H^i(Y_h, I_h\Omega_{Y_h}^\bullet\langle D_h \rangle)), \end{aligned}$$

where $(\cdot)_h$ is the complex analytification functor and I_h is the defining sheaf of D_h . Thus, we conclude by [Deligne 1971, Section 3] (resp. [Peters and Steenbrink 2008, Part II, Example 7.25] for the compact support case). \square

Remark 4.2.2. The degeneracy of the spectral sequence in (ii) of the proposition above can be proved algebraically [Deligne and Illusie 1987]. We don't know an algebraic proof of the isomorphism in (i).

In the sequel, a morphism of pairs (X, Y) as above is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ u \downarrow & & \downarrow v \\ X' & \xrightarrow{g'} & Y'. \end{array}$$

We say that the morphism is strict if the square is cartesian.

The complex $\Omega_Y^\bullet\langle D \rangle$ (resp. $I\Omega_Y^\bullet\langle D \rangle$) is a complex of Zariski sheaves over Y functorial with respect to the pair (X, Y) (resp. strict morphisms of pairs). We can construct two different (generalized) Godement resolutions (see Section 3): one using Zariski points and the other via the K -analytic space associated to Y .

We will write $Pt(Y) = Pt(Y_{\text{zar}})$ for the set of Zariski points of Y with the discrete topology and $Pt(Y_{\text{an}})$ for the discrete site of rigid points (Example 3.2.3) of Y_{an} ; $Pt(Y_{\text{an}}) \sqcup Pt(Y)$ is the direct sum in the category of sites.

Proposition 4.2.3. *With the notation of Proposition 4.2.1, let $w : Y_{\text{an}} \rightarrow Y_{\text{zar}}$ be the canonical map from the rigid analytic site to the Zariski site of Y . Then for any Zariski sheaf Ω on Y , there is a diagram*

$$\text{Gd}_{Pt(Y)}(\Omega) \leftarrow \text{Gd}_{Pt(Y_{\text{an}}) \sqcup Pt(Y)}(\Omega) \rightarrow w_* \text{Gd}_{Pt(Y_{\text{an}})}(w^* \Omega).$$

If we further consider $\Omega = \Omega_Y^\bullet \langle D \rangle$ (resp. $\Omega = I\Omega_Y^\bullet \langle D \rangle$), then the diagram is functorial with respect to the pair (X, Y) (resp. (X, Y) and strict morphisms). The same holds true with Gd_Y^2 instead of Gd_Y .

Proof. The first claim follows from Lemma 3.1.2 applied to the commutative diagram of sites

$$\begin{array}{ccccc} Pt(Y_{\text{an}}) & \longrightarrow & Pt(Y_{\text{an}}) \sqcup Pt(Y) & \longleftarrow & Pt(Y) \\ \downarrow & & \downarrow & & \downarrow \\ Y_{\text{an}} & \xrightarrow{w} & Y_{\text{zar}} & \xleftarrow{\text{id}} & Y_{\text{zar}} \end{array}$$

with respect to the canonical map $\Omega \rightarrow w_* w^* \Omega$. The second claim follows from the functoriality of the complex $\Omega_Y^\bullet \langle D \rangle$ (resp. $I\Omega_Y^\bullet \langle D \rangle$). \square

Proposition 4.2.4. (i) *Let $g : X \rightarrow Y$ be a normal crossings compactification as in Section 4.2. Then there is a quasi-isomorphism of complexes of sheaves*

$$\Omega_Y^\bullet \langle D \rangle \rightarrow \text{Gd}_{Pt(Y)}^2(\Omega_Y^\bullet \langle D \rangle) \quad (\text{resp. } I\Omega_Y^\bullet \langle D \rangle \rightarrow \text{Gd}_{Pt(Y)}^2(I\Omega_Y^\bullet \langle D \rangle)),$$

and the stupid filtration on $\Omega_Y^\bullet \langle D \rangle$ (resp. $I\Omega_Y^\bullet \langle D \rangle$) induces a filtration on the right term of the morphism.

(ii) *Let Sm/K (resp. Sm_c/K) be the category of algebraic and smooth K -schemes (resp. with proper morphisms). Let $D_{\text{dR}}^b(K)$ be the derived category of the exact category of filtered vector spaces. Then there exist two functors*

$$R\Gamma_{\text{dR}}(\cdot) : (\text{Sm}/K)^\circ \rightarrow D_{\text{dR}}^b(K) \quad \text{and} \quad R\Gamma_{\text{dR},c}(\cdot) : (\text{Sm}_c/K)^\circ \rightarrow D_{\text{dR}}^b(K)$$

such that with the same notation as (i), $R\Gamma_{\text{dR}}(X) = \Gamma(Y, \text{Gd}_{Pt(Y)}^2(\Omega_Y^\bullet \langle D \rangle))$ and $R\Gamma_{\text{dR},c}(X) = \Gamma(Y, \text{Gd}_{Pt(Y)}^2(I\Omega_Y^\bullet \langle D \rangle))$.

(iii) *The filtered complexes $R\Gamma_{\text{dR}}(X)$ and $R\Gamma_{\text{dR},c}(X)$ are strict (see Remark 2.0.10).*

Proof. (i) This follows directly from the definition of Godement resolution.

(ii) This follows from the functoriality of the Godement resolution with respect to morphism of pairs and [Deligne 1971, 3.2.11] or [Huber 1995, Lemma 15.2.3] for the compact support case.

(iii) This follows by [Peters and Steenbrink 2008, Part II, Section 4.3, Section 7.3.1]. \square

5. Syntomic cohomology

In this section, we construct the p -adic Hodge complexes needed to define the rigid syntomic cohomology groups (also with compact support) for a smooth algebraic \mathcal{V} -scheme. The functoriality will be a direct consequence of the construction.

5.1. Compactifications.

Lemma 5.1.1 [Grothendieck 1966a, 2.8.5]. *Let $f : \mathcal{X} \rightarrow \mathcal{V}$ be a morphism of schemes, and let $Z \subset \mathcal{X}_K$ be a closed subscheme of the generic fiber of \mathcal{X} . Then there exists a unique closed subscheme $\mathcal{Z} \subset \mathcal{X}$ that is flat over \mathcal{V} and satisfies $\mathcal{Z}_K = Z$. Thus, \mathcal{Z} is the schematic closure of Z in \mathcal{X} .*

Proposition 5.1.2. *Let \mathcal{X} be a smooth scheme over \mathcal{V} . Then there exists a generic normal crossings compactification, that is, an open embedding $g : \mathcal{X} \rightarrow \mathcal{Y}$ such that*

- (i) \mathcal{Y} is proper over \mathcal{V} ,
- (ii) \mathcal{Y}_K is smooth over K and
- (iii) $\mathcal{D}_K \subset \mathcal{Y}_K$ is a normal crossings divisor, where $\mathcal{D} = \mathcal{Y} \setminus \mathcal{X}$.

Proof. First by Nagata [Conrad 2007], there exists an open embedding $\mathcal{X}_K \rightarrow Y$, where Y is a proper K -scheme. By the Hironaka resolution theorem, we can assume that Y is a smooth compactification of \mathcal{X}_K with complement a normal crossings divisor. Now we can define \mathcal{Y}' to be the gluing of \mathcal{X} and Y along the common open subscheme \mathcal{X}_K . These schemes are all of finite type over \mathcal{V} . It follows from the construction that \mathcal{Y}' is a scheme, separated and of finite type over \mathcal{V} , whose generic fiber is Y .

The Nagata compactification theorem works also in a relative setting, namely for a separated and finite type morphism; hence, we can find a \mathcal{V} -scheme \mathcal{Y} that is a compactification of \mathcal{Y}' over \mathcal{V} . Thus, we get an open and dense embedding $h : \mathcal{Y}'_K = Y \rightarrow \mathcal{Y}_K$ of proper K -schemes, so h is the identity, and the statement is proven. \square

Remark 5.1.3. We can also give another proof of the previous proposition assuming an embedding in a smooth \mathcal{V} -scheme. First by Nagata [Conrad 2007], there exists an open embedding $\mathcal{X} \rightarrow \bar{\mathcal{X}}$. Now assume that $\bar{\mathcal{X}}$ is embeddable in a smooth \mathcal{V} -scheme \mathcal{W} ; then by [Włodarczyk 2005, Theorem 1.0.2], we can get a resolution of the K -scheme $\bar{\mathcal{X}}_K$ by making a sequence of blowups with respect to a family of closed subsets in good position with respect to the regular locus of $\bar{\mathcal{X}}_K$, in particular $Z_i \cap \mathcal{X}_K = \emptyset$. One can perform the same construction directly over \mathcal{V} , replacing the closed $Z_i \subset \mathcal{W}_K$ with their Zariski closure \mathcal{Z}_i in \mathcal{W} . By hypothesis, $Z_i \subset \mathcal{W}_K \setminus \mathcal{X}_K$; hence, its closure \mathcal{Z}_i is contained in $\mathcal{W} \setminus \mathcal{X}$, and $(\mathcal{Z}_i)_K = Z_i$ by Lemma 5.1.1. The construction doesn't affect what happens on the generic fiber because the blowup

construction is local and behaves well with respect to open immersions [Hartshorne 1977, Chapter II, 7.15]. This will give \mathcal{Y} as in the proposition.

5.2. Connecting maps. From now on, we keep the notation of Proposition 5.1.2 with $g : \mathcal{X} \rightarrow \mathcal{Y}$ being fixed. To simplify the notation, \mathcal{X} (resp. \mathcal{Y}) denotes the rigid analytic space associated to the generic fiber of \mathcal{X} (resp. \mathcal{Y}), usually denoted $\mathcal{X}_K^{\text{an}}$. Let $w : \mathcal{Y} \rightarrow (\mathcal{Y}_K)_{\text{zar}}$ be the canonical map of sites.

Sometimes we will simply write $\text{Gd}_{\text{an}} = \text{Gd}_{P_t(\mathcal{U})}$ if the rigid space \mathcal{U} is clear from the context. Similarly, we write $\text{Gd}_{\text{zar}} = \text{Gd}_{P_t(X)}$ to denote the Godement resolution with respect to Zariski points of a K -scheme X .

Lemma 5.2.1. *With the notation above, we have the following morphisms of complexes of K -vector spaces:*

$$\begin{aligned} \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} j^\dagger \text{Gd}_{\text{an}} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) &\rightarrow \Gamma(\mathcal{X}, \text{Gd}_{\text{an}} j^\dagger \text{Gd}_{\text{an}} \Omega_{\mathcal{X}}^\bullet) \rightarrow R\Gamma_{\text{rig}}(\mathcal{X}_k/K)_{\mathcal{Y}_k, \widehat{\mathcal{Y}}}, \\ \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} \Gamma_{] \mathcal{X}_k[} \text{Gd}_{\text{an}} w^* I \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) & \\ &\rightarrow \Gamma(\mathcal{X}, \text{Gd}_{\text{an}} \Gamma_{] \mathcal{X}_k[} \text{Gd}_{\text{an}} \Omega_{\mathcal{X}}^\bullet) \rightarrow R\Gamma_{\text{rig}, c}(\mathcal{X}_k/K)_{\mathcal{Y}_k, \widehat{\mathcal{Y}}}. \end{aligned}$$

All the maps are quasi-isomorphisms. We let a and b denote the composition of the maps in the first and second diagrams, respectively.

Proof. By construction, $R\Gamma_{\text{rig}}(\mathcal{X}_k/K)_{\mathcal{Y}_k, \widehat{\mathcal{Y}}}$ (respectively $R\Gamma_{\text{rig}, c}(\mathcal{X}_k/K)_{\mathcal{Y}_k, \widehat{\mathcal{Y}}}$) is a direct limit of complexes indexed over the strict neighborhoods of $] \mathcal{X}_k[_{\widehat{\mathcal{Y}}}$, and \mathcal{X} is one of them (see Remark 4.1.3 and Proposition 4.1.6). Hence, the map on the right (of both diagrams) comes from the universal property of the direct limit.

For the map on the left, consider first the canonical inclusion of algebraic differential forms with log poles into the analytic ones $w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle \rightarrow g_*^{\text{an}} \Omega_{\mathcal{Y}}^\bullet$. By Lemma 3.1.2, we get a map

$$\text{Gd}_{P_t(\mathcal{Y})} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle \rightarrow g_*^{\text{an}} \text{Gd}_{P_t(\mathcal{X})} \Omega_{\mathcal{X}}^\bullet.$$

Then applying the j^\dagger functor and noting that $j^\dagger g_*^{\text{an}} = g_*^{\text{an}} j^\dagger$ [Le Stum 2007, 5.1.14], one obtains a morphism $j^\dagger \text{Gd}_{P_t(\mathcal{Y})} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle \rightarrow g_*^{\text{an}} j^\dagger \text{Gd}_{P_t(\mathcal{X})} \Omega_{\mathcal{X}}^\bullet$. This is what we need to apply Lemma 3.1.2 again and conclude the proof for the first diagram.

For the second diagram, repeat the argument using [Le Stum 2007, 5.2.15]. \square

Lemma 5.2.2. *With the notation above, we have the following morphisms of complexes of K -vector spaces, which we denote by a' and b' , respectively:*

$$\begin{aligned} \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}}^2 w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) &\rightarrow \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} j^\dagger \text{Gd}_{\text{an}} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle), \\ \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}} \Gamma_{] \mathcal{X}_k[} \text{Gd}_{\text{an}} w^* I \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle) &\rightarrow \Gamma(\mathcal{Y}, \text{Gd}_{\text{an}}^2 w^* I \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle). \end{aligned}$$

Proof. The maps a' and b' are induced by the canonical maps $\Omega \rightarrow j^\dagger \Omega$ and $\Gamma_{\mathcal{X}_k} \Omega \rightarrow \Omega$, respectively, where Ω is an abelian sheaf on \mathcal{Y} . In particular, we consider $\Omega = \mathrm{Gd}_{\mathrm{an}} w^* \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle$ (respectively $\Omega = \mathrm{Gd}_{\mathrm{an}} w^* I \Omega_{\mathcal{Y}_K}^\bullet \langle \mathcal{D}_K \rangle$). To conclude the proof, apply the functor $\mathrm{Gd}_{\mathrm{an}}$ again and take global sections. \square

5.3. Syntomic complexes. Now we put together all we have done, getting a diagram, say $R\Gamma'(\mathcal{X})$, of complexes of K -vector spaces:

$$\begin{array}{ccc}
 R\Gamma_{\mathrm{rig}}(\mathcal{X}/K_0) & \xrightarrow{\alpha_1} & R\Gamma_{\mathrm{rig}}(\mathcal{X}/K) \\
 & \searrow^{\alpha_2} & \\
 \widetilde{R}\Gamma_{\mathrm{rig}}(\mathcal{X}/K)_{\mathcal{Y}_k, \hat{\mathcal{Y}}} & \xrightarrow{\alpha_3} & R\Gamma_{\mathrm{rig}}(\mathcal{X}/K)_{\mathcal{Y}_k, \hat{\mathcal{Y}}} \\
 & \searrow^{\alpha_4} & \\
 \Gamma(\mathcal{Y}, \mathrm{Gd}_{\mathrm{an}}^2 w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & \xrightarrow{\alpha_5} & \Gamma(\mathcal{Y}, \mathrm{Gd}_{\mathrm{an}}^2 w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \\
 & \searrow^{\alpha_6} & \\
 \Gamma(\mathcal{Y}_K, \mathrm{Gd}_{\mathrm{an}+\mathrm{zar}}^2 \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & \xrightarrow{\alpha_7} & \Gamma(\mathcal{Y}_K, \mathrm{Gd}_{\mathrm{zar}}^2 \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \\
 & \searrow^{\alpha_8} & \\
 \Gamma(\mathcal{Y}_K, \mathrm{Gd}_{\mathrm{zar}}^2 \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & &
 \end{array}$$

where $\alpha_1, \alpha_5, \alpha_8$ are the identity maps; α_2 and α_3 are the maps of [Proposition 4.1.2](#); α_4 is the composition of $a \circ a'$ (see [Lemma 5.2.1](#) and [Lemma 5.2.2](#)) and α_6 and α_7 are defined in [Proposition 4.2.3](#). By repeatedly applying the quasipushout construction, we obtain a diagram of the shape

$$R\Gamma_{\mathrm{rig}}(\mathcal{X}/K_0) \rightarrow R\Gamma_K(\mathcal{X}) \leftarrow R\Gamma_{\mathrm{dR}}(\mathcal{X}). \quad (3)$$

It represents an object of $p\mathrm{HC}$ that we denote $R\Gamma(\mathcal{X})$.

Similarly, we can construct a p -adic Hodge complex $R\Gamma_c(\mathcal{X})$ associated to the diagram $R\Gamma'_c(\mathcal{X})$ defined as

$$\begin{array}{ccc}
 R\Gamma_{\mathrm{rig},c}(\mathcal{X}/K_0) & \xrightarrow{\beta_1} & R\Gamma_{\mathrm{rig},c}(\mathcal{X}/K) \\
 & \searrow^{\beta_2} & \\
 \widetilde{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}/K)_{\mathcal{Y}_k, \hat{\mathcal{Y}}} & \xrightarrow{\beta_3} & R\Gamma_{\mathrm{rig},c}(\mathcal{X}/K)_{\mathcal{Y}_k, \hat{\mathcal{Y}}} \\
 & \searrow^{\beta_4} & \\
 \Gamma(\mathcal{Y}, \mathrm{Gd}_{\mathrm{an}} \Gamma_{\mathcal{X}|} \mathrm{Gd}_{\mathrm{an}} w^* I \Omega_{Y_K} \langle \mathcal{D} \rangle) & \xrightarrow{\beta_5} & \Gamma(\mathcal{Y}, \mathrm{Gd}_{\mathrm{an}}^2 w^* I \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \\
 & \searrow^{\beta_6} & \\
 \Gamma(\mathcal{Y}_K, \mathrm{Gd}_{\mathrm{an}+\mathrm{zar}}^2 I \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & \xrightarrow{\beta_7} & \Gamma(\mathcal{Y}_K, \mathrm{Gd}_{\mathrm{zar}}^2 I \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \\
 & \searrow^{\beta_8} & \\
 \Gamma(\mathcal{Y}_K, \mathrm{Gd}_{\mathrm{zar}}^2 I \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & &
 \end{array}$$

where β_1 and β_8 are the identity maps; β_2 and β_3 are the maps of [Proposition 4.1.2](#); β_4 is the map b of [Lemma 5.2.1](#); $\beta_5 = b'$ of [Lemma 5.2.2](#) and β_6 and β_7 are defined in [Proposition 4.2.3](#). Note that β_6 is a quasi-isomorphism by GAGA.

Proposition 5.3.1. *Let Sm/R (resp. Sm_c/R) be the category of algebraic and smooth R -schemes (resp. with proper morphisms). The previous construction*

induces the functors

$$R\Gamma(\cdot) : (\mathrm{Sm}/\mathcal{V})^\circ \rightarrow p\mathrm{HD} \quad \text{and} \quad R\Gamma_c(\cdot) : (\mathrm{Sm}_c/\mathcal{V})^\circ \rightarrow p\mathrm{HD}.$$

Proof. Let $f : \mathcal{X} \rightarrow \mathcal{X}'$ be a morphism of smooth \mathcal{V} -schemes. To get the functoriality, we just have to show that can find two gncd compactifications $g : \mathcal{X} \rightarrow \mathcal{Y}$ and $g' : \mathcal{X}' \rightarrow \mathcal{Y}'$ and a map $h : \mathcal{Y} \rightarrow \mathcal{Y}'$ extending f , that is, $hg = g'f$. We argue as in [Deligne 1971, Section 3.2.11]. Fix two gncd compactifications $g' : \mathcal{X}' \rightarrow \mathcal{Y}'$ and $l : \mathcal{X} \rightarrow \mathcal{X}$. Then consider the canonical map $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{Y}'$ induced by l and $g'f$. Let $\bar{\mathcal{X}}$ be the closure of \mathcal{X} in $\mathcal{X} \times \mathcal{Y}'$ and use the same argument as in the proof of Proposition 5.1.2 to get \mathcal{Y} , which is generically a resolution of the singularities of $\bar{\mathcal{X}}$. Then by Proposition 4.2.3 and Proposition 4.1.2, we get the functoriality of $R\Gamma(\cdot)$.

If we further assume f to be proper, then we can apply the same argument in order to get the commutative square $hg = g'f$ as above. Then by properness, this square is also cartesian by [Huber 1995, Lemma 15.2.3]. From this fact and Propositions 4.1.6 and 4.2.3, we obtain the functoriality of $R\Gamma_c(\cdot)$ with respect to proper maps. \square

Definition 5.3.2. Let \mathcal{X} be a smooth algebraic scheme over \mathcal{V} . For any integers n and i , we define the *absolute cohomology* groups of \mathcal{X} as

$$H_{\mathrm{abs}}^n(\mathcal{X}, i) := \mathrm{Hom}_{p\mathrm{HD}}(\mathbb{K}, R\Gamma(\mathcal{X})(i)[n]) = H^n(\Gamma(\mathbb{K}, R\Gamma(\mathcal{X})(i))) \quad (4)$$

and the *absolute cohomology with compact support* groups of \mathcal{X} as

$$H_{\mathrm{abs},c}^n(\mathcal{X}, i) := \mathrm{Hom}_{p\mathrm{HD}}(\mathbb{K}, R\Gamma_c(\mathcal{X})(i)[n]) = H^n(\Gamma(\mathbb{K}, R\Gamma_c(\mathcal{X})(i))). \quad (5)$$

A direct consequence of the definition is the existence of the following long exact sequence, which should be considered to be a p -adic analog of the corresponding sequence for Deligne–Beilinson cohomology [Beilinson 1985, Introduction].

Proposition 5.3.3. *With the notation above, we have the long exact sequences*

$$\rightarrow H_{\mathrm{abs}}^n(\mathcal{X}, i) \rightarrow H_{\mathrm{rig}}^n(\mathcal{X}_k/K_0) \oplus F^i H_{\mathrm{dR}}^n(\mathcal{X}_K) \xrightarrow{h} H_{\mathrm{rig}}^n(\mathcal{X}_k/K_0) \oplus H_{\mathrm{rig}}^n(\mathcal{X}_k/K) \xrightarrow{+},$$

where $h(x_0, x_{\mathrm{dR}}) := (\phi(x_0^\sigma) - p^i x_0, x_0 \otimes 1_K - sp(x_{\mathrm{dR}}))$ and

$$\rightarrow H_{\mathrm{abs},c}^n(\mathcal{X}, i) \rightarrow H_{\mathrm{rig},c}^n(\mathcal{X}_k/K_0) \oplus F^i H_{\mathrm{dR},c}^n(\mathcal{X}_K) \xrightarrow{h_c} H_{\mathrm{rig},c}^n(\mathcal{X}_k/K_0) \oplus H_{\mathrm{dR},c}^n(\mathcal{X}_K) \xrightarrow{+},$$

where $h_c(x_0, x_{\mathrm{dR}}) := (\phi_c(x_0\sigma) - p^i x_0, \mathrm{cosp}(x_0 \otimes 1_K) - x_{\mathrm{dR}})$.

Proof. By Proposition 2.2.2, the absolute cohomology is the cohomology of a mapping cone, namely $\Gamma(\mathbb{K}, R\Gamma(\mathcal{X}))$ (or $\Gamma_c(\mathbb{K}, R\Gamma(\mathcal{X}))$ for the compact support case). The long exact sequences above are easily induced from the distinguished triangle defined by the term of the mapping cone (see Remark 2.2.1(iii)). Indeed, one need only note that the map $c : R\Gamma_{\mathrm{rig}}(\mathcal{X}/K_0) \otimes K \rightarrow R\Gamma_K(\mathcal{X})$ (resp. $s : R\Gamma_{\mathrm{dR}}(\mathcal{X}) \rightarrow R\Gamma_K(\mathcal{X})$) is a quasi-isomorphism. \square

Proposition 5.3.4. *With the notation above, there is a canonical isomorphism between the absolute cohomology we have defined and the (rigid) syntomic cohomology of Besser:*

$$H_{\text{syn}}^n(\mathcal{X}, i) \cong H_{\text{abs}}^n(\mathcal{X}, i).$$

Proof. Besser defines a complex

$$\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i)$$

$$:= \text{Cone}(\mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus \text{Fil}^i \mathbb{R}\Gamma_{\text{dR}}(\mathcal{X}) \rightarrow \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K))[-1]$$

and the syntomic cohomology groups (of degree n and twisted i) $H^n(\mathbb{R}\Gamma_{\text{Bes}}(\mathcal{X}, i))$ (see [Besser 2000, proof of Proposition 6.3]). Note that, modulo the choice of the flasque resolution, $\mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}/K_0) = R\Gamma_{\text{rig}}(\mathcal{X}/K_0)$ (the left-hand side is the notation used by Besser with the bold \mathbb{R}); the complex $\text{Fil}^i \mathbb{R}\Gamma_{\text{dR}}(\mathcal{X})$ is a direct limit over all the normal crossing compactifications of \mathcal{X}_K of complexes of the form $\Gamma(Y, \text{Gd}_{p_t(Y)}^2 \sigma^{\geq i} \Omega_Y \langle D \rangle)$ so that our $F^i R\Gamma_{\text{dR}}(\mathcal{X})$ is an element of this direct limit. To conclude the proof, recall that by Remark 2.2.1(iii), we obtain

$$H_{\text{abs}}^n(\mathcal{X}, i)$$

$$\cong H^n(\text{Cone}(R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus F^i R\Gamma_{\text{dR}}(\mathcal{X}) \rightarrow R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_{\text{rig}}(\mathcal{X}/K))[-1]),$$

and it is easy to check that the maps in the two mapping cones are defined in the same way. \square

Remark 5.3.5. The isomorphism $H^n(R\Gamma_{\text{syn}}(\mathcal{X}, i)) \cong \text{Hom}_{p\text{HD}}(\mathbb{k}, R\Gamma(\mathcal{X})(i)[n])$ can be viewed as a generalization of the result of Bannai [2002], who considers only smooth schemes \mathcal{X} with a fixed compactification \mathcal{Y} and such that $\mathcal{Y} \setminus \mathcal{X} = \mathcal{D}$ is a relative normal crossings divisor over \mathcal{V} . Moreover, Bannai's construction is not functorial with respect to \mathcal{X} ; functoriality holds only with respect to a so-called syntomic datum. We should point out that the category defined by Bannai is endowed with a t -structure whose heart is the category MF_K^f of (weakly) admissible filtered Frobenius modules. In our case, we don't have such a nice picture.

Remark 5.3.6. The category of p -adic Hodge complexes is not endowed with internal Hom. This is due to the fact that the Frobenius is only a quasi-isomorphism; hence, we cannot invert it. In particular, this happens for the complex $R\Gamma_c(\mathcal{X})$; thus, we cannot define the dual $R\Gamma_c(\mathcal{X})^\vee := \mathbb{R}\mathcal{H}om(R\Gamma_c(\mathcal{X}), \mathbb{k})$ and an absolute homology theory as

$$H_n^{\text{abs}}(\mathcal{X}, i) := \text{Hom}_{p\text{HD}}(\mathbb{k}, R\Gamma_c(\mathcal{X})^\vee(-i)[-n]);$$

see [Huber 1995, Section 15.3]. Nevertheless, the usual adjunction between Hom and \otimes should give a natural isomorphism

$$\text{Hom}_{p\text{HD}}(\mathbb{k}, R\Gamma_c(\mathcal{X})^\vee) \cong \text{Hom}_{p\text{HD}}(R\Gamma_c(\mathcal{X}), \mathbb{k}),$$

and the right term does makes sense in our setting. This motivates the following definition:

Definition 5.3.7. With the notation of [Definition 5.3.2](#), we define the *absolute homology* groups of \mathcal{X} as

$$H_n^{\text{abs}}(\mathcal{X}, i) := \text{Hom}_{p\text{HD}}(R\Gamma_c(\mathcal{X}), \mathbb{K}(-i)[-n]) = H^{-n}(\Gamma(R\Gamma_c(\mathcal{X}), \mathbb{K}(-i))). \quad (6)$$

5.4. Cup product and Gysin map. We are going to prove that there is a morphism

$$R\Gamma(\mathcal{X}) \otimes R\Gamma_c(\mathcal{X}) \rightarrow R\Gamma_c(\mathcal{X})$$

of p -adic Hodge complexes. This induces a pairing at the level of the complexes computing absolute cohomology. The key point is the compatibility of the de Rham and rigid pairings with respect to the specialization and cospecialization maps.

Let us start by fixing some notation. Let \mathcal{X} be a smooth algebraic \mathcal{V} -scheme, $g : \mathcal{X} \rightarrow \mathcal{Y}$ be a gncd compactification and $\mathcal{D} = \mathcal{Y} \setminus \mathcal{X}$ be the complement; \mathcal{Y} is the rigid analytic space associated to the K -scheme \mathcal{Y}_K ; as before, we let $w : \mathcal{Y} \rightarrow \mathcal{Y}_{K, \text{zar}}$ denote the canonical morphism of sites (see the notation after [Proposition 5.1.2](#)). For any $\mathcal{O}_{\mathcal{Y}_K}$ -module F , w^* denotes its pullback.

Remark 5.4.1.

(i) The wedge product of algebraic differentials induces the pairing

$$p_{\text{dR}} : \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \otimes I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \rightarrow I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle.$$

(ii) The analytification of p_{dR} gives a pairing

$$w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \otimes w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \rightarrow w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle.$$

Hence, by [\[Berthelot 1997a, Lemma 2.1\]](#), we get the pairing

$$p_{\text{rig}} : j^\dagger w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \otimes \Gamma_{\mathcal{X}_k] w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle \rightarrow \Gamma_{\mathcal{X}_k] w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle.$$

Lemma 5.4.2 (sheaves level). *The diagram*

$$\begin{array}{ccc} \text{Gd}_{\text{an}}^2(w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & \xrightarrow{\quad} & \text{Gd}_{\text{an}}^2(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \\ \otimes \text{Gd}_{\text{an}}^2(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & & \uparrow \\ \text{Gd}_{\text{an}}^2(w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & \xrightarrow{m} & \text{Gd}_{\text{an}} \Gamma_{\mathcal{X}_k] \text{Gd}_{\text{an}}(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \\ \otimes \text{Gd}_{\text{an}} \Gamma_{\mathcal{X}_k] \text{Gd}_{\text{an}}(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & & \downarrow 1 \\ \text{Gd}_{\text{an}} j^\dagger \text{Gd}_{\text{an}}(w^* \Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & \xrightarrow{\quad} & \text{Gd}_{\text{an}} \Gamma_{\mathcal{X}_k] \text{Gd}_{\text{an}}(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) \\ \otimes \text{Gd}_{\text{an}} \Gamma_{\mathcal{X}_k] \text{Gd}_{\text{an}}(w^* I\Omega_{\mathcal{Y}_K} \langle \mathcal{D}_K \rangle) & & \end{array}$$

commutes, where $m := p_{\text{rig}} \circ (j^\dagger \otimes 1)$ by definition.

Proof. The bottom square commutes by construction, and we get $p_{\text{rig}} \circ (j^\dagger \otimes 1) = w^* p_{\text{dR}}$ restricted to $w^* \Omega_{\mathfrak{y}_K} \langle \mathcal{D}_K \rangle \otimes \underline{\Gamma}_{\mathfrak{X}_k} [w^* I \Omega_{\mathfrak{y}_K} \langle \mathcal{D}_K \rangle]$. \square

Proposition 5.4.3. *Let \mathfrak{X} be a smooth \mathcal{V} -scheme. Then there exists a morphism*

$$\pi : R\Gamma(\mathfrak{X}) \otimes R\Gamma_c(\mathfrak{X}) \rightarrow R\Gamma_c(\mathfrak{X})$$

of p -adic Hodge complexes, which is functorial with respect to \mathfrak{X} (as a morphism in $p\text{HD}$). Moreover, taking the cohomology of this map, we get the compatibility

$$\begin{array}{ccc} H_{\text{dR}}^n(\mathfrak{X}_K) \otimes H_{\text{dR},c}^m(\mathfrak{X}_K) & \longrightarrow & H_{\text{dR},c}^{n+m}(\mathfrak{X}_K) \\ \text{id} \otimes \text{cosp} \uparrow & & \text{cosp} \uparrow \\ H_{\text{dR}}^n(\mathfrak{X}_K) \otimes H_{\text{rig},c}^m(\mathfrak{X}_k) & \longrightarrow & H_{\text{rig},c}^{n+m}(\mathfrak{X}_k) \\ \text{sp} \otimes \text{id} \downarrow & & \text{id} \downarrow \\ H_{\text{rig}}^n(\mathfrak{X}_k) \otimes H_{\text{rig},c}^m(\mathfrak{X}_k) & \longrightarrow & H_{\text{rig},c}^{n+m}(\mathfrak{X}_k) \end{array}$$

Proof. It is sufficient to provide a pairing of the enlarged diagrams (see [Remark 2.2.4](#)). Thus, we have to define a morphism of diagrams $\pi' : R\Gamma'(\mathfrak{X}) \otimes R\Gamma'_c(\mathfrak{X}) \rightarrow R\Gamma'_c(\mathfrak{X})$ (notation as in [Section 5.3](#)). It is easy to construct π' using the previous lemma and the compatibility of the Godement resolution with the tensor product. \square

Corollary 5.4.4. *There is a functorial pairing (induced by π of [Proposition 5.4.3](#))*

$$H_{\text{abs}}^n(\mathfrak{X}, i) \otimes H_{\text{abs},c}^m(\mathfrak{X}, j) \rightarrow H_{\text{abs},c}^{n+m}(\mathfrak{X}, i+j).$$

Proof. Consider the pairing of [Proposition 5.4.3](#), which induces a morphism $R\Gamma(\mathfrak{X})(i) \otimes R\Gamma_c(\mathfrak{X})(j) \rightarrow R\Gamma_c(\mathfrak{X})(i+j)$. We then get the corollary by [Lemma 2.2.3](#) and [Definition 5.3.2](#). \square

Proposition 5.4.5 (Poincaré duality). *Let \mathfrak{X} be a smooth and algebraic \mathcal{V} -scheme of dimension d . Then there is a canonical isomorphism*

$$H_{\text{abs}}^n(\mathfrak{X}, i) \cong H_{2d-n}^{\text{abs}}(\mathfrak{X}, d-i).$$

Proof. By definition (Equations (4) and (6)), it is sufficient to prove that the complex $\Gamma(\mathbb{K}, R\Gamma(\mathfrak{X}))$ is quasi-isomorphic to the diagram $\Gamma(R\Gamma_c(\mathfrak{X}), \mathbb{K}[-2d](-d))$.

First recall that $\Gamma(\mathbb{K}, R\Gamma(\mathfrak{X}))$ is defined as

$$\begin{aligned} & \text{Cone}(R\Gamma_{\text{rig}}(\mathfrak{X}/K_0) \oplus R\Gamma_K(\mathfrak{X}) \oplus F^0 R\Gamma_{\text{dR}}(\mathfrak{X})) \\ & \xrightarrow{\psi} R\Gamma_{\text{rig}}(\mathfrak{X}/K_0) \oplus R\Gamma_K(\mathfrak{X}) \oplus R\Gamma_K(\mathfrak{X})[-1], \end{aligned}$$

where $\psi(x_0, x_K, x_{\text{dR}}) := (\phi(x_0) - x_0, c(x_0 \otimes \text{id}_K) - x_K, x_K - s(x_{\text{dR}}))$. To define the desired map, we need to modify this complex, replacing $R\Gamma_K(\mathcal{X})$ with $R\Gamma_{\text{dR}}(\mathcal{X})$ as

$$\begin{aligned} & \text{Cone}(R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_{\text{dR}}(\mathcal{X}) \oplus F^0 R\Gamma_{\text{dR}}(\mathcal{X}) \\ & \xrightarrow{\psi'} R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_K(\mathcal{X}) \oplus R\Gamma_{\text{dR}}(\mathcal{X})[-1], \end{aligned}$$

where $\psi'(x_0, x'_{\text{dR}}, x_{\text{dR}}) := (\phi(x_0) - x_0, c(x_0 \otimes \text{id}_K) - s(x'_{\text{dR}}), x'_{\text{dR}} - x_{\text{dR}})$. It is easy to see that this new complex, call it M^\bullet , is quasi-isomorphic to $\Gamma(\mathbb{K}, R\Gamma(\mathcal{X}))$.

Because the filtered complex $R\Gamma_{\text{dR},c}(\mathcal{X})$ is strict, the truncation $\tau_{\geq 2d} R\Gamma_{\text{dR},c}(\mathcal{X})$ is the usual truncation of complexes of K -vector spaces (see [Lemma 2.1.1\(iii\)](#) and [Remark 2.0.10](#)). Then the cup product induces a morphism of complexes $M^\bullet \rightarrow \Gamma(R\Gamma_c(\mathcal{X}), \tau_{\geq 2d} R\Gamma_c(\mathcal{X}))$ that is a quasi-isomorphism by the Poincaré duality theorems for rigid and de Rham cohomology [[Berthelot 1997a](#); [Huber 1995](#)]. Explicitly, this map is induced by the commutative diagram

$$\begin{array}{ccc} R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_{\text{dR}}(\mathcal{X}) & \xrightarrow{\alpha} & \text{Hom}_{K_0}^\bullet(N_0, \tau_{\geq 2d} N_0) \oplus \text{Hom}_K^\bullet(N_K, \tau_{\geq 2d} N_K) \\ \oplus F^0 R\Gamma_{\text{dR}}(\mathcal{X}) & & \oplus \text{Hom}_K^{\bullet,F}(N_{\text{dR}}, \tau_{\geq 2d} N_{\text{dR}}) \\ \psi' \downarrow & & \downarrow \xi \\ R\Gamma_{\text{rig}}(\mathcal{X}/K_0) \oplus R\Gamma_{\text{rig}}(\mathcal{X}/K) & \xrightarrow{\beta} & \text{Hom}_{K_0}^\bullet(N_0^\sigma, \tau_{\geq 2d} N_0) \oplus \text{Hom}_K^\bullet(N_{\text{rig}}, \tau_{\geq 2d} N_K) \\ \oplus R\Gamma_{\text{dR}}(\mathcal{X}) & & \oplus \text{Hom}_K^\bullet(N_{\text{dR}}, \tau_{\geq 2d} N_K), \end{array}$$

where $N := R\Gamma_c(\mathcal{X})$,

$$\begin{aligned} \xi(f_0, f_K, \text{dR}) &:= (\phi_c \circ f_0^\sigma - f_0 \circ \phi_c, c \circ (f_0 \otimes \text{id}_K) - f_K \circ c, f_K \circ s - s \circ x_{\text{dR}}), \\ \alpha(x_0, x'_{\text{dR}}, x_{\text{dR}}) &: (y_0, y_K, y_{\text{dR}}) \mapsto (x_0 \cup y_0, s(x'_{\text{dR}}) \cup y_K, x_{\text{dR}} \cup y_{\text{dR}}), \\ \beta(x_0, x_{\text{rig}}, x_{\text{dR}}) &: (y_0, y_{\text{rig}}, y_{\text{dR}}) \mapsto (x_0 \cup \phi_c(y_0), x_{\text{rig}} \cup y_{\text{rig}}, x_{\text{dR}} \cup y_{\text{dR}}). \end{aligned}$$

To conclude the proof, it is sufficient to apply the exact functor $\Gamma(R\Gamma_c(\mathcal{X}), \cdot)$ to the quasi-isomorphisms

$$\tau_{\geq 2d} R\Gamma_c(\mathcal{X}) \leftarrow H^{2d}(R\Gamma_c(\mathcal{X}))[-2d] \rightarrow \mathbb{K}(-d)[-2d]. \quad \square$$

Remark 5.4.6. We would like to point out some technical issues regarding Poincaré duality in syntomic cohomology.

(i) If it were possible to define an internal Hom in the category pHC of p -adic Hodge complexes (see [Remark 5.3.6](#)), then by [Proposition 5.4.3](#), one would obtain the natural isomorphism

$$R\Gamma(\mathcal{X})(d)[2d] \cong R\Gamma_c(\mathcal{X})^\vee$$

in the triangulated category pHD , where $R\Gamma_c(\mathcal{X})^\vee := \mathbb{R}\mathcal{H}om_{pHC}(R\Gamma_c(\mathcal{X}), \mathbb{K})$ is the dual of $R\Gamma_c(\mathcal{X})$. Then one would get by adjunction the duality

$$\begin{aligned} \mathrm{Hom}_{pHD}(\mathbb{K}, R\Gamma(X)(i)[n]) &\cong \mathrm{Hom}_{pHD}(\mathbb{K}, R\Gamma_c(X)^\vee(i-d)[n-2d]) \\ &\cong \mathrm{Hom}_{pHD}(R\Gamma_c(X)(d-i)[2d-n], \mathbb{K}). \end{aligned}$$

(ii) The Grothendieck–Leray spectral sequence for absolute homology is

$$E_2^{p,q} = \mathrm{Hom}_{pHD}(H^{-q}(R\Gamma_c(\mathcal{X})(i))[-p], \mathbb{K}),$$

and it degenerates to the short exact sequences

$$\begin{aligned} 0 \rightarrow \mathrm{Hom}_{pHD}(H^n(R\Gamma_c(\mathcal{X})(i)), \mathbb{K}) \rightarrow H_n^{\mathrm{syn}}(\mathcal{X}, i) \\ \rightarrow \mathrm{Hom}_{pHD}(H^{n+1}(R\Gamma_c(\mathcal{X})(i)), \mathbb{K}[1]) \rightarrow 0. \end{aligned}$$

Directly by [Proposition 2.2.2](#), the group $\mathrm{Hom}_{pHD}(H^n(R\Gamma_c(\mathcal{X})(i)), \mathbb{K})$ is

$$\begin{aligned} \{ (x_0, x_{\mathrm{dR}}) \in H_{\mathrm{rig},c}^n(\mathcal{X}_k/K_0)^\vee \oplus (H_{\mathrm{dR},c}^n(\mathcal{X}_K)/F^{i+1})^\vee \\ : (x_0 \otimes 1_K) = x_{\mathrm{dR}} \circ \mathrm{cosp}, x_0 \circ \phi_c = p^i x_0^\sigma \}. \end{aligned}$$

In cohomology, the Frobenius is an invertible; hence, the equation $x_0 \circ \phi_c = p^i x_0^\sigma$ is equivalent to $x_0 = p^i x_0^\sigma \phi_c^{-1}$, where the latter is the formula for the Frobenius of internal $\mathcal{H}om(H_{\mathrm{rig},c}^n(\mathcal{X}_k), K(-i))$ in the category of isocrystals (that is, modules with Frobenius). Hence, by Poincaré duality, we get $\phi((x_0^\vee)^\sigma) = p^{d-i} x_0^\vee$; $x^\vee \in H_{\mathrm{rig}}^{2d-n}(\mathcal{X}_k/K_0)$ is the cohomology class corresponding to x_0 , where ϕ is the Frobenius of $H_{\mathrm{rig}}^{2d-n}(\mathcal{X}_k/K_0)$.

(iii) The absolute homology is defined via the complex $R\Gamma_c(\mathcal{X})$, but it is not clear how to relate it to the dual of the absolute cohomology with compact support.

Corollary 5.4.7 (Gysin map). *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a proper morphism of smooth algebraic \mathcal{V} -schemes of relative dimension d and e , respectively. Then there is a canonical map*

$$f_* : H_{\mathrm{abs}}^n(\mathcal{X}, i) \rightarrow H_{\mathrm{abs}}^{n+2c}(\mathcal{Y}, i+c),$$

where $c = e - d$.

Proof. This is a direct consequence of the previous proposition and the functoriality of $R\Gamma_c(\mathcal{X})$ with respect to proper morphisms. \square

Remark 5.4.8. With the notation above, we get a morphism of spectral sequences

$$\begin{aligned} g : E_2^{p,q}(\mathcal{X}) := \mathrm{Hom}_{pHD}(\mathbb{K}, H^q(\mathcal{X})(i)[p]) \rightarrow E_2^{p,q+2c}(\mathcal{Y}) \\ := \mathrm{Hom}_{pHD}(\mathbb{K}, H^{q+2c}(\mathcal{Y})(i)[p]) \end{aligned}$$

compatible with f_* . The map g is induced by the Gysin morphism in de Rham and rigid cohomology.

We conclude by saying that it is natural to expect that the Gysin map for syntomic cohomology is compatible with the K -theory pushforward under the regulator defined by Besser. We plan to address this open problem in a future work.

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chiarbru@math.unipd.it

*Università degli Studi di Padova, Via Trieste 63,
35100 Padova, Italy*

ciccioni@math.unipd.it

*Università degli Studi di Padova, Via Trieste 63,
35100 Padova, Italy*

nicola.mazzari@math.u-bordeaux1.fr

*IMB, Université Bordeaux 1 351, cours de la Libération,
33405 Talence cedex, France*

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