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F-blowups of normal surface singularities

Nobuo Hara, Tadakazu Sawada and Takehiko Yasuda

We study F-blowups of non-F-regular normal surface singularities. Especially the cases of rational double points and simple elliptic singularities are treated in detail.

1. Introduction

The F-blowup introduced in [Yasuda 2012] is a canonical birational modification of a variety in positive characteristic. For a nonnegative integer e, the e-th F-blowup of a variety X is defined as the blowup at $F^e_*\mathcal{O}_X$, that is, the universal birational flattening of $F^e_*\mathcal{O}_X$. Here $F^e_*\mathcal{O}_X$ is the pushforward of the structure sheaf by the e-iterated Frobenius morphism. It turns out that the F-blowup of a quotient singularity has a connection with the G-Hilbert scheme [Toda and Yasuda 2009; Yasuda 2012]. However, the F-blowup has the advantage that it is canonically defined for arbitrary singularity in positive characteristic, whereas the G-Hilbert scheme is defined only for a quotient singularity. Actually, it is proved in [Yasuda 2012] that the e-th F-blowup of any curve singularity with $e \gg 0$ is normal, and hence resolves singularities in dimension one.

As is naturally expected, the F-blowup is also connected to F-singularities in positive characteristic such as F-pure and F-regular singularities. It is proved that the sequence of F-blowups for an F-pure singularity is monotone [Yasuda 2009] and that the e-th F-blowup of an F-regular surface singularity coincides with the minimal resolution for $e \gg 0$ [Hara 2012]. However, it is too much to ask for F-blowups of normal surface singularities to be the minimal resolution or even smooth in general. Actually, there exist (non-F-regular) rational double points whose F-blowups are singular [Hara and Sawada 2011].

Although some good aspects as well as pathologies of F-blowups have recently been discovered as above, their behavior is a mystery yet, even in dimension two. In this paper, we explore the behavior of F-blowups of certain normal surface

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singularities more in detail. We are mainly concerned with two classes of surface singularities, that is, non-F-regular rational double points (which exist only in characteristics up to five) and simple elliptic singularities. We will discuss F-blowups of these singularities, focusing on the normality, smoothness and stabilization of F-blowup sequences.

For this purpose, we do utilize not only the classical theory of surface singularities, but also computations with `Macaulay2` [Grayson and Stillman 2012], which are complementary to each other. The key to our computations is two `Macaulay2` functions that we will write down. Given a module, the first function computes an ideal such that the blowups at the ideal and module coincide, following Villamayor's description of such an ideal [2006]. Using this together with a built-in function to compute Rees algebras, one can explicitly compute a graded ring describing the blowup at a module. The second function we will write computes the Frobenius pushforward F_*M of a given module M. These functions enable us to investigate F-blowups in detail, especially for hypersurface surface singularities in characteristic two or three.

In the case of rational double points, one can apply general theory of rational surface singularities to show that F-blowups are normal and dominated by the minimal resolution. Then a version of McKay's correspondence [Artin and Verdier 1985] enables us to determine the e-th F-blowup by the direct sum decomposition of F^e_*M into indecomposable modules. For F-regular surface singularities $R = \mathcal{O}_{X,x}$, all indecomposable reflexive R-modules appear as a direct summand of F^e_*R with $e \gg 0$, so that the e-th F-blowup coincides with the minimal resolution [Hara and Sawada 2011; Hara 2012]. Contrary to this we have the following result for non-F-regular Frobenius sandwiches in characteristic $p \le 5$.

Theorem 1.1 (see [Hara and Sawada 2011, Example 4.8]). *Let (X, x) be a rational double point of type D^0_{2n} for $n \ge 2$, E^0_7, E^0_8 in $p = 2$, E^0_6, E^0_8 in $p = 3$ or E^0_8 in $p = 5$; see* [Artin 1977] *for the notation. Then for any $e \ge 1$, the e-th F-blowup* $\mathrm{FB}_e(X)$ *coincides with the normal surface obtained by contracting all but one exceptional curve on the minimal resolution $\widetilde{X}$. The unique exceptional curve on* $\mathrm{FB}_e(X)$ *is indicated by the solid circle in Theorem 3.5.*

We can analyze other types of non-F-regular rational double points by computer-aided calculation. In these cases, computations of the blowups at modules are again useful. For instance, one can see with such computation whether two obtained indecomposable modules are isomorphic. A particularly interesting result is that for $e \ge 2$, the e-th F-blowup of D^1_4- and D^1_5-singularities in characteristic two is the minimal resolution, though D^1_4- and D^1_5-singularities are not F-regular. In our computations so far, there is no other non-F-regular rational double point such that any of its F-blowups is the minimal resolution.

We will investigate F-blowups of simple elliptic singularities in detail as well. Since a simple elliptic singularity (X, x) is quasihomogeneous in general, its minimal resolution $\widetilde{X}$ has the same structure as the conormal bundle over the elliptic exceptional curve E, which is identified with the negative section. We can use this fact to determine the structure of the F-blowups up to normalization, which turns out to be different according to the self-intersection number E^2 and whether the singularity (X, x) is F-pure or not. We summarize the results obtained in Theorems 4.5, 4.7, 4.13 and Proposition 4.18 in the following.

Theorem 1.2. *Let (X, x) be a simple elliptic singularity in characteristic $p > 0$ with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$. Let $\widetilde{\mathrm{FB}}_e(X)$ be the normalization of the e-th F-blowup $\mathrm{FB}_e(X)$ of (X, x) for any $e \geq 1$.*

(1) *If (X, x) is F-pure with $E^2 = -1$, then $\widetilde{\mathrm{FB}}_e(X)$ coincides with the blowup of $\widetilde{X}$ at $p^e - 1$ nontrivial p^e-torsion points on E.*

(2) *If (X, x) is not F-pure with $E^2 = -1$, then $\widetilde{\mathrm{FB}}_e(X)$ coincides with the blowup of $\widetilde{X}$ at an ideal supported at a point $P_0 \in E$ with local expression (t, u^{p^e-1}), where t and u are local coordinates at $P_0 \in \widetilde{X}$.*

(3) *If $E^2 \leq -2$ and $-E^2$ is not a power of p, then $\widetilde{\mathrm{FB}}_e(X) \cong \widetilde{X}$ for all $e \geq 1$. Moreover, if (X, x) is F-pure and $E^2 \leq -3$, then $\mathrm{FB}_e(X) \cong \widetilde{X}$.*

We cannot determine whether or not an F-blowup is normal in general, but we see that an F-blowup is nonnormal in some cases with `Macaulay2` computation. The theorem above tells us that an F-blowup coincides with the minimal resolution in some cases, but in general, F-blowups of simple elliptic singularities behave badly: They are nonnormal, not dominated by the minimal resolution and the sequence of F-blowups does not stabilize. The study of F-blowups for simple elliptic singularities will be pushed further and completed in [Hara 2013].

2. Preliminaries

2a. ***Blowups at modules.*** Let X be a Noetherian integral scheme and $\mathcal{M}$ a coherent sheaf on X. For a modification $f : Y \to X$, we denote the torsion-free pullback $(f^*\mathcal{M})/\mathrm{tors}$ by $f^\star\mathcal{M}$, where tors denotes the subsheaf of torsions.

Definition 2.1. A modification $f : Y \to X$ is called a *flattening* of $\mathcal{M}$ if $f^\star\mathcal{M}$ is flat, or equivalently locally free. A flattening f is said to be *universal* if every flattening $g : Z \to X$ of $\mathcal{M}$ factors as

$$g : Z \to Y \xrightarrow{f} X.$$

(The universal flattening exists and is unique. It can be constructed as a subscheme of a Quot scheme. See for instance [Oneto and Zatini 1991; Villamayor U. 2006].) The universal flattening is also called the *blowup of X at $\mathcal{M}$* and denoted by $\mathrm{Bl}_{\mathcal{M}}(X)$.

The following are basic properties of the blowup at a module, which directly follow from the definition:

(1) The modification $\mathrm{Bl}_{\mathcal{M}}(X) \to X$ is an isomorphism exactly over the locus where $\mathcal{M}$ is flat.

(2) If $\mathcal{N} \subset \mathcal{M}$ is a torsion subsheaf, then $\mathrm{Bl}_{\mathcal{M}}(X) = \mathrm{Bl}_{\mathcal{M}/\mathcal{N}}(X)$.

(3) If $\mathcal{M}$ is an ideal sheaf, then the blowup at $\mathcal{M}$ defined above coincides with the usual blowup with the center $\mathcal{M}$.

The following are examples of blowups at modules. Therefore one can compute them in the method explained below.

Example 2.2. If X is an algebraic variety over a field k, then its Nash blowup is the blowup at $\Omega_{X/k}$, the sheaf of differentials. The higher version of the Nash blowup is also an example of the blowup at a module; see [Yasuda 2007].

Example 2.3. Let Y be a quasiprojective algebraic variety, G a finite group of automorphisms of Y and $X := Y/G$ the quotient variety. Then the G-Hilbert scheme $\mathrm{Hilb}^G(Y)$ is defined to be the closure of the set of free G-orbits in the Hilbert scheme of Y; see [Ito and Nakamura 1996]. One can show that $\mathrm{Hilb}^G(Y)$ is isomorphic to the blowup at $\pi_*\mathcal{O}_Y$, where $\pi : Y \to X$ is the quotient map.

Let r be the rank of $\mathcal{M}$, K the function field of X and fix an isomorphism $\bigwedge^r \mathcal{M} \otimes K \cong K$. Then define a fractional ideal sheaf

$$\mathcal{I}_{\mathcal{M}} := \mathrm{Im}\big(\textstyle\bigwedge^r \mathcal{M} \to \bigwedge^r \mathcal{M} \otimes K \cong K\big).$$

Proposition 2.4 (see [Oneto and Zatini 1991; Villamayor U. 2006]). *The blowup at $\mathcal{M}$ is isomorphic to the blowup at $\mathcal{I}_{\mathcal{M}}$,*

$$\mathrm{Bl}_{\mathcal{I}_{\mathcal{M}}}(X) = \mathrm{Proj}_X\Big(\bigoplus_{n \geq 0} \mathcal{I}_{\mathcal{M}}^n\Big).$$

Note that although $\mathcal{I}_{\mathcal{M}}$ depends on the choice of the isomorphism $\bigwedge^r \mathcal{M} \otimes K \cong K$, the isomorphism class of $\mathcal{I}_{\mathcal{M}}$ and so $\mathrm{Bl}_{\mathcal{I}}(X)$ are independent of it.

We will now recall Villamayor's method [2006] for computing $\mathcal{I}_{\mathcal{M}}$ in the affine case. Suppose that $X = \mathrm{Spec}\, R$. Abusing the notation, we identify the sheaf $\mathcal{M}$ with the corresponding R-module M, the fractional ideal sheaf $\mathcal{I}_{\mathcal{M}}$ with the fractional ideal $I_M \subset K$, and so forth. Let

$$R^m \xrightarrow{A} R^n \to M \to 0$$

be a presentation of M given by an $n \times m$ matrix A. Here and hereafter we think of elements of free modules as column vectors and the map $A : R^m \to R^n$ is given by left multiplication with A, that is, $v \mapsto Av$. We call A a *presentation matrix* of M. Then there exist $n-r$ columns of A such that if A' denotes the submatrix of A formed by these columns, then

$$M' := \mathrm{Coker}(R^{n-r} \xrightarrow{A'} R^n)$$

has rank r. Then M is a quotient of M' by some torsion submodule of M'. Therefore the blowups at M and M' are equal.

Proposition 2.5 [Villamayor U. 2006]. *The ideal generated by $(n-r)$-minors of A', which is by definition the r-th Fitting ideal of M', is equal to I_M for a suitable choice of isomorphism $\bigwedge^r M \otimes K \cong K$.*

The computation of this ideal is implemented in `Macaulay2` as

```
villamayorIdeal = M -> (
  r := rank M;
  P := presentation M;
  s := rank source P;
  t := rank target P;
  I := {};
  for j to s-1 when #I < t-r do (
    J := append(I,j);
    if rank coker P_J == t - #J then I = J;
  );
  fittingIdeal(r,coker P_I);
);
```

Once the ideal I_M was computed, then the blowup at M is computed as the projective spectrum of the Rees algebra of the ideal:

$$\mathrm{Bl}_M(X) = \mathrm{Proj}\, R[I_M t], \quad R[I_M t] := \bigoplus_{i \geq 0} I_M^i t^i \subset R[t].$$

The computation of Rees algebras has been already implemented in `Macaulay2` as `reesAlgebra`.

The computation of blowups at modules is useful for studying modules themselves. For instance, one can see that two given modules are not isomorphic if the associated blowups are not isomorphic.

2b. *F-blowups.* Suppose now that X is a Noetherian integral scheme of characteristic $p > 0$ and that its (absolute) Frobenius morphism $F : X \to X$ is finite.

Definition 2.6 [Yasuda 2012]. For a nonnegative integer e, we define the *e-th F-blowup* of X to be the blowup of X at $F_*^e \mathcal{O}_X$ and denote it by $\mathrm{FB}_e(X)$.

From [Kunz 1969], if $e > 0$, then the flat locus of $F^e_* \mathcal{O}_X$ coincides with the regular locus of X. Therefore the e-th F-blowup is an isomorphism exactly over the regular locus.

If X is an algebraic variety over an algebraically closed field k, then there is a more moduli-theoretic construction of F-blowups, which was actually the original definition of F-blowups in [Yasuda 2012]: The e-th F-blowup is isomorphic (over Z) to the closure of the set

$$\{[(F^e)^{-1}(x)] \mid \text{nonsingular point } x \in X(k)\}$$

in the Hilbert scheme of zero-dimensional subschemes. Here $(F^e)^{-1}(x)$ is the scheme-theoretic inverse image and a closed subscheme of X with length $p^{e \dim X}$, and $[(F^e)^{-1}(x)]$ is the corresponding point in the Hilbert scheme.

2c. ***Computing the Frobenius pushforward.*** Let us now suppose that X is affine, say $X = \operatorname{Spec} R$. In order to compute F-blowups of X along the lines explained above, we need to first compute a presentation of $F^e_* R$. For later use, we will explain more generally how to compute $F^e_* M$ for any finitely generated R-module M in the case where R is finitely generated over the prime field $\mathbb{F}_p$.

2c1. *The case of a polynomial ring.* Set $S = \mathbb{F}_p[x_1, \dots, x_n]$ and $q = p^e$. A monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ defines an S-linear map

$$\mu_{x^a} : S \to S, \quad f \mapsto x^a f.$$

Then we reinterpret this map according to another S-module structure on S by $g \cdot f := g^q f$. We denote this new S-module by S', which is a free S-module of rank q^n and nothing but $F^e_* S$. We also denote the map μ_{x^a} regarded as an endomorphism of S' by μ'_{x^a}, which is nothing but $F^e_* \mu_{x^a}$.

Let $\Lambda := \{0, 1, \dots, q-1\}^n$. Then q^n monomials x^b for $b \in \Lambda$ form a standard basis of S'. For such a monomial x^b, we have

$$\mu_{x^a}(x^b) = x^{a+b} = x^{q((a+b) \div q)} x^{(a+b) \% q}.$$

Here $\div q$ and $\% q$ respectively denote the quotient and the remainder by the component-wise division by q. We rewrite it as

$$\mu'_{x^a}(x^b) = x^{(a+b) \div q} \cdot x^{(a+b) \% q}.$$

Thus we obtain:

Lemma 2.7. *The defining matrix, $U(a, e) = (u_{ij})_{i, j \in \Lambda}$, of μ'_{x^a} with respect to the standard basis is given by*

$$u_{ij} = \begin{cases} x^{(a+j) \div q} & i = (a+j) \% q, \\ 0 & \text{otherwise}. \end{cases}$$

Then for a polynomial $f = \sum_a c_a x^a \in S$, if $\mu_f : S \to S$ denotes the multiplication with f, then $\mu'_f = F^e_* \mu_f$ is defined by the matrix

$$U(f, e) := \sum_a c_a \cdot U(a, e).$$

Note that since the coefficient field is $\mathbb{F}_p$ and the Frobenius map of $\mathbb{F}_p$ is the identity map, we do not have to change the coefficients c_a.

Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{l1} & \cdots & a_{lm} \end{pmatrix}$$

be an $l \times m$ matrix with entries in S, which defines an S-linear map $S^m \to S^l$ denoted again by A. Then the $F^e_* A : (S')^{\oplus m} \to (S')^{\oplus l}$ is given by $q^n l \times q^n m$ matrix

$$U(A, e) = \begin{pmatrix} U(a_{11}, e) & \cdots & U(a_{1m}, e) \\ \vdots & \ddots & \vdots \\ U(a_{l1}, e) & \cdots & U(a_{lm}, e) \end{pmatrix}.$$

Therefore:

Proposition 2.8. *If A is a presentation matrix of an S-module M, then $U(A, e)$ is a presentation matrix of $F^e_* M$.*

2c2. *The general case.* Suppose that R is the quotient ring S/I, $I = (f_1, \dots, f_l)$, and M is a finitely generated R-module. Then we first have to compute a presentation of M as an S-module. Let A be a matrix with entries in S and let $\bar{A}$ be the matrix with entries in R induced from A. Suppose that $\bar{A}$ is a presentation matrix of M:

$$R^m \xrightarrow{\bar{A}} R^n \to M \to 0.$$

Let $\tilde{M}$ be the S-module with the presentation matrix A:

$$S^m \xrightarrow{A} S^n \to \tilde{M} \to 0.$$

Then $M = R \otimes_S \tilde{M}$. The S-module R has a standard presentation

$$S^l \xrightarrow{(f_1, \dots, f_l)} S \to R \to 0.$$

Now a presentation of M as an S-module can be computed from those of R and $\tilde{M}$.

If B is a presentation matrix of M as an S-module, then $U(B, e)$ is one of $F^e_* M$ as an S-module. If $\overline{U(B, e)}$ denotes the matrix with entries in R induced from $U(B, e)$, then $\overline{U(B, e)}$ is a presentation matrix of $F^e_* M$ as an R-module.

2c3. *Implementation in* `Macaulay2`. The following `Macaulay2` function returns the pushforward $F^e_* M$ of the given module M, following the recipe explained above:

```
frobeniusPushForward = (M, e) -> (
  R := ring M;
  p := char R;
  assert(p > 0); q := p^e;
  I := ideal R;
  l := numgens I;
  B := gens ideal R;
  S := ambient R;
  n := numgens S;
  qSequence := i ->
    apply(0..n-1, j -> (i % q^(n-j)) // q^(n-j-1));
  toNumber := i -> sum(n, j ->  i_j * q^(n-j-1) );
  qQuotient := i -> apply(i, j -> j // q);
  qRemainder := i -> apply(i, j -> j % q);
  monoToMatrix := m ->
    (coefficients m)_1_(0,0)
      * map(S^(q^n),S^(q^n),
          (i,j) -> (e = (toList qSequence i) + (exponents m)_0;
     if(toNumber qRemainder e) == j
       then S_(toList qQuotient e)
       else 0));
  polyToMatrix := f ->
    if f == 0_S
      then map(S^(q^n),S^(q^n),0_S)
      else sum(terms f, i -> monoToMatrix i);
  basisToMatrix := b ->
    fold((i, j)->(i | j),
      apply((flatten entries b), polyToMatrix));
  matrixToMatrix := m ->
    fold((i, j)->(i || j),
      apply(apply(entries m, i -> matrix{i}), basisToMatrix));
  ROverS := coker map(S^1,S^l, entries B);
  PresenOverR := presentation minimalPresentation M;
  PresenOverS := presentation minimalPresentation(
coker(sub(PresenOverR,S))**ROverS);
  L := matrixToMatrix PresenOverS;
  minimalPresentation coker sub(L,R)
);
```

Note that in the computations with `Macaulay2`, columns and rows of matrices should be indexed by single indices rather than multiindices. For this aim, the inner functions `qSequence` and `toNumber` above define bijections between the sets $\{0, 1, \dots, q^n - 1\}$ and Λ that are inverses to each other.

Note that one can compute $F^e_* M$ also with the built-in function `PushForward` in the case where the ring and the module are (weighted) homogeneous.

2d. ***Computing the singular and nonnormal loci of a blowup.*** We often would like to know if a given blowup is smooth or normal, or to know where the singular locus or the nonnormal locus is. One way to compute the singular locus of $\mathrm{Bl}_I(X)$ is to compute the singular locus of $\operatorname{Spec} R[It]$. For instance, suppose that we have an expression of $R[It]$ as a quotient of a polynomial ring over R,

$$R[It] = R[t_1, \dots, t_n]/J.$$

Then $\mathrm{Bl}_I(X)$ is smooth if and only if the singular locus of $\operatorname{Spec} R[It]$ is contained in the closed subset $V(t_1, \dots, t_n) \subset \operatorname{Spec} R[It]$. This method is useful when the Rees algebra is relatively simple. Otherwise, the computation may not finish in a reasonable time.

In that case, an alternative way is to compute the singular loci of affine charts. With the notation above, the blowup $\mathrm{Bl}_I(X)$ is covered by n affine charts corresponding to the variables $t_1, \dots, t_n$. Their coordinate rings are

$$R[t_1, \dots, t_n]/(J + (t_i - 1)) \quad \text{for } i = 1, \dots, n.$$

These rings are likely to become simpler than $R[It]$ and easier to compute the singular loci. Computation of these rings is implemented as follows:

```
affineCharts = S -> (
  T := (flattenRing S)_0;
  varsOfS := apply(flatten entries vars S, i->sub(i, T));
  apply(varsOfS, i -> minimalPresentation(T / ideal(i - 1)))
);
```

The same method can apply to find the nonnormal locus.

2e. ***Embedding F-blowups into the Grassmannian and the projective space.*** As already mentioned above, F-blowups are constructed as a subscheme of the Grassmannian. Then further composing with the Plücker embedding, we obtain an embedding into a projective space over X.

To describe this embedding, let $X = \operatorname{Spec} R$ be of dimension n, let K be the function field of X, and let the fractional ideal $I = \operatorname{Im}(\bigwedge^{p^n} R^{1/p^e} \to K)$ be generated by $m+1$ elements $s_0, \dots, s_m$. Then, being the blowup of X at I, the e-th F-blowup $\mathrm{FB}_e(X)$ of X is embedded into the projective space $\mathbb{P}^m_X$ over X.

Suppose now that $f\colon Y \to X$ is any flattening of $R^{1/p^e} \cong F^e_* \mathcal{O}_X$. Then we have a surjection $\mathcal{O}_Y^{\oplus m+1} \to f^\star I = \det f^\star R^{1/p^e}$ induced by $s_0, \dots, s_m$, which gives rise to a morphism $\Phi_e\colon Y \to \mathbb{P}^m_X$ such that $\Phi_e^* \mathcal{O}_{\mathbb{P}}(1) \cong \det f^\star R^{1/p^e}$, and the image $\Phi_e(Y)$ of this morphism is nothing but $\mathrm{FB}_e(X) = \mathrm{Bl}_I(X)$.

In dimension two where the existence of resolution of singularities is established in arbitrary characteristic, we can study F-blowups downwards from a resolution that flattens the $\mathcal{O}_X$-module $F^e_* \mathcal{O}_X \cong \mathcal{O}_X^{1/p^e}$. The following is an immediate consequence of the observation above.

Proposition 2.9. *Let X be a surface over k and let $f\colon \widetilde{X} \to X$ be a resolution with irreducible exceptional curves $E_1, \dots, E_s$. Suppose that $f^\star \mathcal{O}_X^{1/p^e}$ is flat, so that we have a birational morphism $\Phi_e\colon \widetilde{X} \to \mathrm{FB}_e(X)$. Then $\Phi_e(E_i)$ is a curve on $\mathrm{FB}_e(X)$ if $c_1(f^\star \mathcal{O}_X^{1/p^e})E_i > 0$, and E_i contracts to a point on $\mathrm{FB}_e(X)$ if $c_1(f^\star \mathcal{O}_X^{1/p^e})E_i = 0$.*

3. F-blowups of rational surface singularities

Throughout this section we work under the following notation:

k an algebraically closed field of characteristic $p > 0$,

(X, x) a rational surface singularity defined over k with local ring $R = \mathcal{O}_{X,x}$,

$f\colon \widetilde{X} \to X$ the minimal resolution of (X, x) with $\mathrm{Exc}(f) = \bigcup_{i=1}^s E_i$.

The situation is quite simple in this case because of the following fact [Artin and Verdier 1985]: If M is a reflexive $\mathcal{O}_X$-module,[1] then its torsion-free pullback $\widetilde{M} = f^\star M = f^* M/\,$torsion is an f-generated locally free $\mathcal{O}_{\widetilde{X}}$-module such that $f_* \widetilde{M} = M$ and $R^1 f_* \widetilde{M} = 0$. Note that this vanishing of the higher direct image is an easy consequence of the rationality of the singularity (X, x) and the f-generation of $\widetilde{M}$, which gives rise to a surjection $\mathcal{O}_{\widetilde{X}}^{\oplus n} \twoheadrightarrow \widetilde{M}$.

Lemma 3.1 [Hara 2012, Lemma 1.8]. *If M is a reflexive $\mathcal{O}_X$-module of rank r, then the natural map $\bigwedge^r M \to f_*(\det \widetilde{M})$ is surjective.*

Proposition 3.2. *The e-th F-blowup $\mathrm{FB}_e(X)$ of a rational surface singularity (X, x) is dominated by the minimal resolution $\widetilde{X}$ and has only rational singularities for all $e \geq 0$.*

Proof. Because $M := R^{1/p^e}$ is a reflexive R-module, its torsion-free pullback $\widetilde{M} = f^\star R^{1/p^e}$ to $\widetilde{X}$ is flat, so that the minimal resolution $f\colon \widetilde{X} \to X$ factors through the universal flattening $\mathrm{FB}_e(X)$ of R^{1/p^e}. On the other hand, the ideal $I = I_M$ for $M = R^{1/p^e}$ is $I = H^0(\widetilde{X}, \det \widetilde{M})$ by Lemma 3.1, so that we can take I to be an integrally closed ideal in R, or *complete* ideal in the sense of Lipman [1969]. Then the Rees algebra $R[It]$ is normal by [ibid., Proposition 8.1], so

[1] We always assume that M is a finitely generated $\mathcal{O}_X$-module.

$\mathrm{FB}_e(X) = \mathrm{Proj}\, R[It]$ is normal. It then follows from [Artin 1962] that $\mathrm{FB}_e(X)$ has only rational singularities. □

Corollary 3.3. *Let (X, x) be a rational surface singularity over k.*

(1) *For any $e \geq 0$, the e-th F-blowup $\mathrm{FB}_e(X)$ is obtained by contracting some of the exceptional curves $E_1, \dots, E_s$ on the minimal resolution $\widetilde{X}$ to normal points with at most rational singularities.*

(2) *The minimal resolution $\widetilde{X}$ of (X, x) is obtained by finitely many iteration of F-blowups. More explicitly, for any sequence of positive integers $e_1, \dots, e_s$, we have $\widetilde{X} = \mathrm{FB}_{e_s}(\mathrm{FB}_{e_{s-1}}(\cdots \mathrm{FB}_{e_2}(\mathrm{FB}_{e_1}(X))\cdots))$.*

The behavior of F-blowups is especially nice for F-regular surface singularities. Namely, the e-th F-blowup of any F-regular surface singularity is the minimal resolution for $e \gg 0$ [Hara 2012]. We next consider F-blowups of non-F-regular rational double points more in detail. In this case we can use the classification of rational double points in characteristic $p > 0$ [Artin 1977], as well as the following:

Lemma 3.4 [Artin and Verdier 1985]. *Let (X, x) be a rational double point and let $Z_0 = \sum_{i=1}^{s} r_i E_i$ be the fundamental cycle on the minimal resolution $\widetilde{X}$. Then there is a one-to-one correspondence between the exceptional curves E_i of f and the isomorphism classes of nontrivial indecomposable reflexive $\mathcal{O}_X$-modules M_i, satisfying the following properties.*

(1) $\mathrm{rank}\, M_i = r_i$ *for* $1 \leq i \leq s$.

(2) $c_1(\widetilde{M_i})E_j = \delta_{ij}$ *for* $1 \leq i, j \leq s$.

In what follows, we use the notation of [Artin 1977] for rational double points in positive characteristic.

Among non-F-regular rational double points, Frobenius sandwiches have particularly easy to analyze F-blowups. Let X be a Frobenius sandwich of a smooth surface S, that is, the Frobenius morphism of S factors as $F\colon S \xrightarrow{\pi} X \to S$. Then F-blowups of X are also the universal flattening of the reflexive $\mathcal{O}_X$-module $\pi_*\mathcal{O}_S$ [Hara and Sawada 2011, Proposition 4.3]. Thanks to this observation, we can study F-blowups of the Frobenius sandwich X via $\pi_*\mathcal{O}_S$ instead of $F^e_*\mathcal{O}_X$. For example, we find whether the irreducible exceptional curve E_i appears on $\mathrm{FB}_e(X)$ or not by evaluating the intersection number $c_1(f^\star(\pi_*\mathcal{O}_S))E_i$ in Proposition 2.9.

3a. D^0_{2n}-*singularities.* Here we consider a D^0_{2n}-singularity for $n \geq 2$ in $p = 2$ as a Frobenius sandwich. Let $\mathbb{A}^2 = \mathrm{Spec}\, k[x, y]$ and $\pi : \mathbb{A}^2 \to X = \mathbb{A}^2/\delta$ the quotient map by a vector field $\delta = (x^2 + nxy^{n-1})\partial/\partial x + y^n \partial/\partial y \in \mathrm{Der}_k\, \mathcal{O}_{\mathbb{A}^2}$. Here

$$\mathcal{O}_X = k[x, y]^\delta = k[x^2, y^2, x^2y + xy^n] \cong k[X, Y, Z]/(Z^2 + X^2Y + XY^n)$$

and X has a D^0_{2n}-singularity. Then $R = \mathcal{O}_X = k[X, Y, Z]/(Z^2 + X^2Y + XY^n)$ is a graded ring with $\deg X = 2(n-1)$, $\deg Y = 2$ and $\deg Z = 2n-1$. The $4(n-1)$-st Veronese ring of R is

$$R^{(4(n-1))} = k[X^2, Y^{2(n-1)}, XY^{n-1}] \cong k[u, v, w]/(w^2 - uv).$$

Set $x_0 = u^{1/2} = X = x^2$ and $x_1 = v^{1/2} = Y^{n-1} = y^{2(n-1)}$. Then

$$R^{(4(n-1))} \cong k[x_0^2, x_1^2, x_0x_1] = k[x_0, x_1]^{(2)},$$

so that $\operatorname{Proj} R \cong \mathbb{P}^1$ with homogeneous coordinates $(x_0 : x_1) = (x^2 : y^{2(n-1)})$. Let $s = x_1/x_0 = y^{2(n-1)}/x^2$ be the affine coordinate of $U_0 = D_+(x_0) \subset \operatorname{Proj} R \cong \mathbb{P}^1$ and pick a homogeneous element $t = Z/X = y(x + y^{n-1})/x \in R$ of degree 1. Since

$$t^{2(n-1)} = \frac{x_1(x_1 - x_0)^{n-1}}{x_0^{n-1}},$$

the $\mathbb{Q}$-divisor

$$D = \frac{1}{2(n-1)}(0) + \tfrac{1}{2}(1) - \tfrac{1}{2}(\infty)$$

on $\mathbb{P}^1$ gives $R = \bigoplus_{n\geq 0} H^0(\mathbb{P}^1, nD)t^n$ (the Pinkham–Demazure construction).

Let $g\colon X' \to X = \operatorname{Spec} R$ be the weighted blowup with respect to the weight $(2(n-1), 2, 2n-1)$. Then $X' \cong \operatorname{Spec}_{\mathbb{P}^1}(\bigoplus_{n\geq 0} \mathcal{O}_{\mathbb{P}^1}(nD)t^n)$ admits an affine morphism $\rho\colon X' \to \mathbb{P}^1$ that is an $\mathbb{A}^1$-bundle over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and the exceptional curve of g is the negative section $E \cong \mathbb{P}^1$ of ρ. Let $X'_0 = \rho^{-1}U_0$. Then

$$\mathcal{O}_{X'_0} = k\Big[s, t, \frac{t^2}{s-1}, \frac{t^3}{s-1}, \dots, \frac{t^{2(n-1)-2}}{(s-1)^{n-2}}, \frac{t^{2(n-1)-1}}{(s-1)^{n-2}}, \frac{t^{2(n-1)}}{s(s-1)^{n-1}}\Big]$$

and X' has an A_{2n-3}-singularity on $E|_{X'_0} \cong \operatorname{Spec} k[s]$ at $s = 0$.

To resolve the A_{2n-3}-singularity, we may replace

$$X'_0 = \rho^{-1}U_0 \quad \text{by} \quad V = \rho^{-1}(U_0 \setminus \{1\}).$$

The affine coordinate ring of V is

$$\mathcal{O}_V = \mathcal{O}_{X'_0}\Big[\frac{1}{s-1}\Big] = k[s, t, t^{2(n-1)}/s]_{s-1}.$$

The minimal resolution $h\colon \widetilde{V} \to V$ of V is given by

$$\widetilde{V} = \bigcup_{i=1}^{2(n-1)} \widetilde{V}_i, \quad \text{where } \widetilde{V}_i = \operatorname{Spec} k[s/t^{i-1}, t^i/s]_{s-1}.$$

Let $\widetilde{E} \cong \mathbb{P}^1$ be the h-exceptional curve lying on $\widetilde{V}_{n-2} \cup \widetilde{V}_{n-1}$:

$$\widetilde{E} = \operatorname{Spec} k[t^{n-2}/s] \cup \operatorname{Spec} k[s/t^{n-2}] \subset \widetilde{V}_{n-2} \cup \widetilde{V}_{n-1}.$$

Now suppose that n is even; $n = 2k$ with $k \geq 1$. Let $\varphi = t^{n-2}/s(s-1)^{k-1}$ and $\psi = t^{n-1}/s(s-1)^{k-1}$. Then $\varphi, \psi \in \mathbb{O}_{\widetilde{V}_{n-2}}$ and $x = \psi + y\varphi$. Thus

$$(h \circ g)^\star(\pi_* \mathbb{O}_{\mathbb{A}^2})|_{\widetilde{V}_{n-2}} = \mathrm{Im}(\mathbb{O}_{\widetilde{V}_{n-2}} \otimes_{\mathbb{O}_X} \mathbb{O}_{\mathbb{A}^2} \to k(\mathbb{A}^2)) = k[s/t^{n-3}, t^{n-2}/s, x, y]_{s-1}$$

is a free $\mathbb{O}_{\widetilde{V}_{n-2}}$-module with basis $1, y$. Similarly it follows that $(h \circ g)^\star(\pi_* \mathbb{O}_{\mathbb{A}^2})|_{\widetilde{V}_{n-1}}$ is a free $\mathbb{O}_{\widetilde{V}_{n-1}}$-module with basis $1, x$. The transition matrix of the two bases on $\widetilde{V}_{n-2} \cap \widetilde{V}_{n-1}$ is given by

$$(1\ x) = (1\ y) \begin{pmatrix} 1 & t^{n-1}/s(s-1)^{k-1} \\ 0 & t^{n-2}/s(s-1)^{k-1} \end{pmatrix}.$$

Since $s - 1$ is a unit on V, the intersection number of $L = c_1((h \circ g)^\star(\pi_* \mathbb{O}_{\mathbb{A}^2}))$ with $\widetilde{E}$ is $L\widetilde{E} = 1$. In light of Lemma 3.4, this means that the reflexive $\mathbb{O}_X$-module $\pi_* \mathbb{O}_{\mathbb{A}^2}$ of rank 2 is the indecomposable one corresponding to $\widetilde{E}$, which is identified with the exceptional curve E_{n+1} on the minimal resolution $\widetilde{X}$ indicated in the figure below:

$$\begin{array}{c} \quad\ \ E_1 \\ \quad\ \ | \\ E_2 \text{ — } E_3 \text{ — } E_4 - \cdots - E_{n+1} - \cdots - E_{2n} \end{array}$$

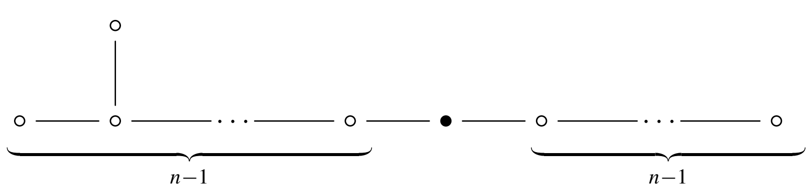
In the case where $n = 2k + 1$ with $k \geq 1$, we obtain the same conclusion that $c_1((h \circ g)^\star(\pi_* \mathbb{O}_{\mathbb{A}^2})) \cdot E_i = \delta_{i,n+1}$.

Thus we conclude that for all $e \geq 1$, the e-th F-blowup $\mathrm{FB}_e(X)$ coincides with the normal surface obtained by contracting all exceptional curves on $\widetilde{X}$ except E_{n+1}.

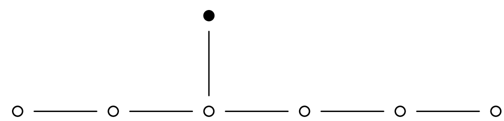
Putting the result above together with [Hara and Sawada 2011, Example 4.8], we obtain the following.

Theorem 3.5. *Let (X, x) be a rational double point of type D^0_{2n} for $n \geq 2$, E^0_7, E^0_8 in $p = 2$, E^0_6, E^0_8 in $p = 3$ or E^0_8 in $p = 5$. Then for any $e \geq 1$, the e-th F-blowup* $\mathrm{FB}_e(X)$ *coincides with the normal surface obtained by contracting the exceptional curves on the minimal resolution $\widetilde{X}$ corresponding to the blank circles in the figure below*:

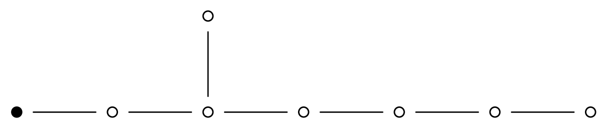
(1) *D^0_{2n}-singularity for $n \geq 2$ in $p = 2$:*



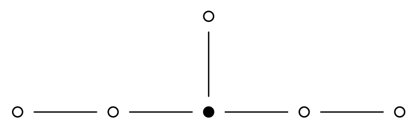
(2) *E_7^0-singularity in $p=2$:*



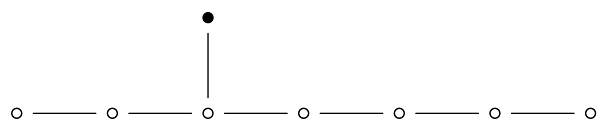
(3) *E_8^0-singularity in $p=2$:*



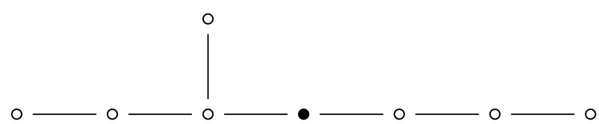
(4) *E_6^0-singularity in $p=3$:*



(5) *E_8^0-singularity in $p=3$:*



(6) *E_8^0-singularity in $p=5$:*



We want to emphasize that all rational double points listed in Theorem 3.5 are non-F-regular Frobenius sandwiches[2] and their F-blowups $\mathrm{FB}_e(X)$ with $e \geq 1$ have only a single exceptional curve corresponding to the solid circle. In particular, their F-blowups do not coincide with the minimal resolution.

We are also able to apply `Macaulay2` to study F-blowups of a few non-F-regular rational double points that are not supposed to be Frobenius sandwiches.

3b. *D_4^1- and D_5^1-singularities in $p = 2$.* First we consider the case of a D_4^1-singularity in $p=2$: Let $X = \operatorname{Spec} R$ with $R = k[x, y, z]/(z^2 + x^2y + xy^2 + xyz)$. Using the `Macaulay2` function `frobeniusPushForward` in Section 2c3, we see that the presentation matrix of F_*R is equivalent to

$$\begin{pmatrix} z & x+y+z \\ xy & z \end{pmatrix} \oplus \begin{pmatrix} z & y \\ x(x+y+z) & z \end{pmatrix} \oplus \begin{pmatrix} z & y(x+y+z) \\ x & z \end{pmatrix} \oplus 0,$$

where 0 is the zero matrix of size 1. Then the cokernel of each matrix of size 2 defines a nontrivial reflexive R-module of rank 1 and those reflexive R-modules are different from each other. Thus $\mathrm{FB}_1(X)$ coincides with the normal surface obtained

[2] We expect that all non-F-regular Frobenius sandwich rational double points are exhausted in Theorem 3.5, although we have not proved it yet. On the other hand, any F-regular Frobenius sandwich double point is an A_{p^e-1}-singularity and its e-th F-blowup is the minimal resolution for $e \gg 0$; see [Hara and Sawada 2011; Yasuda 2012].

by contracting the exceptional curve E_1 on the minimal resolution $\widetilde{X}$ indicated in the figure below:

$$\begin{array}{ccccc} & & E_3 & & \\ & & | & & \\ E_2 & — & E_1 & — & E_4 \end{array}$$

Furthermore, we see that the reflexive R-module corresponding to the central curve E_1 appears as a direct summand of the Frobenius pushforward of each nontrivial rank 1 reflexive module corresponding to E_i with $i = 2, 3, 4$. Thus $\mathrm{FB}_e(X)$ is the minimal resolution for $e \geq 2$, since the D_4^1-singularity is F-pure. A similar result holds for the case of a D_5^1-singularity. Note that D_4^1- and D_5^1-singularities are not F-regular.

Remark 3.6. The D_4^1-singularity in $p = 2$ is a *wild* quotient singularity, that is, there exists a group G of order 2 acting on $Y = \operatorname{Spec} k[[x, y]]$ such that the quotient $X = Y/G$ has the D_4^1-singularity. Although F-blowups of a *tame* quotient singularity are always dominated by the G-Hilbert scheme [Yasuda 2012], this example shows that the same does not hold for wild quotients. Let $R = k[[x, y]]^G \subset S = k[[x, y]]$ be the invariant subring. Then S is an R-module of rank 2. Thus the blowup of X at the R-module S, which coincides with the G-Hilbert scheme $\mathrm{Hilb}^G(Y)$, has at most two irreducible exceptional curves. On the other hand, the F-blowups $\mathrm{FB}_e(X)$ of the D_4^1-singularity have more than three irreducible exceptional curves. Hence the e-th F-blowup $\mathrm{FB}_e(X)$ of the D_4^1-singularity is not dominated by the G-Hilbert scheme $\mathrm{Hilb}^G(Y)$ for all $e \geq 1$.

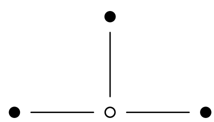
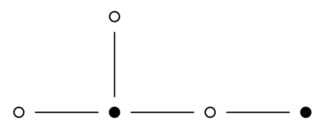
3c. *E_6^0-singularity in $p = 2$.* Let $R = k[x, y, z]/(z^2 + x^3 + y^2 z)$ and $X = \operatorname{Spec} R$. Then X has an E_6^0-singularity in characteristic $p = 2$. Write

$$A_1 = \begin{pmatrix} z & y & x & 0 \\ yz & z & 0 & x \\ x^2 & 0 & z & y \\ 0 & x^2 & yz & z \end{pmatrix}, \quad A_2 = \begin{pmatrix} x & y^2+z & y & 0 \\ z & x^2 & 0 & xy \\ 0 & 0 & x & y^2+z \\ 0 & 0 & z & x^2 \end{pmatrix} \quad \text{and} \quad A_3 = {}^tA_2.$$

Then their cokernels define nontrivial reflexive R-modules of rank 2 and those R-modules are different from each other. Now we see that presentation matrices of $F_* R$ and $F_*^2 R$ are equivalent to $A_1^{\oplus 2}$ and $A_1^{\oplus 4} \oplus A_2^{\oplus 2} \oplus A_3^{\oplus 2}$, respectively. Furthermore, a direct summand other than A_1, A_2 and A_3 does not appear in the presentation matrices of $F_*^e R$ for $e \geq 2$. Since the blowup of X at $\operatorname{Coker} A_1$ has only one singular point, we can specify the exceptional curve on the minimal resolution corresponding to $\operatorname{Coker} A_1$. The resulting descriptions of $\mathrm{FB}_e(X)$ are summarized in the following.

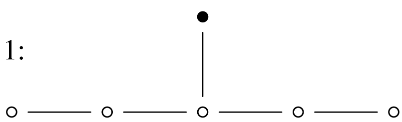
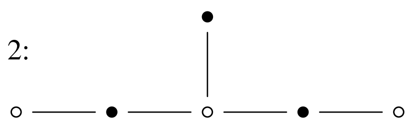
Proposition 3.7. *Let (X, x) be a rational double point of type D_4^1, D_5^1 or E_6^0 in characteristic $p = 2$. Then the e-th F-blowup $\mathrm{FB}_e(X)$ of (X, x) coincides with* *the normal surface obtained by contracting the exceptional curves on the minimal resolution* $\widetilde{X}$ *corresponding to the blank circles in the figure below*:

(1) D_4^1 *and* D_5^1*-singularity in* $p=2$:

$e=1$:  *and* 

For $e \geq 2$*, the* F*-blowups* $\mathrm{FB}_e(X)$ *of both singularities coincide with the minimal resolution.*

(2) E_6^0*-singularity in* $p=2$:

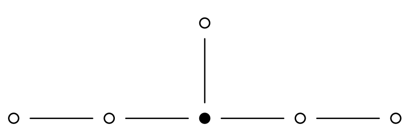
$e=1$:  $e \geq 2$: 

We can also compute the first F-blowup $\mathrm{FB}_1(X)$ of a few other rational double points with `Macaulay2`.

Example 3.8. (1) E_6^1-singularity in $p=2$: Let $R=k[x,y,z]/(z^2+x^2y+xy^2+xyz)$ and $X=\operatorname{Spec} R$. Then X has an E_6^1-singularity. Write

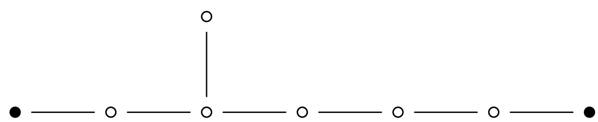
$$A=\begin{pmatrix} z & 0 & 0 & 0 & x & z \\ 0 & z & y & 0 & y & x \\ xy & yz & z & x^2+yz & 0 & 0 \\ 0 & 0 & x & x & y & 0 \\ x^2 & xz & 0 & yz & z & 0 \\ xy+y^2 & x^2 & 0 & xy & 0 & z \end{pmatrix}.$$

Then the cokernel of A defines an indecomposable reflexive R-module of rank 3. The presentation matrix of F_*R is equivalent to $A\oplus 0$, where 0 is the zero matrix of size 1. Thus $\mathrm{FB}_1(X)$ has a unique exceptional curve corresponding to the solid circle in the figure below and has three singular points (an A_1- and two A_2-singularities) on it:



(2) E_8^3-singularity in $p=2$: Let $R=k[x,y,z]/(z^2+x^3+y^5+y^3z)$ and $X=\operatorname{Spec} R$. Then X has an E_8^3-singularity. In this case, F_*R has two kinds of indecomposable reflexive R-modules. Since $\operatorname{rank} F_*R=4$, we see that F_*R is a direct sum of

indecomposable reflexive R-modules of rank 2 corresponding to the solid circles in the figure below:



Thus $\mathrm{FB}_1(X)$ has two exceptional curves corresponding to the solid circles meeting at the unique singular point of type D_6.

4. F-blowups of simple elliptic singularities

In this section (X, x) will denote a simple elliptic singularity defined over an algebraically closed field k of characteristic $p > 0$ unless otherwise noted. Then by a result of Hirokado [2004], (X, x) is quasihomogeneous. So we may assume that $X = \operatorname{Spec} R$ for a graded k-algebra

$$R = R(E, L) = \bigoplus_{n\geq 0} H^0(E, L^n)t^n,$$

where E is an elliptic curve over k, L is an ample line bundle on E and $\deg t = 1$. The minimal resolution $f : \widetilde{X} \to X$ of X is described as follows: $\widetilde{X}$ has an $\mathbb{A}^1$-bundle structure $\pi : \widetilde{X} = \operatorname{Spec}_E(L^n t^n) \to E$ over E, and its zero-section, which we also denote by E, is the exceptional curve of f. Its self-intersection number is $E^2 = -\deg L$. Our situation is summarized in the following diagram:

$$\begin{array}{ccccc} E & \hookrightarrow & \widetilde{X} & \xrightarrow{f} & X \\ & \overset{\mathrm{id}}{\searrow} & \downarrow{\scriptstyle \pi} & & \\ & & E & & \end{array}$$

To compute the F-blowup $\mathrm{FB}_e(X)$ of X, we will look at the structure of the torsion-free pullback $f^\star R^{1/q}$ of $R^{1/q} \cong F^e_* \mathbb{O}_X$, where $q = p^e$. For this purpose we decompose

$$R^{1/q} = \bigoplus_{n\geq 0} H^0(E, F^e_* L^n)t^{n/q} \quad \text{as} \quad R^{1/q} = \bigoplus_{i=0}^{q-1} [R^{1/q}]_{i/q \bmod \mathbb{Z}},$$

where

$$[R^{1/q}]_{i/q \bmod \mathbb{Z}} = \bigoplus_{0\leq n\equiv i \bmod q} H^0(E, F^e_* L^n)t^{n/q} \cong \bigoplus_{m\geq 0} H^0(E, L^m \otimes F^e_* L^i)$$

is an R-summand of $R^{1/q}$ for $i = 0, 1, \ldots, q-1$; see [Smith and Van den Bergh 1997].

In what follows we put $q = p^e$ and $d = \deg L = -E^2$.

Lemma 4.1. *If $1 \le i \le q-1$ and $q \ne di$, then $\widetilde{X}$ is a flattening of $[R^{1/q}]_{i/q \bmod \mathbb{Z}}$.*

Proof. First of all, the locally free sheaf $L^m \otimes F^e_* L^i$ on E is generated by its global sections if $m \ge 1$, or $m = 0$ and $q < di$. To see this, let $P \in E$ and consider the exact sequence

$$0 \to L^m(-P) \otimes F^e_* L^i \to L^m \otimes F^e_* L^i \to \kappa(P) \otimes L^m \otimes F^e_* L^i \to 0. \qquad (1)$$

Since $h^1(L^m(-P) \otimes F^e_* L^i) = h^1(L^{qm+i}(-qP)) = h^0(L^{-qm-i}(qP)) = 0$ by the assumption, the induced map $H^0(E, L^m \otimes F^e_* L^i) \to H^0(E, \kappa(P) \otimes L^m \otimes F^e_* L^i)$ is surjective, that is, $L^m \otimes F^e_* L$ is generated by its global sections at $P \in E$. Hence

$$\begin{aligned} f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}} &= \operatorname{Im}([R^{1/q}]_{i/q \bmod \mathbb{Z}} \otimes_R \mathcal{O}_{\widetilde{X}} \to F^e_* \mathcal{O}_{\widetilde{X}}) \\ &= \operatorname{Im}\Big(\bigoplus_{m \ge 0} H^0(E, L^m \otimes F^e_* L^i) \otimes_k \mathcal{O}_E \xrightarrow{\alpha} \bigoplus_{m \ge 0} L^m \otimes F^e_* L^i\Big) \\ &= \operatorname{Im}(H^0(E, F^e_* L^i) \otimes \mathcal{O}_E \xrightarrow{\alpha_0} F^e_* L^i) \oplus \bigoplus_{m \ge 1} L^m \otimes F^e_* L^i \\ &\subset \bigoplus_{m \ge 0} L^m \otimes F^e_* L^i \cong \pi^* F^e_* L^i, \end{aligned}$$

where α_m ($m \ge 0$) is the graded part of the map α of degree m, and in particular, $f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}} \cong \pi^* F^e_* L^i$ if $q < di$. Since $\pi^* F^e_* L^i$ is a locally free $\mathcal{O}_{\widetilde{X}}$-module, we consider the case $q > di$. Since α_m is surjective for $m \ge 1$, the $\mathcal{O}_{\widetilde{X}}$-module $\operatorname{Coker}(\alpha) = \operatorname{Coker}(\alpha_0)$ is regarded as a coherent sheaf on the exceptional curve $E \subset \widetilde{X}$ of f.

Claim. $\operatorname{Coker}(\alpha) = \operatorname{Coker}(\alpha_0)$ *is a locally free sheaf on E, so that it has depth 1 as an $\mathcal{O}_{\widetilde{X}}$-module at each point on $E \subset \widetilde{X}$.*

To prove the claim, note that $h^0(F^e_* L^i) = h^0(L^i) = di$ by Riemann–Roch and that $F^e_* L^i$ is a locally free sheaf on E of rank q, so that the rank of $\operatorname{Coker}(\alpha) = \operatorname{Coker}(\alpha_0)$ as an $\mathcal{O}_E$-module is at least $q - di$. On the other hand, since

$$H^0(E, \mathcal{O}_E(-P) \otimes F^e_* L^i) = H^0(E, L^i(-qP)) = 0$$

by our assumption, the cohomology long exact sequence of (3) for $m = 0$ turns out to be

$$0 \to H^0(E, F^e_* L^i) \to \kappa(P) \otimes F^e_* L^i \to H^1(E, \mathcal{O}_E(-P) \otimes F^e_* L^i) \to 0,$$

from which we see that the minimal number of local generators of $\operatorname{Coker}(\alpha)$ is $\dim \operatorname{Coker}(\alpha_0) \otimes \kappa(P) = q - di$. Comparing the rank and the minimal number of local generators, we conclude that $\operatorname{Coker}(\alpha) = \operatorname{Coker}(\alpha_0)$ is a locally free sheaf on E of rank $q - di$.

Now we have an exact sequence of $\mathcal{O}_{\widetilde{X}}$-modules

$$0 \to f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}} \to \pi^* F^e_* L^i \to \mathrm{Coker}(\alpha) \to 0,$$

in which $\pi^* F^e_* L^i$ and $\mathrm{Coker}(\alpha)$ have depth 2 and 1, respectively. Thus the depth of $f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}}$ is 2, so that it is locally free on $\widetilde{X}$. □

Remark 4.2. In the case where $1 \le i \le q-1$ and $q = di$, an argument similar to that in the proof of Lemma 4.1 shows that $f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}}$ is *not* flat at $P \in E \subset \widetilde{X}$ if and only if $L^i \cong \mathcal{O}_E(qP)$.

Corollary 4.3. *If $q = p^e > 1$ and $d = -E^2$ is not a power of the characteristic p, then $\widetilde{X}$ is the normalization of the blowup $\mathrm{Bl}_{N_q}(X)$ of $X = \mathrm{Spec}\, R$ at the R-module*

$$N_q = \bigoplus_{i=1}^{q-1} [R^{1/q}]_{i/q \bmod \mathbb{Z}}.$$

Proof. First we will see that N_q is not flat if $q = p^e > 1$. For, if N_q is flat, then the $\mathcal{O}_{X,x}$-module $\mathcal{O}_{X,x}^{1/q}$ has a free summand of rank at least $q(q-1)$. However, the rank of the free summand of $\mathcal{O}_{X,x}^{1/q}$ is exactly equal to 1, since $\mathcal{O}_{X,x}$ is a Gorenstein F-pure local ring with isolated non-F-regular locus; see [Aberbach and Enescu 2005; Sannai and Watanabe 2011, Theorem 5.1].

Now by Lemma 4.1, the minimal resolution $f : \widetilde{X} \to X$ is a flattening of N_q, so it factors as

$$f : \widetilde{X} \xrightarrow{g} \mathrm{Bl}_{N_q}(X) \xrightarrow{h} X.$$

Since N_q is not flat and X is normal, h is not an isomorphism and has an exceptional curve, which is equal to $g(E)$. Hence g is finite (and birational), so that $\widetilde{X}$ is the normalization of $\mathrm{Bl}_{N_q}(X)$. □

Next we consider the structure of $f^\star[R^{1/q}]_{0 \bmod \mathbb{Z}}$, which depends on whether R is F-pure or not. This is equivalent to saying whether the elliptic curve E is ordinary or supersingular, since the section ring $R = R(E, L)$ is F-pure if and only if $E = \mathrm{Proj}\, R$ is F-split.

4a. *The F-pure case.* We first consider the case where R is F-pure, or equivalently, E is an ordinary elliptic curve. In this case, given a fixed point $P_0 \in E$ as the identity element of the group law of E, there are exactly $q = p^e$ distinct q-torsion points $P_0, \dots, P_{q-1}$. In other words, there are exactly q nonisomorphic q-torsion line bundles $L_0, \dots, L_{q-1} \in \mathrm{Pic}^\circ(E)$ given by $L_i = \mathcal{O}_E(P_i - P_0)$. Then $F^e_* \mathcal{O}_E$ splits into line bundles as

$$F^e_* \mathcal{O}_E \cong \bigoplus_{i=0}^{q-1} L_i. \tag{2}$$

Indeed, since $\mathcal{O}_E$ is a direct summand of $F^e_*\mathcal{O}_E$ by F-splitting, each L_i is a direct summand of $L_i \otimes F^e_*\mathcal{O}_E \cong F^e_* F^{e*} L_i \cong F^e_*(L_i^q) \cong F^e_*\mathcal{O}_E$; see [Atiyah 1957].

Lemma 4.4. *Let E be an ordinary elliptic curve.*

(1) *Suppose $d = 1$ and choose the identity element $P_0 \in E$ so that $L \cong \mathcal{O}_E(P_0)$. Then $f^\star[R^{1/q}]_{0 \bmod \mathbb{Z}}$ is not flat exactly at the $q-1$ distinct q-torsion points $P_1, \dots, P_{q-1} \in E \subset \widetilde{X}$ other than P_0. Moreover, $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ is flattened by blowing up the points $P_1, \dots, P_{q-1}$.*

(2) *If $d \geq 2$, then $\widetilde{X}$ is a flattening of $[R^{1/q}]_{0 \bmod \mathbb{Z}}$.*

Proof. Corresponding to the splitting of $F^e_*\mathcal{O}_E$ as in the formula (2) above, the R-module $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ has a splitting $[R^{1/q}]_{0 \bmod \mathbb{Z}} \cong \bigoplus_{i=0}^{q-1} J_i$ into q nonisomorphic reflexive R-modules $R = J_0, J_1, \dots, J_{q-1}$ of rank 1, where

$$J_i = \Gamma_*(L_i) := \bigoplus_{m \in \mathbb{Z}} H^0(E, L_i \otimes L^m) = \bigoplus_{m \geq 0} H^0(E, L_i \otimes L^m).$$

In case (1) where $d = 1$, it is sufficient to show the following:

Claim. *For $i = 1, \dots, q-1$, $f^\star J_i$ is not flat exactly at the single point $P_i \in E \subset \widetilde{X}$. If $\sigma_i : \widetilde{X}_i \to \widetilde{X}$ is the blowup at P_i, then $(f \circ \sigma_i)^\star J_i$ is invertible.*

To prove the claim, note that $\deg L = 1$ and $\deg L_i = 0$. Then the following holds for the linear system $|L_i \otimes L^m|$ on E: $|L_i| = \varnothing$, $|L_i \otimes L| = \mathrm{Bs}\,|L_i \otimes L| = \{P_i\}$ and $|L_i \otimes L^m|$ is base point free for $m \geq 2$. Hence, as in the proof of the previous lemma,

$$\begin{aligned} f^\star J_i &= \mathrm{Im}(J_i \otimes_R \mathcal{O}_{\widetilde{X}} \to F^e_*\mathcal{O}_{\widetilde{X}}) \\ &= \mathrm{Im}\Big(\bigoplus_{m\geq 0} H^0(E, L_i \otimes L^m) \otimes_k \mathcal{O}_E \to \bigoplus_{m \geq 0} L_i \otimes L^m\Big) \\ &= L_i \otimes L(-P_i) \oplus \bigoplus_{m \geq 2} L_i \otimes L^m \subset \bigoplus_{m\geq 1} L_i \otimes L^m \cong \mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^* L_i, \end{aligned}$$

where $L_i \otimes L(-P_i) \cong \mathcal{O}_E \subset L_i \otimes L$ is the graded part of degree $m = 1$. We therefore have the following exact sequence of $\mathcal{O}_{\widetilde{X}}$-modules:

$$0 \to f^\star J_i \to \mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^* L_i \to \kappa(P_i) \to 0,$$

which tells us that $f^\star J_i = \mathcal{I}_{P_i} \cdot \mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^* L_i$, where $\mathcal{I}_{P_i}$ is the ideal sheaf defining the closed point $P_i \in \widetilde{X}$. Now the claim follows immediately.

(2) If $\deg L \geq 2$, then the same argument as in (1) shows that $f^\star J_i$ is isomorphic to $\mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^* L_i$, which is invertible. □

We now state a structure theorem for F-blowups of F-pure $\widetilde{E}_8$-singularities, that is, F-pure simple elliptic singularities with $E^2 = -1$.

Theorem 4.5. *Let (X, x) be an F-pure simple elliptic singularity with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$ such that $E^2 = -1$. Let $P_0, \ldots, P_{q-1} \in E$ be the $q = p^e$ distinct q-torsion points on $E \subset \widetilde{X}$, where the identity element P_0 is chosen so that*

$$\mathcal{O}_{\widetilde{X}}(-E) \otimes \mathcal{O}_E \cong \mathcal{O}_E(P_0),$$

and let $Z = \{P_1, \ldots, P_{q-1}\} \subset \widetilde{X}$. Then for any $e \geq 1$, the normalization of the e-th F-blowup $\mathrm{FB}_e(X)$ coincides with the blowup $\mathrm{Bl}_Z(\widetilde{X})$ of $\widetilde{X}$ at the nontrivial q-torsion points.

In particular, the e-th F-blowup of X is not dominated by the minimal resolution of the singularity (X, x), and the monotonic sequence of F-blowups (see [Yasuda 2009]),

$$\cdots \to \mathrm{FB}_e(X) \to \cdots \to \mathrm{FB}_2(X) \to \mathrm{FB}_1(X) \to X,$$

does not stabilize.

Proof. Since $N_q = \bigoplus_{i=1}^{q-1} [R^{1/q}]_{i/q \bmod \mathbb{Z}}$ is a direct summand of $R^{1/q}$ as an R-module, we have a morphism $\mathrm{FB}_e(X) \to \mathrm{Bl}_{N_q}(X)$ over X. If we denote the normalization of $\mathrm{FB}_e(X)$ by $\widetilde{\mathrm{FB}}_e(X)$, then we have a morphism $\varphi\colon \widetilde{\mathrm{FB}}_e(X) \to \widetilde{X}$ by Corollary 4.3. On the other hand, since $\mathrm{Bl}_Z(\widetilde{X})$ is a flattening of $R^{1/q}$ by Lemmas 4.1 and 4.4, we have a morphism $\mathrm{Bl}_Z(\widetilde{X}) \to \mathrm{FB}_e(X)$ over X, which induces $\psi\colon \mathrm{Bl}_Z(\widetilde{X}) \to \widetilde{\mathrm{FB}}_e(X)$. Thus the blowup $\pi\colon \mathrm{Bl}_Z(\widetilde{X}) \to \widetilde{X}$ at $Z \subset \widetilde{X}$ factors as

$$\pi = \varphi \circ \psi\colon \mathrm{Bl}_Z(\widetilde{X}) \xrightarrow{\psi} \widetilde{\mathrm{FB}}_e(X) \xrightarrow{\varphi} \widetilde{X}.$$

Since $f^\star R^{1/q}$ is not flat exactly at $Z = \{P_1, \ldots, P_{q-1}\}$ by Lemma 4.4, φ has an exceptional curve over every P_i and ψ is finite (and birational), by the same argument as in the proof of Corollary 4.3. Since $\widetilde{\mathrm{FB}}_e(X)$ is normal, ψ is an isomorphism, that is, $\mathrm{Bl}_Z(\widetilde{X}) \cong \widetilde{\mathrm{FB}}_e(X)$ as required. □

The theorem above has nothing to say about the normality of the F-blowups. Let us take a look at a `Macaulay2` computation.

Example 4.6. From [Hirokado 2004, Corollary 4.3], the variety

$$X = \operatorname{Spec} \mathbb{F}_2[x, y, z]/(y^2 + x^3 + xyz + z^6)$$

has a simple elliptic singularity of type $\tilde{E}_8$. Moreover from Fedder's criterion [1983], this is F-pure. Note that since F-blowups are compatible with extensions of perfect fields [Yasuda 2012], the fact that the base field is not algebraically closed does not pose a problem. By `Macaulay2` computation, one can check the following: The first F-blowup $\mathrm{FB}_1(X)$ is nonnormal and its exceptional set consists of two projective lines E_1 and E_2, which intersect transversally at one point. The normalization $\widetilde{\mathrm{FB}}_1(X)$ of $\mathrm{FB}_1(X)$ is smooth. The inverse image of E_1 in $\widetilde{\mathrm{FB}}_1(X)$

is a smooth elliptic curve, which agrees with Theorem 4.5. In particular, this experimental result shows that the normalization in the theorem is really necessary.

Next we consider the case where $E^2 \leq -2$.

Theorem 4.7. *Let (X, x) be an F-pure simple elliptic singularity with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$ such that $E^2 \leq -2$. Assume further that $d = -E^2$ is not a power of the characteristic p. Then $\widetilde{X}$ is the normalization of the e-th F-blowup $\mathrm{FB}_e(X)$ for all $e \geq 1$. Moreover, if $E^2 \leq -3$, then $\widetilde{X} \cong \mathrm{FB}_e(X)$ for all $e \geq 1$.*

Proof. Since $\widetilde{X}$ is a flattening of $R^{1/q}$ by Lemmas 4.1 and 4.4, we see that $\widetilde{X}$ is the normalization of $\mathrm{FB}_e(X)$ as in the proof of Corollary 4.3.

To deduce a stronger conclusion in the special case $E^2 \leq -3$, we need the following:

Lemma 4.8 [Mumford 1970]. *Let V be a projective variety, $\mathcal{F}$ a coherent sheaf on V and let L be a line bundle on V generated by its global sections. Suppose that $H^i(V, \mathcal{F} \otimes L^{-i}) = 0$ for all $i > 0$. Then the natural map*

$$H^0(V, \mathcal{F}) \otimes H^0(V, L)^{\otimes n} \to H^0(V, \mathcal{F} \otimes L^n)$$

is surjective for all $n \geq 1$.

Lemma 4.9. *Let $H_1, \ldots, H_n$ be line bundles on an elliptic curve E of $\deg H_i \geq 3$ for $i = 1, \ldots, n$. Then the natural map*

$$H^0(E, H_1) \otimes \cdots \otimes H^0(E, H_n) \to H^0(E, H_1 \otimes \cdots \otimes H_n)$$

is surjective.

Proof. The case $n \geq 3$ is easily reduced to the case $n = 2$ by induction on n, so let $n = 2$. If $\deg H_1 > \deg H_2$, then $H^1(E, H_1 \otimes H_2^{-1}) = 0$, so that the surjectivity of the map $H^0(E, H_1) \otimes H^0(E, H_2) \to H^0(E, H_1 \otimes H_2)$ immediately follows from Mumford's lemma. Suppose that $\deg H_1 = \deg H_2$ and let $L = H_2 - P$ for any fixed point $P \in E$. Then L is globally generated since $\deg L \geq 2$, and $H^1(E, H_1 \otimes L^{-1}) = 0$ since $\deg(H_1 \otimes L^{-1}) = 1 > 0$. Hence the map

$$H^0(E, H_1) \otimes H^0(E, L) \to H^0(E, H_1 \otimes L)$$

is surjective by Mumford's lemma. We now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 \to & H^0(H_1) \otimes H^0(L) & \to & H^0(H_1) \otimes H^0(H_2) & \to & H^0(H_1) \otimes H^0(H_2 \otimes \kappa(P)) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & H^0(H_1 \otimes L) & \longrightarrow & H^0(H_1 \otimes H_2) & \longrightarrow & H^0(H_1 \otimes H_2 \otimes \kappa(P)), & \end{array}$$

where we have just verified the surjectivity of the vertical map on the left, and the vanishing of the right upper corner comes from $H^1(E, L) = 0$. So, to prove the required surjectivity of the vertical map in the middle, it suffices to show that the vertical map on the right is surjective, by the five-lemma. This map is factorized as

$$H^0(H_1) \otimes H^0(H_2 \otimes \kappa(P)) \xrightarrow{\alpha} H^0(H_1 \otimes \kappa(P)) \otimes H^0(H_2 \otimes \kappa(P)) \xrightarrow{\beta} H^0(H_1 \otimes H_2 \otimes \kappa(P)).$$

Here α is surjective because of the vanishing $H^1(E, H_1(-P)) = 0$, and β is identified with the multiplication map $k^{\otimes 2} \xrightarrow{\sim} k$, which is clearly surjective. Thus $\beta \circ \alpha$ is surjective, and the lemma is proved. □

We continue the proof of Theorem 4.7 in the case $E^2 \leq -3$. Consider the decomposition (2) of $F^e_* \mathbb{O}_E$ into $q = p^e$-torsion line bundles $\mathbb{O}_E = L_0, L_1, \dots, L_{q-1}$ on E. We fix any i with $0 < i \leq q-1$ and let $I \subset R$ be an ideal isomorphic to the reflexive R-module

$$J_i = \Gamma_*(L_i) = \bigoplus_{n \geq 1} H^0(E, L_i \otimes L^n) t^n$$

of rank 1, which is a nontrivial R-summand of $R^{1/q}$. Then the minimal resolution $f : \widetilde{X} \to X = \operatorname{Spec} R$ is factorized as

$$f : \widetilde{X} \to \mathrm{FB}_e(X) \to \mathrm{Bl}_I(X) \to X,$$

where the blowup $\mathrm{Bl}_I(X) = \operatorname{Proj} R[It]$ of X with respect to the ideal I has an exceptional curve that is the image of $E \subset \widetilde{X}$, since $I \cong J_i$ is not a flat R-module. It follows that $\widetilde{X}$ is the normalization of $\mathrm{Bl}_I(X)$. So, to prove the theorem, it is sufficient to show that the Rees algebra $R[It]$ is normal.

To prove the normality of $R[It] = \bigoplus_{m \geq 0} I^m t^m$, note that its normalization is

$$\widetilde{R[It]} = \bigoplus_{m \geq 0} \overline{I^m} t^m,$$

where $\overline{I^m} \subseteq R$ is the integral closure of the ideal I^m; see [Lipman 1969]. Note also that

$$I\mathbb{O}_{\widetilde{X}} \cong f^\star J_i \cong \bigoplus_{n \geq 1} (L_i \otimes L^n) t^n \cong \mathbb{O}_{\widetilde{X}}(-E) \otimes \pi^* L_i$$

is an invertible sheaf on $\widetilde{X}$ by Lemma 4.4, so that

$$\overline{I^m} \cong H^0(\widetilde{X}, \mathbb{O}_{\widetilde{X}}(-mE) \otimes \pi^* L_i^m) \cong \bigoplus_{n \geq m} H^0(E, L_i^m \otimes L^n) t^n \quad \text{for all } m \geq 1.$$

Now, since $\deg L \geq 3$, we can apply Lemma 4.9 to $H_1 = \cdots = H_m := L_i \otimes L$ and $H_{m+1} = \cdots = H_n := L$ to obtain the surjectivity of the map

$$H^0(E, L_i \otimes L)^{\otimes m} \otimes H^0(E, L)^{\otimes n-m} \to H^0(E, L_i^m \otimes L^n)$$

for all $n \geq m \geq 1$. This implies that the multiplication map $\bar{I}^{\otimes m} \to \overline{I^m}$ is surjective in all degree n. Since $I = \bar{I}$ is integrally closed, we conclude that $I^m = \overline{I^m}$, from which the normality of the Rees algebra $R[It]$ follows. □

Example 4.10. Let

$$X = \operatorname{Spec} \mathbb{F}_2[x, y, z]/(y^2 + xyz + x^3z + xz^3).$$

Again from [Hirokado 2004, Corollary 4.3] and Fedder's criterion, X has an F-pure simple elliptic singularity of type $\tilde{E}_7$ at the origin. The exceptional set of $\mathrm{FB}_1(X)$ consists of three projective lines. It shows that it is necessary to suppose in Theorem 4.7 that $d = -E^2$ is not a power of p. The normalization of $\mathrm{FB}_1(X)$ is smooth.

Example 4.11. The variety

$$X = \operatorname{Spec} \mathbb{F}_2[x, y, z]/(y^2z + xyz + x^3 + z^3)$$

has an F-pure simple elliptic singularity of type $\tilde{E}_6$. By `Macaulay2` computations, we can see that $\mathrm{FB}_1(X)$ is smooth and the exceptional set is a smooth elliptic curve, as expected from Theorem 4.7.

4b. ***The non-F-pure case.*** Now we consider the structure of $f^\star[R^{1/q}]_{0 \bmod \mathbb{Z}}$ assuming that R is not F-pure, or equivalently, E is a supersingular elliptic curve. In this case E has no nontrivial q-torsion point under the group law. Then, contrary to the F-pure case, $F^e_*\mathcal{O}_E$ turns out to be indecomposable as we will see below.

For any elliptic curve E and an integer $r > 0$, there exists an indecomposable vector bundle $\mathcal{F}_r$ on E of rank r and degree zero with $h^0(\mathcal{F}_r) = 1$, determined inductively by $\mathcal{F}_1 = \mathcal{O}_E$ and the unique nontrivial extension

$$0 \to \mathcal{F}_{r-1} \to \mathcal{F}_r \to \mathcal{O}_E \to 0. \tag{3}$$

Note that $\mathcal{F}_r$ is self-dual and (3) is the dual sequence of that in [Atiyah 1957, Theorem 5].

Lemma 4.12 (see [Atiyah 1957; Tango 1972]). *If E is a supersingular elliptic curve, then $F^e_*\mathcal{O}_E \cong \mathcal{F}_q$ for all $q = p^e$.*

Proof. Let $F^e_*\mathcal{O}_E = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ be the decomposition of $F^e_*\mathcal{O}_E$ into indecomposable bundles $\mathcal{E}_i$ of rank r_i and degree d_i. Then $d_1 + \cdots + d_n = \chi(F^e_*\mathcal{O}_E) = 0$ by Riemann–Roch. Pick a nontrivial line bundle L of degree zero. Then

$$\sum_{i=1}^{n} h^0(\mathcal{E}_i \otimes L) = h^0(L \otimes F^e_*\mathcal{O}_E) = h^0(L^q) = 0,$$

since there is no nontrivial q-torsion line bundle on a supersingular elliptic curve. Hence $d_i = \deg(\mathcal{E}_i \otimes L) \leq 0$ for all $i = 1, \ldots, n$. Thus the indecomposable summands $\mathcal{E}_i$ of $F^e_*\mathcal{O}_E$ have degree $d_i = 0$, and exactly one of them, say $\mathcal{E}_1$, has a nonzero global section since $h^0(F^e_*\mathcal{O}_E) = 1$. Then by [Atiyah 1957, Theorem 5], we have $\mathcal{E}_1 \cong \mathcal{F}_{r_1}$ and $\mathcal{E}_i \cong \mathcal{F}_{r_i} \otimes L_i$ for $i = 2, \ldots, n$, where $L_2, \ldots, L_n$ are nontrivial line bundles of degree zero. Suppose that $n \geq 2$. Then $L_2^{-1} \otimes F^e_*\mathcal{O}_E$ has a nonzero global section since its direct summand $\mathcal{F}_{r_2}$ does. On the other hand, however, $H^0(E, L_2^{-1} \otimes F^e_*\mathcal{O}_E) = H^0(E, L_2^{-q}) = 0$ since L_2 is not a q-torsion line bundle by our assumption. We thus conclude that $n = 1$, that is, $F^e_*\mathcal{O}_E \cong \mathcal{F}_q$. □

Now for each r, we consider the graded R-module

$$M_r = \bigoplus_{n \geq 0} H^0(E, \mathcal{F}_r \otimes L^n) t^n$$

and regard its torsion-free pullback $\widetilde{M}_r = f^\star M_r$ to the minimal resolution $\widetilde{X}$ of $X = \operatorname{Spec} R$ as a subsheaf of

$$\mathcal{M}_r = \bigoplus_{n \geq 0} (\mathcal{F}_r \otimes L^n) t^n.$$

To obtain information on the flattening of $R^{1/q}$, we consider the torsion-free pullback $f^\star M_r$ of M_r to the minimal resolution, because $[R^{1/q}]_{0 \bmod \mathbb{Z}} \cong M_q$ by Lemma 4.12.

4b1. *Non-F-pure $\widetilde{E}_8$-singularities.* We first consider the case of $\widetilde{E}_8$-singularities, that is, the case $\deg L = -E^2 = 1$. In this case, $L \cong \mathcal{O}_E(P_0)$ for a point $P_0 \in E$.

We fix any point $P \in E$ and let $V \subset E$ be a sufficiently small open neighborhood V of P on which L and $\mathcal{F}_r$ trivialize. We choose a local basis $e_1, \ldots, e_r$ of $\mathcal{F}_r$ on V inductively as follows. For $r = 1$, let e_1 be a (local) basis of $\mathcal{F}_1 = \mathcal{O}_E$ corresponding to its global section $1 \in H^0(E, \mathcal{O}_E)$. For $r \geq 2$, we think of $\mathcal{F}_{r-1}$ as a subbundle of $\mathcal{F}_r$ via the exact sequence (3), and extend the local basis $e_1, \ldots, e_{r-1}$ of $\mathcal{F}_{r-1}$ on V to a local basis $e_1, \ldots, e_r$ of $\mathcal{F}_r$.

Let $U = \pi^{-1}V \subset \widetilde{X}$. Then, with the local trivialization $L|_V \cong \mathcal{O}_V$ and

$$\mathcal{F}_r|_V \cong \bigoplus_{i=1}^{r} \mathcal{O}_V e_i \cong \mathcal{O}_V^{\oplus r}$$

as above, we have

$$\mathcal{M}_r|_U \cong \bigoplus_{i=1}^{r} \mathcal{O}_U e_i \cong \mathcal{O}_U^{\oplus r},$$

where $\mathcal{O}_U = \bigoplus_{n\geq 0}(L|_V)^n t^n \cong \bigoplus_{n\geq 0}\mathcal{O}_V t^n = \mathcal{O}_V[t]$. Note that the fiber coordinate t and a regular parameter u at $P \in E$ form a system of coordinates of U. With this notation we shall express generators of the $\mathcal{O}_U$-module $\widetilde{M}_r|_U \subseteq \mathcal{M}_r|_U$, which come from homogeneous elements of the graded R-module M_r.

First note that the degree zero piece $[M_r]_0 = H^0(E, \mathcal{F}_r) = H^0(E, \mathcal{F}_1)$ of M_r is a one-dimensional k-vector space, so that its contribution to the generation of $\widetilde{M}_r|_U$ is just e_1. It is also easy to see that the graded parts of $\widetilde{M}_r|_U$ and $\mathcal{M}_r|_U$ coincide in degree ≥ 2 and are generated by $t^2e_1, \dots, t^2e_r$, since $\mathcal{F}_r \otimes L^n$ is generated by global sections for $n \geq 2$. It remains to consider the contribution of the degree one piece $[M_r]_1 = H^0(E, \mathcal{F}_r \otimes L)t$ to the generation of $\widetilde{M}_r|_U$. To this end, note that we have an exact sequence

$$0 \to H^0(E, \mathcal{F}_i \otimes L) \to H^0(E, \mathcal{F}_{i+1} \otimes L) \to H^0(E, L) \to 0$$

for $1 \leq i \leq r-1$, via which we regard $H^0(E, \mathcal{F}_i \otimes L)$ as a subspace of $H^0(E, \mathcal{F}_r \otimes L)$. Then, since $h^0(\mathcal{F}_i \otimes L) = i$ by Riemann–Roch, we can choose a basis $s_1, \dots, s_r$ of $H^0(E, \mathcal{F}_r \otimes L)$ so that $s_1, \dots, s_i$ form a basis of $H^0(E, \mathcal{F}_i \otimes L)$ for $1 \leq i \leq r$. It also follows from exact sequence (3)$\otimes L$ that the global sections $s_1, \dots, s_i$ generate $\mathcal{F}_i \otimes L$ on $E \setminus \{P_0\}$, so that they give a basis of $\mathcal{F}_i \otimes L \otimes K$ as a vector space over the function field K of E. On the other hand, $e_1, \dots, e_i$ can also be viewed as a basis of $\mathcal{F}_i \otimes L \otimes K \cong K^{\oplus i}$ under the local trivialization $\mathcal{F}_i \otimes L|_V \cong \bigoplus_{j=1}^{i} \mathcal{O}_V e_i \cong \mathcal{O}_V^{\oplus i}$ induced from $\mathcal{F}_i|_V \cong \mathcal{O}_V^{\oplus i}$ and $L|_V \cong \mathcal{O}_V$. We will compare the basis consisting of $s_i \otimes 1$ and the standard basis $e_1, \dots, e_r$ of $\mathcal{F}_r \otimes L \otimes K \cong K^{\oplus r}$ using the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0(\mathcal{F}_{i-1} \otimes L) \otimes \mathcal{O}_V & \longrightarrow & H^0(\mathcal{F}_i \otimes L) \otimes \mathcal{O}_V & \longrightarrow & H^0(L) \otimes \mathcal{O}_V & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \mathcal{F}_{i-1} \otimes L|_V & \longrightarrow & \mathcal{F}_i \otimes L|_V & \longrightarrow & L|_V & \longrightarrow 0 \\
& \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & \mathcal{O}_V^{\oplus i-1} & \longrightarrow & \mathcal{O}_V^{\oplus i} & \longrightarrow & \mathcal{O}_V & \longrightarrow 0
\end{array}$$

Suppose now that $P = P_0$. Since $\mathrm{Bs}\,|L| = \{P_0\}$, we may choose a regular parameter u at $P_0 \in E$ so that $s_1 \otimes 1 = u$. It then follows from the diagram above that

$$s_i \otimes 1 = ue_i + \sum_{j=1}^{i-1} a_{i,j} e_j,$$

where the a_{ij} are local regular functions on V. We claim that we can replace $s_1, \dots, s_r$ so that they satisfy the following condition:

$$u | a_{i,j} \text{ for } 1 \le j \le i-2 \text{ but } a_{i,i-1} \text{ is not divisible by } u. \tag{4}$$

To prove the claim, there is nothing to do for $i = 1$. So let $i = 2$ and suppose $u|a_{2,1}$. We consider a k-linear map $H^0(E, L) \to H^0(E, \mathcal{F}_2 \otimes L)$ given by $s_1 \mapsto s_2$, which gives rise to a K-linear map $K \cong L \otimes K \to \mathcal{F}_2 \otimes L \otimes K \cong K^2$ sending $1 = u^{-1}(s_1 \otimes 1) \mapsto u^{-1}(s_2 \otimes 1) = e_2 + (a_{21}/u)e_1$. Since $a_{21}/u \in \mathcal{O}_V$, this gives a splitting of the surjective map $\mathcal{O}_V^{\oplus 2} \cong \mathcal{F}_2 \otimes L|_V \to L|_V \cong \mathcal{O}_V$ at $P_0 \in V$, as well as at any other point. Then we have a global splitting of the surjective map $\mathcal{F}_2 \otimes L \to L$, contradicting the nontriviality of the extension (3). Thus $a_{2,1}(P_0) \neq 0$. Next let $i \ge 3$. Then by induction, we may replace s_i by $s_i - \sum_{j=1}^{i-2}(a_{i,j}(P_0)/a_{j+1,j}(P_0))s_{j+1}$ to assume that $u|a_{i,j}$ for $1 \le j \le i-2$. It then follows that $a_{i,i-1}$ is not divisible by u because otherwise, $s_1 \mapsto s_i$ would give a global splitting of $\mathcal{F}_i \otimes L \to L$ as above.

Consequently, local generators of $\widetilde{M}_r$ on a neighborhood U_0 of P_0 are described as

$$\begin{aligned}\widetilde{M}_r|_{U_0} &= \mathcal{O}_{U_0}\langle e_1, tue_i + a_{i,i-1}te_{i-1}, t^2 e_i \mid 2 \le i \le r\rangle \\ &= \mathcal{O}_{U_0}\langle e_1, tue_i + a_{i,i-1}te_{i-1}, t^2 e_r \mid 2 \le i \le r\rangle,\end{aligned}$$

where $a_{i,i-1}(P_0) \neq 0$. Accordingly the ideal $\mathcal{I}_{\widetilde{M}_r} \subset \mathcal{O}_{\widetilde{X}}$ defined in Section 2 has the following local expression:

$$\mathcal{I}_{\widetilde{M}_r}|_{U_0} \cong (t^r, t^{r-1}u^{r-1}) \cong (t, u^{r-1}).$$

If $P_0 \neq P \in U$ then $\widetilde{M_r}|_U = \mathcal{O}_U\langle e_1, te_i \mid 2 \le i \le r\rangle \cong \mathcal{O}_U^{\oplus r}$ by a similar argument.

Summarizing the argument so far, we have

Theorem 4.13. *Let (X, x) be a non-F-pure simple elliptic singularity with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$ such that $E^2 = -1$. Let P_0 be the point on $E \subset \widetilde{X}$ such that $\mathcal{O}_{\widetilde{X}}(-E) \otimes \mathcal{O}_E \cong \mathcal{O}_E(P_0)$ and let $\mathcal{I}_e \subset \mathcal{O}_{\widetilde{X}}$ be the ideal sheaf defining a fat point supported at $P_0 \in \widetilde{X}$ whose local expression at P_0 is*

$$(\mathcal{I}_e)_{P_0} = (t, u^{p^e-1})$$

as above. Then for any $e \ge 1$, the blowup $\mathrm{Bl}_{\mathcal{I}_e}(\widetilde{X})$ of $\widetilde{X}$ at $\mathcal{I}_e$ coincides with the normalization of the e-th F-blowup $\mathrm{FB}_e(X)$.

Proof. We know that $Y = \mathrm{Bl}_{\mathcal{I}_e}(\widetilde{X})$ is a flattening of R^{1/p^e} from the argument above and Corollary 4.3. It is also easy to see that the exceptional curve of the blowup

$\pi : Y \to \widetilde{X}$ is a single $\mathbb{P}^1$. Then the same argument as in the proof of Theorem 4.5 shows that π factors through the normalized F-blowup $\widetilde{\mathrm{FB}}_e(X)$ as

$$\pi = \varphi \circ \psi \colon Y = \mathrm{Bl}_{\mathcal{I}_e}(\widetilde{X}) \xrightarrow{\psi} \widetilde{\mathrm{FB}}_e(X) \xrightarrow{\varphi} \widetilde{X}$$

and that ψ gives an isomorphism $Y \cong \widetilde{\mathrm{FB}}_e(X)$. □

Remark 4.14. Theorem 4.13 says that the e-th normalized F-blowup $\widetilde{\mathrm{FB}}_e(X)$ has the exceptional set consisting of an elliptic curve $E_1 \cong E$ and a smooth rational curve $E_2 \cong \mathbb{P}^1$, and has an A_{p^e-2}-singularity on $E_2 \setminus E_1$. The theorem also says that $\mathrm{FB}_e(X)$ does not dominate $\mathrm{FB}_{e'}(X)$ whenever e and e' are distinct positive integers. In other words, the monotonicity of F-blowup sequences breaks down for non-F-pure $\widetilde{E}_8$-singularities; compare to the F-pure case [Yasuda 2009]. On the other hand, it again has nothing to say about the normality of $\mathrm{FB}_e(X)$.

Let us examine our observation with `Macaulay2` computation.

Example 4.15. The variety

$$X = \operatorname{Spec} \mathbb{F}_3[x, y, z]/(x(x - z^2)(x - 2z^2) - y^2)$$

has a non-F-pure simple elliptic singularity of type $\tilde{E}_8$. The exceptional set of $\mathrm{FB}_1(X)$ is the union of a smooth elliptic curve E_1 and a projective line E_2. We could not check the normality of $\mathrm{FB}_1(X)$ by `Macaulay2` computation only, but we could check the following using `Macaulay2`:

$\mathrm{FB}_1(X)$ is normal at the generic points of E_1 and E_2, and there is a point on $E_2 \setminus E_1$ where $\mathrm{FB}_1(X)$ is normal but singular. The blowup of $\mathrm{FB}_1(X)$ at this point has the projective line as its exceptional locus. $(*)$

It agrees with the fact that $\mathrm{FB}_1(X)$ has an A_1-singularity on $E_2 \setminus E_1$ as stated in the remark above.

Proposition 4.16. *For X as in Example 4.15, if $(*)$ is correct, then $\mathrm{FB}_1(X)$ is normal.*

Proof. We may replace the base field $\mathbb{F}_3$ with an algebraically closed field k. Being quasihomogeneous, X has a k^*-action. From the construction or the universality, the action lifts to F-blowups of X. Every point of the divisor $E_1 \subset \mathrm{FB}_1(X)$, which is a smooth elliptic curve, is fixed by the k^*-action. On the other hand, the divisor $E_2 \cong \mathbb{P}^1$ has exactly two fixed points. One is the singular but normal point mentioned above and the other is the intersection $E_1 \cap E_2$. Since the normal locus is open and there is the k^*-action, $\mathrm{FB}_1(X)$ is normal along E_2 possibly except at $E_1 \cap E_2$. Therefore it is now enough to show that $\mathrm{FB}_1(X)$ is normal along E_1. Let $\tilde{E}_1$ and $\tilde{E}_2$ be the preimages of E_1 and E_2 on the normalization $\widetilde{\mathrm{FB}}_1(X)$ of $\mathrm{FB}_1(X)$. Then for each $i = 1, 2$, since E_i is normal and $\mathrm{FB}_1(X)$ is normal at the generic point of E_i, the map $\tilde{E}_i \to E_i$ is an isomorphism.

Let A be the complete local ring of $\mathrm{FB}_1(X)$ at a point z on E_1. Its normalization is $k[\![s,t]\!]$. We choose local coordinates s,t so that the k^*-action on $k[\![s,t]\!]$ is linear and locally $s=0$ defines $\tilde{E}_1$ and $t=0$ defines the only one-dimensional orbit closure passing through the point over z. Then the k^*-action on t is trivial and the one on s is nontrivial. Since $\tilde{E}_i \to E_i$ for $i=1,2$ are isomorphisms, the composite maps $A \hookrightarrow k[\![s,t]\!] \to k[\![s]\!]$ and $A \hookrightarrow k[\![s,t]\!] \to k[\![t]\!]$ are surjective. Therefore A contains formal power series of the forms

$$f = f_1 s + f_2 t + (\text{higher terms}), \quad (\text{for } f_i \in k,\ f_1 \neq 0),$$
$$g = g_1 s + g_2 t + (\text{higher terms}), \quad (\text{for } g_i \in k,\ g_2 \neq 0).$$

Then by a suitable linear combination of them, we obtain a formal power series

$$h = h_1 s + h_2 t + (\text{higher terms}), \quad (\text{for } h_i \in k,\ h_1 \neq 0, h_2 \neq 0)$$

contained in A. Then for $1 \neq \lambda \in k^*$, $\lambda h \in A$ has a linear part linearly independent of that of h. It follows that $A = k[\![s,t]\!]$ and hence $\mathrm{FB}_1(X)$ is normal. □

Example 4.17. The variety

$$X = \operatorname{Spec} \mathbb{F}_2[x,y,z]/(y^2 + yz^3 + x^3)$$

has a non-F-pure simple elliptic singularity of type $\tilde{E}_8$. The Frobenius pushforward $F_*\mathcal{O}_X$ of the coordinate ring decomposes into the direct sum of two modules, say N_1 and N_2. Then F_*N_i for $i=1,2$ further decomposes as $F_*N_i = N_{i1} \oplus N_{i2}$. By `Macaulay2` computation, we saw that the torsion-free pullbacks $\widetilde{N_1}$ and $\widetilde{N_{11}}$ of N_1 and N_{11} are nonflat at a point and those of the others are flat. Moreover the ideals associated to $\widetilde{N_1}$ and $\widetilde{N_{11}}$ as in Proposition 2.5 are respectively of the forms (u,v) and (u,v^3) around the point with respect to some local coordinates u,v. The last result coincides with Theorem 4.13.

4b2. *Non-F-pure simple elliptic singularities with $E^2 \leq -2$.* In this case, we have $\deg L = -E^2 \geq 2$. Then the argument in Section 4b1 shows that $\widetilde{M_r}$ is flat.

Proposition 4.18. *Let (X,x) be a non-F-pure simple elliptic singularity with elliptic exceptional curve E on the minimal resolution $\widetilde{X}$. Suppose $E^2 \leq -2$ and $d = -E^2$ is not a power of the characteristic p. Then $\widetilde{X}$ is the normalization of the e-th F-blowup $\mathrm{FB}_e(X)$ for all $e \geq 1$.*

Proof. Since $\widetilde{X}$ is a flattening of $R^{1/q} = M_q \oplus N_q$ by Lemmas 4.1 and 4.12 and Section 4b1, the proof goes similarly to that of Theorem 4.7. Note that $\mathcal{O}_{X,x}^{1/q}$ has no free summand in this case, since $\mathcal{O}_{X,x}$ is not F-pure. □

Example 4.19. The variety

$$X = \operatorname{Spec} \mathbb{F}_2[x,y,z]/(y^2 z + yz^2 + x^3)$$

has a non-F-pure simple elliptic singularity of type $\tilde{E}_6$. We could check that $\mathrm{FB}_1(X)$ is the minimal resolution.

Remark 4.20. The behavior of F-blowups remains unsettled in some cases, that is, (i) the case $E^2 \leq -2$ and $-E^2$ is a power of p; and (ii) the normality of F-blowups of non-F-pure simple elliptic singularities with $E^2 \leq -3$. These cases will be treated in [Hara 2013].

References

[Aberbach and Enescu 2005] I. M. Aberbach and F. Enescu, "The structure of F-pure rings", *Math. Z.* **250**:4 (2005), 791–806. MR 2006m:13009 Zbl 1102.13001

[Artin 1962] M. Artin, "Some numerical criteria for contractability of curves on algebraic surfaces", *Amer. J. Math.* **84** (1962), 485–496. MR 26 #3704 Zbl 0105.14404

[Artin 1977] M. Artin, "Coverings of the rational double points in characteristic p", pp. 11–22 in *Complex analysis and algebraic geometry*, edited by J. W. L. Baily and T. Shioda, Iwanami Shoten, Tokyo, 1977. MR 56 #8559 Zbl 0358.14008

[Artin and Verdier 1985] M. Artin and J.-L. Verdier, "Reflexive modules over rational double points", *Math. Ann.* **270**:1 (1985), 79–82. MR 85m:14006 Zbl 0553.14001

[Atiyah 1957] M. F. Atiyah, "Vector bundles over an elliptic curve", *Proc. London Math. Soc.* (3) **7** (1957), 414–452. MR 24 #A1274 Zbl 0084.17305

[Fedder 1983] R. Fedder, "F-purity and rational singularity", *Trans. Amer. Math. Soc.* **278**:2 (1983), 461–480. MR 84h:13031 Zbl 0519.13017

[Grayson and Stillman 2012] D. R. Grayson and M. E. Stillman, "`Macaulay2`, a software system for research in algebraic geometry", 2012, available at http://www.math.uiuc.edu/Macaulay2/.

[Hara 2012] N. Hara, "F-blowups of F-regular surface singularities", *Proc. Amer. Math. Soc.* **140**:7 (2012), 2215–2226. MR 2898685

[Hara 2013] N. Hara, "Structure of the F-blowups of simple elliptic singularities", preprint, 2013.

[Hara and Sawada 2011] N. Hara and T. Sawada, "Splitting of Frobenius sandwiches", pp. 121–141 in *Higher dimensional algebraic geometry*, edited by S. Mukai and N. Nakayama, RIMS Kôkyûroku Bessatsu **B24**, Res. Inst. Math. Sci., Kyoto, 2011. MR 2012f:13009 Zbl 1228.13009

[Hirokado 2004] M. Hirokado, "Deformations of rational double points and simple elliptic singularities in characteristic p", *Osaka J. Math.* **41**:3 (2004), 605–616. MR 2005k:14006 Zbl 1065.14004

[Ito and Nakamura 1996] Y. Ito and I. Nakamura, "McKay correspondence and Hilbert schemes", *Proc. Japan Acad. Ser. A Math. Sci.* **72**:7 (1996), 135–138. MR 97k:14003 Zbl 0881.14002

[Kunz 1969] E. Kunz, "Characterizations of regular local rings for characteristic p", *Amer. J. Math.* **91** (1969), 772–784. MR 40 #5609 Zbl 0188.33702

[Lipman 1969] J. Lipman, "Rational singularities, with applications to algebraic surfaces and unique factorization", *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), 195–279. MR 43 #1986 Zbl 0181.48903

[Mumford 1970] D. Mumford, "Varieties defined by quadratic equations", pp. 29–100 in *Questions on Algebraic Varieties* (Ciclo, Varenna, 1969), Edizioni Cremonese, Rome, 1970. MR 44 #209 Zbl 0198.25801

[Oneto and Zatini 1991] A. Oneto and E. Zatini, "Remarks on Nash blowing-up", *Rend. Sem. Mat. Univ. Politec. Torino* **49**:1 (1991), 71–82. MR 94d:14013 Zbl 0802.14001

[Sannai and Watanabe 2011] A. Sannai and K.-i. Watanabe, “F-signature of graded Gorenstein rings”, *J. Pure Appl. Algebra* **215**:9 (2011), 2190–2195. MR 2012c:13009 Zbl 1223.13003

[Smith and Van den Bergh 1997] K. E. Smith and M. Van den Bergh, “Simplicity of rings of differential operators in prime characteristic”, *Proc. London Math. Soc.* (3) **75**:1 (1997), 32–62. MR 98d:16039 Zbl 0948.16019

[Tango 1972] H. Tango, “On the behavior of extensions of vector bundles under the Frobenius map”, *Nagoya Math. J.* **48** (1972), 73–89. MR 47 #3401 Zbl 0239.14007

[Toda and Yasuda 2009] Y. Toda and T. Yasuda, “Noncommutative resolution, F-blowups and D-modules”, *Adv. Math.* **222**:1 (2009), 318–330. MR 2011b:14005 Zbl 1175.14005

[Villamayor U. 2006] O. Villamayor U., “On flattening of coherent sheaves and of projective morphisms”, *J. Algebra* **295**:1 (2006), 119–140. MR 2006m:13008 Zbl 1087.14011

[Yasuda 2007] T. Yasuda, “Higher Nash blowups”, *Compos. Math.* **143**:6 (2007), 1493–1510. MR 2008j:14029 Zbl 1135.14011

[Yasuda 2009] T. Yasuda, “On monotonicity of F-blowup sequences”, *Illinois J. Math.* **53**:1 (2009), 101–110. MR 2011b:14007 Zbl 1194.14010

[Yasuda 2012] T. Yasuda, “Universal flattening of Frobenius”, *Amer. J. Math.* **134**:2 (2012), 349–378. MR 2905000 Zbl 06029054

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