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Sharp upper bounds for the Betti numbers of a given Hilbert polynomial

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We show that there exists a saturated graded ideal in a standard graded polynomial ring which has the largest total Betti numbers among all saturated graded ideals for a fixed Hilbert polynomial.

1. Introduction

A classical problem consists in studying the number of minimal generators of ideals in a local or a graded ring in relation to other invariants of the ring and of the ideals themselves. In particular, a great amount of work has been done to establish bounds for the number of generators in terms of certain invariants, for instance, multiplicity, Krull dimension, and Hilbert functions (see [Macaulay 1927; Sally 1978]). An important result was proved in [Elías et al. 1991], where the authors established a sharp upper bound for the number of generators $\nu(I)$ of all perfect ideals I in a regular local ring $(R, \boldsymbol{m}, K)$ (or in a polynomial ring over a field K) in terms of their multiplicity and their height.

In a subsequent paper, Valla [1994] provides, under the same hypotheses, sharp upper bounds for every Betti number $\beta_i^R(I) = \dim_K \operatorname{Tor}_i^R(I, K)$; notice that with this notation $\beta_0^R(I) = \nu(I)$. More surprisingly, Valla proved that among all perfect ideals with a fixed multiplicity and height in a formal power series ring over a field K, there exists one which has the largest possible Betti numbers β_i.

The main result of this paper is an extension of Valla's theorem. We will consider both the local and the graded case, although the result we present for the local case follows directly from the graded case.

We first consider the graded case. We show that for every fixed Hilbert polynomial $p(t)$, there exists a point Y in the Hilbert scheme $\mathrm{Hilb}^{p(t)}_{\mathbb{P}^{n-1}}$ such that $\beta_i(I_Y) \geq \beta_i(I_X)$ for all i and for all $X \in \mathrm{Hilb}^{p(t)}_{\mathbb{P}^{n-1}}$. Equivalently, let $S = K[X_1, \ldots, X_n]$ be a standard graded polynomial ring over a field K. We prove:

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Theorem 1.1. *Let $p(t)$ be the Hilbert polynomial of a graded ideal of S. There exists a saturated graded ideal $L \subset S$ with Hilbert polynomial $p(t)$ such that $\beta_i^S(S/L) \geq \beta_i^S(S/I)$ for all i and for all saturated graded ideals $I \subset S$ with Hilbert polynomial $p(t)$.*

Notice that Valla's result corresponds to the special case of the theorem when $p(t)$ is constant.

An important result in the study of upper bounds for Betti numbers is the Bigatti–Hulett–Pardue theorem, which shows that the lex ideal has the largest Betti numbers among all homogeneous ideals in a standard graded polynomial ring for a fixed Hilbert function. By using the Bigatti–Hulett–Pardue theorem, we reduce Theorem 1.1 to a certain combinatorial problem on lex ideals, and prove the theorem by purely combinatorial methods.

We have chosen to not present an explicit formula of the bounds. We are convinced that such a formula, in the general case, would be hard to read and to interpret. Instead, as a part of the proof, we describe the construction of the lex ideal that achieves the bound. Using the Eliahou–Kervaire resolution it is possible to write an explicit formula for the total Betti numbers of every lex ideal in terms of its minimal generators.

In particular, explicit computations of the bounds can be carried out for a given Hilbert polynomial. Thus, it would be possible to describe an explicit formula of the bounds for classes of simple enough Hilbert polynomials. For example, in the special case when the Hilbert polynomials are constant, such a formula was given by Valla [1994].

Theorem 1.1 induces the following upper bounds of Betti numbers of ideals in a regular local ring (see Section 3 for the proof): For a regular local ring $(R, \boldsymbol{m}, K)$ and an ideal $I \subset R$, let $\boldsymbol{p}_{R/I}(t)$ be the Hilbert–Samuel polynomial of R/I with respect to $\boldsymbol{m}$ (see [Bruns and Herzog 1998, §4.6]).

Theorem 1.2. *Let $(R, \boldsymbol{m}, K)$ be a regular local ring of dimension n, and let $\boldsymbol{p}(t)$ be a polynomial such that there is an ideal $J \subset R$ such that $\boldsymbol{p}(t) = \boldsymbol{p}_{R/J}(t)$. There exists an ideal L in $A = K[\![x_1, \dots, x_n]\!]$ with $\boldsymbol{p}_{A/L}(t) = \boldsymbol{p}(t)$ such that $\beta_i^A(A/L) \geq \beta_i^R(R/I)$ for all i and for all ideals $I \subset R$ with $\boldsymbol{p}_{R/I}(t) = \boldsymbol{p}(t)$.*

Unfortunately, the combinatorial part of the proof of Theorem 1.1 is very long and complicated. Moreover, a construction of ideals which achieve the bound is not easy to understand. Thus, it would be desirable to get a simpler proof of the theorem and to get a better understanding for the structure of ideals which attain maximal Betti numbers.

The paper is structured in the following way: In Sections 2 and 3, we reduce a problem of Betti numbers to a problem of combinatorics of lexicographic sets of monomials with a special structure. In Section 4, we introduce key techniques to prove the main result. In particular, we give a new proof of Valla's result. In Section 5, a construction of ideals which attain maximal Betti numbers of Theorem 1.1 will be given. In Section 6, we give a proof of the main combinatorial result about lexicographic sets of monomials, which essentially proves Theorem 1.1. In Section 7, some examples of ideals with maximal Betti numbers are given.

2. Universal lex ideals

In this section, we introduce basic notations which are used in the paper.

Let $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field K. Let M be a finitely generated graded S-module. The *Hilbert function* $H(M, -) : \mathbb{Z} \to \mathbb{Z}$ of M is the numerical function defined by

$$H(M, k) = \dim_K M_k$$

for all $k \in \mathbb{Z}$, where M_k is the graded component of M of degree k. We denote $P_M(t)$ by the Hilbert polynomial of M. Thus $P_M(t)$ is a polynomial in t satisfying $P_M(k) = H(M, k)$ for $k \gg 0$. The numbers

$$\beta^S_{i,j}(M) = \dim_K \operatorname{Tor}^S_i(M, K)_j$$

are called the *graded Betti numbers* of M, and $\beta^S_i(M) = \sum_{j \in \mathbb{Z}} \beta^S_{i,j}(M)$ are called the (*total*) *Betti numbers* of M.

A set of monomials $W \subset S$ is said to be *lex* if, for all monomials $u \in W$ and $v >_{\text{lex}} u$ of the same degree, one has $v \in W$, where $>_{\text{lex}}$ is the lexicographic order induced by the ordering $x_1 >_{\text{lex}} \cdots >_{\text{lex}} x_n$. A monomial ideal $I \subset S$ is said to be *lex* if the set of monomials in I is lex. By the classical Macaulay's theorem [1927], for any graded ideal $I \subset S$ there exists the unique lex ideal $L \subset S$ with the same Hilbert function as I. Moreover, Bigatti [1993], Hulett [1993], and Pardue [1996] proved that lex ideals have the largest graded Betti numbers among all graded ideals having the same Hilbert function.

For any graded ideal $I \subset S$, let

$$\operatorname{sat} I = (I : \mathfrak{m}^\infty)$$

be the *saturation* of $I \subset S$, where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of S. A graded ideal I is said to be *saturated* if $I = \operatorname{sat} I$. It is well-known that I is saturated if and only if $\operatorname{depth}(S/I) > 0$ or $I = S$.

Let $L \subset S$ be a lex ideal. Then $\operatorname{sat} L$ is also a lex ideal. It is natural to ask which lex ideals are saturated. The theory of universal lex ideals gives an answer.

A lex ideal $L \subset S$ is said to be *universal* if $LS[x_{n+1}]$ is also a lex ideal in $S[x_{n+1}]$. The following are fundamental results on universal lex ideals:

Lemma 2.1 [Murai and Hibi 2008]. *Let $L \subset S$ be a lex ideal. The following conditions are equivalent*:

(i) *L is universal.*

(ii) *L is generated by at most n monomials.*

(iii) *$L = S$ or there exist integers $a_1, a_2, \ldots, a_t \geq 0$ with $1 \leq t \leq n$ such that*

$$L = (x_1^{a_1+1}, x_1^{a_1} x_2^{a_2+1}, \ldots, x_1^{a_1} x_2^{a_2} \cdots x_{t-1}^{a_{t-1}} x_t^{a_t+1}). \tag{1}$$

A relation between universal lex ideals and saturated lex ideals is the following:

Lemma 2.2 [Murai and Hibi 2008]. *Let $L \subsetneq S$ be a lex ideal. Then* $\operatorname{depth}(S/L) > 0$ *if and only if L is generated by at most $n-1$ monomials.*

A lex ideal $I \subset S$ is called a *proper universal lex ideal* if I is generated by at most $n-1$ monomials or $I = S$.

Let $I \subset S$ be a graded ideal. Then there exists the unique lex ideal $L \subset S$ with the same Hilbert function as I. Then $\operatorname{sat} L$ is a proper universal lex ideal with the same Hilbert polynomial as I. This construction $I \to \operatorname{sat} L$ gives a one-to-one correspondence between Hilbert polynomials of graded ideals and proper universal lex ideals:

Proposition 2.3. *For any graded ideal $I \subset S$ there exists the unique proper universal lex ideal $L \subset S$ with the same Hilbert polynomial as I.*

Proof. The existence is obvious. What we must prove is that, if L and L' are proper universal lex ideals with the same Hilbert polynomial then $L = L'$.

Since L and L' have the same Hilbert polynomial, their Hilbert functions coincide in sufficiently large degrees. This fact shows $L_d = L'_d$ for $d \gg 0$. Thus $\operatorname{sat} L = \operatorname{sat} L'$. Since L and L' are saturated, $L = \operatorname{sat} L = \operatorname{sat} L' = L$. $\square$

3. 1-lexicographic ideals, Betti numbers and max sequences

In this section, we reduce a problem of Betti numbers of graded ideals to a problem of combinatorics of lex sets of monomials.

Let $S = K[x_1, \ldots, x_n]$ and $\bar{S} = K[x_1, \ldots, x_{n-1}]$. For a monomial ideal $I \subset S$, let $\bar{I} = I \cap \bar{S}$. A monomial ideal $I \subset S$ is said to be 1-*lexicographic* if x_n is a nonzero divisor of S/I and $\bar{I}$ is a lex ideal of $\bar{S}$.

Lemma 3.1 [Iyengar and Pardue 1999, Proposition 4]. *For any saturated graded ideal $I \subset S$, there exists a* 1*-lexicographic ideal $J \subset S$ with the same Hilbert function as I such that $\beta_{i,j}^S(I) \leq \beta_{i,j}^S(J)$ for all i, j.*

Lemma 3.2. *Let $J \subset S$ be a 1-lexicographic ideal. Then:*

(i) $\dim_K J_d = \sum_{k=0}^{d} \dim_K \bar{J}_k$ *for all* $d \geq 0$.

(ii) $\beta_i^S(J) = \beta_i^{\bar{S}}(\bar{J})$ *for all* i.

Proof. Condition (ii) is obvious since x_n is regular on S/J. Also, for all $d \geq 0$, we have a decomposition $J_d = \bigoplus_{k=0}^{d} J_k x_n^{d-k}$ as K-vector spaces. This equality proves (i). □

Corollary 3.3. *Let J and J' be 1-lexicographic ideals in S. If J and J' have the same Hilbert polynomial then $\bar{J}_d = \bar{J}'_d$ for $d \gg 0$.*

Proof. Lemma 3.2(i) says that $\dim_K J_d - \dim_K J_{d-1} = \dim \bar{J}_d$, so

$$\dim_K \bar{J}_d = \dim_K \bar{J}'_d \quad \text{for } d \gg 0.$$

Then the statement follows since $\bar{J}$ and $\bar{J}'$ are lex. □

Next, we describe all 1-lexicographic ideals in S. By Proposition 2.3, fixing a Hilbert polynomial is equivalent to fixing a proper universal lex ideal U. For a proper universal lex ideal $U \subset S$, let

$$\mathscr{L}(U) = \{I \subset \bar{S} : I \text{ is a lex ideal with } I \subset \operatorname{sat} \bar{U} \text{ and } \dim_K(\operatorname{sat} \bar{U})/I = \dim_K(\operatorname{sat} \bar{U})/\bar{U}\}.$$

Note that $\dim_K(\operatorname{sat} J)/J$ is finite for any graded ideal $J \subset S$ since $(\operatorname{sat} J)/J$ is isomorphic to the zeroth local cohomology module $H^0_{\mathfrak{m}}(S/J)$. By using Lemma 3.2, it is easy to see that if $I \in \mathscr{L}(U)$ then IS has the same Hilbert polynomial as U. Actually, the converse is also true.

Lemma 3.4. *Let U be a proper universal lex ideal. If J is a 1-lexicographic ideal such that $P_J(t) = P_U(t)$ then $\bar{J} \in \mathscr{L}(U)$.*

Proof. By Corollary 3.3 we have $\bar{U}_d = \bar{J}_d$ for $d \gg 0$, so $\operatorname{sat} \bar{U} = \operatorname{sat} \bar{J}$. Also, since U and J have the same Hilbert polynomial, for $d \gg 0$, one has

$$\dim_K U_d = \sum_{k=0}^{d} \dim_K \bar{U}_k = \sum_{k=0}^{d} \dim_K(\operatorname{sat} \bar{U}_k) - \dim_K(\operatorname{sat} \bar{U}/\bar{U})$$

and

$$\dim_K J_d = \sum_{k=0}^{d} \dim_K \bar{J}_k = \sum_{k=0}^{d} \dim_K(\operatorname{sat} \bar{J}_k) - \dim_K(\operatorname{sat} \bar{J}/\bar{J}).$$

Since $\operatorname{sat} \bar{J} = \operatorname{sat} \bar{U}$, we have $\dim_K(\operatorname{sat} \bar{J}/\bar{J}) = \dim_K(\operatorname{sat} \bar{U}/\bar{U})$ and $\bar{J} \in \mathscr{L}(U)$. □

By Lemmas 3.1 and 3.4, to prove Theorem 1.1, it is enough to find a lex ideal which has the largest Betti numbers among all ideals in $\mathscr{L}(U)$. We consider a more general setting. For any universal lex ideal $U \subset S$ (not necessarily proper) and for any positive integer $c > 0$, define

$$\mathscr{L}(U; c) = \{I \subset U : I \text{ is a lex ideal with } \dim_K U/I = c\}.$$

We consider the Betti numbers of ideals in $\mathscr{L}(U; c)$.

We first discuss Betti numbers of lex ideals. We need the following notation: For any monomial $u \in S$, let $\max u$ be the largest integer ℓ such that x_ℓ divides u, where $\max(1) = 1$. For a set of monomials (or a K-vector space spanned by monomials) M, let

$$m_{\le i}(M) = \#\{u \in M : \max u \le i\}$$

for $i = 1, 2, \dots, n$, where $\#X$ is the cardinality of a finite set X, and

$$m(M) = \big(m_{\le 1}(M), m_{\le 2}(M), \dots, m_{\le n}(M)\big).$$

These numbers are often used to study Betti numbers of lex ideals. The next formula was proved by Bigatti [1993] and Hulett [1993], by using the famous Eliahou–Kervaire resolution [1990].

Lemma 3.5. *Let $I \subset S$ be a lex ideal. Then, for all i, j,*

$$\beta^S_{i,i+j}(I) = \binom{n-1}{i} \dim_K I_j - \sum_{k=1}^{n} \binom{k-1}{i} m_{\le k}(I_{j-1}) - \sum_{k=1}^{n-1} \binom{k-1}{i-1} m_{\le k}(I_j).$$

For vectors $\boldsymbol{a} = (a_1, \dots, a_n)$, $\boldsymbol{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$, we define

$$\boldsymbol{a} \succeq \boldsymbol{b} \Leftrightarrow a_i \ge b_i \quad \text{for } i = 1, 2, \dots, n.$$

Corollary 3.6. *Let U be a universal lex ideal and $I, J \in \mathscr{L}(U; c)$. Let $\mathcal{M}_I$ (resp. $\mathcal{M}_J$) be the set of all monomials in $U \setminus I$ (resp. $U \setminus J$). If $m(\mathcal{M}_I) \succeq m(\mathcal{M}_J)$ then $\beta^S_i(I) \ge \beta^S_i(J)$ for all i.*

Proof. Observe that $\beta^S_{i,i+j}(I) = \beta^S_{i,i+j}(J) = 0$ for $j \gg 0$. Thus, for $d \gg 0$, we have $\beta^S_i(I) = \sum_{j=0}^{d} \beta^S_{i,i+j}(I)$. Let $I_{\le d} = \bigoplus_{k=0}^{d} I_k$. Then by Lemma 3.5,

$$\beta^S_i(I) = \binom{n-1}{i} \dim_K I_{\le d} - \sum_{k=1}^{n} \binom{k-1}{i} m_{\le k}(I_{\le d-1}) - \sum_{k=1}^{n-1} \binom{k-1}{i-1} m_{\le k}(I_{\le d})$$

and the same formula holds for J. Since, for $d \gg 0$,

$$m(J_{\le d}) = m(U_{\le d}) - m(\mathcal{M}_J) \succeq m(U_{\le d}) - m(\mathcal{M}_I) = m(I_{\le d}),$$

we have $\beta^S_i(I) \ge \beta^S_i(J)$ for all i, as desired. $\square$

Next, we study the structure of $\mathcal{M}_I$. Let

$$U = (x_1^{a_1+1}, x_1^{a_1} x_2^{a_2+1}, \dots, x_1^{a_1} x_2^{a_2} \cdots x_{t-1}^{a_{t-1}} x_t^{a_t+1})$$

be a universal lex ideal, $\delta_i = x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_i^{a_i+1}$, and $b_i = a_1 + \cdots + a_i + 1 = \deg \delta_i$. (If $U = S$ then $t = 1$ and $a_1 = -1$.) Let

$$S^{(i)} = K[x_i, \ldots, x_n].$$

Then, as K-vector spaces, we have a decomposition

$$U = \delta_1 S^{(1)} \oplus \delta_2 S^{(2)} \oplus \cdots \oplus \delta_t S^{(t)}.$$

Definition 3.7. A set of monomials $N \subset S^{(i)}$ is said to be *revlex* if, for all monomials $u \in N$ and $v <_{\text{lex}} u$ of the same degree, one has $v \in N$. Moreover, N is said to be *super-revlex* (in $S^{(i)}$) if it is revlex and $u \in N$ implies $v \in N$ for any monomial $v \in S^{(i)}$ of degree $\leq \deg u - 1$. A *multicomplex* is a set of monomials $N \subset S^{(i)}$ satisfying that $u \in N$ and $v | u$ imply $v \in N$. Thus a multicomplex is the complement of the set of monomials in a monomial ideal. Note that super-revlex sets are multicomplexes.

Let $I \in \mathcal{L}(U; c)$ and $\mathcal{M}_I$ be the set of monomials in $U \setminus I$. Then we can uniquely write

$$\mathcal{M}_I = \delta_1 M_{\langle 1 \rangle} \uplus \delta_2 M_{\langle 2 \rangle} \uplus \cdots \uplus \delta_t M_{\langle t \rangle},$$

where $M_{\langle i \rangle} \subset S^{(i)}$ and $\uplus$ denotes the disjoint union. The following facts are obvious:

Lemma 3.8. (i) *Each $M_{\langle i \rangle}$ is a revlex multicomplex.*

(ii) *If $\delta_i M_{\langle i \rangle}$ has a monomial of degree d then $\delta_{i+1} M_{\langle i+1 \rangle}$ contains all monomials of degree d in $\delta_{i+1} S^{(i+1)}$ for all d.*

Lemma 3.8(ii) is equivalent to saying that if $M_{\langle i \rangle}$ contains a monomial of degree d then $M_{\langle i+1 \rangle}$ contains all monomials of degree $d - a_{i+1}$ in $S^{(i+1)}$.

We say that a set of monomials

$$M = \delta_1 M_{\langle 1 \rangle} \uplus \delta_2 M_{\langle 2 \rangle} \uplus \cdots \uplus \delta_t M_{\langle t \rangle} \subset U,$$

where $M_{\langle i \rangle} \subset S^{(i)}$, is a *ladder set* if it satisfies conditions (i) and (ii) of Lemma 3.8. The next result is the key result in this paper:

Proposition 3.9. *Let $U \subset S$ be a universal lex ideal. For any integer $c \geq 0$, there exists a ladder set $N \subset U$ with $\#N = c$ such that for any ladder set $M \subset U$ with $\#M = c$ one has*

$$m(N) \succeq m(M).$$

We prove Proposition 3.9 in Section 6. Here, we prove Theorem 1.1 by using Proposition 3.9.

Proof of Theorem 1.1. Let $U \subset S$ be a proper universal lex ideal with $P_U(t) = p(t)$ and $\bar{U} = U \cap \bar{S}$. Let $c = \dim_K(\operatorname{sat} \bar{U} / \bar{U})$. For any lex ideal $I \subset \operatorname{sat} \bar{U}$, let $\mathcal{M}_I$ be the set of monomials in $(\operatorname{sat} \bar{U} \setminus I)$.

Let $N \subset \operatorname{sat} \bar{U}$ be a ladder set of monomials with $\#N = c$ given in Proposition 3.9. Consider the ideal $J \subset \bar{S}$ generated by all monomials in $\operatorname{sat} \bar{U} \setminus N$. Then $J \subset \operatorname{sat} \bar{U}$ and $\mathcal{M}_J = N$. In particular, $J \in \mathcal{L}(U)$.

Let $L = JS$. By construction, $P_L(t) = P_U(t) = p(t)$. We claim that L satisfies the desired conditions. Let $I \subset S$ be a saturated graded ideal with $P_I(t) = p(t)$. By Lemmas 3.1 and 3.4, we may assume that I is a 1-lexicographic ideal with $\bar{I} \in \mathcal{L}(U) = \mathcal{L}(\operatorname{sat} \bar{U}; c)$. Since $\mathcal{M}_{\bar{I}}$ is a ladder set, by the choice of J, $m(\mathcal{M}_J) \succeq m(\mathcal{M}_{\bar{I}})$. Then, by Corollary 3.6,

$$\beta_i^S(L) = \beta_i^{\bar{S}}(J) \geq \beta_i^{\bar{S}}(\bar{I}) = \beta_i^S(I)$$

for all i, as desired. □

Another interesting corollary of Proposition 3.9 is:

Corollary 3.10. *Let $U \subset S$ be a universal lex ideal and $c \geq 0$. There exists a lex ideal $L \subset U$ with $\dim_K U/L = c$ such that, for any graded ideal $I \subset U$ with $\dim_K U/I = c$, one has $\beta_i^S(L) \geq \beta_i^S(I)$ for all i.*

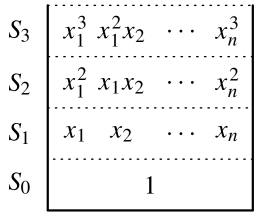
Proof of Theorem 1.2. Let I be an ideal in a regular local ring $(R, \boldsymbol{m}, K)$ such that $\boldsymbol{p}_{R/I}(t) = \boldsymbol{p}(t)$. Then the associated graded ring $\operatorname{gr}_{\boldsymbol{m}}(R/I)$ has the same Hilbert–Samuel polynomial as R/I. Also, we may regard $\operatorname{gr}_{\boldsymbol{m}}(R/I)$ as a quotient of a standard graded polynomial ring $S = K[x_1, \ldots, x_n]$ (see [Bruns and Herzog 1998, Proposition 2.2.5]), and it is known that $\beta_i^R(R/I) \leq \beta_i^S(\operatorname{gr}_{\boldsymbol{m}}(R/I))$ for all i (see [Robbiano 1981; Herzog et al. 1986]).

Let $S' = S[x_{n+1}]$. By adjoining a variable to $\operatorname{gr}_{\boldsymbol{m}}(R/I)$ we obtain a graded ring that is isomorphic to S'/J for a saturated graded ideal $J \subset S'$. Then $\boldsymbol{p}_{\operatorname{gr}_{\boldsymbol{m}}(R/I)}(t)$ is equal to the Hilbert polynomial of S'/J and $\beta_i^S(\operatorname{gr}_{\boldsymbol{m}}(R/I)) = \beta_i^{S'}(S'/J)$ for all i. Let $L' \subset S'$ be the saturated ideal with the same Hilbert polynomial as J given in Theorem 1.1. Observe that L' has no generators which are divisible by x_{n+1} by the construction given in the proof of Theorem 1.1.

Let $L \subset A = K[\![x_1, \ldots, x_n]\!]$ be a monomial ideal having the same generators as L'. We claim that L satisfies the desired conditions. By construction, the Hilbert–Samuel polynomial of A/L is equal to the Hilbert polynomial of S'/L' and $\beta_i^A(A/L) = \beta_i^{S'}(S'/L')$ for all i. Since $\beta_i^R(R/I) \leq \beta_i^{S'}(S'/J) \leq \beta_i^{S'}(S'/L')$ and $\boldsymbol{p}_{R/I}(t) = P_{S'/J}(t) = P_{S'/L'}(t)$, the ideal L satisfies the desired conditions. □

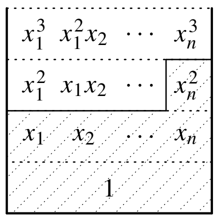
4. Some tools to study max sequence

In this section, we introduce some tools to study $m(-)$. Let $S = K[x_1, \ldots, x_n]$ and $\hat{S} = K[x_2, \ldots, x_n]$. From now on, we identify vector spaces spanned by monomials (such as polynomial rings and monomial ideals) with the set of monomials in the spaces. First, we introduce pictures, which help to understand the proofs. We associate with the set of monomials in S the following picture:



Each block represents a set of monomials in S of a fixed degree ordered by the lex order. We represent a set of monomials $M \subset S$ by a shaded picture so that the set of monomials in the shade is equal to M. For example, here is a representation of the set $M = \{1, x_1, x_2, \ldots, x_n, x_n^2\}$:

$M =$



Definition 4.1. We define the *opposite degree lex order* $>_{\text{opdlex}}$ by $u >_{\text{opdlex}} v$ if

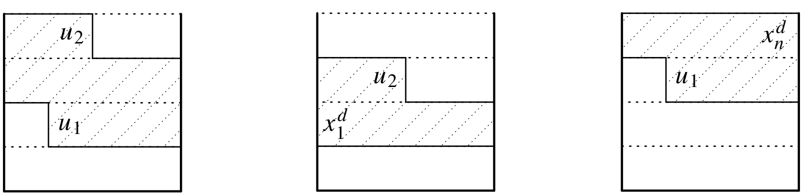
(i) $\deg u < \deg v$ or

(ii) $\deg u = \deg v$ and $u >_{\text{lex}} v$.

For monomials $u_1 \geq_{\text{opdlex}} u_2$, let

$$[u_1, u_2] = \{v \in S : u_1 \geq_{\text{opdlex}} v \geq_{\text{opdlex}} u_2\}.$$

A set of monomials $M \subset S$ is called an *interval* if $M = [u_1, u_2]$ for some monomials $u_1, u_2 \in S$. Moreover, we say that M is a *lower lex set of degree* d if $M = [x_1^d, u_2]$, and that M is an *upper revlex set of degree* d if $M = [u_1, x_n^d]$ (see figure).



Interval Lower lex set Upper rev-lex set

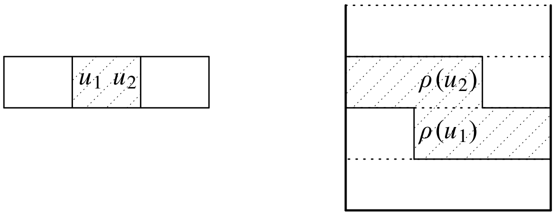
A benefit of considering pictures is that we can visualize the map $\rho : S \to \hat{S}$ defined as follows. For any monomial $x_1^k u \in S$ with $u \in \hat{S}$, let

$$\rho(x_1^k u) = u.$$

This induces a bijection

$$\rho : S_d = \bigoplus_{k=0}^{d} x_1^k \hat{S}_{d-k} \longrightarrow \hat{S}_{\le d} = \bigoplus_{k=0}^{d} \hat{S}_k$$
$$x_1^k u \longrightarrow u.$$

It is easy to see that if $[u_1, u_2] \subset S_d$ then $\rho([u_1, u_2]) = [\rho(u_1), \rho(u_2)]$ is an interval in $\hat{S}$:



$[u_1, u_2] \subset S_d$ $\qquad\qquad$ $\rho([u_1, u_2]) \subset \hat{S}_{\le d}$

In particular:

Lemma 4.2. *Let $M \subset S_d$ be a set of monomials.*

(i) *If M is lex then $\rho(M)$ is a lower lex set of degree* 0 *in $\hat{S}$.*

(ii) *If M is revlex then $\rho(M)$ is an upper revlex set of degree d in $\hat{S}$.*

We define $\max(1) = 1$ in S and $\max(1) = 2$ in $\hat{S}$. For any monomial $u \in S_d$ with $u \ne x_1^d$, one has $\max(u) = \max(\rho(u))$. Hence:

Lemma 4.3. *Let $M \subset S_d$ be a set of monomials. One has $m(M) \succeq m(\rho(M))$. Moreover, if $x_1^d \notin M$ then $m(M) = m(\rho(M))$.*

Lemma 4.4 (Interval Lemma). *Let $[u_1, u_2]$ be an interval in S, $0 \le a \le \deg u_1$, and $b \ge \deg u_2$. Let $L \subset S$ be the lower lex set of degree a and R the upper revlex set of degree b with $\#L = \#R = \#[u_1, u_2]$. Then*

$$m(L) \succeq m\big([u_1, u_2]\big) \succeq m(R).$$

Proof. We use double induction on n and $\#[u_1, u_2]$. The statement is obvious if $n = 1$ or if $\#[u_1, u_2] = 1$. Suppose $n > 1$ and $\#[u_1, u_2] > 1$.

Case 1. We first prove the statement when $[u_1, u_2]$, L, and R are contained in a single component S_d for some degree d. We may assume $L \ne [u_1, u_2]$ and $L \ne R$. Then, since $x_1^d \notin [u_1, u_2]$, $m([u_1, u_2]) = m(\rho([u_1, u_2]))$ and $m(R) = m(\rho(R))$. Since $\rho(L) \subset \hat{S}_{\le d}$ is a lower lex set of degree 0, $\rho([u_1, u_2]) \subset \hat{S}_{\le d}$ is an interval, and $\rho(R) \subset \hat{S}_{\le d}$ is an upper revlex set of degree d in $\hat{S}$. By the induction hypothesis, we have

$$m(L) \succeq m(\rho(L)) \succeq m(\rho([u_1, u_2])) \succeq m(\rho(R)) = m(R).$$

Then the statement follows since $m\big(\rho([u_1, u_2])\big) = m([u_1, u_2])$.

Case 2. Now we prove the statement in general. We first prove the statement for L. We identify S_i with the set of monomials in S of degree i. Suppose $\#[u_1, u_2] > \#S_a$. Then there exist $u_1', u_2' \in S$ such that

$$[u_1, u_2] = [u_1, u_2'] \uplus [u_1', u_2]$$

and $\#[u_1, u_2'] = \#S_a$. Let L' be the lower lex set of degree $a+1$ with $\#L' = \#[u_1', u_2]$. By the induction hypothesis, $m(S_a) \succeq m([u_1, u_2'])$ and $m(L') \succeq m([u_1', u_2])$. Thus

$$m([u_1, u_2]) \preceq m(S_a \uplus L') = m(L).$$

Suppose $\#[u_1, u_2] \le \#S_a$. Then $L \subset S_a$. Let $d = \deg u_1$ and $A \subset S_d$ be the lex set with $\#A = \#[u_1, u_2]$. Then $A = x_1^{d-a} L$. Since $m(A) = m(L)$, what we must prove is:

$$m(A) \succeq m([u_1, u_2]).$$

Since $\#[u_1, u_2] \le \#S_a \le \#S_{d+1}$, we have $\deg u_2 \le d+1$.

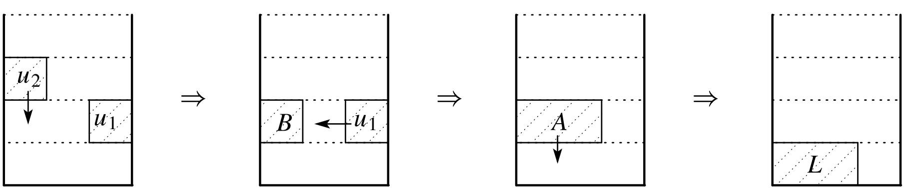
If $\deg u_2 = d$ then $[u_1, u_2] \subset S_d$. Then the desired inequality follows from Case 1. Suppose $\deg u_2 = d+1$. Then

$$[u_1, u_2] = [u_1, x_n^d] \uplus [x_1^{d+1}, u_2].$$

Recall $\#[u_1, u_2] \le \#S_a \le \#S_d$. Let $B \subset S_d$ be the lex set with $\#B = \#[x_1^{d+1}, u_2]$. Then $[x_1^{d+1}, u_2] = x_1 B$. Since $\#B + \#[u_1, x_n^d] = \#[u_1, u_2] \le \#S_d$, $B \cap [u_1, x_n^d] = \varnothing$. Then, by Case 1,

$$m([u_1, u_2]) = m(B) + m\big([u_1, x_n^d]\big) \preceq m(A)$$

(see figure).



$[u_1, u_2]$ $\quad$ $B \uplus [u_1, x_n^d]$ $\quad$ A $\quad$ L

Next, we prove the statement for R. In the same way as in the proof for L, we may assume $\#[u_1, u_2] \le \#S_b$. Let $d = \deg u_2$.

If $\deg u_1 = d$ then $[u_1, u_2] \subset S_d$ and $A = x_1^{b-d}[u_1, u_2]$ is an interval in S_b. Then, by Case 1, we have $m([u_1, u_2]) = m(A) \succeq m(R)$ as desired. Suppose $\deg u_1 < d$. Then

$$[u_1, u_2] = [u_1, x_n^{d-1}] \uplus [x_1^d, u_2].$$

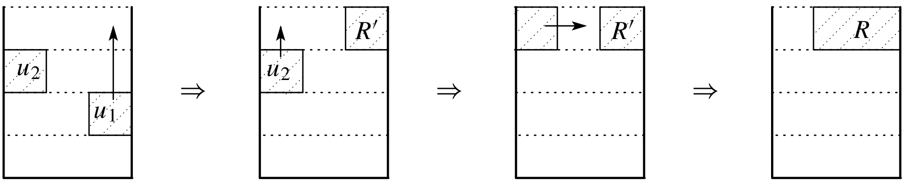
Let R' be the upper revlex set of degree b in S with $\#R' = \#[u_1, x_n^{d-1}]$. Then,

$$m([u_1,u_2]) \succeq m(R') + m\big([x_1^d, u_2]\big) = m(R') + m\big([x_1^b, x_1^{b-d}u_2]\big),$$

where the first inequality follows from the induction hypothesis on the cardinality. Since $R \setminus R' \subset S_b$ is an interval and $[x_1^b, x_1^{b-d}u_2] \subset S_b$ is lex, by Case 1 we have

$$m(R') + m\big([x_1^b, x_1^{b-d}u_2]\big) \succeq m(R') + m(R \setminus R') = m(R),$$

as desired (see figure). □



$[u_1,u_2]$ $\quad$ $R' \uplus [x_1^d, u_2]$ $\quad$ $R' \uplus [x_1^b, x_1^{b-d}u_2]$ $\quad$ R

Recall that a set $M \subset S$ of monomials is said to be super-revlex if it is revlex and $u \in M$ implies $v \in M$ for any monomial $v \in S$ of degree $\le \deg u - 1$.

Corollary 4.5. *Let $R \subset S$ be an upper revlex set of degree d and $M \subset S$ a super-revlex set such that $\#R + \#M \le \#S_{\le d}$. Let $Q \subset S$ be the super-revlex set with $\#Q = \#R + \#M$. Then*

$$m(Q) \succeq m(R) + m(M).$$

Proof. Let $e = \min\{k : x_1^k \notin M\}$ and $F = \{u \in S_e : u \notin M\}$. If $\#F \ge \#R$ then

$$Q = M \uplus (Q \setminus M)$$

and $Q \setminus M \subset F$ is an interval. Thus $m(Q \setminus M) \succeq m(R)$ by the interval lemma.

Suppose $\#F < \#R$. Write

$$R = I \uplus R'$$

such that I is an interval with $\#I = \#F$ and R' is an upper revlex set of degree d. Since F is a lex set, the interval lemma shows

$$m(M) + m(R) = m(M) + m(I) + m(R') \preceq m(F \uplus M) + m(R').$$

Then $F \uplus M$ is a super-revlex set containing x_1^e. By repeating this procedure, we have $m(M) + m(R) \preceq m(Q)$. □

The above corollary proves the next result, which was essentially proved by Elías, Robbiano and Valla [Elías et al. 1991].

Corollary 4.6. *Let $R \subset S$ be a finite revlex set of monomials and $M \subset S$ the super-revlex set with $\#M = \#R$. Then $m(M) \succeq m(R)$.*

Proof. Let $R = \biguplus_{i=0}^{N} R_i$, where R_i is the set of monomials in R of degree i and $N = \max\{i : R_i \neq \varnothing\}$. Let $M_{(\le j)}$ be the super-revlex set with $\#M_{(\le j)} = \#\biguplus_{i=0}^{j} R_i$. We claim $m(M_{(\le j)}) \succeq m\big(\biguplus_{i=0}^{j} R_i\big)$ for all j. This follows inductively from Corollary 4.5 as follows:

$$m\Bigg(\biguplus_{i=0}^{j} R_i\Bigg) = m\Bigg(\biguplus_{i=0}^{j-1} R_i\Bigg) + m(R_j) \preceq m(M_{(\le j-1)}) + m(R_j) \preceq m(M_{(\le j)}).$$

(We use the induction hypothesis for the second step and use Corollary 4.5 for the last step.) Then we have $m(M) = m(M_{(\le N)}) \succeq m\big(\biguplus_{i=0}^{N} R_i\big)$. □

We finish this section by proving the result of Valla, which we mentioned in the introduction.

Corollary 4.7 [Valla 1994]. *Let c be a positive integer and $M \subset S$ the super-revlex set with $\#M = c$. Let $J \subset S$ be the monomial ideal generated by all monomials which are not in M. Then, for any homogeneous ideal $I \subset S$ with $\dim_K(S/I) = c$, we have $\beta_i^S(S/J) \ge \beta_i^S(S/I)$ for all i.*

Proof. The proof is similar to that of Corollary 3.6. By the Bigatti–Hulett–Pardue theorem, we may assume that I is lex. Then Lemma 3.5 says, for $d \gg 0$, we have

$$\beta_i^S(I) = \binom{n-1}{i} \dim_K I_{\le d} - \sum_{k=1}^{n} \binom{k-1}{i} m_{\le k}(I_{\le d-1}) - \sum_{k=1}^{n-1} \binom{k-1}{i-1} m_{\le k}(I_{\le d})$$

and the same formula holds for J. Let $N \subset S$ be the set of monomials which are not in I. Since N is a revlex set with $\#N = c$, for $d \gg 0$, by Corollary 4.6 we have

$$m(J_{\le d}) = m(S_{\le d}) - m(M) \preceq m(S_{\le d}) - m(N) = m(I_{\le d}).$$

Hence $\beta_i^S(J) \ge \beta_i^S(I)$ for all i as desired. □

The proof given in this section provides a new short proof of the above result. The most difficult part in the proof is Corollary 4.6. The original proof given in [Elías et al. 1991] is based on computations of binomial coefficients. On the other hand, our proof is based on moves of interval sets of monomials.

5. Construction

In this section, we give a construction of sets of monomials which satisfy the conditions of Proposition 3.9, and study their properties.

Throughout Sections 5 and 6, we fix the following notation: Let $a_1, a_2, \dots, a_t$ be nonnegative integers, where $t \le n$, and let $b_i = a_1 + \cdots + a_i + 1$ for $i = 1, 2, \dots, t$.

Let $F = S\boldsymbol{e}_1 \oplus S\boldsymbol{e}_2 \oplus \cdots \oplus S\boldsymbol{e}_t$ be a free S-module with $\deg \boldsymbol{e}_i = b_i$ for $i = 1, 2, \dots, t$. We consider the set

$$U = S^{(1)}\boldsymbol{e}_1 \uplus S^{(2)}\boldsymbol{e}_2 \uplus \cdots \uplus S^{(t)}\boldsymbol{e}_t \subset F.$$

Note that we identify each $S^{(k)}$ with the set of monomials in it. For $i = 1, 2, \dots, t$, let $\delta_i = x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_i^{a_i+1}$. Then, by the decomposition given before Definition 3.7, the above set U can be identified with the set of monomials in the universal lex ideal $(\delta_1, \dots, \delta_t) = \delta_1 S^{(1)} \oplus \cdots \oplus \delta_t S^{(t)}$ via the natural correspondence $u\boldsymbol{e}_i \leftrightarrow \delta_i u$.

We call an element $u\boldsymbol{e}_i \in U$ a monomial in U. For each monomial $u\boldsymbol{e}_i \in U$, we define

$$\max(u\boldsymbol{e}_i) = \begin{cases} i & \text{if } u = 1, \\ \max(u) & \text{otherwise.} \end{cases}$$

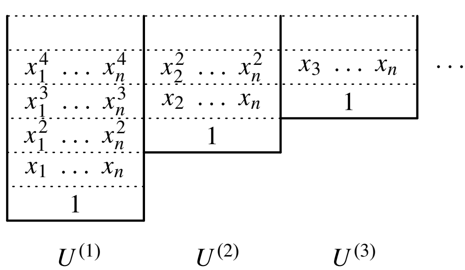
Also, for $M \subset U$, we define $m(M) = (m_{\le 1}(M), m_{\le 2}(M), \dots, m_{\le n}(M))$ in the same way as in Section 3. We say that a subset $M = M_{\langle 1\rangle}\boldsymbol{e}_1 \uplus \cdots \uplus M_{\langle t\rangle}\boldsymbol{e}_t \subset U$ is a *ladder set* if $M_{\langle 1\rangle}, \dots, M_{\langle t\rangle}$ satisfy the conditions (i) and (ii) of Lemma 3.8. Then, considering $m(-)$ of ladder sets in $U = S^{(1)}\boldsymbol{e}_1 \uplus \cdots \uplus S^{(t)}\boldsymbol{e}_t$ is equivalent to considering $m(-)$ of ladder sets in the universal lex ideal $(\delta_1, \dots, \delta_t) = \delta_1 S^{(1)} \oplus \cdots \oplus \delta_t S^{(t)}$. In particular, to prove Proposition 3.9, it is enough to consider ladder sets in U.

Let $M \subset U$. We write

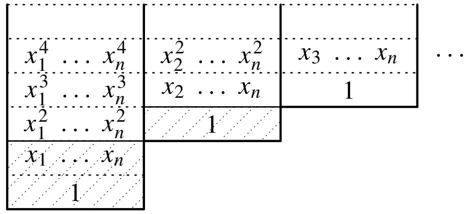
$$U^{(i)} = S^{(i)}\boldsymbol{e}_i,\ M^{(i)} = M \cap U^{(i)},\ U^{(\ge i)} = \biguplus_{k=i}^{t} S^{(k)}\boldsymbol{e}_k,\ \text{and } M^{(\ge i)} = M \cap U^{(\ge i)}.$$

Note that $U^{(\ge i)} = \biguplus_{k \ge i} S^{(k)}\boldsymbol{e}_k$ can be identified with the universal lex ideal in $K[x_i, \dots, x_n]$ generated by $\{(x_i^{b_{i-1}})x_i^{a_i} \cdots x_{k-1}^{a_{k-1}} x_k^{a_k+1} : k = i, i+1, \dots, t\}$. For a subset $M \subset U$, we write M_k for the set of monomials in M of degree k and $M_{\le j} = \biguplus_{k=0}^{j} M_k$.

As in Section 4, we use pictures to help to understand the proofs. We identify U with the following picture:



Note that each low represents the set of monomials in U having the same degree. Thus, in the previous figure, $\deg \boldsymbol{e}_2 = \deg \boldsymbol{e}_1 + 2$ and $\deg \boldsymbol{e}_3 = \deg \boldsymbol{e}_2 + 1$. Also, we present a subset $M \subset U$ by a shaded picture. For example, the following figure represents $M = \{1, x_1, x_2, \dots, x_n\}\boldsymbol{e}_1 \uplus \{1\}\boldsymbol{e}_2$:



M

Also, we define the map $\rho : U \to U$ by extending the map given in Section 4 as follows: For $x_i^k u\boldsymbol{e}_i \in U^{(i)}$ with $u \in K[x_{i+1}, \dots, x_n]$, let

$$\rho(x_i^k u\boldsymbol{e}_i) = \begin{cases} u\boldsymbol{e}_{i+1} & \text{if } i \le t-1, \\ 0 & \text{if } i = t. \end{cases}$$

We call the above map $\rho : U \to U$ the *moving map* of U. The moving map induces a bijection from $U_j^{(i)} = \{u\boldsymbol{e}_i \in U^{(i)} : \deg u = j - b_i\}$ to $U_{\le j+a_{i+1}}^{(i+1)} = \{u\boldsymbol{e}_{i+1} \in U^{(i+1)} : \deg u \le j - b_i\}$ for $i = 1, 2, \dots, t-1$.

Lemma 5.1. *For $N \subset U_j^{(i)}$ with $i \le t-1$, one has $m(N) \succeq m(\rho(N))$. Moreover, if $x_i^{j-b_i}\boldsymbol{e}_i \notin N$ then $m(N) = m(\rho(N))$.*

Next, we define ladder sets $M \subset U$ which attain maximal Betti numbers. Recall that a subset $M \subset U$ is called a ladder set if the following conditions hold:

(i) $\{u \in S^{(i)} : u\boldsymbol{e}_i \in M^{(i)}\}$ is a revlex multicomplex for $i = 1, 2, \dots, t$.

(ii) If $M_j^{(i)} \ne \varnothing$ then $M_j^{(i+1)} = U_j^{(i+1)}$ for $i = 1, 2, \dots, t-1$ and for all $j \ge 0$.

To simplify the notation, we say that $N \subset U^{(i)}$ is a super-revlex set (resp. interval, lower lex set or upper revlex set of degree d) if $N' = \{u \in S^{(i)} : u\boldsymbol{e}_i \in N\}$ is super-revlex (resp. interval, lower lex set or upper revlex set of degree $d - b_i$) in $S^{(i)}$. For monomials $u\boldsymbol{e}_i, v\boldsymbol{e}_i \in U$ and for a monomial order $>$ on $S^{(i)}$, we write $u\boldsymbol{e}_i > v\boldsymbol{e}_i$ if $u > v$.

Definition 5.2. A monomial $f = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\boldsymbol{e}_1 \in U_e^{(1)}$ is said to be *admissible over U* if the following conditions hold:

(i) $\deg \rho^i(f) \le e+1$ or $\rho^i(f) = \boldsymbol{e}_{i+1}$ for $i = 1, 2, \dots, t-2$.

(ii) $\rho^{t-1}(f) = \boldsymbol{e}_t$ or $\rho^{t-1}(f) \ge_{\text{opdlex}} x_t^{e+1-b_t}\boldsymbol{e}_t$.

Note that the second condition in (ii) cannot be satisfied when $e+1-b_t < 0$ and that if $t = 1$ then all monomials in U are admissible. Also, $\rho^{t-1}(f) \ge_{\text{opdlex}} x_t^{e+1-b_t}\boldsymbol{e}_t$ if and only if $\deg \rho^{t-1}(f) \le e$ or $\rho^{t-1}(f) = x_t^{e+1-b_t}\boldsymbol{e}_t$.

We say that $f \in U_e^{(i)}$ is admissible if it is admissible over $U^{(\ge i)}$. Note that $x_i^k\boldsymbol{e}_i \in U^{(i)}$ is admissible for all i and k.

Definition 5.3. Let $>_{\mathrm{dlex}}$ be the degree lex order. Thus for monomials $u, v \in S$, $u >_{\mathrm{dlex}} v$ if $\deg u > \deg v$ or $\deg u = \deg v$ and $u >_{\mathrm{lex}} v$. We extend $>_{\mathrm{dlex}}$ to monomials in U by $u\boldsymbol{e}_i >_{\mathrm{dlex}} v\boldsymbol{e}_j$ if $\delta_i u >_{\mathrm{dlex}} \delta_j v$. Thus, we have $u\boldsymbol{e}_i >_{\mathrm{dlex}} v\boldsymbol{e}_j$ if

(i) $\deg u\boldsymbol{e}_i > \deg v\boldsymbol{e}_j$,

(ii) $\deg u\boldsymbol{e}_i = \deg v\boldsymbol{e}_j$ and $i < j$, or

(iii) $\deg u\boldsymbol{e}_i = \deg v\boldsymbol{e}_j$, $i = j$ and $u >_{\mathrm{dlex}} v$.

Fix an integer $c > 0$. Let

$$f = \max_{>_{\mathrm{dlex}}} \left\{ g \in U^{(1)} : g \text{ is admissible and } \#\{h \in U : h \leq_{\mathrm{dlex}} g\} \leq c \right\}$$

and

$$L_{(c)} = \{h \in U^{(1)} : h \leq_{\mathrm{dlex}} f\}.$$

Let $M = M^{(1)} \uplus \cdots \uplus M^{(t)} \subset U$ be a set of monomials with $\#M = c$. We say that M satisfies the *maximal condition* if $M^{(1)} = L_{(c)}$. Also, we say that M is *extremal* if $M^{(\geq k)} \subset U^{(\geq k)}$ satisfies the maximal condition in $U^{(\geq k)}$ for all k.

Example 5.4. If $t = 1$ then any monomial in $U = S^{(1)}\boldsymbol{e}_1$ is admissible and extremal sets can be identified with super-revlex sets in $S^{(1)}$.

Example 5.5. Suppose $t = 2$. Then $f = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \boldsymbol{e}_1$, where $f \neq x_1^{\alpha_1}\boldsymbol{e}_1$, is admissible in $U = S^{(1)}\boldsymbol{e}_1 \uplus S^{(2)}\boldsymbol{e}_2$ if $\alpha_1 \geq a_2$ or $f = x_1^{a_2-1} x_2^{\alpha_2}\boldsymbol{e}_1$. In other words, a monomial $f \in S_d^{(1)}\boldsymbol{e}_1$ is admissible if and only if $f \geq_{\mathrm{lex}} x_1^{a_2-1} x_2^{d-a_2+1}\boldsymbol{e}_1$ if $a_2 \leq d$ and $f = x_1^d \boldsymbol{e}_1$ if $a_2 > d$. For example, if $\deg \boldsymbol{e}_1 = 2$ and $\deg \boldsymbol{e}_2 = 4$ then the admissible monomials in $U_5^{(1)} = (S_3^{(1)})\boldsymbol{e}_1$ are

$$x_1^3\boldsymbol{e}_1,\ x_1^2x_2\boldsymbol{e}_1,\ x_1^2x_3\boldsymbol{e}_1,\ \ldots,\ x_1^2x_n\boldsymbol{e}_1,\ x_1x_2^2\boldsymbol{e}_1.$$

Example 5.6. Suppose $t = 3$. The situation is more complicated. A monomial $f = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \boldsymbol{e}_1 \in U_e^{(1)}$, where $f \neq x_1^{\alpha_1}\boldsymbol{e}_1$ is admissible in U if and only if

- $\alpha_1 \geq a_2 - 1$ and
- $x_3^{\alpha_3} \cdots x_n^{\alpha_n} \geq_{\mathrm{opdlex}} x_3^{e+1-b_3}$ or $x_3^{\alpha_3} \cdots x_n^{\alpha_n} = 1$.

For example, if $\deg \boldsymbol{e}_1 = 2$, $\deg \boldsymbol{e}_2 = 4$, $\deg \boldsymbol{e}_3 = 6$, and $n = 3$ then the set of the admissible monomials in $U_6^{(1)} = (K[x_1, x_2, x_3]_4)\boldsymbol{e}_1$ are

$$\{x_1^4\boldsymbol{e}_1\} \cup \{x_1^3x_2\boldsymbol{e}_1, x_1^3x_3\boldsymbol{e}_1\} \cup \{x_1^2x_2^2\boldsymbol{e}_1, x_1^2x_2x_3\boldsymbol{e}_1\} \cup \{x_1x_2^3\boldsymbol{e}_1, x_1x_2^2x_3\boldsymbol{e}_1\}.$$

Example 5.7. Let $U = x_1^2 S^{(1)} \uplus x_1x_2^3 S^{(2)}$. Suppose $c = \binom{n+2}{2} + 2$. Then

$$\max_{>_{\mathrm{dlex}}} \left\{ f \in U^{(1)} : f \text{ is admissible and } \#\{h \in U : h \leq_{\mathrm{dlex}} f\} \leq c \right\} = x_1^2\boldsymbol{e}_1.$$

Indeed,

$$\#\{h \in U : h \leq_{\mathrm{dlex}} x_1^2\boldsymbol{e}_1\} = \#S_{\leq 2}^{(1)}\boldsymbol{e}_1 \uplus \{1\}\boldsymbol{e}_2 = \binom{n+2}{2} + 1$$

and

$$\#\{h \in U : h \leq_{\text{dlex}} x_1x_2^2\boldsymbol{e}_1\} = \#\big(S^{(1)}_{\leq 3} \setminus \{x_1^3, x_1^2x_2, \dots, x_1^2x_n\}\big)\boldsymbol{e}_1 \uplus S^{(2)}_{\leq 1}\boldsymbol{e}_2$$
$$= \binom{n+3}{3} > c.$$

By Example 5.5, the lex-smallest admissible monomial in $U^{(1)}_5$ is $x_1x_2^2\boldsymbol{e}_1$. Thus the extremal set $L \subset U$ with $\#L = c$ is

$$L = S^{(1)}_{\leq 2}\boldsymbol{e}_1 \uplus \{1, x_n\}\boldsymbol{e}_2.$$

Example 5.8. In general, it is not easy to understand the shape of extremal sets, but in some special cases they are simple.

If $b_1 = b_2 = \cdots = b_t$ then any monomial in U is admissible. Thus any extremal set M in U is of the form

$$M = \{h \in U : h \leq_{\text{dlex}} f\}$$

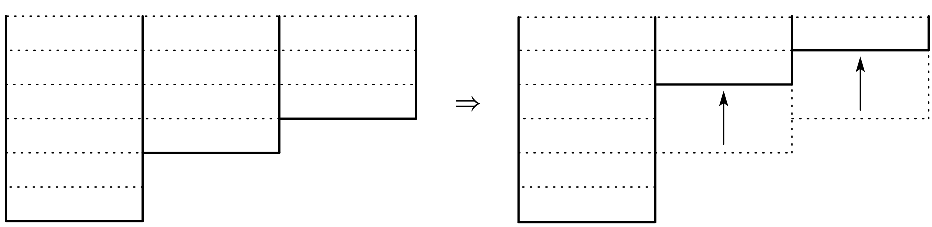
for some $f \in U$.

If $b_2 > e$ then the only admissible monomial in $U^{(1)}_e$ is $x_1^{e-b_1}\boldsymbol{e}_1$. Thus if $b_1 \ll b_2 \ll \cdots \ll b_n$ (for example, if $b_{i+1} - b_i > c$ for all i) then any extremal set M in U with $\#M = c$ is of the form

$$M = S^{(1)}_{\leq d_1}\boldsymbol{e}_1 \uplus S^{(2)}_{\leq d_2}\boldsymbol{e}_2 \uplus \cdots \uplus S^{(t-1)}_{\leq d_{t-1}}\boldsymbol{e}_{t-1} \uplus N,$$

where $N \subset S^{(t)}\boldsymbol{e}_t$ and $\#S^{(i+1)}_{\leq d_{i+1}}\boldsymbol{e}_{i+1} \uplus \cdots \uplus S^{(t-1)}_{\leq d_{t-1}}\boldsymbol{e}_{t-1} \uplus N < \#S^{(i)}_{d_i+1}$ for $i = 1, \dots, t-1$.

In the rest of this section, we study properties of extremal sets. Suppose $t \geq 3$. For an integer $k \geq -a_3$, we write $U^{(i)}[-k] = S^{(i)}\boldsymbol{e}'_i$, where $\boldsymbol{e}'_i$ is a basis element with $\deg \boldsymbol{e}_i = b_i + k$. In the picture, $U^{(i)}[-k]$ is the picture obtained from that of $U^{(i)}$ by moving the blocks k steps above. In particular, for any integer $k \geq -a_3$, $U' = U^{(2)} \uplus \biguplus_{i=3}^t U^{(i)}[-k]$ can be identified with a universal lex ideal in $K[x_2, \dots, x_n]$:



$U^{(\geq 2)}$ $\qquad\qquad$ $U' = U^{(2)} \uplus (\biguplus_{i=3}^t U^{(i)}[-k])$

Lemma 5.9. *Suppose $t \geq 3$. Let $f \in U^{(1)}_e$, $d = \deg \rho(f)$, and $k \geq -a_3$ with $e - d + k \geq 0$. Then f is admissible over U if and only if the following conditions hold*:

- $\deg \rho(f) \le e+1$ *or* $\rho(f) = \boldsymbol{e}_2$.
- $x_2^{e-d+k}\rho(f) \in U^{(2)}_{e+k}$ *is admissible in* $U' = U^{(2)} \uplus \biguplus_{i=3}^t U^{(i)}[-k]$.

Proof. Let $U' = S^{(2)}\boldsymbol{e}_2 \uplus S^{(3)}\boldsymbol{e}'_3 \uplus \cdots \uplus S^{(t)}\boldsymbol{e}'_t$ with $\deg \boldsymbol{e}'_i = \deg \boldsymbol{e}_i + k$ for $k = 3, \ldots, t$, and let ϕ be the moving map of U'. Let $\rho^i(f) = u_{i+1}\boldsymbol{e}_{i+1}$ for $i = 2, \ldots, t-1$. Then $\phi^i(x_2^{e-d+k}\rho(f)) = u_{i+2}\boldsymbol{e}'_{i+2}$ for $i = 1, 2, \ldots, t-2$. Thus $\deg \rho^i(f) \le e+1$ if and only if $\deg \phi^{i-1}(x_2^{e-d+k}\rho(f)) \le e+1+k$ for $i \ge 2$. Also, $\rho^{t-1}(f) \ge_{\mathrm{opdlex}} x_t^{e+1-b_t}\boldsymbol{e}_t$ if and only if $\phi^{t-2}(x_2^{e+d+k}\rho(f)) \ge_{\mathrm{opdlex}} x_t^{e+1-b_t}\boldsymbol{e}'_t$. Since $\deg x_2^{e-d+k}\rho(f) = e+k$, the above facts prove the statement. □

By the definition of the maximal condition, the next result is straightforward:

Lemma 5.10. *Let $M \subset U$ be an extremal set.*

(i) *If $\#M \ge \#U_{\le e}$ then $M \supset U_{\le e}$.*

(ii) *If $\#M \ge \#U^{(1)}_{\le e-1} \uplus U^{(\ge 2)}_{\le e}$ then $M \supset U^{(1)}_{\le e-1} \uplus U^{(\ge 2)}_{\le e}$.*

Proof. Since M is extremal, there exists an $f \in U^{(1)}$ such that

$$M^{(1)} = \{h \in U^{(1)} : h \le_{\mathrm{dlex}} f\}.$$

(i) Since $x_1^{e-b_1}\boldsymbol{e}_1$ is admissible and $\{h \in U : h \le_{\mathrm{dlex}} x_1^{e-b_1}\boldsymbol{e}_1\} = U_{\le e}$, $f \ge_{\mathrm{dlex}} x_1^{e-b_1}\boldsymbol{e}_1$. Then $M^{(1)} \supset \{h \in U^{(1)} : h \le_{\mathrm{dlex}} x_1^{e-b_1}\boldsymbol{e}_1\} = U^{(1)}_{\le e}$. Also, since

$$\#M^{(\ge 2)} = \#M - \#M^{(1)} \ge \#\{h \in U : h \le_{\mathrm{dlex}} f\} - \#\{h \in U^{(1)} : h \le_{\mathrm{dlex}} f\} \ge \#U^{(2)}_{\le e},$$

we have $M^{(\ge 2)} \supset U^{(2)}_{\le e}$ by induction on t.

(ii) It is clear that $M \supset U_{\le e-1}$ by (i). If $\deg f \ge e$ then

$$\#M \ge \#\{h \in U : h \le_{\mathrm{dlex}} f\} \ge \#M^{(1)} \uplus U^{(\ge 2)}_{\le e}.$$

Then $\#M^{(\ge 2)} \ge \#U^{(\ge 2)}_{\le e}$ and $M^{(\ge 2)} \supset U^{(\ge 2)}_{\le e}$ by (i) as desired. If $\deg f < e$ then $M^{(1)} = U^{(1)}_{\le e-1}$ and $\#M^{(\ge 2)} \ge \#U^{(\ge 2)}_{\le e}$ by the assumption. Hence $M^{(\ge 2)} \supset U^{(\ge 2)}_{\le e}$ by (i). □

Corollary 5.11. *Extremal sets are ladder sets.*

Proof. If $M \subset U$ is extremal then $M^{(i)}$ is super-revlex for all i by the maximal condition. It is enough to prove that if $M^{(1)}_e \ne \varnothing$ then $M \supset U^{(\ge 2)}_e$. If $M^{(1)}_e \ne \varnothing$ then there exists an admissible monomial $f \in U^{(1)}_e$ such that

$$\#M \ge \#\{h \in U : h \le_{\mathrm{dlex}} f\} \ge \#U^{(1)}_{\le e-1} \uplus U^{(\ge 2)}_{\le e}.$$

Then the statement follows from Lemma 5.10. □

To simplify notation, for $u\boldsymbol{e}_i, v\boldsymbol{e}_i \in U^{(i)}$ with $u \ge_{\mathrm{opdlex}} v$, we write

$$[u\boldsymbol{e}_i, v\boldsymbol{e}_i] = \{w\boldsymbol{e}_i \in U^{(i)} : u \ge_{\mathrm{opdlex}} w \ge_{\mathrm{opdlex}} v\}.$$

Lemma 5.12. *Suppose $t \geq 2$. Let $M \subset U$ be an extremal set.*

(i) *If $a_2 > 0$ then $M_e^{(1)} \neq 0$ if and only if $\#M \geq \#U_{\leq e}^{(1)}$.*

(ii) *If $a_2 = 0$ and $M_e^{(1)} \neq 0$ then $\#M > \#U_{\leq e}^{(1)}$.*

Proof. Let $f \in U_e^{(1)}$ be the lex-smallest admissible monomial in $U_e^{(1)}$ over U.

(i) It suffices to prove that

$$\#\{h \in U : h \leq_{\text{dlex}} f\} = \#U_{\leq e}^{(1)}. \tag{2}$$

If $f = x_1^{e-b_1}\boldsymbol{e}_1$ then $f' = x_1^{e-b_1-1}x_2\boldsymbol{e}_1$ is not admissible. By the definition of admissibility, one has $\deg \rho(f') = \deg x_2\boldsymbol{e}_2 > e+1$ and $b_2 > e$. In this case we have $\{h \in U : h \leq_{\text{dlex}} f\} = U_{\leq e}^{(1)}$.

Suppose $f \neq x_1^{e-b_1}\boldsymbol{e}_1$. We prove (2) by using induction on t. Suppose $t = 2$. Then $f = x_1^{a_2-1}x_2^{e+1-b_2}\boldsymbol{e}_1$, and

$$\{h \in U : h \leq_{\text{dlex}} f\} = U_{\leq e-1}^{(1)} \uplus [f, x_n^{e-b_1}\boldsymbol{e}_1] \uplus U_{\leq e}^{(2)}.$$

Since $\rho([f, x_n^{e-b_1}\boldsymbol{e}_1]) = \biguplus_{j=e+1}^{e+a_2} U_j^{(2)}$, we have

$$\#\{h \in U : h \leq_{\text{dlex}} f\} = \#U_{\leq e-1}^{(1)} + \#U_{\leq e+a_2}^{(2)} = \#U_{\leq e}^{(1)},$$

where we use $\rho(U_e^{(1)}) = U_{\leq e+a_2}^{(2)}$ for the last equality.

Suppose $t \geq 3$. Since $\rho(f) \neq \boldsymbol{e}_2$, we have $\deg \rho(f) = e+1$. Indeed, by Lemma 5.9, $\deg \rho(f) \leq e+1$. On the other hand, since $x_1^{a_2-1}x_2^{e+1-b_2}\boldsymbol{e}_1$ is admissible over U, $f \leq_{\text{lex}} x_1^{a_2-1}x_2^{e+1-b_2}\boldsymbol{e}_1$. Thus $\deg \rho(f) \geq \deg \rho(x_1^{a_2-1}x_2^{e+1-b_2}\boldsymbol{e}_1) = e+1$.

Consider $U' = U^{(2)} \uplus \biguplus_{i=3}^{t} U^{(i)}[-1]$. By Lemma 5.9 (consider the case when $d = e+1$ and $k = 1$), $\rho(f)$ is the lex-smallest admissible monomial in $U_{e+1}^{(2)}$ over U'. Then

$$\begin{aligned}
\#[\rho(f), x_n^{e+1-b_2}\boldsymbol{e}_2] \uplus U_{\leq e}^{(\geq 2)} &= \#[\rho(f), x_n^{e+1-b_2}\boldsymbol{e}_2] \uplus U_{\leq e}^{(2)} \uplus U'^{(\geq 3)}_{\leq e+1} \\
&= \#\{h \in U' : h \leq_{\text{dlex}} \rho(f)\} \\
&= \#U_{\leq e+1}^{(2)},
\end{aligned} \tag{3}$$

where the last equation follows from the induction hypothesis. On the other hand

$$\{h \in U : h \leq_{\text{dlex}} f\} = [f, x_n^{e-b_1}\boldsymbol{e}_1] \uplus U_{\leq e-1}^{(1)} \uplus U_{\leq e}^{(\geq 2)} \tag{4}$$

and

$$\rho\big([f, x_n^{e-b_1}\boldsymbol{e}_1]\big) = [\rho(f), x_n^{e+1-b_2}\boldsymbol{e}_2] \uplus \biguplus_{j=e+2}^{e+a_2} U_j^{(2)}. \tag{5}$$

Equations (3), (4), and (5) show that

$$\#\{h \in U : h \leq_{\text{dlex}} f\} = \#U_{\leq e-1}^{(1)} \uplus U_{\leq e+a_2}^{(2)} = \#U_{\leq e-1}^{(1)} \uplus U_e^{(1)} = \#U_{\leq e}^{(1)},$$

where the second equality follows since $\rho(U_e^{(1)}) = U_{\leq e+a_2}^{(2)}$.

(ii) It suffices to prove that $\#\{h \in U : h \leq_{\mathrm{dlex}} f\} > \#U^{(1)}_{\leq e}$. Since $a_2 = 0$, $\#U^{(2)}_{\leq e} = \#U^{(1)}_{e}$. Then we have

$$\#\{h \in U : h \leq_{\mathrm{dlex}} f\} > \#U^{(1)}_{\leq e-1} \uplus U^{(2)}_{\leq e} = \#U^{(1)}_{\leq e-1} \uplus U^{(1)}_{e} = U^{(1)}_{\leq e},$$

as desired. $\square$

Corollary 5.13. *Suppose $t \geq 2$. Let $B \subset U^{(1)}_e$ be the revlex set and $N \subset U^{(\geq 2)}$ a ladder set with $\#N \geq \#U^{(\geq 2)}_{\leq e-1}$. Let $Y \subset U$ be the extremal set with*

$$\#Y = \#U^{(1)}_{\leq e-1} \uplus B \uplus N.$$

If $\#B \uplus N < \#U^{(1)}_e$ then

$$Y = U^{(1)}_{\leq e-1} \uplus Y^{(\geq 2)}.$$

Proof. Since $\#Y \geq \#U_{\leq e-1}$, we have $Y \supset U_{\leq e-1}$ by Lemma 5.10. On the other hand, since $\#Y = \#U^{(1)}_{\leq e-1} \uplus B \uplus N < \#U^{(1)}_{\leq e}$ by the assumption, we have $Y^{(1)}_e = \varnothing$ by Lemma 5.12. Hence $Y^{(1)} = U^{(1)}_{\leq e-1}$. $\square$

For monomials $f >_{\mathrm{dlex}} g \in U^{(i)}$, let $[f, g) = [f, g] \setminus \{g\}$.

Lemma 5.14. *Let $f \in U^{(1)}_e$ be the lex-smallest admissible monomial in $U^{(1)}_e$ over U and $g >_{\mathrm{lex}} h \in U^{(1)}_e$ admissible monomials over U such that there are no admissible monomials in $[g, h]$ except for g and h. Then $\#[g, h) \leq \#[f, x_n^{e-b_1}\boldsymbol{e}_1]$.*

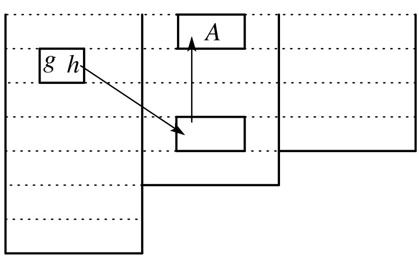
Proof. If $t = 1$ then all monomials are admissible over U. If $t = 2$ then any monomial $w \in U^{(1)}_e$ with $w >_{\mathrm{lex}} f$ is admissible over U. Thus the statement is clear if $t \leq 2$.

Suppose $t \geq 3$. Since $g \neq h$ we have $f \neq x_1^{e-b_1}\boldsymbol{e}_1$. By the definition of admissibility, we have $\deg(\rho(f)) = e$ if $a_2 = 0$ and $\deg(\rho(f)) = e + 1$ if $a_2 > 0$. We consider the case when $a_2 > 0$ (the proof for the case when $a_2 = 0$ is similar).

Consider $U' = U^{(2)} \uplus \biguplus_{i=3}^{t} U^{(i)}[-1]$. Since any monomial $w \in U^{(1)}_e$ such that $\rho(w) = x_2^k\boldsymbol{e}_2$ with $k \leq e + 1 - b_2$ is admissible over U, we have $\rho([g, h)) \subset S_d$ for some $d \leq e + 1$. Let

$$A = x_2^{e+1-d}\rho([g, h)) = \left[x_2^{e+1-d}\rho(g), x_2^{e+1-d}\rho(h)\right) \subset U^{(2)}_{e+1}$$

(see figure).



Let $w \in A$. Then $w = x_2^{e+1-d}\rho(w')$ for some $w' \in [g,h)$. Lemma 5.9 says that w is admissible over U' if and only if w' is admissible over U. Hence A contains no admissible monomial over U' except for $x_2^{e+1-d}\rho(g)$. By Lemma 5.9, $\rho(f) \in U^{(2)}_{e+1}$ is the lex-smallest admissible monomial in $U^{(2)}_{e+1}$ over U'. Then, by the induction hypothesis,

$$\#A \le \#[\rho(f), x_n^{e-b_2}\boldsymbol{e}_2] = \#\rho([f, x_n^{e-b_1}\boldsymbol{e}_1]) \cap U^{(2)}_{e+1} \le \#[f, x_n^{e-b_1}\boldsymbol{e}_1].$$

Then the statement follows since $\#[g,h) = \#\rho([g,h)) = \#A$. □

Lemma 5.15. *Let $M \subset U$ be an extremal set, $e = \min\{k : x_1^{k-b_1}\boldsymbol{e}_1 \notin M\}$, and $H = U_e \setminus M_e$. Let $f \in U^{(1)}_e$ be the lex-smallest admissible monomial in $U^{(1)}_e$ over U. Then*:

(i) $\#U_{\le e} + \#[f, x_n^{e-b_1}\boldsymbol{e}_1] \le \#U^{(1)}_{\le e+1}$.

(ii) $\#M + \#H < \#U^{(1)}_{\le e+1}$.

Proof. We use induction on t. If $t = 1$ then the statements are obvious. Suppose $t > 1$.

(i) If $a_2 > 0$ then by Lemma 5.12

$$\#U_{\le e} + \#[f, x_n^{e-b_1}\boldsymbol{e}_1] = \#\{h \in U : h \le_{\text{dlex}} f\} + \#U^{(1)}_e = \#U^{(1)}_{\le e} + \#U^{(1)}_e < \#U^{(1)}_{\le e+1}$$

as desired. Suppose $a_2 = 0$. Then

$$\rho([f, x_n^{e-b_1}\boldsymbol{e}_1]) = [\rho(f), x_n^{e-b_2}\boldsymbol{e}_1] \subset U^{(2)}_e$$

and $\rho(f)$ is the lex-smallest admissible monomial in $U^{(2)}_e$ over $U^{(\ge 2)}$ by Lemma 5.9. Then by the induction hypothesis,

$$\begin{aligned}\#U_{\le e} + \#[f, x_n^{e-b_1}\boldsymbol{e}_1] &= \#U^{(1)}_{\le e} + \left(\#U^{(\ge 2)}_{\le e} + \#[\rho(f), x_n^{e-b_2}\boldsymbol{e}_2]\right)\\ &\le \#U^{(1)}_{\le e} + \#U^{(2)}_{\le e+1}\\ &= \#U^{(1)}_{\le e+1}\end{aligned}$$

as desired.

(ii) Suppose $M^{(2)}_e \ne U^{(2)}_e$. Then $M^{(1)}_e = \varnothing$. Since $M^{(\ge 2)}$ is extremal over $U^{(\ge 2)}$, by the induction hypothesis,

$$\#M + \#H = \#U^{(1)}_{\le e-1} \uplus M^{(\ge 2)} + \#U^{(1)}_e \uplus H^{(\ge 2)} < \#U^{(1)}_{\le e} + \#U^{(2)}_{\le e+1} \le \#U^{(1)}_{\le e+1},$$

where we use $\#U^{(1)}_{e+1} = \#U^{(2)}_{\le e+1+a_2} \ge \#U^{(2)}_{\le e+1}$ for the last inequality.

Suppose $M^{(2)}_e = U^{(2)}_e$. Let $g = \max_{>_{\text{dlex}}} M^{(1)}$ and let

$$\mu = \min_{>_{\text{dlex}}}\{h \in U^{(1)}_{\le e} : h \text{ is admissible over } U \text{ and } h >_{\text{dlex}} g\}.$$

Then $[\mu, g) \subset U_e^{(1)}$ since $g \geq_{\mathrm{dlex}} x_1^{e-b_1-1}\boldsymbol{e}_1$. Since M is extremal,

$$\#M < \#\{h \in U : h \leq_{\mathrm{dlex}} \mu\}.$$

Since $M^{(1)} = \{h \in U^{(1)} : h \leq_{\mathrm{dlex}} g\}$, $H = [x_1^{e-b_1}\boldsymbol{e}_1, g)$. Thus

$$\begin{aligned}\#M + \#H &< \#\{h \in U : h \leq_{\mathrm{dlex}} \mu\} + \#[x_1^{e-b_1}\boldsymbol{e}_1, g)\\ &= \#U_{\leq e} + \#[\mu, g)\\ &\leq \#U_{\leq e} + \#[f, x_n^{e-b_1}\boldsymbol{e}_1],\end{aligned}$$

where the last inequality follows from Lemma 5.14. Then the desired inequality follows from (i). □

6. Proof of the main theorem

Let $U = S^{(1)}\boldsymbol{e}_1 \uplus S^{(2)}\boldsymbol{e}_2 \uplus \cdots \uplus S^{(t)}\boldsymbol{e}_t$ be as in Section 5. The aim of this section is to prove the next result, which proves Proposition 3.9.

Theorem 6.1. *Let $M \subset U$ be a ladder set and $L \subset U$ the extremal set with $\#L = \#M$. Then $m(L) \succeq m(M)$.*

The proof is by case analysis, and occupies the next three subsections.

In the rest of this section, we fix a ladder set $M \subset U$.

Preliminary of the proof. For two subsets $A, B \subset U$, we define

$$A \gg B \Leftrightarrow \#A = \#B \quad \text{and} \quad m(A) \succeq m(B).$$

Let $X \subset U^{(1)}$ be the super-revlex set with $\#X = \#M^{(1)}$. Then $\left\{k : M_k^{(1)} \neq \varnothing\right\} \supset \{k : X_k \neq \varnothing\}$. Thus $X \cup M^{(\geq 2)}$ is also a ladder set in U. Since $X \gg M^{(1)}$ by Corollary 4.6, we have:

Lemma 6.2. *There exists a ladder set $N \subset U$ such that $N^{(1)}$ is super-revlex and $N \gg M$.*

Thus, in the rest of this section we assume that $M^{(1)}$ is super-revlex. Let

$$e = \min\{k + b_1 : x_1^k\boldsymbol{e}_1 \notin M\}$$

and

$$f = \max_{>_{\mathrm{dlex}}} \left\{g \in U_{\leq e}^{(1)} : g \text{ is admissible over } U \text{ and } \#\{h \in U : h \leq_{\mathrm{dlex}} g\} \leq \#M\right\},$$

where $f = 0$ if $\#\{h \in U : h \leq_{\mathrm{dlex}} \boldsymbol{e}_1\} > \#M$. Since $x_1^{e-b_1-1}\boldsymbol{e}_1$ is admissible over U (when $e \neq b_1$), we have $f = x_1^{e-b_1-1}\boldsymbol{e}_1$ or $\deg f = e$. We will prove:

Proposition 6.3. *With the same notation as above, there exists a ladder set N such that $N \gg M$ and*

$$N^{(1)} = \{h \in U^{(1)} : h \leq_{\mathrm{dlex}} f\},$$

where $\{h \in U^{(1)} : h \leq_{\mathrm{dlex}} f\} = \varnothing$ if $f = 0$.

The above proposition proves Theorem 6.1. Indeed, by applying the above proposition repeatedly, one obtains a set N which satisfies the maximal condition and $N \gg M$. Then apply the induction on t. Also, if $t = 1$ then Proposition 6.3 follows from Corollary 4.6. In the rest of this section, we assume that $t > 1$ and that the statement is true when the number of the free basis of U is at most $t-1$. By the above argument, we may assume that Theorem 6.1 is also true when the number of the free basis of U is at most $t-1$.

Lemma 6.4. *There exists a ladder set $N \subset U$ with $N \gg M$ and* $\min\{k + b_1 : x_1^k \boldsymbol{e}_1 \notin N^{(1)}\} = e$ *satisfying the following conditions*:

(A1) *$N^{(1)}$ is super-revlex and $N^{(\geq 2)}$ is extremal in $U^{(\geq 2)}$.*

(A2) *$\rho(N_e^{(1)}) \cup N^{(2)} \supset U^{(2)}_{\leq e+a_2}$ or $\rho(N_e^{(1)}) \cap N^{(2)} = \varnothing$.*

(A3) *If $t=2$ and $\rho\big(N_e^{(1)}\big) \cap N^{(2)} = \varnothing$ then $N_e^{(1)} = \varnothing$. If $t \geq 3$ and $\rho\big(N_e^{(1)}\big) \cap N^{(2)} = \varnothing$ then $N_e^{(1)} = \varnothing$ or there exists a $d \geq e$ such that $N^{(2)} = U^{(2)}_{\leq d}$ and $N^{(3)}_{d+1} \neq U^{(3)}_{d+1}$.*

Proof. Let $F = M_e^{(1)}$. Then $M = \big(U^{(1)}_{\leq e-1} \uplus F\big) \uplus M^{(2)} \uplus M^{(\geq 3)}$ since $M^{(1)}$ is super-revlex.

Step 1. We first prove that there exits N satisfying (A1). Let X be the extremal set in $U^{(\geq 2)}$ with $\#X = \#M^{(\geq 2)}$. Let

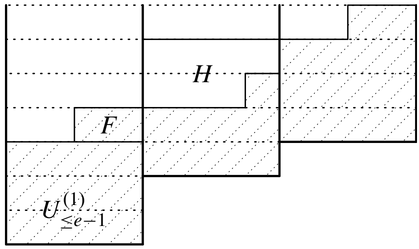
$$N = M^{(1)} \uplus X = U^{(1)}_{\leq e-1} \uplus F \uplus X.$$

Since we assume that Theorem 6.1 is true for $U^{(\geq 2)}$, $N \gg M$. What we must prove is that N is a ladder set. Since $M^{(\geq 2)} \supset U^{(\geq 2)}_{\leq e-1}$, $\#X = \#M^{(\geq 2)} \geq \#U^{(\geq 2)}_{\leq e-1}$. Then Lemma 5.10 says $X \supset U^{(\geq 2)}_{\leq e-1}$, which shows that N is a ladder set if $F = \varnothing$. If $F \neq \varnothing$ then by the definition of ladder sets, $M^{(\geq 2)} \supset U^{(\geq 2)}_{\leq e}$, and $X \supset U^{(\geq 2)}_{\leq e}$ by Lemma 5.10. Hence N is a ladder set.

Step 2. We prove that if M satisfies (A1) but does not satisfy either (A2) or (A3) then there exists an N satisfying (A2) and (A3) such that $N \gg M$ and $\#N^{(1)}$ is strictly smaller than $\#M^{(1)}$. We may assume $\rho(F) \cup M^{(2)} \not\supset U^{(2)}_{\leq e+a_2}$ and $F \neq \varnothing$, otherwise M itself satisfies the desired conditions. Note that $F \neq \varnothing$ implies $M^{(2)} \supset U^{(2)}_{\leq e}$. Let

$$\begin{aligned}
a &= \min\{k : M_k^{(2)} \neq U_k^{(2)}\},\\
b &= \max\{k : k \leq e + a_2,\ \rho(F)_k \neq U_k^{(2)}\},\\
d &= \max\{k : M_k^{(3)} = U_k^{(3)}\},
\end{aligned}$$

where $d = \infty$ if $n = 2$. Let $H = U^{(2)}_{\le d} \setminus M^{(2)}$ (see figure).



M

The set $\rho(F)$ equals $\rho(F)_b \uplus \biguplus_{j=b+1}^{e+a_2} U^{(2)}_j$, since it is an upper revlex set of degree $e+a_2$. Suppose $H = \varnothing$. Then $M^{(2)} = U^{(2)}_{\le d}$. Since $\rho(F) \cup M^{(2)} \not\supset U^{(2)}_{\le e+a_2}$, we have $b > d$ and $\rho(F) \cap M^{(2)} = \varnothing$, which say that M satisfies (A2) and (A3). Suppose $H \ne \varnothing$. Observe that for any super-revlex set L with $U^{(2)}_{\le e} \subset L \subset U^{(2)}_{\le d}$, $M^{(1)} \uplus L \uplus M^{(\ge 3)}$ is a ladder set.

Case 1: Suppose $\#H \ge \#F$. (Note that if $t = 2$ then we always have $\#H \ge \#F$.) Then $M^{(2)}$ is super-revlex since we assume that $M^{(\ge 2)}$ is extremal and $\rho(F)$ is an upper revlex set of degree $e+a_2$ with $\#M^{(2)} + \#\rho(F) \le \#U^{(2)}_{\le d}$. Let $R \subset U^{(2)}$ be the super-revlex set in $U^{(2)}$ with $\#R = \#M^{(2)} + \#\rho(F)$. By Corollary 4.5,

$$m(R) \succeq m(M^{(2)}) + m(\rho(F)) = m(M^{(2)}) + m(F). \tag{6}$$

Also, since R is super-revlex, $U^{(2)}_{\le e} \subset R \subset U^{(2)}_{\le d}$. Thus

$$N = U^{(1)}_{\le e-1} \uplus R \uplus M^{(\ge 3)}$$

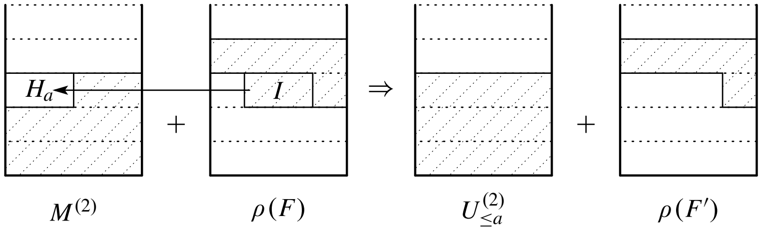
is a ladder set. Then $N^{(1)}_e = \varnothing$ and $N \gg M$ by (6). Hence N satisfies (A2) and (A3).

Case 2: Suppose $\#H < \#F$. Observe that $M^{(2)} \cup \rho(F)$ contains all monomials of degree k in $U^{(2)}$ for $k < a$ and $b < k \le e + a_2$. Since $M \cup \rho(F) \not\supset U^{(2)}_{\le e+a_2}$, we have $a \le b$.

Let $I \subset \rho(F)$ be the interval in $U^{(2)}$ such that $\#I = \#H_a$ and $\rho(F) \setminus I$ is an upper revlex set of degree $e+a_2$, and let $F' \subset F$ be the revlex set with $\rho(F') = \rho(F) \setminus I$. Since H_a is a lower lex set of degree a, the interval lemma gives

$$\begin{aligned} m\big(M^{(2)}\big) + m\big(\rho(F)\big) &\ll m\big(H_a \uplus M^{(2)}\big) + m\big(\rho(F) \setminus I\big) \\ &= m\big(U^{(2)}_{\le a}\big) + m\big(\rho(F')\big). \end{aligned}$$

This is illustrated at the top of the next page.



Suppose $\rho(F') \cup U^{(2)}_{\le a} \supset U^{(2)}_{\le e+a_2}$. Then

$$N = \left(U^{(1)}_{\le e-1} \uplus F'\right) \uplus U^{(2)}_{\le a} \uplus M^{(\ge 3)}$$

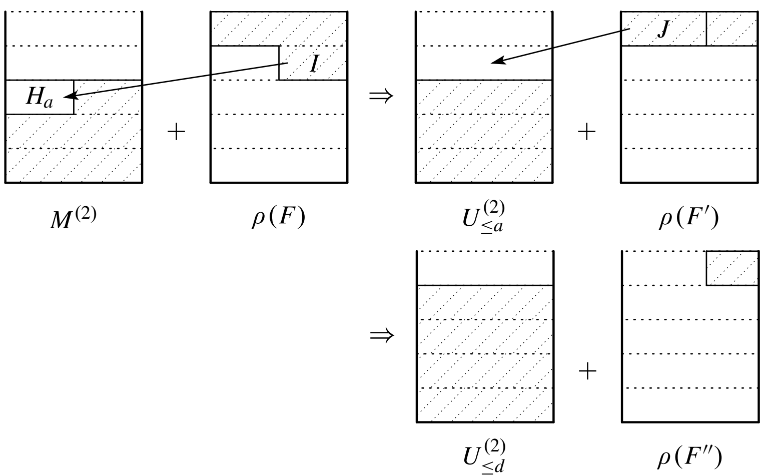
is a ladder set and satisfies $N \gg M$ and conditions (A2) and (A3) since

$$\rho(N^{(1)}_e) \cup N^{(2)} \supset U^{(2)}_{\le e+a_2}.$$

Suppose $\rho(F') \cup U^{(2)}_{\le a} \not\supset U^{(2)}_{\le e+a_2}$. Then $\rho(F') \subset \biguplus_{j=a+1}^{e+a_2} U^{(2)}_j$. Since we assume $\#H < \#F$, $\#F' = \#F - \#H_a > \#(H \setminus H_a)$. Let $J \subset \rho(F')$ be the interval in $U^{(2)}$ such that $\#J = \#(H \setminus H_a)$ and $\rho(F') \setminus J$ is an upper revlex set of degree $e + a_2$, and let $F'' \subset F'$ be the revlex set satisfying $\rho(F'') = \rho(F') \setminus J$. Since $H \setminus H_a = \biguplus_{j=a+1}^{d} U^{(2)}_j$ is a lower lex set of degree $a + 1$, the interval lemma yields

$$m\left(U^{(2)}_{\le a}\right) + m\left(\rho(F')\right) \preceq m\left(M^{(2)} \uplus H\right) + m\left(\rho(F'')\right) = m\left(U^{(2)}_{\le d}\right) + m\left(\rho(F'')\right)$$

(see figure).



Then

$$N = \left(U^{(1)}_{\le e-1} \uplus F''\right) \uplus U^{(2)}_{\le d} \uplus M^{(\ge 3)}$$

is a ladder set and satisfies $N \gg M$ and conditions (A2) and (A3).

Finally, since Step 1 does not change the first component $M^{(1)}$ and Step 2 decreases the first component, by applying Steps 1 and 2 repeatedly, we obtain a set $N \subset U$ satisfying conditions (A1), (A2), and (A3). □

Lemma 6.4 says that to prove Proposition 6.3 we may assume that M satisfies (A1), (A2), and (A3). Thus in the rest of this section we assume that M satisfies these conditions. Also, we may assume $f \neq 0$ since the proposition follows from the induction hypothesis when $f = 0$.

Proof of Proposition 6.3 when $f \neq x_1^{e-b_1-1}\boldsymbol{e}_1$. In this case we have $\deg f = e$. Let

$$f = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \boldsymbol{e}_1$$

and $F = M_e^{(1)}$. Since $x_1^{e-b_1}\boldsymbol{e}_1 \notin F$ by the choice of e, we have $m(F) = m(\rho(F))$. Also, we have

$$M^{(\geq 2)} \supset U_{\leq e}^{(\geq 2)}.$$

Indeed, this is obvious when $F \neq \varnothing$ by the definition of ladder sets. If $F = \varnothing$ then

$$\# M^{(\geq 2)} = \# M - \# U_{\leq e-1}^{(1)} \geq \#\{h \in U : h \leq_{\mathrm{dlex}} f\} - \# U_{\leq e-1}^{(1)} \geq \# U_{\leq e}^{(2)},$$

and since $M^{(\geq 2)}$ is extremal we have $M^{(\geq 2)} \supset U_{\leq e}^{(\geq 2)}$ by Lemma 5.10. Let

$$\epsilon = \deg \rho(f) = \alpha_2 + \cdots + \alpha_n + b_2.$$

Case 1. Suppose $\rho(F) \subset \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$ and $\# F + \# M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)} \leq \# U_{\leq e+a_2}^{(2)}$. Observe that $M^{(2)} \supset \biguplus_{j=\epsilon}^{e} U_j^{(2)}$. Let P be the super-revlex set with

$$\# P = \# M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)},$$

and let $Q \subset U^{(2)}$ be the super-revlex set with $\# Q = \# F + \# M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}$. Since $\rho(F)$ is an upper revlex set of degree $e + a_2$ and $M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}$ is revlex, by Corollaries 4.5 and 4.6, we have

$$m(Q) \succeq m(P) + m\big(\rho(F)\big) \succeq m\left(M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}\right) + m(F) \tag{7}$$

(see the first two steps in Figure 1).

Observe that $Q \subset U_{\leq e+a_2}^{(2)}$ since $\# Q \leq \# U_{\leq e+a_2}^{(2)}$ by the assumption of Case 1. Let $U' = U^{(2)} \uplus \biguplus_{i=3}^{t} U^{(i)}[-a_2]$. Since $M^{(\geq 3)}[-a_2] \supset U_{\leq e}^{(\geq 3)}[-a_2] = U'^{(\geq 3)}_{\leq e+a_2}$,

$$Q \uplus M^{(\geq 3)}[-a_2] \subset U'$$

is a ladder set in U' (see the third step in Figure 1).

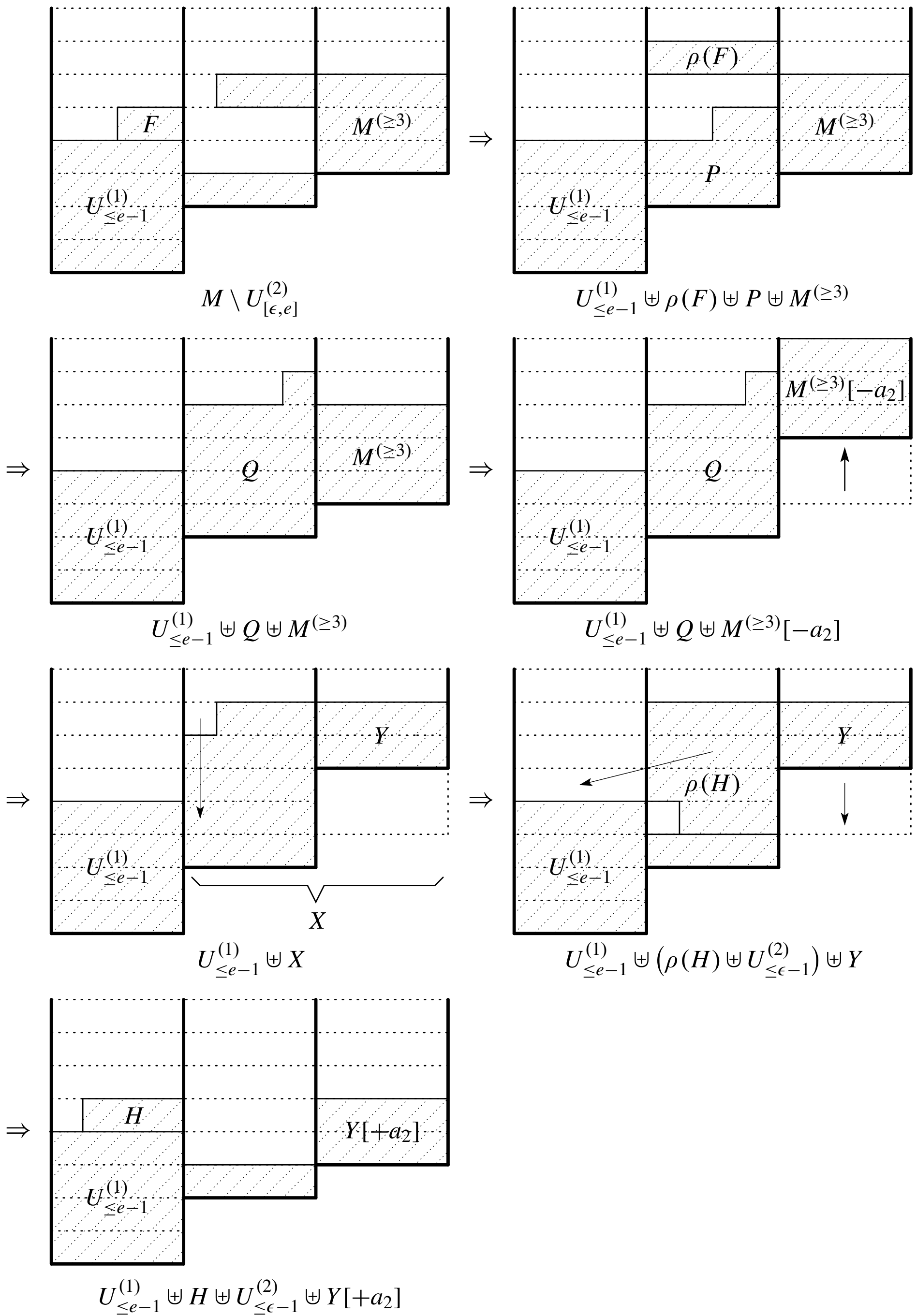


Figure 1. Some steps in the proof of Proposition 6.3 in the case when $f \neq x_1^{e-b_1-1}\boldsymbol{e}_1$. See bottom of previous page and middle and bottom of page 1047.

Let g be the largest admissible monomial in $U^{(2)}_{\le e+a_2}$ over U' with respect to $>_{\mathrm{dlex}}$ satisfying

$$\#\{h \in U' : h \le_{\mathrm{dlex}} g\} \le \#Q \uplus M[-a_2]^{(\ge 3)}.$$

By the induction hypothesis, there exists $Y \subset U'^{(\ge 3)}$ such that

$$X = \{h \in U^{(2)} : h \le_{\mathrm{dlex}} g\} \uplus Y \subset U'$$

is a ladder set in U' and

$$X \gg Q \uplus M^{(\ge 3)}. \tag{8}$$

Lemma 6.5. *Let $d = e + a_2 - \epsilon$. Then $g \ge_{\mathrm{lex}} x_2^d \rho(f)$.*

Proof. Consider

$$L = \{h \in U : h \le_{\mathrm{dlex}} f\}.$$

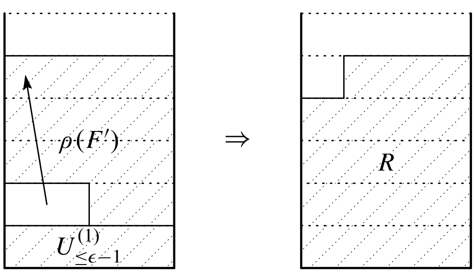
Then $\#M \ge \#L$ and $L^{(\ge 2)} = U^{(\ge 2)}_{\le e}$. Thus $L^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)} = U^{(2)}_{\le \epsilon-1}$. Let $F' = L_e^{(1)} = [f, x_n^{e-b_1}\boldsymbol{e}_1]$. Then $\rho(F') = [\rho(f), x_n^{\epsilon-b_2}\boldsymbol{e}_2] \uplus \biguplus_{j=\epsilon+1}^{e+a_2} U_j^{(2)}$. Also, $\rho(F')$ is disjoint from $L^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}$ and

$$\begin{aligned} m\left(\rho(F') \uplus \left(L^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}\right)\right) &= m\big(U^{(2)}_{\le e+a_2} \setminus \big[x_2^{\epsilon-b_2}\boldsymbol{e}_2, \rho(f)\big)\big) \\ &= m\big(U^{(2)}_{\le e+a_2} \setminus \big[x_2^{e+a_2-b_2}\boldsymbol{e}_2, x_2^d\rho(f)\big)\big). \end{aligned}$$

Let

$$R = U^{(2)}_{\le e+a_2} \setminus \big[x_2^{e+a_2-b_2}\boldsymbol{e}_2, x_2^d\rho(f)\big) = U^{(2)}_{\le e+a_2-1} \uplus \big[x_2^d\rho(f), x_n^{e+a_2-b_2}\boldsymbol{e}_2\big]$$

(see figure).



$U^{(1)}_{\le\epsilon-1} \uplus \rho(F')$ $\qquad\qquad$ R

Then $R \uplus L^{(\ge 3)}[-a_2] \subset U'$ is a ladder set in U' and $x_2^d\rho(f)$ is admissible over U' by Lemma 5.9. On the other hand,

$$\#R \uplus L^{(\ge 3)} = \#L - \#U^{(1)}_{\le e-1} - \#\biguplus_{j=\epsilon}^{e} U_j^{(2)} \le \#M - \#U^{(1)}_{\le e-1} - \#\biguplus_{j=\epsilon}^{e} U_j^{(2)} = \#X.$$

Since $x_2^d\rho(f)$ is admissible over U' and since $R \uplus L^{(\geq 3)}[-a_2] = \{h \in U' : h \leq_{\text{dlex}} x_2^d\rho(f)\}$, by the choice of g, we have

$$g \geq_{\text{lex}} x_2^d\rho(f)$$

as desired. □

By Lemma 6.5, g is divisible by x_2^d. Let $H \subset U_e^{(1)}$ be the revlex set such that

$$\rho(H) = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)} \setminus x_2^{-d}\big[x_2^{e+a_2-b_2}\boldsymbol{e}_2, g\big).$$

Then by Lemma 4.3

$$m(H) + m\big(U_{\leq\epsilon-1}^{(2)}\big) \succeq m\big(U_{\leq e+a_2}^{(2)} \setminus [x_2^{e+a_2-b_2}\boldsymbol{e}_2, g)\big) = m\big(X^{(2)}\big). \tag{9}$$

Let

$$N = \big(U_{\leq e-1}^{(1)} \uplus H\big) \uplus U_{\leq e}^{(2)} \uplus Y[+a_2] \subset U.$$

Since X is a ladder set, $Y \supset U'^{(\geq 3)}_{\leq e+a_2}$ and $Y[+a_2] \supset U_{\leq e}^{(\geq 3)}$. Thus N is a ladder set in U. We claim that N satisfies the desired conditions.

A routine computation shows

$$\#M \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)} = \#U_{\leq e-1}^{(1)} \uplus Q \uplus M^{(\geq 3)} = \#U_{\leq e-1}^{(1)} \uplus X = \#N \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}$$

(see Figure 1). Thus $\#N = \#M$. Let $\mu = \max_{>_{\text{lex}}} H$. Then $x_2^d\rho(\mu) = g$. We claim that $\mu = f$. Since $g \geq_{\text{lex}} x_2^d\rho(f)$, $\mu \geq_{\text{lex}} f$. Since g is admissible over U', μ is admissible over U by Lemma 5.9 (If $t = 2$ then Lemma 5.9 is not applicable; however, if $t = 2$ then any monomial $h \in U_e^{(1)}$ with $h >_{\text{lex}} f$ is admissible). However, since $\#N = \#M$ and $N \supset \{h \in U : h \leq_{\text{dlex}} \mu\}$, by the choice of f, we have $f = \mu$.

It remains to prove $N \gg M$. This follows from (7), (8), and (9) as follows:

$$\begin{aligned} M \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)} &= \big(U_{\leq e-1}^{(1)} \uplus F\big) \uplus \left(M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}\right) \uplus M^{(\geq 3)} \\ &\ll U_{\leq e-1}^{(1)} \uplus Q \uplus M^{(\geq 3)} \\ &\ll U_{\leq e-1}^{(1)} \uplus X \\ &\ll \big(U_{\leq e-1}^{(1)} \uplus H\big) \uplus U_{\leq \epsilon-1}^{(2)} \uplus Y[+a_2] = N \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)} \end{aligned}$$

(see Figure 1).

Case 2. Suppose $\rho(F) \subset \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$ and $\#F + \#M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)} > \#U_{\le e+a_2}^{(2)}$.

Lemma 6.6. *We have $f = x_1^{\alpha_1} x_2^{\alpha_2} \boldsymbol{e}_1$; that is, $\alpha_3 = \cdots = \alpha_n = 0$.*

Proof. Suppose $f \neq x_1^{\alpha_1} x_2^{\alpha_2} \boldsymbol{e}_1$. Let $g = x_1^{\alpha_1} x_2^{\alpha_2+\alpha_3+\cdots+\alpha_n} \boldsymbol{e}_1$. Then $g >_{\mathrm{dlex}} f$ is admissible over U by the definition of admissibility. Also,

$$\#M < \#\{h \in U : h \le_{\mathrm{dlex}} g\} = \#\big(U_{\le e-1}^{(1)} \uplus [g, x_n^{e-b_1}]\boldsymbol{e}_1\big) \uplus U_{\le e}^{(2)} \uplus U_{\le e}^{(\ge 3)}.$$

Since $\rho([g, x_n^{e-b_1}\boldsymbol{e}_1]) = \biguplus_{i=\epsilon}^{e+a_2} U_i^{(2)}$ and $M^{(\ge 3)} \supset U_{\le e}^{(\ge 3)}$,

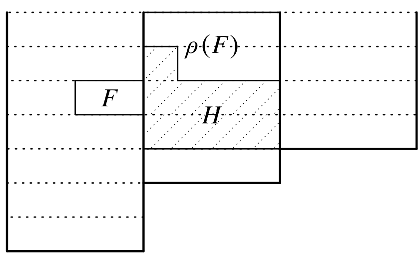
$$\begin{aligned} \#F + \#\Big(M^{(2)} \setminus \biguplus_{j=\epsilon}^{e} U_j^{(2)}\Big) &= \big(\#M - \#U_{\le e-1}^{(1)} - \#M^{(\ge 3)}\big) - \#\biguplus_{j=\epsilon}^{e} U_j^{(2)} \\ &< \#[g, x_n^{e-b_1}\boldsymbol{e}_1] + \#U_{\le e}^{(2)} - \#\biguplus_{j=\epsilon}^{e} U_j^{(2)} = \#U_{\le e+a_2}^{(2)}, \end{aligned}$$

which contradicts the assumption of Case 2. Thus $f = x_1^{\alpha_1} x_2^{\alpha_2} \boldsymbol{e}_1$. $\square$

Lemma 6.6 says that $\rho(f) = x_2^{\epsilon-b_2}\boldsymbol{e}_2$. In particular, $\rho([f, x_n^{e-b_1}\boldsymbol{e}_1]) = \bigcup_{j=\epsilon}^{e+a_2} U_j^{(2)}$. Let

$$H = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)} \setminus \rho(F)$$

(see figure).



H

Since $\rho(F)$ is an upper revlex set of degree $e + a_2$, H is a lower lex set of degree ϵ. Also, since $\#F + \#M^{(2)} > \#U_{\le e+a_2}^{(2)}$, $\rho(F) \cup M^{(2)} \supset U_{\le e+a_2}^{(\ge 2)}$ by (A2). Thus $M^{(2)} \supset H$.

Let R be the super-revlex set in $U^{(2)}$ with $\#R = \#M^{(2)} \setminus H$. Since $M^{(2)} \setminus H$ is revlex, by Corollary 4.6 we have

$$R \gg M^{(2)} \setminus H. \tag{10}$$

Then since $\#R \le \#M^{(2)}$,

$$R \uplus M^{(\ge 3)} \subset U^{(\ge 2)}$$

is a ladder set (see the third picture in Figure 2).

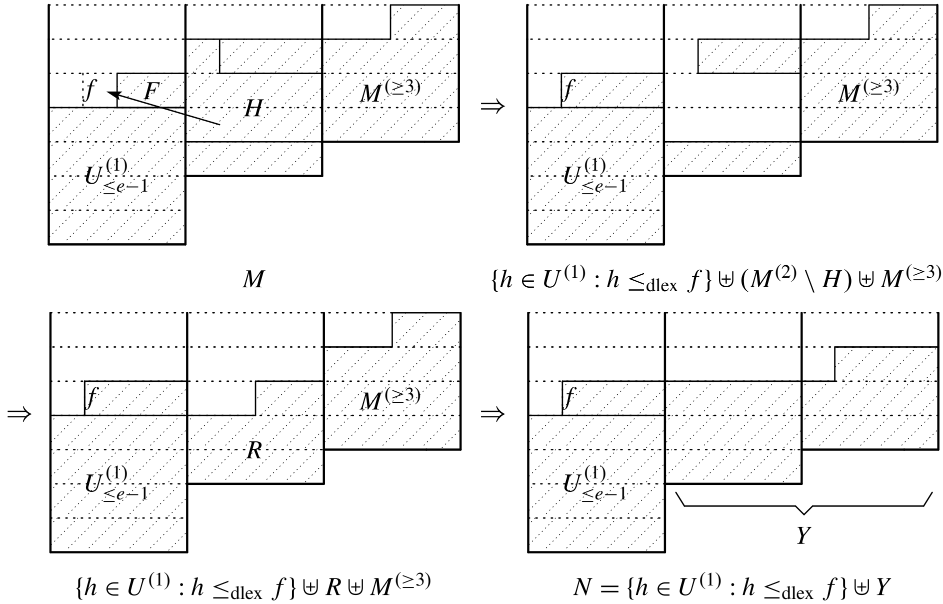


Figure 2. Toward the proof of Case 2.

Let $Y \subset U^{(\ge 2)}$ be the extremal set in $U^{(\ge 2)}$ with $\#Y = \#R \uplus M^{(\ge 3)}$. We claim that

$$N = \{h \in U^{(1)} : h \le_{\mathrm{dlex}} f\} \uplus Y$$

satisfies the desired conditions. Indeed, we have

$$\begin{aligned} M &= \big(U^{(1)}_{\le e-1} \uplus F \uplus H\big) \uplus (M^{(2)} \setminus H) \uplus M^{(\ge 3)} \\ &\ll \big(U^{(1)}_{\le e-1} \uplus [f, x_n^{e-b_1}\boldsymbol{e}_1]\big) \uplus R \uplus M^{(\ge 3)} \\ &\ll \{h \in U^{(1)} : h \le_{\mathrm{dlex}} f\} \uplus Y = N \end{aligned}$$

(see Figure 2) since $\rho(F) \uplus H = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)} = \rho([f, x_n^{e-b_1}\boldsymbol{e}_2])$, by (10).

It remains to prove that N is a ladder set. Since

$$\#Y = \#M - \#\{h \in U^{(1)} : h \le_{\mathrm{dlex}} f\} \ge \#U^{(\ge 2)}_{\le e}$$

by the choice of f, we have $Y \supset U^{(\ge 2)}_{\le e}$ by Lemma 5.10. This fact guarantees that N is a ladder set.

Case 3. Suppose $\rho(F) \not\subset \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$. Then $\rho(F)$ properly contains $\biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$ since $\rho(F)$ is an upper revlex set of degree $e+a_2$. In particular, F properly contains $[f, x_n^{e-b_1}\boldsymbol{e}_1]$. We claim:

Lemma 6.7. *We have $f = x_1^{\alpha_1}x_2^{\alpha_2}\boldsymbol{e}_1$ and $\alpha_2 \neq 0$.*

Proof. If $\alpha_k \neq 0$ for some $k \geq 3$ then $x_1^{\alpha_1} x_2^{\alpha_2+\cdots+\alpha_n} \boldsymbol{e}_1 >_{\mathrm{dlex}} f$ is admissible over U. Then by the choice of f, $F \subset [x_1^{\alpha_1} x_2^{\alpha_2+\cdots+\alpha_n} \boldsymbol{e}_1, x_n^{e-b_1} \boldsymbol{e}_1]$ and

$$\rho(F) \subset \rho\big([x_1^{\alpha_1} x_2^{\alpha_2+\cdots+\alpha_n} \boldsymbol{e}_1, x_n^{e-b_1} \boldsymbol{e}_1]\big) = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)},$$

a contradiction. Also, if $\alpha_2 = 0$ then $\epsilon = \deg \rho(f) = 0$ which implies

$$\rho(F) \subset \rho(U_e^{(1)}) = U_{\leq e+a_2}^{(2)} = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)},$$

a contradiction. □

Recall $\epsilon = \deg \rho(f)$. Thus $\alpha_2 = \epsilon - b_2$ by Lemma 6.7. Let

$$H = \{h \in F : h >_{\mathrm{lex}} f\}$$

and

$$g = \max_{>_{\mathrm{lex}}} H.$$

By the choice of f, H contains no admissible monomials over U. By Lemma 6.7, $\rho(F \setminus H) = \biguplus_{j=\epsilon}^{e+a_2} U_j^{(2)}$. Hence $H \neq \varnothing$ by the assumption of Case 3. Since $x_1^{\alpha_1+1} x_2^{\alpha_2-1} \boldsymbol{e}_1$ is admissible over U,

$$\rho(H) \subset \rho\big([x_1^{\alpha_1+1} x_2^{\alpha_2-1} \boldsymbol{e}_1, x_1^{\alpha_1} x_2^{\alpha_2} \boldsymbol{e}_1)\big) = U_{\epsilon-1}^{(2)}$$

is revlex. Also, $\epsilon - 1 > b_2$ since $U_{b_2}^{(2)} = \{\boldsymbol{e}_2\}$ and $H \neq \varnothing$.

If $t = 2$ then any monomial $h \in U_e^{(1)}$ with $h >_{\mathrm{lex}} f$ is admissible, which implies $H = \varnothing$. Thus we may assume $t \geq 3$.

To prove the statement, it is enough to prove that there exists an extremal set $Z \subset U^{(\geq 3)}$ such that

$$Z \gg H \uplus M^{(\geq 3)}. \tag{11}$$

Indeed, if such a Z exists then $N = (M^{(1)} \setminus H) \uplus M^{(2)} \uplus Z$ satisfies the desired conditions. Recall that $\epsilon \leq e + 1$ by the definition of admissibility.

<u>Subcase 3-1</u>. Suppose $a_3 \geq e - (\epsilon - 1)$.

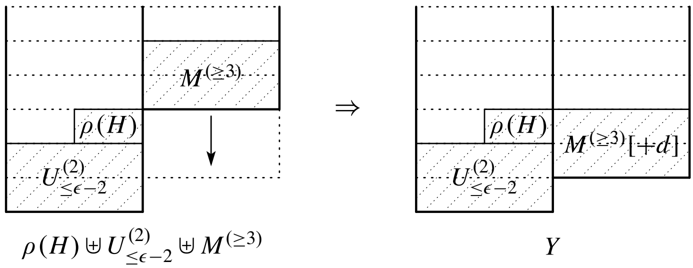
Let $d = e - (\epsilon - 1)$. We consider

$$U' = U^{(2)} \uplus \biguplus_{i=3}^{t} U^{(i)}[+d].$$

This set is well-defined since $a_3 \geq d$. Recall $\rho(H) \subset U_{\epsilon-1}^{(2)}$. Let

$$Y = \rho(H) \uplus U_{\leq \epsilon-2}^{(2)} \uplus M^{(\geq 3)}[+d]$$

(see figure).



Then Y is a ladder set since $M^{(\geq 3)} \supset U^{(\geq 3)}_{\leq \epsilon-1+d} = U^{(\geq 3)}_{\leq e}$. Also, $U^{(2)}_{\leq \epsilon-2} \neq \varnothing$ since $\epsilon - 1 > b_2$.

Let $\mu \in U^{(2)}_{\leq \epsilon-1}$ be the largest admissible monomial in $U^{(2)}_{\leq \epsilon-1}$ over U' with respect to $>_{\mathrm{dlex}}$ satisfying $\#\{h \in U' : h \leq_{\mathrm{dlex}} \mu\} \leq \#Y$. Then since we assume that Proposition 6.3 is true for U', there exists an extremal set $Z \subset U'^{(\geq 3)}$ such that

$$Y \ll \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} \mu\} \uplus Z.$$

To prove (11), it is enough to prove $\{h \in U^{(2)} : h \leq_{\mathrm{dlex}} \mu\} = U^{(2)}_{\leq \epsilon-2}$; in other words:

Lemma 6.8. $$\mu = x_2^{\epsilon-2-b_2} \boldsymbol{e}_2.$$

Proof. Recall that $U^{(2)}_{\leq \epsilon-2} \neq \varnothing$. It is enough to prove that $\deg \mu \neq \epsilon - 1$. Suppose to the contrary that $\deg \mu = \epsilon - 1$. Let $\mu' \in U^{(1)}_{e}$ be a monomial such that $\rho(\mu') = \mu$. Then μ' is admissible over U by Lemma 5.9. Also,

$$\#Y - \#U^{(2)}_{\leq \epsilon-2} \geq \#[\mu, x_n^{\epsilon-1-b_2}\boldsymbol{e}_2] + \#U'^{(\geq 3)}_{\leq \epsilon-1} = \#[\mu, x_n^{\epsilon-1-b_2}\boldsymbol{e}_2] + \#U^{(\geq 3)}_{\leq e}.$$

Since $\#M^{(\geq 3)} + \#H = \#Y - \#U^{(2)}_{\leq \epsilon-2}$ and since $\rho\big([\mu', f)\big) = [\mu, x_n^{\epsilon-1-b_2}\boldsymbol{e}_2]$, we have

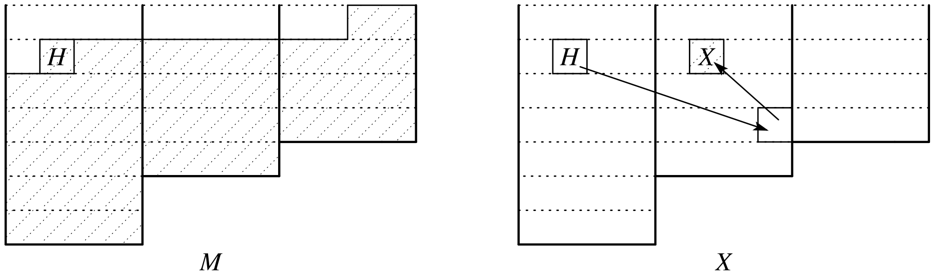
$$\begin{aligned}\#M &= \#(M \setminus H) \uplus M^{(2)} \uplus H \uplus M^{(\geq 3)} \\ &\geq \#(M \setminus H) \uplus U^{(2)}_{\leq e} + \#[\mu, x_n^{\epsilon-1-b_2}\boldsymbol{e}_2] \uplus Z \\ &\geq \#[\mu', f) \uplus (M \setminus H) \uplus U^{(\geq 2)}_{\leq e} = \#\{h \in U : h \leq_{\mathrm{dlex}} \mu'\},\end{aligned}$$

which contradicts the maximality of f since $\mu' >_{\mathrm{lex}} g >_{\mathrm{lex}} f$ and μ' is admissible over U. $\square$

Subcase 3-2. Suppose $a_3 < e - (\epsilon - 1)$. We consider

$$X = x_2^{e-(\epsilon-1)} \rho(H) \subset U^{(2)}_{e},$$

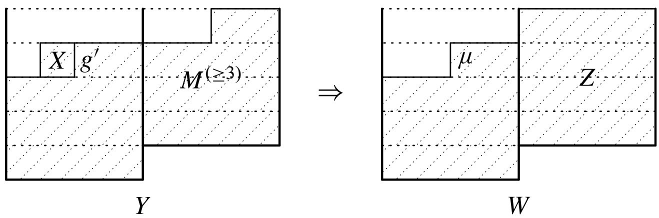
as illustrated at the top of the next page.



Let

$$Y = \left\{h \in U^{(2)} : h \leq_{\text{dlex}} x_2^{e-(\epsilon-1)} \rho(g)\right\} \uplus M^{(\geq 3)},$$

as on the left part of the figure:



Further, let

$$g' = \max_{>_{\text{dlex}}} (Y^{(2)} \setminus X).$$

Since $e - (\epsilon - 1) > a_3$, $e - (\epsilon - 1) \geq 1$. Thus

$$g' = x_2^{e-(\epsilon-1)-1} x_3^{\epsilon - b_2} \boldsymbol{e}_2$$

and

$$Y^{(2)} = X \uplus \{h \in U^{(2)} : h \leq_{\text{dlex}} g'\}.$$

Since $a_3 < e - (\epsilon - 1)$, $\deg \rho(g') = \epsilon + a_3 \leq e$. Thus g' is admissible over $U^{(\geq 2)}$.

Let μ be the largest admissible monomial in $U_{\leq e}^{(2)}$ over $U^{(\geq 2)}$ with respect to $>_{\text{dlex}}$ with $\#\{h \in U^{(\geq 2)} : h \leq_{\text{dlex}} \mu\} \leq \#Y$. Since Lemma 5.9 says that X contains no admissible monomials over $U^{(\geq 2)}$,

$$\mu \geq_{\text{dlex}} g' \text{ and } \mu \notin X.$$

Since we assume that Proposition 6.3 is true for $U^{(\geq 2)}$, there exists an extremal set $Z \subset U^{(\geq 3)}$ such that

$$W = \{h \in U^{(2)} : h \leq_{\text{dlex}} \mu\} \uplus Z$$

is a ladder set and

$$W \gg Y,$$

as shown in the figure immediately above.

Lemma 6.9. $$\mu = g'.$$

Proof. Suppose to the contrary that $\mu \neq g'$. Then $\mu >_{\mathrm{dlex}} g'$ and

$$W = \big[\mu, x_2^{e-(\epsilon-1)}\rho(g)\big) \uplus Y^{(2)} \uplus Z.$$

Then there exists $\mu' \in U_e^{(1)}$ such that

$$x_2^{e-(\epsilon-1)}\rho(\mu') = \mu.$$

By Lemma 5.9, μ' is admissible over U and $\mu' >_{\mathrm{lex}} g >_{\mathrm{lex}} f$. Observe that

$$\#M^{(\geq 3)} + \#H = \#Z \uplus [\mu, g') = \#Z + \#[\mu', f)$$

by the construction of Y and Z. Since $Z \supset U_{\leq e}^{(\geq 3)}$,

$$\begin{aligned}
\#M &\geq \#(M^{(1)} \setminus H) \uplus H \uplus U_{\leq e}^{(2)} \uplus M^{(\geq 3)} \\
&= \#(M^{(1)} \setminus H) \uplus U_{\leq e}^{(2)} \uplus Z \uplus [\mu', f) \\
&\geq \#(M^{(1)} \setminus H) \uplus [\mu', f) \uplus U_{\leq e}^{(2)} \uplus U_{\leq e}^{(\geq 3)} \\
&= \#\{h \in U : h \leq_{\mathrm{dlex}} \mu'\}.
\end{aligned}$$

Since μ' is admissible over U, this contradicts the maximality of f. $\square$

Now

$$W = \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} g'\} \uplus Z$$

and since $W \gg Y$ and $Y = X \uplus \{h \in U^{(2)} : h \leq_{\mathrm{dlex}} g'\} \uplus M^{(\geq 3)}$, we have

$$m(Z) \succeq m(X \uplus M^{(\geq 3)}) = m(H \uplus M^{(\geq 3)}),$$

which proves (11). This completes the proof of Proposition 6.3 when $f \neq x_1^{e-b_1-1}\boldsymbol{e}_1$.

Proof of Proposition 6.3 when $f = x_1^{e-b_1-1}\boldsymbol{e}_1$. Let $F = M_e^{(1)}$. If $F = \varnothing$ then there is nothing to prove. Thus we may assume $F \neq \varnothing$. Then $M \supset U_{\leq e}^{(\geq 2)}$ since M is a ladder set.

Case 1. Suppose $a_2 = 0$. Then $\deg \boldsymbol{e}_1 = \deg \boldsymbol{e}_2 = b_1$. Since $x_2^{e-b_1}\boldsymbol{e}_1$ is admissible over U, $x_2^{e-b_1}\boldsymbol{e}_1 \notin F$. Indeed, if $x_2^{e-b_1}\boldsymbol{e}_1 \in F$ then $M \supset \{h \in U : h \leq_{\mathrm{dlex}} x_2^{e-b_1}\boldsymbol{e}_1\}$, which contradicts the maximality of f. Thus

$$F \subset [x_2^{e-b_1}\boldsymbol{e}_1, x_n^{e-b_1}\boldsymbol{e}_1]$$

and

$$\rho(F) \subset \rho\big([x_2^{e-b_1}\boldsymbol{e}_1, x_n^{e-b_1}\boldsymbol{e}_1]\big) = U_e^{(2)}.$$

Consider

$$X = \rho(F) \uplus U_{\leq e-1}^{(2)} \uplus M^{(\geq 3)} \subset U^{(\geq 2)}$$

and let $Y \subset U^{(\geq 2)}$ be the extremal set with $\#Y = \#X$. Since X is a ladder set in $U^{(\geq 2)}$, by the induction hypothesis we have

$$Y \gg X.$$

Lemma 6.10. $$Y^{(2)} = U^{(2)}_{\leq e-1}.$$

Proof. Suppose to the contrary that $Y^{(2)} \neq U^{(2)}_{\leq e-1}$. Let $g = \bar{g}\boldsymbol{e}_2$ be the largest admissible monomial in $Y^{(2)}_{\leq e}$ over $U^{(\geq 2)}$ with respect to $>_{\mathrm{dlex}}$. Since $X \supset U^{(\geq 2)}_{\leq e-1}$, we have $Y \supset U^{(2)}_{\leq e-1}$ by Lemma 5.10. Thus $\deg g = e$ and $Y \supset U^{(\geq 3)}_{\leq e}$.

Let $g' = \bar{g}\boldsymbol{e}_1$. Since $g = \bar{g}\boldsymbol{e}_2$ is admissible over $U^{(\geq 2)}$ and since $\rho(g') = g$, g' is admissible over U by Lemma 5.9. Observe that $\#Y = \#X \leq \#F + \#M^{(\geq 2)} - \#U^{(2)}_e$. Then

$$\begin{aligned}
\#M &= \#U^{(1)}_{\leq e-1} \uplus F \uplus M^{(\geq 2)} \\
&\geq \#U^{(1)}_{\leq e-1} + \#U^{(2)}_e + \#Y \\
&\geq \#U^{(1)}_{\leq e-1} + \#U^{(2)}_e + \#\{h \in U^{(\geq 2)} : h \leq_{\mathrm{dlex}} g\} \\
&= \#U^{(1)}_{\leq e-1} + \#U^{(2)}_e + \#U^{(2)}_{\leq e-1} \uplus [g, x_n^{e-b_1}\boldsymbol{e}_2] \uplus U^{(\geq 3)}_{\leq e} \\
&= \#U^{(1)}_{\leq e-1} + \#U^{(\geq 2)}_{\leq e} + \#[g', x_n^{e-b_1}\boldsymbol{e}_1] \\
&= \#\{h \in U : h \leq_{\mathrm{dlex}} g'\},
\end{aligned}$$

which contradicts the maximality of f. Hence $Y^{(2)} = U^{(2)}_{\leq e-1}$. □

Then, since $Y \gg X$, we have

$$Y^{(\geq 3)} \gg F \uplus M^{(\geq 3)}. \tag{12}$$

Let

$$N = U^{(1)}_{\leq e-1} \uplus M^{(2)} \uplus Y^{(\geq 3)}.$$

Then N is a ladder set since $\#Y^{(\geq 3)} \geq \#M^{(\geq 3)}$. Also, $N \gg M$ by (12). Thus N satisfies the desired conditions.

Case 2. Suppose $a_2 > 0$. Since $\deg f \neq e$, by Lemma 5.12 we have

$$\#M < \#U^{(1)}_{\leq e}. \tag{13}$$

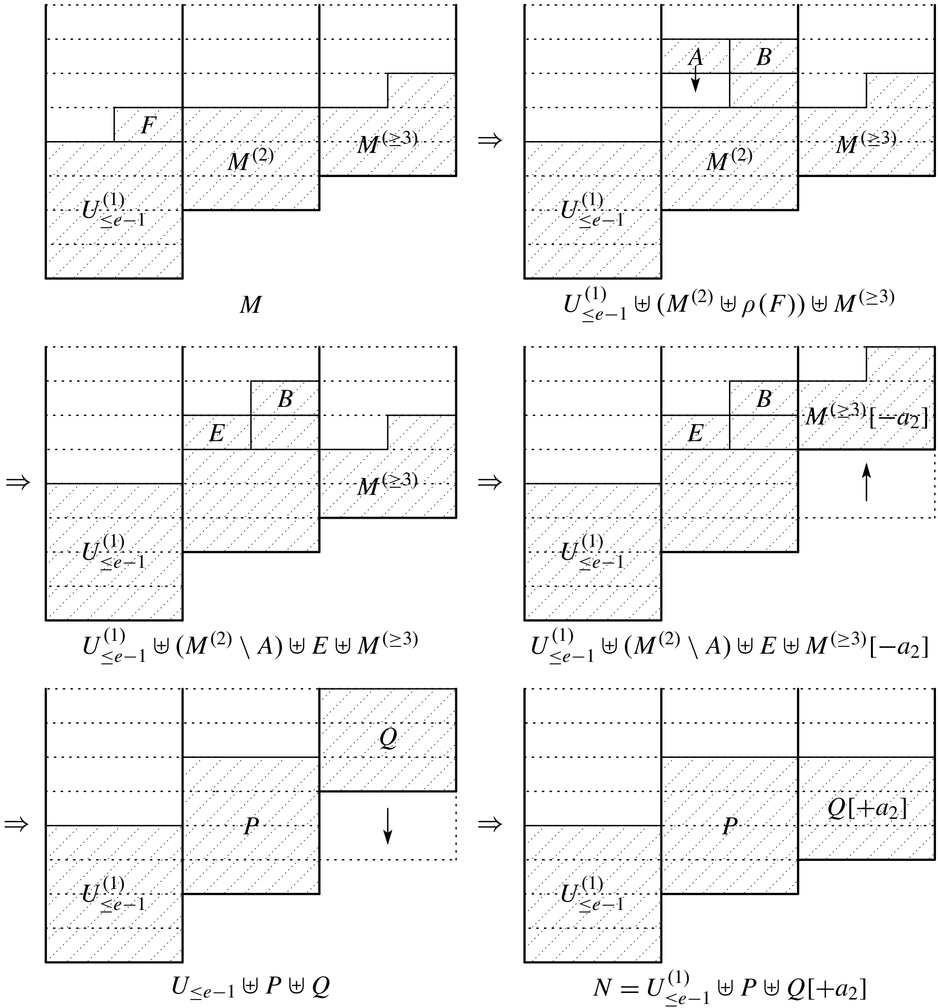
Hence

$$\#F + \#M^{(2)} \leq \#M - \#U^{(1)}_{\leq e-1} < \#U^{(1)}_e \leq \#U^{(2)}_{\leq e+a_2}. \tag{14}$$

Then, by (A2) and (A3), we may assume that $\rho(F) \cap M^{(2)} = \varnothing$, $t \geq 3$, and there exists a $d \geq e$ such that $M^{(2)} = U^{(2)}_{\leq d}$ and $M^{(3)}_{d+1} \neq U^{(3)}_{d+1}$. Let

$$A = \left\{u\boldsymbol{e}_2 \in \rho(F)_{e+a_2} : x_2^{(e+a_2)-(d+1)} \text{ divides } u \text{ and } u/x_2^{(e+a_2)-(d+1)}\boldsymbol{e}_2 \notin \rho(F)_{d+1}\right\},$$

as illustrated in the second picture at the top of the next page.



Also set

$$E = x_2^{-(e+a_2+d+1)} A \subset U^{(2)}_{d+1} \quad \text{and} \quad B = \rho(F)_{e+a_2} \setminus A \subset U^{(2)}_{e+a_2}.$$

Subcase 2-1. Suppose $\#B + \#M^{(\geq 3)} < \#U^{(2)}_{e+a_2}$. Consider

$$U' = U^{(2)} \uplus \biguplus_{i=3}^{t} U^{(i)}[-a_2].$$

Since $M^{(\geq 3)}[-a_2] \supset U'^{(\geq 3)}_{\leq e+a_2}$, by Corollary 5.13 and by the induction hypothesis, there exists the extremal set $Q \subset U'^{(\geq 3)}$ such that

$$Q \gg B \uplus M^{(\geq 3)}. \tag{15}$$

Let P be the super-revlex set in $U^{(2)}$ with $\#P = \#M^{(2)} + \#\rho(F) \setminus B$. Then since $\rho(F)_{\leq e+a_2-1} \uplus E$ is revlex, Corollary 4.6 shows

$$m\big(M^{(2)} \uplus \rho(F) \setminus B\big) = m(M^{(2)}) + m\big(\rho(F)_{\leq e+a_2-1} \uplus E\big) \preceq m(P) \qquad (16)$$

(see the second step in the figure on the previous page). We claim that

$$N = U^{(1)}_{\leq e-1} \uplus P \uplus Q[+a_2] \subset U$$

satisfies the desired conditions. Indeed, by (15) and (16),

$$m(N) \succeq m\big(U^{(1)}_{\leq e-1} \uplus M^{(2)} \uplus (\rho(F) \setminus B) \uplus (B \uplus M^{(\geq 3)})\big) = m(M)$$

(see figure on the previous page).

It remains to prove that N is a ladder set. If $\rho(F) \setminus B = \varnothing$ then $P = M^{(2)}$, and therefore N is a ladder set since $\#Q \geq \#M^{(\geq 3)}$. Suppose $\rho(F) \setminus B \neq \varnothing$. Recall that $\rho(F) \cap M^{(2)} = \varnothing$. Since

$$\#U^{(2)}_{\leq e} \leq \#M^{(2)} \leq \#P = \#\rho(F)_{\leq e+a_2-1} \uplus E \uplus M^{(2)} \leq \#U^{(2)}_{\leq e+a_2-1},$$

we have

$$U^{(2)}_{\leq e} \subset P \subset U^{(2)}_{\leq e+a_2-1}.$$

Then by Lemma 5.10 what we must prove is that

$$\#Q \geq \#U^{(\geq 3)}_{\leq e+a_2-1}.$$

Since $\#S^{(i)}_k = \sum_{j=i}^n \#S^{(j)}_{k-1}$ for all $i > 0$ and $k > 0$, we have

$$\#U^{(3)}_k \geq \sum_{j=3}^{t} \#U^{(j)}_{k-1} = \#U^{(\geq 3)}_{k-1} \qquad (17)$$

for all $k > 0$. Since $\rho(F) \setminus B \neq \varnothing$, $\#B = \#\rho(F)_{e+a_2} \setminus A \geq \#U^{(2)}_{e+a_2} - \#U^{(2)}_{d+1}$. Thus

$$\#B \geq \#U^{(2)}_{e+a_2} - \#U^{(2)}_{d+1} = \# \biguplus_{j=d+2}^{e+a_2} U^{(3)}_{j+a_3} \geq \# \biguplus_{j=d+2}^{e+a_2} U^{(3)}_{j} \geq \sum_{j=d+1}^{e+a_2-1} \#U^{(\geq 3)}_{j},$$

(we use (17) for the last step) and therefore

$$\#Q = \#M^{(\geq 3)} + \#B \geq \#U^{(\geq 3)}_{\leq d} + \sum_{d+1}^{e+a_2-1} U^{(\geq 3)}_{j} \geq \#U^{(\geq 3)}_{\leq e+a_2-1}$$

as desired.

<u>Subcase 2-2</u>. Suppose $\#B + \#M^{(\geq 3)} \geq \#U^{(2)}_{e+a_2}$.

Lemma 6.11. $\rho(F) \not\supset \biguplus_{j=d+2}^{e+a_2} U^{(2)}_j$.

Proof. Suppose to the contrary that $\rho(F) \supset \biguplus_{j=d+2}^{e+a_2} U_j^{(2)}$. Then

$$\#\rho(F) \setminus B = \#\big(\rho(F) \setminus (A \uplus B)\big) \uplus E = \# \biguplus_{j=d+1}^{e+a_2-1} U_j^{(2)}$$

by the choice of E. Then $\#(\rho(F) \setminus B) \uplus M^{(2)} = \#U^{(2)}_{\leq e+a_2-1}$ and

$$\#M = \#U^{(1)}_{\leq e-1} \uplus \rho(F) \uplus M^{(2)} \uplus M^{(\geq 3)} \geq \#U^{(1)}_{\leq e-1} + \#U^{(2)}_{\leq e+a_2-1} + \#U^{(2)}_{e+a_2} = \#U^{(1)}_{\leq e},$$

where we use the assumption $\#B + \#M^{(\geq 3)} \geq \#U^{(2)}_{e+a_2}$ for the second step. However, this contradicts (13). □

The above lemma says that $e + a_2 \geq d + 2$ and $\rho(F)_{d+1} = \varnothing$. Thus B does not contain any monomial $u\boldsymbol{e}_2$ such that u is divisible by $x_2^{(e+a_2)-(d+1)}$. Hence

$$\rho(B) \subset \biguplus_{j=d+2+a_3}^{e+a_2+a_3} U_j^{(3)}. \tag{18}$$

Since $M^{(3)}_{d+1} \neq U^{(3)}_{d+1}$, by Lemma 5.15,

$$\#M^{(\geq 3)} < \#U^{(3)}_{\leq d+2}.$$

Lemma 6.12. $a_3 = 0$.

Proof. If $a_3 > 0$ then

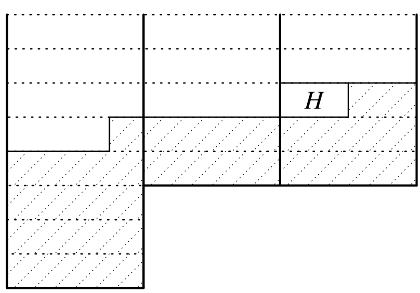
$$\#B + \#M^{(\geq 3)} < \# \biguplus_{j=d+2+a_3}^{e+a_2+a_3} U_j^{(3)} + \#U^{(3)}_{\leq d+2} \leq U^{(3)}_{\leq e+a_2+a_3} = \#U^{(2)}_{e+a_2},$$

which contradicts the assumption of Subcase 2-2. □

Let

$$H = \left\{h \in U^{(\geq 3)}_{d+1} : h \notin M^{(\geq 3)}\right\}$$

(see figure).



M

By Lemma 5.15,

$$\#H + \#M^{(\geq 3)} < \#U^{(3)}_{\leq d+2}.$$

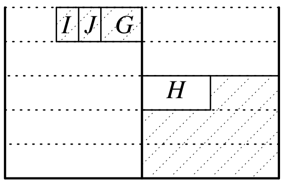
Since $a_3 = 0$, by the assumption of Subcase 2-2,

$$\#B \geq \#U^{(2)}_{e+a_2} - \#M^{(\geq 3)} = \#U^{(3)}_{\leq e+a_2} - \#M^{(\geq 3)} > \#H + \# \biguplus_{j=d+3}^{e+a_2} U^{(3)}_j.$$

Let

$$B = I \uplus J \uplus G,$$

where I is the set of lex-largest $\#H$ monomials in B and G is the revlex set with $\rho(G) = \biguplus_{j=d+3}^{e+a_2} U^{(3)}_j$ (see figure):



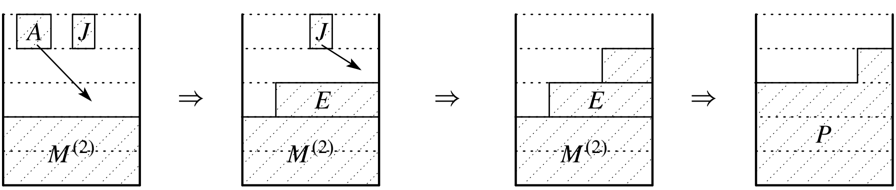
$B \uplus M^{(\geq 3)}$

Since $a_3 = 0$, (18) says $\rho(B) \subset \biguplus_{j=d+2}^{e+a_2} U^{(2)}_j$. Hence $\rho(I) \subset U^{(3)}_{d+2}$. Let $C \subset U^{(3)}_{d+2}$ be the lex set in $U^{(3)}_{d+2}$ with $\#C = \#H$. If we regard $U^{(\geq 3)}$ as a universal lex ideal in $K[x_3, \ldots, x_n]$, then H and C are lex sets in $K[x_3, \ldots, x_n]$ with the same cardinality. Hence $C = x_3 H$. Then, by the interval lemma,

$$m(H) = m(C) \succeq m(\rho(I)) = m(I). \tag{19}$$

Let $P \subset U^{(2)}$ be the super-revlex set with $\#P = \#A + \#J + \#M^{(2)}$. By the choice of G, G is the set of all monomials $u\boldsymbol{e}_2 \in \rho(F)$ such that u is not divisible by $x_2^{e+a_2-(d+2)}$. Also, since B does not contain any monomial $u\boldsymbol{e}_2$ such that u is divisible by $x_2^{e+a_2-(d+1)}$, any monomial in J is divisible by $x_2^{e+a_2-(d+2)}\boldsymbol{e}_2$. Then $x_2^{-(e+a_2)+d+2} J \subset U^{(2)}_{d+2}$ is a revlex set. Since $M^{(2)} \uplus E \uplus (x_2^{-(e+a_2)+(d+2)} J)$ is revlex, we have

$$m(P) \succeq m\big(M^{(2)} \uplus E \uplus x_2^{-(e+a_2)+(d+1)} J\big) = m\big(M^{(2)} \uplus A \uplus J\big). \tag{20}$$



$M^{(2)} \uplus A \uplus J$ $\quad$ $M^{(2)} \uplus E \uplus J$ $\quad$ $M^{(2)} \uplus E \uplus x_2^{-(e+a_2)+d+2} J$ $\quad$ P

Let

$$Q = \rho(F) \setminus (A \uplus B) = \rho(F)_{\leq e+a_2-1}.$$

<u>Subcase 2-2-a</u>. Suppose that $\#P + \#Q \leq \#U^{(2)}_{\leq e+a_2-1}$. Let $R \subset U^{(2)}$ be the super-revlex set with $\#R = \#P + \#Q$. Then since Q is an upper revlex set of degree $e + a_2 - 1$, by Corollary 4.5 and (20)

$$R \gg P \uplus Q \gg M^{(2)} \uplus A \uplus J \uplus Q. \tag{21}$$

On the other hand, by Lemma 5.15,

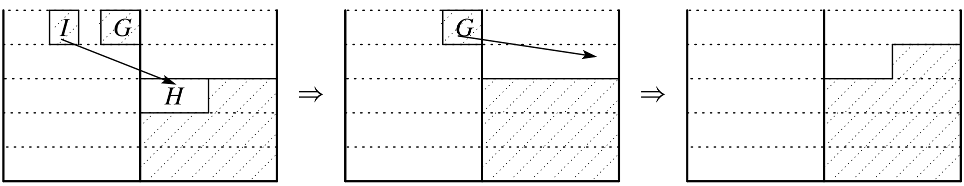
$$\#H + \#M^{(\geq 3)} < \#U^{(3)}_{\leq d+2}.$$

Then since $\rho(G) = \biguplus_{j=d+3}^{e+a_2} U^{(3)}_j$,

$$\#I \uplus G \uplus M^{(\geq 3)} = \#G \uplus H \uplus M^{(\geq 3)} < \#U^{(3)}_{\leq e+a_2} = \#U^{(2)}_{e+a_2}.$$

Let $U' = U^{(2)} \biguplus_{i=3}^{t} U^{(i)}[-a_2]$. Observe that $M^{(3)}[-a_2] \supset U'^{(\geq 3)}_{\leq e+a_2}$. Then Corollary 5.13 and (19) say that there exists an extremal set $Z \subset U^{(\geq 3)}[-a_2]$ such that

$$Z \gg G \uplus H \uplus \left(M^{(\geq 3)}[-a_2]\right) \gg G \uplus I \uplus M^{(\geq 3)}. \tag{22}$$



$I \uplus G \uplus M^{(\geq 3)}$ $\qquad$ $G \uplus H \uplus M^{(\geq 3)}$ $\qquad$ $Z[+a_2]$

We claim that

$$N = U^{(1)}_{\leq e-1} \uplus R \uplus Z[+a_2]$$

satisfies the desired conditions. Indeed, by (21) and (22),

$$\begin{aligned} N &\gg U^{(1)}_{\leq e-1} \uplus (M^{(2)} \uplus A \uplus J \uplus Q) \uplus G \uplus I \uplus M^{(\geq 3)} \\ &\gg U^{(1)}_{\leq e-1} \uplus F \uplus M^{(2)} \uplus M^{(\geq 3)} = M. \end{aligned}$$

(We use $\rho(F) = A \uplus I \uplus J \uplus G \uplus Q$ and $m(F) = m(\rho(F))$ for the second step.)

It remains to prove that N is a ladder set. Since $U^{(2)}_{\leq d} \subset R \subset U^{(2)}_{\leq e+a_2-1}$ it is enough to prove that $Z[+a_2] \supset U^{(\geq 3)}_{\leq e+a_2-1}$. Since $\rho(G) = \biguplus_{j=d+3}^{e+a_2} U^{(3)}_j$,

$$\#Z = \#(H \uplus M^{(\geq 3)} \uplus G) \geq \#U^{(\geq 3)}_{\leq d+1} \uplus \biguplus_{j=d+3}^{e+a_2} U^{(3)}_j \geq \#U^{(\geq 3)}_{\leq e+a_2-1}.$$

(We use $\#U_j^{(3)} \geq \#U_{j-1}^{(\geq 3)}$ for the last step.) Then $Z[+a_2] \supset U^{(\geq 3)}_{\leq e+a_2-1}$ by Lemma 5.10 as desired.

<u>Subcase 2-2-b</u>. Suppose that $\#P + \#Q > \#U^{(2)}_{\leq e+a_2-1}$. Note that

$$\#P + \#Q + \#I + \#G = \#F + \#M^{(2)}.$$

Then $\#M^{(2)} \uplus F > \#U^{(2)}_{\leq e+a_2-1}$. Let R be the super-revlex set with $\#R = \#M^{(2)} + \#F$. Then $\#R = \#M^{(2)} + \#F \leq \#U^{(2)}_{\leq e+a_2}$ by (14). Since $\#R \geq \#P + \#Q > U^{(2)}_{\leq e+a_2-1}$, there exists a revlex set $B' \subset U^{(2)}_{e+a_2}$ such that

$$R = U^{(2)}_{\leq e+a_2-1} \uplus B'.$$

Also by Corollary 4.5,

$$B' \uplus U^{(2)}_{\leq e+a_2-1} = R \gg M^{(2)} \uplus \rho(F). \tag{23}$$

Since $\#F + \#M^{(\geq 2)} < \#U^{(2)}_{\leq e+a_2}$, we have $\#B' + \#M^{(\geq 3)} < \#U^{(2)}_{e+a_2}$. Then by Corollary 5.13 there exists the extremal set $Z \subset U^{(\geq 3)}[-a_2]$ such that

$$B' \uplus (M^{(\geq 3)}[-a_2]) \ll Z. \tag{24}$$

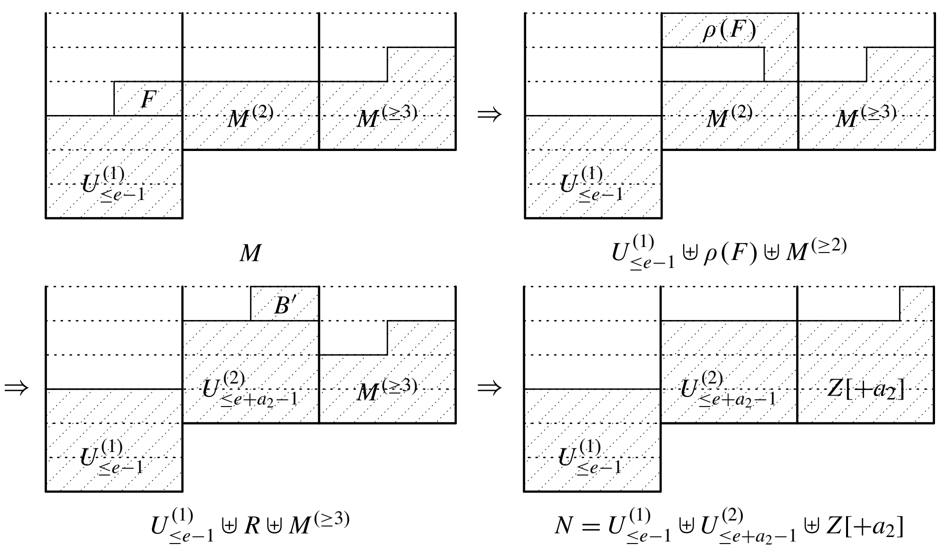
We claim that

$$N = U^{(1)}_{\leq e-1} \uplus U^{(2)}_{\leq e+a_2-1} \uplus Z[+a_2]$$

satisfies the desired conditions.

By (23) and (24),

$$N \gg U^{(1)}_{\leq e-1} \uplus U^{(2)}_{\leq e+a_2-1} \uplus B' \uplus M^{(\geq 3)} \gg U^{(1)}_{\leq e-1} \uplus F \uplus M^{(2)} \uplus M^{(\geq 3)} = M.$$



It remains to prove that N is a ladder set. What we must prove is:

$$Z[+a_2] \supset U^{(\geq 3)}_{\leq e+a_2-1}.$$

By the assumption of Subcase 2-2-b,

$$\#M^{(2)} + \#F - \#(I \uplus G) = \#Q + \#P > \#U^{(2)}_{\leq e+a_2-1}.$$

Then

$$\#B' = \#M^{(2)} + \#F - \#U^{(2)}_{\leq e+a_2-1} > \#I \uplus G.$$

Then in the same way as the computation of $\#Z$ in Subcase 2-2-a, we have

$$\#Z = \#M^{(\geq 3)} \uplus B' \geq \#M^{(\geq 3)} \uplus (I \uplus G) \geq \#U^{(\geq 3)}_{\leq e+a_2-1}.$$

Then by Lemma 5.10, $Z[+a_2] \supset U^{(\geq 3)}_{\leq e+a_2-1}$ as desired.

7. Examples

In this section, we give some examples of saturated graded ideals which attain maximal Betti numbers for a fixed Hilbert polynomial. Observe that, by the decomposition given before Definition 3.7, the Hilbert polynomial of a proper universal lex ideal $I = (\delta_1, \delta_2, \ldots, \delta_t)$ is given by

$$H_I(t) = \binom{t - b_1 + n - 1}{n-1} + \binom{t - b_2 + n - 2}{n-2} + \cdots + \binom{t - b_t + n - t}{n-t},$$

where $b_i = \deg \delta_i$ for $i = 1, 2, \ldots, t$.

Example 7.1. Let $S = K[x_1, \ldots, x_4]$ and $\bar{S} = K[x_1, \ldots, x_3]$. Consider the ideal $I = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^2x_3) \subset S$. Then

$$H_I(t) = \tfrac{1}{6}t^3 + t^2 - \tfrac{19}{6}t + 1 = \binom{t+2}{3} + \binom{t-4}{2} + \binom{t-9}{1}$$

and the proper universal lex ideal with the same Hilbert polynomial as I is

$$L = (x_1, x_2^6, x_2^5x_3^5).$$

Let

$$U = \operatorname{sat} \bar{L} = (\bar{L} : x_3^\infty) = (x_1, x_2^5) \subset \bar{S}$$

and $c = \dim_K U/\bar{L} = 5$. Then the extremal set $M \subset U$ with $\#M = 5$ is

$$M = x_1\{1, x_1, x_2, x_3\} \uplus x_2^5\{1\}.$$

Then the ideal in S generated by all monomials in $U \setminus M$ is

$$J = x_1(x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2) + x_2^5(x_2, x_3) \subset S,$$

and J has the largest total Betti numbers among all saturated graded ideals in S having the same Hilbert polynomial as I.

Example 7.2. Let $S = K[x_1, \dots, x_5]$ and $\bar{S} = K[x_1, \dots, x_4]$. Consider the ideal $I = (x_1, x_2^2, x_2x_3^3, x_2x_3^2x_4^{15})$. Then I is a proper universal lex ideal. Let

$$U = \operatorname{sat} \bar{I} = (\bar{I} : x_4^\infty) = (x_1, x_2^2, x_2x_3^2) \subset \bar{S}$$

and $c = \dim U/\bar{I} = 15$. Then the extremal set $M \subset U$ with $\#M = 15$ is

$$M = x_1\{1, x_1, x_2, x_3, x_4, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2\} \uplus x_2^2\{1, x_2, x_3, x_4\} \uplus x_2x_3^2\{1\}.$$

Then the ideal in S generated by all monomials in $U \setminus M$ is

$$\begin{aligned} J =& x_1(x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3^2, x_2x_3x_4, x_2x_4^2, x_3^3, x_3^2x_4, x_3x_4^2, x_4^3) \\ &+ x_2^2(x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2) + x_2x_3^2(x_3, x_4) \end{aligned}$$

and J has the largest total Betti numbers among all saturated graded ideals in S having the same Hilbert polynomial as I.

Finally, we give an explicit formula of the bounds in Theorem 1.1 for one special case. For positive integers a and d, let

$$a = \binom{a_d + d}{d} + \binom{a_{d-1} + d - 1}{d-1} + \cdots + \binom{a_t + t}{t}$$

be the d-th binomial representation of a. Thus $a_d, \dots, a_t$ are integers satisfying $a_d \ge a_{d-1} \ge \cdots \ge a_t \ge 0$ with $t \ge 1$. We define

$$a_{\langle d \rangle} = \binom{a_d - 1 + d}{d} + \binom{a_{d-1} - 1 + d - 1}{d-1} + \cdots + \binom{a_t - 1 + t}{t}.$$

Also, for $k = 0, 1, \dots, n-1$, we inductively define $a_{\langle d,k \rangle}$ by $a_{\langle d,0 \rangle} = a$ and $a_{\langle d,k \rangle} = (a_{\langle d,k-1 \rangle})_{\langle d \rangle}$ for $k \ge 1$, where $0_{\langle d \rangle} = 0$. The following formula is due to Valla [1994, Proposition 5]:

Lemma 7.3. *Let c be a positive integer, $M \subset S$ the super-revlex set with $\#M = c$, and let $J \subset S$ be the ideal generated by all monomials which are not in M. Let e be the unique integer such that $\binom{e-1+n}{n} \le c < \binom{e+n}{n}$ and let $r = c - \binom{e-1+n}{n}$. Then, for $i \ge 1$, one has*

$$\beta_i^S(S/J) = \binom{e+i-2}{e-1}\binom{e+n-1}{i+e-1} + \sum_{k=1}^{n-1} \binom{k}{i-1} r_{\langle e,n-k \rangle}. \tag{25}$$

The right-hand side of (25) only depends on c, n, and i. Thus we denote it by $B_i(c, n)$.

Let b and c be positive integers. Consider the polynomial

$$p(t) = \binom{t-b+n-1}{n-1} + \cdots + \binom{t-b+2}{2} + \binom{t-b-c+2}{1}. \tag{26}$$

The universal lex ideal having the Hilbert polynomial (26) is

$$L = (x_1^b, \dots, x_1^{b-1}x_{n-2}, x_1^{b-1}x_{n-1}^c).$$

Then $U = \operatorname{sat} \bar{L} = (x_1^{b-1})$ and $\dim_K(\operatorname{sat} \bar{L})/\bar{L} = c$. In this case, an ideal which attains the bound in Theorem 1.1 was considered in Example 5.4. Let $M \subset \bar{S} = K[x_1, \dots, x_{n-1}]$ be the super-revlex set with $\#M = c$ and let $J \subset S$ be the ideal generated by all monomials in $\bar{S}$ which are not in M. Then the ideal $L = x_1^{b-1}J$ attains the bound. In particular, by Lemma 7.3, we have:

Proposition 7.4. *Let $I \subset S$ be a saturated graded ideal whose Hilbert polynomial is of the form (26). Then $\beta_i^S(S/I) \le B_i(c, n-1)$ for all $i \ge 1$.*

Remark 7.5. When $b = 1$, the above proposition is the result of Valla [1994] who considered the case when the Hilbert polynomial of S/I is constant. Indeed, if $P_{S/I}(t)$ is equal to a constant number c then

$$P_I(t) = \binom{t+n-1}{n-1} - c = \binom{t-1+n-1}{n-1} + \cdots + \binom{t-1+2}{2} + \binom{t-1-c+2}{1}.$$

References

[Bigatti 1993] A. M. Bigatti, "Upper bounds for the Betti numbers of a given Hilbert function", *Comm. Algebra* **21**:7 (1993), 2317–2334. MR 94c:13014 Zbl 0817.13007

[Bruns and Herzog 1998] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, 2nd ed., Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998. MR 95h:13020 Zbl 0909.13005

[Eliahou and Kervaire 1990] S. Eliahou and M. Kervaire, "Minimal resolutions of some monomial ideals", *J. Algebra* **129**:1 (1990), 1–25. MR 91b:13019 Zbl 0701.13006

[Elías et al. 1991] J. Elías, L. Robbiano, and G. Valla, "Number of generators of ideals", *Nagoya Math. J.* **123** (1991), 39–76. MR 92h:13023 Zbl 0714.13016

[Herzog et al. 1986] J. Herzog, M. E. Rossi, and G. Valla, "On the depth of the symmetric algebra", *Trans. Amer. Math. Soc.* **296**:2 (1986), 577–606. MR 87f:13009 Zbl 0604.13008

[Hulett 1993] H. A. Hulett, "Maximum Betti numbers of homogeneous ideals with a given Hilbert function", *Comm. Algebra* **21**:7 (1993), 2335–2350. MR 94c:13015 Zbl 0817.13006

[Iyengar and Pardue 1999] S. Iyengar and K. Pardue, "Maximal minimal resolutions", *J. Reine Angew. Math.* **512** (1999), 27–48. MR 2000d:13023 Zbl 0927.13019

[Macaulay 1927] F. S. Macaulay, "Some properties of enumeration in the theory of modular systems", *Proc. London Math. Soc.* (2) **26**:1 (1927), 531–555. MR 1576950 JFM 53.0104.01

[Murai and Hibi 2008] S. Murai and T. Hibi, "The depth of an ideal with a given Hilbert function", *Proc. Amer. Math. Soc.* **136**:5 (2008), 1533–1538. MR 2009b:13028 Zbl 1148.13006

[Pardue 1996] K. Pardue, "Deformation classes of graded modules and maximal Betti numbers", *Illinois J. Math.* **40**:4 (1996), 564–585. MR 97g:13029 Zbl 0903.13004

[Robbiano 1981] L. Robbiano, "Coni tangenti a singolarità razionali", Contribution 6 in *Curve algebriche: atti del Convegno di Geometria Algebrica* (Florence, 1981), edited by F. Gheradelli, Consiglio Nazionale delle Ricerche, Istituto di Analisi Globale ed Applicazioni, Florence, 1981.

[Sally 1978] J. D. Sally, *Numbers of generators of ideals in local rings*, Marcel Dekker, New York, 1978. MR 58 #5654 Zbl 0395.13010

[Valla 1994] G. Valla, "On the Betti numbers of perfect ideals", *Compositio Math.* **91**:3 (1994), 305–319. MR 95d:13012 Zbl 0815.14033

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