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On binary cyclotomic polynomials

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We study the number of nonzero coefficients of cyclotomic polynomials Φ_m, where m is the product of two distinct primes.

1. Presentation of the results

Let $m \geq 1$ be an integer, and let Φ_m be the cyclotomic polynomial defined by

$$\Phi_m(X) := \prod_{\substack{j=1 \\ (j,m)=1}}^{m} (X - \exp(2\pi i j/m)).$$

This monic polynomial belongs to $\mathbb{Z}[X]$, and its degree is equal to $\varphi(m)$, the Euler function of the integer m. Let $\theta(m)$ be the number of nonzero coefficients of Φ_m. Of course, $\theta(m)$ satisfies the trivial inequalities

$$2 \leq \theta(m) \leq \varphi(m) + 1,$$

which are optimal when one considers the case $m = 1$ or $m = p$, a prime number. In these cases, all of the coefficients of Φ_m are equal to 1.

We reserve the letters p and q for prime numbers. We call an integer m *binary* if it is of the form $m = pq$, with p and q distinct. Let $\mathcal{B} = \{6, 10, 14, 15, 21, \dots\}$ be the set of binary integers. For $m \in \mathcal{B}$, we say that the associated cyclotomic polynomial Φ_m is *binary*. The coefficients of the binary cyclotomic polynomial Φ_m are equal to 0, 1 or -1. Furthermore, in that particular case, the function $\theta(m)$ has an explicit expression in terms of p and q that can be exploited by analytic number theory. More precisely:

Proposition A. *Let $m = pq$ be a binary integer with $p \neq q$. Then we have*

$$\theta(m) = 2\bar{p}_q\bar{q}_p - 1, \tag{1}$$

where $\bar{p}_q$ is the unique integer satisfying

$$\bar{p}_q p \equiv 1 \bmod q \quad \textit{and} \quad 1 \leq \bar{p}_q < q$$

and $\bar{q}_p$ is defined similarly.

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For a proof of this basic result, see [Carlitz 1966, Theorem; Bzdęga 2012], and for an interesting characterization of the nonzero coefficients of Φ_{pq}, see [Lam and Leung 1996] for instance.

Recently Bzdęga [2012] started the study of the distribution function of the map

$$m \in \mathcal{B} \mapsto \theta(m).$$

Let us review his results. Let γ and x be real numbers satisfying $0 < \gamma < \frac{1}{2}$ and $x \geq 6$, and let $H_\gamma(x)$ be the counting function

$$H_\gamma(x) := \#\big\{\, m : m \in \mathcal{B}, m \leq x, \theta(m) \leq m^{\frac{1}{2}+\gamma} \,\big\} \tag{2}$$

(because of the inequality (12) below, it is useless to study H_γ for $\gamma \leq 0$). With these conventions, Bzdęga [2012, Theorem] proved the following:

Theorem A. *For every $0 < \gamma < \frac{1}{2}$ and every $\epsilon > 0$, there exist $C(\gamma)$, $c(\epsilon, \gamma) > 0$ and $x_0 = x_0(\epsilon, \gamma)$ such that for $x \geq x_0$ one has the inequalities*

$$c(\epsilon, \gamma) x^{\frac{1}{2}+\gamma-\epsilon} \leq H_\gamma(x) \leq C(\gamma) x^{\frac{1}{2}+\gamma}. \tag{3}$$

The idea of Bzdęga is to relate the integers $m = pq$ contributing to $H_\gamma(x)$ to the solutions of the equations

$$\ell q - np = 1, \tag{4}$$

where ℓ and n are integers satisfying some inequalities depending on p, q and γ. Write $t = np$. By (4) and by ingenious considerations, he is led to counting integers t such that t and $t+1$ both have a large prime factor. Appealing to a deep result of Hildebrand [1985] on p-stable subsets of integers, Bzdęga deduces the inequalities (3).

Our plan is to study (4) in the context of prime number theory and to get three different types of results according to the size of γ. These results suggest that this investigation becomes more and more intricate as γ decreases to 0. The first result gives an asymptotic formula when γ is large. Its proof is mainly based on bounds for Kloosterman–Ramanujan sums over primes (see Lemmas 2 and 3 below) and on the Bombieri–Vinogradov theorem (see Lemma 5).

Theorem 1. *For $0 < \gamma < \frac{1}{2}$, let*

$$C(\gamma) := \frac{2}{1+2\gamma} \log \frac{1+2\gamma}{1-2\gamma}. \tag{5}$$

Then for every $\gamma_0 > 0$, uniformly for γ satisfying $\frac{12}{25} + \gamma_0 \leq \gamma \leq \frac{1}{2} - \gamma_0$, we have

$$H_\gamma(x) \sim C(\gamma) \frac{x^{\frac{1}{2}+\gamma}}{\log x}$$

as $x \to \infty$.

The second result produces a universal upper bound for $H_\gamma(x)$ and is a rather direct consequence of the two-dimensional sieve (see Lemma 4).

Theorem 2. *For every* $\gamma_0 > 0$, *there exists* $C^+(\gamma_0)$ *such that, for every* γ *satisfying* $\gamma_0 \le \gamma \le \frac{1}{2} - \gamma_0$ *and for every* $x \ge 6$, *the following inequality holds*:

$$H_\gamma(x) \le C^+(\gamma_0)\frac{x^{\frac{1}{2}+\gamma}}{\log x}.$$

The last result is a lower bound when γ is large enough. Judging by the tools involved, it is certainly the deepest of our three results (see Lemma 7).

Theorem 3. *For every* $\gamma_0 > 0$, *there exist* $C^-(\gamma_0) > 0$ *and* $x(\gamma_0)$ *such that, for every* γ *satisfying* $\frac{15}{98} + \gamma_0 \le \gamma \le \frac{1}{2} - \gamma_0$ *and for every* $x \ge x(\gamma_0)$, *the following inequality holds*:

$$H_\gamma(x) \ge C^-(\gamma_0)\frac{x^{\frac{1}{2}+\gamma}}{\log x}.$$

When $\gamma = \frac{1}{2}$, $H_\gamma(x)$ counts the number of binary integers less than x, and this number is asymptotic to $x(\log\log x)(\log x)^{-1}$. This explains why the asymptotic formula in Theorem 1 cannot be uniform for $\gamma < \frac{1}{2}$. Finally, we postpone to Section 7 a discussion on a conjectural value of $H_\gamma(x)$.

2. Tools

2.1. *Notation.*

- We reserve the letters p and q for distinct prime numbers. For brevity, we replace the symbols $\bar{p}_q$ and $\bar{q}_p$ (defined in Proposition A) by $\bar{p}$ and $\bar{q}$.
- For $x \ge 1$, $\mathscr{L}$ denotes $\log 2x$, and $\xi := 1 + \mathscr{L}^{-1}$.
- For $N \ge 1$, the notation $n \sim N$ and $n \approx N$ respectively replaces the conditions $N < n \le 2N$ and $N < n \le \xi N$.
- For $N \ge 1$, the notation $n \asymp N$ means that n satisfies $c_1 N < n \le c_2 N$, where $0 < c_1 < c_2$ are absolute constants that are useless to specify.
- For $x \ge 1$, $\pi(x)$ is the number of primes less than x.
- For integers r and s, $\pi(x; r, s)$ is the number of ps less than x and congruent to s modulo r.
- For a real number t, $e(t)$ is the additive character $\exp(2\pi i t)$.
- The number of positive divisors of the integer n is denoted by $\tau(n)$.

2.2. ***Trigonometric sums.*** To detect the oscillations of the fractional part of the quotient $\bar{q}/p$, we shall appeal to the following well known lemma of Vinogradov, which is stated in different ways in the literature:

Lemma 1 [Vinogradov 1954, Lemma 12, page 32]. *Let* $r \geq 1$ *be an integer, and let* β *and* Δ *be real numbers satisfying* $0 < \Delta < \beta/2 < 1/4$. *Then there exist two functions* $\psi^{\pm}$ *with period* 1 *satisfying*

$$\begin{cases} \psi^{+}(t) = 1 & \textit{for}\ 0 \leq t \leq \beta, \\ 0 \leq \psi^{+}(t) \leq 1 & \textit{for}\ -\Delta \leq t \leq 0\ \textit{or}\ \beta \leq t \leq \beta + \Delta, \\ \psi^{+}(t) = 0, & \textit{if}\ t \pmod 1 \notin [-\Delta, \beta + \Delta], \end{cases} \tag{6}$$

$$\begin{cases} \psi^{-}(t) = 1 & \textit{for}\ \Delta \leq t \leq \beta - \Delta, \\ 0 \leq \psi^{-}(t) \leq 1 & \textit{for}\ 0 \leq t \leq \Delta\ \textit{or}\ \beta - \Delta \leq t \leq \beta, \\ \psi^{-}(t) = 0, & \textit{if}\ t \pmod 1 \notin [0, \beta], \end{cases} \tag{7}$$

and

$$\psi^{\pm}(t) = \sum_{m=-\infty}^{\infty} c_m^{\pm} e(mt) \quad \textit{for every real}\ t. \tag{8}$$

The coefficients $c_m^{\pm}$ *satisfy the equalities* $c_0^{\pm} = \beta \pm \Delta$ *and the inequalities*

$$|c_m^{\pm}| \leq 2 \min\Big\{\beta \pm \Delta, \frac{1}{\pi |m|}, \frac{1}{\pi |m|}\Big(\frac{r}{\pi |m| \Delta}\Big)^r\Big\}, \quad m \neq 0.$$

2.3. ***Kloosterman–Ramanujan sums over primes.*** For real $y \geq x \geq 1$ and for a a nonzero integer, we introduce the following trigonometric sum over primes:

$$S_p(a; x, y) := \sum_{x < q < y} e\Big(a \frac{\bar{q}}{p}\Big). \tag{9}$$

This sum differs from a classical Kloosterman–Ramanujan sum by the fact that the summation is restricted to prime values. We will benefit from oscillations of the function $q \mapsto e(a(\bar{q}/p))$ under the form of the two following lemmas extracted from [Fouvry and Shparlinski 2011]. The proofs of these two lemmas are based on the method of Garaev [2010]. For more general results on sums of this type, see [Fouvry and Michel 1998].

The first of these two lemmas considers the case where p is small compared with x and y.

Lemma 2 [Fouvry and Shparlinski 2011, Theorem 3.2]. *The bound*

$$S_p(a; x, y) \ll p^{-\frac{1}{2}} x \mathscr{L}^2 + p^{\frac{1}{4}} x^{\frac{4}{5}} \mathscr{L}^{\frac{3}{2}}$$

holds uniformly for every prime $p \geq 2$, *for every integer* a *not divisible by* p *and for every* $1 \leq x \leq y \leq 2x$.

This bound is interesting for $p \leq x^{\frac{4}{5}}$ only. We will have to deal with sums $S_p(a; x, y)$ for p slightly less than x. Still based on the method of Garaev, we have the following average bound of this sum, which is Theorem 3.3 of [Fouvry and Shparlinski 2011] for the choices $x_p = x$ and $x'_p = 2x$; the extension to the statement given is straightforward.

Lemma 3. *For every $\epsilon > 0$, the inequality*

$$\sum_{p \sim P} \max_{(a,p)=1} \left| S_p(a; x_p, x'_p) \right| \ll_\epsilon \left(x^{\frac{3}{5}} P^{\frac{13}{10}} + x^{\frac{5}{6}} P^{\frac{13}{12}} \right) P^\epsilon$$

holds uniformly for $P^{\frac{3}{2}} \geq x \geq 1$ and for any sequences of integers $(x_p)_{p \sim P}$ and $(x'_p)_{p \sim P}$ satisfying $x \leq x_p \leq x'_p \leq 2x$.

2.4. ***The two-dimensional sieve.*** The following lemma can be obtained by Brun's sieve and will be used in the proof of Theorem 2 since it produces an upper bound for the number of solutions to (4) with a large uniformity over ℓ and n:

Lemma 4 [Friedlander and Iwaniec 2010, Proposition 6.22]. *Let a, b and h be positive integers satisfying*

$$(a, b) = (ab, h) = 1 \quad \textit{and} \quad 2 \mid abh.$$

Let $N_{abh}(x, z)$ be the number of pairs of positive integers m and n satisfying $am \leq x$, $(mn, h) = 1$, $am + h = bn$ and mn has no prime factors less than z. Then, for $z \geq 2$ and

$$x \geq \tau(h) abz (\log z)^4, \tag{10}$$

we have the inequality

$$N_{abh}(x, z) \ll \frac{hx}{\varphi(abh)} (\log z)^{-2},$$

where the implied constant is absolute.

2.5. ***The Bombieri–Vinogradov theorem.*** We now recall this cornerstone of current analytic number theory. It gives the average behavior of the function $\pi(x; r, s)$ and replaces the assumption of the Generalized Riemann Hypothesis for Dirichlet L-functions in many applications. Among the numerous possible references, we give here the version in [Iwaniec and Kowalski 2004, Theorem 17.1, (17.24)].

Lemma 5. *For every $A \geq 0$, there exists $C(A)$ such that, for every $x \geq 1$ and for $R := x^{\frac{1}{2}} \mathscr{L}^{-2A-6}$, one has the inequality*

$$\sum_{r \leq R} \max_{(s,r)=1} \left| \pi(x; r, s) - \frac{\pi(x)}{\varphi(r)} \right| \leq C(A) x \mathscr{L}^{-A-1}.$$

2.6. ***A variant of the Brun–Titchmarsh theorem.*** The proof of Theorem 3 heavily depends on lower bounds for the function $\pi(x; r, s)$ in cases that are not covered by Lemma 5, which means r is larger than $x^{\frac{1}{2}}$. We first recall the original statement of Mikawa [2001, Theorem].

Lemma 6. *Let $L > \frac{32}{17}$ and $A, B > 0$ be given. Let s be an integer and R be large with $0 < |s| \le (\log R)^B$. Then, except possibly for $O(R(\log R)^{-A})$ integers r satisfying $(r, s) = 1$ and $r \sim R$, we have*

$$\inf\{\, p : p \equiv s \mod r\,\} \ll r^L,$$

where the implied constants depend only on A, B and L.

This result can be interpreted as an average version of Linnik's famous theorem concerning the least prime in an arithmetic progression. Actually, Mikawa's proof gives more. For instance, it instantly gives a lower bound with the correct order of magnitude for the function $\pi(r^L; r, s)$ for almost all r as above. Due to the value of L, this result can be viewed as a lower bound of the function $\pi(x; r, s)$ for almost all r coprime with s and slightly larger than $\sqrt{x}$. As far as we know, the first result of that type was due to Rousselet [1988] following techniques of Fouvry [1985], who was dealing with upper bounds of the function $\pi(x; r, s)$ (Brun–Titchmarsh theorem on average). The problem of giving both upper and lower bounds for $\pi(x; r, s)$ for almost r in the interval $[x^{\frac{1}{2}}, x^{\frac{1}{2}+\delta}]$, where δ is a small positive constant, was then treated in several remarkable papers [Bombieri et al. 1987; 1989; Baker and Harman 1996].

We give an improved version of Lemma 6 where we count primes in the interval $]x, 2x]$ with some uniformity over the congruence class $\bar{s} \bmod r$ (as above, $\bar{s}$ is the multiplicative inverse of $s \bmod r$). Such a generalization is necessary for our application and is possible by the structure of the proof of Lemma 6 based on bounds for Kloosterman sums on average (see [Habsieger and Sivak-Fischler 2010, Theorem 1.5] for another reference where this extension is made).

Lemma 7. *For every $K < \frac{17}{32}$, there exist $\alpha_K > 0$, $\beta_K > 0$ and x_K such that for every $x > x_K$, every R satisfying $2 \le R < x^K$ and every s such that $1 \le |s| \le x^{\beta_K}$, the inequality*

$$\pi(2x; r, \bar{s}) - \pi(x; r, \bar{s}) \ge \alpha_K \frac{x}{\varphi(r)\log x},$$

holds for every $r \sim R$ coprime with s with at most $R(\log R)^{-2}$ exceptions.

Remark. Of course, in this lemma, we can suppose that the functions $K \mapsto \alpha_K$ and $K \mapsto \beta_K$ are decreasing and $K \mapsto x_K$ is increasing.

3. Basic transformations

3.1. ***Properties of the function θ.*** We first write the expression of $\theta(pq)$ given by Proposition A in an asymmetrical way. Actually, Bézout's identity and the inequalities $1 \le \bar{p} < q$ and $1 \le \bar{q} < p$ lead to the equality

$$p\bar{p} + q\bar{q} = 1 + pq,$$

which transforms (1) into

$$\theta(pq) = 2pq \cdot \frac{\bar{q}}{p}\Big(1 + \frac{1}{pq} - \frac{\bar{q}}{p}\Big) - 1. \tag{11}$$

Now suppose that $p < q$. From the trivial inequalities

$$\frac{1}{p} \le \frac{\bar{q}}{p} \le 1 - \frac{1}{p}$$

and from the properties of the function $t \mapsto t((1 + 1/pq) - t)$, we deduce

$$\theta(pq) \ge q > (pq)^{\frac{1}{2}}, \tag{12}$$

which implies that $H_\gamma(x) = 0$ for $\gamma \le 0$.

We now want to translate in an efficient manner the inequality

$$\theta(pq) \le (pq)^{\frac{1}{2}+\gamma}.$$

In order to control uniformity aspects, we will frequently assume that we have

$$\gamma_0 \le \gamma \le \tfrac{1}{2} - \gamma_0, \tag{13}$$

where γ_0 is a fixed positive number.

For $t \ge T(\gamma_0)$, let $0 < \theta_0(t) < 1 - \theta_1(t) < 1$ be the solutions of the polynomial equation of degree 2 in the unknown X

$$2tX\Big(1 + \frac{1}{t} - X\Big) - 1 = t^{\frac{1}{2}+\gamma}.$$

For simplicity, we omit in the sequel the dependency on the parameter γ.

Lemma 8. *We suppose that* (13) *holds. Let $m = pq$ be a binary integer with $p < q$ and $m \ge T(\gamma_0)$. Then*

$$\theta(m) \le m^{\frac{1}{2}+\gamma} \iff 0 < \frac{\bar{q}}{p} \le \theta_0(m) \textit{ or } 1 - \theta_1(m) \le \frac{\bar{q}}{p} < 1. \tag{14}$$

The functions $t \mapsto \theta_0(t), \theta_1(t)$ are decreasing for $t > T(\gamma_0)$, are of $\mathscr{C}^\infty$-class and satisfy

$$\theta_0(t),\ \theta_1(t) = \frac{t^{\gamma-\frac{1}{2}}}{2} + O(t^{2\gamma-1}),$$

where the implied constant depends on γ_0 only.

Proof. The proof of (14) is easy; it is only a transcription of (11). Finally, the asymptotic behaviors of the functions $\theta_i(t)$ are consequences of the exact formula

$$\theta_0(t),\, 1-\theta_1(t) = \frac{1+\frac{1}{t}\mp\sqrt{\left(1+\frac{1}{t}\right)^2-2\frac{t^{1/2+\gamma}+1}{t}}}{2}. \qquad \square$$

3.2. ***Decomposition of $H_\gamma(x)$.*** We always suppose that (13) is true. Let $T(\gamma_0)$ be defined as in Lemma 8. We use (14) to split the set contributing to $H_\gamma(x)$

$$\left\{ (p,q) : p < q,\, T(\gamma_0) \le pq \le x,\, \theta(pq) \le (pq)^{\frac{1}{2}+\gamma} \right\}$$

into two disjoint subsets corresponding to $0 < \bar{q}/p \le \theta_0(pq)$ or $1-\theta_1(pq) \le \bar{q}/p < 1$. Let $H_\gamma^0(x)$ and $H_\gamma^1(x)$ be the corresponding cardinalities, which give the equality

$$H_\gamma(x) = H_\gamma^0(x) + H_\gamma^1(x) + O(T(\gamma_0)). \tag{15}$$

We shall concentrate our study on the case of $H_\gamma^0(x)$ since the case of $H_\gamma^1(x)$ is quite similar because the functions θ_0 and θ_1 play the same role (see Lemma 8).

To control the order of magnitude of the variables p and q, we consider, for P, $Q \ge 2$ such that $PQ \ge T(\gamma_0)$, the counting functions

$$R_\gamma(P, Q) := \#\left\{ (p,q) : p < q,\, pq \le x,\, p \approx P,\, q \approx Q,\, 0 < \frac{\bar{q}}{p} \le \Theta_0 \right\}, \tag{16}$$

where

$$\Theta_0 = \theta_0(PQ). \tag{17}$$

Since the function θ_0 is decreasing, we obtain the inequality

$$H_\gamma^0(x) \le \sum_P \sum_Q R_\gamma(P, Q), \tag{18}$$

where the sum is over pairs (P, Q), where P and Q are of the form $2 \cdot \xi^k$ for $k = 0, 1, 2, \ldots$ and satisfy the inequalities

$$T(\gamma_0) \le PQ \le x \quad \text{and} \quad P \le \xi Q. \tag{19}$$

Finally note that (12) implies that we can even restrict the summation to the cases

$$4(PQ)^{\frac{1}{2}+\gamma} \ge Q \tag{20}$$

since otherwise $R_\gamma(P, Q) = 0$. Combining (19) and (20), we deduce that P and Q satisfy the inequalities

$$P \le \xi Q \quad \text{and} \quad \kappa_0 Q^{\frac{1-2\gamma}{1+2\gamma}} \le P \le xQ^{-1} \quad \text{with } \kappa_0 = 4^{-\frac{2}{1+2\gamma}}. \tag{21}$$

The inequality (18) can be easily transformed into a lower bound on $H_\gamma^0(x)$ if one replaces Θ_0 by Θ_0' with $\Theta_0' := \theta_0(\xi^2 PQ)$ in the definition (16) of $R_\gamma(P, Q)$. We note that

$$\Theta_0' - \Theta_0 = O(\Theta_0 \mathscr{L}^{-1}), \tag{22}$$

as a result of Lemma 8 and the fineness of the cutting of the sum $H_\gamma^0(x)$ (see (18)).

4. Proof of Theorem 1

The first purpose of this section is to prove the following:

Proposition 1. *Let* $\gamma_0 > 0$. *Then uniformly for* γ *satisfying*

$$\tfrac{12}{25} + \gamma_0 \le \gamma \le \tfrac{1}{2} - \gamma_0 \tag{23}$$

and for (P, Q) *satisfying the conditions* (21), *one has the equality*

$$R_\gamma(P, Q) = \tfrac{1}{2}(PQ)^{\gamma - \frac{1}{2}}(1 + O(\mathscr{L}^{-1}))\Bigg(\sum_{\substack{p\approx P\\ pq\le x}} \sum_{\substack{q\approx Q\\ p<q}} 1\Bigg) + O(x^{\frac{1}{2}+\gamma}\mathscr{L}^{-6}) + O(Q\mathscr{L}^{-4}).$$

Our proof depends on the size of P compared with Q.

4.1. ***When P is small.*** Let $\mathscr{E}(p, \Theta_0)$ denote the set of congruence classes $s \bmod p$ such that $0 < \bar{s}/p \le \Theta_0$. Of course, $\bar{s}$ is the multiplicative inverse of $s \bmod p$. By the definition (17) and by Lemma 8, its cardinality satisfies

$$\#\mathscr{E}(p, \Theta_0) = (\tfrac{1}{2} + O(\mathscr{L}^{-1}))P^{\frac{1}{2}+\gamma}Q^{\gamma - \frac{1}{2}} + O(1). \tag{24}$$

Let

$$y_p := \max(Q, p) \quad \text{and} \quad z_p := \min(\xi Q, x/p). \tag{25}$$

With this definition, we have the equality

$$R_\gamma(P, Q) = \sum_{\substack{p\approx P\\ y_p \le z_p}} \sum_{s\in\mathscr{E}(p,\Theta_0)} \big(\pi(z_p; p, s) - \pi(y_p; p, s)\big). \tag{26}$$

For (P, Q) satisfying (21), the trivial estimate

$$\big(\pi(z_p; p, s) - \pi(y_p; p, s)\big) \le Q/p + 1 \ll Q/p$$

inserted in (26) gives the bound

$$R_\gamma(P, Q) \ll (PQ)^{\frac{1}{2}+\gamma} + Q \ll (PQ)^{\frac{1}{2}+\gamma} \tag{27}$$

by (20). Hence, for the proof of Proposition 1, we may add the extra condition

$$PQ \ge x\mathscr{L}^{-12}. \tag{28}$$

The equalities (24) and (26) and Lemma 5 allow us to improve (27) by

$$R_\gamma(P,Q)=\bigl[(\tfrac{1}{2}+O(\mathscr{L}^{-1}))P^{\frac{1}{2}+\gamma}Q^{\gamma-\frac{1}{2}}+O(1)\bigr]\biggl(\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}}\frac{1}{\varphi(p)}\biggr)\\ +O((PQ)^{\frac{1}{2}+\gamma}\mathscr{L}^{-6})+O(Q\mathscr{L}^{-6}) \quad (29)$$

provided

$$P\le Q^{\frac{1}{2}}\mathscr{L}^{-100}. \quad (30)$$

The contribution of the $O(1)$-term to the right-hand side of (29) is bounded by $Q\mathscr{L}^{-4}$, up to a multiplicative constant. Recalling the restriction (28), we see that the proof of Proposition 1 is complete in the particular case

$$P\le x^{\frac{1}{3}}\mathscr{L}^{-100}. \quad (31)$$

4.2. ***Medium values of P.*** We apply Lemma 1 with the choices

$$\beta=\Theta_0,\quad \Delta=\Theta_0\mathscr{L}^{-3},\quad r=4.$$

We then have the inequalities

$$\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}}\psi^-\bigl(\tfrac{\bar{q}}{p}\bigr)\le R_\gamma(P,Q)\le\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}}\psi^+\Bigl(\frac{\bar{q}}{p}\Bigr). \quad (32)$$

We only study the upper bound of $R_\gamma(P,Q)$ in (32). We recall the definitions (9) and (25). We apply Lemma 1 (in a slightly weaker form) and decompose the sums according to the values of m and whether p and m are coprime. This gives

$$\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}}\psi^+\Bigl(\frac{\bar{q}}{p}\Bigr)\\ \le(\beta+\Delta)\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}}1\\ +2\sum_{p\approx P}\biggl\{\sum_{\substack{1\le|m|\le\Delta^{-1}\\ p\nmid m}}\frac{1}{\pi|m|}+\sum_{\substack{|m|>\Delta^{-1}\\ p\nmid m}}\frac{256}{\pi^5|m|^5\Delta^4}\\ +\sum_{\substack{1\le|m|\le\Delta^{-1}\\ p\mid m}}\frac{2}{\pi|m|}+\sum_{\substack{|m|>\Delta^{-1}\\ p\mid m}}\frac{256}{\pi^5|m|^5\Delta^4}\biggr\}\bigl|S_p\bigl(m;y_p,z_p\bigr)\bigr|. \quad (33)$$

It remains to apply Lemma 2 when $p\nmid m$, or the trivial inequality $|S_p|\le Q$ otherwise, and to sum over m to obtain the inequality

$$\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}}\psi^+\Big(\frac{\bar q}{p}\Big)\le(\beta+\Delta)\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}}1 \\ +O\Big(\sum_{p\approx P}\big\{(\mathscr{L}+1)(p^{-\frac12}Q+p^{\frac14}Q^{\frac45})\mathscr{L}^2+(p^{-1}\mathscr{L}+p^{-1})Q\big\}\Big). \tag{34}$$

Using the upper bound $\sum_{p\approx P}1\ll_{\gamma_0}P\mathscr{L}^{-2}$, we see that the error term satisfies

$$\text{error term}\ll_{\gamma_0}\big(P^{\frac12}Q+P^{\frac54}Q^{\frac45}\big)\mathscr{L}. \tag{35}$$

By Lemma 8 and (28), we have the equality

$$\Theta_0=\tfrac12(PQ)^{\gamma-\frac12}+O((PQ)^{2\gamma-1})=\tfrac12(PQ)^{\gamma-\frac12}(1+O(\mathscr{L}^{-3})),$$

which, combined with (32), (34) and (35) gives the inequality

$$R_\gamma(P,Q) \\ =\tfrac12(PQ)^{\gamma-\frac12}(1+O(\mathscr{L}^{-3}))\Big(\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}}1\Big)+O\big((P^{\frac12}Q+P^{\frac54}Q^{\frac45})\mathscr{L}\big). \tag{36}$$

Recalling the restrictions (21), we see that (36) implies Proposition 1 as soon as P satisfies the inequalities

$$P\ge x^{1-2\gamma}\mathscr{L}^{14}\quad\text{and}\quad P\le x^{\frac{20}{9}\gamma-\frac23}\mathscr{L}^{-16}. \tag{37}$$

4.3. ***Large values of P.*** Actually, in (33) we may benefit from the summation over $p\approx P$ by appealing to Lemma 3 instead of Lemma 2. By the same technique as in Section 4.2, we arrive at the equality

$$R_\gamma(P,Q)=\Theta_0(1+O(\mathscr{L}^{-3}))\Big(\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}}1\Big)+O_\epsilon\big((P^{\frac{13}{10}}Q^{\frac35}+P^{\frac{13}{12}}Q^{\frac56})x^\epsilon\big) \tag{38}$$

provided $P^{\frac32}\ge Q$ and ϵ is an arbitrary positive number. Hence, by (21) and (28), we see that (38) implies Proposition 1 as soon as P satisfies the extra conditions

$$P\ge x^{\frac25},\qquad P\le x^{\frac{10}{7}\gamma-\frac17-2\epsilon}\quad\text{and}\quad P\le x^{4\gamma-\frac43-5\epsilon}. \tag{39}$$

Suppose now that γ satisfies (23) and that P satisfies $1\le P\le 2\sqrt{x}$. Then we see that P satisfies at least one of the sets of conditions (31), (37) or (39). This completes the proof of Proposition 1. □

4.4. ***Conclusion of the proof of Theorem 1.*** We insert the expansion of $R_\gamma(P,\, Q)$ given in Proposition 1 in the right-hand side of (18) and sum over $(P,\, Q)$ satisfying (21). Recall that the numbers P and Q are of the shape $2\cdot\xi^k$. We first consider the contribution of the term $O(Q\mathscr{L}^{-4})$. By (21), this contribution satisfies

$$\begin{aligned} O(Q\mathscr{L}^{-4}) \text{ term} &\ll \mathscr{L}^{-4}\sum_Q Q \sum_{\kappa_0 Q^{\frac{1-2\gamma}{1+2\gamma}} \le P < xQ^{-1}} 1 \\ &\ll \mathscr{L}^{-3} \sum_{Q\le (\frac{x}{\kappa_0})^{\frac12+\gamma}} Q\Big(\log\Big(\frac{x}{\kappa_0}Q^{-\frac{2}{1+2\gamma}}\Big)+1\Big) \\ &\ll \mathscr{L}^{-3}\Big\{ \sum_{Q\le(\frac{x}{\kappa_0})^{\frac12+\gamma}\mathscr{L}^{-1}} Q\mathscr{L} + \log\mathscr{L} \sum_{(\frac{x}{\kappa_0})^{\frac12+\gamma}\mathscr{L}^{-1}\le Q\le(\frac{x}{\kappa_0})^{\frac12+\gamma}} Q\Big\} \\ &\ll x^{\frac12+\gamma}\mathscr{L}^{-\frac32}. \end{aligned}$$

Since the number of $(P,\, Q)$ satisfying (21) is $O(\mathscr{L}^4)$, the contribution of the term $O(x^{\frac12+\gamma}\mathscr{L}^{-6})$ (coming from Proposition 1) is $O(x^{\frac12+\gamma}\mathscr{L}^{-2})$. From the above considerations, we deduce the inequality

$$H^0_\gamma(x) \le (\tfrac12 + o(1))\sum_P\sum_Q (PQ)^{\gamma-\frac12}\Big(\sum_{\substack{p\approx P\\ pq\le x}}\sum_{\substack{q\approx Q\\ p<q}} 1\Big) + O(x^{\frac12+\gamma}\mathscr{L}^{-2}),$$

where P and Q satisfy (21). We now want to drop the dissection parameters P and Q. To do so, we remark that $(PQ)^{\gamma-\frac12} = (1+o(1))(pq)^{\gamma-\frac12}$ for $p\approx P$ and $q \approx Q$. We gather the rectangles of summation $]P,\, \xi P]\times]Q,\, \xi Q]$ to deduce the inequality

$$H^0_\gamma(x) \le (\tfrac12 + o(1))\Big(\sum_{p<q\le x/p}\sum (pq)^{\gamma-\frac12}\Big) + O(x^{\frac12+\gamma}\mathscr{L}^{-2}). \tag{40}$$

By the prime number theorem, we have

$$\sum_{p<q\le x/p}\sum (pq)^{\gamma-\frac12} \sim \int_{x^{\frac12}}^{x^{\frac12+\gamma}} \frac{y^{\gamma-\frac12}}{\log y}\, dy \int_3^{xy^{-1}} \frac{z^{\gamma-\frac12}}{\log z}\, dz \quad (x\to\infty).$$

Write $y := x^u$ and $z := x^v$ to deduce

$$\begin{aligned} \sum_{p<q\le x/p}\sum (pq)^{\gamma-\frac12} &\sim \int_{\frac12}^{\frac12+\gamma} \frac{x^{u(\gamma+\frac12)}}{u}\, du \int_{\frac{\log 3}{\log x}}^{1-u} \frac{x^{v(\gamma+\frac12)}}{v}\, dv \\ &\sim \int_{\frac12}^{\frac12+\gamma} \frac{x^{u(\gamma+\frac12)}}{u}\cdot\frac{x^{(1-u)(\gamma+\frac12)}}{(1-u)\big(\gamma+\frac12\big)\log x} du \sim C(\gamma)\frac{x^{\frac12+\gamma}}{\log x}, \end{aligned} \tag{41}$$

where $C(\gamma)$ is defined in (5).

The study of $H^1_\gamma(x)$ defined in (15) is similar to the study of $H^0_\gamma(x)$. Combining (15), (40) and (41), we finally arrive at the inequality

$$H_\gamma(x) \le (1+o(1))C(\gamma)\frac{x^{\frac{1}{2}+\gamma}}{\log 2x}. \tag{42}$$

To produce a lower bound for $H^0_\gamma(x)$, we follow the idea presented at the end of Section 3.2, which consists of replacing the constant Θ_0 by Θ_0' in the definition of $R_\gamma(P, Q)$. By (22), we also obtain the inequalities

$$H^0_\gamma(x),\, H^1_\gamma(x) \ge (1-o(1))\frac{C(\gamma)}{2}\cdot\frac{x^{\frac{1}{2}+\gamma}}{\log 2x}$$

as x tends to infinity. Summing these two inequalities, we arrive at

$$H_\gamma(x) \ge (1-o(1))C(\gamma)\frac{x^{\frac{1}{2}+\gamma}}{\log 2x}.$$

Combining with (42), this completes the proof of Theorem 1. □

5. Proof of Theorem 2

We still suppose that (13) is satisfied and that PQ is large enough, which means $PQ \ge T(\gamma_0)$, where $T(\gamma_0)$ is defined in Lemma 8. Since we are searching for an upper bound, it is useless to work with a very thin cutting up as in (16). So let

$$S^0_\gamma(P, Q) := \#\Big\{ (p,q) : p \sim P, q \sim Q, p < q, 0 < \frac{\bar{q}}{p} \le \Theta_0 \Big\}, \tag{43}$$

$$S^1_\gamma(P, Q) := \#\Big\{ (p,q) : p \sim P, q \sim Q, p < q, 1-\Theta_1 < \frac{\bar{q}}{p} < 1 \Big\}, \tag{44}$$

where Θ_0 is still defined by (17) and $\Theta_1 = \theta_1(PQ)$. We then have the inequality

$$H_\gamma(x) \le \sum_{(P,Q)}\!\sum S^0_\gamma(P, Q) + \sum_{(P,Q)}\!\sum S^1_\gamma(P, Q) + O(T(\gamma_0)), \tag{45}$$

where P and Q are powers of 2 and satisfy $P \le 2Q$ and $T(\gamma_0) \le PQ \le x$. We will focus our study on the case of $S^0_\gamma(P, Q)$.

Define

$$L := P^{\gamma+\frac{1}{2}}Q^{\gamma-\frac{1}{2}}. \tag{46}$$

If (p, q) contributes to $S^0_\gamma(P, Q)$, then we have the equality (4) for some ℓ satisfying $1 \le \ell \ll L$. Hence, we have the inequality

$$S^0_\gamma(P, Q) \le \sum_{1\le\ell\ll L}\ \sum_{n\asymp\ell Q/P} F(\ell, n, P, Q), \tag{47}$$

where

- the constants implicit in the symbols $\ll$ and $\asymp$ depend on γ_0 only and
- $F(\ell, n, P, Q)$ is the number of solutions of the equation $\ell q - np = 1$ in primes $p \sim P$ and $q \sim Q$.

By Lemma 4, we have the inequality

$$F(\ell, n, P, Q) \ll \frac{\ell Q}{\varphi(\ell n)} \cdot \log^{-2} z \tag{48}$$

provided $z \leq P^{\frac{1}{2}}$ and $\ell Q \geq \ell n z \log^4 z$. By the order of magnitude of the parameters, this last condition reduces to

$$P \gg \ell z \log^4 z.$$

However, since we have $\ell \ll L$, this inequality is satisfied as soon as

$$(PQ)^{\frac{1}{2}-\gamma} \gg z^2.$$

Choose $z := (PQ)^{\frac{1}{6}-\frac{\gamma}{3}}$. With this choice of z inserted in (48) and by (47), we obtain the inequality

$$S^0_\gamma(P, Q) \ll_{\gamma_0} \frac{Q}{\log^2(PQ)} \sum_{1 \leq \ell \ll L} \ell \sum_{n \asymp \ell Q/P} \frac{1}{\varphi(\ell n)}. \tag{49}$$

Recall the inequality $\varphi(\ell n) \geq \varphi(\ell)\varphi(n)$ and the bound $\sum_{t \sim T} \varphi^{-1}(t) \ll 1$, which is uniform in $T \geq 1$. Then summing over ℓ and n in (49), we deduce the inequality

$$S^0_\gamma(P, Q) \ll_{\gamma_0} LQ \log^{-2}(PQ) \ll_{\gamma_0} (PQ)^{\gamma+\frac{1}{2}} \log^{-2}(PQ).$$

This bound also holds for $S^1_\gamma(P, Q)$. Inserting this bound in (45) and summing over (P, Q) such that $PQ \leq x$, we conclude the proof of Theorem 2. □

6. Proof of Theorem 3

We now suppose that

$$\tfrac{15}{98} + \gamma_0 \leq \gamma \leq \tfrac{13}{27}$$

since the case where γ takes large values is covered by Theorem 1. Define also

$$K_0 := \frac{17 - 49\gamma_0}{32 - 4\gamma_0} \quad (< \tfrac{17}{32}).$$

To deal with the lower bound of $H_\gamma(x)$, we consider

$$T^0_\gamma(P, Q) := \#\{\, (p, q) : p \sim P, q \sim Q, 0 < \frac{\bar{q}}{p} \leq \Theta^\dagger_0 \,\} \tag{50}$$

with

$$\Theta^\dagger_0 := \theta_0(4PQ),$$

where θ_0 is defined in Lemma 8. We have the inequality

$$H_\gamma(x) \geq H_\gamma^0(x) \geq \sum_P \sum_Q T_\gamma^0(P, Q), \tag{51}$$

where $H_\gamma^0(x)$ is defined in (15) and the sum is over the pairs (P, Q) of the form $(2^k, 2^\ell)$ with

$$P \leq Q^{K_0}, \qquad x/16 \leq PQ \leq x/4, \qquad P \leq Q/2 \quad \text{and} \quad 1 \leq L \leq Q^{\beta_{K_0}}, \tag{52}$$

where L is defined in (46) and β_K is the constant introduced in Lemma 7. If the triple (ℓ, p, q) is such that $1 \leq \ell \ll L$, $p \sim P$ and $q \sim Q$ and satisfies $\ell q - np = 1$ for some integer n, then it contributes to $T_\gamma^0(P, Q)$. This leads to the inequality

$$T_\gamma^0(P, Q) \geq \sum_{p \sim P} \sum_{1 \leq \ell \ll L} \bigl(\pi(2Q; p, \bar{\ell}) - \pi(Q; p, \bar{\ell})\bigr).$$

Thanks to (52), we can apply Lemma 7, giving

$$T_\gamma^0(P, Q) \geq \alpha_{K_0} \sum_{p \sim P} L \cdot \frac{Q}{\varphi(p) \log 2Q} - O\Bigl(\frac{P}{\log^2 2P} \cdot L \cdot \frac{Q}{P \log 2Q}\Bigr),$$

which simplifies into

$$T_\gamma^0(P, Q) \geq \frac{\alpha_{K_0}}{2} \cdot \frac{LQ}{\log 2P \log 2Q} \tag{53}$$

for $x \geq x_0$ and (P, Q) satisfying (52).

In terms of P, the conditions (52) and $L \gg 1$ reduce to

$$P \ll x^{\frac{K_0}{1+K_0}} \quad \text{and} \quad x^{\frac{1}{2}-\gamma} \ll P \ll x^{(\frac{1}{2}+\beta_{K_0}-\gamma)/(1+\beta_{K_0})}. \tag{54}$$

The definition of K_0 implies the inequality

$$\frac{K_0}{1+K_0} - \Bigl(\frac{1}{2} - \gamma\Bigr) \geq \frac{K_0}{1+K_0} - \Bigl(\frac{17}{49} - \gamma_0\Bigr) \gg_{\gamma_0} 1.$$

Combining with the inequality $\beta_{K_0} > 0$, we see that there are $\gg_{\gamma_0} \mathscr{L}$ values of P of the form $P = 2^k$ satisfying (54). Since we also have $x/(16P) \leq Q \leq x/(4P)$, we deduce that there are $\gg_{\gamma_0} \mathscr{L}$ pairs (P, Q) satisfying (52). It remains to insert the lower bound (53) in (51) and to sum over the suitable (P, Q) to deduce

$$H_\gamma^0(x) \gg_{\gamma_0} x^{\frac{1}{2}+\gamma} \mathscr{L}^{-1}.$$

This completes the proof of Theorem 3. □

Remark. Not using Lemma 7 but only Lemma 5, one proves Theorem 3 but under the more restrictive condition $\frac{1}{6} + \gamma_0 \leq \gamma \leq 1 - \gamma_0$.

7. A conjectural formula

One may conjecture that for every $\gamma_0 > 0$, one has

$$H_\gamma(x) \sim C(\gamma)\frac{x^{\frac{1}{2}+\gamma}}{\log x} \tag{55}$$

as $x \to \infty$ uniformly under the condition (13). This conjecture, if true, would be an important extension of Theorem 1. However, (55) is a consequence of the Elliott–Halberstam Conjecture (see [Friedlander and Iwaniec 2010, page 406] for instance).

Conjecture 1. *For any* $\epsilon > 0$ *and any* $A > 0$, *one has*

$$\sum_{r \le x^{1-\epsilon}} \max_{(s,r)=1} \left| \pi(x; r, s) - \frac{\pi(x)}{\varphi(r)} \right| = O_{\epsilon,A}(x\mathscr{L}^{-A}). \tag{56}$$

This conjecture can be interpreted as a considerable improvement of Lemma 5 since it gives the average behavior of the function $\pi(x; r, s)$ for almost all $r \le x^{1-\epsilon}$.

We now give some indications on how to deduce (55) from Conjecture 1. First of all, one applies the formula (56) to evaluate $R_\gamma(P, Q)$ as written in (26). This shows that (29) is true uniformly for $P \le Qx^{-\epsilon}$ (compare with (30)). Summing over all these (P, Q), we see that their contribution to $H_\gamma(x)$ is $\sim (C(\gamma) - O(\epsilon))x^{\frac{1}{2}+\gamma}\mathscr{L}^{-1}$ by a computation analogous to (41) and (42) with uniformity given by (13).

For the remaining (P, Q) (those that satisfy $Qx^{-\epsilon} \le P \le \xi \cdot Q$), we apply the two-dimensional sieve as in Section 5. Then one shows that their contribution to $H_\gamma(x)$ is $O_{\gamma_0}(\epsilon x^{\frac{1}{2}+\gamma}\mathscr{L}^{-1})$. Summing up these two contributions and letting ϵ tend to 0, we get (55).

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