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Local and global canonical height functions for affine space regular automorphisms

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In memory of Professor Masaki Maruyama

Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism defined over a number field K. For each place v of K, we construct the v-adic Green functions $G_{f,v}$ and $G_{f^{-1},v}$ (i.e., the v-adic canonical height functions) for f and f^{-1}. Next we introduce for f the notion of good reduction at v, and using this notion, we show that the sum of v-adic Green functions over all v gives rise to a canonical height function for f that satisfies a Northcott-type finiteness property. Using an earlier result, we recover results on arithmetic properties of f-periodic points and non-f-periodic points. We also obtain an estimate of growth of heights under f and f^{-1}, which was independently obtained by Lee by a different method.

Introduction

Height functions are one of the basic tools in diophantine geometry. On abelian varieties defined over a number field, there exist Néron–Tate canonical height functions that behave well relative to the n-th power map. Tate's elegant construction is via a global method using a relation of an ample divisor relative to the n-th power map. Néron's construction is via a local method and gives deeper properties of the canonical height functions. Both constructions are useful in studying arithmetic properties of abelian varieties.

In [Kawaguchi 2006], we showed the existence of canonical height functions for affine plane polynomial automorphisms of dynamical degree at least 2. Our construction was via a global method using the effectiveness of a certain divisor on a certain rational surface. In this paper, we use a local method to construct a canonical height function for affine space regular automorphisms $f : \mathbb{A}^N \to \mathbb{A}^N$, which coincides with the one in [Kawaguchi 2006] when $N = 2$. We note that arithmetic properties of polynomial automorphisms over number fields have been

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studied, for example, by Silverman [1994], Denis [1995], Marcello [2000; 2003], and the author [Kawaguchi 2006].

We recall the definition of regular polynomial automorphisms. Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a polynomial automorphism of degree $d \geq 2$ defined over a field, and let $\overline{f} : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ denote its birational extension to $\mathbb{P}^N$. We write f^{-1} for the inverse of f, d_- for the degree of f^{-1}, and $\overline{f^{-1}}$ for its birational extension to $\mathbb{P}^N$. Then f is said to be *regular* if the intersection of the set of indeterminacy of $\overline{f}$ and that of $\overline{f^{-1}}$ is empty over an algebraic closure of the field (see Definition 2.1 and Remark 2.2). Over $\mathbb{C}$, dynamical properties of affine space regular polynomial automorphisms f are deeply studied, in which the Green function for f plays a pivotal role; see [Sibony 1999, §2].

In Sections 1 and 2, we construct a Green function (a local canonical height function) for f over an algebraically closed field Ω with nontrivial nonarchimedean absolute value $|\cdot|$. For $x = (x_1, \dots, x_N) \in \Omega^N$, we set $\|x\| = \max_{1 \leq i \leq N}\{|x_i|\}$. Our results are put together as follows.

Theorem A (see Proposition 1.1, Lemma 1.3, and Theorem 2.3). *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism of degree $d \geq 2$ defined over Ω.*

(1) *For all $x \in \mathbb{A}^N(\Omega)$, the limits*

$$\lim_{n\to+\infty} \frac{1}{d^n} \log\max\{\|f^n(x)\|, 1\} \quad \textit{and} \quad \lim_{n\to+\infty} \frac{1}{d_-^n} \log\max\{\|f^{-n}(x)\|, 1\}$$

exist and are nonnegative. We respectively write $G_f(x) \geq 0$ and $G_{f^{-1}}(x) \geq 0$ for the limits, which we call Green functions for f and f^{-1}. They satisfy the functional equations $G_f(f(x)) = dG_f(x)$ and $G_{f^{-1}}(f^{-1}(x)) = d_-G_{f^{-1}}(x)$.

(2) *There are constants $c_f, c_{f^{-1}} \in \mathbb{R}$ such that, on $\mathbb{A}^N(\Omega)$,*

$$G_f(\,\cdot\,) \leq \log\max\{\|\cdot\|, 1\} + c_f,$$
$$G_{f^{-1}}(\,\cdot\,) \leq \log\max\{\|\cdot\|, 1\} + c_{f^{-1}}$$

(3) *There are subsets V^+ and V^- of $\mathbb{A}^N(\Omega)$ with $V^+ \cup V^- = \mathbb{A}^N(\Omega)$ and constants $c^+, c^- \in \mathbb{R}$ such that*

$$G_f(\,\cdot\,) \geq \log\max\{\|\cdot\|, 1\} + c^+ \quad \textit{on } V^+,$$
$$G_{f^-}(\,\cdot\,) \geq \log\max\{\|\cdot\|, 1\} + c^- \quad \textit{on } V^-.$$

Over $\mathbb{C}$, Green functions are constructed using compactness arguments [Sibony 1999, §2]. Here we use more algebraic arguments based on Hilbert's Nullstellensatz. Our construction of $V^\pm$ and $c^\pm$ is rather delicate with a choice of two parameters ε and δ, which behaves well when we work over number fields in Sections 6 and 7. We note that over $\mathbb{C}$, our construction gives a different proof of the existence of Green functions with more explicit estimates (see Section 5). In Section 3, we continue to study some basic properties of regular polynomial automorphisms f over Ω, characterizing the set of the points with unbounded orbit by G_f and showing a filtration property for f.

Now we turn our attention to number fields. Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a polynomial automorphism defined over a number field K. For each place v of K, let K_v denote the completion of K with respect to v and $\overline{K}_v$ an algebraic closure of K_v. Then f induces a regular polynomial automorphism over $\overline{K}_v$, so we have Green functions $G_{f,v}$ and $G_{f^{-1},v}$ and estimates with $c_{f,v}$, $c_{f^{-1},v}$, and $c_v^{\pm}$ as in Theorem A. (Here we use the suffix v to indicate that we work over $\overline{K}_v$. See Section 5 when v is archimedean.)

We want to define the canonical height functions $\hat{h}_f^+$ and $\hat{h}_f^-$ for f as the sum of $G_{f,v}$ and $G_{f^{-1},v}$ over all the places v of K. To this end, we introduce the notion of good reduction at a nonarchimedean place v of K. Let $\overline{R}_v$ denote the ring of integers of $\overline{K}_v$ and $\tilde{k}_v$ the residue field. Recall that the notion of good reduction for an endomorphism φ of $\mathbb{P}^1$ over $\overline{K}_v$ is introduced in [Morton and Silverman 1994], which means that φ extends to a morphism over $\overline{R}_v$ and the induced morphism $\tilde{\varphi}$ over $\tilde{k}_v$ has the same degree as φ. Here we say that a regular polynomial automorphism $f : \mathbb{A}^N \to \mathbb{A}^N$ has *good reduction* at v if f extends to an automorphism over $\overline{R}_v$ and the induced morphism $\tilde{f}$ over $\tilde{k}_v$ is again a regular polynomial automorphism such that the degrees of $\tilde{f}$ and $\tilde{f}^{-1}$ are the same as the degrees of f and f^{-1}, respectively (see Definition 4.1 for the precise definition).

Using the notion of good reduction, we show the existence of canonical height functions. Let $h : \mathbb{A}^N(\overline{K}) \to \mathbb{R}$ denote the usual logarithmic Weil height function.

Theorem B (see Proposition 6.2 and Theorem 6.3). *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism of degree $d \geq 2$ over a number field K. Let $d_- \geq 2$ denote the degree of f^{-1}.*

(1) *Then f has good reduction at v except for finitely many places. Further, if this is the case, we can take the constants $c_{f,v} = c_{f^{-1},v} = c_v^{\pm} = 0$ in Theorem A, so*

$$G_f(\,\cdot\,) = \log\max\{\|\cdot\|, 1\} \quad \text{on } V^+,$$
$$G_{f^{-1}}(\,\cdot\,) = \log\max\{\|\cdot\|, 1\} \quad \text{on } V^-.$$

(2) *For all $x \in \mathbb{A}^N(\overline{K})$, the limits*

$$\hat{h}_f^+(x) := \lim_{n\to+\infty} \frac{1}{d^n} h(f^n(x)) \quad \text{and} \quad \hat{h}_f^-(x) := \lim_{n\to+\infty} \frac{1}{d_-^n} h(f^{-n}(x)) \tag{0-1}$$

exist. Further, we have the decomposition into the sum of local Green functions

$$\hat{h}_f^+(x) = \sum_{v\in M_K} n_v G_{f,v}(x) \quad \text{and} \quad \hat{h}_f^-(x) = \sum_{v\in M_K} n_v G_{f^{-1},v}(x).$$

(3) *We define* $\hat{h}_f : \mathbb{A}^N(\overline{K}) \to \mathbb{R}$ *by* $\hat{h}_f := \hat{h}_f^+ + \hat{h}_f^-$. *Then* $\hat{h}_f$ *satisfies* $\hat{h}_f \gg\ll h$ *and*

$$\frac{1}{d}\hat{h}_f \circ f + \frac{1}{d_-}\hat{h}_f \circ f^{-1} = \Bigl(1 + \frac{1}{dd_-}\Bigr)\hat{h}_f.$$

Further, for $x \in \mathbb{A}^N(\overline{K})$ *we have*

$$\hat{h}_f(x) = 0 \iff \hat{h}_f^+(x) = 0 \iff \hat{h}_f^-(x) = 0 \iff x \textit{ is } f\textit{-periodic.}$$

In [Kawaguchi 2006] we have defined $\hat{h}_f^+(x)$ as $\limsup_{n\to\infty} \frac{1}{d^n} h(f^n(x))$, and similarly for $\hat{h}_f^-$. Theorem B shows that $\bigl\{\frac{1}{d^n} h(f^n(x))\bigr\}_{n=0}^{+\infty}$ and $\bigl\{\frac{1}{d_-^n} h(f^{-n}(x))\bigr\}_{n=0}^{+\infty}$ are in fact convergent sequences, i.e., lim sup can be replaced by lim as in (0-1).

Using estimates on local Green functions over all places, we obtain the following estimate on global height functions for all $N \geq 2$ [Kawaguchi 2006, §4; Silverman 2006, Conjecture 3; 2007, Conjecture 7.18]. This result has been independently proved by Chong Gyu Lee [2013]. His proof uses a global method and is based on the effectiveness of a certain divisor (as was done for $N = 2$ in [Kawaguchi 2006]).

Corollary C (see Theorem 7.1). *Let* $f : \mathbb{A}^N \to \mathbb{A}^N$ *be a regular polynomial automorphism over a number field* K. *With the notation as above, there exists a constant* $c \geq 0$ *such that*

$$\frac{1}{d}h(f(x)) + \frac{1}{d_-}h(f^{-1}(x)) \geq \Bigl(1 + \frac{1}{dd_-}\Bigr)h(x) - c \tag{0-2}$$

for all $x \in \mathbb{A}^N(\overline{K})$. *Further, we have*

$$\liminf_{\substack{x\in\mathbb{A}^N(\overline{K})\\ h(x)\to\infty}} \frac{\frac{1}{d}h(f(x)) + \frac{1}{d_-}h(f^{-1}(x))}{h(x)} = 1 + \frac{1}{dd_-}.$$

Since (0-2) holds, by the argument of [Kawaguchi 2006] we recover the results on f-periodic points and refine the results on non-f-periodic points in [Silverman 1994; Denis 1995; Marcello 2000; 2003]. For $x \in \mathbb{A}^N(\overline{K})$, let $O_f(x) := \{ f^n(x) \mid n \in \mathbb{Z} \}$ denote the f-orbit of x. If $O_f(x)$ is infinite, we have the canonical height $\hat{h}(O_f(x))$ of $O_f(x)$ (see Equation (7-6)).

Corollary D (see Equation (7-6) and Corollary 7.4). *Let* $f : \mathbb{A}^N \to \mathbb{A}^N$ *be a regular polynomial automorphism over a number field* K. *With the notation as above,*

(1) *the set of* f*-periodic points in* $\mathbb{A}^N(\overline{K})$ *is a set of bounded height and*

(2) *for any infinite orbit* $O_f(x)$,

$$\#\{\, y \in O_f(x) \mid h(y) \leq T \,\} = \Bigl(\frac{1}{\log d} + \frac{1}{\log d_-}\Bigr)\log T - \hat{h}(O_f(x)) + O(1)$$

as $T \to +\infty$, *where* $O(1)$ *is independent of* T *and* x *but depends on* f.

1. Nonarchimedean Green functions for polynomial maps

Let Ω be an algebraically closed field with nontrivial nonarchimedean absolute value $|\cdot|$ and R its ring of integers. For a point $x = (x_1, \dots, x_N) \in \mathbb{A}^N(\Omega)$, the norm of x is defined by $\|x\| = \max_{i=1,\dots,N}\{|x_i|\}$. We set $\log^+(a) := \log\max\{a, 1\}$ for $a \in \mathbb{R}_{\geq 0}$ as usual so that $\log^+\|x\| = \log\max\{\|x\|, 1\} = \log\|(x, 1)\|$.

Let $f = (f_1, \dots, f_N) : \mathbb{A}^N \to \mathbb{A}^N$ be a polynomial map of degree $d \geq 2$ defined over Ω, where $f_1(X), \dots, f_N(X)$ are polynomials in $\Omega[X_1, \dots, X_N]$ such that $d = \max_{i=1,\dots,N}\{\deg f_i\}$. We write $F_i(X, T) := T^d f_i(X/T) \in \Omega[X_1, \dots, X_N, T]$ for homogenization of f_i. Let $\overline{f} = (F_1 : \cdots : F_N : T^d) : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ denote the extension of f to $\mathbb{P}^N$. We put $F := (F_1, \dots, F_N, T^d) : \mathbb{A}^{N+1} \to \mathbb{A}^{N+1}$, which is a lift of $\overline{f}$.

For the composition $f^n = f \circ \cdots \circ f$, we write $f^n = (f_1^n, \dots, f_N^n)$. Similarly, for the composition $F^n = F \circ \cdots \circ F$, we write $F^n = (F_1^n, \dots, F_N^n, T^{d^n})$. Let d_n denote the degree of f^n, and let $F_{ni}(X, T) = T^{d_n} f_i^n(X/T) \in \Omega[X_1, \dots, X_N, T]$ be homogenization of f_i^n. Since $F_i^n(X, 1) = f_i^n(X) = F_{ni}(X, 1)$, counting degrees gives $F_i^n(X, T) = T^{d^n - d_n} F_{ni}(X, T)$.

Proposition 1.1. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a polynomial map of degree $d \geq 2$ defined over Ω. Then for all $x \in \mathbb{A}^N(\Omega)$, $\frac{1}{d^n}\log^+\|f^n(x)\|$ converges to a nonnegative real number as $n \to +\infty$.*

Proof. We take an $r \in R$ so that $rF_i \in R[X, T]$ for all $i = 1, \dots, N$. We set

$$a_n := \frac{1}{d^n}\log^+\|f^n(x)\|, \quad b_n := \frac{1}{d^n}\log\|F^n(x, 1)\|, \quad c_n := \frac{1}{d^n}\log\|(rF)^n(x, 1)\|,$$

where $rF = (rF_1, \dots, rF_N, rT^d)$. We claim that

$$a_n = b_n = c_n - \frac{1 - d^{-n}}{d - 1}\log|r|. \tag{1-1}$$

Indeed, the first equality follows from $(f^n(x), 1) = (F^n(x, 1))$. The second equality follows from $(rF)^n = r^{1+d+\cdots+d^{n-1}}F^n = r^{(d^n-1)/(d-1)}F^n$. It follows from $\|(rF)(x, 1)\| \leq \|(x, 1)\|^d$ that

$$\frac{1}{d^n}\log\|(rF)^n(x, 1)\| \leq \frac{1}{d^n}\log\|(rF)^{n-1}(x, 1)\|^d = \frac{1}{d^{n-1}}\log\|(rF)^{n-1}(x, 1)\|.$$

In other words, $\{c_n\}_{n=1}^{+\infty}$ is a nonincreasing sequence. Equation (1-1) implies that $\{c_n\}_{n=1}^{+\infty}$ is bounded from below. Indeed, since a_n is nonnegative and $|r| \leq 1$, we have $c_n \geq a_n + \frac{1}{d-1}\log|r| \geq \frac{1}{d-1}\log|r|$. Thus, $\lim_{n\to+\infty} c_n$ exists. Equation (1-1) then gives the existence of $\lim_{n\to+\infty} a_n$, which is nonnegative from the definition. $\square$

Proposition 1.1 allows the following definition:

Definition 1.2. For a polynomial map $f : \mathbb{A}^N \to \mathbb{A}^N$ defined over Ω, we define the nonnegative function $G_f : \mathbb{A}^N(\Omega) \to \mathbb{R}$ by

$$G_f(x) := \lim_{n\to+\infty} \frac{1}{d^n} \log^+\|f^n(x)\| \quad \text{for } x \in \mathbb{A}^N(\Omega)$$

and call it the *Green function* for f.

Lemma 1.3. *Let C'_f be the maximum of the absolute value of all the coefficients of $f_i(X)$ for $1 \le i \le N$, and we set*

$$c_f = \frac{1}{d-1} \log\max\{C'_f, 1\}.$$

Then

$$G_f(\,\cdot\,) \le \log^+\|\cdot\| + c_f \quad \text{on } \mathbb{A}^N(\Omega).$$

Proof. We take $r \in R$ such that $|r| = 1/\max\{C'_f, 1\}$. Then $rF_i \in R[X, T]$ for all $i = 1, \dots, N$. From the proof of Proposition 1.1, we have

$$G_f(x) \le \lim_{n\to+\infty} c_n - \frac{1}{d-1}\log|r| \le c_0 - \frac{1}{d-1}\log|r| = \log^+\|x\| - \frac{1}{d-1}\log|r|.$$

Hence, we get the assertion. □

Lemma 1.4 below shows that for some polynomial maps f, G_f is not interesting. However, we will see in the next section that G_f enjoys nice properties for regular polynomial automorphisms f (see Definition 2.1 and Theorem 2.3).

To state Lemma 1.4, we recall that a polynomial map f is said to be *algebraically stable* if $d_n = d^n$ for all $n \ge 1$ [Sibony 1999, §1.4].

Lemma 1.4. *If f is not algebraically stable, then $G_f(x) = 0$ for all $x \in \mathbb{A}^N(\Omega)$.*

Proof. We take n_0 such that $d_{n_0} < d^{n_0}$, and we put $g = f^{n_0}$. Proposition 1.1 tells us that $(1/d_{n_0}^m)\log^+\|g^m(x)\|$ converges to a nonnegative number as $m \to +\infty$. Hence,

$$\frac{1}{d^{n_0 m}} \log^+\|f^{n_0 m}(x)\| = \left(\frac{d_{n_0}}{d^{n_0}}\right)^m \frac{1}{d_{n_0}^m} \log^+\|g^m(x)\| \to 0 \quad \text{as } m \to +\infty.$$

From Proposition 1.1, we get $G_f(x) = 0$. □

2. Nonarchimedean Green functions for regular automorphisms

In this section, we consider polynomial automorphisms. Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a polynomial automorphism of degree $d \ge 2$ defined over an algebraically closed field Ω with nontrivial nonarchimedean absolute value.

As before, let $\overline{f} = (F_1(X, T) : \cdots : F_N(X, T) : T^d) : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ denote the extension of f to $\mathbb{P}^N$. We denoted by d_- the degree of the inverse $f^{-1} : \mathbb{A}^N \to \mathbb{A}^N$ of f. The integer $d_- \ge 2$ may be different from d. We denote the extension of f^{-1} to $\mathbb{P}^N$ by $\overline{f^{-1}} = (G_1(X, T) : \cdots : G_N(X, T) : T^{d_-}) : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$.

Let I_+ and I_- denote the set of indeterminacy of $\overline{f}$ and $\overline{f^{-1}}$, respectively:

$$I_+ = \{\, (x:0) \in \mathbb{P}^N(\Omega) \mid F_1(x,0) = \cdots = F_N(x,0) = 0 \,\},$$
$$I_- = \{\, (x:0) \in \mathbb{P}^N(\Omega) \mid G_1(x,0) = \cdots = G_N(x,0) = 0 \,\}.$$

Definition 2.1 [Sibony 1999, §2.2]. A polynomial automorphism $f : \mathbb{A}^N \to \mathbb{A}^N$ is called *regular* if $I_+ \cap I_- = \varnothing$.

Remark 2.2. The definition of regular polynomial automorphisms works over any algebraically closed field.

The purpose of this section is to prove the following theorem, which says that the Green functions for regular automorphisms exhibit nice properties:

Theorem 2.3. *Let Ω be an algebraically closed field with nontrivial nonarchimedean valuation and $f : \mathbb{A}^N \to \mathbb{A}^N$ a regular polynomial automorphism over Ω. Then there are open subsets V^+ and V^- of $\mathbb{A}^N(\Omega)$ with respect to the topology induced from the valuation on Ω and constants $c^+, c^- \in \mathbb{R}$ with the properties*

(i) $G_f(\,\cdot\,) \geq \log^+\|\cdot\| + c^+$ *on* V^+,

(ii) $G_{f^{-1}}(\,\cdot\,) \geq \log^+\|\cdot\| + c^-$ *on* V^-, *and*

(iii) $V^+ \cup V^- = \mathbb{A}^N(\Omega)$.

Remark 2.4. Over $\mathbb{C}$, corresponding results (and much more) were established by Sibony [1999, §2.2]. Here since $\mathbb{A}^N(\Omega)$ is not locally compact in general, we give a different proof that is more algebraic in nature based on Hilbert's Nullstellensatz. We also give V^+, V^-, c^+, and c^- with precise estimates so that they work well when we introduce the notion of good reduction in Section 4.

Before proving Theorem 2.3, we will need several lemmas. We begin by introducing some notation. Since $I_+ \cap I_-$ is empty, $F_1(X,0), \ldots, F_N(X,0)$ and $G_1(X,0), \ldots, G_N(X,0)$ have no solutions in common other than 0. Thus, for each $1 \leq i \leq N$, there are polynomials $P_{ij}(X)$, $Q_{ij}(X) \in \Omega[X]$ for $1 \leq j \leq N$ such that

$$\sum_{j=1}^{N} P_{ij}(X) F_j(X,0) + \sum_{j=1}^{N} Q_{ij}(X) G_j(X,0) = X_i^m \tag{2-1}$$

with some $m \geq 1$. Hence, there is a polynomial $R_i(X,T) \in \Omega[X,T]$ such that

$$\sum_{j=1}^{N} P_{ij}(X) F_j(X,T) + \sum_{j=1}^{N} Q_{ij}(X) G_j(X,T) + T R_i(X,T) = X_i^m. \tag{2-2}$$

Here we may and do assume that m is independent of i. Replacing $P_{ij}(X)$ by its homogeneous part with degree $m-d$, $Q_{ij}(X)$ by its homogeneous part with degree $m-d_-$, and $R_i(X,T)$ by its homogeneous part with degree $m-1$, we may and do assume that the $P_{ij}(X)$, $Q_{ij}(X)$, and $R_i(X,T)$ are homogeneous polynomials with degree $m-d$, $m-d_-$, and $m-1$, respectively.

Let C' be the maximum of the absolute value of all the coefficients of $P_{ij}(X)$, $Q_{ij}(X)$, and $R_i(X,T)$ for $1 \le i \le N$ and $1 \le j \le N$. We set

$$C = \max\{C', 1\}. \tag{2-3}$$

We fix real numbers $\varepsilon > 0$ and $\delta > 0$ as follows. First we choose δ to satisfy $\delta \le \frac{1}{C}$. Then choose ε to satisfy

$$\varepsilon \le \min\left\{\frac{\delta^{1/d}}{C}, \frac{\delta^{1/d_-}}{C}\right\}.$$

This ensures $\varepsilon \le \frac{1}{C}$, so in particular, $\varepsilon \le 1$. To sum up, we have

$$\varepsilon \le \tfrac{1}{C}, \quad \delta \le \tfrac{1}{C}, \quad (\varepsilon C)^d \le \delta, \quad \text{and} \quad (\varepsilon C)^{d_-} \le \delta. \tag{2-4}$$

For example,

$$\varepsilon = \frac{1}{C^{\min\{d,d_-\}}} \quad \text{and} \quad \delta = \frac{1}{C^{\min\{d,d_-\}(\min\{d,d_-\}-1)}} \tag{2-5}$$

satisfy (2-4).

We define $N^+_{\delta,\varepsilon}$ and $V^+_{\delta,\varepsilon}$ by

$$\begin{aligned} N^+_{\delta,\varepsilon} &:= \{\, x \in \mathbb{A}^N(\Omega) \mid 1 < \varepsilon\|x\| \text{ and } \|f(x)\| < \delta\|x\|^d \,\}, \\ V^+_{\delta,\varepsilon} &:= \mathbb{A}^N(\Omega) \setminus N^+_{\delta,\varepsilon} = \{\, x \in \mathbb{A}^N(\Omega) \mid \|x\| \le \tfrac{1}{\varepsilon} \text{ or } \|f(x)\| \ge \delta\|x\|^d \,\}. \end{aligned} \tag{2-6}$$

Intuitively, points in $N^+_{\delta,\varepsilon}$ are near to the hyperplane $\{(x:0) \in \mathbb{P}^N(\Omega)\}$ at infinity (measured by ε) and also near to I_+ in "the direction of x" (measured by δ). We note that both $N^+_{\delta,\varepsilon}$ and $V^+_{\delta,\varepsilon}$ are open and closed with respect to the topology induced from the valuation of Ω.

Remark 2.5. We set

$$\overline{N}^+_{\delta,\varepsilon} = \left\{\, (x:t) \in \mathbb{P}^N(\Omega) \;\middle|\; |t| < \varepsilon\|x\| \text{ and } \|(F(x,t), t^d)\| < \delta\|(x,t)\|^d \,\right\}.$$

Then $N^+_{\delta,\varepsilon} = \overline{N}^+_{\delta,\varepsilon} \cap \mathbb{A}^N(\Omega)$. If $(x:t) \in I_+$, then $t = 0$ and $F(x,t) = 0$. Thus, $|t| = 0$ and $\|(F(x,t), t^d)\| = 0$, so we have

$$I_+ \subseteq \overline{N}^+_{\delta,\varepsilon}.$$

The next lemma says that if a point is not too close to I_+, then f maps it to a point that is also not very close to I_+ and that the measurement of "closeness" is uniform with respect to the point.

Lemma 2.6. *We have* $f(V^+_{\delta,\varepsilon}) \subseteq V^+_{\delta,\varepsilon}$.

Proof. Taking the complement, it suffices to show that

$$f^{-1}(N^{+}_{\delta,\varepsilon}) \subseteq N^{+}_{\delta,\varepsilon}.$$

Suppose $x = (x_1, \dots, x_N) \in N^{+}_{\delta,\varepsilon}$. Without loss of generality, we assume $|x_1| = \|x\|$. We note $f(x) = (F_1(x, 1), \dots, F_N(x, 1))$ and $f^{-1}(x) = (G_1(x, 1), \dots, G_N(x, 1))$. Since $\varepsilon \le 1$, we have $\|x\| > 1$. Then the definition of $N^{+}_{\delta,\varepsilon}$ gives

$$\tfrac{1}{\varepsilon} < \|x\|, \tag{2-7}$$

$$\|f(x)\| < \delta\|x\|^{d}. \tag{2-8}$$

We need to show that $f^{-1}(x) \in N^{+}_{\delta,\varepsilon}$, which is equivalent to

$$1 < \varepsilon\|f^{-1}(x)\|, \tag{2-9}$$

$$\|x\| < \delta\|f^{-1}(x)\|^{d}. \tag{2-10}$$

First we show (2-9). To derive a contradiction, we assume that $\|f^{-1}(x)\| \le \frac{1}{\varepsilon}$. Let $\lambda > 0$ be any small number. We have

$$\begin{aligned}
&\left|\sum_{j=1}^{N} P_{1j}(x)F_j(x, 1) + \sum_{j=1}^{N} Q_{1j}(x)G_j(x, 1) + R_1(x, 1)\right| \\
&\qquad < \max\{C\|x\|^{m-d}\cdot\delta\|x\|^{d}, (C+\lambda)\|x\|^{m-d_-}\tfrac{1}{\varepsilon}, (C+\lambda)\|x\|^{m-1}\} \\
&\qquad \le \max\{C\delta\|x\|^{m}, (C+\lambda)\|x\|^{m-d_-+1}, (C+\lambda)\|x\|^{m-1}\} \qquad \text{(from (2-7))} \\
&\qquad \le \max\{C\delta\|x\|^{m}, (C+\lambda)\|x\|^{m-1}\} \qquad \text{(since } d_- \ge 2).
\end{aligned}$$

Since $\lambda > 0$ is arbitrary, (2-2) and the assumption that $|x_1| = \|x\|$ then gives either $\|x\|^m \le C\|x\|^{m-1}$ or $\|x\|^m < C\delta\|x\|^m$. Equivalently, we have either $\|x\| \le C$ or $1 < C\delta$. However, the former contradicts (2-4) and (2-7) while the latter contradicts (2-4). Hence, we get (2-9).

Next we show (2-10). To derive a contradiction, we assume the contrary, i.e., $\|x\| \ge \delta\|f^{-1}(x)\|^{d}$. Letting $\lambda > 0$ be any small number, we have

$$\begin{aligned}
&\left|\sum_{j=1}^{N} P_{1j}(x)F_j(x, 1) + \sum_{j=1}^{N} Q_{1j}(x)G_j(x, 1) + R_1(x, 1)\right| \\
&\quad < \max\bigl\{C\|x\|^{m-d}\cdot\delta\|x\|^{d}, (C+\lambda)\|x\|^{m-d_-}\cdot(\tfrac{1}{\delta})^{1/d}\|x\|^{1/d}, (C+\lambda)\|x\|^{m-1}\bigr\} \\
&\quad \le \max\{C\delta\|x\|^{m}, (C+\lambda)(\tfrac{1}{\delta})^{1/d}\|x\|^{m-d_-+1/d}, (C+\lambda)\|x\|^{m-1}\} \\
&\quad \le \max\{C\delta\|x\|^{m}, (C+\lambda)(\tfrac{1}{\delta})^{1/d}\|x\|^{m-1}\} \qquad \Bigl(\text{since } d_- - \frac{1}{d} \ge 1\Bigr).
\end{aligned}$$

Since $\lambda > 0$ is arbitrary, (2-2) and the assumption that $|x_1| = \|x\|$ gives this time

$$\text{either} \quad \|x\| \le (\tfrac{1}{\delta})^{1/d}C \quad \text{or} \quad 1 < C\delta.$$

However, the former contradicts (2-4) and (2-7) while the latter contradicts (2-4). Hence, we get (2-10), which completes the proof. □

Lemma 2.7. *Set* $C^+_{\delta,\varepsilon} := \min\{\delta, \varepsilon^d\}$. *Then*

$$\max\{\|f(x)\|, 1\} \geq C^+_{\delta,\varepsilon} \cdot \max\{\|x\|^d, 1\} \quad \textit{for all } x \in V^+_{\delta,\varepsilon}.$$

Proof. For $x \in V^+_{\delta,\varepsilon}$, the definition of $V^+_{\delta,\varepsilon}$ gives

$$\text{either} \quad \|x\| \leq \tfrac{1}{\varepsilon} \quad \text{or} \quad \max\{\|f(x)\|, 1\} \geq \delta \max\{\|x\|^d, 1\}.$$

If the latter holds, then we get the assertion since $\delta \geq C^+_{\delta,\varepsilon}$. If the former holds, then $C^+_{\delta,\varepsilon}\|x\|^d \leq 1$. We get $\max\{\|f(x)\|, 1\} \geq 1 \geq C^+_{\delta,\varepsilon} \cdot \max\{\|x\|^d, 1\}$ noting that $C^+_{\delta,\varepsilon} \leq 1$. □

Lemma 2.8. *Set* $c^+_{\delta,\varepsilon} := \frac{1}{d-1} \log C^+_{\delta,\varepsilon}$. *Then*

$$G_f(x) \geq \log^+\|x\| + c^+_{\delta,\varepsilon} \quad \textit{for all } x \in V^+_{\delta,\varepsilon}.$$

Proof. Suppose $x \in V^+_{\delta,\varepsilon}$. It follows from Lemma 2.6 that $f^n(x) \in V^+_{\delta,\varepsilon}$ for all $n \geq 1$. Then Lemma 2.7 gives

$$\log^+\|f^n(x)\| \geq d \log^+\|f^{n-1}(x)\| + \log C^+_{\delta,\varepsilon}.$$

The usual telescoping argument tells us that

$$\begin{aligned} G_f(x) &= \lim_{n\to+\infty} \frac{1}{d^n} \log^+\|f^n(x)\| \\ &= \log^+\|x\| + \sum_{n=1}^{\infty} \frac{1}{d^n}(\log^+\|f^n(x)\| - d\log^+\|f^{n-1}(x)\|) \\ &\geq \log^+\|x\| + c^+_{\delta,\varepsilon}. \end{aligned}$$ □

With f^{-1} in place of f, we define $N^-_{\delta,\varepsilon}$ and $V^-_{\delta,\varepsilon}$ by

$$\begin{aligned} N^-_{\delta,\varepsilon} &:= \left\{ x \in \mathbb{A}^N(\Omega) \;\middle|\; 1 < \varepsilon\|x\| \text{ and } \max\{\|f^{-1}(x)\|, 1\} < \delta \max\{\|x\|^{d_-}, 1\} \right\}, \\ V^-_{\delta,\varepsilon} &:= \mathbb{A}^N(\Omega) \setminus N^-_{\delta,\varepsilon}. \end{aligned} \tag{2-11}$$

Then setting $c^-_{\delta,\varepsilon} := \frac{1}{d_- - 1} \log \min\{\delta, \varepsilon^{d_-}\}$, we have

$$G_{f^{-1}}(x) \geq \log^+\|x\| + c^-_{\delta,\varepsilon} \quad \text{for all } x \in V^-_{\delta,\varepsilon}. \tag{2-12}$$

The next lemma may be seen as a quantified version of the fact that a point cannot be too close to both I^+ and I^- since $I^+ \cap I^- = \varnothing$.

Lemma 2.9. $V^+_{\delta,\varepsilon} \cup V^-_{\delta,\varepsilon} = \mathbb{A}^N(\Omega)$, *or equivalently*, $N^+_{\delta,\varepsilon} \cap N^-_{\delta,\varepsilon} = \varnothing$.

Proof. Taking the complement, it suffices to show that $N^+_{\delta,\varepsilon} \cap N^-_{\delta,\varepsilon} = \varnothing$. To derive a contradiction, we assume that there is an $x \in N^+_{\delta,\varepsilon} \cap N^-_{\delta,\varepsilon}$. Then we have

$$\|x\| > \tfrac{1}{\varepsilon}, \tag{2-13}$$

$$\|f(x)\| < \delta\|x\|^d, \tag{2-14}$$

$$\|f^{-1}(x)\| < \delta\|x\|^{d_-}. \tag{2-15}$$

Without loss of generality, we assume that $|x_1| = \|x\|$. Let $\lambda > 0$ be any small number. By (2-13)–(2-15), we have

$$\begin{aligned}\biggl|\sum_{j=1}^{N} P_{1j}(x)F_j(x,1) + \sum_{j=1}^{N} Q_{1j}(x)G_j(x,1) + R_1(x,1)\biggr| \\ < \max\{C\|x\|^{m-d}\cdot\delta\|x\|^d,\ C\|x\|^{m-d_-}\cdot\delta\|x\|^{d_-},\ (C+\lambda)\|x\|^{m-1}\} \\ \le \max\{C\delta\|x\|^m,\ (C+\lambda)\|x\|^{m-1}\}.\end{aligned}$$

Since λ is arbitrary, it follows from (2-2) that $\|x\|^m < C\delta\|x\|^m$ or $\|x\|^m \le C\|x\|^{m-1}$. Hence, we get

$$\text{either}\quad 1 < C\delta \quad\text{or}\quad \|x\| \le C.$$

However, the former contradicts (2-4) while the latter contradicts (2-4) and (2-13). Thus, we have $N^+_{\delta,\varepsilon} \cap N^-_{\delta,\varepsilon} = \varnothing$. □

Proof of Theorem 2.3. Let δ and ε be constants satisfying (2-4). Then Theorem 2.3 holds with $V^\pm = V^\pm_{\delta,\varepsilon}$ and $c^\pm = c^\pm_{\delta,\varepsilon}$. Indeed, the condition (i) follows from Lemma 2.8 and the condition (ii) from (2-12) while the condition (iii) follows from Lemma 2.9. □

3. Nonarchimedean Green functions and the set of escaping points

In this section, we continue to study basic properties of regular polynomial automorphisms defined over Ω. We keep the notation and the assumption of Section 2. In particular, $f : \mathbb{A}^N \to \mathbb{A}^N$ denotes a regular polynomial automorphism of degree $d \ge 2$ defined over Ω.

In analogy with the field of complex numbers, we define the set W^+ of escaping points and the set $\mathcal{K}^+$ of nonescaping points by

$$W^+ := \{\, x \in \mathbb{A}^N(\Omega) \mid \|f^n(x)\| \to +\infty\ (n \to +\infty) \,\},$$

$$\mathcal{K}^+ := \{\, x \in \mathbb{A}^N(\Omega) \mid \{f^n(x)\}_{n=0}^{+\infty} \text{ is bounded with respect to } \|\cdot\| \,\}.$$

Then the following theorem holds, which is a nonarchimedean version of the results of [Bedford and Smillie 1991, §2 and §3; Sibony 1999, §2]:

Theorem 3.1. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism over Ω, and let G_f be the Green function for f.*

(1) *The set $\mathcal{K}^+$ is exactly the set of points where G_f vanish*:

$$\mathcal{K}^+ = \{\, x \in \mathbb{A}^N(\Omega) \mid G_f(x) = 0 \,\}.$$

(2) $\mathbb{A}^N(\Omega) = W^+ \amalg \mathcal{K}^+$ (*disjoint union*).

To prove Theorem 3.1, we need the following two lemmas. Recall that δ and ε are fixed constants satisfying (2-4).

Lemma 3.2. *For any $x \in N^+_{\delta,\varepsilon/2}$, one has $\|x\| \leq \frac{1}{2}\|f^{-1}(x)\|$.*

Proof. It follows from $x \in N^+_{\delta,\varepsilon/2}$ that

$$\|x\| > \tfrac{2}{\varepsilon} \quad \text{and} \quad \|f(x)\| < \delta\|x\|^d. \tag{3-1}$$

To derive a contradiction, we assume that $\|x\| > \frac{1}{2}\|f^{-1}(x)\|$. Without loss of generality, we assume that $|x_1| = \|x\|$. Then (we take $\lambda = C$ here)

$$\begin{aligned}\left|\sum_{j=1}^{N} P_{1j}(x)F_j(x,1) + \sum_{j=1}^{N} Q_{1j}(x)G_j(x,1) + R_1(x,1)\right|& \\ &< \max\{C\|x\|^{m-d}\cdot\delta\|x\|^d,\, C\|x\|^{m-d_-}\cdot 2\|x\|,\, 2C\|x\|^{m-1}\} \\ &\leq \max\{C\delta\|x\|^m,\, 2C\|x\|^{m-1}\}.\end{aligned}$$

Using (2-2), we get

$$\text{either} \quad 1 < C\delta \quad \text{or} \quad \|x\| < 2C.$$

However, the former contradicts (2-4). If the latter holds, then Equation (3-1) implies $1 < C\varepsilon$, contradicting (2-4). This completes the proof. □

Lemma 3.3. *For any $x \in \mathbb{A}^N(\Omega)$, one has $f^n(x) \in V^+_{\delta,\varepsilon/2}$ for all sufficiently large n.*

Proof. Note that $\frac{\varepsilon}{2}$ and δ satisfy (2-4) with $\frac{\varepsilon}{2}$ in place of ε. Thus, if $x \in V^+_{\delta,\varepsilon/2}$, then Lemma 2.6 gives $f^n(x) \in V^+_{\delta,\varepsilon/2}$ for all $n \geq 0$.

Suppose now that $x \in N^+_{\delta,\varepsilon/2}$. We take a positive integer n_0 so that $\|x\| \leq 2^{n_0+1}/\varepsilon$. We claim that $f^{n_0}(x) \in V^+_{\delta,\varepsilon/2}$. Indeed, if we assume the contrary, then Lemma 3.2 applied to $x, \ldots, f^{n_0}(x) \in N^+_{\delta,\varepsilon/2}$ gives

$$\tfrac{2}{\varepsilon} < \|f^{n_0}(x)\| \leq \tfrac{1}{2}\|f^{n_0-1}(x)\| \leq \cdots \leq \frac{1}{2^{n_0}}\|x\|,$$

which contradicts our choice of n_0. Thus, $f^n(x) \in V^+_{\delta,\varepsilon/2}$ for all $n \geq n_0$. □

Proof of Theorem 3.1. (1) We get $\mathcal{K}^+ \subseteq \{ x \in \mathbb{A}^N(\Omega) \mid G_f(x) = 0 \}$ from Definition 1.2. To show the other inclusion, we assume that $G_f(x) = 0$. Then $G_f(f^n(x)) = d^n G_f(x) = 0$ for all $n \geq 0$. By Lemma 3.3, we take n_0 such that $f^{n_0}(x) \in V^+_{\delta,\varepsilon/2}$. It follows from Lemmas 2.6 and 2.8 (applied to $\frac{\varepsilon}{2}$ in place of ε) that

$$G_f(f^n(x)) \geq \log^+ \| f^n(x) \| + c^+_{\delta,\varepsilon/2}$$

for all $n \geq n_0$. Combined with $G_f(f^n(x)) = 0$, we see that $\| f^n(x) \| \leq \exp(-c^+_{\delta,\varepsilon/2})$ for all $n \geq n_0$. Thus, $\{ x \in \mathbb{A}^N(\Omega) \mid G_f(x) = 0 \} \subseteq \mathcal{K}^+$.

(2) If $x \notin \mathcal{K}^+$, then $G_f(x) > 0$ by (1). Definition 1.2 then gives $\| f^n(x) \| \to +\infty$ as $n \to +\infty$. □

With f^{-1} in place of f, we put

$$\begin{aligned} W^- &:= \{ x \in \mathbb{A}^N(\Omega) \mid \| f^{-n}(x) \| \to +\infty \ (n \to +\infty) \}, \\ \mathcal{K}^- &:= \{ x \in \mathbb{A}^N(\Omega) \mid \{ f^{-n}(x) \}_{n=0}^{+\infty} \text{ is bounded with respect to } \| \cdot \| \}. \end{aligned}$$

Then we have $\mathbb{A}^N(\Omega) = W^- \amalg \mathcal{K}^-$ as in Theorem 3.1.

In the rest of this section, we give filtrations of $\mathbb{A}^N$ relative to f over nonarchimedean fields as in [Bedford and Smillie 1991, §2.2; Shafikov and Wolf 2003, §3] over $\mathbb{C}$.

We set

$$\begin{aligned} B_\varepsilon &= \{ x \in \mathbb{A}^N(\Omega) \mid \|x\| \leq \tfrac{1}{\varepsilon} \}, \\ U^+_{\delta,\varepsilon} &= \{ x \in \mathbb{A}^N(\Omega) \mid \|x\| > \tfrac{1}{\varepsilon} \text{ and } \| f(x) \| \geq \delta \|x\|^d \}, \end{aligned}$$

where δ and ε are constants satisfying (2-4).

Since $\varepsilon \leq 1$ and $\delta / \varepsilon^d \geq C^d \geq 1$ by (2-4), we have

$$U^+_{\delta,\varepsilon} = \left\{ x \in \mathbb{A}^N(\Omega) \;\middle|\; \|x\| > \tfrac{1}{\varepsilon} \text{ and } \max\{\| f(x) \|, 1\} \geq \delta \max\{\|x\|, 1\}^d \right\}$$

so that $B_\varepsilon \amalg U^+_{\delta,\varepsilon} = V^+_{\delta,\varepsilon}$.

Proposition 3.4. *We assume that ε and δ satisfy*

$$\varepsilon^{d-1} \leq \delta \quad \textit{and} \quad \varepsilon^{d_- - 1} \leq \delta \tag{3-2}$$

in addition to (2-4) (*for example*, *if we take ε and δ as* (2-5), *then they also satisfy* (3-2)). *Then we have the following*:

(1) $\mathbb{A}^N(\Omega) = B_\varepsilon \amalg U^+_{\delta,\varepsilon} \amalg N^+_{\delta,\varepsilon}$ (*disjoint union*),

(2) $f(U^+_{\delta,\varepsilon}) \subseteq U^+_{\delta,\varepsilon}$ *and* $f(B_\varepsilon \amalg U^+_{\delta,\varepsilon}) \subseteq B_\varepsilon \amalg U^+_{\delta,\varepsilon}$, *and*

(3) $f^{-1}(N^+_{\delta,\varepsilon}) \subseteq N^+_{\delta,\varepsilon}$ *and* $f^{-1}(B_\varepsilon \amalg N^+_{\delta,\varepsilon}) \subseteq B_\varepsilon \amalg N^+_{\delta,\varepsilon}$.

Proof. (1) This is obvious from the definition.

(2) Since $B_\varepsilon \amalg U^+_{\delta,\varepsilon} = V^+_{\delta,\varepsilon}$, we have $f(B_\varepsilon \amalg U^+_{\delta,\varepsilon}) \subseteq B_\varepsilon \amalg U^+_{\delta,\varepsilon}$ by Lemma 2.6.

Suppose that $x \in U^+_{\delta,\varepsilon}$. Then

$$\|f(x)\| \geq \delta\|x\|^d > \frac{\delta}{\varepsilon^d} \geq \tfrac{1}{\varepsilon}, \tag{3-3}$$

where we have used (3-2) in the last inequality. Also since $x \in U^+_{\delta,\varepsilon} \subseteq V^+_{\delta,\varepsilon}$, we have $f(x) \in V^+_{\delta,\varepsilon}$ by Lemma 2.6. Since $f(x) \notin B_\varepsilon$ by (3-3), we get $f(x) \in V^+_{\delta,\varepsilon} \setminus B_\varepsilon = U^+_{\delta,\varepsilon}$. Hence, $f(U^+_{\delta,\varepsilon}) \subseteq U^+_{\delta,\varepsilon}$.

(3) We put

$$\begin{aligned} U^-_{\delta,\varepsilon} &= \{\, x \in \mathbb{A}^N(\Omega) \mid \|x\| > \tfrac{1}{\varepsilon} \text{ and } \|f^{-1}(x)\| \geq \delta\|x\|^{d_-} \,\} \\ &= \left\{\, x \in \mathbb{A}^N(\Omega) \;\middle|\; \|x\| > \tfrac{1}{\varepsilon} \text{ and } \max\{\|f^{-1}(x)\|, 1\} \geq \delta \max\{\|x\|, 1\}^{d_-} \,\right\}, \end{aligned} \tag{3-4}$$

where the second equality follows from $\delta/\varepsilon^d_- \geq C^{d_-} \geq 1$ by (2-4). Then as in (2), we have $f^{-1}(U^-_{\delta,\varepsilon}) \subseteq U^-_{\delta,\varepsilon}$. Since $B_\varepsilon \amalg U^-_{\delta,\varepsilon} = V^-_{\delta,\varepsilon}$, Lemma 2.9 implies $N^+_{\delta,\varepsilon} \subseteq U^-_{\delta,\varepsilon}$.

Suppose that $x \in N^+_{\delta,\varepsilon}$. Then

$$f^{-1}(x) \in f^{-1}(N^+_{\delta,\varepsilon}) \subseteq f^{-1}(U^-_{\delta,\varepsilon}) \subseteq U^-_{\delta,\varepsilon}.$$

In particular, $\|f^{-1}(x)\| > \frac{1}{\varepsilon}$ so that $f^{-1}(x) \notin B_\varepsilon$. On the other hand, since $x \notin U^+_{\delta,\varepsilon}$ and $f(U^+_{\delta,\varepsilon}) \subseteq U^+_{\delta,\varepsilon}$, we get $f^{-1}(x) \notin U^+_{\delta,\varepsilon}$, so $f^{-1}(x) \in N^+_{\delta,\varepsilon} = \mathbb{A}^N(\Omega) \setminus (B_\varepsilon \amalg U^+_{\delta,\varepsilon})$. We conclude that $f^{-1}(N^+_{\delta,\varepsilon}) \subseteq N^+_{\delta,\varepsilon}$.

Next we show $f^{-1}(B_\varepsilon \amalg N^+_{\delta,\varepsilon}) \subseteq B_\varepsilon \amalg N^+_{\delta,\varepsilon}$. Since $U^+_{\delta,\varepsilon} = \mathbb{A}^N(\Omega) \setminus (B_\varepsilon \amalg N^+_{\delta,\varepsilon})$, it suffices to show that $f^{-1}(U^+_{\delta,\varepsilon}) \supseteq U^+_{\delta,\varepsilon}$, which is obvious from $f(U^+_{\delta,\varepsilon}) \subseteq U^+_{\delta,\varepsilon}$. □

Proposition 3.5. *We assume that* ε *and* δ *satisfy*

$$\varepsilon^{d-1} < \delta \quad \textit{and} \quad \varepsilon^{d_- -1} < \delta \tag{3-5}$$

in addition to (2-4). *Then we have*

(1) $\bigcup_{n=0}^{+\infty} f^{-n}(U^+_{\delta,\varepsilon}) = W^+$ *and*

(2) $\bigcup_{n=0}^{+\infty} f^{n}(N^+_{\delta,\varepsilon}) = W^-$.

Proof. (1) We set $r := \delta/\varepsilon^{d-1} > 1$. We first show that $U^+_{\delta,\varepsilon} \subseteq W^+$. Indeed, if $x \in U^+_{\delta,\varepsilon}$, then

$$\|f(x)\| \geq \delta\|x\|^d > \frac{\delta}{\varepsilon^{d-1}}\frac{1}{\varepsilon} = r\frac{1}{\varepsilon}.$$

Since $f(U^+_{\delta,\varepsilon}) \subseteq U^+_{\delta,\varepsilon}$, we inductively get $\|f^n(x)\| > r^{(d^n-1)/(d-1)}\frac{1}{\varepsilon}$ for all $n \geq 0$. Hence, $x \in W^+$. This completes the proof of $U^+_{\delta,\varepsilon} \subseteq W^+$. Since $f^{-1}(W^+) = W^+$, we get $f^{-n}(U^+_{\delta,\varepsilon}) \subseteq W^+$ for all $n \geq 0$ so that $\bigcup_{n=0}^{+\infty} f^{-n}(U^+_{\delta,\varepsilon}) \subseteq W^+$.

To show the inclusion $\bigcup_{n=0}^{+\infty} f^{-n}(U^+_{\delta,\varepsilon}) \supseteq W^+$, suppose that $x \notin \bigcup_{n=0}^{+\infty} f^{-n}(U^+_{\delta,\varepsilon})$. We need to show that $x \in \mathcal{K}^+$. Since $f^n(x) \notin U^+_{\delta,\varepsilon}$, we have either $f^n(x) \in B_\varepsilon$ or $f^n(x) \in N^+_{\delta,\varepsilon}$.

Case 1. Suppose there is an $n_0 \geq 0$ such that $f^{n_0}(x) \in B_\varepsilon$. Then $f^{n_0+1}(x) \in B_\varepsilon \amalg U^+_{\delta,\varepsilon}$ by Proposition 3.4(2). Since $f^{n_0+1}(x) \notin U^+_{\delta,\varepsilon}$, we obtain $f^{n_0+1}(x) \in B_\varepsilon$. Inductively, $f^n(x) \in B_\varepsilon$ for all $n \geq n_0$, so we conclude that $x \in \mathcal{K}^+$.

Case 2. Suppose that $f^n(x) \in N^+_{\delta,\varepsilon}$ for all $n \geq 0$. By Lemma 3.3, there is an $n_0 \geq 0$ such that $f^n(x) \in V^+_{\delta,\varepsilon/2}$ for all $n \geq n_0$. Then for all $n \geq n_0$, we have

$$f^n(x) \in V^+_{\delta,\varepsilon/2} \cap N^+_{\delta,\varepsilon} \subseteq \{\, y \in \mathbb{A}^N(\Omega) \mid \tfrac{1}{\varepsilon} < \|y\| \leq \tfrac{2}{\varepsilon} \,\}.$$

Hence, $x \in \mathcal{K}^+$.

In both cases, we have $x \in \mathcal{K}^+$, so we get $\bigcup_{n=0}^{+\infty} f^{-n}(U^+_{\delta,\varepsilon}) \supseteq W^+$.

(2) Let $U^-_{\delta,\varepsilon}$ be the set defined by (3-4). Then $\bigcup_{n=0}^{+\infty} f^n(U^-_{\delta,\varepsilon}) = W^-$ by the argument in (1), and so $\bigcup_{n=0}^{+\infty} f^n(N^+_{\delta,\varepsilon}) \subseteq W^-$. To show the other inclusion, suppose that $x \notin \bigcup_{n=0}^{+\infty} f^n(N^+_{\delta,\varepsilon})$. Then we have either $f^{-n}(x) \in B_\varepsilon$ or $f^{-n}(x) \in U^+_{\delta,\varepsilon}$.

Case 1. If there is an $n_0 \geq 0$ such that $f^{-n_0}(x) \in B_\varepsilon$, then the argument of Case 1 of (1) together with Proposition 3.4(3) gives $f^{-n}(x) \in B_\varepsilon$ for all $n \geq n_0$.

Case 2. Suppose that $f^{-n}(x) \in U^+_{\delta,\varepsilon}$ for all $n \geq 0$. Then the argument of Case 2 of (1) together with Lemma 3.3 with f^{-1} in place of f gives $\frac{1}{\varepsilon} < \|x\| < \frac{2}{\varepsilon}$ for sufficiently large n.

In both cases, we get $x \in \mathcal{K}^-$. Hence, $\bigcup_{n=0}^{+\infty} f^n(N^+_{\delta,\varepsilon}) \supseteq W^-$. □

Remark 3.6. If we take

$$0 < \varepsilon < \frac{1}{C^{\min\{d,d_-\}}} \quad \text{and} \quad \delta = \frac{1}{C^{\min\{d,d_-\}(\min\{d,d_-\}-1)}},$$

then they satisfy both (2-4) and (3-5).

4. Regular automorphisms having good reduction

Morton and Silverman [1994] introduced the notion of having good reduction for endomorphisms of $\mathbb{P}^1$ over Ω, which has been useful in studying endomorphisms of $\mathbb{P}^1$ over a global field. For endomorphisms of $\mathbb{P}^N$ having good reduction, see for example [Kawaguchi and Silverman 2007, Remark 12; 2009]. In this section, we introduce the notion of having good reduction for regular polynomial automorphisms of $\mathbb{A}^N$ over Ω. This notion will be useful in studying regular polynomial automorphisms over a global field in Sections 6 and 7.

As in Section 1, R denotes the ring of integers of Ω. Let M be the maximal ideal of R and $\tilde{k} := R/M$ the residue field. Note that $\tilde{k}$ is algebraically closed since Ω is algebraically closed.

Definition 4.1 (Good reduction). Let $f = (f_1, \dots, f_N) : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism over an algebraically closed field Ω with nontrivial nonarchimedean absolute value, and let $f^{-1} = (g_1, \dots, g_N) : \mathbb{A}^N \to \mathbb{A}^N$ denote its inverse. We write d and d_- for the degrees of f and f^{-1}, respectively. We say that f *has good reduction* if the following three conditions are satisfied:

(i) We have that f extends to the polynomial automorphism $f : \mathbb{A}^N_R \to \mathbb{A}^N_R$ over R, so both $f_1(X), \dots, f_N(X)$ and $g_1(X), \dots, g_N(X)$ are in $R[X_1, \dots, X_N]$.

(ii) Let $\tilde{f} = (\widetilde{f_1}, \dots, \widetilde{f_N}) : \mathbb{A}^N_{\tilde{k}} \to \mathbb{A}^N_{\tilde{k}}$ and $\widetilde{f^{-1}} = (\widetilde{g_1}, \dots, \widetilde{g_N}) : \mathbb{A}^N_{\tilde{k}} \to \mathbb{A}^N_{\tilde{k}}$ be the induced polynomial automorphisms over $\tilde{k}$. Then the degrees of $\tilde{f}$ and $\widetilde{f^{-1}}$ are equal to d and d_-, respectively.

(iii) We have that $\tilde{f}$ is regular (see Remark 2.2).

We give some equivalent conditions for regular polynomial automorphisms f to have good reduction. As in Section 1, let $F_i(X, T)$ and $G_j(X, T)$ be the homogenization of $f_i(X)$ and $g_j(X)$. If $F_i(X, T)$ and $G_j(X, T)$ are defined over R, $\widetilde{F_i}(X, T)$ and $\widetilde{G_j}(X, T)$ denote their reductions to $\tilde{k}$. Let $\rho : R \to \tilde{k}$ be the natural map.

Proposition 4.2. *Let f be a regular polynomial automorphism of $\mathbb{A}^N$ over Ω. Assume that f satisfies the conditions (i) and (ii) of Definition 4.1. Then the following are equivalent*:

(1) *We have that f has good reduction*, i.e., *f also satisfies Definition 4.1(iii).*

(2) *As ideals in $R[X_1, \dots, X_N, T]$, one has*

$$(X_1, \dots, X_N, T)^k \subseteq (F_1(X, T), \dots, F_N(X, T), G_1(X, T), \dots, G_N(X, T), T)$$

for some integer $k \geq 1$.

(3) *As ideals in $R[X_1, \dots, X_N]$, one has*

$$(X_1, \dots, X_N)^\ell \subseteq (F_1(X, 0), \dots, F_N(X, 0), G_1(X, 0), \dots, G_N(X, 0))$$

for some integer $\ell \geq 1$.

Proof. (1) $\Longrightarrow$ (3). It suffices to show that

$$(X_1, \dots, X_N)^\ell \subseteq (F_1(X, 0)^{d_-}, \dots, F_N(X, 0)^{d_-}, G_1(X, 0)^d, \dots, G_N(X, 0)^d) \tag{4-1}$$

for some $\ell \geq 1$. We set

$$I = \left\{ r \in R \;\middle|\; \begin{array}{c} \text{there is an } \ell \geq 1 \text{ such that } r(X_1, \dots, X_N)^\ell \subseteq \\ (F_1(X, 0)^{d_-}, \dots, F_N(X, 0)^{d_-}, G_1(X, 0)^d, \dots, G_N(X, 0)^d) \end{array} \right\}.$$

Since f is regular, I is a nonzero ideal of R.

We claim that $\rho(I) \neq 0$. Indeed, suppose that $\rho(I) = 0$. Then elimination theory tells us that there is a point $x = (x_1 : \cdots : x_n) \in \mathbb{P}^{N-1}(\tilde{k})$ such that $\widetilde{F}_i(x, 0) = 0$ and $\widetilde{G}_j(x, 0) = 0$ for all i and j; see [Kawaguchi and Silverman 2007, Theorem 6]. Since f satisfies condition (ii), $\widetilde{F}_i(X, T)$ and $\widetilde{G}_j(X, T)$ are the homogenizations of $\widetilde{f}_i$ and $\widetilde{g}_j$, respectively. Then the existence of such an $x \in \mathbb{P}^{N-1}(\tilde{k})$ contradicts condition (iii), which yields the claim.

Since $\rho(I) \neq 0$, there is an $r \in I$ such that $r \in R^\times = R \setminus M$. Then $I = R$, and we obtain Equation (4-1).

(3) $\Longrightarrow$ (1). The assumption of (3) gives, as ideals in $\tilde{k}[X]$,

$$(X_1, \ldots, X_N)^\ell \subseteq (\rho(F_1(X,0)), \ldots, \rho(F_N(X,0)), \rho(G_1(X,0)), \ldots, \rho(G_N(X,0))).$$

Since $\rho(F_i(X, 0)) = \widetilde{F}_i(X, 0)$ and $\rho(G_j(X, 0)) = \widetilde{G}_j(X, 0)$, we obtain that $\tilde{f}$ is regular.

(2) $\Longrightarrow$ (3). We have only to put $T = 0$.

(3) $\Longrightarrow$ (2). It suffices to show that for any $\alpha = 1, \ldots, N$, there are an integer $k \geq 1$ and polynomials $P_i(X, T)$, $Q_j(X, T)$, and $R(X, T)$ defined over R such that

$$X_\alpha^k = \sum_{i=1}^{N} P_i(X, T) F(X, T) + \sum_{j=1}^{N} Q_j(X, T) G_j(X, T) + T R(X, T). \tag{4-2}$$

By the assumption of (iii), there are an integer $\ell \geq 1$ and polynomials $P_i(X)$ and $Q_j(X)$ defined over R such that

$$X_\alpha^\ell = \sum_{i=1}^{N} P_i(X) F(X, 0) + \sum_{j=1}^{N} Q_j(X) G_j(X, 0).$$

We set $k := \ell$, $P_i(X, T) := P_i(X)$, and $Q_j(X, T) := Q_j(X)$. Then

$$X_\alpha^k - \sum_{i=1}^{N} P_i(X, T) F(X, T) - \sum_{j=1}^{N} Q_j(X, T) G_j(X, T)$$

is a polynomial in $R[X, T]$ that is divisible by T. Hence, there is a polynomial $R(X, T)$ in $R[X, T]$ satisfying Equation (4-2). □

Suppose now that a regular polynomial automorphism f has good reduction. By Proposition 4.2, for each $1 \leq i \leq N$ there are polynomials $P_{ij}(X)$ and $Q_{ij}(X)$ in $R[X]$ that satisfy (2-1). Then the polynomial $R_i(X, T)$ in (2-2) is also defined over R, and the constant C in (2-3) is equal to 1. This means that $\varepsilon = 1$ and $\delta = 1$ satisfy (2-4). It follows that when f has good reduction, G_f and $\log^+\|\cdot\|$ are related simply.

Proposition 4.3. *Suppose that f has good reduction.*

(1) $G_f(\cdot) \le \log^+\|\cdot\|$ *and* $G_{f^{-1}}(\cdot) \le \log^+\|\cdot\|$ *on* $\mathbb{A}^N(\Omega)$.

(2) $\log^+\|\cdot\| = G_f(\cdot)$ *on* $V_{1,1}^+$ *and* $\log^+\|\cdot\| = G_{f^{-1}}(\cdot)$ *on* $V_{1,1}^-$. *Moreover,* $\mathbb{A}^N(\Omega) = V_{1,1}^+ \cup V_{1,1}^-$.

Proof. (1) Since the $f_i(X)$ are defined over R, in the proof of Lemma 1.3 we may take $r = 1$ so that $c_f = 0$. Thus, $G_f(\cdot) \le \log^+\|\cdot\|$ on $\mathbb{A}^N(\Omega)$. The estimate for $G_{f^{-1}}$ is similar.

(2) Since $\varepsilon = 1$ and $\delta = 1$ satisfy (2-4), Lemma 2.9 gives $\mathbb{A}^N(\Omega) = V_{1,1}^+ \cup V_{1,1}^-$. The constant $c_{1,1}^+$ in Lemma 2.8 is equal to 0, and thus, combined with (1), we have $\log^+\|x\| = G_f(x)$ for all $x \in V_{1,1}^+$. The estimate for $G_{f^{-1}}$ is similar. □

5. Green functions for regular automorphisms over $\mathbb{C}$

In this section, we remark that the proof of Theorem 2.3 gives a different proof (more explicit and without compactness arguments) of the corresponding estimates of Green functions over $\mathbb{C}$.

We write the usual absolute value of $\mathbb{C}$ for $|\cdot|_\infty$, and we set $\|x\|_\infty := \max_i\{|x_i|_\infty\}$ for $x = (x_1, \dots, x_N) \in \mathbb{A}^N(\mathbb{C})$.

Let $f = (f_1, \dots, f_N) : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism of degree $d \ge 2$ defined over $\mathbb{C}$. Then the Green function for f is defined by [Sibony 1999, §2]

$$G_f(x) := \lim_{n\to+\infty} \frac{1}{d^n} \log^+\|f^n(x)\| \quad \text{for } x \in \mathbb{A}^N(\mathbb{C}). \tag{5-1}$$

Let $\|f\|_\infty$ be the maximum of the absolute values of all the coefficients of $f_i(X)$ for $1 \le i \le N$, and set $c_{f,\infty} = \frac{1}{d-1}\log\max\bigl\{\binom{N+d-1}{d}\|f\|_\infty, 1\bigr\}$. Note that $\binom{N+d-1}{d}$ is the number of monomials of degree d in the ring of homogeneous polynomials in N variables. Since

$$\log^+\|f(x)\| \le d\log^+\|x\| + \log\max\left\{\binom{N+d-1}{d}\|f\|_\infty, 1\right\}, \tag{5-2}$$

we get

$$G_f(x) \le \log^+\|x\| + c_{f,\infty} \quad \text{for any } x \in \mathbb{A}^N(\mathbb{C}). \tag{5-3}$$

Let $P_{ij}(X), Q_{ij}(X) \in \mathbb{C}[X]$ and $R(X,T) \in \mathbb{C}[X,T]$ be polynomials satisfying (2-2). As before, we may and do assume that the $P_{ij}(X)$, $Q_{ij}(X)$, and $R_i(X,T)$ are homogeneous polynomials with degree $m-d$, $m-d_-$, and $m-1$, respectively. We write $\|P\|_\infty$ for the maximum of the absolute values of all the coefficients of $P_{ij}(X)$ for $1 \le i \le N$ and $1 \le j \le N$, and we write $\|Q\|_\infty$ and $\|R\|_\infty$ similarly. We set

$$C'_\infty = \max\left\{\binom{N+m-d-1}{m-d}\|P\|_\infty, \binom{N+m-d_--1}{m-d_-}\|Q\|_\infty, \binom{N+m}{m-1}\|R\|_\infty, 1\right\}.$$

We note the above formula for C'_∞ is not as explicit as in the nonarchimedean case since it involves the coefficients of P, Q, and R and not only those of F and G. However, $\|P\|_\infty$, $\|Q\|_\infty$, and $\|R\|_\infty$ can be expressed in terms of F and G via an effective version of Hilbert's Nullstellensatz (see [Masser and Wüstholz 1983, Chapter 4] for example).

We put

$$C_\infty = (2N+1)C'_\infty.$$

Fix real numbers $\varepsilon > 0$ and $\delta > 0$ satisfying (2-4) with C_∞ in place of C. We define $N^{\pm}_{\delta,\varepsilon}$ and $V^{\pm}_{\delta,\varepsilon}$ by (2-6) and (2-11) with $\mathbb{C}$ in place of Ω. Then exactly as in Theorem 2.3, we have the following:

Theorem 5.1. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism over $\mathbb{C}$.*

(i) $G_f(\,\cdot\,) \geq \log^+\|\cdot\| + c^+_{\delta,\varepsilon}$ *on* $V^+_{\delta,\varepsilon}$.

(ii) $G_{f^{-1}}(\,\cdot\,) \geq \log^+\|\cdot\| + c^-_{\delta,\varepsilon}$ *on* $V^-_{\delta,\varepsilon}$.

(iii) $V^+_{\delta,\varepsilon} \cup V^-_{\delta,\varepsilon} = \mathbb{A}^N(\Omega)$.

6. Global theory of regular automorphisms

In this section, we turn our attention to regular automorphisms over a number field.

Let K be a number field and O_K its ring of integers. We fix an embedding $K \subset \overline{K}$ into an algebraic closure. Let M_K be the set of absolute values on K. We extend the absolute values on K to those on $\overline{K}$.

Let L be a finite extension field of K. For $x \in \mathbb{A}^N(L)$, we define

$$h(x) = \sum_{v \in M_K} n_v \log^+\|x\|_v, \tag{6-1}$$

where $n_v = [L_v : K_v]/[L : K]$. This gives rise to the logarithmic Weil *height function*

$$h : \mathbb{A}^N(\overline{K}) \to \mathbb{R}.$$

For more details on height functions, we refer the reader to [Bombieri and Gubler 2006; Hindry and Silverman 2000; Lang 1983].

Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism over $\overline{K}$ (see Remark 2.2). If the coefficients of f are all defined over K, then we say that f is a regular polynomial automorphism over K.

Lemma 6.1. *If $f : \mathbb{A}^N \to \mathbb{A}^N$ is a polynomial automorphism over K, then the coefficients of f^{-1} are also all defined over K.*

Proof. We take a finite Galois extension field L of K such that the coefficients of f^{-1} are elements of L. For every $\sigma \in \mathrm{Gal}(L/K)$, the uniqueness of the inverse gives $(f^{-1})^\sigma = f^{-1}$. Thus, the coefficients of f^{-1} are in fact elements of K. $\square$

In [Kawaguchi 2006], we constructed (global) canonical height functions $\hat{h}_f^+$ and $\hat{h}_f^-$ for polynomial automorphisms f over K under the assumption that there exists a constant $c \geq 0$ such that

$$\frac{1}{d}h(f(x)) + \frac{1}{d_-}h(f^{-1}(x)) \geq \Bigl(1 + \frac{1}{dd_-}\Bigr)h(x) - c \tag{6-2}$$

for all $x \in \mathbb{A}^N(\overline{K})$, where d and d_- denote the degrees of f and f^{-1}. (We showed in op. cit. that (6-2) holds for regular polynomial automorphisms in dimension $N = 2$ by a global method, i.e., a method using the effectiveness of a certain divisor on a certain rational surface.)

In the following, using properties of local Green functions studied in the previous sections, we will first construct in Theorem 6.3 (global) canonical height functions h_f^+ and h_f^- for regular polynomial automorphisms. Indeed, we will construct h_f^+ and h_f^- as appropriate sums of local Green functions. Then we show local versions of (6-2) for all places v, and summing them up, we will obtain (6-2) for regular polynomial automorphisms in any dimension $N \geq 2$ in Theorem 7.1.

For a finite subset S of M_K that contains all the archimedean absolute values of K, we let $O_{K,S}$ denote the ring of S-integers:

$$O_{K,S} = \{\, x \in K \mid \|x\|_v \leq 1 \text{ for all } v \notin S \,\}.$$

Proposition 6.2. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism of degree $d \geq 2$ over a number field K. Then there exists a finite subset S of M_K that contains all the archimedean absolute values of K with the following property: for all $v \notin S$, f induces a regular polynomial automorphism over $\overline{K}_v$ that has good reduction.*

Proof. We write $f = (f_1, \dots, f_N)$ and let $F_i(X, T) \in K[X, T]$ be the homogenization of f_i. Let d_- denote the degree of $f^{-1} = (g_1, \dots, g_N)$, and in virtue of Lemma 6.1, let $G_j(X, T) \in K[X, T]$ be the homogenization of g_j. Then there are an integer m and homogeneous polynomials $P_{ij}(X) \in K[X]$ of degree $m - d$, $Q_{ij}(X) \in K[X]$ of degree $m - d_-$, and $R_i(X, T) \in K[X, T]$ of degree $m - 1$ such that (2-2) holds as polynomials in $K[X, T]$.

We take a finite subset S of M_K that contains all the archimedean absolute values of K with the following properties:

(i) The coefficients of $F_i(X, T)$, $G_j(X, T)$, $P_{ij}(X)$, $Q_{ij}(X)$, and $R_i(X, T)$ are all in $O_{K,S}$.

(ii) For $v \notin S$, we let $\rho_v : O_{K,S} \to \tilde{k}_v$ denote the natural map, where $\tilde{k}_v$ is the residue field of $(O_K)_v$. Then $\deg(f) = \deg(\rho_v(f))$ and $\deg(f^{-1}) = \deg(\rho_v(f^{-1}))$.

Then for any $v \notin S$, $f \times_K \overline{K}_v : \mathbb{A}^N_{\overline{K}_v} \to \mathbb{A}^N_{\overline{K}_v}$ satisfies the properties (i) and (ii) of Definition 4.1 and (3) of Proposition 4.2. Hence, $f \times_K \overline{K}_v$ has good reduction. $\square$

Theorem 6.3. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism of degree $d \geq 2$ over a number field K. Let $d_- \geq 2$ denote the degree of f^{-1}.*

(1) *For all $x \in \mathbb{A}^N(\overline{K})$, the limits*

$$\lim_{n\to+\infty} \frac{1}{d^n} h(f^n(x)) \quad \textit{and} \quad \lim_{n\to+\infty} \frac{1}{d_-^n} h(f^{-n}(x))$$

exist. We write $\hat{h}_f^+(x)$ and $\hat{h}_f^-(x)$ for the limits, respectively.

(2) (*Global-to-local decomposition*) *For each place $v \in M_K$, let $G_{f,v}$ and $G_{f^{-1},v}$ be the Green functions for f and f^{-1} at v, respectively. Then for all $x \in \mathbb{A}^N(\overline{K})$,*

$$\hat{h}_f^+(x) = \sum_{v\in M_K} n_v G_{f,v}(x) \quad \textit{and} \quad \hat{h}_f^-(x) = \sum_{v\in M_K} n_v G_{f^{-1},v}(x).$$

(3) *We define $\hat{h}_f : \mathbb{A}^N(\overline{K}) \to \mathbb{R}$ by*

$$\hat{h}_f := \hat{h}_f^+ + \hat{h}_f^-.$$

Then $\hat{h}_f$ satisfies the following two conditions:

(3i) $\frac{1}{d}\hat{h}_f \circ f + \frac{1}{d_-}\hat{h}_f \circ f^{-1} = (1 + \frac{1}{dd_-})\hat{h}_f$ *on $\mathbb{A}^N(\overline{K})$ and*

(3ii) $h + O(1) \leq \hat{h}_f \leq 2h + O(1)$ *on $\mathbb{A}^N(\overline{K})$.*

(4) *The function $\hat{h}_f$ has the following uniqueness property*: *if $h' : \mathbb{A}^N(\overline{K}) \to \mathbb{R}$ is a function satisfying the condition (3i) such that $h' = \hat{h}_f + O(1)$, then $h' = \hat{h}_f$.*

(5) *The functions $\hat{h}_f^+$, $\hat{h}_f^-$, and $\hat{h}_f$ are nonnegative. Further, for $x \in \mathbb{A}^N(\overline{K})$ we have*

$$\hat{h}_f(x) = 0 \iff \hat{h}_f^+(x) = 0 \iff \hat{h}_f^-(x) = 0 \iff x \textit{ is } f\textit{-periodic.}$$

Proof. For each $v \in M_K$, we have estimates of Green functions for f at v as in Lemmas 1.3 and 2.8. We use the suffix v when we work over the absolute value $v \in M_K$. For example, the Green function for f at v is denoted $G_{f,v}$ and constants c_f and $c^{\pm}_{\varepsilon,\delta}$ in Lemmas 1.3 and 2.8 and (2-12) are denoted $c_{f,v}$ and $c^{\pm}_{\varepsilon,\delta,v}$, respectively.

Let S be a finite subset of M_K as in Proposition 6.2.

(1)(2) We fix $x \in \mathbb{A}^N(\overline{K})$. We will show the existence of $h_f^+(x)$ and the decomposition $h_f^+(x) = \sum_{v\in M_K} n_v G_{f,v}(x)$. The existence and decomposition for $h_f^-(x)$ are shown similarly.

For $v \in M_K$ and $n \geq 0$, we set

$$G^+_{v,n}(x) := \frac{1}{d^n} \log^+ \|f^n(x)\|_v.$$

Then the following are true:

- We have $0 \leq G^{+}_{v,n}(x) \leq \log^{+}\|x\|_v + c_{f,v}$ for all $v \in M_K$ and $n \geq 0$ from Proposition 1.1, Lemma 1.3, and Equations (5-2) and (5-3). Indeed, if v is nonarchimedean, then with r in the proof of Proposition 1.1, we have only to set $c_{f,v} = -\frac{1}{d-1}\log|r|$. If v is archimedean, then by (5-2) we have only to set $c_{f,v} = \frac{1}{d-1}\log\max\left\{\binom{N+d-1}{d}\|f\|_\infty, 1\right\}$.
- We have $\lim_{n\to+\infty} G^{+}_{v,n}(x) = G_{f,v}(x)$ from Definition 1.2 and Equation (5-1).
- We have $\frac{1}{d^n}h(f^n(x)) = \sum_{v\in M_K} n_v G^{+}_{v,n}(x)$ from Equation (6-1).
- We may take $c_{f,v} = 0$ for any $v \notin S$ from Propositions 4.3 and 6.2.
- We have $\sum_{v\in M_K} n_v(\log^{+}\|x\|_v + c_{f,v}) = h(x) + \sum_{v\in S} n_v c_{f,v} < +\infty$.

Lebesgue's dominated convergence theorem then implies that $\sum_{v\in M_K} n_v G^{+}_{v,n}(x)$ converges as $n \to +\infty$ and that

$$\begin{aligned}\lim_{n\to+\infty}\frac{1}{d^n}h(f^n(x)) &= \lim_{n\to+\infty}\sum_{v\in M_K} n_v G^{+}_{v,n}(x)\\ &= \sum_{v\in M_K}\lim_{n\to+\infty} n_v G^{+}_{v,n}(x) = \sum_{v\in M_K} n_v G_{f,v}(x).\end{aligned}$$

This completes the proof of (1) and (2)

(3)(4)(5) First we have

$$\begin{aligned}\hat{h}_f(x) &= \sum_{v\in M_K} n_v G_{f,v}(x) + \sum_{v\in M_K} n_v G_{f^{-1},v}(x) \\ &\leq \sum_{v\in M_K} n_v(2\log^{+}\|x\|_v + c_{f,v} + c_{f^{-1},v}) = 2\hat{h}(x) + \sum_{v\in S} n_v(c_{f,v} + c_{f^{-1},v}).\end{aligned} \tag{6-3}$$

On the other hand, we have

- $\min\{c^{+}_{\varepsilon,\delta,v}, c^{-}_{\varepsilon,\delta,v}\} + \log^{+}\|x\| \leq G_{f,v}(x) + G_{f^{-1},v}(x)$ from Lemma 2.8, (2-12), and Theorem 5.1 and
- for any $v \notin S$, we may take $\varepsilon = 1$ and $\delta = 1$ and $\min\{c^{+}_{1,1,v}, c^{-}_{1,1,v}\} = 0$ from Propositions 4.3 and 6.2.

Then

$$\begin{aligned}\hat{h}_f(x) &= \sum_{v\in M_K} n_v G_{f,v}(x) + \sum_{v\in M_K} n_v G_{f^{-1},v}(x) \\ &\geq \sum_{v\in M_K} n_v(\log^{+}\|x\|_v + \min\{c^{+}_{\varepsilon,\delta,v}, c^{-}_{\varepsilon,\delta,v}\}) = \hat{h}_{nv}(x) + \sum_{v\in S} n_v\min\{c^{+}_{\varepsilon,\delta,v}, c^{-}_{\varepsilon,\delta,v}\}.\end{aligned} \tag{6-4}$$

Equations (6-3) and (6-4) give (3ii). For the rest of the proof, see [Kawaguchi 2006, Theorem 4.2(2–4)]. □

Remark 6.4. Theorem 6.3(1) shows that $\left\{\frac{1}{d^n}h(f^n(x))\right\}_{n=0}^{+\infty}$ and $\left\{\frac{1}{d_-^n}h(f^{-n}(x))\right\}_{n=0}^{+\infty}$ are convergent sequences, which gives an improvement of [Kawaguchi 2006] since we replace lim sup by lim in the definition of $\hat{h}_f^{\pm}$.

We now introduce another function

$$\tilde{h}_f(x) := \sum_{v \in M_K} n_v \max\{G_{f,v}(x), G_{f^{-1},v}(x)\} \tag{6-5}$$

for $x \in \mathbb{A}^N(\overline{K})$. The next proposition shows that $\tilde{h}_f$ also behaves well relative to f.

Proposition 6.5. (1) *On* $\mathbb{A}^N(\overline{K})$, $\tilde{h}_f = h + O(1)$.

(2) *For* $x \in \mathbb{A}^N(\overline{K})$, *we have* $\tilde{h}_f(x) = 0$ *if and only if* $\hat{h}_f(x) = 0$.

Proof. (1) We use the notation of the proof of Theorem 6.3. By Lemmas 1.3 and 2.8, Equations (2-12) and (5-3), and Theorem 5.1, we have

$$\log^+\|x\|_v + \min\{c^+_{\varepsilon,\delta,v}, c^-_{\varepsilon,\delta,v}\} \\ \le \max\{G_{f,v}(x), G_{f^{-1},v}(x)\} \le \log^+\|x\|_v + \max\{c_{f,v}, c_{f^{-1},v}\}.$$

Summing up over all places v, we get

$$h(x) + \sum_{v \in M_K} n_v \min\{c^+_{\varepsilon,\delta,v}, c^-_{\varepsilon,\delta,v}\} \le \tilde{h}_f(x) \le h(x) + \sum_{v \in M_K} n_v \max\{c_{f,v}, c_{f^{-1},v}\}.$$

Since we have $c_{f,v} = c_{f^{-1},v} = c^+_{\varepsilon,\delta,v} = c^-_{\varepsilon,\delta,v} = 0$ except for finitely many v (indeed for every $v \notin S$), this gives the assertion.

(2) Since $G_{f,v}$ and $G_{f^{-1},v}$ are nonnegative functions, we see that $\tilde{h}_f(x) = 0$ if and only if $G_{f,v}(x) = G_{f^{-1},v}(x) = 0$ for every $v \in M$ if and only if $\hat{h}_f(x) = 0$. □

7. Arithmetic properties of regular polynomial automorphisms

In this section, we give some applications of local and global canonical height functions. The first application is the following theorem on the usual height function [Kawaguchi 2006, §4; Silverman 2006, Conjecture 3; 2007, Conjecture 7.18], which is independently obtained by Lee [2013] via a different method (global method based on the effectiveness of a certain divisor as in the case of $N = 2$ in [Kawaguchi 2006]).

Theorem 7.1. *Let* $f : \mathbb{A}^N \to \mathbb{A}^N$ *be a regular polynomial automorphism over a number field* K. *Let* d *and* d_- *be the degrees of* f *and* f^{-1}.

(1) *There exists a constant* $c \ge 0$ *such that*

$$\frac{1}{d}h(f(x)) + \frac{1}{d_-}h(f^{-1}(x)) \ge \left(1 + \frac{1}{dd_-}\right)h(x) - c$$

for all $x \in \mathbb{A}^N(\overline{K})$.

(2) *We have*

$$\liminf_{\substack{x\in\mathbb{A}^N(\overline{K})\\ h(x)\to\infty}} \frac{\frac{1}{d}h(f(x))+\frac{1}{d_-}h(f^{-1}(x))}{h(x)} = 1+\frac{1}{dd_-}. \tag{7-1}$$

Proof. (1) We set

$$\widetilde{G}_{f,v} := \max\{G_{f,v}, G_{f^{-1},v}\}.$$

Claim 7.1.1. *For all $x \in \mathbb{A}^N(\overline{K})$, we have*

$$\frac{1}{d}\widetilde{G}_{f,v}(f(x))+\frac{1}{d_-}\widetilde{G}_{f,v}(f^{-1}(x)) \geq \Bigl(1+\frac{1}{dd_-}\Bigr)\widetilde{G}_{f,v}(x). \tag{7-2}$$

We first show that Claim 7.1.1 implies (1). Indeed, we assume Claim 7.1.1. Then summing up over all v, we have

$$\frac{1}{d}\tilde{h}(f(x))+\frac{1}{d_-}\tilde{h}(f^{-1}(x)) \geq \Bigl(1+\frac{1}{dd_-}\Bigr)\tilde{h}(x). \tag{7-3}$$

Since $\tilde{h}_f = h+O(1)$ by Proposition 6.5(1), Equation (7-3) yields (1).

To show Claim 7.1.1, for notational convenience let $A = G_{f,v}(x)$, $B = G_{f^{-1},v}(x)$, and $\gamma = \frac{1}{dd_-}$. Then the definition of $\widetilde{G}_{f,v}$ and $\widetilde{G}_{f^{-1},v}$ and the functional equation of $G_{f,v}(x)$ and $G_{f^{-1},v}(x)$ show that the equality (7-2) is equivalent to

$$\max\{A, \gamma B\}+\max\{\gamma A, B\} \geq (1+\gamma)\max\{A, B\}. \tag{7-4}$$

But the left-hand side of (7-4) is

$$\max\{(1+\gamma)A,\ A+B,\ \gamma(A+B),\ (1+\gamma)B\},$$

which is clearly greater than or equal to the right-hand side of (7-4). This completes the proof of Claim 7.1.1 and hence the proof of Theorem 7.1(1).

(2) From (1), we obtain

$$\liminf_{\substack{x\in\mathbb{A}^N(\overline{K})\\ h(x)\to\infty}} \frac{\frac{1}{d}h(f(x))+\frac{1}{d_-}h(f^{-1}(x))}{h(x)} \geq 1+\frac{1}{dd_-}.$$

On the other hand, it is shown in [Kawaguchi 2006, Proposition 4.4] that for any polynomial automorphism $f : \mathbb{A}^N \to \mathbb{A}^N$, one has

$$\liminf_{\substack{x\in\mathbb{A}^N(\overline{K})\\ h(x)\to\infty}} \frac{\frac{1}{d}h(f(x))+\frac{1}{d_-}h(f^{-1}(x))}{h(x)} \leq 1+\frac{1}{dd_-}. \tag{7-5}$$

Combining these two inequalities gives the assertion. □

Remark 7.2. It is shown in [Kawaguchi 2006, Theorem 4.4] that the equality (7-1) holds in dimension $N = 2$ for regular polynomial automorphisms. Theorem 7.1(2) asserts that the equality holds in any dimension $N \geq 2$ for regular polynomial automorphisms.

Theorem 6.3 recovers the following theorem on f-periodic points.

Corollary 7.3 [Marcello 2000]. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism over a number field K. Then the set of f-periodic points in $\mathbb{A}^N(\overline{K})$ is a set of bounded height. In particular, for any integer $D \geq 1$ the set*

$$\{ x \in \mathbb{A}^N(\overline{K}) \mid x \text{ is } f\text{-periodic},\ [K(x) : K] \leq D \}$$

is finite.

Proof. By Theorem 6.3(3ii)(5), $\hat{h}_f$ satisfies $\hat{h}_f \gg\ll h$, and a point $x \in \mathbb{A}^N(\overline{K})$ is f-periodic if and only if $\hat{h}_f(x) = 0$. Thus, we get the assertion. □

For a non-f-periodic point x, let $O_f(x) := \{ f^n(x) \mid n \in \mathbb{Z} \}$ denote the f-orbit of x. We define the canonical height of the orbit $O_f(x)$ by

$$\hat{h}_f(O_f(x)) = \frac{\log \hat{h}_f^+(x)}{\log d} + \frac{\log \hat{h}_f^-(x)}{\log d_-}. \tag{7-6}$$

We note that for any integer n, Theorem 6.3 implies that

$$\begin{aligned} \frac{\log \hat{h}_f^+(f^n(x))}{\log d} + \frac{\log \hat{h}_f^-(f^n(x))}{\log d_-} &= \frac{\log d^n \hat{h}_f^+(x)}{\log d} + \frac{\log d_-^{-n} \hat{h}_f^-(x)}{\log d_-} \\ &= \frac{\log \hat{h}_f^+(x)}{\log d} + \frac{\log \hat{h}_f^-(x)}{\log d_-}. \end{aligned}$$

Thus, the value $\hat{h}_f(O_f(x))$ depends only on the orbit $O_f(x)$ and not the particular choice of the point x in the orbit. The next corollary gives a refinement of [Marcello 2003, Corollary B].

Corollary 7.4. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a regular polynomial automorphism over a number field K. Let d and d_- be the degrees of f and f^{-1}. Then for any infinite orbit $O_f(x)$,*

$$\#\{ y \in O_f(x) \mid h(y) \leq T \} = \Big(\frac{1}{\log d} + \frac{1}{\log d_-} \Big) \log T - \hat{h}(O_f(x)) + O(1)$$

as $T \to +\infty$. Here the $O(1)$ bound depends only f, independent of the orbit $O_f(x)$.

Proof. Since f satisfies (7-3), we apply [Kawaguchi 2006, Theorem 5.2]. □

In the rest of this section, we consider some global-to-local arithmetic properties. Suppose that f is a regular polynomial automorphism. By Theorem 6.3(2)(5), $x \in \mathbb{A}^N(\overline{K})$ is f-periodic if and only if $G_{f,v}(x) = 0$ for all $v \in M_K$. By Theorem 3.1 for nonarchimedean v and [Sibony 1999, §2] for archimedean v, $G_{f,v}(x) = 0$ is equivalent to $\{f^n(x)\}_{n=0}^{+\infty}$ being bounded with respect to $\|\cdot\|_v$. Thus, we see that $x \in \mathbb{A}^N(\overline{K})$ is f-periodic if and only if $\{f^n(x)\}_{n=0}^{+\infty}$ is bounded with respect to $\|\cdot\|_v$ for all $v \in M_K$.

This actually holds for any polynomial map f, replacing f-periodic points by f-preperiodic points (see [Call and Goldstine 1997, Corollary 6.3] for $N = 1$).

Proposition 7.5. *Let $f : \mathbb{A}^N \to \mathbb{A}^N$ be a polynomial map over a number field K. For $x \in \mathbb{A}^N(\overline{K})$, the following are equivalent*:

(i) *x is f-preperiodic and*

(ii) *for every $v \in M_K$, $\{f^n(x)\}_{n=0}^{+\infty}$ is bounded with respect to the v-adic topology.*

Proof. Taking a finite extension field of K over which x is defined if necessary, we may assume that x is defined over K. It is obvious that (i) implies (ii). We assume (ii) and show (i). We take a finite subset S of M_K containing the set of all archimedean absolute values such that x and f are defined over $O_{K,S}$. Then for any $v \notin S$, we have

$$\|f^n(x)\|_v \leq 1 \quad \text{for all } n \geq 0.$$

Since we assume (ii), there is a constant C_v for each $v \in S$ such that

$$\|f^n(x)\|_v \leq C_v \quad \text{for all } n \geq 0.$$

Then we have

$$h(f^n(x)) = \sum_{v \in M_K} n_v \log^+ \|f^n(x)\| \leq \sum_{v \in S} n_v C_v \quad \text{for all } n \geq 0.$$

Then

$$\{\, f^n(x) \mid n \geq 0 \,\} \subseteq \left\{ y \in \mathbb{A}^N(K) \;\middle|\; h(y) \leq \sum_{v \in S} n_v C_v \right\}.$$

Since the latter set is finite, the set $\{f^n(x)\}_{n \geq 0}$ is finite, so x is f-preperiodic. □

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