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# Multiplicative excellent families of elliptic surfaces of type $E_7$ or $E_8$

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We describe explicit multiplicative excellent families of rational elliptic surfaces with Galois group isomorphic to the Weyl group of the root lattices $E_7$ or $E_8$. The Weierstrass coefficients of each family are related by an invertible polynomial transformation to the generators of the multiplicative invariant ring of the associated Weyl group, given by the fundamental characters of the corresponding Lie group. As an application, we give examples of elliptic surfaces with multiplicative reduction and all sections defined over $\mathbb{Q}$ for most of the entries of fiber configurations and Mordell–Weil lattices described by Oguiso and Shioda, as well as examples of explicit polynomials with Galois group $W(E_7)$ or $W(E_8)$.

## 1. Introduction

For an elliptic curve $E$ over a field $K$, determining its Mordell–Weil group is a fundamental problem in algebraic geometry and number theory. When $K = k(t)$ is a rational function field in one variable, this problem becomes a geometrical one of understanding sections of an elliptic surface with section. Lattice theoretic methods of attack were described in [Shioda 1990]. In particular, when $\mathscr{E} \to \mathbb{P}^1_t$ is a rational elliptic surface given as a minimal proper model of

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

with $a_i(t) \in k[t]$ of degree at most $i$, the possible configurations (types) of bad fibers and Mordell–Weil groups were analyzed by Oguiso and Shioda [1991].

In [Shioda 1991a], the second author studied sections for some families of elliptic surfaces with an additive fiber, by means of the specialization map, and obtained a relation between the coefficients of the Weierstrass equation and the fundamental invariants of the corresponding Weyl groups. Shioda and Usui [1992] expanded

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this by studying families with a bad fiber of additive reduction more exhaustively. They defined the formal notion of an excellent family (see Section 2) and found excellent families for many of the "admissible" types.

The analysis of rational elliptic surfaces of high Mordell–Weil rank, but with a fiber of multiplicative reduction, is much more challenging. However, understanding this situation is arguably more fundamental, since if we write down a "random" elliptic surface, then with probability close to 1 it will have Mordell–Weil lattice $E_8$ and twelve nodal fibers (that is, of multiplicative reduction). To be more precise, if we choose Weierstrass coefficients $a_i(t)$ of degree $i$, with coefficients chosen uniformly at random from among rational numbers (say) of height at most $N$, then as $N \to \infty$ the surface will satisfy the condition above with probability approaching 1. One can make a similar statement for rational elliptic surfaces chosen to have Mordell–Weil lattice $E_7^*$, $E_6^*$, etc.

In [Shioda 2012], this study was carried out for elliptic surfaces with a fiber of type $\mathrm{I}_3$ and Mordell–Weil lattice isometric to $E_6^*$, through a "multiplicative excellent family" of type $E_6$. We will describe this case briefly in Section 3. The main result of this paper shows that two explicitly described families of rational elliptic surfaces with Mordell–Weil lattices $E_7^*$ or $E_8$ are multiplicative excellent. The proof involves a surprising connection with representation theory of the corresponding Lie groups, and in particular, their fundamental characters. In particular, we deduce that the Weierstrass coefficients give another natural set of generators for the multiplicative invariants of the respective Weyl groups, as a polynomial ring. Similar formulas were derived by Eguchi and Sakai [2003] using calculations from string theory and mirror symmetry.

The idea of an excellent family is quite useful and important in number theory. An excellent family of algebraic varieties leads to a Galois extension $F(\mu)/F(\lambda)$ of two purely transcendental extensions of a number field $F$ (say $\mathbb{Q}$), with Galois group a desired finite group $G$. This setup has an immediate number-theoretic application, since one may specialize the parameters $\lambda$ and apply Hilbert's irreducibility theorem to obtain Galois extensions over $\mathbb{Q}$ with the same Galois group. Furthermore, we can make the construction effective if appropriate properties of the group $G$ are known (see Examples 8 and 19 for the case $G = W(E_7)$ or $W(E_8)$). At the same time, an excellent family will give rise to a split situation very easily, by specializing the parameters $\mu$ instead. For examples, in the situation considered in our paper, we obtain elliptic curves over $\mathbb{Q}(t)$ with Mordell–Weil rank 7 or 8 together with explicit generators for the Mordell–Weil group (see Examples 7 and 18). There are also applications to geometric specialization or degeneration of the family. Therefore, it is desirable (but quite nontrivial) to construct explicit excellent families of algebraic varieties. Such a situation is quite rare in general: theoretically, any finite reflection group is a candidate, but it is not generally neatly related to an algebraic geometric family. Hilbert treated the case of the symmetric group $S_n$, corresponding to families of zero-dimensional varieties. Not many examples were known before the (additive) excellent families for the Weyl groups of the exceptional Lie groups $E_6$, $E_7$ and $E_8$ were given in [Shioda 1991a], using the theory of Mordell–Weil lattices. Here, we finish the story for the multiplicative excellent families for these Weyl groups.

## 2. Mordell–Weil lattices and excellent families

Let $X \xrightarrow{\pi} \mathbb{P}^1$ be an elliptic surface with section $\sigma : \mathbb{P}^1 \to X$, that is, a proper relatively minimal model of its generic fiber, which is an elliptic curve. We denote the image of $\sigma$ by $O$, which we take to be the zero section of the Néron model. We let $F$ be the class of a fiber in $\mathrm{Pic}(X) \cong \mathrm{NS}(X)$, and let the reducible fibers of $\pi$ lie over $v_1, \ldots, v_k \in \mathbb{P}^1$. The nonidentity components of $\pi^{-1}(v_i)$ give rise to a sublattice $T_i$ of $\mathrm{NS}(X)$, which is (the negative of) a root lattice (see [Kodaira 1963a; 1963b; Tate 1975]). The *trivial lattice* $T$ is $\mathbb{Z}O \oplus \mathbb{Z}F \oplus (\bigoplus T_i)$, and we have the isomorphism $\mathrm{MW}(X/\mathbb{P}^1) \cong \mathrm{NS}(X)/T$, which describes the Mordell–Weil group. In fact, one can induce a positive definite pairing on the Mordell–Weil group modulo torsion, by inducing it from the negative of the intersection pairing on $\mathrm{NS}(X)$. We refer the reader to [Shioda 1990] for more details. In this paper, we will call $\bigoplus T_i$ the *fibral lattice*.

Next, we recall from [Shioda and Usui 1992] the notion of an *excellent family* with Galois group $G$. Suppose $X \to \mathbb{A}^n$ is a family of algebraic varieties, varying with respect to $n$ parameters $\lambda_1, \ldots, \lambda_n$. The generic member of this family $X_\lambda$ is a variety over the rational function field $k_0 = \mathbb{Q}(\lambda)$. Let $k = \overline{k_0}$ be the algebraic closure, and suppose that $\mathscr{C}(X_\lambda)$ is a group of algebraic cycles on $X_\lambda$ over the field $k$ (in other words, it is a group of algebraic cycles on $X_\lambda \times_{k_0} k$). Suppose in addition that there is an isomorphism $\phi_\lambda : \mathscr{C}(X_\lambda) \otimes \mathbb{Q} \cong V$ for a fixed vector space $V$, and $\mathscr{C}(X_\lambda)$ is preserved by the Galois group $\mathrm{Gal}(k/k_0)$. Then we have the Galois representation

$$\rho_\lambda : \mathrm{Gal}(k/k_0) \to \mathrm{Aut}(\mathscr{C}(X_\lambda)) \to \mathrm{Aut}(V).$$

We let $k_\lambda$ be the fixed field of the kernel of $\rho_\lambda$, that is, it is the smallest extension of $k_0$ over which the cycles of $\mathscr{C}(\lambda)$ are defined. We call it the *splitting field* of $\mathscr{C}(X_\lambda)$.

Now let $G$ be a finite reflection group acting on the space $V$.

**Definition 1.** We say $\{X_\lambda\}$ is an *excellent family* with Galois group $G$ if the following conditions hold:

(1) The image of $\rho_\lambda$ is equal to $G$.

(2) There is a $\mathrm{Gal}(k/k_0)$-equivariant evaluation map $s : \mathscr{C}(X_\lambda) \to k$.

(3) There exists a basis $\{Z_1, \ldots, Z_n\}$ of $\mathscr{C}(X_\lambda)$ such that if we set $u_i = s(Z_i)$, then $u_1, \ldots, u_n$ are algebraically independent over $\mathbb{Q}$.

(4) $\mathbb{Q}[u_1, \ldots, u_n]^G = \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$.

As an example, for $G = W(E_8)$, consider the following family of rational elliptic surfaces over $k_0 = \mathbb{Q}(\lambda)$:

$$y^2 = x^3 + x\Big(\sum_{i=0}^{3} p_{20-6i}t^i\Big) + \Big(\sum_{j=0}^{3} p_{30-6j}t^j + t^5\Big),$$

with $\lambda = (p_2, p_8, p_{12}, p_{14}, p_{18}, p_{20}, p_{24}, p_{30})$. Shioda [1991a] shows that this is an excellent family with Galois group $G$. The $p_i$ are related to the fundamental invariants of the Weyl group of $E_8$, as is suggested by their degrees (subscripts).

We now define the notion of a *multiplicative excellent family* for a group $G$. As before, $X \to \mathbb{A}^n$ is a family of algebraic varieties, varying with respect to $\lambda = (\lambda_1, \ldots, \lambda_n)$, and $\mathscr{C}(X_\lambda)$ is a group of algebraic cycles on $X_\lambda$, isomorphic (via a fixed isomorphism) to a fixed abelian group $M$. The fields $k_0$ and $k$ are as before, and we have a Galois representation

$$\rho_\lambda : \mathrm{Gal}(k/k_0) \to \mathrm{Aut}(\mathscr{C}(X_\lambda)) \to \mathrm{Aut}(M).$$

Suppose that $G$ is a group acting on $M$.

**Definition 2.** We say $\{X_\lambda\}$ is a *multiplicative excellent family* with Galois group $G$ if the following conditions hold:

(1) The image of $\rho_\lambda$ is equal to $G$.

(2) There is a $\mathrm{Gal}(k/k_0)$-equivariant evaluation map $s : \mathscr{C}(X_\lambda) \to k^*$.

(3) There exists a basis $\{Z_1, \ldots, Z_n\}$ of $\mathscr{C}(X_\lambda)$ such that if we set $u_i = s(Z_i)$, then $u_1, \ldots, u_n$ are algebraically independent over $\mathbb{Q}$.

(4) $\mathbb{Q}[u_1, \ldots, u_n, u_1^{-1}, \ldots, u_n^{-1}]^G = \mathbb{Q}[\lambda_1, \ldots, \lambda_n]$.

**Remark 3.** Though we use similar notation, the specialization map $s$ and the $u_i$ in the multiplicative case are quite different from the ones in the additive case. Intuitively, one may think of these as exponentiated versions of the corresponding objects in the additive case. However, we want the specialization map to be an algebraic morphism, and so in general (additive) excellent families specified by Definition 1 will be very different from multiplicative excellent families specified by Definition 2.

In our examples, $G$ will be a finite reflection group acting on a lattice in Euclidean space, which will be our choice for $M$. However, what is relevant here is not the ring of (additive) invariants of $G$ on the vector space spanned by $M$. Instead, note that the action of $G$ on $M$ gives rise to a "multiplicative" or "monomial" action of $G$ on the group algebra $\mathbb{Q}[M]$, and we will be interested in the polynomials on this space that are invariant under $G$. This is the subject of *multiplicative invariant theory* (see, for example, [Lorenz 2005]). In the case when $G$ is the automorphism group of a root lattice or root system, multiplicative invariants were classically studied by using the terminology of "exponentiated" roots $e^\alpha$ (for instance, see [Bourbaki 1968, Section VI.3]).

## 3. The $E_6$ case

We now sketch the construction of multiplicative excellent family in [Shioda 2012]. Consider the family of rational elliptic surfaces $S_\lambda$ with Weierstrass equation

$$y^2 + txy = x^3 + (p_0 + p_1 t + p_2 t^2)\, x + q_0 + q_1 t + q_2 t^2 + t^3$$

with parameter $\lambda = (p_0, p_1, p_2, q_0, q_1, q_2)$. The surface $S_\lambda$ generically only has one reducible fiber at $t = \infty$, of type $\mathrm{I}_3$. Therefore, the Mordell–Weil lattice $M_\lambda$ of $S_\lambda$ is isomorphic to $E_6^*$. There are 54 minimal sections of height $4/3$, and exactly half of them have the property that $x$ and $y$ are linear in $t$. If we have

$$x = at + b \quad \text{and} \quad y = ct + d,$$

then substituting these back in to the Weierstrass equation, we get a system of equations, and we may easily eliminate $b, c, d$ from the system to get a monic equation of degree 27 (subject to a genericity assumption), which we write as $\Phi_\lambda(a) = 0$. Also, note that the specialization of a section of height $4/3$ to the fiber at $\infty$ gives us a point on one of the two nonidentity components of the special fiber of the Néron model (the same component for all the 27 sections). Identifying the smooth points of this component with $\mathbb{G}_m \times \{1\} \subset \mathbb{G}_m \times (\mathbb{Z}/3\mathbb{Z})$, the specialization map $s$ takes the section to $(-1/a, 1)$. Let the specializations be $s_i = -1/a_i$ for $1 \le i \le 27$. We have

$$\begin{aligned}\Phi_\lambda(X) &= \prod_{i=1}^{27}(X - a_i) = \prod_{i=1}^{27}(X + 1/s_i)\\ &= X^{27} + \epsilon_{-1}X^{26} + \epsilon_{-2}X^{25} + \cdots + \epsilon_4 X^4 + \epsilon_3 X^3 + \epsilon_2 X^2 + \epsilon_1 X + 1.\end{aligned}$$

Here $\epsilon_i$ is the $i$-th elementary symmetric polynomial of the $s_i$ and $\epsilon_{-i}$ that of the $1/s_i$. The coefficients of $\Phi_\lambda(X)$ are polynomials in the coordinates of $\lambda$, and we may use the equations for $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_{-1}$ and $\epsilon_{-2}$ to solve for $p_0, \dots, q_3$. However, the resulting solution has $\epsilon_{-2}$ in the denominator. We may remedy this situation as follows. Consider the construction of $E_6^*$ as described in [Shioda 1995]: let $v_1, \dots, v_6$ be vectors in $\mathbb{R}^6$ with $\langle v_i, v_j\rangle = \delta_{ij} + 1/3$, and let $u = (\sum v_i)/3$. The $\mathbb{Z}$-span of $v_1, \dots, v_6, u$ is a lattice $L$ isometric to $E_6^*$. It is clear that $v_1, \dots, v_5, u$

forms a basis of $L$. Here, we choose an isometry between the Mordell–Weil lattice and the lattice $L$, and let the specializations of $v_1, \dots, v_6, u$ be $s_1, \dots, s_6, r$, respectively. These satisfy $s_1 s_2 \dots s_6 = r^3$. The 54 nonzero minimal vectors of $E_6^*$ split up into two cosets (modulo $E_6$) of 27 each, of which we have chosen one. The specializations of these 27 special sections are given by

$$\{s_1, \dots, s_{27}\} := \{s_i : 1 \le i \le 6\} \cup \{s_i/r : 1 \le i \le 6\} \cup \{r/(s_i s_j) : 1 \le i < j \le 6\}.$$

If

$$\delta_1 = r + \frac{1}{r} + \sum_{i \ne j} \frac{s_i}{s_j} + \sum_{i<j<k} \Big( \frac{r}{s_i s_j s_k} + \frac{s_i s_j s_k}{r} \Big),$$

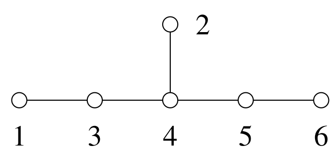
then $\delta_1$ belongs to the $G = W(E_6)$-invariants of $\mathbb{Q}[s_1, \dots, s_5, r, s_1^{-1}, \dots, s_5^{-1}, r^{-1}]$, and explicit computations in [Shioda 2012] show that

$$\begin{aligned} \mathbb{Q}[s_1, \dots, s_5, r, s_1^{-1}, \dots, s_5^{-1}, r^{-1}]^G &= \mathbb{Q}[\delta_1, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_{-1}, \epsilon_{-2}] \\ &= \mathbb{Q}[p_0, p_1, p_2, q_0, q_1, q_2]. \end{aligned}$$

The explicit relation showing the second equality is as follows:

$$\begin{aligned} \delta_1 &= -2p_1, & \epsilon_{-1} &= p_2^2 - q_2, \\ \epsilon_2 &= 13p_2^2 + p_0 - q_2, & \epsilon_{-2} &= -2p_1 p_2 + 6p_2 + q_1, \\ \epsilon_1 &= 6p_2, & \epsilon_3 &= 8p_2^3 + 2p_0 p_2 + p_1^2 - 6p_1 - q_0 + 9. \end{aligned}$$

We make an additional observation. The six fundamental representations of the Lie algebra $E_6$ correspond to the fundamental weights in the following diagram, which displays the standard labeling of these representations.




The dimensions of these representations $V_1, \dots, V_6$ are 27, 78, 351, 2925, 351, 27 respectively, and their characters are related to $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_{-1}, \epsilon_{-2}, \delta_1$ by the nice transformation

$$\begin{aligned} \chi_1 &= \epsilon_1, & \chi_2 &= \delta_1 + 6, & \chi_3 &= \epsilon_2, \\ \chi_4 &= \epsilon_3, & \chi_5 &= \epsilon_{-2}, & \chi_6 &= \epsilon_{-1}. \end{aligned}$$

This explains the reason for bringing in $\delta_1$ into the picture, and also why there is a denominator when solving for $p_0, \dots, q_2$ in terms of $\epsilon_1, \dots, \epsilon_4, \epsilon_{-1}, \epsilon_{-2}$, as remarked in [Shioda 2012]. The coefficients $\epsilon_j$ are essentially the characters of $\bigwedge^j V$, where $V = V_1$ is the first fundamental representation, while $\epsilon_{-j}$ are those of $\bigwedge^j V^*$, where $V_6 = V^*$. Note that $\bigwedge^3 V \cong \bigwedge^3 V^*$. Therefore, from the expressions for $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_{-1}, \epsilon_{-2}$, we may obtain $p_2, q_2, p_0, q_1, q_0$, in terms of the remaining variable $p_1$, without introducing any denominators. However, representation $V_2$ cannot be obtained as a direct summand with multiplicity 1 from a tensor product of $\bigwedge^j V$ (for $1 \le j \le 3$) and $\bigwedge^k V^*$ (for $1 \le k \le 2$). On the other hand, we do have the isomorphism

$$(V_2 \otimes V_5) \oplus V_5 \oplus V_1 \cong \textstyle\bigwedge^4 V_1 \oplus (V_3 \otimes V_6) \oplus (V_6 \otimes V_6).$$

Therefore, we are able to solve for $p_1$ if we introduce a denominator of $\epsilon_{-2}$, which is the character of $V_5$.

## 4. The $E_7$ case

**4.1. *Results.*** Next, we exhibit a multiplicative excellent family for the Weyl group of $E_7$. It is given by the Weierstrass equation

$$y^2 + txy = x^3 + (p_0 + p_1 t + p_2 t^2)\,x + q_0 + q_1 t + q_2 t^2 + q_3 t^3 - t^4.$$

For generic $\lambda = (p_0, \dots, p_2, q_0, \dots, q_3)$, this rational elliptic surface $X_\lambda$ has a fiber of type $\mathrm{I}_2$ at $t = \infty$, and no other reducible fibers. Hence, the Mordell–Weil group $M_\lambda$ is $E_7^*$. We note that any elliptic surface with a fiber of type $\mathrm{I}_2$ can be put into this Weierstrass form (in general over a small degree algebraic extension of the ground field), after a fractional linear transformation of the parameter $t$, and Weierstrass transformations of $x$ and $y$.

**Lemma 4.** *The smooth part of the special fiber is isomorphic to the group scheme $\mathbb{G}_m \times \mathbb{Z}/2\mathbb{Z}$. The identity component is the nonsingular part of the curve*

$$y^2 + xy = x^3.$$

*The $x$- and $y$-coordinates of a section of height* 2 *are polynomials of degrees* 2 *and* 3 *respectively, and its specialization at $t = \infty$ is* $(\lim_{t\to\infty}(y + tx)/y, 0) \in k^* \times \{0, 1\}$. *A section of height* $3/2$ *has $x$ and $y$ coordinates of the form*

$$x = at + b \quad \textit{and} \quad y = ct^2 + dt + e.$$

*and specializes at $t = \infty$ to* $(c, 1)$.

*Proof.* First, to get a local chart for the elliptic surface near $t = \infty$, we set $x = \tilde{x}/u^2$, $y = \tilde{y}/u^3$ and $t = 1/u$, and look for $u$ near 0. Therefore, the special fiber (before blow-up) is given by $\bar{y}^2 + \bar{x}\bar{y} = \bar{x}^3$, where $\bar{x} = \tilde{x}|_{u=0}$ and $\bar{y} = \tilde{y}|_{u=0}$ are the reductions of the coordinates at $u = 0$, respectively. It is an easy exercise to parametrize the smooth locus of this curve: It is given, for instance, by $\bar{x} = s/(s-1)^2$, $\bar{y} = s/(s-1)^3$. We then check that $s = (\bar{y} + \bar{x})/\bar{y}$ and the map from the smooth locus to $\mathbb{G}_m$ that takes the point $(\bar{x}, \bar{y})$ to $s$ is a homomorphism from the secant group law to multiplication in $k^*$. This proves the first half of the lemma. Note that we could just as well have taken $1/s$ to be the parameter on $\mathbb{G}_m$; our choice is a matter of convention. To prove the specialization law for sections of height $3/2$, we may, for instance, take the sum of such a section $Q$ with a section $P$ of height 2 with specialization $(s, 0)$. A direct calculation shows that the $y$-coordinate of the sum has top (quadratic) coefficient $cs$. Therefore the specialization of $Q$ must have the form $\kappa c$, where $\kappa$ is a constant not depending on $Q$. Finally, the sum of two sections $Q_1$ and $Q_2$ of height $3/2$ and having coefficients $c_1$ and $c_2$ for the $t^2$ term of their $y$-coordinates can be checked to specialize to $(c_1c_2, 0)$. It follows that $\kappa = \pm 1$, and we take the plus sign as a convention. (It is easy to see that both choices of sign are legitimate, since the sections of height 2 generate a copy of $E_7$, whereas the sections of height $3/2$ lie in the nontrivial coset of $E_7$ in $E_7^*$). □

There are 56 sections of height $3/2$, with $x$ and $y$ coordinates in the form above. Substituting the formulas above for $x$ and $y$ into the Weierstrass equation, we get the following system of equations.

$$\begin{aligned}
c^2 + ac + 1 &= 0,\\
q_3 + ap_2 + a^3 &= (2c + a)d + bc,\\
q_2 + bp_2 + 3a^2b &= (2c + a)e + (b + d)d,\\
q_1 + bp_1 + ap_0 + 3ab^2 &= (2d + b)e,\\
q_0 + bp_0 + b^3 &= e^2.
\end{aligned}$$

We solve for $a$ and $b$ from the first and second equations, and then $e$ from the third, assuming $c \neq 1$. Substituting these values back into the last two equations, we get two equations in the variables $c$ and $d$. Taking the resultant of these two equations with respect to $d$, and dividing by $c^{30}(c^2-1)^4$, we obtain an equation of degree 56 in $c$, which is monic, reciprocal and has coefficients in $\mathbb{Z}[\lambda] = \mathbb{Z}[p_0, \dots, q_3]$. We denote this polynomial by

$$\Phi_\lambda(X) = \prod_{i=1}^{56} (X - s(P)) = X^{56} + \epsilon_1 X^{55} + \epsilon_2 X^{54} + \cdots + \epsilon_1 X + \epsilon_0,$$

where $P$ ranges over the 56 minimal sections of height $3/2$. It is clear that $a, b, d, e$ are rational functions of $c$ with coefficients in $k_0$.

We have a Galois representation on the Mordell–Weil lattice

$$\rho_\lambda : \mathrm{Gal}(k/k_0) \to \mathrm{Aut}(M_\lambda) \cong \mathrm{Aut}(E_7^*).$$

Here $\mathrm{Aut}(E_7^*) \cong \mathrm{Aut}(E_7) \cong W(E_7)$, the Weyl group of type $E_7$. The *splitting field* of $M_\lambda$ is the fixed field $k_\lambda$ of $\mathrm{Ker}(\rho_\lambda)$. By definition, $\mathrm{Gal}(k_\lambda/k_0) \cong \mathrm{Im}(\rho_\lambda)$. The splitting field $k_\lambda$ is equal to the splitting field of the polynomial $\Phi_\lambda(X)$ over $k_0$, since the Mordell–Weil group is generated by the 56 sections of smallest height $P_i = (a_i t + b_i, c_i t^2 + d_i t + e_i)$. We also have $k_\lambda = k_0(P_1, \ldots, P_{56}) = k_0(c_1, \ldots, c_{56})$. We shall sometimes write $e^\alpha$, (for $\alpha \in E_7^*$ a minimal vector) to refer to the specializations of these sections $c(P_i)$, for convenience.

**Theorem 5.** *Assume that $\lambda$ is generic over $\mathbb{Q}$, i.e., the coordinates $p_0, \ldots, q_3$ are algebraically independent over $\mathbb{Q}$.*

(1) *$\rho_\lambda$ induces an isomorphism* $\mathrm{Gal}(k_\lambda/k_0) \cong W(E_7)$.

(2) *The splitting field $k_\lambda$ is a purely transcendental extension of $\mathbb{Q}$, isomorphic to the function field $\mathbb{Q}(Y)$ of the toric hypersurface*

$$Y \subset \mathbb{G}_m^8 \quad \textit{defined by } s_1 \ldots s_7 = r^3.$$

*There is an action of $W(E_7)$ on $Y$ such that $\mathbb{Q}(Y)^{W(E_7)} = k_\lambda^{W(E_7)} = k_0$.*

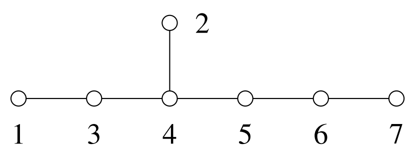
(3) *The ring of $W(E_7)$-invariants in the affine coordinate ring*

$$\mathbb{Q}[Y] = \frac{\mathbb{Q}[s_i, r, 1/s_i, 1/r]}{(s_1 \ldots s_7 - r^3)} \cong \mathbb{Q}[s_1, \ldots, s_6, r, s_1^{-1}, \ldots, s_6^{-1}, r^{-1}]$$

*is equal to the polynomial ring $\mathbb{Q}[\lambda]$:*

$$\mathbb{Q}[Y]^{W(E_7)} = \mathbb{Q}[\lambda] = \mathbb{Q}[p_0, p_1, p_2, q_0, q_1, q_2, q_3].$$

In fact, we shall prove an explicit, invertible polynomial relation between the Weierstrass coefficients $\lambda$ and the fundamental characters for $E_7$. Let $V_1, \ldots, V_7$ be the fundamental representations of $E_7$, and $\chi_1, \ldots, \chi_7$ their characters (on a maximal torus), as labeled below. For a description of the fundamental modules for the exceptional Lie groups see [Carter 2005, Section 13.8].




Note that since the weight lattice $E_7^*$ has been equipped with a nice set of generators $(v_1, \ldots, v_7, u)$ with $\sum v_i = 3u$ (as in [Shioda 1995]), the characters $\chi_1, \ldots, \chi_7$ lie in the ring of Laurent polynomials $\mathbb{Q}[s_i, r, 1/s_i, 1/r]$, where $s_i$ corresponds to $e^{v_i}$ and $r$ to $e^u$, and are obviously invariant under the (multiplicative) action of the Weyl group on this ring of Laurent polynomials. Explicit formulas for the $\chi_i$ are given in the auxiliary files.

We also note that the roots of $\Phi_\lambda$ are given by

$$s_i, \frac{1}{s_i} \quad \text{for } 1 \le i \le 7 \quad \text{and} \quad \frac{r}{s_i s_j}, \frac{s_i s_j}{r} \quad \text{for } 1 \le i < j \le 7.$$

**Theorem 6.** *For generic $\lambda$ over $\mathbb{Q}$, we have*

$$\mathbb{Q}[\chi_1, \ldots, \chi_7] = \mathbb{Q}[p_0, p_1, p_2, q_0, q_1, q_2, q_3].$$

*The transformation between these sets of generators is*

$$\begin{aligned}
\chi_1 &= 6p_2 + 25,\\
\chi_2 &= 6q_3 - 2p_1,\\
\chi_3 &= -q_2 + 13p_2^2 + 108p_2 + p_0 + 221,\\
\chi_4 &= 9q_3^2 - 6p_1q_3 - q_2 - q_0 + 8p_2^3 + 85p_2^2 + (2p_0 + 300)p_2 + p_1^2 + 10p_0 + 350,\\
\chi_5 &= (6p_2 + 26)q_3 + q_1 - 2p_1p_2 - 10p_1,\\
\chi_6 &= -q_2 + p_2^2 + 12p_2 + 27,\\
\chi_7 &= q_3,
\end{aligned}$$

*with inverse*

$$\begin{aligned}
p_2 &= (\chi_1 - 25)/6,\\
p_1 &= (6\chi_7 - \chi_2)/2,\\
p_0 &= -(3\chi_6 - 3\chi_3 + \chi_1^2 - 2\chi_1 + 7)/3,\\
q_3 &= \chi_7,\\
q_2 &= -(36\chi_6 - \chi_1^2 - 22\chi_1 + 203)/36,\\
q_1 &= (24\chi_7 + 6\chi_5 + (-\chi_1 - 5)\chi_2)/6,\\
q_0 &= (27\chi_2^2 - 8\chi_1^3 - 84\chi_1^2 + 120\chi_1 - 136)/108 - (\chi_1 + 2)\chi_6/3 - \chi_4 + (\chi_1 + 5)\chi_3/3.
\end{aligned}$$

Our formulas agree with those of Eguchi and Sakai [2003], who seem to derive these by using an ansatz.

Next, we describe two examples through specialization, one of "small Galois" (in which all sections are defined over $\mathbb{Q}[t]$) and one with "big Galois" (which has Galois group the full Weyl group).

**Example 7.** The values

$$\begin{aligned}
p_0 &= 244655370905444111/(3\mu^2),\\
p_1 &= -4788369529481641525125/(16\mu^2),\\
q_3 &= 184185687325/(4\mu),\\
p_2 &= 199937106590279644475038924955076599/(12\mu^4),\\
q_2 &= 57918534120411335989995011407800421/(9\mu^3),\\
q_1 &= -179880916617213624948875556502808560625/(4\mu^4),\\
q_0 &= 353161437549197551159528020804697629366268908804692 01091/(1728\mu^6),
\end{aligned}$$

where $\mu = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 102102$, give rise to an elliptic surface for which we have $r = 2$, $s_1 = 3$, $s_2 = 5$, $s_3 = 7$, $s_4 = 11$, $s_5 = 13$, $s_6 = 17$, the simplest choice of multiplicatively independent elements. The Mordell–Weil group has a basis of sections for which $c \in \{3, 5, 7, 11, 13, 17, 15/2\}$. We write down their $x$-coordinates below:

$$\begin{aligned}
x(P_1) &= -(10/3)t - 707606695171055129/1563722760600,\\
x(P_2) &= -(26/5)t - 611410735289928023/1563722760600,\\
x(P_3) &= -(50/7)t - 513728975686763429/1563722760600,\\
x(P_4) &= -(122/11)t - 316023939417997169/1563722760600,\\
x(P_5) &= -(170/13)t - 216677827127591279/1563722760600,\\
x(P_6) &= -(290/17)t - 17562556436754779/1563722760600,\\
x(P_7) &= -(229/30)t - 140574879644393807/390930690150.
\end{aligned}$$

In the auxiliary files the $x$-and $y$-coordinates are listed, and it is verified that they satisfy the Weierstrass equation.

**Example 8.** The value $\lambda = \lambda_0 := (1, 1, 1, 1, 1, 1, 1)$ gives rise to an explicit polynomial $f(X) = \Phi_{\lambda_0}(X)$, given by

$$\begin{aligned}
f(X) = X^{56} &- X^{55} + 40X^{54} - 22X^{53} + 797X^{52} - 190X^{51} + 9878X^{50} - 1513X^{49}\\
&+ 82195X^{48} - 17689X^{47} + 496844X^{46} - 175584X^{45} + 2336237X^{44}\\
&- 1196652X^{43} + 8957717X^{42} - 5726683X^{41} + 28574146X^{40}\\
&- 20119954X^{39} + 75465618X^{38} - 53541106X^{37} + 163074206X^{36}\\
&- 110505921X^{35} + 287854250X^{34} - 181247607X^{33} + 420186200X^{32}\\
&- 243591901X^{31} + 518626022X^{30} - 278343633X^{29} + 554315411X^{28}\\
&- 278343633X^{27} + 518626022X^{26} - 243591901X^{25} + 420186200X^{24}\\
&- 181247607X^{23} + 287854250X^{22} - 110505921X^{21} + 163074206X^{20}\\
&- 53541106X^{19} + 75465618X^{18} - 20119954X^{17} + 28574146X^{16}\\
&- 5726683X^{15} + 8957717X^{14} - 1196652X^{13} + 2336237X^{12}\\
&- 175584X^{11} + 496844X^{10} - 17689X^{9} + 82195X^{8} - 1513X^{7}\\
&+ 9878X^{6} - 190X^{5} + 797X^{4} - 22X^{3} + 40X^{2} - X + 1,
\end{aligned}$$

for which we can show that the Galois group is the full group $W(E_7)$, as follows. The reduction of $f(X)$ modulo 7 shows that $\mathrm{Frob}_7$ has cycle decomposition of type $(7)^8$, and similarly, $\mathrm{Frob}_{101}$ has cycle decomposition of type $(3)^2(5)^4(15)^2$.

This implies, as in [Shioda 1991b, Example 7.6], that the Galois group is the entire Weyl group.

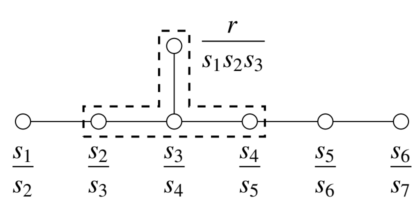
We can also describe degenerations of this family $X_\lambda$ of rational elliptic surfaces by the method of "vanishing roots", where we drop the genericity assumption, and consider the situation where the elliptic fibration might have additional reducible fibers. Let $\psi : Y \to \mathbb{A}^7$ be the finite surjective morphism associated to

$$\mathbb{Q}[p_0, \ldots, q_3] \hookrightarrow \mathbb{Q}[Y] \cong \mathbb{Q}[s_1, \ldots, s_6, r, s_1^{-1}, \ldots, s_6^{-1}, r^{-1}].$$

For $\xi = (s_1, \ldots, s_7, r) \in Y$, let the multiset $\Pi_\xi$ consist of the 126 elements $s_i/r$ and $r/s_i$ for $1 \le i \le 7$, $s_i/s_j$ ((for $1 \le i \ne j \le 7$) and $s_i s_j s_k/r$ and $r/(s_i s_j s_k)$ for $1 \le i < j < k \le 7$, corresponding to the 126 roots of $E_7$. Let $2\nu(\xi)$ be the number of times 1 appears in $\Pi_\xi$, which is also the multiplicity of 1 as a root of $\Psi_\lambda(X)$ (to be defined in Section 4.2), where $\lambda = \psi(\xi)$. We call the associated roots of $E_7$ the *vanishing roots*, in analogy with vanishing cycles in the deformation of singularities. By abuse of notation we label the rational elliptic surface $X_\lambda$ as $X_\xi$.

**Theorem 9.** *The surface $X_\xi$ has new reducible fibers (necessarily at $t \ne \infty$) if and only if $\nu(\xi) > 0$. The number of roots in the root lattice $T_{new}$ is equal to $2\nu(\xi)$, where $T_{new} := \bigoplus_{v \ne \infty} T_v$ is the new part of the trivial lattice.*

We may use this result to produce specializations with trivial lattice including $A_1$, corresponding to the entries in the table of [Oguiso and Shioda 1991, Section 1]. Note that in earlier work [Shioda 1991a; Shioda and Usui 1992], examples of rational elliptic surfaces were produced with a fiber of additive type, for instance, a fiber of type III (which contributes $A_1$ to the trivial lattice) or a fiber of type II. Using our excellent family, we can produce examples with the $A_1$ fiber being of multiplicative type $\mathrm{I}_2$ and all other irreducible singular fibers being nodal (that is, $\mathrm{I}_1$). We list below those types that are not already covered by [Shioda 2012]. To produce these examples, we use an embedding of the new part $T_{\text{new}}$ of the fibral lattice into $E_7$, which gives us any extra conditions satisfied by $s_1, \ldots, s_7, r$. The following multiplicative version of the labeling of simple roots of $E_7$ is useful (compare [Shioda 1995]).




For instance, to produce the example in line 18 of the table (that is, with $T_{\text{new}} = D_4$), we may use the embedding into $E_7$ indicated by embedding the $D_4$ Dynkin diagram within the dashed lines in the figure above. Thus, we must force $s_2 = s_3 = s_4 = s_5$

| Type | Fibral lattice | MW group | $\{s_1, \ldots, s_6, r\}$ |
|---|---|---|---|
| 2 | $A_1$ | $E_7^*$ | 3, 5, 7, 11, 13, 17, 2 |
| 4 | $A_1^2$ | $D_6^*$ | 3, 3, 5, 7, 11, 13, 2 |
| 7 | $A_1^3$ | $D_4^* \oplus A_1^*$ | 3, 3, 5, 5, 7, 11, 2 |
| 10 | $A_1 \oplus A_3$ | $A_1^* \oplus A_3^*$ | 3, 3, 3, 3, 5, 7, 2 |
| 13 | $A_1^4$ | $D_4^* \oplus \mathbb{Z}/2\mathbb{Z}$ | −1, 2, 3, 5, 7, 49/30, 7 |
| 14 | $A_1^4$ | $A_1^{*4}$ | 3, 3, 5, 5, 7, 7, 2 |
| 17 | $A_1 \oplus A_4$ | $\frac{1}{10}\begin{pmatrix} 3 & 1 & -1 \\ 1 & 7 & 3 \\ -1 & 3 & 7 \end{pmatrix}$ | 3, 3, 3, 3, 3, 5, 2 |
| 18 | $A_1 \oplus D_4$ | $A_1^{*3}$ | 2, 3, 3, 3, 3, 5, 18 |
| 21 | $A_1^2 \oplus A_3$ | $A_3^* \oplus \mathbb{Z}/2\mathbb{Z}$ | 3, 5, 60, 30, 30, 30, 900 |
| 22 | $A_1^2 \oplus A_3$ | $A_1^{*2} \oplus \langle 1/4 \rangle$ | 3, 3, 5, 5, 5, 5, 2 |
| 24 | $A_1^5$ | $A_1^{*3} \oplus \mathbb{Z}/2\mathbb{Z}$ | 15/4, 2, 2, 3, 3, 5, 15 |
| 28 | $A_1 \oplus A_5$ | $A_2^* \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, 3, 6, 6, 6, 6, 36 |
| 29 | $A_1 \oplus A_5$ | $A_1^* \oplus \langle 1/6 \rangle$ | 2, 2, 2, 2, 2, 2, 3 |
| 30 | $A_1 \oplus D_5$ | $A_1^* \oplus \langle 1/4 \rangle$ | 2, 2, 2, 2, 2, 3, 8 |
| 33 | $A_1^2 \oplus A_4$ | $\frac{1}{10}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ | 2, 2, 3, 3, 3, 3, 12 |
| 34 | $A_1^2 \oplus D_4$ | $A_1^{*2} \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, 3, 3, 3, 3, 6, 18 |
| 38 | $A_1^3 \oplus A_3$ | $A_1^* \oplus \langle 1/4 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, 2, 3, 3, 3, 4, 12 |
| 42 | $A_1^6$ | $A_1^{*2} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ | 6, −1, −1, 2, 2, 3, 6 |
| 47 | $A_1 \oplus A_6$ | $\langle 1/14 \rangle$ | 8, 8, 8, 8, 8, 8, 128 |
| 48 | $A_1 \oplus D_6$ | $A_1^* \oplus \mathbb{Z}/2\mathbb{Z}$ | 1, 2, 2, 2, 2, 2, 4 |
| 49 | $A_1 \oplus E_6$ | $\langle 1/6 \rangle$ | 2, 2, 2, 2, 2, 2, 8 |
| 52 | $A_1^2 \oplus D_5$ | $\langle 1/4 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, 2, 2, 2, 2, 4, 8 |
| 53 | $A_1^2 \oplus A_5$ | $\langle 1/6 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, 2, 4, 4, 4, 4, 16 |
| 57 | $A_1^3 \oplus D_4$ | $A_1^* \oplus (\mathbb{Z}/2\mathbb{Z})^2$ | −1, 2, 2, 2, 2, −2, −4 |
| 58 | $A_1 \oplus A_3^2$ | $A_1^* \oplus \mathbb{Z}/4\mathbb{Z}$ | $I, I, I, I$, 2, 2, 2 |
| 60 | $A_1^4 \oplus A_3$ | $\langle 1/4 \rangle \oplus (\mathbb{Z}/2\mathbb{Z})^2$ | 2, 2, 2, 2, −1, −1, 4 |
| 65 | $A_1 \oplus E_7$ | $\mathbb{Z}/2\mathbb{Z}$ | 1, 1, 1, 1, 1, 1, 1 |
| 70 | $A_1 \oplus A_7$ | $\mathbb{Z}/4\mathbb{Z}$ | $I, I, I, I, I, I, I$ |
| 71 | $A_1^2 \oplus D_6$ | $(\mathbb{Z}/2\mathbb{Z})^2$ | 1, 1, 1, 1, 1, 1, −1 |
| 74 | $A_1^2 \oplus A_3^2$ | $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$ | $I, I, I, I$, −1, −1, −1 |

**Table 1.** Examples of specializations of the $E_7$ family (types are from [Oguiso and Shioda 1991]).

and $r = s_1 s_2 s_3$, and a simple solution with no extra coincidences is given in the rightmost column (note that $s_7 = 18^3/(2 \cdot 3^4 \cdot 5) = 36/5$).

Here $I = \sqrt{-1}$.

**Remark 10.** For the examples in lines 58, 70 and 74 of the table, one can show that it is not possible to define a rational elliptic surface over $\mathbb{Q}$ in the form we have assumed, such that all the specializations $s_i, r$ are rational. However, there do exist examples with all sections defined over $\mathbb{Q}$, not in the chosen Weierstrass form.

The surface with Weierstrass equation

$$y^2 + xy + \tfrac{1}{16}(c^2-1)(t^2-1)y = x^3 + \tfrac{1}{16}(c^2-1)(t^2-1)x^2$$

has a 4-torsion section $(0,0)$ and a nontorsion section

$$\big((c+1)(t^2-1)/8,\, (c+1)^2(t-1)^2(t+1)/32\big)$$

of height $1/2$, as well as two reducible fibers of type $\mathrm{I}_4$ and a fiber of type $\mathrm{I}_2$. It is an example of type 58.

The surface with Weierstrass equation

$$y^2 + xy + t^2y = x^3 + t^2x^2$$

has a 4-torsion section $(0,0)$, and reducible fibers of types $\mathrm{I}_8$ and $\mathrm{I}_2$. It is an example of type 70.

The surface with Weierstrass equation

$$y^2 + xy - \big(t^2 - \tfrac{1}{16}\big)y = x^3 - \big(t^2 - \tfrac{1}{16}\big)x^2$$

has two reducible fibers of type $\mathrm{I}_4$ and two reducible fibers of type $\mathrm{I}_2$. It also has a 4-torsion section $(0,0)$ and a 2-torsion section $\big((4t-1)/8, (4t-1)^2/32\big)$, which generate the Mordell–Weil group. It is an example of type 74. This last example is the universal elliptic curve with $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ torsion (compare [Kubert 1976]).

**4.2.** ***Proofs.*** We start by considering the coefficients $\epsilon_i$ of $\Phi_\lambda(X)$; we know that $(-1)^i\epsilon_i$ is simply the $i$-th elementary symmetric polynomial in the 56 specializations $s(P_i)$. One shows, either by explicit calculation with Laurent polynomials, or by calculating the decomposition of $\bigwedge^i V$ (where $V = V_7$ is the 56-dimensional representation of $E_7$), and expressing its character as polynomials in the fundamental characters, the following formulas. Some more details are in Section 7 and the auxiliary files in [Kumar and Shioda 2013].

$$\begin{aligned}
\epsilon_1 &= -\chi_7,\\
\epsilon_2 &= \chi_6 + 1,\\
\epsilon_3 &= -(\chi_7 + \chi_5),\\
\epsilon_4 &= \chi_6 + \chi_4 + 1,
\end{aligned}$$

$$\begin{aligned}
\epsilon_5 &= -(\chi_6+\chi_3-\chi_1^2+\chi_1+1)\chi_7+(\chi_1-1)\chi_5-\chi_2\chi_3,\\
\epsilon_6 &= \chi_1\chi_7^2+(\chi_5-(\chi_1+1)\chi_2)\chi_7+\chi_6^2+2(\chi_3-\chi_1^2+\chi_1+1)\chi_6\\
&\qquad -\chi_2\chi_5-(2\chi_1+1)\chi_4+\chi_3^2+2(2\chi_1+1)\chi_3\\
&\qquad +\chi_1\chi_2^2-2\chi_1^3+\chi_1^2+2\chi_1+1,\\
\epsilon_7 &= (-(\chi_1+1)\chi_6+2\chi_4-2(\chi_1+1)\chi_3+\chi_1^3-3\chi_1-1)\chi_7\\
&\qquad -2(\chi_5-\chi_1\chi_2)\chi_6-(\chi_3-\chi_1^2+\chi_1+2)\chi_5+3\chi_2\chi_4\\
&\qquad -(\chi_1+3)\chi_2\chi_3-\chi_2^3+(2\chi_1-1)\chi_1\chi_2.
\end{aligned}$$

On the other hand, we can explicitly calculate the first few coefficients $\epsilon_i$ of $\Phi_\lambda(X)$ in terms of the Weierstrass coefficients, obtaining the following equations. Details for the method are in Section 6.

$$\begin{aligned}
\epsilon_1 &= -q_3,\\
\epsilon_2 &= p_2^2+12p_2-q_2+28,\\
\epsilon_3 &= -3(2p_2+9)q_3-q_1+2p_1(p_2+5),\\
\epsilon_4 &= 9q_3^2-6p_1q_3-2q_2-q_0+8p_2^3+86p_2^2+2(p_0+156)p_2+p_1^2+10p_0+378,\\
\epsilon_5 &= (8q_2-20p_2^2-174p_2-7p_0-351)q_3-2p_1q_2+6(p_2+4)q_1\\
&\qquad +14p_1p_2^2+108p_1p_2+2(p_0+101)p_1,\\
\epsilon_6 &= 12(4p_2+15)q_3^2-(5q_1+38p_1p_2+140p_1)q_3+4q_2^2\\
&\qquad +(16p_2^2+96p_2-4p_0+155)q_2+2p_1q_1+3(4p_2+17)q_0+28p_2^4+360p_2^3\\
&\qquad +(4p_0+1765)p_2^2+2(4p_1^2+21p_0+1950)p_2+29p_1^2+p_0^2+88p_0+3276,\\
\epsilon_7 &= -36q_3^3+42p_1q_3^2+(4q_2-20q_0-56p_2^3-628p_2^2-14(p_0+168)p_2-16p_1^2\\
&\qquad -46p_0-2925)q_3+(3q_1+6p_1p_2+20p_1)q_2+(21p_2^2+162p_2-p_0+323)q_1\\
&\qquad +6p_1q_0+42p_1p_2^3+448p_1p_2^2+2(p_0+799)p_1p_2+2p_1^3+6(p_0+316)p_1.
\end{aligned}$$

Equating the two expressions we have obtained for each $\epsilon_i$, we get a system of seven equations, the first being

$$-\chi_7=-q_3.$$

We label these equations (1) through (7). The last few of these polynomial equations are somewhat complicated, and so to obtain a few simpler ones, we may consider the 126 sections of height 2, which we analyze as follows. Substituting

$$x = at^2 + bt + c,$$
$$y = dt^3 + et^2 + ft + g$$

into the Weierstrass equation, we get another system of equations:

$$\begin{aligned} a^3 &= d^2 + ad, \\ 3a^2 b &= (2d+a)e + bd, \\ a(p_2 + 3ac + 3b^2) &= (2d+a)f + e^2 + be + cd + 1, \\ q_3 + bp_2 + ap_1 + 6abc + b^3 &= (2d+a)g + (2e+b)f + ce, \\ q_2 + cp_2 + bp_1 + ap_0 + 3ac^2 + 3b^2 c &= (2e+b)g + f^2 + cf, \\ q_1 + cp_1 + bp_0 + 3bc^2 &= (2f+c)g, \\ q_0 + cp_0 + c^3 &= g^2. \end{aligned}$$

The specialization of such a section at $t = \infty$ is $1 + a/d$. Setting $d = ar$, we may as before eliminate other variables to obtain an equation of degree 126 for $r$. Substituting $r = 1/(u-1)$, we get a monic polynomial $\Psi_\lambda(X) = 0$ of degree 126 for $u$. Note that the roots are given by elements of the form

$$\begin{aligned} \frac{s_i}{r}, \frac{r}{s_i} \quad & \text{for } 1 \le i \le 7, \\ \frac{s_i}{s_j} \quad & \text{for } 1 \le i \ne j \le 7, \quad \text{and} \\ \frac{s_i s_j s_k}{r}, \frac{r}{s_i s_j s_k} \quad & \text{for } 1 \le i < j < k \le 7. \end{aligned}$$

As before, we can write the first few coefficients $\eta_i$ of $\Psi_\lambda$ in terms of $\lambda = (p_0, \dots, q_3)$, as well as in terms of the characters $\chi_j$, obtaining some more relations. We will only need the first two:

$$-\chi_1 + 7 = \eta_1 = -18 - 6p_2,$$
$$-6\chi_1 + \chi_3 + 28 = \eta_2 = p_0 + 72p_2 + 13p_2^2 - q_2 + 99$$

which we call (1′) and (2′), respectively.

Now we consider the system of six equations (1) through (4), (1′) and (2′). These may be solved for $(p_2, p_0, q_3, q_2, q_1, q_0)$ in terms of the $\chi_j$ and $p_1$. Substituting this solution into the other three relations (5), (6) and (7), we obtain three equations for $p_1$, of degrees 1, 2 and 3, respectively. These have a single common factor, linear in $p_1$, which we then solve. This gives us the proof of Theorem 6.

The proof of Theorem 5 is now straightforward. Part (1) asserts that the image of $\rho_\lambda$ is surjective on to $W(E_7)$. This follows from a standard Galois-theoretic argument: Let $F$ be the fixed field of $W(E_7)$ acting on $k_\lambda = \mathbb{Q}(\lambda)(s_1, \dots, s_6, r) = \mathbb{Q}(s_1, \dots, s_6, r)$, where the last equality follows from the explicit expression of $\lambda = (p_0, \dots, q_3)$ in terms of the $\chi_i$, which are in $\mathbb{Q}(s_1, \dots, s_6, r)$. Then we have that $k_0 \subset F$ since $p_0, \dots, q_3$ are polynomials in the $\chi_i$ with rational coefficients, and the $\chi_i$ are manifestly invariant under the Weyl group. Therefore $[k_\lambda : k_0] \geq [k_\lambda : F] = |W(E_7)|$, where the latter equality is from Galois theory. Finally, $[k_\lambda : k_0] \leq |\mathrm{Gal}(k_\lambda/k_0)| \leq |W(E_7)|$, since $\mathrm{Gal}(k_\lambda/k_0) \hookrightarrow W(E_7)$. Therefore, equality is forced.

Another way to see that the Galois group is the full Weyl group is to show it for a specialization, such as Example 8, and use [Serre 1989, Section 9.2, Proposition 2].

Next, let $Y$ be the toric hypersurface given by $s_1 \dots s_7 = r^3$. Its function field is the splitting field of $\Phi_\lambda(X)$, as we remarked above. We have seen that $\mathbb{Q}(Y)^{W(E_7)} = k_0 = \mathbb{Q}(\lambda)$. Since $\Phi_\lambda(X)$ is a monic polynomial with coefficients in $\mathbb{Q}[\lambda]$, we have that $\mathbb{Q}[Y]$ is integral over $\mathbb{Q}[\lambda]$. Therefore $\mathbb{Q}[Y]^{W(E_7)}$ is also integral over $\mathbb{Q}[\lambda]$, and contained in $\mathbb{Q}(Y)^{W(E_7)} = k_0 = \mathbb{Q}(\lambda)$. Since $\mathbb{Q}[\lambda]$ is a polynomial ring, it is integrally closed in its ring of fractions. Therefore $\mathbb{Q}[Y]^{W(E_7)} \subset \mathbb{Q}[\lambda]$.

We also know $\mathbb{Q}[\chi] = \mathbb{Q}[\chi_1, \dots, \chi_7] \subset \mathbb{Q}[Y]^{W(E_7)}$, since the $\chi_j$ are invariant under the Weyl group. Therefore, we have

$$\mathbb{Q}[\chi] \subset \mathbb{Q}[Y]^{W(E_7)} \subset \mathbb{Q}[\lambda]$$

and Theorem 6, which says $\mathbb{Q}[\chi] = \mathbb{Q}[\lambda]$, implies that all these three rings are equal. This completes the proof of Theorem 5.

**Remark 11.** This argument gives an independent proof of the fact that the ring of multiplicative invariants for $W(E_7)$ is a polynomial ring over $\chi_1, \dots, \chi_7$. See [Bourbaki 1968, Théorème VI.3.1 and Exemple 1] or [Lorenz 2005, Theorem 3.6.1] for the classical proof that the Weyl-orbit sums of a set of fundamental weights are a set of algebraically independent generators of the multiplicative invariant ring; from there to the fundamental characters is an easy exercise.

**Remark 12.** Now that we have found the explicit relation between the Weierstrass coefficients and the fundamental characters, we may go back and explore the "genericity condition" for this family to have Mordell–Weil lattice $E_7^*$. To do this, we compute the discriminant of the cubic in $x$, after completing the square in $y$, and take the discriminant with respect to $t$ of the resulting polynomial of degree 10. A computation shows that this discriminant factors as the cube of a polynomial $A(\lambda)$ (which vanishes exactly when the family has a fiber of additive reduction, generically type II), times a polynomial $B(\lambda)$, whose zero locus corresponds to the occurrence of a reducible multiplicative fiber. In fact, we calculate (for instance, by evaluating the split case) that $B(\lambda)$ is the product of $(e^\alpha - 1)$, where $\alpha$ runs over the 126 roots of $E_7$. We deduce by further analyzing the type II case that the condition to have Mordell–Weil lattice $E_7^*$ is that

$$\prod (e^\alpha - 1) = \Psi_\lambda(1) \neq 0.$$

Note that this is essentially the expression that occurs in Weyl's denominator formula. In addition, the condition for having only multiplicative fibers is that $\Psi_\lambda(1)$ and $A(\lambda)$ both be nonzero.

Finally, the proof of Theorem 9 follows immediately from the discussion in [Shioda 2010a; 2010b] — compare [Shioda 2010b, Section 2.3] for the additive reduction case.

## 5. The $E_8$ case

**5.1. *Results.*** Finally, we show a multiplicative excellent family for the Weyl group of $E_8$. It is given by the Weierstrass equation

$$y^2 = x^3 + t^2 x^2 + (p_0 + p_1 t + p_2 t^2)\, x + (q_0 + q_1 t + q_2 t^2 + q_3 t^3 + q_4 t^4 + t^5).$$

For generic $\lambda = (p_0, \dots, p_2, q_0, \dots, q_4)$, this rational elliptic surface $X_\lambda$ has no reducible fibers, only nodal $\mathrm{I}_1$ fibers at twelve points, including $t = \infty$. We will use the specialization map at $\infty$. The Mordell–Weil lattice $M_\lambda$ is isomorphic to the lattice $E_8$. Any rational elliptic surface with a multiplicative fiber of type $\mathrm{I}_1$ may be put in the form above (over a small degree algebraic extension of the base field), after a fractional linear transformation of $t$ and Weierstrass transformations of $x$, $y$.

**Lemma 13.** *The smooth part of the special fiber is isomorphic to the group scheme $\mathbb{G}_m$. The identity component is the nonsingular part of the curve $y^2 = x^3 + x^2$. The $x$- and $y$-coordinates of a section of height 2 are polynomials of degrees 2 and 3 respectively, and its specialization at $t = \infty$ may be taken as*

$$\lim_{t \to \infty} (y + tx)/(y - tx) \in k^*.$$

The proof of the lemma is similar to that in the $E_7$ case (and simpler!), and we omit it.

There are 240 sections of minimal height 2, with $x$-and $y$-coordinates of the form

$$x = gt^2 + at + b,$$
$$y = ht^3 + ct^2 + dt + e.$$

Under the identification with $\mathbb{G}_m$ of the special fiber of the Néron model, this section goes to $(h + g)/(h - g)$. Substituting the formulas above for $x$ and $y$ into the Weierstrass equation, we get the following system of equations.

$$\begin{aligned}
h^2 &= g^3 + g^2,\\
2ch &= 3ag^2 + 2ag + 1,\\
2dh + c^2 &= q_4 + gp_2 + 3bg^2 + (2b + 3a^2)g + a^2,\\
2eh + 2cd &= q_3 + ap_2 + gp_1 + 6abg + 2ab + a^3,\\
2ce + d^2 &= q_2 + bp_2 + ap_1 + gp_0 + 3b^2 g + b^2 + 3a^2 b,\\
2de &= q_1 + bp_1 + ap_0 + 3ab^2,\\
e^2 &= q_0 + bp_0 + b^3.
\end{aligned}$$

Setting $h = gu$, we eliminate other variables to obtain an equation of degree 240 for $u$. Finally, substituting in $u = (v+1)/(v-1)$, we get a monic reciprocal equation $\Phi_\lambda(X) = 0$ satisfied by $v$, with coefficients in $\mathbb{Z}[\lambda] = \mathbb{Z}[p_0, \dots, p_2, q_0, \dots, q_4]$. We have

$$\Phi_\lambda(X) = \prod_{i=1}^{240}(X - s(P)) = X^{240} + \epsilon_1 X^{239} + \dots + \epsilon_1 X + \epsilon_0,$$

where $P$ ranges over the 240 minimal sections of height 2. It is clear that $a, \dots, h$ are rational functions of $v$, with coefficients in $k_0$.

We have a Galois representation on the Mordell–Weil lattice

$$\rho_\lambda : \mathrm{Gal}(k/k_0) \to \mathrm{Aut}(M_\lambda) \cong \mathrm{Aut}(E_8).$$

Here $\mathrm{Aut}(E_8) \cong W(E_8)$, the Weyl group of type $E_8$. The *splitting field* of $M_\lambda$ is the fixed field $k_\lambda$ of $\mathrm{Ker}(\rho_\lambda)$. By definition, $\mathrm{Gal}(k_\lambda/k_0) \cong \mathrm{Im}(\rho_\lambda)$. The splitting field $k_\lambda$ is equal to the splitting field of the polynomial $\Phi_\lambda(X)$ over $k_0$, since the Mordell–Weil group is generated by the 240 sections of smallest height $P_i = (g_i t^2 + a_i t + b_i, h_i t^3 + c_i t^2 + d_i t + e_i)$. We also have

$$k_\lambda = k_0(P_1, \dots, P_{240}) = k_0(v_1, \dots, v_{240}).$$

**Theorem 14.** *Assume that $\lambda$ is generic over $\mathbb{Q}$, that is, the coordinates $p_0, \dots, q_4$ are algebraically independent over $\mathbb{Q}$.*

(1) *$\rho_\lambda$ induces an isomorphism* $\mathrm{Gal}(k_\lambda/k_0) \cong W(E_8)$.

(2) *The splitting field $k_\lambda$ is a purely transcendental extension of $\mathbb{Q}$, and is isomorphic to the function field $\mathbb{Q}(Y)$ of the toric hypersurface $Y \subset \mathbb{G}_m^9$ defined by $s_1 \cdots s_8 = r^3$. There is an action of $W(E_8)$ on $Y$ such that $\mathbb{Q}(Y)^{W(E_8)} = k_\lambda^{W(E_8)} = k_0$.*

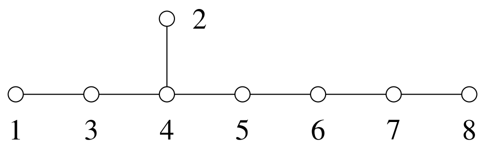
(3) *The ring of $W(E_8)$-invariants in the affine coordinate ring*

$$\mathbb{Q}[Y] = \mathbb{Q}[s_i, r, 1/s_i, 1/r]/(s_1 \dots s_8 - r^3) \cong \mathbb{Q}[s_1, \dots, s_7, r, s_1^{-1}, \dots, s_7^{-1}, r^{-1}]$$

*is equal to the polynomial ring* $\mathbb{Q}[\lambda]$:

$$\mathbb{Q}[Y]^{W(E_8)} = \mathbb{Q}[\lambda] = \mathbb{Q}[p_0, p_1, p_2, q_0, q_1, q_2, q_3, q_4].$$

As in the $E_7$ case, we prove an explicit, invertible polynomial relation between the Weierstrass coefficients $\lambda$ and the fundamental characters for $E_8$. Let $V_1, \dots, V_8$ be the fundamental representations of $E_8$, and $\chi_1, \dots, \chi_8$ their characters as labeled below.




Again, for the set of generators of $E_8$, we choose (as in [Shioda 1995]) vectors $v_1, \dots, v_8, u$ with $\sum v_i = 3u$ and let $s_i$ correspond to $v_i$ and $r$ to $u$, so that $\prod s_i = r^3$. The 240 roots of $\Phi_\lambda(X)$ are given by

$$s_i, \frac{1}{s_i} \quad \text{for } 1 \le i \le 8, \qquad \frac{s_i}{s_j} \quad \text{for } 1 \le i \ne j \le 8,$$

$$\frac{s_i s_j}{r}, \frac{r}{s_i s_j} \quad \text{for } 1 \le i < j \le 8, \quad \text{and} \quad \frac{s_i s_j s_k}{r}, \frac{r}{s_i s_j s_k} \quad \text{for } 1 \le i < j < k \le 8.$$

The characters $\chi_1, \dots, \chi_7$ lie in the ring of Laurent polynomials $\mathbb{Q}[s_i, r, 1/s_i, 1/r]$, and are invariant under the multiplicative action of the Weyl group on this ring of Laurent polynomials. The $\chi_i$ may be explicitly computed using the software `LiE`, as indicated in Section 7 and the auxiliary files.

**Theorem 15.** *For generic $\lambda$ over $\mathbb{Q}$, we have*

$$\mathbb{Q}[\chi_1, \dots, \chi_8] = \mathbb{Q}[p_0, p_1, p_2, q_0, q_1, q_2, q_3, q_4].$$

*The transformation between these sets of generators is*

$$\chi_1 = -1600 q_4 + 1536 p_2 + 3875,$$

$$\chi_2 = 2(-45600 q_4 + 12288 q_3 + 40704 p_2 - 16384 p_1 + 73625),$$

$$\chi_3 = 64(14144 q_4^2 - 72(384 p_2 + 1225) q_4 + 11200 q_3 - 4096 q_2 + 13312 p_2^2 + 87072 p_2 - 17920 p_1 + 16384 p_0 + 104625),$$

$$
\begin{aligned}
\chi_4 &= -91750400q_4^3 + 12288(25600p_2 + 222101)q_4^2 - 256(4530176q_3 - 65536q_2 \\
&\quad + 1392640p_2^2 + 21778944p_2 - 8159232p_1 + 2621440p_0 + 34773585)q_4 \\
&\quad + 32(4718592q_3^2 + 384(80896p_2 - 32768p_1 + 225379)q_3 - 29589504q_2 \\
&\quad + 30408704q_1 - 33554432q_0 + 4194304p_2^3 + 88129536p_2^2 \\
&\quad - 64(876544p_1 - 262144p_0 - 4399923)p_2 + 8388608p_1^2 - 133996544p_1 \\
&\quad + 65175552p_0 + 215596227), \\
\chi_5 &= 24760320q_4^2 - 64(106496q_3 + 738816p_2 - 163840p_1 + 2360085)q_4 \\
&\quad + 12288(512p_2 + 4797)q_3 - 30670848q_2 + 16777216q_1 + 20250624p_2^2 \\
&\quad - 512(16384p_1 - 235911)p_2 - 45154304p_1 + 13631488p_0 + 146325270, \\
\chi_6 &= 110592q_4^2 - 1536(128p_2 + 1235)q_4 + 724992q_3 - 262144q_2 + 65536p_2^2 \\
&\quad + 1062912p_2 - 229376p_1 + 2450240, \\
\chi_7 &= -4(3920q_4 - 1024q_3 - 1152p_2 - 7595), \\
\chi_8 &= -8(8q_4 - 31).
\end{aligned}
$$

**Remark 16.** We omit the inverse for conciseness here; it is easily computed in a computer algebra system and is available in the auxiliary files.

**Remark 17.** As before, our explicit formulas are compatible with those in [Eguchi and Sakai 2003]. Also, the proof of Theorem 14 gives another proof that the multiplicative invariants for $W(E_8)$ are freely generated by the fundamental characters (or by the orbit sums of the fundamental weights).

**Example 18.** Let $\mu = (2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19) = 9699690$. Then

$$
\begin{aligned}
q_4 &= -224337445655936683 4339/(2^5\cdot\mu^2), \\
q_3 &= 43080034312940338834622651824607856 7/(2^{11}\cdot\mu^3), \\
q_2 &= 725551019476490111273917330349841584625731464099057 69/(2^{22}\cdot 3^2\cdot\mu^4), \\
q_1 &= (-1288109930551729133820743237846836849158406377255698116491924 5 \\
&\qquad 30489)/(2^{29}\cdot 3\cdot\mu^5), \\
q_0 &= (882717679332361992942730338148545940191119188371968387097504232 83 \\
&\qquad 443360357992650203)/(2^{42}\cdot 3^3\cdot\mu^6), \\
p_2 &= 1461567739038798710018105 89/(2^9\cdot 3\cdot\mu^2), \\
p_1 &= -2490980504156786698546937977968536001931 3/(2^{20}\cdot\mu^3), \\
p_0 &= 14921071761102637668643191215755039801471771138867387/(2^{23}\cdot 3\cdot\mu^4).
\end{aligned}
$$

These values give an elliptic surface for which we have $r = 2$, $s_1 = 3$, $s_2 = 5$, $s_3 = 7$, $s_4 = 11$, $s_5 = 13$, $s_6 = 17$, $s_7 = 19$, the simplest choice of multiplicatively independent elements. Here, the specializations of a basis are given by $v \in \{3, 5, 7, 11, 13, 17, 19, 15/2\}$. Once again, we list the $x$-coordinates of the corresponding sections, and leave the rest of the verification to the auxiliary files.

$$x(P_1) = 3t^2 - \frac{99950606190359}{620780160}t + \frac{432532755764748812020964981 3}{2642523476911718400},$$

$$x(P_2) = \frac{5}{4}t^2 - \frac{153332163637781}{1655413760}t + \frac{5414114237697608646836821}{5138596941004800},$$

$$x(P_3) = \frac{7}{9}t^2 - \frac{203120672689603}{2793510720}t + \frac{69431643485691306367886386 39}{7927570430735155200},$$

$$x(P_4) = \frac{11}{25}t^2 - \frac{8564057914757}{147804800}t + \frac{1151263722336758003966009 89}{155442557465395200},$$

$$x(P_5) = \frac{13}{36}t^2 - \frac{347479008951469}{6385167360}t + \frac{157133607680949617374030405417}{22197197206058434560 0},$$

$$x(P_6) = \frac{17}{64}t^2 - \frac{1327421017414859}{26486620160}t + \frac{594241929293302141845751730 3}{8901131711702630400},$$

$$x(P_7) = \frac{19}{81}t^2 - \frac{489830985359431}{10056638592}t + \frac{466851372017436964414774549 51}{7134813387661639680 0},$$

$$x(P_8) = \frac{120}{169}t^2 - \frac{30706596009257}{440806080}t + \frac{761644430748287436621654664 09}{5582330844976005120 0}.$$

**Example 19.** The value $\lambda = \lambda_0 := (1, 1, 1, 1, 1, 1, 1, 1)$ gives rise to an explicit polynomial $g(X) = \Phi_{\lambda_0}(X)$, for which we can show that the Galois group is $W(E_8)$, as follows. The reduction of $g(X)$ modulo 79 shows that $\mathrm{Frob}_{79}$ has cycle decomposition of type $(4)^2(8)^{29}$, and similarly, $\mathrm{Frob}_{179}$ has cycle decomposition of type $(15)^{16}$. We deduce, as in [Jouve et al. 2008, Section 3] or [Shioda 2009], that the Galois group is the entire Weyl group. Since the coefficients of $g(X)$ are large, we do not display it here, but it is included in the auxiliary files.

As in the case of $E_7$, we can also describe degenerations of this family of rational elliptic surfaces $X_\lambda$ by the method of "vanishing roots", where we drop the genericity assumption, and consider the situation where the elliptic fibration might have additional reducible fibers. Let $\psi : Y \to \mathbb{A}^8$ be the finite surjective morphism associated to

$$\mathbb{Q}[p_0, \dots, q_4] \hookrightarrow \mathbb{Q}[Y] \cong \mathbb{Q}[s_1, \dots, s_7, r, s_1^{-1}, \dots, s_7^{-1}, r^{-1}].$$

For $\xi = (s_1, \dots, s_8, r) \in Y$, let the multiset $\Pi_\xi$ consist of the 240 elements $s_i$ and $1/s_i$ for $1 \le i \le 8$, $s_i/s_j$ for $1 \le i \ne j \le 8$, $s_i s_j/r$ and $r/(s_i s_j)$ for $1 \le i < j \le 8$, and $s_i s_j s_k/r$ and $r/(s_i s_j s_k)$ for $1 \le i < j < k \le 8$, corresponding to the 240 roots of $E_8$. Let $2\nu(\xi)$ be the number of times 1 appears in $\Pi_\xi$, which is also the

| Type | Fibral lattice | MW group | $\{s_1, \ldots, s_6, r\}$ |
|---|---|---|---|
| 1 | 0 | $E_8$ | 3, 5, 7, 11, 13, 17, 19, 2 |
| 5 | $A_3$ | $D_5^*$ | 2, 2, 2, 2, 5, 7, 11, 3 |
| 8 | $A_4$ | $A_4^*$ | 2, 2, 2, 2, 2, 5, 7, 3 |
| 15 | $A_5$ | $A_2^* \oplus A_1^*$ | 2, 2, 2, 2, 2, 2, 5, 3 |
| 16 | $D_5$ | $A_3^*$ | 2, 3, 3, 3, 3, 3, 5, 18 |
| 25 | $A_6$ | $\frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$ | 2, 2, 2, 2, 2, 2, 2, 3 |
| 26 | $D_6$ | $A_1^{*2}$ | 2, 3, 3, 3, 3, 3, 3, 18 |
| 35 | $A_3^2$ | $A_1^{*2} \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, −1/2, 3, 3, 3, 1, 1, −3 |
| 36 | $A_3^2$ | $\langle 1/4 \rangle$ | 8, 8, 8, 8, 27, 27, 27, 1296 |
| 43 | $E_7$ | $A_1^*$ | 2, 2, 2, 2, 2, 2, 2, 8 |
| 44 | $A_7$ | $A_1^* \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, 2, 2, 2, 2, 2, 2, −8 |
| 45 | $A_7$ | $\langle 1/8 \rangle$ | 8, 8, 8, 8, 8, 8, 8, 256 |
| 46 | $D_7$ | $\langle 1/4 \rangle$ | 2, 4, 4, 4, 4, 4, 4, 32 |
| 54 | $A_3 \oplus D_4$ | $\langle 1/4 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$ | 2, −1, −1, −1, −1, 1, 1, 2 |
| 55 | $A_3 \oplus A_4$ | $\langle 1/20 \rangle$ | 16, 16, 16, 16, 32, 32, 32, 4096 |
| 62 | $E_8$ | 0 | 1, 1, 1, 1, 1, 1, 1, 1 |
| 63 | $A_8$ | $\mathbb{Z}/3\mathbb{Z}$ | 1, 1, 1, 1, 1, 1, 1, $\zeta_3$ |
| 64 | $D_8$ | $\mathbb{Z}/2\mathbb{Z}$ | 1, 1, 1, 1, 1, 1, 1, −1 |
| 67 | $A_4^2$ | $\mathbb{Z}/5\mathbb{Z}$ | 1, 1, 1, 1, $\zeta_5$, $\zeta_5$, $\zeta_5$, $\zeta_5^3$ |
| 72 | $A_3 \oplus D_5$ | $\mathbb{Z}/4\mathbb{Z}$ | 1, 1, 1, $I$, $I$, $I$, $I$, $-I$ |

**Table 2.** Examples of specializations of the $E_8$ family (types are from [Oguiso and Shioda 1991]).

multiplicity of 1 as a root of $\Phi_\lambda(X)$, with $\lambda = \psi(\xi)$. We call the associated roots of $E_8$ the *vanishing roots*, in analogy with vanishing cycles in the deformation of singularities. By abuse of notation we label the rational elliptic surface $X_\lambda$ as $X_\xi$.

**Theorem 20.** *The surface $X_\xi$ has new reducible fibers* (*necessarily at $t \neq \infty$*) *if and only if $\nu(\xi) > 0$. The number of roots in the root lattice $T_{\text{new}}$ is equal to $2\nu(\xi)$, where $T_{\text{new}} := \bigoplus_{v \neq \infty} T_v$ is the new part of the trivial lattice.*

We may use this result to produce specializations with trivial lattice corresponding to most of the entries of [Oguiso and Shioda 1991], and a nodal fiber. We list below those types which are not already covered by [Shioda 1991a; 2012] or our examples for the $E_7$ case, which have an $\mathrm{I}_2$ fiber.

Here $\zeta_3$, $I$ and $\zeta_5$ are primitive third, fourth and fifth roots of unity.

**Remark 21.** As before, for the examples in lines 63, 67 and 72 of the table, one can show it is not possible to define a rational elliptic surface over $\mathbb{Q}$ in the form we have assumed, such that all the specializations $s_i$ and $r$ are rational. However, there do exist examples with all sections defined over $\mathbb{Q}$, not in the chosen Weierstrass form.

The surface with Weierstrass equation

$$y^2 + xy + t^3 y = x^3$$

has a 3-torsion point $(0, 0)$ and a fiber of type $\mathrm{I}_9$. It is an example of type 63.

The surface with Weierstrass equation

$$y^2 + (t+1)xy + ty = x^3 + tx^2$$

has a 5-torsion section $(0, 0)$ and two fibers of type $\mathrm{I}_5$. It is an example of type 67.

The surface with Weierstrass equation

$$y^2 + txy + \frac{t^2(t-1)}{16} y = x^3 + \frac{t(t-1)}{16} x^2$$

has a 4-torsion section $(0, 0)$, and two fibers of types $\mathrm{I}_4$ and $\mathrm{I}_1^*$. It is an example of type 72.

**Remark 22.** Our tables and the one in [Shioda 2012] cover all the cases of [Oguiso and Shioda 1991], except lines 9, 27 and 73 of the table, with trivial lattice $D_4$, $E_6$ and $D_4^2$, respectively. Since these have fibers with additive reduction, examples for them may be directly constructed using the families in [Shioda 1991a]. For instance, the elliptic surface

$$y^2 = x^3 - xt^2$$

has two fibers of type $\mathrm{I}_0^*$ and Mordell–Weil group $(\mathbb{Z}/2\mathbb{Z})^2$. This covers line 73 of the table. For the other two cases, we refer the reader to the original examples of additive reduction in [Shioda 1991a, Section 3].

**5.2. *Proofs.*** The proof proceeds analogously to the $E_7$ case, with two differences: We only have one polynomial $\Phi_\lambda(X)$ to work with (as opposed to having $\Phi_\lambda(X)$ and $\Psi_\lambda(X)$), and the equations are a lot more complicated.

We first write down the relation between the coefficients $\epsilon_i$ for $1 \le i \le 9$, and the fundamental invariants $\chi_j$; as before, we postpone the proofs to the auxiliary files and outline the idea in Section 7. Second, we write down the coefficients $\epsilon_i$ in terms of $\lambda = (p_0, \dots, p_2, q_0, \dots, q_4)$; see Section 6 for an explanation of how this is carried out. In the interest of brevity, we do not write out either of these sets of equations, but relegate them to the auxiliary computer files. Finally, setting the corresponding expressions equal to each other, we obtain a set of equations (1) through (9).

To solve these equations, proceed as follows: first use (1) through (5) to solve for $q_0, \dots, q_4$ in terms of $\chi_j$ and $p_0, p_1, p_2$. Substituting these in to the remaining equations, we obtain (6′) through (9′). These have low degree in $p_0$, which we eliminate, obtaining equations of relatively small degrees in $p_1$ and $p_2$. Finally, we take resultants with respect to $p_1$, obtaining two equations for $p_2$, of which the only common root is the one listed above. Working back, we solve for all the other variables, obtaining the system above and completing the proof of Theorem 15. The deduction of Theorem 14 now proceeds exactly as in the case of $E_7$.

**Remark 23.** As in the $E_7$ case, once we find the explicit relation between the Weierstrass coefficients and the fundamental characters, we may go back and explore the "genericity condition" for this family to have Mordell–Weil lattice isomorphic to $E_8$. To do this we compute the discriminant of the cubic in $x$, after completing the square in $y$, and take the discriminant with respect to $t$ of the resulting polynomial of degree 11. A computation shows that this discriminant factors as the cube of a polynomial $A(\lambda)$ (which vanishes exactly when the family has a fiber of additive reduction, generically type II), and the product of $(e^\alpha - 1)$, where $\alpha$ runs over minimal vectors of $E_8$. Again, the genericity condition to have Mordell–Weil lattice exactly $E_8$ is just the nonvanishing of

$$\Phi_\lambda(1) = \prod (e^\alpha - 1),$$

the expression which occurs in the Weyl denominator formula. Furthermore, the condition to have only multiplicative fibers is that $\Phi_\lambda(1)A(\lambda) \neq 0$.

As before, the proof of Theorem 20 follows immediately from the results of [Shioda 2010a; 2010b], by degeneration from a flat family.

## 6. Resultants, interpolation and computations

We now explain a computational aid, used in obtaining the equations expressing the coefficients of $\Phi_\lambda$ (for $E_8$) or $\Psi_\lambda$ (for $E_7$) in terms of the Weierstrass coefficients of the associated family of rational elliptic surfaces. We illustrate this using the system of equations obtained for sections of the $E_8$ family:

$$\begin{aligned}
h^2 &= g^3 + g^2, \\
2ch &= 3ag^2 + 2ag + 1, \\
c^2 + 2dh &= q_4 + gp_2 + 3bg^2 + (2b + 3a^2)g + a^2, \\
2eh + 2cd &= q_3 + ap_2 + gp_1 + 6abg + 2ab + a^3, \\
2ce + d^2 &= q_2 + bp_2 + ap_1 + gp_0 + 3b^2g + b^2 + 3a^2b, \\
2de &= q_1 + bp_1 + ap_0 + 3ab^2, \\
e^2 &= q_0 + bp_0 + b^3.
\end{aligned}$$

Setting $h = gu$ and solving the first equation for $g$, we have $g = u^2 - 1$. We solve the next three equations for $c$, $d$, $e$, respectively. This leaves us with three equations $R_1(a, b, u) = R_2(a, b, u) = R_3(a, b, u) = 0$. These have degrees 2, 2, 3 respectively in $b$. Taking the appropriate linear combination of $R_1$ and $R_2$ gives us an equation $S_1(a, b, u) = 0$ which is linear in $b$. Similarly, we may use $R_1$ and $R_3$ to obtain another equation $S_2(a, b, u) = 0$, linear in $b$. We write

$$\begin{aligned} S_1(a, b, u) &= s_{11}(a, u)b + s_{10}(a, u), \\ S_2(a, b, u) &= s_{21}(a, u)b + s_{20}(a, u), \\ R_1(a, b, u) &= r_2(a, u)b^2 + r_1(a, u)b + r_0(a, u). \end{aligned}$$

The resultant of the first two polynomials gives us an equation

$$T_1(a, u) = s_{11}s_{20} - s_{10}s_{21} = 0,$$

while the resultant of the first and third gives us

$$T_2(a, u) = r_2 s_{10}^2 - r_1 s_{10} s_{11} + r_0 s_{11}^2 = 0.$$

Finally, we substitute $u = (v + 1)/(v - 1)$ throughout, obtaining two equations $\tilde{T}_1(a, v) = 0$ and $\tilde{T}_2(a, v) = 0$.

Next, we would like to compute the resultant of $\tilde{T}_1(a, v)$ and $\tilde{T}_2(a, v)$, which have degrees 8 and 9 with respect to $a$, to obtain a single equation satisfied by $v$. However, the polynomials $\tilde{T}_1$ and $\tilde{T}_2$ are already fairly large (they take a few hundred kilobytes of memory), and since their degree in $a$ is not too small, it is beyond the current reach of computer algebra systems such as `gp/PARI` or `Magma` to compute their resultant. It would take too long to compute their resultant, and another issue is that the resultant would take too much memory to store, certainly more than is available on the authors' computer systems (it would take more than 16GB of memory).

To circumvent this issue, what we shall do is to use several specializations of $\lambda$ in $\mathbb{Q}^8$. Once we specialize, the polynomials take much less space to store, and the computations of the resultants becomes tremendously easier. Since the resultant can be computed via the Sylvester determinant

$$\begin{vmatrix} a_8 & \dots & a_2 & a_1 & a_0 & 0 & 0 & \dots & 0 \\ 0 & a_8 & \dots & a_2 & a_1 & a_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_8 & \dots & a_2 & a_1 & a_0 & 0 \\ 0 & \dots & 0 & 0 & a_8 & \dots & a_2 & a_1 & a_0 \\ b_9 & b_8 & \dots & b_2 & b_1 & b_0 & 0 & \dots & 0 \\ 0 & b_9 & b_8 & \dots & b_2 & b_1 & b_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_9 & b_8 & \dots & b_2 & b_1 & b_0 \end{vmatrix},$$

where $\tilde{T}_1(a, v) = \sum a_i(v)a^i$ and $\tilde{T}_2(a, v) = \sum b_i(v)a^i$, we see that the resultant is a polynomial $Z(v) = \sum z_i v^i$ with coefficients $z_i$ being polynomials in the coefficients of the $a_i$ and the $b_j$, which happen to be elements of $\mathbb{Q}[\lambda]$ (recall that $\lambda = (p_0, \dots, p_2, q_0, \dots, q_4)$). Furthermore, we can bound the degrees $m_i(j)$ of $z_i(v)$ with respect to the $j$-th coordinate of $\lambda$, by using explicit bounds on the multidegrees of the $a_i$ and $b_i$. Therefore, by using Lagrange interpolation (with respect to the eight variables $\lambda_j$) we can reconstruct $z_i(v)$ from its specializations for various values of $\lambda$. The same method lets us show that $Z(v)$ is divisible by $v^{22}$ (for instance, by showing that $z_0$ through $z_{21}$ are zero), and also by $(v+1)^{80}$ (by first shifting $v$ by 1 and then computing the Sylvester determinant, and proceeding as before), as well as by $(v^2+v+1)^8$ (this time, using cube roots of unity). Finally, it is clear that $Z(v)$ is divisible by the square of the resultant $G(v)$ of $s_{11}$ and $s_{10}$ with respect to $a$. Removing these extraneous factors, we get a polynomial $\Phi_\lambda(v)$ that is monic and reciprocal of degree 240. We compute its top few coefficients by this interpolation method.

Finally, the interpolation method above is in fact completely rigorous. Namely, let $\epsilon_i(\lambda)$ be the coefficient of $v^i$ in $\Phi_\lambda(v)$, with bounds $(m_1, \dots, m_8)$ for its multidegree, and $\epsilon'_i(\lambda)$ the putative polynomial we have computed using Lagrange interpolation on a set $L_1 \times \cdots \times L_8$, where $L_i = \{\ell_{i,0}, \dots, \ell_{i,m_i}\}$ for $1 \le i \le 8$ are sets of integers chosen generically enough to ensure that $G(v)$ has the correct degree and that $Z(v)$ is not divisible by any higher powers of $v$, $v+1$ or $v^2+v+1$ than in the generic case. Then since $\epsilon_j(\ell_{1,i_1}, \dots, \ell_{8,i_8}) = \epsilon'_j(\ell_{1,i_1}, \dots, \ell_{8,i_8})$ for all choices of $i_1, \dots, i_8$, we see that the difference of these polynomials must vanish.

## 7. Representation theory, and some identities in Laurent polynomials

Finally, we demonstrate how to deduce the identities relating the coefficients of $\Phi_{E_7,\lambda}(X)$ or $\Psi_{E_7,\lambda}(X)$ to the fundamental characters for $E_7$ (and similarly, the coefficients of $\Phi_{E_8,\lambda(X)}$ to the fundamental characters of $E_8$).

Conceptually, the simplest way to do this is to express the alternating powers of the 56-dimensional representation $V_7$ or the 133-dimensional representation $V_1$ in terms of the fundamental modules of $E_7$ and their tensor products. We know that the character $\chi_1$ of $V_1$ is $7 + \sum e^\alpha$, where the sum is over the 126 roots of $E_7$. Therefore we have $(-1)\eta_1 = \chi_1 - 7$. For the next example, we consider $\bigwedge^2 V_1 = V_3 \oplus V_1$. This gives rise to the equation

$$\eta_2 + 7 \cdot (-1)\eta_1 + \binom{7}{2} = \chi_3 + \chi_1,$$

which gives the relation $\eta_2 = \chi_3 - 6\chi_1 + 28$.

A similar analysis can be carried out to obtain all the other identities used in our proofs, using the software `LiE` [LiE 2000].

A more explicit method is to compute the expressions for the $\chi_i$ as Laurent polynomials in $s_1, \dots, s_6, r$ (note that $s_7 = r^3/(s_1 \dots s_6)$), and then do the same for the $\epsilon_i$ or $\eta_i$. The latter calculation is simplified by computing the power sums $\sum (e^\alpha)^i$ (for $\alpha$ running over the smallest vectors of $E_7^*$ or $E_7$), for $1 \le i \le 7$ and then using Newton's formulas to convert to the elementary symmetric polynomials, which are $(-1)^i \epsilon_i$ or $(-1)^i \eta_i$. Finally, we check the identities by direct computation in the Laurent polynomial ring (it may be helpful to clear out denominators). This method has the advantage that we obtain explicit expressions for the $\chi_i$ (and then for $\lambda$ by Theorem 6) in terms of $s_1, \dots, s_6, r$, which may then be used to generate examples such as Example 7.

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The auxiliary computer files for checking our calculations are available from the source files in the preprint version of this paper, [Kumar and Shioda 2013].

## References

[Bourbaki 1968] N. Bourbaki, *Groupes et algèbres de Lie, Chapitres IV, V, et VI*, Actualités Scientifiques et Industrielles **1337**, Hermann, Paris, 1968. MR 39 #1590 Zbl 0483.22001

[Carter 2005] R. W. Carter, *Lie algebras of finite and affine type*, Cambridge Studies in Advanced Mathematics **96**, Cambridge University Press, Cambridge, 2005. MR 2006i:17001 Zbl 1110.17001

[Eguchi and Sakai 2003] T. Eguchi and K. Sakai, "Seiberg–Witten curve for $E$-string theory revisited", *Adv. Theor. Math. Phys.* **7**:3 (2003), 419–455. MR 2005g:81234 Zbl 1065.81105

[Jouve et al. 2008] F. Jouve, E. Kowalski, and D. Zywina, "An explicit integral polynomial whose splitting field has Galois group $W(E_8)$", *J. Théor. Nombres Bordeaux* **20**:3 (2008), 761–782. MR 2010e:12005 Zbl 1200.12003

[Kodaira 1963a] K. Kodaira, "On compact analytic surfaces, II", *Ann. of Math.* (2) **77**:3 (1963), 563–626. MR 32 #1730 Zbl 0118.15802

[Kodaira 1963b] K. Kodaira, "On compact analytic surfaces, III", *Ann. of Math.* (2) **78**:1 (1963), 1–40. MR 32 #1730 Zbl 0171.19601

[Kubert 1976] D. S. Kubert, "Universal bounds on the torsion of elliptic curves", *Proc. London Math. Soc.* (3) **33**:2 (1976), 193–237. MR 55 #7910 Zbl 0331.14010

[Kumar and Shioda 2013] A. Kumar and T. Shioda, "Multiplicative excellent families of elliptic surfaces of type $E_7$ or $E_8$", preprint, version 2, 2013. arXiv 1204.1531v2

[LiE 2000] M. A. A. van Leeuwen, A. M. Cohen, and B. Lisser, *LiE: A computer algebra package for Lie group computations, version 2.2.2*, 2000, available at http://www-math.univ-poitiers.fr/~maavl/LiE/.

[Lorenz 2005] M. Lorenz, *Multiplicative invariant theory*, Encyclopaedia of Mathematical Sciences **135**, Springer, Berlin, 2005. MR 2005m:13012 Zbl 1078.13003

[Oguiso and Shioda 1991] K. Oguiso and T. Shioda, "The Mordell–Weil lattice of a rational elliptic surface", *Comment. Math. Univ. St. Paul.* **40**:1 (1991), 83–99. MR 92g:14036 Zbl 0757.14011

[Serre 1989] J.-P. Serre, *Lectures on the Mordell–Weil theorem*, edited by M. Brown, Aspects of Mathematics **E15**, Vieweg, Braunschweig, 1989. MR 90e:11086 Zbl 0676.14005

[Shioda 1990] T. Shioda, "On the Mordell–Weil lattices", *Comment. Math. Univ. St. Paul.* **39**:2 (1990), 211–240. MR 91m:14056 Zbl 0725.14017

[Shioda 1991a] T. Shioda, "Construction of elliptic curves with high rank via the invariants of the Weyl groups", *J. Math. Soc. Japan* **43**:4 (1991), 673–719. MR 92i:11059 Zbl 0751.14018

[Shioda 1991b] T. Shioda, "Theory of Mordell–Weil lattices", pp. 473–489 in *Proceedings of the International Congress of Mathematicians* (Kyoto, 1990), vol. 1, edited by I. Satake, Math. Soc. Japan, Tokyo, 1991. MR 93k:14046 Zbl 0746.14009

[Shioda 1995] T. Shioda, "A uniform construction of the root lattices $E_6$, $E_7$, $E_8$ and their dual lattices", *Proc. Japan Acad. Ser. A Math. Sci.* **71**:7 (1995), 140–143. MR 97e:17015 Zbl 0854.17008

[Shioda 2009] T. Shioda, "Some explicit integral polynomials with Galois group $W(E_8)$", *Proc. Japan Acad. Ser. A Math. Sci.* **85**:8 (2009), 118–121. MR 2011d:12003 Zbl 05651158

[Shioda 2010a] T. Shioda, "Gröbner basis, Mordell–Weil lattices and deformation of singularities, I", *Proc. Japan Acad. Ser. A Math. Sci.* **86**:2 (2010), 21–26. MR 2011e:14068 Zbl 1183.14050

[Shioda 2010b] T. Shioda, "Gröbner basis, Mordell–Weil lattices and deformation of singularities, II", *Proc. Japan Acad. Ser. A Math. Sci.* **86**:2 (2010), 27–32. MR 2011e:14069 Zbl 1186.14039

[Shioda 2012] T. Shioda, "Multiplicative excellent family of type $E_6$", *Proc. Japan Acad. Ser. A Math. Sci.* **88**:3 (2012), 46–51. MR 2908623 Zbl 06051465

[Shioda and Usui 1992] T. Shioda and H. Usui, "Fundamental invariants of Weyl groups and excellent families of elliptic curves", *Comment. Math. Univ. St. Paul.* **41**:2 (1992), 169–217. MR 93m:11047 Zbl 0815.14027

[Tate 1975] J. Tate, "Algorithm for determining the type of a singular fiber in an elliptic pencil", pp. 33–52 in *Modular functions of one variable, IV: Proceedings of the International Summer School* (Antwerp, 1972), edited by B. J. Birch and W. Kuyk, Lecture Notes in Math. **476**, Springer, Berlin, 1975. MR 52 #13850 Zbl 1214.14020

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