Algebra & Number Theory

Volume 7

2013

No. 8

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ALGEBRA AND NUMBER THEORY 7:8 (2013)
dx.doi.org/10.2140/ant.2013.7.1977

The p-adic monodromy theorem in the imperfect residue field case

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Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ and G_K the absolute Galois group of K. In this paper, we will prove the p-adic monodromy theorem for p-adic representations of G_K without any assumption on the residue field of K, for example the finiteness of a p-basis of the residue field of K. The main point of the proof is a construction of (φ, G_K)-module $\widetilde{\mathbb{N}}^{\nabla +}_{\mathrm{rig}}(V)$ for a de Rham representation V, which is a generalization of Pierre Colmez's $\widetilde{\mathbb{N}}^{+}_{\mathrm{rig}}(V)$. In particular, our proof is essentially different from Kazuma Morita's proof in the case when the residue field admits a finite p-basis.

We also give a few applications of the p-adic monodromy theorem, which are not mentioned in the literature. First, we prove a horizontal analogue of the p-adic monodromy theorem. Secondly, we prove an equivalence of categories between the category of horizontal de Rham representations of G_K and the category of de Rham representations of an absolute Galois group of the canonical subfield of K. Finally, we compute H^1 of some p-adic representations of G_K, which is a generalization of Osamu Hyodo's results.

The author is supported by the Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

MSC2010: primary 11F80; secondary 11F85, 11S15, 11S20, 11S25.

Keywords: p-adic Hodge theory, p-adic representations.

Introduction

Let p be a prime and K a complete discrete valuation field of mixed characteristic $(0, p)$ with residue field k_K. Let G_K be the absolute Galois group of K. When k_K is perfect, Jean-Marc Fontaine defined the notions of crystalline, semistable, de Rham, Hodge–Tate representations for p-adic representations of G_K (see [Fontaine 1994a; 1994b] for example). The p-adic monodromy conjecture, which asserts that de Rham representations are potentially semistable, was first proved by Laurent Berger [2002, Théorème 0.7] by using the theory of p-adic differential equations. Precisely speaking, Berger used the p-adic local monodromy theorem for p-adic differential equations with Frobenius structure due to Yves André, Zoghman Mebkhout, and Kiran Kedlaya.

The notions of the above categories of representations were defined by Olivier Brinon [2006] when k_K admits a finite p-basis. In this case, the p-adic monodromy theorem was proved by Kazuma Morita [2011, Corollary 1.2]. Roughly speaking, he proved the p-adic monodromy theorem by studying some differential equations, which are defined by Sen's theory of $\mathbb{B}_{\mathrm{dR}}$ due to Fabrizio Andreatta and Olivier Brinon [2010]. In that reference, Tate–Sen formalism for a quotient Γ_K of G_K is applied to establish Sen's theory of $\mathbb{B}_{\mathrm{dR}}$, where Γ_K is isomorphic to an open subgroup of $\mathbb{Z}_p^\times \ltimes \mathbb{Z}_p(1)^{J_K}$ with $J_K := \dim_{k_K} \Omega^1_{k_K/\mathbb{Z}} < \infty$. To prove Tate–Sen formalism, we iteratively use analogues of the normalized trace map due to John Tate. Hence, we can not use Morita's approach in the case $J_K = \infty$.

Our main theorem in this paper is the p-adic monodromy theorem without any assumption on the residue field k_K. We also give the following applications of the p-adic monodromy theorem, which are not mentioned in the literature: First, we will prove a horizontal analogue of the p-adic monodromy theorem (Theorem 7.4). Secondly, we will prove that the category of horizontal de Rham representations of G_K is canonically equivalent to the category of de Rham representations of $G_{K_{\mathrm{can}}}$ (Theorem 7.6), where K_{can} is the canonical subfield of K. Finally, we will calculate H^1 of horizontal de Rham representations under a certain condition on Hodge–Tate weights (Theorem 7.8). This calculation is a generalization of calculations done by Hyodo for $\mathbb{Z}_p(n)$ with $n \in \mathbb{Z}$ (Theorem 1.16).

Statement of Main Theorem. Let K and G_K be as above. We do not put any assumption on the residue field k_K of K, in particular, we may consider the case that k_K is imperfect with $[k_K : k_K^p] = \infty$. In this setup, the notions of crystalline, semistable, de Rham, Hodge–Tate representations are also defined (see Section 3). Then, our main theorem is the following:

Main Theorem (p-adic monodromy theorem). *Let V be a de Rham representation of G_K. Then, there exists a finite extension L/K such that the restriction $V|_L$ is semistable.*

Note that the converse can be easily proved by using Hilbert 90.

Strategy of proof. As is mentioned above, Kazuma Morita's proof can not be generalized directly. When the residue field k_K is perfect, an alternative proof of the p-adic monodromy theorem due to Pierre Colmez is available, which does not need the theory of p-adic differential equations. We will prove the Main Theorem by generalizing Colmez's method. In the following, we will explain our strategy after recalling Colmez's proof in the case that V is a 2-dimensional de Rham representation. (We can prove the higher-dimensional case in a similar way.) After replacing K by the maximal unramified extension of K and taking a Tate twist of V, we may also assume that we have $\mathbb{D}_{\mathrm{dR}}(V) = (\mathbb{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ and k_K is separably closed.

In this paragraph, assume that the residue field of K is perfect, that is, k_K is algebraically closed. We first fix notation: Let $\tilde{\mathbb{B}}^+_{\mathrm{rig}} := \bigcap_{n\in\mathbb{N}} \varphi^n(\mathbb{B}^+_{\mathrm{cris}})$. For $h \in \mathbb{N}_{>0}$ and $a \in \mathbb{N}$, denote $\mathbb{U}_{h,a} := (\mathbb{B}^+_{\mathrm{cris}})^{\varphi^h=p^a}$ and $\mathbb{U}'_{h,a} := (\mathbb{B}^+_{\mathrm{st}})^{\varphi^h=p^a}$. Note that we have $\mathbb{U}_{h,0} = \mathbb{U}'_{h,0} = \mathbb{Q}_{p^h}$, where $\mathbb{Q}_{p^h}$ denotes the unramified extension of $\mathbb{Q}_p$ with $[\mathbb{Q}_{p^h} : \mathbb{Q}_p] = h$. We will recall Colmez's proof: His proof has the following two key ingredients. One is Dieudonné–Manin classification theorem over $\tilde{\mathbb{B}}^+_{\mathrm{rig}}$. Then, he applies this theorem to construct a rank 2 free $\tilde{\mathbb{B}}^+_{\mathrm{rig}}$-submodule $\tilde{\mathbb{N}}^+_{\mathrm{rig}}(V)$ of $\tilde{\mathbb{B}}^+_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V$ with basis e_1, e_2. Moreover, $\tilde{\mathbb{N}}^+_{\mathrm{rig}}(V)$ is stable by φ and G_K-actions and the following properties are satisfied:

(i) We have an isomorphism of $\mathbb{B}^+_{\mathrm{dR}}[G_K]$-modules

$$\mathbb{B}^+_{\mathrm{dR}} \otimes_{\tilde{\mathbb{B}}^+_{\mathrm{rig}}} \tilde{\mathbb{N}}^+_{\mathrm{rig}}(V) \cong (\mathbb{B}^+_{\mathrm{dR}})^2.$$

(ii) There exist $h \in \mathbb{N}_{>0}$ and a 1-cocycle

$$C : G_K \to \begin{pmatrix} \mathbb{Q}^\times_{p^h} & \mathbb{U}_{h,a} \\ 0 & \mathbb{Q}^\times_{p^h} \end{pmatrix}; \quad g \mapsto C_g := \begin{pmatrix} \chi_1(g) & c_g \\ 0 & \chi_2(g) \end{pmatrix}$$

such that we have $g(e_1, e_2) = (e_1, e_2)C_g$ for all $g \in G_K$.

The second key ingredient is the $H^1_g = H^1_{\mathrm{st}}$-theorem for $\mathbb{U}'_{h,a}$ with $h, a \in \mathbb{N}_{>0}$: Let L/K be a finite extension. If a 1-cocycle $G_L \to \mathbb{U}'_{h,a}$ is a 1-coboundary in $\mathbb{B}^+_{\mathrm{dR}}$, then it is a 1-coboundary in $\mathbb{U}'_{h,a}$. By using these facts, Colmez proved the Main Theorem as follows. When $h = 0$, we may regard C as a p-adic representation of G_K, which is Hodge–Tate of weights 0 by (i). By Sen's theorem on $\mathbb{C}_p$-representations, C has a finite image, which implies the assertion. Therefore, we may assume $h > 0$. By the cocycle condition of C, χ_i for $i = 1, 2$ is a character. By (i), χ_i for $i = 1, 2$ is Hodge–Tate with weights 0 as a p-adic representation. By Sen's theorem again, there exists a finite extension L/K such that $\chi_i(G_L) = 1$ for $i = 1, 2$. By the cocycle condition of C again, $c : G_L \to \mathbb{U}_{h,a}$ is a 1-cocycle, which is a 1-coboundary in $\mathbb{B}^+_{\mathrm{dR}}$

by (i). By the $H^1_g = H^1_{\mathrm{st}}$-theorem, there exists $x \in \mathbb{U}'_{h,a}$ such that $c_g = (g-1)(x)$ for all $g \in G_L$. Therefore,

$$e_1, -xe_1 + e_2 \in \mathbb{B}^+_{\mathrm{st}} \otimes_{\tilde{\mathbb{B}}^+_{\mathrm{rig}}} \tilde{\mathbb{N}}^+_{\mathrm{rig}}(V) \subset \mathbb{B}^+_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$$

form a basis of $\mathbb{D}_{\mathrm{st}}(V|_L)$, which implies that $V|_L$ is semistable.

We will outline our proof of the Main Theorem in the following: For simplicity, we omit some details. We first fix notation: In the imperfect residue field case, we can construct rings of p-adic periods $\mathbb{B}^+_{\mathrm{cris}}$, $\mathbb{B}^+_{\mathrm{st}}$ and $\mathbb{B}^+_{\mathrm{dR}}$, on which connections ∇ act. Let $\mathbb{B}^{\nabla+}_{\mathrm{cris}}$ and $\mathbb{B}^{\nabla+}_{\mathrm{st}}$ be the rings of the horizontal sections of $\mathbb{B}^+_{\mathrm{cris}}$ and $\mathbb{B}^+_{\mathrm{st}}$ respectively. Let $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}} := \bigcap_{n\in\mathbb{N}} \varphi^n(\mathbb{B}^{\nabla+}_{\mathrm{cris}})$. For $h \in \mathbb{N}_{>0}$ and $a \in \mathbb{N}$, let $\mathbb{U}_{h,a} := (\mathbb{B}^{\nabla+}_{\mathrm{cris}})^{\varphi^h=p^a}$ and $\mathbb{U}'_{h,a} := (\mathbb{B}^{\nabla+}_{\mathrm{st}})^{\varphi^h=p^a}$. Even when k_K may not be perfect, we can easily prove a generalization of Sen's theorem (Theorem 2.1) and an analogue of Colmez's Dieudonné–Manin classification theorem in an appropriate setting (see Section 5). By using Dieudonné–Manin theorem, we can also give a functorial construction $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ for a de Rham representation V. Our object $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ is a rank 2 free $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}$-submodule of $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V$ with basis e_1, e_2. Moreover, $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ is stable by φ and G_K-actions and the following properties are satisfied:

(i) We have an isomorphism of $\mathbb{B}^+_{\mathrm{dR}}[G_K]$-modules

$$\mathbb{B}^+_{\mathrm{dR}} \otimes_{\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig}}} \tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V) \cong (\mathbb{B}^+_{\mathrm{dR}})^2.$$

(ii) There exist $h \in \mathbb{N}_{>0}$ and a 1-cocycle

$$C : G_K \to \begin{pmatrix} \mathbb{Q}^\times_{p^h} & \mathbb{U}_{h,a} \\ 0 & \mathbb{Q}^\times_{p^h} \end{pmatrix}; \quad g \mapsto C_g := \begin{pmatrix} \chi_1(g) & c_g \\ 0 & \chi_2(g) \end{pmatrix}$$

such that we have $g(e_1, e_2) = (e_1, e_2)C_g$ for all $g \in G_K$.

By using $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$, we prove the Main Theorem as follows. In the case $h = 0$, the same proof as above is valid, hence we assume $h > 0$. By the cocycle condition of C, χ_i for $i = 1, 2$ is a character, which is Hodge–Tate with weights 0 by (i). By a generalization of Sen's theorem, we may assume that $\chi_i(G_K) = 1$ for $i = 1, 2$ after replacing K by some finite extension. Then, by the cocycle condition of C, $c : G_K \to \mathbb{U}_{h,a}$ is a 1-cocycle, which is a 1-coboundary in $\mathbb{B}^+_{\mathrm{dR}}$. Unfortunately, an analogue of the above $H^1_{\mathrm{st}} = H^1_g$-theorem does not hold in the imperfect residue field case. Instead, we will prove that there exists $x \in (\mathbb{B}^+_{\mathrm{cris}})^{G_{K^{\mathrm{pf}}}}$ and $y \in \mathbb{B}^{\nabla+}_{\mathrm{dR}}$ such that $c_g = (g-1)(x+y)$ for $g \in G_K$ (a special case of Lemma 6.6). Here K^{pf} denotes a "perfection" of K, which is a complete discrete valuation field of mixed characteristic $(0, p)$ with residue field k^{pf}_K and we can regard an absolute Galois group $G_{K^{\mathrm{pf}}}$ of K^{pf} as a closed subgroup of G_K. Since we have a canonical isomorphism $\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)|_{G_{K^{\mathrm{pf}}}} \cong \tilde{\mathbb{N}}^+_{\mathrm{rig}}(V|_{G_{K^{\mathrm{pf}}}})$ by functoriality, we can apply Colmez's $H^1_g = H^1_{\mathrm{st}}$-theorem to the 1-cocycle $c|_{G_{K^{\mathrm{pf}}}}$. As a consequence, there exists $z \in \mathbb{U}'_{h,a}$ such that

$c_g=(g-1)(z)$ for all $g\in G_{K^{\mathrm{pf}}}$. Since we have $c_g=(g-1)(y)$ for all $g\in G_{K^{\mathrm{pf}}}$, we have $z-y\in(\mathbb{B}_{\mathrm{dR}}^{\nabla+})^{G_{K^{\mathrm{pf}}}}$, which is included in $\mathbb{B}_{\mathrm{cris}}^{\nabla+}$ by a calculation. Hence, $e_1,-\{x+(y-z)+z\}e_1+e_2\in\mathbb{B}_{\mathrm{st}}^{+}\otimes_{\tilde{\mathbb{B}}_{\mathrm{rig}}^{\nabla+}}\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V)\subset\mathbb{B}_{\mathrm{st}}^{+}\otimes V$ forms a basis of $\mathbb{D}_{\mathrm{st}}(V|_K)$, which implies that $V|_K$ is semistable.

Structure of paper. In Section 1, we will recall the preliminary facts used in the paper. In Section 2, we will generalize Sen's theorem on $\mathbb{C}_p$-admissible representations, which is a special case of the Main Theorem and will be used in the following. The next two sections are devoted to review rings of p-adic periods in the imperfect residue field case. Although most of the results seem to be well-known, we will give proofs for the convenience of the reader. In Section 3, we will recall basic constructions and algebraic properties of rings of p-adic periods used in p-adic Hodge theory in the imperfect residue field case. In Section 4, we will recall Galois-theoretic properties of rings of p-adic periods constructed in the previous section. In Section 5, we will construct the (φ,G_K)-modules $\widetilde{\mathbb{N}}_{\mathrm{rig}}^{\nabla+}(V)$ for de Rham representations V after Tate twist. In Section 6, we will prove the Main Theorem combining the results proved in the previous sections. In Section 7, we will give applications of the Main Theorem.

Conventions

Throughout this paper, let p be a prime and K a complete discrete valuation field of mixed characteristic $(0,p)$. Denote the integer ring of K by $\mathcal{O}_K$ and a uniformizer of $\mathcal{O}_K$ by π_K. Put $U_K^{(n)}:=1+\pi_K^n\mathcal{O}_K$ for $n\in\mathbb{N}_{>0}$. Denote by k_K the residue field of K. We denote by K^{ur} the p-adic completion of the maximal unramified extension of K. Denote by e_K the absolute ramification index of K. For an extension L/K of complete discrete valuation fields, we define the relative ramification index $e_{L/K}$ of L/K by $e_{L/K}:=e_L/e_K$.

For a field F, fix an algebraic closure (resp. a separable closure) of F, denote it by F^{alg} or $\overline{F}$ (resp. F^{sep}) and let G_F be the absolute Galois group of F. For a field k of characteristic p, let $k^{\mathrm{pf}}:=k^{p^{-\infty}}$ be the perfect closure in a fixed algebraic closure of k. Let $k^{p^{\infty}}:=\bigcap_{n\in\mathbb{N}}k^{p^n}$ be the maximal perfect subfield of k. Denote by $\mathbb{C}_p$ and $\mathcal{O}_{\mathbb{C}_p}$ the p-adic completion of $\overline{K}$ and its integer ring. Let v_p be the p-adic valuation of $\mathbb{C}_p$ normalized by $v_p(p)=1$.

We fix a system of p-power roots of unity $\{\zeta_{p^n}\}_{n\in\mathbb{N}_{>0}}$ in $\overline{K}$, that is, ζ_p is a primitive p-th root of unity and $\zeta_{p^{n+1}}^p=\zeta_{p^n}$ for all $n\in\mathbb{N}_{>0}$. Let $\chi:G_K\to\mathbb{Z}_p^{\times}$ be the cyclotomic character defined by $g(\zeta_{p^n})=\zeta_{p^n}^{\chi(g)}$ for $n\in\mathbb{N}_{>0}$.

For a set S, denote by $|S|$ the cardinality of S. Let J_K be an index set such that we have an isomorphism $\Omega^1_{k_K/\mathbb{Z}}\cong k_K^{\bigoplus J_K}$ as k_K-vector spaces. In this paper, we do not assume $|J_K|<\infty$. Unless a particular mention is stated, we always fix a lift $\{t_j\}_{j\in J_K}$ of a p-basis of k_K and sequences of p-power roots $\{t_j^{p^{-n}}\}_{n\in\mathbb{N}, j\in J_K}$ in $\overline{K}$, that is, we have $(t_j^{p^{-n-1}})^p = t_j^{p^{-n}}$ for $n \in \mathbb{N}_{>0}$.

For a ring R, denote the Witt ring with coefficients in R by $W(R)$. If R has characteristic p, then we denote the absolute Frobenius on R by $\varphi : R \to R$ and also denote the ring homomorphism $W(\varphi) : W(R) \to W(R)$ by φ. Denote by $[x] \in W(R)$ the Teichmüller lift of $x \in R$.

For a p-adically Hausdorff abelian group M in which p is not a nonzero divisor, we define the p-adic semivaluation of M as the map $v : M \to \mathbb{Z} \cup \{\infty\}$ such that $v(0) = \infty$ and $v(x) = n$ if $x \in p^n M \setminus p^{n+1} M$. We have the properties

$$v(px) = 1 + v(x), \quad v(x + y) \geq \inf(v(x), v(y)), \quad v(x) = \infty \iff x = 0,$$

for $x, y \in M$. We can extend v to $v : M[p^{-1}] \to \mathbb{Z} \cup \{\infty\}$, which we call the p-adic semivaluation defined by the lattice M. We also call the topology induced by v the p-adic topology defined by the lattice M.

Let F be a nontrivial nonarchimedean complete valuation field with valuation v_F. Assume that an F-vector space V is endowed with a countable decreasing sequence of valuations $\{v^{(n)} : V \to \mathbb{R} \cup \{\infty\}\}_{n\in\mathbb{N}}$ over F, that is, we have

$$v^{(0)}(x) \geq v^{(1)}(x) \geq \cdots, \quad v^{(n)}(\lambda x) = v_F(\lambda) + v^{(n)}(x),$$
$$v^{(n)}(x + y) \geq \inf(v^{(n)}(x), v^{(n)}(y))$$

for $\lambda \in F$ and $x, y \in V$. We regard V as a topological F-vector space whose topology is generated by $V_r^{(n)} := \{x \in V \mid v^{(n)}(x) \geq r\}$ for $n, r \in \mathbb{N}$. Then, we call V a Fréchet space (over F) if V is complete with respect to this topology (see [Schneider 2002, Section 8]). For Fréchet spaces V and W, we define the completed tensor product $V \widehat{\otimes}_F W$ as the inverse limit $\varprojlim_{n,r\in\mathbb{N}} V/V_r^{(n)} \otimes_F W/W_r^{(n)}$ (see [Schneider 2002, Section 17]).

For a multiset $\{a_i\}_{i\in I}$ of elements in $\mathbb{R} \cup \{\infty\}$, we denote $\{a_i\}_{i\in I} \to \infty$ if the set $\{i \in I, a_i < n\}$ is finite for all $n \in \mathbb{N}_{>0}$. Note that if $|I| < \infty$, then the condition $\{a_i\}_{i\in I} \to \infty$ is always satisfied.

In this paper, we refer to the continuous cohomology group as the group cohomology. For a profinite group G and a topological G-module M, denote by $H^n(G, M)$ the n-th continuous group cohomology with coefficients in M. We also denote $H^0(G, M)$ by M^G. We also consider $H^q(G, M)$ for $q = 0, 1$ if M is a (noncommutative) topological G-group M.

We denote by $\boldsymbol{e}_i \in \mathbb{N}^{\oplus I}$ the element whose i-th component is equal to 1 and zero otherwise. We will use the following multi-index notation: Let M be a monoid. For a subset $\{x_i\}_{i\in I}$ of M and $\boldsymbol{n} = (n_i)_{i\in I} \in \mathbb{N}^{\oplus I}$, we define $\boldsymbol{x}^{\boldsymbol{n}} := \prod_{i\in I} x_i^{n_i}$ and $\boldsymbol{x}^{[\boldsymbol{n}]} := \prod\limits_{i\in I} u_i^{n_i}/n_i!$ when it has a meaning. We denote by $|\boldsymbol{n}|$ the sum $\sum\limits_{i\in I} n_i$ for

$(n_i)_{i\in I} \in \mathbb{N}^{\oplus I}$. If no particular mention is stated, for an index set I, we denote by $\mathbf{u}_I$ or $\mathbf{v}_I$ the formal variables $\{\mathrm{u}_i\}_{i\in I}$ or $\{\mathrm{v}_i\}_{i\in I}$ respectively.

For group homomorphisms $f, g : M \to N$ of abelian groups, we denote by $M^{f=g}$ the kernel of the map $f - g : M \to N$.

1. Preliminaries

This preliminary section is a miscellany of basic definitions, facts, conventions, and remarks used in the paper. Although we will give some proofs for convenience, the reader may skip the proofs by admitting the facts.

1A. ***Cohen ring.*** Let k be a field of characteristic p. Let $C(k)$ be a Cohen ring of k, that is, a complete discrete valuation ring with maximal ideal generated by p and residue field k. This is unique up to a canonical isomorphism if k is perfect (in fact, $C(k) \cong W(k)$) and unique up to noncanonical isomorphisms in general. Denote $J_{C(k)[p^{-1}]}$ by J for a while. For a lift $\{t_j\}_{j\in J} \subset C(k)$ of a p-basis of k, we regard $C(k)$ as a $\mathbb{Z}[T_j]_{j\in J}$-algebra by $T_j \mapsto t_j$. This morphism is formally étale for the p-adic topologies. In fact, we may replace $\mathbb{Z}[T_j]_{j\in J}$ by $R := (\mathbb{Z}[T_j]_{j\in J})_{(p)}$. Since $C(k)/R$ is flat and $k/\mathbb{F}_p(T_j)_{j\in J}$ is formally étale for the discrete topologies, $C(k)/R$ is formally étale by [Grothendieck 1964, 0.19.7.1 and 0.20.7.5].

By the lifting property, we have $C(k_K) \to \mathcal{O}_K$, an injective algebra homomorphism which is totally ramified of degree e_K. We will denote by K_0 the fraction field of the image of $C(k)$ in K. We also note that $\mathcal{O}_{K_0}$ is unique if k_K is perfect and nonunique otherwise. By the lifting property again, we have a lift $\varphi : \mathcal{O}_{K_0} \to \mathcal{O}_{K_0}$ of the absolute Frobenius of k_K: It is unique if k_K is perfect and nonunique otherwise. An example of such a φ is $\varphi(t_j) = t_j^p$ for all $j \in J_{K_0}$. Moreover, when k_K is imperfect, the construction of K_0 cannot be functorial in the following sense: For a finite extension L/K, we cannot always choose $K_0 \subset K$ and $L_0 \subset L$ such that $K_0 \subset L_0$.

Finally, note that for a given lift $\{t_j\}_{j\in J_K} \subset \mathcal{O}_K$ of a p-basis of k_K, we can choose $\mathcal{O}_{K_0}$ such that $\{t_j\}_{j\in J_K} \subset \mathcal{O}_{K_0}$. In fact, we regard $\mathcal{O}_K$ as a $\mathbb{Z}[T_j]_{j\in J_K}$-algebra by sending T_j to t_j. We choose a lift $\{t'_j\}_{j\in J_K} \subset C(k_K)$ of the p-basis $\{\bar{t}_j\}_{j\in J_K} \subset k_K$ and we regard $C(k_K)$ as a $\mathbb{Z}[T_j]_{j\in J_K}$-algebra by $T_j \mapsto t'_j$. Then, we lift the projection $C(k_K) \to k_K$ to a $\mathbb{Z}[T_j]_{j\in J_K}$-algebra homomorphism $C(k_K) \to \mathcal{O}_K$ by the lifting property, whose image satisfies the condition. Thus, if we choose a lift $\{t_j\}_{j\in J_K}$ of a p-basis of k_K, we may always assume that we have $\{t_j\}_{j\in J_K} \subset K_0$.

1B. ***Canonical subfield.*** We first recall the following two lemmas, which are proved in [Epp 1973, 0.4]. We give proofs for the reader.

Lemma 1.1. *Let k be a field of characteristic p.*

(i) *The field k^{p^∞} is algebraically closed in k. In particular, the fields $(k^{p^\infty})^{\mathrm{sep}}$ and k are linearly disjoint over k^{p^∞}.*

(ii) *For a finite extension k'/k^{p^∞}, we have $k' = (kk')^{p^\infty}$.*

Proof.

(i) The assertion follows from the fact that any algebraic extension over a perfect field is perfect.

(ii) As is mentioned in the above proof, k' is perfect. We have $kk' = k \otimes_{k^{p^\infty}} k'$ by (i). Hence, we have $(kk')^{p^n} = k^{p^n} \otimes_{k^{p^\infty}} k'$ and

$$(kk')^{p^\infty} = \bigcap_n (k^{p^n} \otimes_{k^{p^\infty}} k') = k^{p^\infty} \otimes_{k^{p^\infty}} k' = k'. \qquad \square$$

Lemma 1.2. *Let l/k be an algebraic extension of fields of characteristic p.*

(i) *If l/k is a (possibly infinite) Galois extension, then $l^{p^\infty}/k^{p^\infty}$ is also a (possible infinite) Galois extension. Moreover, the canonical map*

$$G_{l/k} \to G_{l^{p^\infty}/k^{p^\infty}}$$

is surjective.

(ii) *If l/k is finite, then $l^{p^\infty}/k^{p^\infty}$ is also a finite extension. Moreover, we have $[l^{p^\infty} : k^{p^\infty}] \leq [l : k]$.*

Proof. (i) We may easily reduce to the case that l/k is finite Galois. Obviously any k-algebra endomorphism on l induces a k^{p^n}-algebra endomorphism on l^{p^n}. In particular, l^{p^n} and l^{p^∞} are $G_{l/k}$-stable. Since the Frobenius commutes with the action of $G_{l/k}$, we have $(l^{p^n})^{G_{l/k}} = (l^{G_{l/k}})^{p^n} = k^{p^n}$. By taking the intersection, we have $(l^{p^\infty})^{G_{l/k}} = k^{p^\infty}$. For $x \in l^{p^\infty}$, let $f(X) \in k[X]$ be the monic irreducible separable polynomial such that $f(x) = 0$. Then all the solutions of f belong to l^{p^∞} and we have $f(X) \in (l^{p^\infty})^{G_{l/k}}[X] = k^{p^\infty}[X]$. This implies that $l^{p^\infty}/k^{p^\infty}$ is a Galois extension. The latter assertion follows from the equality $(l^{p^\infty})^{G_{l/k}} = k^{p^\infty}$.

(ii) We may assume that l/k is purely inseparable or separable. If l/k is purely inseparable, then l is generated by finitely many elements of the form $x^{p^{-n}}$ with $n \in \mathbb{N}$ and $x \in k$ as a k-algebra. Hence we have $l^{p^n} \subset k$ for some n, that is, $k^{p^\infty} = l^{p^\infty}$. Assume that l/k is separable. The first assertion is reduced to the case that l/k is a Galois extension, which follows from (i). Since the canonical k-algebra homomorphism $l^{p^\infty} \otimes_{k^{p^\infty}} k \to l$ is injective by Lemma 1.1(i), we have $[l^{p^\infty} : k^{p^\infty}] \leq [l : k]$. $\square$

Defintion 1.3. (i) (Compare [Hyodo 1986, Theorem 2].) We define the canonical subfield K_{can} of K as the algebraic closure of $W(k_K^{p^\infty})[p^{-1}]$ in K.

(ii) (Compare [Hyodo 1986, (0-5)].) We define condition (H) as follows:

K contains a primitive p^2-th root of unity and we have $e_{K/K_{\mathrm{can}}} = 1$.

Note that K_{can} is a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field $k_K^{p^\infty}$. If k_K is perfect, then we have $K_{\mathrm{can}} = K$. We also note that the restriction $G_K \to G_{K_{\mathrm{can}}}$ is surjective since K_{can} is algebraically closed in K. We will regard $G_{K_{\mathrm{can}}}$ as a quotient of G_K in the rest of the paper.

Remark 1.4. (i) In [Brinon 2006, Notation 2.29], K_{can} is denoted by K^∇ since K_{can} coincides with the kernel of the canonical derivation $d : K \to \widehat{\Omega}^1_K$ (Proposition 1.13 below).

(ii) The canonical morphism

$$K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} K_0 \to K$$

is injective since we have $e_{K_0/K_{\mathrm{can},0}} = 1$ and $K_{\mathrm{can}}/K_{\mathrm{can},0}$ is totally ramified. Note that we have $e_{K/K_{\mathrm{can}}} = 1$ if and only if the above morphism is surjective.

The following are the basic properties of the canonical subfields used in this paper.

Lemma 1.5. *Let L/K be a finite extension.*

(i) *The fields $(K_{\mathrm{can}})^{\mathrm{alg}}$ and K are linearly disjoint over K_{can}.*

(ii) *If L/K is Galois, then $L_{\mathrm{can}}/K_{\mathrm{can}}$ is also a finite Galois extension. Moreover, the canonical map $G_{L/K} \to G_{L_{\mathrm{can}}/K_{\mathrm{can}}}$ is surjective.*

(iii) *The field extension $L_{\mathrm{can}}/K_{\mathrm{can}}$ is finite with $[L_{\mathrm{can}} : K_{\mathrm{can}}] \le [L : K]$.*

(iv) *If K'/K_{can} is a finite extension, then we have $(KK')_{\mathrm{can}} = K'$.*

Proof. (i) Since K_{can} is algebraically closed in K, we have $(K_{\mathrm{can}})^{\mathrm{alg}} \cap K = K_{\mathrm{can}}$, which implies the assertion.

(ii) Since $k_L^{p^\infty}/k_K^{p^\infty}$ is finite by Lemma 1.2(ii), we have $L_{\mathrm{can}} = L \cap (K_{\mathrm{can}})^{\mathrm{alg}}$. Hence we have $L_{\mathrm{can}} \cap K = K_{\mathrm{can}}$. Since $L_{\mathrm{can}}/K_{\mathrm{can}}$ is algebraic, L_{can} and K are linearly disjoint over K_{can} by (i). Let $x \in L_{\mathrm{can}}$ and $f(X) \in K_{\mathrm{can}}[X]$ be the monic irreducible polynomial such that $f(x) = 0$. By the linearly disjointness, $f(X)$ is irreducible in $K[X]$. Since L/K is Galois, all the solutions of $f(X) = 0$ belong to $L \cap (K_{\mathrm{can}})^{\mathrm{alg}} = L_{\mathrm{can}}$. This implies that $L_{\mathrm{can}}/K_{\mathrm{can}}$ is Galois. Since we have $(L_{\mathrm{can}})^{G_{L/K}} = L_{\mathrm{can}} \cap K = K_{\mathrm{can}}$, we have the rest of the assertion.

(iii) The finiteness of $L_{\mathrm{can}}/K_{\mathrm{can}}$ is reduced to the case that L/K is Galois, which follows from (ii). Since the canonical K-algebra homomorphism $L_{\mathrm{can}} \otimes_{K_{\mathrm{can}}} K \to L$ is injective by (i), we have $[L_{\mathrm{can}} : K_{\mathrm{can}}] \le [L : K]$.

(iv) The assertion follows from the inequalities

$$[K' : K_{\mathrm{can}}] \le [(KK')_{\mathrm{can}} : K_{\mathrm{can}}] \le [KK' : K] = [K' : K_{\mathrm{can}}],$$

where the second inequality follows from (iii) and the last equality follows from the linear disjointness of K and K' over K_{can} by (i). □

Theorem 1.6 (the complete case of Epp's theorem [1973]). *There exists a finite Galois extension of K'/K_{can} such that KK' satisfies condition* (*H*).

Proof. By the original Epp's theorem, we have a finite extension K'/K_{can} such that we have $e_{KK'/K'}=1$. We have only to prove that we have $e_{KK''/K''}=1$ for any finite extension K''/K'. In fact, if we choose K'' as the Galois closure of $K'(\mu_{p^2})$ over K_{can}, then K'' satisfies the condition by Lemma 1.5(iv). Since we have $KK''=(KK')\otimes_{K'}K''$ by Lemma 1.5(i) and (iv), we have $e_{KK''/KK'}\le e_{K''/K'}$. By multiplying with $e_{KK'}=e_{K'}$, we have $e_{KK''}\le e_{K''}$, implying the assertion. □

Example 1.7 (the higher-dimensional local fields case). We say that K has a structure of a higher-dimensional local field if K is isomorphic to a finite extension over the fractional field of a Cohen ring of the field

$$\mathbb{F}_q((X_1))((X_2))\dots((X_d))$$

with $q=p^f$ (see [Zhukov 2000] about higher-dimensional local fields). In this case, K_{can} coincides with the algebraic closure of $\mathbb{Q}_p$ in K. In fact, we have only to prove that $k_K^{p^\infty}$ is a finite field. By Lemma 1.2(ii), we may reduce to the case $k_K=\mathbb{F}_q((X_1))\dots((X_d))$. Then, the assertion follows from an iterative use of the following fact: If k is a field of characteristic p, then we have $k((X))^{p^\infty}=k^{p^\infty}$. Obviously, the RHS is contained in the LHS. Let $f=\sum_{n\gg-\infty}a_nX^n\in k((X))^{p^\infty}$ with $a_n\in k$. Since $f\in k((X))^p$, we have $a_n=0$ if $p\nmid n$ and $a_n\in k^p$ otherwise. By repeating this argument, we have $a_n=0$ for $n\neq 0$ and $f=a_0\in k^{p^\infty}$.

1C. *Canonical derivation.*

Defintion 1.8 (Compare [Hyodo 1986, Section 4].). Let $q\in\mathbb{N}$. For a complete discrete valuation ring R with mixed characteristic $(0,p)$, let

$$\hat{\Omega}^q_R:=\varprojlim_n \Omega^q_{R/\mathbb{Z}}/p^n\Omega^q_{R/\mathbb{Z}}$$

and let $d:R\to\hat{\Omega}^1_R$ be the canonical derivation. Let $\hat{\Omega}^q_{R[p^{-1}]}:=\hat{\Omega}^q_R[p^{-1}]$ for $q\in\mathbb{Z}$ and let $d:R[p^{-1}]\to\hat{\Omega}^1_{R[p^{-1}]}$ be the canonical derivation and $d_q:\hat{\Omega}^q_{R[p^{-1}]}\to\hat{\Omega}^{q+1}_{R[p^{-1}]}$ the morphism induced by the exterior derivation, which satisfies the usual formula $d_q(\lambda\omega)=\lambda d_q\omega+(-1)^q\omega\wedge d\lambda$ for $\lambda\in K$ and $\omega\in\hat{\Omega}^q_K$. We endow $\hat{\Omega}^q_{R[p^{-1}]}$ with the p-adic topology defined by the lattice $\mathrm{Im}(\hat{\Omega}^q_R\xrightarrow{\mathrm{can.}}\hat{\Omega}^q_{R[p^{-1}]})$. Obviously, the derivation d_q is continuous.

For $q\in\mathbb{Z}_{<0}$, we put $\hat{\Omega}^q_{R[p^{-1}]}:=0$ as a matter of convention.

The following are the basic properties of the canonical derivations used in the sequel.

Lemma 1.9. *Let R be a discrete valuation ring with uniformizer π_R and $\alpha : M \to M'$ a morphism of R-modules whose kernel and cokernel are killed by π_R^c for $c \in \mathbb{N}$. Then, for any R-module M'', the kernel and cokernel of the morphism* $\mathrm{id} \otimes \alpha : M'' \otimes_R M \to M'' \otimes_R M'$ *are killed by π_R^{2c}. In particular, the kernel and cokernel of $\alpha^{\otimes q} : M^{\otimes q} \to M'^{\otimes q}$ are killed by π_R^{2qc}.*

Proof. We prove the first assertion. If α is injective or surjective, then the cokernel and kernel are killed by π_R^c by the calculation of $\mathrm{Tor}^{\bullet}_R$. The general case follows easily from these cases by writing α as a composition of an injection and a surjection.

The last assertion follows from the following decomposition and induction on q:

$$M^{\otimes(q+1)} = M \otimes_R M^{\otimes q} \xrightarrow{\mathrm{id}\otimes\alpha^{\otimes q}} M \otimes_R M'^{\otimes q} \xrightarrow{\alpha\otimes\mathrm{id}} M' \otimes_R M'^{\otimes q} = M'^{\otimes(q+1)}$$

□

Lemma 1.10 [Hyodo 1986]. *Let $q \in \mathbb{N}$.*

(i) *We have the $\mathbb{O}_{K_0}$-linear isomorphism*

$$\hat{\Omega}^q_{\mathbb{O}_{K_0}} \cong \varprojlim_n \big((\mathbb{O}_{K_0}/p^n\mathbb{O}_{K_0}) \otimes_{\mathbb{Z}} {\textstyle\bigwedge}^q_{\mathbb{Z}} \mathbb{Z}^{\oplus J_K}\big);\;\; dt_{j_1} \wedge \cdots \wedge dt_{j_q} \mapsto 1 \otimes \boldsymbol{e}_{j_1} \wedge \cdots \wedge \boldsymbol{e}_{j_q}.$$

In particular, $\hat{\Omega}^q_{\mathbb{O}_{K_0}}/(p^n)$ is a free $\mathbb{O}_{K_0}/(p^n)$-module.

(ii) *We have a canonical isomorphism*

$$\big({\textstyle\bigwedge}^q_K \hat{\Omega}^1_K\big)^{\wedge} \to \hat{\Omega}^q_K.$$

(iii) *Let L be a finite extension over the completion of an unramified extension of K. Then, we have a canonical isomorphism*

$$L \otimes_K \hat{\Omega}^q_K \to \hat{\Omega}^q_L.$$

Proof. The assertions (i) and (ii) follow from [Hyodo 1986, Lemma (4.4), Remark 3] respectively. The canonical exact sequence

$$0 \to \mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}} \to \Omega^1_{\mathbb{O}_L/\mathbb{Z}} \to \Omega^1_{\mathbb{O}_L/\mathbb{O}_K} \to 0$$

(from [Scholl 1998, Section 3.4, footnote]) induces the exact sequence

$$\Omega^1_{\mathbb{O}_L/\mathbb{O}_K}[p^n] \to \mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}}/(p^n) \xrightarrow{\alpha_n} \Omega^1_{\mathbb{O}_L/\mathbb{Z}}/(p^n) \to \Omega^1_{\mathbb{O}_L/\mathbb{O}_K}/(p^n) \to 0,$$

where $\Omega^1_{\mathbb{O}_L/\mathbb{O}_K}[p^n]$ denotes the kernel of the multiplication by p^n on $\Omega^1_{\mathbb{O}_L/\mathbb{O}_K}$. Fix $c \in \mathbb{N}$ such that $p^c\Omega^1_{\mathbb{O}_L/\mathbb{O}_K} = 0$. Then, the kernel and cokernel of α_n are killed by p^c. Denote by $\mathcal{Q}_n$ and Q_n the kernel of the canonical maps

$$\textstyle\bigotimes^q_{\mathbb{O}_L} \big(\mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}}/(p^n)\big) \to \bigwedge^q_{\mathbb{O}_L} \big(\mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}}/(p^n)\big),$$

$$\textstyle\bigotimes^q_{\mathbb{O}_L} \Omega^1_{\mathbb{O}_L/\mathbb{Z}}/(p^n) \to \Omega^q_{\mathbb{O}_L/\mathbb{Z}}/(p^n).$$

We consider the commutative diagram

$$\begin{array}{ccccc} \mathbb{O}_L \otimes_{\mathbb{O}_K} \dfrac{\bigotimes^q_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}}}{(p^n)} & \underset{\cong}{\xrightarrow{\text{can.}}} & \bigotimes^q_{\mathbb{O}_L} \dfrac{\mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}}}{(p^n)} & \xrightarrow{\alpha_n^{\otimes q}} & \bigotimes^q_{\mathbb{O}_L} \dfrac{\Omega^1_{\mathbb{O}_L/\mathbb{Z}}}{(p^n)} \\ \downarrow{\scriptstyle \text{can.}} & & \downarrow{\scriptstyle \text{can.}} & & \downarrow{\scriptstyle \text{can.}} \\ \mathbb{O}_L \otimes_{\mathbb{O}_K} \dfrac{\Omega^q_{\mathbb{O}_K/\mathbb{Z}}}{(p^n)} & \underset{\cong}{\xrightarrow{\text{can.}}} & \bigwedge^q_{\mathbb{O}_L} \dfrac{\mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K/\mathbb{Z}}}{(p^n)} & \xrightarrow{\bigwedge^q \alpha_n} & \dfrac{\Omega^q_{\mathbb{O}_L/\mathbb{Z}}}{(p^n)}. \end{array}$$

We have only to prove that the kernel and cokernel of $\bigwedge^q \alpha_n$ are killed by p^{3qc}. Indeed, if this is true, then we decompose the canonical map

$$\alpha_n^q : \mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^q_{\mathbb{O}_K/\mathbb{Z}}/(p^n) \to \Omega^q_{\mathbb{O}_L/\mathbb{Z}}/(p^n)$$

into the following exact sequences:

$$0 \longrightarrow \ker \alpha_n^q \xrightarrow{\text{inc.}} \mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^q_{\mathbb{O}_K/\mathbb{Z}}/(p^n) \xrightarrow{\alpha_n^q} \operatorname{Im} \alpha_n^q \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Im} \alpha_n^q \xrightarrow{\text{inc.}} \Omega^q_{\mathbb{O}_L/\mathbb{Z}}/(p^n) \xrightarrow{\text{pr.}} \operatorname{cok} \alpha_n^q \longrightarrow 0.$$

By passing to limits, we obtain the following exact sequences:

$$0 \longrightarrow \varprojlim_n \ker \alpha_n^q \xrightarrow{\text{inc.}} \mathbb{O}_L \otimes_{\mathbb{O}_K} \widehat{\Omega}^q_{\mathbb{O}_K} \xrightarrow{\text{can.}} \varprojlim_n \operatorname{Im} \alpha_n^q \xrightarrow{\delta} \varprojlim_n{}^1 \ker \alpha_n^q,$$

$$0 \longrightarrow \varprojlim_n \operatorname{Im} \alpha_n^q \xrightarrow{\text{inc.}} \widehat{\Omega}^q_{\mathbb{O}_L} \xrightarrow{\text{pr.}} \varprojlim_n \operatorname{cok} \alpha_n^q.$$

Since $\ker \alpha_n^q$ and $\operatorname{cok} \alpha_n^q$ are killed by p^{3qc}, $\varprojlim_n \ker \alpha_n^q$ and $\varprojlim_n^1 \ker \alpha_n^q$, $\varprojlim \operatorname{cok} \alpha_n^q$ are also killed by p^{3qc} [Neukirch et al. 2008, Proposition 2.7.4]. Hence, the kernel and cokernel of the canonical map $\mathbb{O}_L \otimes_{\mathbb{O}_K} \widehat{\Omega}^q_{\mathbb{O}_K} \to \widehat{\Omega}^q_{\mathbb{O}_L}$ are killed by p^{3qc} and p^{6qc} respectively. By inverting p, we obtain the assertion.

Note that the kernel and cokernel of $\alpha_n^{\otimes q}$ are killed by p^{2qc} by Lemma 1.9. By the snake lemma, it suffices to prove that the cokernel of the map $\alpha_n^{\otimes q} : \mathcal{Q}_n \to Q_n$ is killed by p^{qc}. The $\mathbb{O}_L$-module Q_n is generated by the elements of the form $x := x_1 \otimes \cdots \otimes x_q$ with $x_i \in \Omega^1_{\mathbb{O}_L/\mathbb{Z}}/(p^n)$ such that $x_i = x_j$ for some $i \neq j$. Since the cokernel of α_n is killed by p^c, there exist $y_1, \ldots, y_q \in \mathbb{O}_L \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_L/\mathbb{Z}}/(p^n)$ such that $p^c x_i = \alpha_n(y_i)$ and $y_i = y_j$. Hence we have $p^{qc} x = (p^c x_1) \otimes \cdots \otimes (p^c x_q) = \alpha_n^{\otimes q}(y_1 \otimes \cdots \otimes y_q)$ and $y_1 \otimes \cdots \otimes y_q \in \mathcal{Q}_n$, which implies the assertion. $\square$

Remark 1.11. If $[k_K : k_K^p] = p^d < \infty$, then $\dim_K \widehat{\Omega}^q_K = \binom{d}{q} < \infty$ for $q \in \mathbb{N}$ by Lemma 1.10. In particular, the canonical derivation d is K_{can}-linear since the restriction $d|_{K_{\text{can}}}$ factors through $\widehat{\Omega}^1_{K_{\text{can}}} = 0$ by functoriality.

Defintion 1.12. Fix a lift $\{t_j\}_{j\in J_K}\subset\mathbb{O}_{K_0}$ of a p-basis of k_K. By Lemma 1.10(i), dx for $x\in\mathbb{O}_{K_0}$ can be uniquely written in the form $\sum_{j\in J_K}dt_j\otimes\partial_j(x)$, where $\{\partial_j(x)\}_{j\in J_K}\subset\mathbb{O}_{K_0}$ is such that $\{v_p(\partial_j(x))\}_{j\in J_K}\to\infty$. Note that $\{\partial_j\}_{j\in J_K}$ are mutually commutative derivations of $\mathbb{O}_{K_0}$ by the formula $d_1\circ d=0$. We also note that ∂_j is continuous since we have the inequality $v_p(\partial_j(x))\geq v_p(x)$ for $x\in\mathbb{O}_{K_0}$, which we can check by taking modulo p.

The following is another characterization of the canonical subfields.

Proposition 1.13 [Brinon 2006, Proposition 2.28]. *We have the exact sequence*

$$0\longrightarrow K_{\mathrm{can}}\xrightarrow{\text{inc.}}K\xrightarrow{d}\widehat{\Omega}^1_K.$$

Proof. We first reduce to the case $K=K_0$. In the case that K satisfies condition (H), we obtain the exact sequence by applying $K_{\mathrm{can}}\otimes_{K_{\mathrm{can},0}}$ to the exact sequence for $K=K_0$ by Remark 1.4(ii) and Lemma 1.10(iii). In the general case, we choose a finite Galois extension K'/K_{can} such that KK' satisfies condition (H) by Epp's Theorem 1.6. Since we have $(KK')_{\mathrm{can}}=K'$ by Lemma 1.5(iv), $K'\otimes_{K_{\mathrm{can}}}K=KK'$ by Lemma 1.5(i) and $(\widehat{\Omega}^1_{KK'})^{G_{K'/K_{\mathrm{can}}}}=\widehat{\Omega}^1_K$ by Lemma 1.10(iii), the assertion follows from Galois descent.

We will prove the assertion in the case $K=K_0$. We may replace K_{can}, K and $\widehat{\Omega}^1_K$ by $\mathbb{O}_{K_{\mathrm{can}}},\mathbb{O}_K$ and $\widehat{\Omega}^1_{\mathbb{O}_K}$ respectively. Notation is as above. Let φ be the Frobenius on $\mathbb{O}_K$ given by $\varphi(t_j)=t_j^p$ for $j\in J_K$. Let $\varphi_*:\widehat{\Omega}^1_K\to\widehat{\Omega}^1_K$ be the Frobenius induced by φ. Since we have $d\circ\varphi=\varphi_*\circ d$, by a simple calculation, we have $\partial_j\circ\varphi=pt_j^{p-1}\varphi\circ\partial_j$, that is, $(t_j\partial_j)\circ\varphi=p\varphi\circ(t_j\partial_j)$ for $j\in J_K$.

The ring $\varphi(\mathbb{O}_K)$ is a complete discrete valuation ring of mixed characteristic $(0,p)$ and we may regard its residue field as k_K^p. Let $\Lambda:=\{0,\dots,p-1\}^{\oplus J_K}$. Since the image of $\{\boldsymbol{t}^{\boldsymbol{n}}\}_{\boldsymbol{n}\in\Lambda}$ in k_K forms a k_K^p-basis of k_K, by approximation, every element $x\in\mathbb{O}_K$ can be uniquely written in the form $x=\sum_{\boldsymbol{n}\in\Lambda}\varphi(a_{\boldsymbol{n}})\boldsymbol{t}^{\boldsymbol{n}}$, where $a_{\boldsymbol{n}}\in\mathbb{O}_K$ is such that $\{v_p(a_{\boldsymbol{n}})\}_{\boldsymbol{n}\in\Lambda}\to\infty$. We claim that if $\varphi^n(x)\in\ker d$ with $n\in\mathbb{N}$ and $x\in\mathbb{O}_K$, we have $x\in\varphi(\mathbb{O}_K)$. Since the Frobenius φ_* on $\widehat{\Omega}^1_{\mathbb{O}_K}$ is injective by Lemma 1.10(i) and the commutativity $d\circ\varphi=\varphi_*\circ d$, we may assume $n=0$. By definition, we have $\partial_j(x)=0$ for all $j\in J_K$. We have

$$t_j\partial_j(x)=\sum_{\boldsymbol{n}\in\Lambda}(t_j\partial_j\circ\varphi)(a_{\boldsymbol{n}})\boldsymbol{t}^{\boldsymbol{n}}+\sum_{\boldsymbol{n}\in\Lambda}\varphi(a_{\boldsymbol{n}})t_j\partial_j(\boldsymbol{t}^{\boldsymbol{n}})=\sum_{\boldsymbol{n}\in\Lambda}\varphi(pt_j\partial_j(a_{\boldsymbol{n}})+n_ja_{\boldsymbol{n}})\boldsymbol{t}^{\boldsymbol{n}}.$$

Hence, we have $a_{\boldsymbol{n}}=-n_j^{-1}pt_j\partial_j(a_{\boldsymbol{n}})$ if $n_j\neq0$. Therefore, for $\boldsymbol{n}\in\Lambda\setminus\{\boldsymbol{0}\}$, we have $v_p(a_{\boldsymbol{n}})\geq v_p(a_{\boldsymbol{n}})+1$, that is, $a_{\boldsymbol{n}}=0$, which implies the claim. By using the claim, if we have $x\in\ker d$, then we have $x\in\bigcap_{n\in\mathbb{N}}\varphi^n(\mathbb{O}_K)$. Since the complete discrete valuation ring $\bigcap_{n\in\mathbb{N}}\varphi^n(\mathbb{O}_K)$ is absolutely unramified with residue field $k_K^{p^\infty}$, the inclusion $\mathbb{O}_{K_{\mathrm{can}}}\subset\bigcap_{n\in\mathbb{N}}\varphi^n(\mathbb{O}_K)$ is an equality by approximation, which implies the assertion. □

1D. ***A spectral sequence of continuous group cohomology.*** The following lemma is a basic fact when we calculate continuous Galois cohomology whose coefficient is an inverse limit of p-adic Banach spaces with surjective transition maps. For example, we need it later when we calculate cohomology of $\mathbb{B}_{\mathrm{dR}}^{+}$-modules.

Lemma 1.14 (Compare [Neukirch et al. 2008, Theorem 2.7.5].). *Let G be a profinite group and $\{M_n\}_{n\in\mathbb{N}}$ be an inverse system of continuous G-modules* (*each M_n may not be discrete*) *such that the transition map $M_{n+1}\to M_n$ admits a continuous section* (*as topological spaces*) *for all $n\in\mathbb{N}$. Let M_∞ be the continuous G-module $\varprojlim M_n$ with the inverse limit topology. Then, we have a canonical exact sequence*

$$0 \longrightarrow \varprojlim_n{}^1 H^{q-1}(G, M_n) \longrightarrow H^q(G, M_\infty) \longrightarrow \varprojlim_n H^q(G, M_n) \longrightarrow 0$$

for all $q\in\mathbb{N}$, where $\varprojlim^\bullet$ is the derived functor of $\varprojlim$ in the category of inverse systems of abelian groups indexed by $\mathbb{N}$.

Proof. Let $\mathscr{C}_\infty^\bullet := \mathscr{C}^\bullet_{\mathrm{cont.}}(G, M_\infty)$ (resp. $\mathscr{C}_n^\bullet := \mathscr{C}^\bullet_{\mathrm{cont.}}(G, M_n)$) be the continuous cochain complex of G with coefficients in M_∞ (resp. M_n). Then, $\{\mathscr{C}_n^\bullet\}_{n\in\mathbb{N}}$ forms an inverse system of cochain complexes and we have $\mathscr{C}_\infty^\bullet = \varprojlim_n \mathscr{C}_n^\bullet$. Moreover, the transition maps of the inverse system $\{\mathscr{C}_n^\bullet\}_{n\in\mathbb{N}}$ are surjective by the existence of continuous sections, in particular, $\{\mathscr{C}_n^\bullet\}_{n\in\mathbb{N}}$ satisfies the Mittag–Leffler condition. Then, the assertion follows from [Weibel 1994, Variant in pp.84]. □

1E. ***Hyodo's calculations of Galois cohomology.*** We will recall Hyodo's calculations of Galois cohomology. For $n\in\mathbb{Z}$, denote by $\mathbb{Z}_p(n)$ the n-th Tate twist of $\mathbb{Z}_p$. For a $\mathbb{Z}_p[G_K]$-module V, let $V(n) := V\otimes_{\mathbb{Z}_p}\mathbb{Z}_p(n)$.

Theorem 1.15 [Hyodo 1986, Theorem 1]. *For $n\in\mathbb{N}$ and $q\in\mathbb{Z}$, we have canonical isomorphisms*

$$H^n(G_K, \mathbb{C}_p(q)) \cong \begin{cases} 0 & q \neq n, n-1, \\ \hat{\Omega}_K^q & \textit{otherwise}. \end{cases}$$

We will generalize the following theorem as an application of the Main Theorem in Section 7.

Theorem 1.16. (i) [Hyodo 1986, Theorem 2] *We have the exact sequence*

$$0 \longrightarrow H^1(G_{K_{\mathrm{can}}}, \mathbb{Z}_p(1)) \xrightarrow{\mathrm{Inf}} H^1(G_K, \mathbb{Z}_p(1)) \xrightarrow{\mathrm{can.}} H^1(G_K, \mathbb{C}_p(1)).$$

(ii) [Hyodo 1987, Theorem (0-2)] *If k_K is separably closed, then*

$$\mathrm{Inf}: H^1(G_{K_{\mathrm{can}}}, \mathbb{Z}_p(n)) \to H^1(G_K, \mathbb{Z}_p(n))$$

is an isomorphism for $n\neq 1$.

1F. *Closed subgroups of G_K.* Let L be an algebraic extension of K in $\mathbb{C}_p$. Let $\widehat{L}^{\mathrm{alg}}$ be the algebraic closure of $\widehat{L}$ in $\mathbb{C}_p$. Let M be a finite extension of $\widehat{L}$ and choose a polynomial $f(X) \in \widehat{L}[X]$ such that $M \cong \widehat{L}[X]/(f(X))$. Let $f_0(X) \in L[X]$ be a polynomial such that the p-adic valuations of the coefficients of $f - f_0$ are large enough. Then, we have $M \cong \widehat{L}[X]/(f_0(X))$ by Krasner's lemma. In particular, the algebraic extension $(M \cap L^{\mathrm{alg}})/L$ is dense in M. Hence, we have a canonical morphism of profinite groups $G_L \to G_{\widehat{L}}$, which is an isomorphism whose inverse $G_{\widehat{L}} \to G_L$ maps g to $g|_{L^{\mathrm{alg}}}$. In the sequel, we will identify G_L with $G_{\widehat{L}}$ and we also regard $G_{\widehat{L}}$ as a closed subgroup of G_K.

1G. *Perfection.* For a subset J of J_K, we denote the p-adic completion of the field $\bigcup_{n\in\mathbb{N}} K(\{t_j^{p^{-n}}\}_{j\in J})$ by K_J. Then, K_J is a complete discrete valuation field of mixed characteristic $(0, p)$ with $e_{K_J/K} = 1$ and its residue field is isomorphic to $\bigcup_{n\in\mathbb{N}} k_K(\{\bar{t}_j^{p^{-n}}\}_{j\in J})$. We also denote K_{J_K} by K^{pf}, which is referred as a perfection of K since the residue field $k_{K^{\mathrm{pf}}} \cong k_K^{\mathrm{pf}}$ of K^{pf} is perfect. Since we may assume that $\{t_j\}_{j\in J_K}$ is contained in K_0 (Section 1A), we may assume $(K_0)_J = (K_J)_0$, which is denoted by $K_{J,0}$ for simplicity.

Let $\mathcal{P}(J_K)$ be the subsets of J_K consisting of subsets $J \in J_K$ such that $J_K \setminus J$ is finite. Note that we have $[k_{K_J} : k_{K_J}^p] = p^{|J_K\setminus J|} < \infty$ for $J \in \mathcal{P}(J_K)$ since $\{\bar{t}_j\}_{j\in J_K\setminus J}$ forms of a p-basis of k_{K_J}. We regard $\mathcal{P}(J_K)$ as an inverse system with respect to the reverse inclusion. Then, we have

$$K \cong \varprojlim_{J\in\mathcal{P}(J_K)} K_J = \bigcap_{J\in\mathcal{P}(J_K)} K_J,$$

that is, K is represented by an inverse limit of complete discrete valuation fields, whose residue fields admit a finite p-basis. In fact, if we endow J_K with a well-order $\precsim$ by the axiom of choice, then for $J \in \mathcal{P}(J_K)$, the subset

$$\mathscr{E}_J := \{1\} \cup \left\{ t_{j_1}^{a_1 p^{-n_1}} \dots t_{j_m}^{a_m p^{-n_m}} \;\middle|\; \begin{array}{l} j_1 \precnsim \cdots \precnsim j_m \in J, 0 < a_{j_i} < p^{n_{j_i}} \in \mathbb{N}_{>0} \\ (p, a_{j_i}) = 1 \text{ for } 1 \le i \le m \in \mathbb{N}_{>0} \end{array} \right\}$$

of K_J forms a basis of K_J as a K-Banach space. If $J_1 \subset J_2$ are in $\mathcal{P}(J_K)$, then we have $\mathscr{E}_{J_1} \subset \mathscr{E}_{J_2}$ and the assertion follows from the fact $\{1\} = \bigcap_{J\in\mathcal{P}(J_K)} \mathscr{E}_J$.

1H. *G-regular ring.* We will recall basic facts about G-regular rings. For details, see [Fontaine 1994b, Section 1].

Let E be a topological field and G a topological group. A finite-dimensional E-vector space V is an E-representation of G if V has a continuous E-linear action of G. We denote the category of E-representations of G by $\mathrm{Rep}_E G$. We call B an (E, G)-ring if B is a commutative E-algebra and G acts on B by E-algebra automorphisms. Let B be an (E, G)-ring. For $V \in \mathrm{Rep}_E G$, let $D_B(V) := (B \otimes_E V)^G$

and we will call the following canonical homomorphism the comparison map:

$$\alpha_B(V) : B \otimes_{B^G} D_B(V) \to B \otimes_E V.$$

We say that an (E, G)-ring B is G-regular if the following is satisfied:

($G{\cdot}R_1$) The ring B is reduced.

($G{\cdot}R_2$) For all $V \in \mathrm{Rep}_E G$, $\alpha_B(V)$ is injective.

($G{\cdot}R_3$) Every G-stable E-line in B is generated by an invertible element of B.

Here, a G-stable E-line in B means one-dimensional G-stable E-vector space in B. The condition ($G{\cdot}R_3$) implies that B^G is a field. We say that $V \in \mathrm{Rep}_E G$ is B-admissible if $\alpha_B(V)$ is an isomorphism. We denote the category of B-admissible E-representations of G by $\mathrm{Rep}_{B/E} G$, which is a Tannakian full subcategory of $\mathrm{Rep}_E G$ [Fontaine 1994b, Proposition 1.5.2].

Notation. We will call an object of $\mathrm{Rep}_{\mathbb{Q}_p} G_K$ a p-adic representation of G_K. For a $(\mathbb{Q}_p, G_K)$-ring B, we denote $\mathrm{Rep}_{B/\mathbb{Q}_p} G_K$ by $\mathrm{Rep}_B^{\mathrm{adm}} G_K$ if no confusion arises.

We recall the basic facts about G-regular rings.

Lemma 1.17. *Let B be a field and G a group acting on B by ring automorphisms. Let M be a finite-dimensional B-vector space with semilinear G-action. Then, the canonical map*

$$B \otimes_{B^G} M^G \to M$$

is injective. In particular, we have $\dim_{B^G} M^G \le \dim_B M$.

Proof. Suppose that the assertion does not hold. Let $n \in \mathbb{N}$ be the smallest integer such that there exist n elements $v_1, \dots, v_n \in M^G$ which are linearly independent over B^G but not over B. Let $\sum_{1 \le i \le n} \lambda_i v_i = 0$ be a nontrivial relation with $\lambda_i \in B$. Since B is a field, we may assume that $\lambda_1 = 1$. Then, we have

$$0 = (g-1)\Bigl(\sum_{1 \le i \le n} \lambda_i v_i\Bigr) = \sum_{1 < i \le n} (g(\lambda_i) - \lambda_i) v_i.$$

Hence, we have $\lambda_i \in B^G$ by assumption, which is a contradiction. □

Example 1.18 [Fontaine 1994b, Proposition 1.6.1]. All (E, G)-rings which are fields are G-regular. In fact, we have only to verify ($G{\cdot}R_2$), which follows by applying the above lemma to $M := B \otimes_E V$.

Lemma 1.19 [Fontaine 1994b, Proposition 1.4.2]. *Let B be a G-regular (E, G)-ring and V an E-representation of G. Then, we have* $\dim_{B^G} D_B(V) \le \dim_E V$. *Moreover, the equality holds if and only if V is B-admissible.*

Lemma 1.20 [Fontaine 1994b, Proposition 1.6.5]. *Let B be a G-regular (E,G)-ring and B' an E-subalgebra of B stable by G. Assume that B' satisfies $(G\text{-}R_3)$ and that the canonical map $B^G \otimes_{B'^G} B' \to B$ is injective. Then, B' is a G-regular (E,G)-ring. Moreover, if $V \in \mathrm{Rep}_E G$ is B'-admissible, then V is B-admissible and the canonical map*

$$B^G \otimes_{B'^G} D_{B'}(V) \to D_B(V)$$

is an isomorphism.

Lemma 1.21 [Fontaine 1994b, Corollaire 1.6.6]. *Let B' be an integral domain which is an (E,G)-ring, and B the fraction field of B'. If B' satisfies $(G\text{-}R_3)$ and $B'^G = B^G$, then B' is G_K-regular.*

Remark 1.22 (restriction). Let B be a G-regular (E,G)-ring and H a subgroup of G such that B is H-regular as an (E,H)-ring. If $V \in \mathrm{Rep}_E G$ is B-admissible, then $V|_H$ is also B-admissible in $\mathrm{Rep}_E H$. Moreover, we have a canonical isomorphism $B^H \otimes_{B^G} D_B(V) \cong D_B(V|_H)$. Indeed, the admissibility of V implies that we have the comparison isomorphism $B \otimes_{B^G} D_B(V) \cong B \otimes_E V$ as $B[G_K]$-modules. By taking H-invariants, we have $B^H \otimes_{B^G} D_B(V) \cong D_B(V|_H)$. In particular, we have $\dim_{B^H} D_B(V|_H) = \dim_{B^G} D_B(V) = \dim_E V$, which implies the B-admissibility of $V|_H$ by Lemma 1.19.

2. A generalization of Sen's theorem

The aim of this section is to prove the following generalization of Sen's theorem on $\mathbb{C}_p$-admissible representations [Sen 1980, Corollary in (3.2)].

Theorem 2.1. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$. The following are equivalent:*

(i) *There exists a finite extension L over the maximal unramified extension of K such that G_L acts trivially on V.*

(ii) *V is $\mathbb{C}_p$-admissible.*

(iii) *$V|_{K^{\mathrm{pf}}}$ is $\mathbb{C}_p$-admissible as an object of $\mathrm{Rep}_{\mathbb{Q}_p} G_{K^{\mathrm{pf}}}$.*

Lemma 2.2. *Let E be a field of characteristic 0 and $\rho : U_{\mathbb{Q}_p}^{(n)} \ltimes \prod_{i\in I} p^{n_i}\mathbb{Z}_p \to \mathrm{GL}_r(E)$ a group homomorphism with $n, r \in \mathbb{N}_{>0}$ and $(n_i)_{i\in I} \in \mathbb{N}^I$, where the action of $U_{\mathbb{Q}_p}^{(n)}$ on $\prod_{i\in I} p^{n_i}\mathbb{Z}_p$ is given by scalar multiplication. If $\ker\rho$ contains an open subgroup of $U_{\mathbb{Q}_p}^{(n)}$, then the image of ρ is finite.*

Proof. By shrinking $U_{\mathbb{Q}_p}^{(n)}$, we may assume that $\ker\rho$ contains $U_{\mathbb{Q}_p}^{(n)}$. Also, we may assume that E is algebraically closed. Let $x_0 := 1 + p^n \in U_{\mathbb{Q}_p}^{(n)}$, $\boldsymbol{x} \in \prod_{i\in I} p^{n_i}\mathbb{Z}_p$. By the fact that $\ker\rho$ is a normal subgroup of $U := U_{\mathbb{Q}_p}^{(n)} \ltimes \prod_{i\in I} p^{n_i}\mathbb{Z}_p$ and a simple calculation, we have

$$(1,\boldsymbol{x})^{-1}(x_0,\mathbf{0})(1,\boldsymbol{x})(x_0^{-1},\mathbf{0}) = (1,(x_0-1)\boldsymbol{x}) = (1,p^n\boldsymbol{x}) \in \ker\rho.$$

In particular, $\ker\rho$ contains $U_{\mathbb{Q}_p}^{(n)} \ltimes \prod_{i\in I} p^{n+n_i}\mathbb{Z}_p$ as a normal subgroup. By taking the quotient of U by this subgroup, ρ factors through a group homomorphism $\bar{\rho} : (\mathbb{Z}/p^n\mathbb{Z})^I \to \mathrm{GL}_r(E)$.

To prove the assertion, it suffices to prove that for any finite subset S of $\operatorname{Im}\bar{\rho}$, we have $|S| \le p^{nr}$. Any $g \in \operatorname{Im}\bar{\rho}$ is conjugate to a diagonal matrix whose diagonal entries are in $\mu_{p^n}(E)$ since the order of g divides p^n. Since the elements of S commute, S is simultaneously diagonalizable. Hence, up to conjugation, S is contained in the set $\{\mathrm{diag}(a_1, \dots, a_r) \mid a_i \in \mu_{p^n}(E)\}$, whose order is p^{nr}. $\square$

Proof of Theorem 2.1. The implication (i) $\Rightarrow$ (ii) follows from Hilbert 90 and (ii) $\Rightarrow$ (iii) follows from Remark 1.22. We will prove (iii) $\Rightarrow$ (i). Note that if k_K is perfect, then the assertion is a theorem of Sen ([1980, Corollary in (3.2)]).

By replacing K by a finite extension of K^{ur}, we may assume that k_K is separably closed and K satisfies condition (H). In this case, the assertion to prove is that G_K acts on V via a finite quotient. Since the residue field k_K^{pf} of K^{pf} is algebraically closed, $G_{K^{\mathrm{pf}}} = G_{K^{\mathrm{geo}}}$ acts on V via a finite quotient by Sen's theorem, where $K^{\mathrm{geo}} := \bigcup_{n\in\mathbb{N}} K(\{t_j^{p^{-n}}\}_{j\in J_K})$. Hence, there exists a finite extension L/K such that $G_{LK^{\mathrm{geo}}}$ acts trivially on V. In particular, if we put $K_\infty := K^{\mathrm{geo}}(\mu_{p^\infty})$, then G_{LK_∞} acts trivially on V. In the following, we regard V as a p-adic representation of $G_{LK_\infty/L}$. Take a basis of V and let $\rho' : G_{LK_\infty/L} \to \mathrm{GL}_r(\mathbb{Q}_p)$ be the corresponding matrix presentation of V with $r := \dim_{\mathbb{Q}_p} V$. We have only to prove that the image of ρ' is finite.

Since K satisfies condition (H), we have an isomorphism $G_{K_\infty/K} \cong U_{\mathbb{Q}_p}^{(n_0)} \ltimes \mathbb{Z}_p^{J_K}$, where $n_0 \in \mathbb{N}_{>1}$ satisfies $G_{K(\mu_{p^\infty})/K} \cong U_{\mathbb{Q}_p}^{(n_0)}$ via the cyclotomic character and $U_{\mathbb{Q}_p}^{(n_0)}$ acts on $\mathbb{Z}_p^{J_K}$ by scalar multiplication (see [Hyodo 1986, Section 1] for details). We have $G_{LK_\infty/LK^{\mathrm{geo}}} \le \ker\rho' \trianglelefteq_c G_{LK_\infty/L}$. By using the restriction map $\mathrm{Res}^{LK_\infty}_{K_\infty}$ and the above isomorphism, we may regard these groups as subgroups of $U_{\mathbb{Q}_p}^{(n_0)} \ltimes \mathbb{Z}_p^{J_K}$. Since $G_{LK_\infty/L}$ is an open subgroup of $G_{K_\infty/K}$, there exists $n \in \mathbb{N}$ and $(n_j)_{j\in J_K} \in \mathbb{N}^{J_K}$ such that $G_{LK_\infty/L}$ contains $U := U_{\mathbb{Q}_p}^{(n)} \ltimes \prod_{j\in J_K} p^{n_j}\mathbb{Z}_p$ as an open subgroup. Since $G_{LK_\infty/LK^{\mathrm{geo}}}$ is an open subgroup of $G_{K_\infty/K^{\mathrm{geo}}} \cong G_{K(\mu_{p^\infty})/K} \cong U_{\mathbb{Q}_p}^{(n_0)} \cong \mathbb{Z}_p$, $\ker\rho'$ contains an open subgroup of $U_{\mathbb{Q}_p}^{(n)}$. Therefore, the group homomorphism $\rho := \rho'|_U : U \to \mathrm{GL}_r(\mathbb{Q}_p)$ satisfies the assumption of Lemma 2.2, hence, the image of ρ is finite. Since U is open in $G_{LK_\infty/L}$, we obtain the assertion. $\square$

3. Basic construction of rings of p-adic periods

Throughout this section, let $\mathcal{K}$ be a closed subfield of $\mathbb{C}_p$ whose value group $v_p(\mathcal{K}^\times)$ is discrete. We will recall the construction of rings of p-adic periods

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}, \quad \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}, \quad \mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}, \quad \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}, \quad \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$$

due to Fontaine [1994a], which is functorial with respect to $\mathbb{C}_p$ and $\mathcal{K}$. We also recall abstract algebraic properties of these rings as in [Brinon 2006]. Although we do not assume $\mathcal{K}=K$, standard techniques of proofs in the case $\mathcal{K}=K$, which are developed in [Fontaine 1994a; Brinon 2006], can be applied to our situation.

3A. ***Universal pro-infinitesimal thickenings.***

Defintion 3.1 [Fontaine 1994a, Section 1]. A p-adically formal pro-infinitesimal $\mathcal{O}_{\mathcal{K}}$-thickening of $\mathcal{O}_{\mathbb{C}_p}$ is a pair (D,θ_D), where

- D is an $\mathcal{O}_{\mathcal{K}}$-algebra,
- $\theta_D : D\to\mathcal{O}_{\mathbb{C}_p}$ is a surjective $\mathcal{O}_{\mathcal{K}}$-algebra homomorphism such that D is $(p,\ker\theta_D)$-adic Hausdorff complete.

Obviously, p-adically formal $\mathcal{O}_{\mathcal{K}}$-thickenings of $\mathcal{O}_{\mathbb{C}_p}$ form a category.

Theorem 3.2 [Fontaine 1994a, Théorème 1.2.1]. *The category of p-adically formal pro-infinitesimal $\mathcal{O}_{\mathcal{K}}$-thickenings of $\mathcal{O}_{\mathbb{C}_p}$ admits a universal object*, *that is*, *an initial object.*

Such an object is unique up to a canonical isomorphism and we denote it by $(\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}},\theta_{\mathbb{C}_p/\mathcal{K}})$. Note that $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is functorial with respect to $\mathbb{C}_p$ and $\mathcal{K}$. We recall the construction. Let $R_{\mathbb{C}_p}:=\varprojlim_{x\mapsto x^p}\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ be the perfection of the ring $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$. We have the canonical isomorphism

$$\varprojlim_{x\mapsto x^p}\mathcal{O}_{\mathbb{C}_p}\to R_{\mathbb{C}_p};\quad (x^{(n)})_{n\in\mathbb{N}}\mapsto(x^{(n)}\bmod p\mathcal{O}_{\mathbb{C}_p})_{n\in\mathbb{N}},$$

where the addition and the multiplication of the LHS are given by

$$((x^{(n)})+(y^{(n)}))_n=\lim\nolimits_m(x^{(n+m)}+y^{(n+m)})^{p^m},\quad (x^{(n)})\cdot(y^{(n)})=(x^{(n)}y^{(n)}).$$

Let $\theta_{\mathbb{C}_p/\mathbb{Q}_p}:W(R_{\mathbb{C}_p})\to\mathcal{O}_{\mathbb{C}_p}$ be defined by $\sum_{n\in\mathbb{N}}p^n[x_n]\mapsto\sum_{n\in\mathbb{N}}p^nx_n^{(0)}$. This is a surjective $\mathbb{Z}_p$-algebra homomorphism. Let $\theta_{\mathbb{C}_p/\mathcal{K}}:\mathcal{O}_{\mathcal{K}}\otimes_{\mathbb{Z}}W(R_{\mathbb{C}_p})\to\mathcal{O}_{\mathbb{C}_p}$ be the linear extension of $\theta_{\mathbb{C}_p/\mathbb{Q}_p}$. Then, $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is the Hausdorff completion of $\mathcal{O}_{\mathcal{K}}\otimes_{\mathbb{Z}}W(R_{\mathbb{C}_p})$ with respect to the $(p,\ker\theta_{\mathbb{C}_p/\mathcal{K}})$-adic topology. We will give an explicit description of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ later: Note that the description, together with the isomorphism $W(R_{\mathbb{C}_p})\cong\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$ (Remark 3.5), immediately implies that $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is an integral domain (at least) when we have $\mathcal{K}=\mathcal{K}_0$.

We define $\tilde{t}_j:=(t_j,t_j^{p^{-1}},\dots)\in R_{\mathbb{C}_p}$ and $u_j:=t_j-[\tilde{t}_j]\in\ker\theta_{\mathbb{C}_p/\mathcal{K}_0}$. Let $v_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ be the p-adic semivaluation of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. We put

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\}:=\\ \Big\{\sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}}a_{\boldsymbol{n}}\mathbf{u}^{\boldsymbol{n}}\;\Big|\;a_{\boldsymbol{n}}\in\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p},\{v_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{|\boldsymbol{n}|=n}\to\infty\text{ for all }n\in\mathbb{N}\Big\}.$$

If $J_{\mathcal{X}}$ is finite, $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{X}}}\}$ is a ring of formal power series with coefficients in $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$. We extend $\theta_{\mathbb{C}_p/\mathbb{Q}_p}$ to a surjective $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism $\vartheta_{\mathbb{C}_p/\mathcal{X}}:\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{X}}}\}\to\mathbb{O}_{\mathbb{C}_p}$ by $\vartheta_{\mathbb{C}_p/\mathcal{X}}(\mathrm{u}_j)=0$. Then, $(\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{X}}}\},\vartheta_{\mathbb{C}_p/\mathcal{X}})$ is a p-adically formal $\mathbb{Z}_p$-pro-infinitesimal thickening of $\mathbb{O}_{\mathbb{C}_p}$. We have a canonical $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism

$$\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}:\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{X}}}\}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}};\quad \mathbf{u}^{\boldsymbol{n}}\mapsto\boldsymbol{u}^{\boldsymbol{n}}.$$

Lemma 3.3. *If we assume $\mathcal{X}=\mathcal{X}_0$, then $\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}$ is an isomorphism. In particular, we have*

$$v_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}(x)=\inf_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{X}}}}v_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})$$

for $x=\sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{X}}}}a_{\boldsymbol{n}}\boldsymbol{u}^{\boldsymbol{n}}$ with $a_{\boldsymbol{n}}\in\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$.

Proof. Denote $\mathcal{A}=\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{X}}}\}$ and $\vartheta=\vartheta_{\mathbb{C}_p/\mathcal{X}}$. We regard $\mathbb{O}_{\mathcal{X}}$ as a $\mathbb{Z}[T_j]_{j\in J_{\mathcal{X}}}$-algebra as in Section 1A. We recall that since $\mathcal{X}=\mathcal{X}_0$, the map $\mathbb{Z}[T_j]_{j\in J_{\mathcal{X}}}\to\mathbb{O}_{\mathcal{X}}$ is formally étale for the p-adic topology. We also regard $\mathcal{A}$ as a $\mathbb{Z}[T_j]_{j\in J_{\mathcal{X}}}$-algebra by $T_j\mapsto[\tilde{t}_j]+\mathrm{u}_j$. Then, by the lifting property, we can lift the canonical $\mathbb{O}_{\mathcal{X}}$-algebra structure on $\mathcal{A}/(p,\ker\vartheta)\cong\mathbb{O}_{\mathbb{C}_p}/(p)$ to an $\mathbb{O}_{\mathcal{X}}$-algebra structure on $\mathcal{A}\cong\varprojlim_n\mathcal{A}/(p,\ker\vartheta)^n$:

$$\begin{tikzcd} \mathbb{O}_{\mathcal{X}} \arrow[r, "\text{can.}"] \arrow[dr, dashed, "\exists!"] & \mathbb{O}_{\mathbb{C}_p} \\ \mathbb{Z}[T_j]_{j\in J_{\mathcal{X}}} \arrow[u, "\text{str.}"] \arrow[r, "\text{str.}"] & \mathcal{A} \arrow[u, "\vartheta"'] \end{tikzcd}$$

By this structure map, we may regard $\mathcal{A}$ as a pro-infinitesimal $\mathbb{O}_{\mathcal{X}}$-thickening of $\mathbb{O}_{\mathbb{C}_p}$. By universality, we have only to prove that $\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}$ is an $\mathbb{O}_{\mathcal{X}}$-algebra homomorphism. Let $\alpha:\mathbb{O}_{\mathcal{X}}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}$ be the composition of the structure map $\mathbb{O}_{\mathcal{X}}\to\mathcal{A}$ and $\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}$. Since $\iota_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}$ commutes with the projections ϑ and $\theta_{\mathbb{C}_p/\mathcal{X}}$, we have the commutative diagram

$$\begin{tikzcd} \mathbb{O}_{\mathcal{X}} \arrow[r, "\text{can.}"] \arrow[dr, "\alpha"] & \mathbb{O}_{\mathbb{C}_p} \\ \mathbb{Z}[T_j]_{j\in J_{\mathcal{X}}} \arrow[u, "\text{str.}"] \arrow[r, "\text{str.}"] & \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}, \arrow[u, "\theta_{\mathbb{C}_p/\mathcal{X}}"'] \end{tikzcd}$$

where the horizontal structure map is given by $T_j\mapsto t_j$. By this diagram and the lifting property, α coincides with the structure map $\mathbb{O}_{\mathcal{X}}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}$ modulo $(p,\ker\theta_{\mathbb{C}_p/\mathcal{X}})^n$ for all $n\in\mathbb{N}$. Since $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}$ is $(p,\ker\theta_{\mathbb{C}_p/\mathcal{X}})$-adically Hausdorff complete, α coincides with the structure map $\mathbb{O}_{\mathcal{X}}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{X}}$, which implies the assertion. $\square$

For general $\mathscr{K}$, we have:

Lemma 3.4. (i) *The canonical map*

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathscr{K}} \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathscr{K}^{\mathrm{ur}}}$$

is an isomorphism.

(ii) *If $\mathscr{L}/\mathscr{K}$ is a finite extension with $[k_{\mathscr{L}} : k_{\mathscr{K}}]_{\mathrm{sep}} = 1$, then the canonical map*

$$\mathcal{O}_{\mathscr{L}} \otimes_{\mathcal{O}_{\mathscr{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathscr{K}} \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathscr{L}}$$

is an isomorphism.

(iii) *Let $\mathscr{L}$ be a finite extension of the p-adic completion of an unramified extension of $\mathscr{K}$. Then, the canonical map*

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathscr{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathscr{K}})^n \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathscr{L}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathscr{L}})^n$$

is an isomorphism for all $n \in \mathbb{N}$.

Proof. (i) The assertion is equivalent to saying that the category of p-adically formal $\mathcal{O}_{\mathscr{K}}$-pro-infinitesimal thickening of $\mathcal{O}_{\mathbb{C}_p}$ is equivalent to the category of p-adically formal $\mathcal{O}_{\mathscr{K}^{\mathrm{ur}}}$-pro-infinitesimal thickening of $\mathcal{O}_{\mathbb{C}_p}$. Let (D, θ_D) be a p-adically formal $\mathcal{O}_{\mathscr{K}}$-pro-infinitesimal thickening of $\mathcal{O}_{\mathbb{C}_p}$. Then, we have only to prove that there exists a unique $\mathcal{O}_{\mathscr{K}}$-algebra homomorphism $\mathcal{O}_{\mathscr{K}^{\mathrm{ur}}} \to D$ such that θ_D is an $\mathcal{O}_{\mathscr{K}^{\mathrm{ur}}}$-algebra homomorphism. By dévissage, we may replace D by $D/(p, \ker\theta_D)^n$ with $n \in \mathbb{N}$. Since θ_D induces an isomorphism $D/(p, \ker\theta_D) \cong \mathcal{O}_{\mathbb{C}_p}/(p)$ and $\mathcal{O}_{\mathscr{K}^{\mathrm{ur}}}/\mathcal{O}_{\mathscr{K}}$ is p-adically formally étale, the assertion follows from the commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathscr{K}^{\mathrm{ur}}} \arrow[r, "\mathrm{can.}"] \arrow[dr, dashed, "\exists!"] & \mathcal{O}_{\mathbb{C}_p}/(p) \\
\mathcal{O}_{\mathscr{K}} \arrow[u, "\mathrm{can.}"] \arrow[r, "\mathrm{str.}"'] & D/(p, \ker\theta_D)^n, \arrow[u, "(\theta_D)_*"']
\end{tikzcd}$$

where $(\theta_D)_*$ is the ring homomorphism induced by θ_D.

(ii) By assumption, the canonical map $\mathcal{O}_{\mathscr{L}} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathscr{K}^{\mathrm{ur}}} \to \mathcal{O}_{\mathscr{L}^{\mathrm{ur}}}$ is an isomorphism. By using this fact and (i), we may assume that $\mathscr{K} = \mathscr{K}^{\mathrm{ur}}$ and $\mathscr{L} = \mathscr{L}^{\mathrm{ur}}$. In particular, we may consider the case that $k_{\mathscr{K}}$ is separably closed, where the condition $[k_{\mathscr{L}} : k_{\mathscr{K}}]_{\mathrm{sep}} = 1$ is always satisfied. By faithfully flat descent, the assertion is reduced to the case that $\mathscr{L}/\mathscr{K}$ is Galois. Since $\mathscr{L}/\mathscr{K}$ is a solvable extension [Fesenko and Vostokov 2002, Exercise 2, Section 2, Chapter II], we may assume that $\mathscr{L}/\mathscr{K}$ has prime degree.

By universality, we have only to prove that the LHS is a p-adically formal $\mathcal{O}_{\mathscr{L}}$-pro-infinitesimal thickening of $\mathcal{O}_{\mathbb{C}_p}$. Hence, it suffices to verify that $\mathcal{O}_{\mathscr{L}} \otimes_{\mathcal{O}_{\mathscr{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathscr{K}}$ is (p, I)-adically Hausdorff complete, where I denotes the kernel of the canonical map $1 \otimes \theta_{\mathbb{C}_p/\mathscr{K}} : \mathcal{O}_{\mathscr{L}} \otimes_{\mathcal{O}_{\mathscr{K}}} \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathscr{K}} \to \mathcal{O}_{\mathbb{C}_p}$. Since we have an isomorphism of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$-modules $\mathcal{O}_{\mathcal{L}}\otimes_{\mathcal{O}_{\mathcal{K}}}\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}\cong(\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}})^{[\mathcal{L}:\mathcal{K}]}$, we have only to prove that the topologies on $\mathcal{O}_{\mathcal{L}}\otimes_{\mathcal{O}_{\mathcal{K}}}\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ defined by the ideals (p,I) and (p,I') are equivalent, where I' denotes the ideal of $\mathcal{O}_{\mathcal{L}}\otimes_{\mathcal{O}_{\mathcal{K}}}\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ generated by $\ker(\theta_{\mathbb{C}_p/\mathcal{K}}:\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}\to\mathcal{O}_{\mathbb{C}_p})$. By definition, we have $(p,I')\subset(p,I)$. We have only to prove that we have $I^n\subset(\pi_{\mathcal{K}}\otimes 1,I')$ for some $n\in\mathbb{N}$ since p divides $\pi_{\mathcal{K}}^{e_{\mathcal{K}}}$.

In the following, for $x\in\mathcal{O}_{\mathbb{C}_p}$, we denote by $\tilde{x}$ any element $\tilde{x}\in R_{\mathbb{C}_p}$ such that $\tilde{x}^{(0)}=x$. Since we have $\pi_{\mathcal{K}}\otimes 1-1\otimes[\tilde{\pi}_{\mathcal{K}}]\in I'$, we have $(\pi_{\mathcal{K}}\otimes 1,1\otimes[\tilde{\pi}_{\mathcal{K}}])\subset(\pi_{\mathcal{K}}\otimes 1,I')$. Note that if $x\in\mathcal{O}_{\mathcal{L}}$ is primitive, that is, $1,x,\dots,x^{[\mathcal{L}:\mathcal{K}]-1}$ is an $\mathcal{O}_{\mathcal{K}}$-basis of $\mathcal{O}_{\mathcal{L}}$, then we have $I\subset(x\otimes 1-1\otimes[\tilde{x}],I')$. Hence, we have only to prove the existence of a primitive element $x\in\mathcal{O}_{\mathcal{L}}$ satisfying $(x\otimes 1-1\otimes[\tilde{x}])^n\in(\pi_{\mathcal{K}}\otimes 1,I')$ for some $n\in\mathbb{N}$. In the case $[\mathcal{L}:\mathcal{K}]=e_{\mathcal{L}/\mathcal{K}}$, $\pi_{\mathcal{L}}$ is a primitive element of $\mathcal{O}_{\mathcal{L}}$ and we have $(\pi_{\mathcal{L}}\otimes 1-1\otimes[\tilde{\pi}_{\mathcal{L}}])^{2e_{\mathcal{L}/\mathcal{K}}}\in(\pi_{\mathcal{K}}\otimes 1,1\otimes[\tilde{\pi}_{\mathcal{K}}])$. Otherwise, we have $[\mathcal{L}:\mathcal{K}]=[k_{\mathcal{L}}:k_{\mathcal{K}}]_{\mathrm{insep}}=p$. If we choose $x\in\mathcal{O}_{\mathcal{L}}$ whose image in $\mathcal{O}_{\mathcal{L}}/\pi_{\mathcal{K}}\mathcal{O}_{\mathcal{L}}$ does not belong to $k_{\mathcal{K}}$, then x is primitive by Nakayama's lemma. Moreover, if we choose $a\in\mathcal{O}_{\mathcal{K}}$ such that $x^p\equiv a \mod \pi_{\mathcal{K}}\mathcal{O}_{\mathcal{L}}$, then we have

$$(x\otimes 1-1\otimes[\tilde{x}])^p\equiv a\otimes 1-1\otimes[\tilde{a}]\mod(\pi_{\mathcal{K}}\otimes 1,1\otimes[\tilde{\pi}_{\mathcal{K}}])$$

and $a\otimes 1-1\otimes[\tilde{a}]\in I'$, which implies the assertion.

(iii) We denote the map by i and we will construct the inverse. By replacing $\mathcal{K}$ and $\mathcal{L}$ by $\mathcal{K}^{\mathrm{ur}}$ and $\mathcal{L}^{\mathrm{ur}}$, we may assume $[k_{\mathcal{L}}:k_{\mathcal{K}}]_{\mathrm{sep}}=1$. By (ii), we identify $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}$ with $\mathcal{O}_{\mathcal{L}}\otimes_{\mathcal{O}_{\mathcal{K}}}\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. Since $\mathcal{L}/\mathcal{K}$ is étale, by a similar argument as in the proof of (i), we have a unique $\mathcal{K}$-algebra homomorphism

$$j:\mathcal{L}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$$

such that $\theta_{\mathbb{C}_p/\mathcal{K}}:\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n\to\mathbb{C}_p$ is an $\mathcal{L}$-algebra homomorphism. Hence, we have the $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$-algebra homomorphism

$$j\otimes\mathrm{id}:\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{L}})^n\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n.$$

By construction, we have $(j\otimes\mathrm{id})\circ i=\mathrm{id}$. To prove $i\circ(j\otimes\mathrm{id})=\mathrm{id}$, we have only to prove that $i\circ(j\otimes\mathrm{id})$ is an $\mathcal{L}$-algebra homomorphism, which follows from the uniqueness of j. □

Remark 3.5. We may identify $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$ with $W(R_{\mathbb{C}_p})$ [Fontaine 1994a, 1.2.4(e)] and the kernel of $\theta_{\mathbb{C}_p/\mathbb{Q}_p}$ is principal by [Fontaine 1994a, 2.3.3]. Moreover, if $\mathcal{K}=\mathcal{K}_0$ and $k_{\mathcal{K}}$ is perfect, then the canonical map $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ is an isomorphism [Fontaine 1994a, 1.2.4(e)]. Note that we have no canonical choice of an embedding $W(k_K^{\mathrm{alg}})[p^{-1}]\to\mathbb{C}_p$ when k_K is imperfect, since different perfections of K induce different embeddings. Thus, we can not endow $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$ with a canonical $W(k_K^{\mathrm{alg}})$-algebra structure induced by that of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/W(k_K^{\mathrm{alg}})[p^{-1}]}$ via the above isomorphism as in the perfect residue field case.

3B. $\mathbb{B}_{\mathbf{dR}}$ ***and*** $\mathbb{B}_{\mathbf{HT}}$. We define $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} := \varprojlim_n \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$ and

$$t := \log([\varepsilon]) = \sum_{n\in\mathbb{N}_{>0}} (-1)^{n-1}\frac{([\varepsilon]-1)^n}{n} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$$

with $\varepsilon := (1,\zeta_p,\zeta_{p^2},\dots) \in R_{\mathbb{C}_p}$. We also define $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} := \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}[t^{-1}]$. We denote the projection $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \to \mathbb{C}_p$ by $\theta_{\mathbb{C}_p/\mathcal{K}}$ again. Then, $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ is a Hausdorff complete local ring with maximal ideal $\ker\theta_{\mathbb{C}_p/\mathcal{K}}$. Moreover, $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ is an integral domain. In fact, by the following explicit description of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$, it follows from the fact that $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ is a field (Remark 3.6(ii) below).

We define the canonical topology on $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ as follows. We regard

$$\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$$

as a p-adic Banach space whose lattice is given by the image of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. Then, we endow $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with the inverse limit topology, which is a Fréchet complete $\mathcal{K}$-algebra. We also endow $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with a limit of Fréchet topology by regarding $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ as the direct limit of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with respect to the multiplication by t^{-1}. Let $v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ be the semivaluation of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ induced by the p-adic semivaluation of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$ defined by the lattice

$$\mathrm{Im}(\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \xrightarrow{\text{can.}} \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n).$$

Obviously, the semivaluations $\{v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}\}_{n\in\mathbb{N}}$ are decreasing.

We will give an explicit description of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$. Let

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\} := \Big\{ \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}} a_{\boldsymbol{n}}\mathbf{u}^{\boldsymbol{n}} \;\Big|\; a_{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}, \{v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{|\boldsymbol{n}|=n} \to \infty \text{ for all } n,r\in\mathbb{N}\Big\}.$$

This is a $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-algebra. Then, the canonical $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism

$$\iota_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} : \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\{\mathbf{u}_{J_{\mathcal{K}}}\} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}; \quad \mathbf{u}^{\boldsymbol{n}} \mapsto \boldsymbol{u}^{\boldsymbol{n}}$$

is an isomorphism. To prove this, by Remark 3.6(ii) below, we may reduce to the case $\mathcal{K} = \mathcal{K}_0$. In this case, the assertion follows from the explicit description of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$.

For $n\in\mathbb{N}$, let $\mathrm{Fil}^n\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ be the closed ideal of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ generated by the ideal $(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$. We endow $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with the decreasing filtration defined by $\mathrm{Fil}^n\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} := \sum_{i+j=n} t^i\mathrm{Fil}^j\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$. Denote the graded $\mathbb{C}_p$-algebra associated to the filtration by $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$. We also denote by v_j the image of u_j/t in $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K,0}$ for $j\in J_K$. Since the filtration is compatible with the multiplication by t, that is, $t^m\mathrm{Fil}^n\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} = \mathrm{Fil}^{n+m}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$, we have an isomorphism $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}} \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K},0}t^n$.

For $n \in \mathbb{N}$, let

$$\mathbb{C}_p\{\mathbf{v}_{J_{\mathcal{K}}}\}_n := \left\{ \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}:|\boldsymbol{n}|=n} a_{\boldsymbol{n}}\mathbf{v}^{\boldsymbol{n}} \;\middle|\; a_{\boldsymbol{n}} \in \mathbb{C}_p, \{v_p(a_{\boldsymbol{n}})\}_{\boldsymbol{n}} \to \infty \right\}$$

and $\mathbb{C}_p\{\mathbf{v}_{J_{\mathcal{K}}}\} := \bigoplus_{n\in\mathbb{N}} \mathbb{C}_p\{\mathbf{v}_{J_{\mathcal{K}}}\}_n$. We have a $\mathbb{C}_p$-algebra homomorphism

$$\iota_{\mathrm{HT},\mathbb{C}_p/\mathcal{K},0} : \mathbb{C}_p\{\mathbf{v}_{J_{\mathcal{K}}}\} \to \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K},0}; \quad \mathbf{v}^{\boldsymbol{n}} \mapsto \boldsymbol{v}^{\boldsymbol{n}},$$

which is an isomorphism. One reduces to the case $\mathcal{K} = \mathcal{K}_0$ by Remark 3.6(ii) below. Then, the assertion follows from the above explicit description of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and the formula of the semivaluation $v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ (Remark 3.6(iii) below). By this description, $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ is an integral domain.

Remark 3.6. (i) (The perfect residue field case) Assume that $k_{\mathcal{K}}$ is perfect. Then, we have a canonical isomorphism $\mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathcal{K}}$ for $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$. Moreover, $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ is a complete discrete valuation field of equal characteristic 0 with valuation ring $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$, t is a uniformizer and the residue field is $\mathbb{C}_p$. We also have an isomorphism $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p} \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p t^n$. In fact, the first assertion follows from Remark 3.5 and the latter assertion reduces to the case where k_K is perfect by regarding $\mathbb{C}_p$ as the p-adic completion of $(K^{\mathrm{pf}})^{\mathrm{alg}}$ [Fontaine 1994a, 1.5.1].

(ii) (Invariance) The above structures on $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ (ring structure, filtration, topology) are invariant under finite or unramified extensions. As a consequence, we may regard $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ as a $\mathcal{K}^{\mathrm{alg}}$-algebra and a similar invariance for $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ as a graded $\mathbb{C}_p$-algebra also holds. As for a filtered ring, the invariance follows from Lemma 3.4(iii). To prove the rest of the assertion, we have only to prove that for an unramified extension or a finite extension $\mathcal{L}/\mathcal{K}$, the p-adic semivaluations $v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and $v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{L}}$ are equivalent for all $n \in \mathbb{N}$. The unramified case follows from Lemma 3.4(i). In the other case, let $\Lambda^{(n)}_{\mathcal{K}}$ (resp. $\Lambda^{(n)}_{\mathcal{L}}$) be the image of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ (resp. $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}$) in $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}/(\ker\theta_{\mathbb{C}_p/\mathcal{K}})^n$. Replacing $\mathcal{K}$ by the maximal unramified extension of $\mathcal{K}$ in $\mathcal{L}$, we may assume that $\mathcal{L}/\mathcal{K}$ satisfies the assumption in Lemma 3.4(ii). Since $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{L}}$ is a finite $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$-module by Lemma 3.4(ii), there exists $m \in \mathbb{N}$ such that $p^m\Lambda^{(n)}_{\mathcal{L}} \subset \Lambda^{(n)}_{\mathcal{K}}$ by Lemma 3.4(iii). Since we have $\Lambda^{(n)}_{\mathcal{K}} \subset \Lambda^{(n)}_{\mathcal{L}}$ by definition, the two p-adic topologies induced by the lattices $\Lambda^{(n)}_{\mathcal{K}}$ and $\Lambda^{(n)}_{\mathcal{L}}$ respectively are equivalent, which implies the assertion.

(iii) Assume $\mathcal{K} = \mathcal{K}_0$. Then, we have the formula

$$v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(x) = \inf_{|\boldsymbol{n}|<n} v^{(n)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}),$$

where we have $x = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}}\boldsymbol{u}^{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ with $a_{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$. This follows from the explicit description of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$.

3C. ***Connections on*** $\mathbb{B}_{\mathbf{dR}}$ ***and*** $\mathbb{B}_{\mathbf{HT}}$. We denote by $\widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ the direct limit $\varinjlim\widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$, where the transition maps are the multiplication by $1\otimes t^{-1}$. Then, the canonical derivation $d:\mathcal{K}\to\widehat{\Omega}^1_{\mathcal{K}}$ uniquely extends to a $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-linear continuous derivation

$$\nabla:\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}\to\widehat{\Omega}^1_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}.$$

Indeed, the canonical derivation $d:\mathbb{O}_{\mathcal{K}}\to\widehat{\Omega}^1_{\mathbb{O}_{\mathcal{K}}}$ extends to an $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathbb{Q}_p}$-linear derivation $d:\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}\to\widehat{\Omega}^1_{\mathbb{O}_{\mathcal{K}}}\widehat{\otimes}_{\mathbb{O}_{\mathcal{K}}}\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$ by the construction of $\mathbb{A}_{\mathrm{inf}}$. After inverting p, then taking the $\ker\theta_{\mathbb{C}_p/\mathcal{K}}$-adic Hausdorff completion, we obtain a desired derivation. Since the image of $\mathcal{K}\otimes_{\mathbb{Z}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ is dense in $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ by construction, the uniqueness follows. More precisely, if we denote by $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ the derivations on $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ given by $\nabla(x)=\sum_{j\in J_{\mathcal{K}}}dt_j\otimes\partial_j(x)$, then $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ are mutually commutative continuous $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-derivations and we have $\partial_j=\partial/\partial u_j$. More generally, the exterior derivation $d_q:\widehat{\Omega}^q_{\mathcal{K}}\to\widehat{\Omega}^{q+1}_{\mathcal{K}}$ for $q\in\mathbb{N}_{>0}$ uniquely extends to a $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$-linear continuous homomorphism

$$\nabla_q:\widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}\to\widehat{\Omega}^{q+1}_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$$

such that we have $\nabla_q(\omega\otimes x)=\nabla_q(\omega)\otimes x+(-1)^q\omega\wedge\nabla(x)$ for $x\in\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and $\omega\in\widehat{\Omega}^q_{\mathcal{K}}$. Obviously, the connection ∇ satisfies Griffith transversality

$$\nabla(\mathrm{Fil}^n\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}})\subset\widehat{\Omega}^1_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathrm{Fil}^{n-1}\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$$

for $n\in\mathbb{Z}$. These connections are invariant under finite or unramified extensions by Lemma 1.10(iii) and Remark 3.6(ii).

Notation. We will use the following notation:

$$\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}:=(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}})^{\nabla=0},\ \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}:=(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}})^{\nabla=0}$$
$$\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}:=\mathrm{Im}(\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p}\xrightarrow{\text{can.}}\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}).$$

We endow the first two rings with induced filtrations and the last one with an induced graded structure. Note that these rings are invariant under finite or unramified extensions of $\mathcal{K}$ and that $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and $\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ (resp. $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$) have a canonical $(\mathcal{K}_{\mathrm{can}})^{\mathrm{alg}}$-algebra (resp. $\mathbb{C}_p$-algebra) structure. By the above description of the connection and the explicit descriptions of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$, we have:

Lemma 3.7. *The canonical maps*

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}},\quad\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}},\quad\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$$

are isomorphisms. These maps are compatible with filtrations and gradings.

Remark 3.8. Assume that $[k_{\mathcal{K}} : k_{\mathcal{K}}^p] < \infty$. Since $\widehat{\Omega}^1_{\mathcal{K}}$ is a finite-dimensional $\mathcal{K}$-vector space (Remark 1.11), the connection $\nabla : \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \to \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ induces a $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p}$-linear derivation

$$\nabla : \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}} \to \widehat{\Omega}^1_{\mathcal{K}} \otimes_{\mathcal{K}} \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}.$$

More precisely, if we denote by $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ the derivations on $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ defined as above, then, by the explicit description of $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$, $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ are commuting $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathbb{Q}_p}$-linear derivations and we have $\partial_j = t\partial/\partial v_j$. In particular, $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ coincides with $(\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}})^{\nabla=0}$. In the general case, we must handle complicated topologies to define such a connection. To avoid it, we define $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/\mathcal{K}}$ in an ad-hoc way as above.

We also have an analogue of Poincaré lemma.

Lemma 3.9. *The complex*

$$0 \longrightarrow \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \xrightarrow{\text{inc.}} \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \xrightarrow{\nabla} \widehat{\Omega}^1_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}} \xrightarrow{\nabla_1} \widehat{\Omega}^2_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$$

is exact.

Proof. By the invariance of the above complex under a finite extension, we may assume $\mathcal{K} = \mathcal{K}_0$. Recall the explicit description of $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ in Section 3B. Since we have $v_p(\boldsymbol{n}!) \le |\boldsymbol{n}|$ for $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_{\mathcal{K}}}$, $x \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ is written uniquely in the form $x = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}}\boldsymbol{u}^{[\boldsymbol{n}]}$ with $a_{\boldsymbol{n}} \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ such that $\{v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{|\boldsymbol{n}|=n} \to \infty$ for all $r, n \in \mathbb{N}$. Moreover, we have the inequality

$$\inf_{|\boldsymbol{n}|<r} v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}) + r > \inf_{|\boldsymbol{n}|<r} v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(\boldsymbol{n}!\,a_{\boldsymbol{n}}) = v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(x) \tag{1}$$

by Remark 3.6(iii). We have only to prove that there exists $x \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ such that $\nabla(x) = \omega$ for $\omega \in \ker\nabla_1$. Write $\omega = \sum_{j\in J_{\mathcal{K}}} dt_j \otimes \lambda_j$ with $\lambda_j \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ such that $\{v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(\lambda_j)\}_{j\in J_{\mathcal{K}}} \to \infty$ for all $r \in \mathbb{N}$. The assumption $\omega \in \ker\nabla_1$ implies that we have $\partial_{j'}(\lambda_j) = \partial_j(\lambda_{j'})$ for $j, j' \in J_{\mathcal{K}}$. As above, we can write $\lambda_j = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} \lambda_{j,\boldsymbol{n}}\boldsymbol{u}^{[\boldsymbol{n}]}$, where $\lambda_{j,\boldsymbol{n}} \in \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ satisfies the convergence condition as above. We have the relation $\lambda_{j,\boldsymbol{n}+\mathrm{e}_{j'}} = \lambda_{j',\boldsymbol{n}+\mathrm{e}_j}$ for $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_{\mathcal{K}}}$ and $j, j' \in J_{\mathcal{K}}$. We will define a sequence $\{a_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}}$ in $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ as follows: Put $a_{\boldsymbol{0}}$ equal to 0. For $\boldsymbol{n} \neq \boldsymbol{0}$, choose any $j \in J_{\mathcal{K}}$ such that $n_j \neq 0$ and define $a_{\boldsymbol{n}} := \lambda_{j,\boldsymbol{n}-\mathrm{e}_j}$. By the above relation, this is independent of the choice of j. To prove the assertion, it suffices to prove that we have $\{v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{|\boldsymbol{n}|=n} \to \infty$ for all $r, n \in \mathbb{N}$. Indeed, if this is proved, we see that the element $x := \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}}\mathbf{u}^{[\boldsymbol{n}]}$ belongs to $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ and we have $\nabla(x) = \omega$. We have only to prove that, for fixed $r, n, N \in \mathbb{N}$, we have $v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}) \ge N$ for all but finitely many $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_{\mathcal{K}}}$ such that $|\boldsymbol{n}| = n$. We may assume $r \ge n$. Choose a finite subset J of $J_{\mathcal{K}}$ such that $v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(\lambda_j) \ge r + N$

for $j \in J_{\mathcal{K}} \setminus J$. Let $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_K}$ such that $|\boldsymbol{n}| = n$. If there exists $j \in J_{\mathcal{K}} \setminus J$ such that $n_j \neq 0$, then we have

$$v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}) = v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}-\mathbf{e}_j}) > v^{(r)}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}(\lambda_j) - r \geq r + N - r = N,$$

where the first inequality follows from inequality (1). This implies the assertion since our exceptional set $\{\boldsymbol{n} \in \mathbb{N}^J \mid |\boldsymbol{n}| = n\}$ is finite. □

3D. *Universal PD-thickenings.*

Defintion 3.10. A p-adically formal $\mathcal{O}_{\mathcal{K}}$-PD-thickening of $\mathcal{O}_{\mathbb{C}_p}$ is a triple

$$(D, \theta_D, \gamma_D),$$

where

- D is a p-adically Hausdorff complete $\mathcal{O}_{\mathcal{K}}$-algebra,
- $\theta_D : D \to \mathcal{O}_{\mathbb{C}_p}$ is a surjective $\mathcal{O}_{\mathcal{K}}$-algebra homomorphism,
- γ_D is a PD-structure on $\ker\theta_D$, compatible with the canonical PD-structure on the ideal (p).

Obviously, p-adically formal $\mathcal{O}_{\mathcal{K}}$-thickenings of $\mathcal{O}_{\mathbb{C}_p}$ form a category.

Theorem 3.11 [Fontaine 1994b, Théorème 2.2.1]. *The category of p-adically formal $\mathcal{O}_{\mathcal{K}}$-thickenings of $\mathcal{O}_{\mathbb{C}_p}$ admits a universal object*, *that is*, *an initial object.*

Such an object is unique up to a canonical isomorphism and we denote it by $(\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}, \theta_{\mathbb{C}_p/\mathcal{K}}, \gamma)$. Let's recall the construction. Let $(\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}))^{\mathrm{PD}}$ be the PD-envelope of $\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p})$ with respect to the ideal

$$\ker(\theta_{\mathbb{C}_p/\mathcal{K}} : \mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}),$$

compatible with the canonical PD-structure on the ideal (p). Then, $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is the p-adic Hausdorff completion of $(\mathcal{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}))^{\mathrm{PD}}$.

Remark 3.12. (i) By [Fontaine 1994a, Remarques 2.2.3], if we have $\mathcal{K} = \mathcal{K}_0$ and $k_{\mathcal{K}}$ is perfect, then the canonical map $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is an isomorphism.

(ii) By a similar proof as Lemma 3.4(i), the canonical map

$$\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}^{\mathrm{ur}}}$$

is an isomorphism. In general, we have no invariance for $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ as in Remark 3.6(ii) even after inverting p.

If $\mathcal{K} = \mathcal{K}_0$ and $k_{\mathcal{K}}$ is perfect, then we have an explicit description of $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$:

$$\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} = \left\{ \sum_{n\in\mathbb{N}} a_n \frac{\omega^n}{n!} \;\middle|\; a_n \in \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}, \{v_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}(a_n)\}_{n\in\mathbb{N}} \to \infty \right\},$$

where ω denotes a generator of $\ker(\theta_{\mathbb{C}_p/\mathcal{K}} : \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}} \to \mathbb{O}_{\mathbb{C}_p})$. Note that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is not uniquely determined. Moreover, we have $t \in \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ and $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is an integral domain of characteristic 0 whose PD-structure is given by $\gamma_n(x) = x^{[n]} = x^n/n!$ for $x \in \ker\theta_{\mathbb{C}_p/\mathcal{K}}$. In fact, the assertions follow from the case $\mathcal{K} = K_0^{\mathrm{pf}}$ by Remark 3.5 and Remark 3.12(i), and the assertion in this case follows from [Fontaine 1994a, 2.3.3].

We define $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} := \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}[p^{-1}]$ and $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} := \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}[t^{-1}]$. We also define $\mathbb{A}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} := \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}[\mathrm{x}]$, where x is a formal variable, and we set $\mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} := \mathbb{A}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}[p^{-1}]$ and $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} := \mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}[t^{-1}]$. We define a monodromy operator N on $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$ as the $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$-derivation $N := -d/d\mathrm{x}$. We denote by $v_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ the p-adic semivaluation on $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ (or $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$) defined by the lattice $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$.

In the following, we will give an explicit description of $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. Let

$$\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}\langle \mathbf{u}_{J_{\mathcal{K}}}\rangle$$

be the p-adic Hausdorff completion of the PD-polynomial $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-algebra on the indeterminates $\{\mathrm{u}_j\}_{j\in J_{\mathcal{K}}}$. Note that the PD-structure is given by $\gamma_n(\mathrm{u}_j) = \mathrm{u}_j^n/n! = \mathrm{u}_j^{[n]}$ for $n \in \mathbb{N}$ and $j \in J_{\mathcal{K}}$. We also have

$$\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}\langle \mathbf{u}_{J_{\mathcal{K}}}\rangle = \Big\{ \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}}\mathbf{u}^{[\boldsymbol{n}]} \;\Big|\; a_{\boldsymbol{n}} \in \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}, \{v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})\}_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} \to \infty \Big\}.$$

We regard $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ as an $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-algebra by functoriality. Then, by the universal property of PD-polynomial algebras, we have the $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism

$$\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} : \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}\langle \mathbf{u}_{J_{\mathcal{K}}}\rangle \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}; \quad \mathbf{u}^{[\boldsymbol{n}]} \mapsto \boldsymbol{u}^{[\boldsymbol{n}]}.$$

Lemma 3.13. *If $\mathcal{K} = \mathcal{K}_0$, then $\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is an isomorphism. Moreover, we have*

$$v_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}(x) = \inf_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}})$$

for $x = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_{\mathcal{K}}}} a_{\boldsymbol{n}}\boldsymbol{u}^{[\boldsymbol{n}]} \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ with $a_{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$.

We use the following lemma in the proof:

Lemma 3.14. *We also assume that $\mathcal{K} = \mathcal{K}_0$ and we use the notation in Section 1A.*

(i) *If R is a p-adically Hausdorff complete $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$-algebra, then the canonical map*

$$\mathrm{Hom}_{\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}}(\mathbb{O}_{\mathcal{K}}, R) \to \mathrm{Hom}_{\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}, R/(p))$$

is bijective, where the $\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}$*-algebra structure on* $k_{\mathcal{K}}$ *(resp.* $R/(p)$*) is given by* $T_j \mapsto \bar{t}_j$ *(resp. is induced by* $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}} \to R$*). Moreover, the restriction map*

$$|_{k_{\mathcal{K}}^p} : \mathrm{Hom}_{\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}, R/(p)) \to \mathrm{Hom}_{\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}^p, R/(p))$$

is bijective, where the $\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}}$*-algebra structure on* $k_{\mathcal{K}}^p$ *(resp.* $R/(p)$*) is given by* $T_j^p \mapsto t_j^p$ *(resp. the composition of the inclusion* $\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}} \to \mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}$ *and the above structure map* $\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}} \to R/(p)$*).*

(ii) *Let* $\vartheta : S \to R$ *be a surjective homomorphism of* p*-adically Hausdorff complete* $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$*-algebras, whose kernel admits a PD-structure, compatible with the canonical PD-structure on the ideal* (p)*. Then, the canonical map*

$$\vartheta_* : \mathrm{Hom}_{\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}}(\mathcal{O}_{\mathcal{K}}, S) \to \mathrm{Hom}_{\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}}(\mathcal{O}_{\mathcal{K}}, R); \quad f \mapsto \vartheta \circ f$$

is bijective.

Proof. (i) The first claim follows from the p-adic formal étaleness of $\mathcal{O}_{\mathcal{K}}/\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$. The latter assertion follows by using the isomorphism of $k_{\mathcal{K}}^p$-algebras

$$k_{\mathcal{K}}^p[T_j]_{j\in J_{\mathcal{K}}}/(\{T_j^p - \bar{t}_j^p\}_{j\in J_{\mathcal{K}}}) \cong k_{\mathcal{K}}; \quad \overline{T}_j \mapsto \bar{t}_j.$$

(ii) We denote by $\vartheta_1 : S/(p) \to R/(p)$ the ring homomorphism induced by ϑ. By the first assertion of (i), we have only to prove that the canonical map

$$\mathrm{Hom}_{\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}, S/(p)) \to \mathrm{Hom}_{\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}}(k_{\mathcal{K}}, R/(p)); \quad f \mapsto \vartheta_1 \circ f,$$

which is denoted by ϑ_* again, is bijective.

We first note the following: We regard $R/(p)$ as a quotient of $S/(p)$ by ϑ_1. Let $x \in R/(p)$ and let $\hat{x}_1, \hat{x}_2 \in S/(p)$ be lifts of x. Then, we have $\hat{x}_1 - \hat{x}_2 \in \ker\vartheta_1$. Since $a^p = p!\gamma_p(a) \in pS$ for $a \in \ker\vartheta$, where γ denotes a PD-structure on $\ker\vartheta$, we have $\hat{x}_1^p = \hat{x}_2^p$. In particular, if we denote by $\hat{x} \in S/(p)$ a lift of $x \in R/(p)$, then $\hat{x}^p$ depends only on x.

We prove the injectivity. Let $\overline{f} : k_{\mathcal{K}} \to R/(p)$ be an $\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}$-algebra homomorphism and $f,\ f' : k_{\mathcal{K}} \to S/(p)$ lifts of $\overline{f}$, that is, $\vartheta_*(f) = \vartheta_*(f') = \overline{f}$. For $\bar{x} \in k_K$, $f(\bar{x})$ and $f'(\bar{x}) \in S/(p)$ are lifts of $\overline{f}(\bar{x}) \in R/(p)$, hence we have $f(\bar{x}^p) = f(\bar{x})^p = f'(\bar{x})^p = f'(\bar{x}^p)$ by the above remark. Hence, we have $f|_{k_{\mathcal{K}}^p} = f'|_{k_{\mathcal{K}}^p}$, that is, $f = f'$ by the latter assertion of (i).

We prove the surjectivity. Let $\overline{f} : k_{\mathcal{K}} \to R/(p)$ be an $\mathbb{F}_p[T_j]_{j\in J_{\mathcal{K}}}$-algebra homomorphism. We have only to construct an $\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}}$-algebra homomorphism $f : k_{\mathcal{K}}^p \to S/(p)$ such that $\vartheta_*(f)|_{k_{\mathcal{K}}^p}$ coincides with $\overline{f}|_{k_{\mathcal{K}}^p}$, where we endow $k_{\mathcal{K}}^p$ and $S/(p)$ with $\mathbb{F}_p[T_j^p]_{j\in J_{\mathcal{K}}}$-algebra structures by a similar way as in the statement of (i). In fact, we can uniquely extend f to a $\mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}}$-algebra homomorphism $f : k_{\mathcal{K}} \to S/(p)$ by the latter assertion of (i). Moreover, $(\vartheta_*(f))|_{k_{\mathcal{K}}^p} = \vartheta_*(f|_{k_{\mathcal{K}}^p})$ coincides with $\overline{f}|_{k_{\mathcal{K}}^p}$, which implies $\vartheta_*(f) = \overline{f}$ by the latter assertion of (i) again. The set-theoretic map $f : k_{\mathcal{K}}^{p} \to S/(p)$ taking $\bar{y}$ to $\hat{x}^{p}$, where $\hat{x} \in S/(p)$ is any lift of $\bar{f}(\bar{y}^{p^{-1}}) \in R/(p)$, is well-defined by the above remark. Moreover, f is a $\mathbb{Z}[T_j]_{j \in J_{\mathcal{K}}}$-algebra homomorphism by a simple calculation and $\vartheta_*(f)|_{k_{\mathcal{K}}^{p}}$ coincides with $\bar{f}|_{k_{\mathcal{K}}^{p}}$ by construction, which implies the assertion. □

Proof of Lemma 3.13. Obviously, we have only to prove the first assertion. Put $\mathcal{A} = \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}\langle \mathbf{u}_{J_{\mathcal{K}}}\rangle$. Extend $\theta_{\mathbb{C}_p/\mathbb{Q}_p} : \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p} \to \mathcal{O}_{\mathbb{C}_p}$ to a surjective $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-algebra homomorphism $\vartheta : \mathcal{A} \to \mathcal{O}_{\mathbb{C}_p}$ by $\vartheta(\mathbf{u}^{[\boldsymbol{n}]}) = 0$. We first prove that $\mathcal{A}$ has an $\mathcal{O}_{\mathcal{K}}$-algebra structure such that ϑ is an $\mathcal{O}_{\mathcal{K}}$-algebra homomorphism.

Denote by ω a generator of the kernel of $\theta_{\mathbb{C}_p/\mathbb{Q}_p} : \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p} \to \mathcal{O}_{\mathbb{C}_p}$. Then, the PD-structure on the ideal $\ker\theta_{\mathbb{C}_p/\mathbb{Q}_p}$ of $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ canonically extends to a PD-structure δ_1 on the ideal (ω) of $\mathcal{A}$, compatible with the canonical PD-structure on the ideal (p). By construction, the kernel of the map $\xi : \mathcal{A} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ taking $\mathbf{u}^{[\boldsymbol{n}]}$ to 0 is endowed with a PD-structure δ_2, compatible with the canonical PD-structure on the ideal (p). Since $\mathcal{A}$ is an integral domain of characteristic 0, δ_1 and δ_2 induce the same PD-structure on $(\omega) \cap \ker\xi$. Hence, by [Berthelot and Ogus 1978, Proposition 3.12], the ideal $\ker\vartheta = (\omega) + \ker\xi$ admits a PD-structure, compatible with the canonical PD-structure on the ideal (p). Then, the assertion follows by applying Lemma 3.14(ii) to ϑ:

$$\begin{tikzcd} \mathcal{O}_{\mathcal{K}} \arrow[r, "\text{can.}"] \arrow[dr, dashed, "\exists!"] & \mathcal{O}_{\mathbb{C}_p} \\ \mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}} \arrow[u, "\text{str.}"] \arrow[r, "\text{str.}"] & \mathcal{A}, \arrow[u, "\vartheta"'] \end{tikzcd}$$

where the horizontal structure map is given by $T_j \mapsto \mathrm{u}_j + [\tilde{t}_j] \in \mathcal{A}$.

By the above $\mathcal{O}_{\mathcal{K}}$-structure, we may regard $\mathcal{A}$ as a p-adically formal $\mathcal{O}_{\mathcal{K}}$-PD-thickening of $\mathcal{O}_{\mathbb{C}_p}$. By universality, we have only to prove that $\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ is an $\mathcal{O}_{\mathcal{K}}$-algebra homomorphism. Let $\alpha : \mathcal{O}_{\mathcal{K}} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ be the composition of the structure map $\mathcal{O}_{\mathcal{K}} \to \mathcal{A}$ and $\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. Since $\iota_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ commutes with the projections ϑ and $\theta_{\mathbb{C}_p/\mathcal{K}}$, we have the commutative diagram

$$\begin{tikzcd} \mathcal{O}_{\mathcal{K}} \arrow[r, "\text{can.}"] \arrow[dr, "\alpha"] & \mathcal{O}_{\mathbb{C}_p} \\ \mathbb{Z}[T_j]_{j\in J_{\mathcal{K}}} \arrow[u, "\text{str.}"] \arrow[r, "\text{str.}"] & \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}, \arrow[u, "\theta_{\mathbb{C}_p/\mathcal{K}}"'] \end{tikzcd}$$

where the horizontal structure map is given by $T_j \mapsto t_j$. By Lemma 3.14(ii), α coincides with the structure map $\mathcal{O}_{\mathcal{K}} \to \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$, which implies the assertion. □

Finally, we remark that if $\mathcal{K} = \mathcal{K}_0$, then $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ and $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$ are integral domains by the above explicit description of $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$.

3E. ***Connections and Frobenius on $\mathbb{B}_{\mathrm{cris}}$ and $\mathbb{B}_{\mathrm{st}}$.*** In this section, assume $\mathcal{K}=\mathcal{K}_0$. We endow $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ with the p-adic topology defined by the lattice $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. We regard $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ as the direct limit of $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ under the multiplication by t^{-1} and we set

$$\widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}=\varinjlim \widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}.$$

Then, the canonical derivation $d:\mathcal{K}\to\widehat{\Omega}^1_{\mathcal{K}}$ uniquely extends to a $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$-linear continuous derivation $\nabla:\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}\to\widehat{\Omega}^1_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ by the explicit description of $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. Note that $\nabla(x^{[n]})=\nabla(x)\cdot x^{[n-1]}$ for $x\in\ker\theta_{\mathbb{C}_p/\mathcal{K}}$. As in Section 3C, if we denote by $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ the derivations on $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ given by $\nabla(x)=\sum_{j\in J_{\mathcal{K}}}dt_j\otimes\partial_j(x)$, then $\{\partial_j\}_{j\in J_{\mathcal{K}}}$ are commuting continuous $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$-derivations and we have $\partial_j=\partial/\partial u_j$. We also have a canonical extension ∇_q of exterior derivations d_q. Also, we can uniquely extend ∇_q to the map $\nabla_q:\widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}\to\widehat{\Omega}^{q+1}_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$ by putting $\nabla(\mathrm{x})=0$, where we define $\widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}:=(\widehat{\Omega}^q_{\mathcal{K}}\widehat{\otimes}_{\mathcal{K}}\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}})[\mathrm{x}]$.

Let $\varphi:\mathcal{O}_{\mathcal{K}}\to\mathcal{O}_{\mathcal{K}}$ be a lift of the absolute Frobenius on $k_{\mathcal{K}}$. The ring homomorphism $\varphi\otimes\varphi:\mathcal{O}_{\mathcal{K}}\otimes W(R_{\mathbb{C}_p})\to\mathcal{O}_{\mathcal{K}}\otimes W(R_{\mathbb{C}_p})$ induces a ring homomorphism on $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$. Although the resulting map depends on the choice of a Frobenius lift of $\mathcal{O}_{\mathcal{K}}$ in general, we denote it by φ again. By defining $\varphi(\mathrm{x}):=p\mathrm{x}$, we also have a Frobenius on $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$. By construction, the connection and the Frobenius on $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$ commute and we have the relation $N\circ\varphi=p\varphi\circ N$ by a simple calculation.

Notation. We define $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}}:=(\mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}})^{\nabla=0}$ for $\diamondsuit\in\{\mathrm{cris},\mathrm{st}\}$.

By the commutativity of ∇ and φ, these rings are endowed with φ-actions. Obviously, $\mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}}$ is endowed with the monodromy operator N. By the explicit description of $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}}$, we have:

Lemma 3.15. *For $\diamondsuit\in\{\mathrm{cris},\mathrm{st}\}$, the canonical map*

$$\mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}}$$

is an isomorphism. Since this map is compatible with Frobenius, Frobenius on $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}}$ is independent of the choice of a Frobenius lift of $\mathcal{O}_{\mathcal{K}}$. In particular, the Frobenius on $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}}$ is injective.

3F. ***Compatibility with limit.*** When a p-basis of $k_{\mathcal{K}}$ is not finite, some technical difficulties occur. In this case, we will reduce to the finite p-basis case by using the results of Section 1G and the following inverse limits.

Let the notation be as in Section 1G. By functoriality, we have canonical maps

$$\mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}_0}\to\varprojlim_{J\in\mathcal{P}(J_K)}\mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathcal{K}_{J,0}},\quad \mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathcal{K}}\to\varprojlim_{J\in\mathcal{P}(J_K)}\mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathcal{K}_J},$$

where $\diamondsuit\in\{\mathrm{cris},\mathrm{st}\}$, $\heartsuit\in\{\mathrm{dR},\mathrm{HT}\}$. Since these morphisms are compatible with the above explicit descriptions of these rings, it is easy to see that these maps are injective.

3G. ***Embeddings of*** $\mathbb{B}_{\mathrm{cris}}$ ***and*** $\mathbb{B}_{\mathrm{st}}$ ***into*** $\mathbb{B}_{\mathrm{dR}}$**.** Let

$$\mathbb{J}_{\mathbb{C}_p/\mathcal{K}} := \ker(\theta_{\mathbb{C}_p/\mathcal{K}} : \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}] \to \mathbb{C}_p).$$

We endow the ideal $\mathbb{J}_{\mathbb{C}_p/\mathcal{K}}/\mathbb{J}^n_{\mathbb{C}_p/\mathcal{K}}$ of the $\mathbb{Q}$-algebra $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/\mathbb{J}^n_{\mathbb{C}_p/\mathcal{K}}$ with the unique PD-structure. This is compatible with the canonical PD-structure of $\mathbb{O}_{\mathcal{K}}$ on the ideal (p). Hence, the canonical map $\mathbb{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}) \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/\mathbb{J}^n_{\mathbb{C}_p/\mathcal{K}}$ factors through $(\mathbb{O}_{\mathcal{K}} \otimes_{\mathbb{Z}} W(R_{\mathbb{C}_p}))^{\mathrm{PD}} \to \mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}[p^{-1}]/\mathbb{J}^n_{\mathbb{C}_p/\mathcal{K}}$. If we endow the LHS and the RHS with the p-adic topology and the p-adic Banach space topology respectively (see Section 3B), then the above morphism is continuous. In fact, the canonical map times $n!$ factors through the image of $\mathbb{A}_{\mathrm{inf},\mathbb{C}_p/\mathcal{K}}$. By passing to limit, the map extends to $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$. Thus, we have a canonical $\mathcal{K}$-algebra homomorphism $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$. Fixing $\tilde{p} \in R_{\mathbb{C}_p}$ such that $\tilde{p}^{(0)} = p$, we extend this map to $\mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/\mathcal{K}} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$ by sending x to $\log([\tilde{p}]/p) := \sum_{n\in\mathbb{N}_{>0}} (-1)^{n-1}([\tilde{p}]/p-1)^n/n$. Note that these morphisms are compatible with connections.

Proposition 3.16. *Assume that the algebraic closure of* $\mathcal{K}$ *in* $\mathbb{C}_p$ *is dense in* $\mathbb{C}_p$*. Then, the canonical maps*

$$\mathcal{K}_{\mathrm{can}} \otimes_{\mathcal{K}_{\mathrm{can},0}} \mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}_0} \to \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}, \quad \mathcal{K}_{\mathrm{can}} \otimes_{\mathcal{K}_{\mathrm{can},0}} \mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}_0} \to \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}},$$
$$\mathcal{K} \otimes_{\mathcal{K}_0} \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/\mathcal{K}_0} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}, \quad \mathcal{K} \otimes_{\mathcal{K}_0} \mathbb{B}_{\mathrm{st},\mathbb{C}_p/\mathcal{K}_0} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathcal{K}}$$

are injective.

Proof. By identifying $\mathbb{C}_p$ with the p-adic completion of $\mathcal{K}^{\mathrm{alg}}$, we may assume $\mathcal{K} = K$. Note that if k_K is perfect, then this is due to [Fontaine 1994a, 4.2.4]. We consider the general case. We first prove the first two cases. We have only to prove the semistable case. The canonical map $K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} K_0^{\mathrm{pf}} \to K^{\mathrm{pf}}$ is injective since $K_{\mathrm{can}}/K_{\mathrm{can},0}$ is totally ramified and K_0^{pf} is absolutely unramified. Hence, we have the commutative diagram

$$\begin{array}{ccccc}
K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/K_0} & & \xrightarrow{\mathrm{can.}} & & \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K} \\
\downarrow \cong & & & & \downarrow \cong \\
K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{B}_{\mathrm{st},\mathbb{C}_p/K_0^{\mathrm{pf}}} & \xhookrightarrow{\mathrm{can.}} & K^{\mathrm{pf}} \otimes_{K_0^{\mathrm{pf}}} \mathbb{B}_{\mathrm{st},\mathbb{C}_p/K_0^{\mathrm{pf}}} & \xhookrightarrow{\mathrm{can.}} & \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}},
\end{array}$$

where the vertical arrows are induced by base changes and the injectivity of the bottom second arrow follows from the perfect residue field case. Then, the assertion follows from the above diagram. We consider the latter two cases. By passing to limit (Section 3F), we may assume $[k_K : k_K^p] < \infty$. Then, the crystalline case follows from [Brinon 2006, Proposition 2.47], where $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0}$ is denoted by $\mathrm{B}_{\mathrm{cris}}$. We will prove the semistable case. By regarding $K \otimes_{K_0} \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0}$ as a subring of $\mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})$, the assertion is equivalent to saying that x is transcendental over $\mathrm{Frac}(\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})$. Suppose that it is not the case. To deduce a contradiction, we have only to construct a nonzero polynomial in $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{pf}}}[X]$ which has x as a zero. By assumption, we have a nonzero polynomial $f(X) = \sum_i a_i X^i \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0}[X]$ such that $f(\mathrm{x}) = 0$. For $\boldsymbol{m} \in \mathbb{N}^{\bigoplus J_K}$, we denote by $\partial^{\boldsymbol{m}}$ the product $\prod_{j\in J_K} \partial_j^{m_j}$, where $\{\partial_j\}_{j\in J_K}$ are the derivations defined in Section 3C. Denote by $\tilde{f}^{(\boldsymbol{m})}(X) \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{pf}}}[X]$ the image of the polynomial $f^{(\boldsymbol{m})}(X) := \sum_i \partial^{\boldsymbol{m}}(a_i) X^i$ under the canonical homomorphism $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$. Then, $\tilde{f}^{(\boldsymbol{m})}(X)$ has x as a zero since we have $\mathrm{x} \in \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$. Write $a_i = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}} a_{i,\boldsymbol{n}} \boldsymbol{u}^{[\boldsymbol{n}]}$ with $a_{i,\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ by using the explicit description of $\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0}$ given in Section 3D. We have $\partial^{\boldsymbol{m}}(a_i) = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus \mathcal{K}}} a_{i,\boldsymbol{n}+\boldsymbol{m}} \boldsymbol{u}^{[\boldsymbol{n}]}$ and $\tilde{f}^{(\boldsymbol{m})}(X) = \sum_i a_{i,\boldsymbol{m}} X^i$. Hence, we obtain the desired polynomial $\tilde{f}^{(\boldsymbol{m})}(X)$ by choosing $\boldsymbol{m} \in \mathbb{N}^{\bigoplus J_K}$ such that we have $a_{i,\boldsymbol{m}} \neq 0$ for some i. $\square$

4. Basic properties of rings of p-adic periods

We will apply the preceding construction to the cases $\mathcal{K} = \mathbb{Q}_p, K, K^{\mathrm{pf}}$, among others. The resulting rings of p-adic periods will have an appropriate Galois action by the functoriality of the construction: For example, G_K acts on $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$ and $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$, $G_{K^{\mathrm{pf}}}$ acts on $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$. In this section, we will review Galois theoretic properties of these rings. The proofs of the properties are somewhat technical and the reader may skip this section by admitting the results including the G_K-regularities just below. We keep the notation of the previous section.

4A. ***Calculations of H^0 and verification of G_K-regularity.*** In this subsection, we will prove the G_K-regularity of the $(\mathbb{Q}_p, G_K)$-rings

$$\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0},\quad \mathbb{B}_{\mathrm{st},\mathbb{C}_p/K_0},\quad \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K},\quad \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K},$$
$$\mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0},\quad \mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/K_0},\quad \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K},\quad \mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K},$$

which are used later in the paper, and calculate their H^0. Note that these rings are integral domains by their explicit description.

Lemma 4.1. *Let* $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$.

(i) $H^0(G_K, \mathrm{Frac}(\mathbb{B}_{\heartsuit,\mathbb{C}_p/K})) = K$.

(ii) *The* $(\mathbb{Q}_p, G_K)$*-ring* $\mathbb{B}_{\heartsuit,\mathbb{C}_p/K}$ *satisfies condition* $(G\cdot R_3)$ *of Section 1H.*

(iii) *The* $(\mathbb{Q}_p, G_K)$*-ring* $\mathbb{B}_{\heartsuit,\mathbb{C}_p/K}$ *is* G_K*-regular.*

Proof. Assertion (iii) follows from (i), (ii) and Lemma 1.21. We will prove (i) and (ii) separately in the Hodge–Tate case and the de Rham case.

(a) The Hodge–Tate case: We first verify (i). By Theorem 1.15, we have only to prove that if we have nonzero $x, y \in \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$ such that $g(x)y = xg(y)$ for all $g \in G_K$, then we have $x/y \in \mathbb{C}_p$. We first consider the case $|J_K| < \infty$. Note that $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K} \cong \mathbb{C}_p[t, t^{-1}, \{v_j\}_{j\in J_K}]$ is a uniquely factorization domain. Hence we may assume that x and y are relatively prime by dividing x and y by their greatest common divisor. Then we have $g(x) = c_g x$ and $g(y) = c_g y$ for $c_g \in (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^\times \cong \bigcup_{n\in\mathbb{Z}} \mathbb{C}_p^\times t^n$ by assumption. By the explicit description of $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$, we can choose $\boldsymbol{n} \in \mathbb{N}^{J_K}$ such that

$$\partial^{\boldsymbol{n}}(x) \in \mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K} \setminus \{0\} \cong \mathbb{C}_p[t, t^{-1}] \setminus \{0\},$$

where $\partial_j = t\partial/\partial v_j$ and $\partial^{\boldsymbol{n}} := \prod_j \partial_j^{n_j}$ (Remark 3.8). Write $\partial^{\boldsymbol{n}}(x) = \sum_{n\in\mathbb{Z}} a_n t^n$ with $a_n \in \mathbb{C}_p$. Then, we have $g(\partial^{\boldsymbol{n}}(x)) = c_g \partial^{\boldsymbol{n}}(x)$ by the commutativity of ∂_j and the G_K-action. Since c_g is homogeneous with respect to t, we have $c_g \in \mathbb{C}_p$ by comparing degrees. By comparing the leading terms, we have $c_g = g(a_n)/a_n\chi^n(g)$ for all $g \in G_K$, where n is the degree of $\partial^{\boldsymbol{n}}(x)$ with respect to t. Hence, we have $x/a_n t^n \in (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^{G_K}$. Note that we have $(\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^{G_K} = K$. This follows from the facts that we have $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K} = \bigcup_{r\in\mathbb{N}} t^{-r}\mathbb{C}_p[t, \{tv_j\}_{j\in J_K}]$ and

$$H^0(G_K, t^{-r}\mathbb{C}_p[t, \{tv_j\}_{j\in J_K}]) = K$$

by [Brinon 2006, Lemme 2.15], where $\mathbb{C}_p[t, \{tv_j\}_{j\in J_K}]$ is written $\bigoplus_{r\in\mathbb{N}} \mathrm{gr}^r(\mathbb{B}^+_{\mathrm{dR}})$ in the reference. Thus, we have $x \in \mathbb{C}_p^\times t^n$. By the same argument, we have $y \in \mathbb{C}_p^\times t^m$ for some $m \in \mathbb{Z}$. Write $x = at^n$, $y = bt^m$ with $a, b \in \mathbb{C}_p^\times$. Then, we have

$$g(a/b) = \chi^{m-n}(g)(a/b)$$

for $g \in G_K$. Since $H^0(G_K, \mathbb{C}_p(n-m))$ is nonzero if and only if $n = m$ by Theorem 1.15, we must have $n = m$. In particular, we have $x/y = a/b \in \mathbb{C}_p$.

We consider the general case. Recall the notation in Section 1G. Let $J \in \mathcal{P}(J_K)$ and denote by x_J, y_J the image of x, y in $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K_J}$. By applying the above result to $J_K = J$, if x_J and y_J are nonzero, then there exists $\lambda_J \in \mathbb{C}_p^\times$ such that $x_J = \lambda_J y_J$. Since this λ_J is uniquely determined, $\lambda = \lambda_J$ is independent of the choice of J. Since $S_{x,y} := \{J \in \mathcal{P}(J_K) \mid x_J \neq 0 \text{ and } y_J \neq 0\}$ is a cofinal subset of $\mathcal{P}(J_K)$ by the explicit description of $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$, we have $x = \lambda y$ by the injection in Section 1G.

We will verify (ii). Let $x \in \mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$ be a generator of a G_K-stable $\mathbb{Q}_p$-line in $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$. Write $g(x) = c_g x$ with $c_g \in \mathbb{Q}_p^\times$. We use the same notation as above. By a similar argument as above, if $x_J \neq 0$, then we have $x_J = a_J t^{n_J}$ for $a_J \in \mathbb{C}_p^\times$ and $n_J \in \mathbb{N}$. Moreover, a_J and n_J are unique. In particular, $\{a_J\}$ and $\{n_J\}$ are constant on the cofinal subset $S_{x,x}$ of $\mathcal{P}(J_K)$ and we have $x \in \mathbb{C}_p^\times t^n \subset (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^\times$ by the injection in Section 1G.

(b) The de Rham case: To prove assertion (i), we have only to prove that if we have nonzero $x, y \in \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$ such that $g(x)y = xg(y)$ for all $g \in G_K$, then we have $x/y \in K$. Let $J \in \mathcal{P}(J_K)$ and denote by $x_J, y_J \in \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K_J}$ the image of x, y. If $x_J \neq 0$ and $y_J \neq 0$, then we have $x_J/y_J \in H^0(G_{K_J}, \mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K_J})) = K_J$ by [Brinon 2006, Proposition 2.18], where $\mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K_J})$ is denoted by $\mathrm{C_{dR}}$. Since the set $\{J \in \mathcal{P}(J_K) \mid x_J \neq 0 \text{ and } y_J \neq 0\}$ is a cofinal subset of $\mathcal{P}(J_K)$ by the explicit description of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$, we have $x/y \in \bigcap_{J\in\mathcal{P}(J_K)} K_J = K$ by the injection in Section 1G. We will verify (ii). By Remark 3.5(i), we may assume $K = K^{\mathrm{ur}}$. Let V be a G_K-stable $\mathbb{Q}_p$-line in $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$ generated by x. By Lemma 4.2 below and Theorem 2.1, there exist $n \in \mathbb{Z}$ and a finite extension L/K such that $Vt^n \subset (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^{G_L} = (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/L})^{G_L} = L$; in particular, we have $x \in (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^\times$. □

Lemma 4.2. *Let V be a G_K-stable $\mathbb{Q}_p$-line in $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$. Then, up to a Tate twist, V is $\mathbb{C}_p$-admissible as a p-adic representation.*

Proof. We assume $K = K^{\mathrm{ur}}$ by Hilbert 90 and Remark 3.6(ii). Let $x \in \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$ be a generator of V. By multiplying by a power of t, we may assume $x \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$. Let $\rho : G_K \to \mathbb{Q}_p^\times$ be the character defined by $\rho(g) = g(x)/x$. By the explicit description of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$ (Section 3B), we have

$$x = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}} a_{\boldsymbol{n}} \boldsymbol{u}^{\boldsymbol{n}}$$

with $a_{\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$. Choose $\boldsymbol{n} \in \mathbb{N}^{\oplus J_K}$ such that $a_{\boldsymbol{n}} \neq 0$ and write $a_{\boldsymbol{n}} = t^n\lambda$ with $n \in \mathbb{N}$ and $\lambda \in (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^\times$. Since we have $g(a_{\boldsymbol{n}}) = \rho(g)a_{\boldsymbol{n}}$ for $g \in G_{K^{\mathrm{pf}}}$, we have $(\rho\chi^{-n})(g) = g(\lambda)/\lambda$ for $g \in G_{K^{\mathrm{pf}}}$. By taking the $\mathbb{Q}_p$-linear map $\theta_{\mathbb{C}_p/\mathbb{Q}_p}$, we have $(\rho\chi^{-n})(g) = g(\theta_{\mathbb{C}_p/\mathbb{Q}_p}(\lambda))/\theta_{\mathbb{C}_p/\mathbb{Q}_p}(\lambda)$ for $g \in G_{K^{\mathrm{pf}}}$, that is, $\rho\chi^{-n}|_{K^{\mathrm{pf}}}$ is $\mathbb{C}_p$-admissible. Hence, $\rho\chi^{-n}$ is $\mathbb{C}_p$-admissible by Theorem 2.1. □

Corollary 4.3. *We have*

$$\begin{aligned}
(\mathbb{B}^\nabla_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_K} &= (\mathbb{B}^\nabla_{\mathrm{st},\mathbb{C}_p/K_0})^{G_K} = K_{\mathrm{can},0},\\
(\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_K} &= (\mathbb{B}_{\mathrm{st},\mathbb{C}_p/K_0})^{G_K} = K_0,\\
(\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})^{G_K} &= (\mathbb{B}^\nabla_{\mathrm{dR},\mathbb{C}_p/K})^{G_K} = K_{\mathrm{can}},\\
(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{G_K} &= (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^{G_K} = K,\\
(\mathbb{B}^\nabla_{\mathrm{HT},\mathbb{C}_p/K})^{G_K} &= (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^{G_K} = K.
\end{aligned}$$

Proof. Since we have trivial inclusions (such as $K_0 \subset (\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_K}$), we have only to show the converse inclusions. By passing to limit (Section 1G and 3F), we may assume $[k_K : k_K^p] < \infty$. We prove the Hodge–Tate case first. Since we have $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K} \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p(n)$ (Section 3B), the assertion for $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K}$ follows from Theorem 1.15. The assertion for $\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K}$ follows from [Brinon 2006, Lemme 2.15].

We will prove the rest of the assertion. Since we have $K_{\mathrm{can},0} = (K_0)_{\mathrm{can}}$ by comparing the residue fields, the assertions in the horizontal case follow from those in the ∇-less case by taking horizontal sections. The de Rham case follows from Lemma 4.1(i) and the crystalline and semistable cases follow from de Rham case and Proposition 3.16. □

Lemma 4.4. *The $(\mathbb{Q}_p, G_K)$-ring $\mathbb{B}_{\diamond,\mathbb{C}_p/K_0}$ satisfies $(G\cdot R_3)$ for $\diamond \in \{\mathrm{cris}, \mathrm{st}\}$. In particular, $\mathbb{B}_{\diamond,\mathbb{C}_p/K_0}$ is G_K-regular.*

Proof. Note that the last assertion is obtained by applying Lemma 1.20, whose assumptions are satisfied by Proposition 3.16, Lemma 4.1(iii) and Corollary 4.3. By Remark 3.12(ii), we may assume $K = K^{\mathrm{ur}}$. Let V be a G_K-stable $\mathbb{Q}_p$-line in $\mathbb{B}_{\diamond,\mathbb{C}_p/K_0}$ with generator x. By Lemma 4.2, there exists $n \in \mathbb{Z}$ such that Vt^n is $\mathbb{C}_p$-admissible as a p-adic representation of G_K. By Theorem 2.1, the image of the map $\rho : G_K \to \mathbb{Q}_p^\times$ that takes g to $g(xt^n)/(xt^n)$ is included in $(\mathbb{Q}_p^\times)_{\mathrm{tors}}$, which is killed by $2(p-1)$. Therefore, we have $(xt^n)^{2(p-1)} \in (\mathbb{B}_{\diamond,\mathbb{C}_p/K_0})^{G_K} = K_0$, which implies $x \in \mathbb{B}^\times_{\diamond,\mathbb{C}_p/K_0}$. □

Lemma 4.5. *The $(\mathbb{Q}_p, G_K)$-rings*

$$\mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0}, \quad \mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/K_0}, \quad \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}, \quad \mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K}$$

are G_K-regular.

Proof. The G_K-regularity of the field $\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}$ follows from Example 1.18. Since we have a $G_{K^{\mathrm{pf}}}$-equivariant canonical isomorphism $\mathbb{B}^{\nabla}_{\diamond,\mathbb{C}_p/K_0} \cong \mathbb{B}_{\diamond,\mathbb{C}_p/K_0^{\mathrm{pf}}}$ for $\diamond \in \{\mathrm{cris}, \mathrm{st}\}$, the verification of $(G\cdot R_3)$ for $\mathbb{B}^{\nabla}_{\diamond,\mathbb{C}_p/K_0}$ is reduced to that for $\mathbb{B}_{\diamond,\mathbb{C}_p/K_0^{\mathrm{pf}}}$, which follows from [Fontaine 1994b, Proposition 5.1.2(ii)]. By a similar reason, $(G\cdot R_3)$ for $\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}$ is reduced to [Fontaine 1994b, Proposition 3.6]. The $(\mathbb{Q}_p, G_K)$-ring $\mathbb{C}_p((t))$ is a field containing the fractional field of $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K} \cong \mathbb{C}_p[t, t^{-1}]$. By Theorem 1.15 and dévissage, we have $\mathbb{C}_p((t))^{G_K} = K = (\mathbb{B}_{\mathrm{HT},\mathbb{C}_p/K})^{G_K}$, where the last equality follows from Corollary 4.3. By applying Lemma 1.21, $\mathbb{B}^{\nabla}_{\mathrm{HT},\mathbb{C}_p/K}$ is G_K-regular. By Corollary 4.3, the G_K-regularity for $\mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0}$ and $\mathbb{B}^{\nabla}_{\mathrm{st},\mathbb{C}_p/K_0}$ follows from Lemma 1.20 and Proposition 3.16. □

Remark 4.6. For $\bullet \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}, \mathrm{HT}\}$, the $(\mathbb{Q}_p, G_K)$-rings $\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p}$ and $\mathbb{B}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p}$ are G_K-regular We also have

$$(\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p})^{G_K} = (\mathbb{B}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p})^{G_K} \cong (\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K_0})^{G_K}.$$

In fact, the assertion follows from canonical isomorphisms $\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p} = \mathbb{B}_{\bullet,\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K_0}$ as $(\mathbb{Q}_p, G_K)$-rings.

Notation. (i) We define the category of crystalline (resp. horizontal crystalline) representations of G_K as $\mathrm{Rep}^{\mathrm{adm}}_{\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0}} G_K$ (resp. $\mathrm{Rep}^{\mathrm{adm}}_{\mathbb{B}^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0}} G_K$), and we denote it by $\mathrm{Rep}_{\mathrm{cris}} G_K$ (resp. $\mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K$). The corresponding functor $\mathbb{D}_B$ is denoted by $\mathbb{D}_{\mathrm{cris}}$ (resp. $\mathbb{D}^{\nabla}_{\mathrm{cris}}$) and the comparison map α_B by $\alpha_{\mathrm{cris},\mathbb{C}_p/K_0}$ (resp. $\alpha^{\nabla}_{\mathrm{cris},\mathbb{C}_p/K_0}$). We define the category of semistable representations similarly, with "cris" in place of "st".

(ii) We define the category of de Rham (resp. horizontal de Rham) representations of G_K as $\mathrm{Rep}^{\mathrm{adm}}_{\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}} G_K$ (resp. $\mathrm{Rep}^{\mathrm{adm}}_{\mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}} G_K$), and we denote it by $\mathrm{Rep}_{\mathrm{dR}} G_K$ (resp. $\mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$). The corresponding functor $\mathbb{D}_B$ is denoted by $\mathbb{D}_{\mathrm{dR}}$ (resp. $\mathbb{D}^{\nabla}_{\mathrm{dR}}$) and the comparison map α_B (loc. cit.) by $\alpha_{\mathrm{dR},\mathbb{C}_p/K}$ (resp. $\alpha^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}$). We define the category of Hodge–Tate representations similarly, with "dR" in place of "HT".

(iii) We define rings with G_K-actions and automorphisms φ by

$$\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} := \bigcap_{n\in\mathbb{N}} \varphi^n(\mathbb{B}^{\nabla+}_{\mathrm{cris},\mathbb{C}_p/K_0}), \quad \tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0} := \bigcap_{n\in\mathbb{N}} \varphi^n(\mathbb{B}^{\nabla+}_{\mathrm{st},\mathbb{C}_p/K_0}).$$

Note that we have $\tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0} \cong \tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0^{\mathrm{pf}}}$ for $\diamondsuit \in \{\mathrm{rig}, \log\}$.

(iv) In the rest of the paper, when k_K is perfect, we omit hyperscripts ∇ to be consistent with the usual notation; e.g., we write $\tilde{\mathbb{B}}^{+}_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$ instead of $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$.

Remark 4.7. As is explained in Section 1A, there is no canonical choice of a Cohen ring of k_K nor a Frobenius lift when k_K is not perfect. Since some definitions, such as the definition of crystalline representations, involve these choices, we make some remarks on the independence of definitions.

(i) Since we have a canonical isomorphism $\mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathbb{Q}_p} \cong \mathbb{B}^{\nabla}_{\heartsuit,\mathbb{C}_p/K}$ for $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$ (Lemma 3.7), $\mathbb{B}^{\nabla}_{\heartsuit,\mathbb{C}_p/K}$ depend only on $\mathbb{C}_p$ as an abstract ring.

(ii) Since we have a canonical isomorphism $\mathbb{B}^{+}_{\diamondsuit,\mathbb{C}_p/\mathbb{Q}_p} \cong \mathbb{B}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0}$ for $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$ (Lemma 3.15), the category $\mathrm{Rep}^{\nabla}_{\diamondsuit} G_K$ depends only on $\mathbb{C}_p$ but not on the choice of K_0. It also follows that $\tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0}$ for $\diamondsuit \in \{\mathrm{rig}, \log\}$ is independent of the choices of K_0 and φ as a $\mathbb{Q}_p$-algebra with φ-action. Moreover, for a finite extension L/K, $\tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/K_0}$ coincides with $\tilde{\mathbb{B}}^{\nabla+}_{\diamondsuit,\mathbb{C}_p/L_0}$ in $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/L}$.

(iii) By definition, the category $\mathrm{Rep}_{\diamondsuit} G_K$ for $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$ may depend on the choice of K_0. In the case $[k_K : k_K^p] < \infty$ with $\diamondsuit = \mathrm{cris}$, the independence is proved by Brinon [2006, Proposition 3.42]: He proves the assertion by introducing a ring $A_{\max,K}$, which is independent of the choice of K_0 and is slightly bigger than $\mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} \mathbb{A}_{\mathrm{cris},\mathbb{C}_p/K_0}$. Although a similar idea seems to work in the general case, we do not treat this problem in this paper. Instead, we will state a precise version of the Main Theorem later (see Section 6).

Remark 4.8 (Hilbert 90). Let $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$. Then, V is crystalline or semistable if and only if so is $V|_{K^{\mathrm{ur}}}$. In fact, we have $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0} \cong \mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{ur}}}$ by Remark 3.12(i), whose $G_{K^{\mathrm{ur}}}$-invariant is K_0^{ur} by Corollary 4.3. Hence, the assertion in the crystalline case follows from Hilbert 90 and the same proof works also in the semistable case. We can also prove that V is de Rham or Hodge–Tate if and only if so is $V|_L$ for a finite extension L of the completion of an unramified extension of K. This follows from the cases when L/K is finite or unramified and in these cases the claim follows from Remark 3.6(ii) and Hilbert 90.

Algebraic structures of rings of p-adic period, which are compatible with the action of G_K, induce additional structures on the corresponding $\mathbb{D}$. We do not review these structures here since we do not need all of them to prove the Main Theorem. For the reader interested in these structures, see [Brinon 2006, 3.5] for example. We need only the connection on $\mathbb{D}_{\mathrm{dR}}$ for the proof of the Main Theorem: For $V \in \mathrm{Rep}_{\mathrm{dR}} G_K$, the finite-dimensional K-vector space $\mathbb{D}_{\mathrm{dR}}(V)$ has a connection $\nabla : \mathbb{D}_{\mathrm{dR}}(V) \to \widehat{\Omega}^1_K \otimes_K \mathbb{D}_{\mathrm{dR}}(V)$, which is compatible with the canonical derivation on K.

4B. ***Restriction to perfection.*** If we have $V \in \mathrm{Rep}_\bullet G_K$ with $\bullet \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}, \mathrm{HT}\}$, then we have $V|_{K^{\mathrm{pf}}} \in \mathrm{Rep}_\bullet G_{K^{\mathrm{pf}}}$. Moreover, we have canonical isomorphisms

$$K_0^{\mathrm{pf}} \otimes_{K_0} \mathbb{D}_\diamondsuit(V) \to \mathbb{D}_\diamondsuit(V|_{K_0^{\mathrm{pf}}}), \qquad K^{\mathrm{pf}} \otimes_K \mathbb{D}_\heartsuit(V) \to \mathbb{D}_\heartsuit(V|_{K^{\mathrm{pf}}}),$$

induced by the canonical map $\mathbb{B}_{\diamondsuit,\mathbb{C}_p/K_0} \to \mathbb{B}_{\diamondsuit,\mathbb{C}_p/K_0^{\mathrm{pf}}}$ and $\mathbb{B}_{\heartsuit,\mathbb{C}_p/K} \to \mathbb{B}_{\heartsuit,\mathbb{C}_p/K^{\mathrm{pf}}}$ for $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$ and $\heartsuit \in \{\mathrm{dR}, \mathrm{HT}\}$. We first prove the de Rham case. By applying $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}} \otimes_{\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}}$ to the comparison isomorphism $\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V)$, we have a $G_{K^{\mathrm{pf}}}$-equivariant isomorphism

$$\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}} \otimes_K \mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V.$$

By taking $G_{K^{\mathrm{pf}}}$-invariant, we have an isomorphism $K^{\mathrm{pf}} \otimes_K \mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{D}_{\mathrm{dR}}(V|_{K^{\mathrm{pf}}})$. The other cases follow similarly.

5. Construction of $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$

In this section, we construct a (φ, G_K)-module $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ for a de Rham representation V of G_K, possibly after a Tate twist. Our $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}$ coincides with Colmez's $\widetilde{\mathbb{N}}^{+}_{\mathrm{rig}}$ when the residue field k_K is perfect.

We first recall Colmez's Dieudonné–Manin theorem, which is a key ingredient of the construction. Let M be a finite free $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-module of rank $r > 0$. We call N a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-lattice of M if N is a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-submodule of finite type of M such that $N[t^{-1}] = M[t^{-1}]$. Note that a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-lattice of M is finite free of rank r over $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$ since $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$ is a discrete valuation ring.

For $n \in \mathbb{Z}$, denote the composition

$$\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \overset{\varphi^n}{\hookrightarrow} \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \overset{\mathrm{inc.}}{\hookrightarrow} \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$$

by φ^n again. By the commutative diagram

$$\begin{tikzcd} \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \arrow[r, hook, "\varphi^n"] \arrow[d, "\mathrm{can.}"', "\cong"] & \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \arrow[d, "\mathrm{can.}"', "\cong"] \\ \widetilde{\mathbb{B}}^{+}_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}} \arrow[r, hook, "\varphi^n"] & \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}, \end{tikzcd}$$

the proof of the following theorem is reduced to the perfect residue field case [Colmez 2008, Proposition 0.3] (see also the remark below).

Theorem 5.1 (Colmez's Dieudonné–Manin classification theorem). *Let $r \in \mathbb{N}_{>0}$ and M be a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-lattice of $(\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})^r$. Let*

$$M_{\mathrm{rig}} := \{x \in (\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})^r \mid \varphi^n(x) \in M \text{ for all } n \in \mathbb{Z}\}.$$

Then, M_{rig} is a finite free $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$-module of rank r with semilinear φ-action and there exists a basis $e_1, \dots, e_r$ of M_{rig} over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ such that:

(i) *There exist $h \in \mathbb{N}_{>0}$ and $a_1 \le \cdots \le a_r \in \mathbb{N}$ such that $\varphi^h(e_i) = p^{a_i} e_i$ for $1 \le i \le r$;*

(ii) *$e_1, \dots, e_r$ is a basis of M over $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$.*

Remark 5.2. Though our condition (ii) is weaker than that in [Colmez 2008], the conclusions of the theorem are the same for the following reason: By definition, φ acts on M_{rig}. Since φ^h is an automorphism on M_{rig} by (i), φ is also an automorphism on M_{rig}. Hence, (ii) implies that $\varphi^n(e_1), \dots, \varphi^n(e_r)$ is a $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$-basis of M_{rig} for all $n \in \mathbb{Z}$. In particular, $\varphi^n(e_1), \dots, \varphi^n(e_r)$ is a $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-basis of M.

In the rest of this section, let V be a de Rham representation of G_K of dimension r such that $\mathbb{D}_{\mathrm{dR}}(V) = (\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V)^{G_K}$. Note that the last assumption is satisfied for any de Rham representation after some Tate twist. Let

$$\mathbb{N}^{+}_{\mathrm{dR}}(V) := \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_K \mathbb{D}_{\mathrm{dR}}(V).$$

It is a finite free $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K}$-module of rank r with G_K-action and ∇-action which are commuting. By the comparison isomorphism $\alpha_{\mathrm{dR},\mathbb{C}_p/K}$, we have a canonical isomorphism $\mathbb{N}^{+}_{\mathrm{dR}}(V)[t^{-1}] \cong \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V$, in particular, we have

$$t^n \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V \subset \mathbb{N}^{+}_{\mathrm{dR}}(V) \subset \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V$$

for sufficiently large $n \in \mathbb{N}$. Taking horizontal sections, we see that $\mathbb{N}^{\nabla+}_{\mathrm{dR}}(V) := \mathbb{N}^{+}_{\mathrm{dR}}(V)^{\nabla=0}$ is a G_K-stable $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$-lattice of $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V$. By applying Theorem 5.1 to $M = \mathbb{N}^{\nabla+}_{\mathrm{dR}}(V)$, we have the following proposition: (In the following, a (φ, G_K)-module over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ (of rank r) means a finite free module (of rank r)

over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ with a semilinear φ-action and a semilinear G_K-action, which are commuting.)

Proposition 5.3. *The $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$-module*

$$\tilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V) := \{x \in \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \otimes_{\mathbb{Q}_p} V \mid \varphi^n \otimes \mathrm{id}(x) \in \mathbb{N}^{\nabla+}_{\mathrm{dR}}(V) \text{ for all } n \in \mathbb{Z}\}$$

is a (φ, G_K)-module over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ of rank r. Moreover, we have a basis $e_1, \dots, e_r$ of $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ over $\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}$ such that:

(i) *There exist $h \in \mathbb{N}_{>0}$ and $a_1 \le \dots \le a_r \in \mathbb{N}$ such that $\varphi^h(e_i) = p^{a_i} e_i$ for $1 \le i \le r$;*

(ii) *$e_1, \dots, e_r$ is a basis of $\mathbb{N}^{\nabla+}_{\mathrm{dR}}(V)$ over $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$.*

Note that $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ is independent of the choice of K_0 by Remark 4.7(ii). We will use the following property of $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ in the proof of the Main Theorem.

Proposition 5.4. *The canonical map*

$$\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}} \mathbb{N}^{\nabla+}_{\mathrm{dR}}(V) \to \mathbb{N}^{+}_{\mathrm{dR}}(V)$$

is a G_K-equivariant isomorphism. In particular, $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}} \widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ is isomorphic to $(\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K})^r$ as a $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-module by Proposition 5.3(ii).

Proof. Since $V|_{K^{\mathrm{pf}}}$ is de Rham and we have the canonical isomorphism $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$, we have the comparison isomorphism

$$\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}} (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^{G_{K^{\mathrm{pf}}}} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V.$$

By taking the base change of this isomorphism by $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$, we obtain a canonical isomorphism of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}[G_{K^{\mathrm{pf}}}]$-modules

$$\alpha : \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}} (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^{G_{K^{\mathrm{pf}}}} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V. \quad (2)$$

We also have the comparison isomorphism

$$\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V) : \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_K \mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V.$$

Note that we have $(\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}} = (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}$ since we have

$$(t^{-n}\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}/t^{-n+1}\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}} = (\mathbb{C}_p(-n))^{G_{K^{\mathrm{pf}}}} = 0$$

for $n \in \mathbb{N}_{>0}$. We have only to prove that there exists an isomorphism of $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K}$-modules

$$(\mathbb{N}^{+}_{\mathrm{dR}}(V) =)\ \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_K \mathbb{D}_{\mathrm{dR}}(V) \cong \mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{(\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}} (\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^{G_{K^{\mathrm{pf}}}}$$

which is compatible with the injections $\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V)$ and α. Indeed, by taking the horizontal sections of both sides, we have

$$\mathbb{N}^{\nabla+}_{\mathrm{dR}}(V)=\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}},$$

which implies the assertion.

We have

$$\mathbb{D}_{\mathrm{dR}}(V)\hookrightarrow(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}}=\\(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}\otimes_{(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}},$$

where the equality follows by taking $G_{K^{\mathrm{pf}}}$-invariant of (2). Note that we have

$$(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}=(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}.$$

Indeed, if we write $x\in\mathrm{LHS}$ as $x=t^{-n}\sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}}a_{\boldsymbol{n}}\boldsymbol{u}^{\boldsymbol{n}}$ with $a_{\boldsymbol{n}}\in\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}$, since $\{u_j\}_{j\in J_K}$ are invariant by the action of $G_{K^{\mathrm{pf}}}$, we have

$$b_{\boldsymbol{n}}:=a_{\boldsymbol{n}}/t^n\in(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}=(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}.$$

Therefore, we have $x=\sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}}b_{\boldsymbol{n}}\boldsymbol{u}^{\boldsymbol{n}}\in(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}$. Hence we have a canonical map

$$\mathbb{D}_{\mathrm{dR}}(V)\hookrightarrow\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}}.$$

This induces a canonical homomorphism of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$-modules

$$i:\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_K\mathbb{D}_{\mathrm{dR}}(V)\to\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_{(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}},$$

which is compatible with the injections $\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V)$ and α by construction. We have only to prove the surjectivity of i. By Nakayama's lemma, we have only to prove the assertion after applying $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_{\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}}$ (note that $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\to\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$ is a surjective homomorphism of local rings). We have the commutative diagram

$$\begin{tikzcd}
\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_K\mathbb{D}_{\mathrm{dR}}(V) \arrow[r, "\alpha_{\mathrm{dR},\mathbb{C}_p/K}(V)_*"] \arrow[d, "i_*"'] & \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V \arrow[d, equal] \\
\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_{(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\otimes_{\mathbb{Q}_p}V)^{G_{K^{\mathrm{pf}}}} \arrow[r, "\alpha_*"] \arrow[d, "\mathrm{can.}"', "\cong"] & \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V \arrow[d, equal] \\
\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_{K^{\mathrm{pf}}}\mathbb{D}_{\mathrm{dR}}(V|_{K^{\mathrm{pf}}}) \arrow[r, hook, "\alpha_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}(V|_{K^{\mathrm{pf}}})"] & \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}\otimes_{\mathbb{Q}_p}V,
\end{tikzcd}$$

where the left lower arrow is induced by $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/\mathbb{Q}_p}\to\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$, the $G_{K^{\mathrm{pf}}}$-equivariant isomorphism. Denote the composition of the left vertical arrows by i'. Since the canonical map $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$ is $G_{K^{\mathrm{pf}}}$-equivariant, by the diagram, the restriction of i' to $\mathbb{D}_{\mathrm{dR}}(V)$ coincides with the canonical map $\mathbb{D}_{\mathrm{dR}}(V) \to \mathbb{D}_{\mathrm{dR}}(V|_{K^{\mathrm{pf}}})$, which is an isomorphism after tensoring K^{pf} (see Section 4B). Therefore, i' is an isomorphism and we obtain the assertion. □

6. Proof of the Main Theorem

We will restate our main theorem in the point of view of Remark 4.7(iii):

Main Theorem. *Let V be a de Rham representation of G_K. Then, there exists a finite extension L/K such that the restriction $V|_L$ is $\mathbb{B}_{\mathrm{st},\mathbb{C}_p/L_0}$-admissible for any choice of L_0.*

In this section, we give a proof of the Main Theorem in this form. Before the proof, we prepare technical lemmas used in the proof. The reader may go to the proof of the Main Theorem and back to the lemmas if necessary.

We first recall a slightly modified version of [Colmez 2008, Proposition 0.6]. In the rest of this section, denote the unramified extension of $\mathbb{Q}_p$ of degree $h \in \mathbb{N}_{>0}$ by $\mathbb{Q}_{p^h}$.

Proposition 6.1. *Assume that k_K is perfect. Let $\mathbb{U}'_{h,a} := (\tilde{\mathbb{B}}^+_{\log,\mathbb{C}_p/K_0})^{\varphi^h = p^a}$ for h, $a \in \mathbb{N}$. Let M be a (φ, G_K)-module over $\tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K_0}$ of rank $r \in \mathbb{N}_{>0}$ with basis $e_1, \dots, e_r$. Assume that there exists an isomorphism of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K}} M \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^r$ and that $e_1, \dots, e_r$ satisfies the following conditions*:

(i) *There exists $h \in \mathbb{N}_{>0}$ and $a_1 \le \dots \le a_r \in \mathbb{N}$ such that $\varphi^h(e_i) = p^{a_i} e_i$ for $1 \le i \le r$.*

(ii) *For all $g \in G_K$, there exists $c_g \in \mathrm{GL}_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$, a (unique) upper triangular matrix whose diagonal entries are 1, such that $g(e_1, \dots, e_r) = (e_1, \dots, e_r)c_g$.*

Then there exists a $\tilde{\mathbb{B}}^+_{\log,\mathbb{C}_p/K_0}$-basis $f_1, \dots, f_r$ of $\tilde{\mathbb{B}}^+_{\log,\mathbb{C}_p/K_0} \otimes_{\tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K_0}} M$ satisfying the following conditions:

(a) *f_i is fixed by G_K;*

(b) *$f_i = e_i + \sum_{1 \le j \le i-1} \alpha_{ji} e_j$ with $\alpha_{ji} \in \mathbb{U}'_{h,a_i - a_j}$ (hence $\varphi^h(f_i) = p^{a_i} f_i$).*

Proof. Note that we add the extra assumption (ii) and the slightly stronger conclusion (a) to the original proposition. Let U be the subgroup of $\mathrm{GL}_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ consisting of upper triangular matrices whose diagonal entries are 1 and whose (i, j)-component belongs to $\mathbb{U}'_{h,a_j - a_i}$ for $i < j$. We endow U with the subspace topology of $\mathrm{GL}_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$. Then, U is a topological G_K-group and the map $g \mapsto c_g; G_K \to U$ is a continuous 1-cocycle. By [Colmez 2008, Proposition 0.6], there exists a finite Galois extension L/K such that $[c]$ is mapped to the trivial class in $H^1(G_{L^{\mathrm{ur}}},U)$ by the composite $\mathrm{Res}_L^{L^{\mathrm{ur}}}\circ\mathrm{Res}_K^L$, where $[c]$ denotes the class represented by c. Note that for all $a\in\mathbb{N}_{>0}$, we have

$$(\mathbb{U}'_{h,a})^{G_{L^{\mathrm{ur}}}}\subset((\mathbb{B}_{\mathrm{st},\mathbb{C}_p/L_0})^{G_{L^{\mathrm{ur}}}})^{\varphi^h=p^a}=(L_0^{\mathrm{ur}})^{\varphi^h=p^a}=0,$$

where the first equality follows from Remark 3.12(ii) and Corollary 4.3 and the last equality follows from [Colmez 2008, Lemme 10.9]. Hence $U^{G_{L^{\mathrm{ur}}}}=\{1\}$ and $[c]$ is mapped to the trivial class in $H^1(G_L,U)$ by the inflation-restriction exact sequence. Hence, we have only to prove that the inverse image of the trivial element by $\mathrm{Res}_K^L:H^1(G_K,U)\to H^1(G_L,U)$ consists of the trivial element.

We endow U with a G_K-stable decreasing filtration $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ by $\mathscr{F}_n:=\{(x_{ij})\in U\mid x_{ij}=0 \text{ for } 0<j-i\le n\}$. Then, we have $\mathscr{F}_0=U$, $\mathscr{F}_r=\{1\}$, $\mathscr{F}_{n+1}\trianglelefteq\mathscr{F}_n$ and $\mathscr{F}_n/\mathscr{F}_{n+1}$ is isomorphic to a direct sum of copies of $\mathbb{U}'_{h,a}$ with $a\in\mathbb{N}$. We have only to prove that the inverse image of the trivial element under the restriction map $\mathrm{Res}_K^L:H^1(G_K,\mathscr{F}_n)\to H^1(G_L,\mathscr{F}_n)$ for $n\in\mathbb{N}$ consists of the trivial element. Since there exists a G_K-equivariant set-theoretic section of the canonical projection $\mathscr{F}_n\to\mathscr{F}_n/\mathscr{F}_{n+1}$ (for example, we can identify

$$1+\sum_i x_{i,i+n+1}E_{i,i+n+1}\in\mathscr{F}_n$$

with its image in $\mathscr{F}_n/\mathscr{F}_{n+1}$), the canonical maps $\mathscr{F}_n^{G_K}\to(\mathscr{F}_n/\mathscr{F}_{n+1})^{G_K}$ and $\mathscr{F}_n^{G_L}\to(\mathscr{F}_n/\mathscr{F}_{n+1})^{G_L}$ are surjective. By using long exact sequences, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(G_K,\mathscr{F}_{n+1}) \arrow[r, "\text{can.}"] \arrow[d, "\mathrm{Res}_K^L"] & H^1(G_K,\mathscr{F}_n) \arrow[r, "\text{can.}"] \arrow[d, "\mathrm{Res}_K^L"] & H^1(G_K,\mathscr{F}_n/\mathscr{F}_{n+1}) \arrow[d, "\mathrm{Res}_K^L"] \\
0 \arrow[r] & H^1(G_L,\mathscr{F}_{n+1}) \arrow[r, "\text{can.}"] & H^1(G_L,\mathscr{F}_n) \arrow[r, "\text{can.}"] & H^1(G_L,\mathscr{F}_n/\mathscr{F}_{n+1}),
\end{tikzcd}$$

whose rows are exact as pointed sets. To prove the assertion, it suffices to prove the injectivity of the restriction map $H^1(G_K,\mathbb{U}'_{h,a})\to H^1(G_L,\mathbb{U}'_{h,a})$ for $h,a\in\mathbb{N}$. Indeed, it implies the injectivity of the right arrow in the diagram and we obtain the assertion by dévissage and diagram chasing. We first consider the case $a=0$, that is, $\mathbb{U}'_{h,0}=\mathbb{Q}_{p^h}$ (Lemma 6.2 below). Since $H^1(G_{L/K},\mathbb{Q}_{p^h}^{G_L})$ is killed by the multiplication by $[L:K]$ (using the corestriction) which induces an isomorphism on the coefficients, we have $H^1(G_{L/K},\mathbb{Q}_{p^h}^{G_L})=0$. By the inflation-restriction sequence, we obtain the assertion. Consider the case $a>0$. We denote by $\chi:G_K\to\mathbb{Z}_p^\times$ the cyclotomic character. Then, we obtain the assertion by the following commutative diagram:

$$\begin{tikzcd}
H^1(G_K, \mathbb{U}'_{h,a}) \arrow[r, hook, "\Pi(N^k\circ\varphi^{-n})_*"] \arrow[d, "\mathrm{Res}^L_K"] & \prod_{n,k\in\mathbb{N}} H^1(G_K, \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \cong \prod_{n,k\in\mathbb{N}} K\log\chi \arrow[d, hook, "\Pi\mathrm{Res}^L_K"] \\
H^1(G_L, \mathbb{U}'_{h,a}) \arrow[r, hook, "\Pi(N^k\circ\varphi^{-n})_*"] & \prod_{n,k\in\mathbb{N}} H^1(G_L, \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \cong \prod_{n,k\in\mathbb{N}} L\log\chi,
\end{tikzcd}$$

where two isomorphisms follow by dévissage and Lemma 1.14, Theorem 1.15 (a theorem of J. Tate) and the injectivity of the horizontal arrows follow from [Colmez 2008, Proposition 0.4(ii)]. ☐

Lemma 6.2. *We have*

$$(\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})^{\varphi^h=p^{-a}} = (\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0})^{\varphi^h=p^{-a}} = 0 \quad \textit{for } a\in\mathbb{N}_{>0},$$
$$(\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})^{\varphi^h=1} = (\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0})^{\varphi^h=1} = \mathbb{Q}_{p^h}.$$

Proof. We first prove the first assertion. Suppose that we have a nonzero element x in $(\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0})^{\varphi^h=p^{-a}}$. Since $\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0}$ is an integral domain, we may assume that we have $x\in\mathbb{A}_{\mathrm{st},\mathbb{C}_p/K_0}$ by multiplying by some power of p. By assumption and the φ-stability of $\mathbb{A}_{\mathrm{st},\mathbb{C}_p/K_0}$, $x = p^{na}\varphi^{nh}(x)\in p^n\mathbb{A}_{\mathrm{st},\mathbb{C}_p/K_0}$. Hence $x\in\bigcap_n p^n\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/K_0}[\mathrm{x}] = \{0\}$ since $\mathbb{A}_{\mathrm{cris},\mathbb{C}_p/K_0}$ is p-adically separated. Thus $x=0$, which is a contradiction.

We prove the latter assertion. By a simple calculation, we have

$$(\tilde{\mathbb{B}}^{\nabla+}_{\log,\mathbb{C}_p/K_0})^{\varphi^h=1} = (\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})^{\varphi^h=1}.$$

By the canonical isomorphism $\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}\cong\tilde{\mathbb{B}}^{+}_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$, we may reduce to the perfect residue field case, which follows from [Colmez 2002, Proposition 9.2]. ☐

Lemma 6.3. *Let D be a finite free $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$-module with semilinear G_K-action. Then, the canonical map $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_K D^{G_K}\to D$ is injective. In particular, we have* $\dim_K D^{G_K}\le\mathrm{rank}_{\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}} D<\infty$.

Proof. Suppose that we have linearly independent elements $f_1,\dots,f_n\in D^{G_K}$ over K, which have a nontrivial relation $\sum_i\lambda_i f_i=0$ with $\lambda_i\in\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$. Choose the minimum n among such n's. Then for $1\le i\le n$, we have $g(\lambda_i/\lambda_1)=\lambda_i/\lambda_1$ in $\mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K})$. Hence we have both $\lambda_i/\lambda_1\in H^0(G_K,\mathrm{Frac}(\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}))=K$ and $\sum_i(\lambda_i/\lambda_1)f_i=0$, a contradiction. ☐

Lemma 6.4. *Let W be an r-dimensional $\mathbb{Q}_{p^h}$-vector space with semilinear G_K-action. For $0\le i<h$, we define the $\mathbb{Q}_{p^h}$-vector space φ^i_*W with semilinear G_K-action by $\varphi^i_*W := W$ as G_K-module with scalar multiplication*

$$\mathbb{Q}_{p^h}\times W\to W;\quad (\lambda,x)\mapsto\varphi^i(\lambda)x.$$

If we have an isomorphism of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$*-modules*

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_{p^h}} \varphi^i_* W \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^r$$

for $0 \le i < h$, *then* W *is* $\mathbb{C}_p$*-admissible as a* p*-adic representation of* G_K.

Proof. By assumption, we have isomorphisms

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} W \cong \bigoplus_{0\le i<h} \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_{p^h}} \varphi^i_* W \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{hr}$$

of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules, which implies the assertion by tensoring with $\mathbb{C}_p$ over $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$. □

Lemma 6.5. *Assume that* $e_{K/K_{\mathrm{can}}} = 1$. *Then, the complex*

$$K \otimes_{K_0} (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}} \xrightarrow{\nabla} \Omega^1_K \widehat{\otimes}_{K_0} (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}} \xrightarrow{\nabla_1} \Omega^2_K \widehat{\otimes}_{K_0} (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}},$$

which is induced by the inclusion $K \otimes_{K_0} \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$ (*Proposition 3.16*) *and Lemma 3.9*, *is exact. Here*, *we endow* $(\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}}$ *with the* p*-adic topology induced by the* p*-adic semivaluation* $v_{\mathrm{cris},\mathbb{C}_p/K}$.

Proof. Note that the connections are K_{can}-linear by Proposition 1.13. We may reduce to the case $K = K_0$ by Remark 1.4(ii) and Lemma 1.10(iii). Let $\omega \in \ker \nabla_1$. We can write $\omega = \sum_{j\in J_K} dt_j \otimes \lambda_j$ with $\lambda_j \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K}$ such that

$$\{v_{\mathrm{cris},\mathbb{C}_p/K}(\lambda_j)\}_{j\in J_K} \to \infty.$$

We can also write $\lambda_j = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}} \lambda_{j,\boldsymbol{n}} \boldsymbol{u}^{[\boldsymbol{n}]}$ with $\lambda_{j,\boldsymbol{n}} \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}$ such that $\{v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}})\}_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}} \to \infty$. Since u_j is invariant under the action of $G_{K^{\mathrm{pf}}}$, we have $\lambda_{j,\boldsymbol{n}} \in (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p})^{G_{K^{\mathrm{pf}}}}$. Recall the proof of Lemma 3.9: We define $a_{\boldsymbol{0}} = 0$ and $a_{\boldsymbol{n}} = \lambda_{j,\boldsymbol{n}-\mathbf{e}_j}$ if $n_j \neq 0$. Then, we have

$$x = \sum_{\boldsymbol{n}\in\mathbb{N}^{\oplus J_K}} a_{\boldsymbol{n}} \boldsymbol{u}^{[\boldsymbol{n}]} \in \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}$$

and $\nabla(x) = \omega$. Note that we have $x \in (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}$. Hence, we have only to prove $x \in \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K}$. Fix $N \in \mathbb{N}$: we have to show that $v_{\mathrm{cris},\mathbb{C}_p/K}(a_{\boldsymbol{n}}) \ge N$ for all but finitely many $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_K}$. Choose a finite subset J of J_K such that we have $v_{\mathrm{cris},\mathbb{C}_p/K}(\lambda_j) \ge N$ for $j \in J_K \setminus J$. We also choose $n \in \mathbb{N}$ such that we have $v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}}) \ge N$ for $j \in J$ and $|\boldsymbol{n}| \ge n$. Let $\boldsymbol{n} \in \mathbb{N}^{\bigoplus J_K} \setminus \mathbb{N}^J$. Then, we have $v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}) = v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}-\boldsymbol{e}_j}) \ge N$ for some $j \in J_K \setminus J$. Let $\boldsymbol{n} \in \mathbb{N}^J$ with $|\boldsymbol{n}| > n$. Then, we have $v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(a_{\boldsymbol{n}}) = v_{\mathrm{cris},\mathbb{C}_p/\mathbb{Q}_p}(\lambda_{j,\boldsymbol{n}-\boldsymbol{e}_j}) \ge N$ for some $j \in J$. Since the set $\{\boldsymbol{n} \in \mathbb{N}^J \mid |\boldsymbol{n}| \le n\}$ is finite, these inequalities imply the assertion. □

Proof of Main Theorem. Obviously, we may assume $r := \dim_{\mathbb{Q}_p} V > 0$. By Hilbert 90, we may replace K by K^{ur}. Hence, we may assume that k_K is separably closed. After some Tate twist, we may also assume that V satisfies the assumption of Section 5, that is, we have $\mathbb{D}_{\mathrm{dR}}(V) = (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V)^{G_K}$.

We divide the rest of the proof into two steps: We will construct a finite extension L/K in Step 1 and after replacing K by L, we will prove the semistability of V in Step 2. Note that only Step 2 involves the choice of K_0.

Step 1: Set $\mathcal{M} := \widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$ and let $e_1, \dots, e_r$ be as in Proposition 5.3. Also let $\{a'_1 < \cdots < a'_{r'}\}$ be the set of distinct elements in the multiset $\{a_1, \dots, a_r\}$ and m_i the multiplicity of a'_i in the multiset for $1 \le i \le r'$. Let $\{e^{(i)}_1, \dots, e^{(i)}_{m_i}\}$ be the subset of $e_l \in \{e_1, \dots, e_r\}$ satisfying $\varphi^h(e_l) = p^{a'_i} e_l$. We define an exhaustive and separated increasing filtration of $\mathcal{M}$ by

$$\mathcal{M}_n := \begin{cases} 0 & \text{if } n \le 0, \\ \bigoplus_{1 \le i \le n} (\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} e^{(i)}_1 \oplus \cdots \oplus \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} e^{(i)}_{m_i}) & \text{if } 1 \le n < r', \\ \mathcal{M} & \text{otherwise.} \end{cases}$$

The filtration is stable under φ and G_K-actions. In fact, for $1 \le i \le n < r'$ and $g \in G_K$, we have

$$\varphi(e^{(i)}_1), \dots, \varphi(e^{(i)}_{m_i}), g(e^{(i)}_1), \dots, g(e^{(i)}_{m_i}) \in \mathcal{M}^{\varphi^h = p^{a'_i}} \subset \mathcal{M}_n,$$

where the last inclusion follows from Lemma 6.2. We also define

$$W_n := (\mathcal{M}_n/\mathcal{M}_{n-1})^{\varphi^h = p^{a'_n}}$$

for $1 \le n \le r'$. Since we have $W_n = \mathbb{Q}_{p^h} \bar{e}^{(n)}_1 \oplus \cdots \oplus \mathbb{Q}_{p^h} \bar{e}^{(n)}_{m_n}$ by Lemma 6.2 (where $\bar{e}^{(n)}_i$ denotes the image of $e^{(n)}_i$ in $\mathcal{M}_n/\mathcal{M}_{n-1}$), W_n is an m_n-dimensional $\mathbb{Q}_{p^h}$-vector space with continuous semilinear G_K-action. Let

$$D_n := \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}} \mathcal{M}_n.$$

Then, we have the left exact sequence of finite K-vector spaces

$$0 \longrightarrow D^{G_K}_{n-1} \xrightarrow{\text{inc.}} D^{G_K}_n \xrightarrow{\text{pr.}} (D_n/D_{n-1})^{G_K}. \tag{3}$$

Hence, we have the inequalities

$$\dim_K D^{G_K}_n \le \dim_K D^{G_K}_{n-1} + \dim_K (D_n/D_{n-1})^{G_K} \le \dim_K D^{G_K}_{n-1} + m_n$$

for $n \in \mathbb{Z}$ by Lemma 6.3. By Proposition 5.4, we have an isomorphism of $\mathbb{B}^+_{\mathrm{dR}}[G_K]$-modules

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}} \mathcal{M} \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^r, \tag{4}$$

which implies $\dim_K D^{G_K}_n = r$ for $n \ge r'$. Hence, the summation of the above inequalities are equalities. Therefore, the above inequalities are equalities, in particular, the map $\mathrm{pr.}: D_n^{G_K} \to (D_n/D_{n-1})^{G_K}$ in (3) is surjective. Thus, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_K D_{n-1}^{G_K} \arrow[r] \arrow[d, hook] & \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_K D_n^{G_K} \arrow[r] \arrow[d, hook] & \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}\otimes_K (D_n/D_{n-1})^{G_K} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & D_{n-1} \arrow[r] & D_n \arrow[r] & D_n/D_{n-1} \arrow[r] & 0
\end{tikzcd}$$

with exact rows and injective vertical arrows by Lemma 6.3. Since the middle vertical arrow is an isomorphism for $n \geq r'$ by (4), all vertical arrows are isomorphisms. In particular, for $1 \leq n \leq r'$, we have isomorphisms of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_{p^h}} W_n \cong D_n/D_{n-1} \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{m_n}$. Since W_n is stable under the action of φ, the map $W_n \to \varphi^i_* W_n$ taking x to $\varphi^i(x)$ is an isomorphism of $\mathbb{Q}_{p^h}[G_K]$-modules. In particular, we have isomorphisms of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules

$$\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_{p^h}} \varphi^i_* W_n \cong \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_{p^h}} W_n \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{m_n}$$

for $1 \leq n \leq r'$ and $0 \leq i < h$, which implies the $\mathbb{C}_p$-admissibility of W_n by Lemma 6.4. Hence, G_K acts on W_n factoring through a finite quotient by Theorem 2.1. We choose a finite extension L/K such that G_L acts on W_n trivially for all $1 \leq n \leq r'$ and such that L satisfies condition (H).

Step 2: By replacing V by $V|_L$, we will prove that V is semistable by calculating Galois cohomology associated to $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V)$. In the following, we fix K_0 and a lift $\{t_j\}_{j\in J_K}$ of a p-basis of k_K in K_0. We regard $\{t_j\}_{j\in J_K}$ as a lift of a p-basis of k_K in K. We also fix notation: For a commutative ring R, let $U_r(R) \subset \mathrm{GL}_r(R)$ be the group of unipotent upper triangular matrices. Let $N_r(R) \subset M_r(R)$ be the Lie algebra of $U_r(R)$, that is, the group of nilpotent upper triangular matrices. We denote $U^+_{r,\mathrm{dR}} := U_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$, $U^{\nabla+}_{r,\mathrm{dR}} = U_r(\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})$ for simplicity.

By assumption, we have $g(e_1,\ldots,e_r) = (e_1,\ldots,e_r)c_g$ with 1-cocycle

$$c : G_K \to U_r(\widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0}).$$

Since we have $\widetilde{\mathbb{N}}^{\nabla+}_{\mathrm{rig}}(V) \subset \widetilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \otimes_{\mathbb{Q}_p} V$ and

$$(K \otimes_{K_0} \mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0} \otimes_{\mathbb{Q}_p} V)^{G_K} = K \otimes_{K_0} (\mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0} \otimes_{\mathbb{Q}_p} V)^{G_K},$$

we have only to prove that c is a 1-coboundary in $U_r(K \otimes_{K_0} \mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0})$. We have the exact sequence of pointed sets

$$(U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K} \xrightarrow{\delta} H^1(G_K, U^{\nabla+}_{r,\mathrm{dR}}) \xrightarrow{\mathrm{inc.}_*} H^1(G_K, U^+_{r,\mathrm{dR}}), \tag{5}$$

where $U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}}$ denotes the left coset of $U^+_{r,\mathrm{dR}}$ by $U^{\nabla+}_{r,\mathrm{dR}}$, that is, $X \sim Y$ if $X^{-1}Y \in U^{\nabla+}_{r,\mathrm{dR}}$. The class $[c] \in H^1(G_K, U^{\nabla+}_{r,\mathrm{dR}})$ represented by c is mapped to the trivial class in $H^1(G_K, U^+_{r,\mathrm{dR}})$. In fact, since we have $\bar{e}^{(n)}_i \in (D_n/D_{n-1})^{G_K}$ for $1 \le n \le r'$ and $1 \le i \le m_n$ by assumption, there exists an element $\tilde{e}^{(n)}_i \in D_n^{G_K}$ such that $\tilde{e}^{(n)}_i - e^{(n)}_i \in D_{n-1}$ by the exactness of (3). Then,

$$(\tilde{e}^{(1)}_1, \dots, \tilde{e}^{(1)}_{m_1}, \dots, \tilde{e}^{(n)}_1, \dots, \tilde{e}^{(n)}_{m_n})$$

is a $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$-basis of D_n for $1 \le n \le r'$ and we have a unique matrix $U \in U^+_{r,\mathrm{dR}}$ such that

$$(e^{(1)}_1, \dots, e^{(1)}_{m_1}, \dots, e^{(r')}_1, \dots, e^{(r')}_{m_{r'}}) = (\tilde{e}^{(1)}_1, \dots, \tilde{e}^{(1)}_{m_1}, \dots, \tilde{e}^{(r')}_1, \dots, \tilde{e}^{(r')}_{m_{r'}})U.$$

By a simple calculation, we have $c_g = U^{-1}g(U)$ for all $g \in G_K$. Hence, the class $[c]$ is represented by an element of the image of $(U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K}$ under δ by the exact sequence (5). We regard $K \otimes_{K_0} \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0}$ as a subring of $\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}$ by Proposition 3.16. Then, we have the following lemma:

Lemma 6.6. *Every element of $(U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K}$ is represented by an element in $U_r(K \otimes_{K_0} (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}})$.*

We leave the proof of Lemma 6.6 to the end of the proof. Thanks to the lemma, there exist $X_1 \in U_r(K \otimes_{K_0} (\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}})$ and $X_2 \in U^{\nabla+}_{r,\mathrm{dR}}$ such that

$$c_g = X_2^{-1}X_1^{-1}g(X_1)g(X_2) \tag{6}$$

for all $g \in G_K$.

Since the canonical isomorphism $i : \tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0} \to \tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$ is compatible with the actions of φ and $G_{K^{\mathrm{pf}}}$, we may regard $M := i^*\mathcal{M}$ as a $(\varphi, G_{K^{\mathrm{pf}}})$-module over $\tilde{\mathbb{B}}^+_{\mathrm{rig},\mathbb{C}_p/K_0^{\mathrm{pf}}}$. Then, the triple $(M, \{e_1, \dots, e_r\}, i^*c)$ satisfies the assumptions of Proposition 6.1. Indeed, assumption (i) follows from Proposition 5.3, Proposition 5.4 and the functoriality. The image of c is in $U_r(\tilde{\mathbb{B}}^{\nabla+}_{\mathrm{rig},\mathbb{C}_p/K_0})$, which implies assumption (ii). Applying Proposition 6.1 to the above triple, we have $X_3' \in U_r(\mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0^{\mathrm{pf}}})$ such that $i(c_g) = (X_3')^{-1}g(X_3')$. Hence, $X_3 := i^{-1}(X_3') \in U_r(\mathbb{B}^{\nabla+}_{\mathrm{st},\mathbb{C}_p/K_0})$ satisfies $c_g = X_3^{-1}g(X_3)$ for $g \in G_{K^{\mathrm{pf}}}$. Since we have $c_g = X_2^{-1}g(X_2)$ for $g \in G_{K^{\mathrm{pf}}}$ by (6), we have

$$X_2X_3^{-1} \in (U^{\nabla+}_{r,\mathrm{dR}})^{G_{K^{\mathrm{pf}}}} = U_r((\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}).$$

Note that the canonical map

$$K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} (\mathbb{B}^{\nabla+}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}} \to (\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K})^{G_{K^{\mathrm{pf}}}}$$

is an isomorphism. In fact, by using the canonical isomorphisms $\mathbb{B}^{\nabla+}_{\mathrm{cris},\mathbb{C}_p/K_0} \to \mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{pf}}}$ and $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \to \mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$, it follows from the isomorphisms

$$K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} K_0^{\mathrm{pf}} \cong K \otimes_{K_0} K_0^{\mathrm{pf}} \cong K^{\mathrm{pf}},$$

where the first isomorphism easily follows from Remark 1.4(ii) and the second one is trivial. Thus, we have

$$c_g = (X_1 \cdot X_2 X_3^{-1} \cdot X_3)^{-1} g(X_1 \cdot X_2 X_3^{-1} \cdot X_3)$$

for all $g \in G_K$ with X_1, $X_2 X_3^{-1}$, $X_3 \in U_r(K \otimes_{K_0} \mathbb{B}^+_{\mathrm{st},\mathbb{C}_p/K_0})$, which implies the assertion.

Now, we return to the proof of Lemma 6.6. We endow $M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \cong (\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})^{r^2}$ with the product topology. Let

$$d : M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \to \widehat{\Omega}^1_K \widehat{\otimes}_K M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}); \quad (x_{ij}) \mapsto (\nabla(x_{ij})),$$
$$d_1 : \widehat{\Omega}^1_K \widehat{\otimes}_K M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \to \widehat{\Omega}^2_K \widehat{\otimes}_K M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}); \quad (\omega_{ij}) \mapsto (\nabla_1(\omega_{ij}))$$

be the derivations. For $i \in \{1, 2\}$, we endow $\widehat{\Omega}^i_K \widehat{\otimes}_K M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ with the left (resp. right) action of $M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ induced by the left (resp. right) multiplication on $M_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$. We also have the wedge product

$$\wedge : \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(K) \times \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \to \widehat{\Omega}^2_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$$
$$(\omega_{ij}) \times (\omega'_{ij}) \mapsto \Bigl(\sum_{1\le k\le r} \omega_{ik} \wedge \omega'_{kj}\Bigr).$$

Then, we have the formulas $d_1 \circ d = 0$, $d(XX') = dX \cdot X' + X \cdot dX'$, $d_1(\omega \cdot X) = d_1\omega \cdot X - \omega \wedge dX$, and $(\omega \wedge \omega') \cdot X = \omega \wedge (\omega' \cdot X)$, for $X, X' \in \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$, $\omega \in \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(K)$, and $\omega' \in \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$. We define a log differential

$$\mathrm{dlog} : U^+_{r,\mathrm{dR}} \to \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}); \quad X \mapsto dX \cdot X^{-1},$$

which is G_K-equivariant. (Note that it does not preserve the group laws in general.) Since we have $\mathrm{dlog}(XA) = \mathrm{dlog}(X)$ for $A \in U^{\nabla+}_{r,\mathrm{dR}}$ and $X \in U^+_{r,\mathrm{dR}}$ by the above formulas, dlog induces a morphism of G_K-sets

$$\mathrm{dlog}_* : U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}} \to \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}).$$

Moreover, dlog_* is injective. Indeed, let $X, Y \in U^+_{r,\mathrm{dR}}$ such that $\mathrm{dlog}X = \mathrm{dlog}Y$. By $d\boldsymbol{E} = d(Y^{-1}Y) = 0$ and the above formulas, we have $d(Y^{-1}) = -Y^{-1}dY \cdot Y^{-1}$. Hence, we have

$$\begin{aligned}\mathrm{dlog}(Y^{-1}X) &= (d(Y^{-1}) \cdot X + Y^{-1}dX) \cdot X^{-1}Y \\ &= -Y^{-1}(dY \cdot Y^{-1} - dX \cdot X^{-1}) \cdot Y = 0.\end{aligned}$$

Since the inverse image of $\{0\}$ by dlog is $U^{\nabla+}_{r,\mathrm{dR}}$, we have $X \sim Y$. By taking $H^0(G_K, -)$ of dlog_*, we have an injection of sets

$$\mathrm{dlog}_* : (U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K} \hookrightarrow \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(K).$$

We define a decreasing filtration on $N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ by

$$\mathrm{Fil}^n N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) := \{(a_{ij}) \in N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) \mid a_{ij} = 0 \text{ if } j-i \le n\}.$$

Then, we have $\mathrm{Fil}^0 N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) = N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$ and $\mathrm{Fil}^r N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}) = 0$. Let $X \in U^+_{r,\mathrm{dR}}$ such that we have $[X] \in (U^+_{r,\mathrm{dR}}/U^{\nabla+}_{r,\mathrm{dR}})^{G_K}$. Let $\omega := \mathrm{dlog}(X) \in \widehat{\Omega}^1_K \widehat{\otimes}_K N_r(K)$. We will construct $X^{(n)} \in U_r(K \otimes_{K_0} (\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}})$ for $n \in \mathbb{N}$ satisfying

$$\omega \cdot X^{(n)} \equiv dX^{(n)} \mod \widehat{\Omega}^1_K \widehat{\otimes}_K \mathrm{Fil}^n N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}).$$

Set $X^{(0)} := 1$. Suppose that we have constructed $X^{(n)}$. Since we have $\omega \cdot X = dX$, we have $d_1\omega \cdot X = \omega \wedge dX$ by taking d_1. Hence, we have $d_1\omega = (\omega \wedge dX)\cdot X^{-1} = \omega \wedge \omega$. Let $\omega' = (\omega'_{ij}) := \omega \cdot X^{(n)} - dX^{(n)} \in \widehat{\Omega}^1_K \widehat{\otimes}_K \mathrm{Fil}^n N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K})$. Then, by a simple calculation using the above formulas, we have

$$d_1\omega' = \omega \wedge (\omega \cdot X^{(n)} - dX^{(n)}) = \omega \wedge \omega' \equiv 0 \mod \widehat{\Omega}^2_K \widehat{\otimes}_K \mathrm{Fil}^{n+1} N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}),$$

which implies $\nabla_1(\omega'_{i,i+n+1}) = 0$. Since we have

$$\omega'_{ij} \in \widehat{\Omega}^1_K \widehat{\otimes}_K (K \otimes_{K_0} (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}}),$$

by Lemma 6.5, there exists $x'_{i,i+n+1} \in K \otimes_{K_0} (\mathbb{B}^+_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}}$ such that

$$\nabla(x'_{i,i+n+1}) = \omega'_{i,i+n+1}.$$

Let $X^{(n+1)} := X^{(n)} + \sum_i x'_{i,i+n+1} E_{i,i+n+1} \in U_r(K \otimes_{K_0} (\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0})^{G_{K^{\mathrm{pf}}}})$. Then, by a simple calculation, we have

$$\begin{aligned}
&\omega \cdot X^{(n+1)} - dX^{(n+1)} \\
&\quad \equiv \omega \cdot X^{(n)} - dX^{(n)} - d\left(\sum_i x'_{i,i+n+1} E_{i,i+n+1}\right) \\
&\quad \equiv \omega' - \sum_i \nabla(x'_{i,i+n+1}) E_{i,i+n+1} \equiv 0 \mod \widehat{\Omega}_K \widehat{\otimes}_K \mathrm{Fil}^{n+1} N_r(\mathbb{B}^+_{\mathrm{dR},\mathbb{C}_p/K}).
\end{aligned}$$

Hence, we have $\mathrm{dlog}(X^{(r)}) = \omega$, which implies the assertion. $\square$

7. Applications

We will give applications of the Main Theorem. In Section 7A, we will recall linear algebraic structures, which appear in the following. In Section 7B, we will prove a horizontal analogue of the p-adic monodromy theorem. The results of the next two subsections are applications of this theorem. In Section 7C, we will prove an equivalence between the category of horizontal de Rham representations of G_K and the category of de Rham representation of $G_{K_{\mathrm{can}}}$. In Section 7D, we will prove a generalization of Hyodo's Theorem 1.16.

In this section, unless particular mention is stated, we will denote $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/K_0}$ (resp. $\mathbb{B}^{\nabla}_{\heartsuit,\mathbb{C}_p/K}$) by $\mathbb{B}^{\nabla}_{\diamondsuit}$ (resp. $\mathbb{B}^{\nabla}_{\heartsuit}$) for $\diamondsuit\in\{\mathrm{cris},\mathrm{st}\}$ (resp. $\heartsuit\in\{\mathrm{dR},\mathrm{HT}\}$): This notation is justified by the facts that $\mathbb{B}^{\nabla}_{\diamondsuit,\mathbb{C}_p/K_0}$ and $\mathbb{B}^{\nabla}_{\heartsuit,\mathbb{C}_p/K}$ are isomorphic to $\mathbb{B}_{\diamondsuit,\mathbb{C}_p/\mathbb{Q}_p}$ and $\mathbb{B}_{\heartsuit,\mathbb{C}_p/\mathbb{Q}_p}$ as $(\mathbb{Q}_p, G_K)$-rings respectively.

7A. ***Additional structures.*** In the following, let $V\in\mathrm{Rep}_{\mathbb{Q}_p}G_K$. The vector space $\mathbb{D}^{\nabla}_{\bullet}(V)$ has additional structures for $\bullet\in\{\mathrm{cris},\mathrm{st},\mathrm{dR},\mathrm{HT}\}$, which we will recall following [Fontaine 1994b].

- The Hodge–Tate case

We define a graded K-vector space as a finite-dimensional K-vector space D endowed with a decomposition $D=\bigoplus_{n\in\mathbb{Z}}D_n$. Denote by MG_K the category of graded K-vector spaces. The graded ring structure on $\mathbb{B}^{\nabla}_{\mathrm{HT}}$ induces a graded K-vector space structure on $\mathbb{D}^{\nabla}_{\mathrm{HT}}(V)$. Hence, we have a $\otimes$-functor

$$\mathbb{D}^{\nabla}_{\mathrm{HT}}:\mathrm{Rep}^{\nabla}_{\mathrm{HT}}G_K\to MG_K.$$

Assume that we have $V\in\mathrm{Rep}^{\nabla}_{\mathrm{HT}}G_K$. We define the Hodge–Tate weights of V as the multiset consisting of $n\in\mathbb{Z}$ with multiplicity $m_n:=\dim_K(\mathbb{C}_p(-n)\otimes_{\mathbb{Q}_p}V)^{G_K}$. Since the comparison isomorphism $\alpha^{\nabla}_{\mathrm{HT}}$ is compatible with G_K-actions and gradings, by taking the degree zero part, we have an isomorphism of $\mathbb{C}_p[G_K]$-modules

$$\mathbb{C}_p\otimes_{\mathbb{Q}_p}V\cong\bigoplus_{n\in\mathbb{Z}}\mathbb{C}_p(n)^{m_n},$$

which is referred to as the Hodge–Tate decomposition of V. Note that if $V\in\mathrm{Rep}_{\mathbb{Q}_p}G_K$ admits such a decomposition, then it is horizontal Hodge–Tate.

- The de Rham case

We define a filtered K_{can}-module as a finite-dimensional K_{can}-vector space endowed with a decreasing filtration $\{\mathrm{Fil}^nD\}_{n\in\mathbb{Z}}$ of K_{can}-subspaces such that $\mathrm{Fil}^nD=D$ for $n\ll0$ and $\mathrm{Fil}^nD=0$ for $n\gg0$. Denote by $MF_{K_{\mathrm{can}}}$ the category of filtered K_{can}-modules. The filtration $\mathrm{Fil}^n\mathbb{B}^{\nabla}_{\mathrm{dR}}=t^n\mathbb{B}^{\nabla+}_{\mathrm{dR}}$ on $\mathbb{B}^{\nabla}_{\mathrm{dR}}$ induces a filtered K_{can}-module structure on $\mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$. Hence, we have a $\otimes$-functor

$$\mathbb{D}^{\nabla}_{\mathrm{dR}}:\mathrm{Rep}^{\nabla}_{\mathrm{dR}}G_K\to MF_{K_{\mathrm{can}}}.$$

- The crystalline and semistable cases

We first define filtered $(\varphi,N,G_{L/K})$-modules for our later use.

Defintion 7.1. (i) Let L/K be a finite Galois extension. A filtered $(\varphi,N,G_{L/K})$-module is a finite-dimensional $L_{\mathrm{can},0}$-vector space D endowed with

- the Frobenius endomorphism: a bijective φ-semilinear map $\varphi:D\to D$;
- the monodromy operator: an $L_{\mathrm{can},0}$-linear map $N:D\to D$ such that $N\varphi=p\varphi N$;

- the Galois action: an $L_{\mathrm{can},0}$-semilinear action of $G_{L/K}$, which commutes with φ and N;
- the filtration: a decreasing filtration $\{\mathrm{Fil}^n D_{L_{\mathrm{can}}}\}_{n\in\mathbb{Z}}$ of $G_{L/K}$-stable L_{can}-subspaces of $D_{L_{\mathrm{can}}} := L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} D$ satisfying

$$\mathrm{Fil}^n D_{L_{\mathrm{can}}} = D_{L_{\mathrm{can}}} \quad \text{for } n \ll 0 \quad \text{and} \quad \mathrm{Fil}^n D_{L_{\mathrm{can}}} = 0 \quad \text{for } n \gg 0.$$

If $L = K$, then we call D a filtered (φ, N)-module relative to K_{can}. Moreover, if $N = 0$, then we call D a filtered φ-module relative to K_{can}.

A morphism $D_1 \to D_2$ of filtered $(\varphi, N, G_{L/K})$-modules is an $L_{\mathrm{can},0}$-linear map $f : D_1 \to D_2$ such that f commutes with φ and N, $G_{L/K}$-actions and we have $f(\mathrm{Fil}^n D_{1,L_{\mathrm{can}}}) \subset \mathrm{Fil}^n D_{2,L_{\mathrm{can}}}$ for all $n \in \mathbb{Z}$.

Denote by $MF(\varphi, N, G_{L/K})$ (resp. $MF_{K_{\mathrm{can}}}(\varphi, N)$, $MF_{K_{\mathrm{can}}}(\varphi)$) the category of filtered $(\varphi, N, G_{L/K})$-modules (resp. filtered (φ, N)-modules relative to K_{can}, filtered φ-modules relative to K_{can}).

(ii) Let $D \in MF_{K_{\mathrm{can}}}(\varphi, N)$ and $r := \dim_{K_{\mathrm{can},0}} D$. We define $t_N(D)$ and $t_H(D)$ in the following way: First, we consider the case $r = 1$. If we have $v \in D \setminus \{0\}$ and $\varphi(v) = \lambda v$, then $v_p(\lambda) \in \mathbb{Z}$ is independent of the choice of v. We denote it by $t_N(D)$. We denote by $t_H(D)$ the maximum number $n \in \mathbb{Z}$ such that $\mathrm{Fil}^n D_{K_{\mathrm{can}}} \neq 0$. In the general case, we define

$$t_N(D) := t_N\big(\textstyle\bigwedge^r D\big), \quad t_H(D) := t_H\big(\textstyle\bigwedge^r D\big).$$

We say that D is weakly admissible if we have $t_N(D) = t_H(D)$ and $t_N(D') \geq t_H(D')$ for any $K_{\mathrm{can},0}$-subspace D' of D which is stable by φ and N, with $D'_{K_{\mathrm{can}}}$ endowed with the induced filtration of $D_{K_{\mathrm{can}}}$.

Denote by $MF^{\mathrm{wa}}(\varphi, N, G_{L/K})$ the full subcategory of $MF(\varphi, N, G_{L/K})$ whose objects are weakly admissible as object of $MF_{L_{\mathrm{can}}}(\varphi, N)$. We define $MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N)$ and $MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi)$ similarly.

Let $\diamondsuit \in \{\mathrm{cris}, \mathrm{st}\}$. By Proposition 3.16, we have a K_{can}-linear injection

$$K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\diamondsuit}(V) \to \mathbb{D}^{\nabla}_{\mathrm{dR}}(V).$$

We endow $K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\diamondsuit}(V)$ with the induced filtration of $\mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$. Together with the Frobenius endomorphism φ and the monodromy operator N on $\mathbb{B}^{\nabla}_{\mathrm{st}}$, these data induce a structure of a filtered (φ, N)-module over K_{can} relative to $K_{\mathrm{can},0}$ on $\mathbb{D}^{\nabla}_{\diamondsuit}(V)$. Since we have $\mathbb{D}^{\nabla}_{\mathrm{cris}}(V) = (\mathbb{D}^{\nabla}_{\mathrm{st}}(V))^{N=0}$, $\mathbb{D}^{\nabla}_{\mathrm{cris}}(V)$ has a structure of a filtered φ-module over K_{can} relative to $K_{\mathrm{can},0}$. Therefore, we have $\otimes$-functors

$$\mathbb{D}^{\nabla}_{\mathrm{cris}} : \mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K \to MF_{K_{\mathrm{can}}}(\varphi), \quad \mathbb{D}^{\nabla}_{\mathrm{st}} : \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K \to MF_{K_{\mathrm{can}}}(\varphi, N).$$

For $D \in MF_{K_{\mathrm{can}}}(\varphi, N)$, we define

$$\mathbb{V}_{\mathrm{st}}(D) := (\mathbb{B}^{\nabla}_{\mathrm{st}} \otimes_{K_{\mathrm{can},0}} D)^{N=0,\varphi=1} \cap \mathrm{Fil}^0(\mathbb{B}^{\nabla}_{\mathrm{dR}} \otimes_{K_{\mathrm{can}}} D_{K_{\mathrm{can}}}).$$

For $D \in MF_{K_{\mathrm{can}}}(\varphi)$, we define $\mathbb{V}_{\mathrm{cris}}(D) := \mathbb{V}_{\mathrm{st}}(D)$. These are (possibly infinite-dimensional) $\mathbb{Q}_p$-vector spaces with G_K-action.

Remark 7.2. Note that we have the hierarchy of full subcategories of $\mathrm{Rep}_{\mathbb{Q}_p} G_K$

$$\mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K \subset \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K \subset \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K \subset \mathrm{Rep}^{\nabla}_{\mathrm{HT}} G_K.$$

In fact, if we have $V \in \mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K$, then we have $\dim_{\mathbb{Q}_p} V = \dim_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{cris}}(V) \le \dim_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V)$, which implies that V is horizontal semistable by Lemma 1.19. In this case, the canonical injection $\mathbb{D}^{\nabla}_{\mathrm{cris}}(V) \hookrightarrow \mathbb{D}^{\nabla}_{\mathrm{st}}(V)$ is an isomorphism as filtered (φ, N)-modules relative to K_{can}. The inclusion $\mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K \subset \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$ follows from Lemma 1.20, Proposition 3.16 and Corollary 4.3. Moreover, if we have $V \in \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K$, then the canonical map $K_{\mathrm{can}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V) \to \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$ is an isomorphism of filtered K_{can}-modules. Finally, let $V \in \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$. We choose a lift of a K_{can}-basis of $\mathrm{gr}^n \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$ in $\mathrm{Fil}^n \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$ for all $n \in \mathbb{Z}$. We denote these lifts by $\{e_i\}$ and let $n_i \in \mathbb{Z}$ such that $e_i \in \mathrm{Fil}^{n_i} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \setminus \mathrm{Fil}^{n_i+1} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$. Then, $\{e_i\}$ forms a K_{can}-basis of $\mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$. Consider the comparison isomorphism

$$\mathbb{B}^{\nabla}_{\mathrm{dR}} \otimes_{K_{\mathrm{can}}} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \to \mathbb{B}^{\nabla}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V,$$

which is compatible with the filtrations. By taking Fil^n of both sides, we have

$$\sum_i t^{n-n_i} \mathbb{B}^{\nabla+}_{\mathrm{dR}} e_i = t^n \mathbb{B}^{\nabla+}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V.$$

By taking gr^n of both sides and taking $H^0(G_K, -)$, we have

$$K \otimes_{K_{\mathrm{can}}} \mathrm{gr}^n \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \cong \bigoplus_{i : n_i = n} K e_i \cong (\mathbb{C}_p(n) \otimes_{\mathbb{Q}_p} V)^{G_K}$$

by Theorem 1.15. Hence, we have an isomorphism $K \otimes_{K_{\mathrm{can}}} \mathrm{gr} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \cong \mathbb{D}^{\nabla}_{\mathrm{HT}}(V)$ of filtered K-vector spaces, which implies $V \in \mathrm{Rep}^{\nabla}_{\mathrm{HT}} G_K$ by Lemma 1.19. In particular, the multiset of Hodge–Tate weights of V coincides with the multiset consisting of $n \in \mathbb{Z}$ with multiplicity $\dim_{K_{\mathrm{can}}} \mathrm{Fil}^{-n} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)/\mathrm{Fil}^{-n+1} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V)$.

Proposition 7.3. *The functors $\mathbb{D}^{\nabla}_{\mathrm{cris}}$ and $\mathbb{D}^{\nabla}_{\mathrm{st}}$ induce the functors*

$$\mathbb{D}^{\nabla}_{\mathrm{cris}} : \mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K \to MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi), \quad \mathbb{D}^{\nabla}_{\mathrm{st}} : \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K \to MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N).$$

Moreover, these functors are fully faithful.

Proof. We first prove the weak admissibilities of the images. As noted in Remark 7.2, if V is horizontal crystalline, then $\mathbb{D}^{\nabla}_{\mathrm{cris}}(V)$ coincides with $\mathbb{D}^{\nabla}_{\mathrm{st}}(V)$ as a filtered (φ, N)-module relative to K_{can}. Therefore, we may reduce to the case that V is horizontal semistable.

For a filtered (φ, N)-module D relative to K_{can}, we endow the finite K_0^{pf}-vector space $D_{K_0^{\mathrm{pf}}}$ with a structure of a filtered (φ, N)-module relative to K^{pf} as follows.

We extend the Frobenius φ on D to $D_{K_0^{\mathrm{pf}}}$ semilinearly and extend the monodromy operator N on D to $D_{K_0^{\mathrm{pf}}}$ linearly. We also define a filtration of $D_{K^{\mathrm{pf}}}$ as $\mathrm{Fil}^\bullet D_{K^{\mathrm{pf}}} := K^{\mathrm{pf}} \otimes_{K_{\mathrm{can}}} \mathrm{Fil}^\bullet D_{K_{\mathrm{can}}}$. Moreover, the scalar extension

$$K_0^{\mathrm{pf}} \otimes_{K_{\mathrm{can},0}} (-) : MF_{K_{\mathrm{can}}}(\varphi, N) \to MF_{K^{\mathrm{pf}}}(\varphi, N)$$

induces a functor. We have only to prove that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K \arrow[r, "\mathbb{D}^{\nabla}_{\mathrm{st}}"] \arrow[d, "\mathrm{Res}^{K^{\mathrm{pf}}}_K"] & MF_{K_{\mathrm{can}}}(\varphi, N) \arrow[d, "K_0^{\mathrm{pf}} \otimes_{K_{\mathrm{can},0}} (-)"] \\
\mathrm{Rep}_{\mathrm{st}} G_{K^{\mathrm{pf}}} \arrow[r, "\mathbb{D}_{\mathrm{st}}"] & MF_{K^{\mathrm{pf}}}(\varphi, N)
\end{tikzcd}$$

In fact, since $\mathbb{D}_{\mathrm{st}}(V|_{K^{\mathrm{pf}}}) = K_0^{\mathrm{pf}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V)$ is weakly admissible by [Fontaine 1994b, Proposition 5.4.2(i)], $\mathbb{D}^{\nabla}_{\mathrm{st}}(V)$ is weakly admissible by definition.

By functoriality, the canonical map $i : K_0^{\mathrm{pf}} \otimes_{K_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V) \to \mathbb{D}_{\mathrm{st}}(V|_{K^{\mathrm{pf}}})$ is a morphism of filtered (φ, N)-modules relative to K_0^{pf}. Consider the associated graded homomorphism after applying $K^{\mathrm{pf}} \otimes_{K_0^{\mathrm{pf}}}$. The resulting homomorphism coincides with the canonical map $K^{\mathrm{pf}} \otimes_K \mathbb{D}^{\nabla}_{\mathrm{HT}}(V) \to \mathbb{D}_{\mathrm{HT}}(V|_{K^{\mathrm{pf}}})$. Since $V \in \mathrm{Rep}^{\nabla}_{\mathrm{HT}} G_K$ by Remark 7.2, a Hodge–Tate decomposition $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{n \in \mathbb{N}} \mathbb{C}_p(n)^{m_n}$ of V induces a Hodge–Tate decomposition of $V|_{K^{\mathrm{pf}}}$. In particular, $V|_{K^{\mathrm{pf}}}$ is also Hodge–Tate and the above canonical map is an isomorphism. Since the filtrations of $\mathbb{D}^{\nabla}_{\mathrm{st}}(V)$ and $\mathbb{D}_{\mathrm{st}}(V|_{K^{\mathrm{pf}}})$ are separated and exhaustive, i is an isomorphism as filtered (φ, N)-modules relative to K_0^{pf}.

We prove the full faithfulness. We have the fundamental exact sequence

$$0 \longrightarrow \mathbb{Q}_p \xrightarrow{\text{inc.}} (\mathbb{B}^{\nabla}_{\mathrm{cris}})^{\varphi=1} \xrightarrow{\text{can.}} \mathbb{B}^{\nabla}_{\mathrm{dR}}/\mathbb{B}^{\nabla+}_{\mathrm{dR}} \longrightarrow 0.$$

Indeed, the exactness is reduced to [Colmez and Fontaine 2000, Proposition 9.25] by identifying $\mathbb{B}^{\nabla}_{\mathrm{cris}}$ (resp. $\mathbb{B}^{\nabla+}_{\mathrm{dR}}, \mathbb{B}^{\nabla}_{\mathrm{dR}}$) with $\mathbb{B}_{\mathrm{cris},\mathbb{C}_p/K_0^{\mathrm{pf}}}$ (resp. $\mathbb{B}^{+}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}, \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K^{\mathrm{pf}}}$). By the fundamental exact sequence, we have $\mathbb{V}_{\mathrm{st}} \circ \mathbb{D}^{\nabla}_{\mathrm{st}}(V) = V$ (resp. $\mathbb{V}_{\mathrm{cris}} \circ \mathbb{D}^{\nabla}_{\mathrm{cris}}(V) = V$) for $V \in \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_K$ (resp. $V \in \mathrm{Rep}^{\nabla}_{\mathrm{cris}} G_K$). This implies the full faithfulness. $\square$

In Proposition 7.5(ii), we will prove that the above functors induce equivalences of categories, that is, are essentially surjective.

7B. ***A horizontal analogue of the p-adic monodromy theorem.*** The following is a horizontal analogue of the p-adic monodromy theorem. Note that the converse is true by Hilbert 90 and Corollary 4.3.

Theorem 7.4. *Let $V \in \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$. Then, there exists a finite extension K'/K_{can} such that $V|_{KK'}$ is horizontal semistable.*

Proof. First, the comparison isomorphism $\alpha^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K}$ induces an isomorphism of $\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K}[G_K]$-modules

$$\mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{K_{\mathrm{can}}} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V) \to \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} V.$$

By taking $H^0(G_K, -)$, we have $\dim_K \mathbb{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$ by Corollary 4.3, which implies $V \in \mathrm{Rep}_{\mathrm{dR}} G_K$ by Lemma 1.19. Hence, there exists a finite extension L/K such that $V|_L$ is semistable by the Main Theorem. We may assume that L/K is a finite Galois extension satisfying condition (H) by the proof of the Main Theorem (Step 1) and Epp's Theorem 1.6. The extension $L_{\mathrm{can}}/K_{\mathrm{can}}$ is finite Galois by Lemma 1.5(ii). We will prove the assertion for $K' = L_{\mathrm{can}}$.

We have canonical isomorphisms

$$L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}_{\mathrm{st}}(V|_L) \cong L \otimes_{L_0} \mathbb{D}_{\mathrm{st}}(V|_L) \cong \mathbb{D}_{\mathrm{dR}}(V|_L),$$

where the first one is induced by a canonical isomorphism $L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} L_0 \to L$ (Remark 1.4(ii)), the second one follows by using Lemma 1.20 and Proposition 3.16. Moreover, these maps are compatible with the residual $G_{L/K}$-actions and the ∇-actions. By taking the horizontal sections, we have

$$\begin{aligned}\mathbb{D}^{\nabla}_{\mathrm{dR}}(V|_L) &\cong \mathbb{D}_{\mathrm{dR}}(V|_L)^{\nabla=0} \cong (L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}_{\mathrm{st}}(V|_L))^{\nabla=0} \\ &\cong L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}_{\mathrm{st}}(V|_L)^{\nabla=0} \cong L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V|_L),\end{aligned}$$

where the third equality follows from the fact $\nabla|_{L_{\mathrm{can}}} = 0$. By taking $G_{L/K\cdot L_{\mathrm{can}}}$-invariants, we have $\mathbb{D}^{\nabla}_{\mathrm{dR}}(V|_{K\cdot L_{\mathrm{can}}}) = L_{\mathrm{can}} \otimes_{L_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V|_{K\cdot L_{\mathrm{can}}})$. Since $V|_{K\cdot L_{\mathrm{can}}}$ is horizontal de Rham by Remark 1.22 and since $(K\cdot L_{\mathrm{can}})_{\mathrm{can}} = L_{\mathrm{can}}$ by Lemma 1.5(iv), we have

$$\dim_{L_{\mathrm{can}}} \mathbb{D}^{\nabla}_{\mathrm{dR}}(V|_{K\cdot L_{\mathrm{can}}}) = \dim_{\mathbb{Q}_p} V = \dim_{L_{\mathrm{can},0}} \mathbb{D}^{\nabla}_{\mathrm{st}}(V|_{K\cdot L_{\mathrm{can}}}),$$

which implies that $V|_{K\cdot L_{\mathrm{can}}}$ is horizontal semistable. □

7C. *Equivalences of categories.* The surjection of profinite groups $\iota^* : G_K \to G_{K_{\mathrm{can}}}$ induces a $\otimes$-functor of Tannakian categories

$$\iota^* : \mathrm{Rep}_{\mathbb{Q}_p} G_{K_{\mathrm{can}}} \to \mathrm{Rep}_{\mathbb{Q}_p} G_K.$$

Obviously, the functor ι^* is fully faithful. Denote by $\boldsymbol{C}_p$ the p-adic completion of the algebraic closure of K_{can} in $\overline{K}$. For $\bullet \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$, we have a Galois equivariant canonical injection $\mathbb{B}_{\bullet,\boldsymbol{C}_p/K_{\mathrm{can}}} \to \mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K}$ by functoriality and we have $(\mathbb{B}_{\bullet,\boldsymbol{C}_p/K_{\mathrm{can}}})^{G_{K_{\mathrm{can}}}} \cong (\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K})^{G_K}$ ($= K_{\mathrm{can}}$ if $\bullet = \mathrm{dR}$, $K_{\mathrm{can},0}$ otherwise) by Proposition 3.16. Hence, if we have $V \in \mathrm{Rep}_{\bullet} G_{K_{\mathrm{can}}}$, then we have $\iota^* V \in \mathrm{Rep}^{\nabla}_{\bullet} G_K$. In fact, we have a canonical injection $\mathbb{D}_{\bullet}(V) \subset \mathbb{D}^{\nabla}_{\bullet}(\iota^* V)$ of $(\mathbb{B}_{\bullet,\boldsymbol{C}_p/K_{\mathrm{can}}})^{G_{K_{\mathrm{can}}}}$-vector spaces, which implies the $\mathbb{B}^{\nabla}_{\bullet,\mathbb{C}_p/K}$-admissibility of $\iota^* V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$ by

Lemma 1.19. Hence, ι^* induces a fully faithful $\otimes$-functor

$$\iota^*_\bullet : \mathrm{Rep}_\bullet G_{K_{\mathrm{can}}} \to \mathrm{Rep}^\nabla_\bullet G_K .$$

The following proposition is a direct consequence of théorème 4.3 in [Colmez and Fontaine 2000].

Proposition 7.5 (horizontal analogue of Colmez–Fontaine). (i) *The functors ι^*_{cris} and ι^*_{st} are essentially surjective. In particular, ι^*_{cris} and ι^*_{st} induce equivalences of Tannakian categories.*

(ii) *The functors*

$$\mathbb{D}^\nabla_{\mathrm{cris}} : \mathrm{Rep}^\nabla_{\mathrm{cris}} G_K \to MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi), \quad \mathbb{D}^\nabla_{\mathrm{st}} : \mathrm{Rep}^\nabla_{\mathrm{st}} G_K \to MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N)$$

induce equivalences of categories with quasi-inverses $\mathbb{V}_{\mathrm{cris}}$, $\mathbb{V}_{\mathrm{st}}$.

Proof. We first prove the assertion in the semistable case. Together with the full faithfulness of $\mathbb{D}^\nabla_{\mathrm{st}}$, we have only to prove the commutativity of the diagram

$$\begin{tikzcd} \mathrm{Rep}_{\mathrm{st}} G_{K_{\mathrm{can}}} \arrow[r, "\iota^*_{\mathrm{st}}"] \arrow[d, "\mathbb{D}_{\mathrm{st}}", "\cong"'] & \mathrm{Rep}^\nabla_{\mathrm{st}} G_K \arrow[d, "\mathbb{D}^\nabla_{\mathrm{st}}"] \\ MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N) \arrow[r, "\mathrm{id}"] & MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N), \end{tikzcd}$$

where $\mathbb{D}_{\mathrm{st}}$ is an equivalence of categories by Colmez–Fontaine theorem [2000, Théorème 4.3]. As we mentioned above, the canonical map $\mathbb{D}_{\mathrm{st}}(V) \to \mathbb{D}^\nabla_{\mathrm{st}}(\iota^* V)$, which commutes with φ and N-actions, is an isomorphism of $K_{\mathrm{can},0}$-vector spaces. We have only to prove that the map also preserves the filtrations. Obviously, we have $\mathrm{Fil}^\bullet \mathbb{D}_{\mathrm{st}}(V) \subset \mathrm{Fil}^\bullet \mathbb{D}^\nabla_{\mathrm{st}}(\iota^* V)$. To prove the converse, it suffices to prove that the associated graded modules of both sides have the same dimension since the filtrations are exhaustive and separated. Let $\boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{n\in\mathbb{Z}} \boldsymbol{C}_p(n)^{m_n}$ be the Hodge–Tate decomposition of V. Then, it induces the Hodge–Tate decomposition of $\iota^* V$, that is, $\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p(n)^{m_n}$, which implies the assertion.

In the horizontal crystalline case, a similar proof works by replacing $*_{\mathrm{st}}$ and $MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi, N)$ by $*_{\mathrm{cris}}$ and $MF^{\mathrm{wa}}_{K_{\mathrm{can}}}(\varphi)$. ☐

Theorem 7.6. *The functor ι^*_{dR} is essentially surjective. In particular, ι^*_{dR} induces an equivalence of Tannakian categories.*

Proof. For a finite Galois extension L/K such that $K \cdot L_{\mathrm{can}} = L$, let $\mathscr{C}_{L/K}$ be the full subcategory of $\mathrm{Rep}^\nabla_{\mathrm{dR}} G_K$ whose objects consist of $V \in \mathrm{Rep}^\nabla_{\mathrm{dR}} G_K$ such that $V|_L$ is horizontal semistable. Recall the notation in Defintion 7.1. Then, we have an equivalence of categories

$$\mathbb{D}^\nabla_{\mathrm{st},L} : \mathscr{C}_{L/K} \to MF^{\mathrm{wa}}(\varphi, N, G_{L/K}); \quad V \mapsto \mathbb{D}^\nabla_{\mathrm{st}}(V|_L).$$

In fact, we have the following quasi-inverse $\mathbb{V}_{\mathrm{st},L}$: For $D \in MF^{\mathrm{wa}}(\varphi, N, G_{L/K})$, we regard D as an object of $MF^{\mathrm{wa}}_{L_{\mathrm{can}}}(\varphi, N)$ and let $\mathbb{V}_{\mathrm{st},L}(D) := \mathbb{V}_{\mathrm{st}}(D)$. We have $\mathbb{V}_{\mathrm{st},L}(D) \in \mathrm{Rep}^{\nabla}_{\mathrm{st}} G_L$ by Proposition 7.5(ii) and $\mathbb{V}_{\mathrm{st},L}(D)$ has a canonical G_K-action, which is an extension of the action of G_L, induced by the $G_{L/K}$-action on D. We have $D \in \mathscr{C}_{L/K}$ by Remark 4.8 and Remark 7.2. We have $\mathbb{V}_{\mathrm{st},L} \circ \mathbb{D}^{\nabla}_{\mathrm{st},L} \cong \mathrm{id}_{\mathscr{C}_{L/K}}$ and $\mathbb{D}^{\nabla}_{\mathrm{st},L} \circ \mathbb{V}_{\mathrm{st},L} \cong \mathrm{id}_{MF^{\mathrm{wa}}(\varphi,N,G_{L/K})}$ by Proposition 7.5(ii).

The restriction map $\mathrm{Res}^{L}_{L_{\mathrm{can}}} : G_{L/K} \xrightarrow{\cong} G_{L_{\mathrm{can}}/K_{\mathrm{can}}}$ induces the equivalence of categories

$$(\mathrm{Res}^{L}_{L_{\mathrm{can}}})^{*} : MF^{\mathrm{wa}}(\varphi, N, G_{L_{\mathrm{can}}/K_{\mathrm{can}}}) \xrightarrow{\cong} MF^{\mathrm{wa}}(\varphi, N, G_{L/K}).$$

We will prove that the diagram

$$\begin{tikzcd} MF^{\mathrm{wa}}(\varphi, N, G_{L_{\mathrm{can}}/K_{\mathrm{can}}}) \arrow[r, "(\mathrm{Res}^{L}_{L_{\mathrm{can}}})^{*}", "\cong"'] \arrow[d, "\mathbb{V}_{\mathrm{st},L_{\mathrm{can}}}", "\cong"'] & MF^{\mathrm{wa}}(\varphi, N, G_{L/K}) \arrow[d, "\mathbb{V}^{\nabla}_{\mathrm{st},L}", "\cong"'] \\ \mathscr{C}_{L_{\mathrm{can}}/K_{\mathrm{can}}} \arrow[r, "\iota^{*}_{\mathrm{dR}}"] & \mathscr{C}_{L/K} \end{tikzcd}$$

is commutative, where the bottom horizontal arrow is induced by $\iota^{*}_{\mathrm{dR}} : \mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}} \to \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$. Indeed, we have the G_K-equivariant inclusion

$$\iota^{*}_{\mathrm{dR}} \circ \mathbb{V}_{\mathrm{st},L_{\mathrm{can}}}(D) \subset \mathbb{V}^{\nabla}_{\mathrm{st},L} \circ (\mathrm{Res}^{L}_{L_{\mathrm{can}}})^{*}(D)$$

for $D \in MF^{\mathrm{wa}}(\varphi, N, G_{L_{\mathrm{can}}/K_{\mathrm{can}}})$ by construction. Since both sides have the same dimension over $\mathbb{Q}_p$, this inclusion is an equality. By the commutative diagram, the functor $\iota^{*}_{\mathrm{dR}} : \mathscr{C}_{L_{\mathrm{can}}/K_{\mathrm{can}}} \to \mathscr{C}_{L/K}$ is essentially surjective.

Let $V \in \mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$. By Theorem 7.4, we have a finite Galois extension K'/K_{can} such that $V|_{G_{KK'}}$ is horizontal semistable. Let $L := KK'$. By Lemma 1.5(iv), we have $L_{\mathrm{can}} = K'$, that is, L/K satisfies the above assumption. Since we have $V \in \mathscr{C}_{L/K}$, the assertion follows from the essential surjectivity of

$$\iota^{*}_{\mathrm{dR}} : \mathscr{C}_{L_{\mathrm{can}}/K_{\mathrm{can}}} \to \mathscr{C}_{L/K}. \qquad \square$$

The above equivalence induces a $\mathbb{Q}_p$-linear isomorphism of Ext^1 on $\mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}}$ and $\mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K$. Note that for $V \in \mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}}$, we may regard $\mathrm{Ext}^1_{\mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}}}(\mathbb{Q}_p, V)$ and $\mathrm{Ext}^1_{\mathrm{Rep}^{\nabla}_{\mathrm{dR}} G_K}(\mathbb{Q}_p, \iota^{*}V)$ as

$$H^1_g(G_{K_{\mathrm{can}}}, V) := \ker\bigl(H^1(G_K, V) \xrightarrow{(1\otimes \mathrm{id})_*} H^1(G_K, \mathbb{B}_{\mathrm{dR},\mathbb{C}_p/K_{\mathrm{can}}} \otimes_{\mathbb{Q}_p} V)\bigr),$$

$$H^{1,\nabla}_g(G_K, \iota^{*}V) := \ker\bigl(H^1(G_K, \iota^{*}V) \xrightarrow{(1\otimes \mathrm{id})_*} H^1(G_K, \mathbb{B}^{\nabla}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} \iota^{*}V)\bigr)$$

respectively. In particular:

Corollary 7.7. *For $V \in \mathrm{Rep}_{\mathrm{dR}} G_{K_{\mathrm{can}}}$, the inflation map*

$$\mathrm{Inf}: H^1(G_{K_{\mathrm{can}}}, V) \to H^1(G_K, \iota^* V)$$

induces the isomorphism

$$\mathrm{Inf}: H^1_g(G_{K_{\mathrm{can}}}, V) \cong H^{1,\nabla}_g(G_K, \iota^* V).$$

7D. ***A comparison theorem on H^1.*** Notation is as in the previous subsection.

Theorem 7.8 (a generalization of Theorem 1.16). *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_{K_{\mathrm{can}}}$ be a de Rham representation whose Hodge–Tate weights are greater than or equal to* 1. *Then, we have the exact sequence*

$$0 \to H^1(G_{K_{\mathrm{can}}}, V) \xrightarrow{\mathrm{Inf}} H^1(G_K, \iota^* V) \xrightarrow{(1\otimes\mathrm{id})_*} H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V) \tag{7}$$

and a canonical isomorphism

$$(\boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V(-1))^{G_{K_{\mathrm{can}}}} \otimes_{K_{\mathrm{can}}} \widehat{\Omega}^1_K \cong H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V). \tag{8}$$

Moreover, if the Hodge–Tate weights of V are greater than or equal to 2, *then $H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V)$ vanishes, in particular, the inflation map*

$$\mathrm{Inf}: H^1(G_{K_{\mathrm{can}}}, V) \to H^1(G_K, \iota^* V)$$

is an isomorphism.

Proof. We first prove the exactness of (7). Note that the injectivity of the inflation map follows by definition. We have the commutative diagram

$$\begin{tikzcd}
H^1(G_{K_{\mathrm{can}}}, V) \arrow[d, "(1\otimes\mathrm{id})_*"] \arrow[dr, "(1\otimes\mathrm{id})_*\circ\mathrm{Inf}"] & \\
H^1(G_{K_{\mathrm{can}}}, \boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V) \arrow[r, "\mathrm{Inf}"] & H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V).
\end{tikzcd}$$

Since we have a Hodge–Tate decomposition $\boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{n\in\mathbb{N}_{\geq 1}} \boldsymbol{C}_p(n)^{m_n}$, we have $H^1(G_{K_{\mathrm{can}}}, \boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V) = 0$ by Theorem 1.15, which implies $(1\otimes\mathrm{id})_* \circ \mathrm{Inf} = 0$.

Let $\mathscr{H} := \ker\{(1\otimes\mathrm{id})_* : H^1(G_K, \iota^* V) \to H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V)\}$. We have only to prove $\mathscr{H}$ is contained in the image of $\mathrm{Inf}: H^1(G_{K_{\mathrm{can}}}, V) \to H^1(G_K, \iota^* V)$. Consider the exact sequence

$$0 \longrightarrow t\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \xrightarrow{\text{inc.}} \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \xrightarrow{\theta} \mathbb{C}_p \longrightarrow 0$$

with $\theta := \theta_{\mathbb{C}_p/K}$. By applying $\otimes_{\mathbb{Q}_p} \iota^* V$ and taking $H^\bullet(G_K, -)$, we have the commutative diagram with exact row, where S stands for $\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K}$:

$$\begin{tikzcd}
 & H^1(G_K, \iota^* V) \arrow[d, "(1\otimes\mathrm{id})_*"'] \arrow[dr, "(1\otimes\mathrm{id})_*"] & \\
H^1(G_K, tS\otimes_{\mathbb{Q}_p}\iota^* V) \arrow[r, "(\mathrm{inc.}\otimes\mathrm{id})_*"] & H^1(G_K, S\otimes_{\mathbb{Q}_p}\iota^* V) \arrow[r, "(\theta\otimes\mathrm{id})_*"] & H^1(G_K, \mathbb{C}_p\otimes_{\mathbb{Q}_p}\iota^* V).
\end{tikzcd}$$

Since $V(1)$ is de Rham with Hodge–Tate weights ≥ 2, we have

$$H^1(G_K, t\mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} \iota^* V) = 0$$

by Theorem 1.15, Lemma 1.14 and dévissage. Hence, the canonical map

$$(1\otimes\mathrm{id})_* : \mathcal{H} \to H^1(G_K, \mathbb{B}^{\nabla+}_{\mathrm{dR},\mathbb{C}_p/K} \otimes_{\mathbb{Q}_p} \iota^* V)$$

vanishes by the above exact sequence. In particular, we have $\mathcal{H} \subset H^{1,\nabla}_g(G_K, \iota^* V)$. By Corollary 7.7, we have $\mathrm{Inf} : H^1_g(G_{K_{\mathrm{can}}}, V) \cong H^{1,\nabla}_g(G_K, \iota^* V)$, which implies (7).

Then, we will prove the existence of the canonical isomorphism (8). By the inclusion $(\boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V(-1))^{G_{K_{\mathrm{can}}}} \subset (\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V(-1))^{G_K}$ and the canonical isomorphism $\widehat{\Omega}^1_K \to H^1(G_K, \mathbb{C}_p(1))$ in Theorem 1.15, we can define a canonical map f as the composite

$$\begin{aligned}&\big(\boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V(-1)\big)^{G_{K_{\mathrm{can}}}} \otimes_{K_{\mathrm{can}}} \widehat{\Omega}^1_K \xrightarrow{\mathrm{inc.}\otimes\mathrm{can.}} \\ &\qquad (\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V(-1))^{G_K} \otimes_K H^1(G_K, \mathbb{C}_p(1)) \xrightarrow{\mathrm{cup.}} H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V).\end{aligned}$$

We will prove that f is an isomorphism. A Hodge–Tate decomposition of V induces a Hodge–Tate decomposition $\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V \cong \bigoplus_{n\in\mathbb{N}_{\geq 1}} \mathbb{C}_p(n)^{m_n}$ of $\iota^* V$. By replacing $\boldsymbol{C}_p \otimes_{\mathbb{Q}_p} V$ and $\mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V$ by their Hodge–Tate decompositions, we may reduce to the case $V = \mathbb{Q}_p(n)$ with $n \in \mathbb{N}_{\geq 1}$ since the cup product commutes with direct sums. Then, the assertion follows from Theorem 1.15.

We will prove the last assertion. The assumption implies that we have $m_1 = 0$ in the above notation, hence, we have $H^1(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} \iota^* V) = 0$ by the Hodge–Tate decomposition of $\iota^* V$ and Theorem 1.15. □

Remark 7.9. (i) Originally, Theorem 1.16(i) and (ii) are proved separately by using ramification theory in some sense.

(ii) (Finiteness) Suppose that we have $[K_{\mathrm{can}} : \mathbb{Q}_p] < \infty$. For example, consider the case that K has a structure of a higher-dimensional local field (Example 1.7). Let $V \in \mathrm{Rep}_{\mathbb{Q}_p} G_K$ be horizontal de Rham of Hodge–Tate weights greater than or equal to 2. Then we have

$$\dim_{\mathbb{Q}_p} H^1(G_K, V) = [K_{\mathrm{can}} : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} V < \infty.$$

Indeed, by Theorem 7.6 and 7.8, we may reduce to the case $K = K_{\mathrm{can}}$. By a Hodge–Tate decomposition $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \cong \bigoplus_{n \in \mathbb{N}_{\geq 2}} \mathbb{C}_p(n)^{m_n}$ with $m_n \in \mathbb{N}$, we have $H^0(G_K, V) \subset H^0(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} V) = 0$ and $H^2(G_K, V) \cong H^0(G_K, V^\vee(1)) \subset H^0(G_K, \mathbb{C}_p \otimes_{\mathbb{Q}_p} V^\vee(1)) = 0$ by the local Tate duality [Herr 1998, Théorème in Introduction], where $^\vee$ denotes the dual. Then, the assertion follows from the Euler–Poincaré characteristic formula (loc. cit).

Note that $H^1(G_K, V)$ is not finite over $\mathbb{Q}_p$ without the condition on Hodge–Tate weights: For example, $H^1(G_K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n K^\times/(K^\times)^{p^n}$ contains $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_K$, which is infinite-dimensional over $\mathbb{Q}_p$ if k_K is imperfect, via the map $\mathcal{O}_K \hookrightarrow U_K^{(1)}$ that takes x to $\exp(2px)$.

Acknowledgment

The author thanks his advisor Atsushi Shiho for reading earlier manuscripts carefully. The author also thanks Professor Takeshi Tsuji for pointing out errors in an earlier manuscript and thanks Professor Olivier Brinon and Kazuma Morita for helpful discussions. The author thanks the referee for detailed comments.

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Communicated by Brian Conrad
Received 2012-07-01 Revised 2013-04-02 Accepted 2013-05-02

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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Algebra & Number Theory

Volume 7 No. 8 2013

1937-0652(2013)7:8;1-5