Polynomial bounds for Arakelov invariants of Belyi curves

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We explicitly bound the Faltings height of a curve over $\overline{\mathbb{Q}}$ polynomially in its Belyi degree. Similar bounds are proven for three other Arakelov invariants: the discriminant, Faltings' delta invariant and the self-intersection of the dualising sheaf. Our results allow us to explicitly bound these Arakelov invariants for modular curves, Hurwitz curves and Fermat curves in terms of their genus. Moreover, as an application, we show that the Couveignes–Edixhoven–Bruin algorithm to compute coefficients of modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ runs in polynomial time under the Riemann hypothesis for ζ-functions of number fields. This was known before only for certain congruence subgroups. Finally, we use our results to prove a conjecture of Edixhoven, de Jong and Schepers on the Faltings height of a cover of $\mathbb{P}^1_{\mathbb{Z}}$ with fixed branch locus.

1. Introduction and statement of results

We prove that stable Arakelov invariants of a curve over a number field are polynomial in the Belyi degree. We apply our results to give algorithmic, geometric and Diophantine applications.

1.1. *Bounds for Arakelov invariants of three-point covers.* Let $\overline{\mathbb{Q}}$ be an algebraic closure of the field of rational numbers $\mathbb{Q}$. Let X be a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus g. Belyi [1979] proved that there exists a finite morphism $X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ ramified over at most three points. Let $\deg_B(X)$ denote the Belyi degree of X, i.e., the minimal degree of a finite morphism $X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ unramified over $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$. Since the topological fundamental group of the projective line $\mathbb{P}^1(\mathbb{C})$ minus three points is finitely generated, the set of $\overline{\mathbb{Q}}$-isomorphism classes of curves with bounded Belyi degree is finite.

We prove that, if $g \geq 1$, the Faltings height $h_{\mathrm{Fal}}(X)$, the Faltings delta invariant $\delta_{\mathrm{Fal}}(X)$, the discriminant $\Delta(X)$ and the self-intersection of the dualising sheaf $e(X)$ are bounded by a polynomial in $\deg_B(X)$; the precise definitions of these Arakelov invariants of X are given in Section 2.3.

Theorem 1.1.1. *For any smooth projective connected curve X over $\bar{\mathbb{Q}}$ of genus $g \geq 1$,*

$$-\log(2\pi)g \leq h_{\mathrm{Fal}}(X) \leq 13 \cdot 10^6 g \deg_B(X)^5,$$
$$0 \leq \quad e(X) \quad \leq 3 \cdot 10^7 (g-1) \deg_B(X)^5,$$
$$0 \leq \quad \Delta(X) \quad \leq 5 \cdot 10^8 g^2 \deg_B(X)^5,$$
$$-10^8 g^2 \deg_B(X)^5 \leq \delta_{\mathrm{Fal}}(X) \leq 2 \cdot 10^8 g \deg_B(X)^5.$$

The Arakelov invariants in Theorem 1.1.1 all have a different flavour to them. For example, the Faltings height $h_{\mathrm{Fal}}(X)$ plays a key role in Faltings' proof of his finiteness theorem on abelian varieties; see [Faltings 1983]. On the other hand, the strict positivity of $e(X)$ (when $g \geq 2$) is related to the Bogomolov conjecture; see [Szpiro 1990b]. The discriminant $\Delta(X)$ "measures" the bad reduction of the curve $X/\bar{\mathbb{Q}}$ and appears in the discriminant conjecture of Szpiro [1990a] for semistable elliptic curves. Finally, as was remarked by Faltings [1984, Introduction], Faltings' delta invariant $\delta_{\mathrm{Fal}}(X)$ can be viewed as the minus logarithm of a "distance" to the boundary of the moduli space of compact connected Riemann surfaces of genus g.

We were first led to investigate this problem by work of Edixhoven, de Jong and Schepers on covers of complex algebraic surfaces with fixed branch locus; see [Edixhoven et al. 2010]. They conjectured an arithmetic analogue [Edixhoven et al. 2010, Conjecture 5.1] of their main theorem (Theorem 1.1 in [loc. cit.]). We use our results to prove this conjecture; see Section 6 for a more precise statement.

1.2. Outline of proof. To prove Theorem 1.1.1, we will use Arakelov theory for curves over a number field K. To apply Arakelov theory in this context, we will work with *arithmetic surfaces* associated to such curves, i.e., regular projective models over the ring of integers O_K of K. We refer the reader to Section 2.2 for precise definitions and basic properties of Arakelov's intersection pairing on an arithmetic surface. Then, for any smooth projective connected curve X over $\bar{\mathbb{Q}}$ of genus $g \geq 1$, we define the Faltings height $h_{\mathrm{Fal}}(X)$, the discriminant $\Delta(X)$, Faltings' delta invariant $\delta_{\mathrm{Fal}}(X)$ and the self-intersection of the dualising sheaf $e(X)$ in Section 2.3. These are the four Arakelov invariants appearing in Theorem 1.1.1.

We introduce two functions on $X(\bar{\mathbb{Q}})$ in Section 2.3: the canonical Arakelov height function and the Arakelov norm of the Wronskian differential. We show that, to prove Theorem 1.1.1, it suffices to bound the canonical height of some non-Weierstrass point and the Arakelov norm of the Wronskian differential at this point; see Theorem 2.4.1 for a precise statement.

We estimate Arakelov–Green functions and Arakelov norms of Wronskian differentials on finite étale covers of the modular curve $Y(2)$ in Theorem 3.4.5 and Proposition 3.5.1, respectively. In our proof, we use an explicit version of a result of Merkl on the Arakelov–Green function; see Theorem 3.1.2. This version of Merkl's theorem was obtained by Peter Bruin in his master's thesis. The proof of this version of Merkl's theorem is reproduced in the Appendix by Peter Bruin.

In Section 4, we prove the existence of a non-Weierstrass point on X of bounded height; see Theorem 4.5.2. The proof of Theorem 4.5.2 relies on our bounds for Arakelov–Green functions (Theorem 3.4.5), the existence of a "wild" model (Theorem 4.3.2) and a generalisation of Dedekind's discriminant conjecture for discrete valuation rings of characteristic 0 (Proposition 4.1.1), which we attribute to Lenstra.

A precise combination of the above results constitutes the proof of Theorem 1.1.1 given in Section 4.6.

1.3. *Arakelov invariants of covers of curves with fixed branch locus.* We apply Theorem 1.1.1 to prove explicit bounds for the height of a cover of curves. Let us be more precise.

For any finite subset $B \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ and integer $d \geq 1$, the set of smooth projective connected curves X over $\overline{\mathbb{Q}}$ such that there exists a finite morphism $X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ étale over $\mathbb{P}^1_{\overline{\mathbb{Q}}} - B$ of degree d is finite. In particular, the Faltings height of X is bounded by a real number depending only on B and d. In this section, we give an explicit version of this statement. To state our result, we need to define the height of B.

The (exponential) height $H(\alpha)$ of an element α in $\overline{\mathbb{Q}}$ is defined as $H(\alpha) = \left(\prod_v \max(1, \|\alpha\|_v)\right)^{1/[K:\mathbb{Q}]}$. Here K is a number field containing α and the product runs over the set of normalised valuations v of K. (As in [Khadjavi 2002, Section 2], we require our normalisation to be such that the product formula holds.) For any finite set $B \subset \mathbb{P}^1(\overline{\mathbb{Q}})$, define the height of B as $H_B = \max\{H(\alpha) : \alpha \in B\}$.

Theorem 1.3.1. *Let U be a nonempty open subscheme in $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ with complement $B \subset \mathbb{P}^1(\overline{\mathbb{Q}})$. Let N be the number of elements in the orbit of B under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then, for any finite morphism $\pi : Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ étale over U, where Y is a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus $g \geq 1$,*

$$-\log(2\pi)g \leq h_{\mathrm{Fal}}(Y) \leq 13 \cdot 10^6 g (4NH_B)^{45N^3 2^{N-2} N!} (\deg \pi)^5,$$

$$0 \leq e(Y) \leq 3 \cdot 10^7 (g-1)(4NH_B)^{45N^3 2^{N-2} N!} (\deg \pi)^5,$$

$$0 \leq \Delta(Y) \leq 5 \cdot 10^8 g^2 (4NH_B)^{45N^3 2^{N-2} N!} (\deg \pi)^5,$$

$$-10^8 g^2 (4NH_B)^{45N^3 2^{N-2} N!} (\deg \pi)^5$$
$$\leq \delta_{\mathrm{Fal}}(Y) \leq 2 \cdot 10^8 g (4NH_B)^{45N^3 2^{N-2} N!} (\deg \pi)^5.$$

Theorem 1.3.1 is a consequence of Theorem 6.0.4. Note that in Theorem 6.0.4 we consider branched covers of any curve over $\overline{\mathbb{Q}}$ (i.e., not only $\mathbb{P}^1_{\overline{\mathbb{Q}}}$). We use Theorem 1.3.1 to prove [Edixhoven et al. 2010, Conjecture 5.1].

1.4. *Diophantine application.* Explicit bounds for Arakelov invariants of curves of genus $g \geq 2$ over a number field K and with bad reduction outside a finite set S of finite places of K imply famous conjectures in Diophantine geometry such as the *effective Mordell conjecture* and the *effective Shafarevich conjecture*; see [Rémond 1999] and [Szpiro 1985a]. We note that Theorem 1.1.1 shows that one "could" replace Arakelov invariants by the Belyi degree to prove these conjectures. We use this philosophy to deal with cyclic covers of prime degree. In fact, in [Javanpeykar and von Känel 2013], we utilise Theorem 1.1.1 and the theory of logarithmic forms to prove the small points conjecture of Szpiro [1985c, p. 284; 1986] for curves that are cyclic covers of the projective line of prime degree; see [Javanpeykar and von Känel 2013, Theorem 3.1] for a precise statement. In particular, we prove Szpiro's small points conjecture for hyperelliptic curves.

1.5. *Modular curves, Fermat curves, Hurwitz curves and Galois Belyi curves.* Let X be a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus $g \geq 2$. We say that X is a Fermat curve if there exists an integer $n \geq 4$ such that X is isomorphic to the planar curve $\{x^n + y^n = z^n\}$. Moreover, we say that X is a Hurwitz curve if $\# \operatorname{Aut}(X) = 84(g - 1)$. Also, we say that X is a Galois Belyi curve if the quotient $X/\operatorname{Aut}(X)$ is isomorphic to $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ and the morphism $X \to X/\operatorname{Aut}(X)$ is ramified over exactly three points; see [Clark and Voight 2011, Proposition 2.4] or [Wolfart 1997]. Note that Fermat curves and Hurwitz curves are Galois Belyi curves. Finally, we say that X is a modular curve if $X_{\mathbb{C}}$ is a classical congruence modular curve with respect to some (hence any) embedding $\overline{\mathbb{Q}} \to \mathbb{C}$.

If X is a Galois Belyi curve, we have $\deg_B(X) \leq 84(g - 1)$. Zograf [1991] proved that, if X is a modular curve, then $\deg_B(X) \leq 128(g + 1)$. Combining these bounds with Theorem 1.1.1 we obtain the following corollary:

Corollary 1.5.1. *Let X be a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus $g \geq 1$. Suppose that X is a modular curve or Galois Belyi curve. Then*

$$\max\left(h_{\mathrm{Fal}}(X), e(X), \Delta(X), |\delta_{\mathrm{Fal}}(X)|\right) \leq 2 \cdot 10^{19} g^2 (g + 1)^5.$$

Remark 1.5.2. Let $\Gamma \subset \operatorname{SL}_2(\mathbb{Z})$ be a finite-index subgroup, and let X be the compactification of $\Gamma \backslash \mathbb{H}$ obtained by adding the cusps, where Γ acts on the complex upper half-plane $\mathbb{H}$ via Möbius transformations. Let $X(1)$ denote the compactification of $\operatorname{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$. The inclusion $\Gamma \subset \operatorname{SL}_2(\mathbb{Z})$ induces a morphism $X \to X(1)$. For $\overline{\mathbb{Q}} \subset \mathbb{C}$ an embedding, there is a unique finite morphism $Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ of smooth projective connected curves over $\overline{\mathbb{Q}}$ corresponding to $X \to X(1)$. The Belyi

degree of Y is bounded from above by the index d of Γ in $\mathrm{SL}_2(\mathbb{Z})$. In particular,

$$\max\big(h_{\mathrm{Fal}}(Y), e(Y), \Delta(Y), |\delta_{\mathrm{Fal}}(Y)|\big) \leq 10^9 d^7.$$

Remark 1.5.3. Nonexplicit versions of Corollary 1.5.1 were previously known for certain modular curves. Firstly, polynomial bounds for Arakelov invariants of $X_0(n)$ with n squarefree were previously known; see [Ullmo 2000, Théorème 1.1 and Corollaire 1.3; Abbes and Ullmo 1997; Michel and Ullmo 1998, Théorème 1.1; Jorgenson and Kramer 2009]. The proofs of these results rely on the theory of modular curves. Also, similar results for Arakelov invariants of $X_1(n)$ with n squarefree were shown in [Edixhoven and de Jong 2011a; Mayer 2012]. Bounds for the self-intersection of the dualising sheaf of a Fermat curve of prime exponent are given in [Curilla and Kühn 2009; Kühn 2013].

1.6. *The Couveignes–Edixhoven–Bruin algorithm.* Corollary 1.5.1 guarantees that, under the Riemann hypothesis for ζ-functions of number fields, the Couveignes–Edixhoven–Bruin algorithm to compute coefficients of modular forms runs in polynomial time; see Theorem 5.0.1 for a more precise statement.

Conventions. By log, we mean the principal value of the natural logarithm. We define the maximum of the empty set and the product taken over the empty set as 1.

2. Arakelov geometry of curves over number fields

We are going to apply Arakelov theory to smooth projective geometrically connected curves X over number fields K. Arakelov [1974] defined an intersection theory on the *arithmetic surfaces* attached to such curves. Faltings [1984] extended Arakelov's work. In this section, we aim at giving the necessary definitions and results for what we need later (and we need at least to fix our notation).

We start with some preparations concerning Riemann surfaces and arithmetic surfaces. In Section 2.3, we define the (stable) Arakelov invariants of X appearing in Theorem 1.1.1. Finally, we prove bounds for Arakelov invariants of X in the height and the Arakelov norm of the Wronskian differential of a non-Weierstrass point; see Theorem 2.4.1.

2.1. *Arakelov invariants of Riemann surfaces.* Let X be a compact connected Riemann surface of genus $g \geq 1$. The space of holomorphic differentials $\mathrm{H}^0(X, \Omega_X^1)$ carries a natural hermitian inner product

$$(\omega, \eta) \mapsto \frac{i}{2} \int_X \omega \wedge \bar{\eta}.$$

For any orthonormal basis $(\omega_1, \ldots, \omega_g)$ with respect to this inner product, the Arakelov $(1, 1)$-form is the smooth positive real-valued $(1, 1)$-form μ on X given by

$\mu = (i/2g) \sum_{k=1}^{g} \omega_k \wedge \overline{\omega_k}$. Note that μ is independent of the choice of orthonormal basis. Moreover, $\int_X \mu = 1$.

Let gr_X be the Arakelov–Green function on $(X \times X) \setminus \Delta$, where $\Delta \subset X \times X$ denotes the diagonal; see [Arakelov 1974], [de Jong 2005a], [Edixhoven and de Jong 2011b] or [Faltings 1984]. The Arakelov–Green functions determine certain metrics whose curvature forms are multiples of μ, called *admissible metrics*, on all line bundles $\mathcal{O}_X(D)$, where D is a divisor on X, as well as on the holomorphic cotangent bundle Ω_X^1. Explicitly, for $D = \sum_P D_P P$ a divisor on X, the metric $\| \cdot \|$ on $\mathcal{O}_X(D)$ satisfies $\log \|1\|(Q) = \mathrm{gr}_X(D, Q)$ for all Q away from the support of D, where $\mathrm{gr}_X(D, Q) := \sum_P n_P \mathrm{gr}_X(P, Q)$. Furthermore, for a local coordinate z at a point a in X, the metric $\| \cdot \|_{\mathrm{Ar}}$ on the sheaf Ω_X^1 satisfies

$$-\log \|dz\|_{\mathrm{Ar}}(a) = \lim_{b \to a} (\mathrm{gr}_X(a, b) - \log |z(a) - z(b)|).$$

We will work with these metrics on $\mathcal{O}_X(P)$ and Ω_X^1 (as well as on tensor product combinations of them) and refer to them as *Arakelov metrics*. A metrised line bundle $\mathcal{L}$ is called *admissible* if, up to a constant scaling factor, it is isomorphic to one of the admissible bundles $\mathcal{O}_X(D)$. The line bundle Ω_X^1 endowed with the above metric is admissible; see [Arakelov 1974].

For any admissible line bundle $\mathcal{L}$, we endow the determinant of cohomology

$$\lambda(\mathcal{L}) = \det \mathrm{H}^0(X, \mathcal{L}) \otimes \det \mathrm{H}^1(X, \mathcal{L})^\vee$$

of the underlying line bundle with the Faltings metric; see Theorem 1 of [Faltings 1984]. We normalise this metric so that the metric on $\lambda(\Omega_X^1) = \det \mathrm{H}^0(X, \Omega_X^1)$ is induced by the hermitian inner product on $\mathrm{H}^0(X, \Omega_X^1)$ given above.

Let $\mathbb{H}_g$ be the Siegel upper half-space of complex symmetric g-by-g matrices with positive-definite imaginary part. Let τ in $\mathbb{H}_g$ be the period matrix attached to a symplectic basis of $\mathrm{H}_1(X, \mathbb{Z})$, and consider the analytic Jacobian $J_\tau(X) = \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$ attached to τ. On $\mathbb{C}^g$, one has a theta function $\vartheta(z; \tau) = \vartheta_{0,0}(z; \tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i \, {}^t n \tau n + 2\pi i \, {}^t nz)$, giving rise to a reduced effective divisor Θ_0 and a line bundle $\mathcal{O}(\Theta_0)$ on $J_\tau(X)$. The function ϑ is not well-defined on $J_\tau(X)$. Instead, we consider the function

$$\|\vartheta\|(z; \tau) = (\det \Im(\tau))^{1/4} \exp(-\pi \, {}^t y(\Im(\tau))^{-1} y) |\vartheta(z; \tau)| \qquad (1)$$

with $y = \Im(z)$. One can check that $\|\vartheta\|$ descends to a function on $J_\tau(X)$. Now consider on the other hand the set $\mathrm{Pic}_{g-1}(X)$ of divisor classes of degree $g-1$ on X. It comes with a canonical subset Θ given by the classes of effective divisors and a canonical bijection $\mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} J_\tau(X)$ mapping Θ onto Θ_0. As a result, we can equip $\mathrm{Pic}_{g-1}(X)$ with the structure of a compact complex manifold, together with a divisor Θ and a line bundle $\mathcal{O}(\Theta)$. Note that we obtain $\|\vartheta\|$ as a function

on $\mathrm{Pic}_{g-1}(X)$. It can be checked that this function is independent of the choice of τ. Furthermore, note that $\|\vartheta\|$ gives a canonical way to put a metric on the line bundle $\mathbb{O}(\Theta)$ on $\mathrm{Pic}_{g-1}(X)$.

For any line bundle $\mathscr{L}$ of degree $g-1$, there is a canonical isomorphism from $\lambda(\mathscr{L})$ to $\mathbb{O}(-\Theta)[\mathscr{L}]$, the fibre of $\mathbb{O}(-\Theta)$ at the point $[\mathscr{L}]$ in $\mathrm{Pic}_{g-1}(X)$ determined by $\mathscr{L}$. Faltings [1984, Section 3] proves that, when we give both sides the metrics discussed above, the norm of this isomorphism is a constant independent of $\mathscr{L}$. We will write this norm as $\exp(\delta_{\mathrm{Fal}}(X)/8)$ and refer to $\delta_{\mathrm{Fal}}(X)$ as Faltings' delta invariant of X.

Let $S(X)$ be the invariant of X defined in [de Jong 2005a, Definition 2.2]. More explicitly, by [de Jong 2005a, Theorem 2.5],

$$\log S(X) = -\int_X \log \|\vartheta\|(gP - Q) \cdot \mu(P), \tag{2}$$

where Q is any point on X. It is related to Faltings' delta invariant $\delta_{\mathrm{Fal}}(X)$. In fact, let $(\omega_1, \ldots, \omega_g)$ be an orthonormal basis of $\mathrm{H}^0(X, \Omega_X^1)$. Let b be a point on X, and let z be a local coordinate about b. Write $\omega_k = f_k \, dz$ for $k = 1, \ldots, g$. We have a holomorphic function

$$W_z(\omega) = \det \left(\frac{1}{(l-1)!} \frac{d^{l-1} f_k}{dz^{l-1}} \right)_{1 \le k, l \le g}$$

locally about b from which we build the $g(g+1)/2$-fold holomorphic differential $W_z(\omega)(dz)^{\otimes g(g+1)/2}$. It is readily checked that this holomorphic differential is independent of the choice of local coordinate and orthonormal basis. Thus, the holomorphic differential $W_z(\omega)(dz)^{\otimes g(g+1)/2}$ extends over X to give a nonzero global section, denoted by Wr, of the line bundle $\Omega_X^{\otimes g(g+1)/2}$. The divisor of the nonzero global section Wr, denoted by $\mathscr{W}$, is the divisor of Weierstrass points. This divisor is effective of degree $g^3 - g$. We follow [de Jong 2005a, Definition 5.3] and denote the constant norm of the canonical isomorphism of (abstract) line bundles

$$\Omega_X^{g(g+1)/2} \otimes_{\mathbb{O}_X} (\Lambda^g \mathrm{H}^0(X, \Omega_X^1) \otimes_{\mathbb{C}} \mathbb{O}_X)^{\vee} \to \mathbb{O}_X(\mathscr{W})$$

by $R(X)$. Then

$$\log S(X) = \tfrac{1}{8} \delta_{\mathrm{Fal}}(X) + \log R(X). \tag{3}$$

Moreover, for any non-Weierstrass point b in X,

$$\mathrm{gr}_X(\mathscr{W}, b) - \log R(X) = \log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b). \tag{4}$$

2.2. Arakelov's intersection pairing on an arithmetic surface.
Let K be a number field with ring of integers O_K, and let $S = \mathrm{Spec}\, O_K$. Let $p : \mathscr{X} \to S$ be an arithmetic surface, i.e., an integral regular flat projective S-scheme of relative dimension 1 with geometrically connected fibres. For the sake of clarity, let us note that $p : \mathscr{X} \to S$ is

a regular projective model of the generic fibre $\mathscr{X}_K \to \operatorname{Spec} K$ in the sense of [Liu 2006a, Definition 10.1.1].

In this section, we will assume the genus of the generic fibre $\mathscr{X}_K$ to be positive. An Arakelov divisor D on $\mathscr{X}$ is a divisor D_{fin} on $\mathscr{X}$ plus a contribution $D_{\inf} = \sum_\sigma \alpha_\sigma F_\sigma$ running over the embeddings $\sigma : K \to \mathbb{C}$ of K into the complex numbers. Here the α_σ are real numbers and the F_σ are formally the "fibres at infinity", corresponding to the Riemann surfaces $\mathscr{X}_\sigma$ associated to the algebraic curves $\mathscr{X} \times_{O_K,\sigma} \mathbb{C}$. We let $\widehat{\operatorname{Div}}(\mathscr{X})$ denote the group of Arakelov divisors on $\mathscr{X}$. To a nonzero rational function f on $\mathscr{X}$, we associate an Arakelov divisor $\widehat{\operatorname{div}}(f) := (f)_{\mathrm{fin}} + (f)_{\inf}$ with $(f)_{\mathrm{fin}}$ the usual divisor associated to f on $\mathscr{X}$ and $(f)_{\inf} = \sum_\sigma v_\sigma(f) F_\sigma$, where $v_\sigma(f) := -\int_{\mathscr{X}_\sigma} \log|f|_\sigma \cdot \mu_\sigma$. Here μ_σ is the Arakelov $(1,1)$-form on $\mathscr{X}_\sigma$. We will say that two Arakelov divisors on $\mathscr{X}$ are linearly equivalent if their difference is of the form $\widehat{\operatorname{div}}(f)$ for some nonzero rational function f on $\mathscr{X}$. We let $\widehat{\operatorname{Cl}}(\mathscr{X})$ denote the group of Arakelov divisors modulo linear equivalence on $\mathscr{X}$.

Arakelov [1974] showed that there exists a unique symmetric bilinear map $(\cdot,\cdot) : \widehat{\operatorname{Cl}}(\mathscr{X}) \times \widehat{\operatorname{Cl}}(\mathscr{X}) \to \mathbb{R}$ with the following properties:

- If D and E are effective divisors on $\mathscr{X}$ without common component, then

$$(D, E) = (D, E)_{\mathrm{fin}} - \sum_{\sigma : K \to \mathbb{C}} \operatorname{gr}_{\mathscr{X}_\sigma}(D_\sigma, E_\sigma),$$

where σ runs over the complex embeddings of K. Here $(D, E)_{\mathrm{fin}}$ denotes the usual intersection number of D and E as in [Liu 2006a, Section 9.1]; i.e.,

$$(D, E)_{\mathrm{fin}} = \sum_{s \in |S|} i_s(D, E) \log \#k(s),$$

where s runs over the set $|S|$ of closed points of S, $i_s(D, E)$ is the intersection multiplicity of D and E at s and $k(s)$ denotes the residue field of s. Note that, if D or E is vertical, the sum $\sum_{\sigma : K \to \mathbb{C}} \operatorname{gr}_{\mathscr{X}_\sigma}(D_\sigma, E_\sigma)$ is zero.

- If D is a horizontal divisor of generic degree n over S, then $(D, F_\sigma) = n$ for every $\sigma : K \to \mathbb{C}$.

- If $\sigma_1, \sigma_2 : K \to \mathbb{C}$ are complex embeddings, then $(F_{\sigma_1}, F_{\sigma_2}) = 0$.

An *admissible line bundle* on $\mathscr{X}$ is the datum of a line bundle $\mathscr{L}$ on $\mathscr{X}$ together with admissible metrics on the restrictions $\mathscr{L}_\sigma$ of $\mathscr{L}$ to the $\mathscr{X}_\sigma$. Let $\widehat{\operatorname{Pic}}(\mathscr{X})$ denote the group of isomorphism classes of admissible line bundles on $\mathscr{X}$. To any Arakelov divisor $D = D_{\mathrm{fin}} + D_{\inf}$ with $D_{\inf} = \sum_\sigma \alpha_\sigma F_\sigma$, we can associate an admissible line bundle $\mathbb{O}_{\mathscr{X}}(D)$. In fact, for the underlying line bundle of $\mathbb{O}_{\mathscr{X}}(D)$, we take $\mathbb{O}_{\mathscr{X}}(D_{\mathrm{fin}})$. Then, we make this into an admissible line bundle by equipping the pull-back of $\mathbb{O}_{\mathscr{X}}(D_{\mathrm{fin}})$ to each $\mathscr{X}_\sigma$ with its Arakelov metric, multiplied by $\exp(-\alpha_\sigma)$. This

induces an isomorphism

$$\widehat{\mathrm{Cl}}(\mathscr{X}) \xrightarrow{\sim} \widehat{\mathrm{Pic}}(\mathscr{X}).$$

In particular, the Arakelov intersection of two admissible line bundles on $\mathscr{X}$ is well-defined.

Recall that a metrised line bundle $(\mathscr{L}, \|\cdot\|)$ on $\mathrm{Spec}\, O_K$ corresponds to an invertible O_K-module, L, say, with hermitian metrics on the $L_\sigma := \mathbb{C} \otimes_{\sigma, O_K} L$. The *Arakelov degree* of $(\mathscr{L}, \|\cdot\|)$ is the real number defined by

$$\widehat{\deg}(\mathscr{L}) = \widehat{\deg}(\mathscr{L}, \|\cdot\|) = \log \#(L/O_K s) - \sum_{\sigma:K\to\mathbb{C}} \log \|s\|_\sigma,$$

where s is any nonzero element of L (independence of the choice of s follows from the product formula).

Note that the relative dualising sheaf $\omega_{\mathscr{X}/O_K}$ of $p : \mathscr{X} \to S$ is an admissible line bundle on $\mathscr{X}$ if we endow the restrictions $\Omega^1_{\mathscr{X}_\sigma}$ of $\omega_{\mathscr{X}/O_K}$ to the $\mathscr{X}_\sigma$ with their Arakelov metric. Furthermore, for any section $P : S \to \mathscr{X}$, we have

$$\widehat{\deg}\, P^* \omega_{\mathscr{X}/O_K} = (\mathbb{O}_X(P), \omega_{\mathscr{X}/O_K}) =: (P, \omega_{\mathscr{X}/O_K}),$$

where we endow the line bundle $P^* \omega_{\mathscr{X}/O_K}$ on $\mathrm{Spec}\, O_K$ with the pull-back metric.

Definition 2.2.1. We say that $\mathscr{X}$ is *semistable* (*or nodal*) *over S* if every geometric fibre of $\mathscr{X}$ over S is reduced and has only ordinary double singularities; see [Liu 2006a, Definition 10.3.1]. We say that $\mathscr{X}$ is (*relatively*) *minimal* if it does not contain any exceptional divisor; see [Liu 2006a, Definition 9.3.12].

Remark 2.2.2. Suppose that $\mathscr{X}$ is semistable over S and minimal. The blowing-up $\mathscr{Y} \to \mathscr{X}$ along a smooth closed point on $\mathscr{X}$ is semistable over S but no longer minimal.

2.3. *Arakelov invariants of curves.* Let X be a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus $g \geq 1$. Let K be a number field such that X has a semistable minimal regular model $p : \mathscr{X} \to \mathrm{Spec}\, O_K$; see Theorems 10.1.8, 10.3.34.a and 10.4.3 in [Liu 2006a]. (Note that we implicitly chose an embedding $K \to \overline{\mathbb{Q}}$.)

The *Faltings delta invariant* of X, denoted by $\delta_{\mathrm{Fal}}(X)$, is defined as

$$\delta_{\mathrm{Fal}}(X) = \frac{1}{[K:\mathbb{Q}]} \sum_{\sigma:K\to\mathbb{C}} \delta_{\mathrm{Fal}}(\mathscr{X}_\sigma),$$

where σ runs over the complex embeddings of K into $\mathbb{C}$. Similarly, we define

$$\|\vartheta\|_{\max}(X) = \left(\prod_{\sigma:K\to\mathbb{C}} \max_{\mathrm{Pic}_{g-1}(\mathscr{X}_\sigma)} \|\vartheta\| \right)^{1/[K:\mathbb{Q}]}.$$

Moreover, we define

$$R(X) = \left(\prod_{\sigma:K\to\mathbb{C}} R(\mathscr{X}_\sigma) \right)^{1/[K:\mathbb{Q}]} \quad \text{and} \quad S(X) = \left(\prod_{\sigma:K\to\mathbb{C}} S(\mathscr{X}_\sigma) \right)^{1/[K:\mathbb{Q}]}.$$

The *Faltings height* of X is defined by

$$h_{\mathrm{Fal}}(X) = \frac{\widehat{\deg} \det p_* \omega_{\mathscr{X}/O_K}}{[K:\mathbb{Q}]} = \frac{\widehat{\deg} \det R^{\cdot} p_* \mathscr{O}_{\mathscr{X}}}{[K:\mathbb{Q}]},$$

where we endow the determinant of cohomology with the Faltings metric; see Section 2.1. Note that $h_{\mathrm{Fal}}(X)$ coincides with the stable Faltings height of the Jacobian of $\mathscr{X}_K$; see [Szpiro 1985b, Chapter I, Lemma 3.2.1]. Furthermore, we define the *self-intersection of the dualising sheaf* of X, denoted by $e(X)$, as

$$e(X) := \frac{(\omega_{\mathscr{X}/O_K}, \omega_{\mathscr{X}/O_K})}{[K:\mathbb{Q}]},$$

where we use Arakelov's intersection pairing on the arithmetic surface $\mathscr{X}/O_K$. The *discriminant* of X, denoted by $\Delta(X)$, is defined as

$$\Delta(X) = \frac{\sum_{\mathfrak{p}\subset O_K} \delta_\mathfrak{p} \log \#k(\mathfrak{p})}{[K:\mathbb{Q}]},$$

where $\mathfrak{p}$ runs through the maximal ideals of O_K and $\delta_\mathfrak{p}$ denotes the number of singularities in the geometric fibre of $p : \mathscr{X} \to \mathrm{Spec}\, O_K$ over $\mathfrak{p}$. These invariants of X are well-defined; see [Moret-Bailly 1990, Section 5.4].

To bound the above Arakelov invariants, we introduce two functions on $X(\overline{\mathbb{Q}})$: the height and the Arakelov norm of the Wronskian differential. More precisely, let $b \in X(\overline{\mathbb{Q}})$ and suppose that b induces a section P of $\mathscr{X}$ over O_K. Then we define the *height of b*, denoted by $h(b)$, to be

$$h(b) = \frac{\widehat{\deg} P^* \omega_{\mathscr{X}/O_K}}{[K:\mathbb{Q}]} = \frac{(P, \omega_{\mathscr{X}/O_K})}{[K:\mathbb{Q}]}.$$

Note that the height of b is the stable canonical height of a point, in the Arakelov-theoretic sense, with respect to the admissible line bundle $\omega_{\mathscr{X}/O_K}$. We define the Arakelov norm of the Wronskian differential at b as

$$\|\mathrm{Wr}\|_{\mathrm{Ar}}(b) = \left(\prod_{\sigma:K\to\mathbb{C}} \|\mathrm{Wr}\|_{\mathrm{Ar}}(b_\sigma) \right)^{1/[K:\mathbb{Q}]}.$$

These functions on $X(\overline{\mathbb{Q}})$ are well-defined; see [Moret-Bailly 1990, Section 5.4].

Changing the model for X might change the height of a point. Let us show that the height of a point does not become smaller if we take another regular model over O_K.

Lemma 2.3.1. *Let $\mathcal{Y} \to \operatorname{Spec} O_K$ be an arithmetic surface. Assume that $\mathcal{Y}$ is a model for $\mathcal{X}_K$. If Q denotes the section of $\mathcal{Y}$ over O_K induced by $b \in X(\bar{\mathbb{Q}})$, then*

$$h(b) \leq \frac{(Q, \omega_{\mathcal{Y}/O_K})}{[K : \mathbb{Q}]}.$$

Proof. By the minimality of $\mathcal{X}$, there is a unique birational morphism $\phi : \mathcal{Y} \to \mathcal{X}$; see [Liu 2006a, Corollary 9.3.24]. By the factorisation theorem, this morphism is made up of a finite sequence

$$\mathcal{Y} = \mathcal{Y}_n \xrightarrow{\phi_n} \mathcal{Y}_{n-1} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} \mathcal{Y}_0 = \mathcal{X}$$

of blowing-ups along closed points; see [Liu 2006a, Theorem 9.2.2]. For $i = 1, \ldots, n$, let $E_i \subset \mathcal{Y}_i$ denote the exceptional divisor of ϕ_i. Since the line bundles $\omega_{\mathcal{Y}_i/O_K}$ and $\phi_i^* \omega_{\mathcal{Y}_{i-1}/O_K}$ agree on $\mathcal{Y}_i - E_i$, there is an integer a such that

$$\omega_{\mathcal{Y}_i/O_K} = \phi_i^* \omega_{\mathcal{Y}_{i-1}/O_K} \otimes_{\mathcal{O}_{\mathcal{Y}_i}} \mathcal{O}_{\mathcal{Y}_i}(a E_i).$$

Applying the adjunction formula, we see that $a = 1$. Since ϕ_i restricts to the identity morphism on the generic fibre, we have a canonical isomorphism of admissible line bundles

$$\omega_{\mathcal{Y}_i/O_K} = \phi_i^* \omega_{\mathcal{Y}_{i-1}/O_K} \otimes_{\mathcal{O}_{\mathcal{Y}_i}} \mathcal{O}_{\mathcal{Y}_i}(E_i).$$

Let Q_i denote the section of $\mathcal{Y}_i$ over O_K induced by $b \in X(\bar{\mathbb{Q}})$. Then

$$\begin{aligned}
(Q_i, \omega_{\mathcal{Y}_i/O_K}) &= (Q_i, \phi_i^* \omega_{\mathcal{Y}_{i-1}/O_K}) + (Q_i, E_i) \\
&\geq (Q_i, \phi_i^* \omega_{\mathcal{Y}_{i-1}/O_K}) \\
&= (Q_{i-1}, \omega_{\mathcal{Y}_{i-1}/O_K}),
\end{aligned}$$

where we used the projection formula in the last equality. Therefore, we conclude that

$$(Q, \omega_{\mathcal{Y}/O_K}) = (Q_n, \omega_{\mathcal{Y}_n/O_K}) \geq (Q_0, \omega_{\mathcal{Y}_0/O_K}) = (P, \omega_{\mathcal{X}/O_K}) = h(b)[K : \mathbb{Q}]. \quad \square$$

2.4. *Bounding Arakelov invariants in the height of a non-Weierstrass point.* In this section, we prove bounds for Arakelov invariants of curves in the height of a non-Weierstrass point and the Arakelov norm of the Wronskian differential in this point.

Theorem 2.4.1. *Let X be a smooth projective connected curve over $\bar{\mathbb{Q}}$ of genus $g \geq 1$. Let $b \in X(\bar{\mathbb{Q}})$. Then*

$$e(X) \leq 4g(g - 1)h(b),$$
$$\delta_{\operatorname{Fal}}(X) \geq -90g^3 - 4g(2g - 1)(g + 1)h(b).$$

Suppose that b is not a Weierstrass point. Then

$$h_{\mathrm{Fal}}(X) \le \tfrac{1}{2}g(g+1)h(b) + \log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b),$$

$$\delta_{\mathrm{Fal}}(X) \le 6g(g+1)h(b) + 12\log\|\mathrm{Wr}\|_{\mathrm{Ar}}(b) + 4g\log(2\pi),$$

$$\Delta(X) \le 2g(g+1)(4g+1)h(b) + 12\log\|\mathrm{Wr}\|_{\mathrm{Ar}}(b) + 93g^3.$$

This theorem is essential to the proof of Theorem 1.1.1 given in Section 4.5. We give a proof of Theorem 2.4.1 at the end of this section.

Lemma 2.4.2. *For a smooth projective connected curve X over $\overline{\mathbb{Q}}$ of genus $g \ge 1$,*

$$\log \|\vartheta\|_{\max}(X) \le \frac{g}{4}\log\max(1, h_{\mathrm{Fal}}(X)) + (4g^3 + 5g + 1)\log 2.$$

Proof. We kindly thank R. de Jong for sharing this proof with us. We follow the idea of [Graftieaux 2001, Section 2.3.2]; see also [David 1991, Appendice]. Let $\mathscr{F}_g$ be the Siegel fundamental domain of dimension g in the Siegel upper half-space $\mathbb{H}_g$, i.e., the space of complex $(g \times g)$-matrices τ in $\mathbb{H}_g$ such that the following properties are satisfied. Firstly, for every element u_{ij} of $u = \Re(\tau)$, we have $|u_{ij}| \le \frac{1}{2}$. Secondly, for every γ in $\mathrm{Sp}(2g, \mathbb{Z})$, we have $\det \Im(\gamma \cdot \tau) \le \det \Im(\tau)$, and finally, $\Im(\tau)$ is Minkowski-reduced; i.e., for all $\xi = (\xi_1, \ldots, \xi_g) \in \mathbb{Z}^g$ and for all i such that $\xi_i, \ldots, \xi_g$ are nonzero, we have $\xi\Im(\tau)^t\xi \ge (\Im(\tau))_{ii}$ and, for all $1 \le i \le g-1$ we have $(\Im(\tau))_{i,i+1} \ge 0$. One can show that $\mathscr{F}_g$ contains a representative of each $\mathrm{Sp}(2g, \mathbb{Z})$-orbit in $\mathbb{H}_g$.

Let K be a number field such that X has a model X_K over K. For any embedding $\sigma : K \to \mathbb{C}$, let τ_σ be an element of $\mathscr{F}_g$ such that $\mathrm{Jac}(X_{K,\sigma}) \cong \mathbb{C}^g/(\tau_\sigma\mathbb{Z}^g + \mathbb{Z}^g)$ as principally polarised abelian varieties, the matrix of the Riemann form induced by the polarisation of $\mathrm{Jac}(X_{K,\sigma})$ being $\Im(\tau_\sigma)^{-1}$ on the canonical basis of $\mathbb{C}^g$. By a result of Bost (see [Graftieaux 2001, Lemme 2.12] or [Pazuki 2012]), we have

$$\frac{1}{[K : \mathbb{Q}]} \sum_{\sigma:K \to \mathbb{C}} \log \det(\Im(\tau_\sigma)) \le g \log \max(1, h_{\mathrm{Fal}}(X)) + (2g^3 + 2)\log(2). \quad (5)$$

Here we used that the Faltings height of X equals the Faltings height of its Jacobian. Now, let $\vartheta(z; \tau)$ be the Riemann theta function as in Section 2.1, where τ is in $\mathscr{F}_g$ and $z = x + iy$ is in $\mathbb{C}^g$ with $x, y \in \mathbb{R}^g$. Combining (5) with the upper bound

$$\exp(-\pi\,{}^t y(\Im(\tau))^{-1}y)|\vartheta(z; \tau)| \le 2^{3g^3 + 5g} \quad (6)$$

implies the result. Let us prove (6). Note that, if we write $y = \Im(z) = (\Im(\tau)) \cdot b$ for b in $\mathbb{R}^g$,

$$\exp(-\pi\,{}^t g(\Im(\tau))^{-1}y)|\vartheta(z; \tau)| \le \sum_{n \in \mathbb{Z}^g} \exp(-\pi\,{}^t(n+b)(\Im(\tau))(n+b)).$$

Since $\Im(\tau)$ is Minkowski reduced, we have ${}^t m \Im(\tau) m \geq c(g) \sum_{i=1}^{g} m_i^2 (\Im(\tau))_{ii}$ for all m in $\mathbb{R}^g$. Here $c(g) = (4/g^3)^{g-1} \left(\frac{3}{4}\right)^{g(g-1)/2}$. Also, $(\Im(\tau))_{ii} \geq \sqrt{3}/2$ for all $i = 1, \ldots, g$ (see [Igusa 1972, Chapter V.4] for these facts). We deduce that

$$
\begin{aligned}
\sum_{n \in \mathbb{Z}^g} & \exp(-\pi {}^t (n+b)(\Im(\tau))(n+b)) \\
& \leq \sum_{n \in \mathbb{Z}^g} \exp\left(-\sum_{i=1}^{g} \pi c(g)(n_i + b_i)^2 (\Im(\tau))_{ii}\right) \\
& \leq \prod_{i=1}^{g} \sum_{n_i \in \mathbb{Z}} \exp(-\pi c(g)(n_i + b_i)^2 (\Im(\tau))_{ii}) \\
& \leq \prod_{i=1}^{g} \frac{2}{1 - \exp(-\pi c(g)(\Im(\tau))_{ii})} \leq 2^g \left(1 + \frac{2}{\pi \sqrt{3} c(g)}\right)^g.
\end{aligned}
$$

This proves (6). $\qquad\square$

Lemma 2.4.3. *Let $a \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}_{\leq 1}$. Then, for all real numbers $x \geq b$,*

$$
x - a \log \max(1, x) = \tfrac{1}{2}x + \tfrac{1}{2}(x - 2a \log \max(1, x)) \geq \tfrac{1}{2}x + \min\left(\tfrac{1}{2}b, a - a \log(2a)\right).
$$

Proof. It suffices to prove that $x - 2a \log \max(1, x) \geq \min(b, 2a - 2a \log(2a))$ for all $x \geq b$. To prove this, let $x \geq b$. Then, if $2a \leq 1$, we have $x - 2a \log \max(1, x) \geq b \geq \min(b, 2a - 2a \log(2a))$. (To prove that $x - 2a \log \max(1, x) \geq b$, we may assume that $x \geq 1$. It is easy to show that $x - 2a \log x$ is a nondecreasing function for $x \geq 1$. Therefore, for all $x \geq 1$, we conclude that $x - 2a \log x \geq 1 \geq b$.) If $2a > 1$, the function $x - 2a \log(x)$ attains its minimum value at $x = 2a$ on the interval $[1, \infty)$. $\qquad\square$

Lemma 2.4.4 (Bost). *Let X be a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus $g \geq 1$. Then*

$$
h_{\mathrm{Fal}}(X) \geq -\log(2\pi)g.
$$

Proof. See [Gaudron and Rémond 2011, Corollaire 8.4]. (Note that the Faltings height $h(X)$ utilised by Bost, Gaudron and Rémond is bigger than $h_{\mathrm{Fal}}(X)$ due to a difference in normalisation. In fact, we have $h(X) = h_{\mathrm{Fal}}(X) + g \log(\sqrt{\pi})$. In particular, the slightly stronger lower bound $h_{\mathrm{Fal}}(X) \geq -\log(\sqrt{2}\pi)g$ holds.) $\qquad\square$

Lemma 2.4.5. *Let X be a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus $g \geq 1$. Then*

$$
\begin{aligned}
\log S(X) & + h_{\mathrm{Fal}}(X) \\
& \geq \tfrac{1}{2} h_{\mathrm{Fal}}(X) - (4g^3 + 5g + 1) \log 2 + \min\left(-\frac{g}{2} \log(2\pi), \frac{g}{4} - \frac{g}{4} \log\left(\frac{g}{2}\right)\right).
\end{aligned}
$$

Proof. By the explicit formula (2) for $S(X)$ in Section 2.1 and our bounds on theta functions (Lemma 2.4.2),

$$\log S(X) + h_{\text{Fal}}(X) \geq -\frac{g}{4}\log\max(1, h_{\text{Fal}}(X)) - (4g^3 + 5g + 1)\log 2 + h_{\text{Fal}}(X).$$

Since $h_{\text{Fal}}(X) \geq -g\log(2\pi)$, the statement follows from Lemma 2.4.3 (with $x = h_{\text{Fal}}(X)$, $a = g/4$ and $b = -g\log(2\pi)$). □

Lemma 2.4.6. *Let X be a smooth projective connected curve of genus $g \geq 2$ over $\overline{\mathbb{Q}}$. Then*

$$\frac{(2g-1)(g+1)}{8(g-1)}e(X) + \tfrac{1}{8}\delta_{\text{Fal}}(X) \geq \log S(X) + h_{\text{Fal}}(X).$$

Proof. By [de Jong 2005a, Proposition 5.6],

$$e(X) \geq \frac{8(g-1)}{(g+1)(2g-1)}(\log R(X) + h_{\text{Fal}}(X)).$$

Note that $\log R(X) = \log S(X) - \delta_{\text{Fal}}(X)/8$; see (3) in Section 2.1. This implies the inequality. □

Lemma 2.4.7 (Noether formula). *Let X be a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus $g \geq 1$. Then*

$$12h_{\text{Fal}}(X) = e(X) + \Delta(X) + \delta_{\text{Fal}}(X) - 4g\log(2\pi).$$

Proof. This is well-known; see [Faltings 1984, Theorem 6; Moret-Bailly 1989, Théorème 2.2]. □

Proposition 2.4.8. *Let X be a smooth projective connected curve of genus $g \geq 2$ over $\overline{\mathbb{Q}}$. Then*

$$h_{\text{Fal}}(X) \leq \frac{(2g-1)(g+1)}{4(g-1)}e(X) + \tfrac{1}{4}\delta_{\text{Fal}}(X) + 20g^3,$$

$$-g\log(2\pi) \leq \frac{(2g-1)(g+1)}{4(g-1)}e(X) + \tfrac{1}{4}\delta_{\text{Fal}}(X) + 20g^3,$$

$$\Delta(X) \leq \frac{3(2g-1)(g+1)}{g-1}e(X) + 2\delta_{\text{Fal}}(X) + 248g^3.$$

Proof. Firstly, by Lemma 2.4.6,

$$\frac{(2g-1)(g+1)}{8(g-1)}e(X) + \tfrac{1}{8}\delta_{\text{Fal}}(X) \geq \log S(X) + h_{\text{Fal}}(X).$$

To obtain the upper bound for $h_{\text{Fal}}(X)$, we proceed as follows. By Lemma 2.4.5,

$$\log S(X) + h_{\text{Fal}}(X)$$
$$\geq \tfrac{1}{2}h_{\text{Fal}}(X) - (4g^3 + 5g + 1)\log 2 + \min\left(-\frac{g}{2}\log(2\pi), \frac{g}{4} - \frac{g}{4}\log\left(\frac{g}{2}\right)\right).$$

From these two inequalities, we deduce that

$$\tfrac{1}{2}h_{\mathrm{Fal}}(X) \le \frac{(2g-1)(g+1)}{8(g-1)}e(X) + \tfrac{1}{8}\delta_{\mathrm{Fal}}(X) + (4g^3 + 5g + 1)\log 2$$
$$+ \max\!\left(\frac{g}{2}\log(2\pi), \frac{g}{4}\log\!\left(\frac{g}{2}\right) - \frac{g}{4}\right).$$

Finally, it is straightforward to verify the inequality

$$(4g^3 + 5g + 1)\log 2 + \max\!\left(\frac{g}{2}\log(2\pi), \frac{g}{4}\log\!\left(\frac{g}{2}\right) - \frac{g}{4}\right) \le 10g^3.$$

This concludes the proof of the upper bound for $h_{\mathrm{Fal}}(X)$.

The second inequality follows from the first inequality of the proposition and the lower bound $h_{\mathrm{Fal}}(X) \ge -g\log(2\pi)$ of Bost (Lemma 2.4.4).

Finally, to obtain the upper bound of the proposition for the discriminant of X, we eliminate the Faltings height of X in the first inequality using the Noether formula and obtain

$$\Delta(X) + e(X) + \delta_{\mathrm{Fal}}(X) - 4g\log(2\pi) \le \frac{3(2g-1)(g+1)}{(g-1)}e(X) + 3\delta_{\mathrm{Fal}}(X) + 240g^3.$$

Faltings [1984, Theorem 5] showed that $e(X) \ge 0$. Therefore, we conclude that

$$\Delta(X) + \delta_{\mathrm{Fal}}(X) - 4g\log(2\pi) \le \frac{3(2g-1)(g+1)}{(g-1)}e(X) + 3\delta_{\mathrm{Fal}}(X) + 240g^3. \quad \square$$

We are now ready to prove Theorem 2.4.1.

Proof of Theorem 2.4.1. The proof is straightforward. The upper bound $e(X) \le 4g(g-1)h(b)$ is well-known; see [Faltings 1984, Theorem 5].

Let us prove the lower bound for $\delta_{\mathrm{Fal}}(X)$. If $g \ge 2$, the lower bound for $\delta_{\mathrm{Fal}}(X)$ can be deduced from the second inequality of Proposition 2.4.8 and the upper bound $e(X) \le 4g(g-1)h(b)$. When $g = 1$, this follows from a result of Szpiro [de Jong 2005b, Proposition 7.2] and the nonnegativity of $h(b)$.

From now on, we suppose that b is a non-Weierstrass point. The upper bound $h_{\mathrm{Fal}}(X) \le \tfrac{1}{2}g(g+1)h(b) + \log\|\mathrm{Wr}\|_{\mathrm{Ar}}(b)$ follows from Theorem 5.9 in [de Jong 2005a] and (4) in Section 2.1.

We deduce the upper bound $\delta_{\mathrm{Fal}}(X) \le 6g(g+1)h(b) + 12\log\|\mathrm{Wr}\|_{\mathrm{Ar}}(b) + 4g\log(2\pi)$ as follows. Since $e(X) \ge 0$ and $\Delta(X) \ge 0$, the Noether formula implies that

$$\delta_{\mathrm{Fal}}(X) \le 12h_{\mathrm{Fal}}(X) + 4g\log(2\pi).$$

Thus, the upper bound for $\delta_{\mathrm{Fal}}(X)$ follows from the upper bound for $h_{\mathrm{Fal}}(X)$.

The upper bound $\Delta(X) \le 2g(g+1)(4g+1)h(b) + 12\log\|\mathrm{Wr}\|_{\mathrm{Ar}}(b) + 93g^3$ follows from the inequality $\Delta(X) \le 12h_{\mathrm{Fal}}(X) - \delta_{\mathrm{Fal}}(X) + 4g\log(2\pi)$ and the

preceding bounds. (One could also use the last inequality of Proposition 2.4.8 to obtain a similar result.)   ☐

3. Bounds for Arakelov–Green functions of Belyi covers

Our aim is to give explicit bounds for the Arakelov–Green function on a Belyi cover of $X(2)$. Such bounds have been obtained for certain Belyi covers using spectral methods in [Jorgenson and Kramer 2006]. The results in [loc. cit.] do not apply to our situation since the smallest positive eigenvalue of the Laplacian can go to zero in a tower of Belyi covers; see [Long 2008, Theorem 4].

Instead, we use a theorem of Merkl to prove explicit bounds for the Arakelov–Green function on a Belyi cover in Theorem 3.4.5. More precisely, we construct a "Merkl atlas" for an arbitrary Belyi cover. Our construction uses an explicit version of [Jorgenson and Kramer 2004] on the Arakelov $(1, 1)$-form due to Bruin.

We use our results to estimate the Arakelov norm of the Wronskian differential in Proposition 3.5.1.

Merkl's theorem [2011, Theorem 10.1] was used to prove bounds for Arakelov–Green functions of the modular curve $X_1(5p)$ in [Edixhoven and de Jong 2011a].

3.1. *Merkl's theorem.* Let X be a compact connected Riemann surface of positive genus, and recall that μ denotes the Arakelov $(1, 1)$-form on X.

Definition 3.1.1. A *Merkl atlas* for X is a quadruple

$$\big(\{(U_j, z_j)\}_{j=1}^n, r_1, M, c_1\big),$$

where $\{(U_j, z_j)\}_{j=1}^n$ is a finite atlas for X and $\frac{1}{2} < r_1 < 1$, $M \geq 1$ and $c_1 > 0$ are real numbers such that the following properties are satisfied:

(1) Each $z_j U_j$ is the open unit disc.

(2) The open sets $U_j^{r_1} := \{x \in U_j : |z_j(x)| < r_1\}$ with $1 \leq j \leq n$ cover X.

(3) For all $1 \leq j, j' \leq n$, the function $|dz_j/dz_{j'}|$ on $U_j \cap U_{j'}$ is bounded from above by M.

(4) For $1 \leq j \leq n$, write $\mu_{\mathrm{Ar}} = i F_j dz_j \wedge d\overline{z_j}$ on U_j. Then $0 \leq F_j(x) \leq c_1$ for all $x \in U_j$.

Given a Merkl atlas $(\{(U_j, z_j)\}_{j=1}^n, r_1, M, c_1)$ for X, the following result provides explicit bounds for Arakelov–Green functions in n, r_1, M and c_1:

Theorem 3.1.2 (Merkl). *Let* $(\{(U_j, z_j)\}_{j=1}^n, r_1, M, c_1)$ *be a Merkl atlas for* X. *Then*

$$\sup_{(X \times X) \setminus \Delta} \mathrm{gr}_X \leq \frac{330n}{(1-r_1)^{3/2}} \log \frac{1}{1-r_1} + 13.2nc_1 + (n-1) \log M.$$

Furthermore, for every index j and all $x \neq y \in U_j^{r_1}$, we have

$$\left| \mathrm{gr}_X(x,y) - \log|z_j(x) - z_j(y)| \right| \leq \frac{330n}{(1-r_1)^{3/2}} \log \frac{1}{1-r_1} + 13.2nc_1 + (n-1)\log M.$$

Proof. Merkl [2011] proved this theorem without explicit constants and without the dependence on r_1. A proof of the theorem in a more explicit form was given by P. Bruin in his master's thesis. This proof is reproduced, with minor modifications, in the Appendix. □

3.2. *An atlas for a Belyi cover of $X(2)$.* Let $\mathbb{H}$ denote the complex upper half-plane. Recall that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ via Möbius transformations. Let $\Gamma(2)$ denote the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined as

$$\Gamma(2) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \bmod 2 \text{ and } b \equiv c \equiv 0 \bmod 2 \right\}.$$

The Riemann surface $Y(2) = \Gamma(2)\backslash\mathbb{H}$ is not compact. Let $X(2)$ be the compactification of the Riemann surface $Y(2) = \Gamma(2)\backslash\mathbb{H}$ obtained by adding the cusps 0, 1 and ∞. Note that $X(2)$ is known as the *compact modular curve associated to the congruence subgroup $\Gamma(2)$ of $\mathrm{SL}_2(\mathbb{Z})$.* The modular lambda function $\lambda : \mathbb{H} \to \mathbb{C}$ induces an analytic isomorphism $\lambda : X(2) \to \mathbb{P}^1(\mathbb{C})$; see Section 4.4 for details. In particular, the genus of $X(2)$ is zero. For a cusp $\kappa \in \{0, 1, \infty\}$, we fix an element γ_κ in $\mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_\kappa(\kappa) = \infty$.

We construct an atlas for the compact connected Riemann surface $X(2)$. Let $\dot{B}_\infty$ be the open subset given by the image of the strip

$$\dot{S}_\infty := \left\{ x + iy : -1 \leq x < 1, \ y > \tfrac{1}{2} \right\} \subset \mathbb{H}$$

in $Y(2)$ under the quotient map $\mathbb{H} \to \Gamma(2)\backslash\mathbb{H}$ defined by $\tau \mapsto \Gamma(2)\tau$. The quotient map $\mathbb{H} \to \Gamma(2)\backslash\mathbb{H}$ induces a bijection from this strip to $\dot{B}_\infty$. More precisely, suppose that τ and τ' in $\dot{S}_\infty$ lie in the same orbit under the action of $\Gamma(2)$. Then, there exists an element

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma(2)$$

such that $\gamma\tau = \tau'$. If $c \neq 0$, by definition, c is a nonzero integral multiple of 2. Thus, $c^2 \geq 4$. Therefore,

$$\frac{1}{2} < \Im\tau' = \frac{\Im\tau}{|c\tau + d|^2} \leq \frac{1}{4\Im\tau} < \frac{1}{2}.$$

This is clearly impossible. Thus, $c = 0$ and $\tau' = \tau \pm b$. By definition, $b = 2k$ for some integer k. Since τ and τ' lie in the above strip, we conclude that $b = 0$. Thus, $\tau = \tau'$.

Consider the morphism $z_\infty : \mathbb{H} \to \mathbb{C}$ given by $\tau \mapsto \exp(\pi i\tau + \pi/2)$. The image of the strip $\dot{S}_\infty$ under z_∞ in $\mathbb{C}$ is the punctured open unit disc $\dot{B}(0,1)$. Now, for any τ

and τ' in the strip $\dot{S}_\infty$, the equality $z_\infty(\tau) = z_\infty(\tau')$ holds if and only if $\tau' = \tau \pm 2k$ for some integer k. But then $k = 0$ and $\tau = \tau'$. We conclude that z_∞ factors injectively through $\dot{B}_\infty$. Let $z_\infty : B_\infty \to B(0, 1)$ denote, by abuse of notation, the induced chart at ∞, where $B_\infty := \dot{B}_\infty \cup \{\infty\}$ and $B(0, 1)$ is the open unit disc in $\mathbb{C}$. We translate our neighbourhood B_∞ at ∞ to a neighbourhood for κ, where κ is a cusp of $X(2)$. More precisely, for any τ in $\mathbb{H}$, define $z_\kappa(\tau) = \exp(\pi i \gamma_k^{-1} \tau + \pi/2)$. Let $\dot{B}_\kappa$ be the image of $\dot{S}_\infty$ under the map $\mathbb{H} \to Y(2)$ given by $\tau \mapsto \Gamma(2)\gamma_\kappa \tau$. We define $B_\kappa = \dot{B}_\kappa \cup \{\kappa\}$. We let $z_\kappa : B_\kappa \to B(0, 1)$ denote the induced chart (by abuse of notation).

Since the open subsets B_κ cover $X(2)$, we have constructed an atlas $\{(B_\kappa, z_\kappa)\}_\kappa$ for $X(2)$, where κ runs through the cusps 0, 1 and ∞.

Definition 3.2.1. A *Belyi cover* of $X(2)$ is a morphism of compact connected Riemann surfaces $Y \to X(2)$ that is unramified over $Y(2)$. The points of Y not lying over $Y(2)$ are called *cusps*.

Lemma 3.2.2. *Let* $\pi : Y \to X(2)$ *be a Belyi cover with* Y *of genus* g. *Then,* $g \le \deg \pi$.

Proof. This is trivial for $g \le 1$. For $g \ge 2$, the statement follows from the Riemann–Hurwitz formula. $\qquad\square$

Let $\pi : Y \to X(2)$ be a Belyi cover. We are going to "lift" the atlas $\{(B_\kappa, z_\kappa)\}$ for $X(2)$ to an atlas for Y.

Let κ be a cusp of $X(2)$. The branched cover $\pi^{-1}(B_\kappa) \to B_\kappa$ restricts to a finite degree topological cover $\pi^{-1}(\dot{B}_\kappa) \to \dot{B}_\kappa$. In particular, the composed morphism

$$\pi^{-1}\dot{B}_\kappa \to \dot{B}_\kappa \xrightarrow[z_\kappa|_{\dot{B}_\kappa}]{\sim} \dot{B}(0, 1)$$

is a finite degree topological cover of $\dot{B}(0, 1)$.

Recall that the fundamental group of $\dot{B}(0, 1)$ is isomorphic to $\mathbb{Z}$. More precisely, for any connected topological cover of $V \to \dot{B}(0, 1)$ of finite degree, there is a unique integer $e \ge 1$ such that $V \to \dot{B}(0, 1)$ is isomorphic to the cover $\dot{B}(0, 1) \to \dot{B}(0, 1)$ given by $x \mapsto x^e$.

For every cusp y of Y lying over κ, let $\dot{V}_y$ be the unique connected component of $\pi^{-1}\dot{B}_\kappa$ whose closure V_y in $\pi^{-1}(B_\kappa)$ contains y. Then, for any cusp y, there is a positive integer e_y and an isomorphism $w_y : \dot{V}_y \xrightarrow{\sim} \dot{B}(0, 1)$ such that $w_y^{e_y} = z_\kappa \circ \pi|_{\dot{V}_y}$. The isomorphism $w_y : \dot{V}_y \to \dot{B}(0, 1)$ extends to an isomorphism $w_y : V_y \to B(0, 1)$ such that $w_y^{e_y} = z_\kappa \circ \pi|_{V_y}$. This shows that e_y is the ramification index of y over κ. Note that we have constructed an atlas $\{(V_y, w_y)\}$ for Y, where y runs over the cusps of Y.

3.3. *The Arakelov $(1, 1)$-form and the hyperbolic metric.* Let

$$\mu_{\text{hyp}}(\tau) = \frac{i}{2} \frac{1}{\Im(\tau)^2} \, d\tau \, d\bar{\tau}$$

be the hyperbolic metric on $\mathbb{H}$. A Fuchsian group is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. For any Fuchsian group Γ, the quotient space $\Gamma \backslash \mathbb{H}$ is a connected Hausdorff topological space and can be made into a Riemann surface in a natural way. The hyperbolic metric μ_{hyp} on $\mathbb{H}$ induces a measure on $\Gamma \backslash \mathbb{H}$, given by a smooth positive real-valued $(1, 1)$-form outside the set of fixed points of elliptic elements of Γ. If the volume of $\Gamma \backslash \mathbb{H}$ with respect to this measure is finite, we call Γ a *cofinite Fuchsian group*.

Let Γ be a cofinite Fuchsian group, and let X be the compactification of $\Gamma \backslash \mathbb{H}$ obtained by adding the cusps. We assume that Γ has no elliptic elements and that the genus g of X is positive. There is a unique smooth function $F_\Gamma : X \to [0, \infty)$ that vanishes at the cusps of Γ such that

$$\mu = \frac{1}{g} F_\Gamma \mu_{\text{hyp}}. \tag{7}$$

A detailed description of F_Γ is not necessary for our purposes.

Definition 3.3.1. Let $\pi : Y \to X(2)$ be a Belyi cover. Then we define the cofinite Fuchsian group Γ_Y (or simply Γ) associated to $\pi : Y \to X(2)$ as follows. Since the topological fundamental group of $Y(2)$ equals $\Gamma(2)/\{\pm 1\}$, we have $\pi^{-1}(Y(2)) = \Gamma' \backslash \mathbb{H}$ for some subgroup $\Gamma' \subset \Gamma(2)/\{\pm 1\}$ of finite index. We define $\Gamma \subset \Gamma(2)$ to be the inverse image of Γ' under the quotient map $\Gamma(2) \to \Gamma(2)/\{\pm 1\}$. Note that Γ is a cofinite Fuchsian group without elliptic elements.

Theorem 3.3.2 (Jorgenson and Kramer). *For any Belyi cover $\pi : Y \to X(2)$, where Y has positive genus,*

$$\sup_{\tau \in Y} F_\Gamma \leq 64 \max_{y \in Y} (e_y)^2 \leq 64 (\deg \pi)^2.$$

Proof. This is shown in [Bruin 2013]. More precisely, in the notation of [loc. cit.], Bruin shows that, with $a = 1.44$, we have $N_{\mathrm{SL}_2(\mathbb{Z})}(z, 2a^2 - 1) \leq 58$. In particular, $\sup_{z \in Y} N_\Gamma(z, z, 2a^2 - 1) \leq 58$; see Section 8.2 in [loc. cit.]. Now, we apply Proposition 6.1 and Lemma 6.2 (with $\epsilon = 2 \deg \pi$) in [loc. cit.] to deduce the sought inequality. $\qquad \square$

Remark 3.3.3. Jorgenson and Kramer [2004] prove a stronger (albeit nonexplicit) version of Theorem 3.3.2.

3.4. *A Merkl atlas for a Belyi cover of $X(2)$.* In this section, we prove bounds for Arakelov–Green functions of Belyi covers.

Recall that we constructed an atlas $\{(B_\kappa, z_\kappa)\}_\kappa$ for $X(2)$. For a cusp κ of $X(2)$, let

$$y_\kappa : \mathbb{H} \to (0, \infty)$$

be defined by $\tau \mapsto \Im(\gamma_\kappa^{-1}\tau) = \frac{1}{2} - \frac{\log |z_\kappa(\tau)|}{\pi}$. This induces a function $\dot{B}_\kappa \to (0, \infty)$, also denoted by y_κ.

Lemma 3.4.1. *For any two cusps κ and κ' of $X(2)$, we have*

$$\left| \frac{dz_\kappa}{dz_{\kappa'}} \right| \leq 4 \exp(3\pi/2)$$

on $B_\kappa \cap B_{\kappa'}$.

Proof. We work on the complex upper half-plane $\mathbb{H}$. We may and do assume that $\kappa \neq \kappa'$. By applying $\gamma_{\kappa'}^{-1}$, we may and do assume that $\kappa' = \infty$. On $B_\kappa \cap B_\infty$, we have

$$dz_\kappa(\tau) = \pi i \exp(\pi i \gamma_\kappa^{-1}\tau + \pi/2) d(\gamma_\kappa^{-1}\tau) \quad \text{and} \quad dz_\infty(\tau) = \pi i \exp(\pi i \tau + \pi/2) d(\tau).$$

Therefore,

$$\frac{dz_\kappa}{dz_\infty}(\tau) = \exp(\pi i (\gamma_\kappa^{-1}\tau - \tau)) \frac{d(\gamma_\kappa^{-1}\tau)}{d(\tau)}.$$

It follows from a simple calculation that, for $\gamma_\kappa^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$ with $c \neq 0$,

$$\left| \frac{dz_\kappa}{dz_\infty} \right|(\tau) = \frac{1}{|c\tau + d|^2} \exp(\pi (y_\infty(\tau) - y_\kappa(\tau))).$$

For τ and $\gamma_\kappa^{-1}\tau$ in B_∞, one has $y_\infty(\tau) > \frac{1}{2}$ and $y_\kappa(\tau) > \frac{1}{2}$. From $|c\tau + d| \geq y_\infty(\tau) = \Im(\tau)$, it follows that

$$y_\kappa(\tau) = \Im(\gamma_\kappa^{-1}(\tau)) = y_\infty \left(\frac{a\tau + b}{c\tau + d} \right) = \frac{\Im\tau}{|c\tau + d|^2} \leq \frac{\Im\tau}{(\Im\tau)^2} \leq 2,$$

and similarly, $y_\infty(\tau) \leq 2$. The statement follows. $\qquad\square$

Let $\pi : Y \to X(2)$ be a Belyi cover, and let $V = \pi^{-1}(Y(2))$ be the complement of the set of cusps in Y. Recall that we constructed an atlas $\{(V_y, w_y)\}$ for Y. We assume that the genus g of Y is positive, and as usual, we let μ denote the Arakelov $(1, 1)$-form on Y.

Lemma 3.4.2. *Let y be a cusp of $\pi : Y \to X(2)$ with $\kappa = \pi(y)$. Then*

$$i \, dw_y \, d\overline{w}_y = \frac{2\pi^2 y_\kappa^2 |w_y|^2}{e_y^2} \mu_{\text{hyp}} \quad \text{on } \dot{V}_y.$$

Proof. Let $\kappa = \pi(y)$ in $X(2)$. We work on the complex upper half-plane. By the chain rule, we have

$$d(z_\kappa) = d(w_y^{e_y}) = e_y w_y^{e_y - 1} dw_y.$$

Therefore,

$$e_y^2 |w_y|^{2e_y-2} \, dw_y \, d\overline{w}_y = dz_\kappa \, d\overline{z}_\kappa.$$

Note that $dz_\kappa = \pi i z_\kappa \, d(\gamma_\kappa^{-1})$, where we view $\gamma_\kappa^{-1} : \mathbb{H} \to \mathbb{C}$ as a function. Therefore,

$$e_y^2 |w_y|^{2e_y-2} \, dw_y \, d\overline{w}_y = \pi^2 |z_\kappa|^2 \, d(\gamma_\kappa^{-1}) \, d(\overline{\gamma_\kappa^{-1}}).$$

Since $|w_y^{e_y}| = |z_\kappa|$, we have

$$
\begin{aligned}
i \, dw_y \, d\overline{w}_y &= \frac{i\pi^2 |w_y|^2}{e_y^2} \, d(\gamma_\kappa^{-1}) \, d(\overline{\gamma_\kappa^{-1}}) \\
&= \frac{2\pi^2 y_\kappa^2 |w_y|^2}{e_y^2} \frac{i \, d(\gamma_\kappa^{-1}) \, d(\overline{\gamma_\kappa^{-1}})}{2y_\kappa^2} = \frac{2\pi^2 y_\kappa^2 |w_y|^2}{e_y^2} (\mu_{\mathrm{hyp}} \circ \gamma_\kappa^{-1}).
\end{aligned}
$$

Since μ_{hyp} is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, this concludes the proof. $\square$

Proposition 3.4.3. *Let y be a cusp of $\pi : Y \to X(2)$. Write $\mu = i F_y \, dw_y \, d\overline{w}_y$ on V_y. Then F_y is a subharmonic function on V_y and*

$$0 \le F_y \le \frac{128 \exp(3\pi)(\deg \pi)^4}{\pi^2 g}.$$

Proof. The first statement follows from [Jorgenson and Kramer 2004, p. 8]; see also [Bruin 2010, p. 58]. The lower bound for F_y is clear from the definition. Let us prove the upper bound for F_y.

For a cusp κ of $X(2)$, let $\dot{B}_\kappa(2) \subset \dot{B}_\kappa$ be the image of the strip $\{x + iy : -1 \le x < 1, y > 2\}$ in $Y(2)$ under the map $\mathbb{H} \to Y(2)$ given by $\tau \mapsto \Gamma(2)\gamma_\kappa \tau$. For a cusp y of Y lying over κ, define $\dot{V}_y(2) = \pi^{-1}(\dot{B}_\kappa(2))$ and $V_y(2) = \dot{V}_y(2) \cup \{y\}$. Since the boundary $\partial V_y(2)$ of $V_y(2)$ is contained in $V_y - V_y(2)$, by the maximum principle for subharmonic functions,

$$\sup_{V_y} F_y = \max\left(\sup_{V_y(2)} F_y, \sup_{V_y - V_y(2)} F_y \right) = \max\left(\sup_{\partial V_y(2)} F_y, \sup_{V_y - V_y(2)} F_y \right) = \sup_{V_y - V_y(2)} F_y.$$

By Lemma 3.4.2, Definition 3.3.1 and (7) in Section 3.3,

$$F_y = F_\Gamma \frac{e_y^2}{2g\pi^2 y_\kappa^2 |w_y|^2}. \tag{8}$$

Note that $y_\kappa^{-2} < 4$ on V_y. Furthermore,

$$\sup_{V_y - V_y(2)} |w_y|^{-2} \le \sup_{B_\kappa - B_\kappa(2)} |z_\kappa|^{-2} = \exp(-\pi) \sup_{B_\kappa - B_\kappa(2)} \exp(2\pi y_\kappa) \le \exp(3\pi).$$

Thus, the proposition follows from Jorgenson–Kramer's upper bound for F_Γ (Theorem 3.3.2). $\square$

Definition 3.4.4. Define $s_1 = \sqrt{1/2}$. Note that $\frac{1}{2} < s_1 < 1$. For any cusp κ of $X(2)$, let $B_\kappa^{s_1}$ be the open subset of B_κ whose image under z_κ is $\{x \in \mathbb{C} : |x| < s_1\}$. Moreover, define the positive real number r_1 by the equation $r_1^{\deg \pi} = s_1$. Note that $\frac{1}{2} < r_1 < 1$. For all cusps y of $\pi : Y \to X(2)$, define the subset $V_y^{r_1} \subset V_y$ by $V_y^{r_1} = \{x \in V_y : |w_y(x)| < r_1\}$.

Theorem 3.4.5. *Let $\pi : Y \to X(2)$ be a Belyi cover such that Y is of genus $g \geq 1$. Then*

$$\sup_{(Y \times Y) \backslash \Delta} \mathrm{gr}_Y \leq 6378027 \frac{(\deg \pi)^5}{g}.$$

Moreover, for every cusp y and all $x \neq x'$ in $V_y^{r_1}$,

$$\left| \mathrm{gr}_Y(x, x') - \log |w_y(x) - w_y(x')| \right| \leq 6378027 \frac{(\deg \pi)^5}{g}.$$

Proof. Write $d = \deg \pi$. Let s_1 and r_1 be as in Definition 3.4.4. We define real numbers

$$n := \#(Y - V), \quad M := 4d \exp(3\pi) \quad \text{and} \quad c_1 := \frac{128 \exp(3\pi) d^4}{\pi^2 g}.$$

Since n is the number of cusps of Y, we have $n \leq 3d$. Moreover,

$$\frac{1}{1 - r_1} \leq \frac{d}{1 - s_1}.$$

Note that

$$\frac{330n}{(1 - r_1)^{3/2}} \log \frac{1}{1 - r_1} + 13.2nc_1 + (n - 1) \log M \leq 6378027 \frac{d^5}{g}.$$

Therefore, by Theorem 3.1.2, it suffices to show that

$$(\{(V_y, w_y)\}_y, r_1, M, c_1),$$

where y runs over the cusps of $\pi : Y \to X(2)$, constitutes a Merkl atlas for Y.

The first condition of Merkl's theorem is satisfied. That is, $w_y V_y$ is the open unit disc in $\mathbb{C}$.

To verify the second condition of Merkl's theorem, we have to show that the open sets $V_y^{r_1}$ cover Y. For any $x \in V_y$, we have $x \in V_y^{r_1}$ if $\pi(x) \in B_\kappa^{s_1}$. In fact, for any x in V_y, we have $|w_y(x)| < r_1$ if and only if

$$|z_\kappa(\pi(x))| = |w_y(x)|^{e_y} < r_1^{e_y}.$$

Since $r_1 < 1$, we see that $s_1 = r_1^d \leq r_1^{e_y}$. Therefore, if $\pi(x)$ lies in $B_\kappa^{s_1}$, we see that x lies in $V_y^{r_1}$. Now, since $s_1 < \sqrt{3}/2$, we have $X(2) = \bigcup_{\kappa \in \{0,1,\infty\}} B_\kappa^{s_1}$. We conclude that $Y = \bigcup_y V_y^{r_1}$, where y runs through the cusps.

Since we have already verified the fourth condition of Merkl's theorem in Proposition 3.4.3, it suffices to verify the third condition to finish the proof. Let κ and κ' be cusps of $X(2)$. We may and do assume that $\kappa \neq \kappa'$. Now, as usual, we work on the complex upper half-plane. By the chain rule,

$$\left|\frac{dw_y}{dw_{y'}}\right| \leq \frac{d}{|w_y|^{e_y-1}} \sup_{B_\kappa \cap B_{\kappa'}} \left|\frac{dz_\kappa}{dz_{\kappa'}}\right|$$

on $V_y \cap V_{y'}$. Note that $|w_y(\tau)|^{e_y-1} \geq |w_y(\tau)|^{e_y} = |z_\kappa(\tau)|$ for any τ in $\mathbb{H}$. Therefore,

$$\left|\frac{dw_y}{dw_{y'}}\right| \leq \frac{d}{|z_\kappa|} \sup_{B_\kappa \cap B_{\kappa'}} \left|\frac{dz_\kappa}{dz_{\kappa'}}\right| \leq M,$$

where we used Lemma 3.4.1 and the inequality $|z_\kappa| > \exp(-3\pi/2)$ on $B_\kappa \cap B_{\kappa'}$. □

3.5. The Arakelov norm of the Wronskian differential.

Proposition 3.5.1. *Let* $\pi : Y \to X(2)$ *be a Belyi cover with* Y *of genus* $g \geq 1$. *Then*

$$\sup_{Y-\mathrm{Supp}\,\mathcal{W}} \log \|\mathrm{Wr}\|_{\mathrm{Ar}} \leq 6378028 g (\deg \pi)^5.$$

Proof. Let b be a non-Weierstrass point on Y, and let y be a cusp of Y such that b lies in $V_y^{r_1}$. Let $\omega = (\omega_1, \ldots, \omega_g)$ be an orthonormal basis of $\mathrm{H}^0(Y, \Omega_Y^1)$. Then, as in Section 2.1,

$$\log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b) = \log |W_{w_y}(\omega)(b)| + \frac{g(g+1)}{2} \log \|dw_y\|_{\mathrm{Ar}}(b).$$

By Theorem 3.4.5,

$$\frac{g(g+1)}{2} \log \|dw_y\|_{\mathrm{Ar}}(b) \leq 6378027 g (\deg \pi)^5.$$

Let us show that $\log |W_{w_y}(\omega)(b)| \leq g(\deg \pi)^5$. Write $\omega_k = f_k \, dw_y$ on V_y. Note that $\omega_k \wedge \overline{\omega_k} = |f_k|^2 \, dw_y \wedge d\overline{w}_y$. Therefore,

$$\mu = \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \overline{\omega_k} = \frac{i}{2g} \sum_{k=1}^{g} |f_k|^2 \, dw_y \wedge d\overline{w}_y.$$

We deduce that $\sum_{k=1}^{g} |f_k|^2 = 2g F_y$, where F_y is the unique function on V_y such that $\mu = i F_y \, dw_y \wedge d\overline{w}_y$. By our upper bound for F_y (Proposition 3.4.3), for any $j = 1, \ldots, g$,

$$\sup_{V_y} |f_j|^2 \leq \sup_{V_y} \sum_{k=1}^{g} |f_k|^2 = 2g F_y \leq \frac{256 \exp(3\pi)(\deg \pi)^4}{\pi^2}.$$

By Hadamard's inequality,

$$\log |W_{w_y}(\omega)(b)| \leq \sum_{l=0}^{g-1} \log\left(\sum_{k=1}^{g} \left|\frac{d^l f_k}{d w_y^l}\right|^2 (b)\right)^{1/2}.$$

Let $r_1 < r < 1$ be some real number. By Cauchy's integral formula, for any $0 \leq l \leq g - 1$,

$$\left|\frac{d^l f_k}{d w_y^l}\right|(b) = \left|\frac{l!}{2\pi i} \int_{|w_y|=r} \frac{f_k}{(w_y - w_y(b))^{l+1}} d w_y\right|$$

$$\leq \frac{l!}{(r - r_1)^{l+1}} \sup_{V_y} |f_k| \leq \frac{g!}{(1 - r_1)^g} \sup_{V_y} |f_k|.$$

By the preceding estimations, since $g! \leq g^g$ and $1/(1 - r_1) \leq \deg \pi/(1 - s_1)$, we obtain that

$\log |W_{w_y}(\omega)(b)|$

$$\leq \sum_{l=0}^{g-1} \log\left(\frac{g!}{(1 - r_1)^g}\left(\sum_{k=1}^{g} \sup_{V_y} |f_k|^2\right)^{1/2}\right)$$

$$\leq \sum_{l=0}^{g-1} \log\left(\frac{g!}{(1 - r_1)^g}\left(\sum_{k=1}^{g} \frac{256 \exp(3\pi)(\deg \pi)^4}{\pi^2}\right)^{1/2}\right)$$

$$= g \log(g!) + g^2 \log\left(\frac{1}{1 - r_1}\right) + \frac{g}{2} \log\left(\frac{256 g \exp(3\pi)}{\pi^2}\right) + 2g \log(\deg \pi)$$

$$\leq \left(4.5 + \log\left(\frac{1}{1 - s_1}\right) + \tfrac{1}{2} \log\left(\frac{256 \exp(3\pi)}{\pi^2}\right)\right) g^2 \log(\deg \pi)$$

$$\leq 13 g (\deg \pi)^2.$$

Since $g \geq 1$ and $\pi : Y \to X(2)$ is a Belyi cover, the inequality $\deg \pi \geq 3$ holds. Thus,

$$13 g (\deg \pi)^2 \leq \frac{13 g (\deg \pi)^5}{27} \leq g (\deg \pi)^5. \qquad \square$$

4. Points of bounded height

4.1. *Lenstra's generalisation of Dedekind's discriminant bound.* Let A be a discrete valuation ring of characteristic 0 with fraction field K. Let ord_A denote the valuation on A. Let L/K be a finite field extension of degree n, and let B be the integral closure of A in L. Note that L/K is separable, and B/A is finite; see [Serre 1979, Proposition I.4.8].

The inverse different $\mathfrak{D}_{B/A}^{-1}$ of B over A is the fractional ideal

$$\{x \in L : \mathrm{Tr}(x B) \subset A\},$$

where Tr is the trace of L over K. The inverse of the inverse different, denoted by $\mathfrak{D}_{B/A}$, is the different of B over A. Note that $\mathfrak{D}_{B/A}$ is actually an integral ideal of L.

The following proposition (which we would like to attribute to H. W. Lenstra, Jr.) is a generalisation of Dedekind's discriminant bound; see [Serre 1979, Proposition III.6.13].

Proposition 4.1.1 (H. W. Lenstra, Jr.). *Suppose that B is a discrete valuation ring of ramification index e over A. Then, the valuation r of the different ideal $\mathfrak{D}_{B/A}$ on B satisfies the inequality*

$$r \leq e - 1 + e \cdot \mathrm{ord}_A(n).$$

Proof. Let x be a uniformiser of A. Since A is of characteristic 0, we may define $y := 1/nx$; note that y is an element of K. The trace of y (as an element of L) is $1/x$. Since $1/x$ is not in A, this implies that the inverse different $\mathfrak{D}_{B/A}^{-1}$ is strictly contained in the fractional ideal yB. (If not, since A and B are discrete valuation rings, we would have that yB is strictly contained in the inverse different.) In particular, the different $\mathfrak{D}_{B/A}$ strictly contains the fractional ideal (nx). Therefore, the valuation $\mathrm{ord}_B(\mathfrak{D}_{B/A})$ on B of $\mathfrak{D}_{B/A}$ is strictly less than the valuation of nx. Thus,

$$\mathrm{ord}_B(\mathfrak{D}_{B/A}) < \mathrm{ord}_B(nx) = e \cdot \mathrm{ord}_A(nx) = e(\mathrm{ord}_A(n)+1) = e \cdot \mathrm{ord}_A(n) + e.$$

This concludes the proof of the inequality. $\square$

Remark 4.1.2. If the extension of residue fields of B/A is separable, Proposition 4.1.1 follows from the remark following Proposition III.6.13 in [Serre 1979]. (The result in that proposition was conjectured by Dedekind and proved by Hensel when $A = \mathbb{Z}$.) The reader will see that, in the proof of Proposition 4.2.4, we have to deal with imperfect residue fields.

Proposition 4.1.3. *Suppose that the residue characteristic p of A is positive. Let m be the biggest integer such that $p^m \leq n$. Then, for $\beta \subset B$ a maximal ideal of B with ramification index e_β over A, the valuation r_β of the different ideal $\mathfrak{D}_{B/A}$ at β satisfies the inequality*

$$r_\beta \leq e_\beta - 1 + e_\beta \cdot \mathrm{ord}_A(p^m).$$

Proof. To compute r_β, we localise B at β and then take the completions $\widehat{A}$ and $\widehat{B_\beta}$ of A and B_β, respectively. Let d be the degree of $\widehat{B_\beta}$ over $\widehat{A}$. Then, by Lenstra's result (Proposition 4.1.1), the inequality

$$r_\beta \leq e_\beta - 1 + e_\beta \cdot \mathrm{ord}_{\widehat{A}}(d)$$

holds. By definition, $\mathrm{ord}_{\widehat{A}}(d) = \mathrm{ord}_A(d) \leq \mathrm{ord}_A(p^m)$. This concludes the proof. $\square$

4.2. *Covers of arithmetic surfaces with fixed branch locus.* Let K be a number field with ring of integers O_K, and let $S = \mathrm{Spec}\, O_K$. Let D be a reduced effective divisor on $\mathcal{X} = \mathbb{P}^1_S$, and let U denote the complement of the support of D in $\mathcal{X}$.

Let $\mathcal{Y} \to S$ be an integral normal two-dimensional flat projective S-scheme with geometrically connected fibres, and let $\pi : \mathcal{Y} \to \mathcal{X}$ be a finite surjective morphism of S-schemes that is étale over U. Let $\psi : \mathcal{Y}' \to \mathcal{Y}$ be the minimal resolution of singularities [Liu 2006a, Proposition 9.3.32]. Note that we have the following diagram of morphisms:

$$\mathcal{Y}' \xrightarrow{\psi} \mathcal{Y} \xrightarrow{\pi} \mathcal{X} \to S.$$

Consider the prime decomposition $D = \sum_{i \in I} D_i$, where I is a finite-index set. Let D_{ij} be an irreducible component of $\pi^{-1}(D)$ mapping onto D_i, where j is in the index set J_i. We define r_{ij} to be the valuation of the different ideal of $\mathcal{O}_{\mathcal{Y}, D_{ij}} / \mathcal{O}_{\mathcal{X}, D_i}$. We define the ramification divisor R to be $\sum_{i \in I} \sum_{j \in J_i} r_{ij} D_{ij}$. We define $B := \pi_* R$.

We apply [Liu 2006a, 6.4.26] to obtain that there exists a dualising sheaf $\omega_{\mathcal{Y}/S}$ for $\mathcal{Y} \to S$ and a dualising sheaf ω_π for $\pi : \mathcal{Y} \to \mathcal{X}$ such that the adjunction formula

$$\omega_{\mathcal{Y}/S} = \pi^* \omega_{\mathcal{X}/S} \otimes \omega_\pi$$

holds. Since the local ring at the generic point of a divisor on $\mathcal{X}$ is of characteristic 0, basic properties of the different ideal imply that ω_π is canonically isomorphic to the line bundle $\mathcal{O}_{\mathcal{Y}}(R)$. We deduce the *Riemann–Hurwitz* formula

$$\omega_{\mathcal{Y}/S} = \pi^* \omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}}(R).$$

Let $K_{\mathcal{X}} = -2 \cdot [\infty]$ be the divisor defined by the tautological section of $\omega_{\mathcal{X}/O_K}$. Let $K_{\mathcal{Y}'}$ denote the Cartier divisor on $\mathcal{Y}'$ defined by the rational section $d(\pi \circ \psi)$ of $\omega_{\mathcal{Y}'/S}$. We define the Cartier divisor $K_{\mathcal{Y}}$ on $\mathcal{Y}$ analogously; i.e., $K_{\mathcal{Y}}$ is the Cartier divisor on $\mathcal{Y}$ defined by $d\pi$. Note that $K_{\mathcal{Y}} = \psi_* K_{\mathcal{Y}'}$. Also, the Riemann–Hurwitz formula implies the following equality of Cartier divisors:

$$K_{\mathcal{Y}} = \pi^* K_{\mathcal{X}} + R.$$

Let $E_1, \ldots, E_s$ be the exceptional components of $\psi : \mathcal{Y}' \to \mathcal{Y}$. Note that the pull-back of the Cartier divisor $\psi^* K_{\mathcal{Y}}$ coincides with $K_{\mathcal{Y}'}$ on

$$\mathcal{Y}' - \bigcup_{i=1}^{s} E_i.$$

Therefore, there exist integers c_i such that

$$K_{\mathcal{Y}'} = \psi^* K_{\mathcal{Y}} + \sum_{i=1}^{s} c_i E_i,$$

where this is an equality of Cartier divisors (*not only* modulo linear equivalence). Note that $(\psi^* K_{\mathcal{Y}}, E_i) = 0$ for all i. In fact, $K_{\mathcal{Y}}$ is linearly equivalent to a Cartier divisor with support disjoint from the singular locus of $\mathcal{Y}$.

Lemma 4.2.1. *For all $i = 1, \ldots, s$, we have $c_i \leq 0$.*

Proof. We have the following local statement. Let y be a singular point of $\mathcal{Y}$, and let $E_1, \ldots, E_r$ be the exceptional components of ψ lying over y. We define

$$V_+ = \sum_{\substack{i=1 \\ c_i > 0}}^{r} c_i E_i$$

as the sum on the $c_i > 0$. To prove the lemma, it suffices to show that $V_+ = 0$. Since the intersection form on the exceptional locus of $\mathcal{Y}' \to \mathcal{Y}$ is negative-definite [Liu 2006a, Proposition 9.1.27], to prove $V_+ = 0$, it suffices to show that $(V_+, V_+) \geq 0$. Clearly, to prove the latter inequality, it suffices to show that, for all i such that $c_i > 0$, we have $(V_+, E_i) \geq 0$. To do this, fix $i \in \{1, \ldots, r\}$ with $c_i > 0$. Since $\mathcal{Y}' \to \mathcal{Y}$ is minimal, we have that E_i is not a (-1)-curve. In particular, by the adjunction formula, the inequality $(K_{\mathcal{Y}'}, E_i) \geq 0$ holds. We conclude that

$$(V_+, E_i) = (K_{\mathcal{Y}'}, E_i) - \sum_{\substack{j=1 \\ c_j < 0}}^{r} c_j (E_j, E_i) \geq 0,$$

where, in the last inequality, we used that, for all j such that $c_j < 0$, we have that $E_j \neq E_i$. $\square$

Proposition 4.2.2. *Let $P' : S \to \mathcal{Y}'$ be a section, and let $Q : S \to \mathcal{X}$ be the induced section. If the image of P' is not contained in the support of $K_{\mathcal{Y}'}$, then*

$$(K_{\mathcal{Y}'}, P')_{\mathrm{fin}} \leq (B, Q)_{\mathrm{fin}}.$$

Proof. By the Riemann–Hurwitz formula, we have $K_{\mathcal{Y}} = \pi^* K_{\mathcal{X}} + R$. Therefore, by Lemma 4.2.1, we get that

$$(K_{\mathcal{Y}'}, P')_{\mathrm{fin}} = (\psi^* K_{\mathcal{Y}} + \sum c_i E_i, P')_{\mathrm{fin}}$$

$$= \left(\psi^* \pi^* K_{\mathcal{X}} + \psi^* R + \sum_{i=1}^{s} c_i E_i, P' \right)_{\mathrm{fin}}$$

$$\leq (\psi^* \pi^* K_{\mathcal{X}}, P')_{\mathrm{fin}} + (\psi^* R, P')_{\mathrm{fin}}.$$

Since the image of P' is not contained in the support of $K_{\mathcal{Y}'}$, we can apply the projection formula for the composed morphism $\pi \circ \psi : \mathcal{Y}' \to \mathcal{X}$ to $(\psi^* \pi^* K_{\mathcal{X}}, P')_{\mathrm{fin}}$ and $(\psi^* R, P')_{\mathrm{fin}}$; see [Liu 2006a, Section 9.2]. This gives

$$(K_{\mathcal{Y}'}, P')_{\mathrm{fin}} \leq (\psi^* \pi^* K_{\mathcal{X}}, P')_{\mathrm{fin}} + (\psi^* R, P')_{\mathrm{fin}} = (K_{\mathcal{X}}, Q)_{\mathrm{fin}} + (\pi_* R, Q)_{\mathrm{fin}}.$$

Since $K_{\mathcal{X}} = -2 \cdot [\infty]$, the inequality $(K_{\mathcal{X}}, Q)_{\mathrm{fin}} \leq 0$ holds. By definition, $B = \pi_* R$. This concludes the proof. $\square$

We introduce some notation. For i in I and j in J_i, let e_{ij} and f_{ij} be the ramification index and residue degree of π at the generic point of D_{ij}, respectively. Moreover, let $\mathfrak{p}_i \subset O_K$ be the maximal ideal corresponding to the image of D_i in $\mathrm{Spec}\, O_K$. Then, note that e_{ij} is the multiplicity of D_{ij} in the fibre of $\mathcal{Y}$ over $\mathfrak{p}_i$. Now, let $e_{\mathfrak{p}_i}$ and $f_{\mathfrak{p}_i}$ be the ramification index and residue degree of $\mathfrak{p}_i$ over $\mathbb{Z}$, respectively. Finally, let p_i be the residue characteristic of the local ring at the generic point of D_i and, if $p_i > 0$, let m_i be the biggest integer such that $p_i^{m_i} \leq \deg \pi$, i.e., $m_i = \lfloor \log(\deg \pi)/\log(p_i) \rfloor$.

Lemma 4.2.3. *Let i be in I such that $0 < p_i \leq \deg \pi$. Then, for all j in J_i,*

$$r_{ij} \leq 2e_{ij} m_i e_{\mathfrak{p}_i}.$$

Proof. Let ord_{D_i} be the valuation on the local ring at the generic point of D_i. Then, by Proposition 4.1.3, the inequality

$$r_{ij} \leq e_{ij} - 1 + e_{ij} \cdot \mathrm{ord}_{D_i}(p_i^{m_i})$$

holds. Note that $\mathrm{ord}_{D_i}(p_i^{m_i}) = m_i e_{\mathfrak{p}_i}$. Since $p_i \leq \deg \pi$, we have that $m_i \geq 1$. Therefore,

$$r_{ij} \leq e_{ij} - 1 + e_{ij} m_i e_{\mathfrak{p}_i} \leq 2e_{ij} m_i e_{\mathfrak{p}_i}. \qquad \square$$

Let us introduce a bit more notation. Let I_1 be the set of i in I such that D_i is horizontal (i.e., $p_i = 0$) or $p_i > \deg \pi$. Let $D_1 = \sum_{i \in I_1} D_i$. We are now finally ready to combine our results to bound the "nonarchimedean" part of the height of a point.

Proposition 4.2.4. *Let $P' : S \to \mathcal{Y}'$ be a section, and let $Q : S \to \mathcal{X}$ be the induced section. If the image of P' is not contained in the support of $K_{\mathcal{Y}'}$, then*

$$(K_{\mathcal{Y}'}, P')_{\mathrm{fin}} \leq \deg \pi (D_1, Q)_{\mathrm{fin}} + 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbb{Q}].$$

Proof. Note that

$$B = \sum_{i \in I} \left(\sum_{j \in J_i} r_{ij} f_{ij} \right) D_i.$$

Let I_2 be the complement of I_1 in I. Let $D_2 = \sum_{i \in I_2} D_i$, and note that $D = D_1 + D_2$. In particular,

$$\begin{aligned}
(B, Q)_{\mathrm{fin}} &= \sum_{i \in I} \sum_{j \in J_i} r_{ij} f_{ij} (D_i, Q)_{\mathrm{fin}} \\
&= \sum_{i \in I_1} \sum_{j \in J_i} r_{ij} f_{ij} (D_i, Q)_{\mathrm{fin}} + \sum_{i \in I_2} \sum_{j \in J_i} r_{ij} f_{ij} (D_i, Q)_{\mathrm{fin}}.
\end{aligned}$$

Note that, for all i in I_1 and j in J_i, the ramification of D_{ij} over D_i is tame; i.e., the equality $r_{ij} = e_{ij} - 1$ holds. Note that, for all i in I, we have $\sum_{j \in J_i} e_{ij} f_{ij} = \deg \pi$. Thus,

$$\sum_{i \in I_1} \sum_{j \in J_i} r_{ij} f_{ij} (D_i, Q)_{\text{fin}} \leq \sum_{i \in I_1} \sum_{j \in J_i} e_{ij} f_{ij} (D_i, Q)_{\text{fin}} = \deg \pi (D_1, Q)_{\text{fin}}.$$

We claim that

$$\sum_{i \in I_2} \sum_{j \in J_i} r_{ij} f_{ij} (D_i, Q)_{\text{fin}} \leq 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbb{Q}].$$

In fact, since, for all i in I_2 and j in J_i, by Lemma 4.2.3, the inequality

$$r_{ij} \leq 2 e_{ij} m_i e_{\mathfrak{p}_i}$$

holds, we have that

$$\sum_{i \in I_2} \sum_{j \in J_i} r_{ij} f_{ij} (D_i, Q)_{\text{fin}} \leq 2 \sum_{i \in I_2} m_i e_{\mathfrak{p}_i} (D_i, Q)_{\text{fin}} \left(\sum_{j \in J_i} e_{ij} f_{ij} \right)$$

$$= 2(\deg \pi) \sum_{i \in I_2} m_i e_{\mathfrak{p}_i} (D_i, Q)_{\text{fin}}.$$

Note that $(D_i, Q) = \log(\#k(\mathfrak{p}_i)) = f_{\mathfrak{p}_i} \log p_i$. We conclude that

$$\sum_{i \in I_2} m_i e_{\mathfrak{p}_i} (D_i, Q)_{\text{fin}} = \sum_{p \text{ prime}} \left(\sum_{i \in I_2, \; p_i = p} e_{\mathfrak{p}_i} f_{\mathfrak{p}_i} \right) \left\lfloor \frac{\log(\deg \pi)}{\log p} \right\rfloor \log(p)$$

$$= [K : \mathbb{Q}] \sum_{\mathscr{X}_p \cap |D_2| \neq \varnothing} \left\lfloor \frac{\log(\deg \pi)}{\log p} \right\rfloor \log(p),$$

where the last sum runs over all prime numbers p such that the fibre $\mathscr{X}_p$ contains an irreducible component of the support of D_2. Thus,

$$(B, Q)_{\text{fin}} \leq (\deg \pi)(D_1, Q)_{\text{fin}} + 2(\deg \pi)[K : \mathbb{Q}] \sum_{\mathscr{X}_p \cap D_2 \neq \varnothing} \left\lfloor \frac{\log(\deg \pi)}{\log p} \right\rfloor \log(p).$$

Note that

$$\sum_{\mathscr{X}_p \cap D_2 \neq \varnothing} \left\lfloor \frac{\log(\deg \pi)}{\log p} \right\rfloor \log(p) \leq \sum_{\mathscr{X}_p \cap D_2 \neq \varnothing} \log(\deg \pi) \leq \deg \pi \log(\deg \pi),$$

where we used that $\mathscr{X}_p \cap D_2 \neq \varnothing$ implies that $p \leq \deg \pi$. In particular,

$$(B, Q)_{\text{fin}} \leq (\deg \pi)(D_1, Q)_{\text{fin}} + 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbb{Q}].$$

By Proposition 4.2.2, we conclude that

$$(K_{\mathcal{Y}'}, P')_{\mathrm{fin}} \le (\deg \pi)(D_1, Q)_{\mathrm{fin}} + 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbb{Q}]. \qquad \square$$

4.3. *Models of covers of curves.* In this section, we give a general construction for a model of a cover of the projective line. Let K be a number field with ring of integers O_K, and let $S = \mathrm{Spec}\, O_K$.

Proposition 4.3.1. *Let $\mathcal{Y} \to \mathrm{Spec}\, O_K$ be a flat projective morphism with geometrically connected fibres of dimension 1, where $\mathcal{Y}$ is an integral normal scheme. Then, there exists a finite field extension L/K such that the minimal resolution of singularities of the normalisation of $\mathcal{Y} \times_{O_K} O_L$ is semistable over O_L.*

Proof. This follows from [Liu 2006b, Corollary 2.8]. $\qquad \square$

The main result of this section reads as follows.

Theorem 4.3.2. *Let K be a number field, and let Y be a smooth projective geometrically connected curve over K. Then, for any finite morphism $\pi_K : Y \to \mathbb{P}^1_K$, there exists a number field L/K such that*

- *the normalisation $\pi : \mathcal{Y} \to \mathbb{P}^1_{O_L}$ of $\mathbb{P}^1_{O_L}$ in the function field of Y_L is finite flat surjective,*
- *the minimal resolution of singularities $\psi : \mathcal{Y}' \to \mathcal{Y}$ is semistable over O_L and*
- *each irreducible component of the vertical part of the branch locus of the finite flat morphism $\pi : \mathcal{Y} \to \mathbb{P}^1_{O_L}$ is of characteristic less than or equal to $\deg \pi$. (The characteristic of a prime divisor D on $\mathbb{P}^1_{O_L}$ is the residue characteristic of the local ring at the generic point of D.)*

Proof. By Proposition 4.3.1, there exists a finite field extension L/K such that the minimal resolution of singularities $\psi : \mathcal{Y}' \to \mathcal{Y}$ of the normalisation of $\mathbb{P}^1_{O_L}$ in the function field of Y_L is semistable over O_L. Note that the finite morphism $\pi : \mathcal{Y} \to \mathbb{P}^1_{O_L}$ is flat. (The source is normal of dimension 2, and the target is regular.) Moreover, since the fibres of $\mathcal{Y}' \to \mathrm{Spec}\, O_L$ are reduced, the fibres of $\mathcal{Y}$ over O_L are reduced. Let $\mathfrak{p} \subset O_L$ be a maximal ideal of residue characteristic strictly bigger than $\deg \pi$, and note that the ramification of $\pi : \mathcal{Y} \to \mathbb{P}^1_{O_L}$ over (each prime divisor of $\mathbb{P}^1_{O_L}$ lying over) $\mathfrak{p}$ is tame. Since the fibres of $\mathcal{Y} \to \mathrm{Spec}\, O_L$ are reduced, we see that the finite morphism π is unramified over $\mathfrak{p}$. In fact, since $\mathbb{P}^1_{O_L} \to \mathrm{Spec}\, O_L$ has reduced (even smooth) fibres, the valuation of the different ideal $\mathfrak{D}_{\mathcal{O}_D/\mathcal{O}_{\pi(D)}}$ on $\mathcal{O}_D$ of an irreducible component D of $\mathcal{Y}_{\mathfrak{p}}$ lying over $\pi(D)$ in $\mathcal{X}$ is precisely the multiplicity of D in $\mathcal{Y}_{\mathfrak{p}}$. (Here we let $\mathcal{O}_D$ denote the local ring at the generic point of D and $\mathcal{O}_{\pi(D)}$ the local ring at the generic point of $\pi(D)$.) Thus, each irreducible component of the vertical part of the branch locus of $\pi : \mathcal{Y} \to \mathbb{P}^1_{O_L}$ is of characteristic less or equal to $\deg \pi$. $\qquad \square$

4.4. *The modular lambda function.* The modular function $\lambda : \mathbb{H} \to \mathbb{C}$ is defined as

$$\lambda(\tau) = \frac{\wp\left(\frac{1}{2} + \frac{\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right)}{\wp\left(\frac{\tau}{2}\right) - \wp\left(\frac{1}{2}\right)},$$

where $\wp$ denotes the Weierstrass elliptic function for the lattice $\mathbb{Z} + \tau\mathbb{Z}$ in $\mathbb{C}$. The function λ is $\Gamma(2)$-invariant. More precisely, λ factors through the $\Gamma(2)$-quotient map $\mathbb{H} \to Y(2)$ and an analytic isomorphism $Y(2) \xrightarrow{\sim} \mathbb{C} \setminus \{0, 1\}$. Thus, the modular function λ induces an analytic isomorphism $X(2) \to \mathbb{P}^1(\mathbb{C})$. Let us note that $\lambda(i\infty) = 0$, $\lambda(1) = \infty$ and $\lambda(0) = 1$.

The restriction of λ to the imaginary axis $\{iy : y > 0\}$ in $\mathbb{H}$ induces a homeomorphism, also denoted by λ, from $\{iy : y > 0\}$ to the open interval $(0, 1)$ in $\mathbb{R}$. In fact, for α in the open interval $(0, 1)$,

$$\lambda^{-1}(\alpha) = i\,\frac{M(1, \sqrt{\alpha})}{M(1, \sqrt{1 - \alpha})},$$

where M denotes the arithmetic-geometric mean.

Lemma 4.4.1. *For τ in $\mathbb{H}$, let $q(\tau) = \exp(\pi i \tau)$ and let $\lambda(\tau) = \sum_{n=1}^{\infty} a_n q^n(\tau)$ be the q-expansion of λ on $\mathbb{H}$. Then, for any real number $\frac{4}{5} \leq y \leq 1$,*

$$-\log\left|\sum_{n=1}^{\infty} n a_n q^n(iy)\right| \leq 2.$$

Proof. Note that

$$\sum_{n=1}^{\infty} n a_n q^n = q\,\frac{d\lambda}{dq}.$$

It suffices to show that $|q\,d\lambda/dq| \geq \frac{3}{20}$. We will use the product formula for λ. Namely,

$$\lambda(q) = 16q \prod_{n=1}^{\infty} f_n(q) \quad \text{and} \quad f_n(q) := \frac{1 + q^{2n}}{1 + q^{2n-1}}.$$

Write $f_n'(q) = df_n(q)/dq$. Then,

$$q\,\frac{d\lambda}{dq} = \lambda\left(1 + q\sum_{n=1}^{\infty} \frac{f_n'(q)}{f_n(q)}\right) = \lambda\left(1 + q\sum_{n=1}^{\infty} \frac{d}{dq} \log f_n(q)\right).$$

Note that, for any positive integer n and $\frac{4}{5} \leq y \leq 1$,

$$\left(\frac{d}{dq} \log f_n(q)\right)(iy) \leq 0.$$

Moreover, since $\lambda(i) = \frac{1}{2}$ and $\lambda(0) = 1$, the inequality $\lambda(iy) \geq \frac{1}{2}$ holds for all $0 \leq y \leq 1$. Also, for $\frac{4}{5} \leq y \leq 1$,

$$\left(-q \sum_{n=1}^{\infty} \frac{d}{dq} \log f_n(q)\right)(iy) \leq \frac{7}{10}.$$

In fact,

$$\sum_{n=1}^{\infty} \frac{d}{dq} \log f_n(q) = \sum_{n=1}^{\infty} \frac{2nq^{2n-1}}{1+q^{2n}} - \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-2}}{1+q^{2n-1}}.$$

It is straightforward to verify that, for all $\frac{4}{5} \leq y \leq 1$, the inequality

$$\sum_{n=1}^{\infty} \frac{2nq^{2n-1}(iy)}{1+q^{2n}(iy)} - \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-2}(iy)}{1+q^{2n-1}(iy)}$$
$$\geq \frac{100}{109} \sum_{n=1}^{\infty} 2nq^{2n-1}(iy) - \sum_{n=1}^{\infty} (2n-1)q^{2n-2}(iy)$$

holds. Finally, utilising classical formulas for geometric series, for all $\frac{4}{5} \leq y \leq 1$,

$$q(iy) \sum_{n=1}^{\infty} \frac{d}{dq}(\log f_n(q))(iy) \geq q(iy)\left(\frac{200q(iy)}{109(1-q^2(iy))^2} - \frac{1+q^2(iy)}{(1-q^2(iy))^2}\right) \geq \frac{7}{10}.$$

We conclude that

$$\left| q \frac{d\lambda}{dq} \right| \geq \frac{1}{2}\left(1 - \frac{7}{10}\right) = \frac{3}{20}. \qquad \square$$

4.5. *A non-Weierstrass point with bounded height.* The logarithmic height of a nonzero rational number $a = p/q$ is given by

$$h_{\text{naive}}(a) = \log \max(|p|, |q|),$$

where p and q are coprime integers and $q > 0$.

Theorem 4.5.1. *Let $\pi_{\overline{\mathbb{Q}}} : Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ be a finite morphism of degree d, where $Y/\overline{\mathbb{Q}}$ is a smooth projective connected curve of positive genus $g \geq 1$. Assume that $\pi_{\overline{\mathbb{Q}}} : Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is unramified over $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$. Then, for any rational number $0 < a \leq \frac{2}{3}$ and any $b \in Y(\overline{\mathbb{Q}})$ lying over a,*

$$h(b) \leq 3h_{\text{naive}}(a)d^2 + 6378031\frac{d^5}{g}.$$

Proof. By Theorem 4.3.2, there exist a number field K and a model

$$\pi_K : Y \to \mathbb{P}^1_K$$

for $\pi_{\bar{\mathbb{Q}}} : Y \to \mathbb{P}^1_{\bar{\mathbb{Q}}}$ with the following three properties: the minimal resolution of singularities $\psi : \mathcal{Y}' \to \mathcal{Y}$ of the normalisation $\pi : \mathcal{Y} \to \mathbb{P}^1_{O_K}$ of $\mathbb{P}^1_{O_K}$ in $\mathcal{Y}$ is semistable over O_K, each irreducible component of the vertical part of the branch locus of $\pi : \mathcal{Y} \to \mathbb{P}^1_{O_K}$ is of characteristic less than or equal to $\deg \pi$ and every point in the fibre of π_K over a is K-rational. Also, the morphism $\pi : \mathcal{Y} \to \mathbb{P}^1_{O_K}$ is finite flat surjective.

Let $b \in Y(K)$ lie over a. Let P' be the closure of b in $\mathcal{Y}'$. By [Lemma 2.3.1](), the height of b is "minimal" on the minimal regular model. That is,

$$h(b) \leq \frac{(P', \omega_{\mathcal{Y}'/O_K})}{[K : \mathbb{Q}]}.$$

Recall the following notation from [Section 4.2](). Let $\mathscr{X} = \mathbb{P}^1_{O_K}$. Let $K_{\mathscr{X}} = -2 \cdot [\infty]$ be the divisor defined by the tautological section. Let $K_{\mathcal{Y}'}$ be the divisor on $\mathcal{Y}'$ defined by $d(\pi_K)$ viewed as a rational section of $\omega_{\mathcal{Y}'/O_K}$. Since the support of $K_{\mathcal{Y}'}$ on the generic fibre is contained in $\pi_K^{-1}(\{0, 1, \infty\})$, the section P' is not contained in the support of $K_{\mathcal{Y}'}$. Therefore, we get that

$$h(b)[K : \mathbb{Q}] \leq (P', \omega_{\mathcal{Y}'/O_K}) = (P', K_{\mathcal{Y}'})_{\text{fin}} + \sum_{\sigma:K\to\mathbb{C}} (-\log \|d\pi_K\|_\sigma)(\sigma(b)).$$

Let D be the branch locus of $\pi : \mathcal{Y} \to \mathscr{X}$ endowed with the reduced closed subscheme structure. Write $D = 0 + 1 + \infty + D_{\text{ver}}$, where D_{ver} is the vertical part of D. Note that, in the notation of [Section 4.2](), we have that $D_1 = 0 + 1 + \infty$. Thus, if Q denotes the closure of a in $\mathscr{X}$, by [Proposition 4.2.4](), we get

$$(P', K_{\mathcal{Y}'})_{\text{fin}} \leq (\deg \pi)(0 + 1 + \infty, Q)_{\text{fin}} + 2(\deg \pi)^2 \log(\deg \pi)[K : \mathbb{Q}].$$

Write $a = p/q$, where p and q are coprime positive integers with $q > p$. Note that

$$\begin{aligned}(0 + 1 + \infty, Q)_{\text{fin}} &= [K : \mathbb{Q}] \log(pq(q - p)) \\ &\leq 3 \log(q)[K : \mathbb{Q}] \\ &= 3h_{\text{naive}}(a)[K : \mathbb{Q}].\end{aligned}$$

We conclude that

$$\frac{(P', K_{\mathcal{Y}'})_{\text{fin}}}{[K : \mathbb{Q}]} \leq 3h_{\text{naive}}(a)(\deg \pi)^2 + 2(\deg \pi)^3.$$

It remains to estimate $\sum_{\sigma:K\to\mathbb{C}}(-\log \|d\pi_K\|_\sigma)(\sigma(b))$. We will use our bounds for Arakelov–Green functions.

Let $\sigma : K \to \mathbb{C}$ be an embedding. The composition

$$Y_\sigma \xrightarrow{\pi_\sigma} \mathbb{P}^1(\mathbb{C}) \xrightarrow{\lambda^{-1}} X(2)$$

is a Belyi cover (Definition 3.2.1). By abuse of notation, let π denote the composed morphism $Y_\sigma \to X(2)$. Note that $\lambda^{-1}(\frac{2}{3}) \approx 0.85i$. In particular, $\Im(\lambda^{-1}(a)) \geq \Im(\lambda^{-1}(\frac{2}{3})) > s_1$. (Recall that $s_1 = \sqrt{1/2}$.) Therefore, the element $\lambda^{-1}(a)$ lies in $\dot{B}^{s_1}_\infty$. Since $V^{r_1}_y \supset V_y \cap \pi^{-1} B^{s_1}_\infty$, there is a unique cusp y of $Y_\sigma \to X(2)$ lying over ∞ such that $\sigma(b)$ lies in $V^{r_1}_y$.

Note that $q = z_\infty \exp(-\pi/2)$. Therefore, since $\lambda = \sum\limits_{j=1}^{\infty} a_j q^j$ on $\mathbb{H}$,

$$\lambda \circ \pi = \sum_{j=1}^{\infty} a_j \exp(-j\pi/2)(z_\infty \circ \pi)^j = \sum_{j=1}^{\infty} a_j \exp(-j\pi/2) w_y^{e_y j}$$

on V_y. Thus, by the chain rule,

$$d(\lambda \circ \pi) = e_y \sum_{j=1}^{\infty} j a_j \exp(-j\pi/2) w_y^{e_y j - 1} d(w_y).$$

By the trivial inequality $e_y \geq 1$, the inequality $|w_y| \leq 1$ and Lemma 4.4.1,

$-\log \|d(\lambda \circ \pi)\|_{\mathrm{Ar}}(\sigma(b))$

$$= -\log \|dw_y\|_{\mathrm{Ar}}(\sigma(b)) - \log \left| e_y \sum_{j=1}^{\infty} j a_j \exp(-j\pi/2) w_y^{e_y j - 1} (\sigma(b)) \right|$$

$$\leq -\log \|dw_y\|_{\mathrm{Ar}}(\sigma(b)) - \log \left| \sum_{j=1}^{\infty} j a_j \exp(-j\pi/2) w_y^{e_y j} (\sigma(b)) \right|$$

$$\leq -\log \|dw_y\|_{\mathrm{Ar}}(\sigma(b)) + 2.$$

Thus, by Theorem 3.4.5, we conclude that

$$\frac{\sum_{\sigma:K\to\mathbb{C}} (-\log \|d\pi_K\|_\sigma)(\sigma(b))}{[K:\mathbb{Q}]} \leq 6378027 \frac{(\deg \pi)^5}{g} + 2. \qquad \square$$

Theorem 4.5.2. *Let Y be a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus $g \geq 1$. For any finite morphism $\pi : Y \to \mathbb{P}^1_{\mathbb{Q}}$ ramified over exactly three points, there exists a non-Weierstrass point b on Y such that*

$$h(b) \leq 6378033 \frac{(\deg \pi)^5}{g}.$$

Proof. Define the sequence $(a_n)_{n=1}^{\infty}$ of rational numbers by $a_1 = \frac{1}{2}$ and $a_n = n/(2n-1)$ for $n \geq 2$. Note that $\frac{1}{2} \leq a_n \leq \frac{2}{3}$ and that $h_{\mathrm{naive}}(a_n) \leq \log(2n)$. We may and do assume that $\pi : Y \to \mathbb{P}^1_{\mathbb{Q}}$ is unramified over $\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. By Theorem 4.5.1, for all $x \in \pi^{-1}(\{a_n\})$,

$$h(x) \leq 3 \log(2n)(\deg \pi)^2 + 6378031 \frac{(\deg \pi)^5}{g}. \qquad (9)$$

Since the number of Weierstrass points on Y is at most $g^3 - g$, there exists an integer $1 \leq i \leq (\deg \pi)^2$ such that the fibre $\pi^{-1}(a_i)$ contains a non-Weierstrass point, say b. Applying (9) to b, we conclude that

$$h(b) \leq 3 \log(2(\deg \pi)^2)(\deg \pi)^2 + 6378031 \frac{(\deg \pi)^5}{g}$$
$$\leq 2 \frac{(\deg \pi)^5}{g} + 6378031 \frac{(\deg \pi)^5}{g}. \qquad \square$$

4.6. For a smooth projective connected curve X over $\overline{\mathbb{Q}}$, we let $\deg_B(X)$ denote the Belyi degree of X.

Proof of Theorem 1.1.1. The inequality $\Delta(X) \geq 0$ is trivial, the lower bound $e(X) \geq 0$ is due to Faltings [1984, Theorem 5] and the lower bound $h_{\mathrm{Fal}}(X) \geq -g \log(2\pi)$ is due to Bost (Lemma 2.4.4).

For the remaining bounds, we proceed as follows. By Theorem 4.5.2, there exists a non-Weierstrass point b in $X(\overline{\mathbb{Q}})$ such that

$$h(b) \leq 6378033 \frac{\deg_B(X)^5}{g}.$$

By our bound on the Arakelov norm of the Wronskian differential in Proposition 3.5.1, we have $\log \|\mathrm{Wr}\|_{\mathrm{Ar}}(b) \leq 6378028 g \deg_B(X)^5$. To obtain the theorem, we combine these bounds with Theorem 2.4.1. $\qquad \square$

5. Computing coefficients of modular forms

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, and let k be a positive integer. A modular form f of weight k for the group Γ is determined by k and its q-expansion coefficients $a_m(f)$ for $0 \leq m \leq k \cdot [\mathrm{SL}_2(\mathbb{Z}) : \{\pm 1\}\Gamma]/12$. In this section, we follow [Bruin 2011] and give an algorithmic application of the main result of this paper. More precisely, the goal of this section is to complete the proof of the following theorem. The proof is given at the end of this section.

Theorem 5.0.1 (Couveignes, Edixhoven, Bruin). *Assume the Riemann hypothesis for ζ-functions of number fields. There exists a probabilistic algorithm that, given*

- *a positive integer k,*
- *a number field K,*
- *a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$,*
- *a modular form f of weight k for Γ over K, and*
- *a positive integer m in factored form,*

computes $a_m(f)$ and whose expected running time is bounded by a polynomial in the length of the input.

Remark 5.0.2. We should make precise how the number field K, the congruence subgroup Γ and the modular form f should be given to the algorithm and how the algorithm returns the coefficient $a_m(f)$. We should also explain what "probabilistic" means in this context. For the sake of brevity, we refer the reader to [Bruin 2011, p. 20] for the precise definitions. Following the definitions there, the above theorem becomes a precise statement.

Remark 5.0.3. The algorithm in Theorem 5.0.1 is due to Bruin, Couveignes and Edixhoven. Assuming the Riemann hypothesis for ζ-functions of number fields, it was shown that the algorithm runs in polynomial time for *certain* congruence subgroups; see [Bruin 2011, Theorem 1.1]. Bruin did not have enough information about the semistable bad reduction of the modular curve $X_1(n)$ at primes p such that p^2 divides n to show that the algorithm runs in polynomial time. Nevertheless, our bounds on the discriminant of a curve can be used to show that the algorithm runs in polynomial time for *all* congruence subgroups.

Proof of Theorem 5.0.1. We follow Bruin's strategy [2010, Chapter V.1, p. 165]. He notes that, to assure that the algorithm runs in polynomial time for all congruence subgroups, it suffices to show that, for all positive integers n, the discriminant $\Delta(X_1(n))$ is polynomial in n (or equivalently the genus of $X_1(n)$). The latter follows from Corollary 1.5.1. In fact, the Belyi degree of $X_1(n)$ is at most the index of $\Gamma_1(n)$ in $\mathrm{SL}_2(\mathbb{Z})$. Since

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(n)] = n^2 \prod_{p|n}(1 - 1/p^2) \leq n^2,$$

we conclude that $\Delta(X_1(n)) \leq 5 \cdot 10^8 n^{14}$. $\square$

6. Bounds for heights of covers of curves

Let X be a smooth projective connected curve over $\overline{\mathbb{Q}}$. We prove that Arakelov invariants of (possibly ramified) covers of X are polynomial in the degree. Let us be more precise.

Theorem 6.0.4. *Let X be a smooth projective connected curve over $\overline{\mathbb{Q}}$, let U be a nonempty open subscheme of X, let $B_f \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be a finite set and let $f : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ be a finite morphism unramified over $\mathbb{P}^1_{\overline{\mathbb{Q}}} - B_f$. Define*

$$B := f(X \setminus U) \cup B_f.$$

Let N be the number of elements in the orbit of B under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and let H_B be the height of B as defined in Section 1.3. Define

$$c_B := (4NH_B)^{45N^3 2^{N-2}N!}.$$

Then, for any finite morphism $\pi : Y \to X$ étale over U, where Y is a smooth projective connected curve over $\overline{\mathbb{Q}}$ of genus $g \geq 1$,

$$-\log(2\pi)g \leq h_{\mathrm{Fal}}(Y) \leq 13 \cdot 10^6 g c_B (\deg f)^5 (\deg \pi)^5,$$
$$0 \leq \ e(Y) \ \leq 3 \cdot 10^7 (g-1) c_B (\deg f)^5 (\deg \pi)^5,$$
$$0 \leq \ \Delta(Y) \ \leq 5 \cdot 10^8 g^2 c_B (\deg f)^5 (\deg \pi)^5,$$
$$-10^8 g^2 c_B (\deg f)^5 (\deg \pi)^5 \leq \delta_{\mathrm{Fal}}(Y) \ \leq 2 \cdot 10^8 g c_B (\deg f)^5 (\deg \pi)^5.$$

Proof. We apply Khadjavi's effective version of Belyi's theorem. More precisely, by [Khadjavi 2002, Theorem 1.1.c], there exists a finite morphism $R : \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ étale over $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$ such that $\deg R \leq (4N H_B)^{9N^3 2^{N-2} N!}$ and $R(B) \subset \{0, 1, \infty\}$. Note that the composed morphism

$$R \circ f \circ \pi : Y \xrightarrow{\pi} X \xrightarrow{f} \mathbb{P}^1_{\overline{\mathbb{Q}}} \xrightarrow{R} \mathbb{P}^1_{\overline{\mathbb{Q}}}$$

is unramified over $\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}$. We conclude by applying Theorem 1.1.1 to the composition $R \circ f \circ \pi$. $\qquad\square$

Note that Theorem 6.0.4 implies Theorem 1.3.1 (with $X = \mathbb{P}^1_{\overline{\mathbb{Q}}}$, B_f the empty set and $f : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ the identity morphism).

In the proof of Theorem 6.0.4, we used Khadjavi's effective version of Belyi's theorem. Khadjavi's bounds are not optimal; see [Liţcanu 2004, Lemme 4.1] and [Khadjavi 2002, Theorem 1.1.b] for better bounds when B is contained in $\mathbb{P}^1(\mathbb{Q})$. Actually, the use of Belyi's theorem makes the dependence on the branch locus enormous in Theorem 6.0.4. It should be possible to avoid the use of Belyi's theorem and improve the dependence on the branch locus in Theorem 6.0.4. This is not necessary for our present purposes.

Remark 6.0.5. Let us mention the quantitative Riemann existence theorem due to Bilu and Strambi [2010]. Bilu and Strambi give explicit bounds for the naive logarithmic height of a cover of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ with fixed branch locus. Although their bound on the naive height is exponential in the degree, the dependence on the height of the branch locus in their result is logarithmic.

Let us show that Theorem 1.3.1 implies the following:

Theorem 6.0.6 [Edixhoven et al. 2010, Conjecture 5.1]. *Let $U \subset \mathbb{P}^1_{\mathbb{Z}}$ be a nonempty open subscheme. Then there are integers a and b with the following property. For any prime number ℓ and for any connected finite étale cover $\pi : V \to U_{\mathbb{Z}[1/\ell]}$, the Faltings height of the normalisation of $\mathbb{P}^1_{\mathbb{Q}}$ in the function field of V is bounded by $(\deg \pi)^a \ell^b$.*

Proof. We claim that this conjecture holds with $b = 0$ and an integer a depending only on the generic fibre $U_{\mathbb{Q}}$ of U. In fact, let $\pi : Y \to \mathbb{P}^1_{\mathbb{Q}}$ denote the normalisation of $\mathbb{P}^1_{\mathbb{Q}}$

in the function field of V. Note that π is étale over $U_\mathbb{Q}$. Let $B = \mathbb{P}^1_\mathbb{Q} - U_\mathbb{Q} \subset \mathbb{P}^1(\overline{\mathbb{Q}})$, and let N be the number of elements in the orbit of B under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By Theorem 1.3.1,

$$h_{\mathrm{Fal}}(Y) := \sum_{X \subset Y_{\overline{\mathbb{Q}}}} h_{\mathrm{Fal}}(X) \leq (\deg \pi)^a,$$

where the sum runs over all connected components X of $Y_{\overline{\mathbb{Q}}} := Y \times_\mathbb{Q} \overline{\mathbb{Q}}$, and

$$a = 6 + \log\left(13 \cdot 10^6 N (4N H_B)^{45 N^3 2^{N-2} N!}\right).$$

Here we used that $g \leq N \deg \pi$ and

$$13 \cdot 10^6 g (4N H_B)^{45 N^3 2^{N-2} N!} \leq (\deg \pi)^{1 + \log(13 \cdot 10^6 N (4N H_B)^{45 N^3 2^{N-2} N!})}.$$

This concludes the proof.   □

Let us briefly mention the context in which these results will hopefully be applied. Let S be a smooth projective geometrically connected surface over $\mathbb{Q}$. As is explained in Section 5 of [Edixhoven et al. 2010], it seems reasonable to suspect that there exists an algorithm that, on input of a prime ℓ, computes the étale cohomology groups $\mathrm{H}^i(S_{\overline{\mathbb{Q}}, \text{ét}}, \mathbb{F}_\ell)$ with their $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action in time *polynomial* in ℓ for all $i = 0, \dots, 4$.

Appendix: Merkl's method of bounding Green functions
by Peter Bruin

The goal of this appendix is to prove Theorem 3.1.2. Let X be a compact connected Riemann surface, and let μ be a smooth nonnegative $(1, 1)$-form on X such that $\int_X \mu = 1$. Let $*$ denote the star operator on 1-forms on X, given with respect to a holomorphic coordinate $z = x + iy$ by

$$*dx = dy \quad \text{and} \quad *dy = -dx,$$

or equivalently

$$*dz = -i \, d\bar{z} \quad \text{and} \quad *d\bar{z} = i \, dz.$$

The *Green function* for μ is the unique smooth function

$$\mathrm{gr}_\mu : (X \times X) \setminus \Delta \to \mathbb{R},$$

with a logarithmic singularity along the diagonal Δ, such that for fixed $w \in X$ we have, in a distributional sense,

$$\frac{1}{2\pi} d * d \, \mathrm{gr}_\mu(z, w) = \delta_w(z) - \mu(z) \quad \text{and} \quad \int_{z \in X \setminus \{w\}} \mathrm{gr}_\mu(z, w) \mu(z) = 0.$$

For all $a, b \in X$, we write $g_{a,b}$ for the unique smooth function on $X \setminus \{a, b\}$ satisfying

$$d * dg_{a,b} = \delta_a - \delta_b \quad \text{and} \quad \int_{X \setminus \{a,b\}} g_{a,b}\mu = 0. \tag{1}$$

Then for all $a \in X$, we consider the function $g_{a,\mu}$ on $X \setminus \{a\}$ defined by

$$g_{a,\mu}(x) = \int_{b \in X \setminus \{x\}} g_{a,b}(x)\mu(b). \tag{2}$$

A straightforward computation using Fubini's theorem shows that this function satisfies

$$d * dg_{a,\mu} = \delta_a - \mu \quad \text{and} \quad \int_{X \setminus \{a\}} g_{a,\mu}\mu = 0.$$

This implies that $2\pi g_{a,\mu}(b) = \mathrm{gr}_\mu(a, b)$, where gr_μ is the Green function for μ defined above.

We begin by restricting our attention to one of the charts of our atlas, say (U, z). By assumption, z is an isomorphism from U to the open unit disc in $\mathbb{C}$. Let r_2 and r_4 be real numbers with

$$r_1 < r_2 < r_4 < 1,$$

and write

$$r_3 = (r_2 + r_4)/2.$$

We choose a smooth function

$$\tilde{\chi} : \mathbb{R}_{\geq 0} \to [0, 1]$$

such that $\tilde{\chi}(r) = 1$ for $r \leq r_2$ and $\tilde{\chi}(r) = 0$ for $r \geq r_4$. We also define a smooth function χ on X by putting

$$\chi(x) = \tilde{\chi}(|z(x)|) \quad \text{for } x \in U$$

and extending by 0 outside U. Furthermore, we put

$$\chi^c = 1 - \chi.$$

For $0 < r < 1$, we write

$$U^r = \{x \in U : |z(x)| < r\}.$$

For all $a, b \in U^{r_1}$, the function

$$f_{a,b} = \frac{1}{2\pi} \log \left| \frac{(z - z(a))(\overline{z(a)}z - r_4^2)}{(z - z(b))(\overline{z(b)}z - r_4^2)} \right|$$

is defined on $U \setminus \{a, b\}$. Moreover, $f_{a,b}$ is harmonic on $U \setminus \{a, b\}$ since the logarithm of the modulus of a holomorphic function is harmonic. We extend $\chi^c f_{a,b}$ to a smooth function on U by defining it to be zero in a and b.

We consider the open annulus

$$A = U^{r_4} \setminus \overline{U^{r_2}}.$$

Let (ρ, ϕ) be polar coordinates on A such that $z = \rho \exp(i\phi)$. A straightforward calculation shows that in these coordinates the star operator is given by

$$*d\rho = \rho \, d\phi \quad \text{and} \quad *d\phi = -\frac{d\rho}{\rho}.$$

We consider the inner product

$$\langle \alpha, \beta \rangle_A = \int_A \alpha \wedge *\beta$$

on the $\mathbb{R}$-vector space of square-integrable real-valued 1-forms on A. Furthermore, we write

$$\|\alpha\|_A^2 = \langle \alpha, \alpha \rangle_A.$$

Lemma A.1. *For every real harmonic function g on A such that $\|dg\|_A$ exists,*

$$\max_{|z|=r_3} g - \min_{|z|=r_3} g \leq \frac{2\sqrt{\pi}}{r_4 - r_2} \|dg\|_A.$$

Proof. By the formula for the star operator in polar coordinates,

$$dg \wedge *dg = (\partial_\rho g \, d\rho + \partial_\phi g \, d\phi) \wedge (\rho \partial_\rho g \, d\phi - \rho^{-1} \partial_\phi g \, d\rho)$$
$$= ((\partial_\rho g)^2 + (\rho^{-1} \partial_\phi g)^2) \rho \, d\rho \, d\phi.$$

Using the mean value theorem, we can bound the left-hand side of the inequality we need to prove by

$$\max_{|z|=r_3} g - \min_{|z|=r_3} g \leq \pi \max_{|z|=r_3} |\partial_\phi g|$$
$$= \pi |\partial_\phi g|(x) \quad \text{for some } x \text{ with } |z(x)| = r_3.$$

We write $R = (r_4 - r_2)/2$, and we consider the open disc

$$D = \{z \in U : |z - z(x)| < R\}$$

of radius R around x; this lies in A because $r_3 = (r_4 + r_2)/2$. Let (σ, ψ) be polar coordinates on D such that $z - z(x) = \sigma \exp(i\psi)$. Because g is harmonic, so is $\partial_\phi g$, and Gauss's mean value theorem implies that

$$\partial_\phi g(x) = \frac{1}{\pi R^2} \int_D \partial_\phi g \, \sigma \, d\sigma \, d\psi.$$

On the space of real continuous functions on D, we have the inner product

$$(h_1, h_2) \mapsto \int_D h_1 h_2 \, \sigma \, d\sigma \, d\psi.$$

Applying the Cauchy–Schwarz inequality with $h_1 = \rho^{-1} \partial_\phi g$ and $h_2 = \rho$ gives

$$\left| \int_D \partial_\phi g \, \sigma \, d\sigma \, d\psi \right| \leq \left[\int_D (\rho^{-1} \partial_\phi g)^2 \sigma \, d\sigma \, d\psi \right]^{1/2} \cdot \left[\int_D \rho^2 \sigma \, d\sigma \, d\psi \right]^{1/2}$$

$$\leq \left[\int_A (\rho^{-1} \partial_\phi g)^2 \rho \, d\rho \, d\phi \right]^{1/2} \cdot \left[\int_D \sigma \, d\sigma \, d\psi \right]^{1/2}$$

$$\leq \left[\int_A dg \wedge *dg \right]^{1/2} [\pi R^2]^{1/2}$$

$$= \sqrt{\pi} \, R \|dg\|_A.$$

Combining the above results finishes the proof. $\qquad\qquad\square$

Lemma A.2. *For all $a, b \in U^{r_1}$, there exists a smooth function $\tilde{g}_{a,b}$ on X such that*

$$d*d\tilde{g}_{a,b} = \begin{cases} d*d(\chi^c f_{a,b}) & \text{on } U, \\ 0 & \text{on } X \setminus \overline{U}. \end{cases}$$

It is unique up to an additive constant and fulfils

$$\|d\tilde{g}_{a,b}\|_A \leq \|d(\chi^c f_{a,b})\|_A.$$

Proof. First we note that the expression on the right-hand side of the equality defines a smooth 2-form on X because $d*d(\chi^c f_{a,b})(z)$ vanishes for $|z| > r_4$; this follows from the choice of χ and the fact that $f_{a,b}$ is harmonic for $|z| > r_1$. Since moreover $\chi^c f_{a,b} = 0$ on U^{r_2}, we see that the support of this 2-form is contained in the closed annulus $\overline{A}$. By Stokes's theorem,

$$\int_{\overline{A}} d*d(\chi^c f_{a,b}) = \int_{\partial \overline{A}} *d(\chi^c f_{a,b}).$$

Notice that $f_{a,b}$ is invariant under the substitution $z \mapsto r_4^2/\bar{z}$; this implies that $\partial_\rho f_{a,b}(z) = 0$ for $|z| = r_4$. Furthermore, $\chi^c(z) = 1$ and $d\chi^c(z) = 0$ for $|z| = r_4$, so we see that

$$d(\chi^c f_{a,b})(z) = \chi^c(z) \, df_{a,b}(z) = (\partial_\phi f_{a,b} \, d\phi)(z) \quad \text{if } |z| = r_4.$$

Likewise, since $\chi^c = 0$ and $d\chi^c(z) = 0$ for $|z| = r_2$,

$$d(\chi^c f_{a,b})(z) = \chi^c(z) df_{a,b}(z) = 0 \quad \text{if } |z| = r_2.$$

This means that, for z on the boundary of $\bar{A}$,

$$*d(\chi^{\mathrm{c}} f_{a,b})(z) = \begin{cases} -(\partial_\phi f_{a,b} \, d\rho)(z) & \text{if } |z| = r_4, \\ 0 & \text{if } |z| = r_2. \end{cases}$$

In particular, $*d(\chi^{\mathrm{c}} f_{a,b})$ vanishes when restricted to the submanifold $\partial\bar{A}$ of X. From this, we conclude that

$$\int_{\bar{A}} d*d(\chi^{\mathrm{c}} f_{a,b}) = \int_{\partial\bar{A}} *d(\chi^{\mathrm{c}} f_{a,b}) = 0.$$

This implies that a function $\tilde{g}_{a,b}$ with the required property exists.

To prove the inequality $\|d\tilde{g}_{a,b}\|_A \le \|d(\chi^{\mathrm{c}} f_{a,b})\|_A$, we note that

$$\|d(\chi^{\mathrm{c}} f_{a,b})\|_A^2 = \|d\tilde{g}_{a,b} + d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\|_A^2$$
$$= \|d\tilde{g}_{a,b}\|_A^2 + 2\langle d\tilde{g}_{a,b}, d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\rangle_A + \|d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\|_A^2.$$

The last term is clearly nonnegative. Furthermore, integration by parts using Stokes's theorem gives

$$\langle d\tilde{g}_{a,b}, d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\rangle_A = \int_A d\tilde{g}_{a,b} \wedge *d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})$$
$$= \int_{\partial\bar{A}} \tilde{g}_{a,b} *d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b}) - \int_A \tilde{g}_{a,b} \, d*d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b}).$$

The second term vanishes because $d*d\tilde{g}_{a,b} = d*d(\chi^{\mathrm{c}} f_{a,b})$ on A. From our earlier expression for $*d(\chi^{\mathrm{c}} f_{a,b})(z)$ on the boundary of A, we see that

$$\int_{\partial\bar{A}} \tilde{g}_{a,b} *d(\chi^{\mathrm{c}} f_{a,b}) = 0.$$

Finally, because $\partial\bar{A}$ is also the (negatively oriented) boundary of $X \setminus A$ and because $d*d\tilde{g}_{a,b} = 0$ on $X \setminus A$,

$$-\int_{\partial\bar{A}} \tilde{g}_{a,b} *d\tilde{g}_{a,b} = \int_{X \setminus A} d\tilde{g}_{a,b} \wedge *d\tilde{g}_{a,b} \ge 0.$$

Thus, we have

$$\langle d\tilde{g}_{a,b}, d(\chi^{\mathrm{c}} f_{a,b} - \tilde{g}_{a,b})\rangle_A \ge 0,$$

which proves the inequality.   □

Lemma A.3. *Let* $\lambda = \max\limits_{r_2 \le r \le r_4} |\tilde{\chi}'(r)|$. *Then*

$$\max_X \tilde{g}_{a,b} - \min_X \tilde{g}_{a,b} \le c_3(r_1, r_2, r_4, \lambda),$$

where

$$c_3(r_1, r_2, r_4, \lambda) =$$

$$4\sqrt{\frac{r_4+r_2}{r_4-r_2}}\left(\frac{\lambda}{2}\log\frac{(r_1+r_4)^2}{(r_2-r_1)(r_4-r_1)} + \frac{1}{r_2-r_1} + \frac{r_1}{r_4(r_4-r_1)}\right) + \frac{2}{\pi}\log\frac{(r_1+r_4)^2}{(r_2-r_1)(r_4-r_1)}.$$

Proof. First, we note that

$$\max_X \tilde{g}_{a,b} = \max\left\{\sup_{U^{r_3}} \tilde{g}_{a,b}, \sup_{X\setminus U^{r_3}} \tilde{g}_{a,b}\right\}, \quad \min_X \tilde{g}_{a,b} = \min\left\{\inf_{U^{r_3}} \tilde{g}_{a,b}, \inf_{X\setminus U^{r_3}} \tilde{g}_{a,b}\right\}.$$

Furthermore,

$$\sup_{U^{r_3}} \tilde{g}_{a,b} \leq \sup_{U^{r_3}}(\tilde{g}_{a,b} - \chi^c f_{a,b}) + \sup_{U^{r_3}} \chi^c f_{a,b} = \max_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) + \max_{r_2\leq|z|\leq r_3} \chi^c f_{a,b}$$

because of the maximum principle ($\tilde{g}_{a,b} - \chi^c f_{a,b}$ is harmonic on U) and because $\chi^c(z) = 0$ for $|z| < r_2$. In the same way, we find

$$\inf_{U^{r_3}} \tilde{g}_{a,b} \geq \min_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) + \min_{r_2\leq|z|\leq r_3} \chi^c f_{a,b}.$$

We extend $\chi f_{a,b}$ to a smooth function on $X \setminus \{a, b\}$ by putting $(\chi f_{a,b})(x) = 0$ for $x \notin U$. Then $\tilde{g}_{a,b} + \chi f_{a,b}$ is harmonic on $X \setminus \{a, b\}$, and the same method as above gives us

$$\sup_{X\setminus U^{r_3}} \tilde{g}_{a,b} \leq \max_{|z|=r_3}(\tilde{g}_{a,b} + \chi f_{a,b}) - \min_{r_3\leq|z|\leq r_4} \chi f_{a,b}$$

$$\leq \max_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) + \max_{|z|=r_3} f_{a,b} - \min_{r_3\leq|z|\leq r_4} \chi f_{a,b}$$

and

$$\inf_{X\setminus U^{r_3}} \tilde{g}_{a,b} \geq \min_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) + \min_{|z|=r_3} f_{a,b} - \max_{r_3\leq|z|\leq r_4} \chi f_{a,b}.$$

These bounds imply that

$$\max_X \tilde{g}_{a,b} \leq \max_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) + 2\sup_A|f_{a,b}|,$$

$$\min_X \tilde{g}_{a,b} \geq \min_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) - 2\sup_A|f_{a,b}|$$

and hence

$$\max_X \tilde{g}_{a,b} - \min_X \tilde{g}_{a,b} \leq \max_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) - \min_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) + 4\sup_A|f_{a,b}|.$$

By Lemmas A.1 and A.2,

$$\max_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) - \min_{|z|=r_3}(\tilde{g}_{a,b} - \chi^c f_{a,b}) \leq \frac{2\sqrt{\pi}}{r_4-r_2}\|d\tilde{g}_{a,b} - d(\chi^c f_{a,b})\|_A$$

$$\leq \frac{2\sqrt{\pi}}{r_4-r_2}(\|d\tilde{g}_{a,b}\|_A + \|d(\chi^c f_{a,b})\|_A)$$

$$\leq \frac{4\sqrt{\pi}}{r_4-r_2}\|d(\chi^c f_{a,b})\|_A.$$

We have

$$\|d(\chi^c f_{a,b})\|_A \le \|d(\chi^c) f_{a,b}\|_A + \|\chi^c d f_{a,b}\|_A$$
$$\le \|\tilde{\chi}'(\rho) f_{a,b}\, d\rho\|_A + \|d f_{a,b}\|_A$$
$$\le \lambda \|d\rho\|_A \sup_A |f_{a,b}| + \|d f_{a,b}\|_A.$$

Now

$$\|d\rho\|_A^2 = \int_A d\rho \wedge *d\rho = \int_A \rho\, d\rho \wedge d\phi = \pi(r_4^2 - r_2^2).$$

Furthermore, for all $a, b \in U^{r_1}$, we have

$$|f_{a,b}(z)| = \frac{1}{2\pi}\big|\log|z - z(a)| + \log|\overline{z(a)}z - r_4^2| - \log|z - z(b)| - \log|\overline{z(b)}z - r_4^2|\big|.$$

For all $a \in U^{r_1}$ and all $z \in A$, the triangle inequality gives

$$r_2 - r_1 < |z - z(a)| < r_4 + r_1 \quad \text{and} \quad r_4(r_4 - r_1) < |\overline{z(a)}z - r_4^2| < r_4(r_4 + r_1).$$

From this, we deduce that, for all $a, b \in U^{r_1}$,

$$\sup_A |f_{a,b}| \le \frac{1}{2\pi} \log \frac{(r_1 + r_4)^2}{(r_2 - r_1)(r_4 - r_1)}.$$

Finally, we bound the quantity $\|d f_{a,b}\|_A$. Because $f_{a,b}$ is a real function, we have $d f_{a,b} = \partial_z f_{a,b}\, dz + \overline{\partial_z f_{a,b}}\, d\bar{z}$. Therefore,

$$\|d f_{a,b}\|_A^2 = \int_A d f_{a,b} \wedge *d f_{a,b} = 2i \int_A |\partial_z f_{a,b}|^2\, dz \wedge d\bar{z}$$
$$= 4 \int_0^{2\pi} \int_{r_2}^1 |\partial_z f_{a,b}|^2 \rho\, d\rho\, d\phi \le 4\pi(1 - r_2^2) \sup_A |\partial_z f_{a,b}|^2.$$

A straightforward computation gives

$$\partial_z f_{a,b} = \frac{1}{4\pi}\left(\frac{1}{z - z(a)} + \frac{\overline{z(a)}}{\overline{z(a)}z - r_4^2} - \frac{1}{z - z(b)} - \frac{\overline{z(b)}}{\overline{z(b)}z - r_4^2} \right).$$

Our previous bounds for $|z - z(a)|$ and $|\overline{z(a)}z - 1|$ yield

$$\sup_A |\partial_z f_{a,b}| \le \frac{1}{2\pi}\left(\frac{1}{r_2 - r_1} + \frac{r_1}{r_4(r_4 - r_1)} \right).$$

From this, we obtain

$$\|d f_{a,b}\|_A \le \sqrt{\frac{r_4^2 - r_2^2}{\pi}}\left(\frac{1}{r_2 - r_1} + \frac{r_1}{r_4(r_4 - r_1)} \right).$$

Combining the bounds for $\sup_A |f_{a,b}|$ and $\|d f_{a,b}\|_A$ yields the lemma. $\qquad\square$

From now on, we impose the normalisation condition

$$\int_X \tilde{g}_{a,b}\mu = 0$$

on $\tilde{g}_{a,b}$ for all $a, b \in U^{r_1}$; this can be attained by adding a suitable constant to $\tilde{g}_{a,b}$.
Then for all $a, b \in U^{r_1}$, the function $g_{a,b}$ defined earlier is equal to

$$g_{a,b} = \tilde{g}_{a,b} + \chi f_{a,b} - \int_X \chi f_{a,b}\mu. \tag{3}$$

Indeed, by the definition of $\tilde{g}_{a,b}$, the right-hand side satisfies (1). Furthermore, for
all $a \in U^{r_1}$, we define a smooth function l_a on $X \setminus \{a\}$ by

$$l_a = \begin{cases} (\chi/2\pi)\log|z - z(a)| & \text{on } U, \\ 0 & \text{on } X \setminus \overline{U}; \end{cases}$$

this is bounded from above by $(1/2\pi)\log(r_4 + r_1)$.

Lemma A.4. *For all $a, b \in U^{r_1}$, we have*

$$\max_X |g_{a,b} - l_a + l_b| < c_4(r_1, r_2, r_4, \lambda, c_1),$$

where

$$c_4(r_1, r_2, r_4, \lambda, c_1) = c_3(r_1, r_2, r_4, \lambda) + \frac{1}{2\pi}\log\frac{r_4 + r_1}{r_4 - r_1} + \left(\tfrac{8}{3}\log 2 - \tfrac{1}{4}\right)\frac{c_1}{r_4^2}.$$

Proof. By (3) and the definitions of $f_{a,b}$ and l_a, we get

$$g_{a,b} - l_a + l_b = \tilde{g}_{a,b} - \int_X \chi f_{a,b}\mu + \frac{\chi}{2\pi}\log\left|\frac{\overline{z(a)}z - r_4^2}{\overline{z(b)}z - r_4^2}\right|,$$

where the last term is extended to zero outside U. We bound each of the terms on
the right-hand side. From $\int_X \tilde{g}_{a,b}\mu = 0$ and the nonnegativity of μ, it follows that

$$\max_X \tilde{g}_{a,b} \geq 0 \geq \min_X \tilde{g}_{a,b}.$$

Together with the bound for $\max_X \tilde{g}_{a,b} - \min_X \tilde{g}_{a,b}$ from Lemma A.3, this implies

$$\max_X |\tilde{g}_{a,b}| \leq c_3(r_1, r_2, r_4, \lambda, c_1).$$

Because the support of χ is contained in U^{r_4}, the hypothesis (4) of Definition 3.1.1
together with the definition of $f_{a,b}$ gives

$$\int_X \chi f_{a,b}\mu$$

$$= \int_{U^{r_4}} \frac{\chi}{2\pi}\left(\log\left|\frac{z - z(a)}{r_4}\right| + \log\left|\frac{\overline{z(a)}z}{r_4^2} - 1\right| - \log\left|\frac{z - z(b)}{r_4}\right| - \log\left|\frac{\overline{z(b)}z}{r_4^2} - 1\right|\right)\mu.$$

Writing $w = z/r_4$ and $t = z(a)/r_4$, we have

$$\int_{U^{r_4}} \frac{\chi}{2\pi} \log\left|\frac{z - z(a)}{r_4}\right| \mu \leq \frac{c_1}{2\pi r_4^2} \int_{\substack{|w|<1 \\ |w-t|>1}} \log|w - t|\, i\, dw \wedge d\bar{w}.$$

We note that t satisfies $|t| < r_1/r_4$; for simplicity, we relax this to $|t| \leq 1$. Then it is easy to see that the above expression attains its maximum for $|t| = 1$; by rotational symmetry, we can take $t = 1$. We now have to integrate over the crescent-shaped domain $\{w \in \mathbb{C} : |w| < 1 \text{ and } |w - 1| > 1\}$, which is contained in $\{1 + r \exp(i\phi) : 1 < r < 2,\ 2\pi/3 < \phi < 4\pi/3\}$. We get

$$\int_{U^{r_4}} \frac{\chi}{2\pi} \log\left|\frac{z - z(a)}{r_4}\right| \mu < \frac{c_1}{\pi} \int_{2\pi/3}^{4\pi/3} \int_1^2 \log(r)\, r\, dr\, d\phi$$
$$= \left(\tfrac{4}{3}\log 2 - \tfrac{1}{2}\right)c_1.$$

In a similar way, we obtain

$$\int_{U^{r_4}} \frac{\chi}{2\pi} \log\left|\frac{z - z(a)}{r_4}\right| \mu \geq -\frac{c_1}{2r_4^2},$$

$$\int_{U^{r_4}} \frac{\chi}{2\pi} \log\left|\frac{\overline{z(a)}z}{r_4^2} - 1\right| \mu < \left(\tfrac{4}{3}\log 2 - \tfrac{1}{2}\right)\frac{c_1}{r_4^2},$$

$$\int_{U^{r_4}} \frac{\chi}{2\pi} \log\left|\frac{\overline{z(a)}z}{r_4^2} - 1\right| \mu \geq -\frac{c_1}{4r_4^2}.$$

The same bounds hold for b. Combining everything, we get

$$\left|\int_X \chi f_{a,b}\mu\right| \leq \left(\tfrac{8}{3}\log 2 - \tfrac{1}{4}\right)\frac{c_1}{r_4^2}.$$

Finally, we have

$$\max_X \frac{\chi}{2\pi} \log\left|\frac{\overline{z(a)}z - r_4^2}{\overline{z(b)}z - r_4^2}\right| \leq \frac{1}{2\pi} \sup_{U^{r_4}} \log\left|\frac{r_4 - \overline{z(a)}z/r_4}{r_4 - \overline{z(b)}z/r_4}\right|$$
$$\leq \frac{1}{2\pi} \log\frac{r_4 + r_1}{r_4 - r_1},$$

which finishes the proof.   □

We will now apply Lemma A.4, which holds for any chart (U, z) satisfying the hypotheses (1) and (4) of Definition 3.1.1, to our atlas $\{(U_j, z_j) : 1 \leq j \leq n\}$. Besides including the index j in the notation for the coordinates, we denote by $l_a^{(j)}$ and $\chi^{(j)}$ the functions l_a and χ defined for the coordinate (U_j, z_j). We obtain the following generalisation of Lemma A.4 to the situation where a and b are arbitrary points of X:

Lemma A.5. *For all $a, b \in X$ and all j and k such that $a \in U_j^{r_1}$ and $b \in U_k^{r_1}$,*

$$\sup_X \left| g_{a,b} - l_a^{(j)} + l_b^{(k)} \right| \leq c_5(r_1, r_2, r_4, \lambda, n, c_1, M),$$

where

$$c_5(r_1, r_2, r_4, \lambda, c_1, n, M) = nc_4(r_1, r_2, r_4, \lambda, c_1) + \frac{n-1}{2\pi} \log \left(M \frac{r_4 + r_1}{r_2 - r_1} \right).$$

Proof. We first show that for any two coordinate indices j and k and for all $a \in U_k^{r_1} \cap U_j^{r_1}$,

$$\sup_X \left| l_a^{(k)} - l_a^{(j)} \right| \leq \frac{1}{2\pi} \log \left(M \frac{r_4 + r_1}{r_2 - r_1} \right). \tag{4}$$

To prove this, let $y \in X$. We distinguish three cases to prove that $l_a^{(k)}(y) - l_a^{(j)}(y)$ is bounded from above by the right-hand side of (4); the inequality then follows by interchanging j and k.

Case 1. Suppose $y \in U_j$ with $|z_j(y) - z_j(a)| < (r_2 - r_1)/M$. In this case, we have

$$|z_j(y)| < |z_j(a)| + \frac{r_2 - r_1}{M} < r_2;$$

hence, $a, y \in U_j^{r_2}$. Let $[a, y]^j$ denote the line segment between a and y in the z_j-coordinate, i.e., the curve in $U_j^{r_2}$ whose z_j-coordinate is parametrised by

$$\hat{z}_j(t) = (1-t)z_j(a) + tz_j(y) \quad \text{for } 0 \leq t \leq 1.$$

We claim that this line segment also lies inside $U_k^{r_2}$. Suppose this is not the case; then, because the "starting point" $z_j^{-1}(\hat{z}_j(0)) = a$ does lie in $U_k^{r_2}$, there exists a smallest $t \in (0, 1)$ for which the point

$$y' = z_j^{-1}(\hat{z}_j(t)) \in U_j^{r_2}$$

lies on the boundary of $U_k^{r_2}$. It follows from hypothesis (3) of Definition 3.1.1 that

$$|z_k(y') - z_k(a)| \leq M |z_j(y') - z_j(a)|.$$

On the other hand,

$$|z_j(y') - z_j(a)| = t |z_j(y) - z_j(a)|$$
$$< (r_2 - r_1)/M$$

by assumption, and

$$|z_k(y') - z_k(a)| > r_2 - r_1$$

by the triangle inequality. This implies

$$|z_k(y') - z_k(a)| > M |z_j(y') - z_j(a)|,$$

a contradiction. Therefore, the line segment $[a, y]^j$ lies inside $U_j^{r_2} \cap U_k^{r_2}$. By hypothesis (3) of Definition 3.1.1, we have

$$|z_k(y) - z_k(a)| \le M|z_j(y) - z_j(a)|.$$

Because $\chi^{(j)}(y) = \chi^{(k)}(y) = 1$, we find

$$l_a^{(k)}(y) - l_a^{(j)}(y) = \frac{1}{2\pi} \log \left| \frac{z_k(y) - z_k(a)}{z_j(y) - z_j(a)} \right| \le \frac{1}{2\pi} \log M,$$

which is bounded by the right-hand side of (4).

Case 2. Suppose $y \notin U_j$. Then $l_a^{(j)}(y) = 0$, and thus,

$$l_a^{(k)}(y) - l_a^{(j)}(y) = l_a^{(k)}(y) \le \frac{\log(r_4 + r_1)}{2\pi}.$$

Case 3. Suppose $y \in U_j$ and $|z_j(y) - z_j(a)| \ge (r_2 - r_1)/M$. Then

$$l_a^{(k)}(y) - l_a^{(j)}(y) \le \frac{\log(r_4 + r_1)}{2\pi} - \frac{\chi^{(j)}(y)}{2\pi} \log \frac{r_2 - r_1}{M},$$

which is also bounded by the right-hand side in (4).

By hypothesis (2) of Definition 3.1.1, the open sets $U_j^{r_1}$ cover X. Furthermore, X is connected. For arbitrary $a, b \in X$ and indices j and k such that $a \in U_j^{r_1}$ and $b \in U_k^{r_1}$, we can therefore choose a finite sequence of indices $j = j_1, j_2, \ldots, j_m = k$ with $m \le n$ and points $a = a_0, a_1, \ldots, a_m = b$ such that $a_i \in U_{j_i}^{r_1} \cap U_{j_{i+1}}^{r_1}$ for $1 \le i \le m - 1$. Using $g_{a,b} = \sum_{i=1}^{m} g_{a_{i-1}, a_i}$, we get

$$\sup_X \left| g_{a,b} - l_a^{(j)} + l_b^{(k)} \right| = \sup_X \left| \sum_{i=1}^{m} \left(g_{a_{i-1}, a_i} - l_{a_{i-1}}^{(j_i)} + l_{a_i}^{(j_i)} \right) + \sum_{i=1}^{m-1} \left(l_{a_i}^{(j_{i+1})} - l_{a_i}^{(j_i)} \right) \right|$$

$$\le \sum_{i=1}^{m} \sup_X \left| g_{a_{i-1}, a_i} - l_{a_{i-1}}^{(j_i)} + l_{a_i}^{(j_i)} \right| + \sum_{i=1}^{m-1} \sup_X \left| l_{a_i}^{(j_{i+1})} - l_{a_i}^{(j_i)} \right|.$$

The lemma now follows from Lemma A.4 and the inequality (4). $\qquad\square$

Proof of Theorem 3.1.2. We choose a continuous partition of unity $\{\phi^j\}_{j=1}^n$ subordinate to the covering $\{U_j^{r_1}\}_{j=1}^n$. Let $a \in X$, and let j be an index such that $a \in U_j^{r_1}$. By the definition of $g_{a,\mu}$, we have

$$g_{a,\mu}(x) - l_a^{(j)}(x) = \int_{b \in X} g_{a,b}(x)\mu(b) - l_a^{(j)}(x)$$

$$= \sum_{k=1}^{n} \int_{b \in U_k^{r_1}} \phi^k(b) \left(g_{a,b}(x) - l_a^{(j)}(x) \right) \mu(b)$$

$$= \sum_{k=1}^{n} \int_{b \in U_k^{r_1}} \phi^k(b)\big(g_{a,b}(x) - l_a^{(j)}(x) + l_b^{(k)}(x)\big)\mu(b) - \sum_{k=1}^{n} \int_{b \in U_k^{r_1}} \phi^k(b)l_b^{(k)}(x)\mu(b).$$

In a similar way to in the proof of Lemma A.4, one can check that, for every index k and all $x \in X$, we have

$$-\frac{c_1}{2} \leq \int_{b \in U_k^{r_1}} \phi^k(b)l_b^{(k)}(x)\mu(b) \leq \big(\tfrac{4}{3}\log 2 - \tfrac{1}{2}\big)c_1$$

so that

$$\sup_{x \in X}\left| \int_{b \in U_k^{r_1}} \phi^k(b)l_b^{(k)}(x)\mu(b) \right| \leq \frac{c_1}{2}.$$

Together with Lemma A.5, this gives the inequality

$$\sup_X\big|g_{a,\mu} - l_a^{(j)}\big| \leq c_5(r_1, r_2, r_4, \lambda, c_1, n, M) \sum_{j=1}^{n} \int_{b \in U_j^{r_1}} \phi^j(b)\mu(b) + \sum_{j=1}^{n} \frac{c_1}{2}$$

$$= c_5(r_1, r_2, r_4, \lambda, c_1, n, M) + \frac{nc_1}{2}.$$

We also have

$$\sup_X g_{a,\mu} \leq \sup_X\big(g_{a,\mu} - l_a^{(j)}\big) + \sup_X l_a^{(j)} \leq \sup_X\big(g_{a,\mu} - l_a^{(j)}\big) + \frac{\log(r_4 + r_1)}{2\pi}.$$

By varying the choice of r_4 and $\tilde{\chi}$, we can let r_4 tend to 1 and λ to $\frac{1}{1-r_2}$. This leads to

$$c_3\left(r_1, r_2, 1, \frac{1}{1-r_2}\right) = 4\sqrt{\frac{1+r_2}{1-r_2}} \left(\frac{1}{2(1-r_2)} \log \frac{(r_1+1)^2}{(r_2-r_1)(1-r_1)} + \frac{1}{r_2-r_1} + \frac{r_1}{1-r_1}\right)$$

$$+ \frac{2}{\pi} \log \frac{(r_1+1)^2}{(r_2-r_1)(1-r_1)},$$

which implies successively

$$c_4\left(r_1, r_2, 1, \frac{1}{1-r_2}, c_1\right) = c_3\left(r_1, r_2, 1, \frac{1}{1-r_2}\right) + \frac{1}{2\pi} \log \frac{1+r_1}{1-r_1} + \big(\tfrac{8}{3}\log 2 - \tfrac{1}{4}\big)c_1,$$

$$c_5 = nc_4\left(r_1, r_2, r_4, \frac{1}{1-r_2}, c_1\right) + \frac{n-1}{2\pi} \log\left(M\frac{1+r_1}{r_2-r_1}\right).$$

We take $r_2 = 0.39 + 0.61r_1$. Then, for $r_1 > \frac{1}{2}$, one can check numerically that

$$c_5 \leq 52.4\frac{n}{(1-r_1)^{3/2}} \log \frac{1}{1-r_1} + 1.60nc_1 + \frac{n-1}{2\pi} \log M.$$

From this, the theorem follows. $\qquad\square$

Acknowledgements

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