(φ, Γ)-modules over noncommutative
overconvergent and Robba rings

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We construct noncommutative multidimensional versions of overconvergent power series rings and Robba rings. We show that the category of étale (φ, Γ)-modules over certain completions of these rings is equivalent to the category of étale (φ, Γ)-modules over classical overconvergent or Robba rings as the case may be (hence also to the category of p-adic Galois representations of $\mathbb{Q}_p$). In the case of Robba rings, the assumption of étaleness is not necessary, so there exists a notion of trianguline objects in this sense.

1. Introduction

In recent years it has become increasingly clear that some kind of p-adic version of the local Langlands correspondence should exist. In fact, Colmez [2010c; 2010b] constructed such a correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. His construction is done in several steps using (φ, Γ)-modules (the category of which is well known [Fontaine 1990] to be equivalent to the category of p-adic Galois representations of $\mathbb{Q}_p$). We briefly recall Colmez's correspondence here. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers o_K and uniformizer p_K.

The "Montreal functor" associates to a smooth o_K-torsion representation of the standard Borel subgroup $B_2(\mathbb{Q}_p)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ an o_K-torsion (φ, Γ)-module over Fontaine's ring $\mathbb{O}_{\mathscr{E}}$. If we are given a unitary Banach space representation Π over the field K of the group $\mathrm{GL}_2(\mathbb{Q}_p)$, then it admits an o_K-lattice $L(\Pi)$ that is invariant under $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence $L(\Pi)/p_K^r$ is a smooth o_K-torsion representation that we restrict now to $B_2(\mathbb{Q}_p)$. The (φ, Γ)-module associated to Π is the projective limit (as $r \to \infty$) of the (φ, Γ)-modules associated to $L(\Pi)/p_K^r$ via the Montreal functor. This is generalized in [Schneider and Vignéras 2011] to general reductive groups over $\mathbb{Q}_p$.

The reverse direction, how one adjoins a unitary continuous p-adic representation to a 2-dimensional (φ, Γ)-module D over Fontaine's ring, is even more subtle. One first constructs a unitary p-adic Banach space representation $\Pi(D)$ to each

2-dimensional *trianguline* (φ, Γ)-module D over $\mathscr{E} = \mathbb{O}_\mathscr{E}[p^{-1}]$ using a kind of parabolic induction. This Banach space is well described as a quotient of the space of p-adic functions satisfying certain properties by a certain $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant subspace (see [Colmez 2010a; Breuil 2004] for details); however, a priori it is not clear whether or not it is nontrivial. On the other hand, there is a general construction of a (somewhat bigger) $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $D \boxtimes_\delta \mathbb{P}^1$ that is in fact the space of global sections of a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant sheaf $U \mapsto D \boxtimes_\delta U$ ($U \subseteq \mathbb{P}^1$ open) on the projective space $\mathbb{P}^1(\mathbb{Q}_p) \cong \mathrm{GL}_2(\mathbb{Q}_p)/B_2(\mathbb{Q}_p)$ for any (not necessarily 2-dimensional) (φ, Γ)-module D and any unitary character $\delta\colon \mathbb{Q}_p^\times \to o_K^\times$. This sheaf has the following properties: (i) the center of $\mathrm{GL}_2(\mathbb{Q}_p)$ acts via δ on $D \boxtimes_\delta \mathbb{P}^1$; (ii) we have $D \boxtimes_\delta \mathbb{Z}_p \cong D$ as a module over the monoid

$$\begin{pmatrix} \mathbb{Z}_p \backslash \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

(where we regard $\mathbb{Z}_p$ as an open subspace in $\mathbb{P}^1 = \mathbb{Q}_p \cup \{\infty\}$). (See [Schneider et al. 2012] for a generalization of this construction to general reductive groups.) Then Colmez shows that in case D is 2-dimensional and trianguline, there exists a unitary character δ (namely $\delta = \chi^{-1} \det D$, where χ is the cyclotomic character and $\det D$ is the character associated to the 1-dimensional (φ, Γ)-module $\bigwedge^2 D$ via Fontaine's equivalence composed with class field theory) such that a certain subspace $D^\natural \boxtimes_\delta \mathbb{P}^1$ (for the definition see [Colmez 2010b]) of $D \boxtimes_\delta \mathbb{P}^1$ is isomorphic to the dual of the Banach space representation $\Pi(\check{D})$ associated earlier to the dual (φ, Γ)-module $\check{D}$ — therefore showing in particular that the previous construction is nonzero. This subspace makes sense also when D is not trianguline (nor of rank 2), but a priori only known to be $B_2(\mathbb{Q}_p)$-invariant. Also, whenever D is indecomposable and 2-dimensional, then the above δ is unique [Paškūnas 2013], and whenever D is absolutely irreducible and at least 3-dimensional, then there does not exist such a character δ (so that the subspace $D^\natural \boxtimes_\delta \mathbb{P}^1$ is $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant) [Paškūnas 2013]. Since the construction of $D \mapsto D^\natural \boxtimes_\delta \mathbb{P}^1$ behaves well in families (see Chapter II in [Colmez 2010c]) and the trianguline Galois representations are Zariski-dense in the deformation space of 2-dimensional (φ, Γ)-modules with given reduction mod p [Kisin 2010], Colmez [2010c] shows that this subspace is not only $B_2(\mathbb{Q}_p)$, but also $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant for general 2-dimensional (φ, Γ)-modules. For $\delta = \chi^{-1} \det D$ (in this case we omit the subscript δ from the notation), we have a short exact sequence

$$0 \to \Pi(\check{D}) \to D \boxtimes \mathbb{P}^1 \to \Pi(D) \to 0,$$

where $\Pi(D)$ is the unitary Banach-space representation associated to D via the p-adic Langlands correspondence.

Colmez [2010c, Chapters V and VI] also identifies the space $\Pi(D)^{\mathrm{an}}$ of locally analytic and the space $\Pi(D)^{\mathrm{alg}}$ of locally algebraic vectors in the Banach-space representation $\Pi(D)$. These play a crucial role in the proof of the compatibility of the p-adic and classical local Langlands correspondences. In fact, we have $\Pi(D)^{\mathrm{an}} = (D^{\dagger} \boxtimes \mathbb{P}^1)/K \cdot (D^{\natural} \boxtimes \mathbb{P}^1)$, where $D^{\dagger} \boxtimes \mathbb{P}^1$ is the subspace of elements $x \in D \boxtimes \mathbb{P}^1$ such that both $\mathrm{Res}_{\mathbb{Z}_p}^{\mathbb{P}^1}(x)$ and

$$\mathrm{Res}_{\mathbb{Z}_p}^{\mathbb{P}^1}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x\right)$$

lie in the subspace of overconvergent elements $D^{\dagger} \subset D \cong D \boxtimes \mathbb{Z}_p$. $D^{\dagger}$ is an étale (φ, Γ)-module over the ring $\mathscr{E}^{\dagger}$ of overconvergent power series with coefficients in K such that $D \cong \mathscr{E} \otimes_{\mathscr{E}^{\dagger}} D^{\dagger}$ [Cherbonnier and Colmez 1998].

Let now G be the group of $\mathbb{Q}_p$-points of a connected $\mathbb{Q}_p$-split reductive group and let $P = TN$ be a Borel subgroup of G. Further denote by Φ^+ the set of positive roots with respect to P and $\Delta \subset \Phi^+$ the set of simple roots. The above noted generalizations of Colmez's work [Schneider and Vignéras 2011; Schneider et al. 2012] both use a certain microlocalization $\Lambda_{\ell}(N_0)$ (constructed originally in [Schneider and Venjakob 2010]) of the Iwasawa algebra $\Lambda(N_0)$ of a compact open subgroup N_0 of N. This can be thought of as the noncommutative analogue of Fontaine's ring $\mathbb{O}_{\mathscr{E}}$. On the other hand, Colmez's p-adic Langlands correspondence heavily relies on the theory of trianguline (φ, Γ)-modules. A (φ, Γ)-module over the Robba ring is a free module $D_{\mathrm{rig}}^{\dagger}$ over $\mathscr{R}$ together with commuting semilinear actions of the operator φ and the group Γ such that φ takes a basis of the free module to another basis. Such a (φ, Γ)-module $D_{\mathrm{rig}}^{\dagger}$ is said to be étale (or of slope 0) if there is a basis of $D_{\mathrm{rig}}^{\dagger}$ such that the matrix of φ in this basis is an invertible matrix over the subring $\mathbb{O}_{\mathscr{E}}^{\dagger} \subset \mathscr{R}$ of overconvergent Laurent series. An étale (φ, Γ)-module over $\mathscr{R}$ is *trianguline* if it admits a filtration of (not necessarily étale) (φ, Γ)-modules over $\mathscr{R}$ with subquotients of rank 1 possibly after a finite base change $E \otimes_K \cdot$. The fact that the Robba ring and the ring of overconvergent Laurent series play such a role in the construction of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and also in the identification of the locally analytic vectors is the motivation for the construction of noncommutative analogues of these rings — as they will most probably be needed for a future correspondence for reductive groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$.

The motivation of this paper is twofold. On the one hand, we reinterpret the ring $\Lambda_{\ell}(N_0)$ as follows. Instead of localizing and completing the Iwasawa algebra $\Lambda(N_0)$, one may construct $\Lambda_{\ell}(N_0)$ as the projective limit of certain skew group rings over $\mathbb{O}_{\mathscr{E}}$. The only assumptions on the ring $R = \mathbb{O}_{\mathscr{E}}$ such that this new construction of $\Lambda_{\ell}(N_0)$ can be carried out are that R admits an inclusion $\chi: \mathbb{Z}_p \to R^{\times}$ of the

additive group $\mathbb{Z}_p$ into its group of invertible elements and an étale action of an operator φ that is compatible with χ. The noncommutative ring that is constructed is a completed skew group ring $R[\![H_1, \ell]\!]$ of a closed normal subgroup H_1 of a pro-p group H_0 such that $\ell \colon H_0 \twoheadrightarrow \mathbb{Z}_p$ is a homomorphism with kernel H_1 (hence $H_0/H_1 \cong \mathbb{Z}_p$). The main result in this direction is Proposition 3.1, showing that the category of φ-modules over R is equivalent to the category of φ-modules over the completed skew group ring $R[\![H_1, \ell]\!]$. This can be applied also to the ring $R = \mathbb{O}_{\mathscr{E}}^{\dagger}$ of overconvergent Laurent series with coefficients in o_K and the Robba ring $\mathscr{R}$. The other motivation (probably the more important one) is the construction of the right noncommutative analogues of $\mathbb{O}_{\mathscr{E}}^{\dagger}$ and $\mathscr{R}$. The elements of the rings $\mathscr{R}[\![H_1, \ell]\!]$ and $\mathbb{O}_{\mathscr{E}}^{\dagger}[\![H_1, \ell]\!]$, however, are not necessarily convergent in any open annulus since they are obtained by taking an inverse limit. Therefore we shall we construct the rings $\mathscr{R}(H_1, \ell)$ and $\mathscr{R}^{\mathrm{int}}(H_1, \ell)$ (pages 221 ff. and 231–232, with N_1 for H_1) as direct limits of certain microlocalizations of the distribution algebra. The elements of these rings are convergent in a region of the form

$$\{\rho_2 < |b_\alpha| < 1, \ |b_\beta| < |b_\alpha|^r \text{ for } \beta \in \Phi^+ \setminus \{\alpha\}\}$$

for some $p^{-1} < \rho_2 < 1$ and $1 \le r \in \mathbb{Z}$. We will show (pages 233 ff.) that $\mathscr{R}[\![N_1, \ell]\!]$ and $\mathbb{O}_{\mathscr{E}}^{\dagger}[\![N_1, \ell]\!]$ are in a certain sense completions of $\mathscr{R}(N_1, \ell)$ and $\mathscr{R}^{\mathrm{int}}(N_1, \ell)$, respectively. Note that, although the natural map $j_{\mathrm{int}} \colon \mathscr{R}^{\mathrm{int}}(N_1, \ell) \to \mathbb{O}_{\mathscr{E}}^{\dagger}[\![N_1, \ell]\!]$ is injective, the map $j \colon \mathscr{R}(N_1, \ell) \to \mathscr{R}[\![N_1, \ell]\!]$ is not. Both the ring $\mathscr{R}(N_1, \ell)$ and its integral version admit an étale action of the monoid $T_+ = \{t \in T \mid tN_0 t^{-1} \subseteq N_0\}$. However, it is an open question whether the categories of étale T_+-modules over these rings are equivalent to the étale T_+-modules over their completions.

In my opinion, the right noncommutative analogue of the ring $\mathscr{R}$ (resp. $\mathbb{O}_{\mathscr{E}}^{\dagger}$) is $\mathscr{R}(N_1, \ell)$ (resp. $\mathscr{R}^{\mathrm{int}}(N_1, \ell)$) in the context of $\mathbb{Q}_p$-split reductive groups G over $\mathbb{Q}_p$, as both rings admit an étale action of the monoid T_+ and their elements converge in certain polyannuli. However, it might still be useful to also consider the rings $\mathscr{R}[\![H_1, \ell]\!]$ and $\mathbb{O}_{\mathscr{E}}^{\dagger}[\![H_1, \ell]\!]$, as they can help us compare the category of usual (φ, Γ)-modules with the category of T_+-modules over $\mathscr{R}(N_1, \ell)$ (resp. over $\mathscr{R}^{\mathrm{int}}(N_1, \ell)$) using the equivalence of categories in Proposition 3.1. Only one variable is inverted in these rings, in contrast to the rings constructed in [Zábrádi 2012]. The reasons for this are the following: (i) this way $\mathscr{R}^{\mathrm{int}}(N_1, \ell)$ is a subring of $\Lambda_\ell(N_0)$; (ii) the equivalence of categories in Proposition 3.1 holds for rings in which only one variable is inverted; and (iii) all the usual (φ, Γ)-modules are overconvergent, that is, they descend to $\mathbb{O}_{\mathscr{E}}^{\dagger}$ already in one variable. However, if $\mathbb{Q}_p$ is replaced by a finite unramified extension F, then one might have to consider Lubin–Tate (φ, Γ_F)-modules (with $\Gamma_F \cong o_F^{\times}$) instead so that the monoid $\varphi^{\mathbb{N}} \Gamma_F$ is isomorphic to $o_F \setminus \{0\}$. These (φ, Γ_F)-modules are not overconvergent in general but they

might still correspond to objects over certain multivariable Robba rings (in which all the variables are inverted). For a first result in this direction see [Berger 2013]. It is plausible to expect that for general reductive groups G over F one has to invert exactly $|F : \mathbb{Q}_p|$ ($\mathbb{Q}_p$-)variables that correspond to the root subgroup $N_\alpha \cong F \cong \mathbb{Q}_p^{|F:\mathbb{Q}_p|}$ for a given simple root α.

2. Completed skew group rings

Let R be a commutative ring (with identity) with the following properties:

 (i) There exists a group homomorphism $\chi \colon \mathbb{Z}_p \hookrightarrow R^\times$.

 (ii) The ring R admits an étale action of the p-Frobenius φ that is compatible with χ. More precisely, there is an injective ring homomorphism $\varphi \colon R \hookrightarrow R$ such that $\varphi(\chi(x)) = \chi(px)$ and

$$R = \bigoplus_{i=0}^{p-1} \chi(i)\varphi(R).$$

In particular, R is free of rank p over $\varphi(R)$.

We remark first of all that one may iterate (ii) c times for any positive integer c to obtain

$$R = \bigoplus_{i=0}^{p^c-1} \chi(i)\varphi^c(R). \tag{1}$$

Indeed, by induction, we may assume that (1) holds for $c - 1$ and obtain

$$R = \bigoplus_{k=0}^{p^{c-1}-1} \chi(k)\varphi^{c-1}(R) = \bigoplus_{k=0}^{p^{c-1}-1} \chi(k)\varphi^{c-1}\left(\bigoplus_{j=0}^{p-1} \chi(j)\varphi(R)\right)$$

$$= \bigoplus_{k=0}^{p^{c-1}-1} \bigoplus_{j=0}^{p-1} \chi(k + p^{c-1}j)\varphi^c(R),$$

since φ^{c-1} takes direct sums to direct sums as it is injective. Now the claim follows from noting that any integer $0 \le i \le p^c - 1$ can be uniquely written in the form $i = k + p^{c-1}j$ with $0 \le k \le p^{c-1} - 1$ and $0 \le j \le p - 1$.

For any $x \in \mathbb{Z}_p$ we have $\chi(p^c x) = \varphi^c(\chi(x)) \in \varphi^c(R)^\times$. Hence $\chi(i)\varphi^c(R) = \chi(i + p^c x)\varphi^c(R)$ and we may replace each value of i in the formula (1) by any element in the coset $i + p^c \mathbb{Z}_p$.

Definition 2.1. We call a ring R with the above properties (i) and (ii) a φ-ring over $\mathbb{Z}_p$ or often just a φ-ring.

For example, if $K/\mathbb{Q}_p$ is a finite extension with ring of integers o and uniformizer p_K, then the Iwasawa algebra $o[\![T]\!]$ is a φ-ring with the homomorphism

$$\chi: \mathbb{Z}_p \to o[\![T]\!], \quad 1 \mapsto 1 + T$$

and Frobenius $\varphi(T) = (T + 1)^p - 1$. Similarly, with the same χ and φ, Fontaine's ring $\mathbb{O}_{\mathscr{E}}$, its field of fractions $\mathscr{E}$, the Robba ring $\mathscr{R}$ and the rings $\mathscr{E}^\dagger$, $\mathbb{O}_{\mathscr{E}}^\dagger$ of over-convergent power series are also φ-rings. (For the definitions of $\mathbb{O}_{\mathscr{E}}$ and $\mathscr{E}$ see the paragraph before Lemma 2.13, and for those of $\mathscr{R}$, $\mathbb{O}_{\mathscr{E}}^\dagger$, and $\mathscr{E}^\dagger$ see (12) and subsequent paragraph.)

Lemma 2.2. *For any positive integer c we have a ring isomorphism*

$$\varphi^c(R)[X]/\big(X^{p^c} - \chi(p^c)\big) \xrightarrow{\sim} R, \quad X \mapsto \chi(1).$$

Proof. Since the polynomial ring $\varphi^c(R)[X]$ is a free object in the category of commutative $\varphi^c(R)$-algebras, we may extend the natural inclusion homomorphism $f: \varphi^c(R) \hookrightarrow R$ given by (ii) to a ring homomorphism $\tilde{f}: \varphi^c(R)[X] \to R$ by any free choice for the value $\tilde{f}(X)$, in particular such that $\tilde{f}(X) := \chi(1) \in R$ and, of course, $\tilde{f}_{|\varphi^c(R)} := f$. We need to show that $\tilde{f}$ is surjective with kernel equal to the ideal generated by $X^{p^c} - \chi(p^c)$. Note that $\chi(p^c) = \varphi^c(\chi(1))$ lies in $\varphi^c(R)$, so the claim makes sense.

By (i), the map χ is a group homomorphism, so $\chi(r) = \chi(1)^r = \tilde{f}(X)^r = \tilde{f}(X^r)$ lies in the image of $\tilde{f}$ for any positive integer r. Hence we obtain the surjectivity from (1) by noting that $\varphi^c(R)$ also lies in the image of $\tilde{f}$.

Using again $\chi(r) = \tilde{f}(X^r)$ with the choice of $r = p^c$, we see immediately that $X^{p^c} - \chi(p^c)$ lies in the kernel of $\tilde{f}$. Moreover,

$$\varphi^c(R)[X]/\big(X^{p^c} - \chi(p^c)\big)$$

is a free module of rank p^c over $\varphi^c(R)$ with generators the classes of $\{X^r\}_{r=0}^{p^c-1}$ in the quotient. On the other hand, R is also a free module of rank p^c with generators $\{\chi(r)\}_{r=0}^{p^c-1}$, by (1), and these two sets of generators correspond to each other under the map $\tilde{f}$; hence the isomorphism. $\square$

Let H_0 be a pro-p group of finite rank (therefore a compact p-adic Lie group by Corollary 4.3 and Theorem 8.18 in [Dixon et al. 1999]) without elements of order p admitting a continuous surjective group homomorphism $\ell: H_0 \twoheadrightarrow \mathbb{Z}_p$ with kernel $H_1 := \text{Ker}(\ell)$. We further assume the following:

(A) H_0 also admits an injective group endomorphism $\varphi: H_0 \hookrightarrow H_0$ with finite cokernel and compatible with ℓ in the sense that $\ell(\varphi(h)) = \varphi(\ell(h)) = p\ell(h)$. In particular, we have $\varphi(H_1) \subseteq H_1$.

(B) $\bigcap_{n \geq 1} \varphi^n(H_0) = \{1\}$ and the subgroups $\varphi^n(H_0)$ form a system of neighborhoods of 1 in H_0.

We remark first of all that by a theorem of Serre [Dixon et al. 1999, Theorem 1.17], any finite index subgroup in H_0 is open. Hence the homomorphism φ is automatically continuous and the subgroups $\varphi^n(H_0)$ are open.

H_1 is a closed subgroup of H_0; hence it is also a pro-p group of finite rank. By assumption (B), we also have in particular that the subgroups $\varphi^n(H_1)$ form a system of open neighborhoods of 1 in H_1. The subgroups $\varphi^n(H_1)$ may not be normal in either H_1 or H_0. Hence for $k > 1$ we define the normal subgroup $H_k \lhd H_0$ as the normal subgroup of H_0 generated by $\varphi^{k-1}(H_1)$. Since H_1 is normal in H_0 we automatically have $H_k \subseteq H_1$ for any $k \geq 1$. Also, since the p-adic Lie group H_1 has a system of neighborhoods of 1 containing only characteristic subgroups, the H_k also form a system of neighborhoods of 1 in H_1. On the other hand, we have by definition that $\varphi(H_k) \subseteq H_{k+1} \subseteq H_k$ for each $k \geq 1$. In particular, we have an induced φ action on the quotient group H_0/H_k. This is, of course, no longer injective.

Since the group $\mathbb{Z}_p$ is topologically generated by one element, we may find a splitting $\iota : \mathbb{Z}_p \hookrightarrow H_0$ for the group homomorphism ℓ. We fix this splitting ι, too. Assume further that:

(C) the group homomorphism ι is φ-equivariant, that is, we have $\iota(\varphi(x)) = \varphi(\iota(x))$ for all $x \in \mathbb{Z}_p$.

We define the skew group ring $R[H_1/H_k, \ell, \iota]$ as follows. We put

$$R[H_1/H_k, \ell, \iota] := \bigoplus_{h \in H_1/H_k} Rh \qquad (2)$$

as left R-modules. Since H_1 is a normal subgroup in H_0, we also have

$$H_1/H_k \lhd H_0/H_k.$$

Therefore we obtain a conjugation action of $\mathbb{Z}_p$ on H_1/H_k given by

$$\rho : \mathbb{Z}_p \to \mathrm{Aut}(H_1/H_k), \quad z \mapsto \big(h \mapsto \iota(z)h\iota(z)^{-1}, h \in H_1/H_k\big).$$

Since H_1/H_k is a finite p-group, $\mathrm{Aut}(H_1/H_k)$ is finite and we have an integer $c_k \geq 1$ such that $p^{c_k}\mathbb{Z}_p \subseteq \mathrm{Ker}(\rho)$. The multiplication is defined so that $\varphi^{c_k}(R)$ commutes with elements h in H_1/H_k and $\chi(i)$ acts on H_1/H_k via $\iota \circ \chi^{-1}$ and conjugation. More precisely, for $r_1, r_2 \in R$ and $h_1, h_2 \in H_1/H_k$ we may write

$$r_2 = \sum_{i=0}^{p^{c_k}-1} \chi(i)\varphi^{c_k}(r_{i,2}) \qquad (3)$$

and put

$$(r_1 h_1)(r_2 h_2) := \sum_{i=0}^{p^{c_k}-1} r_1 \chi(i)\varphi^{c_k}(r_{i,2})\big((\iota(i)^{-1}h_1\iota(i))h_2\big) \in \bigoplus_{h \in H_1/H_k} Rh. \qquad (4)$$

In case $r_2 = 1$ we have $(r_1 h_1) h_2 = r_1 (h_1 h_2)$, and in case $h_1 = 1$ we have $r_1 (r_2 h_2) = (r_1 r_2) h_2$. By the choice of c_k, $\iota(p^{c_k} \mathbb{Z}_p)$ lies in the center of H_0/H_k. So we may use any set of representatives of $\mathbb{Z}_p / p^{c_k} \mathbb{Z}_p$ instead of $\{0, 1, \ldots, p^{c_k} - 1\}$ in (3) in order to compute (4). Indeed, if $i \equiv i'$ (mod p^{c_k}), then

$$\chi(i) \varphi^{c_k}(r_{i,2}) = \chi(i') \varphi^{c_k} \left(\chi\left(\frac{i - i'}{p^{c_k}} \right) r_{i,2} \right)$$

and $\iota(i)^{-1} h_1 \iota(i) = \iota(i')^{-1} h_1 \iota(i')$.

Lemma 2.3. *The multiplication* (4) *equips* $R[H_1/H_k, \ell, \iota]$ *with a ring structure.*

Proof. There exists an easy, but rather long computation showing this. However, there is another, more conceptual description of the ring $R[H_1/H_k, \ell, \iota]$ pointed out by Torsten Schoeneberg that proves this lemma without any computations. Let S be the group ring $S := \varphi^{c_k}(R)[H_1/H_k]$ and σ be the automorphism of S trivial on $\varphi^{c_k}(R)$ and acting by conjugation with $\iota(1)$ on H_1/H_k, that is, for $h \in H_1/H_k$ put $\sigma(h) := \iota(1)^{-1} h \iota(1)$. Now define the skew polynomial ring $S[X, \sigma]$ by the relation $aX = X\sigma(a)$ for $a \in S$. By the definition of σ, the subring $\varphi^{c_k}(R)$ lies in the center of $S[X, \sigma]$; therefore, so does $\chi(p^{c_k}) = \varphi^{c_k}(\chi(1)) \in \varphi^{c_k}(R)$. On the other hand, we have

$$aX^{p^{c_k}} = X^{p^{c_k}} \sigma^{p^{c_k}}(a) = X^{p^{c_k}} a$$

for all $a \in S$, since $\sigma^{p^{c_k}}$ is the conjugation by the central element $\iota(1)^{p^{c_k}} = \iota(p^{c_k})$ of H_0/H_k on H_1/H_k and is trivial by definition on $\varphi^{c_k}(R)$; hence $\sigma^{p^{c_k}} = \mathrm{id}_S$. This shows that $X^{p^{c_k}} - \chi(p^{c_k})$ is central and that

$$S[X, \sigma]\big(X^{p^{c_k}} - \chi(p^{c_k})\big) = \big(X^{p^{c_k}} - \chi(p^{c_k})\big)S[X, \sigma]$$

is a two-sided ideal in $S[X, \sigma]$. So we may form the quotient ring and compute (as left $\varphi^{c_k}(R)$-modules)

$$S[X, \sigma]/\big(X^{p^{c_k}} - \chi(p^{c_k})\big) \cong \left(\bigoplus_{r=0}^{\infty} \bigoplus_{h \in H_1/H_k} X^r \varphi^{c_k}(R)h \right) \Big/ \big(X^{p^{c_k}} - \chi(p^{c_k})\big)$$

$$\cong \bigoplus_{h \in H_1/H_k} \big(\varphi^{c_k}(R)[X]/(X^{p^{c_k}} - \chi(p^{c_k}))\big)h$$

$$\cong \bigoplus_{h \in H_1/H_k} Rh,$$

using Lemma 2.2 in the middle. On the component $h = 1$ in the above direct sum, the identification is even multiplicative as Lemma 2.2 gives an isomorphism of rings, not just $\varphi^{c_k}(R)$-modules. Hence $S[X, \sigma]/\big(X^{p^{c_k}} - \chi(p^{c_k})\big)$ contains R as a subring and the isomorphism above is an isomorphism of left R-modules. The

transport of ring structure gives back the definition (4) of multiplication on the right side. Indeed, we have

$$(r_1h_1)(r_2h_2) = \sum_{i=0}^{p^{c_k}-1} r_1h_1\chi(i)\varphi^{c_k}(r_{i,2})h_2 = \sum_{i=0}^{p^{c_k}-1} r_1h_1X^i\varphi^{c_k}(r_{i,2})h_2$$

$$= \sum_{i=0}^{p^{c_k}-1} r_1X^i\sigma^i(h_1)\varphi^{c_k}(r_{i,2})h_2 = \sum_{i=0}^{p^{c_k}-1} r_1\chi(i)\varphi^{c_k}(r_{i,2})\big((\iota(i)^{-1}h_1\iota(i))h_2\big),$$

since $\chi(i)$ corresponds to X^i under the isomorphism in Lemma 2.2. $\qquad\square$

We further have a natural action of φ on $R[H_1/H_k, \ell, \iota]$ coming from the φ-action on both R and H_1/H_k by putting $\varphi(rh) := \varphi(r)\varphi(h)$ for $r \in R$ and $h \in H_1/H_k$.

Lemma 2.4. *The map* $\varphi\colon R[H_1/H_k, \ell, \iota] \to R[H_1/H_k, \ell, \iota]$ *defined above is a ring homomorphism.*

Proof. The additivity is clear, so it suffices to check the multiplicativity. Using (4) we compute

$$\varphi((r_1h_1)(r_2h_2)) = \sum_{i=0}^{p^{c_k}-1} \varphi\big(r_1\chi(i)\varphi^{c_k}(r_{i,2})\big)\varphi\big((\iota(i)^{-1}h_1\iota(i))h_2\big)$$

$$= \sum_{i=0}^{p^{c_k}-1} \varphi(r_1)\chi(pi)\varphi^{c_k+1}(r_{i,2})\big((\iota(pi)^{-1}\varphi(h_1)\iota(pi))\varphi(h_2)\big)$$

$$= \sum_{i=0}^{p^{c_k}-1} \varphi(r_1)\varphi^{c_k+1}(r_{i,2})\varphi(h_1)\chi(pi)\varphi(h_2)$$

$$= \varphi(r_1)\varphi(h_1) \sum_{i=0}^{p^{c_k}-1} \chi(pi)\varphi^{c_k+1}(r_{i,2})\varphi(h_2)$$

$$= \varphi(r_1h_1)\varphi\left(\sum_{i=0}^{p^{c_k}-1} \chi(i)\varphi^{c_k}(r_{i,2})h_2 \right) = \varphi(r_1h_1)\varphi(r_2h_2). \quad\square$$

The map χ and the inclusion of the group H_1/H_k in the multiplicative group of $R[H_1/H_k, \ell, \iota]$ are compatible in the sense that they glue together to a φ-equivariant group homomorphism $\chi_k\colon H_0 \to R[H_1/H_k, \ell, \iota]^{\times}$ (with kernel $\operatorname{Ker}\chi_k = H_k$), making the diagram

$$\begin{array}{ccc}
\mathbb{Z}_p & \xrightarrow{\;\iota\;} & H_0 \\
{\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi_k} \\
R & \xrightarrow[\iota_{R,k}]{} & R[H_1/H_k, \ell]
\end{array} \qquad (5)$$

commutative, where $\iota_{R,k}$ is the natural inclusion of R in $R[H_1/H_k, \ell]$. Indeed, $H_0 \cong \iota(\mathbb{Z}_p) \ltimes H_1$, so we put

$$\chi_k(\iota(i)h) := \chi(i)(hH_k)$$

for $i \in \mathbb{Z}_p$, $h \in H_1$ and compute

$$\begin{aligned}
\chi_k\big(\iota(i_1)h_1\iota(i_2)h_2\big) &= \chi_k\big(\iota(i_1+i_2)\iota(i_2)^{-1}h_1\iota(i_2)h_2\big) \\
&= \chi(i_1+i_2)\big(\iota(i_2)^{-1}h_1\iota(i_2)h_2\big)H_k \\
&= \chi(i_1)(h_1H_k)\chi(i_2)(h_2H_k) = \chi_k(\iota(i_1)h_1)\chi_k(\iota(i_2)h_2),
\end{aligned}$$

showing that χ_k is indeed a group homomorphism. The commutativity of the diagram (5) is clear by definition. Moreover, χ_k is φ-equivariant, since we have

$$\begin{aligned}
\chi_k \circ \varphi(\iota(i)h) &= \chi_k(\iota(pi)\varphi(h)) = \chi(pi)\varphi(h)H_k \\
&= \varphi(\chi(i)hH_k) = \varphi \circ \chi_k(\iota(i)h).
\end{aligned}$$

Lemma 2.5. *The above definition of $R[H_1/H_k, \ell, \iota]$ does not depend on the choice of the section ι up to natural isomorphism.*

Proof. Let $\iota' \colon \mathbb{Z}_p \hookrightarrow H_0$ be another section of ℓ. The integer c_k depends on ι, but we also have another integer c'_k such that $\iota'(p^{c'_k})$ acts trivially by conjugation on H_1/H_k, that is, $\iota'(p^{c'_k})$ lies in the center of H_0/H_k. On the other hand, we may choose $m_k \geq 0$ so that $H_1^{p^{m_k}} \subseteq H_k$, since H_1/H_k is a finite p-group. From $\ell \circ \iota = \mathrm{id}_{\mathbb{Z}_p} = \ell \circ \iota'$, we see that $\iota^{-1}\iota'(\mathbb{Z}_p) \subseteq \mathrm{Ker}(\ell) = H_1$, and hence for any $x \in \mathbb{Z}_p$ we have

$$\begin{aligned}
\iota^{-1}(p^{m_k+\max(c_k,c'_k)}&x)\iota'(p^{m_k+\max(c_k,c'_k)}x) \\
&= \iota^{-1}(p^{\max(c_k,c'_k)}x)^{p^{m_k}}\iota'(p^{\max(c_k,c'_k)}x)^{p^{m_k}} \\
&= \big(\iota^{-1}(p^{\max(c_k,c'_k)}x)\iota'(p^{\max(c_k,c'_k)}x)\big)^{p^{m_k}} \in H_1^{p^{m_k}} \subseteq H_k.
\end{aligned}$$

Therefore for $m \geq m_k + \max(c_k, c'_k)$, the map $\iota'_k \colon R \hookrightarrow R[H_1/H_k, \ell, \iota]$ given by

$$\iota'_k\left(\sum_{i=0}^{p^m-1}\chi(i)\varphi^m(r_i)\right) := \sum_{i=0}^{p^m-1}\chi(i)\varphi^m(r_i)\big(\iota(i)^{-1}\iota'(i)\big) \tag{6}$$

extends to an isomorphism

$$\iota'_k \colon R[H_1/H_k, \ell, \iota'] \to R[H_1/H_k, \ell, \iota], \quad rh \mapsto \iota'_k(r)h,$$

of φ-rings. Indeed, the map ι'_k is clearly additive and bijective. We claim that it is multiplicative and φ-equivariant. We first show the latter statement and compute

$$\varphi \circ \iota'_k \left(\sum_{i=0}^{p^m-1} \chi(i)\varphi^m(r_i) \right) = \sum_{i=0}^{p^m-1} \varphi\big(\chi(i)\varphi^m(r_i)\big(\iota(i)^{-1}\iota'(i)\big)\big)$$

$$= \sum_{i=0}^{p^m-1} \chi(pi)\varphi^{m+1}(r_i)\big(\iota(pi)^{-1}\iota'(pi)\big)$$

$$= \iota'_k \left(\sum_{i=0}^{p^m-1} \chi(pi)\varphi^{m+1}(r_i) \right) = \iota'_k \circ \varphi \left(\sum_{i=0}^{p^m-1} \chi(i)\varphi^m(r_i) \right).$$

Since $m \geq \max(c_k, c'_k)$, the subring $\varphi^m(R)$ lies in the center of both $R[H_1/H_k, \ell, \iota]$ and $R[H_1/H_k, \ell, \iota']$. Therefore — in view of the associativity (Lemma 2.3) — we may compute the multiplication (4) by expanding elements of R to degree m. So we write

$$r_1 = \sum_{j=0}^{p^m-1} \chi(j)\varphi^m(r_{j,1}), \quad r_2 = \sum_{i=0}^{p^m-1} \chi(i)\varphi^m(r_{i,2}).$$

We may compute (6) using any set of representatives of $\mathbb{Z}_p/p^m\mathbb{Z}_p$ (for example $\{j, j+1, \ldots, j+p^m-1\}$ instead of $\{0, 1, \ldots, p^m-1\}$) since $\iota^{-1}\iota'(p^m\mathbb{Z}_p) \subseteq H_k$. Hence we obtain

$$\iota'_k\big((r_1h_1)(r_2h_2)\big)$$

$$= \iota'_k \left(\sum_{i=0}^{p^m-1} r_1\chi(i)\varphi^m(r_{i,2})\big(\iota'(i)^{-1}h_1\iota'(i)h_2\big) \right)$$

$$= \iota'_k \left(\sum_{i,j=0}^{p^m-1} \chi(j)\varphi^m(r_{j,1})\chi(i)\varphi^m(r_{i,2})\big(\iota'(i)^{-1}h_1\iota'(i)h_2\big) \right)$$

$$= \sum_{i=0}^{p^m-1} \iota'_k \left(\sum_{j=0}^{p^m-1} \chi(i+j)\varphi^m(r_{j,1}r_{i,2}) \right) \iota'(i)^{-1}h_1\iota'(i)h_2$$

$$= \sum_{i,j=0}^{p^m-1} \chi(i+j)\varphi^m(r_{j,1}r_{i,2})\iota(i+j)^{-1}\iota'(i+j)\iota'(i)^{-1}h_1\iota'(i)h_2$$

$$= \sum_{i,j=0}^{p^m-1} \chi(i+j)\varphi^m(r_{j,1}r_{i,2})\iota(i+j)^{-1}\iota'(j)h_1\iota'(i)h_2$$

$$= \sum_{i,j=0}^{p^m-1} \chi(j)\varphi^m(r_{j,1})\chi(i)\varphi^m(r_{i,r})\iota(i)^{-1}\big(\iota(j)^{-1}\iota'(j)h_1\big)\iota(i)\big(\iota(i)^{-1}\iota'(i)h_2\big)$$

$$= \left(\sum_{j=0}^{p^m-1} \chi(j)\varphi^m(r_{j,1})\big(\iota(j)^{-1}\iota'(j)h_1\big) \right)\left(\sum_{i=0}^{p^m-1} \chi(i)\varphi^m(r_{i,2})\big(\iota(i)^{-1}\iota'(i)h_2\big) \right)$$

$$= \left(\sum_{j=0}^{p^m-1} \varphi^m(r_{j,1})\iota'(j)h_1 \right)\left(\sum_{i=0}^{p^m-1} \varphi^m(r_{i,2})\iota'(i)h_2 \right) = \iota'_k(r_1h_1)\iota'_k(r_2h_2). \quad \square$$

In view of this lemma we omit ι from the notation from now on. This construction is compatible with the natural surjective homomorphisms $H_1/H_{k+1} \twoheadrightarrow H_1/H_k$; therefore the rings $R[H_1/H_k, \ell]$ form an inverse system for the induced maps. So we may define the completed skew group ring $R[\![H_1, \ell]\!]$ as the projective limit

$$R[\![H_1, \ell]\!] := \varprojlim_k R[H_1/H_k, \ell].$$

We denote by I_k the kernel of the canonical surjective homomorphism from $R[\![H_1, \ell]\!]$ to $R[H_1/H_k, \ell]$.

Whenever R is a topological ring, we equip $R[\![H_1, \ell]\!]$ with the projective limit topology of the product topologies on each $\bigoplus_{h \in H_1/H_k} Rh$.

The augmentation map $H_1 \twoheadrightarrow 1$ induces a ring homomorphism

$$\ell := \ell_R : R[\![H_1, \ell]\!] \twoheadrightarrow R.$$

This also has a section $\iota := \iota_R = \varprojlim \iota_{R,k} : R \hookrightarrow R[\![H_1, \ell]\!]$ (whenever clear we omit the subscript $_R$), that is, $\ell_R \circ \iota_R = \mathrm{id}_R$. By (5), the group homomorphism $\chi : \mathbb{Z}_p \to R^\times$ extends to a group homomorphism $\chi_{H_0} : H_0 \to R[\![H_1, \ell]\!]^\times$, making the diagram

$$
\begin{array}{ccc}
\mathbb{Z}_p & \xrightarrow{\ \iota\ } & H_0 \\
\chi \downarrow & & \downarrow \chi_{H_0} \\
R & \xrightarrow[\iota_R]{} & R[\![H_1, \ell]\!]
\end{array}
$$

commutative.

The operator φ acts naturally on this projective limit. If R is a topological ring and φ acts continuously on R, then φ also acts continuously on each $R[H_1/H_k, \ell]$ by taking the limit also on $R[\![H_1, \ell]\!]$. For an open subgroup H' of a profinite group H we use the notation $J(H/H')$ for a set of representatives of the left cosets of H' in H. Similarly, we use $J(H' \setminus H)$ for a set of representatives of the right cosets $H' \setminus H$.

Lemma 2.6. (a) *Let $L \leq K \leq H$ be groups. Then the set $J(H/K)J(K/L)$ (resp. $J(L \setminus K)J(K \setminus H)$) is a set of representatives for the cosets H/L (resp. for $L \setminus H$).*

(b) *Let $K \leq H$ be groups and $N \lhd H$ a* normal *subgroup. Then $J((K \cap N) \setminus N)$ is also a set of representatives for $K \setminus KN$.*

Proof. These are well known facts in group theory; however, for the convenience of the reader, we recall their proofs here. In (b) we need N to be a normal subgroup so that KN is a subgroup of H. Moreover, $J(K \setminus KN)$ might not lie in N in general.

(a) Let $h_1, h_2 \in J(H/K)$ and $k_1, k_2 \in J(K/L)$. Suppose we have $h_1 k_1 L = h_2 k_2 L$. Then we also have $h_1^{-1} h_2 \in K$ and so $h_1 = h_2$, whence $k_1^{-1} k_2 \in L$ and so $k_1 = k_2$.

So the elements of the set $J(H/K)J(K/L)$ are in distinct left cosets of L. On the other hand, if $hL \in H/L$ is a left coset, then we may first choose $h_1 \in J(H/K)$ so that $h_1^{-1}h \in K$ and then $k_1 \in J(K/L)$ so that $k_1^{-1}h_1^{-1}h \in L$; that is, $hL = h_1 k_1 L$.

(b) If $n_1 \neq n_2 \in J((K \cap N) \setminus N)$ are distinct, then $Kn_1 \neq Kn_2$, as $n_1 n_2^{-1}$ does not lie in $K \cap N$, but it lies in N. On the other hand, if $kn \in KN$, then we may find $n_1 \in J((K \cap N) \setminus N)$ such that $nn_1^{-1} \in K \cap N$, and hence $knn_1^{-1} \in K$. $\quad\square$

Proposition 2.7. *The map* $\varphi \colon R[\![H_1, \ell]\!] \to R[\![H_1, \ell]\!]$ *is injective. Also*

$$R[\![H_1, \ell]\!] = \bigoplus_{h \in J(\varphi(H_0) \setminus H_0)} \varphi(R[\![H_1, \ell]\!])h.$$

In particular, $R[\![H_1, \ell]\!]$ *is a free (left) module of rank* $[H_0 : \varphi(H_0)]$ *over itself via* φ.

Proof. Step 1. Let k be an integer and denote by A_k the kernel of the map $\varphi \colon R[H_1/H_k, \ell] \to R[H_1/H_k, \ell]$ so that we have a short exact sequence of abelian groups

$$0 \to A_k \to R[H_1/H_k, \ell] \xrightarrow{\varphi} \varphi(R[H_1/H_k, \ell]) \to 0.$$

We show that the sequence A_k satisfies the trivial Mittag-Leffler condition. From this the injectivity of φ follows, and we obtain

$$\varprojlim_k \varphi(R[H_1/H_k, \ell]) \cong \varphi(\varprojlim_k R[H_1/H_k, \ell]) = \varphi(R[\![H_1, \ell]\!]). \qquad (7)$$

Take a fixed positive integer k. Since $\varphi \colon H_1 \to H_1$ is an open map (bijective and continuous between the compact sets H_1 and $\varphi(H_1)$, and hence a homeomorphism) and the subgroups H_l form a system of neighborhoods, we find an integer $l > k$ such that $H_k \supseteq \varphi^{-1}(H_l)$. In view of Lemma 2.6 we put

$$J(H_1/H_l) := J(H_1/\varphi^{-1}(H_l))J(\varphi^{-1}(H_l)/H_l)$$

for $J(H_1/\varphi^{-1}(H_l))$ and $J(\varphi^{-1}(H_l)/H_l)$ arbitrarily fixed sets of representatives for the cosets of $H_1/\varphi^{-1}(H_l)$ and of $\varphi^{-1}(H_l)/H_l$, respectively.

Now let $\sum_{h \in J(H_1/H_l)} r_h \chi_l(h)$ be an element in A_l and denote by $f_{k,l}$ the natural

surjection from $R[H_1/H_l, \ell] \twoheadrightarrow R[H_1/H_k, \ell]$. We have

$$
\begin{aligned}
0 &= \varphi\left(\sum_{h \in J(H_1/H_l)} r_h \chi_l(h)\right) \\
&= \sum_{h_1 \in J(H_1/\varphi^{-1}(H_l))} \sum_{h_2 \in J(\varphi^{-1}(H_l)/H_l)} \varphi(r_{h_1 h_2}) \chi_l(\varphi(h_1 h_2)) \\
&= \sum_{h_1} \sum_{h_2} \varphi(r_{h_1 h_2}) \chi_l(\varphi(h_1)) \\
&= \sum_{h_1 \in J(H_1/\varphi^{-1}(H_l))} \varphi\left(\sum_{h \in J(H_1/H_l) \cap h_1 \varphi^{-1}(H_l)} r_h\right) \chi_l(\varphi(h_1)).
\end{aligned}
$$

For $h_1 \neq h_1' \in J(H_1/\varphi^{-1}(H_l))$ we have $\varphi(h_1) H_l \neq \varphi(h_1') H_l$. Since $R[H_1/H_l, \ell]$ is defined as a direct sum, we obtain

$$
\varphi\left(\sum_{h \in J(H_1/H_l) \cap h_1 \varphi^{-1}(H_l)} r_h\right) = 0, \quad \text{whence} \quad \sum_{h \in J(H_1/H_l) \cap h_1 \varphi^{-1}(H_l)} r_h = 0
$$

for any fixed $h_1 \in J(H_1/\varphi^{-1}(H_l))$, as φ is injective on R. On the other hand, we have

$$
\begin{aligned}
f_{k,l}\left(\sum_{h \in J(H_1/H_l)} r_h \chi_l(h)\right) &= \sum_{h_1 \in J(H_1/H_k)} \left(\sum_{h \in J(H_1/H_l) \cap h_1 H_k} r_h\right) \chi_k(h_1) \\
&= \sum_{h_1 \in J(H_1/H_k)} 0 \chi_k(h_1) = 0,
\end{aligned}
$$

as $h_1 H_k$ is a disjoint union of cosets of $\varphi^{-1}(H_l)$ by the choice of l. This shows that $f_{k,l}(A_l) = 0$ as claimed. Therefore (7) follows as discussed above.

Step 2. Since $\varphi(H_0) \cap H_1$ is open in H_1, there exists an integer $k_0 \geq 2$ such that for $k \geq k_0$ we have $H_k \subseteq \varphi(H_0)$. (We may not be able to take $k_0 = 2$ because H_k is the *normal* subgroup generated by $\varphi(H_1)$, which does have elements outside $\varphi(H_0)$ in general.) We claim now the decomposition

$$
R[H_1/H_k, \ell] = \bigoplus_{h \in J(\varphi(H_0) \backslash H_0)} \varphi(R[H_1/H_k, \ell]) \chi_k(h) \tag{8}
$$

for $k \geq k_0$. Since H_k is a normal subgroup of H_0 contained in $\varphi(H_0)$, the elements $\chi_k(h)$ above are distinct.

For the proof of (8) we apply Lemma 2.6(b) in the situation $K := \varphi(H_0)$, $N := H_1$, and $H := H_0$ to be able to choose

$$
J(\varphi(H_0) \backslash \varphi(H_0) H_1) := J((\varphi(H_0) \cap H_1) \backslash H_1).
$$

Also, by the injectivity of φ on H_0/H_1, we see that $\varphi(H_0) \cap H_1 = \varphi(H_1)$. On the other hand, $\iota(\{0, 1, \ldots, p-1\})$ is a set of representatives for the cosets $H_1\varphi(H_0) \setminus H_0$. Therefore (using Lemma 2.6(a) with $L := \varphi(H_0)$, $K := \varphi(H_0)H_1$, and $H := H_0$) we may choose

$$J(\varphi(H_0) \setminus H_0) := J(\varphi(H_0) \setminus \varphi(H_0)H_1)J(\varphi(H_0)H_1 \setminus H_0)$$
$$= J(\varphi(H_1) \setminus H_1)\iota(\{0, 1, \ldots, p-1\}).$$

We are going to use this specific set $J(\varphi(H_0) \setminus H_0)$ in order to compute the right side of (8). Let $\sum_{h \in J(H_1/H_k)} r_h \chi_k(h)$ be an arbitrary element in $R[H_1/H_k, \ell]$. By the étaleness of the action of φ on R (noting that R is commutative), we may uniquely decompose

$$r_h = \sum_{i=0}^{p-1} \chi(i)\varphi(r_{i,h}) = \sum_{i=0}^{p-1} \varphi(r_{i,h})\chi(i).$$

On the other hand, we write $\iota(i)h\iota(i)^{-1} = \varphi(u_{i,h})v_{i,h}$ with unique $u_{i,h} \in H_1$ and $v_{i,h} \in J(\varphi(H_1) \setminus H_1)$. Therefore we have

$$\sum_{h \in J(H_1/H_k)} r_h \chi_k(h) = \sum_{h \in J(H_1/H_k)} \sum_{i=0}^{p-1} \varphi(r_{i,h})\chi(i)\chi_k\big(\iota(i)^{-1}\varphi(u_{i,h})v_{i,h}\iota(i)\big)$$

$$= \sum_{h \in J(H_1/H_k)} \sum_{i=0}^{p-1} \varphi(r_{i,h}\chi_k(u_{i,h}))\chi_k(v_{i,h}\iota(i))$$

$$\in \sum_{h \in J(\varphi(H_0) \setminus H_0)} \varphi(R[H_1/H_k, \ell])\chi_k(h),$$

as $\chi(i) = \chi_k(\iota(i))$ and $\chi_k \circ \varphi = \varphi \circ \chi_k$, by (5).

It remains to show that the sum in (8) is indeed direct. For this we may expand any element $x_{i,h} \in R[H_1/H_k, \ell]$ as

$$x_{i,h} = \sum_{m \in J(H_1/H_k)} r_{i,h,m}\chi_k(m)$$

and compute

$$\sum_{i=0}^{p-1} \sum_{h \in J(\varphi(H_1) \setminus H_1)} \varphi(x_{i,h})\chi_k(h\iota(i))$$

$$= \sum_{i,h,m} \varphi(r_{i,h,m})\chi_k(\varphi(m)h\iota(i))$$

$$= \sum_{i,h,m} \chi(i)\varphi(r_{i,h,m})\chi_k\big(\iota(i)^{-1}\varphi(m)h\iota(i)\big)$$

$$= \sum_{i,h} \sum_{m_0 \in J(\varphi(H_1)/H_k)} \chi(i) \left(\sum_{m \in J(H_1/H_k) \cap \varphi^{-1}(m_0 H_k)} \varphi(r_{i,h,m}) \right) \chi_k \left(\iota(i)^{-1} m_0 h \iota(i) \right). \quad (9)$$

Assume now that the left side of (9) is 0. The set $J(\varphi(H_1)/H_k) J(\varphi(H_1) \setminus H_1)$ is a set of representatives of $H_k \setminus H_1$ because H_k is normal in H_1, whence $\varphi(H_1)/H_k = H_k \setminus \varphi(H_1)$. This shows that the elements $m_0 h$ are distinct in H_1/H_k on the right side of (9). The conjugation by $\iota(i)$ is an automorphism of H_1/H_k; therefore the elements $\iota(i)^{-1} m_0 h \iota(i)$ are also distinct for any fixed $i \in \{0, 1, \ldots, p-1\}$. On the other hand, by the étaleness of φ on R and by (2), we obtain

$$R[H_1/H_k, \ell] = \bigoplus_{i=0}^{p-1} \bigoplus_{h_1 \in H_1/H_k} \chi(i) \varphi(R) h.$$

Hence we have

$$\sum_{m \in J(H_1/H_k) \cap \varphi^{-1}(m_0 H_k)} \varphi(r_{i,h,m}) = 0$$

for any fixed m_0, i, and h. In particular, we also have

$$\varphi(x_{i,h}) = \sum_{m \in J(H_1/H_k)} \varphi(r_{i,h,m}) \chi_k (\varphi(m))$$

$$= \sum_{m_0 \in J(\varphi(H_1)/H_k)} \left(\sum_{m \in J(H_1/H_k) \cap \varphi^{-1}(m_0 H_k)} \varphi(r_{i,h,m}) \right) \chi_k(m_0) = 0,$$

showing that the sum in (8) is direct.

Step 3. The result follows by taking the projective limit of (8) using (7). □

Remark 2.8. This lemma also holds if we interchange left and right, that is,

$$R[\![H_1, \ell]\!] = \bigoplus_{h \in J(H_0/\varphi(H_0))} h \varphi(R[\![H_1, \ell]\!]).$$

Let S be a (not necessarily commutative) ring (with identity) with the following properties:

(i) There exists a group homomorphism $\chi: H_0 \hookrightarrow S^\times$.

(ii) The ring S admits an étale action of the p-Frobenius φ that is compatible with χ. More precisely, there is an injective ring homomorphism $\varphi: S \hookrightarrow S$ such that $\varphi(\chi(x)) = \chi(\varphi(x))$ and

$$S = \bigoplus_{h \in \varphi(H_0) \setminus H_0} \varphi(S) \chi(h) = \bigoplus_{h \in H_0/\varphi(H_0)} \chi(h) \varphi(S).$$

In particular, S is free of rank $|H_0 : \varphi(H_0)|$ as a left as well as a right module over $\varphi(S)$.

Definition 2.9. We call a ring S with the properties (i) and (ii) a φ-ring over H_0.

Corollary 2.10. *The map $R \mapsto R[\![N_1, \ell]\!]$ is a functor from the category of φ-rings over $\mathbb{Z}_p$ to the category of φ-rings over H_0.*

Remark 2.11. We have $\varphi(I_k) \subseteq I_{k+1}$ for all $k \geq 1$.

Proof. Take $x \in I_k$ and write $x + I_{k+1} \in R[H_1/H_{k+1}, \ell]$, as

$$x + I_{k+1} = \sum_{h \in J(H_1/H_{k+1})} r_h \chi_{k+1}(h).$$

Since $x \in I_k$, we have

$$0 = \sum_{h \in J(H_1/H_{k+1})} r_h \chi_k(h) = \sum_{h_1 \in J(H_1/H_k)} \sum_{h \in J(H_1/H_{k+1}) \cap h_1 H_k} r_h \chi_k(h_1),$$

and hence $\sum_{h \in J(H_1/H_{k+1}) \cap h_1 H_k} r_h = 0$ for any fixed $h_1 \in J(H_1/H_k)$. So we compute

$$\begin{aligned}
\varphi(x) + I_{k+1} &= \sum_{h_1 \in J(H_1/H_k)} \sum_{h \in J(H_1/H_{k+1}) \cap h_1 H_k} \varphi(r_h) \varphi(\chi_k(h)) \\
&= \sum_{h_1 \in J(H_1/H_k)} \sum_{h \in J(H_1/H_{k+1}) \cap h_1 H_k} \varphi(r_h) \chi_k(\varphi(h_1)) \\
&= \sum_{h_1 \in J(H_1/H_k)} 0 \chi_k(\varphi(h_1)) = 0,
\end{aligned}$$

since $\varphi(H_k) \subseteq H_{k+1}$, whence $\varphi(h_1) = \varphi(h)$ above. $\qquad\square$

Recall that Fontaine's ring $\mathbb{O}_{\mathscr{E}} := \varprojlim_n (o[\![T]\!][T^{-1}])/p_K^n$ is defined as the p-adic completion of the ring of formal Laurent series over o. It is a complete discrete valuation ring with maximal ideal $p_K \mathbb{O}_{\mathscr{E}}$, residue field $k(\!(T)\!)$, and field of fractions $\mathscr{E} = \mathbb{O}_{\mathscr{E}}[p_K^{-1}]$. We show that the completed skew group ring $\mathbb{O}_{\mathscr{E}}[\![H_1, \ell]\!]$ is isomorphic to the previously constructed microlocalized ring $\Lambda_\ell(H_0)$ of the Iwasawa algebra $\Lambda(H_0)$ ([Schneider and Venjakob 2010]; see also [Schneider and Vignéras 2011, Section 8; Schneider et al. 2012; Zábrádi 2011]). ($H_0 = N_0$ in the notations of [Schneider and Vignéras 2011; Schneider et al. 2012; Zábrádi 2011].) For the convenience of the reader we recall the definition here. Let $\Lambda(H_0) := o[\![H_0]\!]$ be the Iwasawa algebra of the pro-p group H_0. It is shown in [Coates et al. 2005] that $S := \Lambda(H_0) \setminus (p_K, H_1 - 1)$ is a left and right Ore set in $\Lambda(H_0)$ so that the localization $\Lambda(H_0)_S$ exists. The ring $\Lambda_\ell(H_0)$ is defined as the $(p_K, H_1 - 1)$-adic completion of $\Lambda(H_0)_S$ (the so-called "microlocalization"). Since $\varphi: H_0 \to H_0$ is a continuous group homomorphism, it induces a continuous ring homomorphism $\varphi: \Lambda(H_0) \to \Lambda(H_0)$ of the Iwasawa algebra. Since $\varphi(S) \subset S$, φ extends to a

ring homomorphism $\varphi \colon \Lambda(H_0)_S \to \Lambda(H_0)_S$ and, by continuity, to its completion $\Lambda_\ell(H_0)$ (see Section 8 of [Schneider and Vignéras 2011] for more details).

Remark 2.12. Let R be a φ-ring containing (as a φ-subring) the Iwasawa algebra $o[\![T]\!] \cong \Lambda(\mathbb{Z}_p)$. Then using (1) we compute

$$
\begin{aligned}
R[H_1/H_k, \ell] &\cong \big(R \otimes_{\Lambda(\mathbb{Z}_p)} \Lambda(\mathbb{Z}_p)\big)[H_1/H_k, \ell] \cong R \otimes_{\Lambda(\mathbb{Z}_p)} \big(\Lambda(\mathbb{Z}_p)[H_1/H_k, \ell]\big) \\
&\cong R \otimes_{\Lambda(\mathbb{Z}_p), \iota} \Lambda(H_0/H_k) \\
&\cong \big(\varphi^{c_k}(R) \otimes_{\varphi^{c_k}(\Lambda(\mathbb{Z}_p))} \Lambda(\mathbb{Z}_p)\big) \otimes_{\Lambda(\mathbb{Z}_p), \iota} \Lambda(H_0/H_k) \\
&\cong \varphi^{c_k}(R) \otimes_{\varphi^{c_k}(\Lambda(\mathbb{Z}_p)), \iota} \Lambda(H_0/H_k),
\end{aligned}
$$

for any $k \geq 1$.

Lemma 2.13. *We have a φ-equivariant ring-isomorphism $\mathbb{O}_{\mathscr{E}}[\![H_1, \ell]\!] \cong \Lambda_\ell(H_0)$.*

Proof. The ring $\Lambda_\ell(H_0)$ is complete and Hausdorff with respect to the filtration by the ideals generated by $(H_k - 1)$, since these ideals are closed with intersection zero in the pseudocompact ring $\Lambda_\ell(H_0)$ (compare Theorem 4.7 in [Schneider and Venjakob 2010]). So it remains to show that $\Lambda_\ell(H_0/H_k)$ is naturally isomorphic to the skew group ring $\mathbb{O}_{\mathscr{E}}[H_1/H_k, \ell]$. First we show that $\Lambda(H_0/H_k) \cong \Lambda(\mathbb{Z}_p)[H_1/H_k, \ell]$. Both sides are free modules of rank $|H_1/H_k|$ over $\Lambda(\mathbb{Z}_p)$ with generators $h \in H_1/H_k$, so there is an obvious isomorphism between them as $\Lambda(\mathbb{Z}_p)$-modules. Moreover, $\varphi^{c_k}(\Lambda(\mathbb{Z}_p))$ lies in the center of both rings. However, the obvious map above is also multiplicative, since the multiplication on $\Lambda(\mathbb{Z}_p)[H_1/H_k, \ell]$ is uniquely determined by (4), so that (5) is satisfied and $\varphi^{c_k}(\Lambda(\mathbb{Z}_p))$ lies in the center.

Now by Remark 2.12, we have

$$
\mathbb{O}_{\mathscr{E}}[H_1/H_k, \ell] \cong \varphi^{c_k}(\mathbb{O}_{\mathscr{E}}) \otimes_{\varphi^{c_k}(\Lambda(\mathbb{Z}_p)), \iota} \Lambda(H_0/H_k)
$$

for any $k \geq 1$.

Since $\iota(\varphi^{p^{c_k}}(\Lambda(\mathbb{Z}_p)))$ lies in the center of $\Lambda(H_0/H_k)$, the right side above is the localization of $\Lambda(H_0/H_k)$, inverting the central element $\varphi^{p^{c_k}}(T)$ and taking the p-adic completion afterwards (that is, "microlocalization" at $\varphi^{p^{c_k}}(T)$). However, in a p-adically complete ring, T is invertible if and only if $\varphi^{p^{c_k}}(T)$ is too. Indeed, we have

$$
T \mid \varphi^{p^{c_k}}(T) = (T+1)^{p^{c_k}} - 1 = \sum_{i=1}^{p^{c_k}} \binom{p^{c_k}}{i} T^i \in T^{p^{c_k}}\big(1 + po[\![T]\!][T^{-1}]\big).
$$

Hence we obtain

$$
\varphi^{c_k}(\mathbb{O}_{\mathscr{E}}) \otimes_{\varphi^{c_k}(\Lambda(\mathbb{Z}_p)), \iota} \Lambda(H_0/H_k) \cong \Lambda_\ell(H_0/H_k),
$$

as both sides are the microlocalization of $\Lambda(H_0/H_k)$ at T. $\qquad\square$

3. Equivalence of categories

Let S be a φ-ring over any pro-p group H_0 satisfying (A), (B), and (C) (for now it would suffice to assume that S has an injective ring-endomorphism $\varphi \colon S \to S$). We define a φ-module over S to be a free S-module D of finite rank together with a semilinear action of φ such that the map

$$1 \otimes \varphi \colon S \otimes_{S, \varphi} D \to D, \quad r \otimes d \mapsto r\varphi(d), \tag{10}$$

is an isomorphism. For rings S in which p is not invertible (such as $S = \mathbb{O}_{\mathscr{E}}$ and $\mathbb{O}_{\mathscr{E}}^{\dagger}$), this is the definition of an *étale* φ-module. However, for rings in which p is invertible (such as the Robba ring $\mathscr{R}$), this is the usual definition of a φ-module. We use this definition for both $S = R$ and $S = R[\![H_1, \ell]\!]$ — the former being a φ-ring over $\mathbb{Z}_p$ and the latter being a φ-ring over H_0. We denote the category of φ-modules over R (resp. over $R[\![H_1, \ell]\!]$) by $\mathfrak{M}(R, \varphi)$ (resp. by $\mathfrak{M}(R[\![H_1, \ell]\!], \varphi)$). These are clearly additive categories. However, they are not abelian in general, as the kernel and cokernel might not be a free module over R (resp. over $R[\![H_1, \ell]\!]$).

For modules M over $R[\![H_1, \ell]\!]$, saying that (10) (with $D = M$) is an isomorphism is equivalent to saying that each element $m \in M$ is uniquely decomposed as

$$m = \sum_{u \in J(H_0 / \varphi^k(H_0))} u \varphi^k(m_{u,k})$$

for $k = 1$, or equivalently, for all $k \geq 1$.

There is an obvious functor in both directions induced by ℓ_R and ι_R that we denote by

$$\mathbb{D} := R \otimes_{R[\![H_1, \ell]\!], \ell} \cdot \colon \mathfrak{M}(R[\![H_1, \ell]\!], \varphi) \to \mathfrak{M}(R, \varphi),$$

$$\mathbb{M} := R[\![H_1, \ell]\!] \otimes_{R, \iota} \cdot \colon \mathfrak{M}(R, \varphi) \to \mathfrak{M}(R[\![H_1, \ell]\!], \varphi).$$

The following is a generalization of Theorem 8.20 in [Schneider et al. 2012]. The proof is also similar, but we include it here for the convenience of the reader.

Proposition 3.1. *The functors $\mathbb{D}$ and $\mathbb{M}$ are quasi-inverse equivalences of categories.*

Proof. We first note that since $\ell \circ \iota = \mathrm{id}_R$, we also have $\mathbb{D} \circ \mathbb{M} \cong \mathrm{id}_{\mathfrak{M}(R, \varphi)}$. So it remains to show that $\mathbb{D}$ is full and faithful.

For the faithfulness of $\mathbb{D}$, let $f \colon M_1 \to M_2$ be a morphism in $\mathfrak{M}(R[\![H_1, \ell]\!], \varphi)$ such that $\mathbb{D}(f) = 0$, which means that $f(M_1) \subseteq I_1 M_2$. Let $m \in M_1$. For any $k \in \mathbb{N}$, we write $m = \sum_{u \in J(H_0 / \varphi^k(H_0))} u \varphi^k(m_{u,k})$ and

$$f(m) = \sum_{u \in J(H_0 / \varphi^k(H_0))} u \varphi^k f(m_{u,k}) \in \varphi^k(I_1 M_2) \subseteq I_{k+1} M_2,$$

by Remark 2.11. Therefore $f(M_1) \subseteq I_{k+1}M_2$ for any $k \geq 0$, and therefore $f = 0$ since M_2 is a finitely generated free module over $R[\![H_1, \ell]\!]$, and $\bigcap_{k \geq 0} I_{k+1} = 0$ since $R[\![H_1, \ell]\!] \cong \varprojlim R[\![H_1, \ell]\!]/I_k$.

Now we prove that for any object M in $\mathfrak{M}(R[\![H_1, \ell]\!], \varphi)$ we have an isomorphism $\mathbb{M} \circ \mathbb{D}(M) \to M$. We start with an arbitrary finite $R[\![H_1, \ell]\!]$-basis $(\epsilon_i)_{1 \leq i \leq d}$ of M (where d is the rank of M). As R-modules we have

$$M = \left(\bigoplus_{1 \leq i \leq d} \iota(R)\epsilon_i \right) \oplus \left(\bigoplus_{1 \leq i \leq d} I_1\epsilon_i \right).$$

Clearly, the $R[\![H_1, \ell]\!]$-linear map from M to $\mathbb{M}(\mathbb{D}(M))$ sending ϵ_i to $1 \otimes (1 \otimes \epsilon_i)$ is bijective. It is φ-equivariant if and only if $\bigoplus_{1 \leq i \leq d} \iota(R)\epsilon_i$ is φ-stable, which is, of course, not true in general. We always have

$$\varphi(\epsilon_i) = \sum_{1 \leq j \leq d} (a_{i,j} + b_{i,j})\epsilon_j, \text{ where } a_{i,j} \in \iota(R), \ b_{i,j} \in I_1.$$

If the $b_{i,j}$ are not all 0, we will find elements $x_{i,j} \in I_1$ such that

$$\eta_i := \epsilon_i + \sum_{1 \leq j \leq d} x_{i,j}\epsilon_j$$

satisfies

$$\varphi(\eta_i) = \sum_{1 \leq j \leq d} a_{i,j}\eta_j \ \text{ for } i \in I.$$

The conditions on the matrix $X := (x_{i,j})_{1 \leq i,j \leq d}$ are

$$\varphi(\mathrm{id} + X)(A + B) = A(\mathrm{id} + X)$$

for the matrices $A := (a_{i,j})_{1 \leq i,j \leq d}$, $B := (b_{i,j})_{1 \leq i,j \leq d}$. The coefficients of A belong to the commutative ring $\iota(R)$. The matrix $A + B$ is invertible because the $R[\![H_1, \ell]\!]$-endomorphism f of M defined by

$$f(\epsilon_i) = \varphi(\epsilon_i) \text{ for } 1 \leq i \leq d$$

is an automorphism of M as M lies in $\mathfrak{M}(R[\![H_1, \ell]\!], \varphi)$. Therefore the matrix $A = \ell(A + B)$ is also invertible. We have reduced the proof to solving the equation

$$A^{-1}B + A^{-1}\varphi(X)(A + B) = X$$

in the indeterminate X. We are looking for the solution X in the form of an infinite sum

$$X = A^{-1}B + \cdots$$
$$+ \left(A^{-1}\varphi(A^{-1}) \cdots \varphi^{k-1}(A^{-1})\varphi^k(A^{-1}B)\varphi^{k-1}(A+B) \cdots \varphi(A+B)(A+B) \right)$$
$$+ \cdots.$$

The coefficients of $A^{-1}B$ belong to the two-sided ideal I_1 of $R[\![H_1, \ell]\!]$, and the coefficients of the k-th term of the series

$$\left(A^{-1}\varphi(A^{-1})\cdots\varphi^{k-1}(A^{-1})\varphi^k(A^{-1}B)\varphi^{k-1}(A+B)\cdots\varphi(A+B)(A+B)\right)$$

belong to $\varphi^k(I_1) \subseteq I_{k+1}$. Hence the series converges, since

$$R[\![H_1, \ell]\!] \cong \varprojlim_k R[\![H_1, \ell]\!]/I_k.$$

Its limit X is the unique solution of the equation. The coefficients of every term in the series belong to I_1 and I_1 is closed in $R[\![H_1, \ell]\!]$, and hence $x_{i,j} \in I_1$ for $1 \leq i$, $j \leq d$.

We still need to show that the set $(\eta_i)_{1 \leq i \leq d}$ is an $R[\![H_1, \ell]\!]$-basis of M. Similarly to the above equation, we may find a matrix Y with coefficients in I_1 such that

$$(A + B)(\mathrm{id} + Y) = \varphi(\mathrm{id} + Y)A.$$

Therefore we obtain

$$(A + B)(\mathrm{id} + Y)(\mathrm{id} + X) = \varphi((\mathrm{id} + Y)(\mathrm{id} + X))(A + B),$$

which means that the map

$$(\mathrm{id} + Y)(\mathrm{id} + X)\colon M \to M, \quad \epsilon_i \mapsto (\mathrm{id} + Y)(\mathrm{id} + X)\epsilon_i$$

is a φ-equivariant map such that $\mathbb{D}((\mathrm{id}+Y)(\mathrm{id}+X)) = \mathrm{id}$, so $(\mathrm{id} + Y)(\mathrm{id} + X) = \mathrm{id}$ by the faithfulness of $\mathbb{D}$. By a similar computation, we also obtain

$$A(\mathrm{id} + X)(\mathrm{id} + Y) = \varphi((\mathrm{id} + X)(\mathrm{id} + Y))A,$$

showing that $(\mathrm{id} + X)(\mathrm{id} + Y)$ is a φ-equivariant endomorphism of $\mathbb{M} \circ \mathbb{D}(M)$ reducing to the identity modulo I_1. Hence $(\mathrm{id} + Y)$ is a two-sided inverse to the map $(\mathrm{id} + X)$, and in particular $(\eta_i)_{1 \leq i \leq d}$ is an $R[\![H_1, \ell]\!]$-basis of M. So we obtain an isomorphism in $\mathfrak{M}(R[\![H_1, \ell]\!], \varphi)$,

$$\Theta : M \to M(\mathbb{D}(M)), \ \Theta(\eta_i) = 1 \otimes (1 \otimes \eta_i) \text{ for } 1 \leq i \leq d,$$

such that $\mathbb{D}(\Theta)$ is the identity morphism of $\mathbb{D}(M)$.

Now if $f : \mathbb{D}(M_1) \to \mathbb{D}(M_2)$, then for

$$\mathbb{M}(f) : M_1 \cong \mathbb{M} \circ \mathbb{D}(M_1) \to \mathbb{M} \circ \mathbb{D}(M_2) \cong M_2,$$

we have $\mathbb{D} \circ \mathbb{M}(f) = f$; therefore $\mathbb{D}$ is full. $\qquad\square$

Remark 3.2. There is a small mistake in Lemma 1 of [Zábrádi 2011]. The map ω is in fact not a p-valuation, since assertion (iii) states that $\omega(g^p) = \omega(g) + 1$ is false. It is only true in the weaker form $\omega(g^p) \geq \omega(g) + 1$. However, this does not influence the validity of the rest of the paper, as $N_{0,n} := \{g \in N_0 \mid \omega(g) \geq n\}$ is

still a subgroup satisfying Lemma 2. Alternatively, it is possible to modify ω so that one truly obtains a p-valuation. I would like to take this opportunity to thank Torsten Schoeneberg for pointing this out to me.

Remark 3.3. In the case of $R = \mathbb{O}_{\mathscr{C}}$ we may end the proof of Proposition 3.1 by saying that $\mathrm{id} + X$ is invertible since X lies in $I_1^{d \times d}$ and $\mathbb{O}_{\mathscr{C}}[\![H_1, \ell]\!] \cong \Lambda_\ell(H_0)$ is I_1-adically complete. However, in the general situation $R[\![H_1, \ell]\!]$ may not be complete I_1-adically. The reason for this is that the ideals $(I_k)_{k \geq 1}$ are only cofinal with the ideals I_1^k whenever R is killed by a power of p. Therefore if R is not p-adically complete, we do not have $R[\![H_1, \ell]\!] \cong \varprojlim R[\![H_1, \ell]\!]/I_1^k$ in general. In the case of $R = \mathbb{O}_{\mathscr{C}}$, Proposition 3.1 holds for not necessarily free modules as well. See [Schneider et al. 2012] for the proof of this.

Remark 3.4. The matrix Y in the proof of Proposition 3.1 is given by a convergent sum of the terms

$$-(A+B)^{-1}\varphi((A+B)^{-1})\cdots\varphi^{k-1}((A+B)^{-1})\varphi^k((A+B)^{-1}B)\varphi^{k-1}(A)\cdots\varphi(A)A$$

for $k \geq 0$, and a direct computation also shows that $(\mathrm{id} + Y)(\mathrm{id} + X) = \mathrm{id} = (\mathrm{id} + X)(\mathrm{id} + Y)$.

Reductive groups over $\mathbb{Q}_p$ and Whittaker functionals. Let p be a prime number and let $\mathbb{Q}_p \subseteq K$ be a finite extension with ring of integer o_K, uniformizer p_K, and residue field $k = o_K/p_K$. This field will only play the role of coefficients; the reductive groups will all be defined over $\mathbb{Q}_p$. Following [Schneider and Vignéras 2011], let G be the $\mathbb{Q}_p$-rational points of a $\mathbb{Q}_p$-split connected reductive group over $\mathbb{Q}_p$. In particular, G is a locally $\mathbb{Q}_p$-analytic group. We also assume that the center of G is connected. We fix a Borel subgroup $P = TN$ in G with maximal split torus T and unipotent radical N. Let Φ^+ denote, as usual, the set of positive roots of T with respect to P and let $\Delta \subseteq \Phi^+$ be the subset of simple roots. For any $\alpha \in \Phi^+$ we have the root subgroup $N_\alpha \subseteq N$. We recall that $N = \prod_{\alpha \in \Phi^+} N_\alpha$ (set-theoretically) for any total ordering of Φ^+. Let $T_0 \subseteq T$ be the maximal compact subgroup. We fix a compact open subgroup $N_0 \subseteq N$ that is totally decomposed; in other words, $N_0 = \prod_\alpha (N_0 \cap N_\alpha)$ for any total ordering of Φ^+. Hence $P_0 := T_0 N_0$ is a group. We introduce the submonoid $T_+ \subseteq T$ of all $t \in T$ such that $t N_0 t^{-1} \subseteq N_0$, or equivalently, such that $|\alpha(t)| \leq 1$ for any $\alpha \in \Delta$. Obviously, $P_+ := N_0 T_+ = P_0 T_+ P_0$ is then a submonoid of P.

We fix once and for all isomorphisms of algebraic groups

$$\iota_\alpha \colon N_\alpha \xrightarrow{\cong} \mathbb{Q}_p$$

for $\alpha \in \Delta$, such that

$$\iota_\alpha(tnt^{-1}) = \alpha(t)\iota_\alpha(n)$$

for any $n \in N_\alpha$ and $t \in T$. We normalize these isomorphisms so that $\iota_\alpha(N_0 \cap N_\alpha) = \mathbb{Z}_p \subset \mathbb{Q}_p$. Since $\prod_{\alpha \in \Delta} N_\alpha$ is naturally a quotient of $N/[N, N]$ we may view any homomorphism

$$\ell : \prod_{\alpha \in \Delta} N_\alpha \to \mathbb{Q}_p$$

as a functional on N. We fix once and for all a homomorphism ℓ such that $\ell(N_0) = \mathbb{Z}_p$. Let $X^*(T) := \mathrm{Hom}_{\mathrm{alg}}(T, \mathbb{G}_m)$ (resp. $X_*(T) := \mathrm{Hom}_{\mathrm{alg}}(\mathbb{G}_m, T)$) be the group of algebraic characters (resp. cocharacters) of T. Since we assume that the center of G is connected, the quotient $X^*(T)/\bigoplus_{\alpha \in \Delta} \mathbb{Z}\alpha$ is free. Hence we find a cocharacter ξ in $X_*(T)$ such that $\alpha \circ \xi = \mathrm{id}_{\mathbb{G}_m}$ for any α in Δ. It is injective and uniquely determined up to a central cocharacter. We fix such a ξ. It satisfies

$$\xi(\mathbb{Z}_p \setminus \{0\}) \subseteq T_+$$

and

$$\ell(\xi(a) n \xi(a^{-1})) = a\ell(n) \tag{11}$$

for any a in $\mathbb{Q}_p^\times$ and n in N, since ℓ is a linear functional on the space $\prod_{\alpha \in \Delta} N_\alpha$ and therefore can be written as a linear combination of the isomorphisms $\iota_\alpha : N_\alpha \to \mathbb{Q}_p$.

For example, if $G = \mathrm{GL}_n(\mathbb{Q}_p)$, T is the group of diagonal matrices, and N is the group of unipotent upper triangular matrices, then we could choose

$$\xi : \mathbb{G}_m(\mathbb{Q}_p) = \mathbb{Q}_p^\times \to T = (\mathbb{Q}_p^\times)^n, \quad \xi(x) := \begin{pmatrix} x^{n-1} & & & \\ & x^{n-2} & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Put $\Gamma := \xi(\mathbb{Z}_p^\times)$ and $s := \xi(p)$. The element s acts by conjugation on the group N_0 such that $\bigcap_k s^k N_0 s^{-k} = \{1\}$. We denote this action by $\varphi := \varphi_s$. This is compatible with the functional ℓ in the sense $\ell \circ \varphi = p\ell$ (see Section 2) by (11). Therefore we may apply the theory of the preceding sections to any φ-ring R with the homomorphism $\ell : N_0 \to \mathbb{Z}_p$ and $N_1 := \mathrm{Ker}(\ell_{|N_0})$. We are going to apply the theory of Section 2 in the setting $H_0 := N_0$ and $H_1 := N_1$.

In [Schneider and Vignéras 2011] and [Zábrádi 2011] ℓ is assumed to be generic— we do not assume this here, though. For any $\alpha \in \Delta$ the restriction of ℓ to a fixed N_α is either zero or an isomorphism of N_α with $\mathbb{Q}_p$, and we put $a_\alpha := \ell(\iota_\alpha^{-1}(1))$. By the assumption $\ell(N_0) = \mathbb{Z}_p$ we obtain $a_\alpha \in \mathbb{Z}_p$ for all $\alpha \in \Delta$, and $a_\alpha \in \mathbb{Z}_p^\times$ for at least one α in Δ. We put $T_{+,\ell} := \{t \in T_+ \mid t N_1 t^{-1} \subseteq N_1\}$. The monoid $T_{+,\ell}$ acts on the group $\mathbb{Z}_p$ via $\ell : N_0 \to \mathbb{Z}_p$, too.

A (φ, Γ)-ring R is by definition a φ-ring (in the sense of Section 2) together with an action of $\Gamma \cong \mathbb{Z}_p^\times$ commuting with φ and satisfying $\gamma(\chi(x)) = \chi(\xi^{-1}(\gamma)x)$. For example, $\mathbb{O}_{\mathscr{E}}, \mathbb{O}_{\mathscr{E}}^\dagger, \mathscr{E}^\dagger, \mathscr{R}$ are (φ, Γ)-rings. The endomorphism ring $\mathrm{End}(\mathbb{Z}_p)$ of the

p-adic integers (as a topological abelian group) is isomorphic to $\mathbb{Z}_p$. On the other hand, the multiplicative monoid $\mathbb{Z}_p \setminus \{0\}$ is isomorphic to $\varphi^{\mathbb{N}}\Gamma$. Now having an action of φ and Γ on R we obtain an action of $T_{+,\ell}$ on R since the map $\ell\colon N_0 \to \mathbb{Z}_p$ induces a monoid homomorphism $T_{+,\ell} \to \mathbb{Z}_p \setminus \{0\} \cong \varphi^{\mathbb{N}}\Gamma$. We denote the kernel of this monoid homomorphism by $T_{0,\ell}$. Similarly, we have a natural action of $T_{+,\ell}$ on the ring $R[\![N_1, \ell]\!]$ by conjugation. Indeed, if $t \in T_{+,\ell}$, then since T is commutative, we have

$$t\varphi^k(N_1)t^{-1} = t s^k N_1 s^{-k} t^{-1} = s^k t N_1 t^{-1} s^{-k} = \varphi^k(t N_1 t^{-1}) \subseteq \varphi^k(N_1),$$

whence $t N_k t^{-1} \subseteq N_k$. Hence t acts naturally on the skew group ring $R[N_1/N_k, \ell]$, and by taking the limit, we also obtain an action on $R[\![N_1, \ell]\!]$. We denote the map on both R and $R[\![N_1, \ell]\!]$ induced by the action of $t \in T_{+,\ell}$ by φ_t.

Now a $T_{+,\ell}$-module over R (resp. over $R[\![N_1, \ell]\!]$) is a finitely generated free R-module D (resp. $R[\![N_1, \ell]\!]$-module M) with a semilinear action of $T_{+,\ell}$ (denoted by $\varphi_t\colon D \to D$, resp. $\varphi_t\colon M \to M$ for any $t \in T_{+,\ell}$) such that the restriction of the $T_{+,\ell}$-action to $s \in T_{+,\ell}$ defines a φ-module over R (resp. over $R[\![N_1, \ell]\!]$). We denote the category of $T_{+,\ell}$-modules over R (resp. over $R[\![N_1, \ell]\!]$) by $\mathfrak{M}(R, T_{+,\ell})$ (resp. by $\mathfrak{M}(R[\![N_1, \ell]\!], T_{+,\ell})$).

Lemma 3.5. *Let M be in $\mathfrak{M}(R[\![N_1, \ell]\!], T_{+,\ell})$ and D be in $\mathfrak{M}(R, T_{+,\ell})$. Then the maps*

$$1 \otimes \varphi_t\colon R[\![N_1, \ell]\!] \otimes_{R[\![N_1,\ell]\!], \varphi_t} M \to M, \quad r \otimes m \mapsto r\varphi_t(m)$$

and

$$1 \otimes \varphi_t\colon R \otimes_{R, \varphi_t} D \to D, \quad r \otimes d \mapsto r\varphi_t(d)$$

are isomorphisms for any $t \in T_{+,\ell}$.

Proof. We only prove the statement for M (the statement for D is entirely analogous). First note that the subgroups $s^k N_0 s^{-k}$ (resp. $s^k N_1 s^{-k}$) form a system of neighborhoods of 1 in N (resp. in $\mathrm{Ker}(\ell)$). On the other hand, if t is in $T_{+,\ell}$, then

$$t \,\mathrm{Ker}(\ell_{|N}) t^{-1} = t\left(\bigcup_{k \in \mathbb{Z}} s^k N_1 s^{-k}\right) t^{-1} = \bigcup_{k \in \mathbb{Z}} s^k t N_1 t^{-1} s^{-k} = \mathrm{Ker}(\ell_{|N}),$$

since $t N_1 t^{-1}$ has finite index in N_1. Now since $t^{-1} N_0 t$ and $t^{-1} N_1 t$ are compact, we find $k_0 > 0$, so that $t^{-1} N_0 t \subseteq s^{-k_0} N_0 s^{k_0}$ and $t^{-1} N_1 t \subseteq s^{-k_0} N_1 s^{k_0}$, whence $s^{k_0} t^{-1}$ lies in $T_{+,\ell}$. Since M is a φ-module over $R[\![N_1, \ell]\!]$, the map

$$1 \otimes \varphi_{s^{k_0}}\colon R[\![N_1, \ell]\!] \otimes_{R[\![N_1,\ell]\!], \varphi_{s^{k_0}}} M \to M, \quad r \otimes m \mapsto r\varphi_{s^{k_0}}(m)$$

is an isomorphism. Under the identifications

$$R[\![N_1, \ell]\!] \otimes_{R[\![N_1, \ell]\!], \varphi_t} \left(R[\![N_1, \ell]\!] \otimes_{R[\![N_1, \ell]\!], \varphi_{s^{k_0}t-1}} M \right)$$
$$\cong R[\![N_1, \ell]\!] \otimes_{R[\![N_1, \ell]\!], \varphi_{s^{k_0}}} M$$
$$\cong R[\![N_1, \ell]\!] \otimes_{R[\![N_1, \ell]\!], \varphi_{s^{k_0}t-1}} \left(R[\![N_1, \ell]\!] \otimes_{R[\![N_1, \ell]\!], \varphi_t} M \right)$$

we have

$$(1 \otimes \varphi_t) \circ (1 \otimes (1 \otimes \varphi_{s^{k_0}t-1})) = 1 \otimes \varphi_{s^{k_0}} = (1 \otimes \varphi_{s^{k_0}t-1}) \circ (1 \otimes (1 \otimes \varphi_t)),$$

so $1 \otimes \varphi_t$ is surjective by the equality on the left and injective by the equality on the right. □

Remark 3.6. The action of $T_{0,\ell}$ on a $T_{+,\ell}$-module D over R is linear, since $T_{0,\ell}$ acts trivially on R. Therefore this action extends (uniquely) to the subgroup $T_\ell \leq T$ generated by the monoid $T_{0,\ell}$.

Proof. By the étaleness of the action of φ_t for $t \in T_{0,\ell}$ we see immediately that φ_t is an automorphism of D since $\varphi_t \colon R \to R$ is the identity map. Therefore φ_t has a (left and right) inverse (as a linear transformation of the R-module D), which we denote by φ_{t-1}. The remark follows if we note that T_ℓ consists of the quotients of elements of $T_{0,\ell}$. □

In the case when $\ell = \ell_\alpha$ given by the projection of $\prod_{\beta \in \Delta} N_\beta$ to N_α for some fixed simple root $\alpha \in \Delta$, it is clear that $T_{+,\ell} = T_+$, as N_β is T_+-invariant for each $\beta \in \Phi^+$ and $\mathrm{Ker}(\ell) = \prod_{\alpha \neq \beta \in \Phi^+} N_\beta$. Therefore $T_\ell \cong (\mathbb{Q}_p^\times)^{n-1}$, where $n = \dim T$. This is the case in which a G-equivariant sheaf on G/P is constructed in [Schneider et al. 2012] associated to any object D in $\mathfrak{M}(\mathbb{O}_\mathscr{E}, T_{+,\ell})$. So an object in $\mathfrak{M}(\mathbb{O}_\mathscr{E}, T_{+,\ell})$ is nothing other than a (φ, Γ)-module over $\mathbb{O}_\mathscr{E}$ with an additional linear action of the group T_ℓ (once we fixed the cocharacter ξ). In case of $G = \mathrm{GL}_2(\mathbb{Q}_p)$ this additional action is just an action of the center $Z = T_\ell$ of G. In the work of Colmez [2010c; 2010b] on the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, the action of Z on an irreducible 2-dimensional étale (φ, Γ)-module D is given by the determinant (that is, the action of $\mathbb{Q}_p^\times \cong Z$ on $\bigwedge^2 D$). It is unclear at this point whether the action of T_ℓ can be chosen canonically (in a similar fashion) for a given n-dimensional irreducible étale (φ, Γ)-module D.

As a corollary of Proposition 3.1, we obtain:

Proposition 3.7. *The functors* $\mathbb{D} = R \otimes_{R[\![N_1, \ell]\!], \ell} \cdot$ *and* $\mathbb{M} = R[\![N_1, \ell]\!] \otimes_{R, \iota} \cdot$ *are quasi-inverse equivalences of categories between* $\mathfrak{M}(R[\![N_1, \ell]\!], T_{+,\ell})$ *and* $\mathfrak{M}(R, T_{+,\ell})$.

Proof. Since we clearly have $\mathbb{D} \circ \mathbb{M} \cong \mathrm{id}_{\mathfrak{M}(R, T_{+,\ell})}$ and the faithfulness of $\mathbb{D}$ is a formal consequence of Proposition 3.1, it suffices to show that the isomorphism

$\Theta \colon M \to \mathbb{M} \circ \mathbb{D}(M)$ is $T_{+,\ell}$-equivariant whenever M lies in $\mathfrak{M}(R[\![N_1, \ell]\!], T_{+,\ell})$. Let $t \in T_{+,\ell}$ be arbitrary and for an $m \in M$ write $m = \sum_{u \in J(N_0/\varphi_s^k(N_0))} u\varphi_s^k(m_{u,k})$. Since $\mathbb{D}(\Theta) = \mathrm{id}_{\mathbb{D}(M)}$, we have $(\Theta \circ \varphi_t - \varphi_t \circ \Theta)(M) \subseteq I_1 \mathbb{M} \circ \mathbb{D}(M)$. We compute

$$(\Theta \circ \varphi_t - \varphi_t \circ \Theta)(m) = \sum_{u \in J(N_0/\varphi_s^k(N_0))} \varphi_t(u)\varphi_s^k \circ (\Theta \circ \varphi_t - \varphi_t \circ \Theta)(m_{u,k})$$

$$\subseteq \varphi_s^k(I_1 \mathbb{M} \circ \mathbb{D}(M)) \subseteq I_{k+1} \mathbb{M} \circ \mathbb{D}(M)$$

for all $k \geq 0$, showing that Θ is φ_t-equivariant. $\qquad\qquad\qquad\qquad \square$

4. The case of overconvergent and Robba rings

The locally analytic distribution algebra. Let p be a prime and put $\epsilon_p = 1$ if p is odd and $\epsilon_p = 2$ if $p = 2$. If H is a compact locally $\mathbb{Q}_p$-analytic group, then we denote by $D(H, K)$ the algebra of K-valued locally analytic distributions on H. Recall that $D(H, K)$ is equal to the strong dual of the locally convex vector space $C^{\mathrm{an}}(H, K)$ of K-valued locally $\mathbb{Q}_p$-analytic functions on H with the convolution product.

Recall that a topologically finitely generated pro-p group H is uniform if it is powerful (that is, $H/\overline{H^{p^{\epsilon_p}}}$ is abelian) and $|P_i(H) : P_{i+1}(H)| = |H : P_2(H)|$ for all $i \geq 1$, where $P_1(H) = H$ and $P_{i+1}(H) = \overline{P_i(H)^p[P_i(H), H]}$ (see [Dixon et al. 1999] for more details). Now if H is uniform, it has a bijective global chart

$$\mathbb{Z}_p^d \to H, \quad (x_1, \dots, x_d) \mapsto h_1^{x_1} \cdots h_d^{x_d},$$

where $h_1, \dots, h_d$ is a fixed (ordered) minimal set of topological generators of H. Putting $b_i := h_i - 1 \in \mathbb{Z}[G]$, $\boldsymbol{b^k} := b_1^{k_1} \cdots b_d^{k_d}$ for $\boldsymbol{k} = (k_i) \in \mathbb{N}^d$, we can identify $D(H, K)$ with the ring of all formal series

$$\lambda = \sum_{\boldsymbol{k} \in \mathbb{N}^d} d_{\boldsymbol{k}} \boldsymbol{b^k},$$

with $d_{\boldsymbol{k}}$ in K such that the set $\{|d_{\boldsymbol{k}}| \rho^{\epsilon_p |\boldsymbol{k}|}\}_{\boldsymbol{k}}$ is bounded for all $0 < \rho < 1$. Here the first $|\cdot|$ is the normalized absolute value on K and the second one denotes the degree of $\boldsymbol{k}$, that is, $\sum_i k_i$. For any ρ in $p^{\mathbb{Q}}$ with $p^{-1} < \rho < 1$, we have a multiplicative norm $\|\cdot\|_\rho$ on $D(H, K)$ [Schneider and Teitelbaum 2003] given by

$$\|\lambda\|_\rho := \sup_{\boldsymbol{k}} |d_{\boldsymbol{k}}| \rho^{\epsilon_p |\boldsymbol{k}|}.$$

The family of norms $\|\cdot\|_\rho$ defines the Fréchet topology on $D(H, K)$. The completion with respect to the norm $\|\cdot\|_\rho$ is denoted by $D_{[0,\rho]}(H, K)$.

Microlocalization. Let G be the group of $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split connected reductive group with a fixed Borel subgroup $P = TN$. We also choose a simple root α for the Borel subgroup P and let $\ell = \ell_\alpha$ be the functional given by the projection

$$\ell_\alpha : N \to N/[N, N] \to \prod_{\beta \in \Delta} N_\beta \to N_\alpha \xrightarrow{\iota_\alpha} \mathbb{Q}_p .$$

Therefore we have $T_{+,\ell} = T_+$, as N_β is T_+-invariant for each $\beta \in \Phi^+$. We assume further that N_0 is *uniform*.

Let us begin by recalling the definition of the classical Robba ring for the group $\mathbb{Z}_p$. The distribution algebra $D(\mathbb{Z}_p, K)$ of $\mathbb{Z}_p$ can clearly be identified with the ring of power series (in variable T) with coefficients in K that are convergent in the p-adic open unit disc. Now put

$$\mathcal{A}_{[\rho,1)} := \text{the ring of all Laurent series } \sum_{n \in \mathbb{Z}} a_n T^n \text{ that converge for } \rho \le |T| < 1.$$

For $\rho \le \rho'$ we have a natural inclusion $\mathcal{A}_{[\rho,1)} \hookrightarrow \mathcal{A}_{[\rho',1)}$, so we can form the inductive limit

$$\mathcal{R} := \varinjlim_{\rho \to 1} \mathcal{A}_{[\rho,1)} \tag{12}$$

defining the Robba ring. $\mathcal{R}$ is a (φ, Γ)-ring over $\mathbb{Z}_p$ with the maps $\chi \colon \mathbb{Z}_p \to \mathcal{R}^\times$ and $\varphi \colon \mathcal{R} \to \mathcal{R}$ such that $\chi(1) = 1 + T$, $\varphi(T) = (T+1)^p - 1$, and $\gamma(T) = (1+T)^{\xi^{-1}(\gamma)} - 1$ for $\gamma \in \Gamma$.

Recall that the ring

$$\mathcal{O}_{\mathscr{E}}^\dagger := \left\{ \sum_{n \in \mathbb{Z}} a_n T^n \;\middle|\; a_n \in o_K \text{ and there exists a } \rho < 1 \text{ such that } \lim_{n \to -\infty} |a_n| \rho^n = 0 \right\}$$

is called the ring of overconvergent power series. It is a subring of both $\mathcal{O}_{\mathscr{E}}$ and $\mathcal{R}$. We put $\mathscr{E}^\dagger := K \otimes_{o_K} \mathcal{O}_{\mathscr{E}}^\dagger$, which is also a subring of the Robba ring. These rings are also (φ, Γ)-rings.

The rings $\mathcal{O}_{\mathscr{E}}^\dagger[\![N_1, \ell]\!]$ and $\mathcal{R}[\![N_1, \ell]\!]$ constructed in the previous sections are only overconvergent or Robba in the variable b_α for the fixed simple root α. In all the other variables b_β, they behave like the Iwasawa algebra $\Lambda(N_1)$, since we took the completion with respect to the ideals generated by $(N_k - 1)$. Moreover, in the projective limit $\mathcal{O}_{\mathscr{E}}^\dagger[\![N_1, \ell]\!] \cong \varprojlim_k \mathcal{O}_{\mathscr{E}}^\dagger[N_1/N_k, \ell]$, the terms are not forced to share a common region of convergence. In this section we construct the rings $\mathcal{R}^{\mathrm{int}}(N_1, \ell)$ and $\mathcal{R}(N_1, \ell)$ with better analytic properties.

We start by constructing a ring $\mathfrak{R}_0 = \mathfrak{R}_0(N_0, K, \alpha)$ as a certain microlocalization of the distribution algebra $D(N_0, K)$. We fix the topological generator n_α of $N_0 \cap N_\alpha$ such that $\ell_\alpha(n_\alpha) = 1$. This is possible since we normalized $\iota_\alpha \colon N_\alpha \xrightarrow{\sim} \mathbb{Q}_p$ so that $\iota_\alpha(N_0 \cap N_\alpha) = \mathbb{Z}_p$. We also fix topological generators n_β of $N_0 \cap N_\beta$ for each

$\alpha \neq \beta \in \Phi^+$. Since N_0 is uniform of dimension $|\Phi^+|$, the set $A := \{n_\beta \mid \beta \in \Phi^+\}$ is a minimal set of topological generators of the group N_0. Moreover, $A \setminus \{n_\alpha\}$ is a minimal set of generators of the group $N_1 = \mathrm{Ker}(\ell) \cap N_0$. Further, we put $b_\beta := n_\beta - 1$. For any real number $p^{-1} < \rho < 1$ in $p^{\mathbb{Q}}$, the formula $\|b_\beta\|_\rho := \rho$ (for all $\beta \in \Phi^+$) defines a multiplicative norm on $D(N_0, K)$. The completion of $D(N_0, K)$ with respect to this norm is a Banach algebra that we denote by $D_{[0,\rho]}(N_0, K)$. Let now $p^{-1} < \rho_1 < \rho_2 < 1$ be real numbers in $p^{\mathbb{Q}}$. We take the generalized microlocalization (see the Appendix of [Zábrádi 2012]) of the Banach algebra $D_{[0,\rho_2]}(N_0, K)$ at the multiplicatively closed set $\{(n_\alpha - 1)^i\}_{i \geq 1}$ with respect to the pair of norms (ρ_1, ρ_2). This provides us with the Banach algebra $D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$. Recall that the elements of this Banach algebra are equivalence classes of Cauchy sequences $((n_\alpha - 1)^{-k_n} x_n)_n$ (with $x_n \in D_{[0,\rho_2]}(N_0, K)$) with respect to the norm $\|\cdot\|_{\rho_1,\rho_2} := \max(\|\cdot\|_{\rho_1}, \|\cdot\|_{\rho_2})$.

Letting ρ_2 tend to 1, we define

$$D_{[\rho_1,1)}(N_0, K, \alpha) := \varprojlim_{\rho_2 \to 1} D_{[\rho_1,\rho_2]}(N_0, K, \alpha).$$

This is a Fréchet–Stein algebra (the proof is completely analogous to that of Theorem 5.5 in [Zábrádi 2012], but it is not a formal consequence of that). However, we will not need this fact in the sequel, so we omit the proof. Now the partial Robba ring $\mathfrak{R}_0 := \mathfrak{R}_0(N_0, K, \alpha) := \varinjlim_{\rho_1 \to 1} D_{[\rho_1,1)}(N_0, K, \alpha)$ is defined as the injective limit of these Fréchet–Stein algebras. We equip $\mathfrak{R}_0$ with the inductive limit topology of the Fréchet topologies of $D_{[\rho_1,1)}(N_0, K, \alpha)$. By the following parametrization, the partial Robba ring can be thought of as a skew Laurent series ring on the variables b_β ($\beta \in \Phi^+$) with certain convergence conditions such that only the variable b_α is invertible. In [Zábrádi 2012], a "full" Robba ring is constructed such that all the variables b_β are invertible. We denote the corresponding "fully" microlocalized Banach algebras by $D_{[\rho_1,\rho_2]}(N_0, K)$. In all these rings, we will often omit K from the notation if it is clear from the context.

Remark 4.1. The microlocalization of quasiabelian normed algebras (Appendix of [Zábrádi 2012]) is somewhat different from the microlocalization constructing $\Lambda_\ell(N_0)$, where first a localization (with respect to an Ore set) is constructed and then the completion is taken. The set we are inverting here does not satisfy the Ore property, so the localization in the usual sense does not exist. However, we may complete and localize at the same time in order to obtain a microlocalized ring directly.

In order to be able to work with these rings we will show that their elements can be viewed as Laurent series. The discussion below is completely analogous to the discussion before Proposition A.24 in [Zábrádi 2012]. However, for the

convenience of the reader, we explain the method specialized to our case here. We introduce the affinoid domain

$$A_\alpha[\rho_1, \rho_2] := \{(z_\beta)_{\beta \in \Phi^+} \in \mathbb{C}_p^{\Phi^+} \mid \rho_1 \leq |z_\alpha| \leq \rho_2, 0 \leq |z_\beta/z_\alpha| \leq 1 \text{ for } \alpha \neq \beta \in \Phi^+\}.$$

This has the affinoid subdomain

$$X_{[\rho_1, \rho_2]}^{\Phi^+} := \{(z_\beta)_{\beta \in \Phi^+} \in \mathbb{C}_p^{\Phi^+} \mid \rho_1 \leq |z_{\beta_1}| = \cdots = |z_{\beta_{|\Phi^+|}}| \leq \rho_2\}$$

(where $\{\beta_1, \ldots, \beta_{|\Phi^+|}\} = \Phi^+$) as defined in [Zábrádi 2012, Proposition A.24].

Lemma 4.2. *The ring $\mathbb{O}_K(A_\alpha[\rho_1, \rho_2])$ of K-analytic functions on $A_\alpha[\rho_1, \rho_2]$ is the ring of all Laurent series*

$$f(\mathbf{Z}) = \sum_{\mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}} d_{\mathbf{k}} \mathbf{Z}^{\mathbf{k}}$$

with $d_{\mathbf{k}} \in K$ and such that $\lim_{\mathbf{k} \to \infty} |d_{\mathbf{k}}| \rho^{\mathbf{k}} = 0$ for any $\rho_1 \leq \rho \leq \rho_2$. (Here $\mathbf{Z}^{\mathbf{k}} := \prod_{\beta \in \Phi^+} Z_\beta^{k_\beta}$ and $\rho^{\mathbf{k}} := \rho^{\sum_{\beta \in \Phi^+} k_\beta}$, and $\mathbf{k} \to \infty$ means that $\sum_{\beta \in \Phi^+} |k_\beta| \to \infty$.) This is the subring of $\mathbb{O}_K(X_{[\rho_1, \rho_2]}^{\Phi^+})$ consisting of elements in which the variables Z_β appear only with nonnegative exponent for all $\alpha \neq \beta \in \Phi^+$.

Proof. Since $X_{[\rho_1, \rho_2]}^{\Phi^+} \subseteq A_\alpha[\rho_1, \rho_2]$, we clearly have $\mathbb{O}_K(A_\alpha[\rho_1, \rho_2]) \subseteq \mathbb{O}_K(X_{[\rho_1, \rho_2]}^{\Phi^+})$. Also, the power series in $\mathbb{O}_K(A_\alpha[\rho_1, \rho_2])$ converge for $z_\beta = 0$ ($\beta \neq \alpha$), and hence these variables appear with nonnegative exponent. On the other hand, if we have a power series $f(\mathbf{Z}) \in \mathbb{O}_K(X_{[\rho_1, \rho_2]}^{\Phi^+})$ such that the variables Z_β have nonnegative exponent for all $\alpha \neq \beta \in \Phi^+$, then it also converges in the region $A_\alpha[\rho_1, \rho_2]$, as we have in this case the trivial estimate

$$\left| \prod_{\beta \in \Phi^+} z_\beta^{k_\beta} \right| \leq |z_\alpha|^{\sum_{\beta \in \Phi^+} k_\beta}. \qquad \square$$

Since $\rho^{\mathbf{k}} \leq \max(\rho_1^{\mathbf{k}}, \rho_2^{\mathbf{k}})$ for any $\rho_1 \leq \rho \leq \rho_2$ and any $\mathbf{k} \in \mathbb{Z}^{(\alpha)} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}$, the convergence condition on f is equivalent to

$$\lim_{\mathbf{k} \to \infty} |d_{\mathbf{k}}| \rho_1^{\mathbf{k}} = \lim_{\mathbf{k} \to \infty} |d_{\mathbf{k}}| \rho_2^{\mathbf{k}} = 0.$$

The spectral norm on the affinoid algebra $\mathbb{O}_K(A_\alpha[\rho_1, \rho_2])$ (for the definition of these notions see [Fresnel and van der Put 2004]) is given by

$$\|f\|_{A_\alpha[\rho_1, \rho_2]} = \sup_{\rho_1 \leq \rho \leq \rho_2} \max_{\mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}} |d_{\mathbf{k}}| \rho^{\mathbf{k}}$$

$$= \max\left(\max_{\mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}} |d_{\mathbf{k}}| \rho_1^{\mathbf{k}}, \max_{\mathbf{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}} |d_{\mathbf{k}}| \rho_2^{\mathbf{k}} \right).$$

Setting $\mathbf{b}^{\mathbf{k}} := \prod_{\beta \in \Phi^+} b_\beta^{k_\beta}$ for some fixed ordering of Φ^+ and for any $\mathbf{k} = (k_\beta)_{\beta \in \Phi^+}$

in $\mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}$, we claim that

$$f(\boldsymbol{b}) := \sum_{\boldsymbol{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}} d_{\boldsymbol{k}} \boldsymbol{b}^{\boldsymbol{k}}$$

converges in $D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$ for $f \in \mathbb{O}_K(A_\alpha[\rho_1, \rho_2])$. As a consequence of Proposition A.21 and Lemma A.7.iii in [Zábrádi 2012], we have

$$\|\boldsymbol{b}^{\boldsymbol{k}}\|_{\rho_1,\rho_2} = \max(\rho_1^{\boldsymbol{k}}, \rho_2^{\boldsymbol{k}})$$

for any $\boldsymbol{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}$. Hence

$$\lim_{\boldsymbol{k}\to\infty} \|d_{\boldsymbol{k}} \boldsymbol{b}^{\boldsymbol{k}}\|_{\rho_1,\rho_2} = \lim_{\boldsymbol{k}\to\infty} \max(|d_{\boldsymbol{k}}|\rho_1^{\boldsymbol{k}}, |d_{\boldsymbol{k}}|\rho_2^{\boldsymbol{k}})$$

$$= \max\left(\lim_{\boldsymbol{k}\to\infty} |d_{\boldsymbol{k}}|\rho_1^{\boldsymbol{k}}, \lim_{\boldsymbol{k}\to\infty} |d_{\boldsymbol{k}}|\rho_2^{\boldsymbol{k}}\right)$$

$$= 0.$$

Therefore

$$\mathbb{O}_K(A_\alpha[\rho_1, \rho_2]) \to D_{[\rho_1,\rho_2]}(N_0, K, \alpha), \quad f \mapsto f(\boldsymbol{b}),$$

is a well defined K-linear map. In order to investigate this map we introduce the filtration

$$F^i D_{[\rho_1,\rho_2]}(N_0, K, \alpha) := \left\{e \in D_{[\rho_1,\rho_2]}(N_0, K, \alpha) : \|e\|_{\rho_1,\rho_2} \leq |p|^i\right\} \quad \text{for } i \in \mathbb{R},$$

on $D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$. Since K is discretely valued and $\rho_1, \rho_2 \in p^{\mathbb{Q}}$, this filtration is quasi-integral in the sense of [Schneider and Teitelbaum 2003, §1]. The corresponding graded ring $\operatorname{gr}^{\cdot} D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$, by Proposition A.21 in [Zábrádi 2012], is commutative. We let $\sigma(e) \in \operatorname{gr}^{\cdot} D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$ denote the principal symbol of any element $e \in D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$.

Proposition 4.3. (i) $\operatorname{gr}^{\cdot} D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$ *is a free* $\operatorname{gr}^{\cdot} K$-*module with basis*

$$\left\{\sigma(\boldsymbol{b}^{\boldsymbol{k}}) : \boldsymbol{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}\right\}.$$

(ii) *The map*

$$\mathbb{O}_K(A_\alpha[\rho_1, \rho_2]) \xrightarrow{\cong} D_{[\rho_1,\rho_2]}(N_0, K, \alpha), \quad f \mapsto f(\boldsymbol{b}),$$

is a K-*linear isometric bijection.*

Proof. Since $\{b_\alpha^{-l}\mu : l \geq 0, \mu \in D_{[0,\rho_1]}(N_0, K)\}$ is dense in $D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$, every element in the graded ring $\operatorname{gr}^{\cdot} D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$ is of the form $\sigma(b_\alpha^{-l}\mu)$. Suppose that $\mu = \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} d_{\boldsymbol{k}} \boldsymbol{b}^{\boldsymbol{k}}$. Then $b_\alpha^{-l}\mu = \sum_{\boldsymbol{k} \in \mathbb{N}_0^d} d_{\boldsymbol{k}} b_\alpha^{-l} \boldsymbol{b}^{\boldsymbol{k}}$ and, using [Zábrádi

2012, Lemma A.7.iii], we compute

$$\|b_\alpha^{-l}\mu\|_{\rho_1,\rho_2} = \max\big(\|b_\alpha\|_{\rho_1}^{-l}\|\mu\|_{\rho_1}, \|b_\alpha\|_{\rho_2}^{-l}\|\mu\|_{\rho_2}\big)$$
$$= \max\big(\max_{\boldsymbol{k}\in\mathbb{N}_0^d}|d_{\boldsymbol{k}}|\rho_1^{\boldsymbol{k}-l}, \max_{\boldsymbol{k}\in\mathbb{N}_0^d}|d_{\boldsymbol{k}}|\rho_2^{\boldsymbol{k}-l}\big)$$
$$= \max_{\boldsymbol{k}\in\mathbb{N}_0^d}|d_{\boldsymbol{k}}|\max\big(\rho_1^{\boldsymbol{k}-l}, \rho_2^{\boldsymbol{k}-l}\big) = \max_{\boldsymbol{k}\in\mathbb{N}_0^d}|d_{\boldsymbol{k}}||b_\alpha^{-l}\boldsymbol{b}^{\boldsymbol{k}}|_{\rho_1,\rho_2}.$$

It follows that $\mathrm{gr}^{\cdot} D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$ as a $\mathrm{gr}^{\cdot} K$-module is generated by the principal symbols $\sigma(b_\alpha^{-l}\boldsymbol{b}^{\boldsymbol{k}})$ with $\boldsymbol{k} \in \mathbb{N}_0^d$, $l \geq 0$. But it also follows that, for a fixed $l \geq 0$, the principal symbols $\sigma(b_\alpha^{-l}\boldsymbol{b}^{\boldsymbol{k}})$ with $\boldsymbol{k}$ running over $\mathbb{N}_0^d$ are linearly independent over $\mathrm{gr}^{\cdot} K$. By Proposition A.21 in [Zábrádi 2012], we may permute the factors in $\sigma(b_\alpha^{-l}\boldsymbol{b}^{\boldsymbol{k}})$ arbitrarily. Hence $\mathrm{gr}^{\cdot} D_{[\rho_1,\rho_2]}(N_0, K, \alpha)$ is a free $\mathrm{gr}^{\cdot} K$-module with basis $\{\sigma(\boldsymbol{b}^{\boldsymbol{k}}) : \boldsymbol{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+\setminus\{\alpha\}}\}$.

On the other hand, we of course have

$$\|f(\boldsymbol{b})\|_{\rho_1,\rho_2} \leq \max_{\boldsymbol{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}}|d_{\boldsymbol{k}}||\boldsymbol{b}^{\boldsymbol{k}}|_{\rho_1,\rho_2} = \max_{\boldsymbol{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}}|d_{\boldsymbol{k}}|\max(\rho_1^{\boldsymbol{k}}, \rho_2^{\boldsymbol{k}})$$
$$= \max\big(\max_{\boldsymbol{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}}|d_{\boldsymbol{k}}|\rho_1^{\boldsymbol{k}}, \max_{\boldsymbol{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}}|d_{\boldsymbol{k}}|\rho_2^{\boldsymbol{k}}\big) = |f|_{A_\alpha[\rho_1,\rho_2]}.$$

This means that if we introduce on $\mathcal{O}_K(A_\alpha[\rho_1, \rho_2])$ the filtration defined by the spectral norm, then the asserted map respects the filtrations, and by the above reasoning, it induces an isomorphism between the associated graded rings. Hence, by completeness of these filtrations, it is an isometric bijection. □

Now we turn to the construction of $\mathcal{R}(N_1, \ell)$. The problem with (naïve) microlocalization is that the ring $\mathfrak{R}_0$ is not finitely generated over $\varphi(\mathfrak{R}_0)$. The reason for this is that φ improves the order of convergence for a power series in $\mathfrak{R}_0$. In the case $G \neq \mathrm{GL}_2(\mathbb{Q}_p)$, the operator $\varphi = \varphi_s$ acts by conjugation on N_β by raising to the $\beta(s)$-th power. Whenever $\beta \in \Phi^+ \setminus \Delta$ is not a simple root, $\beta(s) = p^{m_\beta} > \alpha(s) = p$, where m_β is the degree of the map $\beta \circ \xi \colon \mathbb{G}_m \to \mathbb{G}_m$.

Lemma 4.4. *We have* $\|b_\beta\|_\rho = \|b_\alpha\|_\rho = \rho$ *and*

$$\|\varphi(b_\beta)\|_\rho = \max_{0\leq j\leq m_\beta}(\rho^{p^j}p^{j-m_\beta}) < \max(\rho^p, p^{-1}\rho) = \|\varphi(b_\alpha)\|_\rho$$

for any $p^{-1} < \rho < 1$. *In general,* $\|\varphi_t(b_\beta)\|_\rho = \max_{0\leq j\leq \mathrm{val}_p(\beta(t))}\big(\rho^{p^j}p^{j-\mathrm{val}_p(\beta(t))}\big).$

Proof. We compute

$$\|\varphi_t(b_\beta)\|_\rho = \|(1+b_\beta)^{\beta(t)} - 1\|_\rho = \left\|\sum_{i=1}^\infty \binom{\beta(t)}{i} b_\beta^i\right\|_\rho$$
$$= \max_{0\leq j\leq \mathrm{val}_p(\beta(t))}\big(\rho^{p^j}p^{j-\mathrm{val}_p(\beta(t))}\big).$$

Here we use the trivial estimate

$$\text{val}_p \binom{n}{k} = \text{val}_p \left(\frac{n}{k} \binom{n-1}{k-1} \right) \geq \text{val}_p(n) - \text{val}_p(k)$$

for $n := \beta(t) \in \mathbb{Z}_p$ and $k \in \mathbb{N}$. We see immediately that whenever $m_\beta > 1$, we have $\rho^{p^j} p^{j-m_\beta} < \rho^p$ for $1 \leq j \leq m_\beta$ and $p^{-m_\beta} \rho < p^{-1}\rho$. $\qquad\square$

Now choose an ordering $<$ on Φ^+ such that (i) $m_{\beta_1} < m_{\beta_2}$ implies $\beta_1 > \beta_2$ and (ii) $\alpha > \beta$ for any $\alpha \neq \beta, \beta_1, \beta_2 \in \Phi^+$. Then by Proposition 4.3, any element in $\mathfrak{R}_0$ has a skew Laurent-series expansion

$$f(\boldsymbol{b}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}} c_{\boldsymbol{k}} \boldsymbol{b}^{\boldsymbol{k}}$$

such that there exists $p^{-1} < \rho < 1$ with $|c_{\boldsymbol{k}}|_p \rho_1^{\sum k_\beta} \to 0$ as $\sum |k_\beta| \to \infty$ for all $\rho < \rho_1 < 1$. By Lemma 4.4 and the discussion above, we clearly have:

Example 4.5. Let $\beta \in \Phi^+ \setminus \Delta$ be a nonsimple root. Then the series $\sum_{n=1}^\infty b_\beta^n b_\alpha^{-n}$ does not belong to $\mathfrak{R}_0(N_0)$. However, the series $\sum_{n=1}^\infty \varphi(b_\beta^n b_\alpha^{-n})$ converges in each $D_{[\rho_1,\rho_2]}(N_0, \alpha)$ (for arbitrary $p^{-1} < \rho_1 < \rho_2 < 1$); hence it defines an element in $\mathfrak{R}_0(N_0)$. Therefore we cannot have a continuous left inverse ψ to φ on $\mathfrak{R}_0(N_0)$, as otherwise $\psi\left(\sum_{n=1}^\infty \varphi(b_\beta^n b_\alpha^{-n})\right) = \sum_{n=1}^\infty b_\beta^n b_\alpha^{-n}$ would converge. In particular, we cannot write $\mathfrak{R}_0(N_0)$ as the topological direct sum $\bigoplus_{u \in N_0/\varphi(N_0)} u\varphi(\mathfrak{R}_0(N_0))$ of closed subspaces in $\mathfrak{R}_0(N_0)$, as otherwise the operator

$$\psi: \mathfrak{R}_0(N_0) \to \mathfrak{R}_0(N_0), \qquad \sum_{u \in J(N_0/\varphi(N_0))} u\varphi(f_u) \mapsto \varphi^{-1}(u_0) f_{u_0}$$

for the unique $u_0 \in J(N_0/\varphi(N_0)) \cap \varphi(N_0)$ would be a continuous left inverse to φ. In fact, we even have $\mathfrak{R}_0(N_0) \neq \bigoplus_{u \in N_0/\varphi(N_0)} u\varphi(\mathfrak{R}_0(N_0))$ algebraically; however, the proof of this requires the forthcoming machinery (see Remark 4.10).

In order to overcome this counterexample, we are going to consider the ring $\mathfrak{R}(N_1, \ell)$ of all the skew power series of the form $f(\boldsymbol{b})$ such that $f(\varphi_t(\boldsymbol{b}))$ is convergent in $\mathfrak{R}_0$ for some $t \in T_+$. A priori it is not clear that these series form a ring, so we are going to give a more conceptual construction.

Take an arbitrary element $t \in T_+$. The conjugation by t on N_0 gives an isomorphism $\varphi_t: N_0 \to \varphi_t(N_0)$ of pro-p groups (since it is injective). Hence $\varphi_t(N_0)$ is also a uniform pro-p group with minimal set of generators $\{\varphi_t(n_\beta)\}_{\beta \in \Phi^+}$. So we may define the distribution algebra $D(\varphi_t(N_0)) := D(\varphi_t(N_0), K)$. The inclusion $\varphi_t(N_0) \hookrightarrow N_0$ induces an injective homomorphism of Fréchet algebras $\iota_{1,t}: D(\varphi_t(N_0)) \hookrightarrow D(N_0)$. It is well known [Schneider and Teitelbaum 2003] that

$$D(N_0) = \bigoplus_{n \in J(N_0/\varphi_t(N_0))} n\iota_{1,t}\big(D(\varphi_t(N_0))\big)$$

as right $D(\varphi_t(N_0))$-modules. Moreover, the direct summands are closed in $D(N_0)$. For each real number $p^{-1} < \rho < 1$, the ρ-norm on $D(N_0)$ defines a norm $r_t(\rho)$ on $D(\varphi_t(N_0))$ by restriction. This is different from the ρ-norm on $D(\varphi_t(N_0))$ (using the uniform structure on $\varphi_t(N_0)$). However, the family $(r_t(\rho))_\rho$ of norms defines the Fréchet topology on $D(\varphi_t(N_0))$. On the other hand, whenever r is a norm on $D(\varphi_t(N_0))$, we may extend r to a norm $q_t(r)$ on $D(N_0)$ by putting

$$\left\| \sum_{n \in J(N_0/\varphi_t(N_0))} n \iota_{1,t}(x_n) \right\|_{q_t(r)} := \max(\|x_n\|_r).$$

These norms define the Fréchet topology on $D(N_0)$. More precisely, if $\beta(t) = p^{m(\beta,t)} u(\beta,t)$ with $m(\beta,t) := \mathrm{val}_p(\beta(t)) \geq 0$ integer and $u(\beta,t) \in \mathbb{Z}_p^\times$, then:

Lemma 4.6.

$$\|x\|_\rho \leq \|x\|_{q_t(r_t(\rho))} \leq \rho^{-\sum_{\beta \in \Phi^+} (p^{m(\beta,t)} - 1)} \|x\|_\rho$$

for any

$$p^{-\frac{1}{\max_{\beta \in \Phi^+} p^{m(\beta,t)}}} < \rho < 1$$

and $x \in D(N_0)$. In particular, the norms ρ and $q_t(r_t(\rho))$ define the same topology.

Proof. The inequality on the left is clear from the triangle inequality. For the other inequality, note that our assumption on ρ implies in particular that

$$\rho^{p^{m(\beta,t)}} = \rho^{p^j} \rho^{p^{m(\beta,t)} - p^j} > \rho^{p^j} p^{-\frac{p^{m(\beta,t)} - p^j}{p^{m(\beta,t)}}} > \rho^{p^j} p^{j - m(\beta,t)}$$

for all $0 \leq j < m(\beta,t)$. Hence, by Lemma 4.4, we have

$$\rho^{p^{m(\beta,t)}} = \left\| \binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}} \right\|_\rho = \|\varphi_t(b_\beta)\|_\rho.$$

Moreover, there exists an *invertible* element y in the Iwasawa algebra $\Lambda(N_{0,\beta})$ such that

$$y \varphi_t(b_\beta) \equiv \binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}} \pmod{p}$$

(as both sides have the same principal term). But by the choice of ρ, $|p| = 1/p < \rho^{p^{m(\beta,t)}} = \|\varphi_t(b_\beta)\|_\rho = \|\varphi_t(b_\beta)\|_{q_t(r_t(\rho))}$. Therefore we also have

$$\rho^{p^{m(\beta,t)}} = \left\| b_\beta^{p^{m(\beta,t)}} \right\|_\rho = \left\| \binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}} \right\|_\rho = \|\varphi_t(b_\beta)\|_\rho$$

$$= \|\varphi_t(b_\beta)\|_{q_t(r_t(\rho))} = \|y \varphi_t(b_\beta)\|_{q_t(r_t(\rho))}$$

$$= \left\| \binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}} \right\|_{q_t(r_t(\rho))} = \left\| b_\beta^{p^{m(\beta,t)}} \right\|_{q_t(r_t(\rho))},$$

whence

$$\left\| b_\beta^{k_\beta} \right\|_{q_t(r_t(\rho))} = \left\| b_\beta^{p^{m(\beta,t)}} \right\|_{q_t(r_t(\rho))}^{k_{1,\beta}} \left\| b_\beta^{k_{2,\beta}} \right\|_{q_t(r_t(\rho))}$$

$$\leq \rho^{k_{1,\beta} p^{m(\beta,t)}} \leq \rho^{-p^{m(\beta,t)}+1} \left\| b_\beta^{k_\beta} \right\|_\rho, \tag{13}$$

where $k_\beta = p^{m(\beta,t)} k_{1,\beta} + k_{2,\beta}$ with $0 \leq k_{2,\beta} \leq p^{m(\beta,t)} - 1$ and $k_{1,\beta}$ nonnegative integers.

Now consider an element of $D(N_0)$ of the form

$$x = \sum_{\boldsymbol{k}=(k_\beta)\in\mathbb{N}^{\Phi^+}} c_{\boldsymbol{k}} \prod_{\beta\in\Phi^+} b_\beta^{k_\beta}.$$

We may assume without loss of generality that

$$J(N_0/\varphi_t(N_0)) = \left\{ \prod_{\beta\in\Phi^+} n_\beta^{j_\beta} \mid 0 \leq j_\beta \leq p^{m(\beta,t)} - 1 \right\},$$

where the product is taken in the reverse order. Let $\eta \in \Phi^+$ be the largest root (with respect to the ordering $<$ defined after Lemma 4.4) such that there exists a $\boldsymbol{k} \in \mathbb{N}^{\Phi^+}$ with $c_{\boldsymbol{k}} \neq 0$ and $k_\eta \neq 0$. We are going to show the estimate

$$\|x\|_{q_t(r_t(\rho))} \leq \rho^{-\sum_{\beta\leq\eta}(p^{m(\beta,t)}-1)} \|x\|_\rho$$

by induction on η. This induction has in fact finitely many steps since $|\Phi^+| < \infty$. At first we write $b_\eta^{k_\eta} = \sum_{j_\eta=0}^{p^{m(\eta,t)}-1} n_\eta^{j_\eta} f_{\boldsymbol{k},j_\eta}(\varphi_t(b_\eta))$ for each $\boldsymbol{k} \in \mathbb{N}^{\Phi^+}$. By the choice of the ordering on Φ^+, for any fixed η, the set $\prod_{\beta<\eta} N_{0,\beta}$ is a normal subgroup of N_0. Moreover, the conjugation by any element of N_0 preserves the ρ-norm on $D(N_0)$. Therefore we may write

$$\prod_{\beta\leq\eta} b_\beta^{k_\beta} = \sum_{j_\eta=0}^{p^{m(\eta,t)}-1} n_\eta^{j_\eta} x_{\boldsymbol{k},j_\eta} f_{\boldsymbol{k},j_\eta}(\varphi_t(b_\eta))$$

such that

$$x_{\boldsymbol{k},j_\eta} := n_\eta^{-j_\eta} \left(\prod_{\beta<\eta} b_\beta^{k_\beta} \right) n_\eta^{j_\eta} \in D\left(\prod_{\beta<\eta} N_{0,\beta} \right).$$

By (13), we have

$$\left\| f_{\boldsymbol{k},j_\eta}(\varphi_t(b_\eta)) \right\|_{q_t(r_t(\rho))} = \left\| f_{\boldsymbol{k},j_\eta}(\varphi_t(b_\eta)) \right\|_\rho$$

$$\leq \|b_\eta^{k_\eta}\|_{q_t(r_t(\rho))} \leq \rho^{-p^{m(\eta,t)}+1} \|b_\eta^{k_\eta}\|_\rho.$$

Since the $r_t(\rho)$-norm is multiplicative on $D(\varphi_t(N_0))$, for any $a \in D(N_0)$ and $b \in D(\varphi_t(N_0))$ we also have $\|a\iota_{1,t}(b)\|_{q_t(r_t(\rho))} = \|a\|_{q_t(r_t(\rho))} \|b\|_{r_t(\rho)}$. Indeed, if we decompose a as $a = \sum_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}(a_n)$, then we have $a\iota_{1,t}(b) =$

$\sum_{n \in J(N_0/\varphi_t(N_0))} n \iota_{1,t}(a_n b)$. Now $f_{\boldsymbol{k}, j_\eta}(\varphi_t(b_\eta))$ lies in $\iota_{1,t}(D(\varphi_t(N_0)))$, so we see that

$$\left\| x_{\boldsymbol{k}, j_\eta} f_{\boldsymbol{k}, j_\eta}(\varphi_t(b_\eta)) \right\|_{q_t(r_t(\rho))} = \left\| x_{\boldsymbol{k}, j_\eta} \right\|_{q_t(r_t(\rho))} \left\| f_{\boldsymbol{k}, j_\eta}(\varphi_t(b_\eta)) \right\|_{q_t(r_t(\rho))}.$$

On the other hand, the inductional hypothesis tells us that

$$\left\| x_{\boldsymbol{k}, j_\eta} \right\|_{q_t(r_t(\rho))} \leq \rho^{-\sum_{\beta < \eta}(p^{m(\beta, t)} - 1)} \left\| x_{\boldsymbol{k}, j_\eta} \right\|_\rho = \rho^{-\sum_{\beta < \eta}(p^{m(\beta, t)} - 1)} \left\| \prod_{\beta < \eta} b_\beta^{k_\beta} \right\|_\rho.$$

Hence we compute

$$
\begin{aligned}
\| x \|_{q_t(r_t(\rho))} &= \left\| \sum_{\boldsymbol{k}} c_{\boldsymbol{k}} \sum_{j_\eta = 0}^{p^{m(\eta, t)} - 1} n_\eta^{j_\eta} x_{\boldsymbol{k}, j_\eta} f_{\boldsymbol{k}, j_\eta}(\varphi_t(b_\eta)) \right\|_{q_t(r_t(\rho))} \\
&\leq \max_{\boldsymbol{k}, j_\eta} \left(|c_{\boldsymbol{k}}| \left\| x_{\boldsymbol{k}, j_\eta} f_{\boldsymbol{k}, j_\eta}(\varphi_t(b_\eta)) \right\|_{q_t(r_t(\rho))} \right) \\
&\leq \max_{\boldsymbol{k}} \left(|c_{\boldsymbol{k}}| \rho^{-\sum_{\beta < \eta}(p^{m(\beta, t)} - 1)} \left\| \prod_{\beta < \eta} b_\beta^{k_\beta} \right\|_\rho \rho^{-p^{m(\eta, t)} + 1} \| b_\eta^{k_\eta} \|_\rho \right) \\
&= \rho^{-\sum_{\beta \leq \eta}(p^{m(\beta, t)} - 1)} \| x \|_\rho. \qquad \square
\end{aligned}
$$

In particular, for each ρ in the range $p^{-1/\max_\beta p^{m(\beta, t)}} < \rho < 1$, the completions of $D(N_0)$ with respect to the topologies defined by $\| \cdot \|_\rho$ and by $\| \cdot \|_{q_t(r_t(\rho))}$ are the same, that is,

$$D_{[0, \rho]}(N_0) = \bigoplus_{n \in J(N_0/\varphi_t(N_0))} n \iota_{1,t} \left(D_{r_t([0, \rho])}(\varphi_t(N_0)) \right), \tag{14}$$

where $D_{r_t([0, \rho])}(\varphi_t(N_0))$ denotes the completion of $D(\varphi_t(N_0))$ with respect to the norm $r_t(\rho)$.

Now we turn to the microlocalization and note first that $\varphi_t(b_\alpha) = (b_\alpha + 1)^{\alpha(t)} - 1$ is divisible by b_α. So if $\varphi_t(b_\alpha)$ is invertible in a ring, then so is b_α. On the other hand, if $p^{-1/p^{m(\alpha, t)}} < \rho < 1$, then by Lemma 4.4 we have

$$\left\| \varphi_t(b_\alpha) - \binom{\alpha(t)}{p^{m(\alpha, t)}} b_\alpha^{p^{m(\alpha, t)}} \right\|_\rho < \left\| \binom{\alpha(t)}{p^{m(\alpha, t)}} b_\alpha^{p^{m(\alpha, t)}} \right\|_\rho.$$

Hence $\varphi_t(b_\alpha)$ is invertible in the Banach algebra $D_{[\rho_1, \rho_2]}(N_0, \alpha)$ for any ρ_1, ρ_2 with $p^{-1/p^{m(\alpha, t)}} < \rho_1 < \rho_2 < 1$, since it is close to the invertible element

$$\binom{\alpha(t)}{p^{m(\alpha, t)}} b_\alpha^{p^{m(\alpha, t)}}$$

(as the binomial coefficient in this expression is not divisible by p). This shows that

the microlocalization of $D_{[0,\rho_2]}(N_0)$ with respect to the multiplicative set $\varphi_t(b_\alpha)^{\mathbb{N}}$ and norm $\max(\rho_1, \rho_2)$ equals $D_{[\rho_1,\rho_2]}(N_0, \alpha)$. Therefore for each ρ_1 and ρ_2 with $p^{-1/p^{m(\alpha,t)}} < \rho_1 < \rho_2 < 1$ we obtain

$$D_{[\rho_1,\rho_2]}(N_0, \alpha) = \bigoplus_{n \in J(N_0/\varphi_t(N_0))} n \iota_{0,1}\big(D_{r_t([\rho_1,\rho_2])}(\varphi_t(N_0), \alpha)\big)$$

by microlocalizing both sides of (14). Now letting ρ_2 tend to 1 and then also $\rho_1 \to 1$, we get

$$\mathfrak{R}_0(N_0, \alpha) = \bigoplus_{n \in J(N_0/\varphi_t(N_0))} n \iota_{1,t}\big(\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0), \alpha)\big) \tag{15}$$

for all $t \in T_+$. Here we define

$$\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0), \alpha) := \varinjlim_{\rho_1 \to 1} \varprojlim_{\rho_2 \to 1} D_{r_t([\rho_1, r(\rho_2)])}(\varphi_t(N_0), \alpha),$$

which is in general different from $\mathfrak{R}_0(\varphi_t(N_0), \alpha)$ (in which, by definition, we use norms ρ such that $\|\varphi_t(b_\beta)\|_\rho = \|\varphi_t(b_\alpha)\|_\rho$), by Example 4.5. Indeed, for $t = s$ the sum $\sum_{n=1}^\infty \varphi(b_\beta^n b_\alpha^{-n})$ converges in $\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0), \alpha)$, but not in $\mathfrak{R}_0(\varphi_t(N_0), \alpha)$.

By entirely the same proof, we also obtain

$$\mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0), \alpha) = \bigoplus_{n \in J(\varphi_{t_1}(N_0)/\varphi_{t_1 t_2}(N_0))} n \iota_{t_1, t_1 t_2}\big(\mathfrak{R}_{0,r_{t_1 t_2}(\cdot)}(\varphi_{t_1 t_2}(N_0), \alpha)\big) \tag{16}$$

for each pair $t_1, t_2 \in T_+$, where $\iota_{t_1, t_1 t_2}$ is the inclusion of the rings above induced by the natural inclusion $\varphi_{t_1 t_2}(N_0) \hookrightarrow \varphi_{t_1}(N_0)$.

Now we would like to define continuous homomorphisms

$$\varphi_{t_2 t_1, t_1} : \mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0), \alpha) \to \mathfrak{R}_{0,r_{t_1 t_2}(\cdot)}(\varphi_{t_1 t_2}(N_0), \alpha), \quad \varphi_{t_1}(b_\beta) \mapsto \varphi_{t_1 t_2}(b_\beta)$$

induced by the group isomorphism $\varphi_{t_2} : \varphi_{t_1}(N_0) \to \varphi_{t_1 t_2}(N_0)$ so that we can take the injective limit

$$\mathfrak{R}(N_1, \ell) := \varinjlim_t \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0), \alpha)$$

with respect to the maps $\varphi_{t_2 t_1, t_1}$. This is not possible for all t_2 since the map φ_{t_2} will not always be norm-decreasing on monomials b^k for $k \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}$. To overcome this, we define the pre-ordering $\leq_\alpha$ (depending on the choice of the simple root α) on T_+ the following way: $t_1 \leq_\alpha t_2$ if and only if $|\beta(t_2 t_1^{-1})| \leq |\alpha(t_2 t_1^{-1})| \leq 1$ for all $\beta \in \Phi^+$. (That is, if and only if we have $m(\beta, t_2 t_1^{-1}) \geq m(\alpha, t_2 t_1^{-1}) \geq 0$.) In particular, $t_1 \leq_\alpha t_2$ implies $t_2 t_1^{-1} \in T_+$ and it is equivalent to $1 \leq_\alpha t_2 t_1^{-1}$. We also have $1 \leq_\alpha s$ for any $\alpha \in \Delta$. It is clear that $\leq_\alpha$ is transitive and reflexive. If $t_2 \leq_\alpha t_1 \leq_\alpha t_2$, then $|\beta(t_2 t_1^{-1})| = 1$ for all $\beta \in \Phi^+$, whence $t_2 t_1^{-1}$ lies in T_0. Therefore $\leq_\alpha$ defines a partial ordering on the quotient monoid T_+/T_0.

Lemma 4.7. *The partial ordering $\leq_\alpha$ on T_+/T_0 is right filtered, that is, any finite subset of T_+/T_0 has a common upper bound with respect to $\leq_\alpha$.*

Proof. Take any $t_1, t_2 \in T_+$ with $|\alpha(t_1)| \leq |\alpha(t_2)|$. Since the simple roots $\beta \in \Delta$ are linearly independent in $X^*(T) = \mathrm{Hom}_{\mathrm{alg}}(T, \mathbb{G}_m)$ and the pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ is perfect, we may choose $s_{\overline{\alpha}} \in T$ so that $|\beta(s_{\overline{\alpha}})| < |\alpha(s_{\overline{\alpha}})| = 1$ for all $\alpha \neq \beta \in \Delta$. Since all the positive roots are positive linear combinations of the simple roots, we see immediately that $s_{\overline{\alpha}} \in T_+$. If $\alpha \neq \gamma \in \Phi^+$, then γ is not a scalar multiple of α; hence, writing $\gamma = \sum_{\beta \in \Delta} m_{\beta,\gamma} \beta$, there is an $\alpha \neq \beta \in \Delta$ with $m_{\beta,\gamma} > 0$, whence $|\gamma(s_{\overline{\alpha}})| < 1$. So we have $t_1 \leq_\alpha t_1 s_{\overline{\alpha}}^k$ for any $k \geq 0$ and $t_2 \leq_\alpha t_1 s_{\overline{\alpha}}^k$ for k big enough. $\square$

Fix an element $1 \leq_\alpha t \in T_+$ and let ρ_1, ρ_2 be real numbers in $p^{\mathbb{Q}}$ such that

$$p^{-1/\max_{\beta \in \Phi^+} p^{m(\beta,t)+m(\alpha,t)}} < \rho_1 < \rho_2 < 1.$$

Note that $\varphi_t \colon N_0 \to \varphi_t(N_0)$ is an isomorphism of pro-p groups. Hence it induces an isometric isomorphism

$$\varphi_t \colon D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(N_0) \to D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(\varphi_t(N_0)),$$

$$\sum_{\mathbf{k}} c_{\mathbf{k}} \prod_\beta b_\beta^{k_\beta} \mapsto \sum_{\mathbf{k}} c_{\mathbf{k}} \prod_\beta \varphi_t(b_\beta)^{k_\beta}$$

of Banach algebras, where

$$D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(\varphi_t(N_0))$$

denotes the completion of $D(\varphi_t(N_0))$ with respect to the $\rho_2^{p^{m(\alpha,t)}}$-norm defined by the set of generators $\{\varphi_t(n_\beta)\}_{\beta \in \Phi^+}$ of $\varphi_t(N_0)$. To avoid confusion, from now on we denote by the subscript ρ, N_0 the ρ-norm (as before) on $D(N_0)$ and by the subscript $\rho, \varphi_t(N_0)$ the ρ-norm on $D(\varphi_t(N_0))$. By Lemma 4.4, we have

$$\|\varphi_t(b_\beta)\|_{\rho,N_0} = \rho^{p^{m(\beta,t)}} \leq \rho^{p^{m(\alpha,t)}} = \|\varphi_t(b_\alpha)\|_{\rho,N_0}$$

for any $\beta \in \Phi^+$ and $\rho = \rho_1$ or $\rho = \rho_2$ because of our assumption $1 \leq_\alpha t$. This shows that for any monomial $\prod_{\beta \in \Phi^+} \varphi_t(b_\beta)^{k_\beta}$ (with $k_\beta \geq 0$ for all $\beta \in \Phi^+$), we have

$$\left\| \prod_{\beta \in \Phi^+} \varphi_t(b_\beta)^{k_\beta} \right\|_{r_t(\rho)} = \left\| \prod_{\beta \in \Phi^+} \varphi_t(b_\beta)^{k_\beta} \right\|_{\rho,N_0}$$

$$\leq \rho^{p^{m(\alpha,t)}k} = \left\| \prod_{\beta \in \Phi^+} \varphi_t(b_\beta)^{k_\beta} \right\|_{\rho^{p^{m(\alpha,t)}},\varphi_t(N_0)},$$

since both norms are multiplicative on $D(\varphi_t(N_0))$. We obtain a norm-decreasing

homomorphism

$$D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(N_0) \xrightarrow{\sim} D_{[0,\rho_2^{p^{m(\alpha,t)}}]}(\varphi_t(N_0)) \to D_{r_t([0,\rho_2])}(\varphi(N_0))$$
$$\to D_{r_t([\rho_1,\rho_2])}(\varphi(N_0),\alpha).$$

The element $\varphi_t(b_\alpha)$ is invertible in $D_{r_t([\rho_1,\rho_2])}(\varphi_t(N_0),\alpha)$, and for each $\rho_1 \le \rho \le \rho_2$ and $x \in D_{[0,\rho_2]}(N_0)$, we have

$$\left\| \varphi_t(x)\varphi_t(b_\alpha)^{-k} \right\|_{r_t(\rho)} \le \|x\|_{\rho^{p^{m(\alpha,t)}},N_0} \|b_\alpha^{-k}\|_{\rho^{p^{m(\alpha,t)}},N_0}.$$

Therefore by the universal property of microlocalization [Zábrádi 2012, Proposition A.18], we obtain a norm-decreasing homomorphism

$$\varphi_{t,1} \colon D_{[\rho_1^{p^{m(\alpha,t)}},\rho_2^{p^{m(\alpha,t)}}]}(N_0,\alpha) \to D_{r_t([\rho_1,\rho_2])}(\varphi_t(N_0),\alpha), \quad b_\beta \mapsto \varphi_t(b_\beta). \quad (17)$$

This map is not surjective in general, by Example 4.5.

Lemma 4.8. *The map* (17) *is injective.*

Proof. Take an element

$$f(\boldsymbol{b}) = \sum_{\boldsymbol{k}} d_{\boldsymbol{k}} \boldsymbol{b}^{\boldsymbol{k}} \in D_{[\rho_1^{p^{m(\alpha,t)}},\rho_2^{p^{m(\alpha,t)}}]}(N_0,\alpha)$$

and pairwise distinct $\boldsymbol{k}_1,\dots,\boldsymbol{k}_r \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}$. Note that

$$\|\varphi_t(b_\beta)\|_{\rho,N_0} > \left\| \varphi_t(b_\beta) - \binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}} \right\|_{\rho,N_0};$$

hence we obtain

$$\left\| \sum_{j=1}^{r} d_{\boldsymbol{k}_j} \varphi_t(\boldsymbol{b})^{\boldsymbol{k}_j} \right\|_{r_t(\rho_1),r_t(\rho_2)} = \left\| \sum_{j=1}^{r} d_{\boldsymbol{k}_j} \prod_{\beta \in \Phi^+} \left(\binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}} \right)^{k_{j,\beta}} \right\|_{\rho_1,\rho_2}$$

$$= \max_j \left\| d_{\boldsymbol{k}_j} \prod_{\beta \in \Phi^+} \left(\binom{\beta(t)}{p^{m(\beta,t)}} b_\beta^{p^{m(\beta,t)}} \right)^{k_{j,\beta}} \right\|_{\rho_1,\rho_2}$$

$$= \max_j \left\| d_{\boldsymbol{k}_j} \varphi_t(\boldsymbol{b})^{\boldsymbol{k}_j} \right\|_{r_t(\rho_1),r_t(\rho_2)}$$

using Proposition 4.3, as we have $\prod_{\beta \in \Phi^+} b_\beta^{p^{m(\beta,t)}k_{j_1,\beta}} \ne \prod_{\beta \in \Phi^+} b_\beta^{p^{m(\beta,t)}k_{j_2,\beta}}$ for $1 \le j_1 \ne j_2 \le r$.

Since the map $\varphi_{t,1}$ is norm-decreasing, we have $\|d_{\boldsymbol{k}} \varphi_t(\boldsymbol{b})^{\boldsymbol{k}}\|_{r_t(\rho_1),r_t(\rho_2)} \to 0$ as $\boldsymbol{k} \to \infty$. Therefore we also have

$$\left\| \sum_{\boldsymbol{k}} d_{\boldsymbol{k}} \varphi_t(\boldsymbol{b})^{\boldsymbol{k}} \right\|_{r_t(\rho_1),r_t(\rho_2)} = \max_{\boldsymbol{k}} \left\| d_{\boldsymbol{k}} \varphi_t(\boldsymbol{b})^{\boldsymbol{k}} \right\|_{r_t(\rho_1),r_t(\rho_2)},$$

which is nonzero if there exists a $\boldsymbol{k}$ with $d_{\boldsymbol{k}} \ne 0$. Therefore the injectivity. $\square$

Taking projective and injective limits, we obtain an injective ring homomorphism

$$\varphi_{t,1} \colon \mathfrak{R}_0(N_0, \alpha) \hookrightarrow \mathfrak{R}_{0, r_t(\cdot)}(\varphi_t(N_0), \alpha)$$

for any $1 \leq_\alpha t \in T_+$.

Remark 4.9. Note that $\mathfrak{R}_{0, r_t(\cdot)}(\varphi_t(N_0), \alpha)$ is a subring of $\mathfrak{R}_0(N_0, \alpha)$ via the map $\iota_{1,t}$ (for all $t \in T_+$). Hence for $1 \leq_\alpha t$ we obtain a ring homomorphism $\varphi_t = \iota_{1,t} \circ \varphi_{t,1} \colon \mathfrak{R}_0(N_0, \alpha) \to \mathfrak{R}_0(N_0, \alpha)$. However, if $1 \not\leq_\alpha t$ for some $t \in T_+$, then we in fact do not have a continuous ring homomorphism $\varphi_t \colon \mathfrak{R}_0(N_0, \alpha) \to \mathfrak{R}_0(N_0, \alpha)$. Indeed, in this case there exists a $\beta \in \Phi^+$ such that $|\beta(t)| > |\alpha(t)|$, so there exist integers $k_\beta > k_\alpha$ such that for any $p^{-|\alpha(t)|} < \rho < 1$,

$$\left\| \varphi_t \left(b_\beta^{k_\beta} b_\alpha^{-k_\alpha} \right) \right\|_\rho = \rho^{\frac{k_\beta}{|\beta(t)|} - \frac{k_\alpha}{|\alpha(t)|}} > 1;$$

therefore

$$\sum_{n=1}^\infty \varphi_t \left(b_\beta^{n k_\beta} b_\alpha^{-n k_\alpha} \right)$$

does not converge in $\mathfrak{R}_0(N_0, \alpha)$ even though $\sum_{n=1}^\infty b_\beta^{n k_\beta} b_\alpha^{-n k_\alpha}$ does.

Remark 4.10. If $\Phi^+ \neq \Delta$ (for example, if $G = \mathrm{GL}_n(\mathbb{Q}_p)$, $n > 2$), then we have

$$\mathfrak{R}_0(N_0, \alpha) = \bigoplus_{u \in J(N_0/\varphi_t(N_0))} u \iota_{1,s} \big(\mathfrak{R}_{0, r_s(\cdot)}(\varphi(N_0), \alpha) \big)$$

$$\supsetneq \bigoplus_{n \in J(N_0/\varphi(N_0))} u \varphi(\mathfrak{R}_0(N_0, \alpha)),$$

by (15) (with the choice $t = s$) and Example 4.5 (which shows that $\varphi_{s,1}$ is not surjective).

In a similar fashion, we get for $t_1 \in T_+$ (and $1 \leq_\alpha t \in T_+$) an injective homomorphism

$$\varphi_{tt_1, t_1} \colon \mathfrak{R}_{0, r_{t_1}(\cdot)}(\varphi_{t_1}(N_0), \alpha) \to \mathfrak{R}_{0, r_{tt_1}(\cdot)}(\varphi_{tt_1}(N_0), \alpha).$$

In view of Lemma 4.7, we define

$$\mathcal{R}(N_1, \ell) := \varinjlim_{t \in T_+} \mathfrak{R}_{0, r_t(\cdot)}(\varphi_t(N_0), \alpha)$$

with respect to the maps φ_{t_1, t_2} for $t_2 \leq_\alpha t_1$.

Now take any $t \in T_+$ (not necessarily satisfying $1 \leq_\alpha t$). The map

$$\varphi_t := \varinjlim_{t_1} \iota_{t_1, tt_1} \colon \mathcal{R}(N_1, \ell) \to \mathcal{R}(N_1, \ell)$$

is defined as the direct limit of the inclusion maps

$$\iota_{t_1,tt_1}\colon \mathfrak{R}_{0,r_{tt_1}(\cdot)}(\varphi_{tt_1}(N_0),\alpha)\hookrightarrow \mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0),\alpha)$$

induced by $\varphi_{tt_1}(N_0)\subseteq \varphi_{t_1}(N_0)$. By definition, the ring $\mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$ for any $t\in T_+$ consists of formal power series $\sum_{\boldsymbol{k}} c_{\boldsymbol{k}}\varphi_t(\boldsymbol{b})^{\boldsymbol{k}}$ that converge in $\mathfrak{R}_0(N_0,\alpha)$. Therefore the map

$$\bigcup_{t\in T_+}\left\{\sum_{\boldsymbol{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} c_{\boldsymbol{k}}\boldsymbol{b}^{\boldsymbol{k}}\ \Big|\ \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}\varphi_t(\boldsymbol{b})^{\boldsymbol{k}} \text{ converges in } \mathfrak{R}_0(N_0,\alpha)\right\}\to \mathscr{R}(N_1,\ell),$$

$$\sum_{\boldsymbol{k}} c_{\boldsymbol{k}}\boldsymbol{b}^{\boldsymbol{k}}\mapsto \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}\varphi_t(\boldsymbol{b})^{\boldsymbol{k}}\in \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)\hookrightarrow \mathscr{R}(N_1,\ell)$$

is well-defined and bijective, since $\sum_{\boldsymbol{k}} c_{\boldsymbol{k}}\varphi_t(\boldsymbol{b})^{\boldsymbol{k}}$ converges for some $t\in T_+$ and the connecting homomorphisms in the injective limit defining $\mathscr{R}(N_1,\ell)$ are injective and given by φ_{t_1,t_2} for $t_2\leq_\alpha t_1$.

Hence we may identify

$$\mathscr{R}(N_1,\ell)$$
$$=\bigcup_{t\in T_+}\left\{\sum_{\boldsymbol{k}\in\mathbb{Z}^{\{\alpha\}}\times\mathbb{N}^{\Phi^+\setminus\{\alpha\}}} c_{\boldsymbol{k}}\boldsymbol{b}^{\boldsymbol{k}}\ \Big|\ \sum_{\boldsymbol{k}} c_{\boldsymbol{k}}\varphi_t(\boldsymbol{b})^{\boldsymbol{k}} \text{ convergent in } \mathfrak{R}_0(N_0,\alpha)\right\}\quad (18)$$

and obtain:

Proposition 4.11. *The natural map $\varphi_t\colon \mathscr{R}(N_1,\ell)\to \mathscr{R}(N_1,\ell)$ is injective for all $t\in T_+$, and we have the decomposition*

$$\mathscr{R}(N_1,\ell)=\bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\varphi_t(\mathscr{R}(N_1,\ell)).$$

In particular, $\mathscr{R}(N_1,\ell)$ is a free (right) module over itself via φ_t and it is a φ-ring over N_0 with $\varphi=\varphi_s$ in the sense of Definition 2.9.

Proof. By (16), we have

$$\mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0),\alpha)=\bigoplus_{n\in J(N_0/\varphi_t(N_0))} \varphi_{t_1}(n)\iota_{t_1,tt_1}\big(\mathfrak{R}_{0,r_{tt_1}(\cdot)}(\varphi_{tt_1}(N_0),\alpha)\big)$$

for any $t_1\in T_+$. The statement follows by taking the injective limit of both sides (with respect to t_1) and noting that

$$\varphi_{t_1,1}(n)=\varphi_{t_1}(n)\in \varphi_{t_1}(N_0)\subseteq \mathfrak{R}_{0,r_{t_1}(\cdot)}(\varphi_{t_1}(N_0),\alpha)$$

for $n\in N_0\subseteq \mathfrak{R}_0(N_0)$ and $1\leq_\alpha t_1$; therefore n corresponds to $\varinjlim_{1\leq_\alpha t_1}(\varphi_{t_1}(n))_{t_1}$ via the identification (18). $\square$

Remark 4.12. The ring $\mathfrak{R}(N_1, \ell)$ via the description (18) consists of exactly those Laurent-series

$$x = \sum_{\boldsymbol{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}} c_{\boldsymbol{k}} b^{\boldsymbol{k}}$$

that converge on the open annulus of the form

$$\{\rho_2 < |z_\alpha| < 1, \ |z_\beta| \le |z_\alpha|^r \text{ for } \beta \in \Phi^+ \setminus \{\alpha\}\}, \tag{19}$$

for some $p^{-1} < \rho_2 < 1$ and $1 \le r \in \mathbb{Z}$.

Proof. If $x \in \mathfrak{R}(N_1, \ell)$, then there exists a $t \in T_+$ such that $\varphi_t(x)$ converges in $\mathfrak{R}_0(N_0)$, that is, it converges in the norm $\|b_\beta\|_\rho = \rho$ for all $\beta \in \Phi^+$ for some fixed $p^{-1} < \rho_0 < 1$ and all $\rho \in (\rho_0, 1)$. By Lemma 4.7, we may assume that $|\alpha(t)| = 1$, whence $\|\varphi_t(b_\alpha)\|_\rho = \rho$ for all $\rho < 1$, as we may take $t = s_{\frac{k}{\alpha}}$ for k large enough. Now let $\rho_2 := \rho_0$ and $r := \max_{\beta \in \Phi^+}([|1/\beta(t)|] + 1) \in \mathbb{Z}$. Then x converges on the annulus (19), as we have $\rho^r \le \rho^{1/|\beta(t)|} \le \|\varphi_t(b_\beta)\|_\rho$ for all $\beta \in \Phi^+ \setminus \{\alpha\}$, by Lemma 4.4.

Conversely, for any fixed $p^{-1} < \rho_2 < 1$ and integer $r \ge 1$, we need to find a $t \in T_+$ and a $\rho_0 \in (p^{-1}, 1)$ such that for all $\rho \in (\rho_0, 1)$ we have $\rho_2 < \|\varphi_t(b_\alpha)\|_\rho < 1$ and $\|\varphi_t(b_\beta)\|_\rho \le \|\varphi_t(b_\alpha)\|_\rho^r$. We take $t := s_{\frac{k}{\alpha}}$ and $\rho_0 := \max(\rho_2, p^{-|\beta(t)|} \,|\, \beta \in \Phi^+ \setminus \{\alpha\})$, where

$$k := \max_{\beta \in \Phi^+ \setminus \{\alpha\}} \left(\left[-\frac{\log r}{\log |\beta(s_{\overline{\alpha}})|} \right] + 1 \right)$$

(for the definition of $s_{\overline{\alpha}}$ see the proof of Lemma 4.7). Indeed, since $|\alpha(s_{\frac{k}{\alpha}})|$ equals 1, we have

$$\rho_2 < \rho = \|\varphi_{s_{\frac{k}{\alpha}}}(b_\alpha)\|_\rho < 1$$

(for any k). On the other hand, we have $|\beta(s_{\overline{\alpha}})| < 1$ for all $\alpha \ne \beta \in \Phi^+$ (whence, in particular, the definition of k makes sense), so we obtain

$$\|\varphi_t(b_\beta)\|_\rho = \max_{0 \le j \le \mathrm{val}_p(\beta(t))} \left(\rho^{p^j} p^{j - \mathrm{val}_p(\beta(t))} \right) = \rho^{p^{\mathrm{val}_p(\beta(t))}} = \rho^{1/|\beta(s_{\overline{\alpha}})|^k} \le \rho^r$$

for all $\beta \in \Phi^+ \setminus \{\alpha\}$ by Lemma 4.4, with a choice of k such that $r \le 1/|\beta(s_{\overline{\alpha}})|^k$, and a choice of $p^{-|\beta(t)|} < \rho$ such that $\max_{0 \le j \le \mathrm{val}_p(\beta(t))} \left(\rho^{p^j} p^{j - \mathrm{val}_p(\beta(t))} \right) = \rho^{p^{\mathrm{val}_p(\beta(t))}}$. $\square$

Bounded rings. We write $\mathfrak{R}_0^b$ for the set of elements $x \in \mathfrak{R}_0$ such that $\lim_{\rho \to 1} \|x\|_{\rho, \rho}$ exists, and by $\mathfrak{R}_0^{\mathrm{int}}$ the subset for which that limit is at most 1. By Proposition A.28 in [Zábrádi 2012], these are subrings of $\mathfrak{R}_0$. Moreover, since φ_t is norm-decreasing for any $1 \le_\alpha t$ (see (17)), these subrings are stable under the action of φ_t ($1 \le_\alpha t \in T_+$).

We put

$$\mathfrak{R}^b_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) := \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) \cap \mathfrak{R}^b_0(N_0,\alpha),$$

$$\mathfrak{R}^{\mathrm{int}}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) := \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) \cap \mathfrak{R}^{\mathrm{int}}_0(N_0,\alpha),$$

where the intersection is taken inside $\mathfrak{R}_0$ under the inclusion

$$\iota_{1,t} \colon \mathfrak{R}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha) \hookrightarrow \mathfrak{R}_0.$$

Hence

$$\mathscr{R}^b(N_1,\ell) := \varinjlim_t \mathfrak{R}^b_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$$

and

$$\mathscr{R}^{\mathrm{int}}(N_1,\ell) := \varinjlim_t \mathfrak{R}^{\mathrm{int}}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)$$

are T_+-stable subrings of $\mathscr{R}(N_1,\ell)$ (the injective limit is taken with respect to the maps φ_{t_1,t_2} for $t_1 \leq_\alpha t_2 \in T_+$ as in the construction of $\mathscr{R}(N_1,\ell)$). Further, Lemma 4.6 shows that for any $t \in T_+$ and $x \in \mathfrak{R}_0$, we have

$$\lim_{\rho \to 1} \|x\|_\rho = \lim_{\rho \to 1} \|x\|_{q_t(r_t(\rho))}. \tag{20}$$

Indeed, we may use Lemma 4.6 in the context of $\mathfrak{R}_0$ the following way. The elements of $D_{[\rho_1,\rho_2]}(N_0,\alpha)$ are Cauchy sequences $(a_n\varphi_t(b_\alpha)^{-k_n})_{n\in\mathbb{N}}$ (in the norm $\max(\|\cdot\|_{\rho_1}, \|\cdot\|_{\rho_2})$) with $a_n \in D_{[0,\rho_2]}(N_0)$ and $k_n \geq 0$. Since $\|\cdot\|_\rho$ is multiplicative for any $\rho_1 \leq \rho \leq \rho_2$ in $p^{\mathbb{Q}}$ and so is its restriction to $D(\varphi_t(N_0))$, we compute

$$\left\| a_n\varphi_t(b_\alpha)^{-k_n} \right\|_\rho \rho^{\sum_{\beta\in\Phi^+}(p^{m(\beta,t)}-1)} = \frac{\|a_n\|_\rho}{\|\varphi_t(b_\alpha)^{k_n}\|_\rho \rho^{-\sum_{\beta\in\Phi^+}(p^{m(\beta,t)}-1)}}$$

$$\leq \frac{\|a_n\|_{q_t(r_t(\rho))}}{\|\varphi_t(b_\alpha)^{k_n}\|_{q_t(r_t(\rho))}}$$

$$= \left\| a_n\varphi_t(b_\alpha)^{-k_n} \right\|_{q_t(r_t(\rho))}$$

$$\leq \frac{\|a_n\|_\rho \rho^{-\sum_{\beta\in\Phi^+}(p^{m(\beta,t)}-1)}}{\|\varphi_t(b_\alpha)^{k_n}\|_\rho}$$

$$\leq \left\| a_n\varphi_t(b_\alpha)^{-k_n} \right\|_\rho \rho^{-\sum_{\beta\in\Phi^+}(p^{m(\beta,t)}-1)}.$$

If $\rho \to 1$ and $n \to \infty$, we obtain (20). Combining this observation with (15), we obtain

$$\mathfrak{R}^b_0(N_0,\alpha) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}\big(\mathfrak{R}^b_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)\big),$$

$$\mathfrak{R}^{\mathrm{int}}_0(N_0,\alpha) = \bigoplus_{n\in J(N_0/\varphi_t(N_0))} n\iota_{1,t}\big(\mathfrak{R}^{\mathrm{int}}_{0,r_t(\cdot)}(\varphi_t(N_0),\alpha)\big).$$

So by a similar argument as for $\mathscr{R}(N_1, \ell)$, we also obtain

$$\mathscr{R}^b(N_1, \ell) = \bigoplus_{n \in J(N_0/\varphi_t(N_0))} n\varphi_t\big(\mathscr{R}^b(N_1, \ell)\big),$$

$$\mathscr{R}^{\mathrm{int}}(N_1, \ell) = \bigoplus_{n \in J(N_0/\varphi_t(N_0))} n\varphi_t\big(\mathscr{R}^{\mathrm{int}}(N_1, \ell)\big);$$

in other words, these are φ-rings over N_0 in the sense of Definition 2.9.

Remark 4.13. By [Zábrádi 2012, Lemma A.27], an element $\sum_{\boldsymbol{k} \in \mathbb{N}^{\Phi^+ \setminus \{\alpha\}} \times \mathbb{Z}} c_{\boldsymbol{k}} \boldsymbol{b}^{\boldsymbol{k}}$ of $\mathscr{R}(N_1, \ell)$ (under the parametrization (18)) lies in $\mathscr{R}^b(N_1, \ell)$ (resp. in $\mathscr{R}^{\mathrm{int}}(N_1, \ell)$) if and only if $|c_{\boldsymbol{k}}|$ is bounded (resp. ≤ 1) for $\boldsymbol{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}$.

Relation with the completed Robba ring and overconvergent ring.

Lemma 4.14. *There exists a continuous (in the weak topology of $\Lambda_\ell(N_0)$) injective ring homomorphism $j_{\mathrm{int}} \colon \mathscr{R}^{\mathrm{int}}(N_1, \ell) \to \Lambda_\ell(N_0)$ respecting Laurent series expansions. The image of j_{int} is contained in $\mathbb{O}_{\mathscr{E}}^\dagger[\![N_1, \ell]\!] \subset \Lambda_\ell(N_0)$.*

Proof. We proceed in 3 steps. In Step 1 we construct a map $j_{\mathrm{int},0} = j_{\mathrm{int}|\mathfrak{R}_0^{\mathrm{int}}} \colon \mathfrak{R}_0^{\mathrm{int}} \to \Lambda_\ell(N_0)$ that is a priori continuous and o_K-linear. In Step 2 we show that $j_{\mathrm{int},0}$ is multiplicative, and hence a ring homomorphism. In Step 3 we extend it to $\mathscr{R}^{\mathrm{int}}(N_1, \ell)$ and show that the image lies in $\mathbb{O}_{\mathscr{E}}^\dagger[\![N_1, \ell]\!] \subset \mathbb{O}_{\mathscr{E}}[\![N_1, \ell]\!] = \Lambda_\ell(N_0)$.

Step 1. By Proposition 4.3 and Remark 4.13, we may write any element in $\mathfrak{R}_0^{\mathrm{int}}$ in a Laurent series expansion $\sum_{\boldsymbol{k} \in \mathbb{N}^{\Phi^+ \setminus \{\alpha\}} \times \mathbb{Z}} c_{\boldsymbol{k}} \boldsymbol{b}^{\boldsymbol{k}}$ with coefficients $c_{\boldsymbol{k}}$ in o_K. So we may collect all the terms containing $b_\alpha^{k_\alpha}$ for some fixed k_α into an element of the Iwasawa algebra $\Lambda(N_1)$ to obtain an expansion $\sum_{n \in \mathbb{Z}} b_\alpha^n f_n$ with $f_n \in \Lambda(N_1)$. These power series satisfy the convergence property that there exists a real number $p^{-1} < \rho_1 < 1$ such that $\rho^n \|f_n\|_\rho \to 0$ as $|n| \to \infty$ for all $\rho_1 < \rho < 1$. In particular, if $n \to -\infty$, then $f_n \to 0$ in the compact topology of $\Lambda(N_1)$. Hence the sum $\sum_n b_\alpha^n f_n$ also converges in $\Lambda_\ell(N_0)$. In this way we have obtained a right $\Lambda(N_0)$-linear injective map $j_{\mathrm{int},0} \colon \mathfrak{R}_0^{\mathrm{int}} \to \Lambda_\ell(N_0)$.

Recall that the weak topology (see [Schneider and Venjakob 2010; Schneider and Vignéras 2011; Schneider et al. 2012] for instance) on $\Lambda_\ell(N_0)$ is defined by the open neighborhoods of 0 of the form $\mathcal{M}(r) = \mathcal{M}_\ell(N_0)^r + \mathcal{M}(N_0)^r$, where $\mathcal{M}_\ell(N_0) = \Lambda_\ell(N_0)\mathcal{M}(N_1)$ denotes the maximal ideal of $\Lambda_\ell(N_0)$ and $\mathcal{M}(N_i)$ denotes the maximal ideal of $\Lambda(N_i) \subseteq \Lambda_\ell(N_0)$ ($i = 0, 1$). For any fixed $p^{-1} < \rho_1 < \rho < 1$, the preimage of $\mathcal{M}(r)$ in $\mathfrak{R}_0^{\mathrm{int}} \cap D_{[\rho_1, 1)}(N_1, \alpha)$ contains the open ball

$$\{x \mid \|x\|_\rho < p^{-r}\}.$$

Indeed, if $x = \sum_{n \in \mathbb{Z}} b_\alpha^n f_n$, then for any $n < 0$, we have $\|f_n\|_\rho < p^{-r}$, and hence $f_n \in \mathcal{M}(N_1)^r$ and $b_\alpha^n f_n \in \mathcal{M}_\ell(N_0)$. On the other hand, the positive part

$\sum_{n \geq 0} b_\alpha^n f_n$ lies in $\Lambda(N_0)$ and has ρ-norm smaller than p^{-r}, and therefore lies in $\mathcal{M}(N_0)^r$. Hence the continuity.

Step 2. Now by the continuity and linearity of $j_{\text{int},0}$, it suffices to show that it is multiplicative on monomials $\boldsymbol{b^k}$. Moreover, each monomial is a linear combination of elements of the form $b_\alpha^n g$ with $g \in N_0$. In order to expand the product $(b_\alpha^{n_1} g_1)(b_\alpha^{n_2} g_2)$ into a skew Laurent series, it suffices to expand $g_1 b_\alpha^{n_2}$ with $n_2 < 0$. However, if $g_1 b_\alpha^{n_2} = \sum_n b_\alpha^n h_n$ is the expansion in $\mathfrak{R}^{\text{int}}(N_1, \ell)$, then $\sum_{|n| < n_0} b_\alpha^n h_n b_\alpha^{-n_2}$ tends to g_1 (as $n_0 \to +\infty$) in the topology of $\mathfrak{R}_0^{\text{int}}$ (induced by the norms), and hence also in the weak topology. Therefore the expansion in $\Lambda_\ell(N_0)$ is also $g_1 b_\alpha^{n_2} = \sum_n b_\alpha^n h_n$. So the above constructed map $j_{\text{int},0}$ is indeed a ring homomorphism as claimed.

Step 3. Finally, take an element $x \in \mathfrak{R}^{\text{int}}(N_1, \ell)$. There exists an element $1 \leq_\alpha t \in T_+$ such that $\varphi_t(x)$ lies in the image of the composite map

$$\mathfrak{R}_{0, r_t(\cdot)}^{\text{int}}(\varphi_t(N_0), \alpha) \hookrightarrow \mathfrak{R}_0^{\text{int}}(N_0, \alpha) \hookrightarrow \mathfrak{R}^{\text{int}}(N_1, \ell),$$

where the first arrow is induced by the inclusion $\varphi_t(N_0) \subseteq N_0$. Now if we reduce $j_{\text{int},0}(\varphi_t(x)) \in \Lambda_\ell(N_0)$ modulo the ideal generated by $N_l - 1$ for some integer $l \geq 1$, then we obtain an element in $\varphi_t(\mathbb{O}_{\mathscr{C}}^\dagger[N_1/N_l, \ell])$. Indeed, $\varphi_t(\mathbb{O}_{\mathscr{C}}[N_1/N_l, \ell])$ is a closed subspace in $\mathbb{O}_{\mathscr{C}}[N_1/N_l, \ell]$ and all the monomials $j_{\text{int},0}(\varphi_t(\boldsymbol{b^k}))$ map into this subspace under the reduction modulo $(N_l - 1)$. Hence the image lies in $\varphi_t(\mathbb{O}_{\mathscr{C}}[N_1/N_l, \ell])$. By the convergence property of elements in $\mathfrak{R}_0^{\text{int}}$, we may expand

$$\varphi_t(x) = \sum_{n \in \mathbb{Z}} b_\alpha^n f_n$$

with $f_n \in \Lambda(N_1)$ and $\rho^n \|f_n\|_\rho \to 0$ as $n \to \infty$ for all $\rho_1 < \rho < 1$ and a fixed $p^{-1} < \rho_1 < 1$ depending on x. Since the reduction map $\Lambda(N_1) \to o[N_1/N_l]$ is continuous in the ρ-norm, we obtain that the reduction of $j_{\text{int},0}(\varphi_t(x))$ modulo $(N_l - 1)$ also lies in $\mathbb{O}_{\mathscr{C}}^\dagger[N_1/N_l, \ell]$. Hence we have $j_{\text{int},0}(\varphi_t(x)) \pmod{N_l - 1} \in \varphi_t(\mathbb{O}_{\mathscr{C}}^\dagger[N_1/N_l, \ell]) = \varphi_t(\mathbb{O}_{\mathscr{C}}[N_1/N_l, \ell]) \cap \mathbb{O}_{\mathscr{C}}^\dagger[N_1/N_l, \ell]$. Taking the limit, we see (using (7)) that $j_{\text{int},0}(\varphi_t(x))$ lies in

$$\varprojlim_l \varphi_t(\mathbb{O}_{\mathscr{C}}^\dagger[N_1/N_l, \ell]) = \varphi_t(\mathbb{O}_{\mathscr{C}}^\dagger[\![N_1, \ell]\!]),$$

so we put $j_{\text{int}}(x) := \varphi_t^{-1}\big(j_{\text{int},0}(\varphi_t(x))\big)$. This extends the ring homomorphism $j_{\text{int},0}$ to a continuous ring homomorphism $j_{\text{int}} \colon \mathfrak{R}^{\text{int}}(N_1, \ell) \hookrightarrow \mathbb{O}_{\mathscr{C}}^\dagger[\![N_1, \ell]\!] \subset \Lambda_\ell(N_0)$, by Lemma 4.7. This map is T_+-equivariant, as it respects power series expansions. $\square$

Now the following proposition compares $\mathfrak{R}(N_1, \ell)$ with the previous construction $\mathfrak{R}[\![N_1, \ell]\!]$.

Proposition 4.15. *There exists a natural T_+-equivariant ring homomorphism*

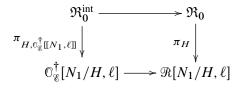$$j : \mathcal{R}(N_1, \ell) \to \mathcal{R}[\![N_1, \ell]\!]$$

with dense image.

Proof. At first we construct the map $j_0 = j_{|\mathfrak{R}_0}$ on $\mathfrak{R}_0 \subset \mathcal{R}(N_1, \ell)$ with dense image. We are going to show that for any open characteristic subgroup $H \leq N_1$, we have an isomorphism $\mathfrak{R}_0/\mathfrak{R}_0(H - 1) \cong \mathcal{R}[N_1/H, \ell]$. Note that N_1 being a compact p-adic Lie group, N_1 has a system of neighborhoods of 1 consisting of open uniform characteristic subgroups (in fact N_1 is uniform — since so is N_0 by assumption — and one can take repeatedly the Frattini subgroups of N_1 that are characteristic subgroups, that is, stable under all the continuous automorphisms of N_1). So we may assume without loss of generality that H is uniform with topological generators $h_1, h_2, \ldots, h_d$ with $d = \dim N_1$ as a p-adic Lie group.

Under the parametrization in Proposition 4.3, the elements of $\mathfrak{R}_0$ can be written as power series $\sum_{n \in \mathbb{Z}} b_\alpha^n f_n$ with $f_n \in D(N_1, K)$ and the convergence property that there exists a real number $\rho_1 < 1$ such that $\rho^n \|f_n\|_\rho \to 0$ (as $|n| \to \infty$) for all $\rho_1 \leq \rho < 1$. Now we have $D(N_1, K) = \bigoplus_{u \in J(N_1/H)} u D(H, K)$. Hence the right ideal $D(N_1, K)(H - 1)$ in $D(N_1, K)$ is generated by the elements $h_i - 1$ for $1 \leq i \leq d$ and it is the kernel of the natural projection $\pi_H : D(N_1, K) \to D(N_1/H) = K[N_1/H]$. This quotient map factors through the inclusion $D(N_1, K) \hookrightarrow D_{[0,\rho]}(N_1, K)$ for any $p^{-1} < \rho < 1$. Hence $\rho^n \|\pi_H(f_n)\| \to 0$, where $\|x\| := \max_u |x_u|$ with $x = \sum_{u \in N_1/H} x_u u$ and $x_u \in K$. Therefore we obtain a map

$$\pi_H : \mathfrak{R}_0 \to \mathcal{R}[N_1/H, \ell] = \bigoplus_{u \in N_1/H} \mathcal{R}u, \quad \sum_{n \in \mathbb{Z}} b_\alpha^n f_n \mapsto \sum_{u \in N_1/H} \sum_{n \in \mathbb{Z}} \pi_H(f_n)_u T^n u.$$

A priori this map is only known to be K-linear, continuous, and surjective between topological K-vector spaces. So for the multiplicativity, it suffices to show that $\pi_H(b^{k_1} b^{k_2}) = \pi_H(b^{k_1}) \pi_H(b^{k_2})$ for monomials b^{k_i} with $k_i \in \mathbb{N} \times \mathbb{Z}^d$ ($i = 1, 2$). On the other hand, these monomials are contained in the subring $\mathfrak{R}_0^{\mathrm{int}}$. By Lemma 4.14, we have a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{R}_0^{\mathrm{int}} & \longrightarrow & \mathfrak{R}_0 \\
{\scriptstyle \pi_{H, \mathcal{O}_{\mathscr{E}}^\dagger[\![N_1, \ell]\!]}} \downarrow & & \downarrow {\scriptstyle \pi_H} \\
\mathcal{O}_{\mathscr{E}}^\dagger[N_1/H, \ell] & \longrightarrow & \mathcal{R}[N_1/H, \ell]
\end{array}
$$

of o-modules such that all the maps are ring homomorphisms except possibly for π_H. However, from the commutativity of the diagram, it follows that π_H is also multiplicative on monomials and therefore a ring homomorphism. By taking the

projective limit of maps π_H, we obtain a ring homomorphism $j_0 \colon \mathfrak{R}_0 \to \mathfrak{R}[\![N_1, \ell]\!]$ with dense image and extending $j_{\text{int},0} \colon \mathfrak{R}_0^{\text{int}} \hookrightarrow \mathbb{O}_{\mathscr{E}}^{\dagger}$.

Finally, the homomorphism j_0 is extended to $\mathfrak{R}(N_1, \ell)$ as in the proof of Lemma 4.14. The T_+-equivariance is clear on monomials by Lemma 4.14 and follows in general from the continuity and linearity. $\qquad\square$

Remark 4.16. The map $j \colon \mathfrak{R}(N_1, \ell) \to \mathfrak{R}[\![N_1, \ell]\!]$ constructed above is not injective in general. Indeed, for any root $\beta \neq \alpha$ in Φ^+, the element $\log(n_\beta) = \log(1 + b_\beta)$ lies in $D(N_1) \subset \mathfrak{R}(N_1, \ell)$. It is easy to see that $\log(1 + b_\beta)$ is divisible by $\varphi^r(b_\beta)$ for any nonnegative integer r. Indeed, we clearly have $b_\beta \mid \log(1 + b_\beta)$. Applying φ^r on the both sides of the divisibility, we obtain

$$\varphi^r(b_\beta) \mid \varphi^r(\log(1 + b_\beta)) = \log(1 + b_\beta)^{p^{r m_\beta}} = p^{r m_\beta} \log(1 + b_\beta) \mid \log(1 + b_\beta),$$

as $p^{r m_\beta}$ is invertible in $\mathfrak{R}$. Therefore $\log(1 + b_\beta)$ lies in the kernel of π_H for all $H = N_r$, and hence also in the kernel of j.

Remark 4.17. Via the inclusion $\mathbb{O}_{\mathscr{E}}^{\dagger} \subseteq \mathfrak{R}$, we also have $\mathbb{O}_{\mathscr{E}}^{\dagger}[\![N_1, \ell]\!] \subseteq \mathfrak{R}[\![N_1, \ell]\!]$. However, if $N_1 \neq 1$, then we have $j_{\text{int}}(\mathfrak{R}^{\text{int}}(N_1, \ell)) \neq j(\mathfrak{R}(N_1, \ell)) \cap \mathbb{O}_{\mathscr{E}}^{\dagger}[\![N_1, \ell]\!] \subset \mathfrak{R}[\![N_1, \ell]\!]$.

Proof. Assume $N_1 \neq 1$ so that we have a positive root $\beta \neq \alpha \in \Phi^+$. We proceed in three steps. In Step 1, we construct an element $x \in \mathfrak{R}(N_1, \ell)$ with several properties. In Step 2, we show that $j(x)$ lies in $\mathbb{O}_{\mathscr{E}}^{\dagger}[\![N_1, \ell]\!] \subset \mathfrak{R}[\![N_1, \ell]\!]$. In Step 3, we prove that $j(x)$ does not lie in $j_{\text{int}}(\mathfrak{R}^{\text{int}}(N_1, \ell))$. The other inclusion, $j_{\text{int}}(\mathfrak{R}^{\text{int}}(N_1, \ell)) \subset j(\mathfrak{R}(N_1, \ell)) \cap \mathbb{O}_{\mathscr{E}}^{\dagger}[\![N_1, \ell]\!]$, is obvious.

Step 1. We denote by $s_n := \sum_{i=1}^n (-1)^{i+1} (b_\beta^i / i)$ the n-th estimating sum of $\log(1 + b_\beta) \in \mathfrak{R}(N_1, \ell)$. Note that $k_n := [\log_p n]$ is the smallest positive integer such that

$$p^{k_n} s_n \in \mathbb{Z}_p[N_{\beta,0}] \subseteq \mathfrak{R}(N_1, \ell), \tag{21}$$

where $[\,\cdot\,]$ denotes the integer part of a real number. We further choose a sequence of real numbers $p^{-1} < \rho_1 < \cdots < \rho_n < \cdots < 1$ in $p^{\mathbb{Q}}$ such that $\lim_{n \to \infty} \rho_n = 1$. Now for any fixed positive integer n, let i_n be the smallest positive integer satisfying the properties

$$\log_{\rho_{n-1}}\big(\|p^{k_{i_{n-1}}} \log(1 + b_\beta)\|_{\rho_{n-1}}\big) + 1 < \log_{\rho_n}\big(\|p^{k_{i_n}} \log(1 + b_\beta)\|_{\rho_n}\big),$$

$$\frac{\|\log(1 + b_\beta)\|_{\rho_n}}{p^n} > \|\log(1 + b_\beta) - s_{i_n}\|_{\rho_n}, \tag{22}$$

$$p^{k_{i_n}/2} > \|\log(1 + b_\beta)\|_{\rho_n},$$

$$\|\varphi^i(\log(1 + b_\beta))\|_{\rho_j} > \|\varphi^i(\log(1 + b_\beta) - s_{i_n})\|_{\rho_j}$$

for all $1 \leq i, j \leq n$. Such an i_n exists, as for any fixed $1 \leq i, j \leq n$, we have

$\lim_{k \to \infty} \|\varphi^i (\log(1 + b_\beta) - s_k)\|_{\rho_j} = 0$. The first condition in (22) makes the definition of i_n inductive. As a consequence, we have $\|\log(1 + b_\beta)\|_{\rho_n} = \|s_{i_n}\|_{\rho_n}$ by the ultrametric inequality. Now define $j_n \in \mathbb{Z}$ such that

$$\rho_n^{j_n+1} < \frac{\|s_{i_n}\|_{\rho_n}}{p^{k_{i_n}}} = \left\| p^{k_{i_n}} s_{i_n} \right\|_{\rho_n} = \left\| p^{k_{i_n}} \log(1 + b_\beta) \right\|_{\rho_n} \leq \rho_n^{j_n}. \qquad (23)$$

(In other words, $j_n = [\log_{\rho_n}(\|p^{k_{i_n}} \log(1 + b_\beta)\|_{\rho_n})]$.) By (21), we have $j_n \geq 0$. By the first condition in (22), the sequence $(j_n)_n$ is strictly increasing: $j_{n-1} < j_n$ for all $n > 1$. On the other hand,

$$(-1)^{p^{k_{in}}} b_\beta^{p^{k_{in}}}$$

is a summand in $p^{k_{in}} s_{i_n}$; therefore we have

$$\rho_n^{p^{k_{in}}} \leq \| p^{k_{in}} s_{i_n} \|_{\rho_n} \leq \rho_n^{j_n},$$

whence

$$j_n \leq p^{k_{in}} \leq i_n. \qquad (24)$$

Put $x := \sum_{n=1}^{\infty} p^{k_{in}} (\log(1 + b_\beta) - s_{i_n}) b_\alpha^{-j_n}$. Our goal in this step is to show that the sum x converges in $\mathfrak{R}_0(N_0, \alpha) \subset \mathfrak{R}(N_1, \ell)$. For this it suffices to verify that for any fixed $k \geq 1$, we have $\| p^{k_{in}} (\log(1 + b_\beta) - s_{i_n}) b_\alpha^{-j_n} \|_{\rho_k} \to 0$ as $n \to \infty$. In the power series expansion of $\log(1 + b_\beta) - s_{i_n}$, all the terms have degree $> i_n \geq j_n$ by (24). Therefore in the power series expansion of x all the terms have positive degree. In particular, for $k < n$ we have $\|y\|_{\rho_k} \leq \|y\|_{\rho_n}$ whenever y is a monomial in the expansion of x. By (22) and (23) we obtain

$$\left\| p^{k_{in}} (\log(1 + b_\beta) - s_{i_n}) b_\alpha^{-j_n} \right\|_{\rho_k} \leq \left\| p^{k_{in}} (\log(1 + b_\beta) - s_{i_n}) b_\alpha^{-j_n} \right\|_{\rho_n}$$

$$< \frac{\| p^{k_{in}} \log(1 + b_\beta)\|_{\rho_n}}{p^n} \rho_n^{-j_n} \leq \frac{1}{p^n}$$

for $k < n$. Hence we have $x \in \mathfrak{R}_0(N_0, \alpha) \subset \mathfrak{R}(N_1, \ell)$.

Step 2. By Remark 4.16, $\log(1 + b_\beta)$ lies in the kernel of π_H for all open normal subgroups $H \leq N_1$. Hence, by the continuity of π_H, we obtain $\pi_H(x) = \sum_{n=1}^{\infty} \pi_H(-p^{k_{in}} s_{i_n} b_\alpha^{-j_n}) \in \mathbb{O}_{\mathscr{E}}^\dagger[N_1/H, \ell] \subseteq \mathfrak{R}[N_1/H, \ell]$, as we have $-p^{k_{in}} s_{i_n} \in \mathbb{Z}_p[N_1]$ and $\mathbb{O}_{\mathscr{E}}^\dagger$ is closed in $\mathfrak{R}$.

Step 3. Assume finally that $j_{\text{int}}(z) = j(x)$ for some $z \in \mathfrak{R}^{\text{int}}(N_1, \ell)$. Both z and $j(x) \in \mathbb{O}_{\mathscr{E}}^\dagger[[N_1, \ell]] \subset \mathbb{O}_{\mathscr{E}}[[N_1, \ell]]$ have a power series expansion. By the injectivity of j_{int}, these expansions are equal. Hence, put $z = \sum_{\boldsymbol{k} \in \mathbb{Z} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}} d_{\boldsymbol{k}} \boldsymbol{b}^{\boldsymbol{k}}$ with $d_{\boldsymbol{k}} \in \mathbb{Z}_p$. By the definition of $\mathfrak{R}^{\text{int}}(N_1, \ell)$, there exists an element $t \in T_+$ such that $\varphi_t(z)$ lies in $\mathfrak{R}_0^{\text{int}}$. This means that there exists a positive integer K_0 such that for

all fixed $k \geq K_0$ and $\varepsilon > 0$ we have $\|\varphi_t(d_{\boldsymbol{k}} \boldsymbol{b^k})\|_{\rho_k} < \varepsilon$ for all but finitely many $\boldsymbol{k} \in \mathbb{Z} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}$. In particular, for any fixed $k \geq K_0$ we have

$$\left\|\varphi_t(-p^{k_{i_n}} s_{i_n} b_\alpha^{-j_n})\right\|_{\rho_k} < \varepsilon$$

for all but finitely many positive integers n since the sequence j_n is strictly increasing by construction; therefore the terms in $x = \sum_{n=1}^{\infty} p^{k_{i_n}}(\log(1 + b_\beta) - s_{i_n})b_\alpha^{-j_n}$ cannot cancel each other. Now we clearly have $\|\varphi_t(b_\alpha)\|_{\rho_k} \leq \rho_k$. On the other hand, we compute (for $n > \max(k, m(\beta, t))$ large enough)

$$\begin{aligned}
\|\varphi_t(-p^{k_{i_n}} s_{i_n})\|_{\rho_k} &= \frac{\|\varphi^{m(\beta,t)}(s_{i_n})\|_{\rho_k}}{p^{k_{i_n}}} \\
&= \frac{\|\varphi^{m(\beta,t)}(\log(1 + b_\beta))\|_{\rho_k}}{p^{k_{i_n}}} \\
&= \frac{\|\log(1 + b_\beta)\|_{\rho_k}}{p^{m(\beta,t)+k_{i_n}}}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\varepsilon &> \|\varphi_t(-p^{k_{i_n}} s_{i_n} b_\alpha^{-j_n})\|_{\rho_k} \\
&\geq \frac{\|\log(1 + b_\beta)\|_{\rho_k}}{p^{m(\beta,t)+k_{i_n}} \rho_k^{j_n}} \\
&> \frac{\rho_k \|\log(1 + b_\beta)\|_{\rho_k}}{p^{m(\beta,t)+k_{i_n}} \|p^{k_{i_n}} \log(1 + b_\beta)\|_{\rho_n}^{\log_{\rho_n} \rho_k}} \\
&= \frac{\rho_k \|\log(1 + b_\beta)\|_{\rho_k}}{p^{m(\beta,t)}} \frac{p^{k_{i_n}(\log_{\rho_n} \rho_k - 1)}}{\|\log(1 + b_\beta)\|_{\rho_n}^{\log_{\rho_n} \rho_k}} \\
&> \frac{\rho_k \|\log(1 + b_\beta)\|_{\rho_k}}{p^{m(\beta,t)}} p^{k_{i_n}(\frac{1}{2} \log_{\rho_n} \rho_k - 1)},
\end{aligned}$$

using (22) and (23). This is a contradiction, as the right side tends to ∞ as $n \to \infty$. Therefore $j(x)$ is not in the image of j_{int} as claimed. $\qquad\square$

Remark 4.18. The elements of $\mathfrak{R}[\![N_1, \ell]\!]$ cannot be expanded as a skew Laurent series of the form $\sum_{\boldsymbol{k} \in \mathbb{Z}^{\Phi^+}} d_{\boldsymbol{k}} \boldsymbol{b^k}$ in general. Indeed, the sum

$$\sum_{n=1}^{\infty} \varphi^n(b_\beta)/p^{2^n} = \sum_{n=1}^{\infty} ((b_\beta + 1)^{p^n} - 1)/p^{2^n}$$

converges in $\mathfrak{R}[\![N_1, \ell]\!]$ for any simple root $\beta \neq \alpha$ but does not have a skew Laurent-series expansion, as the coefficient of b_β in its expansion would be the nonconvergent sum $\sum_{n=1}^{\infty} p^{n-2^n}$.

We end this section with a diagram showing all the rings constructed.

$$
\begin{array}{ccccccc}
\mathbb{O}_{\mathcal{E}} & \lhook\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\longrightarrow & & & & \mathbb{O}_{\mathcal{E}}[\![N_1, \ell]\!] = \Lambda_\ell(N_0) \\
\uparrow & & & & & \downarrow \\
\mathbb{O}_{\mathcal{E}}^{\dagger} & \hookrightarrow \mathfrak{R}_0^{int}(N_0, \alpha) \hookrightarrow \mathcal{R}^{int}(N_1, \ell) & \xrightarrow{j_{int}} & \mathbb{O}_{\mathcal{E}}^{\dagger}[\![N_1, \ell]\!] \\
\mathcal{E}^{\dagger} & \hookrightarrow \mathfrak{R}_0^{bd}(N_0, \alpha) \hookrightarrow \mathcal{R}^{bd}(N_1, \ell) & \xrightarrow{j_{int} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p} & \mathcal{E}^{\dagger}[\![N_1, \ell]\!] \\
\mathcal{R} & \hookrightarrow \mathfrak{R}_0(N_0, \alpha) \hookrightarrow \mathcal{R}(N_1, \ell) & \xrightarrow{j} & \mathcal{R}[\![N_1, \ell]\!]
\end{array}
\tag{25}
$$

Here $\mathcal{R}(N_1, \ell)$ consists of Laurent series $\sum_{\boldsymbol{k}} c_{\boldsymbol{k}} \boldsymbol{b}^{\boldsymbol{k}}$ with $c_{\boldsymbol{k}} \in K$ that converge on the open annulus of the form

$$
\{ \rho_2 < |z_\alpha| < 1, \ |z_\beta| \le |z_\alpha|^r \text{ for } \beta \in \Phi^+ \setminus \{\alpha\} \}
$$

for some $0 < \rho_2 < 1$ and $1 \le r \in \mathbb{Z}$. The elements of $\mathfrak{R}_0(N_0, \alpha)$ are exactly those for which we can take $r = 1$. Their analogous integral (resp. bounded) versions consist of those Laurent series having the same convergence condition for which $c_{\boldsymbol{k}} \in o_K$ for all $\boldsymbol{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}}$ (resp. for which $\{ c_{\boldsymbol{k}} \mid \boldsymbol{k} \in \mathbb{Z}^{\{\alpha\}} \times \mathbb{N}^{\Phi^+ \setminus \{\alpha\}} \} \subset K$ bounded).

4.1. *Towards an equivalence of categories for overconvergent and Robba rings.* Propositions 3.1 and 3.7 apply in both the cases $R = \mathbb{O}_{\mathcal{E}}$ and $R = \mathbb{O}_{\mathcal{E}}^{\dagger}$. In both cases the category $\mathfrak{M}(R, \varphi)$ is the category of *étale φ-modules* over R. By the main result of [Cherbonnier and Colmez 1998] (see also [Kedlaya 2012]), we also have an equivalence of categories between finite free étale (φ, Γ)-modules over $\mathbb{O}_{\mathcal{E}}^{\dagger}$ and finite free étale (φ, Γ)-modules over $\mathbb{O}_{\mathcal{E}}$ given by the base change $\mathbb{O}_{\mathcal{E}} \otimes_{\mathbb{O}_{\mathcal{E}}^{\dagger}} \cdot \,$. On the other hand, T_ℓ acts by automorphisms on an object D in $\mathfrak{M}(\mathbb{O}_{\mathcal{E}}, T_+)$ and also on an object $D^{\dagger}$ in $\mathfrak{M}(\mathbb{O}_{\mathcal{E}}^{\dagger}, T_+)$. Since automorphisms correspond to automorphisms in an equivalence of categories, we obtain:

Proposition 4.19. *The functors*

$$
\mathbb{O}_{\mathcal{E}} \otimes_{\mathbb{O}_{\mathcal{E}}^{\dagger}} \cdot : \mathfrak{M}(\mathbb{O}_{\mathcal{E}}^{\dagger}, T_+) \to \mathfrak{M}(\mathbb{O}_{\mathcal{E}}, T_+), \quad \cdot^{\dagger} : \mathfrak{M}(\mathbb{O}_{\mathcal{E}}, T_+) \to \mathfrak{M}(\mathbb{O}_{\mathcal{E}}^{\dagger}, T_+)
$$

are quasi-inverse equivalences of categories.

For the Robba ring $\mathcal{R}$, étaleness is stronger than what we assumed for a module $D_{rig}^{\dagger}$ to belong to $\mathfrak{M}(\mathcal{R}, \varphi)$. The category $\mathfrak{M}(\mathcal{R}, \varphi)$ is just the category of φ-modules over the Robba ring. Recall that an object $D_{rig}^{\dagger}$ in $\mathfrak{M}(\mathcal{R}, \varphi)$ is *étale* (or unit-root, or pure of slope zero) whenever it comes from an overconvergent

étale φ-module $D^\dagger$ over the ring of "overconvergent" power series $\mathbb{O}_{\mathscr{E}}^\dagger$ by base extension. We denote by $\mathfrak{M}^0(\mathscr{R}, \varphi)$ the category of étale φ-modules over the Robba ring $\mathscr{R}$. We consequently define the categories $\mathfrak{M}^0(\mathscr{R}, T_+)$, $\mathfrak{M}^0(\mathscr{R}[\![N_1, \ell]\!], \varphi)$, and $\mathfrak{M}^0(\mathscr{R}[\![N_1, \ell]\!], T_+)$ as the full subcategory of étale objects in the corresponding categories without superscript 0. Via the equivalence of categories 3.7, étale objects correspond to each other. Combining this observation with the main result of [Berger 2002] leads to:

Corollary 4.20. *We have a commutative diagram of equivalences of categories*

$$
\begin{array}{ccccc}
\mathfrak{M}^0(\mathscr{R}, T_+) & \longleftarrow & \mathfrak{M}(\mathscr{E}^\dagger, T_+) & \longrightarrow & \mathfrak{M}(\mathscr{E}, T_+) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{M}^0(\mathscr{R}[\![N_1, \ell]\!], T_+) & \longleftarrow & \mathfrak{M}(\mathscr{E}^\dagger[\![N_1, \ell]\!], T_+) & \longrightarrow & \mathfrak{M}(\mathscr{E}[\![N_1, \ell]\!], T_+).
\end{array}
$$

Proof. The left horizontal arrows are also equivalences of categories by [Berger 2002], noting that T_ℓ acts via automorphisms on both types of objects in the upper row. $\square$

Remark 4.21. The category $\mathfrak{M}^0(\mathscr{R}[\![N_1, \ell]\!], T_+)$ of étale T_+-modules is embedded into the bigger category $\mathfrak{M}(\mathscr{R}[\![N_1, \ell]\!], T_+)$. So we may speak of *trianguline* objects in $\mathfrak{M}^0(\mathscr{R}[\![N_1, \ell]\!], T_+)$ as in the classical case (see for instance [Berger 2011]). Indeed, we call an object $M_{\mathrm{rig}}^\dagger$ in $\mathfrak{M}^0(\mathscr{R}[\![N_1, \ell]\!], T_+)$ trianguline if it becomes a successive extension of objects in $\mathfrak{M}(\mathscr{R}[\![N_1, \ell]\!], T_+)$ of rank 1 after a finite base extension $L \otimes_K \cdot$. It is clear that trianguline objects correspond to trianguline objects via the first vertical arrow in Corollary 4.20.

Remark 4.22. It would be interesting to construct a noncommutative version of the "big" rings $\tilde{A}_{\mathbb{Q}_p}$ and $\tilde{A}_{\mathbb{Q}_p}^\dagger$ in [Kedlaya 2012] and generalize (the proofs of) Theorems 2.3.5, 2.4.5, and 2.6.2 to this noncommutative setting. For this, one would need a generalization for results in the present paper to base fields other than $\mathbb{Q}_p$.

Remark 4.23. Since we have the natural inclusions $\mathbb{O}_{\mathscr{E}}^\dagger \hookrightarrow \mathscr{R}^{\mathrm{int}}(N_1, \ell) \hookrightarrow \mathbb{O}_{\mathscr{E}}^\dagger[\![N_1, \ell]\!]$ we have a fully faithful functor

$$
\Theta := \big(\mathscr{R}^{\mathrm{int}}(N_1, \ell) \otimes_{\mathbb{O}_{\mathscr{E}}^\dagger} \cdot\big) \circ \big(\mathbb{O}_{\mathscr{E}}^\dagger \otimes_{\ell, \mathbb{O}_{\mathscr{E}}^\dagger[\![N_1, \ell]\!]} \cdot\big) \circ \big(\mathbb{O}_{\mathscr{E}}^\dagger[\![N_1, \ell]\!] \otimes_{\mathscr{R}^{\mathrm{int}}(N_1, \ell)} \cdot\big)
$$

from the category $\mathfrak{M}(\mathscr{R}^{\mathrm{int}}(N_1, \ell), T_+)$ to itself. Whether or not it is essentially surjective (or equivalently, naturally isomorphic to the identity functor) is not clear. However, we have $\Theta \cong \Theta \circ \Theta$ naturally.

Proof. The faithfulness is clear since the objects in the category $\mathfrak{M}(\mathscr{R}^{\mathrm{int}}(N_1, \ell), T_+)$ are free modules, the maps $\mathbb{O}_{\mathscr{E}}^\dagger \hookrightarrow \mathscr{R}^{\mathrm{int}}(N_1, \ell) \hookrightarrow \mathbb{O}_{\mathscr{E}}^\dagger[\![N_1, \ell]\!]$ are injective, and the

functor

$$\mathbb{O}_{\mathscr{E}}^{\dagger} \otimes_{\ell, \mathbb{O}_{\mathscr{E}}^{\dagger}[[N_1, \ell]]} {}^{\bullet}$$

in the middle is an equivalence of categories by Proposition 3.7. The assertion $\Theta \cong \Theta \circ \Theta$ is also clear by Proposition 3.7. For the fullyness let $f : \Theta(\mathcal{M}_1) \to \Theta(\mathcal{M}_2)$ be a morphism in $\mathfrak{M}(\mathscr{R}^{\mathrm{int}}(N_1, \ell))$. Then we have $\Theta(f - \Theta(f)) = 0$, and by the faithfulness of Θ, we obtain $f = \Theta(f)$. □

Acknowledgements

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zger@cs.elte.hu   *Department of Algebra and Number Theory,*
                                    Institute of Mathematics, Eötvös Loránd University,
                                    Pázmány Péter sétány 1/C, Budapest H-1117, Hungary

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