**K3 surfaces and equations
for Hilbert modular surfaces**

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We outline a method to compute rational models for the Hilbert modular surfaces $Y_-(D)$, which are coarse moduli spaces for principally polarized abelian surfaces with real multiplication by the ring of integers in $\mathbb{Q}(\sqrt{D})$, via moduli spaces of elliptic K3 surfaces with a Shioda–Inose structure. In particular, we compute equations for all thirty fundamental discriminants D with $1 < D < 100$, and analyze rational points and curves on these Hilbert modular surfaces, producing examples of genus-2 curves over $\mathbb{Q}$ whose Jacobians have real multiplication over $\mathbb{Q}$.

1. Introduction

Hilbert modular surfaces have been objects of extensive investigation in complex and algebraic geometry, and more recently in number theory. Since Hilbert modular varieties are moduli spaces for abelian varieties with real multiplication by an order in a totally real field, they have intrinsic arithmetic content. Their geometry is enriched by the presence of modular subvarieties.

In [Hirzebruch 1973; Hirzebruch and van de Ven 1974; Hirzebruch and Zagier 1977] the geometric invariants of many of these surfaces were computed, and they were placed within the Enriques–Kodaira classification. A chief aim of the present work is to compute equations for birational models of some of these surfaces over the field of rational numbers.

More precisely, let D be a positive fundamental discriminant, i.e., the discriminant of the ring of integers $\mathcal{O}_D$ of the real quadratic field $\mathbb{Q}(\sqrt{D})$. The quotient $\mathrm{PSL}_2(\mathcal{O}_D) \backslash (\mathcal{H}^+ \times \mathcal{H}^-)$ (where $\mathcal{H}^+$ and $\mathcal{H}^-$ are the complex upper and lower half-planes) parametrizes abelian surfaces with an action of $\mathcal{O}_D$. It has a natural

compactification $Y_-(D)$, obtained by adding finitely many points and desingularizing these cusps.

These surfaces $Y_-(D)$ have models defined over $\mathbb{Q}$, and the main goal of this paper is to describe a method to compute explicit equations for these models, as well as to carry out this method for all fundamental discriminants D with $1 < D < 100$. This felt like a good place to stop for now, though these calculations may be extended to some higher D, as well as to non-fundamental discriminants.

We briefly summarize the method, which we describe in more detail in later sections. The method relies on being able to explicitly parametrize K3 surfaces that are related by Shioda–Inose structure to abelian surfaces with real multiplication by some $\mathcal{O}_D$. The K3 surface corresponding to such an abelian surface has Néron–Severi lattice containing L_D, a specific indefinite lattice of signature $(1, 17)$ and discriminant $-D$. In all our examples, we obtain the moduli space $\mathcal{M}_D$ of L_D-polarized K3 surfaces as a family of elliptic surfaces with a specific configuration of reducible fibers and sections.

We then use the 2- and 3-neighbor method to transform to another elliptic fibration, with two reducible fibers of types II* and III* respectively. This lets us read off the map (generically one-to-one) of moduli spaces from $\mathcal{M}_D$ into the 3-dimensional moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces, using the formulae from [Kumar 2008]. The image of $\mathcal{M}_D$ is the Humbert surface corresponding to discriminant D. The Hilbert modular surface $Y_-(D)$ itself is a double cover of the Humbert surface, branched along a union of modular curves. We use simple lattice arguments to obtain the branch locus, and pin down the exact twist for the double cover by counting points on reductions of the related abelian surfaces modulo several primes. In all our examples, the Humbert surface happens to be a rational surface (i.e., birational to $\mathbb{P}^2$ over $\mathbb{Q}$), and we display the equation of $Y_-(D)$ as a double cover of $\mathbb{P}^2$ branched over a curve of small degree. We analyze these equations in some detail, attempting to produce rational or elliptic curves on them, with the intent of producing several (possibly infinitely many) examples of genus-2 curves whose Jacobians have real multiplication. When $Y_-(D)$ is a K3 surface, it often has very high Picard number (19 or 20), and we attempt to compute generators for the Picard group. When $Y_-(D)$ is an honestly elliptic surface, we analyze the singular fibers and the Mordell–Weil group, and attempt to compute a basis for the sections.

To our knowledge, this is the first algebraic description of most of these surfaces by explicit equations. We outline some related work in the literature. Wilson [1998; 2000] obtained equations for the Hilbert modular surface $Y_-(5)$ corresponding to the smallest fundamental discriminant $D > 1$. Van der Geer [1988] gives a few examples of algebraic equations for Hilbert modular surfaces corresponding to a congruence subgroup of the full modular group (in other words, abelian surfaces

with some level structure). Humbert surfaces have also been well-studied in the literature, and Runge [1999] and Gruenewald [2008] have obtained equations for some of these. However, these equations are quite complicated, and do not shed as much light on the geometry of Hilbert modular surfaces. While the methods are simpler, involving theta functions and q-expansions, the result is analogous to exhibiting the modular polynomial whose zero locus in $\mathbb{A}^1 \times \mathbb{A}^1$ is a singular model of the complement of the cusps in the modular curve $X_0(N)$. The coefficients of these polynomials can quickly become enormous. We believe that our approach, giving simpler equations for these surfaces together with their maps to $\mathcal{A}_2$, is more conducive to an investigation of arithmetic properties.

It is our hope that these equations will be of much help in subsequent arithmetic investigation of these surfaces. For instance, they should provide a testing ground for many conjectures in Diophantine geometry, because of the abundance of rational curves and points. Another direction of future investigation is to use these equations to investigate modularity of the corresponding abelian surfaces. Modularity of abelian varieties with real multiplication over $\mathbb{Q}$ is now proven, by combining results of Ribet [2004] with the recent proof of Serre's conjecture by Khare and Wintenberger [2009a; 2009b]. However, unlike the case of dimension 1, where one has modular parametrizations and very good control of the moduli spaces, the situation in dimensions 2 and above is much less clear. For instance, it is not at all clear how to find a modular form corresponding to a given abelian surface with real multiplication.[1] We hope that the abundance of examples provided by these equations will help pave the path for a better understanding of the 2-dimensional case. For example, in [Dembélé and Kumar 2013], our formulas are combined with efficient computation of Hilbert modular forms to find examples of simple abelian surfaces over real quadratic fields, with everywhere good reduction. An example of such an abelian surface is the Jacobian of the genus-2 curve

$$2y^2 = x^6 - \tau x^5 + 74x^4 - 14\tau x^3 + 267x^2 - 13\tau x + 46$$
$$= \left(x^3 - \frac{1+\tau}{2}x^2 + 13x + \frac{3-\tau}{2}\right)\left(x^3 + \frac{1-\tau}{2}x^2 + 13x - \frac{3+\tau}{2}\right),$$

where $\tau = \sqrt{193}$ (the curve and its Jacobian can in fact be defined over $\mathbb{Q}$, but the Jacobian attains everywhere good reduction and real multiplication by $\mathcal{O}_{17}$ only over $\mathbb{Q}(\tau)$).

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[1] Suppose $A/\mathbb{Q}$ is an abelian surface with $\mathbb{Q}$-endomorphisms by $\mathcal{O}_D$, and let $\varphi = \sum_n a_n q^n$ be an eigenform with every $a_n \in \mathcal{O}_D$. If φ corresponds to A then counting points over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ determines each a_p up to Galois conjugation. But conceivably there might be some eigenform $\varphi' = \sum_n a'_n q^n$, different from both φ and its Galois conjugate, such that each a'_p equals either a_p or the Galois conjugate of a_p; if that happens, we do not know how to decide which eigenform corresponds to A. Likewise for abelian varieties of dimension 3 and higher.

The outline for the rest of the paper is as follows. In Section 2, we describe the relevant background on K3 surfaces and their moduli spaces, and their connection to moduli spaces of abelian surfaces via Shioda–Inose structures. In Section 3, we describe the Hilbert modular surfaces and the corresponding moduli spaces of K3 surfaces. In Section 4, we precisely describe our methods to compute their equations. Section 5 describes the 2- and 3-neighbor method in more detail. The rest of the paper consists of detailed examples of Hilbert modular surfaces for the discriminants less than 100, as well as an arithmetic investigation of these surfaces.

An accompanying online supplement contain formulas for the Igusa–Clebsch invariants, as well as a description of the parametrizations exhibited for the moduli spaces of K3 surfaces in the paper, and the details of the neighbor steps to transform to a fibration with II^* and III^* fibers. The online supplement can be obtained from this article's publication page. The files are also available as part of the source package for the arXiv version of this article (arXiv 1209.3527). See the file README.txt for an overview.

2. Moduli spaces of abelian surfaces and lattice polarized K3 surfaces

Throughout this section, we work with K3 surfaces over a field k of characteristic 0. When convenient, we will suppose $k \subseteq \mathbb{C}$, and use transcendental methods.

2.1. *K3 surfaces.* A K3 surface X over k is a projective algebraic nonsingular surface with $h^1(X, \mathcal{O}_X) = 0$ and $K_X \cong \mathcal{O}_X$. For such a surface, $H^2(X, \mathbb{Z})$ is torsion-free, and when endowed with the cup-product form becomes a 22-dimensional lattice, abstractly isomorphic with $\Lambda := E_8(-1)^2 \oplus U^3$. Here E_8 is the even unimodular lattice in eight dimensions, U is the hyperbolic plane with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and for any lattice Λ and real number α, the lattice $\Lambda(\alpha)$ consists of the same underlying abelian group with the form multiplied by α. The Néron–Severi group $NS(X)$ of algebraic divisors defined over $\bar{k}$ modulo algebraic equivalence, which for a K3 surface is the same as linear or numerical equivalence, is a primitive sublattice of Λ of signature $(1, \rho - 1)$, where $\rho \in \{1, \ldots, 20\}$ is the Picard number of X. The orthogonal complement of $NS(X)$ is the transcendental lattice T_X. There is a Torelli theorem for K3 surfaces, due to Piatetski-Shapiro and Shafarevich [1971] and Friedman [1984], which describes the moduli space of K3 surfaces with a fixed polarization. More generally, let L be an even nondegenerate lattice of signature $(1, r - 1)$, with $r \in \{1, \ldots, 20\}$. Assume that L has a unique primitive embedding in Λ, up to isometries of Λ. Then there is a coarse moduli space $\mathcal{F}_L$ of L-polarized K3 surfaces (X, j), where $j : L \to NS(X)$ is a primitive lattice embedding such that $j(L)$ contains a pseudo-ample class on X. The space $\mathcal{F}_L$ is isomorphic to the quotient of an appropriate fundamental

domain

$$\Omega_L = \mathbb{P}\big(\{\omega \in L^{\perp} \otimes \mathbb{C} \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0\}\big)$$

by an arithmetic subgroup Γ_L, which is the image of

$$\Gamma(L) = \{\sigma \in O(\Lambda) \mid \forall x \in L, \sigma(x) = x\}$$

in $O(\Lambda^{\perp})$. (Here, we have fixed an embedding $i : L \to \Lambda$, so we may use its orthogonal complement $L^{\perp}$.) Therefore $\mathcal{F}_L$ is a quasiprojective variety.

In fact, there is a fine moduli space $\mathcal{K}_L$ of marked pseudo-ample L-polarized K3 surfaces, i.e., (X, ϕ), where $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ is an isomorphism (a *marking*) such that $\phi^{-1}(L) \subseteq \mathrm{NS}(X)$. There is a period map which associates to such a marked K3 surface the class of the global algebraic 2-form up to scaling, giving a point $[\omega] \in \mathbb{P}(L^{\perp} \otimes \mathbb{C})$. Furthermore, $\omega \cup \omega = 0$ and $\omega \cup \overline{\omega} > 0$. This domain Ω_L consists of two copies of a bounded Hermitian domain of type IV_{20-r}. The period map $(X, \phi) \to [\omega_X]$ sets up an isomorphism between the moduli space $\mathcal{K}_L$ and the period domain Ω_L, using the Torelli theorem and the surjectivity of the period map [Kulikov 1977; Persson and Pinkham 1981]. The quotient $\Gamma_L \backslash \mathcal{K}_L \cong \Gamma_L \backslash \Omega_L$ forgets the marking, and describes a coarse moduli space of L-polarized K3 surfaces. For details, the reader may consult [Nikulin 1979a; Dolgachev 1996].

2.2. *Elliptic K3 surfaces.* We shall be especially interested in moduli spaces of elliptic K3 surfaces. In this paper, an elliptic K3 surface will be a K3 surface X with a relatively minimal genus-1 fibration $\pi : X \to \mathbb{P}^1$, together with a section. In other words, we may write a Weierstrass equation of X over $\mathbb{P}^1_t$ as

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t), \tag{1}$$

with $a_i(t)$ a polynomial in t of degree at most $2i$. (More canonically, each a_i is a homogeneous polynomial of degree $2i$ in the two homogeneous coordinates of $\mathbb{P}^1_t$.) Of course, this Weierstrass equation describes the generic fiber of X; to understand the reducible special fibers, one can use Tate's algorithm [1975] to blow up the singular points and describe the minimal proper model. (This will also detect when a Weierstrass equation (1) is equivalent to one with each a_i vanishing to order at least i at some t_0, that is, when the equation gives not a K3 elliptic surface but a rational or constant one.) The singular fibers are classified by Kodaira and Néron, and the non-identity components of any reducible fiber $\pi^{-1}(v)$ contribute an irreducible root lattice (scaled by -1), say L_v, to the Néron–Severi lattice. The *trivial lattice* is defined to be

$$T = \mathbb{Z}O \oplus \mathbb{Z}F \oplus \left(\bigoplus_v L_v\right)$$

(note that O and F span a copy of the hyperbolic plane U).

A theorem of Shioda and Tate [Shioda 1972; 1990; Tate 1966b] shows that as long as the elliptic fibration is nontrivial (equivalently, it has at least one singular fiber), the Mordell–Weil group of X over $\mathbb{P}^1$ is isomorphic to $\mathrm{NS}(X)/T$. In particular, we have the Shioda–Tate formula

$$\rho(X) = 2 + \mathrm{rank}\, \mathrm{MW}(X/\mathbb{P}^1) + \sum_v \mathrm{rank}\, L_v.$$

One may also compute the discriminant of the Néron–Severi lattice:

$$|\mathrm{disc}(\mathrm{NS}(X))| = \frac{\det(H(X/\mathbb{P}^1)) \cdot \prod_v \mathrm{disc}(L_v)}{|\mathrm{MW}(X/\mathbb{P}^1)_{\mathrm{tors}}|^2},$$

where $H(X/\mathbb{P}^1)$ is the height pairing matrix for a basis of the torsion-free part of the Mordell–Weil group of X over $\mathbb{P}^1$.

If F is the class of the fiber for an elliptic K3 surface X, then F is primitive and nef, with $F^2 = 0$. Conversely, suppose that $F \in \mathrm{NS}(X)$ is a nonzero divisor class which is primitive and nef with $F^2 = 0$. Then a simple application of the Riemann–Roch theorem shows that F or $-F$ must be effective, and since F is nef, it must be represented by an effective divisor. Then a theorem of Piatetski-Shapiro and Shafarevich [1971, p. 559] shows that F defines a genus-1 fibration.

Lemma 1. *Let $F = \sum a_i E_i$ be a positive linear combination of smooth rational curves on a K3 surface X such that $F \cdot E_i = 0$ for all i, and such that F is a primitive class in $\mathrm{NS}(X)$. Then F defines a genus-1 fibration.*

Proof. We have $F^2 = \sum a_i (F \cdot E_i) = 0$. By the above discussion, it is enough to show that F is nef. Let E' be an irreducible curve on X. If E' is distinct from the E_i, then $E_i \cdot E' \geq 0$ for every i, and so $F \cdot E' \geq 0$. On the other hand, if $E' = E_i$, say, then $F \cdot E' = F \cdot E_i = 0$. Therefore F is nef. $\square$

Let us define an *elliptic divisor* to be a divisor satisfying the conditions of Lemma 1. We will frequently use this lemma, displaying an elliptic divisor by finding a subdiagram of the set of roots of $\mathrm{NS}(X)$, represented by smooth rational curves, which is an extended Dynkin diagram for a root lattice. Then the class of the appropriate linear combination of roots F will define a genus-1 fibration. We need to know when such a fibration has a section.

Lemma 2. *Suppose F is an elliptic divisor defining a genus-1 fibration $\pi : X \to \mathbb{P}^1$. Suppose $D \in \mathrm{NS}(X)$ satisfies $D \cdot F = 1$. Then π has a section.*

Proof. Consider the divisor $D' = D + mF$, for some large integer m. Then $(D')^2 = D^2 + 2m$, while $K \equiv 0$, so the Riemann–Roch theorem implies

$$h^0(D') - h^1(D') + h^2(D') = \frac{(D')^2}{2} + \chi(\mathcal{O}_X) = D^2 + m + 2.$$

Also, $h^2(D') = h^0(K - D') = h^0(-D') = 0$ by Serre duality, and since $(-D') \cdot H = -D \cdot H - mF \cdot H < 0$ for any ample divisor H, as long as m is large enough. Therefore we see that for large m, the divisor class D' can be represented by an effective divisor, which we may call D', by abuse of notation. Note that we still have $D' \cdot F = 1$. Decompose D' as $D'_{\text{vert}} + D'_{\text{hor}}$, where the first term contains all the components which lie along fibers of the genus-1 fibration defined by F, and the second contains the other components. Then $D'_{\text{hor}} \cdot F = 1$. Therefore, D'_{hor} must be reduced and irreducible, and thus defines a section of the genus-1 fibration. $\square$

Corollary 3. *Let F be an elliptic divisor, and let D_1 and D_2 be two divisor classes such that $D_1 \cdot F$ and $D_2 \cdot F$ are coprime. Then the fibration has a section.*

Proof. There exist integers a_1, a_2 such that $(a_1 D_1 + a_2 D_2) \cdot F = 1$. Now take $D = a_1 D_1 + a_2 D_2$ in Lemma 1. $\square$

Finally, we note a lattice-theoretic result which allows us to deduce that in all of the cases studied in this paper, the genus-1 fibration defined by an elliptic divisor F has a section.

Proposition 4. *Let D be a fundamental discriminant, and let $L = U \oplus N(-1)$, where U is the hyperbolic plane and N a positive definite lattice of rank 16 and discriminant D. Suppose in addition that N contains a sublattice isomorphic to $E_8 \oplus E_7$. If $v \in L$ is a primitive vector with $v \cdot v = 0$, then there exists $w \in L$ such that $v \cdot w = 1$.*

Proof. Suppose not. Then $\{v \cdot w : w \in L\} = c\mathbb{Z}$ for some integer $c > 1$. Since v is primitive, we can take a basis $v_1 = v, \ldots, v_{16}$ of L. Then

$$L' = \mathbb{Z}(v/c) + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_{16}$$

is an integral lattice containing L with index $c > 1$. Since $L' \supset L \supset U$, we have $L' = U \oplus N'(-1)$ (since U is unimodular), with N' a positive definite lattice containing N with index c. Then N' must be generated by $E_8 \oplus E_7$ and a vector x whose projection $x^\perp$ to the orthogonal complement of $E_8 \oplus E_7$ has norm $D/(2c^2)$. The dual lattice of $E_8 \oplus E_7$ has norms congruent to 0 or $\frac{3}{2}$ modulo 2, so $x^\perp$ has norm 0 or $\frac{1}{2}$ mod 2. Therefore D/c^2 must be an integer congruent to 0 or 1 modulo 4. Since D is a fundamental discriminant, this is impossible. $\square$

For the examples in this paper, we will often draw a Dynkin-type diagram indicating some of the roots of $\text{NS}(X)$ for an elliptic K3 surface X, which will always be nodal classes (i.e., classes of smooth rational curves on X). We will outline an elliptic divisor F by drawing a subdiagram in bold which cuts out an extended Dynkin diagram of an irreducible root lattice (the multiplicities will be omitted). Where convenient, we will also indicate the class of a divisor D such that $D \cdot F = 1$ (in some cases, there is an obvious node in the Dynkin diagram satisfying

this property), or the classes of two divisors D_1 and D_2 such that $D_1 \cdot F$ and $D_2 \cdot F$ are coprime. This is not strictly necessary, because Proposition 4 guarantees the existence of such a divisor D, but having an explicit divisor might be useful for further calculations. Then F defines another elliptic fibration with section on X, and we may proceed as in Section 5 to write down its Weierstrass equation.

2.3. *Kummer surfaces, Nikulin involutions and Shioda–Inose structures.* Let A be an abelian surface, and consider the involution ι on A defined by multiplication by -1 in the group law. The quotient of A by the group $\{1, \iota\}$ is a surface Y' with sixteen singularities, the images of the 2-torsion points of A. In fact, Y' may be realized as a quartic surface in $\mathbb{P}^3$ with sixteen ordinary double points (which is the maximum number of singularities possible for a quartic surface in $\mathbb{P}^3$ [Hudson 1990, p. 15]), by considering the linear system on A corresponding to twice the theta divisor. The corresponding map is two-to-one from A to Y'.

Taking the minimal desingularization of Y' gives a K3 surface Y, the Kummer surface of A, which contains sixteen disjoint nodal classes coming from the blowups of the singular points. Note that if A is defined over some number field k, then so is $Y = \text{Km}(A)$. The Néron–Severi lattice of the surface Y contains the saturation of the lattice spanned by the sixteen special nodal classes; this is a lattice Λ_{Km} of signature $(0, 16)$ and discriminant 2^6. Conversely, Nikulin showed that a K3 surface whose Néron–Severi lattice contains Λ_{Km} must be the Kummer surface of some complex torus. Of course, since A is an abelian variety, Y is a projective surface, so $\text{NS}(Y)$ contains an ample divisor as well.

We will be especially concerned with the case when $A = J(C)$ is the Jacobian of a curve of genus 2. Let $x_0, \ldots, x_5$ be the Weierstrass points of C. The embedding $\eta_0 : C \to A$ given by $x \mapsto [x] - [x_0]$ gives a particular theta divisor on A, and the translates $\eta_0(C) + [x_i] - [x_j]$ with $0 \le i < j \le 5$ give fifteen more special divisors. The images of these sixteen divisors (tropes) on the Kummer surface of A are disjoint rational curves, and each intersects six rational curves coming from the blowups of the singular points (i.e., the nodes). This classical configuration of tropes and nodes on the Kummer surface is called the $(16, 6)$ configuration, and the intersection pairing describes a vertex- and edge-transitive bipartite graph of degree 6 on 32 vertices, isomorphic with the quotient of the 6-cube by central reflection.

Next, consider a K3 surface X with a symplectic involution ι, i.e., an involution ι that multiplies the algebraic 2-forms on X by $+1$ (such an involution of X is also known as a Nikulin involution). Then ι has eight fixed points on X, and the minimal desingularization Y of the quotient $X/\{1, \iota\}$ is again a K3 surface. If in addition Y is a Kummer surface $\text{Km}(A)$ and the quotient map $\pi : X \to Y$ induces a Hodge isometry $\pi_* : T_X(2) \cong T_Y$, we say that X and A are related by a Shioda–Inose

structure. We have a diagram
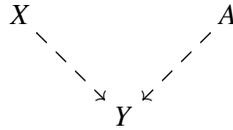
of rational maps of degree 2, and Hodge isometries $T_X(2) \cong T_Y \cong T_A(2)$, thus inducing a Hodge isometry $T_X \cong T_A$. (Note: A Hodge isometry is an isometry of cohomology lattices compatible with the Hodge decomposition.)

Conversely, a theorem of Morrison [1984] shows that any Hodge isometry between T_X and T_A for a K3 surface X and an abelian surface A is induced by a Shioda–Inose structure.

2.4. *Elliptic K3 surfaces with* II* *and* III* *fibers, and curves of genus* 2. We shall exploit such a Shioda–Inose correspondence between Jacobians of genus-2 curves and elliptic K3 surfaces with singular fibers of type II* and III*; equivalently, whose root lattices L_v are E_8 and E_7 respectively. Let C be a curve of genus 2 over k, and let

$$y^2 = f(x) = f_6 x^6 + \cdots + f_0 = f_6 \prod (x - \alpha_i)$$

be a Weierstrass equation for C, with $f_i \in k$ and $\alpha_i \in \bar{k}$ (though in general $\alpha_i \notin k$). There exist polynomial functions $I_2(f)$, $I_4(f)$, $I_6(f)$ and $I_{10}(f) = \operatorname{disc}(f)$ of degrees $2, 4, 6, 10$ in the coefficients of f (the Igusa–Clebsch invariants of f), giving a well-defined point $(I_2 : I_4 : I_6 : I_{10})$ in weighted projective space $\mathbb{P}^3_{1,2,3,5}$ which does not depend on the choice of Weierstrass equation. The complement of the hyperplane $z_4 = 0$ (where z_4 is the last coordinate on $\mathbb{P}^3_{1,2,3,5}$) yields a coarse moduli space $\mathcal{M}_2$ of curves of genus 2 [Igusa 1960]. Note that I_{10} is the discriminant of the sextic polynomial f, and therefore it cannot vanish. Also, the space $\mathcal{M}_2$ has a singular point at $(0:0:0:1)$, corresponding to the curve $y^2 = x^5 + 1$. Given $\alpha_1, \alpha_2, \alpha_3, \alpha_5 \in k$, with $\alpha_5 \neq 0$, it is not necessarily the case that one can construct a genus-2 curve over k with invariants $I_d = \alpha_{2d}$. There is an obstruction in $\operatorname{Br}_2(k)$: when it vanishes, the construction of C is made explicit by Mestre [1991]. In any case, C may always be defined over a quadratic extension of k. When k is a finite field, the Brauer obstruction vanishes, and we may define C over k. Also note that such a curve C is unique only up to $\bar{k}$-isomorphism, since $\mathcal{M}_2$ is only a coarse moduli space.

The main result of [Kumar 2008] is the following.

Theorem 5. *The elliptic* K3 *surface given by the Weierstrass equation*

$$y^2 = x^3 - t^3\left(\frac{I_4}{12}t + 1\right)x + t^5\left(\frac{I_{10}}{4}t^2 + \frac{I_2 I_4 - 3I_6}{108}t + \frac{I_2}{24}\right),$$

which has elliptic fibers of type E_8 and E_7 respectively at $t = \infty$ and $t = 0$, is
related by a Shioda–Inose structure to the Jacobian of the genus-2 curve C whose
Igusa–Clebsch invariants are $(I_2 : I_4 : I_6 : I_{10})$.

Let L be $U \oplus E_8(-1) \oplus E_7(-1)$. Then the above theorem gives an isomorphism

$$\psi : \mathcal{M}_2 \to \mathcal{E}_{E_8, E_7}$$

between the coarse moduli space $\mathcal{M}_2$ of genus-2 curves and the moduli space
of elliptic K3 surfaces with an E_8 fiber at ∞ and an E_7 fiber at 0. Furthermore,
this correspondence is Galois-invariant: the Igusa–Clebsch invariants of C and the
Weierstrass coefficients of the K3 surface are defined over the same field. This is
the key fact which leads to number-theoretic applications, such as computation of
models of Shimura curves over $\mathbb{Q}$ in [Elkies 2008] or of Hilbert modular surfaces
in this paper. However, note that C may not be itself defined over the ground field,
even though its Igusa–Clebsch invariants are.

Now, let $\mathcal{A}_2$ be the moduli space of principally polarized abelian surfaces. Note
that the space $\mathcal{M}_2$ is the complement of the divisor in $\mathcal{A}_2$ consisting of points
corresponding to the product of two elliptic curves. On the other hand, the moduli
space $\mathcal{E}_{E_8, E_7}$ is an open subset of the moduli space $\mathcal{F}_L$ of K3 surfaces polarized by
$L = U \oplus E_8(-1) \oplus E_7(-1)$. We may write such a K3 surface in Weierstrass form

$$y^2 = x^3 + t^3(at + a')x + t^5(b''t^2 + bt + b').$$

It has an E_8 fiber at $t = \infty$ and at least an E_7 fiber at $t = 0$. The discriminant of
the cubic polynomial is

$$d = -t^9 \big(27b''^2 t^5 + 54bb'' t^4 + (4a^3 + 27b^2 + 54b'b'')t^3$$
$$+ (12a^2 a' + 54bb')t^2 + (12aa'^2 + 27b'^2)t + 4a'^3 \big).$$

By Tate's algorithm, $b'' \neq 0$ (otherwise we would have a rational elliptic surface),
and the fiber at $t = \infty$ must be of type E_8, while the fiber at $t = 0$ is of type E_7 if and
only if $a' \neq 0$, in which case one may set $a' = -1$ by scaling (x, y, t) appropriately.
If on the other hand $a' = 0$, then the E_7 fiber gets promoted to an E_8 fiber (and
no further, since b' cannot vanish, else we would have a rational elliptic surface).
Therefore, the moduli space $\mathcal{E}_{E_8, E_7}$ is the complement in $\mathcal{F}_L$ of the hypersurface
$a' = 0$ that corresponds to polarization by $U \oplus E_8(-1) \oplus E_8(-1)$.

When we do have $a' = 0$, it was shown by Inose that the K3 surface

$$y^2 = x^3 + at^4 x + t^5(b''t^2 + bt + b')$$

has a Shioda–Inose structure, making it 2-isogenous with a product of two elliptic
curves [Inose 1978; Shioda 2006; Kuwata and Shioda 2008; Clingher and Doran
2007; Elkies $\geq$ 2015]. Recall that $b'b'' \neq 0$, and it follows from the formulas in

[Inose 1978; Shioda 2006; Kumar 2008] that the j-invariants j_1, j_2 of the two elliptic curves are determined by

$$\frac{j_1}{1728}\frac{j_2}{1728} = \frac{-a^3}{27b'b''}, \quad \left(1 - \frac{j_1}{1728}\right)\left(1 - \frac{j_2}{1728}\right) = \frac{b^2}{4b'b''}.$$

Note that again the map is Galois invariant and invertible, since a is only defined up to a cube root of unity (one may scale x by a cube root of unity), and b is only defined up to sign (one may scale t by -1).

Putting everything together, we have the following proposition.

Proposition 6. *There is a Galois invariant isomorphism $\phi : \mathcal{F}_L \to \mathcal{A}_2$, which on the open subset $\mathcal{E}_{E_8,E_7}$ restricts to the inverse of the explicit isomorphism $\psi : \mathcal{M}_2 \to \mathcal{E}_{E_8,E_7}$ given by Theorem 5.*

For a Hodge-theoretic approach to this isomorphism of moduli spaces, see [Gritsenko and Nikulin 1997, pp. 186–188].

3. Humbert surfaces and Hilbert modular surfaces

We next discuss moduli spaces of abelian surfaces with real multiplication. As above, let $D > 0$ be a fundamental discriminant, i.e., $D = d$ for $d \equiv 1 \pmod 4$ or $D = 4d$ for $d \equiv 2, 3 \pmod 4$, where $d > 1$ is squarefree in both cases. Then D is the discriminant of the ring of integers $\mathcal{O}_D = \mathbb{Z} + \mathbb{Z}(D + \sqrt{D})/2$ of the real quadratic field $K = \mathbb{Q}(\sqrt{D})$. Let σ_1, σ_2 be the two embeddings of K into $\mathbb{C}$. Then $\mathrm{SL}_2(\mathcal{O}_D)/\{\pm 1\}$ acts on $\mathcal{H}^+ \times \mathcal{H}^-$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z_1, z_2) \mapsto \left(\frac{\sigma_1(a)z + \sigma_1(b)}{\sigma_1(c)z + \sigma_1(d)}, \frac{\sigma_2(a)z + \sigma_2(b)}{\sigma_2(c)z + \sigma_2(d)}\right),$$

where $\mathcal{H}^+ = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ is the complex upper half-plane and $\mathcal{H}^- = -\mathcal{H}^+$ is the lower half-plane.

Let

$$\mathrm{SL}_2(\mathcal{O}_D, \mathcal{O}_D^*) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(K) \,\middle|\, a, d \in \mathcal{O}_D, \, c \in \mathcal{O}_D^*, \, b \in (\mathcal{O}_D^*)^{-1}\right\}.$$

We claim that this action is equivalent to the action of $\mathrm{SL}_2(\mathcal{O}_D, \mathcal{O}_D^*)$ on $\mathcal{H}^+ \times \mathcal{H}^+$. Here $\mathcal{O}_D^*$ is the dual of $\mathcal{O}_D$ with respect to the trace form on K (that is, $\mathcal{O}_D^*$ is the inverse different of K). It is an invertible $\mathcal{O}_D$-module of rank 1; in fact, it is easily checked to be $(1/\sqrt{D})\mathcal{O}_D$. Assume, without loss of generality, that $\sigma_1(\sqrt{D}) > 0$ and $\sigma_2(\sqrt{D}) < 0$. Then if we let

$$\psi : \mathcal{H}^+ \times \mathcal{H}^- \to \mathcal{H}^+ \times \mathcal{H}^+$$

be the biholomorphic map $(z_1, z_2) \mapsto (z_1\sigma_1(\sqrt{D}), z_2\sigma_2(\sqrt{D}))$, and

$$\phi : \mathrm{SL}_2(\mathcal{O}_D) \to \mathrm{SL}_2(\mathcal{O}_D, \mathcal{O}_D^*)$$

be the group isomorphism given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b\sqrt{D} \\ c/\sqrt{D} & d \end{pmatrix},$$

an easy check shows that the following diagram commutes.

$$
\begin{array}{ccc}
\mathrm{SL}_2(\mathcal{O}_D) \times \mathcal{H}^+ \times \mathcal{H}^- & \longrightarrow & \mathcal{H}^+ \times \mathcal{H}^- \\
\downarrow{\scriptstyle \phi \times \psi} & & \downarrow{\scriptstyle \psi} \\
\mathrm{SL}_2(\mathcal{O}_D, \mathcal{O}_D^*) \times \mathcal{H}^+ \times \mathcal{H}^+ & \longrightarrow & \mathcal{H}^+ \times \mathcal{H}^+
\end{array}
$$

Therefore, ψ induces an isomorphism on the quotients, as desired.

Next, we outline the proof that $\mathrm{SL}_2(\mathcal{O}_D, \mathcal{O}_D^*)$ is the coarse moduli space of principally polarized abelian surfaces with real multiplication by $\mathcal{O}_D$, closely following [Hirzebruch and van der Geer 1981]. Let $M = \mathcal{O}_D \oplus \mathcal{O}_D^*$, and define an alternating $\mathbb{Z}$-valued form on M by

$$E_M((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = \mathrm{Tr}_{K/\mathbb{Q}}(\alpha_1\beta_2 - \alpha_2\beta_1).$$

Now, for $z = (z_1, z_2) \in \mathcal{H}^2$, consider the embedding

$$L_z : K \oplus K \to V = \mathbb{C}^2,$$

$$(\alpha, \beta) \mapsto \alpha z + \beta = \big(\sigma_1(\alpha)z + \sigma_1(\beta), \sigma_2(\alpha)z + \sigma_2(\beta)\big),$$

which gives us a lattice $L_z(M)$ in V. We use this to transfer E_M to $L_z(M)$ and extend the form $\mathbb{R}$-linearly. This gives an alternating form

$$E_{M,z} : V \times V \to \mathbb{R},$$

which can be described in coordinates on $\mathbb{C}^2$ as

$$E_{M,z}((\zeta_1, \zeta_2), (\eta_1, \eta_2)) = \frac{\mathrm{Im}\, \zeta_1\overline{\eta_1}}{\mathrm{Im}\, z_1} + \frac{\mathrm{Im}\, \zeta_2\overline{\eta_2}}{\mathrm{Im}\, z_2}.$$

This gives a Riemann form on the resulting complex torus $V/L_z(M)$, and since the form E_M on the lattice $L_z(M)$ is unimodular, we obtain an abelian variety with principal polarization. The action of $\mathcal{O}_D$ is as follows:

$$\iota(\alpha)(\zeta_1, \zeta_2) \mapsto (\sigma_1(\alpha)\zeta_1, \sigma_2(\alpha)\zeta_2).$$

Conversely, it is not hard to show that any principally polarized abelian surface with real multiplication by $\mathcal{O}_D$ can be identified with some $V/L_z(M)$. Finally, we note that the abelian surfaces corresponding to two different $z \in \mathcal{H}^+ \times \mathcal{H}^+$ are isomorphic (with an $\mathcal{O}_D$-equivariant isomorphism) exactly when these two points differ by an element of $\mathrm{SL}_2(\mathcal{O}_D, \mathcal{O}_D^*)$. Therefore, it follows that the moduli space

of abelian surfaces with real multiplication is $\mathrm{SL}_2(\mathcal{O}_D, \mathcal{O}_D^*) \backslash (\mathcal{H}^+ \times \mathcal{H}^+)$, which as we have shown above, is biholomorphic with $\mathrm{SL}_2(\mathcal{O}_D) \backslash (\mathcal{H}^+ \times \mathcal{H}^-)$.

The construction above yields a map from $\mathrm{SL}_2(\mathcal{O}_D) \backslash (\mathcal{H}^+ \times \mathcal{H}^-)$ to the quotient of $\mathcal{S}_2$, the Siegel upper half-space of degree 2, by the arithmetic group $\mathrm{Sp}_4(\mathbb{Z})$. In other words, we get a holomorphic map to $\mathcal{A}_2$, the moduli space of principally polarized abelian surfaces. Its image (or its closure in $\mathcal{A}_2$) is the *Humbert surface* $\mathcal{H}_D$ for discriminant D. We next show that the map from the Hilbert modular surface $\mathrm{SL}_2(\mathcal{O}_D) \backslash (\mathcal{H}^+ \times \mathcal{H}^-)$ to the Humbert surface is generically two-to-one. We may compute this degree above a very general point on the Humbert surface. Such a point corresponds to a principally polarized abelian surface A, where one has forgotten the action of $\mathcal{O}_D$ by endomorphisms. Generically, there are exactly two ways to extend the obvious map $\mathbb{Z} \to \mathrm{End}(A) \cong \mathcal{O}_D$ to $\mathcal{O}_D$, corresponding to the choice of image of $(D + \sqrt{D})/2$.

Our approach to computing equations of Hilbert modular surfaces begins as follows. Fix a discriminant D, which we assume to be a fundamental discriminant. First, we need to compute a model of the Humbert surface, i.e., the subvariety of $\mathcal{A}_2$ corresponding to abelian surfaces with real multiplication by $\mathcal{O}_D$. Via the inverse of the isomorphism $\phi : \mathcal{F}_L \to \mathcal{A}_2$ of Section 2.4 above, the Humbert surface corresponds to a surface inside the 3-dimensional moduli space of L-polarized K3 surfaces.

Define a pairing on $\mathcal{O}_D$ by $(\alpha, \beta) \mapsto \mathrm{tr}(\alpha\beta^*)$, where β^* is the Galois conjugate of β. This gives $\mathcal{O}_D$ the structure of an indefinite lattice, which we next identify with the Néron–Severi lattice of A.

Proposition 7. *Let A be a principally polarized abelian surface with $\mathrm{End}(A) \cong \mathcal{O}_D$. Then $\mathrm{NS}(A) \cong \mathrm{End}(A)$. The lattice $\mathrm{NS}(A)$ has a basis with Gram matrix*

$$\begin{pmatrix} 2 & D \\ D & (D^2 - D)/2 \end{pmatrix} \tag{2}$$

of signature $(1, 1)$ and discriminant $-D$.

Proof. For a principally polarized abelian surface A, there is an isomorphism

$$\mathrm{NS}(A) \to (\mathrm{End}(A))^\dagger$$

induced naturally by the polarization, where $\dagger$ is the Rosati involution (also arising from the polarization). Note that for a general polarization one only gets a weaker isomorphism, between the $\mathbb{Q}$-spans of both sides. For the proof of both assertions, see Proposition 5.2.1 in [Birkenhake and Lange 2004]. Now, the Rosati involution is a positive involution of the real quadratic field $\mathrm{End}(A) \otimes \mathbb{Q}$, and hence cannot be the nontrivial element of the Galois group. It must therefore be the identity, whence the subring of $\mathrm{End}(A)$ fixed by $\dagger$ is $\mathrm{End}(A)$ itself, proving the first statement. The

isomorphism of groups $NS(A) \to End(A)$ is an isometry, taking the intersection form on $NS(A)$ to the form $(\phi, \psi) \mapsto tr(\phi\psi^\vee)$ on $End(A)$, which becomes $(\alpha, \beta) \mapsto tr(\alpha\beta^*)$ on $\mathcal{O}_D$. Computing the matrix of this form on the basis $(1, (D + \sqrt{D})/2)$ of $\mathcal{O}_D$, we obtain the claimed Gram matrix (2). $\square$

We henceforth use $\mathcal{O}_D$ also to denote the lattice with underlying group $\mathcal{O}_D \cong \mathbb{Z}^2$ and form $(\alpha, \beta) \mapsto tr(\alpha\beta^*)$, with Gram matrix (2).

Proposition 8. *There is a primitive embedding, unique up to isomorphism, of the lattice $\mathcal{O}_D$ into U^3. Let T_D be the orthogonal complement of $\mathcal{O}_D$ in U^3. Then there is a primitive embedding, unique up to isomorphism, of T_D into the K3 lattice Λ.*

Remark. In the present paper we analyze only fundamental discriminants D, for which every embedding into an even lattice of $\mathcal{O}_D$ and T_D (and of the lattice L_D, to be introduced before the next theorem) is automatically primitive. But the condition of primitivity is necessary to extend our theoretical analysis also to non-fundamental D.

Proof. Since $\mathcal{O}_D$ is 2-dimensional, the number of generators $\ell(\mathcal{O}_D^*/\mathcal{O}_D)$ of the discriminant group is at most 2. In fact, the discriminant group is cyclic of order D if D is odd, and isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \oplus \left(\mathbb{Z}/\left(\frac{D}{2}\mathbb{Z}\right)\right)$ if D is even. Therefore, by [Nikulin 1979b, Theorem 1.14.4], $\mathcal{O}_D$ has a unique embedding into the even unimodular lattice U^3. Let the orthogonal complement be T_D; it has signature $(2, 2)$ and the same discriminant group. By another application of [Nikulin 1979b, Theorem 1.14.4], we see that T_D has a unique embedding into Λ_{K3}. $\square$

Let q_D be the discriminant form of $\mathcal{O}_D$, and let L_D be the orthogonal complement of T_D in Λ. Then L_D has signature $(1, 17)$ and discriminant form q_D. By [Nikulin 1979b, Theorem 1.13.2], L_D is characterized uniquely by its signature and discriminant form. Since $\Lambda \cong U^3 \oplus E_8(-1)^2$, it is clear that $L_D \cong E_8(-1)^2 \oplus \mathcal{O}_D$. Finally, L_D has a primitive embedding in Λ, which (again by [Nikulin 1979b, Theorem 1.14.4]) is unique up to isomorphism.

Theorem 9. *Let $\mathcal{F}_{L_D}$ be the moduli space of K3 surfaces that are lattice polarized by L_D. Then the isomorphism $\phi : \mathcal{F}_L \to \mathcal{A}_2$ of Section 2.4 induces a birational surjective morphism $\mathcal{F}_{L_D} \to \mathcal{H}_D$.*

Proof. First, we note that $\mathbb{Z} \subset \mathcal{O}_D$ induces an embedding of lattices $\langle 2 \rangle \subset \mathcal{O}_D$, and therefore an embedding $T_D \subset U^2 \oplus \langle -2 \rangle$ of orthogonal complements. Taking orthogonal complements once more in Λ_{K3}, we deduce $L \subset L_D$.

Fix embeddings $L \subset L_D \subset \Lambda_{K3}$. Then we have $L_D^\perp \hookrightarrow L^\perp$, which induces a map $\Omega_{L_D} \hookrightarrow \Omega_L$ of period domains. We also have a map $\Gamma_{L_D} \subset \Gamma_L$. These induce a map $\mathcal{F}_{L_D} = \Gamma_{L_D} \backslash \Omega_{L_D} \xrightarrow{\beta_D} \Gamma_L \backslash \Omega_L = \mathcal{F}_L$. We will use the fact that this morphism has degree 1 onto the image (i.e., it is an embedding on the generic point of $\mathcal{F}_{L_D}$). We postpone the proof of this fact until the conclusion of the present argument.

Let $\phi_D : \mathcal{F}_{L_D} \overset{\beta_D}{\to} \mathcal{F}_L \overset{\phi}{\to} \mathcal{A}_2$ be the composition of the maps above. Now suppose X is a K3 surface corresponding to a point p in the image of β_D. Then we have $\mathrm{NS}(X) \supset L_D \cong E_8(-1)^2 \oplus \mathcal{O}_D$, and therefore $T_X \subset T_D$. If A (corresponding to $\phi(p)$) is the abelian surface connected to X through a Shioda–Inose structure, we have $T_A \cong T_X \subset T_D$. Therefore, $\mathrm{NS}(A) \supset \mathcal{O}_D$, the orthogonal complement of T_D in U^3. Therefore $\mathrm{End}(A)^\dagger \supset \mathcal{O}_D$, and $\phi(p)$ must lie on the Humbert surface $\mathcal{H}_D$. This proves that the image of $\mathcal{F}_{L_D}$ lands in $\mathcal{H}_D$.

Conversely, suppose q is a point on $\mathcal{H}_D$ corresponding to an abelian surface A with real multiplication by $\mathcal{O}_D$. Then $\mathrm{End}(A) \supset \mathcal{O}_D$, and since the Rosati involution (being positive) can only act as the trivial element of $\mathbb{Q}(\sqrt{D})$, we must have $\mathrm{End}(A)^\dagger \supset \mathcal{O}_D$. Retracing the argument in the previous paragraph, we see $p = \phi^{-1}(q)$ must have $\mathrm{NS}(X) \supset L_D$, and therefore is in the image of β_D. This proves that $\phi \circ \beta_D$ is surjective onto $\mathcal{H}_D$.

Since ϕ is an isomorphism and β_D has degree 1 onto its image, so does $\phi \circ \beta_D$. $\square$

We now prove that fact used above: β_D is generically an embedding.

Proposition 10. *The map $\mathcal{F}_{L_D} \to \mathcal{F}_L$ is generically an embedding.*

Proof. To see this, we claim that it is enough to show that there is an element $\gamma_0 \in O(\Lambda)$ such that γ_0 fixes L pointwise, preserves the sublattice L_D, and acts by -1 on $L^\perp$. For suppose we have $x = \gamma y$ with $x, y \in \Omega_{L_D}$ and $\gamma \in \Gamma_L$, with x and y very general. Then we would like to show that in fact we already have $x = \gamma' y$ with $\gamma' \in \Gamma_{L_D}$. Then $\Lambda_x^{1,1} := \{z \in x \mid \langle z, x \rangle = 0\}$ and $\Lambda_y^{1,1}$ both equal L_D, since x and y are very general, and therefore γ preserves L_D (though it might not fix this lattice pointwise). If γ acts trivially on L_D, we may take $\gamma' = \gamma$, and are done. If not, then γ must act by -1 on $L_D/L \cong \mathbb{Z}$. Then $\gamma' = \gamma_0 \gamma$ will suffice.

To prove the claim, we may consider specific embeddings $L \subset L_D \subset \Lambda$. If $D \equiv 0 \pmod 4$, then $L_D \cong E_7(-1) \oplus \langle -D/2 \rangle \oplus U \oplus E_8(-1)$. We may embed it inside $(E_8(-1) \oplus U \oplus U) \oplus (U \oplus E_8(-1))$ by embedding $E_7(-1)$ inside $E_8(-1)$ as the orthogonal complement of a root δ, and $\langle -D/2 \rangle$ primitively inside $U \oplus U$. Let γ_0 be the automorphism of Λ which acts on the first copy of $E_8(-1)$ as the reflection s_δ in the hyperplane orthogonal to δ, by -1 on $U \oplus U$, and as the identity on the part $U \oplus E_8(-1)$.

If $D \equiv 1 \pmod 4$, consider a system of simple roots in $E_8(-1)$ whose intersections give the standard Dynkin matrix. Let α be the simple root such that if we remove the corresponding vertex from the Dynkin diagram, we obtain the Dynkin diagram for $E_7(-1)$. The orthogonal complement of this copy of $E_7(-1)$ is some root δ. Let $\beta \in U$ be any element of norm $(1-D)/2$. It is easy to check that the copy of $E_7(-1)$ and $\alpha + \beta$ generate a negative definite lattice M of discriminant D, and that $U \oplus M \oplus E_8(-1)$ is isometric to L_D (since it is has the right signature and discriminant). In other words, the diagram of embeddings

$$E_7(-1) \hookrightarrow M \subset E_8(-1) \oplus U \hookrightarrow E_8(-1) \oplus U^2$$

extends on adding a factor of $U \oplus E_8(-1)$ to

$$L = E_7(-1) \oplus U \oplus E_8(-1) \hookrightarrow M \oplus U \oplus E_8(-1) = L_D \subset E_8(-1) \oplus U^2 \oplus E_8(-1)$$
$$\to E_8(-1) \oplus U^3 \oplus E_8(-1) = \Lambda.$$

Now, consider the automorphism γ_0 of $\Lambda \cong E_8(-1) \oplus U^3 \oplus E_8(-1)$ given by $(s_\delta, -1|_U, -1|_U, 1_U, 1_{E_8(-1)})$. By construction, γ_0 fixes L pointwise, and acts by -1 on $L^\perp$. We need to show that γ_0 preserves L_D, or equivalently, that it preserves M. We know that M is spanned by $E_7(-1)$ (which is fixed by γ_0) and $\alpha + \beta$. Now, γ_0 takes β to $-\beta$, by construction. Also, $\alpha + \gamma_0(\alpha) = \alpha + s_\delta(\alpha) \in E_8(-1)$ is invariant by s_δ, and is therefore in the orthogonal complement $E_7(-1)$ of δ. Therefore $\gamma_0(\alpha) \in -\alpha + E_7(-1)$ and $\gamma_0(\alpha + \beta) \equiv -(\alpha + \beta) \mod E_7(-1)$. So γ_0 preserves the lattice M. $\square$

Next, we want to understand the Hilbert modular surface $Y_-(D)$, which is a double cover of $\mathcal{H}_D$. First, we must identify the branch locus. Since the map $Y_-(D) \to \mathcal{H}_D$ is obtained by simply forgetting the action of $e_D = (D + \sqrt{D})/2$ on the abelian surface, the branch locus is the subvariety W of $\mathcal{H}_D$ corresponding to abelian surfaces A such that $e_D = (D + \sqrt{D})/2$ and $e'_D = (D - \sqrt{D})/2$ are conjugate in the endomorphism ring, say by $\iota \in \text{End}(A)$. It follows that ι^2 fixes $\mathcal{O}_D$ pointwise. Generically this implies $\iota^2 = \pm 1$, and $\iota e_D = e'_D \iota$. This shows that $\text{End}(A)$ for such A is generically an order in a quaternion algebra B.

In fact, the branch locus corresponds to the case when we have a split quaternion algebra, i.e., $\iota^2 = 1$. Then A is isogenous over $\overline{\mathbb{Q}}$ to the square of an elliptic curve. To see this fact, observe that the map $\Gamma \backslash \mathcal{H}^2 \to \text{Sp}_4(\mathbb{Z}) \backslash \mathcal{S}_2$ (where $\Gamma = \text{SL}_2(\mathcal{O}_D, \mathcal{O}_D^*)$) factors through the quotient $(\Gamma \cup \Gamma \sigma) \backslash \mathcal{H}^2$, where σ is the involution $(z_1, z_2) \mapsto (z_2, z_1)$ exchanging the two factors of $\mathcal{H}^2$, and the induced map on this quotient is generically one-to-one [Hirzebruch and van der Geer 1981, p. 158]. Therefore, to understand the branch locus, we need to understand the 1-dimensional part of the fixed point set of σ on $\Gamma \backslash \mathcal{H}^2$. This (and more) was done by Hausmann [1982, p. 35]. The result is that the fixed point set consists of a small number of explicit modular curves F_w for $w \in \{1, 4, D, D/4\}$ (some of these may be empty, and when they are nonempty, these curves are irreducible). A simple explicit analysis of the condition relating z_1 and z_2 on these curves reveals that the generic point on each of these corresponds to an abelian surface whose ring of endomorphisms contains zero divisors, and is therefore a split quaternion algebra. For instance, the image of the diagonal of $\mathcal{H}^2$ (given by $z_1 = z_2$) in $\Gamma \backslash \mathcal{H}^2$ is an obvious component of the branch locus. For the corresponding abelian surface, as constructed earlier in this section, the map $(\zeta_1, \zeta_2) \mapsto (\zeta_2, \zeta_1)$ is a holomorphic involution, showing that the endomorphism algebra is split. For completeness, we prove this next.

Proposition 11. *Let* $\Gamma = \mathrm{SL}_2(\mathcal{O}_D, \mathcal{O}_D^*)$ *and let C be one of the curves F_w for $w \in \{1, D\}$ if D is odd, $w \in \{1, 4, D/4, D\}$ if D is even. The generic point on C corresponds to an abelian surface whose algebra of endomorphisms is a split quaternion algebra.*

Proof. We use the notation of [van der Geer 1988, Chapter V]. The ideal $\mathfrak{a} = \mathcal{O}_D^*$ has norm $A = 1/D$. In the reference, it is assumed that $\mathfrak{a}$ is an integral ideal, but we may for instance replace $\mathcal{O}_D^*$ by the integral $D\mathcal{O}_D^* = \sqrt{D}\mathcal{O}_D$ without loss of generality in Hausmann's proof. This would replace A by $D^2 A = D$, and would not affect any of the arguments below, which depend only on the square class of A. Proposition V.1.5 of [van der Geer 1988] states that a point on F_N corresponds to an abelian surface whose endomorphism algebra is isomorphic to the indefinite quaternion algebra

$$Q_N = \left(\frac{D, -N/(AD)}{\mathbb{Q}} \right),$$

while Lemma V.1.4 in the book says that F_N is nonempty if and only if for each prime q dividing D and not dividing N, we have $\chi_{D(q)}(N) = (A, D)_q$. For an explanation of the notation, see [van der Geer 1988, pp. 2–3]. In our situation, we have $AD = 1$. If $N = D$, we get the algebra $\left(\frac{D, -D}{\mathbb{Q}} \right)$, which is obviously split. This argument also takes care of $N = D/4$ when D is even. Next, suppose $N = 1$. Then F_1 is nonempty if and only if for every prime dividing D, we have $(D, D)_q = 1$. So

$$1 = (D, D)_q = (D, -D)_q \cdot (D, -1)_q = (D, -1)_q.$$

It follows that the quaternion algebra $Q_1 = \left(\frac{D, -1}{\mathbb{Q}} \right)$ is split. The proof for $N = 4$ is similar. $\square$

Remark 12. A further analysis of the proof above reveals the number of components of the branch locus, which is corroborated by the calculations in this paper.

Coming back to our analysis of the endomorphisms of the abelian surfaces corresponding to points on the branch locus, we note that the Rosati involution must fix $\mathcal{O}_D$ and ι, by positivity. Consider the form $(x, y) \mapsto \mathrm{Tr}(x\bar{y})$, where $\bar{\ } : B \to B$ is the natural involution taking $\sqrt{D}$ to its negative and ι to its negative, and Tr is the reduced trace. The matrix of this form acting on $\mathcal{O}_D + \mathbb{Z}\iota$ is

$$\begin{pmatrix} 2 & D & 0 \\ D & (D^2 - D)/2 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

which therefore gives an even lattice of signature $(1, 2)$ and discriminant $2D$. We claim that when D is fundamental, the index, call it c, of $\mathcal{O}_D \oplus \mathbb{Z}\iota$ as a sublattice of $\mathrm{End}(A)$ is 1 or 2, with $c = 2$ possible only if $4 \mid D$. Indeed $\mathrm{End}(A)$ is an even

lattice of rank 3 and discriminant $2D/c^2$. But the discriminant is an even integer because the rank is odd, and D has no repeated prime factors except possibly 2^2 or 2^3. Thus as claimed the index c must be either 1 or 2, with $c = 2 \Rightarrow 4 \mid D$. Therefore, the only possibilities for the discriminant of the resulting Néron–Severi group for a point corresponding to the branch locus are $2D$ and $D/2$. This is also the discriminant of the transcendental lattice of the abelian surface, and by the Shioda–Inose structure, the corresponding K3 surface also has a Néron–Severi lattice of rank 19 and discriminant $2D$ or $D/2$, and contains $L_D \cong E_8(-1)^2 \oplus \mathcal{O}_D$.

Finally, we mention that a group of involutions acts naturally on the Hilbert modular surface $Y_-(D)$. This group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{t-1}$, where t is the number of distinct primes dividing D. These extra involutions arise from the Hurwitz–Maass extension of $\mathrm{PSL}_2(\mathcal{O}_D, \mathcal{O}_D^*)$. For more details see [van der Geer 1988, Section I.4].

4. Method of computation

We now outline our method to obtain equations for the Hilbert modular surfaces $Y_-(D)$ for fundamental discriminants D.

Step 1. We compute the unique lattice L_D, writing it in the form $U \oplus N$ (we know L_D contains a copy of U, since $L = U \oplus E_8(-1) \oplus E_7(-1)$ does, and $L \supset L_D$). Note that N is not uniquely determined, and it turns out to be useful to choose N such that its root lattice R has small codimension in N, and such that N/R has a basis consisting of vectors of small (fractional) norm. One may then attempt to realize L_D as the Néron–Severi lattice of an elliptic K3 surface with reducible fibers corresponding to the irreducible root lattices in R. The remaining generators of the Néron–Severi lattice should then arise as sections (of small height) in the Mordell–Weil group of the elliptic fibration. Of course, we need to make sure that N contains $E_8 \oplus E_7$, but this is easily achieved by showing that the dual lattice N^* contains a vector of norm $2/D$.

For most of the examples in this paper, we were able to describe a family of elliptic K3 surfaces with Néron–Severi lattice L_D and with Mordell–Weil rank 0 or 1. This construction gives us a Zariski-open subset of $\mathcal{F}_{L_D}$, with some possible missing curves corresponding to jumps in the Mordell–Weil ranks or the ranks of the reducible fibers, or to denominators introduced in the parametrization process.

For all the examples we considered, the moduli space $\mathcal{F}_{L_D}$ turns out to be rational (i.e., birational with $\mathbb{P}^2$ over $\mathbb{Q}$). This property will fail to hold once D is large enough, but there is still some hope of writing down usable equations for $\mathcal{F}_{L_D}$ if it is not too complicated.

Suppose that we have exhibited a Zariski open subset of $\mathcal{F}_{L_D}$ as a Zariski open subset $U = \mathbb{P}^2_{r,s} \setminus V$ of the projective plane.

Step 2. We find a different elliptic fibration on the generic member of the family of K3 surfaces in Step 1, with reducible fibers of types E_8 at $t = \infty$ and E_7 at $t = 0$. This is accomplished by 2- and 3-neighbor steps, which we shall describe in the next section.

Once we obtain the alternate elliptic fibration, we may compute the map from U to $\mathcal{A}_2$ by the explicit formulas of Section 2.4. This gives us the Igusa–Clebsch invariants of the associated genus-2 curve, in terms of the two parameters r and s.

Step 3. Consider the base change diagram

$$
\begin{array}{ccc}
\widetilde{Y}_-(D) & \xrightarrow{\ \widetilde{\eta}\ } & \mathcal{F}_{L_D} \\
{\scriptstyle \widetilde{\phi}}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\
Y_-(D) & \xrightarrow{\ \eta\ } & \mathcal{H}_D
\end{array}
$$

Since ϕ is a birational map, so is $\widetilde{\phi}$. We want to describe the degree-2 map $\widetilde{\eta}$. In the current situation, we must have a model for $\widetilde{Y}_-(D)$ (which is birational to $Y_-(D)$) of the form

$$z^2 = f(r, s),$$

where $f(r, s) = 0$ describes the branch locus of $\widetilde{\eta}$. As we have seen, this locus $f(r, s) = 0$ must be the union of some irreducible curves, which are either components of the excluded locus V above, or belong to the sublocus of U where the Picard number jumps by 1, the discriminant changing to $2D$ or $D/2$. There are finitely many possibilities for how this may happen. For instance, the elliptic fibration may have an extra A_1 fiber, or a D_6 fiber may get promoted to an E_7 fiber, or the surface may have a new section that raises the Mordell–Weil rank. Similarly, in any instance, we may easily list the allowed changes in the reducible fibers, or the allowed heights and intersections with components of reducible fibers of a new section of the fibration. Hence we get a list of polynomials $f_1(r, s), f_2(r, s), \ldots, f_k(r, s) \in \mathbb{Z}[r, s]$ whose zero loci are the possible components of the branch locus. We may assume that each $f_i(r, s)$ has content 1.

We therefore have the following model for $\widetilde{Y}_-(D)$ as a double cover of $\mathbb{P}^2_{r,s}$:

$$z^2 = f(r, s) = C f_{i_1}(r, s) f_{i_2}(r, s) \ldots f_{i_m}(r, s)$$

for some subset $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, k\}$ and some squarefree integer C.

Step 4. In the final step, we show how to compute the subset $I = \{i_1, \ldots, i_m\}$ and the twist C. First, we note that the Hilbert modular surface $Y_-(D)$ has good reduction outside primes dividing D, by [Rapoport 1978; Deligne and Pappas 1994]. Therefore, there are only finitely many choices for C as well. Now we check each of these choices by computer, using the method described in the next paragraph, and rule out all but one.

So suppose we have a putative choice of C and I. We proceed to check whether this choice of twist is correct by reduction modulo several odd primes p that do not divide the discriminant D. We specialize r, s to elements of $\mathbb{F}_p$, and by the formulas of Step 2, obtain Igusa–Clebsch invariants for in weighted projective space over $\mathbb{F}_p$. Since the Brauer obstruction vanishes for finite fields, we can construct a curve $C_{r,s}$ over $\mathbb{F}_p$.

We may use the following lemma to detect whether the Jacobian of $C_{r,s}$ has real multiplication defined over $\mathbb{F}_p$ or not.

Lemma 13. *Let A be an abelian surface over $\mathbb{F}_p$, and let ϕ be the Frobenius endomorphism of A relative to $\mathbb{F}_p$. Suppose that the characteristic polynomial $P(T)$ of ϕ is irreducible over $\mathbb{Q}$. Let Q be the symmetric characteristic polynomial of $\phi + p\phi^{-1}$, defined by $P(T) = T^2 Q(T + pT^{-1})$.*

(1) *If A has real multiplication by an order in $\mathcal{O}_D$ defined over $\mathbb{F}_p$, then Q is a quadratic polynomial of discriminant $c^2 D$ for some integer c.*

(2) *If A has real multiplication by an order in $\mathcal{O}_D$, defined over $\mathbb{F}_{p^2}$, but not over $\mathbb{F}_p$, then we have $Q(X) = X^2 - n$ for some $n \in \mathbb{Z}$ not of the form $c^2 D$ for any integer c.*

(3) *If $Q(X)$ is a quadratic polynomial of discriminant D (resp. $c^2 D$ for some positive integer c), then A has real multiplication by $\mathcal{O}_D$ (resp. an order in $\mathcal{O}_D$ of conductor dividing c), defined over $\mathbb{F}_p$.*

Note that A might simultaneously have real multiplication $\iota : \mathcal{O} \hookrightarrow \text{End}(A)$ and $\iota' : \mathcal{O}' \hookrightarrow \text{End}(A)$ by two orders of $\mathcal{O}_D$, if its ring of endomorphisms is a quaternion algebra. Furthermore, if one of these is defined over $\mathbb{F}_p$ and the other only over $\mathbb{F}_p^2$, then by parts (1) and (2) of the lemma, $Q(X)$ must be of the form $X^2 - c^2 D$ for some integer c.

We will give the proof of the lemma below, but first we indicate how to use it to find the correct twist. We choose a suitable prime $p \nmid D$, and for some r, s in $\mathbb{F}_p$, we calculate the number of points mod p and p^2 of the resulting curve $C_{r,s}$. This is enough to describe the polynomial $Q(X)$ for the abelian surface $A = J(C)$, by the Lefschetz–Grothendieck trace formula. Suppose $P(T) = T^2 Q(T + pT^{-1})$ is irreducible. Then the first hypothesis of the lemma is satisfied. If in addition $Q(X) = X^2 + aX + b$ has discriminant D and nonzero linear term a, then A must have real multiplication defined over $\mathbb{F}_p$, by part (3) of the lemma (and the comments following it). In fact, most of the time we can even make do with the weaker assumption that $Q(X)$ has discriminant in the square class of D, since (3) guarantees that A has real multiplication by an order in $\mathbb{Q}(\sqrt{D})$. We can compute the discriminant of $\text{NS}(A)$ by computing that of $\text{NS}(Y)$, where Y is the elliptic K3 surface related to A by a Shioda–Inose structure. If this discriminant is D, it follows that A must have real multiplication by the full ring of integers $\mathcal{O}_D$.

Now, by the property of a coarse moduli space, A gives rise to an $\mathbb{F}_p$-point (r, s) of the Hilbert modular surface $Y_-(D)$, i.e., the corresponding point on the Humbert surface must actually lift to the double cover. Therefore, if $Cf_{i_1}(r, s) \ldots f_{i_m}(r, s)$ is not a square, we must have the wrong quadratic twist. We can run this test for many such $(r, s) \in \mathbb{F}_p \times \mathbb{F}_p$. Note that a single large prime is usually enough to pin down the correct choice of $\{i_1, \ldots, i_m\}$, by eliminating all but one possibility. However, to pin down the correct choice of C, we may need to use several primes, until we find one for which C/C' is not a quadratic residue, where C' is the correct twist. However, in any case, this procedure is guaranteed to terminate, and in practice, it terminates fairly quickly.

At the end of Step 4 of the algorithm, we have determined a birational model of $Y_-(D)$ over $\mathbb{Q}$. We now give the proof of the lemma above.

Proof of Lemma 13. We will use [Tate 1966a, Theorem 2]. Since $P(T)$ is irreducible, it follows that $F = \mathbb{Q}[\phi]$ is simple, and also that $F = E := \mathrm{End}_{\mathbb{F}_p}(A) \otimes \mathbb{Q}$ is a quartic number field. First, assume that real multiplication by an order $\mathcal{O} \subset \mathcal{O}_D$ is defined over $\mathbb{F}_{p^2}$ but not over $\mathbb{F}_p$. Say $\mathcal{O}$ contains $f\sqrt{D}$ for some positive integer f. Then consider the base change of A to $\mathbb{F}_{p^2}$. The Frobenius over the new field is ϕ^2, and therefore, ϕ^2 commutes with $f\sqrt{D}$, but ϕ must anticommute with $f\sqrt{D}$. Therefore $\mathbb{Q}(\phi^2)$ is a strict subfield of $\mathbb{Q}(\phi)$, and must be quadratic. Hence ϕ^2 satisfies a quadratic equation $\phi^4 + a\phi^2 + b = 0$, and since this must be the characteristic polynomial of ϕ, we must have $b = p^2$. Therefore

$$\phi^2 + p^2\phi^{-2} + a = (\phi + p\phi^{-1})^2 + a - 2p = 0,$$

proving the assertion (1).

Now, assume that A has real multiplication by $\mathcal{O} \subset \mathcal{O}_D$, defined over $\mathbb{F}_p$, and let $f\sqrt{D} \subset \mathcal{O}$, as before. Then $\mathbb{Q}(\phi + p\phi^{-1})$ is a quadratic subfield of $\mathbb{Q}(\phi)$. Also, $f\sqrt{D} \in \mathbb{Q}(\phi)$, so we have $f\sqrt{D} = g(\phi)$ for some polynomial $g \in \mathbb{Q}[T]$. Then $f\sqrt{D}$ is its own dual isogeny, and so $f\sqrt{D} = g(p\phi^{-1})$ as well. Therefore, $f\sqrt{D} = \frac{1}{2}(g(\phi) + g(p\phi^{-1}))$ can be expressed as $h(\phi + p\phi^{-1})$ for some $h \in \mathbb{Q}(T)$. It follows that $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\phi + p\phi^{-1})$ and therefore the minimal polynomial $Q(T)$ of $\phi + p\phi^{-1}$ has discriminant equal to D times a square. This proves (2).

Finally, if Q is a quadratic polynomial of discriminant $c^2 D$, then $\eta = \phi + p\phi^{-1}$ is an endomorphism of A satisfying $Q(\eta) = 0$. Therefore $\mathbb{Z}[\eta] \cong \mathcal{O} := \mathbb{Z} + c\,\mathcal{O}_D$, and we conclude that A has real multiplication by $\mathcal{O}$ defined over $\mathbb{F}_p$, proving (3). $\square$

We conclude this section with a few comments on the models of Hilbert modular surfaces computed in this paper. We first give formulas for the family of elliptic K3 surfaces over $\mathcal{F}_{L_D}$, and then describe the 2- and 3-neighbor steps necessary to reach the alternate fibration of Step 2. The details of the parametrization will not be included in the paper, but they are available in the online supplement. Steps

3 and 4 are relatively easy to automate. We simply write down the result, which is the equation of $Y_-(D)$ as a double cover of the Humbert surface. We give a table of points of small height for which the Brauer obstruction vanishes, and the associated curves of genus 2. We also analyze the Hilbert modular surface further in the cases that we can describe it as a K3 or honestly elliptic surface. In particular, we determine the (geometric) Picard number and generators for the Mordell-Weil group of sections in most of the cases. We also analyze the branch locus of Step 3, identifying it with a union of quotients of classical modular curves in several cases, with the help of explicit formulas given in [Elkies 1998] or obtained by the methods of that paper. Finally, for many discriminants, we are able to exhibit curves of low genus on the surface, possessing infinitely many rational points.

We found the method of [van Luijk 2007] quite useful in determining the ranks of the Néron–Severi lattices of these surfaces. Briefly, the method is as follows. Let X be a smooth projective surface over $\mathbb{Q}$. For a prime p of good reduction, let X_p be the reduction of a good model of X at p. By counting points on $X_p(\mathbb{F}_q)$ for a small number of prime powers $q = p^e$, we obtain the characteristic polynomial of the Frobenius ϕ_p on some ℓ-adic étale cohomology group $H^2(X \times \overline{\mathbb{F}}_p, \mathbb{Q}_l)$. The number of roots $\rho_0(p)$ of this polynomial which are p times a root of unity is an upper bound on the geometric Picard number of X_p. Therefore $\rho(X) \leq \rho_0(p)$ for such primes. If we have $\rho_0(p_1) = \rho_0(p_2) = \rho_0$, but the (expected) square classes of the discriminant of the Néron–Severi groups modulo these primes (as predicted by the Artin–Tate formula [Tate 1966b]) are distinct, we may even deduce $\rho(X) < \rho_0$. For if X_p does not satisfy the Tate conjecture for some $p \in \{p_1, p_2\}$, then $\rho(X) \leq \rho(X_p) < \rho_0$. On the other hand, if both these reductions satisfy the Tate conjecture, then they also satisfy the Artin–Tate conjecture [Milne 1975a, 1975b], and since the size of the Brauer group of X_p is a square [Milne 1975a; Liu et al. 2005], the Néron–Severi groups must have discriminants in the same square class.

5. Neighbor method

Finally, we describe how to transform from one elliptic fibration to another, using 2- and 3-neighbor steps. We start with an elliptic K3 surface over a field k, which we assume has characteristic different from 2 or 3. Let F be the class of the fiber, and O be the class of the zero section. The surface X is a minimal proper model of a given Weierstrass equation

$$y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t).$$

Here the elliptic fibration is $\pi : X \to \mathbb{P}^1_t$. Now, let F' be an elliptic divisor (i.e., F' is effective, $F'^2 = 0$, the components of F' are smooth rational curves, and F' is primitive).

We would like to write down the Weierstrass equation for this new elliptic fibration on X. The space of global sections $H^0(X, \mathcal{O}_X(F'))$ has dimension 2 over k. The ratio of any two linearly independent sections gives us the new elliptic parameter u. To compute the space of global sections, we proceed as follows. Any global section gives a section of the generic fiber $E = \pi^{-1}(\eta)$, which is an elliptic curve over $k(t) = k(\eta)$. Therefore, if we have a basis of global sections of $D = F'_{\text{hor},\eta}$ over $k(t)$, say $\{s_1, \ldots, s_r\}$ (where $r = h^0(\mathcal{O}_E(D)) = \deg(D) = F'_{\text{hor}} \cdot F = F' \cdot F$), we can assume that any global section of $\mathcal{O}_X(F')$ is of the form

$$b_1(t)s_1 + \cdots + b_r(t)s_r.$$

We can now use the information from F_{ver}, which gives us conditions about the zeroes and poles of the functions b_i, to find the linear conditions cutting out $H^0(X, \mathcal{O}_X(F'))$, which will be 2-dimensional.

When $F \cdot F' = r$, we say that going between these elliptic fibrations is an r-neighbor step. We will explain the reason for this terminology shortly. In this paper we use only 2- and 3-neighbor steps. First, we describe how to convert to a genus-1 curve in the case when $E = F_{\text{hor},\eta} = 2O$ or $3O$. These are the most familiar cases of the 2- and 3-neighbor steps, and the other cases of the 2-neighbor step that are needed are more exhaustively described in [Kumar 2014].

Suppose $D = 2O$. Then $\{1, x\}$ is a basis of global sections of $\mathcal{O}_E(D)$. Therefore, on X we obtain two global sections, 1 and $c(t) + d(t)x$ for some $c(t), d(t) \in k(t)$. The ratio between the two gives the elliptic parameter u. We set $x = (u - c(t))/d(t)$, and substitute into the Weierstrass equation to obtain an equation

$$y^2 = g(t, u).$$

Because F' is an elliptic divisor, the generic fiber of this surface over $\mathbb{P}^1_u$ is a curve of genus 1. Thus, once we absorb square factors into y^2 we obtain an equivalent g that is a polynomial of degree 3 or 4 in t. Then $y^2 = g(t, u)$ is standard form of the equation of a genus-1 curve as a branched double cover of $\mathbb{P}^1$.

Suppose $D = 3O$. Then $\{1, x, y\}$ is a basis of global sections of $\mathcal{O}_E(D)$. On X we obtain two global sections, 1 and $c(t) + d(t)x + e(t)y$. We set the ratio equal to u, solve for y, and substitute into the Weierstrass equation to obtain

$$\left(u - c(t) + d(t)x\right)^2 = e(t)^2\left(x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)\right).$$

This equation is of degree 3 in x, and after some simple algebra (scaling and shifting x), we may arrange it to have degree 3 in t as well. We end up with a plane cubic curve, which is the standard model of a genus-1 curve with a degree-3 line bundle.

Finally, if we also know that the elliptic fibration corresponding to F' has a section (which follows in each of our examples by Proposition 4), then it is isomorphic

to the Jacobian of the genus-1 curve we have computed. We may use standard formulas to write down the Weierstrass equation of the Jacobian (see [An et al. 2001] and the references cited therein), and this is the desired Weierstrass equation for the elliptic fibration with fiber F'.

This neighbor step from F to F' corresponds to computing an explicit isomorphism between two presentations $\mathbb{Z}O + \mathbb{Z}F + T$ and $\mathbb{Z}O' + \mathbb{Z}F' + T'$ of NS(X). Note that $T \cong F^\perp/\mathbb{Z}F$, where $\perp$ refers to the orthogonal complement in NS(X). The sublattice $(\mathbb{Z}F + \mathbb{Z}F')^\perp$ projects to an index-r sublattice of both $F^\perp/\mathbb{Z}F$ and $F'^\perp/\mathbb{Z}F'$, where $r = F \cdot F'$. Therefore these lattices are r-neighbors.

6. Discriminant 5

6.1. Parametrization. This is the smallest fundamental discriminant for real multiplication, and it is small enough that we do not need any 2- or 3-neighbor steps: we can instead just start with a K3 surface with E_8 and E_7 fibers, and ask for the extra condition which allows a section of height $\frac{5}{2} = 4 - \frac{3}{2}$.

Proposition 14. *The moduli space of* K3 *surfaces lattice polarized by* L_5, *the unique even lattice of signature* $(1, 17)$ *and discriminant 5 containing*

$$U \oplus E_8(-1) \oplus E_7(-1),$$

is birational to the projective plane $\mathbb{P}^2_{g,h}$. *The family of* K3 *surfaces is given by the Weierstrass equation*

$$y^2 = x^3 + \tfrac{1}{4}t^3(-3g^2t + 4)x - \tfrac{1}{4}t^5\big(4h^2t^2 + (4h + g^3)t + (4g + 1)\big).$$

Proof. We start with a family of elliptic K3 surfaces with E_8 and E_7 fibers:

$$y^2 = x^3 + xt^3(a_0 + a_1t) + t^5(b_0 + b_1t + b_2t^2).$$

To have discriminant -5 for the Picard group, the elliptic K3 surfaces in this family must have a section of height $\frac{5}{2} = 4 - \frac{3}{2}$. The x-coordinate for such a section must have the form $t^2(x_0 + x_1t + h^2t^2)$. Substituting x into the right-hand side and completing the square, we may solve for b_0, b_1, b_2 and a_1 in terms of a_0, h, x_0 and x_1. We then set $x_1 = eh$ and $x_0 = g + e^2/4$ to simplify the expressions. Finally, we note that scaling x, y, t by $\lambda^2, \lambda^3, \mu/\lambda$ gives a_0, e, g, h weights $(1, 3), (0, 1)$, $(0, 2)$ and $(-1, 2)$ respectively with respect to (λ, μ). Therefore, we may scale a_0 and e to equal 1 independently (at most removing hypersurfaces in the moduli space), and get the parametrization in the statement of the proposition. We note that the section P of height $\frac{5}{2}$ is given by

$$x(P) = \tfrac{1}{4}t^2((1 + 2ht)^2 + 4g), \quad y(P) = \tfrac{1}{8}t^3(1 + 2ht)((1 + 2ht)^2 + 6g). \quad \square$$

Corollary 15. *The Humbert surface $\mathcal{H}_5$ is birational to $\mathbb{P}^2_{g,h}$, with the map to $\mathcal{A}_2$ given by the Igusa–Clebsch invariants*

$$(I_2 : I_4 : I_6 : I_{10}) = \big(6(4g+1), 9g^2, 9(4h+9g^3+2g^2), 4h^2\big).$$

Proof. This follows immediately from Theorem 9. The Igusa–Clebsch invariants may be read out directly from the Weierstrass equation above. $\qquad\square$

Theorem 16. *A birational model over $\mathbb{Q}$ of the Hilbert modular surface $Y_-(5)$ is given by the following double cover of $\mathbb{P}^2_{g,h}$:*

$$z^2 = 2\big(6250h^2 - 4500g^2h - 1350gh - 108h - 972g^5 - 324g^4 - 27g^3\big).$$

It is a rational surface (i.e., birational to $\mathbb{P}^2$).

Proof. We follow the method of Section 4. The possible factors of the branch locus are g, h, $8h - 9g^2$ (the zero locus of this polynomial defines a subvariety of the moduli space for which the corresponding elliptic K3 surfaces acquire an extra I_2 fiber), $64h^2 + 48g^2h + 48g^5 + 9g^4$ (extra II fiber), and $6250h^2 - 4500g^2h - 1350gh - 108h - 972g^5 - 324g^4 - 27g^3$ (extra I_2 fiber). By Step 4 of the method, we deduce that only the last factor occurs, and the correct quadratic twist is by 2. It was already well-known that the Hilbert modular surface is a rational surface, but we give an explicit parametrization in the following analysis. $\qquad\square$

6.2. Analysis. This is a rational surface: to obtain a parametrization, we complete the square in h, writing $h = k + 9(250g^2 + 75g + 6)/6250$. Following this up with $k = 3m(10g + 3)(15g + 2)/6250$, and removing square factors by writing $z = 3n(10g+3)(15g+2)/25$, we obtain the equation $5n^2 - m^2 + 9 + 30g = 0$, which we can solve for g. This gives an explicit birational map between $Y_-(5)$ and $\mathbb{P}^2_{m,n}$.

The branch locus is the curve obtained by setting $z = 0$, or alternatively $n = 0$ in the above parametrization. It is parametrized by one variable m; we have

$$(g, h) = \left(\frac{(m^2 - 9)}{30}, \frac{(m - 2)^2(m + 3)^3}{12500}\right).$$

6.3. Examples. We list in Table 1 some points of small height and corresponding genus-2 curves. The second entry in the table is the modular curve $X_0(67)/\langle w \rangle$, where w is the Atkin–Lehner involution. We find several other modular curves with real multiplication by $\mathcal{O}_5$ (and also a few for discriminants other than 5) corresponding to points of larger height.

6.4. Brumer's and Wilson's families of genus-2 curves. Genus-2 curves whose Jacobians have real multiplication by $\mathcal{O}_5$ have been studied by Brumer [1995] and Wilson [1998; 2000]. Brumer describes a 3-dimensional family of genus-2 curves

Rational point (g, h)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{8}{3}, \frac{47}{2}\right)$	$-x^5 + x^4 - x^3 - x^2 + 2x - 1$
$\left(-\frac{37}{6}, 67\right)$	$x^6 + 2x^5 + x^4 - 2x^3 + 2x^2 - 4x + 1$
$\left(0, \frac{27}{50}\right)$	$3x^5 + 5x^3 + 1$
$\left(-\frac{1}{24}, \frac{1}{100}\right)$	$4x^6 + 4x^5 + 5x^4 - 5x^2 - 2x - 2$
$\left(-\frac{25}{54}, \frac{59}{81}\right)$	$-x^6 - 2x^4 - 6x^3 - 5x^2 - 6x - 1$
$\left(-\frac{2}{3}, -\frac{14}{25}\right)$	$-7x^6 - 7x^5 - 5x^4 + 5x^2 - x - 1$
$\left(\frac{47}{54}, \frac{71}{81}\right)$	$3x^6 - 6x^5 + 7x^4 - 2x^3 - 2x^2 - 1$
$\left(-\frac{1}{6}, \frac{1}{25}\right)$	$x^6 - 4x^5 + 10x^4 - 10x^3 + 5x^2 + 2x - 3$
$\left(-\frac{4}{3}, \frac{16}{25}\right)$	$-x^6 - x^5 + 5x^2 - 7x - 12$
$\left(-\frac{4}{3}, \frac{49}{22}\right)$	$7x^5 + 5x^4 + 3x^3 - 9x^2 - 14x - 7$
$\left(0, -\frac{54}{11}\right)$	$2x^6 - 6x^5 - 6x^4 + 3x^3 - 18x^2 - 6x - 2$
$\left(\frac{11}{6}, \frac{53}{11}\right)$	$9x^6 - 14x^5 + 13x^4 - 2x^3 - 22x^2 - 8x - 7$
$\left(-\frac{5}{24}, \frac{1}{64}\right)$	$6x^6 - 2x^5 - 15x^4 - 16x^3 - 25x^2 - 8x - 4$
$\left(\frac{11}{6}, -\frac{89}{25}\right)$	$-x^6 + 2x^5 - 5x^4 + 30x^3 - 10x^2 + 8x - 1$
$\left(-\frac{2}{3}, \frac{68}{11}\right)$	$-2x^6 - 2x^5 - 11x^4 - 29x^3 - 31x^2 - 26x - 6$
$\left(-\frac{2}{3}, \frac{26}{25}\right)$	$-2x^6 + 36x^5 + 5x^4 + 35x^3 - 10x^2 - 21x - 17$

Table 1. Some points of height ≤ 100 on the surface of Theorem 16 and the corresponding genus-2 curves.

given by

$$y^2 + \left(1 + x + x^3 + c(x + x^2)\right)y$$
$$= -bdx^4 + (b - d - 2bd)x^3 + (1 - 3b - bd)x^2 + (1 + 3b)x + b.$$

In the online supplement, we give formulas for the corresponding values of our parameters g and h.

Wilson describes a family of genus-2 curves by their Igusa–Clebsch invariants. His moduli space is 2-dimensional, though he uses three coordinates z_6, s_2 and σ_5 of weights 1, 2, 5 respectively, in a weighted projective space. These coordinates are related to ours via

$$(g, h) = \left(-\frac{2z_6^2 + s_2}{12z_6^2}, \frac{\sigma_5}{64z_6^5}\right).$$

7. Discriminant 8

7.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type D_9 and E_7. A Weierstrass equation for such a family is given by

$$y^2 = x^3 + t\left((2r + 1)t + r\right)x^2 + 2rst^4(t + 1)x + rs^2t^7.$$

We then identify a fiber of type E_8, and transform to it by a 2-neighbor step.

The resulting elliptic fibration has E_8 and E_7 fibers, and we may read out the Igusa–Clebsch invariants, and then compute the branch locus of the double cover that defines the Hilbert modular surface. It corresponds to elliptic K3 surfaces with an extra I_2 fiber. We obtain the following result for $Y_-(8)$.

Theorem 17. *The Humbert surface $\mathcal{H}_8$ is birational to $\mathbb{P}^2_{r,s}$, with the explicit map to $\mathcal{A}_2$ given by the Igusa–Clebsch invariants*

$$I_2 = -4(3s + 8r - 2),$$
$$I_4 = 4(9rs + 4r^2 + 4r + 1),$$
$$I_6 = -4(36rs^2 + 94r^2s - 35rs + 4s + 48r^3 + 40r^2 + 4r - 2),$$
$$I_{10} = -8s^2r^3.$$

A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(8)$ as a double cover

Rational point (r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{2}{25}, \frac{83}{50}\right)$	$x^6 - x^5 + 3x^4 + x^3 + x^2 + 2x + 1$
$\left(-\frac{34}{9}, \frac{50}{9}\right)$	$x^6 - 2x^5 - 2x^4 - 4x^3 + 2x^2 + 4x - 3$
$\left(-22, \frac{59}{2}\right)$	$x^6 + 2x^5 - 3x^4 + 5x^3 + x^2 - x - 1$
$\left(\frac{2}{49}, \frac{58}{49}\right)$	$-x^6 - 4x^5 - 6x^4 + 2x^2 + 2x - 1$
$\left(-52, \frac{83}{2}\right)$	$-4x^6 + 2x^5 + 5x^4 - 7x^3 - x + 1$
$\left(\frac{1}{8}, \frac{59}{32}\right)$	$-x^6 - x^5 - 7x^2 + 5x - 4$
$\left(-\frac{4}{9}, \frac{23}{8}\right)$	$-2x^6 + 6x^5 - x^4 - 7x^3 - x^2 - 3x$
$\left(80, \frac{83}{2}\right)$	$x^6 + 5x^5 + 8x^4 + 5x^3 + 5x^2 - 8x + 4$
$\left(\frac{19}{2}, 22\right)$	$-x^6 - 4x^5 - 8x^4 - 8x^3 - 8x^2 + 4x - 4$
$(94, -54)$	$-x^6 + 6x^4 - 8x^3 + 6x^2 + 6x - 9$
$\left(\frac{13}{8}, -\frac{2}{9}\right)$	$-x^6 + 6x^5 - 9x^4 + 3x^2 - 6x - 2$
$\left(\frac{86}{9}, -\frac{13}{2}\right)$	$x^6 - 7x^4 - 7x^3 - x^2 + 9x + 9$
$\left(-\frac{3}{8}, \frac{9}{4}\right)$	$-3x^6 - 9x^5 - 2x^4 + 10x^3 + x^2 + 3x$
$\left(\frac{1}{14}, \frac{10}{7}\right)$	$-x^6 - 2x^5 - 7x^4 + 4x^3 - 3x^2 + 10x - 5$
$\left(\frac{1}{18}, \frac{31}{18}\right)$	$-3x^6 - 10x^5 - 7x^4 - 5x^2 + 2x - 1$
$\left(\frac{1}{32}, \frac{19}{16}\right)$	$4x^6 - 7x^5 - 3x^4 - 2x^3 + 10x^2 + 5x + 1$

Table 2. Some points of small height on the surface of Theorem 17 and the corresponding genus-2 curves.

of $\mathbb{P}^2$ *is given by the equation*

$$z^2 = 2\,(16rs^2 + 32r^2s - 40rs - s + 16r^3 + 24r^2 + 12r + 2).$$

It is also a rational surface.

7.2. Analysis. This Hilbert modular surface is a rational surface. To see this, we may complete the square in s, setting $s = s_1 - r + 5/4 + 1/(32r)$. Then setting $s_1 = m(16r - 1)/(32r)$ and $z = n(16r - 1)$, we remove square factors from the equation, which becomes linear in r. We find $r = (m^2 - 1)/(32n^2 - 16)$, thus obtaining an explicit parametrization of $Y_-(8)$ by $\mathbb{P}^2_{m,n}$.

The branch locus is a genus-0 curve; we obtain a parametrization by setting $z = 0$:

$$(r, s) = \left(\frac{1 - t^2}{16}, \frac{(t + 3)^3}{16(t + 1)} \right).$$

7.3. Examples. Table 2 lists some points of small height and their genus-2 curves.

8. Discriminant 12

8.1. Parametrization. We start with an elliptic K3 surface with fibers of types E_8, D_6 and A_2. The Weierstrass equation for such a family is given by

$$y^2 = x^3 + \big((1 - f^2)(1 - t) + t\big)tx^2 + 2et^3(t - 1)x + e^2(t - 1)^2t^5.$$

We identify a fiber of type E_7, and move to the associated elliptic fibration by a 2-neighbor step.
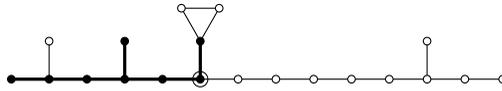
The new elliptic fibration has E_8 and E_7 fibers, and so we may read out the Igusa–Clebsch invariants, and determine the branch locus for the Hilbert modular surface.

Theorem 18. *A birational model over* $\mathbb{Q}$ *for the Hilbert modular surface* $Y_-(12)$ *as a double cover of* $\mathbb{P}^2_{e,f}$ *is given by the equation*

$$z^2 = (f - 1)(f + 1)(f^6 - f^4 - 18ef^2 + 27e^2 + 16e).$$

It is a rational surface.

8.2. Analysis. Note the extra involution $(e, f) \mapsto (e, -f)$ arising from the Hurwitz–Maass extension (as described at the end of Section 3), since 12 is not a prime power. In fact, there are two independent involutions evident in the diagram above, but one of them has been used up to fix the Weierstrass scaling of x (namely, the coefficient of x^2 evaluates to 1 at $t = 1$). The other involution is reflected in the

Weierstrass equation for the universal family of elliptic K3 surfaces as $f \mapsto -f$. It preserves the branch locus of the map $Y_-(12) \to \mathcal{H}_{12}$, and therefore lifts to $Y_-(12)$.

The branch locus has three components; the two simple components $f = \pm 1$ correspond to the D_6 fiber getting promoted to an E_7 fiber, while the remaining component corresponds to having an extra I_2 fiber. This last component is a rational curve; completing the square with respect to e, we find after some easy algebraic manipulation the parametrization

$$(e, f) = \left(\frac{16(h^2 - 1)}{(h^2 + 3)^3}, \frac{-4h}{h^2 + 3} \right).$$

This Hilbert modular surface is rational as well. To obtain an explicit parametrization, note that the right-hand side of the above equation is quadratic in e, and $e = 0$ makes it a square. Therefore the conic bundle over $\mathbb{P}^1_f$ has a section. Setting $z = ge + f^2(f^2 - 1)$, we may solve for e, obtaining a birational parametrization by $\mathbb{P}^2_{f,g}$.

We will not list the Igusa–Clebsch invariants for this (and higher) discriminants, as they are complicated expressions. They are available in the online supplement.

8.3. *Examples.* Table 3 lists some points of small height and their genus-2 curves.

Rational point (e, f)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{34}{27}, \frac{5}{3} \right)$	$-2x^6 - 2x^5 + x^4 - 3x^2 + 2x - 1$
$\left(\frac{34}{27}, -\frac{5}{3} \right)$	$x^5 - x^4 + x^3 - 3x^2 - x + 5$
$\left(\frac{51}{100}, 2 \right)$	$-3x^6 + 6x^5 + 4x^4 - 2x^3 - 8x^2 - 6x - 6$
$\left(-\frac{11}{3}, -2 \right)$	$-x^6 - 8x^3 + 12x - 12$
$\left(-\frac{11}{3}, 2 \right)$	$-x^6 + 12x^4 + 8x^3 - 12x^2 + 12x - 4$
$\left(\frac{4}{3}, -2 \right)$	$-8x^6 + 12x^4 + 8x^3 - 6x^2 - 12x - 3$
$\left(\frac{4}{3}, 2 \right)$	$-5x^6 + 12x^5 - 6x^4 + 8x^3 - 12x^2 - 8$
$\left(-\frac{14}{27}, \frac{1}{3} \right)$	$-x^5 + 13x^4 - 6x^3 - 2x^2 + 3x - 7$
$\left(-\frac{29}{14}, -\frac{15}{14} \right)$	$x^6 + 2x^5 + 13x^4 - 16x^3 + 17x^2 + 4x + 8$
$\left(\frac{80}{81}, 2 \right)$	$-8x^6 - 12x^5 + 15x^4 + 5x^3 - 21x^2 + 15x - 5$
$\left(\frac{51}{100}, -2 \right)$	$-x^6 - 6x^5 - 11x^4 - 14x^3 - 23x^2 + 6x + 3$
$\left(-\frac{5}{2}, -\frac{3}{2} \right)$	$-5x^6 - 10x^5 - x^4 + 24x^3 - 5x^2 - 8x - 20$
$\left(-\frac{23}{54}, -\frac{1}{2} \right)$	$x^6 - 6x^5 - 3x^4 + 24x^3 - 3x^2 - 4$
$\left(\frac{25}{18}, -\frac{3}{2} \right)$	$4x^6 - 24x^5 + 27x^4 - 28x^3 + 21x^2 - 5$
$\left(-\frac{5}{54}, \frac{2}{3} \right)$	$3x^6 - 26x^5 + 31x^4 + 12x^3 - 3x^2 - 10x - 15$
$\left(\frac{13}{64}, -\frac{5}{4} \right)$	$-5x^6 + 3x^5 - 12x^4 + 28x^3 + 12x^2 - 36x$

Table 3. Some points of small height on the surface of Theorem 18 and the corresponding genus-2 curves.

We mention a few special curves on the Hilbert modular surface, which one may recognize by looking at a plot of the rational points. First, specializing f to some $f_0 \in \mathbb{Q}$ gives a conic in z and e, which always has a rational point (since, as we noted above, $e = 0$ makes the right-hand side a square). These may be used to produce many rational points and examples of genus-2 curves when the Brauer obstruction happens to vanish.

Another noticeable feature of the plot is the parabola $e = 4(f^2 - 1)/9$. We check that the Brauer obstruction vanishes identically on this locus, giving a family of genus-2 curves whose Jacobians have real multiplication by $\mathcal{O}_{12}$. Note that this is a non-modular curve: generically the endomorphism ring is no larger (i.e., not a quaternion algebra).

9. Discriminant 13

9.1. *Parametrization.* We start with an elliptic K3 surface with reducible fibers of types E_8, E_6 and A_1, and with a section of height $\frac{13}{6} = 4 - \frac{4}{3} - \frac{1}{2}$. A Weierstrass equation for such a family is given by

$$y^2 = x^3 + (4g+1)t^2 x^2 - 4g(h-g-1)(t-1)t^3 x + 4g^2(t-1)^2 t^4 (h^2 t + 1).$$

We identify an E_7 fiber below and move to it by a 2-neighbor step.

The resulting elliptic fibration has E_8 and E_7 fibers, so we may read out the Igusa–Clebsch invariants and work out the branch locus of the double cover defining the Hilbert modular surface. It corresponds to the elliptic K3 surface having an extra I_2 fiber.

Theorem 19. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(13)$ as a double cover of $\mathbb{P}^2_{g,h}$ is given by the equation*

$$z^2 = 108gh^3 - (27g^2 + 468g - 4)h^2 + 8(82g^2 + 71g - 2)h - 16(2g-1)^3.$$

It is a rational surface.

9.2. *Analysis.* The surface $Y_-(13)$ is rational. To show this, we proceed as follows. The substitutions $g = g_1 - \frac{1}{54}$ and $h = h_1 + \frac{64}{27}$, followed by $h_1 = mg_1$, make the right-hand side quadratic in g_1, up to a square factor. Removing this factor by setting $z = z_1 g_1/18$, and then setting $g_1 = g_2/54$, we get the conic bundle over $\mathbb{Q}(m)$

$$z_1^2 = 3m^2(4m-1)g_2^2 - 6(2m^3 - 301m^2 - 528m + 128)g_2 - 507(m^2 - 96m - 1024).$$

Furthermore, $g_2 = 1$ makes the resulting expression a square, giving us a rational point on the conic over $\mathbb{Q}(m)$. Therefore, we may set $z_1 = 36(m + 20) + n(g_2 - 1)$ and solve for g_2. This gives us a birational parametrization of $Y_-(13)$ by $\mathbb{P}^2_{m,n}$.

We can also deduce that the branch locus is a rational curve, as follows. By setting $z_1 = 0$, we obtain a quadratic equation in g_2 whose discriminant, up to a square factor, is $m^2 - 44m + 16$. Setting it equal to n^2 and noting that $m = 0$ makes the expression a square, we obtain a parametrization of this conic, as

$$m = -4 \frac{2r + 11}{r^2 - 1}.$$

Working backwards, we obtain a parametrization of the branch locus as

$$(g, h) = \left(\frac{-2(r - 2)^2(r + 1)}{27(r + 7)}, \frac{2(2r + 5)^3}{27(r + 1)(r + 7)} \right).$$

9.3. *Examples.* Table 4 lists some points of small height and their genus-2 curves.

On a plot of rational points we observe the line $h = (g + 4)/3$, along which the Brauer obstruction vanishes, leading to a family of genus-2 curves whose Jacobians have "honest" real multiplication by $\mathcal{O}_{13}$.

| Rational point (g, h)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-17, \frac{1}{3}\right)$	$-3x^6 - 6x^5 + 4x^3 + 3x^2 + 6x - 5$
$\left(-\frac{13}{2}, -\frac{11}{2}\right)$	$x^5 + 5x^4 + 5x^3 - 5x^2 + 6x - 1$
$\left(-\frac{17}{2}, \frac{7}{2}\right)$	$-x^5 - 2x^4 - 3x^3 - 6x^2 - 7$
$\left(\frac{11}{5}, \frac{1}{5}\right)$	$x^6 + 4x^5 + 2x^4 - 8x^3 - 5x^2 - 5$
$\left(-\frac{1}{3}, \frac{11}{9}\right)$	$-x^6 + 3x^4 + 12x^3 + 6x^2 - 11$
$\left(-\frac{14}{11}, -\frac{2}{11}\right)$	$13x^6 + 12x^5 + 6x^4 + 10x^3 - 7x^2 - 2x + 1$
$\left(-\frac{2}{17}, \frac{2}{17}\right)$	$-x^6 - 2x^5 - x^4 + 14x^3 + 2x^2 - 8x - 9$
$\left(\frac{1}{5}, \frac{17}{15}\right)$	$3x^6 + 12x^5 + 6x^4 - 4x^3 - 15x^2 + 5$
$\left(-\frac{10}{3}, -\frac{10}{9}\right)$	$-x^6 + 18x^5 + 3x^4 - 6x^3 + 6x^2 - 5$
$(-10, 6)$	$-x^6 - 6x^4 - 10x^3 - 9x^2 - 30x + 11$
$\left(-\frac{18}{5}, \frac{14}{5}\right)$	$-x^6 - 2x^4 - 10x^3 + 7x^2 - 30x - 9$
$\left(-\frac{7}{13}, \frac{3}{13}\right)$	$-31x^6 + 12x^5 - 30x^4 + 4x^3 - 33x^2 - 12x - 1$
$\left(\frac{10}{3}, \frac{10}{3}\right)$	$7x^6 + 18x^5 - 9x^4 - 34x^3 + 18x^2 - 5$
$\left(-11, -\frac{11}{9}\right)$	$-5x^6 + 6x^5 + 3x^4 - 4x^3 + 18x^2 - 36x - 9$
$\left(-\frac{7}{8}, \frac{10}{3}\right)$	$-x^6 + 3x^4 - 20x^3 + 30x^2 - 36x + 9$
$\left(-\frac{13}{9}, \frac{13}{9}\right)$	$-x^6 - 9x^4 + 8x^3 - 30x^2 + 36x - 3$

Table 4. Some points of small height on the surface of Theorem 19 and the corresponding genus-2 curves.

10. Discriminant 17

10.1. *Parametrization.* We start with an elliptic K3 surface with a I_{17} fiber. A Weierstrass equation for such a family of surfaces is given by

$$y^2 = x^3$$
$$+ \left(1 + 2gt + (2h + (g+1)^2)t^2 + 2(gh + g + 2g^2 + h)t^3 + ((g+h)^2 + 2g^3)t^4\right)x^2$$
$$- 4h^2t^5\left(1 + gt + (h + 2g + 1)t^2 + (h + 2g^2 + g)t^3\right)x + 4h^4t^{10}\left((2g+1)t^2 + 1\right).$$

We first identify an E_7 fiber and go to the associated elliptic fibration via a 2-neighbor step.
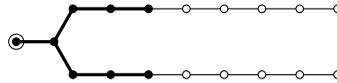
The resulting elliptic fibration has E_7 and A_8 fibers, and a section P of height $\frac{17}{18} = 4 - \frac{3}{2} - \frac{14}{9}$. We next identify a fiber F' of type E_8 and perform a 3-neighbor step to move to the associated elliptic fibration.
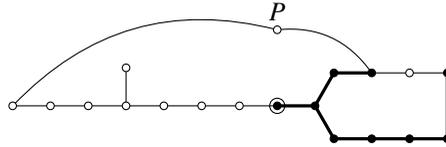
Since $P \cdot F' = 2$, while the remaining component of the A_8 fiber intersects F' with multiplicity 3, the new elliptic fibration has a section. We may therefore convert to the Jacobian; this has E_8 and E_7 fibers, and we may read out the Igusa–Clebsch invariants.

Theorem 20. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_{-}(17)$ as a double cover of $\mathbb{P}^2$ is given by the equation*

$$z^2 = -256h^3 + (192g^2 + 464g + 185)h^2$$
$$- 2(2g+1)(12g^3 - 65g^2 - 54g - 9)h + (g+1)^4(2g+1)^2.$$

It is a rational surface.

10.2. *Analysis.* This is evidently a rational elliptic surface over $\mathbb{P}^1_g$. In fact, it is a rational surface over $\mathbb{Q}$. We exhibit a birational parametrization as follows: set $h = \left(4(2g+1) + m(2g+1)(27g+13)\right)/27$ and absorb square factors by setting $z = z_1(2g+1)(27g+13)/243$, to get (after scaling $g = g_1/3$) the equation

$$z_1^2 = -9(8m-1)^3 g_1^2 - 2(16m+7)(424m^2 - 385m + 1)g_1$$
$$- 3(3328m^3 - 1923m^2 - 3138m - 803)$$

which is a conic bundle over $\mathbb{P}^1_m$. We see that $g_1 = -\frac{3}{2}$ makes the right hand side a

square, and so setting $z_1 = 9(2m+11)/2 + n(g_1 + \frac{3}{2})$ and solving for g_1, we get a birational map from $\mathbb{P}^2_{m,n}$.

The branch locus is a curve of genus 0. To produce a parametrization, we set $z_1 = 0$ and note that the discriminant of the resulting quadratic equation in g_1 is a square times $64m^2 + 218m - 8$. Setting $64m^2 + 218m - 8 = (8m + r)^2$ and solving for m, we ultimately obtain:

$$(g, h) = \left(\frac{-(8r^3 - 111r^2 + 1212r - 8146)}{2(2r-7)^3}, \frac{(r-17)^2(r+10)^4}{4(2r-7)^6} \right).$$

10.3. Examples. Table 5 lists some points of small height and their genus-2 curves.

A plot of the rational points on $Y_-(17)$ reveals two special curves on the surface. First, there is the line $h = -g/2$. Substituting this into the equation for the Hilbert modular surface, we obtain a conic, which can be parametrized as

$$(g, h) = \left(-\frac{m^2 - 4}{2(m-6)}, \frac{m^2 - 4}{4(m-6)} \right).$$

The Brauer obstruction vanishes along this locus. However, this curve is modular: the endomorphism ring is a split quaternion algebra, strictly containing $\mathcal{O}_{17}$. The

Rational point (g, h)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(0, \frac{13}{32}\right)$	$-2x^6 - x^5 - 6x^4 - 5x^3 - 4x^2 - 4x$
$\left(-\frac{5}{11}, \frac{1}{88}\right)$	$3x^6 + 4x^5 + 4x^4 - 6x^3 - 5x^2 - 4x + 4$
$(6, 26)$	$-2x^6 + 2x^5 + x^4 + 8x^3 + 7x^2 + 4x$
$\left(\frac{9}{4}, \frac{77}{64}\right)$	$-2x^6 - x^5 + 8x^4 - 5x^3 - 4x^2 + 4x - 4$
$\left(\frac{3}{5}, -\frac{11}{50}\right)$	$-8x^6 + 8x^5 + 7x^4 + 2x^3 - x$
$(5, -11)$	$4x^5 + 9x^4 + 2x^3 - 8x^2 - 2x$
$(5, 22)$	$2x^6 - 4x^5 + 9x^4 - 10x^3 + 4x^2 - 4x$
$\left(\frac{1}{5}, \frac{28}{25}\right)$	$-10x^6 - 10x^5 - 2x^4 - 7x^3 - x$
$\left(-\frac{1}{4}, -\frac{11}{64}\right)$	$4x^5 + 3x^4 + 11x^3 - 7x^2 + x$
$\left(\frac{5}{4}, -\frac{35}{64}\right)$	$x^5 - 7x^3 + 2x^2 - 8x + 12$
$\left(-\frac{5}{2}, -\frac{13}{8}\right)$	$4x^5 - x^4 - 13x^3 - 3x^2 + 13x$
$\left(\frac{1}{5}, -\frac{7}{20}\right)$	$3x^6 + 7x^5 + 6x^4 + 16x^3 + 14x^2 - 8x - 8$
$\left(-12, -\frac{23}{2}\right)$	$-7x^6 - 19x^5 - 7x^4 + 14x^3 - x$
$\left(0, -\frac{1}{16}\right)$	$-4x^6 + 19x^5 - 20x^4 + 11x^3 + 15x^2 - 8x - 4$
$\left(2, -\frac{35}{4}\right)$	$-4x^5 + 20x^4 - 12x^3 - 15x^2 - 4x$
$\left(4, \frac{51}{8}\right)$	$-4x^6 - 12x^5 - 27x^4 + 2x^3 + 12x^2 + 18x$

Table 5. Some points of small height on the surface of Theorem 20 and the corresponding genus-2 curves.

second curve is the parabola $h = -(6g^2 + g - 1)/8$. The Brauer obstruction vanishes along this curve too, giving a 1-parameter family of genus-2 curves whose Jacobians have real multiplication by $\mathcal{O}_{17}$.

11. Discriminant 21

11.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type E_8, A_6 and A_2 at $t = \infty$, 0 and 1 respectively.
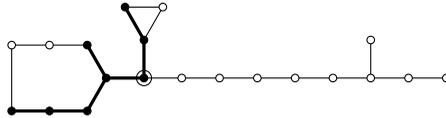
A Weierstrass equation for such a family is

$$y^2 = x^3 + (a_0 + a_1 t + a_2 t^2)x^2 + 2t^2(t-1)(b_0 + b_1 t)x + t^4(t-1)^2(c_0 + c_1 t)$$

with

$$
\begin{array}{lll}
a_0 = 1, & a_1 = -r^2 + 2rs - 1, & a_2 = (r-s)^2; \\
b_0 = (r^2 - 1)(s-r)^2, & b_1 = (r^2 - 1)(s-r)^2(rs - 1); \\
c_0 = (r^2 - 1)^2(s-r)^4, & c_1 = (r^2 - 1)^3(s-r)^4.
\end{array}
$$

We identify a fiber of type E_7, and a 3-neighbor step gives us the desired $E_8 E_7$ fibration.

We read out the Igusa–Clebsch invariants, and the branch locus of $Y_-(21)$ as a double cover of $\mathbb{P}^2_{r,s}$ corresponds to the subvariety of the moduli space where the elliptic K3 surfaces have an extra I_2 fiber.

Theorem 21. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(21)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$
\begin{aligned}
z^2 = 16s^4 - 8r(27r^2 - 23)s^3 + (621r^4 - 954r^2 + 349)s^2 \\
- 18(r^3 - r)(33r^2 - 29)s + (r^2 - 1)(189(r^4 - r^2) + 16).
\end{aligned}
$$

It is a singular K3 *surface (i.e., of Picard number 20).*

11.2. *Analysis.* The extra involution (corresponding to $21 = 3 \cdot 7$) here is given by $(r, s) \mapsto (-r, -s)$.

The branch locus is a rational curve; a parametrization is given by

$$(r, s) = \left(\frac{h^4 + 72h^2 - 81}{18h(h^2 + 3)}, \frac{(h^2 - 9)(h^4 - 126h^2 + 189)}{432h(h^2 + 3)} \right).$$

The equation of $Y_-(21)$ above expresses it as a surface fibered by genus-1 curves over $\mathbb{P}^1_r$. In fact, since the coefficient of s^4 is a square, this fibration has a section,

and we may convert it to its Jacobian form, which after some simple Weierstrass transformations may be written as

$$z^2 = w^3 + (-27r^4 + 43)w^2 + 4(r^2 - 1)(8127r^4 - 18459r^2 + 9740)w$$
$$+ 4(r^2 - 1)^2 w(-186624r^6 + 1320813r^4 - 1817964r^2 + 705679).$$

This is a K3 surface with an elliptic fibration to $\mathbb{P}^1_r$. The discriminant of the cubic polynomial is

$$(r^2 - 1)^3(27r^2 - 25)^2(27r^4 + 342r^2 - 289)^3,$$

from which we deduce that we have three I_2 fibers (including $r = \infty$) and six I_3 fibers, which contribute 15 to the Picard number. We find the following sections.

$$P_0 = \big(6(r^2 - 1)(6r^2 - 7), 4(r - 1)(r + 1)(27r^4 + 342r^2 - 289)\big),$$
$$P_1 = \big(2(324r^4 - 1503r^2 + 1019)/21, 8(27r^2 - 25)(27r^4 + 342r^2 - 289)/(21\mu)\big),$$
$$P_2 = \big((-102 + 32\nu)(r^2 - 1), 32\nu r (r^2 - 1)(-27r^2 + 31 + 22\nu)\big),$$
$$P_3 = \big((-102 - 32\nu)(r^2 - 1), -32\nu r (r^2 - 1)(-27r^2 + 31 - 22\nu)\big).$$

Here $\mu = \sqrt{21}$ and $\nu = \sqrt{-1}$. Note that P_0 is a 3-torsion section, whereas the height matrix for P_1, P_2, P_3 is

$$\frac{1}{6}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 13 & 1 \\ 0 & 1 & 13 \end{pmatrix}.$$

Therefore, the Picard number of the surface is 20, and the discriminant of the lattice spanned by these sections and the trivial lattice is $1008 = 2^4 3^2 7$. We showed that this is the entire Néron–Severi group by checking that the above subgroup of the Mordell–Weil group is 2- and 3-saturated.

Using P_0, we may rewrite the equation in the much simpler form

$$z^2 + (9r^2 - 13)wz + (r^2 - 1)(27r^4 + 342r^2 - 289)z = w^3.$$

The quotient of the Hilbert modular surface by the involution

$$(r, s, z) \mapsto (-r, -s, -z)$$

is the rational elliptic surface

$$z^2 = w^3 + (-27t^2 + 43)w^2 + 4(t - 1)(8127t^2 - 18459t + 9740)w$$
$$+ 4(t - 1)^2(-186624t^3 + 1320813t^2 - 1817964t + 705679).$$

It has three reducible fibers of type I_3 and one of type I_2, a 3-torsion section, and Mordell–Weil rank 1, generated by the following section of height $\frac{1}{6}$:

$$(z, w) = \left(2(324r^2 - 1503r + 1019)/21,\; 8(27r - 25)(27r^2 + 342r - 289)/(21\mu)\right).$$

11.3. Examples. Table 6 lists some points of small height and their genus-2 curves. The torsion section P_0 on the Jacobian model of $Y_-(21)$ pulls back to the curve

$$s = \frac{45r^3 + 9r^2 - 45r - 17}{2(27r^2 + 6r - 17)}.$$

The Brauer obstruction always vanishes for this family, and yields a family of genus-2 curves over $\mathbb{Q}(r)$, whose Jacobians have real multiplication by $\mathcal{O}_{21}$.

Another special curve which we observe from a plot of the rational points is the hyperbola $s^2 = (3r^2 + 1)/4$. It may be parametrized as

$$(r, s) = \left(\frac{4m}{4m^2 - 3},\; \frac{3 + 4m^2}{2(3 - 4m^2)}\right).$$

The Brauer obstruction also vanishes for this family.

Rational point (r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{3}{2}, \frac{1}{44}\right)$	$-5x^6 - 8x^5 + 20x^4 + 5x^3 - 13x$
$\left(\frac{21}{34}, -\frac{43}{68}\right)$	$13x^6 + 26x^5 + 33x^4 + 9x^3 - 5x^2 - 11x$
$\left(-\frac{36}{13}, -\frac{49}{26}\right)$	$7x^6 - 42x^5 - 44x^4 - 12x^3 - 11x^2 - 14x - 7$
$\left(0, \frac{1}{2}\right)$	$13x^6 + 54x^5 + 32x^4 - 28x^3 - 25x^2 + 14x - 1$
$\left(0, -\frac{1}{2}\right)$	$-x^6 + 2x^5 + 4x^4 + 36x^3 + 25x^2 + 42x - 59$
$\left(\frac{45}{46}, -\frac{25}{92}\right)$	$9x^6 + 16x^5 + 12x^4 - 73x^3 - 41x^2 + 14x + 77$
$\left(-\frac{3}{2}, -\frac{1}{44}\right)$	$-13x^5 + 39x^4 + 31x^3 - 115x^2 - 50x - 125$
$\left(-\frac{21}{34}, \frac{43}{68}\right)$	$-13x^5 + 58x^4 - 83x^3 - 90x^2 + 100x + 125$
$\left(-\frac{45}{46}, \frac{25}{92}\right)$	$5x^5 + 30x^4 - 61x^3 - 122x^2 + 112x + 161$
$\left(3, -\frac{1}{28}\right)$	$28x^6 + 52x^5 - 149x^4 - 174x^3 + 235x^2 - 60x - 100$
$\left(\frac{55}{63}, -\frac{4}{63}\right)$	$8x^6 + 192x^5 + 237x^4 + 238x^3 - 15x^2 - 60x - 76$
$\left(-\frac{1}{3}, -\frac{11}{9}\right)$	$-56x^6 + 132x^5 + 102x^4 + 195x^3 - 240x^2 - 204x - 178$
$\left(-3, \frac{1}{28}\right)$	$4x^6 - 52x^5 + 133x^4 + 34x^3 - 171x^2 - 308x - 292$
$\left(\frac{1}{9}, \frac{11}{9}\right)$	$-2x^6 - 54x^5 + 135x^4 + 120x^3 - 135x^2 - 324x - 108$
$\left(-4, -\frac{7}{2}\right)$	$-15x^6 + 36x^5 + 5x^4 + 52x^3 + 50x^2 + 156x + 375$
$\left(-\frac{35}{69}, \frac{11}{69}\right)$	$4x^6 - 12x^5 - 219x^4 - 460x^3 + 15x^2 - 246x - 230$

Table 6. Some points of small height on the surface of Theorem 21 and the corresponding genus-2 curves.
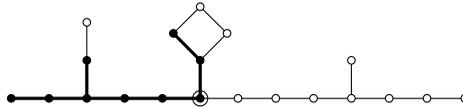
12. Discriminant 24

12.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type E_6, E_7 and A_3 at $t = \infty$, 0 and 1 respectively.

A Weierstrass equation for such a family is given by

$$y^2 = x^3 + t^2 x^2 + a(t-1)t^3\big(2 + (d^2 - a + 1)(t-1)\big)x + a^2(t-1)^2 t^5\big(1 + d^2(t-1)\big).$$

We identify a fiber of type E_8, and this leads us by a 3-neighbor step to an $E_8 E_7$ fibration.

From the new elliptic fibration we read out the Igusa–Clebsch invariants as usual, and then obtain the branch locus of $Y_-(24)$ as a double cover of $\mathbb{P}^2_{a,d}$, which is a union of two curves: one corresponding to an extra I_2 fiber, and one corresponding to the locus where the E_7 fiber promotes to an E_8 fiber.

Theorem 22. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(24)$ as a double cover of $\mathbb{P}^2_{a,d}$ is given by the equation*

$$z^2 = (d^2 - a - 1)(16ad^4 - 8a^2d^2 - 20ad^2 + d^2 + a^3 - 3a^2 + 3a - 1).$$

It is a singular K3 surface.

12.2. *Analysis.* Note the extra involution $(a, d) \mapsto (a, -d)$.

The branch locus has two components. The first is the zero locus of $d^2 - a - 1$, and is obviously a rational curve (i.e., of genus 0) in the moduli space. It parametrizes the K3 surfaces in the family for which the E_7 fiber gets promoted to an E_8 fiber. The other component parametrizes elliptic K3 surfaces for which there is an extra I_2 fiber. It is also a genus-0 curve, though this fact is less obvious. A parametrization is given by

$$(a, d) = \left(\frac{1 - g^2}{2g^2 - 1}, \frac{g^3}{2g^2 - 1} \right).$$

The equation of the Hilbert modular surface describes it as a family of curves of genus 1 fibered over $\mathbb{P}^1_d$. In fact, we readily check that $(a, z) = (0, d^2 - 1)$ gives a section. So in fact, we have an elliptic surface, and by using the formula for the Jacobian, we can write it in Weierstrass form as

$$y^2 = x^3 - xd^2(144d^6 - 324d^4 + 235d^2 - 54)/48$$
$$- d^2(3456d^{10} - 22032d^8 + 50625d^6 - 54866d^4 + 28647d^2 - 5832)/1728.$$

This is an elliptic K3 surface E, and in fact, it is the base change (by $d \mapsto d^2$) of a rational elliptic surface with reducible fibers of types I_2, I_3 and I_4 (and therefore with Mordell–Weil rank 2). We can use this to readily compute one section

$$(x_0, y_0) = \left(11d^4 - 239d^2/12 + 9, (d^2 - 1)(9d^2 - 8)(32d^2 - 27)/8\right)$$

of height $\frac{1}{6}$. Translating x by x_0 and scaling to get rid of denominators, we get the following nicer form for E:

$$y^2 = x^3 + (132d^4 - 239d^2 + 108)x^2$$
$$+ 2(d^2 - 1)(9d^2 - 8)(32d^2 - 27)(10d^2 - 9)x + \left((d^2 - 1)(9d^2 - 8)(32d^2 - 27)\right)^2.$$

This has reducible fibers of type I_2 at $d = \infty$ and $d^2 = \frac{8}{9}$, type I_3 at $d^2 = \frac{27}{32}$, type IV at $d = 0$, and type I_4 at $d = \pm 1$. This gives a root system of type $A_1^3 \oplus A_2^3 \oplus A_3^2$, which has rank 15 and discriminant 1152.

In addition to the section $P_1 = \left(0, (d^2 - 1)(9d^2 - 8)(32d^2 - 27)\right)$, we also find the sections

$$P_2 = \left(-5(d^2 - 1)(9d^2 - 8), \sqrt{-1}(d^2 - 1)^2(9d^2 - 8)\right),$$
$$P_3 = \left(-(32d^2 - 27)(9d^2 - 8)/6, d(32d^2 - 27)(9d^2 - 8)/\sqrt{216}\right).$$

This shows that the elliptic K3 surface $Y_-(24)$ has geometric Picard number 20, i.e., is a singular K3 surface. The discriminant of the span of the algebraic divisors exhibited is 96. We showed that this is the entire Néron–Severi group by checking that our subgroup of the Mordell–Weil group is 2-saturated.

As mentioned, the quotient by the involution $d \mapsto -d$ is a rational elliptic surface

$$y^2 = x^3 + (132t^2 - 239t + 108)x^2$$
$$+ 2(t - 1)(9t - 8)(32t - 27)(10t - 9)x + \left((t - 1)(9t - 8)(32t - 27)\right)^2.$$

This surface has an I_2 fiber at $t = \frac{8}{9}$, an I_3 fiber at $t = \frac{27}{32}$ and an I_4 fiber at $t = 1$.

The Mordell–Weil group is generated by the sections

$$P_1 = \left(0, (t - 1)(9t - 8)(32t - 27)\right),$$
$$P_2 = \left(-5(t - 1)(9t - 8), \sqrt{-1}(t - 1)^2(9t - 8)\right),$$

with height matrix

$$\begin{pmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

12.3. Examples. Table 7 lists some points of small height and their genus-2 curves.

For instance, for the genus-2 curves corresponding to $\left(1, \frac{3}{2}\right)$ and $\left(1, -\frac{3}{2}\right)$, the point counts modulo p match the twist by $\mathbb{Q}(\sqrt{2})$ of a modular form of level 2592.

Rational point (a, d)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{77}{36}, -\frac{1}{6}\right)$	$-7x^6 - 18x^4 - 10x^3 + 3x^2 + 10x + 22$
$\left(\frac{21}{16}, \frac{1}{8}\right)$	$-12x^5 + 8x^4 + 24x^3 - 16x^2 - 21x + 14$
$\left(\frac{9}{4}, -\frac{1}{2}\right)$	$3x^6 - 12x^5 + 20x^4 - 79x^3 + 11x^2 - 84x + 60$
$\left(-\frac{7}{81}, -\frac{17}{18}\right)$	$27x^6 - 56x^5 - 21x^4 + 52x^3 + 57x^2 - 84x + 49$
$\left(\frac{77}{36}, \frac{1}{6}\right)$	$-32x^6 - 80x^5 + 94x^4 + 115x^3 - 91x^2 - 55x + 33$
$\left(\frac{9}{4}, -\frac{5}{4}\right)$	$-24x^6 + 48x^5 + 100x^4 + 120x^3 - 50x^2 - 72x - 41$
$\left(-\frac{3}{4}, -\frac{1}{4}\right)$	$25x^6 + 70x^4 + 24x^3 + 124x^2 + 48x + 72$
$\left(\frac{9}{4}, \frac{1}{2}\right)$	$-72x^6 + 84x^5 + 127x^4 - 123x^3 - 83x^2 + 51x + 25$
$\left(1, \frac{3}{2}\right)$	$9x^6 + 9x^4 - 60x^3 - 45x^2 + 132x - 53$
$\left(\frac{33}{50}, -\frac{1}{5}\right)$	$50x^5 - 50x^4 + 35x^3 - 35x^2 - 31x + 139$
$\left(-\frac{3}{4}, \frac{1}{4}\right)$	$33x^6 - 36x^5 + 110x^4 - 120x^3 + 140x^2 - 96x + 72$
$\left(\frac{21}{16}, -\frac{1}{8}\right)$	$7x^5 + 3x^4 + 78x^3 - 2x^2 + 63x + 147$
$\left(21, -\frac{5}{2}\right)$	$-8x^6 - 24x^5 + 80x^4 - 100x^3 + 170x^2 - 84x + 63$
$\left(-\frac{4}{9}, \frac{2}{3}\right)$	$-3x^6 + 14x^5 - 63x^4 + 96x^3 - 171x^2 - 48x - 188$
$\left(-\frac{7}{81}, \frac{17}{18}\right)$	$-7x^6 + 56x^5 - 95x^4 - 20x^3 - 205x^2 - 44x - 93$
$\left(21, \frac{5}{2}\right)$	$-24x^6 - 24x^5 - 280x^4 - 260x^3 + 170x^2 - 24x + 1$

Table 7. Some points of small height on the surface of Theorem 22 and the corresponding genus-2 curves.

We describe a few curves on the Hilbert modular surface, which can be used to produce rational points. Setting $a = -\frac{1}{9}$ gives a rational curve of genus 0, with infinitely many points. Sections of the fibration will also lead to curves birational to $\mathbb{P}^1$ over $\mathbb{Q}$. For instance, P_1 and $2P_1$ describe the rational curves given by $a = 4(d^2 - 1)/5$ and $a = (4d^2 - 5)/13$, respectively. The Brauer obstruction does not vanish identically on any of these.

13. Discriminant 28

13.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type E_6, D_5 and A_4 at $t = \infty$, 0 and 1 respectively, and a section of height $\frac{28}{60} = \frac{7}{15} = 4 - \frac{6}{5} - 1 - \frac{4}{3}$.
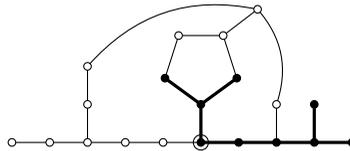
A Weierstrass equation for this family is given by

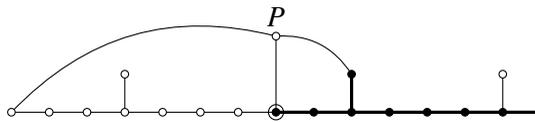$$y^2 = x^3 + atx^2 + bt^2(t - 1)^2 x + ct^3(t - 1)^4,$$

where

$$a = 2(f^2 - g^2)(t - 1) + t,$$
$$b = (f^2 - g^2)^2(1 - t) - 2(f^2 - g^2)(f + 1)t,$$
$$c = (f + 1)^2(f^2 - g^2)t.$$

We identify the class of a D_8 fiber and carry out a 2-neighbor step to convert to an elliptic fibration with D_8 and E_7 fibers, and a section of height $\frac{7}{2} = 4 + 2 \cdot 1 - 1 - \frac{3}{2}$.

Then we identify the class of an E_8 fiber, and carry out a 2-neighbor step to get the desired $E_8 E_7$ fibration.

The new elliptic fibration has a section, since $P \cdot F' = 5$, while the remaining component of the D_8 fiber has intersection number 2 with F'. We now read out the Igusa–Clebsch invariants.

Theorem 23. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(28)$ as a double cover of $\mathbb{P}^2_{g,h}$ is given by the equation*

$$z^2 = -(g - f - 2)(g + f + 2)$$
$$\times (8g^4 + 92f^2g^2 + 180fg^2 + 71g^2 - 100f^4 - 180f^3 - 71f^2 + 4f + 4).$$

It is a singular K3 *surface.*

13.2. Analysis. It has a second involution $(f, g) \mapsto (f, -g)$. The branch locus consists of three components. The factors $g \pm (f + 2)$ correspond to the locus where the Picard number of the K3 surface jumps to 19, due to the presence of an I_2 fiber, but the discriminant decreases to 14, because the nontrivial section of height $\frac{7}{15}$ becomes divisible by 2. The more complicated factor corresponds to just the presence of an extra I_2 factor, which makes the discriminant 56. This component of the branch locus is also a genus 0 curve; a parametrization is given by

$$(f, g) = \left(-\frac{2m^4 + 17m^3 + 57m^2 + 85m + 47}{2(m + 1)(m + 2)(m^2 + 6m + 11)}, -\frac{(m^2 + 6m + 7)(2m^2 + 7m + 7)}{2(m + 1)(m + 2)(m^2 + 6m + 11)} \right).$$

Rational point (f, g)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{31}{16}, -\frac{9}{16}\right)$	$9x^6 - 15x^5 + 39x^4 - 25x^3 + 36x^2 + 11$
$\left(-\frac{56}{65}, -\frac{61}{65}\right)$	$-37x^6 + 42x^5 + 17x^4 - 33x^3 - 7x^2 - 3x - 3$
$\left(-\frac{31}{16}, \frac{9}{16}\right)$	$15x^5 - 39x^4 - 11x^3 + 45x^2 + 27x$
$\left(-\frac{35}{24}, -\frac{31}{24}\right)$	$-27x^6 + 54x^5 - 36x^4 + 30x^3 - 24x^2 - 6x - 2$
$\left(-\frac{12}{11}, -\frac{1}{11}\right)$	$x^6 + 14x^5 + 61x^4 + 73x^3 - 49x^2 + 3x + 1$
$\left(-\frac{14}{9}, -\frac{5}{9}\right)$	$37x^6 + 69x^5 + 75x^4 + 79x^3 + 69x^2 + 12x + 20$
$\left(-\frac{2}{5}, -\frac{7}{5}\right)$	$-4x^6 + 12x^5 - x^4 + 87x^3 + 68x^2 + 48x + 3$
$\left(-\frac{13}{8}, -\frac{7}{8}\right)$	$3x^6 - 6x^5 + 5x^4 - 10x^3 + 41x^2 + 30x + 87$
$\left(-\frac{42}{41}, 1\right)$	$-92x^6 - 76x^5 + 21x^4 + 9x^3 + 9x^2 - x - 1$
$\left(-\frac{13}{8}, \frac{7}{8}\right)$	$18x^6 + 78x^5 + 44x^4 - 94x^3 - 76x^2 + 30x + 25$
$\left(\frac{5}{16}, -\frac{19}{16}\right)$	$-48x^6 + 45x^5 + 11x^4 + 87x^3 - 97x^2 - 63x + 56$
$\left(-\frac{30}{41}, -\frac{36}{41}\right)$	$-27x^6 + 57x^5 - 100x^4 - 68x^3 - 76x^2 + 36x$
$\left(-\frac{11}{3}, \frac{7}{3}\right)$	$16x^6 - 24x^5 - 111x^4 + 9x^3 + 102x^2 - 27x - 33$
$\left(\frac{26}{31}, -\frac{64}{31}\right)$	$-9x^6 - 6x^5 + 32x^4 + 32x^3 - 112x^2 - 6x + 99$
$\left(\frac{13}{80}, -\frac{15}{16}\right)$	$-9x^6 - 15x^5 + 85x^3 - 135x + 54$
$\left(\frac{26}{31}, \frac{64}{31}\right)$	$-25x^6 + 30x^5 - 64x^4 + 72x^3 - 136x^2 + 102x - 69$

Table 8. Some points of small height on the surface of Theorem 23 and the corresponding genus-2 curves.

Since the double cover is branched along a sextic, the Hilbert modular surface is itself a K3 surface. Setting $f = h - 2$ and then using the invertible substitution $h = t(1 + 1/x)$, $g = t(1 - 1/x)$ (and absorbing square factors) converts it to an elliptic fibration over $\mathbb{P}_t^1$, which we can write in Weierstrass form as

$$y^2 = x^3 - t(108t^3 - 176t^2 + 63t + 4)x^2$$
$$+ 32(t-1)^2 t^3(135t^2 - 36t - 106)x - 64(t-1)^4 t^4(6075t^2 - 6075t - 196).$$

This has fibers of type I_2 at $t = \infty$, $\frac{4}{5}$ and $\frac{28}{27}$, I_1^* at $t = 0$, I_5 at $t = 1$, and I_3 at $t = (19 \pm 7\sqrt{7})/36$, giving a contribution of $D_5 \oplus A_4 \oplus A_1^3 \oplus A_2^2$ to the Néron–Severi lattice. Therefore the Picard number is at least 18. We identify the sections

$$P_1 = \left(12t^2(t-1)(36t-37), 4t^2(t-1)(27t-28)(72t^2 - 76t + 1)\right),$$
$$P_2 = \big(t(1080t^3 - 2064t^2 + 953t + 28)/7,$$
$$2t^2(5t-4)(27t-28)(72t^2 - 76t + 1)/7^{3/2}\big),$$

with height matrix

$$\begin{pmatrix} \frac{7}{60} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}.$$

Therefore the Picard number is 20. An easy lattice-theoretic argument (see the online supplement) shows that these sections must generate the Mordell–Weil group, and therefore the Néron–Severi lattice has discriminant 112.

The quotient by the involution $g \mapsto -g$ has equation

$$y^2 = x^3 - (84f^2 + 148f + 39)x^2/4 - (96f^4 + 364f^3 + 615f^2 + 500f + 140)x$$
$$- (f+2)^2(5f+2)^2(4f^2+4f-1).$$

This is a rational elliptic surface with an I_4 fiber at $f = -1$, an I_3 fiber at $f = -\frac{17}{18}$, and I_2 fibers at $t = -\frac{26}{27}$ and $t = \infty$. The Mordell–Weil group is generated by the 2-torsion section $(-2(f+2)^2, 0)$ and the non-torsion section $\left(-2(f^2+3f+3), (f+1)(18f+17)\right)$ of height $\frac{1}{12}$.

13.3. *Examples.*
Table 8 lists some points of small height and their genus-2 curves.

Next, we describe some special curves on the surface, which may be used to produce rational points. First, $f = -\frac{17}{18}$ gives a rational curve, which can be parametrized as $g = -19(h^2 - 2)/(18(h^2 + 2))$. The Brauer obstruction vanishes identically for this family, giving a family of genus-2 curves with real multiplication by $\mathcal{O}_{28}$. Next, the specialization $f = -\frac{26}{27}$ gives another rational curve, which can be parametrized as

$$g = -2\,\frac{13h^2 + 729h - 75816}{27(h^2 + 5832)}.$$

The Brauer obstruction does not vanish identically for this family. Finally, the section P_1 on the Jacobian of $Y_-(28)$ can be described as

$$(f, g) = \left(\frac{4t^2 - 8t + 1}{4t - 1}, \frac{4t^2 - 2t + 1}{4t - 1}\right).$$

The Brauer obstruction vanishes identically on this family as well.

From a plot of the rational points, we observe many rational points on the lines $g = \pm(5f + 2)/3$. However, the Brauer obstruction does not vanish identically along these lines.

14. Discriminant 29

14.1. *Parametrization.*
We start with an elliptic K3 surface with fibers of type E_7 and A_8, and a section P of height $\frac{29}{18} = 4 - \frac{3}{2} - \frac{8}{9}$. A Weierstrass equation for this family is given by

$$y^2 = x^3 + \left(-(4f - 1)t^2 + (g - 2)t + 1\right)x^2$$
$$- 2gt^3\left(2f^2t^2 + (-g + 2f + 1)t - 1\right)x + g^2t^6\left((g - 4f)t + 1\right).$$

We identify a fiber F of type E_8, and go to the associated elliptic fibration by a 3-neighbor step. Note that $P \cdot F = 4$, while the remaining component of the A_8 fiber intersects F with multiplicity 3. Therefore the new fibration has a section.
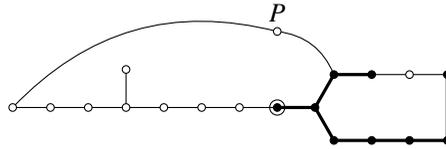
The new elliptic fibration has E_8 and E_7 fibers, and we may read out the Igusa–Clebsch invariants. The branch locus for the double cover of $\mathbb{P}^2_{f,g}$ corresponds to elliptic K3 surfaces having an extra I_2 fiber.

Theorem 24. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(29)$ as a double cover of $\mathbb{P}^2$ is given by the equation*

$$z^2 = -g^4 + (6f + 11)g^3 - (27f^4 - 18f^3 + 5f^2 + 102f - 1)g^2$$
$$+ 8f(36f^3 - 25f^2 + 35f - 1)g - 16f^2(4f - 1)^3.$$

It is a K3 surface of Picard number 19.

14.2. *Analysis.* The branch locus is a rational curve, and a parametrization is given by

$$(f, g) = \left(\frac{(m + 2)^2(m^2 + 2m - 4)}{m^2(m^2 + 6m + 12)}, \frac{4(m + 2)^2(m + 4)^3}{m(m^2 + 6m + 12)^2} \right).$$

The substitution $g = hf$ makes the right-hand side of the equation a quartic in f, after absorbing a factor of f^2 into z^2 on the left. Also, $f = 0$ makes the right-hand side a square (namely $(h - 4)^2$), giving us a point on the genus-1 curve over $\mathbb{P}^1_h$. We may then replace the genus-1 curve by its Jacobian, which has the following equation after the change of parameter $h = 4 - t$, and some Weierstrass transformations:

$$y^2 = x^3 - (t^4 - 10t^3 - t^2 - t - 16)x^2$$
$$- t(20t^5 - 140t^4 - 3740t^3 + 14234t^2 + 3349t - 1120)x$$
$$- t^2(3475t^6 - 46300t^5 + 180355t^4 - 10362t^3 - 849193t^2 + 276269t - 19600).$$

This is an elliptic K3 surface, with bad fibers of type I_4 at $t = \infty$, I_3 at $t = 0$ and at the roots of the polynomial $15t^3 + 12t^2 + 160t - 64$ (which gives the cubic field of discriminant -87), and I_2 at the roots of $2t^3 - 38t^2 + 255t - 28$ (which gives the cubic field of discriminant -116). The trivial lattice therefore has rank 16, leaving room for Mordell–Weil rank at most 4. We find the linearly independent sections

$$P_1 = \left(t(125t^3 - 700t^2 + 800t - 76)/4, \right.$$

$$t(1375t^5 - 11450t^4 + 29240t^3 - 23616t^2 + 7296t - 512)/8),$$

$$P_2 = \big((30t^4 - 306t^3 + 636t^2 + 1833t - 448)/29,$$
$$(2t^3 - 38t^2 + 255t - 28)(15t^3 + 12t^2 + 160t - 64)/(29b)\big),$$

$$P_3 = \big(2(3a - 5)t^2 - 3(8a - 23)t,$$
$$at\big(-99t^3 + 18(a + 37)t^2 - 144(3a + 7)t + 416a - 288\big)/3\big).$$

Here $a = \sqrt{-3}$ and $b = \sqrt{29}$. Therefore, the Picard number of the Hilbert modular surface is at least 19. We showed by counting points modulo 11 and 13 that the Picard number must be exactly 19. This agrees with the calculations in [Oda 1982, p. 109]. The height matrix of the sections above is

$$\frac{1}{6}\begin{pmatrix} 20 & 0 & -10 \\ 0 & 3 & 0 \\ -10 & 0 & 14 \end{pmatrix}.$$

Therefore, the sublattice of the Néron–Severi lattice spanned by the sections above together with the trivial lattice has discriminant $2^4 \cdot 3^4 \cdot 5 = 6480$. By checking that it is 2- and 3-saturated, we showed that it is the entire Néron–Severi lattice, and therefore the sections P_1, P_2, P_3 generate the Mordell-Weil group.

14.3. *Examples.* Table 9 lists some points of small height and their genus-2 curves. The section P_1 corresponds to the rational curve given by $g = \frac{4}{5}uf$, with

$$f = \frac{22u^5 - 321u^4 + 1651u^3 - 3377u^2 + 1980u + 50}{25u^6 - 400u^5 + 2375u^4 - 6050u^3 + 4813u^2 + 2325u + 225}.$$

The Brauer obstruction vanishes identically, yielding a 1-parameter family of genus-2 curves with real multiplication by $\mathcal{O}_{29}$.
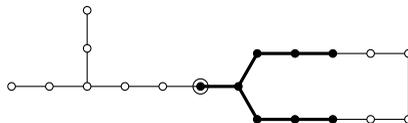
15. Discriminant 33

15.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type A_{10} and E_6 at $t = \infty$ and $t = 0$ respectively. A Weierstrass equation for such a family is given by

$$y^2 = x^3 + (c + 2d + 1)t^2x^2 + 2(c + d)t^4x + ct^4,$$

with

$$c = (s^2 - r^2)^2t^2 - (s^2 - r^2)(s^2 - r^2 + 2r)t + s^2, \quad d = (s^2 - r^2)(1 - t) + r.$$

We identify the class of an E_7 fiber below, and perform a 2-neighbor step.

The new elliptic fibration has fibers of type E_6, E_7 and A_2, as well as a section P of height $\frac{33}{18} = \frac{11}{6} = 4 - 0 - \frac{3}{2} - \frac{2}{3}$. We then identify the class of an E_8 fiber and carry out a 3-neighbor step to an $E_8 E_7$ elliptic fibration.

The new elliptic fibration has a section, since $P \cdot F' = 5$, while the intersection number of the remaining component of the E_6 fiber with F' is 3, and these are coprime.

From the Weierstrass equation of the $E_8 E_7$ fibration, we determine the Igusa–Clebsch invariants, and then the equation of the branch locus.

Theorem 25. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(33)$ as a double cover of $\mathbb{P}^2$ is given by the equation*

$$z^2 = 9s^6 - (26r^2 - 80r + 104)s^4 + (25r^4 - 152r^3 + 400r^2 - 408r + 432)s^2$$
$$- (8r^6 - 72r^5 + 280r^4 - 472r^3 + 336r^2 - 64r - 16).$$

Rational point (f, g)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-37, -\frac{37}{2}\right)$	$-98x^5 - 56x^4 - 131x^3 - 114x^2 + 10x - 68$
$\left(\frac{20}{67}, \frac{16}{67}\right)$	$264x^6 + 760x^5 + 183x^4 - 630x^3 - 53x^2 - 20x - 4$
$\left(\frac{6}{7}, \frac{48}{7}\right)$	$12x^6 - 12x^5 - 409x^4 + 1062x^3 + 287x^2 - 588x - 252$
$(-40, 40)$	$-53x^6 + 227x^5 + 374x^4 + 1191x^3 + 669x^2 + 680x + 900$
$\left(-1, -\frac{29}{10}\right)$	$1210x^5 - 110x^4 + 511x^3 - 17x^2 + 53x + 1$
$\left(-4, -\frac{16}{5}\right)$	$-200x^6 + 1360x^5 - 995x^4 + 242x^3 - 191x^2 - 4x - 12$
$\left(-\frac{1}{7}, -\frac{37}{98}\right)$	$-1588x^6 + 986x^5 - 122x^4 + 221x^3 - 68x^2 - 2x - 8$
$(-6, 8)$	$540x^6 + 2052x^5 - 1149x^4 + 1724x^3 - 39x^2 - 894x - 506$
$\left(\frac{5}{43}, \frac{19}{43}\right)$	$-2x^6 + 80x^5 - 786x^4 + 2265x^3 + 74x^2 + 80x - 2$
$\left(\frac{8}{9}, \frac{16}{9}\right)$	$-4x^6 + 204x^5 - 837x^4 - 160x^3 + 2451x^2 + 1620x - 228$
$\left(-\frac{6}{13}, -\frac{16}{13}\right)$	$236x^6 - 796x^5 + 2293x^4 - 2178x^3 + 1525x^2 + 2492x - 764$
$\left(-\frac{4}{9}, \frac{16}{9}\right)$	$552x^6 - 2232x^5 + 3183x^4 + 562x^3 - 4713x^2 - 1248x - 88$
$\left(-3, \frac{36}{5}\right)$	$1024x^6 - 1920x^5 - 2252x^4 + 1065x^3 - 2288x^2 - 6195x + 66$
$\left(\frac{18}{19}, \frac{48}{19}\right)$	$-768x^6 - 2560x^5 + 1571x^4 + 7838x^3 - 2133x^2 - 6912x + 2376$
$\left(-\frac{4}{5}, -\frac{32}{25}\right)$	$-1412x^6 - 1372x^5 + 2149x^4 + 8226x^3 + 4889x^2 - 896x - 4096$
$\left(\frac{2}{9}, \frac{7}{9}\right)$	$3189x^6 + 4599x^5 - 6897x^4 - 9331x^3 + 5424x^2 + 5040x - 1968$

Table 9. Some points of small height on the surface of Theorem 24 and the corresponding genus-2 curves.

It is a singular K3 *surface.*

15.2. *Analysis.* This is a double cover of the (r, s)-plane branched along a sextic, and is therefore a K3 surface. The extra involution is given by $(r, s) \mapsto (r, -s)$. The equation of the branch locus may be transformed as follows: setting $t = s^2$, we have

$$-8r^6 + 72r^5 + (25t - 280)r^4 + (-152t + 472)r^3 + (-26t^2 + 400t - 336)r^2$$
$$+ (80t^2 - 408t + 64)r + (9t^3 - 104t^2 + 432t + 16) = 0$$

which, after resolution of singularities, becomes a genus-0 curve, parametrized by

$$r = \frac{m^3 + 4m^2 + 4m + 4}{m^2(m+1)}, \quad t = \frac{8(m^3 + 4m^2 + 4m + 2)}{m^4(m+1)^2}.$$

Then the branch locus can be written as a double cover

$$s^2 = 8(m^3 + 4m^2 + 4m + 2)/(m^4(m+1)^2).$$

After removing square factors and performing a Weierstrass transformation, it is converted to the elliptic curve $y^2 + y = x^3 - x^2$. It is isomorphic to $X_1(11) \cong X_0(33)/\langle w_{33} \rangle$, where w_{33} is the Atkin–Lehner involution.

For the equation of the Hilbert modular surface, the transformation $s = r + t$ makes the right-hand side of the equation a quartic in r, with the coefficient of r^4 being a square. Converting to the Jacobian form, and applying a Weierstrass transformation as well as scaling t, we get an elliptic fibration

$$y^2 = x^3 + (t^4 + 24t^3 + 58t^2 + 84t + 1)x^2$$
$$+ (280t^5 + 5488t^4 + 1376t^3 + 2192t^2 + 72t)x$$
$$+ 4608t^7 + 95632t^6 + 32576t^5 + 26848t^4 + 14656t^3 + 1296t^2.$$

This has bad fibers of type I_5 at $t = \infty$, type II at $t = 1$, type I_3 at $t = 0$, $\frac{1}{2}$ and $(-17 \pm 3\sqrt{33})/2$, and type I_2 at $t = -\frac{21}{2} \pm \frac{11\sqrt{33}}{6}$. These contribute $A_1^3 \oplus A_2^4 \oplus A_4$ to the Néron–Severi lattice.

By finding sections modulo a small prime and attempting to lift them to $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{33})$, we find the following sections of small height:

$$P_1 = \left(4t(3t^2 + 60t - 13), \; 4t(t^2 + 17t - 2)(3t^2 + 63t - 2)\right),$$

$$P_2 = \left(-4t(3t + 11), \; 4t(t - 1)(3t^2 + 63t - 2)\right),$$

$$P_3 = \left((4\sqrt{33} + 12)t^2 - (2\sqrt{33} + 34)t, \right.$$
$$\left. (\sqrt{33} + 3)t(2t - 1)(2t + 3\sqrt{33} + 17)(6t + 63 - 11\sqrt{33})/6\right).$$

Rational point (r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{11}{3}, \frac{8}{3}\right)$	$-9x^6 - 6x^5 - 7x^4 + 7x^3 + 2x^2 + 3x - 2$
$(1, 3)$	$-4x^6 + 15x^4 - x^3 - 9x^2 + 12x + 5$
$\left(-\frac{13}{5}, \frac{27}{5}\right)$	$-14x^5 + 20x^4 + 2x^3 - 15x^2 - 4x$
$\left(\frac{28}{3}, \frac{23}{3}\right)$	$-5x^6 + 6x^5 - 5x^4 + 27x^3 - 11x^2 + 12x - 24$
$\left(\frac{73}{19}, -\frac{41}{19}\right)$	$-7x^6 + 15x^5 + x^4 - 31x^3 - 2x^2 + 12x + 12$
$(1, -3)$	$-x^6 - 3x^5 + 9x^4 + 34x^3 - 30x^2 - 9x + 8$
$\left(\frac{41}{51}, \frac{7}{51}\right)$	$-9x^6 - 9x^5 - 35x^4 + 11x^3 - 8x^2 + 12x$
$\left(\frac{16}{15}, \frac{1}{15}\right)$	$2x^6 - 3x^5 + 7x^4 + 13x^3 - 20x^2 + 36x - 15$
$\left(-\frac{13}{5}, -\frac{27}{5}\right)$	$-5x^5 - 3x^4 + 13x^3 - 17x^2 - 40x$
$\left(\frac{46}{3}, \frac{41}{3}\right)$	$-8x^6 + 36x^5 - 23x^4 - 21x^3 - 47x^2 - 18x - 9$
$\left(-\frac{17}{10}, \frac{27}{10}\right)$	$-20x^5 + 28x^4 - 37x^3 + 60x^2 + 44x$
$\left(\frac{13}{3}, -\frac{11}{3}\right)$	$7x^6 - 48x^5 + 68x^4 - 2x^3 - 25x^2 + 24x - 36$
$\left(-\frac{38}{11}, \frac{61}{11}\right)$	$-9x^6 - 6x^5 + 69x^4 - 19x^3 - 39x^2 + 12x - 8$
$\left(-\frac{29}{3}, -11\right)$	$-31x^5 + 71x^4 - 32x^3 + 23x^2 - 40x - 8$
$\left(\frac{1}{33}, \frac{56}{33}\right)$	$3x^6 + 24x^5 + 8x^4 + 41x^3 + 32x^2 - 72x - 36$
$\left(\frac{13}{3}, \frac{11}{3}\right)$	$-12x^6 + 24x^5 - 31x^4 + 81x^3 - 67x^2 + 39x - 70$

Table 10. Some points of small height on the surface of Theorem 25 and the corresponding genus-2 curves.

These are linearly independent in the Mordell–Weil group, and the matrix of Néron–Tate heights is

$$\frac{1}{30}\begin{pmatrix} 6 & -2 & 3 \\ -2 & 19 & -1 \\ 3 & -1 & 9 \end{pmatrix}.$$

It has determinant $\frac{11}{360}$, and so the sublattice of the Néron–Severi group generated by these sections and the trivial lattice has rank 20 and discriminant $-\frac{11}{360} \cdot 2^3 \cdot 3^4 \cdot 5 = -99$. We show that this is the full Picard group by checking that our subgroup of the Mordell–Weil group is 3-saturated. We deduce that $Y_-(33)$ is a singular K3 surface with Picard lattice of discriminant -99.

The quotient of the $Y_-(33)$ by the involution $s \mapsto -s$ has Weierstrass equation

$$y^2 = x^3 - 2(13r^2 - 40r + 52)x^2 + 9(25r^4 - 152r^3 + 400r^2 - 408r + 432)x$$
$$- 648(r-1)^3(r^3 - 6r^2 + 14r + 2).$$

It is a rational elliptic surface, with reducible fibers of types I_5, I_3 and I_2 at $r = \infty$, 19 and 23 respectively. The Mordell–Weil group is generated by the section $\left(3(3t^2 - 16t + 112), 12(r-23)(r-19)\right)$ of height $\frac{1}{30}$.

15.3. *Examples.* Table 10 on the previous page lists some points of small height and their genus-2 curves.

We may attempt to match up these examples with eigenforms in the tables of modular forms. For instance, for the points $(1, 3)$ and $(1, -3)$, the corresponding genus-2 curves (with isogenous Jacobians)

$$y^2 = -(x^3 - 3x - 1)(4x^3 - 3x + 5),$$
$$y^2 = -(x^3 - 3x^2 + 1)(x^3 + 6x^2 + 9x - 8)$$

have the property that their traces match those of a newform of weight $1296 = 2^4 \cdot 3^4$ in the modular forms database.

We also see some simple rational curves on the Hilbert modular surface: the specialization $r = 19$ gives a rational curve parametrized by $s = -(16m^2 + 41)/(3m)$, while $r = 23$ is also a rational curve, parametrized by $s = -(16m^2 + 7)/m$. The Brauer obstructions do not identically vanish for points on these curves.

We have slightly better luck with sections of the elliptic fibration: for instance, the sections $P_1, 2P_1, 3P_1$ give rise to rational curves with respective parametrizations

$$\left(-\frac{3t^2 - 112}{2(3t + 8)}, \frac{3t^2 + 16t + 112}{2(3t + 8)}\right), \quad \left(-\frac{t^2 - 12}{2(t + 2)}, \frac{t^2 + 4t + 12}{2(t + 2)}\right),$$

$$\left(-\frac{3t^2 - 20}{6(t + 2)}, \frac{3t^2 + 12t + 20}{6(t + 2)}\right).$$

The Brauer obstruction vanishes identically on each of these.

16. Discriminant 37

16.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type E_6, D_5 and A_4 at $t = \infty$, 0 and 1 respectively, and a section of height $\frac{37}{60} = 4 - \frac{4}{3} - \frac{5}{4} - \frac{4}{5}$.

A Weierstrass equation for this family is

$$y^2 = x^3 + atx^2 + bt^2(t - 1)^2 x + ct^3(t - 1)^4,$$

with

$$a = (2g - f + 1)(t - 1) + g^2 t/4,$$
$$b = (f - g - 1)\big((f + g - 1)(t - 1) + (f - 2)gt/2\big),$$
$$c = (g - f + 1)^2\big(f^2(t - 1) + (f - 2)^2\big)/4.$$

We identify the class of an D_8 fiber and carry out a 2-neighbor step to convert to an elliptic fibration with D_8 and E_6 fibers.
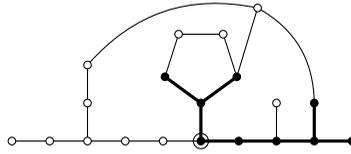
This new elliptic fibration has Mordell–Weil rank 2. In fact, it is quite easy to exhibit one non-torsion section P (we do so in the online supplement) of height $\frac{2}{3} = 4 - \frac{4}{3} - 2$. Then we find an E_8 fiber as shown below, and proceed to it by a 2-neighbor step. The section P combined with most of the components of the E_6 fiber gives a disjoint E_7 configuration, and therefore the new elliptic fibration has reducible fibers of type E_8 and E_7. The fact that $(-P)$ intersects the multiplicity-1 component of the E_8 fiber shown below implies that the new fibration has a section.
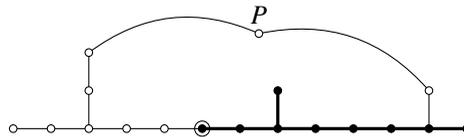
We then read out the Igusa–Clebsch invariants and write down the equation of the Hilbert modular surface.

Theorem 26. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(37)$ as a double cover of $\mathbb{P}^2$ is given by the equation*

$$z^2 = f^2 g^4 + 2f(14f - 1)g^3 - (126 f^3 - 142 f^2 + 44f - 1)g^2$$
$$+ (f - 1)(54 f^3 - 34 f^2 + 17f - 10)g - (f - 1)^2(27 f^2 - 8f + 8).$$

It is a K3 *surface of Picard number* 19.

16.2. *Analysis.* The branch locus corresponds to the locus where the elliptic K3 surface acquires an extra I_2 fiber. The transformation

$$(f, g) = \left(\frac{2x^2 y + 4xy + y + x^4 + x^3 - 3x^2 - x}{x^3(x + 2)}, \frac{2y + x^2 - 2x - 1}{x^2} \right)$$

converts it to the elliptic curve

$$y^2 + y = x^3 - x,$$

which is 37a in Cremona's tables. This is an elliptic curve of rank 1, isomorphic to $X_0(37)/\langle w \rangle$, where w is the Atkin–Lehner involution.

To analyze this Hilbert modular surface, note that we have a genus-1 fibration over $\mathbb{P}^1_f$, which has a section because the coefficient of g^4 is a square. Hence $Y_-(37)$ is an elliptic K3 surface. Taking the Jacobian of this genus-1 curve over $\mathbb{Q}(f)$, and

Rational point (f, g)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{3}{2}, \frac{11}{7}\right)$	$17x^6 - 24x^5 - 66x^4 + 68x^3 + 81x^2 - 54x - 27$
$\left(\frac{3}{4}, \frac{11}{9}\right)$	$15x^6 - 6x^5 - 71x^4 + 35x^3 + 94x^2 - 48x - 21$
$\left(\frac{3}{2}, \frac{1}{3}\right)$	$11x^6 - 54x^5 - 125x^4 - 52x^3 - 32x^2 - 42x - 75$
$\left(\frac{3}{2}, \frac{59}{65}\right)$	$-135x^6 + 108x^5 + 45x^4 - 44x^3 + 130x^2 - 12x + 95$
$\left(\frac{1}{4}, \frac{17}{5}\right)$	$81x^5 - 135x^4 + 13x^3 - 9x^2 + 70x - 15$
$\left(-\frac{13}{3}, \frac{16}{13}\right)$	$-13x^6 + 156x^5 - 24x^4 - 132x^3 - 45x^2 + 108x - 27$
$\left(-\frac{4}{9}, -\frac{13}{6}\right)$	$36x^6 - 108x^5 + 165x^4 - 124x^3 + 21x^2 + 36x - 28$
$\left(\frac{5}{3}, \frac{13}{10}\right)$	$-54x^6 + 54x^5 - 9x^4 - 84x^3 + 141x^2 - 180x + 100$
$\left(\frac{3}{16}, \frac{13}{8}\right)$	$52x^6 + 156x^5 - 39x^4 - 180x^3 + 9x^2 + 72x - 16$
$\left(-\frac{1}{2}, \frac{15}{13}\right)$	$-31x^6 + 156x^5 - 195x^4 - 260x^3 + 210x^2 + 156x + 23$
$\left(3, \frac{81}{11}\right)$	$-18x^6 + 122x^5 - 135x^4 - 268x^3 - 25x^2 + 144x + 176$
$\left(\frac{1}{4}, 3\right)$	$x^6 + 72x^5 - 18x^4 - 189x^3 - 117x^2 + 270x + 45$
$\left(\frac{31}{15}, \frac{16}{31}\right)$	$-53x^6 + 6x^5 - 21x^4 + 208x^3 - 258x^2 - 276x + 259$
$\left(\frac{34}{27}, \frac{14}{51}\right)$	$-2x^6 + 36x^5 - 138x^4 + 105x^3 - 33x^2 - 153x - 289$
$\left(3, \frac{7}{2}\right)$	$-108x^6 - 324x^5 - 207x^4 - 116x^3 + 105x^2 - 12x - 12$
$\left(\frac{22}{3}, \frac{38}{11}\right)$	$-22x^6 + 72x^5 + 84x^4 - 341x^3 - 441x^2 + 417x + 473$

Table 11. Some points of small height on the surface of Theorem 26 and the corresponding genus-2 curves.

reparametrizing $f = t/(t+1)$, we get, after some Weierstrass transformations, the equation

$$y^2 = x^3 - (t+1)(27t^3 + 21t^2 - 19t - 1)x^2 - 8t^2(t+1)^2(30t^2 - 235t - 1)x$$
$$- 16t^3(t+1)^2(3136t^4 + 5484t^3 + 1024t^2 - 2161t - 108).$$

This surface has reducible fibers of type IV at $t = -1$, type I_2 at $t = -\frac{1}{28}$, and type I_3 at $t = 0, \infty$ and the four roots of $(27t^4 + 45t^3 + 10t^2 + 22t + 3)$. This quartic polynomial describes a quadratic extension of $\mathbb{Q}(\sqrt{37})$. The bad fibers contribute $A_2^7 \oplus A_1$ to the trivial lattice. We also have a 3-torsion section

$$P_0 = \left(4t(t+1)(9t^2 + 7t - 3), 4t(t+1)(27t^4 + 45t^3 + 10t^2 + 22t + 3)\right)$$

and two non-torsion sections

$$P_1 = \left(4t(t+1)(49t^2 + 28t - 1), 4t(t+1)(637t^4 + 854t^3 + 276t^2 + 33t + 1)\right),$$
$$P_2 = \left(4(252t^4 + 457t^3 + 118t^2 - 177t - 9)/37,\right.$$
$$\left. 4(t+3)(28t+1)(27t^4 + 45t^3 + 10t^2 + 22t + 3)/37^{3/2}\right).$$

These two sections have height $\frac{8}{3}$ and $\frac{5}{6}$ respectively and are orthogonal with respect

to the Néron–Tate height pairing. Therefore, the Néron–Severi lattice contains a sublattice of rank 19 and discriminant 1080. Counting points modulo 11 and 13 shows that the Picard number must be exactly 19. This is again confirmed by Oda's tables [1982, p. 109]. We checked that the Mordell–Weil subgroup generated by P_0, P_1 and P_2 is saturated at 2 and 3, and thus that we have the full Néron–Severi lattice.

16.3. *Examples.* Table 11 on the previous page lists some points of small height and their genus-2 curves.

Next, we describe some curves of small genus on the surface, which may be used to produce rational points. The specialization $f = -\frac{1}{27}$ gives a rational curve, with parametrization $g = 7(h^2 - 8h + 19)/(3(h^2 - 1))$. The Brauer obstruction does not vanish identically for this rational curve.

The sections P_1 and $-P_1$ give rational curves, parametrized by

$$g = \frac{13 f^2 - 7 f + 3}{f(3 f + 2)} \quad \text{and} \quad g = \frac{9 f^2 - 2 f + 2}{7 f + 1}$$

respectively. The Brauer obstruction vanishes on both these loci, yielding families of genus-2 curves whose Jacobians have real multiplication by $\mathcal{O}_{37}$.

17. Discriminant 40

17.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type E_7, D_5 and A_4 at $t = \infty, 0$ and 1 respectively.

A Weierstrass equation for this family is given by

$$y^2 = x^3 + t\big(e^2 t + (4d + 1)(1 - t)\big)x^2$$
$$+ 2t^2(t - 1)^2\big(2det + 2d(d + 1)(1 - t)\big)x + 4d^2 t^3(t - 1)^4$$

with

$$d = (f - e + 1)(f + e - 1)/2.$$

We first identify the class of a D_8 fiber, and perform a 2-neighbor step to an elliptic fibration with D_8 and E_7 fibers.
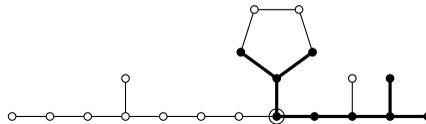
This fibration has a section P of height $5 = 4 + 2 - 1$. Next, we take a 2-neighbor step to go to a fibration with E_8 and E_7 fibers. We have $P \cdot F' = 9$, while the intersection number of the F' with the remaining component of the D_8 fiber is 2.

Since these are coprime, the new elliptic fibration has a section.

Finally, we read out the Igusa–Clebsch invariants, and compute the branch locus of the double cover, which is a union of two curves, one corresponding to an extra I_2 fiber, the other to a promotion of the fiber at $t = \infty$ from E_7 to E_8.

Theorem 27. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(40)$ as a double cover of $\mathbb{P}^2_{e,f}$ is given by the equation*

$$z^2 = -(f^2 - e^2 + 1)\big(8f^4 + (-17e^2 + 12e - 8)f^2 + 9e^4 - 12e^3 + 7e^2 + 10e + 2\big).$$

It is a K3 *surface of Picard number* 19.

17.2. Analysis. The extra involution is $(e, f) \mapsto (e, -f)$. The branch locus is a union of two curves. The first, $f^2 - e^2 + 1 = 0$, is a rational curve, parametrized by say $(e, f) = \big((t^2 + 1)/(2t), (t^2 - 1)/(2t)\big)$. It corresponds to the subfamily of elliptic K3 surfaces for which the E_7 fiber is promoted to an E_8 fiber. The other component is also a rational curve, parametrized by

$$(e, f) = \left(-\frac{(m^2 - 2)^2}{m(m-2)(m^2 - 4m + 2)}, \ \frac{2(m^2 - 2m + 2)(m^2 - m - 1)}{m(m-2)(m^2 - 4m + 2)}\right).$$

It corresponds to elliptic K3 surfaces with an extra I_2 fiber.

The Hilbert modular surface is a double cover of a plane branched along a sextic, and is therefore a K3 surface. The transformation $f = e + t$ makes it a quartic in e, whose leading coefficient is a square. We thus get an elliptic K3 surface whose Weierstrass equation may be obtained by taking the Jacobian of this genus-1 curve. After some elementary algebra, we get the elliptic K3 surface

$$y^3 = x^3 + (t^4 + 24t^3 + 98t^2 + 16t + 1)x^2$$
$$+ (128t^5 + 2352t^4 + 1088t^3 + 64t^2)x - (512t^6 + 9216t^5 + 704t^4).$$

It has reducible fibers of types I_6 at $t = \infty$, I_4 at $t = 0$, I_3 at $t = -8 \pm \frac{5\sqrt{10}}{2}$, and I_2 at $t = \frac{1}{3}$ and $t = -9 \pm 4\sqrt{5}$, giving a contribution of $A_5 \oplus A_3 \oplus A_1^3 \oplus A_2^2$ to the Néron–Severi lattice. The trivial lattice thus has rank 17 and discriminant 1728.

We also have the two sections

$$P_1 = \big(4t(2t^2 + 36t + 3), -4t(t^2 + 18t + 1)(2t^2 + 32t + 3)\big),$$
$$P_2 = \big(-(6t^4 + 124t^3 + 303t^2 + 138t + 9)/10,$$
$$\sqrt{10}(2t + 3)(3t - 1)(t^2 + 18t + 1)(2t^2 + 32t + 3)/100\big).$$

These have heights $\frac{1}{12}$ and $\frac{7}{6}$ respectively, and are orthogonal with respect to the height pairing. Therefore, the Picard number is at least 19. Counting points modulo 7 and 11 proves that the Picard number is 19 (we thank Ronald van Luijk for carrying out such a calculation), in agreement with Oda's calculations [1982, p. 109]. The part of the Néron–Severi lattice spanned by the above sections with the trivial lattice has rank 19 and discriminant 168. We showed that this lattice is 2-saturated, so it is the full Néron–Severi lattice.

The quotient of the Hilbert modular surface by the involution $f \mapsto -f$ is the rational elliptic surface

$$y^2 = (x + 8e^2 - 8)\big(x^2 + (17e^2 - 12e + 8)x + 8(3e + 1)^2(e^2 - 2e + 2)\big).$$

It has reducible fibers of types I_6, I_3, I_2 at $e = \infty, 8, 9$, respectively, and the Mordell–Weil group is generated by the 6-torsion section

$$\big(-4(2e^2 + e - 11), 4(e - 9)(e - 8)\big);$$

indeed this is the universal elliptic curve over $X_1(6)$.

17.3. *Examples.* Table 12 lists some points of small height and their genus-2 curves.

Rational point (e, f)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{49}{8}, -\frac{47}{8}\right)$	$-12x^6 - 12x^5 - 21x^4 + 14x^3 + 39x^2 + 48x - 64$
$\left(\frac{31}{6}, 5\right)$	$108x^6 + 108x^5 - 81x^4 + x^3 + 63x^2 - 33x + 9$
$\left(-\frac{1}{12}, -\frac{5}{6}\right)$	$36x^5 + 78x^4 - 41x^3 - 129x^2 + 45x + 27$
$\left(-\frac{13}{10}, -\frac{4}{5}\right)$	$-72x^6 + 108x^5 + 135x^4 - 135x^3 - 219x^2 + 135x + 53$
$\left(\frac{49}{8}, \frac{47}{8}\right)$	$-72x^6 + 216x^5 - 315x^4 + 162x^3 + 21x^2 - 72x - 16$
$\left(-\frac{23}{14}, -\frac{13}{7}\right)$	$-8x^6 + 60x^5 - 87x^4 - 163x^3 + 288x^2 + 324x + 27$
$\left(-\frac{29}{12}, -\frac{25}{12}\right)$	$12x^6 + 132x^5 + 355x^4 - 90x^3 - 245x^2 + 36x - 44$
$\left(\frac{87}{55}, \frac{12}{55}\right)$	$44x^5 + 200x^4 - 422x^3 + 180x^2 - 81x$
$\left(-\frac{75}{28}, \frac{18}{7}\right)$	$-77x^6 + 147x^5 - 45x^4 - 335x^3 + 186x^2 - 180x - 432$
$\left(\frac{49}{23}, \frac{43}{23}\right)$	$46x^6 - 24x^5 + 252x^4 - 29x^3 + 468x^2 + 24x + 366$
$\left(-\frac{37}{36}, \frac{35}{36}\right)$	$-18x^6 - 258x^5 - 475x^4 + 220x^3 - 325x^2 + 72x - 48$
$\left(\frac{19}{8}, \frac{13}{8}\right)$	$432x^6 + 216x^5 - 27x^4 - 502x^3 - 87x^2 + 36x + 116$
$\left(\frac{55}{28}, -\frac{43}{28}\right)$	$-496x^6 + 48x^5 - 545x^4 + 90x^3 - 257x^2 + 120x - 80$
$\left(-\frac{1}{28}, \frac{15}{28}\right)$	$314x^6 + 426x^5 + 555x^4 + 140x^3 - 195x^2 - 264x - 176$
$\left(-\frac{23}{15}, \frac{22}{15}\right)$	$-586x^6 + 330x^5 - 512x^4 + 150x^3 - 110x^2 - 24x - 1$
$\left(-\frac{5}{4}, -\frac{1}{4}\right)$	$8x^6 - 168x^5 - 269x^4 + 466x^3 + 451x^2 - 624x - 376$

Table 12. Some points of small height on the surface of Theorem 27 and the corresponding genus-2 curves.

The section P_1 gives a rational curve

$$(e, f) = \left(-\frac{2g^2 + 11}{4g - 1}, \frac{2g^2 - g - 11}{4g - 1} \right).$$

However, the Brauer obstruction does not vanish identically on this locus. The section $2P_1$ gives a rational curve

$$(e, f) = \left(-\frac{2g^2 + 3}{4g}, \frac{2g^2 - 3}{4g} \right)$$

on the surface. Here, the Brauer obstruction does vanish identically, yielding a family of genus-2 curves with real multiplication by $\mathcal{O}_{40}$.

18. Discriminant 41

18.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type A_5 at $t = 0$ and A_{10} at $t = \infty$, with a section of height $\frac{41}{66} = 4 - \frac{4 \cdot 7}{11} - \frac{5}{6}$.

A Weierstrass equation for this family is given by

$$y^2 = x^3 + (t^2 + 2dft + cf^2)x^2 + 2t^2(dt + cf)x + ct^4,$$

with

$$c = r^2 s^2 \big(16t^2 - 8(4rs - 16s - r)t + (4rs - 16s + r)^2 \big),$$

$$d = rs(4t - 12rs + 16s + r),$$

$$f = (t + 4s)/(4rs).$$

We identify the class of an E_8 fiber, and perform a 3-neighbor step to an elliptic fibration with E_8 and A_7 fibers.
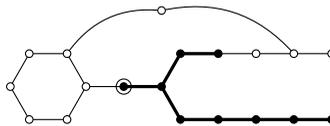
This fibration has a section P of height $\frac{41}{8} = 4 + 2 - \frac{7}{8}$. Next, we take a 2-neighbor step to go to a fibration with E_8 and E_7 fibers.
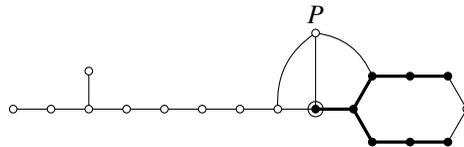
The intersection number of the new fiber F' with the remaining component of the A_7 fiber is 2 and with the section P is 5. Since these number are coprime, the new genus-1 fibration defined by F' has a section.

Reading out the map to $\mathcal{M}_2$ from the $E_8 E_7$ fibration, we obtain the following result.

Theorem 28. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(41)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$z^2 = (4s+1)^4 r^4 - 16(8s^2 + 42s - 1)(4s+1)^2 r^3$$
$$+ 32(128s^4 + 4672s^3 + 1976s^2 + 248s + 3)r^2$$
$$- 256(2688s^3 + 872s^2 + 82s - 1)r + 256(16s+1)^3.$$

It is a K3 *surface of Picard number* 19.

18.2. *Analysis.* The branch locus is a rational curve, with a parametrization given by

$$(r, s) = \left(-\frac{4(u-1)(u+1)^3}{(u^2 - u - 1)^2}, -\frac{1}{4u^2(u-1)^2} \right).$$

The Hilbert modular surface has a genus-1 fibration to $\mathbb{P}^1_r$, which is in fact an elliptic fibration, since the coefficient of r^4 is a perfect square. Converting to the Jacobian form, we get the Weierstrass equation

$$y^2 = x^3 + (t-1)(t^3 + 23t^2 + 96t - 32)x^2$$
$$+ 16(t-1)(4t^4 + 53t^3 - 217t^2 + 112t - 16)x.$$

This is an elliptic K3 surface, with bad fibers of type I_6 at $t = \infty$, I_4 at $t = 0$, III at $t = 1$, I_3 at $t = -15$, and I_2 at the four roots of $q(t) = 4t^4 + 53t^3 - 217t^2 + 112t - 16$. This quartic $q(t)$ describes a dihedral Galois extension K of $\mathbb{Q}$, quadratic over $\mathbb{Q}(\sqrt{41})$. Thus we get a trivial lattice of rank 17 and discriminant 2304. We also have a 2-torsion point $(0, 0)$ and the two non-torsion sections

$$P_1 = \left(-164(t-1), 4\mu(t-1)(t+15)(5t-6) \right),$$
$$P_2 = \left((5+\mu)(t-1)(8t^2 + 53t + 9t\mu - 185 - 29\mu)/16, \right.$$
$$\left. (5+\mu)(t-1)(t+15)(4t - 11 + \mu)(8t^2 + 53t + 9t\mu - 185 - 29\mu)/64 \right),$$

where $\mu = \sqrt{41}$. The height matrix for P_1 and P_2 is

$$\frac{1}{3}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}.$$

Therefore the Picard number is at least 19. Counting points modulo 7 and 11 shows that the Picard number must be 19, in agreement with [Oda 1982]. The sections above and the trivial lattice therefore generate a lattice of rank 19 and discriminant 512. We showed that it is 2-saturated, and is thus the full Néron–Severi lattice.

Rational point (r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{52}{3}, -\frac{1}{5}\right)$	$-6x^6 + 3x^5 - 4x^4 + 3x^3 + 20x^2 - 36x + 20$
$\left(\frac{7}{2}, \frac{13}{12}\right)$	$-x^6 - 15x^5 - 65x^4 - 57x^3 + 40x^2 + 6x + 20$
$\left(\frac{56}{9}, -\frac{43}{40}\right)$	$-56x^6 - 42x^5 + 64x^4 - 78x^3 - 29x^2 + 36x - 28$
$\left(\frac{52}{7}, -\frac{2}{3}\right)$	$-24x^6 + 36x^5 - 34x^4 - 51x^3 + 86x^2 - 87x + 18$
$\left(13, -\frac{7}{12}\right)$	$-15x^6 + 33x^5 + 23x^4 + 73x^3 - 72x^2 - 54x - 108$
$\left(\frac{32}{5}, -\frac{47}{48}\right)$	$-18x^6 - 3x^5 + 46x^4 - 111x^3 + 22x^2 + 48x - 84$
$\left(\frac{92}{9}, -\frac{11}{14}\right)$	$-4x^6 + 24x^5 - 17x^4 + 3x^3 - 125x^2 - 72x - 95$
$\left(\frac{60}{13}, 3\right)$	$-10x^6 + 33x^5 + 40x^4 - 111x^3 - 152x^2 + 60x + 140$
$\left(\frac{76}{13}, -\frac{4}{3}\right)$	$-24x^6 - 12x^5 - 50x^4 + 111x^3 + 10x^2 + 129x - 153$
$\left(28, -\frac{23}{24}\right)$	$-46x^6 + 69x^5 - 14x^4 + 169x^3 - 134x^2 - 84x - 72$
$\left(\frac{68}{15}, -\frac{11}{4}\right)$	$48x^6 - 48x^5 - 119x^4 - 84x^3 + 145x^2 + 180x + 100$
$\left(\frac{23}{6}, \frac{21}{4}\right)$	$168x^5 + 85x^4 + 70x^3 + 229x^2 + 24x - 72$
$\left(\frac{40}{3}, -\frac{55}{56}\right)$	$-39x^6 + 36x^5 - 116x^4 + 186x^3 - 107x^2 + 240x - 200$
$\left(\frac{52}{25}, -\frac{1}{24}\right)$	$-112x^6 - 264x^5 + 25x^4 + 240x^3 - 125x^2 - 114x + 62$
$\left(\frac{4}{13}, -\frac{11}{96}\right)$	$16x^6 + 24x^5 - 223x^4 + 274x^3 - 7x^2 + 216x - 120$
$\left(\frac{27}{2}, \frac{11}{76}\right)$	$12x^6 - 132x^5 - 219x^4 + 286x^3 + 201x^2 - 264x - 56$

Table 13. Some points of small height on the surface of Theorem 28
and the corresponding genus-2 curves.

18.3. _Examples._ Table 13 lists some points of small height and their genus-2
curves.

Next, we describe some curves which are a source of many rational points.
The specialization $s = -4$ gives a curve of genus 0, with a parametrization $r = -2(m^2 + 343)/(9(m - 13))$. The specialization $r = \frac{108}{25}$ also gives a rational curve,
parametrized by $s = -(4m - 169)(4m + 169)/(16(108m - 10813))$. The Brauer
obstruction vanishes on both these loci, giving families of curves whose Jacobians
have real multiplication by $\mathcal{O}_{41}$.

Finally, sections of the elliptic fibration also give rational curves on the surface.
For instance, the section P_0 is parametrized by

$$r = \frac{4(256s^3 - 48s^2 - 16s - 1)}{(4s + 1)^2(16s - 7)}.$$

The Brauer obstruction vanishes here as well.

19. Discriminant 44

19.1. _Parametrization._ We start with an elliptic K3 surface with fibers of type A_{10}
at $t = 0$ and D_6 at $t = \infty$. This has Néron–Severi lattice of discriminant $11 \cdot 4 = 44$

and rank $2 + 10 + 6 = 18$. The Weierstrass equation for this universal family can be written as

$$y^2 = x^3 + ax^2 + 2bt^4x + ct^8,$$

with

$$a = s(r^2s^2 - s^2 + 2s + 2)t^3 - s(2r^2s - 3s + 2)t^2 + s(r^2 - 3)t + 1,$$
$$b = s^2\big((2r^2s^2 - 2s^2 + 4s + 1)/2t^2 - (r^2s - 2s + 1)t - 1\big),$$
$$c = s^4((r^2s - s + 2)t + 1).$$

We identify the class of an E_8 fiber and move to this fibration via a 3-neighbor step.
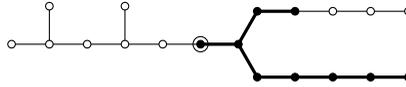
The new elliptic fibration has E_8, D_6 and A_1 fibers, and a section P of height $\frac{11}{2} = 4 + 2 \cdot 1 - \frac{1}{2}$. We identify an E_7 fiber F' below. Note that $P \cdot F' = 5$, while the excluded component of the D_6 fiber intersects F' in 2. Since 2 and 5 are coprime, we see that the fibration defined by F' has a section. We move to it by a 2-neighbor step.
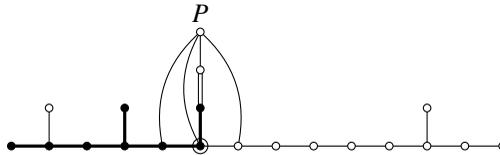
Calculating the Igusa–Clebsch invariants and following the rest of the algorithm in Section 4, we obtain the following result.

Theorem 29. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(44)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$z^2 = (rs + s - 1)(rs - s + 1)(r^6s^2 - r^4s^2 + 18r^2s - 16s + 27).$$

It is an honestly elliptic surface, with arithmetic genus 2 and Picard number 29.

19.2. Analysis. The extra involution is $\iota : (r, s) \mapsto (-r, s)$. The branch locus has three components. The two simpler components $rs \pm (s - 1) = 0$ are obviously rational curves, and a simple calculation shows that they correspond to elliptic K3 surfaces for which the D_6 fiber gets promoted to an E_7 fiber. The more complicated component of the branch locus, corresponding to elliptic fibrations with an extra I_2 fiber, is also a rational curve; a parametrization is given by

$$(r, s) = \left(\frac{4m}{m^2 + 3}, -\frac{(m^2 + 3)^3}{16(m^2 - 1)} \right).$$

The right-hand side of the equation defining the Hilbert modular surface is quartic in s, whence the surface $Y_-(44)$ is elliptically fibered over $\mathbb{P}^1_r$. The two linear factors (or the fact that the coefficient of s^4 is a square) imply that there are sections, so we may convert to the Jacobian form:

$$y^2 = x^3 + 2(r^6 - r^4 - 9r^2 + 11)x^2 + (r^2 - 1)^3(r^6 + r^4 + 91r^2 - 121)x.$$

This is an honestly elliptic surface, with $\chi = 3$. It has reducible fibers of type I_2 at $r = 0$, I_6 at $r = \pm 1$, I_4 at $r = \infty$, I_3 at $r = \pm\frac{2}{\sqrt{3}}$, and type I_2 at the six roots of $(r^3 - 3r^2 + 5r + 11)(r^3 + 3r^2 + 5r - 11)$ (both factors generate the cubic field k_{-44} of discriminant -44). The trivial lattice has rank 26, leaving room for Mordell–Weil rank at most 4. We find the following sections, of which P_0 is 2-torsion, while P_1, P_2 and P_3 are linearly independent non-torsion sections, orthogonal with respect to the height pairing and of heights $\frac{7}{6}$, $\frac{3}{2}$ and $\frac{11}{6}$ respectively:

$P_0 = (0, 0)$,
$P_1 = \left(11(r^2 - 1), 4\sqrt{11}\, r(r^2 - 1)^3(3r^2 - 4)\right)$,
$P_2 = \left(-(r + 1)^3(r^3 - 3r^2 + 5r + 11), 6\sqrt{-3}\, r(r + 1)^3(r^3 - 3r^2 + 5r + 11)\right)$,
$P_3 = \left((r - 1)(r + 1)^2(r^3 - 3r^2 + 5r + 11), 2r^3(r - 1)(r + 1)^2(r^3 - 3r^2 + 5r + 11)\right)$.

Therefore, the Picard number is at least 29. Analysis of the associated quotient elliptic K3 surface and its twist (see below) shows that the Mordell–Weil rank is exactly 3 and therefore the Picard number is exactly 29. The sections above together with the trivial lattice generate a lattice of discriminant $133056 = 2^6 \cdot 3^3 \cdot 7 \cdot 11$. We checked that it is 2- and 3-saturated, and so it is the entire Néron–Severi lattice. Therefore these sections generate the Mordell–Weil group.

We next analyze the quotient of $Y_-(44)$ by the involution ι. Taking $t = r^2$, we find the equation

$$z^2 = s^4 t^4 - s^2(2s^2 - 2s + 1)t^3 + s^2(s^2 + 16s + 1)t^2$$
$$-s(2s - 3)(17s - 6)t + (s - 1)^2(16s - 27).$$

This has an elliptic fibration over $\mathbb{P}^1_s$, and since the coefficient of t^4 is a square, we may convert to the Jacobian, which is

$$y^2 = x^3 + s^2(s + 2)(s + 8)x^2 + 2s^3(6s^2 + 47s + 9)x + s^4(36s^2 + 268s - 27).$$

This is an elliptic K3 surface, with reducible fibers of type E_6 at $s = 0$, I_7 at $s = \infty$, I_3 at $s = -\frac{27}{4}$, and I_2 at the roots of $2s^3 + 14s^2 - 6s + 1$, which generates the cubic field k_{-44}. The trivial lattice has rank 19. We find a non-torsion section

$$P = (1 - 4s, 2s^3 + 14s^2 - 6s + 1)$$

of height $4 - \frac{12}{7} - \frac{3}{2} = \frac{11}{14}$. It is easy to show that P generates the Mordell–Weil group: the configuration of reducible fibers does not allow for either nontrivial torsion or a section of height $11/14n^2$ for any integer $n > 1$. Therefore the K3 surface is singular, with Néron–Severi lattice of discriminant $-396 = -4 \cdot 9 \cdot 11$.

We may also analyze the quotient by considering it as an elliptic surface over $\mathbb{P}^1_t$, and since the coefficient of s^4 is $t^2(t-1)^2$, which is a square, there is a section. Converting to the Jacobian, we get the elliptic K3 surface

$$y^2 = x^3 + 2(t^3 - t^2 - 9t + 11)x^2 + (t-1)^3(t^3 + t^2 + 91t - 121)x$$

which is also obtained by replacing r^2 by t in the Weierstrass equation of $Y_-(44)$. This elliptic fibration has bad fibers of type I_6^*, I_6 and I_3 at $t = \infty$, 1 and $\frac{4}{3}$ respectively, and of type I_2 at the roots of $t^3 + t^2 + 91t - 121$, which generates k_{-44}. Therefore, the root lattice has rank 18. We find the sections

$P_0 = (0, 0),$

$P_1 = \left((1-t)(t - 3\sqrt{-3})^2, 2(3 - \sqrt{-3})(t-1)(t - 3\sqrt{-3})(9t - 6 - 2\sqrt{-3})/3\right)$

$P_2 = \left((1-t)(t + 3\sqrt{-3})^2, 2(3 + \sqrt{-3})(t-1)(t + 3\sqrt{-3})(9t - 6 + 2\sqrt{-3})/3\right),$

with P_0 being 2-torsion, and P_1 and P_2 having height pairing matrix

$$\begin{pmatrix} \frac{5}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{3} \end{pmatrix}.$$

These sections, along with the trivial lattice, generate a lattice of rank 20 and discriminant -396, which must therefore be the entire Néron–Severi lattice. The Mordell–Weil rank of this elliptic surface is 2.

Finally, we consider the quadratic twist of the above elliptic K3 surface, which is the quotient of $Y_-(44)$ by the involution $\iota' : (r, s, z) \mapsto (-r, s, -z)$. It is given by the equation

$$y^2 = x^3 + 2t(t^3 - t^2 - 9t + 11)x^2 + t^2(t-1)^3(t^3 + t^2 + 91t - 121)x.$$

This is an elliptic K3 surface with reducible fibers of type I_1^*, I_6, I_3 and I_2 at $t = 0$, 1, $\frac{4}{3}$ and ∞ respectively, and I_2 at the roots of $t^3 + t^2 + 91t - 121$. The trivial lattice has rank 18. We find the sections

$$P_0 = (0, 0),$$
$$P_1 = \left(11t(t-1)^3, 4\sqrt{11}t^2(t-1)^3(3t-4)\right),$$

the first being 2-torsion, and the second of height $\frac{7}{12}$. The Picard number is therefore at least 19. Counting points modulo 5 and 7 shows that it is exactly 19; therefore the Mordell–Weil rank is exactly 1. These sections and the trivial lattice span

a sublattice of discriminant 168 of the Néron–Severi lattice. This sublattice is 2-saturated, since the configuration of fibers does not allow for a section of height $\frac{7}{48}$, and we can easily check that the elliptic surface does not have 4-torsion or other 2-torsion sections. Therefore, we have the entire Néron–Severi lattice, and P_0 and P_1 generate the Mordell–Weil group.

The calculation of the Mordell–Weil ranks of the quotient elliptic surface and its quadratic twist allows us to conclude that the Mordell–Weil rank of the original (honestly) elliptic surface is $1 + 2 = 3$.

19.3. *Examples*. Table 14 lists some points of small height and their genus-2 curves.

The specializations $r = \pm 1$ give rational curves on the surface, but points on these correspond to decomposable abelian surfaces, which therefore have an endomorphism ring strictly larger than $\mathcal{O}_{44}$. The section P_0 is a rational curve, given by $s = (r^4 + 27)/(2(r^2 - 1)(r^2 - 8))$. However, the Brauer obstruction does not vanish identically on it.

(r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-2, -\frac{7}{6}\right)$	$101x^6 - 60x^5 + 2x^4 + 92x^2 + 48x + 24$
$\left(2, -\frac{7}{6}\right)$	$15x^6 - 168x^5 + 170x^4 - 112x^3 + 20x^2 - 8$
$\left(4, -\frac{7}{15}\right)$	$-24x^6 + 48x^5 + 52x^4 + 144x^3 + 238x^2 + 588x - 161$
$\left(-4, -\frac{7}{15}\right)$	$56x^6 + 196x^4 - 320x^3 + 250x^2 - 480x + 723$
$\left(-5, -\frac{13}{12}\right)$	$-144x^6 + 120x^5 + 265x^4 + 700x^3 - 425x^2 - 750x - 1750$
$\left(2, \frac{7}{12}\right)$	$696x^6 - 2112x^5 + 7492x^4 - 7032x^3 + 10234x^2 - 756x + 5103$
$\left(5, -\frac{13}{12}\right)$	$12x^6 - 60x^5 - 145x^4 + 400x^3 - 1225x^2 - 1500x + 11500$
$\left(2, \frac{13}{36}\right)$	$-5193x^6 - 5124x^5 - 16906x^4 - 11576x^3 - 17212x^2 - 6240x - 5304$
$\left(-\frac{3}{4}, \frac{36}{7}\right)$	$-4744x^6 - 15552x^5 + 7596x^4 + 42048x^3 - 20310x^2 - 32112x + 33553$
$\left(-2, \frac{13}{36}\right)$	$-18261x^6 + 13668x^5 + 65210x^4 - 41512x^3 - 74284x^2 + 18816x + 24696$
$\left(\frac{3}{4}, \frac{36}{7}\right)$	$-6504x^6 + 22608x^5 + 28428x^4 - 94288x^3 - 66510x^2 + 103092x + 73009$
$\left(-2, \frac{7}{12}\right)$	$25389x^6 - 97062x^5 - 511x^4 + 240860x^3 - 5989x^2 - 127758x - 83849$
$\left(-5, -\frac{9}{32}\right)$	$-691156x^6 + 20220x^5 - 232521x^4 + 19406x^3 - 22521x^2 + 2484x - 564$
$\left(2, -\frac{61}{42}\right)$	$-629624x^6 + 272400x^5 - 383596x^4 - 704000x^3 - 60778x^2$ $- 194100x - 32193$
$\left(-2, -\frac{61}{42}\right)$	$-550872x^6 + 1549296x^5 - 1810124x^4 - 2005984x^3$ $+ 3719134x^2 - 1321788x - 4862401$
$\left(5, -\frac{9}{32}\right)$	$1781676x^6 - 5240052x^5 + 5462991x^4 - 5705734x^3$ $+ 1769571x^2 + 1002576x - 1011776$

Table 14. Some rational points (r, s) of small height on the surface of Theorem 29 and the corresponding genus-2 curves.

20. Discriminant 53

20.1. *Parametrization.* We start with an elliptic K3 surface with A_8, A_6 and A_1 fibers, and a section of height $\frac{53}{126} = 4 - \frac{1}{2} - \frac{6}{7} - \frac{20}{9}$. A Weierstrass equation for this family is given by

$$y^2 = x^3 + ax^2 + 2bt(t-h)x + ct^2(t-h)^2,$$

with

$$a = g^2 t^4 + (4(h+1)^2 - 2g)t^3 + (4h^2 + 4gh + 6g - 3)t^2$$
$$+ 2(8h^2 - 4gh + 8h + 1)t + (2h+1)^2,$$
$$b = -4(h - g + 1)\big(g(2h+1)t^2 + (6h^2 - 2gh + 6h + 1)t + (2h+1)^2\big),$$
$$c = 16(h - g + 1)^2(2h+1)^2.$$

To transform to a fibration with E_8 and E_7 fibers, we first identify an E_7 fiber, and move to the associated elliptic fibration via a 2-neighbor step.
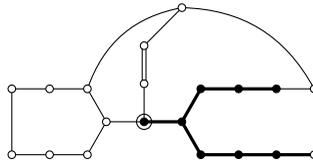
The resulting elliptic fibration has E_7 and A_8 fibers, and a section P of height $\frac{53}{18} = 4 + 2\cdot 1 - \frac{3}{2} - 4 \cdot \frac{5}{9}$. We identify a fiber F of type E_8 and perform a 3-neighbor step to move to the associated elliptic fibration. Note that $P \cdot F = 4$, while the remaining component of the A_8 fiber intersects F with multiplicity 3. Therefore the new elliptic fibration has a section.
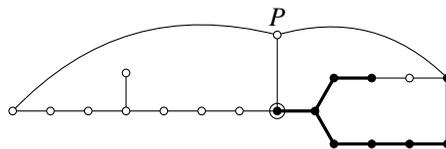
The new elliptic fibration has the requisite E_8 and E_7 fibers, and therefore we may read out the Igusa–Clebsch invariants, and describe the branch locus, which corresponds to elliptic K3 surfaces having an extra I_2 fiber.

Theorem 30. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(53)$ as a double cover of $\mathbb{P}^2_{g,h}$ is given by the equation*

$$z^2 = -27h^4 g^4 - 2h^3(13h^2 + 9h + 9)g^3$$
$$- (11h^6 + 138h^5 + 383h^4 + 506h^3 + 353h^2 + 120h + 16)g^2$$
$$- 2(h+1)^2(52h^4 + 99h^3 + 65h^2 + 19h + 2)g - (h+1)^4(44h^3 + 76h^2 + 40h + 7).$$

It is an honestly elliptic surface, with arithmetic genus 2 *and Picard number* 28.

20.2. *Analysis.* The branch locus is a curve of genus 1, and the change of coordinates

$$g = -\frac{(x^2 - x + 1)(x^2 + 3x + 3)}{(x + 1)(2x + 1)y + x^4 + 3x^3 + 5x^2 + 3},$$

$$h = \frac{(x^5 - x^4 + 4x^3 - 4x^2 + x + 1)y + x^2(x^6 - x^5 + 2x^4 + 2x^3 - x + 1)}{(x^2 - x + 1)^2\big((x + 1)(2x + 1)y + x^4 + 3x^3 + 5x^2 + 3\big)}$$

transforms it to the elliptic curve $y^2 + xy + y = x^3 - x^2$ of conductor 53, which is isomorphic to $X_0(53)/\langle w \rangle$, where w is the Atkin–Lehner involution.

This surface has a genus-1 fibration over $\mathbb{P}^1_h$, making it honestly elliptic with $\chi = 3$. We could find no sections over $\mathbb{Q}$, although there certainly exist sections over $\overline{\mathbb{Q}}$, since the coefficient of g^4 is a square in $\mathbb{Q}(\sqrt{-3})$. The Jacobian of this fibration has the following equation (after making the linear fractional transformation $h = -t/(t + 1)$ on the base, and some Weierstrass transformations):

$$\begin{aligned}
y^2 = x^3 &+ (t^6 - 18t^5 + 55t^4 + 106t^3 - 179t^2 + 24t - 16)x^2 \\
&- 8(t - 1)t^2(37t^5 - 471t^4 - 140t^3 + 1121t^2 - 309t + 248)x \\
&- 16(t - 1)^2 t^4(196t^5 - 2797t^4 + 2712t^3 + 8606t^2 - 3084t + 3115).
\end{aligned}$$

This surface has reducible fibers of type I_7, I_4 and I_3 at $t = \infty, 0, 1$ respectively, type I_2 at the roots of $7t^3 - 99t^2 + 104t - 32$ (which generates the cubic field of discriminant $-2^2 \cdot 53$), and I_3 at the roots of $t^5 - 11t^4 - 11t^3 + 6t^2 - 3t - 9$ (which generates the quintic field of discriminant $-3^2 \cdot 53^2$, and whose roots generate a dihedral D_{10} extension unramified over its quadratic subfield $\mathbb{Q}(\sqrt{-3 \cdot 53})$). The trivial lattice has rank 26, leaving room for Mordell–Weil rank at most 4. We find the sections

$$\begin{aligned}
P_1 = \big(&-4(7t^6 - 106t^5 + 189t^4 + 202t^3 - 778t^2 + 342t - 216)/53, \\
&4(5t - 3)(7t^3 - 99t^2 + 104t - 32)(t^5 - 11t^4 - 11t^3 + 6t^2 - 3t - 9)/(53\sqrt{53})\big) \\
P_2 = \big(&-4(49t^6 - 63t^5 + 99t^4 + 162t^3 - 351t^2 + 162t - 108)/27, \\
&(637t^8 - 378t^7 - 1485t^6 + 3186t^5 - 1755t^4 - 1782t^3 + 3942t^2 - 243t - 972) \\
&\hspace{6cm} \times 4t/(81\sqrt{-3})\big)
\end{aligned}$$

of heights $\frac{7}{6}$ and $\frac{21}{4}$ respectively, orthogonal with respect to the height pairing. Therefore, the Mordell–Weil rank is least 2. By Oda's calculations [1982], we deduce that the Mordell–Weil rank is exactly 2. The sections P_1 and P_2 and the trivial lattice span a lattice of discriminant $1000188 = 2^2 \cdot 3^6 \cdot 7^3$. Checking that it is saturated at 2, 3 and 7, we deduce that it must be the full Néron–Severi lattice.

(g, h)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{11}{152}, -\frac{16}{19}\right)$	$-2332x^6 - 902x^5 - 5060x^4 - 17111x^3 - 5995x^2 - 17545x - 27951$
$\left(\frac{8}{21}, -\frac{2}{3}\right)$	$-8788x^6 + 34200x^5 - 22425x^4 - 11907x^3 - 16230x^2 - 35604x + 34024$
$\left(\frac{72}{49}, -\frac{13}{7}\right)$	$816x^6 - 1944x^5 - 13459x^4 + 24712x^3 + 37733x^2 + 7596x - 2300$
$\left(\frac{75}{26}, -\frac{19}{4}\right)$	$-106605x^6 - 62661x^5 + 467345x^4 + 193313x^3 - 691816x^2$ $- 149592x + 346546$
$\left(\frac{12}{5}, -10\right)$	$139968x^6 + 471744x^5 - 1301409x^4 - 77363x^3 + 671633x^2$ $+ 236496x + 18756$
$\left(-\frac{72}{31}, -4\right)$	$-138482x^6 + 1643417x^5 + 2645029x^4 - 1507309x^3 - 2429567x^2$ $+ 1188320x + 36288$
$\left(\frac{2}{135}, -\frac{10}{11}\right)$	$2175200x^6 - 2750760x^5 + 2725545x^4 + 7678368x^3 - 5205621x^2$ $+ 3781674x + 9014158$
$\left(\frac{16}{63}, -5\right)$	$3829988x^6 + 11621820x^5 - 19617225x^4 - 25097450x^3 + 29201451x^2$ $+ 14626080x - 14560512$
$\left(\frac{18}{103}, -\frac{8}{11}\right)$	$-4788022x^6 - 21151494x^5 - 18288935x^4 - 20340320x^3 - 61042325x^2$ $+ 10128456x - 40124160$
$\left(-\frac{39}{76}, -\frac{29}{16}\right)$	$394632000x^6 - 1964113200x^5 - 1523778060x^4 + 4757784967x^3$ $+ 148400811x^2 - 3811137819x + 540964447$
$\left(\frac{55}{117}, -\frac{79}{144}\right)$	$-5511986931x^6 + 20881501795x^5 + 17115817125x^4 + 19864594645x^3$ $+ 1353729618x^2 - 16117938900x + 5833685448$
$\left(\frac{49}{135}, -\frac{61}{75}\right)$	$-26148549648x^6 - 278797809744x^5 - 748507062651x^4 + 329438683288x^3$ $+ 1420002530997x^2 - 1751808944796x + 1934174962804$
$\left(-\frac{50}{117}, -\frac{13}{9}\right)$	$-159682912000x^6 - 1077016472800x^5 - 2039981245815x^4$ $- 762577047304x^3 + 6811301171385x^2 - 4055008902300x + 449504680500$
$\left(\frac{81}{91}, -\frac{13}{7}\right)$	$134236157214x^6 + 962817170858x^5 - 12198892111873x^4 + 23659009829816x^3$ $+ 9649525790385x^2 - 12776814846900x - 8264106337500$
$\left(\frac{59}{153}, -\frac{17}{27}\right)$	$-687158622928816x^6 + 23483931596064x^5 - 14038441316573x^4$ $- 893569395800x^3 - 20141231607x^2 - 200112822x - 748062$

Table 15. Some rational points (g, h) of small height on the surface of Theorem 30 and the corresponding genus-2 curves.

20.3. *Examples.* Table 15 lists some points of small height and their genus-2 curves.

We could not find any curves of genus 0 which were not contained in the "bad locus" where the abelian surfaces have a strictly larger ring of endomorphisms.

21. Discriminant 56

21.1. *Parametrization.* We start with a K3 elliptic surface with D_6, A_8 and A_1 fibers, and a section P of height $\frac{56}{72} = \frac{7}{9} = 4 - 1 - \frac{20}{9}$. The Weierstrass equation of this family is

$$y^2 = x^3 + ax^2 + 2bt^2(\lambda t - \mu)x + ct^4(\lambda t - \mu)^2,$$

with

$$\lambda = (2h - g^2 + 1), \quad \mu = -2h,$$
$$c = -(g^2 - 1)^2 \big(4(g^2 - 1)t - (h + 2)^2\big),$$
$$b = (g^2 - 1)\big(2(g^2 - 1)t^2 - (h^2 + 4h - 4g^2 + 8)t - (h + 2)^2\big),$$
$$a = (g^2 - 1)^2 t^3 + (h + 2)(h - 2g^2 + 4)t^2 + 2(h^2 + 4h - 2g^2 + 6)t + (h + 2)^2.$$

To find an $E_8 E_7$ fibration on such a K3 surface, we first identify the class of an E_7 fiber below.
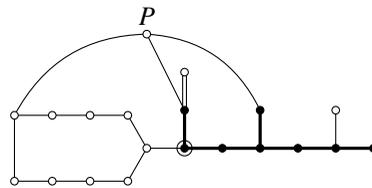
This converts it to an $A_8 E_7$ fibration. Note that this fibration has a section, because the section P intersects F' in 3, whereas (for instance) the non-identity component of the A_1 fiber intersects F' in 2, and these two numbers are coprime.

The new fibration has a section of height $\frac{56}{18} = \frac{28}{9} = 4 - \frac{8}{9}$, which intersects the identity component of the E_7 fiber and component 1 of the A_8 fiber. We then do a 3-neighbor step to get an $E_8 E_7$ fibration.
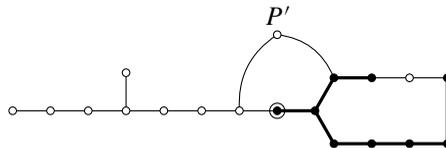
The new fiber F'' satisfies $P' \cdot F'' = 2$, while the excluded component of the A_8 fiber intersects F'' in 3. Since these are coprime, the fibration defined by F'' has a section.

Now we can read out the Igusa–Clebsch invariants, and describe the branch locus of $Y_-(56)$ as a double over of $\mathbb{P}^2_{g,h}$.

Theorem 31. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(56)$ as a double cover of $\mathbb{P}^2_{g,h}$ is given by the equation*

$$z^2 = (2h - g^2 + 1)\big(2h^5 + 27g^2 h^4 - 11h^4 + 72g^2 h^3 - 24h^3 - 40g^2 h^2$$
$$+ 104h^2 - 160g^2 h + 192h + 64g^4 - 144g^2 + 80\big).$$

It is a surface of general type.

21.2. *Analysis.* The extra involution is $g \mapsto -g$. The branch locus has two components. Points on the simpler component $2h - g^2 + 1 = 0$ (which is clearly a rational curve) correspond to elliptic K3 surfaces for which the A_1 and D_6 fibers merge and get promoted to a D_8 fiber. The other component corresponds to elliptic K3 surfaces with an extra I_2 fiber. It is also of genus 0, and a parametrization is given by

$$(g, h) = \left(\frac{s(s^4 + 22s^2 - 7)}{(3s^2 + 1)^2}, -\frac{(s^2 - 1)(s^2 - 5)}{2(3s^2 + 1)} \right).$$

The Hilbert modular surface $Y_-(56)$ is of general type.

We now analyze the quotient of the Hilbert modular surface by the involution $(g, h, z) \mapsto (-g, h, z)$. Setting $f = g^2$, the right-hand side becomes a cubic in f. After some elementary Weierstrass transformations, we get the equation

$$y^2 = x^3 - (27h^4 + 72h^3 - 40h^2 + 96h - 16)x^2 + 512h^3(7h^2 + 20h - 4)x.$$

This is an elliptic K3 surface, with reducible fibers of type I_6, I_6, I_4, I_3 at $h = 0, \infty, -2, \frac{2}{9}$ respectively, and I_2 fibers at $h = (-10 \pm 8\sqrt{2})/7$. The trivial lattice has rank 19 and discriminant 1728. There is an obvious 2-torsion section $P_0 = (0, 0)$, and we find a non-torsion section

$$P_1 = (128h, 128h(h + 2)^2)$$

of height $\frac{2}{3}$. We checked that the group generated by P_0 and P_1 is saturated at 2 and 3. Therefore, this is a singular K3 surface, with Néron–Severi lattice of rank 20 and discriminant -288.

Next, we analyze the twist of the elliptic K3 surface above, obtained by substituting $z = wg$ in the equation of the Hilbert modular surface, and then setting $f = g^2$ (it is the quotient of $Y_-(56)$ by the involution $(g, h, z) \mapsto (-g, h, -z)$). This twist is an honestly elliptic surface, with $\chi = 3$. After some simple algebra, the Weierstrass equation can be written as

$$y^2 = x^3 + (58h^5 + 149h^4 - 56h^3 - 152h^2 - 64h + 16)x^2$$
$$+ 8h^3(2h + 5)(h^2 - 4h - 4)^2(7h^2 + 20h - 4)x.$$

It has reducible fibers of type $I_0^*, I_6, I_4, I_3, I_2, I_2$ at $h = \infty, 0, -2, \frac{2}{9}, -\frac{1}{2}, -\frac{5}{2}$ respectively, I_4 fibers at $h = (-10 \pm 8\sqrt{2})/7$ and I_2 fibers at $h = 2 \pm 2\sqrt{2}$. The trivial lattice has rank 26. In addition to the 2-torsion section $P_0 = (0, 0)$, we find the sections

$$P_1 = \left(-56h^3(h^2 - 4h - 4), 16\sqrt{14}h^3(2h + 1)(9h - 2)(h^2 - 4h - 4) \right),$$
$$P_2 = \left(-8h^3(7h^2 + 20h - 4), 64\sqrt{-1}h^3(2h + 1)(7h^2 + 20h - 4) \right),$$
$$P_3 = \left(4h^2(7h^2 + 20h - 4), 4h^2(h + 2)^2(2h + 1)(7h^2 + 20h - 4) \right)$$

of heights $\frac{5}{6}$, 2 and $\frac{7}{6}$ respectively, orthogonal with respect to the height pairing. On the other hand, counting points on the reductions modulo 11 and 29 shows that the Picard number is at most 29. Therefore, it is exactly 29. The lattice spanned by these sections and the trivial lattice has discriminant $35840 = 2^{10} \cdot 5 \cdot 7$. We checked that it is 2-saturated, and therefore it must equal the entire Néron–Severi lattice.

21.3. *Examples*. Table 16 lists some points of small height and their genus-2 curves.

Next, we analyze curves of low genus on the Hilbert modular surface. The specialization $h = \frac{2}{9}$ gives a rational curve, parametrized by $g = (m^2 - 4m - 9)/(m^2 + 9)$. The Brauer obstruction vanishes identically for rational points on this curve, giving a 1-parameter family of genus-2 curves whose Jacobians have real multiplication by $\mathcal{O}_{56}$.

The specializations $h = -\frac{1}{2}$ and $h = -\frac{5}{2}$ give genus-1 curves with rational points, both of whose Jacobians have rank 1. The Brauer obstruction does not vanish identically on either of these loci.

We also obtain some genus-1 curves by pulling back some sections from the quotient K3 surface. For instance, the section $P_0 + P_1$ gives the genus-1 curve $g^2 = -(7h^4 + 20h^3 - 4h^2 - 32h - 16)/16$ which has rational points (such as

(g, h)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{79}{61}, \frac{28}{61}\right)$	$-2000x^6 + 2040x^5 - 565x^4 + 628x^3 - 349x^2 - 36x - 68$
$\left(\frac{23}{19}, \frac{24}{95}\right)$	$480x^6 + 1200x^5 + 2657x^4 + 1264x^3 + 497x^2 - 2220x + 660$
$\left(\frac{2}{13}, -\frac{6}{13}\right)$	$-600x^6 - 360x^5 + 2660x^4 + 256x^3 - 2698x^2 - 222x + 639$
$\left(-\frac{2}{13}, -\frac{6}{13}\right)$	$1096x^6 - 24x^5 - 3388x^4 + 608x^3 + 2750x^2 - 930x - 225$
$\left(\frac{79}{61}, \frac{28}{61}\right)$	$1350x^6 + 270x^5 + 3375x^4 - 3944x^3 + 1669x^2 - 5328x + 3392$
$\left(-\frac{3}{7}, -\frac{8}{7}\right)$	$-1340x^6 + 5900x^5 - 2227x^4 + 5096x^3 + 2707x^2 + 10x + 1950$
$\left(\frac{3}{7}, -\frac{8}{7}\right)$	$1440x^6 - 4720x^5 - 13227x^4 + 20x^3 + 7389x^2 - 1080x - 432$
$\left(-\frac{1}{7}, \frac{24}{7}\right)$	$-12600x^6 - 3192x^5 - 16975x^4 - 4442x^3 + 5717x^2 - 516x + 4$
$\left(\frac{1}{91}, -\frac{40}{91}\right)$	$3740x^6 - 6420x^5 - 11789x^4 + 18160x^3 + 7315x^2 - 13356x + 2268$
$\left(-\frac{11}{7}, \frac{9}{7}\right)$	$35220x^6 + 10548x^5 + 43345x^4 - 10038x^3 + 3313x^2 - 228x + 52$
$\left(-\frac{23}{19}, \frac{24}{95}\right)$	$47824x^6 + 45048x^5 + 13973x^4 - 11016x^3 + 9341x^2 - 2040x + 400$
$\left(\frac{2}{3}, \frac{2}{9}\right)$	$-6883x^6 + 10038x^5 + 62514x^4 + 31744x^3 - 21780x^2 + 3720x - 200$
$\left(\frac{37}{31}, \frac{9}{31}\right)$	$-1548x^6 - 7732x^5 - 33547x^4 - 51202x^3 - 71163x^2 + 65988x - 11772$
$\left(-\frac{4}{17}, -\frac{3}{8}\right)$	$-316x^6 + 4764x^5 + 21121x^4 - 11666x^3 - 75071x^2 + 20364x + 49716$
$\left(\frac{11}{7}, \frac{9}{7}\right)$	$25092x^6 + 70500x^5 + 71881x^4 - 29834x^3 - 80543x^2 - 25908x + 38700$
$\left(\frac{55}{13}, \frac{28}{3}\right)$	$-94x^6 - 114x^5 - 2497x^4 - 660x^3 - 29263x^2 - 10920x - 170352$

Table 16. Some rational points (g, h) of small height on the surface of Theorem 31 and the corresponding genus-2 curves.

$(h, g) = (1, \pm 1))$, with a Jacobian of conductor $2^4 \, 211$ and Mordell–Weil group $\cong \mathbb{Z}^2$. The section $2P_1$ gives the genus-1 curve $g^2 = -(49h^4 - 112h^3 + 64h^2 - 32h - 16)/16$ which also has rational points $(h, g) = (1, \pm 1)$, with Jacobian of conductor $2^6 \, 7 \cdot 23$ and Mordell–Weil group $\cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$. The Brauer obstruction does not vanish identically on either of these loci.

22. Discriminant 57

22.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type E_6, A_7 and A_2 at $t = \infty, 0, 1$ respectively, and a section of height $\frac{57}{72} = \frac{19}{24} = 4 - \frac{4}{3} - \frac{3 \cdot 5}{8}$.

The Weierstrass equation for this family is

$$y^2 = x^3 + ax^2 + 2bt^2(t-1)x + ct^4(t-1)^2,$$

with

$$a = t(t-1)\big((g^2 - \tfrac{1}{4})(h^2 - 3) + 2gh\big) + \big((g^2 + \tfrac{1}{4})h - g\big)^2 t^2 - t + 1,$$

$$b = -\big(g^2 - \tfrac{1}{4}\big)^2 (h^2 - 1)$$
$$\big((1-t) - \big((g^2 - \tfrac{1}{4})(h^2 - 2) + 2gh\big)t(1-t) - \big(g - \tfrac{1}{2}h\big)\big((g^2 + \tfrac{1}{4})h - g\big)t^2\big),$$

$$c = \big(g^2 - \tfrac{1}{4}\big)^4 (h^2 - 1)^2\big((1-t) + \big(g - \tfrac{1}{2}h\big)^2 t^2 + t(1-t)\big((g - \tfrac{1}{2}h)^2 - (gh + \tfrac{1}{2})^2\big)\big).$$

We identify an E_8 fiber below, and the resulting 3-neighbor step takes us to an elliptic fibration with E_8 and A_7 fibers.
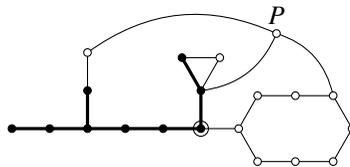
Since $P \cdot F' = 2$ for the new fiber F', while the intersection number of the remaining component of the E_6 fiber with F' is 3, we deduce that the new fibration has a section.

The new fibration has a section of height $\frac{57}{8} = 4 + 2 \cdot 2 - \frac{1 \cdot 7}{8}$. Now we can identify an E_7 fiber F'' and move to the $E_8 E_7$ fibration by a 2-neighbor step. Note that since $Q \cdot F'' = 7$, while the remaining component of the A_7 fiber intersects F'' in 2, the new fibration will have a section.
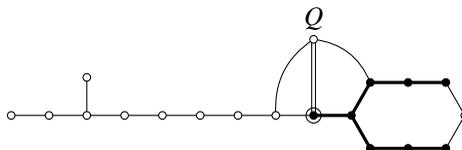
Now we can read out the Igusa–Clebsch invariants and compute the branch locus, which is the subfamily with an extra I_2 fiber.

Theorem 32. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(57)$ as a double cover of $\mathbb{P}^2_{g,h}$ is given by the equation*

$$z^2 = (256g^6 - 176g^4 + 40g^2 - 3)h^4 + (2176g^5 - 960g^3 + 104g)h^3$$
$$+ (4320g^4 - 688g^2 - 34)h^2 + (-3456g^5 + 576g^3 + 328g)h - 2160g^4 - 648g^2 + 361.$$

It is an honestly elliptic surface, with arithmetic genus 2 and Picard number 29.

22.2. Analysis. The extra involution is $(g, h) \mapsto (-g, -h)$. The branch locus has genus 2; the transformation

$$(g, h) = \left(\frac{y + x^2 + 1}{4x^3}, \frac{x(4y - x^4 - 14x^2 - 1)}{(x^2 - 1)^2} \right)$$

converts it to Weierstrass form

$$y^2 = 3x^6 + 11x^4 + x^2 + 1.$$

It is isomorphic to the quotient of $X_0(57)$ by the Atkin–Lehner involution w_{57}.

The Hilbert modular surface $Y_-(57)$ is an honestly elliptic surface, with a genus-1 fibration over $\mathbb{P}^1_g$, and in fact, setting $h = 1$ gives a section. Therefore, we may use the Jacobian form, which has the Weierstrass equation

$$y^2 = x^3 + 4(12g^2 - 1)(28g^2 - 5)x^2$$
$$- 4(2g - 1)^3(2g + 1)^3(12g^2 - 5)(108g^2 - 19)x + (2g - 1)^6(2g + 1)^6(108g^2 - 19)^2.$$

It has reducible fibers of type I_7 at $g = \pm\frac{1}{2}$, IV at $g = \infty$, I_2 at $g = \pm\frac{\sqrt{57}}{18}$, and I_3 at the four roots of $432g^4 + 216g^2 - 49$ (which generate a dihedral extension containing $\mathbb{Q}(\sqrt{57})$). The trivial lattice contributes 26 to the rank of the Néron–Severi lattice, leaving room for Mordell–Weil rank at most 4. We find the sections

$$P_1 = \big(0, (2g - 1)^3(2g + 1)^3(108g^2 - 19) \big),$$
$$P_2 = \big((2g - 1)^2(2g + 1)^2(6g - 1)(6g + 1),$$
$$2g(2g - 1)^2(2g + 1)^2(432g^4 + 216g^2 - 49) \big),$$
$$P_3 = \big((\mu + 3)(18g - \mu)(2g - 1)(2g + 1)^3/3,$$
$$(36g^2 + 9 - 2\mu)(2g + 6 + \mu)(18g - \mu)(2g - 1)(2g + 1)^3/3 \big)$$

(where $\mu = \sqrt{57}$), with nondegenerate height pairing matrix

$$\frac{1}{42} \begin{pmatrix} 38 & 0 & -19 \\ 0 & 20 & -10 \\ -19 & -10 & 39 \end{pmatrix}.$$

Therefore, the Picard number is at least 29. In fact, counting points modulo 7 and 11 shows that the Picard number must be exactly 29. Alternatively, analysis of the quotients below gives another proof. The sections above together with the trivial lattice span a lattice of discriminant $11970 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 19$. We checked that it is 3-saturated, and thus it is all of the Néron–Severi lattice. Therefore, these sections generate the Mordell–Weil group.

Next, we analyze the quotient by the involution $(g, h, z) \mapsto (-g, -h, z)$. This turns out to have a genus-1 fibration with section over the t-line, where $t = g^2$. The Weierstrass equation may be written (after a linear change of the base parameter and a Weierstrass transformation)

$$y^2 = x^3 + 4(t+1)(3t+2)(7t+2)x^2$$
$$- 4t^3(t+1)^2(3t-2)(27t+8)x + t^6(t+1)^3(27t+8)^2.$$

This is an elliptic K3 surface with fibers of type I_7 at $t = 0$, I_0^* at $t = -1$, I_2 at $t = -\frac{8}{27}$, II at $t = \infty$, and I_3 at $t = -2 \pm \frac{2\sqrt{57}}{9}$. Thus the trivial lattice has rank 17, leaving room for at most three independent sections.

We find the independent sections

$$P_1 = \left(t^2(t+1)(9t+8),\, t^2(t+1)^2(27t^2+108t+32)\right),$$
$$P_2 = \left((t+1)(27t+8)(3t^2-64t-64)/57,\right.$$
$$\left. (t+1)^2(t+40)(27t+8)(27t^2+108t+32)/57^{3/2}\right)$$

of heights $\frac{5}{21}$ and $\frac{7}{6}$ respectively, and orthogonal with respect to the height pairing. Therefore the Picard number is at least 19. Counting points modulo 11 and 13 shows that the Picard number cannot be 20. These sections together with the trivial lattice generate a lattice of discriminant 140. We check that it is 2-saturated and must therefore be the full Néron–Severi lattice.

Replacing g^2 by t in the Weierstrass equation for $Y_-(57)$ gives a quadratic twist of the above quotient K3 surface, given by the Weierstrass equation

$$y^2 = x^3 + 4(12t-1)(28t-5)x^2$$
$$- 4(4t-1)^3(12t-5)(108t-19)x + (4t-1)^6(108t-19)^2.$$

This is an elliptic K3 surface with fibers of type I_7 at $t = \frac{1}{4}$, I_2 at $t = \frac{19}{108}$, I_3 at $t = -\frac{1}{4} \pm \frac{\sqrt{57}}{18}$ and IV^* at $t = \infty$. The section $P = (0, (4t-1)^3(108t-19))$ has height $\frac{19}{42}$. Therefore, this K3 surface is singular. Together with the trivial lattice, the section P spans a lattice of discriminant $171 = 3^2 \cdot 19$. Since there is no 3-torsion section, and we cannot have a section of height $19/(3^2 \cdot 42)$ due to the configuration of fibers, this must be the full Néron–Severi lattice.

Therefore, the Mordell–Weil rank of the original surface must be $2 + 1 = 3$.

(g, h)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{1}{10}, -\frac{5}{3}\right)$	$-2x^6 + 15x^5 + 131x^4 + 240x^3 - 61x^2 - 8$
$\left(\frac{7}{10}, -\frac{29}{3}\right)$	$81x^6 + 54x^5 - 286x^4 - 186x^3 + 323x^2 + 120x - 130$
$\left(\frac{7}{18}, 9\right)$	$-108x^6 + 324x^5 - 243x^4 + 186x^3 - 279x^2 - 80$
$\left(-\frac{7}{6}, \frac{21}{5}\right)$	$-100x^6 - 390x^5 - 204x^4 + 74x^3 - 69x^2 - 12x + 8$
$\left(-\frac{1}{18}, -3\right)$	$-220x^6 + 420x^5 - 111x^4 + 238x^3 + 381x^2 + 168x + 24$
$\left(-\frac{1}{4}, -\frac{16}{3}\right)$	$95x^6 - 114x^5 + 325x^4 + 35x^3 + 10x^2 - 429x - 234$
$\left(\frac{5}{14}, \frac{35}{3}\right)$	$120x^6 - 192x^5 + 122x^4 + 286x^3 - 448x^2 + 357x - 63$
$\left(\frac{19}{18}, -\frac{3}{5}\right)$	$58x^6 + 39x^5 - 129x^4 + 132x^3 + 519x^2 + 240x - 40$
$\left(-\frac{7}{10}, \frac{29}{3}\right)$	$-390x^6 + 451x^5 - 230x^4 - 593x^3 + 682x^2 + 220x - 200$
$\left(\frac{4}{9}, 18\right)$	$-540x^6 + 729x^5 - 135x^4 + 255x^3 - 225x^2 - 36x - 52$
$\left(-\frac{5}{2}, -\frac{10}{3}\right)$	$-60x^6 - 156x^5 + 137x^4 + 310x^3 - 351x^2 + 108x + 756$
$\left(-\frac{17}{10}, -\frac{47}{33}\right)$	$60x^5 + 839x^4 + 278x^3 - 652x^2 - 36x - 489$
$\left(\frac{1}{18}, 3\right)$	$-60x^6 + 60x^5 - 3x^4 + 184x^3 + 669x^2 + 132x + 868$
$\left(\frac{5}{2}, \frac{10}{3}\right)$	$-819x^5 - 1042x^4 + 61x^3 + 248x^2 - 48x$
$\left(-\frac{1}{18}, \frac{9}{5}\right)$	$-40x^6 - 72x^5 - 45x^4 - 534x^3 - 297x^2 - 324x - 1188$
$\left(-\frac{5}{2}, \frac{5}{3}\right)$	$-36x^6 + 84x^5 + 491x^4 - 750x^3 - 337x^2 - 912x - 1200$

Table 17. Some rational points (g, h) of small height on the surface of Theorem 32 and the corresponding genus-2 curves.

22.3. *Examples.* Table 17 lists some points of small height and their genus-2 curves.

The specialization $h = 0$ gives a genus-1 curve with rational points, whose Jacobian has rank 1. Of course, there is a large supply of genus-1 curves, simply by specializing g, since we have an elliptic surface.

The sections P_1 and P_2 give rational curves $h = -16g/(4g^2 - 1)$ and $h = (72g^3 - 36g^2 + 78g - 7)/((4g^2 - 1)(6g - 7))$ respectively. The Brauer obstruction vanishes on these curves.

23. Discriminant 60

23.1. *Parametrization.* We start with a K3 elliptic surface with E_6, D_6, A_4 fibers at $\infty, 0, 1$ respectively. The Weierstrass equation for this family is

$$y^2 = x^3 + atx^2 + 2bt^3(t - 1)x + ct^5(t - 1)^2,$$

with

$$a = (h^2 - g^2 - 1)^2 - 4(2h^2 - 3g^2 + 6)(t - 1),$$
$$b = 4\big(-4(h^2 - g^2 - 1)^2 + 16(h^2 - 2g^2 + 1)(t - 1)\big),$$
$$c = 256\big((h^2 - g^2 - 1)^2 + 4g^2(t - 1)\big).$$

We identify an E_8 fiber below, and move to this elliptic fibration by a 3-neighbor step.
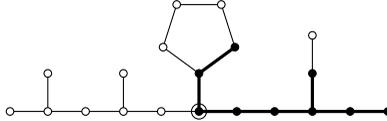
The new elliptic fibration has fibers of type E_8, D_6 and A_1, and a section P of height $\frac{60}{8} = \frac{15}{2} = 4 + 2 \cdot 2 - \frac{1}{2}$. We now identify an E_7 fiber F' and perform a 2-neighbor step to go to the new fibration. Note that it has a section, since $P \cdot F' = 7$, while the remaining component of the D_6 fiber has intersection number 2 with F'.
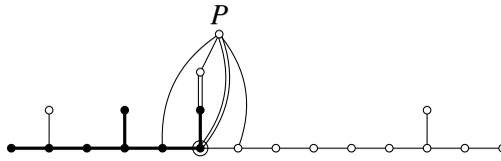
From the resulting $E_8 E_7$ fibration, we can read out the map to $\mathcal{A}_2$.

Theorem 33. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(60)$ as a double cover of $\mathbb{P}^2_{g,h}$ is given by the equation*

$$z^2 = -(h^2 - 2h - g^2 + 5)(h^2 + 2h - g^2 + 5)$$
$$\times (8h^6 - 25g^2h^4 + 24h^4 + 26g^4h^2 - 86g^2h^2 + 24h^2 - 9g^6 + 66g^4 + 47g^2 + 8).$$

It is a surface of general type.

23.2. *Analysis.* The surface $Y_-(60)$ has two extra commuting involutions, $\iota_1 : (g, h) \mapsto (-g, h)$ and $\iota_2 : (g, h) \mapsto (g, -h)$. The two simpler components

$$h^2 \pm 2h - g^2 + 5 = 0$$

of the branch locus correspond to the subfamily of elliptic K3 surfaces where the D_6 fiber gets promoted to an E_7 fiber, while the more complicated component corresponds to an extra I_2 fiber. The simpler components are easily seen to be rational curves, as they define conics in the (g, h)-plane, with rational points. The last component is also a rational curve; a parametrization is given by

$$(g, h) = \left(\frac{(t^2 + 1)^3}{t(t^2 - 1)(t^2 - 2t - 1)}, \frac{(t^2 + 2t - 1)(t^4 - t^3 + 2t^2 + t + 1)}{t(t^2 - 1)(t^2 - 2t - 1)} \right).$$

This Hilbert modular surface is of general type.

We now analyze some of the quotients of this surface by the involutions. The quotient under both involutions is given by

$$z^2 = -(h^2 - 2gh + 6h + g^2 - 10g + 25)$$
$$(8h^3 - 25gh^2 + 24h^2 + 26g^2h - 86gh + 24h - 9g^3 + 66g^2 + 47g + 8),$$

and this is actually a rational surface; the transformation

$$(g, h) = (h' + g'^2 + 2g' + 5, h' + g'^2)$$

converts the above equation into a conic bundle over $\mathbb{P}^1_{h'}$ with a section.

The quotient by the involution ι_1 turns out to be an elliptic K3 surface, with Weierstrass equation given by

$$y^2 = x^3 + 2t^2(215t^2 + 356t + 140)x^2 - t^3(t + 2)^3(5t^2 + 874t + 864)x/3$$
$$-8t^4(t + 2)^6(163t^2 - 54t - 216)/27.$$

It has reducible fibers of type E_6 at $t = 0$, I_6 at $t = -2$, I_2 at $t = -1$ and $t = -\frac{2}{9}$, and I_3 at $t = (-7 \pm 5\sqrt{5})/19$. The trivial lattice therefore has rank 19. We find a 3-torsion section with x-coordinate $11t^2(t + 2)^2/3$ and a non-torsion section with $x = -t^2(t + 2)^2/3$. Therefore the K3 surface is singular. These sections, together with the trivial lattice, generate a lattice of rank 20 and discriminant 60. It must be the full Néron–Severi lattice, since otherwise there would have to be a 6-torsion section or section of height $\frac{5}{24}$, neither of which is possible with our configuration of reducible fibers.

The quotient by the involution ι_2 is also an elliptic K3 surface, with Weierstrass equation

$$y^2 = x^3 - (11t^4 - 20t^2 + 8)\, x^2 + 16(t - 1)^3(t + 1)^3(4t^2 - 5)x.$$

This has bad fibers of type I_6 at $t = \pm 1$, I_2 at $t = \pm\frac{\sqrt{5}}{2}$, and I_3 at $t = \pm\frac{2}{\sqrt{3}}$. Therefore the trivial lattice has rank 18, leaving room for at most two independent sections. We find the following sections, of which the first is 6-torsion.

$$P_0 = \left(2(t^2 - 1)(4t^2 - 5),\ 4(t^2 - 1)(3t^2 - 4)(4t^2 - 5)\right),$$
$$P_1 = \left((11 - 3\mu)t^2(t^2 - 1)/2,\ (3 + 5\mu)t(t^2 - 1)/12(18t^2 - 15 + \mu)\right),$$
$$P_2 = \left((11 + 3\mu)t^2(t^2 - 1)/2,\ (3 - 5\mu)t(t^2 - 1)/12(18t^2 - 15 - \mu)\right),$$

where $\mu = \sqrt{-15}$. The height pairing matrix of P_1 and P_2 is

$$\frac{1}{3}\begin{pmatrix} 7 & -2 \\ -2 & 7 \end{pmatrix}.$$

(g, h)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{17}{6}, -\frac{13}{6}\right)$	$468x^6 + 1332x^5 + 1345x^4 - 20x^3 + 1051x^2 - 150x + 186$
$\left(-\frac{17}{6}, \frac{13}{6}\right)$	$-12x^6 + 132x^5 + 371x^4 + 1506x^3 + 1391x^2 - 528x - 1872$
$\left(\frac{17}{6}, \frac{13}{6}\right)$	$-942x^6 + 3150x^5 - 869x^4 - 4220x^3 + 745x^2 + 2244x + 468$
$\left(-\frac{17}{6}, -\frac{7}{6}\right)$	$48x^6 - 360x^5 + 1907x^4 - 4000x^3 + 5195x^2 + 828x + 2556$
$\left(-\frac{17}{6}, -\frac{13}{6}\right)$	$-4500x^6 - 9300x^5 - 5365x^4 + 4106x^3 + 3335x^2 - 2112x - 2648$
$\left(-\frac{17}{6}, \frac{7}{6}\right)$	$-4116x^6 - 6468x^5 + 8617x^4 + 11086x^3 - 12239x^2 - 3708x + 4212$
$\left(\frac{17}{6}, \frac{7}{6}\right)$	$72x^6 - 2136x^5 + 15869x^4 - 258x^3 - 1759x^2 + 108x - 4$
$\left(\frac{17}{6}, -\frac{7}{6}\right)$	$12x^6 - 36x^5 + 929x^4 - 1458x^3 + 16361x^2 - 4452x + 1476$
$\left(\frac{51}{5}, -\frac{54}{5}\right)$	$9248x^6 - 2312x^5 + 12427x^4 - 29852x^3 - 21811x^2 + 26690x + 21270$
$\left(\frac{57}{10}, \frac{43}{10}\right)$	$2272x^6 + 35064x^5 + 12877x^4 - 24234x^3 - 37079x^2 + 29700x - 3500$
$\left(-\frac{51}{5}, -\frac{54}{5}\right)$	$-6368x^6 - 20760x^5 + 11991x^4 + 29560x^3 - 61443x^2 + 39870x - 12150$
$\left(-\frac{46}{15}, -\frac{49}{15}\right)$	$-575x^6 - 3075x^5 - 12269x^4 - 16401x^3 - 56024x^2 - 21792x - 73242$
$\left(-\frac{61}{10}, \frac{49}{10}\right)$	$-36450x^6 - 10530x^5 + 6327x^4 + 78760x^3 - 29879x^2 - 17700x + 2612$
$\left(\frac{61}{10}, \frac{49}{10}\right)$	$8612x^6 - 4020x^5 - 52381x^4 - 4290x^3 + 91787x^2 + 47220x - 11540$
$\left(\frac{61}{10}, -\frac{49}{10}\right)$	$-17092x^6 + 13812x^5 - 101885x^4 + 63210x^3 - 89229x^2 + 69580x - 15092$
$\left(-\frac{51}{5}, \frac{54}{5}\right)$	$-100572x^6 - 102884x^5 - 147679x^4 - 25432x^3 + 27727x^2 + 35870x - 11890$

Table 18. Some rational points (g, h) of small height on the surface of Theorem 33 and the corresponding genus-2 curves.

Therefore the Picard number of this K3 surface is 20. The discriminant of the sublattice of $\mathrm{NS}(X)$ generated by the trivial lattice and these sections is 180. We checked that this lattice is 2- and 3-saturated, which proves that it is the entire Picard group.

23.3. *Examples.* Table 18 lists some points of small height and their genus-2 curves.

We now describe some curves of genus 1, possessing infinitely many rational points, on the Hilbert modular surface. These were obtained by pulling back rational curves on the quotients by ι_1 and ι_2 obtained as sections of the elliptic fibrations. In each case we give the curve as a double cover of $\mathbb{P}^1$, exhibit a coordinate of a point on $\mathbb{P}^1$ that lifts to a rational point, and give the conductor and Mordell–Weil group.

- $v = \infty$, conductor $2^4\, 3^2\, 11 \cdot 97$, Mordell–Weil group $(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$, equation

$$g^2 = \frac{v^4 + 132v^3 + 11784v^2 + 566280v + 20175732}{36(v+9)^2}, \quad h = \frac{v^2 + 72v + 4560}{6(v+9)}.$$

- $v = 1$, conductor $2 \cdot 3 \cdot 7 \cdot 17 \cdot 19$, Mordell–Weil group $(\mathbb{Z}/2\mathbb{Z})^2 \oplus \mathbb{Z}$, equation

$$g^2 = \frac{4913v^4 + 1990v^2 + 153}{36(v^2 - 1)^2}, \quad h = -\frac{25v^2 + 3}{2(v^2 - 1)}.$$

- $t = -1$, conductor $5^2 \, 11 \cdot 17 \cdot 47$, Mordell–Weil group $\mathbb{Z}^2$, equation

$$g^2 = \frac{14049t^4 - 57248t^3 + 87462t^2 - 59840t + 15657}{(t+1)^2(9t-11)^2}, \quad h = -\frac{2(t-1)(54t-67)}{(t+1)(9t-11)}.$$

- $v = 0$, conductor $238 = 2 \cdot 7 \cdot 17$, Mordell–Weil group $(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$, equation

$$h^2 = \frac{833v^4 + 190v^2 + 1}{4(v^2 - 1)^2}, \quad g = \frac{27v^2 + 5}{2(v^2 - 1)}.$$

We can obtain a few more such curves from these, by applying the involution ι_2 to the first three curves, and the involution ι_1 to the last. If two genus-2 curves are parametrized by points related by such an involution, then the curves' Jacobians are isogenous.

24. Discriminant 61

24.1. *Parametrization.* Start with an elliptic surface with D_7, A_6 and A_2 fibers and a section of height $\frac{61}{84} = 4 - \frac{2}{3} - \frac{6}{7} - \frac{7}{4}$.

The Weierstrass equation for this family can be written as

$$y^2 = x^3 + ax^2 + 2bt(1-1)x + ct^2(t-1)^2,$$

where

$$a = 4h^3(h-g)^3t^3 + (h-g)^2(g^2h^2 - 4gh^2 - 8h^2 - 2g^2h + g^2 + 12g + 12)t^2$$
$$- 2(g+1)(h-g)(g^2h + 4h - g^2 - 6g)t + g^2(g+1)^2,$$

$$b = -4(g+1)(h-g)^2\big((h-g)^2(2gh^2 + 4h^2 + g^2h - g^2 - 6g - 6)t^2$$
$$+ (g+1)(h-g)(g^2h + 2h - 2g^2 - 6g)t - g^2(g+1)^2\big),$$

$$c = 16(g+1)^2(h-g)^4\big((g+2)(h-g)t + g(g+1)\big)^2.$$

We first perform a 2-neighbor step to move to an elliptic fibration with E_7 and A_7 fibers, by locating an E_7 fiber, as follows.
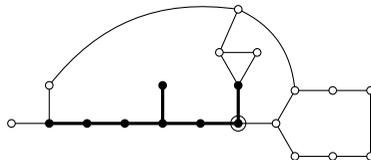
This elliptic fibration has Mordell–Weil rank 2, and we can in fact write down two generators of the Mordell–Weil group, which intersect the components of reducible fibers as shown below. Next, we identify the class of an E_8 fiber, and use it to perform 2-neighbor step to an elliptic fibration with E_8, A_5 and A_1 fibers, and Mordell–Weil rank 2.
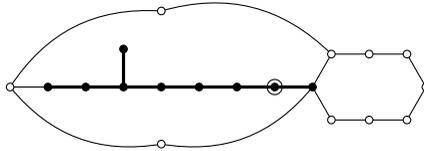
We show one of the generators P of the Mordell–Weil lattice, which has height $4 - 2 \cdot \frac{4}{6} - \frac{1}{2}$. Next, we go to a fibration with E_8 and E_6 fibers using the fiber class F' of E_6 below. Since $P \cdot F' = 1$, the new elliptic fibration has a section.
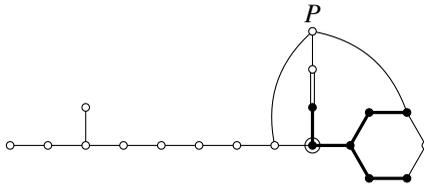
We can find a section P' of this elliptic fibration with E_8 and E_6 fibers, of height $\frac{8}{3} = 4 - \frac{4}{3}$. We use it to go to a fibration with E_8 and E_7 fibers as shown.
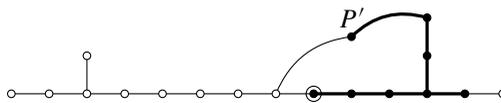
From the resulting $E_8 E_7$ fibration we read out the Igusa–Clebsch invariants and calculate the equation of $Y_-(61)$ as a double cover of $\mathbb{P}^2_{g,h}$.

Theorem 34. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(61)$ as a double cover of $\mathbb{P}^2_{g,h}$ is given by the equation*

$$z^2 = (h-1)^4 g^4 - 2(h-1)h(h^3 - 14h^2 - 20h - 21)g^3$$
$$+ h(h^5 - 46h^4 - 19h^3 + 42h^2 + 39h - 44)g^2$$
$$+ 2h^2(10h^4 + 5h^3 - 13h^2 - h + 12)g - h^2(8h^4 - 13h^2 + 16).$$

It is an honestly elliptic surface, with arithmetic genus 2 and Picard number 28.

24.2. *Analysis.* The branch locus has genus 1. The transformation

$$g = -\frac{8x^3y - 34x^2y + 37xy - 14y + x^6 - 10x^5 + 44x^4 - 89x^3 + 98x^2 - 58x + 14}{x^2(x-1)^3(x-7)},$$

$$h = \frac{12x^2y - 35xy + 26y + 3x^4 + 11x^3 - 61x^2 + 74x - 26}{x^2(9x^2 - 24x + 13)}$$

converts it to the Weierstrass form

$$y^2 + xy = x^3 - 2x + 1,$$

an elliptic curve of conductor 61. It is isomorphic to $X_0(61)/\langle w \rangle$, where w is the Atkin–Lehner involution.

The Hilbert modular surface $Y_-(61)$ is an honestly elliptic surface, since we have an evident genus-1 fibration over $\mathbb{P}^1_h$. Since the coefficient of g^4 is a square, we convert to the Jacobian. In Weierstrass form, we obtain

$$\begin{aligned}
y^2 = x^3 &+ h(h^5 + 14h^4 + 23h^3 - 102h^2 + 88)x^2 \\
&- h^2(110h^6 + 908h^5 - 2854h^4 - 1028h^3 + 4795h^2 + 120h - 2000)x \\
&+ h^4\big(1728h^7 + 16849h^6 - 24666h^5 - 50145h^4 \\
&\qquad\qquad\qquad\qquad + 52138h^3 + 50406h^2 - 29200h - 20000\big).
\end{aligned}$$

This elliptic surface S has $\chi(\mathcal{O}_S) = 3$, with bad fibers of type I_0^* at $h = 0$, I_7 at $h = \infty$, I_2 at $h = 1$ and at the roots of $h^3 + 13h^2 + 24h + 16$ (which generates the cubic field of discriminant -244), and I_3 at $h = -1$ and at the roots of $3h^4 + 23h^3 - 64h^2 + 22h + 25$ (which generates the quartic field of discriminant $-3 \cdot 61^2$, a quadratic extension of $\mathbb{Q}(\sqrt{61})$). The trivial lattice therefore has rank 26, leaving room for at most 4 independent sections. We find the two sections

$$P_1 = \big(h^2(-36h + 55),\ 12h^2(3h^4 + 23h^3 - 64h^2 + 22h + 25)\big),$$

$$\begin{aligned}
P_2 = \big(&-(36h^6 + 444h^5 + 472h^4 - 1492h^3 - 799h^2 + 1048h - 400)/61, \\
&4(15h^2 - 2h - 5)(h^3 + 13h^2 + 24h + 16)(3h^4 + 23h^3 - 64h^2 + 22h + 25)/61^{3/2}\big)
\end{aligned}$$

of heights $\frac{13}{21}$ and $\frac{11}{6}$ respectively, and orthogonal to each other under the height pairing. By Oda's calculations, the Picard number is 28, and therefore the Mordell–Weil rank is exactly 2. The sublattice of the Picard group generated by the above sections and the trivial lattice has discriminant $41184 = 2^5 \cdot 3^2 \cdot 11 \cdot 13$. We checked that it is 2- and 3-saturated, and therefore it must be the entire Néron–Severi lattice. Therefore the Mordell–Weil group is generated by P_1 and P_2.

24.3. *Examples.* Table 19 lists some points of small height and their genus-2 curves.

(g, h)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{3}{2}, \frac{1}{2}\right)$	$-32x^6 - 144x^5 - 229x^4 - 24x^3 - 157x^2 - 84x + 92$
$\left(\frac{47}{72}, -\frac{1}{8}\right)$	$-10x^6 - 42x^5 + 39x^4 + 728x^3 + 489x^2 + 1260x - 1068$
$\left(-\frac{47}{12}, -\frac{11}{4}\right)$	$756x^6 - 756x^5 + 1953x^4 - 282x^3 - 327x^2 + 1404x - 1444$
$\left(-\frac{86}{9}, -8\right)$	$-348x^6 - 972x^5 - 2661x^4 - 2326x^3 - 2205x^2 + 1260x - 140$
$\left(-\frac{17}{18}, -\frac{1}{2}\right)$	$467x^6 + 1551x^5 + 3906x^4 + 3027x^3 - 495x^2 - 1800x + 400$
$\left(\frac{15}{4}, 5\right)$	$-325x^6 + 1410x^5 + 1045x^4 - 3993x^3 - 2636x^2 + 3456x + 2143$
$\left(-\frac{69}{58}, -\frac{20}{29}\right)$	$-128x^6 + 96x^5 - 2519x^4 + 4362x^3 + 1321x^2 + 456x + 32$
$\left(\frac{31}{45}, \frac{14}{5}\right)$	$-204x^6 - 108x^5 + 2553x^4 - 946x^3 - 4683x^2 - 360x + 1200$
$\left(1, \frac{4}{3}\right)$	$-188x^6 - 1812x^5 - 4707x^4 - 130x^3 + 6189x^2 - 1620x + 108$
$\left(\frac{13}{18}, -\frac{1}{2}\right)$	$-612x^6 + 3708x^5 - 6501x^4 + 5656x^3 + 693x^2 - 318x - 1126$
$\left(-\frac{29}{6}, -\frac{29}{8}\right)$	$-148x^6 + 2472x^5 - 1481x^4 + 5001x^3 - 6980x^2 + 1425x - 6625$
$\left(-\frac{37}{18}, -\frac{1}{2}\right)$	$-16x^6 - 264x^5 + 477x^4 - 2268x^3 + 4029x^2 - 6156x + 10372$
$\left(\frac{23}{84}, -\frac{1}{3}\right)$	$-4671x^6 - 6660x^5 - 10362x^4 + 11195x^3 + 1287x^2 + 1590x + 7621$
$\left(\frac{5}{6}, \frac{5}{3}\right)$	$821x^6 - 1896x^5 - 4922x^4 + 8588x^3 + 11341x^2 - 7674x - 8247$
$\left(-\frac{23}{2}, -10\right)$	$2716x^6 + 84x^5 + 7107x^4 + 10642x^3 + 4803x^2 + 13764x + 10204$
$\left(\frac{65}{18}, \frac{5}{2}\right)$	$-3756x^6 - 12012x^5 + 9297x^4 + 18116x^3 - 10335x^2 - 7560x + 3600$

Table 19. Some rational points (g, h) of small height on the surface of Theorem 34 and the corresponding genus-2 curves.

Next, we list some rational curves on the surface. The specialization $h = -1$ gives a rational curve, but the curves of genus 2 corresponding to the points on this curve have Jacobians with endomorphism ring larger than just $\mathcal{O}_{61}$ (they are isogenous to the symmetric squares of elliptic curves). The section P_1 gives the rational curve $g = (3h^2 - 7h + 5)/(3(h - 1))$, for which the Brauer obstruction vanishes identically, yielding a 1-parameter family of genus-2 curves whose Jacobian have real multiplication by $\mathcal{O}_{61}$.

25. Discriminant 65

25.1. *Parametrization.* We start with an elliptic surface with E_7, A_4 and A_4 fibers at $t = \infty, 0, 1$ respectively, and a section of height $\frac{65}{50} = \frac{13}{10} = 4 - \frac{3}{2} - \frac{2 \cdot 3}{5}$. The Weierstrass equation of such a family is

$$y^2 = x^3 + \left(a_0(1 - t) + a_1 t(1 - t) + t^2\right)x^2$$
$$+ 2t^2(t - 1)e\left(b_0(1 - t) + b_1 t(1 - t) + t^2\right)x + e^2 t^4(t - 1)^2(c_0(1 - t) + t),$$

with

$$a_0 = (s^2 - 5)^2(2rs^2 + s + 2r)^2/4,$$

$$a_1 = 4(5s^6 - 8s^4 - 7s^2 - 6)r^2 + 4s(5s^4 - 13s^2 - 4)r + 5s^4 - 18s^2 + 5,$$
$$b_0 = (s^2 - 5)(2rs^2 + s + 2r)(2rs^4 + s^3 + 4rs^2 + s - 6r)/4,$$
$$b_1 = 8(s^2 - 1)(s^2 + 1)^2 r^2 + 8s(s^4 - 1)r + 2s^4 - 2s^2 + 1,$$
$$c_0 = (2rs^4 + s^3 + 4rs^2 + s - 6r)^2/4,$$
$$e = -(s - 1)(s + 1)(2rs^2 - 4rs + s + 2r - 2)(2rs^2 + 4rs + s + 2r + 2).$$

To describe an elliptic fibration with E_8 and E_7 fibers, we identify the class of an E_8 fiber and move by a 2-neighbor step to an $E_8 A_4 A_2$ fibration.
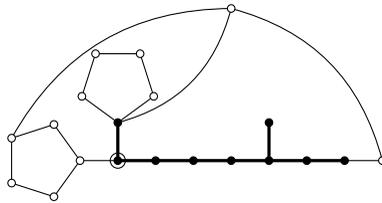
The new elliptic fibration has Mordell–Weil rank 2, and we compute two generators P and Q, each of height $\frac{32}{15} = 4 - \frac{2}{3} - \frac{6}{5}$, with intersection pairing $\frac{7}{15}$. We draw P in the figure below, as well as the class of an A_7 fiber F'. Because $Q \cdot F' = 1$, the new fibration has a section.
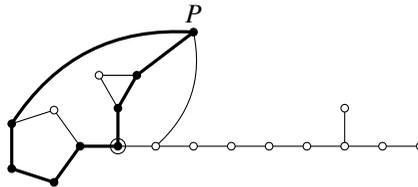
The new elliptic fibration has A_7 and E_8 fibers, and a section P' of height $\frac{65}{8} = 4 + 2 \cdot 3 - 3 \cdot \frac{5}{8}$. We now go to $E_8 E_7$ via a 2-neighbor step. Note that the section P' intersects the new E_7 fiber F'' in 7, while the remaining component of the A_7 fiber intersects F'' in 2. Since these are coprime, the genus-1 fibration defined by F'' has a section.
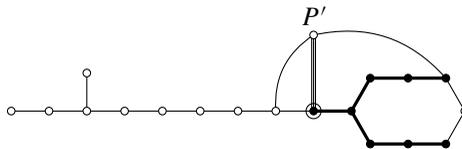
We now read out the Igusa–Clebsch invariants, and compute the equation of $Y_-(65)$ as a double cover of $\mathbb{P}^2_{r,s}$. It is branched over the locus where the K3 surfaces acquire an extra I_2 fiber.

Theorem 35. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(65)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$z^2 = -16(s^4 + 2s^2 + 13)^2(4s^6 + 3s^4 - 10s^2 - 13)r^4$$
$$- 32s(4s^{12} + 15s^{10} + 127s^8 - 10s^6 - 494s^4 - 1253s^2 - 949)r^3$$
$$- 8(12s^{12} + 33s^{10} + 408s^8 - 898s^6 - 2672s^4 - 2023s^2 + 404)r^2$$
$$- 8s(4s^{10} + 7s^8 + 149s^6 - 627s^4 - 641s^2 + 148)r$$
$$- (4s^{10} + 3s^8 + 166s^6 - 997s^4 + 328s^2 - 80).$$

It is an honestly elliptic surface, with arithmetic genus 2 *and Picard number* 28.

25.2. Analysis. This is an honestly elliptic surface, with the extra involution ι : $(r, s, z) \mapsto (-r, -s, z)$ corresponding to the factorization $65 = 5 \cdot 13$.

The branch locus is a curve of genus 1, isomorphic to the elliptic curve

$$y^2 + xy = x^3 - x$$

of conductor 65. For lack of space we do not write down the explicit isomorphism here, relegating the relevant formulae to the online supplement. This elliptic curve is isomorphic to the quotient of $X_0(65)$ by the Atkin–Lehner involution w_{65}.

We were unable to find a section of this genus-1 fibration. However, for purposes of analyzing the Picard number, we study the Jacobian of this elliptic curve over $\mathbb{Q}(s)$, given by

$$y^2 = x^3 + (8s^6 + 13s^4 - 106s^2 + 101)x^2$$
$$+ (16s^{12} + 52s^{10} - 564s^8 + 1416s^6 - 1624s^4 + 900s^2 - 196)x.$$

This has reducible fibers of type I_8 at $s = \pm 1$, I_4 at $s = \infty$, I_3 at $s = \pm\frac{\sqrt{13}}{3}$, and I_2 at the four roots of $4s^4 + 29s^2 - 49$ (a dihedral extension containing $\sqrt{65}$). The trivial lattice therefore has rank 27, leaving room for Mordell–Weil rank at most 3. There is the obvious 2-torsion section $(0, 0)$, and we find a non-torsion section of height $\frac{2}{3}$:

$$P = \big((73 + 9\mu)/2(s^2 - 1)^2(s^2 + (29/8 - 5/8\mu)),$$
$$(657 + 81\mu)/2s(s^2 - 1)^2(s^2 - 13/9)(s^2 + (29/8 - 5/8\mu))\big)$$

with $\mu = \sqrt{65}$. Analysis of the quotient by ι, and its twist, shows that the Mordell–Weil rank is exactly 1. Therefore the Picard number of $Y_-(65)$ is 28. The discriminant of the sublattice of the Néron–Severi group generated by the trivial lattice and these two sections is $6144 = 2^{11} \cdot 3$. We checked that it is 2-saturated, and so it equals the entire Néron–Severi lattice. Therefore the sections above generate the Mordell–Weil group.

The quotient by the involution ι is given by the equation

$$w^2 = -16t^2(t^2 + 2t + 13)^2(4t^3 + 3t^2 - 10t - 13)r^4$$
$$- 32t^2(4t^6 + 15t^5 + 127t^4 - 10t^3 - 494t^2 - 1253t - 949)r^3$$
$$- 8t(12t^6 + 33t^5 + 408t^4 - 898t^3 - 2672t^2 - 2023t + 404)r^2$$
$$- 8t(4t^5 + 7t^4 + 149t^3 - 627t^2 - 641t + 148)r$$
$$- (4t^5 + 3t^4 + 166t^3 - 997t^2 + 328t - 80),$$

where $t = s^2$. Once again we study the Jacobian elliptic fibration: it has the equation

$$y^2 = x^3 + (8t^4 + 13t^3 - 106t^2 + 101t)x^2$$
$$+ (16t^8 + 52t^7 - 564t^6 + 1416t^5 - 1624t^4 + 900t^3 - 196t^2)x.$$

This is an elliptic K3 surface with bad fibers of type I_0^* at $t = 0$, I_8 at $t = 1$, I_3 at $t = \frac{13}{9}$, and I_2 at $t = \infty$ and $t = (-29 \pm 5\sqrt{65})/8$. Therefore the trivial lattice has rank 18, and the Mordell–Weil rank can be at most 2. As before we have a 2-torsion

(r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{40}{41}, -\frac{1}{5}\right)$	$-396x^6 + 216x^5 + 281x^4 - 889x^3 + 50x^2 + 939x + 315$
$\left(-\frac{40}{41}, \frac{1}{5}\right)$	$-648x^5 + 3015x^4 - 422x^3 - 4369x^2 + 2216x - 752$
$\left(\frac{35}{136}, \frac{1}{5}\right)$	$-72x^6 + 969x^5 - 3509x^4 + 847x^3 + 9373x^2 + 816x - 3724$
$\left(-\frac{40}{143}, -\frac{2}{5}\right)$	$-240x^6 - 384x^5 + 695x^4 + 2724x^3 + 5543x^2 - 10992x - 2736$
$\left(\frac{2}{15}, -2\right)$	$-16200x^5 + 1125x^4 - 8972x^3 - 30493x^2 + 14186x - 18974$
$\left(\frac{40}{143}, \frac{2}{5}\right)$	$-4368x^6 + 420x^5 + 28144x^4 - 13235x^3 - 35846x^2 + 10080x + 14112$
$\left(-\frac{5}{64}, \frac{7}{5}\right)$	$800x^6 + 6480x^5 + 19405x^4 + 35306x^3 - 39491x^2 - 2688x - 48$
$\left(-\frac{49}{197}, \frac{1}{2}\right)$	$5088x^6 - 48648x^5 + 85307x^4 + 9352x^3 - 59071x^2 - 15690x + 730$
$\left(\frac{49}{197}, -\frac{1}{2}\right)$	$546x^6 + 9798x^5 + 24115x^4 - 25228x^3 - 98531x^2 + 58920x + 38880$
$\left(-\frac{35}{136}, -\frac{1}{5}\right)$	$2744x^6 - 22344x^5 - 45297x^4 - 16942x^3 + 100440x^2 + 89910x - 72900$
$\left(\frac{5}{64}, -\frac{7}{5}\right)$	$-332100x^6 + 344220x^5 - 54545x^4 + 106126x^3 - 68117x^2 + 3528x - 16464$
$\left(-\frac{2}{15}, 2\right)$	$-216000x^6 + 506400x^5 - 283195x^4 - 70483x^3 + 13883x^2 + 3456x + 300$
$\left(-\frac{1}{65}, \frac{1}{2}\right)$	$-366600x^6 - 2197788x^5 - 64538x^4 + 11447529x^3 + 133360x^2$ $- 19021554x + 9447840$
$\left(\frac{1}{65}, -\frac{1}{2}\right)$	$-412287975x^6 - 3236837061x^5 + 5479876697x^4 + 3156545763x^3$ $+ 1177706300x^2 - 7413585000x - 1103500000$

Table 20. Some rational points (r, s) of small height on the surface of Theorem 35 and the corresponding genus-2 curves.

point $P_1 = (0, 0)$. We also find a non-torsion point

$$P_2 = \big((73 - 9\mu)t\,(t - 1)^2(8t + 29 + 5\mu)/16,$$
$$(-73 + 9\mu)t^2(t - 1)^2(9t - 13)(8t + 29 + 5\mu)/16\big)$$

of height $\frac{1}{3}$, with $\mu = \sqrt{65}$ as before. Therefore, the Picard number is at least 19, and point counting modulo 11 and 23 shows that the Picard number must be 19. We verified by checking 2-saturation that the sections P_1 and P_2 and the trivial lattice span the Néron–Severi group, which has discriminant 64.

We next analyze the quotient by $\iota' : (r, s, z) \mapsto (-r, -s, -z)$, which is the quadratic twist of the elliptic K3 surface above by $\sqrt{t}$:

$$y^2 = x^3 + (8t^3 + 13t^2 - 106t + 101)x^2$$
$$+ (16t^6 + 52t^5 - 564t^4 + 1416t^3 - 1624t^2 + 900t - 196)x.$$

This is also an elliptic K3 surface, with reducible fibers of type I_2^* at $t = \infty$, I_8 at $t = 1$, I_3 at $t = \frac{13}{9}$, and I_2 at $t = (-29 \pm 5\sqrt{65})/8$. The trivial lattice has rank 19. Again, point counting modulo 11 and 23 shows that the Picard number is 19, and the Mordell–Weil group therefore has rank 0, with only the 2-torsion section $(0, 0)$.

25.3. *Examples.* Table 20 on the previous page lists some points of small height and their genus-2 curves.

26. Discriminant 69

26.1. *Parametrization.* Start with an elliptic K3 surface with fibers of type E_6, A_8 and A_1 and a section of height $\frac{69}{54} = \frac{23}{18} = 4 - \frac{1}{2} - \frac{20}{9}$. We can write down the Weierstrass equation of this family as

$$y^2 = x^3 + (a_0 + a_1 t + a_2 t^2)x^2 + t^3(b_0 + b_1 t + b_2 t^2)x + t^6(c_0 + c_1 t + c_2 t^2),$$

with

$$a_0 = \tfrac{1}{4}(\sigma_2 - \sigma_1 + 2)^2, \qquad\qquad a_1 = \tfrac{1}{2}(\sigma_2^2 - \sigma_2(\sigma_1 - 4) - 2(\sigma_1 - 1)),$$
$$a_2 = \tfrac{1}{4}(\sigma_2^2 + 4\sigma_2 - 8), \qquad\qquad b_0 = \tfrac{1}{2}\sigma_2(\sigma_2 - \sigma_1 + 2)^2,$$
$$b_1 = \tfrac{1}{2}(\sigma_2^3 - \sigma_2^2(\sigma_1 - 4) - \sigma_1^2 - \sigma_2\sigma_1 + 2\sigma_1), \quad b_2 = \tfrac{1}{2}(\sigma_2(\sigma_1 - 6) + 2(\sigma_1 - 1)),$$
$$c_0 = \tfrac{1}{4}\sigma_2^2(\sigma_2 - \sigma_1 + 2)^2, \qquad c_1 = \tfrac{1}{2}\sigma_2(\sigma_1 - 2)(\sigma_2 - \sigma_1), \qquad c_2 = \tfrac{1}{4}(\sigma_1^2 - 4\sigma_2),$$

where

$$\sigma_1 = r + s, \qquad \sigma_2 = rs.$$

First we find the class of another E_6 fiber below and go to that elliptic fibration via a 2-neighbor step.
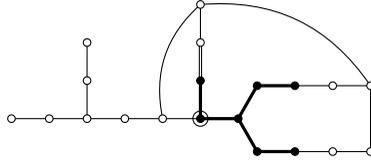
The resulting elliptic fibration has E_6, E_6 and A_3 fibers, and a section of height $\frac{23}{12} = 4 - \frac{4}{3} - \frac{3}{4}$. Now we find the class of an A_7 fiber in the diagram below.
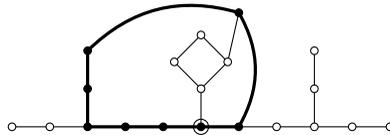
The resulting fibration has A_7, A_5, A_2 and A_1 fibers, a 2-torsion section, and a non-torsion section P of height $\frac{69}{72} = \frac{23}{24} = 4 - \frac{2}{3} - \frac{3 \cdot 3}{6} - \frac{1 \cdot 7}{8}$. We next identify the class F' of an E_7 fiber, and go to it via a 2-neighbor step. Note that $P \cdot F' = 1$, so the new fibration has a section.
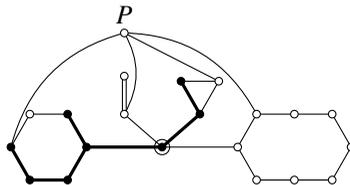
The new elliptic fibration has E_7, A_7 and A_1 fibers, a 2-torsion section Q', and a non-torsion section P' of height $\frac{69}{8} = 4 + 2 \cdot 3 - \frac{1}{2} - \frac{7}{8}$. We identify a fiber F'' of type E_7 below.
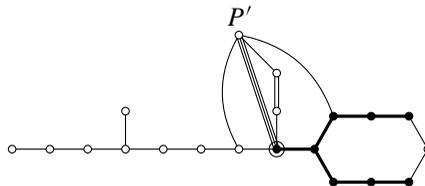
Note that $P' \cdot F'' = 2 \cdot 3 + 3 = 9$, while the remaining component of the A_7 fiber intersects F'' with multiplicity 2. Therefore the new elliptic fibration has a section. Converting to the Jacobian, we read out the Weierstrass coefficients of the $E_8 E_7$ form, which give us the Igusa–Clebsch invariants.

Theorem 36. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(69)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$z^2 = (r-1)^2 r^4 s^6 - 2(r-1)r^2(r^3 + 13r^2 - 37r + 22)s^5$$
$$+ (r^6 + 100r^5 - 439r^4 + 640r^3 - 357r^2 + 72r - 16)s^4$$
$$- 2(59r^5 - 320r^4 + 590r^3 - 436r^2 + 133r - 32)s^3$$
$$+ (44r^5 - 357r^4 + 872r^3 - 830r^2 + 314r - 83)s^2$$
$$+ 2(36r^4 - 133r^3 + 157r^2 - 65r + 19)s - 16r^4 + 64r^3 - 83r^2 + 38r - 11.$$

It is a surface of general type.

26.2. Analysis. This is a surface of general type, with an extra involution $(r, s, z) \mapsto (s, r, z)$, corresponding to $69 = 3 \cdot 23$.

The branch locus is a curve of genus 2; the transformation

$$r = -\frac{x^3 y + x^2 y - y - 3x^6 - 4x^5 + x^4 + 6x^3 - 2x^2 - 2x + 1}{2x^2(x^2 + x - 1)^2},$$

$$s = \frac{x^3 y + x^2 y - y + 3x^6 + 4x^5 - x^4 - 6x^3 + 2x^2 + 2x - 1}{2x^2(x^2 + x - 1)^2}$$

converts it into Weierstrass form

$$y^2 = (x^3 + x^2 - 1)(5x^3 - 7x^2 + 4x - 1).$$

It is isomorphic to $X_0(69)/\langle w \rangle$, where w is the Atkin–Lehner involution.

The quotient surface is (with $m = r + s, n = rs$)

$$z^2 = -16m^4 + 4(11n^2 + 18n + 16)m^3 + (n^4 - 118n^3 - 357n^2 - 202n - 83)m^2$$
$$- 2(n^5 - 50n^4 - 254n^3 - 328n^2 - 61n - 19)m$$
$$+ n^6 - 26n^5 - 203n^4 - 466n^3 - 330n^2 + 36n - 11.$$

The substitution $m = n + k$ makes the right-hand side quartic in n, with highest coefficient a square. Converting to the Jacobian, we get (after some Weierstrass transformations and change of the parameter on the base) the elliptic K3 surface

$$y^2 = x^3 - (88t^3 + 15t^2 + 6t - 1)x^2 + 8t^3(250t^3 + 57t^2 + 45t - 8)x$$
$$+ 16t^5(1125t^3 + 552t^2 + 208t - 36).$$

This has fibers of type I_5 at $t = 0$, I_0^* at $t = \infty$, I_2 at $t = (-3 \pm 2\sqrt{3})/4$, and I_3 at the roots of $25t^3 + 17t^2 + 2t - 1$ (which generates the cubic field of discriminant -23). The trivial lattice has rank 18, leaving room for at most two independent sections. We find the non-torsion section

$$P_1 = \left(4t(1-t),\, 4t(25t^3 + 17t^2 + 2t - 1)\right)$$

of height $\frac{1}{5}$. Counting points modulo 7 and 13 then shows that the Picard number must be exactly 19. Therefore, the discriminant of the sublattice spanned by P_1 and the trivial lattice is $432 = 2^4 \cdot 3^3$. Looking at the contributions to the Néron–Tate height from the fiber configuration, one easily sees that there cannot be any 2- or 3-torsion. Similarly, it is impossible to have a section of height $\frac{1}{20}$ or $\frac{1}{45}$. Therefore, this sublattice must be the entire Néron–Severi lattice, and P_1 is a generator of the Mordell–Weil group.

26.3. *Examples.* Table 21 lists some points of small height and their genus-2 curves.

(r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{5}{6}, \frac{5}{2}\right)$	$-144x^6 - 336x^5 + 491x^4 - 274x^3 + 4919x^2 - 23076x - 6476$
$\left(\frac{5}{7}, \frac{10}{3}\right)$	$-108456x^6 + 89940x^5 + 3518x^4 + 11915x^3 + 29021x^2 - 40515x + 2841$
$\left(\frac{10}{3}, \frac{5}{7}\right)$	$-7146x^6 + 26076x^5 + 26698x^4 - 128487x^3 - 87881x^2 + 140967x + 106899$
$\left(\frac{9}{14}, 3\right)$	$-205648x^6 - 71112x^5 + 4931x^4 - 3219x^3 - 1369x^2 + 336x + 64$
$\left(3, \frac{9}{14}\right)$	$-47792x^6 + 212184x^5 - 134731x^4 - 131082x^3 + 58025x^2 + 39900x + 3500$
$\left(\frac{5}{2}, \frac{5}{6}\right)$	$111132x^6 + 308700x^5 + 150199x^4 - 166350x^3 - 85877x^2 + 37080x + 3208$
$\left(\frac{5}{4}, -\frac{25}{3}\right)$	$-203124x^6 + 537156x^5 - 1147529x^4 - 958036x^3$ $- 185681x^2 + 583356x - 97236$
$\left(\frac{31}{18}, 3\right)$	$-4138876x^6 - 12791196x^5 - 14043627x^4 - 2580588x^3$ $- 2332545x^2 - 7239150x - 962750$
$\left(\frac{97}{34}, 3\right)$	$-4774900x^6 + 11612125x^5 - 1487685x^4 - 13117009x^3$ $+ 29039993x^2 + 24527448x - 2106844$
$\left(3, -\frac{13}{9}\right)$	$-1749188x^6 + 9004404x^5 - 5544841x^4 - 13022828x^3$ $- 36459313x^2 + 31091676x + 62509084$
$\left(3, \frac{31}{18}\right)$	$1490720x^6 + 34810248x^5 + 203477725x^4 - 39952362x^3$ $- 392594159x^2 + 53751372x - 121943204$
$\left(\frac{31}{5}, \frac{5}{6}\right)$	$354444x^6 + 2968308x^5 - 37732823x^4 + 4713146x^3$ $+ 223323505x^2 + 714572220x - 955946700$
$\left(-\frac{25}{3}, \frac{5}{4}\right)$	$-28660432x^6 - 9277032x^5 - 367100597x^4 + 64181262x^3$ $- 1142233133x^2 + 827398968x - 146839168$
$\left(\frac{9}{8}, \frac{29}{13}\right)$	$953161100x^6 - 1709768900x^5 + 57159815x^4 - 997590336x^3$ $- 322970431x^2 - 72177962x + 2546122$
$\left(\frac{5}{6}, \frac{31}{5}\right)$	$6865596x^6 - 82041816x^5 + 58608103x^4 + 1250964773x^3$ $+ 921256891x^2 - 4495870224x - 3609058132$
$\left(-\frac{53}{7}, \frac{71}{57}\right)$	$-125838448x^6 + 33513120x^5 - 1068122125x^4 + 1220630640x^3$ $- 2407159591x^2 + 4627695870x - 2358802782$

Table 21. Some rational points (r, s) of small height on the surface of Theorem 36 and the corresponding genus-2 curves.

By pulling back some of the low-height sections of the quotient, we produce curves of low genus on $Y_-(69)$. Since $r + s = m$ and $rs = n$, the appropriate condition is that $m^2 - 4n = (r - s)^2$ be a square (then we can take a square root and solve for r, s). In other words, $(n + k)^2 - 4n$ must be a square. The section P_1 is given by $n = -(5k^2 - 17k + 15)/(k - 2)$; it gives rise to a genus-1 curve

$$y^2 = 16k^4 - 100k^3 + 237k^2 - 254k + 105,$$

which has rational points (for instance, at infinity). It is therefore an elliptic curve, and we calculate that it has conductor 1711 (prime) and Mordell–Weil group $\mathbb{Z}^2$.

Similarly, the section $-P_1$ is defined by $n = -(25k^2 - 92k + 89)/(10k - 21)$. It also gives rise to a genus-1 curve

$$y^2 = 225k^4 - 1130k^3 + 1931k^2 - 1350k + 445$$

with rational points (as at $k = \infty$). It is an elliptic curve of conductor $50435 = 5 \cdot 7 \cdot 11 \cdot 131$, with trivial torsion and rank 1.

27. Discriminant 73

27.1. *Parametrization.* Start with a K3 elliptic surface with fibers of type A_6, A_9 and a section of height $\frac{73}{70} = 4 - \frac{21}{10} - \frac{6}{7}$. The Weierstrass equation is

$$y^2 = x^3 + (a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4)x^2$$
$$+ 2\mu t(b_0 + b_1t + b_2t^2 + b_3t^3)x + \mu^2t^2(c_0 + c_1t + c_2t^2),$$

with

$$\mu = -2s^4(s + 2r)^2, \qquad a_0 = s^2(rs - 2r - 2)^2, \qquad a_4 = 16r^2(r + 1)^2,$$
$$b_0 = s(rs - 2r - 2)^2, \qquad b_3 = 4r(r + 1)^2(s + 2r + 2), \quad c_0 = (rs - 2r - 2)^2,$$
$$c_2 = (r + 1)^2(s + 2r + 2)^2,$$
$$a_3 = 8r(r + 1)\big((r + 1)s^2 + 2(4r + 1)s + 4r(r - 1)\big),$$
$$c_1 = -2(s + 2r + 2)\big(s^2 + (r + 2)(r - 1)s - 2(r^2 - 1)\big),$$
$$b_1 = -3s^4 - 2r(r + 5)s^3 - 2(r^3 + 4r^2 - 2r - 2)s^2 - 8(r + 1)s + 8r(r + 1)^2,$$
$$a_1 = -2s\big(3s^4 + r(r + 11)s^3 + 2(5r^2 - r - 1)s^2 + 4(r + 1)(r^2 + 1)s - 8r(r + 1)^2\big),$$
$$b_2 = (r + 1)^2s^3 + 2(r^3 + 7r^2 + 6r + 2)s^2$$
$$+ 4(4r^3 + 8r^2 + 7r + 1)s + 8(r - 1)r(r + 1)(r + 2),$$
$$a_2 = (r + 1)^2s^4 + 4(4r^2 + 3r + 1)s^3 + 4(4r^3 + 10r^2 + 10r + 1)s^2$$
$$+ 16r(r + 1)(r - 2)s + 16r^2(r + 1)^2.$$

We first identify the class of an E_8 fiber, and perform a 3-neighbor step to move to an elliptic fibration with E_8 and A_7 fibers.
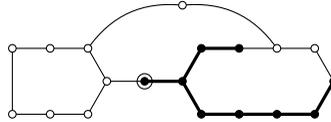
This fibration has a section of height $\frac{73}{8} = 4 + 2 \cdot 3 - \frac{7}{8}$. Next, we locate an E_7 fiber and compute a 2-neighbor step to go to a fibration with E_8 and E_7 fibers.
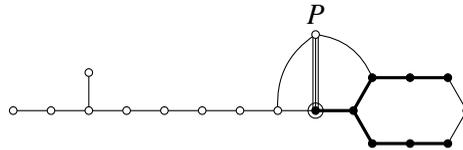
The intersection number of the new fiber F' with the remaining component of the A_7 fiber is 2 and with the section P is 9. Therefore, the new genus-1 fibration defined by F' has a section since 9 and 2 are coprime.

Theorem 37. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(73)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$z^2 = 16(s-2)^2 r^4 + 8(s-2)(17s^3 - 52s^2 + 36s - 8)r^3$$
$$+ (s^6 + 56s^5 - 384s^4 + 448s^3 + 432s^2 - 512s + 64)r^2$$
$$+ 2s(s+2)(s^4 - 34s^3 + 108s^2 - 64s + 16)r + s^2(s+2)^4.$$

It is an honestly elliptic surface, with arithmetic genus 2 and Picard number 28.

27.2. Analysis. The branch locus is a curve of genus 2; the transformation of coordinates

$$r = \frac{3x^3y - 3xy + y - x^6 - 2x^5 - 4x^4 - 3x^2 + 4x - 1}{2x^2(x^2 + 2x - 1)^2},$$

$$s = -2\frac{(2xy - y - x^3 - 7x^2 + 3x)}{(x+1)^2(x^2 + 2x - 1)}$$

converts it to Weierstrass form

$$y^2 = x^6 + 4x^5 + 2x^4 - 6x^3 + x^2 - 2x + 1.$$

This is a genus-2 curve, isomorphic to the quotient of $X_0(73)$ by the Atkin–Lehner involution.

The Hilbert modular surface $Y_-(73)$ is an elliptic surface. Since the coefficient of r^4 is a square, this genus-1 curve over $\mathbb{P}^1_s$ has a section. Computing the Jacobian,

we get the following Weierstrass equation after a change of parameter on the base and some simple Weierstrass transformations:

$$y^2 = x^3 - (83t^6 - 316t^5 + 390t^4 - 158t^3 - 21t^2 + 22t - 1)x^2$$
$$+ 8(t-1)^4 t^3 (287t^5 - 1040t^4 + 960t^3 - 73t^2 - 217t + 67)x$$
$$- 16(t-1)^7 t^6 (1323t^5 - 5887t^4 + 7110t^3 - 1426t^2 - 2201t + 1033).$$

This is an honestly elliptic surface with $\chi = 3$. It has reducible fibers of type I_7 at $t = 1$, I_6 at $t = 0$, I_5 at $t = \infty$, I_3 at $t = (-5 \pm \sqrt{73})/4$, and I_2 at $t = -1$ and $(13 \pm \sqrt{73})/24$. The trivial lattice therefore has rank 24, leaving room for Mordell–Weil rank at most 6.

We find the following four independent sections.

$$P_1 = \left(4t^3(t-1)^2(9t-7),\ 4t^3(t-1)^2(t+1)(3t-2)(2t^2+5t-6)\right),$$
$$P_2 = \left(4t^3(t-1)(7t^2-17t+8),\ 4t^3(t-1)(t+1)(9t^2-13t+5)\right),$$
$$P_3 = \left(4t^2(t-1)^3(19t-16),\ 4t^2(t-1)^3(7t^2-12t+8)(12t^2-13t+2)\right),$$
$$P_4 = \left(4t^3(7t^3-24t^2-2\mu t+42t-2\mu+9),\right.$$
$$\left.(-13+\mu)t^3(t+1)(-24t+13+\mu)(-17-16t+3\mu)(-4t-5+\mu)/192\right)$$

(where $\mu = \sqrt{73}$), with nondegenerate height pairing matrix

$$\begin{pmatrix} \frac{26}{21} & \frac{5}{7} & -\frac{1}{7} & -\frac{2}{3} \\ \frac{5}{7} & \frac{68}{35} & \frac{11}{7} & -\frac{1}{5} \\ -\frac{1}{7} & \frac{11}{7} & \frac{41}{21} & \frac{1}{2} \\ -\frac{2}{3} & -\frac{1}{5} & \frac{1}{2} & \frac{49}{30} \end{pmatrix}.$$

Therefore, the Mordell–Weil rank is at least 4. From Oda's calculations [1982, p. 109], the Picard number is 28, so the Mordell–Weil rank is $28 - 24 = 4$ and our sections generate a subgroup of finite index in the full Mordell–Weil group. The sublattice of the Néron–Severi lattice generated by the trivial lattice and these sections has discriminant $3916 = 2^2 \cdot 11 \cdot 89$. We checked that this sublattice is 2-saturated, and therefore it is the entire Néron–Severi lattice.

27.3. *Examples.* Table 22 lists some points of small height and their genus-2 curves.

We get many curves of genus 0 on the surface by taking sections of the elliptic fibration. For instance, the Brauer obstruction vanishes for the two curves defined by $r = -(s-4)(s+2)/(4(s-2))$ and $r = s(s+2)/((s-2)(3s-2))$, yielding families of genus-2 curves parametrized by s, whose Jacobians have real multiplication by $\mathcal{O}_{73}$.

(r,s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{10}{3}, \frac{4}{3}\right)$	$-4x^6 - 12x^5 - 23x^4 + 4x^3 + 31x^2 + 57x - 18$
$(-3, 1)$	$4x^6 - 3x^4 - 35x^3 + 12x + 76$
$(5, 3)$	$-4x^6 - 24x^5 + 7x^4 + 83x^3 + 25x^2 - 75x - 40$
$\left(\frac{9}{4}, 3\right)$	$-15x^5 + 73x^4 - 41x^3 - 158x^2 - 12x + 36$
$\left(-5, \frac{3}{2}\right)$	$-50x^6 + 45x^5 - 2x^4 - 159x^3 + 70x^2 + 12x - 120$
$\left(\frac{5}{6}, -2\right)$	$-48x^6 + 168x^5 - 149x^4 + 56x^3 - 53x^2 + 12x - 4$
$\left(-\frac{5}{12}, -1\right)$	$195x^6 + 82x^5 - 75x^4 - 186x^3 - 233x^2 - 96x - 87$
$\left(\frac{5}{3}, \frac{1}{2}\right)$	$-60x^6 - 105x^5 + 55x^4 + 49x^3 + 13x^2 + 252x + 116$
$\left(-\frac{2}{3}, -\frac{2}{5}\right)$	$-36x^6 + 39x^5 - 217x^4 + 129x^3 - 271x^2 - 108x + 84$
$\left(\frac{5}{4}, -6\right)$	$20x^6 + 84x^5 - 15x^4 - 162x^3 + 225x^2 + 324x - 180$
$\left(\frac{9}{5}, -8\right)$	$-204x^6 + 348x^5 + 27x^4 + 34x^3 + 3x^2 + 108x - 36$
$\left(-\frac{1}{3}, \frac{4}{5}\right)$	$-25x^6 + 135x^5 + 139x^4 - 383x^3 + 82x^2 + 252x - 162$
$\left(\frac{9}{14}, -2\right)$	$-48x^6 - 72x^5 - 219x^4 - 319x^3 - 159x^2 - 432x + 192$
$\left(-\frac{14}{5}, 12\right)$	$-440x^6 + 90x^5 + 324x^4 + 38x^3 - 9x^2 - 36x - 8$
$\left(-\frac{9}{4}, 1\right)$	$-455x^6 - 420x^5 + 66x^4 - 167x^3 - 15x^2 - 3x - 6$
$\left(\frac{6}{5}, 4\right)$	$-160x^6 + 450x^5 - 114x^4 - 474x^3 + 171x^2 + 162x - 27$

Table 22. Some rational points (r, s) of small height on the surface of Theorem 37 and the corresponding genus-2 curves.

28. Discriminant 76

28.1. *Parametrization.* Start with a K3 elliptic surface with fibers of type A_6, A_2 and D_7, and a section of height $\frac{76}{84} = \frac{19}{21} = 4 - 1 - \frac{2}{3} - \frac{10}{7}$. The Weierstrass equation of this family is

$$y^2 = x^3 + (a_0 + a_1t + a_2t^2 + a_3t^3)x^2 + t^2(b_0 + b_1t + b_2t^2)x + t^4(c_0 + c_1t + c_2t^2),$$

with

$c_0 = (r^2 - 1)^2 s^4 (2s - r + 1)^2 (2s + r + 1)^2 / 16,$

$b_0 = (r^2 - 1)s^2(2s - r + 1)(2s + r + 1)/2,$

$a_3 = (r^2 - 1)(s + 1)^6,$

$a_1 = r^2s^2 - 5s^2 + r^2s - 7s + r^2 - 3,$

$c_2 = (r^2 - 1)^2 r^2 s^6 (s + 1)^6,$

$a_2 = (s + 1)^2(r^2s^2 + 3s^2 - 3r^2s + 7s - 2r^2 + 3),$

$a_0 = 1,$

$$b_2 = (r^2 - 1)s^2(s+1)^4(2r^2s^2 + 6s^2 - 2r^2s + 6s + r^4 - 2r^2 + 1)/2,$$

$$c_1 = (r^2 - 1)^2 s^4 (s+1)^2 (2s - r + 1)(2s + r + 1)$$
$$\times (4r^2s^2 - 12s^2 - 8s - r^4 + 2r^2 - 1)/16,$$

$$b_1 = -(r^2 - 1)s^2\big((2s^2 + 3s + 2)r^4 - 2(4s^4 + 8s^3 + 10s^2 + 8s + 3)r^2$$
$$+ (2s+1)(16s^3 + 32s^2 + 21s + 4)\big)/4.$$

We identify the class of an E_7 fiber:
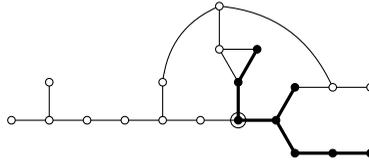
The resulting 3-neighbor step gives us an elliptic fibration with D_8 and E_7 fibers, and also a section P of height $\frac{19}{2} = 4 + 2 \cdot 4 - \frac{3}{2} - 1$. Next we take a 2-neighbor step to go from D_8 to E_8, keeping the E_7 fiber intact.
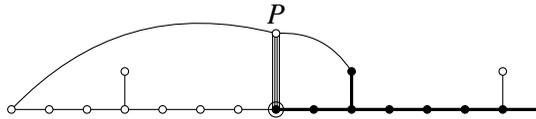
The intersection number of F' with the remaining component of the D_8 fiber is 2, whereas $P \cdot F' = 11$. Therefore the new fibration has a section.

Now we may read out the Igusa–Clebsch invariants from the Weierstrass equation of this $E_8 E_7$ fibration, and thence compute the equation of $Y_-(76)$ as a double cover of the r, s-plane, following our general method of Section 4.

Theorem 38. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(76)$ as a double cover of $\mathbb{P}^2$ is given by the equation*

$$z^2 = -(rs - 3s - 2)(rs + 3s + 2)(32s^4 + 80s^3 - 13r^2s^2 + 85s^2 - 4r^2s + 32s + 4r^4).$$

It is a surface of general type.

28.2. *Analysis.* The branch locus has three components; the more complicated one is where the elliptic K3 surface has an extra I_2 fiber, while the two simpler components correspond to an extra I_2 fiber as well as the section becoming divisible by 2, giving a section of height $\frac{19}{84} = 4 - \frac{1}{2} - \frac{2}{3} - \frac{6}{7} - \frac{7}{4}$. The simpler components are easily seen to be curves of genus 0. The last component is a curve of genus 1; the transformation

$$(r, s) = \left(\frac{2y + x + 1}{x^2 + x + 2}, \frac{-2}{x^2 + x + 2} \right)$$

converts it to Weierstrass form

$$y^2 + xy + y = x^3 + x^2 + 1,$$

which is an elliptic curve of conductor 38. It is isomorphic to $X_0(76)/\langle w_4, w_{19} \rangle$.

The Hilbert modular surface $Y_-(76)$ is a surface of general type. The extra involution is $\iota : (r, s) \mapsto (-r, s)$. Next, we analyze the quotient of the surface by ι. This turns out to be an elliptic K3 surface, and after some Weierstrass transformations and linear shift of parameter on the base, its Weierstrass equation may be written as

$$y^2 = x^3 + (13t^4 - 48t^3 - 6t^2 + 8t + 1)x^2 + 64t^4(2t - 1)(t^3 - 5t^2 + 7t + 1)x.$$

It has fibers of type I_8 at $t = 0$, I_5 at $t = 1$, I_3 at $t = -\frac{1}{7}$, and I_2 at $t = \frac{1}{2}$ and at the roots of $t^3 - 5t^2 + 7t + 1$ (which generates the cubic field of discriminant -76). The trivial lattice has rank 19. In addition to the obvious 2-torsion section $P_0 = (0, 0)$, we find a section $P_1 = (16t^3(2t - 1), 16t^3(t - 1)(2t - 1)(7t + 1))$ of height $\frac{19}{120}$. Therefore the K3 surface is singular. These sections and the trivial lattice generate a sublattice of the Néron–Severi lattice of discriminant -76. It must be the entire Néron–Severi lattice, since otherwise, we would have either another 2-torsion section, a 4-torsion section, or a section of height $\frac{19}{480}$, none of which is possible with this configuration of reducible fibers.

The quotient of $Y_-(76)$ by the involution $(r, s, z) \mapsto (-r, s, -z)$ is an honestly elliptic surface with $\chi = 3$. Its Weierstrass equation may be written as follows:

$$\begin{aligned}
y^2 = x^3 &+ (t - 1)(64t^5 - 160t^4 + 53t^3 + 33t^2 - 5t - 1)x^2 \\
&+ 16(t - 1)t^4(2t - 1)(t^3 - 5t^2 + 7t + 1)(32t^3 - 16t^2 + 21t - 5)x.
\end{aligned}$$

It has bad fibers of type I_8 at $t = 0$, I_4 at $t = \infty$ and $t = \frac{1}{3}$, III at $t = 1$, I_3 at $t = -\frac{1}{7}$, and I_2 at $t = \frac{1}{2}$, at the roots of $t^3 - 5t^2 + 7t + 1$ seen above, and at the roots of $32t^3 - 16t^2 + 21t - 5$ (which generates the cubic field of discriminant -152). Hence the trivial lattice has rank 25, leaving room for Mordell–Weil rank at most 5. Counting points on the reduction modulo 11 and 23 shows that the Picard number is at most 29. On the other hand, we find three independent sections in addition to the 2-torsion section $P_0 = (0, 0)$:

$$P_1 = \left(152(t - 1)t^4(2t - 1), \; 8\mu(t - 1)t^4(2t - 1)(3t - 1)(4t - 3)(7t + 1)\right),$$
$$P_2 = \left(4(t^3 - 5t^2 + 7t + 1)t^3, \; 4t^3(2t - 1)(3t - 1)^2(t^3 - 5t^2 + 7t + 1)\right),$$
$$P_3 = \left(-(t - 1)^3(32t^3 - 16t^2 + 21t - 5), \; 2\nu(t - 1)^2(3t - 1)^2(32t^3 - 16t^2 + 21t - 5)\right).$$

Here $\mu = \sqrt{19}$ and $\nu = \sqrt{-1}$. These sections have heights $\frac{23}{12}$, $\frac{9}{8}$ and $\frac{13}{8}$ respectively, and are orthogonal with respect to the height pairing. Therefore, the Mordell–Weil rank is either 3 or 4; we have not been able to determine it exactly.

(r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{4}{11}, -\frac{8}{11}\right)$	$-8x^6 + 48x^5 - 196x^4 + 324x^3 - 340x^2 - 330x - 65$
$(2, -4)$	$375x^6 + 300x^5 + 230x^4 - 224x^3 - 76x^2 - 48x + 72$
$\left(\frac{7}{3}, -\frac{1}{3}\right)$	$-80x^6 - 120x^5 - 109x^4 - 348x^3 - 469x^2 - 120x + 80$
$(-2, -4)$	$-225x^6 + 600x^5 + 650x^4 + 400x^3 + 20x^2 - 8$
$\left(\frac{13}{23}, -\frac{19}{23}\right)$	$-228x^6 + 684x^5 - 1029x^4 - 432x^3 + 525x^2 + 150x + 10$
$\left(-\frac{7}{3}, -\frac{1}{3}\right)$	$100x^6 - 220x^5 + 621x^4 - 528x^3 + 1699x^2 - 234x + 1478$
$\left(\frac{19}{33}, -\frac{9}{11}\right)$	$256x^6 - 1056x^5 - 2335x^4 + 1480x^3 + 1715x^2 - 1386x + 126$
$\left(\frac{19}{11}, \frac{1}{11}\right)$	$-2232x^6 - 2016x^5 + 2581x^4 + 2802x^3 - 983x^2 - 660x - 180$
$\left(\frac{47}{37}, -\frac{43}{37}\right)$	$3100x^6 - 540x^5 - 271x^4 - 1742x^3 + 161x^2 + 84x + 252$
$\left(\frac{43}{27}, \frac{29}{27}\right)$	$-592x^6 + 372x^5 + 1003x^4 + 1328x^3 - 1406x^2 - 132x - 3709$
$\left(-\frac{19}{33}, -\frac{9}{11}\right)$	$-4404x^6 - 540x^5 - 1697x^4 - 980x^3 - 257x^2 - 240x - 64$
$\left(\frac{22}{17}, -\frac{20}{17}\right)$	$600x^6 - 3360x^5 + 3604x^4 + 2256x^3 + 4546x^2 + 1440x + 775$
$\left(-\frac{22}{13}, -\frac{20}{13}\right)$	$-367x^6 - 618x^5 - 1539x^4 + 316x^3 + 1839x^2 + 4662x + 3507$
$\left(-\frac{1}{23}, -\frac{31}{46}\right)$	$-2245x^6 - 137x^5 - 5393x^4 - 1675x^3 - 3618x^2 - 1728x - 675$
$\left(\frac{22}{13}, -\frac{20}{13}\right)$	$1425x^6 - 2610x^5 + 6333x^4 - 4948x^3 + 8271x^2 - 4242x + 5971$
$\left(-\frac{1}{23}, -\frac{17}{23}\right)$	$24x^6 + 552x^5 + 2075x^4 - 1970x^3 - 9925x^2 + 9072x + 1216$

Table 23. Some rational points (r, s) of small height on the surface of Theorem 38 and the corresponding genus-2 curves.

28.3. *Examples.*

Table 23 lists some points of small height and their genus-2 curves.

We now describe some curves on $Y_-(76)$, which are useful in producing rational points.

The specialization $s = -\frac{2}{3}$ gives a genus-1 curve $y^2 = -81r^4 + 63r^2 + 19$. It has rational points, such as $(r, y) = (1, 1)$. It is thus an elliptic curve; we find that it has conductor 760 and Mordell–Weil group $(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$. The specialization $s = -\frac{8}{7}$ gives a rational curve, which we can parametrize as $r = -5(m^2 - 1)/(4(m^2 + 1))$.

The sections P_1, $P_1 + P_0$, $2P_1 + P_0$ and $3P_1 + P_0$ of the K3 quotient give the following genus-1 curves, which all have rational points.

Equation	conductor	Mordell–Weil group
$r^2 = -(8s^3 + 28s^2 + 27s + 8)/s$	$2 \cdot 29$	$\mathbb{Z}$
$r^2 = -(s^3 - s^2 - 11s - 8)/s$	$2^4 \, 11 \cdot 191$	$\mathbb{Z}^2$
$r^2 = -(8s^4 + 4s^3 - 33s^2 - 44s - 16)/(s+2)^2$	$3^5 \, 19$	$\mathbb{Z}^2$
$r^2 = -(s^4 + s^3 - 3s^2 - s + 1)$	$2^2 \, 5 \cdot 29$	$(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$

The section $2P_1$ gives a genus-0 curve $r^2 + s^2 + 6s + 4 = 0$, which we can parametrize as

$$(r, s) = \left(-\frac{m^2 + 4m - 1}{m^2 + 1}, -\frac{m^2 + 2m + 5}{m^2 + 1} \right).$$

The Brauer obstruction always vanishes on this locus, giving us a 1-parameter family of genus-2 curves whose Jacobians have real multiplication by $\mathcal{O}_{76}$.

29. Discriminant 77

29.1. _Parametrization._ We start with a family of K3 surfaces with fibers of type A_1, A_3, A_5 and D_5, a 2-torsion section T, and two orthogonal sections P, Q of height $\frac{11}{12} = 4 - 0 - \frac{2 \cdot 2}{4} - \frac{1 \cdot 5}{6} - (1 + \frac{1}{4})$ and $\frac{7}{4} = 4 - 0 - \frac{1 \cdot 3}{4} - \frac{3 \cdot 3}{6} - 0$. The orthogonality comes from $0 = 2 - 0 - \frac{1 \cdot 2}{4} - \frac{1 \cdot 3}{6} - 0 - 1$, where the last term comes from the intersection number $(P) \cdot (Q)$ on the surface. We can write the Weierstrass equation of this family as

$$y^2 = x^3 + \big((rs - 1)^2 - (s^2 - 1)(rs - 4r^2 - 1)t$$
$$+ r(s^2 - 1)(rs^2 + 8s - 57r)t^2/4 + 8r^2(s^2 - 1)t^3 \big)x^2$$
$$+ r^2(s^2 - 1)^2 t^3 (t - 1)^2 \big(16r^2 t + (rs - 1)^2 - (s - 5r)^2 \big)x.$$

We go to $E_8 E_7$ form in three steps, via $D_8 E_6$ and $E_8 E_6$.

First, we identify a D_8 fiber F' in the figure below (we omit drawing the node representing Q and the edges connecting it to the rest of the diagram, as it would clutter up the picture).

The section P intersects F' once, and so the new fibration has a section. It has D_8 and E_6 fibers and rank 2.

Next, we identify the class of an E_8 fiber below, and go to an elliptic fibration with E_8 and E_6 fibers, by a 2-neighbor step.

The resulting elliptic fibration has Mordell–Weil lattice of rank 2 and discriminant $\frac{77}{3}$. We can relatively easily describe a section P'' of height $\frac{8}{3}$, which intersects a non-identity component of the E_6 fiber.

Now the E_7 fiber F''' drawn below defines an elliptic fibration with a section and with E_8 and E_7 fibers.

We now read out the Igusa–Clebsch invariants and proceed as in Section 4 to compute the equation of $Y_-(77)$ as a double cover of the Humbert surface $\mathcal{H}_{77}$.

Theorem 39. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(77)$ as a double cover of $\mathbb{P}^2$ is given by the equation*

$$z^2 = r^2(r-1)^2(r+1)^2 s^6 + 2r(r-1)(r+1)(21r^2 - 13)s^5$$
$$- (14r^6 - 433r^4 + 328r^2 + 27)s^4 - 4r(281r^4 - 522r^2 - 79)s^3$$
$$- (343r^6 + 6268r^4 + 1763r^2 - 54)s^2 + 2r(3997r^4 + 2446r^2 - 171)s$$
$$- (1372r^6 + 4531r^4 - 362r^2 + 27).$$

It is a surface of general type.

29.2. *Analysis.* It is a surface of general type. It has an extra involution $(r, s) \mapsto (-r, -s)$. The branch locus is a curve of genus 2. The change of coordinates

$$(r, s) = \left(\frac{4xy - 5x^4 - 22x^2 - 21}{x(x^2 + 7)^2}, \frac{-(2x^2 y + 2y - x^5 - 6x^3 - 9x)}{(x^2 - 1)^2} \right)$$

(r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{1}{2}, -\frac{23}{5}\right)$	$1581x^6 - 25965x^5 + 128199x^4 - 124655x^3 - 282240x^2 - 92400x - 9478$
$\left(-\frac{3}{5}, -\frac{47}{9}\right)$	$926711x^6 - 351913x^5 - 2531791x^4 + 699677x^3 + 2176646x^2$ $- 359536x - 579882$
$\left(\frac{1}{2}, \frac{23}{5}\right)$	$2038350x^6 + 1601640x^5 - 6288456x^4 + 116705x^3 - 5729115x^2$ $- 8845053x - 2931103$
$\left(-\frac{13}{2}, 6\right)$	$10433826x^6 - 16243110x^5 + 25749477x^4 - 25899800x^3 + 5297523x^2$ $- 6454140x - 9577876$
$\left(\frac{13}{2}, -6\right)$	$-749865564x^6 + 4317895148x^5 + 1178682897x^4 - 6739621816x^3$ $- 800208729x^2 + 2973824982x + 128797182$
$\left(-\frac{33}{65}, -\frac{239}{63}\right)$	$-5738303278944x^6 + 6551435295576x^5$ $+ 28045528925148x^4 - 5100723398753x^3 - 23656013198837x^2$ $+ 3333165270637x + 3904336668117$

Table 24. Some rational points (r, s) of small height on the surface of Theorem 39 and the corresponding genus-2 curves.

converts it to Weierstrass form

$$y^2 = x^6 + 5x^4 + 3x^2 + 7.$$

It is isomorphic to the quotient of $X_0(77)$ by the Atkin–Lehner involution w_{77}.

29.3. *Examples*. Table 24 on the previous page lists some points of small height and their genus-2 curves.

30. Discriminant 85

30.1. *Parametrization*. We start with a K3 elliptic surface with fibers of type E_6, D_5 and A_4, with a section of height $\frac{85}{60} = \frac{17}{12} = 4 - \frac{4}{3} - \frac{5}{4}$.
  The Weierstrass equation is

$$y^2 = x^3 + t(a_0 + a_1 t) x^2 + 2t^2(t-1)(b_0 + b_1 t) x + t^3(t-1)^4(c_0 + c_1 t),$$

with

$$a_0 = -4(e^2 - 1)(3ef^2 + 2e^2 f + 2f - 2e^2 + e + 2),$$

$$c_0 = -64e^2(e^2 - 1)^3(f^2 - 1)^2(f + 2e + 1)(ef - e + 2),$$

$$a_1 = 4(e^2 - 1)^2 f^2,$$

$$b_0 = 8e(e^2 - 1)^2(f^2 - 1)(3ef^2 + 4e^2 f + 4f - 4e^2 + 5e + 4),$$

$$b_1 = -8f(e^2 - 1)^2(f^2 - 1)(e^4 f + e^3 f - ef + f - e^4 + e^3 + e + 1),$$

$$c_1 = 16(e^2 - 1)^2(f^2 - 1)^2(e^4 f + e^3 f - ef + f - e^4 + e^3 + e + 1)^2.$$

We identify the class of a D_8 fiber below, and move to the new elliptic fibration via a 2-neighbor step.
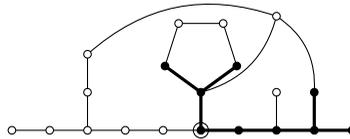
The new elliptic fibration has reducible fibers of type D_8 and E_6, and Mordell–Weil rank 2. Next, we move to an elliptic fibration with E_8 and E_6 fibers by another 2-neighbor step, using the E_8 fiber shown below.

The new elliptic fibration has E_8 and E_6 fibers. We can find an explicit section P of the $E_8 E_6$ fibration which intersects a non-identity component of the E_6 fiber and does not intersect the zero section.

Let F be an E_7 fiber enclosed by the box in the picture below. We move to it by a 2-neighbor step, to recover an elliptic fibration with E_8 and E_7 fibers.
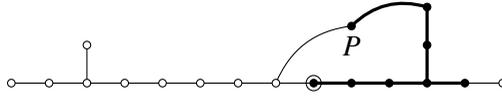
We may now read out the Igusa–Clebsch invariants, and work out the equation of $Y_-(85)$ as a double cover of $\mathbb{P}^2_{e,f}$.

Theorem 40. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(85)$ as a double cover of $\mathbb{P}^2_{e,f}$ is given by the equation*

$$
\begin{aligned}
z^2 = {}& -(e^2 - e - 1)^2(8e^4 + 11e^2 + 8)\,f^4 \\
&+ 4(e^4 - 1)(18e^4 - 11e^3 + 27e^2 + 11e + 18)\,f^3 \\
&- 2(82e^8 - 118e^7 - 9e^6 - 60e^5 - 173e^4 + 60e^3 - 9e^2 + 118e + 82)\,f^2 \\
&+ 4(e^4 - 1)(4e^2 - 5e - 4)(9e^2 - 13e - 9)\,f \\
&- (2e^2 - e - 2)^2(11e^4 - 34e^3 + 5e^2 + 34e + 11).
\end{aligned}
$$

It is an honestly elliptic surface, of arithmetic genus 3 and Picard number 37 or 38.

30.2. *Analysis.* It is an honestly elliptic surface, with extra involution $\iota : (e, f, z) \mapsto (-1/e, -f, z/e^4)$.

The branch locus is a hyperelliptic curve of genus 3. The change of coordinates

$$
(e, f) = \left(\frac{y + x^4 + x^3 - x + 1}{2x^2 + 2x - 1},\ \frac{(3x^4 + x^3 - 3x^2 - x + 3)y - (x^2 - 1)(x^2 + 1)^3}{(x^2 + x - 1)^2(2x^4 - x^2 + 2)} \right)
$$

converts it to Weierstrass form:

$$
y^2 = x^8 + 2x^7 - x^6 - 8x^5 + x^4 + 8x^3 - x^2 - 2x + 1.
$$

It is isomorphic to the quotient of $X_0(85)$ by the Atkin–Lehner involution w_{85}.

The equation of $Y_-(85)$ describes it as an elliptic surface over $\mathbb{P}^1_e$. So far, we are unable to find a section.

The Jacobian of this genus-1 curve over $\mathbb{Q}(e)$ can be written (after some Weierstrass transformations) as

$$
\begin{aligned}
y^2 = {}& x^3 + (e^8 - 10e^7 + 3e^6 + 84e^5 + 85e^4 - 84e^3 + 3e^2 + 10e + 1)x^2 \\
&- 8e^5(23e^6 - 133e^5 - 420e^4 - 64e^3 + 420e^2 - 133e - 23)x \\
&- 16e^9(108e^6 - 637e^5 - 1944e^4 - 26e^3 + 1944e^2 - 637e - 108).
\end{aligned}
$$

It has $\chi = 4$. There are bad fibers of type I_9 at $e = 0$ and $e = \infty$, I_3 at $e = \pm 1$, I_2 at the roots of $e = 4 \pm \sqrt{17}$, and I_3 at the roots of $3e^6 - 16e^5 - 54e^4 + 5e^3 + 54e^2 - 16e - 3$

(which generates the compositum of $\mathbb{Q}(\sqrt{85})$ and the cubic field of discriminant $-3 \cdot 5 \cdot 17$). The trivial lattice has rank 36, leaving room for at most four independent sections. So far, we can only say from the ensuing analysis that the rank is either 1 or 2.

Next, we analyze the quotient of this surface by the involution ι. In terms of $g = e - 1/e$ and $h = f/(e + 1/e)$ (which are invariant under ι), its equation is

$$z^2 = -(g-1)^2(g^2+4)^2(8g^2+27)h^4 + 4g(g^2+4)^2(18g^2-11g+63)h^3$$
$$- 2(g^2+4)(82g^4-118g^3+319g^2-414g-27)h^2$$
$$+ 4g(4g-5)(9g-13)(g^2+4)h - (2g-1)^2(11g^2-34g+27).$$

This is also an honestly elliptic surface, this time with $\chi = 3$. Its Jacobian is

$$y^2 = x^3 + (g^2+4)(g^4-10g^3+7g^2+66g-27)x^2$$
$$+8g(g-8)(g^2+4)^2(g^3-12g^2-12g+27)x+16g^2(g-8)^2(g^2+4)^3(g^2-14g-27).$$

It has bad fibers of type I_9 at $g = \infty$, I_3 at $g = 0$, I_2 at $g = 8$, I_0^* at $g = \pm 2\sqrt{-1}$, and I_3 at the roots of $3g^3 - 16g^2 - 45g - 27$ (which generates the cubic field of discriminant -255). The trivial lattice has rank 27, leaving room for Mordell–Weil rank at most 3. Counting points modulo 11 and 19 shows that the Picard number is at most 29. On the other hand, we are able to find the non-torsion section

$$P_1 = \big(-4(g-8)(g^2+4)(9g^3+3g^2+g+72)/85,$$
$$4(g-8)(21g-4)(g^2+4)^2(3g^3-16g^2-45g-27)/85^{3/2}\big)$$

of height $\frac{3}{2}$. Therefore, the Mordell–Weil rank is either 1 or 2.

Next, we consider the quadratic twist of the quotient elliptic surface, which is obtained by simply removing the factors of (g^2+4) in the Weierstrass equation above (recalling that $g^2+4 = (e+1/e)^2$). We get a K3 surface with a genus-1 fibration, whose Jacobian has Weierstrass equation

$$y^2 = x^3 + (g^4-10g^3+7g^2+66g-27)x^2$$
$$+ 8g(g-8)(g^3-12g^2-12g+27)x + 16g^2(g-8)^2(g^2-14g-27).$$

It has reducible fibers of type I_9 at $g = \infty$, I_2 at $g = 8$, and I_3 at $g = 0$ and at the roots of $3g^3 - 16g^2 - 45g - 27$. Therefore the trivial lattice has rank 19, and the Mordell–Weil rank can be 0 or 1. We find a 3-torsion section

$$P_0 = (8g+36, 4(3g^3-16g^2-45g-27)).$$

Counting points modulo 7 and 19 shows that the Picard number is exactly 19. The 3-torsion section and trivial lattice span a sublattice of discriminant $162 = 2 \cdot 3^4$ of the Néron–Severi lattice of discriminant. It is easy to check that this sublattice is 3-saturated, and therefore must form the entire Néron–Severi lattice.

(e, f)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{1}{2}, -\frac{29}{15}\right)$	$576x^6 + 432x^5 + 927x^4 + 81x^3 + 171x^2 - 72x - 208$
$\left(\frac{4}{3}, \frac{1}{7}\right)$	$-1344x^6 - 672x^5 - 2233x^4 - 3026x^3 - 997x^2 - 2196x - 548$
$\left(\frac{7}{2}, \frac{8}{17}\right)$	$-1566x^6 - 7704x^5 - 4056x^4 - 8581x^3 - 5841x^2 - 2055x - 2395$
$\left(-2, \frac{29}{15}\right)$	$-3500x^6 - 2100x^5 + 11205x^4 + 2422x^3 - 11295x^2 + 1080x + 2160$
$\left(-2, \frac{23}{21}\right)$	$-316x^6 + 3048x^5 + 14649x^4 + 10547x^3 - 13509x^2 - 1296x + 1728$
$\left(-\frac{6}{7}, -13\right)$	$-5028x^6 - 10620x^5 - 2605x^4 - 16750x^3 + 5255x^2 - 6600x + 2832$
$\left(-\frac{1}{2}, -7\right)$	$8964x^6 - 3132x^5 + 18927x^4 + 6286x^3 + 6655x^2 + 11300x - 500$
$\left(\frac{2}{5}, -\frac{7}{3}\right)$	$21006x^6 - 45414x^5 + 16263x^4 - 20048x^3 - 7227x^2 + 960x - 3200$
$\left(-\frac{3}{4}, -\frac{3}{5}\right)$	$5500x^6 + 30300x^5 + 19835x^4 + 20174x^3 - 46885x^2 + 2340x - 380$
$\left(\frac{5}{2}, \frac{7}{3}\right)$	$-12852x^6 - 15876x^5 + 40383x^4 + 49976x^3 - 30231x^2 - 43650x + 2250$
$\left(\frac{11}{20}, -\frac{71}{49}\right)$	$-66020x^6 + 43980x^5 + 10001x^4 + 1154x^3 - 5899x^2 - 1464x + 1096$
$\left(-\frac{2}{5}, -\frac{7}{3}\right)$	$-72620x^6 + 37884x^5 - 12135x^4 + 29302x^3 - 4107x^2 + 1848x - 2672$
$\left(-\frac{5}{2}, \frac{7}{3}\right)$	$20x^6 + 180x^5 - 3879x^4 - 34668x^3 + 44937x^2 + 62856x - 73296$
$\left(\frac{7}{6}, 13\right)$	$-5442x^6 + 3630x^5 - 7079x^4 - 93460x^3 + 35059x^2 - 420x + 9212$
$\left(-4, \frac{11}{9}\right)$	$-16964x^6 - 33804x^5 + 53325x^4 + 100170x^3 - 35163x^2 - 81540x - 1116$
$\left(\frac{11}{2}, \frac{9}{13}\right)$	$-102046x^6 + 130482x^5 + 61857x^4 + 9504x^3 - 74697x^2 - 38412x - 14036$

Table 25. Some rational points (e, f) of small height on the surface of Theorem 40 and the corresponding genus-2 curves.

30.3. *Examples.* Table 25 lists some points of small height and their genus-2 curves.

31. Discriminant 88

31.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type A_9, D_4 and A_2, and a section of height $\frac{11}{15} = 4 - \frac{2}{3} - 1 - \frac{16}{10}$. The Weierstrass equation for this family is

$$y^2 = x^3 + (a_0 + a_1 t + a_2 t^2 + a_3 t^3)x^2$$
$$+ 2t^2(\lambda t - \mu)(b_0 + b_1 t + b_2 t^2 + b_3 t^3)x + t^4(\lambda t - \mu)^2(c_0 + c_1 t)^2,$$

with

$$a_0 = 1, \qquad \mu = rs + 2s + 1, \qquad \lambda = s(r+2)^2(2s+1)^2,$$

$$c_0 = -2, \qquad b_0 = 2, \qquad a_1 = -8s(rs + 2r + 1),$$

$$c_1 = 8(r+2)s^2 + 8(r+1)s + r^2,$$

$$b_3 = 2r(r+2)s(2s+1)(8s+r^2)(8s^2+r),$$

$$b_2 = 32r(r+2)s^4 + 32(3r^2+4r+2)s^3 - 4(r^3 - 20r^2 - 24r - 8)s^2 + 4(r-1)r^2s,$$

$$b_1 = -16(r+1)s^2 - 8(3r+2)s - r^2,$$
$$a_3 = 4rs\big(64rs^4 + 16(r^3+3r^2+12r+4)s^3$$
$$+ 2r(r^2+4)s^2 + r(r^3+12r^2+12r+16)s + r^3\big),$$
$$a_2 = 4s\big(4r^2s^3 + 8r(2r-1)s^2 - 4(r^3-4r^2-4r-1)s - r^2(r+4)\big).$$

First we identify an E_7 fiber, and make a 3-neighbor move to it.
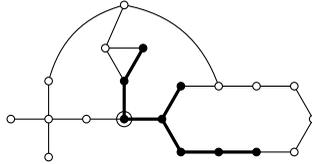
This gives us an elliptic fibration with E_7, D_5 and A_3 fibers, and a section of height $\frac{88}{32} = \frac{11}{4} = 4 - \frac{5}{4}$. Then we can identify a D_8 fiber F' below, and move to the associated genus 1 fibration by a 2-neighbor step.
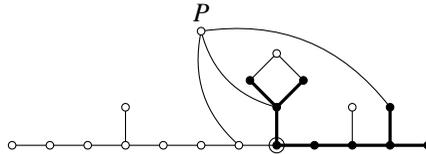
To see that the genus-1 fibration defined by this fiber F' has a section, note that $P \cdot F' = 3$, while F' intersects the near leaf of the D_5 fiber with multiplicity 2. Therefore we may replace the genus-1 fibration by its Jacobian.

Finally, we go by another 2-neighbor move to a fibration with E_8 and E_7 fibers. We identify the class of an E_8 fiber F'' below. The Mordell–Weil group is generated by a section P' of height $88/(2 \cdot 4) = 11 = 4 + 2 \cdot 4 - 1$, so the section must intersect the zero section with multiplicity 4, and it must intersect the near leaf of the D_8 fiber. Therefore $P' \cdot F'' = 2 \cdot 4 + 3 = 11$, whereas the omitted far leaf of the D_8 fiber intersects F'' with multiplicity 2. So the new fibration has a section.
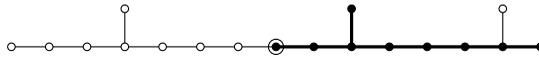
We may now read out the Igusa–Clebsch invariants and compute the equation of the branch locus for $Y_-(88) \mapsto \mathbb{P}^2_{r,s}$.

Theorem 41. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(88)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$z^2 = (8rs^2 + 16s^2 + 8s + r^2)\big(8r^3s^4 + 16r^2s^4 + 96r^3s^3 + 472r^2s^3 + 544rs^3$$
$$- 27r^4s^2 - 120r^3s^2 + 64r^2s^2 + 472rs^2 + 16s^2 - 46r^3s - 120r^2s + 96rs + 8s - 27r^2\big).$$

It is a surface of general type.

31.2. Analysis. The branch locus has two components. Both correspond to elliptic K3 surfaces with an extra I_2 fiber, and the simpler component to having a 2-torsion section in addition. The simpler component of the branch locus has genus 1; the change of coordinates $r = 2y/x^2$, $s = -1/(2x)$ converts it to Weierstrass form

$$y^2 + y = x^3 - x^2,$$

which is an elliptic curve of conductor 11 (isomorphic to $X_1(11)$).

The other component has genus 2. The transformation

$$(r, s) = \left(\frac{-(x-1)y + (x+1)(x^3 - 3x^2 - 3x - 1)}{3x^2 + 2x + 1}, \right.$$

$$\left. \frac{(3x+1)y - (x+1)(3x^3 + 3x^2 + 3x + 1)}{4x(x^2 - 2x - 1)} \right)$$

converts it to Weierstrass form

$$y^2 = x^6 - 2x^5 + 11x^4 + 20x^3 + 15x^2 + 6x + 1.$$

The Hilbert modular surface $Y_-(88)$ is a surface of general type. We now analyze its quotient by the involution $\iota : (r, s, z) \mapsto (1/s, 1/r, z/(rs)^3)$. Writing $h = -(s + 1/r)$, $m = (s - 1/r)^2$, we find the equation

$$z^2 = (h^4 - 2h^3 - 2mh^2 + 2mh + m^2 + 1)(9h^6 - 30h^5 - 26mh^4 + 16h^4 + 58mh^3$$

$$+ 30h^3 + 25m^2h^2 - 8mh^2 - 25h^2 - 28m^2h - 30mh - 8m^3 - 8m^2 - 2m).$$

The invertible transformation $m = 2 + 1/t + 8/(nt) + 4/(nt)^2$, $h = -1 - 2/(nt)$ makes this a quartic in n,

$$z^2 = (4t - n - 4)\big(t(2t+1)(6t^2 - 13t + 8)n^3 + 4t(2t-3)^2n^2 - 4(2t-1)^2n + 16(t-1)\big),$$

with an obvious section $n = 4t - 4$. Converting to the Jacobian, we get an elliptic K3 surface with the following equation (after some Weierstrass transformations and a change of parameter $t \mapsto 1 - t$ on the base):

$$y^2 = x^3 + (28t^4 - 24t^3 - 8t^2 + 4t + 1)x^2$$

$$- 16t^3(t-1)^2(t^3 - 10t^2 + 4t + 1)x + 64t^6(t-1)^4(29t^2 - 10t - 3).$$

This has bad fibers of type I_6 at $t = 0$, I_5 at $t = 1$, I_2 at $t = \frac{1}{2}$ and $t = -\frac{1}{6}$, and I_3 at $t = \frac{1}{4} \pm \frac{\sqrt{33}}{12}$. The trivial lattice has rank 17. We find the independent sections

$$P_1 = \big(-4t(t-1)(7t^2 - 2t - 1),\ 4t(t-1)(t+1)(2t-1)(6t^2 - 3t - 1)\big),$$

$$P_2 = \big(4t(t-1)^2(5t+1),\ 4t(t-1)^2(6t+1)(6t^2 - 3t - 1)\big),$$

$$P_3 = \left(4t^3(6 - 13t),\ 12\sqrt{-3}\,t^4(2t - 1)(6t + 1)\right),$$

with height matrix

$$\begin{pmatrix} \frac{8}{15} & \frac{1}{10} & 0 \\ \frac{1}{10} & \frac{2}{15} & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}.$$

Therefore the K3 surface is singular, and an easy argument shows that these sections and the trivial lattice must span the Néron–Severi lattice, which therefore has rank 20 and discriminant -99.

31.3. *Examples.* Table 26 lists some points of small height and their genus-2 curves.

We describe some curves on the surface which are a source of rational points (some more may be produced by applying the involution ι). The specialization

(r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{2}{3}, -\frac{7}{10}\right)$	$25x^6 + 120x^5 - 291x^4 - 1292x^3 + 987x^2 + 588x - 497$
$\left(-4, \frac{3}{2}\right)$	$486x^6 - 810x^5 + 1323x^4 - 800x^3 + 585x^2 - 48x + 64$
$\left(-\frac{10}{7}, -\frac{3}{2}\right)$	$515x^6 + 1314x^5 - 3120x^4 - 1332x^3 + 2292x^2 + 720x - 200$
$\left(-\frac{4}{7}, -\frac{3}{10}\right)$	$20x^6 + 180x^5 - 159x^4 - 3276x^3 + 249x^2 + 1980x - 1100$
$\left(\frac{2}{3}, -\frac{1}{4}\right)$	$4608x^6 + 6048x^5 + 3771x^4 - 1026x^3 + 351x^2 - 36x + 4$
$\left(-\frac{20}{7}, -\frac{5}{6}\right)$	$356x^6 - 4980x^5 + 6373x^4 + 2580x^3 - 4409x^2 - 4170x - 790$
$\left(\frac{8}{21}, -\frac{2}{5}\right)$	$1664x^6 + 624x^5 + 3747x^4 - 5222x^3 + 5511x^2 - 1140x + 15020$
$\left(-\frac{6}{5}, -\frac{7}{20}\right)$	$6260x^6 - 21060x^5 + 7009x^4 - 1254x^3 - 239x^2 - 540x - 100$
$\left(-\frac{10}{3}, -\frac{7}{4}\right)$	$2x^6 + 54x^5 + 45x^4 + 1080x^3 - 2961x^2 - 44352$
$\left(-\frac{4}{3}, -\frac{13}{6}\right)$	$-14388x^6 - 86076x^5 - 115441x^4 + 70272x^3 + 86417x^2 - 10794x + 314$
$\left(-\frac{22}{21}, -\frac{13}{70}\right)$	$7865x^6 - 9750x^5 + 62049x^4 - 2788x^3 + 162759x^2 - 4350x + 119375$
$\left(-\frac{5}{2}, \frac{21}{8}\right)$	$363300x^6 - 50652x^5 + 128541x^4 + 2266x^3 + 19257x^2 + 1008x + 896$
$\left(\frac{38}{65}, -\frac{13}{40}\right)$	$-1106244x^6 + 336780x^5 + 23283x^4 + 248770x^3 - 101625x^2 - 33072x - 28736$
$\left(-\frac{8}{39}, -\frac{1}{42}\right)$	$-48600x^6 + 483840x^5 - 1386285x^4 - 264482x^3 - 282489x^2 + 883404x - 1658988$
$\left(-\frac{70}{13}, -\frac{21}{22}\right)$	$599697x^6 - 445662x^5 + 824913x^4 - 838612x^3 + 2057823x^2 - 1620774x + 957519$
$\left(-\frac{10}{7}, -\frac{13}{36}\right)$	$-1112220x^6 + 2309556x^5 - 397465x^4 - 269262x^3 - 847153x^2 - 265908x + 612$

Table 26. Some rational points (r, s) of small height on the surface of Theorem 41 and the corresponding genus-2 curves.

$s = -\frac{5}{6}$ gives a genus-1 curve

$$y^2 = -(9r^2 + 50r + 40)(243r^2 + 670r - 40).$$

It has rational points, such as $(r, y) = (0, 40)$. It is thus an elliptic curve; we find that it has conductor $2 \cdot 3 \cdot 5^2 \cdot 29 \cdot 53$ and Mordell–Weil group $(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}^2$.

Pulling back sections of the elliptic fibration on the quotient surface gives us some more curves of genus 1, each with a rational point and rank 1:

Equation		point	conductor	group
$h =$	$g^2 =$			
$-\dfrac{4t^2 - 5t + 2}{2t(t-1)}$	$\dfrac{(2t-1)(14t^3 - 25t^2 + 16t - 4)}{4t^2(t-1)^2}$	$t = \frac{1}{2}$	53	$\mathbb{Z}$
$-\dfrac{12t^2 - 5t + 2}{2t(3t-2)}$	$\dfrac{252t^4 - 192t^3 + 73t^2 - 20t + 4}{4t^2(3t-2)^2}$	$t = 0$	$2 \cdot 3 \cdot 4391$	$\mathbb{Z}$
$-\dfrac{8t^2 - 15t + 8}{2t(t-1)}$	$\dfrac{92t^4 - 320t^3 + 433t^2 - 268t + 64}{4t^2(t-1)^2}$	$t = 0$	$7 \cdot 977$	$(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$
$-\dfrac{3t^2 - 3t + 2}{3t(t-1)}$	$\dfrac{18t^4 - 27t^3 + 24t^2 - 15t + 4}{9t^2(t-1)^2}$	$t = 0$	$2^5 3^2 7$	$(\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$

32. Discriminant 89

32.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type $A_8 A_7$, and a section of height $\frac{89}{72} = 4 - \frac{1 \cdot 8}{9} - \frac{3 \cdot 5}{8}$.

The Weierstrass equation for this family is

$$y^2 = x^3 + (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4)x^2$$
$$+ 2\mu t^2 (b_0 + b_1 t + b_2 t^2 + b_3 t^3)x + \mu^2 t^4 (c_0 + c_1 t + c_2 t^2),$$

with

$a_0 = (rs + 1)^2,$ $\qquad b_0 = -(rs + 1)^2,$ $\qquad c_0 = (rs + 1)^2,$

$\mu = 4rs(r+1)^2,$ $\qquad b_3 = s(s^2 - rs - 2s + 1)^2,$ $\qquad a_4 = s^2(s^2 - rs - 2s + 1)^2,$

$c_2 = (s^2 - rs - 2s + 1)^2,$

$c_1 = 2rs^3 - 2(r^2 + 4r + 1)s^2 + 4(r+1)^2 s - 2,$

$b_2 = (2r - 1)s^4 - (3r^2 + 7r - 2)s^3 + (r^3 + 6r^2 + 7r - 1)s^2 - 2r(r+1)s + r,$

$a_1 = -2(r^2 - r)s^3 + 2(r^3 - 6r - 1)s^2 + 2(4r^2 + 6r + 1)s + 2r,$

$b_1 = r(r-2)s^3 - (r^3 - r^2 - 10r - 2)s^2 - (6r^2 + 10r + 3)s - r + 1,$

$a_3 = 2s\big((r-1)s^4 - (2r^2 + 3r - 3)s^3 + (r^3 + 4r^2 + 3r - 3)s^2 - (2r^2 + 2r - 1)s + r\big),$

$$a_2 = (r^2 - 4r + 1)s^4 - 2(r^3 - 2r^2 - 9r)s^3$$
$$+ (r^4 - 16r^2 - 22r - 3)s^2 - 2(r^3 - r - 1)s + r^2.$$

To obtain an $E_8 E_7$ elliptic fibration on these K3 surfaces, we first move by a 2-neighbor step to one with E_7 and A_8 fibers.
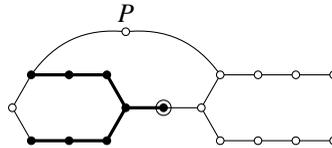
The elliptic fibration defined by this new fiber F' has a section, since $P \cdot F' = 1$. Also, the new elliptic fibration must have a section P' of height $\frac{89}{18} = 4 + 2 \cdot 2 - \frac{3}{2} - \frac{2 \cdot 7}{9}$.

Finally, we go to $E_8 E_7$ by a 3-neighbor step.
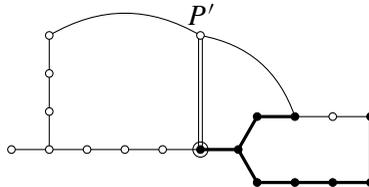
The new fiber F'' satisfies $P' \cdot F'' = 2 + 2 \cdot 3 = 8$, and the identity component of the E_7 fiber intersects F' in 3. Since these have greatest common divisor 1, the genus-1 fibration defined by F' has a section.

Theorem 42. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(89)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$z^2 = s^4 r^6 - 2s^3(2s^2 + 3s + 2)r^5 + s^2(6s^4 + 16s^3 - 49s^2 - 26s + 6)r^4$$
$$- 2s(2s^6 + 6s^5 - 50s^4 + 26s^3 + 73s^2 - 35s + 2)r^3$$
$$+ (s^8 - 36s^6 + 26s^5 + 273s^4 - 514s^3 + 271s^2 - 38s + 1)r^2$$
$$+ 2(s - 1)^2 s(s^5 - 4s^4 + 25s^3 - 107s^2 + 147s - 44)r + (s - 4)^3(s - 1)^4 s.$$

It is a surface of general type.

32.2. *Analysis.* The branch locus has genus 1; one can give an explicit isomorphism (see the online supplement) to the elliptic curve of conductor 89 given by the Weierstrass equation

$$y^2 + xy + y = x^3 + x^2 - x.$$

It is isomorphic to $X_0(89)/\langle w \rangle$, where w is the Atkin–Lehner involution.

The Hilbert modular surface $Y_-(89)$ is a surface of general type. Note that the change of coordinates $r = s + g$ simplifies the equation a bit further, making the

(r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(-\frac{31}{3}, -\frac{1}{6}\right)$	$-334084x^6 + 65892x^5 + 847841x^4 - 156012x^3 - 1036555x^2 - 453867x - 525$
$\left(-\frac{49}{9}, -\frac{2}{3}\right)$	$632x^6 - 480x^5 + 43475x^4 - 97578x^3 - 1030393x^2 + 855708x - 1045044$
$(-19, -1)$	$126905x^6 + 2388081x^5 - 2600778x^4 - 3075787x^3 - 5448045x^2$ $\qquad\qquad\qquad\qquad - 3683352x - 709200$
$\left(-\frac{31}{10}, \frac{5}{2}\right)$	$-83300x^6 + 168420x^5 + 5079215x^4 - 6586832x^3 + 584735x^2$ $\qquad\qquad\qquad\qquad\qquad + 70020x - 8100$
$\left(-\frac{5}{6}, \frac{11}{6}\right)$	$2185004x^6 - 12346980x^5 + 10798163x^4 + 732660x^3 + 47975267x^2$ $\qquad\qquad\qquad\qquad\qquad + 21406020x + 27911916$
$\left(\frac{13}{9}, -\frac{5}{9}\right)$	$-14966100x^6 - 43598124x^5 + 25890735x^4 + 105396908x^3 - 44422995x^2$ $\qquad\qquad\qquad\qquad\qquad - 65750574x + 34674550$
$\left(-\frac{40}{7}, -\frac{1}{2}\right)$	$2754000x^6 + 86434200x^5 + 150411025x^4 - 14830346x^3 - 49970411x^2$ $\qquad\qquad\qquad\qquad\qquad + 242599308x + 131021492$
$\left(\frac{16}{33}, \frac{11}{3}\right)$	$25329267x^6 - 96789717x^5 + 223774305x^4 - 449560367x^3 - 46904988x^2$ $\qquad\qquad\qquad\qquad\qquad - 772810308x + 413626230$

Table 27. Some rational points (r, s) of small height on the surface of Theorem 42 and the corresponding genus-2 curves.

degree of the right-hand side equal to 6 in each variable. However, it complicates the original defining Weierstrass equation of the family of K3 surfaces, so we have chosen the (r, s) coordinate system.

32.3. *Examples.* Table 27 lists some points of small height and their genus-2 curves.

We find two elliptic curves of positive rank on the surface. The specialization $s = \frac{25}{22}$ gives a curve of genus 1

$$y^2 = 1210000r^4 - 19157600r^3 - 17065736r^2 + 678600r - 8575$$

with rational points (as at infinity), conductor $3 \cdot 5 \cdot 11 \cdot 163 \cdot 191 \cdot 881$, and rank at least 2. The locus $s = r + \frac{101}{50}$ gives another curve of genus 1,

$$y^2 = -3739190000r^4 - 21451957600r^3 - 43018833576r^2$$
$$- 36551728152r - 11227811551,$$

with rational points (as at $(r, y) = (-1, \pm 65^2)$), conductor $2^4 \cdot 5 \cdot 17 \cdot 19 \cdot 463 \cdot 58787$, and rank at least 3. We were not able to determine the exact rank of either curve, but the global root numbers indicate that the rank should be even for the former curve and odd for the latter, so one might guess that the lower bounds 2 and 3 on their ranks are sharp.

33. Discriminant 92

33.1. *Parametrization.* We start with an elliptic K3 surface with fibers of type A_8, A_1 and D_6, and a section of height $\frac{92}{72} = \frac{23}{18} = 4 - \frac{1\cdot1}{2} - \frac{4\cdot5}{9}$.

The Weierstrass equation may be written as

$$y^2 = x^3 + (a_0 + a_1 t + a_2 t^2 + a_3 t^3)x^2$$
$$+ 2t^2(\lambda t - \mu)(b_0 + b_1 t + b_2 t^2)x + t^4(\lambda t - \mu)^2(c_0 + c_1 t),$$

with

$$\lambda = (r+s)^2, \qquad\qquad\qquad \mu = r + 2s,$$
$$a_3 = -4rs(s+r^2+r)(s^2+rs-r), \qquad a_0 = (rs-r-1)^2,$$
$$b_0 = -4rs^2(rs-r-1)^2, \qquad\qquad c_0 = 16r^2s^4(rs-r-1)^2,$$
$$a_1 = -2r((r+1)^2(s-1)^2+s^2), \qquad b_1 = 4r^2s^2(rs+2s-r-1)^2,$$
$$c_1 = -64r^3(r+1)(s-1)s^5,$$
$$b_2 = -8r^2s^3\big((r+2)s^2+(2r^2+2r-1)s-2r(r+1)\big),$$
$$a_2 = r(rs+4s-r-1)(4s^2+r^2s+4rs-r^2-r).$$

As in the case of discriminant 56, we first go to an $E_7 A_8$ fibration using the E_7 fiber F' identified below. Note that $F' \cdot P = 3$, while the component of the D_6 fiber which is not included in F' intersects F' with multiplicity 2. Since $\gcd(2,3) = 1$, the fibration defined by F' has a section.
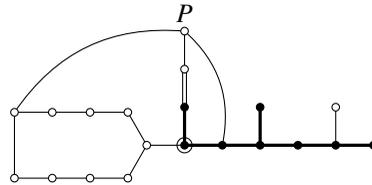
The new elliptic fibration has a section P' of height $\frac{92}{2\cdot9} = \frac{46}{9} = 4 + 2\cdot1 - \frac{8}{9}$, which must therefore intersect the zero section, the identity component of the E_7 fiber and component 1 of the A_8 fiber.

We identify an E_8 fiber and compute its Weierstrass equation by a 3-neighbor move. Note that it intersects P' in 7 and the excluded component of the A_8 fiber in 3. Therefore the fibration it defines has a section, and we may convert to the Jacobian.
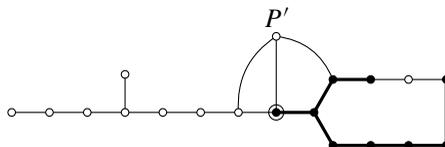
Now we read out the Igusa–Clebsch invariants and compute the branch locus.

Theorem 43. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(92)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$z^2 = (s + r^2 + r)(s^2 + rs - r)$$
$$\times \big((s-1)^3 r^5 + (s-1)^2(s^2 - 15s - 3)r^4 - (s-1)(42s^3 - 27s^2 - 31s - 3)r^3$$
$$- (27s^5 - 30s^4 - 77s^3 + 69s^2 + 17s + 1)r^2 + s(46s^3 - 30s^2 - 42s - 1)r - 27s^3\big).$$

It is a surface of general type.

33.2. Analysis. The branch locus has three components. Points of the two simpler components correspond to elliptic K3 surfaces where the D_6 fiber is promoted to an E_7 fiber, while the more complicated component corresponds to an extra I_2 fiber. All the three components are rational (genus 0). This is obvious from inspection for the simpler components, and for the last we have the parametrization

$$(r, s) = \left(\frac{t(t^2 + 2)(t+1)^2}{(t+2)(t^3 + 2t^2 + 2t + 2)}, \frac{-t(t+1)(t^3 + 2t^2 + 4t + 4)}{(t+2)^2(t^2 + 2)} \right).$$

The surface $Y_-(92)$ is a surface of general type. The extra involution is $(r, s, z) \mapsto (-1/s, -1/r, z/(rs)^4)$. We now analyze the quotient of the Hilbert modular surface by this involution. Because the involution fixes r/s, we obtain the quotient by setting $r = st$ and writing everything in terms of $m = s - 1/(ts)$. We find the equation

$$y^2 = \big(t^2(t+1)m - t^3 + t^2 + 2t + 1\big)\big(t^3(t+1)m^3 - t(3t^3 + 17t^2 + 42t + 27)m^2$$
$$+ t(3t^3 + 31t^2 + 72t + 30)m - (t^4 + 15t^3 + 30t^2 + 7t + 8)\big),$$

which expresses the quotient as a genus-1 curve over $\mathbb{Q}(t)$. Since there is an obvious section (where the first factor vanishes), we may convert to the Jacobian, which has the Weierstrass equation (after shifting t by 1 and performing some Weierstrass transformations)

$$y^2 = x^3 - (2t+1)(8t^3 + 8t^2 - 6t - 1)x^2$$
$$- 8(t-1)t^4(t+1)(10t^2 - 32t - 5)x - 16(t-1)^2 t^8(t+1)(47t+7).$$

This is an elliptic K3 surface. It has reducible fibers of type I_8 at $t = 0$, I_3 at 1 and $(-7 \pm 3\sqrt{3})/11$, and I_2 at the roots of $11t^3 - 10t^2 + 5t + 1$ (which generates the cubic field of discriminant -23). Therefore the trivial lattice has rank 18. We easily identify a non-torsion section P of height $\frac{5}{8}$ with x-coordinate $4t^3(6t+1)$. On the other hand, counting points modulo 13 and 17 shows that the Picard number cannot be 20. Therefore the Picard number of this quotient surface is 19. The sublattice of the Néron–Severi group spanned by P and the trivial lattice has discriminant $360 = 2^3 \cdot 3^2 \cdot 5$. It is easy to see from height calculations that there cannot be any 2-

(r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{3}{10}, -\frac{39}{70}\right)$	$5981584x^6 - 4016376x^5 + 1699985x^4 + 313485x^3 - 168322x^2$ $+ 49665x + 21175$
$\left(\frac{70}{39}, -\frac{10}{3}\right)$	$2916x^6 + 591516x^5 + 6670933x^4 + 12740602x^3 - 44084051x^2$ $+ 8704740x + 16105100$

Table 28. Some rational points (r, s) of small height on the surface of Theorem 43 and the corresponding genus-2 curves.

or 3-torsion sections, and that P cannot be divisible by 2 or 3 in the Mordell–Weil group. Hence, this sublattice is the entire Néron–Severi lattice.

33.3. *Examples.* Table 28 lists some points of small height and their genus-2 curves.

Specializations of r or s, and pullbacks of sections of the quotient, do not seem to yield any genus 0 or 1 curves on the surface (at any rate, none corresponding to abelian surfaces with "honest" real multiplication by $\mathcal{O}_{92}$ and not a larger endomorphism ring).

34. Discriminant 93

34.1. *Parametrization.* Start with an elliptic K3 surface with fibers of type A_{10}, A_3 and A_2, with a section of height $\frac{31}{44} = 4 - \frac{3}{4} - \frac{28}{11}$. The extra involution comes from flipping the A_2 fiber.

This family has the Weierstrass equation

$$y^2 = x^3 + (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4)x^2$$
$$+ 2t(\lambda t - \mu)(b_0 + b_1 t + b_2 t^2 + b_3 t^3)x + t^2(\lambda t - \mu)^2(c_0 + c_1 t + c_2 t^2),$$

with

$$\lambda = -(n^2 - 1), \qquad\qquad \mu = (n^2 - mn - n - m)(n^2 + mn + n - m),$$

$$a_0 = (m+1)^2 n^8, \qquad\qquad b_0 = m^4(m+1)^2 n^8,$$

$$c_0 = m^8(m+1)^2 n^8, \qquad\qquad a_4 = 1,$$

$$c_2 = m^8 n^4, \qquad\qquad b_3 = m^4 n^2,$$

$$a_3 = (m^2 + 2m + 4)n^2 - 3m^2, \qquad c_1 = m^8 n^4((m^2 + 2m + 2)n^2 - m^2),$$

$$b_2 = m^4 n^2\big((m^2 + 2m + 3)n^2 - 2m^2\big),$$

$$a_2 = 3(m^2 + 2m + 2)n^4 - 2m^2(m^2 + 3m + 3)n^2 + 3m^4,$$

$$b_1 = m^4 n^2\big((2m^2 + 4m + 3)n^4 - m^2(m+1)(m+2)n^2 + m^4\big),$$

$$a_1 = (3m^2 + 6m + 4)n^6 - 3m^2(m+1)^2 n^4 + m^4(m+1)(m+3)n^2 - m^6.$$

We first identify the class of an E_8 fiber below, and move to it by a 3-neighbor step.
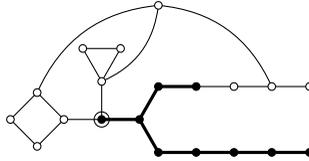
This gives us an elliptic fibration with E_8, A_5 and A_2 fibers, and a section P of height $\frac{93}{18} = \frac{31}{6} = 4 + 2 \cdot 1 - \frac{5}{6}$. We then identify an E_7 fiber and move to it by a 2-neighbor step. Since the new fiber intersects the section P in 7 and the excluded component of the A_5 fiber in 3, we see that the new fibration has a section.
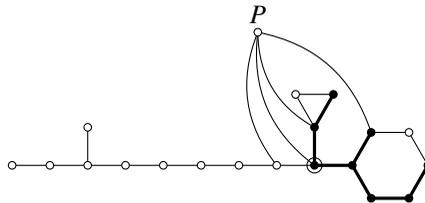
Theorem 44. *A birational model over* $\mathbb{Q}$ *for the Hilbert modular surface* $Y_-(93)$ *as a double cover of* $\mathbb{P}^2_{m,n}$ *is given by the equation*

$$z^2 = 16(n^2 - 1)^2 n^2 m^6 + 8(n^2 - 1)(21n^4 + 22n^2 - 27)m^5$$
$$- (27n^8 - 684n^6 - 1246n^4 + 1620n^2 + 27)m^4$$
$$- 8n^2(27n^6 - 109n^4 - 471n^2 + 41)m^3 - 8n^2(81n^6 + 135n^4 - 273n^2 - 7)m^2$$
$$- 96n^4(9n^2 - 1)(n^2 + 3)m - 16n^4(n^2 + 3)(27n^2 + 1).$$

It is a surface of general type.

34.2. *Analysis.* The extra involution is $\iota : (m, n) \mapsto (m, -n)$.

The branch locus is a curve of genus 4, isomorphic to $X_0(93)/\langle w_{93} \rangle$, where w_{93} is the Atkin–Lehner involution. We do not give the explicit isomorphism here, but the formulas are available in the online supplement. Setting $k = n^2$, we can write it as a double cover of a genus-2 curve, which can be transformed to the Weierstrass form

$$y^2 - (9x^3 + 11x - 3)y + x^2(69x^3 - 56x^2 + 81x - 22) = 0.$$

This Hilbert modular surface is a surface of general type. The quotient by this involution ι has the equation (with $k = n^2$)

$$z^2 = -27(m+2)^4 k^4 + 4(4m^6 + 42m^5 + 171m^4 + 218m^3 - 270m^2 - 624m - 328)k^3$$
$$- 2(16m^6 - 4m^5 - 623m^4 - 1884m^3 - 1092m^2 - 144m + 24)k^2$$
$$+ 4m^2(4m^4 - 98m^3 - 405m^2 - 82m + 14)\,k + 27m^4(8m - 1).$$

This has a genus-1 fibration over $\mathbb{Q}(m)$. The fibration has a section at infinity defined over $\mathbb{Q}(\sqrt{-3})$; we do not know whether there is a section defined over $\mathbb{Q}$. The Jacobian has the Weierstrass equation

$$y^2 = x^3 + (m^6 + 20m^5 + 118m^4 + 186m^3 + 33m^2 + 18m - 3)x^2$$
$$+ 8m^2(m + 10)(9m^4 + 39m^3 + 57m^2 - 1)x$$
$$+ 16m^4(m + 10)^2(4m^3 + 13m^2 + 18m - 3).$$

This is an honestly elliptic surface with $\chi = 3$. It has reducible fibers of type I_9 at $m = \infty$, I_4 at $m = 0$, I_2 at $m = -10$, and I_3 at the roots of

$$m^6 + 11m^5 + 16m^4 + 32m^3 + 17m^2 - 9m + 1$$

(whose splitting field is a dihedral extension of degree 12 containing $\sqrt{93}$). The trivial lattice has rank 26, leaving room for Mordell–Weil rank up to 4. Counting points modulo 13 and 17 shows that the Picard number is at most 29. On the other hand, we are (so far) able to produce the sections

$$P_0 = \big(-16(m^3 + m^2 + 6m - 1),\, 32(m^6 + 11m^5 + 16m^4 + 32m^3 + 17m^2 - 9m + 1)\big),$$
$$P_1 = \big(-16(m + 10)(m^5 + m^4 + 6m^3 + 3m^2 + 18m - 3)/31,$$
$$288(m + 10)(3m^2 + 2m + 9)(m^6 + 11m^5 + 16m^4 + 32m^3 + 17m^2 - 9m + 1)/93^{3/2}\big),$$

of which P_0 is 3-torsion, while P_1 has height $\frac{3}{2}$. Therefore, the Mordell–Weil rank is between 1 and 3.

34.3. Examples. Table 29 lists some points of small height and their genus-2 curves.

(m, n)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$(2, \frac{1}{3})$	$-2112x^6 - 5184x^5 + 5451x^4 + 2593x^3 - 4596x^2 - 2223x - 101$
$(-10, \frac{5}{3})$	$-7452x^6 - 4860x^5 - 24039x^4 - 4540x^3 - 17205x^2 + 4686x - 302$
$(\frac{1}{5}, \frac{1}{7})$	$-24786x^6 + 25272x^5 + 90900x^4 - 73885x^3 - 107482x^2 + 54020x + 40286$
$(-10, 5)$	$-31752x^5 - 48825x^4 + 52868x^3 - 175537x^2 + 91124x - 80644$
$(2, -\frac{1}{3})$	$-594x^6 + 10962x^5 - 154233x^4 + 391936x^3 + 265521x^2 + 330228x - 71068$
$(-10, -5)$	$43756x^6 + 110088x^5 + 463887x^4 - 609201x^3 + 208770x^2 - 6300x - 211000$
$(-10, -\frac{5}{3})$	$-3008x^6 + 270048x^5 - 773739x^4 - 611989x^3 - 2150523x^2$ $+ 631152x - 342144$
$(\frac{1}{5}, -\frac{1}{7})$	$-253800x^6 - 1186380x^5 - 1627302x^4 + 4611739x^3 + 1795017x^2$ $- 2139291x + 2480233$

Table 29. Some rational points (m, n) of small height on the surface of Theorem 44 and the corresponding genus-2 curves.

35. Discriminant 97

35.1. *Parametrization.* Start with an elliptic K3 surface with fibers of type D_5, A_4 and A_6, with a section of height $\frac{97}{140} = 4 - \left(1 + \frac{1}{4}\right) - \frac{6}{5} - \frac{6}{7}$.

We can write the Weierstrass equation as

$$y^2 = x^3 + (a_0 + a_1t + a_2t^2 + a_3t^3)x^2 + 2t^2(t-1)^2(b_0 + b_1t + b_2t^2)x$$
$$+ t^4(t-1)^4(c_0 + c_1t),$$

with

$$a_0 = (r+1)^2(rs^2 + s^2 + r^2s + r)^2,$$

$$a_1 = 2(r+1)\big((r+1)^2s^5 + 2(r+1)(r^2 - r - 1)s^4 + r(r^3 - 4r^2 - 4r + 2)s^3$$
$$- r(2r^3 + r^2 + 2r + 5)s^2 - r^2(r^2 + 3r + 3)s - r^2(r+1)\big),$$

$$a_2 = (r+1)^2s^6 + 2(r+1)(r^2 - 4r - 2)s^5 + (r^4 - 16r^3 - 6r^2 + 18r + 4)s^4$$
$$- 2r(4r^3 - 6r^2 - 4r + 11)s^3 + r(6r^3 - 6r^2 - 3r + 20)s^2$$
$$+ 2r^2(r+3)(2r+3)s + r^2(r+1)^2,$$

$$a_3 = -4r(s-1)s(s+r)\big(s^3 + (r-2)s^2 - (r-2)s - 2r - 3\big),$$

$$b_0 = 4r(r+1)^2(s-1)^2s(s+r)(rs^2 + s^2 + r^2s + r)^2,$$

$$b_1 = 4r(r+1)(s-1)^2s(s+r)$$
$$\times \big((r+1)^2s^5 + 2(r+1)(r^2 - 2r - 1)s^4 + r(r^3 - 8r^2 - 6r + 4)s^3$$
$$- r(4r^3 + r^2 + 2r + 7)s^2 - r^2(r^2 + 5r + 5)s - r^2(r+1)\big),$$

$$b_2 = -8r^2(r+1)(s-1)^2s^2(s+r)^2\big(2s^3 + 2(r-2)s^2 - 2(r-2)s - r - 3\big),$$

$$c_0 = 16r^2(r+1)^2(s-1)^4s^2(s+r)^2(rs^2 + s^2 + r^2s + r)^2,$$

$$c_1 = -64r^3(r+1)^2(s-1)^4s^3(s+r)^3(s^2 + rs - s + 1).$$

First we identify a D_8, and move to the associated elliptic fibration (which also has an A_6 fiber) by a 2-neighbor step.
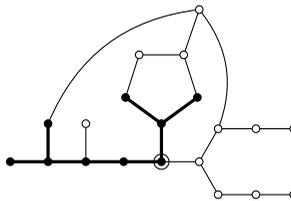
This elliptic fibration has D_8 and A_6 fibers, and two independent sections P, Q with height matrix

$$\begin{pmatrix} \frac{8}{7} & -\frac{5}{14} \\ -\frac{5}{14} & \frac{22}{7} \end{pmatrix}.$$

We identify a fiber F' of type E_8 and move to the associated fibration by a 2-neighbor step. Note that $Q \cdot F' = 3$, while the remaining component of the D_8 fiber has intersection 2 with F'. Therefore the new genus-1 fibration has a section.
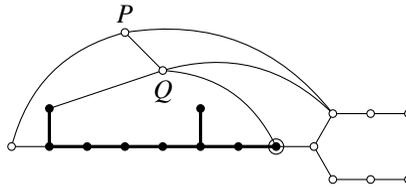
The new elliptic fibration has bad fibers of types E_8 and A_7, and a section P' of height $\frac{97}{8} = 2 + 2 \cdot 6 - 3 \cdot \frac{5}{8}$. We identify a fiber F'' of type E_7, and move to the associated elliptic fibration by a 2-neighbor step. Note that $P' \cdot F'' = 13$, while the remaining component of the A_7 fiber has intersection 2 with F''. Therefore the elliptic fibration associated to F'' has a section, and is the of the desired type $E_8 E_7$.
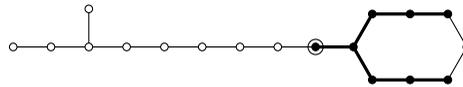
We now read out the Igusa–Clebsch invariants and work out the equation of the branch locus of $Y_-(97)$ as a double cover of $\mathcal{H}_{97}$.

Theorem 45. *A birational model over $\mathbb{Q}$ for the Hilbert modular surface $Y_-(97)$ as a double cover of $\mathbb{P}^2_{r,s}$ is given by the equation*

$$
\begin{aligned}
z^2 = {} & s^2(s^2 + 14s + 1)r^6 + 2s(2s^4 + 27s^3 - 13s^2 + 15s + 1)r^5 \\
& + (6s^6 + 80s^5 - 75s^4 + 128s^3 - 54s^2 + 18s + 1)r^4 \\
& + 2(2s^7 + 28s^6 - 32s^5 + 84s^4 - 74s^3 + 48s^2 - 13s + 1)r^3 \\
& + (s^8 + 18s^7 - 11s^6 + 68s^5 - 101s^4 + 112s^3 - 69s^2 + 22s + 1)r^2 \\
& + 2s^2(s^6 + 3s^5 - 5s^4 + 7s^3 + 3s^2 - 16s + 12)r + (s - 2)^4 s^4.
\end{aligned}
$$

It is a surface of general type.

35.2. *Analysis.* The branch locus is a genus-3 curve, isomorphic to the quotient of $X_0(97)$ by the Atkin–Lehner involution. We omit the formulas for the isomorphism, but they are available in the online supplement.

The Hilbert modular surface itself is of general type. The substitution $r = u - s$ simplifies the equation of the double cover somewhat, making it degree-6 in each variable. However, it complicates the original Weierstrass equation of the K3 family, so we have chosen the (r, s)-coordinates on the moduli space.

(r, s)	Sextic polynomial $f_6(x)$ defining the genus-2 curve $y^2 = f_6(x)$
$\left(\frac{5}{2}, -2\right)$	$-100x^6 + 180x^5 + 3x^4 + 12x^3 - 207x^2 + 54x + 54$
$\left(-\frac{5}{3}, \frac{2}{3}\right)$	$115x^6 - 120x^5 - 692x^4 - 42x^3 + 643x^2 + 36x - 156$
$\left(\frac{8}{15}, -\frac{6}{5}\right)$	$-418x^6 - 99x^5 + 700x^4 + 130x^3 - 401x^2 - 45x + 81$
$\left(-\frac{14}{3}, 6\right)$	$528x^6 - 792x^5 + 311x^4 - 26x^3 - 205x^2 + 60x - 20$
$\left(-\frac{20}{7}, \frac{5}{14}\right)$	$-72x^6 - 72x^5 + 669x^4 + 706x^3 - 1623x^2 - 60x + 500$
$\left(\frac{13}{6}, -\frac{3}{2}\right)$	$1236x^6 - 852x^5 - 1919x^4 + 1702x^3 + 1473x^2 - 940x - 700$
$\left(\frac{17}{6}, -\frac{1}{3}\right)$	$200x^6 - 420x^5 + 1918x^4 - 1455x^3 + 2968x^2 + 1740x - 1175$
$\left(-\frac{13}{7}, \frac{11}{21}\right)$	$-1872x^5 - 3540x^4 + 1021x^3 + 2331x^2 - 1185x + 145$
$\left(\frac{23}{35}, -\frac{5}{14}\right)$	$370x^6 + 1084x^5 - 2510x^4 - 683x^3 - 32x^2 - 752x - 3822$
$\left(-\frac{11}{30}, \frac{5}{6}\right)$	$-1225x^6 + 3570x^5 - 3266x^4 - 176x^3 + 3463x^2 + 1446x + 5868$
$\left(\frac{1}{2}, \frac{5}{2}\right)$	$-1938x^6 + 3132x^5 + 1730x^4 - 855x^3 + 609x^2 - 9065x + 5145$
$\left(-\frac{19}{30}, \frac{3}{10}\right)$	$4900x^6 + 5320x^5 - 11751x^4 - 4255x^3 + 2867x^2 + 4515x - 1596$
$\left(-\frac{29}{18}, \frac{10}{9}\right)$	$16048x^6 - 7524x^5 - 11096x^4 + 16107x^3 - 4244x^2 - 5652x - 864$
$\left(\frac{13}{4}, -\frac{1}{4}\right)$	$-1140x^6 + 4820x^5 - 3105x^4 + 3366x^3 - 16681x^2 - 6468x - 12348$
$\left(-\frac{7}{2}, 6\right)$	$14076x^6 - 20748x^5 + 11899x^4 + 1252x^3 - 125x^2 + 2676x + 380$
$\left(\frac{9}{10}, -\frac{2}{5}\right)$	$-6688x^6 + 9840x^5 - 8271x^4 + 24640x^3 - 5373x^2 + 12150x - 7290$

Table 30. Some rational points (r, s) of small height on the surface of Theorem 45 and the corresponding genus-2 curves.

35.3. *Examples.* Table 30 lists some points of small height and their genus-2 curves.

We find a few curves with infinitely many rational points. For instance, $r = 1 - s$ gives a rational curve, with parametrization

$$(r, s) = \left(\frac{(m+1)(m+3)}{m^2 + 7}, \frac{-4(m-1)}{m^2 + 7} \right).$$

The Brauer obstruction vanishes identically along this curve. However, it turns out to be a modular curve: the corresponding abelian surfaces have endomorphism ring a (split) quaternion algebra.

Another curve of genus 0 is given by $r = -(3s^2 + 8s + 4)/(3s)$. Again, the Brauer obstruction vanishes, and this time we get a family of abelian surfaces with "honest" real multiplication.

The locus $r = \frac{1}{2} - s$ gives a genus-1 curve

$$y^2 = (2s + 1)(2s^3 - 39s^2 + 28s + 36)$$

with conductor $5862 = 2 \cdot 3 \cdot 977$ and Mordell–Weil group $\mathbb{Z}^2$.

Acknowledgements

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elkies@math.harvard.edu   *Department of Mathematics, Harvard University, Cambridge, MA 02138, United States*

abhinav@math.mit.edu   *Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, United States*

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