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Intermediate co-t-structures, two-term silting objects, τ-tilting modules, and torsion classes

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If (A, B) and $(\mathsf{A}', \mathsf{B}')$ are co-t-structures of a triangulated category, then $(\mathsf{A}', \mathsf{B}')$ is called intermediate if $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$. Our main results show that intermediate co-t-structures are in bijection with two-term silting subcategories, and also with support τ-tilting subcategories under some assumptions. We also show that support τ-tilting subcategories are in bijection with certain finitely generated torsion classes. These results generalise work by Adachi, Iyama, and Reiten.

Introduction

The aim of this paper is to discuss the relationship between the following objects:

- Intermediate co-t-structures.
- Two-term silting subcategories.
- Support τ-tilting subcategories.
- Torsion classes.

The motivation is that if T is a triangulated category with suspension functor Σ and (X, Y) is a t-structure of T with heart $\mathsf{H} = \mathsf{X} \cap \Sigma\mathsf{Y}$, then there is a bijection between "intermediate" t-structures $(\mathsf{X}', \mathsf{Y}')$ with $\Sigma\mathsf{X} \subseteq \mathsf{X}' \subseteq \mathsf{X}$ and torsion pairs of H. This is due to [Beligiannis and Reiten 2007, Theorem 3.1] and [Happel et al. 1996, Proposition 2.1]; see [Woolf 2010, Proposition 2.3].

We will study a co-t-structure analogue of this which also involves silting subcategories, that is, full subcategories $\mathsf{S} \subseteq \mathsf{T}$ with thick closure equal to T which satisfy $\mathrm{Hom}_{\mathsf{T}}(\mathsf{S}, \Sigma^i\mathsf{S}) = 0$ for $i \geq 1$. Silting subcategories are a useful generalisation of tilting subcategories.

The next theorem follows from the bijection between bounded co-t-structures and silting subcategories in [Mendoza Hernández et al. 2013, Corollary 5.9]. See [Pauksztello 2008] and [Aihara and Iyama 2012] for background on co-t-structures

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and silting subcategories. Note that the *co-heart* of a co-t-structure (A, B) is $\mathsf{A} \cap \Sigma^{-1}\mathsf{B}$. If F, G are full subcategories of a triangulated category, then $\mathsf{F} * \mathsf{G}$ denotes the full subcategory of objects e which permit a distinguished triangle $f \to e \to g$ with $f \in \mathsf{F}$, $g \in \mathsf{G}$.

Theorem 0.1 (Theorem 2.2). *Let* T *be a triangulated category*, (A, B) *a bounded co-t-structure of* T *with co-heart* S. *Then we have a bijection between the following sets*:

(i) *Co-t-structures* $(\mathsf{A}', \mathsf{B}')$ *of* T *with* $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$.

(ii) *Silting subcategories of* T *which are in* $\mathsf{S} * \Sigma\mathsf{S}$.

The co-t-structures in (i) are called *intermediate*. The silting subcategories in (ii) are called *two-term*, motivated by the existence of a distinguished triangle $s_1 \to s_0 \to s'$ with $s_i \in \mathsf{S}$ for each $s' \in \mathsf{S}'$. The theorem reduces the study of intermediate co-t-structures to the study of two-term silting subcategories.

Our main results on two-term silting subcategories and τ-tilting theory can be summed up as follows. We extend the notion of support τ-tilting modules for finite-dimensional algebras over fields given in [Adachi et al. 2014] to essentially small additive categories; see Definitions 1.3 and 1.5. For a commutative ring $\Bbbk$, we say that a $\Bbbk$-linear category is *Hom-finite* if each Hom-set is a finitely generated $\Bbbk$-module.

Theorem 0.2 (Theorems 3.4 and 4.6). *Let* T *be a triangulated category with a silting subcategory* S. *Assume that each object of* $\mathsf{S} * \Sigma\mathsf{S}$ *can be written as a direct sum of indecomposable objects unique up to isomorphism. Then there is a bijection between the following sets*:

(i) *Silting subcategories of* T *which are in* $\mathsf{S} * \Sigma\mathsf{S}$.

(ii) *Support τ-tilting pairs of* $\operatorname{mod} \mathsf{S}$.

If T *is Krull–Schmidt*, $\Bbbk$*-linear and Hom-finite over a commutative ring* $\Bbbk$, *and* $\mathsf{S} = \operatorname{add} s$ *for a silting object* s, *then there is a bijection between the following sets*:

(iii) *Basic silting objects of* T *which are in* $\mathsf{S} * \Sigma\mathsf{S}$, *modulo isomorphism.*

(iv) *Basic support τ-tilting modules of* $\operatorname{mod} E$, *modulo isomorphism*, *where* $E = \operatorname{End}_{\mathsf{T}}(s)$.

Note that in this case, *there is a bijection between* (i) *and* (iii) *by* [Aihara and Iyama 2012, Proposition 2.20, Lemma 2.22(a)].

Note that Theorem 0.2 is a much stronger version of Theorem 3.2 of [Adachi et al. 2014], where T is assumed to be the homotopy category of bounded complexes of finitely generated projective modules over a finite-dimensional algebra Λ over a field, and s is assumed to be Λ.

Moreover, we give the following link between τ-tilting theory and torsion classes. Our main result shows that support τ-tilting pairs correspond bijectively with certain finitely generated torsion classes, which is a stronger version of [Adachi et al. 2014, Theorem 2.7]. Note that $\mathsf{Fac}\,\mathsf{M}$ is the subcategory of $\mathsf{Mod}\,\mathsf{C}$ consisting of factor objects of finite direct sums of objects of M, and $\mathsf{P}(\mathsf{T})$ denotes the Ext-projective objects of T; see Definition 1.7.

Theorem 0.3 (Theorem 5.1). *Let $\Bbbk$ be a commutative noetherian local ring and C an essentially small, Krull–Schmidt, $\Bbbk$-linear Hom-finite category. There is a bijection* $\mathsf{M} \mapsto \mathsf{Fac}\,\mathsf{M}$ *from the first of the following sets to the second*:

(i) *Support τ-tilting pairs* (M, E) *of* $\mathsf{mod}\,\mathsf{C}$.

(ii) *Finitely generated torsion classes* T *of* $\mathsf{Mod}\,\mathsf{C}$ *such that each finitely generated projective* C*-module has a left* $\mathsf{P}(\mathsf{T})$*-approximation.*

1. Basic definitions

Let C be an additive category. When we say that U is a *subcategory* of C, we always assume U is full and closed under finite direct sums and direct summands. For a collection U of objects of C, we denote by $\mathsf{add}\,\mathsf{U}$ the smallest subcategory of C containing U.

Let C be an essentially small additive category. We write $\mathsf{Mod}\,\mathsf{C}$ for the abelian category of contravariant additive functors from C to the category of abelian groups, and $\mathsf{mod}\,\mathsf{C}$ for the full subcategory of finitely presented functors; see [Auslander 1974, pp. 184, 204].

The suspension functor of a triangulated category is denoted by Σ.

We first recall the notions of co-t-structures and silting subcategories.

Definition 1.1. Let T be a triangulated category. A *co-t-structure* on T is a pair (A, B) of full subcategories of T such that:

(i) $\Sigma^{-1}\mathsf{A} \subseteq \mathsf{A}$ and $\Sigma\mathsf{B} \subseteq \mathsf{B}$.

(ii) $\mathrm{Hom}_{\mathsf{T}}(a, b) = 0$ for $a \in \mathsf{A}$ and $b \in \mathsf{B}$.

(iii) For each $t \in \mathsf{T}$ there is a triangle $a \to t \to b \to \Sigma a$ in T with $a \in \mathsf{A}$ and $b \in \mathsf{B}$.

The *co-heart* is defined as the intersection $\mathsf{A} \cap \Sigma^{-1}\mathsf{B}$. See [Pauksztello 2008; Bondarko 2010].

Definition 1.2. Let T be a triangulated category.

(i) A subcategory U of T is called a *presilting subcategory* if $\mathsf{T}(u, \Sigma^{\geq 1}u') = 0$ for any $u, u' \in \mathsf{U}$.

(ii) A presilting subcategory $\mathsf{S} \subseteq \mathsf{T}$ is a *silting subcategory* if $\mathsf{thick}(\mathsf{S}) = \mathsf{T}$; see [Aihara and Iyama 2012, Definition 2.1(a)]. Here $\mathsf{thick}(\mathsf{S})$ denotes the smallest thick subcategory of T containing S.

(iii) An object $u \in \mathsf{T}$ is called a *presilting object* if it satisfies $\mathsf{T}(u, \Sigma^{\geq 1}u) = 0$, namely, if $\mathsf{add}(u)$ is a presilting subcategory. Similarly an object $u \in \mathsf{T}$ is called a *silting object* if $\mathsf{add}(u)$ is a silting subcategory.

Next we introduce the notion of support τ-tilting subcategories.

Definition 1.3. Let C be an essentially small additive category.

(i) Let M be a subcategory of $\mathsf{mod}\,\mathsf{C}$. A class $\left\{P_1 \xrightarrow{\pi^m} P_0 \to m \to 0 \mid m \in \mathsf{M}\right\}$ of projective presentations in $\mathsf{mod}\,\mathsf{C}$ is said to have *property* (S) if

$$\mathrm{Hom}_{\mathsf{mod}\,\mathsf{C}}(\pi^m, m') : \mathrm{Hom}_{\mathsf{mod}\,\mathsf{C}}(P_0, m') \to \mathrm{Hom}_{\mathsf{mod}\,\mathsf{C}}(P_1, m')$$

is surjective for any $m, m' \in \mathsf{M}$.

(ii) A subcategory M of $\mathsf{mod}\,\mathsf{C}$ is said to be *τ-rigid* if there is a class of projective presentations $\{P_1 \to P_0 \to m \to 0 \mid m \in \mathsf{M}\}$ which has property (S).

(iii) A *τ-rigid pair* of $\mathsf{mod}\,\mathsf{C}$ is a pair (M, E), where M is a τ-rigid subcategory of $\mathsf{mod}\,\mathsf{C}$ and $\mathsf{E} \subseteq \mathsf{C}$ is a subcategory with $\mathsf{M}(\mathsf{E}) = 0$, that is, $m(e) = 0$ for each $m \in \mathsf{M}$ and $e \in \mathsf{E}$.

(iv) A τ-rigid pair (M, E) is *support τ-tilting* if $\mathsf{E} = \mathrm{Ker}(\mathsf{M})$ and for each $s \in \mathsf{C}$ there exists an exact sequence $\mathsf{C}(-, s) \xrightarrow{f} m^0 \to m^1 \to 0$ with $m^0, m^1 \in \mathsf{M}$ such that f is a left M-approximation.

It is useful to recall the notion of Krull–Schmidt categories:

Definition 1.4. An additive category C is called *Krull–Schmidt* if each of its objects is the direct sum of finitely many objects with local endomorphism rings. It follows that these finitely many objects are indecomposable and determined up to isomorphism; see [Bass 1968, Theorem I.3.6]. It also follows that C is *idempotent complete*; that is, for an object c of C and an idempotent $e \in \mathsf{C}(c, c)$, there exist objects c_1 and c_2 such that $c = c_1 \oplus c_2$ and $e = \mathrm{id}_{c_1}$; see [Keller 2013, 5.1].

(i) An object $c \in \mathsf{C}$ is *basic* if it has no repeated indecomposable direct summands.

(ii) For an object $c \in \mathsf{C}$, let $\#_{\mathsf{C}}(c)$ denote the number of pairwise nonisomorphic indecomposable direct summands of c.

The following is a version of Definition 1.3 for rings:

Definition 1.5. Let E be a ring such that $\mathsf{mod}\,E$ is Krull–Schmidt.

(i) A module $U \in \mathsf{mod}\,E$ is called *τ-rigid* if there is a projective presentation $P_1 \xrightarrow{\pi} P_0 \to U \to 0$ in $\mathsf{mod}\,E$ such that $\mathrm{Hom}_E(\pi, U)$ is surjective.

(ii) A τ-rigid module $U \in \mathsf{mod}\,E$ is called *support τ-tilting* if there is an idempotent $e \in E$ which satisfies $Ue = 0$ and $\#_{\mathsf{mod}\,E}(U) = \#_{\mathsf{prj}(E/EeE)}(E/EeE)$.

Remark 1.6. Part (ii) of the definition makes sense because $\mathsf{prj}(E/EeE)$ is Krull–Schmidt. Namely, since $\mathsf{mod}\, E$ is Krull–Schmidt, it follows that $\mathsf{prj}\, E$ is Krull–Schmidt with additive generator E_E. The same is hence true for $(\mathsf{prj}\, E)/[\mathsf{add}\, eE]$ for each idempotent $e \in E$, and it is not hard to check that the endomorphism ring of E_E in $(\mathsf{prj}\, E)/[\mathsf{add}\, eE]$ is E/EeE, so there is an equivalence of categories

$$(\mathsf{prj}\, E)/[\mathsf{add}\, eE] \xrightarrow{\sim} \mathsf{prj}(E/EeE).$$

Hence $\mathsf{prj}(E/EeE)$ is Krull–Schmidt.

If E is a finite-dimensional algebra over a field, then the definition coincides with the original definition of basic support τ-tilting modules by Adachi, Iyama and Reiten [Adachi et al. 2014, Definition 0.1(c)].

Finally we introduce the notion of torsion classes:

Definition 1.7. Let C be an essentially small additive category and T a full subcategory of $\mathsf{Mod}\,\mathsf{C}$.

(i) We say that T is a *torsion class* if it is closed under factor modules and extensions.

(ii) For a subcategory M of $\mathsf{Mod}\,\mathsf{C}$, we denote by $\mathsf{Fac}\,\mathsf{M}$ the subcategory of $\mathsf{Mod}\,\mathsf{C}$ consisting of factor objects of objects of M.

(iii) We say that a torsion class T is *finitely generated* if there exists a full subcategory M of $\mathsf{mod}\,\mathsf{C}$ such that $\mathsf{T} = \mathsf{Fac}\,\mathsf{M}$. Clearly the objects in $\mathsf{Fac}\,\mathsf{M}$ are finitely generated C-modules, which are not necessarily finitely presented.

(iv) An object t of a torsion class T is *Ext-projective* if $\mathrm{Ext}^1_{\mathsf{Mod}\,\mathsf{C}}(t, \mathsf{T}) = 0$. We denote by $\mathsf{P}(\mathsf{T})$ the full subcategory of T consisting of all Ext-projective objects of T.

2. Silting subcategories and co-t-structures

In this section, T is an essentially small, idempotent complete triangulated category.

Let (A, B) be a co-t-structure on T. It follows from the definition that

$$\mathsf{A} = \{t \in \mathsf{T} \mid \mathrm{Hom}(t, b) = 0 \text{ for all } b \in \mathsf{B}\},$$
$$\mathsf{B} = \{t \in \mathsf{T} \mid \mathrm{Hom}(a, t) = 0 \text{ for all } a \in \mathsf{A}\}.$$

In particular, both A and B are idempotent complete and extension closed. Hence so is the co-heart $\mathsf{S} = \mathsf{A} \cap \Sigma^{-1}\mathsf{B}$. Set

$$\mathsf{S} * \Sigma\mathsf{S} = \{t \in \mathsf{T} \mid \text{there is a triangle } s_1 \to s_0 \to t \to \Sigma s_1 \text{ with } s_0, s_1 \in \mathsf{S}\} \subseteq \mathsf{T}.$$

The following lemma will often be used without further remark:

Lemma 2.1. *There is an equality* $\mathsf{S} * \Sigma\mathsf{S} = \Sigma\mathsf{A} \cap \Sigma^{-1}\mathsf{B}$. *As a consequence*, $\mathsf{S} * \Sigma\mathsf{S}$ *is idempotent complete and extension closed.*

Proof. The inclusion $\mathsf{S} * \Sigma\mathsf{S} \subseteq \Sigma\mathsf{A} \cap \Sigma^{-1}\mathsf{B}$ is clear, because both S and $\Sigma\mathsf{S}$ are contained in $\Sigma\mathsf{A} \cap \Sigma^{-1}\mathsf{B}$, which is extension closed. Next we show the opposite inclusion. Let $t \in \Sigma\mathsf{A} \cap \Sigma^{-1}\mathsf{B}$. Then by Definition 1.1(iii) there is a triangle $a \to t \to b \to \Sigma a$ with $a \in \mathsf{A}$ and $b \in \mathsf{B}$. Since both t and Σa are in $\Sigma\mathsf{A}$, so is b due to the fact that A is extension closed. Thus $b \in \Sigma\mathsf{A} \cap \mathsf{B} = \Sigma\mathsf{S}$. Similarly, one shows that $a \in \mathsf{S}$. Thus we obtain a triangle $\Sigma^{-1}b \to a \to t \to b$ with $\Sigma^{-1}b$ and a in S, meaning that $t \in \mathsf{S} * \Sigma\mathsf{S}$. □

It is easy to see that $\mathrm{Hom}(s, \Sigma^{\geq 1}s') = 0$ for any $s, s' \in \mathsf{S}$. That is, S is a presilting subcategory of T. The co-t-structure (A, B) is said to be *bounded* if

$$\bigcup_{n\in\mathbb{Z}} \Sigma^n\mathsf{B} = \mathsf{T} = \bigcup_{n\in\mathbb{Z}} \Sigma^n\mathsf{A}.$$

Theorem 2.2 [Mendoza Hernández et al. 2013, Corollary 5.9]. *There is a bijection* $(\mathsf{A}, \mathsf{B}) \mapsto \mathsf{A} \cap \Sigma^{-1}\mathsf{B}$ *from the first of the following sets to the second*:

(i) *Bounded co-t-structures on* T.

(ii) *Silting subcategories of* T.

This result has the following consequence:

Theorem 2.3. *Let* (A, B) *be a bounded co-t-structure on* T *with co-heart* S. *Then there is a bijection* $(\mathsf{A}', \mathsf{B}') \mapsto \mathsf{A}' \cap \Sigma^{-1}\mathsf{B}'$ *from the first of the following sets to the second*:

(i) *Bounded co-t-structures* $(\mathsf{A}', \mathsf{B}')$ *on* T *with* $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$.

(ii) *Silting subcategories of* T *which are in* $\mathsf{S} * \Sigma\mathsf{S}$.

Proof. Let $(\mathsf{A}', \mathsf{B}')$ be a bounded co-t-structure on T with $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$. Then $\mathsf{B} \supseteq \mathsf{B}' \supseteq \Sigma\mathsf{B}$. It follows that $\mathsf{A}' \cap \Sigma^{-1}\mathsf{B}' \subseteq \Sigma\mathsf{A} \cap \Sigma^{-1}\mathsf{B} = \mathsf{S} * \Sigma\mathsf{S}$. The last equality is by Lemma 2.1.

Let S' be a silting subcategory of T which is in $\mathsf{S} * \Sigma\mathsf{S}$. Let A' be the smallest extension closed subcategory of T containing $\Sigma^{\leq 0}\mathsf{S}'$ and B' the smallest extension closed subcategory of T containing $\Sigma^{\geq 1}\mathsf{S}'$. Then $(\mathsf{A}', \mathsf{B}')$ is the bounded co-t-structure corresponding to S' as in Theorem 2.2; see [Mendoza Hernández et al. 2013, Corollary 5.9]. Since $\mathsf{S}' \subseteq \mathsf{S} * \Sigma\mathsf{S}$, it follows that A' is contained in the smallest extension closed subcategory of T containing $\Sigma^{\leq 1}\mathsf{S}$, which is exactly $\Sigma\mathsf{A}$. Similarly, one shows that B' is contained in B, implying that A' contains A. Thus, $\mathsf{A} \subseteq \mathsf{A}' \subseteq \Sigma\mathsf{A}$. □

The co-t-structures in (i) are called *intermediate* with respect to (A, B). The silting subcategories in (ii) are called *2-term* with respect to S. Clearly, if $(\mathsf{A}', \mathsf{B}')$ is intermediate with respect to (A, B), then (A, B) is intermediate with respect to $(\Sigma^{-1}\mathsf{A}', \Sigma^{-1}\mathsf{B}')$. The next result is a corollary of Theorems 2.2 and 2.3:

Corollary 2.4. *Let* S *and* S' *be two silting subcategories of* T. *If* S' *is* 2*-term with respect to* S, *then* S *is* 2*-term with respect to* $\Sigma^{-1}\mathsf{S}'$.

3. Two-term silting subcategories and support τ-tilting pairs

In this section, T is an essentially small, idempotent complete triangulated category, and $\mathsf{S} \subseteq \mathsf{T}$ is a silting subcategory.

Remark 3.1. (i) There is a functor

$$F : \mathsf{T} \to \operatorname{Mod} \mathsf{S}, \quad t \mapsto \mathsf{T}(-, t)|_{\mathsf{S}},$$

sometimes known as the restricted Yoneda functor.

(ii) By Yoneda's lemma, for $M \in \operatorname{Mod} \mathsf{S}$ and $s \in \mathsf{S}$, there is a natural isomorphism

$$\operatorname{Hom}_{\operatorname{Mod} \mathsf{S}}(\mathsf{S}(-, s), M) \xrightarrow{\sim} M(s);$$

see [Auslander 1974, p. 185].

(iii) By [Iyama and Yoshino 2008, Proposition 6.2(3)], the functor F from (i) induces an equivalence

$$(\mathsf{S} * \Sigma\mathsf{S})/[\Sigma\mathsf{S}] \xrightarrow{\sim} \operatorname{mod} \mathsf{S}. \tag{1}$$

This follows from that proposition by setting $\mathscr{X} = \mathsf{S}$, $\mathscr{Y} = \Sigma\mathsf{S}$, and observing that the proof works in the generality of the present paper.

Lemma 3.2. *Let* U *be a full subcategory of* $\mathsf{S} * \Sigma\mathsf{S}$. *For* $u \in \mathsf{U}$ *let*

$$s_1^u \xrightarrow{\sigma} s_0^u \longrightarrow u \longrightarrow \Sigma s_1^u \tag{2}$$

be a distinguished triangle in T *with* $s_0^u, s_1^u \in \mathsf{S}$. *Applying the functor* F *gives a projective presentation*

$$P_1^U \xrightarrow{\pi^u} P_0^U \longrightarrow U \longrightarrow 0 \tag{3}$$

in $\operatorname{mod} \mathsf{S}$, *and*

$$\mathsf{U} \textit{ is a presilting subcategory} \iff \textit{the class } \{\pi^u \mid u \in \mathsf{U}\} \textit{ has property } (S).$$

Proof. Clearly, F applied to the distinguished triangle (2) gives the projective presentation (3).

To get the bi-implication in the last line of the lemma, first note that for $u, u' \in \mathsf{U}$ we have

$$\mathsf{T}(u, \Sigma^{\geq 2}u') = 0 \tag{4}$$

since $u, u' \in \mathsf{S} * \Sigma\mathsf{S}$.

By Remark 3.1(ii), the map $\mathrm{Hom}_{\mathsf{mod}\,\mathsf{S}}(\pi, F(u'))$ is the same as

$$\mathsf{T}(s_0^u, u') \to \mathsf{T}(s_1^u, u'). \tag{5}$$

So the class $\{\pi^u \mid u \in \mathsf{U}\}$ has property (S) if and only if the morphism (5) is surjective for all $u, u' \in \mathsf{U}$. However, the distinguished triangle (2) induces an exact sequence

$$\mathsf{T}(s_0^u, u') \longrightarrow \mathsf{T}(s_1^u, u') \longrightarrow \mathsf{T}(\Sigma^{-1}u, u') \longrightarrow \mathsf{T}(\Sigma^{-1}s_0^u, u'),$$

where the last module is 0 since $u' \in \mathsf{S} * \Sigma\mathsf{S}$. So (5) is surjective if and only if $\mathsf{T}(\Sigma^{-1}u, u') \cong \mathsf{T}(u, \Sigma u') = 0$. This happens for all $u, u' \in \mathsf{U}$ if and only if U is presilting, because of (4). □

Theorem 3.3. *The functor $F : \mathsf{T} \to \mathsf{Mod}\,\mathsf{S}$ induces a surjection*

$$\Phi : \mathsf{U} \mapsto (F(\mathsf{U}), \mathsf{S} \cap \Sigma^{-1}\mathsf{U})$$

from the first of the following sets to the second:

(i) *Presilting subcategories of T which are contained in $\mathsf{S} * \Sigma\mathsf{S}$.*

(ii) *τ-rigid pairs of $\mathsf{mod}\,\mathsf{S}$.*

It restricts to a surjection Ψ from the first of the following sets to the second:

(iii) *Silting subcategories of T which are contained in $\mathsf{S} * \Sigma\mathsf{S}$.*

(iv) *Support τ-tilting pairs of $\mathsf{mod}\,\mathsf{S}$.*

Proof. We need to prove

(a) The map Φ has values in τ-rigid pairs of $\mathsf{mod}\,\mathsf{S}$.

(b) The map Φ is surjective.

(c) The map Ψ has values in support τ-tilting pairs of $\mathsf{mod}\,\mathsf{S}$.

(d) The map Ψ is surjective.

(a) Let U be a presilting subcategory of T which is contained in $\mathsf{S} * \Sigma\mathsf{S}$. For each $u \in \mathsf{U}$, there is a distinguished triangle $s_1 \to s_0 \to u \to \Sigma s_1$ with $s_0, s_1 \in \mathsf{S}$. Lemma 3.2 says that F sends the set of these triangles to a set of projective presentations (3) which has property (S), because U is presilting. It remains to show that for $u \in \mathsf{U}$ and $u' \in \mathsf{S} \cap \Sigma^{-1}\mathsf{U}$ we have $F(u)(u') = 0$. This is again true because $F(u)(u') = \mathsf{T}(u', u)$ and U is presilting.

(b) Let (M, E) be a τ-rigid pair of $\operatorname{mod}\mathsf{S}$. For each $m \in \mathsf{M}$ take a projective presentation

$$P_1 \xrightarrow{\pi^m} P_0 \longrightarrow m \longrightarrow 0 \tag{6}$$

such that the class $\{\pi^m \mid m \in \mathsf{M}\}$ has property (S). By Remark 3.1(ii) there is a unique morphism $f_m : s_1 \to s_0$ in S such that $F(f_m) = \pi^m$. Moreover, $F(\operatorname{cone}(f_m)) \cong m$. Since (6) has property (S), it follows from Lemma 3.2 that the category

$$\mathsf{U}_1 := \{\operatorname{cone}(f_m) \mid m \in \mathsf{M}\}$$

is a presilting subcategory, and the inclusion $\mathsf{U}_1 \subseteq \mathsf{S} * \Sigma\mathsf{S}$ is clear. Let U be the additive hull of U_1 and $\Sigma\mathsf{E}$ in $\mathsf{S}*\Sigma\mathsf{S}$. Now we show that U is a presilting subcategory of T. Let $e \in \mathsf{E}$. Clearly we have $\mathsf{T}(\operatorname{cone}(f_m) \oplus \Sigma e, \Sigma^2 e) = 0$. Applying $\mathsf{T}(e, -)$ to a triangle $s_1 \xrightarrow{f_m} s_0 \to \operatorname{cone}(f_m) \to \Sigma s_1$, we have an exact sequence

$$\mathsf{T}(e, s_1) \xrightarrow{f_m} \mathsf{T}(e, s_0) \longrightarrow \mathsf{T}(e, \operatorname{cone}(f_m)) \longrightarrow 0,$$

which is isomorphic to $P_1(e) \xrightarrow{\pi^m} P_0(e) \to m(e) \to 0$ by Remark 3.1(ii). The condition $\mathsf{M}(\mathsf{E}) = 0$ implies that $\mathsf{T}(e, \operatorname{cone}(f_m)) = 0$. Thus the assertion follows. It is clear that $\Phi(\mathsf{U}) = (\mathsf{M}, \mathsf{E})$.

(c) Let U be a silting subcategory of T which is contained in $\mathsf{S} * \Sigma\mathsf{S}$.

Let $s \in \mathsf{S}$ be an object of $\operatorname{Ker} F(\mathsf{U})$, i.e., $\mathsf{T}(s, u) = 0$ for each $u \in \mathsf{U}$. This implies that $\mathsf{U} \oplus \operatorname{add}(\Sigma s)$ is also a silting subcategory of T in $\mathsf{S} * \Sigma\mathsf{S}$. It follows from [Aihara and Iyama 2012, Theorem 2.18] that Σs belongs to U, whence s belongs to $\Sigma^{-1}\mathsf{U}$ and hence to $\mathsf{S} \cap \Sigma^{-1}\mathsf{U}$. This shows the inclusion $\operatorname{Ker} F(\mathsf{U}) \subseteq \mathsf{S} \cap \Sigma^{-1}\mathsf{U}$. The reverse inclusion was shown in (a), so $\operatorname{Ker} F(\mathsf{U}) = \mathsf{S} \cap \Sigma^{-1}\mathsf{U}$.

By Corollary 2.4, we have $\mathsf{S} \subseteq (\Sigma^{-1}\mathsf{U}) * \mathsf{U}$. In particular, for $s \in \mathsf{S}$, there is a distinguished triangle

$$s \longrightarrow u^0 \longrightarrow u^1 \longrightarrow \Sigma s. \tag{7}$$

Applying F, we obtain an exact sequence

$$F(s) \xrightarrow{f} F(u^0) \longrightarrow F(u^1) \longrightarrow 0. \tag{8}$$

For each $u \in \mathsf{U}$, we have the commutative diagram

$$\begin{tikzcd}
\mathsf{T}(u^0, u) \arrow[r] \arrow[d] & \mathsf{T}(s, u) \arrow[r] \arrow[d] & \mathsf{T}(u^1, \Sigma u) = 0 \\
\operatorname{Hom}_{\operatorname{mod}\mathsf{S}}(F(u^0), F(u)) \arrow[r, "f^*"'] & \operatorname{Hom}_{\operatorname{mod}\mathsf{S}}(F(s), F(u)) &
\end{tikzcd}$$

The right vertical map is induced from the Yoneda embedding, so it is bijective. It follows that f^* is surjective, that is, f is a left $F(\mathsf{U})$-approximation. Altogether, we have shown that $\Phi(\mathsf{U})$ is a support τ-tilting pair of $\operatorname{mod}\mathsf{S}$.

(d) Let (M, E) be a support τ-tilting pair of $\operatorname{mod}\mathsf{S}$, and let U be the preimage of (M, E) under the map Φ constructed in (b).

By definition, for each $s \in \mathsf{S}$ there is an exact sequence $F(s) \xrightarrow{f} F(u_s^0) \to F(u_s^1) \to 0$ such that $u_s^0, u_s^1 \in \mathsf{U}$ and f is a left $F(\mathsf{U})$-approximation. By Yoneda's lemma, there is a unique morphism $\alpha : s \to u_s^0$ such that $F(\alpha) = f$. Form the distinguished triangle

$$s \xrightarrow{\alpha} u_s^0 \longrightarrow t_s \longrightarrow \Sigma s. \tag{9}$$

Let $\widetilde{\mathsf{U}}$ be the additive closure of U and $\{t_s \mid s \in \mathsf{U}\}$. We claim that $\widetilde{\mathsf{U}}$ is a silting subcategory of T contained in $\mathsf{S} * \Sigma\mathsf{S}$ such that $\Phi(\widetilde{\mathsf{U}}) = (\mathsf{M}, \mathsf{E})$.

First, $t_s \in u_s^0 * \Sigma s \subseteq \mathsf{S} * \Sigma\mathsf{S}$. Therefore, $\widetilde{\mathsf{U}} \subseteq \mathsf{S} * \Sigma\mathsf{S}$.

Second, by applying F to the triangle (9), we see that $F(t_s)$ and $F(u_s^1)$ are isomorphic in $\operatorname{mod}\mathsf{S}$. For $u \in \mathsf{U}$, consider the following commutative diagram.

$$\begin{tikzcd}
\mathsf{T}(u_s^0, u) \arrow[r, "\alpha^*"] \arrow[d, "F(-)"'] & \mathsf{T}(s, u) \arrow[r] \arrow[d, "\cong"] & \mathsf{T}(t_s, \Sigma u) \arrow[r] & \mathsf{T}(u_s^0, \Sigma u) = 0 \\
\operatorname{Hom}_{\operatorname{mod}\mathsf{S}}(F(u_s^0), F(u)) \arrow[r, "f^*"'] & \operatorname{Hom}_{\operatorname{mod}\mathsf{S}}(F(s), F(u)) & &
\end{tikzcd}$$

By Remark 3.1(iii), the map $F(-)$ is surjective. Because f is a left $F(\mathsf{U})$-approximation, f^* is also surjective. So α^* is surjective too, implying that $\mathsf{T}(t_s, \Sigma u) = 0$. On the other hand, applying $\mathsf{T}(u, -)$ to the triangle (9), we obtain an exact sequence

$$\mathsf{T}(u, \Sigma u_s^0) \longrightarrow \mathsf{T}(u, \Sigma t_s) \longrightarrow \mathsf{T}(u, \Sigma^2 s).$$

The two outer terms are trivial, hence so is the middle term. Moreover, if $s' \in \mathsf{S}$, then applying $\mathsf{T}(t_{s'}, -)$ to the triangle (9) gives an exact sequence

$$\mathsf{T}(t_{s'}, \Sigma u_s^0) \longrightarrow \mathsf{T}(t_{s'}, \Sigma t_s) \longrightarrow \mathsf{T}(t_{s'}, \Sigma^2 s).$$

The two outer terms are trivial, hence so is the middle term. It follows that $\widetilde{\mathsf{U}}$ is presilting. It is then silting because it generates S.

Thirdly, $F(\widetilde{\mathsf{U}}) = F(\mathsf{U})$ because $F(t_s) \cong F(u_s^1)$.

Finally, $\mathsf{S} \cap \Sigma^{-1}\widetilde{\mathsf{U}} = \mathsf{E}$. This is because $\mathsf{S} \cap \Sigma^{-1}\widetilde{\mathsf{U}} \supseteq \mathsf{S} \cap \Sigma^{-1}\mathsf{U} = \mathsf{E}$ and $\mathsf{S} \cap \Sigma^{-1}\widetilde{\mathsf{U}} \subseteq \operatorname{Ker} F(\mathsf{U}) = \mathsf{E}$. □

Theorem 3.4. *Assume that each object of* $\mathsf{S} * \Sigma\mathsf{S}$ *can be written as the direct sum of indecomposable objects which are unique up to isomorphism. Then the maps* Φ *and* Ψ *defined in* Theorem 3.3 *are bijective.*

Proof. It suffices to show the injectivity of Φ.

By Remark 3.1(iii), when we apply the functor $F : \mathsf{S} * \Sigma\mathsf{S} \to \operatorname{mod}\mathsf{S}$, we are in effect forgetting the indecomposable direct summands which are in $\Sigma\mathsf{S}$. So if $F(u) \cong F(u')$ for $u, u' \in \mathsf{S} * \Sigma\mathsf{S}$, then there is an isomorphism $u \oplus \Sigma s \cong u' \oplus \Sigma s'$ for some $s, s' \in \mathsf{S}$. By the assumption in the theorem, if we assume that u and u' do not have direct summands in $\Sigma\mathsf{S}$, then $u \cong u'$.

Now let U and U' be two presilting subcategories of T contained in $\mathsf{S} * \Sigma\mathsf{S}$ such that $\Phi(\mathsf{U}) = \Phi(\mathsf{U}')$. Let U_1 and U'_1 be respectively the full subcategories of U and U' consisting of objects without direct summands in $\Sigma\mathsf{S}$. Then $\mathsf{U} = \mathsf{U}_1 \oplus (\mathsf{U} \cap \Sigma\mathsf{S})$ and $\mathsf{U}' = \mathsf{U}'_1 \oplus (\mathsf{U}' \cap \Sigma\mathsf{S})$. Since $\Phi(\mathsf{U}) = \Phi(\mathsf{U}')$, it follows that $F(\mathsf{U}_1) = F(\mathsf{U}'_1)$ and $\mathsf{U} \cap \Sigma\mathsf{S} = \mathsf{U}' \cap \Sigma\mathsf{S}$. The first equality, by the above argument, implies that $\mathsf{U}_1 = \mathsf{U}'_1$. Therefore $\mathsf{U} = \mathsf{U}'$, which shows the injectivity of Φ. □

4. The Hom-finite Krull–Schmidt silting object case

In this section, $\Bbbk$ is a commutative ring, T is a triangulated category which is essentially small, Krull–Schmidt, $\Bbbk$-linear and Hom-finite, and $s \in \mathsf{T}$ is a basic silting object.

We write $E = \mathsf{T}(s, s)$ for the endomorphism ring and $\mathsf{S} = \operatorname{add}(s)$ for the associated silting subcategory.

Remark 4.1. (i) We write $\operatorname{Mod} E$ for the abelian category of right E-modules, $\operatorname{mod} E$ for the full subcategory of finitely presented modules, and $\operatorname{prj} E$ for the full subcategory of finitely generated projective modules.

(ii) Since s is an additive generator of S, there is an equivalence

$$G : \operatorname{Mod}\mathsf{S} \xrightarrow{\sim} \operatorname{Mod} E, \quad M \mapsto M(s),$$

which restricts to an equivalence

$$\operatorname{mod}\mathsf{S} \xrightarrow{\sim} \operatorname{mod} E, \quad M \mapsto M(s).$$

This permits us to move freely between the “E-picture” and the “S-picture” which was used in the previous section.

(iii) The restricted Yoneda functor F from the S-picture corresponds to the functor

$$\mathsf{T} \to \operatorname{Mod} E, \quad t \mapsto \mathsf{T}(s, t)$$

in the E-picture.

(iv) By [Auslander 1974, Proposition 2.2(e)] the functor $t \mapsto \mathsf{T}(s, t)$ from (iii) restricts to an equivalence

$$Y : \mathsf{S} \xrightarrow{\sim} \operatorname{prj} E.$$

Since $\mathsf{S} = \mathsf{add}(s)$ is closed under direct sums and summands, it is Krull–Schmidt, and it follows that so is $\mathsf{prj}\, E$.

(v) By Remark 3.1(iii) the functor $t \mapsto \mathsf{T}(s, t)$ from (iii) induces an equivalence

$$(\mathsf{S} * \Sigma\mathsf{S})/[\Sigma\mathsf{S}] \xrightarrow{\sim} \mathsf{mod}\, E. \tag{10}$$

Since $\mathsf{S} * \Sigma\mathsf{S}$ is obviously closed under direct sums, and under direct summands by Lemma 2.1, it is Krull–Schmidt. Hence so is $(\mathsf{S} * \Sigma\mathsf{S})/[\Sigma\mathsf{S}]$ and it follows that so is $\mathsf{mod}\, E$.

(vi) The additive category $\mathsf{prj}\, E$ is Krull–Schmidt by part (iv) and has additive generator E_E. The same is hence true for $(\mathsf{prj}\, E)/[\mathsf{add}\, eE]$ for each idempotent $e \in E$. It is not hard to check that the endomorphism ring of E_E in $(\mathsf{prj}\, E)/[\mathsf{add}\, eE]$ is E/EeE, so there is an equivalence of categories

$$(\mathsf{prj}\, E)/[\mathsf{add}\, eE] \xrightarrow{\sim} \mathsf{prj}(E/EeE).$$

In particular, $\mathsf{prj}(E/EeE)$ is Krull–Schmidt.

The following result is essentially already in [Aihara 2013, Proposition 2.16], [Fei and Derksen 2011, start of Section 5], and [Wei 2013, Proposition 6.1], all of which give triangulated versions of Bongartz's classic proof:

Lemma 4.2 (Bongartz completion). *Let $u \in \mathsf{S} * \Sigma\mathsf{S}$ be a presilting object. Then there exists an object $u' \in \mathsf{S} * \Sigma\mathsf{S}$ such that $u \oplus u'$ is a silting object.*

Proof. This has essentially the same proof as classic Bongartz completion: Since T is Hom-finite over the commutative ring $\Bbbk$, there is a right $\mathsf{add}(u)$-approximation $u_0 \to \Sigma s$. This gives a distinguished triangle $s \to u' \to u_0 \to \Sigma s$, and it is straightforward to check that u' has the desired properties. □

The following result is essentially already contained in [Fei and Derksen 2011, Theorem 5.4]:

Proposition 4.3. *Let $u \in \mathsf{S} * \Sigma\mathsf{S}$ be a basic presilting object. Then*

$$u \text{ is a silting object} \iff \#_{\mathsf{T}}(u) = \#_{\mathsf{T}}(s).$$

Proof. The implication $\Longrightarrow$ is immediate from [Aihara and Iyama 2012, Theorem 2.27], and $\Longleftarrow$ is a straightforward consequence of that theorem and Lemma 4.2. □

As a consequence, we have:

Corollary 4.4. *Let U be a presilting subcategory of T contained in $\mathsf{S} * \Sigma\mathsf{S}$. Then there exists $u \in \mathsf{U}$ such that $\mathsf{U} = \mathsf{add}(u)$.*

Proof. Suppose on the contrary that $\mathsf{U} \neq \mathsf{add}(u)$ for each $u \in \mathsf{U}$. Then U contains infinitely many isomorphism classes of indecomposable objects. In particular, there is a basic presilting object $u \in \mathsf{U}$ such that $\#_{\mathsf{T}}(u) = \#_{\mathsf{T}}(s) + 1$. By Lemma 4.2, there is an object $u' \in \mathsf{T}$ such that $u \oplus u'$ is a basic silting object of T. Therefore, $\#_{\mathsf{T}}(s) + 1 = \#_{\mathsf{T}}(u) \leq \#_{\mathsf{T}}(u \oplus u') = \#_{\mathsf{T}}(s)$, a contradiction. Here the last equality follows from Proposition 4.3. □

Theorem 3.3 in the current setting combined with Corollary 4.4 immediately yields the following result. For an object u of $\mathsf{S} * \Sigma\mathsf{S}$, let Σu_1 be its maximal direct summand in $\Sigma\mathsf{S}$.

Theorem 4.5. *The assignment*

$$u \mapsto (\mathsf{add}(F(u)), \mathsf{add}(u_1))$$

defines a bijection from the first of the following sets to the second:

(i) *Basic presilting objects of* T *which are in* $\mathsf{S} * \Sigma\mathsf{S}$, *modulo isomorphism.*

(ii) τ*-rigid pairs of* $\mathsf{mod}\,\mathsf{S}$.

It restricts to a bijection from the first of the following sets to the second:

(iii) *Basic silting objects of* T *which are in* $\mathsf{S} * \Sigma\mathsf{S}$, *modulo isomorphism.*

(iv) *Support* τ*-tilting pairs of* $\mathsf{mod}\,\mathsf{S}$.

As a consequence, if (M, E) is a τ-rigid pair of $\mathsf{mod}\,\mathsf{S}$, then there is an S-module M such that $\mathsf{M} = \mathsf{add}(M)$.

Next we move to the E-picture. Recall from Remark 4.1(ii) and (iv) that there are equivalences $G : \mathsf{Mod}\,\mathsf{S} \xrightarrow{\sim} \mathsf{Mod}\,E$ and $Y : \mathsf{S} \xrightarrow{\sim} \mathsf{prj}\,E$.

Theorem 4.6. *An E-module U is a support τ-tilting module if and only if the pair*

$$\big(G^{-1}(\mathsf{add}(U)), Y^{-1}(\mathsf{add}(eE))\big)$$

is a support τ-tilting pair of $\mathsf{mod}\,\mathsf{S}$ *for some idempotent $e \in E$.*

Consequently, the functor $\mathsf{T}(s, -) : \mathsf{T} \to \mathsf{Mod}\,E$ *induces a bijection from the first of the following sets to the second*:

(i) *Basic silting objects of* T *which are in* $\mathsf{S} * \Sigma\mathsf{S}$, *modulo isomorphism.*

(ii) *Basic support* τ*-tilting modules of* $\mathsf{mod}\,E$, *modulo isomorphism.*

Proof. We only prove the first assertion. The proof is divided into three parts. Let $u_p \in \mathsf{S} * \Sigma\mathsf{S}$ be such that u_p has no direct summand in $\Sigma\mathsf{S}$ and $F(u_p) = G^{-1}(U)$.

(a) It is clear that U is a τ-rigid E-module if and only if $G^{-1}(\mathsf{add}(U))$ is a τ-rigid subcategory of $\mathsf{mod}\,\mathsf{S}$.

(b) Let e be an idempotent of E and let $u_1 \in \mathsf{S}$ be such that $Y(u_1) = eE$. We have

$$\begin{aligned} Ue &\cong \mathrm{Hom}_E(eE, U) \\ &= \mathrm{Hom}_{\mathsf{Mod}\,\mathsf{S}}(\mathsf{S}(-, u_1), F(u_p)) \\ &\cong F(u_p)(u_1) && \text{Remark 3.1(ii).} \end{aligned}$$

Therefore $Ue = 0$ if and only if $M(u') = 0$ for each $M \in \mathsf{add}(F(u_p)) = G^{-1}(\mathsf{add}(U))$ and each $u' \in \mathsf{add}(u_1) = Y^{-1}(\mathsf{add}(eE))$.

(c) Suppose that $(G^{-1}(\mathsf{add}(U)), Y^{-1}(\mathsf{add}(eE)))$ is a τ-rigid pair. Let u be the corresponding basic presilting object of T as in Theorem 4.5. More precisely, let $u = u_p \oplus \Sigma u_1$, where u_p and u_1 are as above. Then

$$\begin{aligned} &\big(G^{-1}(\mathsf{add}(U)), Y^{-1}(\mathsf{add}(eE))\big) \text{ is a support } \tau\text{-tilting pair} \\ &\iff u \text{ is a silting object} && \text{Theorem 4.5} \\ &\iff \#_{\mathsf{T}}(u) = \#_{\mathsf{T}}(s) && \text{Proposition 4.3} \\ &\iff \#_{\mathsf{S}*\Sigma\mathsf{S}}(u) = \#_{\mathsf{S}}(s) \\ &\iff \#_{\mathsf{S}*\Sigma\mathsf{S}}(u) = \#_{\mathsf{prj}\,E}(E) && \text{Remark 4.1(iv)} \\ &\iff \#_{(\mathsf{S}*\Sigma\mathsf{S})/[\Sigma\mathsf{S}]}(u) + \#_{\mathsf{S}*\Sigma\mathsf{S}}(\Sigma u_1) = \#_{\mathsf{prj}\,E}(E) \\ &\iff \#_{\mathsf{mod}\,E}(U) + \#_{\mathsf{prj}\,E}(eE) = \#_{\mathsf{prj}\,E}(E) && \text{Remark 4.1(iv), (v)} \\ &\iff \#_{\mathsf{mod}\,E}(U) = \#_{\mathsf{prj}\,E}(E) - \#_{\mathsf{prj}\,E}(eE) \\ &\iff \#_{\mathsf{mod}\,E}(U) = \#_{(\mathsf{prj}\,E)/[\mathsf{add}\,eE]}(E) \\ &\iff \#_{\mathsf{mod}\,E}(U) = \#_{\mathsf{prj}(E/EeE)}(E/EeE) && \text{Remark 4.1(vi)} \\ &\iff U \text{ is a support } \tau\text{-tilting module.} && \square \end{aligned}$$

5. Support τ-tilting pairs and torsion classes

In this section $\Bbbk$ is a commutative noetherian local ring and C is an essentially small, Krull–Schmidt, $\Bbbk$-linear and Hom-finite category.

The main result in this section is the following:

Theorem 5.1. *There is a bijection* $\mathsf{M} \mapsto \mathsf{Fac}\,\mathsf{M}$ *from the first of the following sets to the second*:

(i) *Support τ-tilting pairs* (M, E) *of* $\mathsf{mod}\,\mathsf{C}$.

(ii) *Finitely generated torsion classes* T *of* $\mathsf{Mod}\,\mathsf{C}$ *such that each finitely generated projective* C*-module has a left* $\mathsf{P}(\mathsf{T})$*-approximation.*

We start with the following observation:

Lemma 5.2. *Let* M *be a subcategory of* $\operatorname{mod}\mathsf{C}$. *The following conditions are equivalent*:

(i) M *is* τ*-rigid.*

(ii) $\operatorname{Ext}^1_{\operatorname{Mod}\mathsf{C}}(\mathsf{M}, \operatorname{Fac}\mathsf{M}) = 0$.

(iii) *Each* $m \in \mathsf{M}$ *has a minimal projective presentation*

$$0 \longrightarrow \Omega^2 m \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow m \longrightarrow 0$$

such that for each $m' \in \mathsf{M}$ *and each morphism* $f : P_1 \to m'$, *there exist morphisms* $a : P_0 \to m'$ *and* $b : P_1 \to \Omega^2 m$ *such that* $f = ad_1 + fd_2b$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 m \arrow[r, shift left, "d_2"] & P_1 \arrow[l, shift left, "b"] \arrow[r, "d_1"] \arrow[d, "f"] & P_0 \arrow[r] \arrow[dl, "a"] & m \arrow[r] & 0 \\
& & m' & & &
\end{tikzcd}$$

Proof. (i) $\Longrightarrow$ (ii): For each $m \in \mathsf{M}$, there exists a projective presentation $P_1 \xrightarrow{\pi} P_0 \to m \to 0$ such that $\operatorname{Hom}_{\operatorname{Mod}\mathsf{C}}(\pi, m')$ is surjective for each $m' \in \mathsf{M}$. Let $n \in \operatorname{Fac}\mathsf{M}$ be given and pick an epimorphism $p : m' \to n$ with $m' \in \mathsf{M}$. To show $\operatorname{Ext}^1_{\operatorname{Mod}\mathsf{C}}(m, n) = 0$, it is enough to show that each $f \in \operatorname{Hom}_{\operatorname{Mod}\mathsf{C}}(P_1, n)$ factors through π. Since p is an epimorphism and P_1 is projective, there exists $g : P_1 \to m'$ such that $f = pg$. Then there exists $h : P_0 \to m'$ such that $g = h\pi$, by the property of π.

$$\begin{tikzcd}
P_1 \arrow[r, "\pi"] \arrow[d, "g"'] \arrow[dr, "f"] & P_0 \arrow[r] \arrow[dl, dashed, "h"'] & m \arrow[r] & 0 \\
m' \arrow[r, "p"'] & n & &
\end{tikzcd}$$

Thus $f = ph\pi$, and we have the assertion.

(ii) $\Longrightarrow$ (iii): For each $m \in \mathsf{M}$, take a minimal projective presentation $0 \to \Omega^2 m \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to m \to 0$. Let $m' \in \mathsf{M}$ and $f : P_1 \to m'$ be given, set $n := \operatorname{Im}(fd_2)$ and let $0 \to n \xrightarrow{\iota} m' \xrightarrow{\pi} n' \to 0$ be an exact sequence. Then $\pi f : P_1 \to n'$ factors through $P_1 \to \operatorname{Im} d_1$. Since $n' \in \operatorname{Fac}\mathsf{M}$ and $\operatorname{Ext}^1_{\operatorname{Mod}\mathsf{C}}(m, \operatorname{Fac}\mathsf{M}) = 0$, there exists $g : P_0 \to n'$ such that $gd_1 = \pi f$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 m \arrow[r, "d_2"] \arrow[d, "f'"'] & P_1 \arrow[r] \arrow[d, "f"] & \operatorname{Im} d_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & n \arrow[r, "\iota"'] & m' \arrow[r, "\pi"'] & n' \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & \operatorname{Im} d_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[dl, "g"] & m \arrow[r] & 0 \\
& n' & & &
\end{tikzcd}$$

Since π is an epimorphism and P_0 is projective, there exists $a : P_0 \to m'$ such that $g = \pi a$. Since $\pi(f - ad_1) = 0$, there exists $h : P_1 \to n$ such that $f = ad_1 + \iota h$. Since f' is surjective (by definition of n) and P_1 is projective, there exists $b : P_1 \to \Omega^2 m$ such that $h = f'b$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2 m \arrow[r, shift left, "d_2"] \arrow[d, "f'"'] & P_1 \arrow[l, shift left, "b"] \arrow[r, "d_1"] \arrow[d, "f"] \arrow[dl, "h"] & P_0 \arrow[r] \arrow[d, "g"] \arrow[dl, "a"] & m \arrow[r] & 0 \\
0 \arrow[r] & n \arrow[r, "\iota"'] & m' \arrow[r, "\pi"'] & n' \arrow[r] & 0 &
\end{tikzcd}$$

Then we have $f = ad_1 + \iota f'b = ad_1 + fd_2 b$.

(iii) $\Longrightarrow$ (i): For each $m \in \mathsf{M}$, take a minimal projective presentation $0 \to \Omega^2 m \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to m \to 0$ satisfying the assumption in (iii). We need to show that each $f : P_1 \to m'$ with $m' \in \mathsf{M}$ factors through d_1. By our assumption, there exist $a : P_0 \to m'$ and $b : P_1 \to \Omega^2 m$ such that $f = ad_1 + fd_2 b$. Applying our assumption to $fd_2 b : P_1 \to m'$, there exist $a' : P_0 \to m'$ and $b' : P_1 \to \Omega^2 m$ such that $fd_2 b = a'd_1 + fd_2 bd_2 b'$. Thus $f = (a + a')d_1 + fd_2 bd_2 b'$. Repeating a similar argument gives

$$\mathrm{Hom}_{\mathsf{Mod}\,\mathsf{C}}(P_1, m) = \mathrm{Hom}_{\mathsf{Mod}\,\mathsf{C}}(P_0, m)d_1 + \mathrm{Hom}_{\mathsf{Mod}\,\mathsf{C}}(P_1, m)(\mathrm{rad}\,\mathrm{End}_{\mathsf{Mod}\,\mathsf{C}}(P_1))^n$$

for each $n \geq 1$, since $d_2 \in \mathrm{rad}\,\mathrm{Hom}_{\mathsf{Mod}\,\mathsf{C}}(\Omega^2 m, P_1)$. Since C is Hom-finite over $\Bbbk$, we have $(\mathrm{rad}\,\mathrm{End}_{\mathsf{Mod}\,\mathsf{C}}(P_1))^\ell \subset \mathrm{End}_{\mathsf{Mod}\,\mathsf{C}}(P_1)(\mathrm{rad}\,\Bbbk)$ for sufficiently large ℓ. Thus we have

$$\mathrm{Hom}_{\mathsf{Mod}\,\mathsf{C}}(P_1, m) = \bigcap_{n \geq 0} \bigl(\mathrm{Hom}_{\mathsf{Mod}\,\mathsf{C}}(P_0, m)d_1 + \mathrm{Hom}_{\mathsf{Mod}\,\mathsf{C}}(P_1, m)(\mathrm{rad}\,\Bbbk)^n\bigr).$$

The right-hand side is equal to $\mathrm{Hom}_{\mathsf{Mod}\,\mathsf{C}}(P_0, m)d_1$ itself by Krull's intersection theorem [Matsumura 1989]. $\square$

Proposition 5.3. *Let* (M, E) *be a support* τ*-tilting pair of* $\mathsf{mod}\,\mathsf{C}$*. Then* $\mathsf{Fac}\,\mathsf{M}$ *is a finitely generated torsion class with* $\mathsf{P}(\mathsf{Fac}\,\mathsf{M}) = \mathsf{M}$.

Proof. (i) We show that $\mathsf{Fac}\,\mathsf{M}$ is a torsion class. Clearly $\mathsf{Fac}\,\mathsf{M}$ is closed under factor modules. We show that $\mathsf{Fac}\,\mathsf{M}$ is closed under extensions. Let $0 \to x \to y \xrightarrow{f} z \to 0$ be an exact sequence in $\mathsf{Mod}\,\mathsf{C}$ such that $x, z \in \mathsf{Fac}\,\mathsf{M}$. Take an epimorphism $p : m \to z$ with $m \in \mathsf{M}$. Since $\mathrm{Ext}^1_{\mathsf{Mod}\,\mathsf{C}}(m, x) = 0$ by Lemma 5.2(ii), we have that p factors through f. Thus we have an epimorphism $x \oplus m \to y$, and $y \in \mathsf{Fac}\,\mathsf{M}$ holds. Hence $\mathsf{Fac}\,\mathsf{M}$ is a torsion class.

(ii) Since $\mathrm{Ext}^1_{\mathsf{Mod}\,\mathsf{C}}(\mathsf{M}, \mathsf{Fac}\,\mathsf{M}) = 0$ by Lemma 5.2(ii), each object in M is Ext-projective in $\mathsf{Fac}\,\mathsf{M}$. It remains to show that if n is an Ext-projective object in $\mathsf{Fac}\,\mathsf{M}$, then $n \in \mathsf{M}$. Let $P_1 \xrightarrow{f} P_0 \xrightarrow{e} n \to 0$ be a projective presentation. Since M is support τ-tilting, there exist exact sequences $P_i \xrightarrow{g_i} m_i \xrightarrow{h_i} m_i' \to 0$ with $m_i, m_i' \in \mathsf{M}$ and a left M-approximation g_i for $i = 0, 1$.

Let $\overline{\mathsf{C}} := \mathsf{C}/\operatorname{ann}\mathsf{M}$ for the annihilator ideal $\operatorname{ann}\mathsf{M}$ of M and $\overline{P_i} := P_i \otimes_{\mathsf{C}} \overline{\mathsf{C}}$. Then we have induced exact sequences $0 \to \overline{P_i} \xrightarrow{g_i} m_i \xrightarrow{h_i} m_i' \to 0$ for $i = 0, 1$ and $\overline{P_1} \xrightarrow{f} \overline{P_0} \xrightarrow{e} n \to 0$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{P_1} \arrow[r, "g_1"] \arrow[d, "f"] & m_1 \arrow[r, "h_1"] \arrow[d, "a"] & m_1' \arrow[r] \arrow[d, "b"] & 0 \\
0 \arrow[r] & \overline{P_0} \arrow[r, "g_0"'] & m_0 \arrow[r, "h_0"'] & m_0' \arrow[r] & 0
\end{tikzcd}$$

of exact sequences. Taking a mapping cone, we have an exact sequence

$$0 \longrightarrow \overline{P_1} \xrightarrow{\left[\begin{smallmatrix} g_1 \\ f \end{smallmatrix}\right]} m_1 \oplus \overline{P_0} \xrightarrow{\left[\begin{smallmatrix} h_1 & 0 \\ a & -g_0 \end{smallmatrix}\right]} m_1' \oplus m_0 \xrightarrow{[\, b - h_0 \,]} m_0' \longrightarrow 0.$$

Since $\mathrm{Ext}^1_{\mathsf{Mod}\,\mathsf{C}}(m_0', n) = 0$ by Lemma 5.2(ii), we have the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[r] & \overline{P_1} \arrow[r, "\left[\begin{smallmatrix} g_1 \\ f \end{smallmatrix}\right]"] \arrow[d, equal] & m_1 \oplus \overline{P_0} \arrow[r, "\left[\begin{smallmatrix} h_1 & 0 \\ a & -g_0 \end{smallmatrix}\right]"] \arrow[d, "{[0\ 1]}"] & m_1' \oplus m_0 \arrow[r, "{[b\ \ -h_0]}"] \arrow[d, dotted] & m_0' \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker} f \arrow[r] & \overline{P_1} \arrow[r, "f"'] & \overline{P_0} \arrow[r, "e"'] & n \arrow[rr] & & 0
\end{tikzcd}$$

Taking a mapping cone, we have an exact sequence

$$0 \longrightarrow \overline{P_1} \oplus \operatorname{Ker} f \longrightarrow m_1 \oplus \overline{P_0} \oplus \overline{P_1} \longrightarrow m_1' \oplus m_0 \oplus \overline{P_0} \longrightarrow m_0' \oplus n \longrightarrow 0.$$

Cancelling a direct summand of the form $\overline{P_1} \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} \overline{P_0} \oplus \overline{P_1} \xrightarrow{[1\ 0]} \overline{P_0}$, we have an exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow m_1 \xrightarrow{c} m_1' \oplus m_0 \xrightarrow{d} m_0' \oplus n \longrightarrow 0.$$

Since $\operatorname{Im} c \in \mathsf{Fac}\,\mathsf{M}$ and $m_0' \oplus n$ is Ext-projective in $\mathsf{Fac}\,\mathsf{M}$, the epimorphism d splits. Thus $n \in \mathsf{M}$ as desired. $\square$

Now we are ready to prove Theorem 5.1.

Let M be a support τ-tilting subcategory of $\operatorname{mod}\mathsf{C}$. By definition, each representable C-module has a left M-approximation. Since $\mathsf{P}(\operatorname{Fac}\mathsf{M}) = \mathsf{M}$ by Proposition 5.3, the map $\mathsf{M} \mapsto \operatorname{Fac}\mathsf{M}$ is well-defined from the set (i) to the set (ii), and it is injective.

We show that the map is surjective. For T in the set described in (ii), let $\mathsf{E} := \bigcap_{m\in\mathsf{T}} \operatorname{Ker} m$ and $\mathsf{M} := \mathsf{P}(\mathsf{T})$. We will show that (M, E) is a support τ-tilting pair of $\operatorname{mod}\mathsf{C}$. Since $\operatorname{Ext}^1_{\operatorname{Mod}\mathsf{C}}(\mathsf{M}, \mathsf{T}) = 0$ and $\operatorname{Fac}\mathsf{M} \subset \mathsf{T}$, it follows from Lemma 5.2 that M is τ-rigid. For $s \in \mathsf{C}$, take a left M-approximation $\mathsf{C}(-, s) \xrightarrow{f} m$.

It remains to show $\operatorname{Coker} f \in \mathsf{M}$. Since $\operatorname{Coker} f \in \mathsf{T}$, we only have to show $\operatorname{Ext}^1_{\operatorname{Mod}\mathsf{C}}(\operatorname{Coker} f, m') = 0$ for each $m' \in \mathsf{M}$. Let $f = \iota\pi$ for $\pi : \mathsf{C}(-, s) \to \operatorname{Im} f$ and $\iota : \operatorname{Im} f \to m$. Applying $\operatorname{Hom}_{\operatorname{Mod}\mathsf{C}}(-, m')$ to the exact sequence $0 \to \operatorname{Im} f \xrightarrow{\iota} m \to \operatorname{Coker} f \to 0$, we have an exact sequence

$$\operatorname{Hom}_{\operatorname{Mod}\mathsf{C}}(m, m') \xrightarrow{\iota^*} \operatorname{Hom}_{\operatorname{Mod}\mathsf{C}}(\operatorname{Im} f, m') \to \operatorname{Ext}^1_{\operatorname{Mod}\mathsf{C}}(\operatorname{Coker} f, m') \to \operatorname{Ext}^1_{\operatorname{Mod}\mathsf{C}}(m, m') = 0.$$

Let $g : \operatorname{Im} f \to m'$ be a morphism in $\operatorname{Mod}\mathsf{C}$. Since f is a left M-approximation, there exists $h : m \to m'$ such that $g\pi = hf$. Then $g = h\iota$. Thus $\iota^* : \operatorname{Hom}_{\operatorname{Mod}\mathsf{C}}(m, m') \to \operatorname{Hom}_{\operatorname{Mod}\mathsf{C}}(\operatorname{Im} f, m')$ is surjective, and we have $\operatorname{Ext}^1_{\operatorname{Mod}\mathsf{C}}(\operatorname{Coker} f, m') = 0$. Consequently we have $\operatorname{Coker} f \in \mathsf{P}(\mathsf{T}) = \mathsf{M}$. Thus the assertion follows. $\square$

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