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A p-adic Eisenstein measure for vector-weight automorphic forms

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We construct a p-adic Eisenstein measure with values in the space of vector-weight p-adic automorphic forms on certain unitary groups. This measure allows us to p-adically interpolate special values of certain vector-weight C^{∞} automorphic forms, including Eisenstein series, as their weights vary. This completes a key step toward the construction of certain p-adic L-functions.

We also explain how to extend our methods to the case of Siegel modular forms and how to recover Nicholas Katz's p-adic families of Eisenstein series for Hilbert modular forms.

1. Introduction

The significance of p-adic families of Eisenstein series as a tool in number theory, especially for the construction of p-adic L-functions, is well established. For example, p-adic families of Eisenstein series play a key role in constructions of p-adic L-functions completed in [Serre 1973; Katz 1978; Deligne and Ribet 1980]. In a completely different direction, p-adic families of Eisenstein series also play a role in homotopy theory [Hopkins 1995; 2002; Ando et al. 2010].

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Each of the constructions mentioned above concerns only automorphic forms of scalar weight. Automorphic forms on groups of rank 1 (for example, modular forms and Hilbert modular forms, which are the forms with which Katz, Deligne, Ribet, and Serre worked) can only have scalar weights. Automorphic forms on groups of higher rank, however, need not have scalar weights.

By a vector-weight automorphic form, we mean an automorphic form whose weight is an irreducible representation with highest weight $\lambda_n \geq \cdots \geq \lambda_1$ is not required to have $\lambda_i = \lambda_{i+1}$ for all i, i.e., an automorphic form whose weight is not required to be a one-dimensional representation. In order to complete a construction of p-adic L-functions for automorphic forms on unitary groups in full generality as in [Eischen et al. $\geq$ 2014], one needs a p-adic Eisenstein measure that takes values in the space of p-adic vector-weight automorphic forms. (By an *Eisenstein measure*, we mean a p-adic measure valued in a space of p-adic automorphic forms and whose values at locally constant functions are Eisenstein series.)

The main result of this paper is the construction in Section 5 of a p-adic measure that takes values in the space of automorphic forms on unitary groups of signature (n, n). In particular, Theorem 14 gives a p-adic Eisenstein measure with values in the space of vector-weight automorphic forms. As explained in Theorem 15, this measure, together with the results of Section 4, allows us to p-adically interpolate the values of certain vector-weight C^∞ (not necessarily holomorphic) automorphic forms, including Eisenstein series, as the (highest) weights of these automorphic forms vary. Note that this is the first ever construction of a p-adic Eisenstein measure taking values in the space of *vector*-weight automorphic forms on unitary groups.

We follow the approach of [Katz 1978, Chapters 4 and 5] more closely than we did in [Eischen 2013]. (There, we constructed a p-adic Eisenstein measure for scalar-weight automorphic forms on unitary groups of signature (n, n).) As a consequence, in Section 6, we easily recover Katz's Eisenstein measure from [1978, Chapters 4 and 5] as a special case of our results.

We also explain in Section 6 how to generalize the results of Section 5 to the case of Siegel modular forms, i.e., automorphic forms on symplectic groups. In that setting, in the case where $n = 1$, we are in exactly the situation in which Katz [1978] constructs a p-adic Eisenstein measure for Hilbert modular forms. As demonstrated in Section 6.1, the setup in the earlier sections of the paper makes the connection between our Eisenstein measure and the Eisenstein measure in [Katz 1978, Definition (4.2.5) and Equation (5.5.7)] almost transparent.

1.1. ***Applications and context.*** The main anticipated application of this paper is to the construction of p-adic L-functions for unitary groups, most immediately and crucially to [Eischen et al. $\geq$ 2014]. In particular, the L-functions in that paper are obtained through the "doubling method" (an approach described in [Gelbart et al.

1987, Part A; Cogdell 2006, Section 2]), which expresses values of L-functions in terms of values of Eisenstein series and values of cusp forms. The p-adic Eisenstein measure in [Eischen 2013, Section 4] suffices in the case of scalar weights, but if one does not restrict to scalar weights, one needs the results of the present paper.

The behavior of certain L-functions (for example, for unitary groups) is strongly tied to the behavior of certain Eisenstein series. For instance, Shimura [2000, Introduction] uses the algebraicity (up to a well-determined period) of values of Eisenstein series at CM points to prove the algebraicity (up to a well-determined period) of certain values of corresponding L-functions (normalized by a period). Analogously, Katz [1978, Introduction] uses the p-adic interpolation of values of certain Eisenstein series (normalized by a period) at CM points to p-adically interpolate certain values of L-functions (normalized by a period). Similarly, the p-adic families of Eisenstein series in the present paper play a key role in determining the behavior of the L-functions in [Eischen et al. $\geq$ 2014].

1.2. ***Overview and structure of the paper.*** In Section 2, we introduce the conventions with which we will work, as well as standard background results necessary for this paper. The conventions and background are similar to those in [Eischen 2012; 2013, Section 2]. The background is quite technical; we have summarized just what is needed for this paper. For the reader seeking further details, we recommend [Shimura 1997; 2000] for the theory of C^∞ automorphic forms and Eisenstein series on unitary groups, [Lan 2012; 2013] for the algebraic geometric background and a discussion of algebraically defined q-expansions, and [Hida 2004; 2005] for the theory of p-adic automorphic forms.

In Section 3, which relies in part on the results of [Eischen 2013, Section 2], we define certain scalar-weight Eisenstein series and automorphic forms on unitary groups of signature (n, n). This set includes the Eisenstein series defined in [Eischen 2013, Section 2] but also includes other automorphic forms. We need this larger space of automorphic forms in order to construct a p-adic measure with values in the space of vector-weight automorphic forms in Section 5, whereas in [Eischen 2013] we only were concerned with p-adic families of scalar-weight automorphic forms. Like in [Eischen 2013], we work adelically. The formulation of the main result of the section (Theorem 2) is closer to that of [Katz 1978, Theorem (3.2.3)], though, so that the reader can see parallels with the analogous construction in [Katz 1978, Section 3], which is useful in Section 6.1 when we compare our Eisenstein measure to the measure obtained in [Katz 1978, Definition (4.2.5) and Equation (5.5.7)].

Section 4 discusses differential operators that are necessary for comparing the values of certain C^∞ automorphic forms and certain p-adic automorphic forms. These differential operators are closely related to the differential operators discussed in [Eischen 2012, Sections 8 and 9]. Note that because we work with vector-weight automorphic forms, and not just scalar-weight automorphic forms, we need more differential operators than we did in [Eischen 2013], which handled only the case of scalar-weight automorphic forms.

Section 5 contains the main results of the paper, namely the construction of a p-adic Eisenstein measure and the p-adic interpolation of values of certain automorphic forms. This is the heart of the paper. The format of Section 5 closely parallels the construction of a p-adic Eisenstein measure in [Katz 1978, Sections 3.4 and 4.2]. We also explain in Remark 16 precisely how the Eisenstein measure of [Eischen 2013, Section 4] and the Eisenstein measure given in Theorem 14 are related. For $n \geq 2$, the measure in Theorem 14 is on a larger group than the measure in [Eischen 2013, Section 4]. In order to construct a measure with values in the space of *vector*-weight automorphic forms without fixing a partition of n, this larger group is necessary. (The approach in [Eischen 2013] relied on a choice of a partition of n, but it turns out that with this larger group we do not need to fix a partition of n and can consider a larger class of automorphic forms all at once.) We also note that the construction of the measures in [Eischen 2014, Section 4] uses this measure as a starting point.

In Section 6, we comment on how to extend the results of this paper to the case of Siegel modular forms, i.e., automorphic forms on symplectic groups. The fact that our presentation in Section 5 closely follows the approach in [Katz 1978, Sections 3.4 and 4.2] also allows us to recover the Eisenstein measure of [Katz 1978, Definition (4.2.5) and Equation (5.5.7)] with ease in Section 6.1.

2. Conventions and background

In Section 2.1, we introduce the conventions that we will use throughout the paper. In Section 2.2, we briefly summarize the necessary background on automorphic forms on unitary groups. (See the start of Section 2.2 for references.)

2.1. ***Conventions.*** Once and for all, fix a CM field K with maximal totally real subfield E. Fix a prime p that is unramified in K and such that each prime of E dividing p splits completely in K. Fix embeddings

$$\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C},$$
$$\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p,$$

and fix an isomorphism

$$\iota : \bar{\mathbb{C}}_p \xrightarrow{\sim} \mathbb{C}$$

satisfying $\iota \circ \iota_p = \iota_\infty$. From here on, we identify $\overline{\mathbb{Q}}$ with $\iota_p(\overline{\mathbb{Q}})$ and $\iota_\infty(\overline{\mathbb{Q}})$. Let $\mathbb{O}_{\mathbb{C}_p}$ denote the ring of integers in $\mathbb{C}_p$.

Fix a CM type Σ for $K/\mathbb{Q}$. For each element $\sigma \in \operatorname{Hom}(E, \overline{\mathbb{Q}})$, we also write σ to denote the unique element of Σ prolonging $\sigma : E \hookrightarrow \overline{\mathbb{Q}}$ (when no confusion can arise). For each element $x \in K$, denote by $\bar{x}$ the image of x under the unique nontrivial element $\epsilon \in \mathrm{Gal}(K/E)$, and let $\bar{\sigma} = \sigma \circ \epsilon$.

Given an element a of E, we identify it with an element of $E \otimes \mathbb{R}$ via the embedding

$$\begin{aligned} E &\hookrightarrow E \otimes \mathbb{R} \\ a &\mapsto (\sigma(a))_{\sigma \in \Sigma}. \end{aligned} \tag{1}$$

We identify $a \in K$ with an element of $K \otimes \mathbb{C} \xrightarrow{\sim} (E \otimes \mathbb{C}) \times (E \otimes \mathbb{C})$ via the embedding

$$\begin{aligned} K &\hookrightarrow K \otimes \mathbb{C} \\ a &\mapsto \big((\sigma(a))_{\sigma \in \Sigma}, (\bar{\sigma}(a))_{\sigma \in \Sigma}\big). \end{aligned} \tag{2}$$

Let $d = (d_v)_{v \in \Sigma} \in \mathbb{Z}^\Sigma$, and let $a = (a_v)_{v \in \Sigma}$ be an element of $\mathbb{C}^\Sigma$ or $\mathbb{C}_p^\Sigma$. We denote by a^d the element of $\mathbb{C}$ or $\mathbb{C}_p$ defined by

$$a^d := \prod_{v \in \Sigma} a_v^{d_v}.$$

If $e = (e_v)_{v \in \Sigma} \in \mathbb{Z}^\Sigma$, we denote by $d + e$ the tuple defined by

$$d + e = (d_v + e_v)_{v \in \Sigma} \in \mathbb{Z}^\Sigma.$$

If $k \in \mathbb{Z}$, we denote by $k + d$ or $d + k$ the element

$$k + d = d + k = (d_v + k)_{v \in \Sigma} \in \mathbb{Z}^\Sigma.$$

For any ring R, we denote the ring of $n \times n$ matrices with coefficients in R by $M_{n \times n}(R)$. We denote by 1_n the multiplicative identity in $M_{n \times n}(R)$. Also, for any subring R of $K \otimes_E E_v$, with v a place of E, let $\mathrm{Her}_n(R)$ denote the space of Hermitian $n \times n$ matrices with entries in R. Given $x \in \mathrm{Her}_n(E)$,

$$x > 0$$

if $\sigma(x)$ is positive definite for every $\sigma \in \Sigma$.

2.1.1. *Adelic norms.* Let $|\cdot|_E$ denote the adelic norm on $E^\times \backslash \mathbb{A}_E^\times$ such that, for all $a \in \mathbb{A}_E^\times$,

$$|a|_E = \prod_v |a|_v,$$

where the right-hand product is over all places of E and where the absolute values are normalized so that

$$\begin{aligned} |v|_v &= q_v^{-1}, \\ q_v &= \text{the cardinality of } \mathcal{O}_{E v}/v\mathcal{O}_{E v} \end{aligned}$$

for all nonarchimedean primes v of the totally real field E. Consequently, for all $a \in E$,

$$\prod_{v \nmid \infty} |a|_v^{-1} = \prod_{v \in \Sigma} \sigma_v(a)\mathrm{Sign}(\sigma_v(a)),$$

where the product is over all archimedean places v of the totally real field E. We denote by $|\cdot|_K$ the adelic norm on $K^\times \backslash \mathbb{A}_K^\times$ such that, for all $a \in \mathbb{A}_K^\times$,

$$|a|_K = |a\bar{a}|_E.$$

For $a \in K$ and v a place of E, we let

$$|a|_v = |a\bar{a}|_v^{1/2}.$$

Given an element $a \in K$, we associate a with an element of $K \otimes \mathbb{R}$ via the embedding

$$a \mapsto (\sigma(a))_{\sigma \in \Sigma}.$$

For any field extension L/M, we write $\boldsymbol{N}_{L/M}$ to denote the norm from L to M. Given an $\mathbb{O}_M$-algebra R, the norm map $\boldsymbol{N}_{L/M}$ on L provides a group homomorphism

$$(\mathbb{O}_L \otimes R)^\times \to R^\times$$

in which $a \otimes r \mapsto \boldsymbol{N}_{L/M}(a)r$. When the fields are clear, we shall just write $\boldsymbol{N}$.

2.1.2. *Exponential characters.* For each archimedean place $v \in \Sigma$, denote by $\boldsymbol{e}_v$ the character of E_v (i.e., $\mathbb{R}$) defined by

$$\boldsymbol{e}_v(x_v) = e^{2\pi i x_v}$$

for all x_v in E_v. Denote by $\boldsymbol{e}_\infty$ the character of $E \otimes \mathbb{R}$ defined by

$$\boldsymbol{e}_\infty((x_v)_{v \in \Sigma}) = \prod_{v \mid \infty} \boldsymbol{e}_v(x_v).$$

Following our convention from (1), we put

$$\boldsymbol{e}_\infty(a) = \boldsymbol{e}_\infty\big((\sigma(a))_{\sigma \in \Sigma}\big) = e^{2\pi i \,\mathrm{tr}_{E/\mathbb{Q}}(a)}$$

for all $a \in E$. For each finite place v of E dividing a prime q of $\mathbb{Z}$, denote by $\boldsymbol{e}_v$ the character of E_v defined, for each $x_v \in E_v$, by

$$\boldsymbol{e}_v(x_v) = e^{-2\pi i y},$$

where $y \in \mathbb{Q}$ is the fractional part of $\mathrm{tr}_{E_v/\mathbb{Q}_q}(x_v) \in \mathbb{Q}_p$; that is, if we write $\mathrm{tr}_{E_v/\mathbb{Q}_q}(x_v) = \sum_{i=k}^{\infty} a_i p^i$ for some integer $k \leq 0$ and $a_i \in \{0, \ldots, p-1\}$, then $y = \sum_{i=k}^{0} a_i p^i$. We denote by $\boldsymbol{e}_{\mathbb{A}_E}$ the character of $\mathbb{A}_E$ defined by

$$\boldsymbol{e}_{\mathbb{A}_E}(x) = \prod_v \boldsymbol{e}_v(x_v) \quad \text{for all } x = (x_v) \in \mathbb{A}_E.$$

Remark 1. We identify $a \in E$ with the element $(\sigma_v(a))_v \in \mathbb{A}_E$, where $\sigma_v : E \hookrightarrow E_v$ is the embedding corresponding to v. Following this convention, we put

$$\boldsymbol{e}_{\mathbb{A}_E}(a) = \prod_v \boldsymbol{e}_v(\sigma_v(a)) \tag{3}$$

for all $a \in E$.

2.1.3. *Spaces of functions.* Given topological spaces X and Y, we let

$$\mathscr{C}(X, Y)$$

denote the space of continuous functions from X to Y.

2.2. *Background concerning automorphic forms on unitary groups.*

2.2.1. *Unitary groups of signature* (n, n). We now recall basic information about unitary groups and automorphic forms on unitary groups. (A more detailed discussion of unitary groups and automorphic forms on unitary groups appears in [Shimura 1997; 2000; Hida 2004; Harris et al. 2006; Eischen 2012; Lan 2013]; the analogous background for the case of Hilbert modular forms is the main subject of [Katz 1978, Section 1].)

The material in this section is similar to [Eischen 2013, Section 2.1]. Although we discussed embeddings of nondefinite unitary groups of various signatures into unitary groups of signature (n, n) there, we are primarily concerned only with unitary groups of signature (n, n) and definite unitary groups in this paper; in the sequel [Eischen 2014] we discuss pullbacks to various products of unitary groups occurring as subgroups.

Let V be a vector space of dimension n over the CM field K, and let $\langle\, ,\rangle_V$ denote a positive definite hermitian pairing on V. Let $-V$ denote the vector space V with the negative definite hermitian pairing $-\langle\, ,\rangle_V$. Let

$$W = 2V = V \oplus -V$$
$$\langle (v_1, v_2), (w_1, w_2)\rangle_W = \langle v_1, w_1\rangle_V + \langle v_2, w_2\rangle_{-V}.$$

The hermitian pairing $\langle\, ,\rangle_W$ defines an involution $g \mapsto \tilde{g}$ on $\mathrm{End}_K(W)$ by

$$\langle g(w), w'\rangle_W = \langle w, \tilde{g}(w')\rangle_W$$

(where w and w' denote elements of W). This involution extends to an involution on $\mathrm{End}_{K\otimes_E R}(W \otimes_E R)$ for any E-algebra R. We denote by U the algebraic group such that, for any E-algebra R, the R-points of U are given by

$$U(R) = U(R, W) = \{g \in \mathrm{GL}_{K\otimes_E R}(W \otimes_E R) \mid g\tilde{g} = 1\}.$$

Similarly, we define $U(R, V)$ to be the algebraic group associated to $\langle\, ,\rangle_V$ and $U(R, -V)$ to be the algebraic group associated to $\langle\, ,\rangle_{-V}$. Note that $U(\mathbb{R})$ is of signature (n, n). Also, the canonical embedding

$$V \oplus V \hookrightarrow W$$

induces an embedding

$$U(R, V) \times U(R, -V) \hookrightarrow U(R, W)$$

for all E-algebras R. When the E-algebra R over which we are working is clear from context or does not matter, we shall write $U(W)$ for $U(R, W)$, $U(V)$ for $U(R, V)$, and $U(-V)$ for $U(R, -V)$. We also sometimes write just U to denote $U(W)$.

We also have groups

$$GU(R) = GU(R, W) = \{g \in \mathrm{GL}_{K\otimes_E R}(W \otimes_E R) \mid g\tilde{g} \in R^\times\}.$$

We use the notation ω to denote the similitude character

$$\begin{aligned} \omega : GU(R) &\to R^\times \\ g &\mapsto g\tilde{g}. \end{aligned}$$

When the E-algebra R over which we are working is clear from context or does not matter, we shall write $GU(W)$ for $GU(R, W)$. We shall also use the notation

$$G(R) = GU(R, W)$$

or write simply G or GU when the ring R is clear from context or does not matter. When $R = \mathbb{A}_E$ or $R = \mathbb{R}$, we write

$$G_+ := GU_+$$

to denote the subgroup of $G = GU$ consisting of elements such that the similitude factor at each archimedean place of E is positive.

For the space $W = V \oplus -V$ defined above, $U(W)$ and $GU(W)$ have signature (n, n). So we will sometimes write $U(n, n)$ and $GU(n, n)$, respectively, to refer to these groups.

We write $W = V_d \oplus V^d$, where V_d and V^d denote the maximal isotropic subspaces

$$\begin{aligned} V^d &= \{(v, v) | v \in V\}, \\ V_d &= \{(v, -v) | v \in V\}. \end{aligned}$$

Let P be the Siegel parabolic subgroup of $U(W)$ stabilizing V^d in $V_d \oplus V^d$ under the action of $U(W)$ on the right. Denote by M the Levi subgroup of P and by N the unipotent radical of P. Similarly, denote by GP the Siegel parabolic subgroup of $GU(W)$ stabilizing V^d in $V_d \oplus V^d$ under the action of $GU(W)$ on the right, and denote by GM the Levi subgroup of GP and by N the unipotent radical of GP. We also, similarly, denote by GP_+ the Siegel parabolic subgroup of GU_+ stabilizing

V^d in $V_d \oplus V^d$ under the action of GU_+ on the right, and denote by GM_+ the Levi subgroup of GP_+ and by N the unipotent radical of GP_+.

A choice of a basis $e_1, \ldots, e_n$ for V over K gives an identification of V with V^d (via $e_i \mapsto (e_i, e_i)$) and with V_d (via $e_i \mapsto (e_i, -e_i)$). The choice of a basis for V also identifies $\mathrm{GL}_K(V)$ with $\mathrm{GL}_n(K)$. With respect to the ordered basis $(e_1, e_1) \ldots, (e_n, e_n), (e_1, -e_1) \ldots, (e_n, -e_n)$ for W, M consists of the block diagonal matrices of the form

$$m(h) := ({}^t\bar{h}^{-1}, h)$$

with $h \in \mathrm{GL}_n(K \otimes R)$, and GM consists of the block diagonal matrices of the form

$$m(h, \lambda) := ({}^t\bar{h}^{-1}, \lambda h)$$

with $h \in \mathrm{GL}_n(K)$ and $\lambda \in E^\times$. Thus, the choice of basis $e_1, \ldots, e_n$ for V over K fixes identifications

$$M \xrightarrow{\sim} \mathrm{GL}_K(V),$$
$$GM \xrightarrow{\sim} \mathrm{GL}_K(V) \times E^\times.$$

These isomorphisms extend to isomorphisms

$$M(R) \xrightarrow{\sim} \mathrm{GL}_{K\otimes_E R}(V \otimes_E R), \tag{4}$$
$$GM(R) \xrightarrow{\sim} \mathrm{GL}_{K\otimes_E R}(V \otimes_E R) \times R^\times \tag{5}$$

for each E-algebra R.

We fix a Shimura datum $(G, X(W))$ and a corresponding Shimura variety $\mathrm{Sh}(W) = \mathrm{Sh}(U(n, n))$ according to the conditions in [Harris et al. 2006, Section 1.2] and [Eischen 2012, Section 2.2]. The symmetric domain $X(W)$ is holomorphically isomorphic to the tube domain consisting of $[E : \mathbb{Q}]$ copies of

$$\mathcal{H}_n = \{z \in M_{n\times n}(\mathbb{C}) \mid i({}^t\bar{z} - z) > 0\}.$$

When we need to emphasize over which ring R we work, we sometimes write $\mathrm{Sh}(R)$. Let $\mathcal{K}_\infty$ be the stabilizer in $G(\mathbb{R})$ of the point $i \cdot 1_n$. So $\prod_{\sigma\in\Sigma} \mathcal{K}_\infty$ is the stabilizer in $\prod_{\sigma\in\Sigma} G(\mathbb{R})$ of the point

$$\boldsymbol{i} = (i \cdot 1_n)_{\sigma\in\Sigma} \in \prod_{\sigma\in\Sigma} \mathcal{H}_n. \tag{6}$$

We can identify $G_+(\mathbb{R})/\mathcal{K}_\infty$ with $\mathcal{H}_n$. Given a compact open subgroup $\mathcal{K}$ of $G(\mathbb{A}_f)$, denote by ${}_{\mathcal{K}}\mathrm{Sh}(W)$ the Shimura variety whose complex points are given by

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\mathcal{K}.$$

This Shimura variety is a moduli space for abelian varieties together with a polarization, an endomorphism, and a level structure (dependent upon the choice of $\mathcal{K}$). Note that ${}_{\mathcal{K}}\mathrm{Sh}(W)$ consists of copies of quotients of spaces isomorphic to $\mathcal{H}_n$.

When we work with some other group H, we write $\mathrm{Sh}(H)$ instead of $\mathrm{Sh}(W)$.

2.2.2. *Automorphic forms on unitary groups.* Automorphic forms on unitary groups are typically discussed from any of the following three perspectives (which are equivalent over $\mathbb{C}$):

(1) Functions on a unitary group that satisfy an automorphy condition.

(2) C^∞ (or holomorphic) functions on a hermitian symmetric space (analogue of the upper half plane) that satisfy an automorphy condition.

(3) Sections of a certain vector bundle over a moduli space (a Shimura variety) parametrizing abelian varieties together with a polarization, endomorphism, and level structure.

Which perspective is most natural depends upon context. In this paper, we shall need all three perspectives. (In [Eischen 2012, Section 2], we provided a detailed discussion of automorphic forms and the relationships between different approaches to defining them.)

The relationship between the first two approaches to automorphic forms is reviewed in [Eischen 2013, p. 9; Shimura 2000, A8]. The relationship between the second two approaches to automorphic forms is discussed in [Eischen 2012, Section 2] and is similar to the analogous relationship for modular forms given in [Katz 1973, A1.1].

An automorphic form f on $U(n, n)$ has a weight, which is a representation ρ of $\mathrm{GL}_n \times \mathrm{GL}_n$. In the special case where this representation is of the form

$$\rho(a, b) = \det(a)^{k+\nu} \det(b)^{-\nu},$$

we shall say f *is an automorphic form of weight* (k, ν).

As explained in [Lan 2012; 2013], for the unitary groups of signature (n, n) there is a higher-dimensional analogue of the Tate curve (which we call the "Mumford object" in [Eischen 2012, Section 4.2; 2013, Section 2.2.11]), and so in analogue with the case for modular forms evaluated at the Tate curve, one obtains an algebraic q-expansion by evaluating an automorphic form at the Mumford object. Like in the case of modular forms, the coefficients of an algebraically defined q-expansion of a holomorphic automorphic form f of over $\mathbb{C}$ agree with the (analytically defined) Fourier coefficients of f [Lan 2012]. Also, like in the case of modular forms, there is a q-expansion principle for automorphic forms on unitary groups [Lan 2013, Proposition 7.1.2.15]; note that the q-expansion principle for automorphic forms over a Shimura variety requires the evaluation of an automorphic form at one cusp of each connected component. To apply the q-expansion principle, it is enough [Hida 2004, Section 8.4] to check the cusps parametrized by points of $GM_+(\mathbb{A}_E)$. (The author is grateful to thank Kai-Wen Lan for explaining this to her.) We shall say "a cusp $m \in GM_+(\mathbb{A}_E)$" to mean "the cusp corresponding to the point m." The q-expansion of an automorphic form at a cusp $m(h, \lambda)$ is a sum of the form

$$\sum_{\beta \in L_{m(h,\lambda)}} a(\beta) q^\beta,$$

where $L_{m(h,\lambda)}$ is a lattice in $\mathrm{Her}_n(E)$ dependent upon the choice of the cusp $m(h, \lambda)$ and $a(\beta) \in \mathbb{C}$ for all β (or, more generally, if f is a V-valued automorphic form for some $\mathbb{C}$-vector space V, $a(\beta) \in V$ for all β). We sometimes also write

$$\sum_{\beta \in \mathrm{Her}_n(E)} a(\beta) q^\beta,$$

when we do not need to make the cusp explicit; in this case, we know that the coefficients $a(\beta)$ are zero outside of some lattice in $\mathrm{Her}_n(E)$ (namely, the lattice corresponding to the unspecified cusp).

Throughout the paper, all cusps m and corresponding lattices $L_m \subseteq \mathrm{Her}_n(K)$ determined by m are chosen so that the elements of L_m have p-integral coefficients.[1]

3. Eisenstein series on unitary groups

In this section, we introduce certain Eisenstein series on unitary groups of signature (n, n). These Eisenstein series are related to the ones discussed in [Eischen 2013, Section 2; Shimura 1997, Section 18; Katz 1978, Section (3.2)].

For $k \in \mathbb{Z}$ and $\nu = (\nu(\sigma))_{\sigma \in \Sigma} \in \mathbb{Z}^\Sigma$, we denote by $\boldsymbol{N}_{k,\nu}$ the function

$$\begin{aligned} \boldsymbol{N}_{k,\nu} : K^\times &\to K^\times \\ b &\mapsto \prod_{\sigma \in \Sigma} \sigma(b)^{k+2\nu(\sigma)} (\sigma(b)\bar{\sigma}(b))^{-(\nu(\sigma))}. \end{aligned}$$

For all $b \in \mathbb{O}_E^\times$,

$$\boldsymbol{N}_{k,\nu}(b) = \boldsymbol{N}_{E/\mathbb{Q}}^k(b).$$

Theorem 2. *Let R be an $\mathbb{O}_K$-algebra, let $\nu = (\nu(\sigma)) \in \mathbb{Z}^\Sigma$, and let $k \geq n$ be an integer. Let*

$$F : (\mathbb{O}_K \otimes \mathbb{Z}_p) \times M_{n \times n}(\mathbb{O}_E \otimes \mathbb{Z}_p) \to R$$

be a locally constant function supported on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n \times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$ that satisfies

$$F(ex, \boldsymbol{N}_{K/E}(e^{-1}) y) = \boldsymbol{N}_{k,\nu}(e) F(x, y) \tag{7}$$

[1] Even without this choice for m and L_m, which we did not make a priori in [Eischen 2013], we could force the Fourier coefficients at all the non-p-integral elements of $\mathrm{Her}_n(K)$ to be zero, simply by our choice of a Siegel section at p later in this paper. In fact, in [Eischen 2013, Section 2.2], our choice of Siegel sections at p forced the Fourier coefficients at all the non-p-integral elements of $\mathrm{Her}_n(K)$ to be zero.

for all $e \in \mathbb{O}_K^\times$, $x \in \mathbb{O}_K \otimes \mathbb{Z}_p$, and $y \in M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$. There is an automorphic form $G_{k,\nu,F}$ (on $U(n,n)$) of weight (k,ν) defined over R whose q-expansion at a cusp $m \in GM_+(\mathbb{A}_E)$ is of the form $\sum_{0<\beta\in L_m} c(\beta)q^\beta$ (where L_m is the lattice in $\mathrm{Her}_n(K)$ *determined by m), with $c(\beta)$ a finite $\mathbb{Z}$-linear combination of terms of the form*

$$F(a, \boldsymbol{N}_{K/E}(a)^{-1}\beta)\boldsymbol{N}_{k,\nu}(a^{-1}\det\beta)\boldsymbol{N}_{E/\mathbb{Q}}(\det\beta)^{-n}$$

(where the linear combination is a sum over a finite set of p-adic units $a \in K$ dependent upon β and the choice of cusp $m \in GM$). When $R = \mathbb{C}$, these are the Fourier coefficients at $s = \frac{1}{2}k$ of the C^∞ automorphic form $G_{k,\nu,F}(z,s)$ (which is holomorphic at $s = \frac{1}{2}k$) that will be defined in Lemma 9.

(Above, the elements of $(\mathbb{O}_E \otimes \mathbb{Z}_p)^\times$ in $M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$ are viewed as homomorphisms, i.e., multiplication by an element of $(\mathbb{O}_E \otimes \mathbb{Z}_p)^\times$, so as diagonal matrices in $M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$. Also note that, when $\det\beta = 0$, the coefficient of q^β is 0, so we can restrict the discussion to F with support in $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p)$.)

Proof. By an argument similar to Katz's argument at the beginning of the proof of [1978, Theorem (3.2.3)], every locally constant R-valued function F supported on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$ that satisfies (7) is an R-linear combination of $\mathbb{O}_K$-valued functions F supported on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$ that satisfy (7). So it is enough to prove the theorem for $\mathbb{O}_K$-valued functions F.

Now, if we can construct an automorphic form satisfying the conditions of the theorem over $R = \mathbb{C}$, then by the q-expansion principle [Lan 2013, Proposition 7.1.2.15], the case over R will follow for any $\mathbb{O}_K$-subalgebra R (in particular, for $R = \mathbb{O}_K$) of $\mathbb{C}$. By [Lan 2012], it sufficient to show that there is a $\mathbb{C}$-valued C^∞ automorphic form $G_{k,\nu,F}$ of weight (k,ν) holomorphic at $s = \frac{1}{2}k$ whose Fourier coefficients (at $s = \frac{1}{2}k$) are as in the statement of the theorem. We will devote Section 3.1 to the construction of such an automorphic form. □

3.1. ***Construction of a C^∞ automorphic form over $\mathbb{C}$ whose Fourier coefficients meet the conditions of Theorem 2.*** In this section, we construct the C^∞ automorphic form $G_{k,\nu,F}$ necessary to complete the proof of Theorem 2.

Let $\mathfrak{m}$ be an ideal that divides p^∞. Let χ be a unitary Hecke character of type A_0,

$$\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times,$$

of conductor $\mathfrak{m}$, i.e.,

$$\chi_v(a) = 1$$

for all finite primes v in K and all $a \in K_v^\times$ such that

$$a \in 1 + \mathfrak{m}_v \mathbb{O}_{K_v}.$$

Let $\nu(\sigma)$ and $k(\sigma)$, $\sigma \in \Sigma$, denote integers such that the infinity type of χ is

$$\prod_{\sigma\in\Sigma} \sigma^{-k(\sigma)-2\nu(\sigma)}(\sigma\cdot\bar{\sigma})^{\frac{1}{2}k(\sigma)+\nu(\sigma)}. \tag{8}$$

For any $s \in \mathbb{C}$, we view $\chi\cdot|\cdot|_K^{-s}\otimes|\cdot|_E^{-ns}$ as a character of the parabolic subgroup $GP_+(\mathbb{A}_E) = GM_+(\mathbb{A}_E)N(\mathbb{A}_E) \subseteq G_+(\mathbb{A}_E)$ via the composition of maps

$$GP(\mathbb{A}_E) \xrightarrow{\bmod N(\mathbb{A}_E)} GM(\mathbb{A}_E) \xrightarrow{(5)} \mathrm{GL}_{\mathbb{A}_K}(V\otimes_E \mathbb{A}_E)\times \mathrm{GL}_1(\mathbb{A}_E) \longrightarrow \mathbb{C}^\times,$$

where the last one is the map

$$(h,\lambda) \longmapsto |\lambda|_E^{-ns}\cdot\chi(\det h)|\det h|_K^{-s}.$$

Consider the induced representation

$$\begin{aligned} I(\chi,s) &= \mathrm{Ind}_{GP_+(\mathbb{A}_E)}^{G_+(\mathbb{A}_E)}\big(\chi\cdot|\cdot|_K^{-s}\otimes|\omega(\,\cdot\,)|_K^{-ns/2}\big)\\ &\cong \bigotimes_v \mathrm{Ind}_{GP_+(E_v)}^{G_+(E_v)}\big(\chi_v\cdot|\cdot|_v^{-2s}\otimes|\omega(\,\cdot\,)|_v^{-ns}\big), \end{aligned} \tag{9}$$

where the product is over all places of E.

Given a section $f \in I(\chi,s)$, the Siegel Eisenstein series associated to f is the $\mathbb{C}$-valued function of G defined by

$$E_f(g) = \sum_{\gamma\in GP_+(E)\backslash G_+(E)} f(\gamma g).$$

This function converges for $\Re(s) > 0$ and can be continued meromorphically to the entire complex plane.

Remark 3. As in [Eischen 2013], if we were working with normalized induction, then the function would converge for $\Re(s) > \frac{1}{2}n$, but we have absorbed the exponent $\frac{1}{2}n$ into the exponent s. (Our choice not to include the modulus character at this point is equivalent to shifting the plane on which the function converges by $\frac{1}{2}n$.)

All the poles of E_f are simple and there are at most finitely many of them. Details about the poles are given in [Tan 1999].

As we noted in [Eischen 2013, Section 2.2.4], if the Siegel section f factors as $f = \bigotimes_v f_v$, then E_f has a Fourier expansion such that, for all $h \in \mathrm{GL}_n(K)$ and $m \in \mathrm{Her}_n(K)$,

$$E_f\left(\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}{}^t\bar{h}^{-1} & 0\\ 0 & h\end{pmatrix}\right) = \sum_{\beta\in\mathrm{Her}_n(K)} c(\beta,h;f)\boldsymbol{e}_{\mathbb{A}_E}(\mathrm{tr}(\beta m))$$

with $c(\beta,h;f)$ a complex number dependent only on the choice of section f, the hermitian matrix $\beta\in\mathrm{Her}_n(K)$, h_v for finite places v, and $(h\cdot{}^t\bar{h})_v$ for archimedean places v of E.

By [Shimura 1997, Sections 18.9, 18.10], the Fourier coefficients of the Siegel sections $f = \bigotimes_v f_v$ that we will choose below are products of local Fourier coefficients determined by the local sections f_v. More precisely, for each $\beta \in \mathrm{Her}_n(K)$,

$$c(\beta, h; f) = C(n, K)\prod_v c_v(\beta, h; f),$$

where

$$c_v(\beta, h; f) = \int_{\mathrm{Her}_n(K\otimes E_v)} f_v\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & m_v\\ 0 & 1\end{pmatrix}\begin{pmatrix}{}^t\bar{h}_v^{-1} & 0\\ 0 & h_v\end{pmatrix}\right)\boldsymbol{e}_v(-\operatorname{tr}(\beta_v m_v))\, dm_v, \tag{10}$$

$$C(n, K) = 2^{n(n-1)[E:\mathbb{Q}]/2}|D_E|^{-n/2}|D_K|^{-n(n-1)/4}, \tag{11}$$

D_E and D_K are the discriminants of K and E, respectively, $\beta_v = \sigma_v(\beta)$ for each place v of E, and d_v denotes the Haar measure on $\mathrm{Her}_n(K_v)$ such that

$$\int_{\mathrm{Her}_n(\mathbb{O}_K\otimes_E E_v)} d_v x = 1 \quad \text{for each finite place } v \text{ of } E \tag{12}$$

and

$$d_v x := \left|\textstyle\bigwedge_{j=1}^n dx_{jj}\bigwedge_{j<k}(2^{-1}dx_{jk}\wedge d\bar{x}_{jk})\right| \quad \text{for each archimedean place } v \text{ of } E.$$

(Here x denotes the matrix whose ij-th entry is x_{ij}.)

Below, we recall [Eischen 2013, Lemma 19], which explains how the Fourier coefficients $c(\beta, h; f)$ transform when we change the point h. For each $h \in \mathrm{GL}_n(\mathbb{A}_K)$ and $\lambda \in \mathbb{A}_E^\times$, let $m(h, \lambda)$ denote the matrix

$$\begin{pmatrix}{}^t\bar{h}^{-1} & 0\\ 0 & \lambda h\end{pmatrix}.$$

Generalizing (10), we define

$$c_v(\beta, m(h,\lambda); f) = \int_{\mathrm{Her}_n(K\otimes E_v)} f_v\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & m_v\\ 0 & 1\end{pmatrix} m(h,\lambda)\right)\boldsymbol{e}_v(-\operatorname{tr}(\beta_v m_v))\, dm_v.$$

We also define $c(\beta, m(h,\lambda); f) = C(n, K)\prod_v c_v(\beta, m(h,\lambda); f)$.

Lemma 4 [Eischen 2013, Lemma 19]. *For each $h \in \mathrm{GL}_n(\mathbb{A}_K)$, $\lambda \in \mathbb{A}_E^\times$, and $\beta \in \mathrm{Her}_n(K)$,*

$$c\left(\beta, \begin{pmatrix}{}^t\bar{h}^{-1} & 0\\ 0 & \lambda h\end{pmatrix}; f\right) = \chi(\det(\overline{\lambda h})^{-1})\left|\det\left((\overline{\lambda h})^{-1}\cdot(\lambda h)^{-1}\right)\right|_E^{n-s}|\lambda|_E^{-ns} c(\lambda^{-1}h^{-1}\beta{}^t\bar{h}^{-1}, 1_n; f). \tag{13}$$

Proof. Let $\eta = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$. Observe that, for any $n\times n$ matrix m,

$$\eta\cdot m(h,\lambda)\cdot\eta^{-1} = m(\lambda^{-1}\,{}^t\bar{h}^{-1},\lambda)$$
$$m(h,\lambda)^{-1}\cdot\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\cdot m(h,\lambda) = \begin{pmatrix}1 & \lambda\,{}^t\bar{h}mh\\ 0 & 1\end{pmatrix}.$$

Therefore,

$$\begin{aligned}\eta\cdot\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\cdot m(h,\lambda) &= (\eta\cdot m(h,\lambda)\cdot\eta^{-1})\eta\left(m(h,\lambda)^{-1}\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}m(h,\lambda)\right)\\ &= m(\lambda^{-1}\,{}^t\bar{h}^{-1},\lambda)\eta\begin{pmatrix}1 & \lambda\,{}^t\bar{h}mh\\ 0 & 1\end{pmatrix}.\end{aligned}$$

So, for any place v of E and section $f_v\in\mathrm{Ind}_{GP(E_v)}^{G_+(E_v)}(\chi,s)$,

$$\begin{aligned}&f_v\left(\eta\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}m(h_v,\lambda)\right)\\ &\quad= \chi_v(\det(\overline{\lambda_v h_v})^{-1})\left|\det(\overline{\lambda_v h_v})^{-1}\right|_v^{-2s}|\lambda|_v^{-ns}f_v\left(\eta\begin{pmatrix}1 & \lambda\,{}^t\bar{h}_v m h_v\\ 0 & 1\end{pmatrix}\right).\end{aligned}\tag{14}$$

The lemma now follows from (14) and the fact that the Haar measure d_v satisfies $d_v(\lambda h_v x\,{}^t\bar{h}_v) = |\det(\lambda_v{}^t\bar{h}_v\cdot h_v)|_v^n d_v(x)$ for each place v of E. □

So,

$$\begin{aligned}c\left(\beta,\begin{pmatrix}\lambda^{-1}\,{}^t\bar{h}^{-1} & 0\\ 0 & h\end{pmatrix};f\right) &= \chi(\lambda^n)|\lambda^{2n}|_E^{n-s}|\lambda|_E^{2ns}\left(\beta,\begin{pmatrix}{}^t\bar{h}^{-1} & 0\\ 0 & \lambda h\end{pmatrix};f\right)\\ &= |\lambda^{2n^2}|_E\,\chi(\lambda^n)c\left(\beta,\begin{pmatrix}{}^t\bar{h}^{-1} & 0\\ 0 & \lambda h\end{pmatrix};f\right).\end{aligned}\tag{15}$$

Below, we choose more specific Siegel sections $f=\bigotimes_v f_v$ and compute the corresponding Fourier coefficients.

3.1.1. *The Siegel section at* ∞. We now define a section $f_\infty^{k,\nu} = f_\infty^{k,\nu}(\bullet\,;i\cdot 1_n,\chi,s)$ in $\bigotimes_{v\mid\infty}\mathrm{Ind}_{GP_+(E_v)}^{G_+(E_v)}(\chi_v\cdot|\cdot|_v^{-2s}\otimes|\omega(\cdot)|_E^{-ns})$.

For each $\alpha=\prod_{v\mid\infty}\alpha_v\in\prod_{v\mid\infty}G(E_v)$, we write α_v in the form $\left(\begin{smallmatrix} a_v & b_v\\ c_v & d_v\end{smallmatrix}\right)$ with a_v, b_v, c_v, and d_v $n\times n$ matrices. Each element $\alpha\in G(E_v)$ acts on $z=\prod_{v\mid\infty}z_v\in\prod_{v\mid\infty}\mathcal{H}_n$ by

$$\alpha_v(z_v) = (a_v z_v+b_v)(c_v z_v+d_v)^{-1},$$
$$\alpha(z) = \prod_{v\mid\infty}\alpha_v(z_v).$$

Let

$$\lambda_{\alpha_v}(z_v) = \lambda(\alpha_v, z_v) = \overline{c_v} \cdot {}^t z_v + \overline{d_v},$$

$$\lambda_\alpha(z) = \lambda(\alpha, z) = \prod_{v|\infty} \lambda_{\alpha_v}(z_v),$$

$$\mu_{\alpha_v}(z_v) = \mu(\alpha_v, z_v) = c_v \cdot z_v + d_v,$$

$$\mu_\alpha(z) = \mu(\alpha, z) = \prod_{v|\infty} \mu_{\alpha_v}(z_v).$$

(These are the canonical automorphy factors. Properties of them are discussed in [Shimura 2000, Section 3.3], for example.) We write

$$j_{\alpha_v}(z_v) = j(\alpha_v, z_v) = \det \mu_{\alpha_v}(z_v),$$

$$j_\alpha(z) = j(\alpha, z) = \prod_{v|\infty} j_{\alpha_v}(z_v).$$

Note that

$$\det(\lambda_{\alpha_v}(z_v)) = \det(\overline{\alpha_v})\omega(\alpha_v)^{-n} j_{\alpha_v}(z_v) \tag{16}$$

$$= \det(\alpha_v)^{-1}\omega(\alpha_v)^n j_{\alpha_v}(z_v), \tag{17}$$

so

$$|\det(\lambda_{\alpha_v}(z_v))| = |j_{\alpha_v}(z_v)|.$$

Consistent with the notation in [Shimura 1997, Equation (10.4.3)], we define

$$j_\alpha^{k,\nu}(z) := j_\alpha(z)^{k+\nu} \det(\lambda_\alpha(z))^{-\nu}.$$

By (16) and (17), we see that

$$j_\alpha^{k,\nu}(z) = (\det(\bar{\alpha})\omega(\alpha)^{-n})^{-\nu} j_\alpha(z)^k$$
$$= (\det(\alpha)^{-1}\omega(\alpha)^n)^{-\nu} j_\alpha(z)^k.$$

If $\beta = \prod_{v|\infty} \beta_v$ is also an element of $\prod_{v|\infty} G(E_v)$, then

$$\lambda(\beta_v\alpha_v, z_v) = \lambda(\beta_v, \alpha_v z_v)\lambda(\alpha_v, z_v), \tag{18}$$

$$\mu(\beta_v\alpha_v, z_v) = \mu(\beta_v, \alpha_v z_v)\mu(\alpha_v, z_v). \tag{19}$$

Consistent with the notation in [Shimura 2000, Section 3], we define functions η and δ on $\mathcal{H}_n$ by

$$\eta(z) = i({}^t\bar{z} - z),$$

$$\delta(z) = \det\left(\tfrac{1}{2}\eta(z)\right)$$

for each $z \in \mathcal{H}_n$. So

$$\eta(i \cdot 1_n) = 2 \cdot 1_n,$$

$$\delta(i \cdot 1_n) = 1.$$

We also write η and δ to denote the functions $\prod_{\sigma\in\Sigma}\eta$ and $\prod_{\sigma\in\Sigma}\delta$, respectively, on $\prod_{\sigma\in\Sigma}\mathcal{H}_n$. So $\delta(\boldsymbol{i})=1$. Also, note that

$$\delta(\alpha z)=\omega(\alpha)^n|j_\alpha(z)|^{-2}\delta(z)=\omega(\alpha)^n|j_\alpha(z)\det(\lambda_\alpha(z))|^{-1}\delta(z).$$

Following [Shimura 2000, Sections 3 and 5], given $(k,\nu)=\prod_{v\mid\infty}(k_v,\nu_v)\in(\mathbb{Z}\times\mathbb{Z})^\Sigma$, we define functions $f\|_{k,\nu}$ and $f|_{k,\nu}$ on $\prod_{\sigma\in\Sigma}\mathcal{H}_n$ by

$$(f\|_{k,\nu}\alpha)(z)=j_\alpha^{k,\nu}(z)^{-1}f(\alpha z),$$
$$f|_{k,\nu}\alpha=f\|_{k,\nu}(\omega(\alpha)^{-\frac{1}{2}}\alpha)$$

for each $\mathbb{C}$-valued function f on $\mathcal{H}_n$, point $z\in\mathcal{H}_n$, and element $\alpha\in G$. Note that $\omega(\alpha)^{-1/2}\alpha\in U(\eta_n)$ and, if $\omega(\alpha_v)=1$ for all $v\in\Sigma$, then

$$f|_{k,\nu}\alpha=f\|_{k,\nu}\alpha.$$

More generally, for each function f on $\prod_{\sigma\in\Sigma}\mathcal{H}_n$ with values in some representation (V,ρ) of $\prod_{\sigma\in\Sigma}\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C})$, we define functions $f\|_\rho$ and $f|_\rho$ on $\mathcal{H}_n$ by

$$(f\|_\rho\alpha)(z)=\rho(\mu_\alpha(z),\lambda_\alpha(z))^{-1}f(\alpha z),$$
$$f|_\rho\alpha=f\|_\rho(\omega(\alpha)^{-\frac{1}{2}}\alpha).$$

We also use the notation $f\|$ and $f|$ when we are working with just one copy of $\mathcal{H}_n$, rather than $[E:\mathbb{Q}]$ copies of $\mathcal{H}_n$ at once.

We define

$$f_\infty^{k,\nu}=\bigotimes_{v\mid\infty}f_v^{k,\nu}(\bullet\,;i\cdot 1_n,\chi,s)\in\bigotimes_{v\mid\infty}\mathrm{Ind}_{GP_+(E_v)}^{G_+(E_v)}(\chi_v\cdot|\cdot|_v^{-2s}\otimes|\omega(\cdot)|_E^{-ns})$$

by

$$\begin{aligned}
f_\infty^{k,\nu}&(\alpha;i\cdot 1_n,\chi,s)\\
&=(\delta^{s-\frac{1}{2}k}|_{k,\nu}\alpha)(i\cdot 1_n)\\
&=j^{k,\nu}_{\omega(\alpha)^{-1/2}\alpha}(i\cdot 1_n)^{-1}\left|j_{\omega(\alpha)^{-1/2}\alpha}(i\cdot 1_n)^{-2}\omega(\omega(\alpha)^{-1/2}\alpha)^n\right|^{s-\frac{1}{2}k(\sigma_v)}\\
&=j^{k,\nu}_{\omega(\alpha)^{-1/2}\alpha}(i\cdot 1_n)^{-1}\left|j_{\omega(\alpha)^{-1/2}\alpha}(i\cdot 1_n)^{-2}\right|^{s-\frac{1}{2}k}.
\end{aligned}$$

Given $\alpha\in G$, we also define a function $f_\infty^{k,\nu}(\alpha;\bullet,\chi,s)$ on $\mathcal{H}_n$ by

$$\begin{aligned}
f_\infty^{k,\nu}(\alpha;z,\chi,s)&=(\delta^{s-\frac{1}{2}k}|_{k,\nu}\alpha)(z)\\
&=j^{k,\nu}_{\omega(\alpha)^{-1/2}\alpha}(z)^{-1}\left|j_{\omega(\alpha)^{-1/2}\alpha}(z)^{-2}\right|^{s-\frac{1}{2}k}\delta(z)^{s-\frac{1}{2}k}.
\end{aligned}$$

By (18) and (19), we see that if $g\in G$ is such that $g(\boldsymbol{i})=z$ then, for each $\alpha\in G$,

$$f_\infty^{k,\nu}(\alpha g;i\cdot 1_n,\chi,s)=f_\infty^{k,\nu}(\alpha;z,\chi,s)f_\infty^{k,\nu}(g;i\cdot 1_n,\chi,s)\delta(z)^{\frac{1}{2}k-s}.$$

For $k \in \mathbb{Z}$ and $\nu = (\nu_v)_{v\in\Sigma} \in \mathbb{Z}^{\Sigma}$, $f_\infty^{k,\nu}(\alpha; \bullet, \chi, s)$ is a holomorphic function on $\mathcal{H}_n$ at $s = \frac{1}{2}k$.

3.1.2. *The Fourier coefficients at archimedean places of E.* When there is an integer k such that

$$s = \tfrac{1}{2}k = \tfrac{1}{2}k(\sigma) \quad \text{for all } \sigma \in \Sigma$$

(i.e., when $f_\infty^{k,\nu}(\alpha; z, \chi, s)$ is a holomorphic function of $z \in \mathcal{H}_n$), [Shimura 1983, Equation (7.12)] describes the archimedean Fourier coefficients precisely:

$$\begin{aligned} &c_v\big(\beta, 1_n;\, f_v^{k,\nu}\big(\bullet\,; i1_n, \chi, \tfrac{1}{2}k\big)\big) \\ &= 2^{(1-n)n} i^{-nk} (2\pi)^{nk} \left(\pi^{n(n-1)/2} \prod_{t=0}^{n-1} \Gamma(k-t)\right)^{-1} \sigma_v(\det\beta)^{k-n} \boldsymbol{e}\big(i\,\mathrm{tr}(\sigma_v(\beta))\big) \end{aligned} \tag{20}$$

for each archimedean place v of E. Observe that, when $k \geq n$,

$$\prod_{v|\infty} c_v\big(\beta, h;\, f_v^{k,\nu}\big(\bullet\,; i1_n\chi, \tfrac{1}{2}k\big)\big) = 0$$

unless $\det(\beta) \neq 0$ and $\det(h) \neq 0$, i.e., unless β is of rank n. Also, note that in our situation β will be in $\mathrm{Her}_n(K)$, so $\prod_{v\in\Sigma} \boldsymbol{e}\big(i\,\mathrm{tr}(\sigma_v(\beta))\big) = \boldsymbol{e}(ib)$ for some $b \in \mathbb{Q}$, so $\prod_{v\in\Sigma} \boldsymbol{e}\big(i\,\mathrm{tr}(\sigma_v(\beta))\big) = \boldsymbol{e}(ib)$ is a root of unity.

3.1.3. *Siegel sections at p.* We work with Siegel sections at p that are similar to the ones in [Eischen 2013, Section 2.2.8] (we multiply those by $|\omega(g)|_p^{-ns}$ to account for a similitude factor).

Lemma 5 [Eischen 2013, Lemma 10]. *Let Γ be a compact and open subset of $\prod_{v\in\Sigma} \mathrm{GL}_n(\mathbb{O}_{Ev})$, and let $\widetilde{F}$ be a locally constant Schwartz function*

$$\begin{aligned} \widetilde{F} : \prod_{v\in\Sigma} (\mathrm{Hom}_{K_v}(V_v, V_{d,v}) \oplus \mathrm{Hom}_{K_v}(V_v, V_v^d)) &\to R \\ (X_1, X_2) &\mapsto \widetilde{F}(X_1, X_2) \end{aligned}$$

(*with R a subring of* $\mathbb{C}$) *whose support in the first variable is Γ and such that*

$$\widetilde{F}(X, {}^tX^{-1}Y) = \prod_{v\in\Sigma} \chi_v(\det(X)) \widetilde{F}(1, Y) \tag{21}$$

for all X in Γ and Y in $\prod_{v\in\Sigma} M_{n\times n}(E_v)$.[2] *There is a Siegel section $f^{P\widetilde{F}(-X,Y)}$ at p whose Fourier coefficient at $\beta \in M_{n\times n}(E_v)$ is*

$$c(\beta, 1;\, f^{P\widetilde{F}(-X,Y)}) = \mathrm{volume}(\Gamma) \cdot \widetilde{F}(1, {}^t\beta).$$

[2]The version of the right-hand side of (21) appearing in [Eischen 2013, Lemma 10] reads "$\chi_1\chi_2^{-1}(\det(X))F(1, Y)$". The characters denoted χ_1 and χ_2 in [Eischen 2013] have the property that $\chi_1\chi_2^{-1}(a) = \prod_{v\in\Sigma} \chi_v(a)$ for all $a \in \prod_{v\in\Sigma} \mathbb{O}_{Ev}$. The function denoted by $\widetilde{F}$ in the current paper is denoted by F in [Eischen 2013].

We use the notation *PF*, for "partial Fourier transform", to be consistent with [Katz 1978, Section 3.1; Eischen 2013, Section 2.2.8], but we do not need to discuss partial Fourier transforms here.

As a direct consequence of Lemma 5, we obtain the following corollary:

Corollary 6. *For any locally constant Schwartz function $\widetilde{F}$ satisfying the conditions of Lemma 5 for some Γ with positive volume, there is a Siegel section $f_{\widetilde{F}}$ in $\bigotimes_{v\in\Sigma}\operatorname{Ind}_{P(E_v)}^{G(E_v)}(\chi_v\cdot|\cdot|^{-2s})$ whose local (at p) Fourier coefficient at β is $\widetilde{F}(1,{}^t\beta)$.*

Furthermore, we can significantly weaken the conditions placed on $\widetilde{F}$:

Corollary 7. *Let k be a positive integer. Let $\widetilde{F}$ be a locally constant Schwartz function*

$$\widetilde{F}:\prod_{v\in\Sigma}(M_{n\times n}(\mathbb{O}_{E_v})\times M_{n\times n}(\mathbb{O}_{E_v}))\to R$$

whose support lies in $\prod_{v\in\Sigma}(\mathrm{GL}_n(\mathbb{O}_{E_v})\times M_{n\times n}(\mathbb{O}_{E_v}))$ and which satisfies

$$\widetilde{F}(e,{}^te^{-1}y)=\boldsymbol{N}_{E/\mathbb{Q}}(\det e)^k\widetilde{F}(1,y)$$

for all $e\in\mathrm{GL}_n(\mathbb{O}_E)$ contained in the support Γ in the first variable of $\widetilde{F}$. Suppose, furthermore, that Γ has positive volume. Then there is a Siegel section $f_{\widetilde{F}}\in\bigotimes_{v\in\Sigma}\operatorname{Ind}_{P(E_v)}^{G(E_v)}(\chi_v\cdot|\cdot|^{-2s})$ whose local (at p) Fourier coefficient at β is $\widetilde{F}(1,{}^t\beta)$.

Proof. Let $\widetilde{F}$ be a locally constant Schwartz function

$$\widetilde{F}:\prod_{v\in\Sigma}(M_{n\times n}(\mathbb{O}_{E_v})\times M_{n\times n}(\mathbb{O}_{E_v}))\to R$$

whose support lies in $\prod_{v\in\Sigma}(\mathrm{GL}_n(\mathbb{O}_{E_v})\times M_{n\times n}(\mathbb{O}_{E_v}))$ and which satisfies

$$\widetilde{F}(e,{}^te^{-1}y)=\boldsymbol{N}_{E/\mathbb{Q}}(\det e)^k\widetilde{F}(1,y),\tag{22}$$

for all $e\in\mathrm{GL}_n(\mathbb{O}_E)$ contained in the support in the first variable of $\widetilde{F}$. Then, since $\widetilde{F}$ is locally constant, has compact support, and satisfies (22), there is a unitary Hecke character χ whose infinity type is as in (8) and such that the conductor $\mathfrak{m}=p^d$ for d a sufficiently large positive integer, so that

$$\widetilde{F}=a_1F_1+\cdots+a_lF_l$$

for some positive integer l and $a_1,\dots,a_l\in R$, and functions $F_1,\dots,F_l$ meeting the conditions of Corollary 6 (all for this *same character* χ but possibly with *different supports* $\Gamma_1,\dots,\Gamma_l$, respectively, in the first variable).

Now, we define

$$f_{\widetilde{F}}:=a_1f_{F_1}+\cdots+a_lf_{F_l},$$

where $f_{F_1},\dots,f_{F_l}$ are the Siegel sections obtained in Corollary 6. Then $f_{\widetilde{F}}$ is a linear combination of elements of the module $\bigotimes_{v\in\Sigma}\operatorname{Ind}_{P(E_v)}^{G(E_v)}(\chi_v\cdot|\cdot|^{-2s})$. So,

$f_{\widetilde{F}}$ is itself an element of $\bigotimes_{v\in\Sigma}\mathrm{Ind}_{P(E_v)}^{G(E_v)}(\chi_v\cdot|\cdot|^{-2s})$. Now, the Fourier coefficient of a sum of Siegel sections is the sum of the Fourier coefficients of these Siegel sections. So, the Fourier coefficient at β of $f_{\widetilde{F}}$ is

$$a_1F_1(1,{}^t\beta)+\cdots+a_lF_l(1,{}^t\beta)=\widetilde{F}(1,{}^t\beta). \qquad \square$$

3.1.4. *Siegel sections away from p and ∞.* We use the same Siegel sections at places $v\nmid p\infty$ as in [Eischen 2013, Section 2.2.9]. We now recall the key properties of these Siegel sections, which are described in more detail in [Shimura 1997, Section 18].

Let $\mathfrak{b}$ be an ideal in $\mathcal{O}_E$ prime to p. For each finite place v prime to p, there is a Siegel section $f_v^{\mathfrak{b}}=f_v^{\mathfrak{b}}(\bullet;\chi_v,s)\in\mathrm{Ind}_{P(E_v)}^{G(E_v)}(\chi_v,s)$ with the following property: by [Shimura 1997, Proposition 19.2], whenever the Fourier coefficient $c(\beta,m(1);f_v^{\mathfrak{b}})$ is nonzero,

$$\prod_{v\nmid p\infty}c(\beta,m(1);f_v^{\mathfrak{b}})$$
$$=N_{E/\mathbb{Q}}(\mathfrak{b}\mathcal{O}_E)^{-n^2}\prod_{i=0}^{n-1}L^p(2s-i,\chi_E^{-1}\tau^i)^{-1}\prod_{v\nmid p\infty}P_{\beta,v,\mathfrak{b}}(\chi_E(\pi_v)^{-1}|\pi_v|_v^{2s}), \quad (23)$$

where:

(1) the product is over primes of E;

(2) the Hecke character χ_E is the restriction of χ to E;

(3) the function $P_{\beta,v,\mathfrak{b}}$ is a polynomial that is dependent only on β, v, and $\mathfrak{b}$ and has coefficients in $\mathbb{Z}$ and constant term 1;

(4) the polynomial $P_{\beta,v,\mathfrak{b}}$ is identically 1 for all but finitely many v;

(5) τ is the Hecke character of E corresponding to K/E;

(6) π_v is a uniformizer of $O_{E,v}$, viewed as an element of $K^\times$ prime to p;

(7) $L^p(r,\chi_E^{-1}\tau^i)=\prod_{v\nmid p\infty\,\mathrm{cond}\,\tau}\big(1-\chi_v(\pi_v)^{-1}\tau^i(\pi_v)|\pi_v|_v^r\big)^{-1}$.

3.1.5. *Global Fourier coefficients.* Recall that, by Lemma 4, the Fourier coefficients $c(\beta,h;f)$ are completely determined by the coefficients $c(\beta,1_n;f)$. In Proposition 8, we combine the results of Sections 3.1.2, 3.1.3, and 3.1.4 in order to give the global Fourier coefficients of the Eisenstein series E_f.

Let χ be a unitary Hecke character as above and, furthermore, suppose the infinity type of χ is

$$\prod_{\sigma\in\Sigma}\sigma^{-k-2\nu(\sigma)}(\sigma\bar{\sigma})^{\frac{1}{2}k+\nu(\sigma)} \quad (24)$$

(i.e., $k(\sigma)=k\in\mathbb{Z}$ for all $\sigma\in\Sigma$). Let $C(n,K)$ be the constant dependent only upon n and K defined in (11).

Proposition 8. *Let $k \geq n$, let $\nu = (\nu(\sigma)) \in \mathbb{Z}^{\Sigma}$, and let*

$$f_{k,\nu,\chi,\widetilde{F}} := f_{k,\nu,\chi,\mathfrak{b},\widetilde{F}} := \bigotimes_{v \in \Sigma} f_{\widetilde{F},v} \otimes f_{\infty}^{k,\nu}(\bullet\,; i1_n, \chi, s) \otimes f^{\mathfrak{b}} \in \mathrm{Ind}_{P(\mathbb{A}_E)}^{G(\mathbb{A}_E)}(\chi \cdot |\cdot|_K^{-s}) \tag{25}$$

with χ as in (24), *$\bigotimes_{v\in\Sigma} f_{\widetilde{F},v}$ the section at p from Corollary 6, $f_{\infty}^{k,\nu}$ the section at ∞ defined in Section 3.1.2, and $f^{\mathfrak{b}}$ the section away from p and ∞ defined in Section 3.1.4.*

Then, at $s = \frac{1}{2}k$, all the nonzero Fourier coefficients $c(\beta, 1_n; f_{k,\nu,\chi,\widetilde{F}})$ are given by

$$D(n, K, \mathfrak{b}, p, k) \prod_{v \nmid p\infty} P_{\beta,v,\mathfrak{b}}(\chi_E(\pi_v)^{-1}|\pi_v|_v^k)\widetilde{F}(1, {}^t\beta) \prod_{v\in\Sigma} \sigma_v(\det\beta)^{k-n} \boldsymbol{e}(i\,\mathrm{tr}_{E/\mathbb{Q}}(\beta)), \tag{26}$$

where

$$\begin{aligned} D(n, K, \mathfrak{b}, p, k) &= C(n, K)N(\mathfrak{b}\mathbb{O}_E)^{-n^2}\left(2^{(1-n)n} i^{-nk}(2\pi)^{nk}\left(\pi^{n(n-1)/2}\prod_{t=0}^{n-1}\Gamma(k-t)\right)^{-1}\right)^{[E:\mathbb{Q}]} \\ &\qquad \times \prod_{i=0}^{n-1} L^p(k-i, \chi_E^{-1}\tau^i)^{-1}. \end{aligned}$$

Proof. This follows directly from (11), Corollary 6, (23), and (20). □

Given $\widetilde{F}$ as above, define

$$\widetilde{F}_{\chi} : (\mathbb{O}_K \otimes \mathbb{Z}_p) \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p) \to R$$

to be the locally constant function whose support lies in

$$(\mathbb{O}_K \otimes \mathbb{Z}_p)^{\times} \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$$

and which is defined on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^{\times} \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$ by

$$\widetilde{F}_{\chi}(x, y) = \prod_{v\in\Sigma} \chi_v(x)\widetilde{F}(1, \boldsymbol{N}_{K/E}(x)^t y), \tag{27}$$

where the product is over the primes in Σ dividing p. Then, for all $e \in \mathbb{O}_K^{\times}$,

$$\widetilde{F}_{\chi}(ex, \boldsymbol{N}_{K/E}(e^{-1})y) = \boldsymbol{N}_{k,\nu}(e)\widetilde{F}_{\chi}(x, y)$$

for all $x \in \mathbb{O}_K \otimes \mathbb{Z}_p$ and $y \in M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$. On the other hand, any locally constant function

$$F : (\mathbb{O}_K \otimes \mathbb{Z}_p) \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p) \to R$$

supported on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^{\times} \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$ which satisfies

$$F(ex, \boldsymbol{N}_{K/E}(e)^{-1}y) = \boldsymbol{N}_{E/\mathbb{Q}}(e)^k F(x, y)$$

for all $e \in \mathbb{O}_K^\times$, $x \in \mathbb{O}_K \otimes \mathbb{Z}_p$, and $y \in M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$ can be written as a linear combination of such functions $\widetilde{F}_\chi$ for Hecke characters χ of infinity type (k, ν) and conductor dividing p^∞ and functions $\widetilde{F}$ as above.

Now, let

$$G_{k,\nu,\chi,\widetilde{F}} = D(n, K, \mathfrak{b}, p, k)^{-1} E_{f_{k,\nu,\chi,\widetilde{F}}}.$$

Applying Proposition 8, we see that the Fourier coefficients of the holomorphic function $G_{k,\nu,\chi,\widetilde{F}}\big(z, \frac{1}{2}k\big)$ on $\mathcal{H}_n$ are all finite $\mathbb{Z}$-linear combinations (over a finite set of p-adic units $a \in K$) of terms of the form

$$\widetilde{F}_\chi(a, \boldsymbol{N}_{K/E}(a)^{-1}\beta)\boldsymbol{N}_{k,\nu}(a^{-1}\det\beta)\boldsymbol{N}_{E/\mathbb{Q}}(\det\beta)^{-n} \tag{28}$$

(Although π_v from Proposition 8 is a place of E for all v, the element a from (28) might be in K but not $\mathbb{O}_E$, depending on our choice of cusp. The effect of the change of a cusp $m \in GM_+(\mathbb{A}_E)$ on q-expansions is given in Lemma 4.)

Thus, we obtain the following result:

Lemma 9. *Let $k \in \mathbb{Z}_{\geq n}$ and $\nu \in \mathbb{Z}^\Sigma$. Let F be a locally constant function*

$$F : (\mathbb{O}_K \otimes \mathbb{Z}_p) \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p) \to R$$

supported on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$ which satisfies

$$F(ex, \boldsymbol{N}_{K/E}(e)^{-1}y) = \boldsymbol{N}_{k,\nu}(e)F(x, y)$$

for all $e \in \mathbb{O}_K^\times$, $x \in \mathbb{O}_K \otimes \mathbb{Z}_p$, and $y \in M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$. Then there is a C^∞ automorphic form $G_{k,\nu,F}(z, s)$ (on $U(n, n)$) of weight (k, ν) that is holomorphic at $s = \frac{1}{2}k$ and whose Fourier expansion at $s = \frac{1}{2}k$ at a cusp $m \in GM_+(\mathbb{A}_E)$ is of the form $\sum_{0<\beta\in L_m} c(\beta)q^\beta$ (where L_m is the lattice in $\mathrm{Her}_n(K)$ determined by m) with $c(\beta)$ a finite $\mathbb{Z}$-linear combination of terms of the form given in (28).

(We obtain $G_{k,\nu,F}$ as a linear combination of the automorphic forms $G_{k,\nu,\chi,\widetilde{F}}$.)

4. Differential operators

4.1. ***C^∞ differential operators.*** In this section, we summarize results on C^∞ differential operators that were studied extensively by Shimura [1984a; 1984b; 1997, Section 23; 2000, Section 12]. Let $T = M_{n\times n}(\mathbb{C})$; we identify T with the tangent space of $\mathcal{H}_n$. For each nonnegative integer d, let $\mathfrak{S}_d(T)$ denote the vector space of $\mathbb{C}$-valued homogeneous polynomial functions on T of degree d. (For instance, the e-th power of the determinant function, $\det^e$, is in $\mathfrak{S}_{ne}(T)$.) We denote by τ^d the representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ on $\mathfrak{S}_d(T)$ defined by

$$\tau^d(a, b)g(z) = g({}^t azb)$$

for all $a, b \in \mathrm{GL}_n(\mathbb{C})$, $z \in T$, and $g \in \mathfrak{S}_d(T)$.

The classification of the irreducible subspaces of polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ and of irreducible subspaces of τ^r for each r is provided in [Shimura 1984b, Section 2; 1997, Sections 12.6 and 12.7]. We summarize the key features needed for our results; further details can be found in those two references. Given a matrix $a \in M_{n\times n}(\mathbb{C})$, let $\det_j(a)$ denote the determinant of the upper left $j \times j$ submatrix of a. Each polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ can be composed into a direct sum of irreducible representations of $\mathrm{GL}_n(\mathbb{C})$. Each irreducible representation ρ of $\mathrm{GL}_n(\mathbb{C})$ contains a unique eigenvector p of highest weight $r_1 \geq \cdots r_n \geq 0$ (for a unique ordered n-tuple $r_1 \geq \cdots \geq r_n \geq 0$ of integers dependent on ρ), which is a common eigenvector of the upper triangular matrices of $\mathrm{GL}_n(\mathbb{C})$ and satisfies

$$\rho(a)p = \prod_{j=1}^{n} \det{}_j(a)^{e_j} p,$$

$$e_j = r_j - r_{j+1}, \quad 1 \leq j \leq n-1, \tag{29}$$

$$e_n = r_n \tag{30}$$

for all a in the subgroup of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. Also, for each ordered n-tuple $r_1 \geq \cdots \geq r_n \geq 0$, there is a unique corresponding irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$. If ρ and σ are irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ then, by [Shimura 2000, Theorem 12.7], $\rho \otimes \sigma$ occurs in τ^r if and only if ρ and σ are representations of the same highest weights $r_1 \geq \cdots \geq r_n$ as each other and $r_1 + \cdots + r_n = r$. In this case, $\rho \otimes \sigma$ occurs with multiplicity one in τ^r, and the corresponding irreducible subspace of τ^r contains the polynomial $p(x) = \prod_{j=1}^{n} \det_j(x)^{e_j}$ (where e_j is defined as in (29) and (30)); this polynomial $p(x)$ is an eigenvector of highest weight with respect to both ρ and σ.

Let (Z, τ_Z) be an irreducible subspace of $(\mathfrak{S}_d, \tau)$ of highest weight $r_1 \geq \cdots \geq r_n$, and let $\zeta \in Z$. By [Shimura 1984b; 1997, Section 23; 2000, Section 13], there are C^∞ differential operators $D_k(\zeta)$ that act on C^∞ functions on $\mathcal{H}_n$ and have the property that, for all $\alpha \in U(\eta_n)$, $\zeta \in Z \subseteq \mathfrak{S}_d(T)$, and complex numbers s,

$$D_k(\zeta)(\delta^s \|_{k,\nu}\alpha) = i^d \psi_Z(-k-s)(\delta^s \|_{k,\nu}\alpha) \cdot \zeta({}^t\eta^{-1}\,{}^t\overline{\lambda_\alpha}\,{}^t\mu_\alpha^{-1}), \tag{31}$$

where (as proved in [Shimura 1984b, Theorem 4.1])

$$\psi_Z(s) = \prod_{h=1}^{n} \prod_{j=1}^{r_h} (s - j + h).$$

If ρ is the representation of $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$, there is a differential operator D_ρ^Z (defined in [Eischen 2012, p. 222; Shimura 2000, Equation (12.20)]) such that for all C^∞ functions f on $\mathcal{H}_n$, $D_\rho^Z f$ is a $\mathrm{Hom}(Z, \mathbb{C})$-valued C^∞ function on $\mathcal{H}_n$ with the property that

$$(D_\rho^Z f)\|_{\rho\otimes\tau_Z}\alpha = D_\rho^Z(f\|_\rho\alpha) \tag{32}$$

for all $\alpha \in G$. Furthermore, if ρ is defined by $\rho(a,b) = \det(b)^k$ then, as the proof of [Shimura 1997, Lemma 23.4] explains,

$$D_k(\zeta)f = (D_\rho^Z f)(\zeta).$$

When Z is a Σ-tuple $(Z_v)_{v\in\Sigma}$, we also use ψ_Z to denote $\prod_{v\in\Sigma}\psi_{Z_v}$.

So, for example, if $d \in \mathbb{Z}_{\geq 0}$ and $\zeta = \det^d$, then (31) becomes

$$\begin{aligned} D_k(\det^d)(\delta^s\|_{k,\nu}\alpha) &= i^{nd}\psi_Z(-k-s)\delta^s\|_{k,\nu}\alpha\cdot\det^d({}^t\eta^{-1}\,{}^t\overline{\lambda_\alpha}\cdot{}^t\mu_\alpha^{-1}) \\ &= \left(\tfrac{1}{2}i\right)^{nd}\prod_{h=1}^{n}\prod_{j=1}^{d}(-k-s-j+h)\delta^{s-d}\|_{k+2d,\nu-d}\alpha. \end{aligned}$$

Consequently, if $d = (d(\sigma))_{\sigma\in\Sigma} \in \mathbb{Z}_{\geq 0}^{\Sigma}$, then

$$\begin{aligned} &\left(\prod_{\sigma\in\Sigma} D_k(\det^{d(\sigma)})\right)\left(G_{k,\nu,F}\left(z,\tfrac{1}{2}k\right)\right) \\ &\qquad = \prod_{\sigma\in\Sigma}\left(\tfrac{1}{2}i\right)^{nd(\sigma)}\prod_{h=1}^{n}\prod_{j=1}^{d(\sigma)}(-k-j+h)G_{k+2d,\nu-d,F}\left(z,\tfrac{1}{2}k\right) \end{aligned}$$

as in [Eischen 2013, Equation (43)].

As noted in [Shimura 1984b, Section 6], $G_{k,\nu,F}(z,s)$ is a special case of the automorphic form $G_{k,\nu,\zeta,F}(z,s)$ that satisfies

$$D_k(\zeta)\left(G_{k,\nu,F}\left(z,\tfrac{1}{2}k\right)\right) = \prod_{v\in\Sigma} i^{d_v}\psi_{Z_v}(-k)G_{k,\nu,\zeta,F}\left(z,\tfrac{1}{2}k\right),$$

where

$$D_k(\zeta) = \prod_{v\in\Sigma} D_k(\zeta_v).$$

The case where ζ is a highest-weight vector will be of particular interest to us.

4.2. ***Rational representations.*** In order to generalize our discussion from the C^∞ setting to the p-adic setting, we introduce rational representations, following [Hida 2004, Section 8.1.2] (which, in turn, summarizes relevant results from [Hida 2000; Jantzen 1987]).

Let A be a ring or a sheaf of rings over a scheme. Let B denote the Borel subgroup of GL_n consisting of upper triangular matrices in GL_n. Let N denote the unipotent radical of B. Let $T \cong B/N$ denote the torus. Following the notation of

[Hida 2004, Section 8.1.2], for each character κ of T we define

$$\begin{aligned} R_A[\kappa] &= \mathrm{Ind}_B^{\mathrm{GL}_n}(\kappa) \\ &= \{ f : \mathrm{GL}_n/N \to \mathbf{A}^1 \mid f(ht) = \kappa(t) f(h) \text{ for all } t \in T,\ h \in \mathrm{GL}_n/N \}. \end{aligned}$$

The group GL_n acts on $R_A[\kappa]$ via

$$(g \cdot f)(x) = f(g^{-1}x).$$

As noted in [Hida 2004, p. 332], there is a unique (up to an A-unit multiple) N-invariant linear form ℓ_{can} in the dual space $R_A[\kappa]^\vee$ that generates $(R_A[\kappa]^\vee)^N$ and can be normalized so that, for all $f \in R_A[\kappa]$,

$$\ell_{\mathrm{can}}(f) = f(1_n),$$

where 1_n denotes the origin in GL_n/N.

Note that, for each C^∞ automorphic form f on $\prod_{v\in\Sigma} \mathcal{H}_n$ such that $f\|_{k,\nu}\alpha = f$ (for all α in some congruence subgroup) and each highest-weight vector ζ in an irreducible representation of highest weight κ, we may view $D_k(\zeta)f$ as an $R_{\mathbb{C}}[\det^{k+\nu} \cdot \kappa] \otimes R_{\mathbb{C}}[\det^{-\nu} \cdot \kappa]$-valued function on $\mathcal{H}_n$. We define a corresponding character $\kappa_{k,\nu}(t_1, \dots, t_n, t_{n+1}, \dots, t_{2n}) = \prod_{i=1}^n t_i^{k+\nu} t_{i+n}^{-\nu}$ on $T(\mathbb{C}) \times T(\mathbb{C})$.

4.3. ***The algebraic geometric setting.*** As explained in detail in [Eischen 2012, Section 8.4], which generalizes [Katz 1978, Section 2.3], the C^∞ differential operators discussed by Shimura have a geometric interpretation in terms of the Gauss–Manin connection. C^∞ automorphic forms can [Eischen 2012, Section 2] be interpreted as sections of a vector bundle on (the complex analytification of) the moduli spaces $\mathcal{M}_{n,n} = \mathrm{Sh}(W)$. Applying a differential operator (as discussed in [Eischen 2012, Sections 6–9]) to an automorphic form of weight ρ on $\mathcal{M}_{n,n}$ sends it to an automorphic form of weight $\rho \otimes \tau$ on $\mathcal{M}_{n,n}$.

We now recall the setting of [Eischen 2013, Section 3], as we will momentarily be in a similar (but not identical) situation. For any $\mathbb{O}_K$-algebra R, the R-valued points of ${}_{\mathcal{K}}\mathrm{Sh}(R)$ parametrize tuples $\underline{A}$ consisting of an abelian variety together with a polarization, endomorphism, and level structure. (We shall not need further details of these points here; see [Lan 2013, Chapter 1; Hida 2004, Chapter 7; Eischen 2012, Section 2] for more details.) Given a point $\underline{A}$ in ${}_{\mathcal{K}}\mathrm{Sh}(R)$, we write $\underline{\omega}_{\underline{A}/R} = \underline{\omega}^+_{\underline{A}/R} \oplus \underline{\omega}^-_{\underline{A}/R}$ for the sheaf of one-forms on $\underline{A}$. (As in [Eischen 2012, Section 2], $\underline{\omega}^+_{\underline{A}/R}$ and $\underline{\omega}^-_{\underline{A}/R}$ are the rank-n submodules determined by the action of $\mathbb{O}_K$.) We identify $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/\mathcal{K}$ (which we identify with copies of $\mathcal{H}_n$) with the points of ${}_{\mathcal{K}}\mathrm{Sh}(\mathbb{C})$; we shall write $\underline{A}(z)$ to mean the point of $\underline{A}$ identified with $z \in \prod_{v\in\Sigma} \mathcal{H}_n$ under this identification. Under this identification, if we fix an ordered basis of differentials $u_1^\pm, \dots, u_n^\pm$ for $\underline{\omega}^\pm_{A_{\mathrm{univ}}/\mathcal{H}_n}$, then an automorphic form

f on $\mathcal{H}_n$ corresponds to an automorphic form $\tilde{f}$ on ${}_{\mathcal{K}}\mathrm{Sh}(\mathbb{C})$ via

$$f(z) = \tilde{f}(\underline{A}(z), u_1^{\pm}(z), \ldots, u_n^{\pm}(z)),$$

Any other ordered basis of differentials for $\underline{\omega}^{\pm}_{\underline{A}/\mathbb{C}}$ is simply obtained by the linear action of $\mathrm{GL}_n(\mathbb{O}_K \otimes \mathbb{C}) \cong \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ on $\underline{\omega}(z) = \underline{\omega}(z)^+ \oplus \underline{\omega}(z)^-$, and

$$\tilde{f}\big(\underline{A}(z), g \cdot (u_1^{\pm}(z), \ldots, u_n^{\pm}(z))\big) = g \cdot \big(f(\underline{A}(z), u_1^{\pm}(z), \ldots, u_n^{\pm}(z))\big)$$

4.3.1. *A p-adic analogue.* In [Eischen 2012, Section 9], we discussed a p-adic analogue θ_ρ^Z of the differential operators D_ρ^Z. The differential operators θ_ρ^Z act on sections of certain vector bundles on the Igusa tower $T_{\infty,\infty}$ (a formal scheme over the ordinary locus of ${}_{\mathcal{K}}\mathrm{Sh}(R)$ for R a mixed characteristic discrete valuation ring with residue characteristic p); for details on the Igusa tower, see [Hida 2004, Section 8]. More precisely, θ_ρ^Z acts on sections of $R_{T_{\infty,\infty}}[\kappa]$ for various weights κ. By [Hida 2004, map (8.4)],

$$\ell_{\mathrm{can}} : H^0(T_{\infty,\infty}, R_{T_{\infty,\infty}}[\kappa]) \to V^N[\kappa] \tag{33}$$

is an injective map into the space $V^N[\kappa] = V^N_{\infty,\infty}[\kappa]$ of p-adic modular forms of weight κ. Given a highest-weight vector ζ in Z, we define $\theta(\zeta) := \theta_k := \ell_{\mathrm{can}} \circ \theta_\rho^Z$, where $\rho(a, b) := \det(b)^k$.

In [Eischen 2012, Section 9], we gave a formula for the action of p-adic differential operators θ_ρ^Z on q-expansions. In particular, if the q-expansion of a scalar weight form $f \in H^0(T_{\infty,\infty}, R_{T_{\infty,\infty}}[\kappa])$ at a cusp $m \in GM$ is

$$f(q) = \sum_\beta a(\beta) q^\beta,$$

and ζ is a highest-weight vector, then it follows from the formulas in [Eischen 2012, Section 9] that

$$(\theta(\zeta) f)(q) = \sum_\beta a(\beta) \cdot \zeta(\beta) q^\beta. \tag{34}$$

5. A p-adic Eisenstein measure with values in the space of vector-weight automorphic forms

5.1. ***p*-adic Eisenstein series.** As we explain in Theorem 10, when R is a (profinite) p-adic ring, we can extend Theorem 2 to the case of continuous (not necessarily locally constant) functions F. For the remainder of the paper, let N be as in Section 4.2.

Theorem 10. *Let R be a* (*profinite*) *p-adic $\mathbb{O}_K$-algebra. Fix an integer $k \geq n$, and let $\nu = (\nu(\sigma))_{\sigma \in \Sigma} \in \mathbb{Z}^\Sigma$. Let*

$$F : (\mathbb{O}_K \otimes \mathbb{Z}_p) \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p) \to R$$

be a continuous function supported on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p)$ *which satisfies*

$$F(ex, \boldsymbol{N}_{K/E}(e)^{-1} y) = \boldsymbol{N}_{k,\nu}(e) F(x, y)$$

for all $e \in \mathbb{O}_K^\times$, $x \in \mathbb{O}_K \otimes \mathbb{Z}_p$ *and* $y \in \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p)$. *Then there exists a* p*-adic automorphic form* $G_{k,\nu,F}$ *whose* q*-expansion at a cusp* $m \in GM$ *is of the form* $\sum_{0<\beta\in L_m} c(\beta) q^\beta$ (*where* L_m *is the lattice in* $\mathrm{Her}_n(K)$ *determined by* m), *with* $c(\beta)$ *a finite* $\mathbb{Z}$*-linear combination of terms of the form*

$$F(a, \boldsymbol{N}_{K/E}(a)^{-1}\beta) \boldsymbol{N}_{k,\nu}(a^{-1} \det \beta) \boldsymbol{N}_{E/\mathbb{Q}}(\det \beta)^{-n}$$

(*where the linear combination is the sum over a finite set of* p*-adic units* $a \in K$ *dependent upon* β *and the choice of cusp* $m \in GM$).

Proof. The proof is similar to the proof of [Katz 1978, Theorem (3.4.1)]. We remind the reader of the idea of that result. For each integer $j \geq 1$, define

$$F_j : (\mathbb{O}_K \otimes \mathbb{Z}_p) \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p) \to R/p^j R$$
$$F_j(x, y) = F(x, y) \bmod p^j R.$$

Then F_j is a locally constant function satisfying the conditions of Theorem 2. So, by the q-expansion principle for p-adic forms [Hida 2005, Corollary 10.4; Hida 2004, Section 8.4], there is a p-adic automorphic form $G_{k,\nu,F}$ whose q-expansion satisfies the conditions in the statement of the theorem. □

Corollary 11. *Let* R *be a* (*profinite*) p*-adic* $\mathbb{O}_K$*-algebra, let* $\nu = (\nu(\sigma))_{\sigma\in\Sigma} \in \mathbb{Z}^\Sigma$, *and let* $k \geq n$ *be an integer. Let*

$$F : (\mathbb{O}_K \otimes \mathbb{Z}_p) \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p) \to R$$

be a continuous function supported on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p)$ *which satisfies*

$$F(ex, \boldsymbol{N}_{K/E}(e)^{-1} yz) = \boldsymbol{N}_{k,\nu}(e) F(x, y)$$

for all $e \in \mathbb{O}_K^\times$, $x \in \mathbb{O}_K \otimes \mathbb{Z}_p$, *and* $y \in M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$. *Then*

$$G_{k,\nu,F} = G_{n,0,\boldsymbol{N}_{k-n,\nu}(x^{-1}\boldsymbol{N}_{K/E}(x)^n \det y) F(x,y)}, \tag{35}$$

where

$$\boldsymbol{N}_{k-n,\nu}(x^{-1} \boldsymbol{N}_{K/E}(x)^n \det y) F(x, y),$$

denotes the function defined by

$$(x, y) \mapsto \boldsymbol{N}_{k-n,\nu}(x^{-1} \boldsymbol{N}_{K/E}(x)^n \det y) F(x, y).$$

on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$ *and extended by* 0 *to* $(\mathbb{O}_K \otimes \mathbb{Z}_p) \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$.

Proof. This follows from the q-expansion principle [Hida 2005, Corollary 10.4]. □

Remark 12. We comment now on the relationship between the weight of $G_{n,0,F}$ and the p-adically continuous function F appearing in the subscript. By Corollary 11 and Theorem 2, we have that, if F is a locally constant function satisfying the conditions of Corollary 11, then the p-adic automorphic form $G_{n,0,N_{k-n,\nu}(x^{-1}N_{K/E}(x)^n \det y)F(x,y)}$ is the weight-(k,ν) p-adic automorphic form $G_{k,\nu,F}$. More generally, by (34), the p-adic automorphic form $G_{n,0,N_{k-n,\nu}(x^{-1}N_{K/E}(x)^n \det y)F(x,y)\zeta_\kappa(N_{K/E}(x)y^{-1})}$ is the weight-$(\kappa\cdot\kappa_{k,\nu})$ p-adic automorphic form $\theta(\zeta_\kappa)G_{k,\nu,F}$, where ζ_κ is a highest-weight vector for the representation of weight κ. In particular, the p-adic automorphic form $G_{n,0,N_{k-n,\nu}(x^{-1}N_{K/E}(x)^n \det y)F(x,y)\det(N_{K/E}(x)y^{-1})^d}$ is the p-adic automorphic form $\theta(\det^d)G_{n,0,N_{k-n,\nu}(x^{-1}N_{K/E}(x)^n \det y)F(x,y)\zeta_\kappa(N_{K/E}(x)y^{-1})}$ of weight $(k+2d,\nu-d)$.

5.1.1. *CM points and pullbacks.* In this section, we compare the values of certain p-adic automorphic forms and C^∞ automorphic forms at CM points.[3] This material extends [Eischen 2013, Section 3.0.1] beyond the case of scalar weights. Let R be an $\mathbb{O}_K$-subalgebra of $\overline{\mathbb{Q}}\cap\iota_\infty^{-1}(\mathbb{O}_{\mathbb{C}_p})$ in which p splits completely. Note that the embeddings ι_∞ and ι_p restrict to R to give embeddings

$$\iota_\infty : R \hookrightarrow \mathbb{C},$$
$$\iota_p : R \hookrightarrow R_0 = \varprojlim_m R/p^m R.$$

Let $\underline{A}$ be a CM abelian variety with PEL structure over R, i.e., a CM point of the moduli space ${}_K\mathrm{Sh}(R)$ or, equivalently, a point of $\mathrm{Sh}(U(n)\times U(n))\hookrightarrow \mathrm{Sh}(U(n,n))$. By extending scalars we may also view $\underline{A}$ as an abelian variety over $\mathbb{C}$ or R_0.

By an argument similar to [Eischen 2013, Section 3.0.1], there are complex and p-adic periods $\Omega=(\Omega^+,\Omega^-)\in(\mathbb{C}^\times)^n\times(\mathbb{C}^\times)^n$ and $c=(c^+,c^-)\in(\mathbb{O}_{\mathbb{C}_p}^\times)^n\times(\mathbb{O}_{\mathbb{C}_p}^\times)^n$, respectively, attached to each CM abelian variety $\underline{A}$ over R such that (if F is R-valued, so $G_{k,\nu,F}$ arises over R)

$$(\kappa\cdot\kappa_{k,\nu})^{-1}(\Omega)\prod_{\sigma\in\Sigma}\kappa_\sigma(2\pi i)\psi_Z(-k)G_{k,\nu,\zeta,F}\big(z;h,\chi,\mu,\tfrac{1}{2}k\big) = (\kappa\cdot\kappa_{k,\nu})^{-1}(c)\theta(\zeta)G_{k,\nu,F}(\underline{A}), \quad (36)$$

where z is a point in $\prod_{\sigma\in\Sigma}\mathcal{H}_n$ corresponding to the CM abelian variety $\underline{A}$ viewed as an abelian variety over $\mathbb{C}$ (by extending scalars to $\mathbb{C}$). Here, Z is the irreducible subrepresentation of $\prod_{v\in\Sigma}\mathrm{GL}_n(\mathbb{C})\times\mathrm{GL}_n(\mathbb{C})$ of highest weight $\kappa\in(\mathbb{Z}^n)^\Sigma$ and has ζ as a highest-weight vector; by $\kappa(a)$ with a a scalar, we mean κ evaluated at the n-tuple $(a,\dots,a)$ in the torus. (The periods Ω and c can be defined uniformly

[3] The significance of CM points is that they correspond to points of $U(n)\times U(n)\subseteq U(n,n)$, which are the points used (for instance, by Shimura) to study algebraicity of values of Eisenstein series, which are used in turn to study algebraicity of values of certain L-functions (through the doubling method, or "pull back method", a construction of L-functions described in various sources, including [Gelbart et al. 1987, Part A; Cogdell 2006, Section 2]). Determining the *precise* values of these Eisenstein series at CM points is neither necessary nor generally computationally feasible at this time.

for all CM points at once [Katz 1978, Section 5.1]. For the present paper, though, this is not necessary.) Note that when $\kappa = \det^d$ (i.e., is the highest weight for a one-dimensional representation), we recover [Eischen 2013, Equation (45)].

5.2. ***Eisenstein measures.*** In analogue with [Katz 1978, Lemma (4.2.0)] (which handles the case of Hilbert modular forms), we have the following lemma (which applies to all integers $n \geq 1$):

Lemma 13. *Let R be a p-adic $\mathbb{O}_K$-algebra. Then the inverse constructions*

$$H(x, y) = \frac{1}{\boldsymbol{N}_{n,0}(x\boldsymbol{N}_{K/E}(x)^{-n} \det y)} F(x, y^{-1}), \tag{37}$$

$$F(x, y) = \frac{1}{\boldsymbol{N}_{n,0}(x^{-1}\boldsymbol{N}_{K/E}(x)^{n} \det y)} H(x, y^{-1}) \tag{38}$$

give an R-linear bijection between the set of continuous R-valued functions

$$F : (\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p) \to R$$

satisfying

$$F(ex, \boldsymbol{N}_{K/E}(e)^{-1} y) = \boldsymbol{N}_{n,0}(e) F(x, y) \quad \textit{for all } e \in \mathbb{O}_K^\times$$

and the set of continuous R-valued functions

$$H : (\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p) \to R$$

satisfying

$$H(ex, \boldsymbol{N}_{K/E}(e) y) = H(x, y) \quad \textit{for all } e \in \mathbb{O}_K^\times.$$

Proof. The proof follows immediately from the properties of F and H. □

Let

$$\mathcal{G}_n = ((\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p))/\overline{\mathbb{O}_K^\times}, \tag{39}$$

where $\overline{\mathbb{O}_K^\times}$ denotes the p-adic closure of $\mathbb{O}_K^\times$ embedded diagonally, as $(e, \boldsymbol{N}_{K/E}(e))$, in $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p)$ (and, as before, $(\mathbb{O}_E \otimes \mathbb{Z}_p)^\times$ is embedded diagonally inside of $\mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p)$). Then Lemma 13 gives a bijection between the R-valued continuous functions H on $\mathcal{G}_n$ and the R-valued continuous functions F on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p)$ satisfying $F(ex, \boldsymbol{N}_{K/E}(e)^{-1} y) = \boldsymbol{N}_{n,0}(e) F(x, y)$ for all $e \in \mathbb{O}_K^\times$.

For any (profinite) p-adic ring R, an R-valued p-adic measure on a (profinite) compact, totally disconnected topological space Y is a $\mathbb{Z}_p$-linear map

$$\mu : \mathcal{C}(Y, \mathbb{Z}_p) \to R$$

or, equivalently [Katz 1978, Section 4.0], an R'-linear map

$$\mu : \mathcal{C}(Y, R') \to R$$

for any p-adic ring R' such that R is an R'-algebra. Instead of $\mu(f)$, one typically writes

$$\int_Y f\, d\mu.$$

In Theorem 14, we specialize to the case where R is the ring $\mathcal{V}_{n,n}$ of p-adic automorphic forms on $U(n,n)$ and Y is the group $\mathcal{G}_n$ defined in (39).

Theorem 14 (a p-adic Eisenstein measure for vector-weight automorphic forms). *Let R be a profinite p-adic ring. There is a $\mathcal{V}_{n,n}$-valued p-adic measure $\mu = \mu_{\mathfrak{b},n}$ on $\mathcal{G}_n$ defined by*

$$\int_{\mathcal{G}_n} H\, d\mu_{\mathfrak{b},n} = G_{n,0,F}$$

for all continuous R-valued functions H on $\mathcal{G}_n$, with

$$F(x,y) = \frac{1}{\boldsymbol{N}_{n,0}(x^{-1}\boldsymbol{N}_{K/E}(x)^n \det y)} H(x, y^{-1})$$

extended by 0 *to all of* $(\mathbb{O}_K \otimes \mathbb{Z}_p) \times M_{n\times n}(\mathbb{O}_E \otimes \mathbb{Z}_p)$.

Proof. F is the function corresponding to H under the bijection in Lemma 13. The theorem then follows immediately from Theorem 10, Corollary 11, Lemma 13, and the q-expansion principle. □

Note that the measure $\mu_{\mathfrak{b},n}$ depends only upon n and $\mathfrak{b}$. In Section 6, we relate the measure $\mu_{\mathfrak{b},n}$ to the Eisenstein measure in [Katz 1978, Definition (4.2.5) and Equation (5.5.7)] and comment on how $\mu_{\mathfrak{b},n}$ can be modified to the case of Siegel modular forms (i.e., automorphic forms on symplectic groups).

It follows from the definition of the measure $\mu_{\mathfrak{b},n}$ in Theorem 14 that, for each highest-weight vector ζ_κ of highest weight κ,

$$\int_{\mathcal{G}_n} H(x,y)\zeta_\kappa(\boldsymbol{N}_{K/E}(x)y^{-1})\, d\mu_{\mathfrak{b},n} = \theta(\zeta_\kappa) G_{n,0,F(x,y)}.$$

Now, let $\underline{A}$ be an ordinary CM abelian variety with PEL structure over a subring R of $\overline{\mathbb{Q}} \cap \mathbb{O}_{\mathbb{C}_p}$, i.e., a CM point of the moduli space ${}_K\mathrm{Sh}(R)$, or equivalently, a point of $\mathrm{Sh}(U(n)\times U(n)) \hookrightarrow \mathrm{Sh}(U(n,n))$. As discussed above, by extending scalars, we may also view $\underline{A}$ as an abelian variety over $\mathbb{C}$ or over $R_0 = \varprojlim_m R/p^m R$. It follows from (36) and Corollary 11 that, for $F(x,y)$ locally constant, supported on $(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathbb{O}_E \otimes \mathbb{Z}_p)$ and satisfying

$$F(ex, \boldsymbol{N}_{K/E}(e)^{-1}y) = \boldsymbol{N}_{k,\nu}(e) F(x,y)$$

for all $e \in \mathcal{O}_K^\times$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in \mathrm{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$,

$$(\kappa \cdot \kappa_{k,\nu})^{-1}(c) \int_{\mathcal{G}_n} \frac{1}{\boldsymbol{N}_{k,\nu}(x\boldsymbol{N}_{K/E}(x)^{-n} \det y)} F(x, y^{-1}) \zeta_\kappa(\boldsymbol{N}_{K/E}(x) y^{-1})\, d\mu_{\mathfrak{b},n}(\underline{A}) \\ = (\kappa \cdot \kappa_{k,\nu})^{-1}(\Omega) \prod_{\sigma \in \Sigma} \kappa_\sigma(2\pi i) \psi_Z(-k) G_{k,\nu,\zeta_\kappa,F}\big(z, \tfrac{1}{2}k\big), \quad (40)$$

and, for any $d = (d_\nu)_{\nu \in \Sigma} \in \mathbb{Z}_{\geq 0}^\Sigma$,

$$(\kappa_{k+2d,\nu-d})^{-1}(c) \\ \times \int_{\mathcal{G}_n} \frac{1}{\boldsymbol{N}_{k,\nu}(x\boldsymbol{N}_{K/E}(x)^{-n} \det y)} F(x, y^{-1}) \det(\boldsymbol{N}_{K/E}(x) y^{-1})^d\, d\mu_{\mathfrak{b},n}(\underline{A}) \\ = (\kappa_{k+2d,\nu-d})^{-1}(\Omega) \prod_{\sigma \in \Sigma} (2\pi i)^{nd} \psi_Z(-k) G_{k+2d,\nu-d,F(x,y)}\big(z, \tfrac{1}{2}k\big),$$

where z is a point in $\prod_{\sigma \in \Sigma} \mathcal{H}_n$ corresponding to the CM abelian variety $\underline{A}$ viewed as an abelian variety over $\mathbb{C}$ (by extending scalars to $\mathbb{C}$) and Ω and c are the periods from (36). Here, Z is the irreducible subrepresentation of $\prod_{\sigma \in \Sigma} \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ of highest weight κ and has ζ_κ as a highest-weight vector; by $\kappa(a)$ with a a scalar, we mean κ evaluated at the n-tuple $(a, \dots, a)$ in the torus.

In other words, the p-adic measure $\mu_{\mathfrak{b},n}$ allows us to p-adically interpolate the values of the C^∞ (not necessarily holomorphic) function $G_{k,\nu,\zeta_\kappa,F}\big(z, \tfrac{1}{2}k\big)$ at CM points z.

Theorem 15. *For each ordinary abelian variety $\underline{A}$ defined over a (profinite) p-adic $\mathcal{O}_K$-algebra R_0, there is an R_0-valued p-adic measure $\mu(\underline{A}) := \mu_{\mathfrak{b},n}(\underline{A})$ defined by*

$$\int_{\mathcal{G}_n} H\, d\mu_{\mathfrak{b},n}(\underline{A}) = G_{n,0,F}(\underline{A})$$

for all continuous R-valued functions H on $\mathcal{G}_n$, with

$$F(x, y) = \frac{1}{\boldsymbol{N}_{n,0}(x^{-1}\boldsymbol{N}_{K/E}(x)^n \det y)} H(x, y^{-1})$$

extended by 0 to all of $(\mathcal{O}_K \otimes \mathbb{Z}_p) \times M_{n\times n}(\mathcal{O}_E \otimes \mathbb{Z}_p)$. If $R_0 = \varprojlim_m R/p^m R$ with $R \subseteq \overline{\mathbb{Q}}$, $\underline{A}$ is an ordinary CM point defined over R, and F is a locally constant function supported on $(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \times \mathrm{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$ satisfying

$$F(ex, \boldsymbol{N}_{K/E}(e)^{-1} y) = \boldsymbol{N}_{k,\nu}(e) F(x, y)$$

for all $e \in \mathcal{O}_K^\times$, $x \in \mathcal{O}_K \otimes \mathbb{Z}_p$, and $y \in \mathrm{GL}_n(\mathcal{O}_E \otimes \mathbb{Z}_p)$, then

$$(\kappa \cdot \kappa_{k,\nu})^{-1}(c) \int_{\mathcal{G}_n} \frac{1}{\boldsymbol{N}_{k,\nu}(x\boldsymbol{N}_{K/E}(x)^{-n} \det y)} F(x, y^{-1}) \zeta_\kappa(\boldsymbol{N}_{K/E}(x) y^{-1})\, d\mu_{\mathfrak{b},n}(\underline{A}) \\ = (\kappa \cdot \kappa_{k,\nu})^{-1}(\Omega) \prod_{\sigma \in \Sigma} \kappa_\sigma(2\pi i) \psi_Z(-k) G_{k,\nu,\zeta_\kappa,F}\big(z, \tfrac{1}{2}k\big),$$

with $z \in \prod_{v\in\Sigma} \mathcal{H}_n$ *corresponding to the ordinary CM abelian variety* $\underline{A}$ *viewed as an abelian variety over* $\mathbb{C}$.

The pullback of an automorphic form on $U(n,n)$ to $U(n)\times U(n)$ is automatically an automorphic form on the product of definite unitary groups $U(n)\times U(n)$. So Theorem 14 also gives a p-adic measure with values in the space of automorphic forms on the product of definite unitary groups $U(n)\times U(n)$. In [Eischen 2014, Section 4], we explain how to modify our construction to obtain p-adic measures with values in the space of automorphic forms on certain nondefinite groups.

Remark 16 (relationship to the Eisenstein measures in [Eischen 2013, Section 4]). For the curious reader, we briefly explain the relationship between the measure $\mu_{\mathfrak{b},n}$ defined in Theorem 14 and the measure ϕ defined in [Eischen 2013, Theorem 20]. For each $v\in\Sigma$, let $r_v = r(v)\le n$ be a positive integer and let $r=(r_v)_v\in\mathbb{Z}^\Sigma$. As in [Eischen 2013, Equation (33)], let

$$T(r)=\prod_{v\in\Sigma}\underbrace{\mathcal{O}_{E_v}^\times\times\cdots\times\mathcal{O}_{E_v}^\times}_{r_v\text{ copies}}. \tag{41}$$

Let $\rho=\prod_{v\in\Sigma}(\rho_{1,v},\ldots,\rho_{r(v),v})$ be a p-adic character on $T(r)$ (i.e., $\rho((\alpha_v)_{v\in\Sigma}) := \prod_{v\in\Sigma}\prod_{i=1}^{r(v)}\rho_{i,v}(\alpha_v)$ for all $\alpha=(\alpha_v)_{v\in\Sigma}\in T(r)$), let $n=n_{1,v}+\cdots+n_{r_v,v}$ be a partition of n for each $v\in\Sigma$, and let F_ρ be the function on $M_{n\times n}(E)$ defined by

$$F_\rho(x) := \prod_{v\in\Sigma}\prod_{i=1}^{r(v)}\rho_{i,v}(\det_{n_i}(x)),$$

with $\det_j$ defined as on page 2455. Let χ be a p-adic function supported on $(\mathcal{O}_K\otimes\mathbb{Z}_p)^\times/\overline{\mathcal{O}_K^\times}$ and extended by 0 to all of $\mathcal{O}_K\otimes\mathbb{Z}_p$. Let $H_{\rho,\chi}$ be the function corresponding via the bijection in Lemma 13 to the function $F_{\rho,\chi}$ supported on $\mathcal{G}_n$ (and extended by 0) defined by

$$F_{\rho,\chi}(x,y)=\chi(x)\mathbb{N}_{n,0}(x)F_\rho(N_{K/E}(x)^t y).$$

Then

$$\int_{\mathcal{G}_n} H_{\rho,\chi}\,d\mu_{\mathfrak{b},n}=\int_{(\mathcal{O}_K\otimes\mathbb{Z}_p)^\times/\overline{\mathcal{O}_K^\times}\times T(r)}(\chi,\rho)\,d\phi.$$

Note that the measure ϕ is dependent upon the choice of r and the choice of the partition of n, while the measure $\mu_{\mathfrak{b},n}$ is independent of both of these choices.

6. Remarks about the case of symplectic groups, Siegel modular forms, and Katz's Eisenstein measure for Hilbert modular forms

The case of Siegel modular forms is quite similar. We essentially just need to replace the CM field K with the totally real field E throughout. Once we have replaced

K by E, $\boldsymbol{N}_{k,\nu}$ becomes $\boldsymbol{N}_{E/\mathbb{Q}}^k$ and $\boldsymbol{N}_{K/E}$ becomes the identity map. Consequently, (37) and (38) become

$$H(x, y) = \frac{1}{\boldsymbol{N}_{E/\mathbb{Q}}(x^{1-n}\det y)^n} F(x, y^{-1}),$$
$$F(x, y) = \frac{1}{\boldsymbol{N}_{E/\mathbb{Q}}(x^{-1+n}\det y)^n} H(x, y^{-1}).$$

To highlight the similarity with [Katz 1978, Section 4.2] we note that, when $n = 1$, these equations become

$$H(x, y) = \frac{1}{\boldsymbol{N}_{E/\mathbb{Q}}(y)} F(x, y^{-1}),$$
$$F(x, y) = \frac{1}{\boldsymbol{N}_{E/\mathbb{Q}}(y)} H(x, y^{-1}).$$

This relationship between H and F is similar to the relationship between the similar functions denoted H and F by Katz [1978, Section 4.2]. (The minor difference is due to the fact that, throughout the paper, his $F(x, y)$ is our $F(y, x)$.)

The differential operators are developed from the C^∞ perspective simultaneously for both unitary and symplectic groups in [Shimura 2000, Section 12]. As noted in [Eischen 2012, p. 4; 2012, Section 3.1.1; Panchishkin 2005; Courtieu and Panchishkin 2004], the algebraic geometric and p-adic formulation of the operators for Siegel modular forms (i.e., for symplectic groups) is similar. In the case of Siegel modular forms, the algebraic geometric formulation of the differential operators is discussed in [Harris 1981, Section 4]. Also, the case of symplectic groups is handled directly alongside the case of unitary groups in Hida's discussion [2004, Chapter 8] of p-adic automorphic forms. So the construction in this paper carries over with only minor changes (essentially, replacing K by E throughout) to the case of symplectic groups over a totally real field E and automorphic forms (Siegel modular forms) on those groups.

6.1. *The case $n = 1$.* Continuing with the symplectic case with $n = 1$, Theorem 2 becomes:

Theorem 17. *Let R be an $\mathcal{O}_E$-algebra and let $k \geq 1$ be an integer. For each locally constant function*

$$F : (\mathcal{O}_E \otimes \mathbb{Z}_p) \times (\mathcal{O}_E \otimes \mathbb{Z}_p) \to R$$

supported on $(\mathcal{O}_E \otimes \mathbb{Z}_p)^\times \times (\mathcal{O}_E \otimes \mathbb{Z}_p)^\times$ which satisfies

$$F(ex, e^{-1}y) = \boldsymbol{N}_{E/\mathbb{Q}}(e)^k F(x, y) \tag{42}$$

for all $e \in \mathcal{O}_E^\times$, $x \in \mathcal{O}_E \otimes \mathbb{Z}_p$, and $y \in \mathcal{O}_E \otimes \mathbb{Z}_p$, there is a Hilbert modular form $G_{k,F}$ of weight k defined over R whose q-expansion at a cusp $m \in GM$ is of the form

$\sum_{\beta>0} c(\beta)q^\beta$ (*where* L_m *is the lattice in* E *determined by* m) *with* $c(\beta)$ *a finite* $\mathbb{Z}$*-linear combination of terms of the form*

$$F(a, (a)^{-1}\beta)\boldsymbol{N}(a^{-1}\beta)^k \boldsymbol{N}_{E/\mathbb{Q}}(\beta)^{-1}$$

(*where the linear combination is a sum over a finite set of* p*-integral* $a \in E$ *dependent upon* β *and the choice of cusp* $m \in GM$).

Still continuing with the symplectic case with $n = 1$, Theorem 14 becomes:

Theorem 18. *There is a measure* μ *on*

$$\mathscr{G} = ((\mathbb{O}_E \otimes \mathbb{Z}_p)^\times \times (\mathbb{O}_E \otimes \mathbb{Z}_p)^\times)/\overline{\mathbb{O}_E^\times}$$

(*with values in the space of* p*-adic Hilbert modular forms*), *defined by*

$$\int_{\mathscr{G}} H \, d\mu = G_{1,F}$$

for all continuous R*-valued functions* H *on* $\mathscr{G}$, *with*

$$F(x, y) = \frac{1}{\boldsymbol{N}_{E/\mathbb{Q}}(y)} H(x, y^{-1})$$

extended by 0 *to all of* $(\mathbb{O}_E \otimes \mathbb{Z}_p) \times (\mathbb{O}_E \otimes \mathbb{Z}_p)$.

Note that we have essentially recovered the Eisenstein series and measure from [Katz 1978, Definition (4.2.5)]. (Again, the difference between Katz's order of the variables x and y and ours is due to the fact that, throughout the paper, his $F(x, y)$ is our $F(y, x)$.) The reader might notice the similarities with [Katz 1978, (5.5.1)–(5.5.7)]. In particular, let χ be a Grössencharacter of the CM field K whose conductor divides p^∞ and whose infinity type is

$$-k \sum_{\sigma\in\Sigma} \sigma - \sum_{\sigma\in\Sigma} d(\sigma)(\sigma - \bar{\sigma})$$

with $d(\sigma) \geq 0$ for all $\sigma \in \Sigma$ and $k \geq n$. We view χ as an $\mathbb{O}_{\mathbb{C}_p}$-valued character on $\mathbb{A}^{\infty,\times} \times \prod_{v\in\Sigma} \overline{\mathbb{Q}}$ (by restricting it to this group) and consider its restriction to the subring consisting of elements $((1_v)_{v\nmid p\infty}, a, a)$, with $a \in \mathbb{O}_K \otimes \mathbb{Z}_{(p)}$, which is a subring of

$$(\mathbb{O}_K \otimes \mathbb{Z}_p)^\times \xrightarrow{\sim} (\mathbb{O}_E \otimes \mathbb{Z}_p)^\times \times (\mathbb{O}_E \otimes \mathbb{Z}_p)^\times.$$

Then we have

$$\chi(\alpha) = \chi_{\text{finite}}(\alpha) \cdot \frac{\prod_{\sigma\in\Sigma} \sigma(\overline{\alpha})^{d(\sigma)}}{\prod_{\sigma\in\Sigma} \sigma(\alpha)^{k+d(\sigma)}},$$

$$\chi(x, y) = \chi_{\text{finite}}(x, y) \cdot \frac{\prod_{\sigma\in\Sigma} \sigma(x)^{d(\sigma)}}{\prod_{\sigma\in\Sigma} \sigma(y)^{k+d(\sigma)}},$$

with χ_{finite} a locally constant function. If

$$F(x, y) = \frac{1}{\boldsymbol{N}(y)}\chi\left(x, \frac{1}{y}\right) = \chi_{\text{finite}}\left(x, \frac{1}{y}\right) \cdot \boldsymbol{N}(y)^{k-1} \prod_{\sigma\in\Sigma} \sigma(xy)^{d(\sigma)}, \tag{43}$$

then

$$\int_{\mathscr{G}} \chi(x, y)\, d\mu_{\mathfrak{b},1} = G_{1,F} \tag{44}$$

$$= G_{1,\chi_{\text{finite}}(x,1/y)\boldsymbol{N}(y)^{k-1}\prod_{\sigma\in\Sigma}\sigma(xy)^{d(\sigma)}} \tag{45}$$

$$= G_{k,\chi_{\text{finite}}(x,1/y)\prod_{\sigma\in\Sigma}\sigma(xy)^{d(\sigma)}} \tag{46}$$

$$= \left(\prod_{\sigma\in\Sigma} \theta(\sigma)^{d(\sigma)}\right)(G_{k,\chi_{\text{finite}(x,1/y)}}), \tag{47}$$

where $\theta(\sigma)$ denotes the (σ component of the) differential operator $\theta(\det)$ acting on automorphic forms in the one-dimensional, symplectic case. Note the similarity of (43) through (47) with [Katz 1978, Equations (5.5.6)–(5.5.7)].

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