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Vladimir Chernousov and Alexander Merkurjev

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Essential dimension of spinor and Clifford groups

Vladimir Chernousov and Alexander Merkurjev

We conclude the computation of the essential dimension of split spinor groups, and an application to algebraic theory of quadratic forms is given. We also compute essential dimension of the split even Clifford group or, equivalently, of the class of quadratic forms with trivial discriminant and Clifford invariant.

1. Introduction

We recall briefly the definition of the essential dimension.

Let F be a field, and let $\mathcal{F} : \mathit{Fields}/F \to \mathit{Sets}$ be a functor from the category of field extensions over F to the category of sets. Let $E \in \mathit{Fields}/F$ and $K \subset E$ a subfield over F. We say that K is a *field of definition* of an element $\alpha \in \mathcal{F}(E)$ if α belongs to the image of the map $\mathcal{F}(K) \to \mathcal{F}(E)$. The *essential dimension of* α, denoted $\operatorname{ed}^{\mathcal{F}}(\alpha)$, is the least transcendence degree $\operatorname{tr.deg}_F(K)$ over all fields of definition K of α. The *essential dimension of the functor* $\mathcal{F}$ is

$$\operatorname{ed}(\mathcal{F}) = \sup\{\operatorname{ed}^{\mathcal{F}}(\alpha)\},$$

where the supremum is taken over all fields $E \in \mathit{Fields}/F$ and all $\alpha \in \mathcal{F}(E)$ (see [Berhuy and Favi 2003, Definition 1.2] or [Merkurjev 2009, §1]). Informally, the essential dimension of $\mathcal{F}$ is the smallest number of algebraically independent parameters required to define $\mathcal{F}$ and may be thought of as a measure of complexity of $\mathcal{F}$.

Let p be a prime integer. The *essential p-dimension of* $\alpha \in \mathcal{F}(E)$, denoted $\operatorname{ed}_p^{\mathcal{F}}(\alpha)$, is defined as the minimum of $\operatorname{ed}^{\mathcal{F}}(\alpha_{E'})$, where E' ranges over all finite field extensions of E of degree prime to p and $\alpha_{E'}$ is the image of α under the map $\mathcal{F}(E) \to \mathcal{F}(E')$. The *essential p-dimension of* $\mathcal{F}$ is

$$\operatorname{ed}_p(\mathcal{F}) = \sup\{\operatorname{ed}_p^{\mathcal{F}}(\alpha)\},$$

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where the supremum ranges over all fields $E \in \mathit{Fields}/F$ and all $\alpha \in \mathcal{F}(E)$. By definition, $\mathrm{ed}(\mathcal{F}) \geq \mathrm{ed}_p(\mathcal{F})$ for all p.

For convenience, we write $\mathrm{ed}_0(\mathcal{F}) = \mathrm{ed}(\mathcal{F})$, so $\mathrm{ed}_p(\mathcal{F})$ is defined for $p = 0$ and all prime p.

Let G be an algebraic group scheme over F. Write $\mathcal{F}_G$ for the functor taking a field extension E/F to the set $H^1_{\text{ét}}(E, G)$ of isomorphism classes of principal homogeneous G-spaces (G-torsors) over E. The essential (p-)dimension of $\mathcal{F}_G$ is called the *essential* (p-)*dimension of* G and is denoted by $\mathrm{ed}(G)$ and $\mathrm{ed}_p(G)$ (see [Reichstein 2000; Reichstein and Youssin 2000]). Thus, the essential dimension of G measures complexity of the class of principal homogeneous G-spaces.

In this paper, we conclude the computation of the essential dimension of the split spinor groups $\mathbf{Spin}_n$ originated in [Brosnan et al. 2010; Garibaldi 2009] and continued in [Merkurjev 2009] (Theorem 2.2). In the missing case $n = 4m \geq 16$, we prove that

$$\mathrm{ed}_2(\mathbf{Spin}_n) = \mathrm{ed}(\mathbf{Spin}_n) = 2^{(n-2)/2} + 2^m - \frac{n(n-1)}{2},$$

where 2^m is the largest power of 2 dividing n. The value of $\mathrm{ed}(\mathbf{Spin}_n)$ is surprisingly large. Recall a striking consequence of this (see [Brosnan et al. 2010, Theorem 1-1]): the Pfister number $\mathrm{Pf}(3, n)$ is at least exponential in n.

In Theorem 4.2, we give an application in algebraic theory of quadratic forms. Precisely, we determine all pairs (n, b) of natural numbers (with two possible exceptions) such that, for every field F, any quadratic form in $I^3(F)$ of dimension n contains a subform of trivial discriminant of dimension b. This result, stated entirely in terms of algebraic theory of quadratic forms, is proved using the tools of the essential dimension!

Theorem 4.2 is applied later in the paper for the computation of the essential dimension of split even Clifford group $\mathbf{\Gamma}_n^+$ or, equivalently, of the functor given by n-dimensional quadratic forms with trivial discriminant and Clifford invariant (Theorem 7.1).

We use heavily the work [Popov 1987], where the base field is assumed to be of characteristic zero. This explains the characteristic restriction in most of our results.

2. Essential dimension of $\mathbf{Spin}_n$

Let G be an algebraic group over F, and let $C \subset G$ be a normal subgroup over F. For a torsor $E \to \mathrm{Spec}(F)$ of the group $H := G/C$, consider the stack $[E/G]$ (see [Vistoli 2005]). Recall that an object of the category $[E/G](K)$ for a field extension K/F is a pair (E', φ), where E' is a G-torsor over K and $\varphi : E'/C \xrightarrow{\sim} E_K$ is an isomorphism of H-torsors over K. The essential dimension $\mathrm{ed}[E/G]$ of the stack $[E/G]$ is the essential dimension of the functor $K \mapsto$ set of isomorphism classes of objects in $[E/G](K)$.

The following was proven independently by R. Lötscher [2013, Example 3.4]:

Proposition 2.1. *Let C be a normal subgroup of an algebraic group G over F and $H = G/C$. Then*

$$\mathrm{ed}(G) \le \mathrm{ed}(H) + \max \mathrm{ed}[E/G],$$

where the maximum is taken over all field extensions L/F and all H-torsors E over L.

Proof. Let I' be a G-torsor over a field extension K/F. Then $I := I'/C$ is an H-torsor over K. There is a subextension K_0/F of K/F and an H-torsor E over K_0 such that there is an isomorphism $\varphi : I \xrightarrow{\sim} E_K$ of H-torsors and $\mathrm{tr.deg}(K_0/F) \le \mathrm{ed}(H)$.

Consider the stack $[E/G]$ over K_0. The pair (I', φ) is an object of $[E/G](K)$. There is a subextension K_1/K_0 of K/K_0 such that (I', φ) is defined over K_1 and $\mathrm{tr.deg}(K_1/K_0) \le \mathrm{ed}[E/G]$. It follows that I' is defined over the field K_1 with

$$\mathrm{tr.deg}(K_1/F) = \mathrm{tr.deg}(K_0/F) + \mathrm{tr.deg}(K_1/K_0) \le \mathrm{ed}(H) + \mathrm{ed}[E/G]. \qquad \square$$

The following theorem concludes computation of the essential dimension of the spinor groups initiated in [Brosnan et al. 2010; Garibaldi 2009] and continued in [Merkurjev 2009]. We write $\mathbf{Spin}_n$ for the split spinor group of a nondegenerate quadratic form of dimension n and maximal Witt index.

If $\mathrm{char}(F) \ne 2$, then the essential dimension of $\mathbf{Spin}_n$ has the following values for $n \le 14$ (see [Garibaldi 2009, §23]):

n	≤ 6	7	8	9	10	11	12	13	14
$\mathrm{ed}_2(\mathbf{Spin}_n) = \mathrm{ed}(\mathbf{Spin}_n)$	0	4	5	5	4	5	6	6	7

In the following theorem, we give the values of $\mathrm{ed}_p(\mathbf{Spin}_n)$ for $n \ge 15$ and $p = 0$ and 2. Note that $\mathrm{ed}_p(\mathbf{Spin}_n) = 0$ if $p \ne 0, 2$ as every $\mathbf{Spin}_n$-torsor over a field is split over an extension of degree a power of 2.

Theorem 2.2. *Let F be a field of characteristic zero. For every integer $n \ge 15$, we have*

$$\mathrm{ed}_2(\mathbf{Spin}_n) = \mathrm{ed}(\mathbf{Spin}_n) = \begin{cases} 2^{(n-1)/2} - n(n-1)/2 & \text{if } n \text{ is odd,} \\ 2^{(n-2)/2} - n(n-1)/2 & \text{if } n \equiv 2 \pmod 4, \\ 2^{(n-2)/2} + 2^m - n(n-1)/2 & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

where 2^m is the largest power of 2 dividing n.

Proof. The case $n \ge 15$ and n not divisible by 4 has been considered in [Brosnan et al. 2010, Theorem 3-3].

Now assume that $n > 15$ and n is divisible by 4. The inequality "$\geq$" was obtained in [Merkurjev 2009, Theorem 4.9], so we just need to prove the inequality "$\leq$". The case $n = 16$ was considered in [Merkurjev 2009, Corollary 4.10]. Assume that $n \geq 20$ and n is divisible by 4.

Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{Spin}_n & \longrightarrow & \mathbf{Spin}_n^+ & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{O}_n^+ & \longrightarrow & \mathbf{PGO}_n^+ & \longrightarrow & 1 \end{array}$$

where $\mathbf{Spin}_n^+$ is the semispinor group, $\mathbf{O}_n^+$ is the split special orthogonal group and $\mathbf{PGO}_n^+$ is the split special projective orthogonal group. We see from the diagram that the image of the connecting map

$$\delta_K : H^1_{\text{ét}}(K, \mathbf{Spin}_n^+) \to H^2_{\text{ét}}(K, \boldsymbol{\mu}_2) \subset \text{Br}(K)$$

is contained in the image of the other connecting map

$$H^1_{\text{ét}}(K, \mathbf{PGO}_n^+) \to H^2_{\text{ét}}(K, \boldsymbol{\mu}_2) \subset \text{Br}(K)$$

for every field extension K/F. The image of the last map consists of the classes $[A]$ of all central simple K-algebras A of degree n admitting orthogonal involutions (see [Knus et al. 1998, §31]). As $\text{ind}(A)$ is a power of 2 dividing n, we have $\text{ind}(A) \leq 2^m$, where 2^m is the largest power of 2 dividing n.

Let E be a $\mathbf{Spin}_n^+$-torsor over K. We have shown that, if $\delta_K([E]) = [A]$ for a central simple K-algebra A, then $\text{ind}(A) \leq 2^m$. It follows from [Brosnan et al. 2011, Theorem 4.1] that $\text{ed}[E/\mathbf{Spin}_n] = \text{ind}(A) \leq 2^m$.

It is shown in [Brosnan et al. 2010, Remark 3-10] that

$$\text{ed}(\mathbf{Spin}_n^+) = 2^{(n-2)/2} - \frac{n(n-1)}{2}$$

for every integer $n \geq 20$ divisible by 4. Finally, by Proposition 2.1,

$$\text{ed}(\mathbf{Spin}_n) \leq \text{ed}(\mathbf{Spin}_n^+) + 2^m = 2^{(n-2)/2} + 2^m - \frac{n(n-1)}{2}. \qquad \square$$

3. The functors I_n^k

We use the following notation. Let F be a field of characteristic different from 2 and K/F a field extension. We define

$$I_n^1(K) = \boxed{\begin{array}{l}\text{Set of isomorphism classes of nondegenerate} \\ \text{quadratic forms over } K \text{ of dimension } n\end{array}}$$

and recall from [Knus et al. 1998, §29.E] the existence of a natural bijection $I_n^1(K) \simeq H^1_{\text{ét}}(K, \mathbf{O}_n)$.

Recall that the *discriminant* $\operatorname{disc}(q)$ of a form $q \in I_n^1(K)$ is equal to

$$(-1)^{n(n-1)/2} \det(q) \in K^\times / K^{\times 2}.$$

Set

$$I_n^2(K) = \{q \in I_n^1(K) : \operatorname{disc}(q) = 1\}.$$

We have a natural bijection $I_n^2(K) \simeq H^1_{\text{ét}}(K, \mathbf{O}_n^+)$ (see [Knus et al. 1998, §29.E]).

The *Clifford invariant* $c(q)$ of a form $q \in I_n^2(K)$ is the class in the Brauer group $\operatorname{Br}(K)$ of the Clifford algebra of q if n is even and the class of the even Clifford algebra if n is odd [Knus et al. 1998, §8.B]. Define

$$I_n^3(K) = \{q \in I_n^2(K) : c(q) = 0\}.$$

Remark 3.1. Our notation of the functors I_n^k for $k = 1, 2, 3$ is explained by the following property: $I_n^k(K)$ consists of all classes of quadratic forms $q \in W(K)$ of dimension n such that $q \in I(K)^k$ if n is even and $q \perp \langle -1 \rangle \in I(K)^k$ if n is odd, where $I(K)$ is the fundamental ideal in the Witt ring $W(K)$ of K.

The functor I_n^3 is related to $\mathbf{Spin}_n$-torsors as follows. The short exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}_n \to \mathbf{O}_n^+ \to 1$$

yields an exact sequence

$$H^1_{\text{ét}}(K, \boldsymbol{\mu}_2) \to H^1_{\text{ét}}(K, \mathbf{Spin}_n) \to H^1_{\text{ét}}(K, \mathbf{O}_n^+) \xrightarrow{c} H^2_{\text{ét}}(K, \boldsymbol{\mu}_2), \tag{1}$$

where c is the Clifford invariant. Thus, $\operatorname{Ker}(c) = I_n^3(K)$.

The essential dimensions of I_n^1 and I_n^2 were computed in [Reichstein 2000, Theorems 10.3 and 10.4]: we have $\operatorname{ed}(I_n^1) = n$ and $\operatorname{ed}(I_n^2) = n - 1$. In Section 7, we compute $\operatorname{ed}(I_n^3)$. We will need the following lemma, which was proven in [Brosnan et al. 2010, Lemma 5-1]:

Lemma 3.2. *We have* $\operatorname{ed}_p(I_n^3) \le \operatorname{ed}_p(\mathbf{Spin}_n) \le \operatorname{ed}_p(I_n^3) + 1$ *for every* $p \ge 0$.

Proof. Let K/F be a field extension. The group $H^1_{\text{ét}}(K, \boldsymbol{\mu}_2) = K^\times / K^{\times 2}$ acts transitively on the fibers of the second map in the sequence (1). It follows that the natural map $\mathbf{Spin}_n$-*Torsors* $\to I_n^3$ is a surjection with $\mathbf{G}_m$ acting surjectively on the fibers. The statement follows from [Berhuy and Favi 2003, Proposition 1.13]. □

Let $\mathbf{\Gamma}_n^+$ be the split even Clifford group (see [Knus et al. 1998, §23]). The commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{Spin}_n & \longrightarrow & \mathbf{O}_n^+ & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \boldsymbol{\Gamma}_n^+ & \longrightarrow & \mathbf{O}_n^+ & \longrightarrow & 1 \\
 & & {\scriptstyle 2}\downarrow & & {\scriptstyle \text{spinor}}\downarrow{\scriptstyle \text{norm}} & & & & \\
 & & \mathbf{G}_m & = & \mathbf{G}_m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

yields a bijection $H^1_{\text{ét}}(K, \boldsymbol{\Gamma}_n^+) \simeq I_n^3(K)$ for any field extension K/F (see [Knus et al. 1998, §28]). In particular, $\mathrm{ed}_p(\boldsymbol{\Gamma}_n^+) = \mathrm{ed}_p(I_n^3)$.

4. Subforms of forms in I_n^3

In this section, we study the following problem in quadratic form theory, which will be used in Section 7 in order to compute the essential dimension of I_n^3. Note that the problem is stated entirely in terms of quadratic forms while in the solution we use the essential dimension. We don't know how to solve the problem by means of quadratic form theory.

Problem 4.1. *Given a field F, determine all integers n such that every form in $I_n^3(K)$ contains a nontrivial subform in $I^2(K)$ for any field extension K/F.*

All forms in $I_n^3(K)$ for $n \leq 14$ are classified (see [Garibaldi 2009, Example 17.8, Theorems 17.13 and 21.3]). Inspection shows that for such n the problem has positive solution.

In the following theorem, we show that in the range $n \geq 15$ the problem has negative solution (with possibly two exceptions):

Theorem 4.2. *Let F be a field of characteristic zero, let $n \geq 15$ and let b be an even integer with $0 < b < n$. Then there is a field extension K/F and a form in $I_n^3(K)$ that does not contain a subform in $I_b^2(K)$ (with possible exceptions $(n, b) = (15, 8)$ or $(16, 8)$).*

Let $a := n - b$. Write $H_{a,b}$ for the image of the natural homomorphism

$$\mathbf{Spin}_a \times \mathbf{Spin}_b \to \mathbf{Spin}_n. \tag{2}$$

Note that the kernel of (2) is contained in

$$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 = \mathrm{Ker}(\mathbf{Spin}_a \times \mathbf{Spin}_b \to \mathbf{O}_a^+ \times \mathbf{O}_b^+)$$

and therefore is the cyclic group of order 2 generated by $(-1, -1)$. Hence, we have an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to H_{a,b} \to \mathbf{O}_a^+ \times \mathbf{O}_b^+ \to 1$$

and therefore a map

$$H^1_{\text{ét}}(R, H_{a,b}) \to H^1_{\text{ét}}(R, \mathbf{O}_a^+ \times \mathbf{O}_b^+) = H^1_{\text{ét}}(R, \mathbf{O}_a^+) \times H^1_{\text{ét}}(R, \mathbf{O}_b^+)$$

for a commutative F-algebra R.

We write $q(\eta) := (q_a, q_b)$ for the image of an element $\eta \in H^1_{\text{ét}}(R, H_{a,b})$ under this map, where $q_a \in H^1_{\text{ét}}(R, \mathbf{O}_a^+)$ and $q_b \in H^1_{\text{ét}}(R, \mathbf{O}_b^+)$.

Consider the commutative diagram with the exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & H_{a,b} & \longrightarrow & \mathbf{O}_a^+ \times \mathbf{O}_b^+ & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \tau} & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{Spin}_n & \longrightarrow & \mathbf{O}_n^+ & \longrightarrow & 1 \end{array}$$

The image of an element $\xi \in H^1_{\text{ét}}(R, \mathbf{Spin}_n)$ in $H^1_{\text{ét}}(R, \mathbf{O}_n^+)$ will be denoted by $q(\xi)$.

If $\xi \in H^1_{\text{ét}}(R, \mathbf{Spin}_n)$ is the image of an element $\eta \in H^1_{\text{ét}}(R, H_{a,b})$, then $q(\xi) = q_a \perp q_b$, the image of $(q_a, q_b) = q(\eta)$ under the map induced by τ. We can reverse this statement as follows.

Lemma 4.3. *Let $\xi \in H^1_{\acute{e}t}(R, \mathbf{Spin}_n)$ with $q(\xi) = q_a \perp q_b$, where $q_a \in H^1_{\acute{e}t}(R, \mathbf{O}_a^+)$ and $q_b \in H^1_{\acute{e}t}(R, \mathbf{O}_b^+)$. Then ξ is the image of an element η under the map $H^1_{\acute{e}t}(R, H_{a,b}) \to H^1_{\acute{e}t}(R, \mathbf{Spin}_n)$ such that $q(\eta) = (q_a, q_b)$.*

Proof. The diagram above yields a commutative diagram with the exact rows

$$\begin{array}{ccccc} H^1_{\text{ét}}(R, H_{a,b}) & \longrightarrow & H^1_{\text{ét}}(R, \mathbf{O}_a^+) \times H^1_{\text{ét}}(R, \mathbf{O}_b^+) & \xrightarrow{c'} & H^2_{\text{ét}}(R, \boldsymbol{\mu}_2) \\ \downarrow & & \downarrow & & \| \\ H^1_{\text{ét}}(R, \mathbf{Spin}_n) & \longrightarrow & H^1_{\text{ét}}(R, \mathbf{O}_n^+) & \xrightarrow{c} & H^2_{\text{ét}}(R, \boldsymbol{\mu}_2) \end{array}$$

Moreover, the group $H^1_{\text{ét}}(R, \boldsymbol{\mu}_2)$ acts transitively on the fibers of the left maps in the two rows. The result follows. □

For nonnegative integers a, b and a field extension K/F, set

$$I^3_{a,b}(K) := \{(q_a, q_b) \in I^2_a(K) \times I^2_b(K) : q_a \perp q_b \in I^3_n(K)\}.$$

Corollary 4.4. *For any $\eta \in H^1_{ét}(K, H_{a,b})$, we have $q(\eta) \in I^3_{a,b}(K)$. The morphism of functors $q : H_{a,b}$-Torsors $\to I^3_{a,b}$ is surjective. In particular*, $\mathrm{ed}_p(I^3_{a,b}) \le \mathrm{ed}_p(H_{a,b})$ *for every $p \ge 0$.*

Proof. Note that the map c' in the proof of Lemma 4.3 when $R = K$ takes a pair (q_a, q_b) to the Clifford invariant of $q_a \perp q_b$ in $\mathrm{Br}(K)$. The pair $(q_a, q_b) \in I^2_a(K) \times I^2_b(K)$ comes from $H^1_{ét}(K, H_{a,b})$ if and only if the Clifford invariant of $q_a \perp q_b$ is split, i.e., $q_a \perp q_b \in I^3_n(K)$. □

Lemma 4.5. *For an even a and any b*,

$$\mathrm{ed}_p(I^3_{a,b}) \le \mathrm{ed}_p(I^3_{a-1,b}) + 1$$

for every $p \ge 0$.

Proof. Consider the morphism of functors

$$\alpha : \mathbf{G}_m \times I^3_{a-1,b} \to I^3_{a,b}, \quad (\lambda;\, f, g) \mapsto (\lambda(f \perp \langle -1\rangle), g).$$

Every form h in $I^2_a(K)$ can be written in the form $h = \lambda(f \perp \langle -1\rangle)$ for a value λ of h and a form $f \in I^2_{a-1}(K)$; i.e., α is a surjection, whence the result. □

Write V_n and W_n for the (semi)spinor and regular representations, respectively, of the group $\mathbf{Spin}_n$. We have

$$\dim(V_n) = \begin{cases} 2^{(n-1)/2} & \text{if } n \text{ is odd,} \\ 2^{(n-2)/2} & \text{if } n \text{ is even} \end{cases}$$

and $\dim(W_n) = n$. We consider the tensor product $V_{a,b} := V_a \otimes V_b$ as the representation of the group $H_{a,b}$. We also view W_a and W_b as $H_{a,b}$-representations via the natural homomorphisms $H_{a,b} \to \mathbf{O}^+_a$ and $H_{a,b} \to \mathbf{O}^+_b$, respectively.

A representation V of an algebraic group H is *generically free* if the stabilizer of a generic vector in V is trivial. In this case, by [Reichstein and Youssin 2000],

$$\mathrm{ed}(H) \le \dim(V) - \dim(H).$$

Lemma 4.6. *Let a be odd and b even. Suppose that $V_{a,b}$ is a generically free representation of the image of the homomorphism $H_{a,b} \to \mathbf{GL}(V_{a,b})$. Then $V_{a,b} \oplus W_b$ is a generically free representation of $H_{a,b}$. In particular*,

$$\mathrm{ed}(H_{a,b}) \le \dim(V_{a,b}) + \dim(W_b) - \dim(H_{a,b}).$$

Proof. Write C_n for the kernel of $\mathbf{Spin}_n \to \mathbf{PGO}^+_n$ and C'_n for the kernel of $\mathbf{Spin}_n \to \mathbf{O}^+_n$, so $C'_n = \{\pm 1\} \subset C_n$. By assumption, the generic stabilizer H of the action of $\mathbf{Spin}_a \times \mathbf{Spin}_b$ on $V_{a,b}$ is contained in the center $C_a \times C_b$. Since $C_b/C'_b = \boldsymbol{\mu}_2$ acts on W_b by multiplication by -1, we have $H \subset C_a \times C'_b \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. Note that $\boldsymbol{\mu}_2 \times 1$ and $1 \times \boldsymbol{\mu}_2$ act by multiplication by -1 on $V_{a,b}$; hence, H is generated by $(-1, -1)$. It follows that $H_{a,b} = (\mathbf{Spin}_a \times \mathbf{Spin}_b)/H$ acts generically freely on $V_{a,b} \oplus W_b$. □

Proposition 4.7. *Let* $\mathrm{char}(F) = 0$. *If* $n = a + b \geq 15$ *with* $a \leq b$, *then* $V_{a,b}$ *is a generically free representation of the image of* $H_{a,b} \to \mathbf{GL}(V_{a,b})$ *if and only if* $(a, b) \neq (3, 12), (4, 11), (4, 12), (6, 10)$ *and* $(8, 8)$.

Proof. All the cases of infinite generic stabilizers H are listed in [Èlašvili 1972, §3, Row 7 of Table 6]: H is infinite if and only if $(a, b) = (3, 12)$ and $(4, 12)$.

If H is finite, by [Popov 1987, Theorem 1, Rows 1, 12 and 13 of Table 1], H is nontrivial if and only if $(a, b) = (4, 11), (6, 10)$ and $(8, 8)$. □

Proof of Theorem 4.2. Note that the case (n, b) with n even implies the case $(n-1, b)$. Indeed, suppose that every form in I^3_{n-1} for an even n contains a subform from I^2_b. Take any form $q \in I^3_n(K)$ for a field extension K/F, and write $q = \lambda(f \perp \langle -1\rangle)$ for a $\lambda \in K^\times$ and $f \in I^3_{n-1}(K)$. If f contains a subform $h \in I^2_b(K)$, then q contains λh.

We need to show that the natural morphism of functors $I^3_{a,b} \to I^3_n$ is not surjective. It suffices to prove that $\mathrm{ed}(I^3_{a,b}) < \mathrm{ed}(I^3_n)$. We may assume that n (and hence also a) is even. Moreover, we may assume that $a \leq b$.

Suppose that $n \geq 18$. By Proposition 4.7, Lemmas 4.5 and 4.6 and Corollary 4.4,

$$\begin{aligned}
\mathrm{ed}(I^3_{a,b}) &\leq \mathrm{ed}(I^3_{a-1,b}) + 1\\
&\leq \mathrm{ed}(H_{a-1,b}) + 1\\
&\leq \dim(V_{a-1,b}) + \dim(W_b) - \dim(H_{a-1,b}) + 1\\
&= 2^{n/2-2} + b - (a-1)(a-2)/2 - b(b-1)/2 + 1\\
&= 2^{n/2-2} - (a^2 + b^2 - 3a - 3b)/2\\
&\leq 2^{n/2-2} - (n^2 - 6n)/4
\end{aligned}$$

as $a^2 + b^2 \geq n^2/2$. The last integer is strictly less than

$$2^{n/2-1} - n(n-1)/2 - 1 \leq \mathrm{ed}(\mathbf{Spin}_n) - 1 \leq \mathrm{ed}(I^3_n)$$

by Theorem 2.2 and Lemma 3.2.

It remains to consider the case $n = 16$. Note that, by Theorem 2.2 and Lemma 3.2,

$$\mathrm{ed}(I^3_{16}) \geq \mathrm{ed}(\mathbf{Spin}_{16}) - 1 = 23. \tag{3}$$

We shall prove that $\mathrm{ed}(I^3_{a,b}) < 23$. All possible values of b are 8, 10, 12 and 14.

Case $\boldsymbol{(n, b) = (16, 10)}$. Consider the representation $V := W_6 \oplus V_{6,10} \oplus W_{10}$ of $H_{6,10}$. We claim that V is generically free. The stabilizer in $\mathbf{Spin}_6$ of a point in general position in W_6 is $\mathbf{Spin}_5$. Hence, the stabilizer in $H_{6,10}$ of a point in general position in W_6 is $H_{5,10}$. Note that the restriction of $V_{6,10}$ to $H_{5,10}$ is isomorphic to $V_{5,10}$. Finally, the $H_{5,10}$-representation $V_{5,10} \oplus W_{10}$ is generically free by Proposition 4.7.

It follows from (3) and Corollary 4.4 that

$$\mathrm{ed}(I^3_{6,10}) \le \mathrm{ed}(H_{6,10}) \le \dim(V) - \dim(H_{6,10}) = 80 - 60 = 20.$$

***Case* $(n, b) = (16, 12)$.** Consider the representation $V := W_3 \oplus W_3 \oplus V_{3,12} \oplus W_{12}$ of $H_{3,12}$. We claim that V is generically free as the representation of $H_{3,12}$. Indeed, the stabilizer in $H_{3,12}$ of a generic vector in W_{12} is $H_{3,11}$. We are reduced to showing that $W_3 \oplus W_3 \oplus V_{3,11}$ is a generically free representation of $H_{3,11}$. By [Popov 1987, §5, p. 246], the generic stabilizer S of $H_{3,11}$ in $V_{3,11}$ is finite (isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$), and the restriction to S of the natural projection $H_{3,11} \to \mathbf{O}_3^+$ is injective. It remains to notice that the representation $W_3 \oplus W_3$ of $\mathbf{O}_3^+ = \mathbf{PGL}_2$ is generically free.

It follows from Lemmas 4.5 and 4.6 and Corollary 4.4 that

$$\begin{aligned}\mathrm{ed}(I^3_{4,12}) \le \mathrm{ed}(I^3_{3,12}) + 1 \le \mathrm{ed}(H_{3,12}) + 1 \\ \le \dim(V) - \dim(H_{3,12}) + 1 = 82 - 69 + 1 = 14.\end{aligned}$$

***Case* $(n, b) = (16, 14)$.** As every form in I^3_2 is hyperbolic, we have $I^3_{2,14} = I^3_{14}$ and $\mathrm{ed}(I^3_{14}) = 7$ by Theorem 2.2. □

5. Unramified principal homogeneous spaces

Let G be an algebraic group over F, and let K/F be a field extension with a discrete valuation v trivial on F. Write O for the valuation ring of v. It is a local F-algebra. We say that a class $\xi \in H^1_{\text{ét}}(K, G)$ is *unramified* (with respect to v) if ξ belongs to the image of the map $H^1_{\text{ét}}(O, G) \to H^1_{\text{ét}}(K, G)$.

Let $\bar{K}$ be the residue field of v. The ring homomorphism $O \to \bar{K}$ yields a map $H^1_{\text{ét}}(O, G) \to H^1_{\text{ét}}(\bar{K}, G)$. This map is a bijection if K is complete (see [SGA 3 1970, Exposé XXIV, Proposition 8.1]). Hence, we have the map

$$H^1_{\text{ét}}(\bar{K}, G) \xrightarrow{\sim} H^1_{\text{ét}}(O, G) \to H^1_{\text{ét}}(K, G). \tag{4}$$

Example 5.1. Let $\mathrm{char}(F) \neq 2$ and $G = \mathbf{O}_n$. Then $H^1_{\text{ét}}(K, G)$ is the set of isomorphism classes of nondegenerate quadratic forms of dimension n over K. A quadratic form q over a field K with a discrete valuation is unramified if and only if $q \simeq \langle a_1, a_2, \dots, a_n\rangle$, where a_i are units in the valuation ring O in K. In general, every q can be written $q = q_1 \perp \pi q_2 \perp h$, where π is a prime element, q_1 and q_2 are unramified anisotropic quadratic forms and h is a hyperbolic form. The form q is unramified if and only if $q_2 = 0$. It follows that, if two forms q and πq are both unramified, then q is hyperbolic. If K is complete, then the map (4) takes $f = \langle \bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\rangle$ over $\bar{K}$, where a_i are units in O, to $f_K := \langle a_1, a_2, \dots, a_n\rangle$.

6. Essential dimension of PI_n^3

Two quadratic forms f and g over a field K are called *similar* if $f = \lambda g$ for some $\lambda \in K^\times$. If n is even, we write $PI_n^3(K)$ for the set of similarity classes of forms in $I_n^3(K)$. The group $K^\times$ acts transitively on the fibers of the natural surjective map $I_n^3(K) \to PI_n^3(K)$. Hence,

$$\mathrm{ed}_p(PI_n^3) \le \mathrm{ed}_p(I_n^3) \le \mathrm{ed}_p(PI_n^3) + 1$$

for any $p \ge 0$ by [Berhuy and Favi 2003, Proposition 1.13].

Proposition 6.1. *Let* $\mathrm{char}(F) \ne 2$. *For an even* $n \ge 8$, *and* $p = 0$ *or* 2, *we have*

$$\mathrm{ed}_p(PI_n^3) = \mathrm{ed}_p(I_n^3) - 1.$$

Proof. Let K/F be a field extension, and let $q \in I_n^3(K)$ be a nonhyperbolic form. Consider the form tq over the field $K((t))$. It suffices to show that

$$\mathrm{ed}_p^{I_n^3}(tq) \ge \mathrm{ed}_p^{PI_n^3}(q) + 1.$$

Let $M/K((t))$ be a finite field extension of degree prime to p (i.e., $M = K((t))$ if $p = 0$ and $[M : K((t))]$ is odd if $p = 2$), let L/F be a subextension of M/F and let $f \in I_n^3(L)$ be such that $\mathrm{tr.deg}(L/F) = \mathrm{ed}_p^{I_n^3}(tq)$ and $tq_M \simeq f_M$.

Let v be the (unique) extension on M of the discrete valuation of $K((t))$, and let w be the restriction of v on L. The residue field $\overline{M}$ is a finite extension of K of degree prime to p. As the form q is not hyperbolic, q_M is not hyperbolic, and therefore, the form $tq_M \simeq f_M$ is ramified by Example 5.1. It follows that w is nontrivial, i.e., w is a discrete valuation on L.

Let $\hat{L}$ be the completion of L. Note that, as M is complete, we can identify $\hat{L}$ with a subfield of M. Write $f_{\hat{L}} \simeq (f_1)_{\hat{L}} \perp \pi(f_2)_{\hat{L}}$, where f_1 and f_2 are quadratic forms over the residue field $\overline{L}$ and $\pi \in L$ is a prime element (see Example 5.1). Note that $f_1, f_2 \in I^2(\overline{L})$ by [Elman et al. 2008, Lemma 19.4]. If the ramification index e of M/L is even, then π is a unit in the valuation ring O of M modulo squares in $M^\times$; hence, f_M is unramified, a contradiction. It follows that e is odd. Writing $\pi = ut^e$ with a unit $u \in O^\times$, we have

$$tq_M \simeq f_M \simeq (f_1)_M \perp \pi(f_2)_M \simeq (f_1)_M \perp ut(f_2)_M;$$

hence, $(f_1)_M = 0$ and $q_M = u(f_2)_M$ in $W(M)$. It follows that $(f_1)_{\overline{M}} = 0$ and $q_{\overline{M}} = \bar{u}(f_2)_{\overline{M}}$ in $W(\overline{M})$, and therefore,

$$q_{\overline{M}} = \bar{u}(f_2)_{\overline{M}} = \bar{u}g_{\overline{M}}, \tag{5}$$

where $g := f_1 \perp f_2$ is the form over $\overline{L}$ of dimension n. Note that $f_{\hat{L}} - g_{\hat{L}} = \langle \pi, -1\rangle (f_2)_{\hat{L}} \in I^3(\hat{L})$; hence, $g_{\hat{L}} \in I^3(\hat{L})$ and $g \in I^3(\overline{L})$.

It follows from (5) that $q_{\overline{M}}$ is similar to $g_{\overline{M}}$, i.e., the form q is p-defined over $\bar{L}$ for the functor PI_n^3 (see [Merkurjev 2009, §1.1]), and therefore,

$$\operatorname{ed}_p^{I_n^3}(tq) = \operatorname{tr.deg}(L/F) \geq \operatorname{tr.deg}(\bar{L}/F) + 1 \geq \operatorname{ed}_p^{PI_n^3}(q) + 1. \qquad \square$$

7. Essential dimension of $\mathbf{\Gamma}_n^+$

In this section, we compute the essential dimension of $\mathbf{\Gamma}_n^+$ and I_n^3.

Theorem 7.1. *Let F be a field of characteristic zero. Then for every integer $n \geq 15$ and $p = 0$ or 2, we have*

$$\operatorname{ed}_p(\mathbf{\Gamma}_n^+) = \operatorname{ed}_p(I_n^3) = \begin{cases} 2^{(n-1)/2} - 1 - n(n-1)/2 & \text{if } n \text{ is odd}, \\ 2^{(n-2)/2} - n(n-1)/2 & \text{if } n \equiv 2 \pmod 4, \\ 2^{(n-2)/2} + 2^m - 1 - n(n-1)/2 & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

where 2^m is the largest power of 2 dividing n.

If $\operatorname{char}(F) \neq 2$, then the essential dimension of I_n^3 has the following values for $n \leq 14$:

n	≤ 6	7	8	9	10	11	12	13	14
$\operatorname{ed}_2(I_n^3) = \operatorname{ed}(I_n^3)$	0	3	4	4	4	5	6	6	7

Proof. We will prove the theorem case by case.

Case $n \equiv 2 \pmod 4$ and $n \geq 10$. The exact sequence

$$1 \to \boldsymbol{\mu}_4 \to \mathbf{Spin}_n \to \mathbf{PGO}_n^+ \to 1$$

yields a surjective map $\mathbf{Spin}_n$-$\mathit{Torsors}(K) \to PI_n^3(K)$ for any K/F, with the group $K^\times$ acting transitively on the fibers of this map. It follows from Theorem 2.2, Proposition 6.1 and Lemma 3.2 that

$$\operatorname{ed}_2(I_n^3) = \operatorname{ed}_2(PI_n^3) + 1 \geq \operatorname{ed}_2(\mathbf{Spin}_n) = \operatorname{ed}(\mathbf{Spin}_n) \geq \operatorname{ed}(I_n^3) \geq \operatorname{ed}_2(I_n^3).$$

Hence, $\operatorname{ed}_2(I_n^3) = \operatorname{ed}(I_n^3) = \operatorname{ed}(\mathbf{Spin}_n)$. The latter value is known by Theorem 2.2.

Case $n \not\equiv 2 \pmod 4$ and $n \geq 15$. Let $n = a + b$ with even $b \neq 2$. Let Z be the trivial group if $b = 0$ and the image of the center C_b of $\mathbf{Spin}_b$ in $H_{a,b}$ if $b \geq 4$. Then Z is central in $H_{a,b}$; hence, the group $H^1_{\text{ét}}(K, Z)$ acts on $H^1_{\text{ét}}(K, H_{a,b})$.

Lemma 7.2. *Let $\xi, \eta \in H^1_{ét}(K, H_{a,b})$ with even $b \neq 2$. Suppose that $q(\xi) = q_a \perp q_b$ and $q(\eta) = q_a \perp \lambda q_b$ with the forms $q_a \in I_a^2(K)$ and $q_b \in I_b^2(K)$ and $\lambda \in K^\times$. Then $\eta = \alpha\xi$ for some $\alpha \in H^1_{ét}(K, Z)$.*

Proof. The statement is trivial if $b = 0$, so assume that $b \geq 4$. The restriction of the natural homomorphism $H_{a,b} \to \mathbf{O}_b^+$ to the subgroup Z yields a surjection

$\varphi : Z \to \boldsymbol{\mu}_2 = \mathrm{Center}(\mathbf{O}_b^+)$. The kernel of φ coincides with the kernel C of the canonical homomorphism $H_{a,b} \to \mathbf{O}_a^+ \times \mathbf{O}_b^+$.

As Z is isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ or $\boldsymbol{\mu}_4$, the homomorphism $\varphi^* : H^1_{\text{ét}}(K, Z) \to H^1_{\text{ét}}(K, \boldsymbol{\mu}_2) = K^\times / K^{\times 2}$ is surjective. Let $\gamma \in H^1_{\text{ét}}(K, Z)$ be such that $\varphi^*(\gamma) = \lambda K^{\times 2}$. Then $q(\gamma\xi) = q_a \perp \lambda q_b = q(\eta)$. Then there is $\beta \in H^1_{\text{ét}}(K, C)$ such that $\eta = \beta(\gamma\xi)$. Hence, $\eta = \alpha\xi$, where $\alpha = \beta'\gamma$ with β' the image of β under the map $H^1_{\text{ét}}(K, C) \to H^1_{\text{ét}}(K, Z)$ induced by the inclusion of C into Z. □

Let $\xi \in H^1_{\text{ét}}(K, \mathbf{Spin}_n)$ be such that the form $q = q(\xi) \in I^3_n(K)$ is generic for the functor I^3_n (see [Merkurjev 2009, §2.2]). In particular, $\mathrm{ed}^{I^3_n}(q) = \mathrm{ed}(I^3_n)$. Note that q is anisotropic.

Identifying $\boldsymbol{\mu}_2$ with the kernel of $\mathbf{Spin}_n \to \mathbf{O}_n^+$, we have an action of $H^1_{\text{ét}}(E, \boldsymbol{\mu}_2) = E^\times / E^{\times 2}$ on $H^1_{\text{ét}}(E, \mathbf{Spin}_n)$, where $E = K((t))$. Consider the element $t\xi_E \in H^1_{\text{ét}}(E, \mathbf{Spin}_n)$ over E. We claim that $t\xi_E$ is ramified. Suppose not, i.e., $t\xi_E$ comes from an element $\rho \in H^1_{\text{ét}}(O, \mathbf{Spin}_n)$, where $O = K[[t]]$. Let $q' \in H^1_{\text{ét}}(O, \mathbf{O}_n^+)$ be the image of ρ viewed as a quadratic form over O. We have

$$q'_E = q(t\xi_E) = q(\xi_E) = q_E;$$

hence, $q' = q_O$. Then ρ and ξ_O belong to the same fiber of the map

$$H^1_{\text{ét}}(O, \mathbf{Spin}_n) \to H^1_{\text{ét}}(O, \mathbf{O}_n^+).$$

As the group $H^1_{\text{ét}}(O, \boldsymbol{\mu}_2) = O^\times / O^{\times 2}$ acts transitively on the fiber, there is a unit $u \in O^\times$ satisfying $t\xi_E = u\xi_E$. It follows from [Knus et al. 1998, Proposition 28.11] that tu^{-1} is in the image spinor norm map

$$\mathbf{O}^+(q_E) \to H^1_{\text{ét}}(E, \boldsymbol{\mu}_2) = E^\times / E^{\times 2}$$

for the form q_E; hence, q is isotropic by [Elman et al. 2008, Theorem 18.3], a contradiction. The claim is proven.

Let L/F be a subextension of E/F, and let $\eta \in H^1_{\text{ét}}(L, \mathbf{Spin}_n)$ be such that $\mathrm{tr.deg}(L/F) = \mathrm{ed}^{\mathbf{Spin}_n}(t\xi)$ and $\eta_E \simeq t\xi_E$. We have $q(\eta)_E = q(t\xi) = q(\xi_E) = q_E$; hence, the form $q(\eta)_E$ is anisotropic.

Let v be the restriction on L of the discrete valuation of E. As $t\xi$ is ramified, v is nontrivial; hence, v is a discrete valuation. Let $\pi \in L$ be a prime element.

Consider the completion $\hat{L}$ of L. As E is complete, we can view $\hat{L}$ as a subfield of E. Write $q(\eta_{\hat{L}}) = (q_a)_{\hat{L}} \perp \pi(q_b)_{\hat{L}}$, where q_a and q_b are anisotropic quadratic forms over the residue field $\bar{L}$ of dimension a and b, respectively. As $q(\eta) \in I^3(\hat{L})$, we have $q_b \in I^2(\bar{L})$, and therefore, b is even and $b \neq 2$. By Lemma 4.3, there is $\eta' \in H^1_{\text{ét}}(\hat{L}, H_{a,b})$ that maps to η with $q(\eta') = ((q_a)_{\hat{L}}, \pi(q_b)_{\hat{L}})$.

We claim that the ramification index e of the extension $E/\hat{L}$ is odd. Suppose e is even. Note that $q_a \perp q_b \in I^3_n(\bar{L})$. Lemma 4.3 allows us to choose an unramified element $\nu \in H^1_{\text{ét}}(\hat{L}, H_{a,b})$ with $q(\nu) = ((q_a)_{\hat{L}}, (q_b)_{\hat{L}})$. By Lemma 7.2, there is $\alpha \in H^1_{\text{ét}}(\hat{L}, Z)$ such that $\eta' = \alpha\nu$. If b is divisible by 4, we have $Z \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. As e is even, α is unramified over E; hence, η'_E is unramified. It follows that $\eta_E \simeq t\xi$ is also unramified, a contradiction.

Suppose that $b \equiv 2 \pmod 4$. Note that $0 < b < n$ since $n \not\equiv 2 \pmod 4$. Write $\pi = ut^k$ with a unit $u \in O^\times$ and even k. Then

$$(q_a \perp uq_b)_E \simeq (q_a \perp \pi q_b)_E \simeq q(\eta_E) \simeq q(t\xi_E) = q(\xi_E) = q_E.$$

It follows that $q \simeq (q_a)_K \perp (\bar{u}q_b)_K$, i.e., q contains the subform $(\bar{u}q_b)_K$ in $I^2(K)$ of dimension b. This contradicts Theorem 4.2. The claim is proven.

Thus, e is odd. We have

$$(q_a \perp utq_b)_E \simeq (q_a \perp \pi q_b)_E \simeq q(\eta_E) \simeq q(t\xi_E) = q(\xi_E) = q_E.$$

It follows that $(q_b)_K$ is hyperbolic and hence $(q_a \perp q_b)_K = (q_a)_K = q$ in $W(K)$, i.e., $(q_a \perp q_b)_K \simeq q$.

Note that $(q_a)_{\hat{L}} = (q_a)_{\hat{L}} + \pi(q_b)_{\hat{L}} = q(\eta_{\hat{L}}) \in I^3(\hat{L})$; hence, $q_a \in I^3(\bar{L})$ and $q_a \perp q_b \in I^3_n(\bar{L})$. Therefore, q is defined over $\bar{L}$ for the functor I^3_n; hence,

$$\operatorname{ed}^{\mathbf{Spin}_n}(t\xi) = \operatorname{tr.deg}(L/F) \geq \operatorname{tr.deg}(\bar{L}/F) + 1 \geq \operatorname{ed}^{I^3_n}(q) + 1 = \operatorname{ed}(I^3_n) + 1.$$

It follows that $\operatorname{ed}(\mathbf{Spin}_n) \geq \operatorname{ed}(I^3_n) + 1$; hence, $\operatorname{ed}(I^3_n) = \operatorname{ed}(\mathbf{Spin}_n) - 1$ by Lemma 3.2. The value of $\operatorname{ed}(\mathbf{Spin}_n)$ is given in Theorem 2.2.

In what follows, we use the following observation (see [Berhuy and Favi 2003]): if a functor $\mathcal{F}$ admits a nontrivial cohomological invariant of degree d with values in $\mathbb{Z}/2\mathbb{Z}$, then $\operatorname{ed}_2(\mathcal{F}) \geq d$.

Case n = 7. Every form q in $I^3_7(K)$ is the pure subform of a 3-fold Pfister form $\langle\langle a, b, c\rangle\rangle$; hence, $\operatorname{ed}(I^3_7) \leq 3$. On the other hand, the Arason invariant $e_3(q \perp \langle -1\rangle) = (a) \cup (b) \cup (c) \in H^3(K, \mathbb{Z}/2\mathbb{Z})$ is nontrivial (see [Garibaldi 2009, §18.6]); hence, $\operatorname{ed}_2(I^3_7) \geq 3$.

Case n = 8. Every form q in $I^3_8(K)$ is a multiple $e\langle\langle a, b, c\rangle\rangle$ of a 3-fold Pfister form; hence, $\operatorname{ed}(I^3_8) \leq 4$. The invariant $a_4(q) = (e) \cup (a) \cup (b) \cup (c) \in H^4(K, \mathbb{Z}/2\mathbb{Z})$ is nontrivial; hence, $\operatorname{ed}_2(I^3_8) \geq 4$.

Case n = 9 and 10. Every form q in $I^3_9(K)$ or $I^3_{10}(K)$ is equal to $f \perp \langle 1\rangle$ or $f \perp \langle 1, -1\rangle$, respectively, where f is a multiple of a 3-fold Pfister form over K, by [Lam 2005, XII.2.8]. Hence, $I^3_8 \simeq I^3_9 \simeq I^3_{10}$.

Case n = 11. The degree-5 cohomological invariant a_5 of $\mathbf{Spin}_{11}$ defined in [Garibaldi 2009, §20.8] factors through a nontrivial invariant of I^3_{11}; hence $\operatorname{ed}_2(I^3_{11}) \geq 5$. On the other hand, $\operatorname{ed}(I^3_{11}) \leq \operatorname{ed}(\mathbf{Spin}_{11}) = 5$.

Case n = 12. The degree-6 cohomological invariant a_6 of $\mathbf{Spin}_{12}$ defined in [Garibaldi 2009, §20.13] factors through a nontrivial invariant of I^3_{12}, so $\mathrm{ed}_2(I^3_{12}) \geq 6$. On the other hand, $\mathrm{ed}(I^3_{12}) \leq \mathrm{ed}(\mathbf{Spin}_{12}) = 6$.

Case n = 13 and 14. We know from the beginning of the proof (case $n \equiv 2 \pmod 4$ and $n \geq 10$) and from Theorem 2.2 that $\mathrm{ed}_2(I^3_{14}) = \mathrm{ed}(I^3_{14}) = \mathrm{ed}(\mathbf{Spin}_{14}) = 7$. By Lemma 4.5, $\mathrm{ed}_2(I^3_{13}) = \mathrm{ed}_2(I^3_{13,0}) \geq \mathrm{ed}_2(I^3_{14,0}) - 1 = 6$. On the other hand, $\mathrm{ed}(I^3_{13}) \leq \mathrm{ed}(\mathbf{Spin}_{13}) = 6$. □

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chernous@math.ualberta.ca *Department of Mathematical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada*

merkurev@math.ucla.edu *Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095-1555, United States*

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