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Derived invariants of irregular varieties and Hochschild homology

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We study the behavior of cohomological support loci of the canonical bundle under derived equivalence of smooth projective varieties. This is achieved by investigating the derived invariance of a generalized version of Hochschild homology. Furthermore, using techniques coming from birational geometry, we establish the derived invariance of the Albanese dimension for varieties having nonnegative Kodaira dimension. We apply our machinery to study the derived invariance of the holomorphic Euler characteristic and of certain Hodge numbers for special classes of varieties. Further applications concern the behavior of particular types of fibrations under derived equivalence.

1. Introduction

It is now well-known that derived equivalent varieties share quite a few invariants. For instance, the dimension, the Kodaira dimension, the numerical dimension and the canonical ring are examples of derived invariants. By describing the behavior under derived equivalence of the Picard variety, Popa and Schnell [2011] establish the derived invariance of the number of linearly independent holomorphic one-forms. In this paper, we study the behavior under derived equivalence of other fundamental objects in the geometry of *irregular* varieties, i.e., those with positive *irregularity* $q(X) := h^0(X, \Omega^1_X)$, such as the cohomological support loci and the Albanese dimension. Applications of our techniques concern the derived invariance of the holomorphic Euler characteristic of varieties with large Albanese dimension and the derived invariance of some of the Hodge numbers of fourfolds again with large Albanese dimension. A further application concerns the behavior of fibrations of derived equivalent threefolds onto irregular varieties. This work is motivated by a well-known conjecture predicting the derived invariance of all Hodge numbers and by a conjecture of Popa (see Conjectures 1.2 and 1.3 and [Popa 2013]).

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The main tool we use to approach the problems described above is the comparison of the cohomology groups of twists by topologically trivial line bundles of the canonical bundles of the varieties in play. This is achieved by studying a generalized version of Hochschild homology that takes into account an important isomorphism due to Rouquier related to derived autoequivalences (see [Rouquier 2011, Théorème 4.18]). In this way, we obtain a theoretical result of independent interest in the study of derived equivalences of smooth projective varieties, which we now present. To begin with, we recall the *Hochschild cohomology and homology* of a smooth projective variety X:

$$HH^*(X) := \bigoplus_k \operatorname{Ext}^k_{X\times X}(i_*\mathbb{O}_X, i_*\mathbb{O}_X),$$
$$HH_*(X) := \bigoplus_k \operatorname{Ext}^k_{X\times X}(i_*\mathbb{O}_X, i_*\omega_X),$$

where $i : X \hookrightarrow X \times X$ is the diagonal embedding of X. The space $HH^*(X)$ has a structure of ring under composition of morphisms, and $HH_*(X)$ is a graded $HH^*(X)$-module with the same operation. Results of Căldăraru [2003, Theorem 8.1] and Orlov [2003, Theorem 2.1.8] show that the Hochschild cohomology and homology are derived invariants. More precisely, if $\Phi : \boldsymbol{D}(X) \to \boldsymbol{D}(Y)$ is an equivalence of derived categories of smooth projective varieties, then it induces an isomorphism of rings $HH^*(X) \cong HH^*(Y)$ and an isomorphism of graded modules $HH_*(X) \cong HH_*(Y)$ compatible with the isomorphism $HH^*(X) \cong HH^*(Y)$. We now present the generalization of Hochschild homology mentioned above. For a triple $(\varphi, L, m) \in \operatorname{Aut}^0(X) \times \operatorname{Pic}^0(X) \times \mathbb{Z}$, we define the graded $HH^*(X)$-module

$$HH_*(X, \varphi, L, m) := \bigoplus_k \operatorname{Ext}^k_{X\times X}\big(i_*\mathbb{O}_X, (1, \varphi)_*(\omega_X^{\otimes m} \otimes L)\big)$$

with module structure given by composition of morphisms. We think of these spaces as a "twisted" version of the Hochschild homology of X. Lastly, we recall that a derived equivalence $\boldsymbol{D}(X) \cong \boldsymbol{D}(Y)$ induces an isomorphism of algebraic groups, called *Rouquier's isomorphism*

$$F : \operatorname{Aut}^0(X) \times \operatorname{Pic}^0(X) \to \operatorname{Aut}^0(Y) \times \operatorname{Pic}^0(Y). \tag{1}$$

(An explicit description of F is given in (3) (see [Rouquier 2011, Théorème 4.18; Huybrechts 2006, Proposition 9.45; Rosay 2009, Theorem 3.1]; cf. [Popa and Schnell 2011, footnote, p. 531]).) The following theorem describes the behavior of the twisted Hochschild homology under derived equivalence. Its proof follows the general strategy of the proofs of Orlov and Căldăraru, but further technicalities appear due to the possible presence of nontrivial automorphisms of X and Y; see Section 2 for its proof.

Theorem 1.1. *Let $\Phi : \boldsymbol{D}(X) \to \boldsymbol{D}(Y)$ be an equivalence of derived categories of smooth projective varieties defined over an algebraically closed field, and let $m \in \mathbb{Z}$.*

If $F(\varphi, L) = (\psi, M)$ *(where* F *is the Rouquier isomorphism), then* Φ *induces an isomorphism of graded modules*

$$HH_*(X, \varphi, L, m) \cong HH_*(Y, \psi, M, m)$$

compatible with the isomorphism $HH^*(X) \cong HH^*(Y)$.

We now move our attention to the main application of Theorem 1.1, namely, the behavior of *cohomological support loci* under derived equivalence. These loci are defined as

$$V^k(\omega_X) := \{L \in \mathrm{Pic}^0(X) \mid h^k(X, \omega_X \otimes L) > 0\}$$

where X is a smooth projective variety and $k \geq 0$ is an integer. From here on, we work over the field of the complex numbers. The $V^k(\omega_X)$'s have been studied for instance in [Green and Lazarsfeld 1987; 1991; Ein and Lazarsfeld 1997; Arapura 1992; Hacon 2004; Pareschi and Popa 2011]. They are one of the most important tools in the birational study of irregular varieties; roughly speaking, they control the geometry of the Albanese map and the fibrations onto lower-dimensional irregular varieties. The following conjecture, and its weaker variant, predicts the behavior of cohomological support loci under derived equivalence. As a matter of notation, we denote by $V^k(\omega_X)_0$ the union of the irreducible components of $V^k(\omega_X)$ passing through the origin.

Conjecture 1.2 [Popa 2013, Conjecture 1.2]. *If* X *and* Y *are smooth projective derived equivalent varieties, then*

$$V^k(\omega_X) \cong V^k(\omega_Y) \quad \textit{for all } k \geq 0.$$

Conjecture 1.3 [Popa 2013, Variant 1.3]. *Under the assumptions of Conjecture 1.2, there exist isomorphisms*

$$V^k(\omega_X)_0 \cong V^k(\omega_Y)_0 \quad \textit{for all } k \geq 0.$$

It is important to emphasize that for all the applications we are interested in (e.g., invariance of the Albanese dimension, invariance of the holomorphic Euler characteristic and invariance of Hodge numbers) it is in fact enough to verify Conjecture 1.3. We also remark that Conjecture 1.2 holds for varieties of general type since the cohomological support loci are birational invariants while derived equivalent varieties of general type are birational by [Kawamata 2002, Theorem 1.4]. Moreover, in [Popa 2013, §2], it has been shown that Conjecture 1.2 holds for surfaces as well.

In Section 3, we try to attack the above conjectures for varieties of arbitrary dimension. To begin with, we show that Theorem 1.1 implies the derived invariance of $V^0(\omega_X)$ (see Proposition 3.1). On the other hand, due to the possible presence of nontrivial automorphisms, the study of the derived invariance of the higher

cohomological support loci is more involved. Nonetheless, by using a version of the Hochschild–Kostant–Rosenberg isomorphism and Brion's structural results on the actions of nonaffine groups on smooth varieties, we are able to show the derived invariance of $V^1(\omega_X)_0$ (see Corollary 3.4). The next theorem summarizes the main results on the derived invariance of these loci.

Theorem 1.4. *Let X and Y be smooth projective derived equivalent varieties. Then the Rouquier isomorphism induces isomorphisms of algebraic sets*

(i) $V^0(\omega_X) \cong V^0(\omega_Y)$,

(ii) $V^0(\omega_X) \cap V^1(\omega_X) \cong V^0(\omega_Y) \cap V^1(\omega_Y)$ *and*

(iii) $V^1(\omega_X)_0 \cong V^1(\omega_Y)_0$.

We note that (i) also holds if we consider arbitrary powers of the canonical bundle (see Proposition 3.1). We point out also that cases in which the Rouquier isomorphism induces the full isomorphism $V^1(\omega_X) \cong V^1(\omega_Y)$ occur for instance when either X is of maximal Albanese dimension (see Corollary 5.2) or when the neutral component of the automorphism group, $\mathrm{Aut}^0(X)$, is affine (see Remark 3.6); Theorem 1.4 is proved in Section 3.

Next we study Conjectures 1.2 and 1.3 for varieties of dimension three. In the process, we recover Conjecture 1.2 in dimension two as well, making the isomorphisms on cohomological support loci explicit. In the following theorem, we collect all results concerning the behavior of cohomological support loci of derived equivalent threefolds. We denote by $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ the Albanese map of X, and we say that X is of *maximal Albanese dimension* if $\dim \mathrm{alb}_X(X) = \dim X$, i.e., alb_X is generically finite onto its image.

Theorem 1.5. *Let X and Y be smooth projective irregular derived equivalent threefolds. Then*:

(i) *Conjecture 1.3 holds.*

(ii) *Conjecture 1.2 holds if one of the following hypotheses is satisfied*:

  (a) *X is of maximal Albanese dimension.*

  (b) *$V^k(\omega_X) = \mathrm{Pic}^0(X)$ for some $k \geq 0$ (for instance, by* [Pareschi and Popa 2011, Theorem E], *$V^0(\omega_X) = \mathrm{Pic}^0(X)$ whenever $\mathrm{alb}_X(X)$ is not fibered in subtori and $V^0(\omega_X) \neq \varnothing$).*

  (c) *$\mathrm{Aut}^0(X)$ is affine (for instance, by a theorem of Nishi* [Matsumura 1963, Theorem 2], *this again happens when $\mathrm{alb}_X(X)$ is not fibered in subtori).*

(iii) *If $q(X) \geq 2$, then $\dim V^k(\omega_X) = \dim V^k(\omega_Y)$ for all $k \geq 0$.*

Point (iii) brings evidence to a further variant of Conjecture 1.2 predicting the invariance of the dimensions of cohomological support loci [Popa 2013, Variant 1.4]; partial results for the case $q(X) = 1$ are described in Remark 6.10. Since the proofs

of Theorems 1.4 and 1.5 extend to analogous results regarding cohomological support loci of bundles of holomorphic p-forms, when possible, we will prove them in such generality. Please refer to Theorem 4.2 and Section 6 for the proof of Theorem 1.5.

Finally, we move our attention to applications of Theorems 1.4 and 1.5. The first regards the behavior of the Albanese dimension, $\dim \operatorname{alb}_X(X)$, under derived equivalence. According to Conjecture 1.3, the Albanese dimension is expected to be preserved under derived equivalence as it can be read off from the dimensions of the $V^k(\omega_X)_0$'s (see (5)), which is the case in dimension three thanks to Theorem 1.5. In higher dimension, we establish this invariance for varieties having nonnegative Kodaira dimension $\kappa(X)$ by using the derived invariance of the irregularity and an extension of a result due to Chen, Hacon and Pardini [Hacon and Pardini 2002, Proposition 2.1; Chen and Hacon 2004, Corollary 3.6] on the study of the geometry of the Albanese map via the Iitaka fibration; see Section 5.

Theorem 1.6. *Let X and Y be smooth projective derived equivalent varieties. If $\dim X \leq 3$, or if $\dim X > 3$ and $\kappa(X) \geq 0$, then*

$$\dim \operatorname{alb}_X(X) = \dim \operatorname{alb}_Y(Y).$$

The second application concerns the holomorphic Euler characteristic. This is expected to be the same for arbitrary derived equivalent smooth projective varieties since the Hodge numbers are expected to be preserved (which is known to hold in dimension up to three [Popa and Schnell 2011, Corollary C]). We deduce this for varieties of large Albanese dimension as a consequence of the previous results and generic vanishing.

Corollary 1.7. *Let X and Y be smooth projective derived equivalent varieties. If $\dim \operatorname{alb}_X(X) = \dim X$, or if $\dim \operatorname{alb}_X(X) = \dim X - 1$ and $\kappa(X) \geq 0$, then*

$$\chi(\omega_X) = \chi(\omega_Y).$$

An immediate consequence is the derived invariance of two of the Hodge numbers for fourfolds satisfying the hypotheses of Corollary 1.7.

Corollary 1.8. *Let X and Y be smooth projective derived equivalent fourfolds. If $\dim \operatorname{alb}_X(X) = 4$, or if $\dim \operatorname{alb}_X(X) = 3$ and $\kappa(X) \geq 0$, then*

$$h^{0,2}(X) = h^{0,2}(Y) \quad \text{and} \quad h^{1,3}(X) = h^{1,3}(Y).$$

We remark that Popa and Schnell [2011, Corollary 3.4] establish the invariance of $h^{0,2}$ and $h^{1,3}$ under different hypotheses, namely, when $\operatorname{Aut}^0(X)$ is not affine (we recall that $h^{0,4}$, $h^{0,3}$, $h^{0,1}$ and $h^{1,2}$ are always known to be invariant; see [Popa and Schnell 2011]). Corollaries 1.7 and 1.8 are proved in Section 7.

We now present our last application in a direction that is one of the main motivations for Conjectures 1.2 and 1.3 as explained in [Popa 2013]. From the classification of Fourier–Mukai equivalences for surfaces [Kawamata 2002; Bridgeland and Maciocia 2001], it is known that, if X admits a fibration $f : X \to C$ onto a smooth curve of genus ≥ 2, then any of its Fourier–Mukai partners admits a fibration onto the same curve. Here we use our analysis, and a theorem of Green and Lazarsfeld regarding the properties of positive-dimensional irreducible components of the cohomological support loci, to investigate the behavior of fibrations of derived equivalent threefolds onto irregular varieties. Recall that a smooth variety X is called of *Albanese general type* if alb_X is nonsurjective and generically finite onto its image. The proof of the next corollary is contained in Proposition 7.3 and Remark 7.4.

Corollary 1.9. *Let X and Y be smooth projective derived equivalent threefolds. There exists a morphism $f : X \to W$ with connected fibers onto a normal variety W of dimension ≤ 2 such that any smooth model of W is of Albanese general type if and only if Y has a fibration of the same type. Moreover, there exists a morphism $f : X \to C$ with connected fibers onto a smooth curve C of genus ≥ 2 if and only if there exists a morphism $h : Y \to D$ with connected fibers onto a smooth curve D of genus ≥ 2.*

To conclude, we remark that, while the approach in this paper relies in part on techniques of [Popa and Schnell 2011], the key new ingredient is their interaction with the twisted Hochschild homology, introduced and studied here. We are hopeful that this general method will find further applications in the future.

2. Derived invariance of the twisted Hochschild homology

In this section, we aim to prove Theorem 1.1. Its proof is based on a technical lemma extending previous computations carried out by Căldăraru [2003, Proposition 8.1] and Orlov [2003, Isomorphism (10)].

Let X and Y be smooth projective varieties defined over an algebraically closed field K, and let p and q be the projections from $X \times Y$ onto the first and second factor, respectively. We denote by $\boldsymbol{D}(X) := \boldsymbol{D}^b(\mathscr{C}oh(X))$ the bounded derived category of coherent sheaves on a smooth projective variety X. When there is no possibility of ambiguity, we use the same symbol to denote a functor and its associated derived functor. An object $\mathscr{E}$ in $\boldsymbol{D}(X \times Y)$ defines *Fourier–Mukai functors* with *kernel* $\mathscr{E}$ as

$$\Phi_{\mathscr{E}} : \boldsymbol{D}(X) \to \boldsymbol{D}(Y), \quad \mathscr{F} \mapsto q_*(p^*\mathscr{F} \otimes \mathscr{E}),$$
$$\Psi_{\mathscr{E}} : \boldsymbol{D}(Y) \to \boldsymbol{D}(X), \quad \mathscr{G} \mapsto p_*(q^*\mathscr{G} \otimes \mathscr{E}).$$

We say that X and Y are *derived equivalent* if there exists a K-linear exact equivalence of triangulated categories $\Phi : \boldsymbol{D}(X) \to \boldsymbol{D}(Y)$. By a fundamental result of

Orlov, any such equivalence is of Fourier–Mukai type; i.e., there exists an object $\mathscr{E}$ in $\boldsymbol{D}(X \times Y)$ such that $\Phi \cong \Phi_{\mathscr{E}}$. Furthermore, the object $\mathscr{E}$ is unique up to isomorphism.

We recall that an equivalence $\Phi_{\mathscr{E}} : \boldsymbol{D}(X) \to \boldsymbol{D}(Y)$ induces an equivalence

$$\Phi_{\mathscr{E}^* \boxtimes \mathscr{E}} : \boldsymbol{D}(X \times X) \to \boldsymbol{D}(Y \times Y)$$

with kernel

$$\mathscr{E}^* \boxtimes \mathscr{E} := p_{13}^* \mathscr{E}^* \otimes p_{24}^* \mathscr{E},$$

where $\mathscr{E}^* := \boldsymbol{R}\mathscr{H}om(\mathscr{E}, \mathbb{O}_{X \times Y}) \otimes p^* \omega_X[\dim X]$ and p_{rs} is the projection from $X \times X \times Y \times Y$ onto the (r, s)-factor [Orlov 2003, Proposition 2.1.7]. Moreover, for any automorphisms $\varphi \in \mathrm{Aut}^0(X)$ and $\psi \in \mathrm{Aut}^0(Y)$ (here the superscript 0 denotes the neutral component of the corresponding group), we define the embeddings $(1, \varphi) : X \hookrightarrow X \times X,\ x \mapsto (x, \varphi(x))$ and $(1, \psi) : Y \hookrightarrow Y \times Y,\ y \mapsto (y, \psi(y))$. Finally, we denote by i and j the diagonal embeddings of X and Y, respectively.

Lemma 2.1. *Let X and Y be smooth projective varieties defined over an algebraically closed field, and let $\Phi_{\mathscr{E}} : \boldsymbol{D}(X) \to \boldsymbol{D}(Y)$ be an equivalence. Denote by F the induced Rouquier isomorphism (see (1)), and let $m \in \mathbb{Z}$. If $F(\varphi, L) = (\psi, M)$, then*

$$\Phi_{\mathscr{E}^* \boxtimes \mathscr{E}}\big((1, \varphi)_*(\omega_X^{\otimes m} \otimes L)\big) \cong (1, \psi)_*(\omega_Y^{\otimes m} \otimes M).$$

Proof. We denote by t_r and t_{rs} the projections from $Y \times X \times Y$ onto the r-th and (r, s)-th factors, respectively. Moreover, we define the morphism $\lambda : Y \times X \times Y \to X \times X \times Y \times Y$ as $(y_1, x, y_2) \mapsto (x, \varphi(x), y_1, y_2)$, and we look at the fiber product diagram

$$\begin{array}{ccc} Y \times X \times Y & \xrightarrow{\lambda} & X \times X \times Y \times Y \\ \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle p_{12}} \\ X & \xrightarrow{(1,\varphi)} & X \times X \end{array}$$

so that, by base change and the projection formula, we get

$$\begin{aligned} \Phi_{\mathscr{E}^* \boxtimes \mathscr{E}}\big((1, \varphi)_*(\omega_X^{\otimes m} \otimes L)\big) &\cong p_{34*}\big(p_{12}^*(1, \varphi)_*(\omega_X^{\otimes m} \otimes L) \otimes (\mathscr{E}^* \boxtimes \mathscr{E})\big) \\ &\cong p_{34*}\big(\lambda_* t_2^*(\omega_X^{\otimes m} \otimes L) \otimes p_{13}^* \mathscr{E}^* \otimes p_{24}^* \mathscr{E}\big) \\ &\cong p_{34*}\lambda_*\big(t_2^*(\omega_X^{\otimes m} \otimes L) \otimes \lambda^* p_{13}^* \mathscr{E}^* \otimes \lambda^* p_{24}^* \mathscr{E}\big) \\ &\cong t_{13*}\big(t_2^*(\omega_X^{\otimes m} \otimes L) \otimes t_{21}^* \mathscr{E}^* \otimes t_{23}^*(\varphi \times 1)^* \mathscr{E}\big). \end{aligned} \tag{2}$$

By [Orlov 2003, p. 535], the equivalence $\Phi_{\mathscr{E}}$ induces an isomorphism $\mathscr{E} \otimes p^* \omega_X \cong \mathscr{E} \otimes q^* \omega_Y$. Moreover, by [Popa and Schnell 2011, Lemma 3.1], the condition $F(\varphi, L) = (\psi, M)$ is equivalent to an isomorphism of objects in $\boldsymbol{D}(X \times Y)$

$$(\varphi \times 1)^* \mathscr{E} \otimes p^* L \cong (1 \times \psi)_* \mathscr{E} \otimes q^* M. \tag{3}$$

Therefore, we get an isomorphism of objects

$$p^*(\omega_X^{\otimes m} \otimes L) \otimes (\varphi \times 1)^*\mathscr{E} \cong q^*(\omega_Y^{\otimes m} \otimes M) \otimes (1 \times \psi)_*\mathscr{E},$$

and by pulling it back via $t_{23} : Y \times X \times Y \to X \times Y$, we finally obtain

$$t_2^*(\omega_X^{\otimes m} \otimes L) \otimes t_{23}^*(\varphi \times 1)^*\mathscr{E} \cong t_3^*(\omega_Y^{\otimes m} \otimes M) \otimes t_{23}^*(1 \times \psi)_*\mathscr{E}. \tag{4}$$

At this point, we rewrite the morphism $t_3 : Y \times X \times Y \to Y$ as $t_3 = \sigma_2 \circ t_{13}$, where $\sigma_2 : Y \times Y \to Y$ is the projection onto the second factor. Moreover, we denote by $\rho : Y \times X \to X \times Y$ the inversion morphism $(y, x) \mapsto (x, y)$. Then by (2) and (4), we obtain

$$\begin{aligned}
\Phi_{\mathscr{E}^* \boxtimes \mathscr{E}}\big((1, \varphi)_*(\omega_X^{\otimes m} \otimes L)\big) &\cong t_{13*}\big(t_3^*(\omega_Y^{\otimes m} \otimes M) \otimes t_{21}^*\mathscr{E}^* \otimes t_{23}^*(1 \times \psi)_*\mathscr{E}\big) \\
&\cong t_{13*}\big(t_{13}^*\sigma_2^*(\omega_Y^{\otimes m} \otimes M) \otimes t_{21}^*\mathscr{E}^* \otimes t_{23}^*(1 \times \psi)_*\mathscr{E}\big) \\
&\cong \sigma_2^*(\omega_Y^{\otimes m} \otimes M) \otimes t_{13*}\big(t_{21}^*\mathscr{E}^* \otimes t_{23}^*(1 \times \psi)_*\mathscr{E}\big) \\
&\cong \sigma_2^*(\omega_Y^{\otimes m} \otimes M) \otimes t_{13*}\big(t_{12}^*\rho^*\mathscr{E}^* \otimes t_{23}^*(1 \times \psi)_*\mathscr{E}\big).
\end{aligned}$$

Finally, by [Orlov 2003, Proposition 2.1.2], the object $t_{13*}(t_{12}^*\rho^*\mathscr{E}^* \otimes t_{23}^*(1 \times \psi)_*\mathscr{E})$ in $\boldsymbol{D}(Y \times Y)$ is the kernel of the composition

$$\Phi_{(1\times\psi)_*\mathscr{E}} \circ \Phi_{\rho^*\mathscr{E}^*} \cong \psi_* \circ \Phi_{\mathscr{E}} \circ \Psi_{\mathscr{E}^*} \cong \psi_* \circ \mathrm{id}_{\boldsymbol{D}(Y)} \cong \psi_*,$$

where we used the fact that $\Psi_{\mathscr{E}^*}$ is the right adjoint to $\Phi_{\mathscr{E}}$. On the other hand, since the kernel of the derived functor $\psi_* : \boldsymbol{D}(Y) \to \boldsymbol{D}(Y)$ is the structure sheaf of the graph of ψ, i.e., $\mathcal{O}_{\Gamma_\psi} \cong (1, \psi)_*\mathcal{O}_Y$ [Huybrechts 2006, Example 5.4], we have an isomorphism

$$t_{13*}\big(t_{12}^*\rho^*\mathscr{E}^* \otimes t_{23}^*(1 \times \psi)_*\mathscr{E}\big) \cong (1, \psi)_*\mathcal{O}_Y$$

as the kernel of an equivalence is unique up to isomorphism. To recap,

$$\begin{aligned}
\Phi_{\mathscr{E}^* \boxtimes \mathscr{E}}\big((1, \varphi)_*(\omega_X^{\otimes m} \otimes L)\big) &\cong \sigma_2^*(\omega_Y^{\otimes m} \otimes M) \otimes (1, \psi)_*\mathcal{O}_Y \\
&\cong (1, \psi)_*\big((1, \psi)^*\sigma_2^*(\omega_Y^{\otimes m} \otimes M)\big) \\
&\cong (1, \psi)_*\big(\psi^*(\omega_Y^{\otimes m} \otimes M)\big) \\
&\cong (1, \psi)_*(\omega_Y^{\otimes m} \otimes M).
\end{aligned}$$

The last isomorphism follows as the action of $\mathrm{Aut}^0(X)$ on $\mathrm{Pic}^0(X)$ is trivial [Popa and Schnell 2011, Footnote, p. 531]. □

Proof of Theorem 1.1. Let $\mathscr{E}$ be the kernel of the equivalence Φ so that $\Phi \cong \Phi_{\mathscr{E}}$. By Lemma 2.1, the equivalence $\Phi_{\mathscr{E}^* \boxtimes \mathscr{E}}$ induces isomorphisms between the graded

components of $HH_*(X, \varphi, L, m)$ and $HH_*(Y, \psi, M, m)$ as follows:

$$\begin{aligned}\operatorname{Ext}^k_{X\times X}\big(i_*\mathbb{O}_X, (1,\varphi)_*(\omega_X^{\otimes m}\otimes L)\big) &\cong \operatorname{Ext}^k_{Y\times Y}\big(\Phi_{\mathscr{E}^*\boxtimes\mathscr{E}}(i_*\mathbb{O}_X), \Phi_{\mathscr{E}^*\boxtimes\mathscr{E}}\big((1,\varphi)_*(\omega_X^{\otimes m}\otimes L)\big)\big)\\ &\cong \operatorname{Ext}^k_{Y\times Y}\big(j_*\mathbb{O}_Y, (1,\psi)_*(\omega_Y^{\otimes m}\otimes M)\big).\end{aligned}$$

Moreover, since $\Phi_{\mathscr{E}^*\boxtimes\mathscr{E}}$ is a functor, it follows that it induces an isomorphism of graded modules. □

Theorem 1.1 will be often used in the following weaker form:

Corollary 2.2. *Let X and Y be smooth projective derived equivalent varieties defined over an algebraically closed field of characteristic zero. If $F(1, L) = (1, M)$, then for any integers m and $k \geq 0$ there exist isomorphisms*

$$\bigoplus_{q=0}^{k} H^{k-q}(X, \Omega_X^{\dim X-q}\otimes\omega_X^{\otimes m}\otimes L) \cong \bigoplus_{q=0}^{k} H^{k-q}(Y, \Omega_Y^{\dim Y-q}\otimes\omega_Y^{\otimes m}\otimes M).$$

Proof. The corollary is a consequence of Theorem 1.1 and of the general fact that the groups $\operatorname{Ext}^k_{X\times X}(i_*\mathbb{O}_X, i_*\mathscr{F})$ decompose as $\bigoplus_{q=0}^k H^{k-q}(X, \Omega_X^{\dim X-q}\otimes\omega_X^{-1}\otimes\mathscr{F})$ for any coherent sheaf $\mathscr{F}$ and for all $k \geq 0$ [Yekutieli 2003, Corollary 4.7; Swan 1996, Corollary 2.6]. □

3. Behavior of cohomological support loci under derived equivalence

In this section, we study the behavior of cohomological support loci under derived equivalence. Applications of our analysis will be provided in Section 7. From now on, we work over the field of the complex numbers.

3A. ***Cohomological support loci.*** Let X be a complex smooth projective irregular variety. Given a coherent sheaf $\mathscr{F}$ on X, we define the *cohomological support loci of* $\mathscr{F}$ as

$$V^k_r(\mathscr{F}) := \{L \in \operatorname{Pic}^0(X) \mid h^k(X, \mathscr{F}\otimes L) \geq r\}$$

for all integers $k \geq 0$ and $r \geq 1$. By semicontinuity, these loci are algebraic closed subsets in $\operatorname{Pic}^0(X)$. We set $V^k(\mathscr{F}) := V^k_1(\mathscr{F})$, and we denote by $V^k_r(\mathscr{F})_0$ the union of all the irreducible components of $V^k_r(\mathscr{F})$ passing through the origin of $\operatorname{Pic}^0(X)$. By following the work of Pareschi and Popa [2011], we say that $\mathscr{F}$ is a *GV-sheaf* if

$$\operatorname{codim}_{\operatorname{Pic}^0(X)} V^k(\mathscr{F}) \geq k \quad \text{for all } k > 0.$$

In the following, we study the behavior of the loci $V^k_r(\mathscr{F})$ under equivalence of derived categories where $\mathscr{F} = \omega_X, \omega_X^{\otimes m}, \Omega^p_X\otimes\omega_X^{\otimes m}$ with $m, p \in \mathbb{Z}$ and $p \geq 0$. We recall that the cohomological support loci $V^k(\omega_X)$ associated to the canonical bundle are invariant under birational modifications for all $k \geq 0$. Furthermore, they

detect the Albanese dimension of X, namely, the dimension of the image of the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$, thanks to the following formula [Popa 2013, p. 7] deduced from results of [Green and Lazarsfeld 1987; Lazarsfeld and Popa 2010]:

$$\dim \mathrm{alb}_X(X) = \min_{k=0,\dots,\dim X} \{\dim X - k + \operatorname{codim} V^k(\omega_X)_0\}. \tag{5}$$

Finally, we point out that, if $\dim \mathrm{alb}_X(X) = \dim X - k$, then there are inclusions

$$V^k(\omega_X) \supset V^{k+1}(\omega_X) \supset \cdots \supset V^{\dim X}(\omega_X) = \{\mathcal{O}_X\} \tag{6}$$

(see [Pareschi and Popa 2011, Proposition 3.14; Green and Lazarsfeld 1987, Theorem 1] or [Ein and Lazarsfeld 1997, Lemma 1.8] for the case $k = 0$).

3B. ***Derived invariance of the zeroth cohomological support locus.*** The following proposition proves and extends Theorem 1.4(i):

Proposition 3.1. *Let X and Y be smooth projective varieties, and let $\Phi_{\mathscr{E}} : \boldsymbol{D}(X) \to \boldsymbol{D}(Y)$ be an equivalence. Denote by F the induced Rouquier isomorphism, and let m and r be integers such that $r \geq 1$. If $L \in V^0_r(\omega_X^{\otimes m})$ and $F(1, L) = (\psi, M)$, then $\psi = 1$ and $M \in V^0_r(\omega_Y^{\otimes m})$. Moreover, F induces an isomorphism of algebraic sets*

$$V^0_r(\omega_X^{\otimes m}) \cong V^0_r(\omega_Y^{\otimes m}).$$

Proof. Let L be a line bundle in $V^0_r(\omega_X^{\otimes m})$, and suppose that $F(1, L) = (\psi, M)$ for some $\psi \in \mathrm{Aut}^0(Y)$ and $M \in \mathrm{Pic}^0(Y)$. By Theorem 1.1 and the adjunction formula, we have

$$\begin{aligned} r \leq h^0(X, \omega_X^{\otimes m} \otimes L) &= \dim \mathrm{Hom}_{X\times X}\big(i_*\mathcal{O}_X, i_*(\omega_X^{\otimes m} \otimes L)\big) \\ &= \dim \mathrm{Hom}_{Y\times Y}\big(j_*\mathcal{O}_Y, (1,\psi)_*(\omega_Y^{\otimes m} \otimes M)\big) \\ &= \dim \mathrm{Hom}_{Y}\big((1,\psi)^* j_*\mathcal{O}_Y, \omega_Y^{\otimes m} \otimes M\big). \end{aligned}$$

Since $(1, \psi)^* j_* \mathcal{O}_Y$ is supported on the locus of fixed points of ψ (which is of codimension ≥ 1 if $\psi \neq 1$) and since there are no nonzero morphisms from a torsion sheaf to a locally free sheaf, we must have that ψ is the identity automorphism on Y and consequently that $M \in V^0_r(\omega_Y^{\otimes m})$. Therefore, we have an inclusion of algebraic sets $F(1, V^0_r(\omega_X^{\otimes m})) \subset (1, V^0_r(\omega_Y^{\otimes m}))$.

In order to show the reverse inclusion, we consider the right adjoint $\Psi_{\mathscr{E}^*}$ to $\Phi_{\mathscr{E}}$ so that $\Psi_{\mathscr{E}^*} \circ \Phi_{\mathscr{E}} \cong 1_{\boldsymbol{D}(X)}$ and $\Phi_{\mathscr{E}} \circ \Psi_{\mathscr{E}^*} \cong 1_{\boldsymbol{D}(Y)}$. An easy computation shows that, if F' is the Rouquier isomorphism induced by $\Psi_{\mathscr{E}^*}$, then $F' = F^{-1}$ [Lombardi 2013, Lemma 2.1.9]. Hence, by repeating the previous argument, we get an inclusion $F^{-1}(1, V^0_r(\omega_Y^{\otimes m})) \subset (1, V^0_r(\omega_X^{\otimes m}))$ inducing the wanted isomorphism. □

3C. ***Behavior of higher cohomological support loci under derived equivalence.*** In this section, we establish the isomorphism $V^1(\omega_X)_0 \cong V^1(\omega_Y)_0$ of Theorem 1.4. It turns out that, by using the same techniques (i.e., invariance of twisted Hochschild homology and Brion's results on actions of nonaffine groups), one can show a more general result involving cohomological support loci associated to bundles of holomorphic p-forms, which we now present.

Theorem 3.2. *Let X and Y be smooth projective varieties of dimension d, and let $\Phi_{\mathscr{E}} : \boldsymbol{D}(X) \to \boldsymbol{D}(Y)$ be an equivalence. Denote by F be the induced Rouquier isomorphism, and let m be an integer. If $L \in \bigcup_{p,q\geq 0} V^p(\Omega_X^q \otimes \omega_X^{\otimes m})_0$ and $F(1, L) = (\psi, M)$, then $\psi = 1$ and $M \in \bigcup_{p,q\geq 0} V^p(\Omega_Y^q \otimes \omega_Y^{\otimes m})_0$. Moreover, F induces isomorphisms of algebraic sets*

$$\bigcup_{q=0}^{k} V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m})_0 \cong \bigcup_{q=0}^{k} V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes m})_0 \quad \textit{for any } k \geq 0.$$

Proof. To begin with, we recall some notation and facts from [Popa and Schnell 2011, Theorem A]. Let $\alpha : \mathrm{Pic}^0(Y) \to \mathrm{Aut}^0(X)$ and $\beta : \mathrm{Pic}^0(X) \to \mathrm{Aut}^0(Y)$ be morphisms defined as

$$\alpha(M) = \mathrm{pr}_1(F^{-1}(1, M)) \quad \text{and} \quad \beta(L) = \mathrm{pr}_1(F(1, L))$$

(pr_1 denotes the projection onto the first factor from the product $\mathrm{Aut}^0(\,\cdot\,) \times \mathrm{Pic}^0(\,\cdot\,)$). We denote by A and B the images of α and β, respectively. We recall that A and B are isogenous abelian varieties.

We first consider the case when A is trivial. Then $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$, and by Corollary 2.2, we get inclusions

$$F\big(1, \textstyle\bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m} r)\big) \subset \big(1, \textstyle\bigcup_q V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes m} r)\big) \quad \text{for any } k \geq 0.$$

In order to prove the reverse inclusions, we note that B is trivial as well and that the Rouquier isomorphism induced by the right adjoint $\Psi_{\mathscr{E}^*}$ to $\Phi_{\mathscr{E}}$ is F^{-1}. Therefore, a second application of Corollary 2.2 yields inclusions

$$F^{-1}\big(1, \textstyle\bigcup_q V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes m} r)\big) \subset \big(1, \textstyle\bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m})\big) \quad \text{for any } k \geq 0,$$

concluding the proof of this case.

We suppose now that both A and B are nontrivial. We first show the following:

Claim 3.3. *There are inclusions $F\big(1, \bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m})_0\big) \subset (1, \mathrm{Pic}^0(Y))$ for all integers m and $k \geq 0$.*

Proof. Brion's results on actions of nonaffine algebraic groups imply that X is an étale locally trivial fibration $\xi : X \to A/H$ where H is a finite subgroup of A (the proof of this fact is analogous to the one of [Popa and Schnell 2011, Lemma 2.4];

see also [Brion 2010]). Let Z be the smooth and connected fiber of ξ over the origin of A/H. Via base change, we get a commutative diagram

$$\begin{array}{ccc} A \times Z & \xrightarrow{g} & X \\ \downarrow & & \downarrow \xi \\ A & \longrightarrow & A/H \end{array}$$

where $g(\varphi, z) = \varphi(z)$. Let $(z_0, y_0) \in Z \times Y$ be an arbitrary point, and let

$$f = (f_1 \times f_2) : A \times B \to X \times Y$$

be the orbit map $(\varphi, \psi) \mapsto (\varphi(z_0), \psi(y_0))$. In [Popa and Schnell 2011, p. 533], it is shown that

$$L \in (\operatorname{Ker} f_1^*)_0 \implies F(1, L) = (1, M) \quad \text{for some } M \in \operatorname{Pic}^0(Y)$$

(here $(\operatorname{Ker} f_1^*)_0$ denotes the neutral component of $\operatorname{Ker} f_1^*$). So it is enough to show the inclusion

$$\bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m})_0 \subset (\operatorname{Ker} f_1^*)_0 \quad \text{for any } k \geq 0. \tag{7}$$

This is achieved by computing cohomology groups on $A \times Z$ via the étale morphism g and by using the fact that these computations are straightforward on A. Let p_1 and p_2 be the projections from the product $A \times Z$ onto the first and second factors, respectively. By denoting by $\nu : A \times \{z_0\} \hookrightarrow A \times Z$ the inclusion morphism, we have $g \circ \nu = f_1$. Moreover, via the isomorphism $\operatorname{Pic}^0(A \times Z) \cong \operatorname{Pic}^0(A) \times \operatorname{Pic}^0(Z)$, we obtain $g^*L \cong p_1^*L_1 \otimes p_2^*L_2$, where $L_1 \in \operatorname{Pic}^0(A)$ and $L_2 \in \operatorname{Pic}^0(Z)$. Note also that $f_1^*L \cong \nu^* g^* L \cong L_1$. Finally, for all $L \in \bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m})$, there are inclusions

$$0 \neq \bigoplus_{q=0}^{k} H^{k-q}(X, \Omega_X^{d-q} \otimes \omega_X^{\otimes m} \otimes L) \subset \bigoplus_{q=0}^{k} H^{k-q}(A \times Z, \Omega_{A\times Z}^{d-q} \otimes \omega_{A\times Z}^{\otimes m} \otimes g^*L) \tag{8}$$

[Lazarsfeld 2004, Injectivity Lemma 4.1.14]. Therefore, thanks to Künneth's formula, the sum on the right-hand side of (8) is nonzero only if $f_1^*L \cong \mathbb{O}_A$, i.e., $L \in \operatorname{Ker} f_1^*$. This shows (7). □

By Claim 3.3 and Corollary 2.2, we obtain that for any $k \geq 0$ the Rouquier isomorphism maps

$$1 \times \bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m})_0 \mapsto 1 \times \bigcup_q V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes m})_0.$$

In complete analogy, one can also show that

$$M \in \bigcup_q V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes m})_0 \implies F^{-1}(1, M) = (1, L) \quad \text{for some } L \in \operatorname{Pic}^0(X).$$

This concludes the proof since, by Corollary 2.2, F^{-1} maps

$$1 \times \bigcup_q V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes m})_0 \mapsto 1 \times \bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m})_0$$

for any $k \geq 0$. □

The following corollaries yield the proofs of Theorem 1.4(iii) and (ii):

Corollary 3.4. *Under the assumptions of Theorem 3.2, the Rouquier isomorphism F induces isomorphisms of algebraic sets*

$$V_r^1(\omega_X)_0 \cong V_r^1(\omega_Y)_0 \quad \textit{for any } r \geq 1.$$

Proof. Let $L \in V_r^1(\omega_X)_0$. By Theorem 3.2, we have $F(1, L) = (1, M)$ for some $M \in \mathrm{Pic}^0(Y)$, and by Corollary 2.2, we get an isomorphism

$$H^1(X, \omega_X \otimes L) \oplus H^0(X, \Omega_X^{d-1} \otimes L) \cong H^1(Y, \omega_Y \otimes M) \oplus H^0(Y, \Omega_Y^{d-1} \otimes M).$$

Moreover, by Serre duality and the Hodge linear-conjugate isomorphism, we obtain equalities

$$h^0(X, \Omega_X^{d-1} \otimes L) = h^1(X, \omega_X \otimes L) \quad \text{and} \quad h^0(Y, \Omega_Y^{d-1} \otimes M) = h^1(Y, \omega_Y \otimes M).$$

Hence, $h^1(X, \omega_X \otimes L) = h^1(Y, \omega_Y \otimes M) \geq r$, and therefore, F induces the wanted isomorphisms as in the proof of Theorem 3.2. □

Corollary 3.5. *Under the assumptions of Theorem 3.2, and for any integers l, m, r and s with $r, s \geq 1$, the Rouquier isomorphism F induces isomorphisms of algebraic sets*

$$V_r^0(\omega_X^{\otimes m}) \cap \Big(\bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes l})\Big) \cong V_r^0(\omega_Y^{\otimes m}) \cap \Big(\bigcup_q V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes l})\Big),$$
$$V_r^0(\omega_X^{\otimes m}) \cap V_s^1(\omega_X) \cong V_r^0(\omega_Y^{\otimes m}) \cap V_s^1(\omega_Y).$$

Proof. In Proposition 3.1, we have seen that, if $L \in V_r^0(\omega_X^{\otimes m})$, then $F(1, L) = (1, M)$ for some $M \in V_r^0(\omega_Y^{\otimes m})$. We argue then as in the proofs of Theorem 3.2 and Corollary 3.4. □

Remark 3.6. It is important to note that, whenever $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$, the proofs of Theorem 3.2 and Corollary 3.4 yield full isomorphisms

$$\bigcup_q V^{k-q}(\Omega_X^{d-q} \otimes \omega_X^{\otimes m}) \cong \bigcup_q V^{k-q}(\Omega_Y^{d-q} \otimes \omega_Y^{\otimes m}) \quad \text{for any } k \geq 0,$$
$$V_r^1(\omega_X) \cong V_r^1(\omega_Y).$$

By Theorem 3.2, this occurs either if $V^p(\Omega_X^q \otimes \omega_X^{\otimes m}) = \mathrm{Pic}^0(X)$ for some $p, q \geq 0$ and $m \in \mathbb{Z}$ or if $\mathrm{Aut}^0(X)$ is affine (since in this case the abelian variety A in the proof of Theorem 3.2 is trivial).

4. Popa's conjectures in dimensions two and three

In this section, we aim to prove Theorem 1.5(i). In other words, we show that Conjecture 1.3, predicting the derived invariance of cohomological support loci of type $V^k(\omega_X)_0$, holds in dimension three. The proofs of (ii) and (iii) of the same theorem are postponed to Section 6 since they use the derived invariance of the Albanese dimension, which will be proved in Section 5. Before starting with the proof of Theorem 1.5(i), we make a couple of considerations regarding the case of surfaces.

4A. ***The case of surfaces.*** In dimension two, Popa [2013, Theorem 2.1] proves the derived invariance of the full cohomological support loci $V^k(\omega_X)$. His proof is based on an explicit computation of cohomological support loci according to the classification of surfaces up to Fourier–Mukai equivalences [Bridgeland and Maciocia 2001]. As an application of Proposition 3.1 and Corollary 3.4, we recover this result by making the isomorphisms between cohomological support loci explicit. More precisely, if F is the Rouquier isomorphism induced by an equivalence of derived categories, then $F(1, V_r^k(\omega_X)) = (1, V_r^k(\omega_Y))$ for all integers $k \geq 0$ and $r \geq 1$. Moreover, by using the same techniques, it is possible to show that F induces further isomorphisms $V_r^1(\Omega_X^1) \cong V_r^1(\Omega_Y^1)$ for all $r \geq 1$ (see [Lombardi 2013, Theorem 5.1.2] for a detailed analysis).

Example 4.1 (Elliptic surfaces). Let X be an elliptic surface of Kodaira dimension one and of maximal Albanese dimension (i.e., an isotrivial elliptic surface fibered onto a curve of genus ≥ 2). By following [Beauville 1992], we recall an invariant attached to this type of surfaces. First of all, we note that X admits a unique fibration $f : X \to C$ onto a curve of genus ≥ 2 (see for instance [Popa 2013, p. 5]). We then denote by G the general fiber of f and by $\mathrm{Pic}^0(X, f)$ the kernel of the pull-back of the inclusion $u : G \hookrightarrow X$

$$0 \to \mathrm{Pic}^0(X, f) \to \mathrm{Pic}^0(X) \xrightarrow{u^*} \mathrm{Pic}^0(G).$$

In [Beauville 1992, (1.6)], it is shown that there exists a finite group $\Gamma^0(f)$ and an isomorphism

$$\mathrm{Pic}^0(X, f) \cong f^* \mathrm{Pic}^0(C) \times \Gamma^0(f).$$

The group $\Gamma^0(f)$ is the invariant mentioned above; it is identified with the group of the connected components of $\mathrm{Pic}^0(X, f)$.

We now consider another smooth projective surface Y such that $\boldsymbol{D}(X) \cong \boldsymbol{D}(Y)$. Then, by [Bridgeland and Maciocia 2001, Proposition 4.4], Y is an elliptic surface fibered onto C. Moreover, Y is of maximal Albanese dimension as well. To see this, we observe that, since the cohomological support loci are derived invariant in dimension two, we have $\dim \mathrm{alb}_Y(Y) = \dim \mathrm{alb}_X(X) = 2$ thanks to (5). Hence, we

denote by $g : Y \to C$ the unique fibration of Y and by $\Gamma^0(g)$ its invariant. Pham [2011, Theorem 5.2.7] proves that the invariant $\Gamma^0(\cdot)$ attached to this kind of surface is a derived invariant; in other words, he proves that

$$\Gamma^0(f) \cong \Gamma^0(g). \tag{9}$$

Here we note that (9) also follows from the derived invariance of the zeroth cohomological support locus. In fact, by results of Popa [2013, p. 5], we know that

$$V^0(\omega_X) = \mathrm{Pic}^0(X, f) \cong f^* \mathrm{Pic}^0(C) \times \Gamma^0(f)$$

and similarly for $V^0(\omega_Y)$. Therefore, Proposition 3.1 implies

$$f^* \mathrm{Pic}^0(C) \times \Gamma^0(f) \cong V^0(\omega_X) \cong V^0(\omega_Y) \cong g^* \mathrm{Pic}^0(C) \times \Gamma^0(g),$$

which in particular yields (9).

4B. *Proof of Theorem 1.5(i).*

Theorem 4.2. *Let X and Y be smooth projective threefolds and $\Phi_{\mathscr{E}} : \boldsymbol{D}(X) \to \boldsymbol{D}(Y)$ an equivalence, and let F be the induced Rouquier isomorphism. Then F induces isomorphisms of algebraic sets*

$$V_r^p(\Omega_X^q)_0 \cong V_r^p(\Omega_Y^q)_0 \quad \text{for any } p, q \geq 0 \text{ and } r \geq 1.$$

Proof. The isomorphisms $V_r^0(\omega_X) \cong V_r^0(\omega_Y)$ and $V_r^1(\omega_X)_0 \cong V_r^1(\omega_Y)_0$ have been proved in Proposition 3.1 and Corollary 3.4, respectively. On the other hand, the isomorphisms $V_r^3(\omega_X) \cong V_r^3(\omega_Y)$ are trivial and follow by Serre duality. We now show the isomorphisms $V_r^2(\omega_X)_0 \cong V_r^2(\omega_Y)_0$. To begin with, we note that, by Claim 3.3, if $L \in V_r^2(\omega_X)_0$, then necessarily $F(1, L) = (1, M)$ for some line bundle $M \in \mathrm{Pic}^0(Y)$. Moreover, for $k = 0, 1$, we have equalities $h^k(X, \omega_X \otimes L) = h^k(Y, \omega_Y \otimes M)$ whenever $L \in V_r^2(\omega_X)_0$ and $F(1, L) = (1, M)$ (see Corollary 2.2). Finally, since the holomorphic Euler characteristic is both a derived invariant in dimension three [Popa and Schnell 2011, Corollary C] and invariant under deformation, we have equalities $\chi(\omega_X \otimes L) = \chi(\omega_X) = \chi(\omega_Y) = \chi(\omega_Y \otimes M)$, from which we easily deduce $h^2(X, \omega_X \otimes L) = h^2(Y, \omega_Y \otimes M)$. Thus, if $L \in V_r^2(\omega_X)_0$, then $M \in V_r^2(\omega_Y)_0$ and consequently F induces inclusions $F(1, V_r^2(\omega_X)_0) \subset (1, V_r^2(\omega_Y)_0)$. Since F^{-1} is the Rouquier isomorphism induced by the right adjoint $\Psi_{\mathscr{E}^*}$ to $\Phi_{\mathscr{E}}$, we can repeat the previous argument to obtain the reverse inclusions $F^{-1}(1, V_r^2(\omega_Y)_0) \subset (1, V_r^2(\omega_X)_0)$. This in turn yields isomorphisms $V_r^0(\Omega_X^1)_0 \cong V_r^0(\Omega_Y^1)_0$ thanks to Serre duality and the Hodge linear-conjugate isomorphism.

We now prove the isomorphisms $V_r^1(\Omega_X^q)_0 \cong V_r^1(\Omega_Y^q)_0$ for $q = 1, 2$. By Claim 3.3, we have $F(1, V_r^1(\Omega_X^q)_0) \subset (1, \mathrm{Pic}^0(Y))$. By Serre duality and the Hodge linear-conjugate isomorphism, $h^0(X, \Omega_X^1 \otimes L) = h^2(X, \omega_X \otimes L)$ and $h^0(Y, \Omega_Y^1 \otimes M) = h^2(Y, \omega_Y \otimes M)$ for all line bundles $L \in \mathrm{Pic}^0(X)$ and $M \in \mathrm{Pic}^0(Y)$. Consequently, if

$L \in V^0(\Omega^1_X)_0$ and $F(1, L) = (1, M)$, then by Corollary 2.2 with $m = 0$ and $k = 2$ we have $h^1(X, \Omega^2_X \otimes L) = h^1(Y, \Omega^2_Y \otimes M)$. At this point, in order to prove the wanted isomorphisms, it is enough to proceed as before. In complete analogy, one can also prove the isomorphisms $V^1_r(\Omega^1_X)_0 \cong V^1_r(\Omega^1_Y)_0$, this time by using Corollary 2.2 with $m = 0$ and $k = 3$. □

5. Behavior of the Albanese dimension under derived equivalence

In this section, we prove Theorem 1.6. Our main tool is a generalization of a result due to Chen, Hacon and Pardini saying that, if $f : X \to Z$ is a nonsingular representative of the Iitaka fibration of a smooth projective variety X of maximal Albanese dimension, then

$$q(X) - q(Z) = \dim X - \dim Z$$

[Hacon and Pardini 2002, Proposition 2.1; Chen and Hacon 2004, Corollary 3.6]. We generalize this fact in two ways: (i) we consider all possible values of the Albanese dimension of X, and (ii) we replace the Iitaka fibration with a more general class of morphisms.

Lemma 5.1. *Let X and Z be smooth projective varieties and $f : X \to Z$ a surjective morphism with connected fibers. If the general fiber of f is a smooth variety with surjective Albanese map, then*

$$q(X) - q(Z) = \dim \mathrm{alb}_X(X) - \dim \mathrm{alb}_Z(Z).$$

Proof. We follow [Hacon and Pardini 2002, Proposition 2.1; Chen and Hacon 2004, Corollary 3.6]. Due to the functoriality of the Albanese map, we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{alb}_X} & \mathrm{Alb}(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_*} \\ Z & \xrightarrow{\mathrm{alb}_Z} & \mathrm{Alb}(Z) \end{array}$$

where f_* is surjective since f is [Beauville 1996, Remark V.14]. Furthermore, f_* has connected fibers. To see this, we denote by K the connected component of $\mathrm{Ker}\, f_*$ through the origin and set $A := \mathrm{Alb}(X)/K$. Then the natural map $\nu : A \to \mathrm{Alb}(Y)$ is étale and f factors through the induced map $Y \times_{\mathrm{Alb}(Y)} A \to Y$, which is étale of the same degree as ν. Since f has connected fibers, we see that ν is an isomorphism and $K = \mathrm{Ker}\, f_*$.

We now show that the image of a general fiber P of f via alb_X is a translate of $\mathrm{Ker}\, f_*$. Since alb_P is surjective, the image of P via alb_X is a translate of a subtorus of $\mathrm{Ker}\, f_*$. Furthermore, since P moves in a continuous family, such images are all translates of a fixed subtorus $T \subset \mathrm{Ker}\, f_*$. Our next step is to show $T = \mathrm{Ker}\, f_*$. By

setting $B := \mathrm{Alb}(X)/T$, we see that the induced morphism $X \to B$ maps a general fiber of f to a point. Therefore, it induces a rational map $h : Z \dashrightarrow B$, which is a morphism since B is an abelian variety. Furthermore, $h(Z)$ generates the abelian variety B since the image of the Albanese map generates the Albanese variety. This leads to the inequality

$$\dim B \leq q(Z) = q(X) - \dim \mathrm{Ker}\, f_*,$$

which in turn yields $\dim T \geq \dim \mathrm{Ker}\, f_*$ as $\dim B = q(X) - \dim T$. For dimension reasons, we get then $T = \mathrm{Ker}\, f_*$. In particular, this says that $\mathrm{alb}_X(X)$ is fibered in tori of dimension $q(X) - q(Z)$ over $\mathrm{alb}_Z(Z)$, and by the theorem on the dimension of the fibers of a morphism, we get the stated equality. □

Proof of Theorem 1.6. We begin with the case $\dim X \leq 3$. In Sections 4A and 4B, we have seen that in dimension up to three the cohomological support loci associated to the canonical bundle around the origin are derived invariant, i.e., $V^k(\omega_X)_0 \cong V^k(\omega_Y)_0$ for all $k \geq 0$. Therefore, (5), in combination with the fact that derived equivalent varieties have the same dimension, immediately leads to $\dim \mathrm{alb}_X(X) = \dim \mathrm{alb}_Y(Y)$.

We now assume $\dim X > 3$ and $\kappa(X) \geq 0$. If $\kappa(X) = \kappa(Y) = 0$, then the Albanese maps of X and Y are surjective by [Kawamata 1981, Theorem 1]. Thus, the Albanese dimensions of X and Y are $q(X)$ and $q(Y)$, respectively, which are equal by work of Popa and Schnell [2011, Corollary B].

We now suppose $\kappa(X) = \kappa(Y) > 0$. Since the problem is invariant under birational modification, with a little abuse of notation, we consider nonsingular representatives $f : X \to Z$ and $g : Y \to W$ of the Iitaka fibrations of X and Y, respectively [Mori 1987, (1.10)]. As the canonical rings of X and Y are isomorphic [Orlov 2003, Corollary 2.1.9], it turns out that Z and W are birational varieties (see [Mori 1987, Proposition 1.4] or [Toda 2006, p. 13]). By [Kawamata 1981, Theorem 1], the morphisms f and g satisfy the hypotheses of Lemma 5.1, which yields

$$\begin{aligned} q(X) - \dim \mathrm{alb}_X(X) &= q(Z) - \dim \mathrm{alb}_Z(Z) \\ &= q(W) - \dim \mathrm{alb}_W(W) = q(Y) - \dim \mathrm{alb}_Y(Y). \end{aligned}$$

We conclude as $q(X) = q(Y)$. □

As an application of Theorem 1.6, we have the following:

Corollary 5.2. *Let X and Y be smooth projective derived equivalent varieties with X of maximal Albanese dimension. If F denotes the induced Rouquier isomorphism and $F(1, L) = (\psi, M)$ with $L \in V^1_r(\omega_X)$, then $\psi = 1$ and $M \in V^1_r(\omega_Y)$. Moreover, F induces isomorphisms of algebraic sets*

$$V^1_r(\omega_X) \cong V^1_r(\omega_Y) \quad \textit{for any } r \geq 1.$$

Proof. We have $\kappa(X) \geq 0$ since X is of maximal Albanese dimension. Hence, Theorem 1.6 ensures that Y is of maximal Albanese dimension as well. We apply then Corollary 3.5 after having noted the inclusions $V_r^1(\omega_X) \subset V^0(\omega_X)$ and $V_r^1(\omega_Y) \subset V^0(\omega_Y)$ (see (6)). □

6. End of the proof of Theorem 1.5

6A. ***Proof of Theorem 1.5(ii).*** The following two propositions prove and extend Theorem 1.5(ii):

Proposition 6.1. *Let X and Y be smooth projective derived equivalent threefolds, and let F be the induced Rouquier isomorphism. Assume that either* $\mathrm{Aut}^0(X)$ *is affine or that $V^p(\Omega_X^q \otimes \omega_X^{\otimes m}) = \mathrm{Pic}^0(X)$ for some $m, p, q \in \mathbb{Z}$ with $p, q \geq 0$. Then F induces isomorphisms of algebraic sets*

$$V_r^p(\Omega_X^q) \cong V_r^p(\Omega_Y^q) \quad \text{for all } p, q \geq 0 \text{ and } r \geq 1.$$

Proof. By Remark 3.6, we have $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$. The isomorphisms $V_r^0(\omega_X) \cong V_r^0(\omega_Y)$ and $V_r^1(\omega_X) \cong V_r^1(\omega_Y)$ hold by Proposition 3.1 and Remark 3.6, respectively. The isomorphisms $V_r^2(\omega_X) \cong V_r^2(\omega_Y)$ follow since in dimension three $\chi(\omega_X) = \chi(\omega_Y)$ [Popa and Schnell 2011, Corollary C].

We now establish the isomorphisms $V_r^1(\Omega_X^2) \cong V_r^1(\Omega_Y^2)$. Let $L \in V_r^1(\Omega_X^2)$ so that $F(1, L) = (1, M)$ for some $M \in \mathrm{Pic}^0(Y)$. By Corollary 2.2 with $m = 0$ and $k = 2$, Serre duality and the Hodge linear-conjugate isomorphism, we get $h^1(X, \Omega_X^2 \otimes L) = h^1(Y, \Omega_Y^2 \otimes M)$. This shows that F maps $1 \times V_r^1(\Omega_X^2) \mapsto 1 \times V_r^1(\Omega_Y^2)$, inducing the wanted isomorphisms as in Proposition 3.1. Finally, the isomorphisms $V_r^1(\Omega_X^1) \cong V_r^1(\Omega_Y^1)$ are deduced in the same way by using Corollary 2.2 with $m = 0$ and $k = 3$. □

Proposition 6.2. *Let X and Y be smooth projective derived equivalent threefolds, and let F be the induced Rouquier isomorphism. If X is of maximal Albanese dimension, then F induces isomorphisms of algebraic sets*

$$V_r^k(\omega_X) \cong V_r^k(\omega_Y) \quad \text{for all } k \geq 0 \text{ and } r \geq 1.$$

Proof. Proposition 3.1 and Corollary 5.2 yield the isomorphisms $V_r^k(\omega_X) \cong V_r^k(\omega_Y)$ for any $k \neq 2$, so we only focus on the remaining case. Since X is of maximal Albanese dimension, we obtain an inclusion $V_r^2(\omega_X) \subset V^0(\omega_X)$ (see (6)) leading to a further inclusion $F(1, V_r^2(\omega_X)) \subset (1, \mathrm{Pic}^0(Y))$ thanks to Proposition 3.1. Hence, by Corollary 2.2, $h^k(X, \omega_X \otimes L) = h^k(Y, \omega_Y \otimes M)$ whenever $F(1, L) = (1, M)$ with $L \in V_r^2(\omega_X)$ and $k = 0, 1$. Moreover, we get $h^2(X, \omega_X \otimes L) = h^2(Y, \omega_Y \otimes M)$ since $\chi(\omega_X) = \chi(\omega_Y)$ [Popa and Schnell 2011, Corollary C]. Therefore, F maps $1 \times V_r^2(\omega_X) \mapsto 1 \times V_r^2(\omega_Y)$, and by arguing as in Proposition 3.1, F^{-1} maps $1 \times V_r^2(\omega_Y) \mapsto 1 \times V_r^2(\omega_X)$, finishing the proof. □

6B. ***Proof of Theorem 1.5(iii).*** We show now the proof of Theorem 1.5(iii). Before jumping into technicalities, we first present the plan of its proof.

Thanks to Propositions 6.1 and 6.2, we can assume that X is a threefold with $\dim \operatorname{alb}_X(X) \leq 2$ and $V^0(\omega_X) \subsetneq \operatorname{Pic}^0(X)$ and with nonaffine automorphism group $\operatorname{Aut}^0(X)$. In particular, we can suppose that X is not of general type and that $\chi(\omega_X) = 0$ [Popa and Schnell 2011, Corollary 2.6]. Thanks to Proposition 3.1, Theorem 1.6 and [Popa and Schnell 2011, Theorem A(1)], the Fourier–Mukai partner Y of X satisfies the same hypotheses as X. Hence, Theorem 1.5(iii) follows as soon as we classify $\dim V^i(\omega_X)$ in terms of derived invariants. This classification is carried out in the following Propositions 6.5–6.9 where $\dim V^1(\omega_X)$ and $\dim V^2(\omega_X)$ are computed in terms of $\kappa(X)$, $q(X)$, $\dim \operatorname{alb}_X(X)$ and $\dim V^0(\omega_X)$.

The main tools we use towards the proofs of Propositions 6.5–6.9 are generic vanishing theorems [Green and Lazarsfeld 1987, Theorem 1; Pareschi and Popa 2011, Theorem 5.8], Kollár's result on higher direct images of the canonical bundle [Kollár 1986b, Theorem 3.1; 1986a, Theorem 2.1 and Proposition 7.6] and the classification of smooth projective surfaces (see for instance [Beauville 1996]). The following two lemmas will be useful to our analysis:

Lemma 6.3. *Let X and Y be smooth projective varieties and $f : X \to Y$ be a surjective morphism with connected fibers. If h denotes the dimension of the general fiber of f, then*

$$f^* V^k(\omega_Y) \subset V^{k+h}(\omega_X) \quad \text{for any } k = 0, \ldots, \dim Y.$$

Proof. By [Kollár 1986a, Theorem 2.1 and Proposition 7.6], we have $R^h f_* \omega_X \cong \omega_Y$ and $R^k f_* \omega_X = 0$ for $k > h$. Moreover, by [Kollár 1986b, Theorem 3.1], we obtain decompositions

$$H^{k+h}(X, \omega_X \otimes f^* L) \cong H^k(Y, \omega_Y \otimes L) \oplus \bigoplus_{l \neq k} H^l(Y, R^{h+k-l} f_* \omega_X \otimes L)$$

for any $L \in \operatorname{Pic}^0(Y)$. At this point, it is enough to note that the pull-back homomorphism $f^* : \operatorname{Pic}^0(Y) \to \operatorname{Pic}^0(X)$ is injective as the fibers of f are connected. □

Lemma 6.4. *Let X be a smooth projective variety with $\kappa(X) = -\infty$. Then $V^0(\omega_X^{\otimes m}) = \varnothing$ for any $m > 0$.*

Proof. Suppose that $L \in V^0(\omega_X^{\otimes m})$ for some $m > 0$. By [Chen and Hacon 2004, Theorem 3.2], we can assume that L is a line bundle of finite order, say, of order e. If $\mathcal{O}_X \to \omega_X^{\otimes m} \otimes L$ is a nonzero section of $\omega_X^{\otimes m} \otimes L$, then it induces a nonzero section $\mathcal{O}_X \to \omega_X^{\otimes me}$; this yields a contradiction as $\kappa(\omega_X) = -\infty$. □

Proposition 6.5. *Let X be a smooth projective threefold such that $\kappa(X) = 2$, $\dim \operatorname{alb}_X(X) = 2$, $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \operatorname{Pic}^0(X)$. If $q(X) = 2$, then we have* (i) $\dim V^2(\omega_X) = 0$, (ii) $\dim V^1(\omega_X) = 1$ if and only if $\dim V^0(\omega_X) = 1$ and

(iii) $\dim V^1(\omega_X) = 0$ if and only if $\dim V^0(\omega_X) \leq 0$. If $q(X) > 2$, then we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = q(X) - 1$.

Proof. Since the problem is invariant under birational modification, with a little abuse of notation, we consider a nonsingular representative $f : X \to S$ of the Iitaka fibration of X [Mori 1987, (1.10)] so that X and S are smooth varieties and f is an algebraic fiber space. We divide the proof into three cases according to the values of the Albanese dimension of S.

Case I: $\dim \mathrm{alb}_S(S) = 2$. By the classification theory of smooth projective surfaces, S is either a surface of general type, or birational to an abelian surface, or birational to an elliptic surface fibered onto a curve of genus ≥ 2. Moreover, by Lemma 5.1, we have $q(X) = q(S)$.

If S is of general type, then by Castelnuovo's theorem [Beauville 1996, Theorem X.4] we have $\chi(\omega_S) > 0$ and hence $V^0(\omega_S) = \mathrm{Pic}^0(S)$. Therefore, by Lemma 6.3, we get $V^1(\omega_X) = \mathrm{Pic}^0(X)$, and consequently, $V^0(\omega_X) = \mathrm{Pic}^0(X)$ since $\chi(\omega_X) = 0$ and $V^2(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ (see (5)). This contradicts our hypotheses, and hence, this case does not occur.

If S is birational to an abelian surface, then we have $q(X) = q(S) = 2$ and $f^* \mathrm{Pic}^0(S) = \mathrm{Pic}^0(X)$. By using [Kollár 1986b, Theorem 3.1], we obtain decompositions

$$H^2(X, \omega_X \otimes f^*L) \cong H^2(S,\, f_*\omega_X \otimes L) \oplus H^1(S,\, R^1 f_*\omega_X \otimes L)$$

for any $L \in \mathrm{Pic}^0(S)$. Moreover, we note that $R^1 f_*\omega_X \cong \omega_S$ and $R^2 f_*\omega_X = 0$ [Kollár 1986a, Proposition 7.6 and Theorem 2.1]. Therefore, since by [Pareschi and Popa 2011, Theorem 5.8] $f_*\omega_X$ is a GV-sheaf on S (i.e., $\mathrm{codim}_{\mathrm{Pic}^0(S)} V^k(f_*\omega_X) \geq k$ for $k > 0$), we get $\dim V^2(\omega_X) = 0$. At this point, the statements (ii) and (iii) of the proposition follow as $\chi(\omega_X) = 0$ and $\dim V^1(\omega_X) \geq 0$ (note that $\mathcal{O}_X \in V^1(\omega_X)$ since $q(X) = 2$).

If S is birational to an elliptic surface $h : S \to C$ fibered onto a curve C of genus $g(C) = q(S) - 1 = q(X) - 1 \geq 2$, then X is fibered onto C as well. Therefore, we have $V^0(\omega_C) = \mathrm{Pic}^0(C)$, and consequently, $V^2(\omega_X)$ is of codimension one in $\mathrm{Pic}^0(X)$ by Lemma 6.3 and (5). Since $\chi(\omega_X) = 0$, $V^1(\omega_X)$ is of codimension one as well.

Case II: $\dim \mathrm{alb}_S(S) = 1$. We have $q(X) = q(S) + 1$ by Lemma 5.1. Moreover, alb_S has connected fibers, and by [Beauville 1996, Proposition V.15], $\mathrm{alb}_S(S)$ is a smooth curve of genus $q(S)$. We distinguish two subcases: $q(S) = 1$ and $q(S) \geq 2$.

If $q(S) = 1$, then $q(X) = 2$ and alb_X is surjective. Let $X \xrightarrow{b} Z \to \mathrm{Alb}(X)$ be the Stein factorization of alb_X, and let $b' : X' \to Z'$ be a nonsingular representative of b. We note that Z' is a smooth surface with $q(Z') = 2$ and of maximal Albanese dimension. Therefore, either Z' is of general type or it is birational to an abelian surface. However, we have just seen that Z' cannot possibly be of general type;

therefore, Z' is birational to an abelian surface, and the same calculations of the previous case apply.

If $q(S) \geq 2$, then the Albanese map of S induces a fibration of S onto a smooth curve C of genus $g(C) = q(S)$. Therefore, X is fibered onto C as well, and we conclude as in the previous case.

Case III: $\dim \operatorname{alb}_S(S) = 0$. As we have seen in the proof of Lemma 5.1, the image of a general fiber of f is mapped via alb_X onto a fiber of the induced morphism $f_* : \operatorname{Alb}(X) \to \operatorname{Alb}(S)$. On the other hand, if $\dim \operatorname{alb}_S(S) = 0$, then $\operatorname{Alb}(S)$ is trivial. This yields a contradiction, and therefore, this case does not occur. □

Proposition 6.6. *Let X be a smooth projective threefold such that $\kappa(X) = 2$, $\dim \operatorname{alb}_X(X) = 1$, $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \operatorname{Pic}^0(X)$. If $q(X) = 1$, then we have $\dim V^1(\omega_X) \leq 0$ and $\dim V^2(\omega_X) = 0$. On the other hand, if $q(X) > 1$, then we have $V^1(\omega_X) = V^2(\omega_X) = \operatorname{Pic}^0(X)$.*

Proof. As in the previous proof, we denote by $f : X \to S$ a nonsingular representative of the Iitaka fibration of X. We distinguish two cases: $\dim \operatorname{alb}_S(S) = 0$ and $\dim \operatorname{alb}_S(S) = 1$.

If $\dim \operatorname{alb}_S(S) = 0$, then we have $q(S) = 0$ and therefore $q(X) = 1$ by Lemma 5.1. Moreover, by [Ueno 1973, Lemma 2.11], alb_X is surjective and has connected fibers. We set $E := \operatorname{Alb}(X)$ and $a := \operatorname{alb}_X$ and note that by [Kollár 1986a, Proposition 7.6] there is an isomorphism $R^2 a_* \omega_X \cong \mathcal{O}_E$. Finally, by [Kollár 1986b, Theorem 3.1], we get isomorphisms

$$H^2(X, \omega_X \otimes a^* L) \cong H^1(E, R^1 a_* \omega_X \otimes L) \oplus H^0(E, L)$$

for any $L \in \operatorname{Pic}^0(E) \cong \operatorname{Pic}^0(X)$. By [Hacon 2004, Corollary 4.2], $R^1 a_* \omega_X$ is a GV-sheaf on E. Hence, $\dim V^2(\omega_X) = 0$, and consequently, $V^1(\omega_X)$ is either empty or zero-dimensional as $V^0(\omega_X) \subsetneq \operatorname{Pic}^0(X)$ and $\chi(\omega_X) = 0$.

We now suppose $\dim \operatorname{alb}_S(S) = 1$. In this case, alb_S has connected fibers and its image is a smooth curve B of genus $g(B) = q(S) > 1$. Moreover, we have $q(X) = q(S)$ by Lemma 5.1. We distinguish two subcases: $q(S) = 1$ and $q(S) > 1$. If $q(S) = 1$, then the image of alb_X is an elliptic curve and the same argument of the previous case applies. If $q(S) = g(B) > 1$, then we get $V^0(\omega_B) = \operatorname{Pic}^0(B)$ and $\operatorname{Pic}^0(X) \cong \operatorname{Pic}^0(S) \cong \operatorname{Pic}^0(B)$. Hence, by Lemma 6.3, there are inclusions

$$\operatorname{alb}_S^* \operatorname{Pic}^0(B) = \operatorname{alb}_S^* V^0(\omega_B) \subset V^1(\omega_S) \subset \operatorname{Pic}^0(S)$$

leading to $V^1(\omega_S) = \operatorname{Pic}^0(S)$. Moreover, a second application of Lemma 6.3 gives

$$f^* V^1(\omega_S) \subset V^2(\omega_X) \subset \operatorname{Pic}^0(X),$$

showing that $V^2(\omega_X) = \operatorname{Pic}^0(X)$. Finally, we also have $V^1(\omega_X) = \operatorname{Pic}^0(X)$ as $\chi(\omega_X) = 0$. □

Proposition 6.7. *Let X be a smooth projective threefold such that $\kappa(X) = 1$, $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$.*

(i) *Assume* $\dim \mathrm{alb}_X(X) = 2$. *If $q(X) = 2$, then we have* (i) $\dim V^2(\omega_X) = 0$, (ii) $\dim V^1(\omega_X) = 1$ if and only if $\dim V^0(\omega_X) = 1$ and (iii) $\dim V^1(\omega_X) = 0$ if and only if $\dim V^0(\omega_X) \leq 0$. If $q(X) \geq 3$, then we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = q(X) - 1$.

(ii) Assume $\dim \mathrm{alb}_X(X) = 1$. If $q(X) = 1$, then we have $\dim V^1(\omega_X) \leq 0$ and $\dim V^2(\omega_X) = 0$. If $q(X) \geq 2$, then we obtain $V^1(\omega_X) = V^2(\omega_X) = \mathrm{Pic}^0(X)$.

Proof. We start with the case $\dim \mathrm{alb}_X(X) = 2$. Let $f : X \to C$ be a nonsingular representative of the Iitaka fibration of X where C is a smooth curve.

If $g(C) \geq 2$, then by Lemma 5.1 we have $q(X) = g(C) + 1 \geq 3$, and by Lemma 6.3, we obtain a series of inclusions $f^* \mathrm{Pic}^0(C) = f^* V^0(\omega_C) \subset V^2(\omega_X) \subset \mathrm{Pic}^0(X)$. We conclude that

$$\dim V^2(\omega_X) = q(X) - 1$$

since $V^2(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ by (5). Therefore, we see that $V^1(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ as $\chi(\omega_X) = 0$ and $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$. Finally, thanks to the inclusion $V^1(\omega_X) \supset V^2(\omega_X)$ of (6), we obtain $\dim V^1(\omega_X) = q(X) - 1$.

If $g(C) \leq 1$, then $q(X) = 2$ and $a := \mathrm{alb}_X$ is surjective. Let $b : X' \to Z'$ be a nonsingular representative of the Stein factorization of a. Then, as we have seen in the proof of Proposition 6.5, Z' is birational to an abelian surface, and therefore, $\dim V^2(\omega_X) = 0$. Since $\mathcal{O}_X \in V^1(\omega_X)$, we obtain the statements (ii) and (iii) of part (i).

We now study the case $\dim \mathrm{alb}_X(X) = 1$. If $g(C) \geq 2$, then $q(X) = g(C)$ and $f^* \mathrm{Pic}^0(C) = \mathrm{Pic}^0(X)$. Therefore, by Lemma 6.3, we get $V^2(\omega_X) = \mathrm{Pic}^0(X)$, and hence, we have $V^1(\omega_X) = \mathrm{Pic}^0(X)$. On the other hand, if $g(C) \leq 1$, then $q(X) = 1$ and $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is an algebraic fiber space onto an elliptic curve. We conclude then as in the proof of Proposition 6.6. □

Proposition 6.8. *Let X be a smooth projective threefold such that $\kappa(X) = 0$ and $\chi(\omega_X) = 0$. If $\dim \mathrm{alb}_X(X) = 2$, then we have $\dim V^1(\omega_X) = \dim V^2(\omega_X) = 0$. On the other hand, if $\dim \mathrm{alb}_X(X) = 1$, then we have $\dim V^1(\omega_X) \leq 0$ and $\dim V^2(\omega_X) = 0$.*

Proof. We recall that, by [Chen and Hacon 2002, Lemma 3.1], $V^0(\omega_X)$ consists of at most one point. We start with the case $\dim \mathrm{alb}_X(X) = 2$. By [Kawamata 1981, Theorem 1], alb_X is surjective and has connected fibers. Therefore, we have $q(X) = h^2(X, \omega_X) = 2$ and hence $\mathcal{O}_X \in V^1(\omega_X)$ since $\chi(\omega_X) = 0$. We set $a := \mathrm{alb}_X$, and we note that, by [Hacon 2004, Corollary 4.2], $a_* \omega_X$ is a GV-sheaf, i.e.,

$$\mathrm{codim}\, V^1(a_* \omega_X) \geq 1 \quad \text{and} \quad \mathrm{codim}\, V^2(a_* \omega_X) \geq 2.$$

By using that $R^1a_*\omega_X \cong \mathcal{O}_{\mathrm{Alb}(X)}$ and $R^2a_*\omega_X = 0$ [Kollár 1986a, Proposition 7.6 and Theorem 2.1] and by using [Kollár 1986b, Theorem 3.1], we get isomorphisms

$$H^1(X, \omega_X \otimes a^*L) \cong H^1(\mathrm{Alb}(X), a_*\omega_X \otimes L) \oplus H^0(\mathrm{Alb}(X), L)$$

for any $L \in \mathrm{Pic}^0(\mathrm{Alb}(X)) \cong \mathrm{Pic}^0(X)$. Therefore, we have

$$\operatorname{codim} V^1(\omega_X) \geq 1 \quad \text{and} \quad \operatorname{codim} V^2(\omega_X) \geq 2,$$

and consequently, the hypothesis $\chi(\omega_X) = 0$ implies $\dim V^1(\omega_X) = 0$.

If $\dim \mathrm{alb}_X(X) = 1$, then as in the previous case we have $\dim V^2(\omega_X) = 0$. Therefore, $V^1(\omega_X)$ is either empty or of dimension zero since $\chi(\omega_X) = 0$. □

Proposition 6.9. *Let X be a smooth projective threefold such that $\kappa(X) = -\infty$ and $\chi(\omega_X) = 0$.*

(i) *Suppose* $\dim \mathrm{alb}_X(X) = 2$. *If* $q(X) = 2$, *then* $V^1(\omega_X) = V^2(\omega_X) = \{\mathcal{O}_X\}$. *If* $q(X) > 2$, *then we obtain* $\dim V^1(\omega_X) = \dim V^2(\omega_X) = q(X) - 1$.

(ii) *Suppose* $\dim \mathrm{alb}_X(X) = 1$. *If* $q(X) = 1$, *then we have* $\dim V^1(\omega_X) \leq 0$ *and* $\dim V^2(\omega_X) = 0$. *If* $q(X) > 1$, *then we obtain* $V^1(\omega_X) = V^2(\omega_X) = \mathrm{Pic}^0(X)$.

Proof. We start with the case $\dim \mathrm{alb}_X(X) = 2$. Let $a : X \to S \subset \mathrm{Alb}(X)$ be the Albanese map of X and $b : X \to S'$ be the Stein factorization of a, and let $c : X' \to S''$ be a nonsingular representative of b. We can easily check that $q(X') = q(S'')$ and $\dim \mathrm{alb}_S(S) = 2$ and hence that $\kappa(S'') \geq 0$. Furthermore, we have $c_*\omega_{X'} = 0$. To see this, we point out that by [Pareschi and Popa 2011, Theorem 5.8] $c_*\omega_{X'}$ is a GV-sheaf on S'' and moreover that, by Lemma 6.4, $V^0(c_*\omega_{X'}) = V^0(\omega_{X'}) = V^0(\omega_X) = \varnothing$. This immediately implies $c_*\omega_{X'} = 0$ as a GV-sheaf $\mathcal{F}$ is nonzero if and only if $V^0(\mathcal{F}) \neq \varnothing$. We distinguish now three cases according to the values of $\kappa(S'')$.

If $\kappa(S'') = 0$, then S'' is birational to an abelian surface. This forces $q(X) = q(X') = q(S'') = 2$ and $c^* \mathrm{Pic}^0(S'') = \mathrm{Pic}^0(X')$. By [Kollár 1986b, Theorem 3.1; 1986a, Theorem 2.1 and Proposition 7.6], we obtain isomorphisms

$$H^2(X', \omega_{X'} \otimes c^*L) \cong H^1(S'', \omega_{S''} \otimes L) \quad \text{and} \quad H^1(X', \omega_{X'} \otimes c^*L) \cong H^0(S'', \omega_{S''} \otimes L)$$

for any $L \in \mathrm{Pic}^0(S'')$. Therefore, we have $V^2(\omega_X) \cong V^2(\omega_{X'}) = c^*V^1(\omega_{S''}) = \{\mathcal{O}_{X'}\}$ and $V^1(\omega_X) \cong V^1(\omega_{X'}) = c^*V^0(\omega_{S''}) = \{\mathcal{O}_{X'}\}$.

If $\kappa(S'') = 1$, then S'' is birational to an elliptic surface of maximal Albanese dimension fibered onto a curve of genus $g(C) \geq 2$. Thus, X is fibered onto C as well and $q(X') = q(S'') = g(C) + 1$. By Lemma 6.3 and (5), we deduce $\dim V^2(\omega_{X'}) = g(C) = q(X') - 1$, and therefore, we get $\dim V^1(\omega_{X'}) = q(X') - 1$ as $\chi(\omega_{X'}) = 0$ and $V^0(\omega_{X'}) = \varnothing$.

If $\kappa(S'') = 2$, then by Castelnuovo's theorem we have $\chi(\omega_{S''}) > 0$, which immediately yields $V^0(\omega_{S''}) = \mathrm{Pic}^0(S'')$. By using Lemma 6.3, we see that $\dim V^0(\omega_{X'}) > 0$. This contradicts Lemma 6.4, and hence, this case does not occur.

We now suppose $\dim \mathrm{alb}_X(X) = 1$. Let $a : X \to C \subset \mathrm{Alb}(X)$ be the Albanese map of X where $C := \mathrm{Im}\, a$. Then a has connected fibers and $q(X) = g(C)$ by [Ueno 1973, Lemma 2.11]. As in the previous case, we note that $a_*\omega_X = 0$. Moreover, by [Kollár 1986b, Theorem 3.1; 1986a, Proposition 7.6], we obtain isomorphisms

$$H^1(X, \omega_X \otimes a^*L) \cong H^0(C, R^1a_*\omega_X \otimes L),$$
$$H^2(X, \omega_X \otimes a^*L) \cong H^1(C, R^1a_*\omega_X \otimes L) \oplus H^0(C, \omega_C \otimes L)$$

for any $L \in \mathrm{Pic}^0(C)$. At this point, we distinguish two cases: $g(C) = 1$ and $g(C) > 1$. If $g(C) = q(X) > 1$, then we have $V^0(\omega_C) = \mathrm{Pic}^0(C)$, and by Lemma 6.3, we get $V^2(\omega_X) = V^1(\omega_X) = \mathrm{Pic}^0(X)$. On the other hand, if $g(C) = q(X) = 1$, then by [Hacon 2004, Corollary 4.2] $R^1a_*\omega_X$ is a GV-sheaf on $C = \mathrm{Alb}(X)$. Hence, we obtain $\dim V^2(\omega_X) = 0$, and consequently, we see that $\dim V^1(\omega_X) \leq 0$ since $\chi(\omega_X) = 0$ and $V^0(\omega_X) = \varnothing$. □

Remark 6.10. In the case $q(X) = 1$, the previous propositions yield the following statement: for each k, $\dim V^k(\omega_X) = 1$ if and only if $\dim V^k(\omega_Y) = 1$. In general, we have not been able to show that, if a locus $V^k(\omega_X)$ is empty or of dimension zero, then the corresponding locus $V^k(\omega_Y)$ is empty or of dimension zero, respectively. This ambiguity is mainly caused by the possible presence of nontrivial automorphisms.

An application of a sheafified version of the derivative complex [Ein and Lazarsfeld 1997, Theorem 3; Lazarsfeld and Popa 2010] can be shown to yield Conjecture 1.2 for threefolds having $q(X) = 2$ [Lombardi 2013, Proposition 5.2.15].

7. Applications

In this final section, we prove Corollaries 1.7, 1.8 and 1.9. Moreover, we present a further result regarding the invariance of the Euler characteristic of powers of the canonical bundle for derived equivalent smooth minimal varieties of maximal Albanese dimension.

7A. *Holomorphic Euler characteristic and Hodge numbers.*

Proof of Corollary 1.7. Let $d := \dim X = \dim Y$. We begin with the case $\dim \mathrm{alb}_X(X) = d$. By Theorem 1.6, Y is of maximal Albanese dimension, and by (5), we get inequalities

$$\mathrm{codim}\, V^1(\omega_X) \geq 1 \quad \text{and} \quad \mathrm{codim}\, V^1(\omega_Y) \geq 1.$$

We distinguish two cases: $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ and $V^0(\omega_X) = \mathrm{Pic}^0(X)$. If $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$, then we also have $V^0(\omega_Y) \subsetneq \mathrm{Pic}^0(Y)$ by Proposition 3.1. Moreover, there are inclusions $\mathrm{Pic}^0(X) \supsetneq V^0(\omega_X) \supset V^1(\omega_X) \supset \cdots \supset V^d(\omega_X) = \{\mathcal{O}_X\}$ and similarly for the loci $V^k(\omega_Y)$ (see (6)). Therefore, if $L \notin V^0(\omega_X)$ and $M \notin V^0(\omega_Y)$, then

$h^k(X, \omega_X \otimes L) = h^k(Y, \omega_Y \otimes M) = 0$ for all $k \geq 0$. Since the holomorphic Euler characteristic is invariant under deformation, we finally obtain

$$\chi(\omega_X) = \chi(\omega_X \otimes L) = 0 = \chi(\omega_Y \otimes M) = \chi(\omega_Y).$$

On the other hand, if $V^0(\omega_X) = \mathrm{Pic}^0(X)$, then, by Proposition 3.1, $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$, and thus,

$$\text{there exists } L_0 \in V^0(\omega_X) \setminus \left(\bigcup_{k=1}^{d} V^k(\omega_X) \right)$$
$$\text{such that } F(1, L_0) = (1, M_0) \text{ with } M_0 \in V^0(\omega_Y) \setminus \left(\bigcup_{k=1}^{d} V^k(\omega_Y) \right).$$

Hence, by using Corollary 2.2 with $m = k = 0$, we have

$$\chi(\omega_X) = \chi(\omega_X \otimes L_0) = h^0(X, \omega_X \otimes L_0) = h^0(Y, \omega_Y \otimes M_0) = \chi(\omega_Y \otimes M_0) = \chi(\omega_Y).$$

We suppose now $\dim \mathrm{alb}_X(X) = d - 1$ and $\kappa(X) \geq 0$. By Theorem 1.6, we have $\dim \mathrm{alb}_Y(Y) = d - 1$, and therefore, there are inclusions $V^1(\omega_X) \supset V^2(\omega_X) \supset \cdots \supset V^d(\omega_X)$ and $V^1(\omega_Y) \supset V^2(\omega_Y) \supset \cdots \supset V^d(\omega_Y)$. We distinguish four cases.

The first case is when $V^0(\omega_X) = V^1(\omega_X) = \mathrm{Pic}^0(X)$. By Proposition 3.1 and Corollary 3.4, it turns out that $V^0(\omega_Y) = V^1(\omega_Y) = \mathrm{Pic}^0(Y)$ as well. We claim that

$$\text{there exists } \mathcal{O}_X \neq L_1 \in V^0(\omega_X) \setminus V^2(\omega_X)$$
$$\text{such that } F(1, L_1) = (1, M_1) \text{ with } \mathcal{O}_Y \neq M_1 \in V^0(\omega_Y) \setminus V^2(\omega_Y).$$

In fact, the Rouquier isomorphism maps $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$ by Remark 3.6, and therefore, it is enough to choose the image under F^{-1} of a generic element $(1, M)$ with $M \notin V^2(\omega_Y)$. By using Corollary 2.2 twice, first with $k = 0$ and then with $k = 1$, we obtain

$$\begin{aligned} \chi(\omega_X) = \chi(\omega_X \otimes L_1) &= h^0(X, \omega_X \otimes L_1) - h^1(X, \omega_X \otimes L_1) \\ &= h^0(Y, \omega_Y \otimes M_1) - h^1(Y, \omega_Y \otimes M_1) = \chi(\omega_Y \otimes M_1) = \chi(\omega_Y). \end{aligned}$$

The second case is when $V^0(\omega_X) = \mathrm{Pic}^0(X)$ and $V^1(\omega_X) \subsetneq \mathrm{Pic}^0(X)$. By Proposition 3.1 and Corollary 3.4, $V^0(\omega_Y) = \mathrm{Pic}^0(Y)$ and $V^1(\omega_Y) \subsetneq \mathrm{Pic}^0(Y)$. As before, $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$, and hence, we can pick an element

$$\mathcal{O}_X \neq L_2 \in V^0(\omega_X) \setminus V^1(\omega_X)$$
$$\text{such that } F(1, L_2) = (1, M_2) \text{ with } \mathcal{O}_Y \neq M_2 \in V^0(\omega_Y) \setminus V^1(\omega_Y).$$

Hence, equalities $\chi(\omega_X) = \chi(\omega_X \otimes L_2) = h^0(X, \omega_X \otimes M_2) = h^0(Y, \omega_Y \otimes M_2) = \chi(\omega_Y \otimes M_2) = \chi(\omega_Y)$ hold.

The third case is when $V^0(\omega_X) \subsetneq \mathrm{Pic}^0(X)$ and $V^1(\omega_X) = \mathrm{Pic}^0(X)$. By using Proposition 3.1 and Corollary 3.4, it is easy to see that $V^0(\omega_Y) \subsetneq \mathrm{Pic}^0(Y)$ and

$V^1(\omega_Y) = \mathrm{Pic}^0(Y)$. Moreover, Remark 3.6 yields $F(1, \mathrm{Pic}^0(X)) = (1, \mathrm{Pic}^0(Y))$. Therefore, similarly to the previous cases, there exists a pair $(L_3, M_3) \neq (\mathcal{O}_X, \mathcal{O}_Y)$ such that

$$F(1, L_3) = (1, M_3) \text{ with } L_3 \notin V^0(\omega_X) \cup V^2(\omega_X) \text{ and } M_3 \notin V^0(\omega_Y) \cup V^2(\omega_Y),$$

and by Corollary 2.2, we have $\chi(\omega_X) = \chi(\omega_X \otimes L_3) = -h^1(X, \omega_X \otimes L_3) = -h^1(Y, \omega_Y \otimes M_3) = \chi(\omega_Y \otimes M_3) = \chi(\omega_Y)$.

The last case is when both $V^0(\omega_X)$ and $V^1(\omega_X)$ are proper subvarieties of $\mathrm{Pic}^0(X)$. Then $V^0(\omega_Y)$ and $V^1(\omega_Y)$ are proper subvarieties as well, and hence, $\chi(\omega_X) = \chi(\omega_Y) = 0$. □

Proof of Corollary 1.8. By the derived invariance of Hochschild homologies $HH_0(X) \cong HH_0(Y)$ and $HH_1(X) \cong HH_1(Y)$, we have $h^0(X, \omega_X) = h^0(Y, \omega_Y)$ and $h^1(X, \omega_X) = h^1(Y, \omega_Y)$. Therefore, Corollary 1.7 implies $h^{0,2}(X) = h^{0,2}(Y)$ since $h^3(X, \omega_X) = q(X) = q(Y) = h^3(Y, \omega_Y)$ and $h^4(X, \omega_X) = 1 = h^4(Y\omega_Y)$.

For the second equality, we apply Corollary 2.2 with $(L, M) = (\mathcal{O}_X, \mathcal{O}_Y)$ and $k = 2$ so that $h^2(X, \omega_X) + h^1(X, \Omega_X^3) + h^0(X, \Omega_X^2) = h^2(Y, \omega_Y) + h^1(Y, \Omega_Y^3) + h^0(Y, \Omega_Y^2)$. Therefore, we obtain $h^{1,3}(X) = h^{1,3}(Y)$ since Serre duality and the Hodge linear-conjugate isomorphism yield equalities $h^2(X, \omega_X) = h^0(X, \Omega_X^2)$ and $h^2(Y, \omega_Y) = h^0(Y, \Omega_Y^2)$. □

By using a result in [Pareschi and Popa 2011], we can also derive a consequence about pluricanonical bundles.

Corollary 7.1. *Let X and Y be smooth projective derived equivalent varieties with X of maximal Albanese dimension and minimal. Then*

$$\chi(\omega_X^{\otimes m}) = \chi(\omega_Y^{\otimes m}) \quad \text{for all } m \geq 2.$$

Proof. By [Pareschi and Popa 2011, Corollary 5.5], $\omega_X^{\otimes m}$ and $\omega_Y^{\otimes m}$ are GV-sheaves on X and Y, respectively, for any $m \geq 2$.[1] In particular, this implies that $\mathrm{codim}\, V^1(\omega_X^{\otimes m}) \geq 1$ and $\mathrm{codim}\, V^1(\omega_Y^{\otimes m}) \geq 1$. At this point, we argue as in the first part of the proof of Corollary 1.7 after having noted the inclusions $V^0(\omega_X^{\otimes m}) \supset V^1(\omega_X^{\otimes m})$ and $V^0(\omega_Y^{\otimes m}) \supset V^1(\omega_Y^{\otimes m})$ [Pareschi and Popa 2011, Proposition 3.14]. □

7B. ***Fibrations.*** In this subsection, we study the behavior of particular types of fibrations under derived equivalence. We begin by recalling some terminology from [Catanese 1991; Lazarsfeld and Popa 2010].

A smooth projective variety X is of *Albanese general type* if it is of maximal Albanese dimension and has nonsurjective Albanese map. An *irregular fibration* or a *higher irrational pencil* is a surjective morphisms with connected fibers $f : X \to Z$

[1] The minimality condition is necessary; see [Pareschi and Popa 2011, Example 5.6].

onto a normal variety Z with $0 < \dim Z < \dim X$ and such that any smooth model of Z is of maximal Albanese dimension or Albanese general type, respectively.

Popa [2013, Corollary 3.4] observes that a consequence of Conjecture 1.3 is that, if X admits a fibration onto a variety having nonsurjective Albanese map, then any Fourier–Mukai partner of X admits an irregular fibration. With Theorem 1.4 at hand, we can verify this statement under an additional hypothesis on X.

Proposition 7.2. *Let X and Y be smooth projective derived equivalent varieties with $\dim \operatorname{alb}_X(X) \geq \dim X - 1$. If X admits a surjective morphism $f : X \to Z$ with connected fibers onto a normal variety Z having nonsurjective Albanese map and such that $\dim X > \dim Z$, then Y admits an irregular fibration.*

Proof. Let $Z \xrightarrow{f'} Z' \to \operatorname{alb}_Z(Z)$ be the Stein factorization of alb_Z. By taking a nonsingular representative of f', we can assume Z' smooth. We can easily check that Z' is of maximal Albanese dimension (so that $\mathcal{O}_{Z'} \in V^0(\omega_{Z'})$) and that $\operatorname{alb}_{Z'}$ is not surjective. Hence, by [Ein and Lazarsfeld 1997, Proposition 2.2], there exists a positive-dimensional irreducible component V of $V^0(\omega_{Z'})$ passing through the origin. Moreover, by Lemma 6.3, we have $(f \circ f')^* V \subset V^k(\omega_X)_0$ where $k = \dim X - \dim Z'$, and by (5), we get $(f \circ f')^* V \subset V^k(\omega_X)_0 \subset V^1(\omega_X)_0$. Finally, by Theorem 1.4(iii), there exists a positive-dimensional irreducible component $V' \subset V^1(\omega_Y)_0$ We conclude then by applying [Green and Lazarsfeld 1991, Theorem 0.1]. □

We point out that, thanks to Theorem 1.5, we can remove the hypothesis "$\dim \operatorname{alb}_X(X) \geq \dim X - 1$" from the above proposition in the case of threefolds. The following proposition, together with the subsequent remark, provides the proof of Corollary 1.9:

Proposition 7.3. *Let X and Y be smooth projective derived equivalent threefolds. Fix k to be either 1 or 2. Then X admits a higher irrational pencil $f : X \to Z$ with $0 < \dim Z \leq k$ if and only if Y admits a higher irrational pencil $g : Y \to W$ with $0 < \dim W \leq k$.*

Proof. We start with the case $k = 1$, and therefore, we consider a higher irrational pencil $f : X \to Z$ onto a smooth curve Z of genus $g(Z) \geq 2$. By Lemma 6.3, we have $f^* V^0(\omega_Z) = f^* \operatorname{Pic}^0(Z) \subset V^2(\omega_X)_0$, and by Theorem 1.5(i), there exists a component $T \subset V^2(\omega_Y)_0$ such that

$$\dim T \geq q(Z) \geq 2. \tag{10}$$

Moreover, by [Green and Lazarsfeld 1991, Theorem 0.1] or by [Beauville 1992, Corollaire 2.3], there exists an irrational fibration $g : Y \to W$ onto a smooth curve W such that $T \subset g^* \operatorname{Pic}^0(W) + \gamma$ for some $\gamma \in \operatorname{Pic}^0(Y)$. Therefore, we obtain the inequality

$$q(W) = g(W) \geq \dim T \geq 2 \tag{11}$$

ensuring that g is a higher irrational pencil.

We suppose now $k=2$, and we consider a higher irrational pencil $f: X \to Z$ onto a surface. It is a general fact that, by possibly replacing Z with a lower-dimensional variety, one can furthermore assume $\chi(\omega_{Z'}) > 0$ for any smooth model Z' of Z (see [Pareschi and Popa 2009, p. 271]). If $\dim Z = 1$, then we apply the argument of the previous case. On the other hand, if $\dim Z = 2$ then by Lemma 6.3 we get

$$f^* V^0(\omega_Z) = f^* \operatorname{Pic}^0(Z) \subset V^1(\omega_X)_0.$$

Moreover, by Theorem 1.5, there exists a component $T \subset V^1(\omega_Y)_0$ such that $\dim T \geq q(Z') \geq 3$, and by [Green and Lazarsfeld 1991, Theorem 0.1], there exists an irregular fibration $g: Y \to W$ such that $T \subset g^* \operatorname{Pic}^0(W) + \gamma$ for some $\gamma \in \operatorname{Pic}^0(Y)$. Therefore, $q(W) \geq \dim T \geq 3$ and g is a higher irrational pencil. □

Remark 7.4. We can slightly improve the statement of Proposition 7.3 in the case of fibrations onto curves. In fact, by going back to the proof of Proposition 7.3 in the case $k = 1$, we see that from the inequalities (10) and (11) we obtain the inequality $q(W) \geq q(Z)$. Then the following holds. Fix an integer $g \geq 2$. The variety X admits a higher irrational pencil $f: X \to C$ onto a curve of genus $g(C) \geq g$ if and only if Y admits a higher irrational pencil $h: Y \to D$ onto a curve of genus $g(D) \geq g$.

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