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Consider tuples $(K_1, \dots, K_r)$ of separable algebras over a common local or global number field F, with the K_i related to each other by specified resolvent constructions. Under the assumption that all ramification is tame, simple group-theoretic calculations give best possible divisibility relations among the discriminants of K_i/F. We show that for many resolvent constructions, these divisibility relations continue to hold even in the presence of wild ramification.

1. Overview

Let G be a finite group and let $\phi_1, \dots, \phi_r$ be permutation characters of G. We say that a tuple $(K_1, \dots, K_r)$ of separable algebras over a common ground field F has type $(G, \phi_1, \dots, \phi_r)$ if for a joint splitting field K^{gal} one can identify $\mathrm{Gal}(K^{\mathrm{gal}}/F)$ with a subgroup of G such that the action of $\mathrm{Gal}(K^{\mathrm{gal}}/F)$ on $\mathrm{Hom}_F(K_i, K^{\mathrm{gal}})$ has character ϕ_i.

When F is a local or global number field, one has discriminants $\mathfrak{D}_{K_i/F}$ which are ideals in the ring of integers $\mathbb{O}_F$ of F. One can ask for the strongest divisibility relations among these discriminants which hold as $(K_1, \dots, K_r)/F$ varies over all possibilities of a given type $(G, \phi_1, \dots, \phi_r)$. This question has a simple group-theoretic answer if one restricts attention to tuples for which all ramification in each K_i/F is tame.

This paper focuses on the following phenomenon: *for many* $(G, \phi_1, \dots, \phi_r)$*, the divisibility relations for tame* $(K_1, \dots, K_r)/F$ *of type* $(G, \phi_1, \dots, \phi_r)$ *hold also for arbitrary* $(K_1, \dots, K_r)/F$ *of type* $(G, \phi_1, \dots, \phi_r)$. In this case, we say that the tame-wild principle holds for $(G, \phi_1, \dots, \phi_r)$. Our terminology "tame-wild principle" is intended to be reminiscent of the standard terminology "local-global principle": we are showing in this paper that simple tame computations can often solve a complicated wild problem, just as simple local computations can often solve a complicated global problem.

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Section 2 provides an introductory example. Section 3 reviews some ramification theory centering on Artin characters, placing it in a framework which will be convenient for us. Section 4 states the tame-wild principle and gives two simple methods for proving instances of it.

If the tame-wild principle holds for a fixed G and any $(\phi_1, \ldots, \phi_r)$ then we say it holds universally for G. Section 5 proves that the tame-wild principle holds universally for G in a small class of groups we call U-groups. Section 6 considers the remaining groups, called N-groups, finding that the tame-wild principle usually does not hold universally for them.

Sections 7 and 8 return to the more practical situation where one is given not only G but also a small list of naturally arising ϕ_i. Our theme is that the tame-wild principle is likely to hold, despite the negative results on N-groups. Section 7 focuses on comparing an arbitrary algebra K/F with its splitting field K^{gal}/F, proving that one of the two divisibility statements coming from the tame-wild principle holds for arbitrary G. Section 8 gives a collection of examples exploring the range of $(G, \phi_1, \ldots, \phi_r)$ for which the tame-wild principle holds.

This paper is written with applications to tabulating number fields of small discriminant in mind. The topics in Section 2B, Section 7E, and Section 8A all relate to this application. Moreover, as we will make clear, the theory we present here still applies when permutation characters ϕ_i are replaced by general characters χ_i, and discriminants are correspondingly generalized to conductors. Applications to Artin L-functions of small conductor will be presented elsewhere.

2. An introductory example

In Section 2A, we provide an introductory instance of the tame-wild principle that we will revisit later to provide simple illustrations of general points. In Section 2B, we illustrate how this instance of the tame-wild principle gives an indirect but efficient way of solving a standard problem in tabulating number fields.

2A. *The tame-wild principle for* (S_5, ϕ_5, ϕ_6).

The Cayley–Weber type. For our introductory example, we take the type (S_5, ϕ_5, ϕ_6), where ϕ_5 is the character of the given degree five permutation representation, and ϕ_6 is the character of the degree six representation $S_5 \xrightarrow{\sim} \mathrm{PGL}_2(5) \subset S_6$. A pair of algebras (K_5, K_6) has type (S_5, ϕ_5, ϕ_6) exactly when K_6 is the Cayley–Weber resolvent of K_5 (see [Jensen et al. 2002, §2.3], for instance, for this notion). An explicit example over $\mathbb{Q}$ is given by $K_n = \mathbb{Q}[x]/f_n(x)$ with

$$f_5(x) = x^5 - 2x^4 + 4x^3 - 4x^2 + 2x - 4, \qquad D_5 = 2^8 3^4 5^1, \tag{2-1}$$

$$f_6(x) = x^6 - 2x^5 + 4x^4 - 4x^3 + 2x^2 - 4x - 6, \qquad D_6 = 2^{10} 3^4 5^3. \tag{2-2}$$

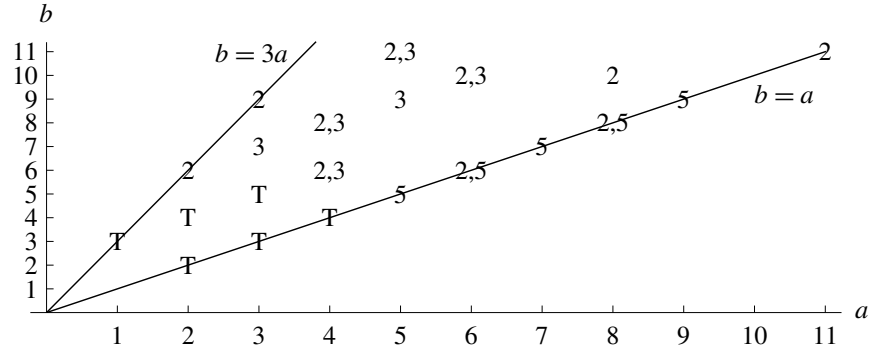


Figure 1. The complete list of pairs $(a, b) \neq (0, 0)$ which occur as $(a_\mathfrak{p}, b_\mathfrak{p})$ for (S_5, ϕ_5, ϕ_6) over $\mathbb{Q}$. The pairs labeled T can occur with tame ramification, while the others can only occur for wild p-adic ramification as indicated.

In this example, the Galois group is all of S_5, discriminants $\mathfrak{D}_{K_n/\mathbb{Q}} = (D_n)$ are as indicated, and ramification is wild at 2 and tame at 3 and 5. We are concerned with exponent pairs $(a_\mathfrak{p}, b_\mathfrak{p})$ on discriminants. Here $(a_2, b_2) = (8, 10)$, $(a_3, b_3) = (4, 4)$, $(a_5, b_5) = (1, 3)$, and otherwise $(a_\mathfrak{p}, b_\mathfrak{p}) = (0, 0)$.

All possibilities for $(a_\mathfrak{p}, b_\mathfrak{p})$. Figure 1 gives all nonzero possibilities for $(a_\mathfrak{p}, b_\mathfrak{p})$ over $\mathbb{Q}$. The fact that the tame list is complete is immediate from the general formalism of the next section. A brute force proof that the wild list is complete goes as follows: there are 113, 57, and 51 quintic algebras K_5 over $\mathbb{Q}_p$ for $p = 2$, 3, and 5 respectively [Jones and Roberts 2006]; for each, one can compute K_6 and thus the pairs $(a_\mathfrak{p}, b_\mathfrak{p})$; the lists arising are the ones drawn in Figure 1. For larger number fields F, the list of possibilities for tame $(a_\mathfrak{p}, b_\mathfrak{p})$ is exactly the same, but the list of possibilities for wild $(a_\mathfrak{p}, b_\mathfrak{p})$ grows without bound.

The tame-wild principle. Figure 1 and the comment about general base fields F clearly illustrate two general phenomena about exponent vectors $(a_\mathfrak{p}, b_\mathfrak{p})$. First, in absolute terms, the exponents can be much larger in wild cases than they are in all tame cases. But second, in relative terms, one can hope that the ratios $a_\mathfrak{p}/b_\mathfrak{p}$ are quite similar in the wild and tame cases. We are interested in this paper only in the second phenomenon and so we systematically consider ratios.

In our example, the tame-wild principle is the statement that

$$\tfrac{1}{3} b_\mathfrak{p} \leq a_\mathfrak{p} \leq b_\mathfrak{p} \tag{2-3}$$

holds for all $(K_5, K_6)/F$ of type (S_5, ϕ_5, ϕ_6) and all primes $\mathfrak{p}$ of F. In other words, when $b_\mathfrak{p} \neq 0$ one must have $a_\mathfrak{p}/b_\mathfrak{p} \in \left[\frac{1}{3}, 1\right]$. We have summarized a proof that (2-3) holds when one restricts F to be $\mathbb{Q}$ or one of the $\mathbb{Q}_p$. We will see by a

group-theoretic argument in Section 4C, not involving inspecting wild ramification at all, that (2-3) holds for general F. However the situation is subtle, as the analog of (2-3) holds for many (G, ϕ_n, ϕ_m) but not for all.

2B. ***Application to number field tabulation.*** The example of this section provides a convenient illustration of the application of tame-wild inequalities to number field tabulation. The right inequality of (2-3) globalizes to the divisibility relation $\mathfrak{D}_{K_5/F} \mid \mathfrak{D}_{K_6/F}$ which on the level of magnitudes becomes

$$|\mathfrak{D}_{K_5/F}| \leq |\mathfrak{D}_{K_6/F}|. \tag{2-4}$$

Consider the problem of finding sextic field extensions K_6/F with Galois group either $\mathrm{PSL}_2(5)$ or $\mathrm{PGL}_2(5)$. These all arise as Cayley–Weber resolvents of K_5/F with Galois group either A_5 or S_5. From (2-4), one sees that to find all K_6/F with $|\mathfrak{D}_{K_6/F}| \leq B$ it suffices to find all K_5/F with $|\mathfrak{D}_{K_5/F}| \leq B$, apply the Cayley–Weber resolvent, and keep those K_6/F with $|\mathfrak{D}_{K_6/F}| \leq B$. This indirect quintic method is enormously faster for large B, but the direct sextic method over $F = \mathbb{Q}$ was used in [Ford and Pohst 1992] and [Ford et al. 1998] for the $\mathrm{PSL}_2(5)$ and $\mathrm{PGL}_2(5)$ cases, respectively.

3. Character theory and discriminants

In this section, we review how Artin characters underlie discriminants. Each of the subsections introduces concepts and notation which play an important role in the rest of the paper. The notions we emphasize are slightly different from the most standard representation-theoretic notions. However they are appropriate here because all our characters are rational-valued.

3A. ***Class sets.*** Let G be a finite group. We say that two elements of G are *power-conjugate* if each is conjugate to a power of the other. Let $G^\sharp$ be the set of power-conjugacy classes. Thus one has a natural surjection $G \to G^\sharp$, with the fiber $C_\sigma \subset G$ above $\sigma \in G^\sharp$ being its set of representatives. The *order* $\bar{\sigma}$ of an element $\sigma \in G^\sharp$ is the order of a representing element $g \in C_\sigma$. Similarly the *power* σ^k of a class σ is the class of g^k for any representing element $g \in C_\sigma$.

When dealing with explicit examples, we most commonly indicate an element of $G^\sharp$ by giving its order and an extra identifying label, as in, e.g., 2B. To emphasize the role of order, we say that a class τ divides a class σ if some power of σ is τ. Thus divisibility of classes $\tau \mid \sigma$ implies divisibility of integers $\bar{\tau} \mid \bar{\sigma}$, but not conversely. In connection with divisibility, the quantity $[\sigma] = |C_\sigma|/\phi(\bar{\sigma})$ is useful, with $\phi(n) = |(\mathbb{Z}/n)^\times|$ the Euler ϕ-function. This quantity is integral because C_σ consists of $[\sigma]$ power-classes, each of size $\phi(\bar{\sigma})$. Alternatively, one can think of $G^\sharp$ as indexing conjugacy classes of cyclic subgroups of G, and then $[\sigma]$ is the number of cyclic subgroups of type σ.

Sections 5 and 6 systematically reason with class sets using diagrams based on the divisibility relation and the quantities $[\sigma]$. In general, G itself often recedes into the background of our considerations and the focus is on $G^\sharp$ and its inherited structures.

3B. ***Characters.*** Our calculations take place mainly in the ring $\mathbb{Q}(G^\sharp)$ of $\mathbb{Q}$-valued functions on $G^\sharp$. We also use the larger ring $\mathbb{R}(G^\sharp)$ of real-valued functions, so that we can use standard terms like cone, hull, and interval with their usual meaning. We make extensive use of the natural inner product on $\mathbb{Q}(G^\sharp)$, given by

$$(f_1, f_2) = \sum_{\sigma \in G^\sharp} \frac{|C_\sigma|}{|G|} f_1(\sigma) f_2(\sigma).$$

Important elements in $\mathbb{Q}(G^\sharp)$ for us include the characters ϕ_X of G-sets X. By definition, these characters are obtained by counting fixed points: $\phi_X(\sigma) = |X^g|$, for g any representative of σ. Both the identity class $e \in G^\sharp$ and the constant function $1 \in \mathbb{Q}(G^\sharp)$ usually play trivial roles in our situation. To efficiently remove these quantities from our attention, we define $G^{\sharp 0} = G^\sharp - \{e\}$ and let $\mathbb{Q}(G^\sharp)^0 \subset \mathbb{Q}(G^\sharp)$ be the orthogonal complement to 1.

The characters $\phi_{G/H}$ and a_H. Let H be a subgroup of G. Then the character of the G-set G/H is given by

$$\phi_{G/H}(\sigma) = \frac{|G||C_\sigma \cap H|}{|H||C_\sigma|}. \tag{3-1}$$

Taking $H = \{e\}$ gives the regular character ϕ_G with value $|G|$ at e and 0 elsewhere. We define the formal Artin character of H to then be the difference

$$a_H = \phi_G - \phi_{G/H}, \tag{3-2}$$

which lies in $\mathbb{Q}(G^\sharp)^0$. Here we use the adjective "formal" because often one talks about Artin characters only in the presence of fields, while currently we are in a purely group-theoretic setting.

The case that H is cyclic. The case where H is cyclic is particularly important to us. The generators of H all represent the same class $\tau \in G^\sharp$ and we use the alternative notation $a_\tau = a_H$, calling the a_τ tame characters for reasons which will be clear shortly in Section 3C.

To study the a_τ explicitly, it is convenient to make use of what we call *precharacters* $\hat{a}_\tau$, for $\tau \in G^\sharp$. By definition, $\hat{a}_e$ is the 0 function and otherwise $\hat{a}_\tau$ has two nonzero values:

$$\hat{a}_\tau(e) = |G|, \quad \hat{a}_\tau(\tau) = -\frac{|G|}{|C_\tau|}. \tag{3-3}$$

Tame characters and precharacters are related to each other via

$$a_\tau = \sum_{k|\bar{\tau}} \frac{\phi(\bar{\tau}/k)}{\bar{\tau}} \hat{a}_{\tau^k}, \quad \hat{a}_\tau = \sum_{k|\bar{\tau}} \frac{\bar{\tau}\mu(k)}{\phi(\bar{\tau})k} a_{\tau^k}, \tag{3-4}$$

where μ is the Möbius μ-function taking values in $\{-1, 0, 1\}$. Thus, $a_e = \hat{a}_e = 0$, $a_\tau = \frac{\bar{\tau}-1}{\bar{\tau}}\hat{a}_\tau$ if $\bar{\tau}$ is prime, and otherwise a_τ and $\hat{a}_\tau$ are not proportional to each other. As τ ranges over $G^{\sharp 0}$, the $\hat{a}_\tau$ clearly form a basis for $\mathbb{Q}(G^\sharp)^0$. So the a_τ also form a basis for $\mathbb{Q}(G^\sharp)^0$

3C. ***Artin characters.*** Let F be a local or global number field. Let L/F be a Galois extension with Galois group identified with a subgroup of G. A permutation representation ρ of G gives an F-algebra K split by L. For $\mathfrak{p}$ a prime ideal of F, the discriminant exponent $c_\mathfrak{p}(K)$ depends only on the character $\phi \in \mathbb{Q}(G^\sharp)$ of ρ and in fact depends linearly on ϕ. The associated Artin character $a_{L/F,\mathfrak{p}}$ is the unique element of $\mathbb{Q}(G^\sharp)$ such that one has the general formula

$$c_\mathfrak{p}(K) = (a_{L/F,\mathfrak{p}}, \phi). \tag{3-5}$$

From $c_\mathfrak{p}(F) = 0$, one gets $(a_{L/F,\mathfrak{p}}, 1) = 0$ and so $a_{L/F,\mathfrak{p}} \in \mathbb{Q}(G^\sharp)^0$. One can completely compute $a_{L/F,\mathfrak{p}}$ by computing $c_\mathfrak{p}(K)$ for any $|G^{\sharp 0}|$ different K having characters which are linearly independent in $\mathbb{Q}(G^\sharp)/\mathbb{Q}$.

Before continuing, we note a subtlety that disappears in the Artin character formalism that we are reviewing. Namely, it can happen that nonisomorphic algebras K' and K'' give rise to the same permutation character ϕ. In this case K' and K'' are called arithmetically equivalent. They are indeed equivalent from the point of view of this paper, and any occurrence of K' can simply be replaced by K''.

An Artin conductor $a_{L/F,\mathfrak{p}}$ can be expressed directly in terms of inertia groups in their upper numbering as follows. Let $\mathfrak{P}$ be a prime of L above $\mathfrak{p}$ and let $I_{L/F,\mathfrak{P}} \subseteq \mathrm{Gal}(L/F) \subseteq G$ be the corresponding inertia group. Then one has rational numbers $1 \le s_1 < s_2 < \cdots < s_k$ and normal subgroups

$$I_{L/F,\mathfrak{P}} = I^{s_1} \supset I^{s_2} \supset \cdots \supset I^{s_k} \supset \{e\} \tag{3-6}$$

satisfying

$$a_{L/F,\mathfrak{p}} = \sum_{i=1}^{k} (s_i - s_{i-1}) a_{I^{s_i}}. \tag{3-7}$$

Here, for the sake of the conciseness of formulas, we put $s_0 = 0$. As a similar convention, we put $I^{s_{k+1}} = \{e\}$. The upper numbers s_i we are using here are called *slopes* in [Jones and Roberts 2006] and are designed to capture tame and wild ramification simultaneously; one has $s_i = u_i + 1$ where the u_i are the upper numbers used in the standard reference [Serre 1979].

If $s_1 = 1$ then I^{s_1}/I^{s_2} is cyclic of order prime to p. Otherwise, all the $I^{s_u}/I^{s_{u+1}}$ are abelian groups of exponent p. In particular, $I_{L/F,\mathfrak{P}}$ itself is a *p-inertial* group in the sense that it is an extension of a prime-to-p cyclic group by a p-group. In general, we say that a group is *inertial* if it is p-inertial for some prime p.

The prime $\mathfrak{p}$ is unramified in L/F if and only if $k = 0$ in which case $a_{L/F,\mathfrak{p}}$ is zero. The cases where $\mathfrak{p}$ is ramified but only tamely are those with $k = 1$ and $s_1 = 1$. In both these two settings, (3-7) becomes $a_{L/F,\mathfrak{p}} = a_\tau$ with τ being the class of any generator of any $I_{L/F,\mathfrak{P}}$. Thus the tame characters of the previous subsection are exactly the Artin characters which arise when ramification is tame.

3D. ***Bounds on Artin characters.*** Define cones in $\mathbb{R}(G^\sharp)^0$ spanned by characters or precharacters as follows:

$$\begin{aligned} \text{the tame cone} \quad & T_+(G) = \langle a_\tau \rangle, \\ \text{the wild cone} \quad & W_+(G) = \langle a_{L/F,\mathfrak{p}} \rangle, \\ \text{the inertial cone} \quad & \tilde{T}_+(G) = \langle a_I \rangle, \\ \text{the broad cone} \quad & \hat{T}_+(G) = \langle \hat{a}_\tau \rangle. \end{aligned}$$

The tame and broad cones are the simplest of these objects, as their generators are indexed by the small and explicit set $G^{\sharp 0}$. The inertial cone is also a purely group-theoretic object, although now more complicated as its generators are indexed by conjugacy classes of inertial subgroups. Finally, $W_+(G)$ is much more complicated in nature: its definition depends on the theory of p-adic fields, with $a_{L/F,\mathfrak{p}}$ running over all possible Artin characters, as above.

Our considerations in this section have established the following inclusions:

$$T_+(G) \subseteq W_+(G) \subseteq \tilde{T}_+(G) \subseteq \hat{T}_+(G). \tag{3-8}$$

The first inclusion holds because tame characters are special cases of Artin characters, the second by the expansion (3-7), and the third because all a_I take only positive values on $G^{\sharp 0}$.

4. The tame-wild principle

We begin in Section 4A by giving a formulation of the tame-wild principle in a somewhat abstract context, so that its motivation and structural features can be seen clearly. Next, Section 4B observes that the bounds from the previous section give techniques for group-theoretically proving instances of the tame-wild principle. Finally, Section 4C details one way of introducing coordinates to render everything explicit and Section 4D sketches alternative approaches.

4A. ***Abstract formulation.*** We seek settings where general ramification is governed by tame ramification. The statement that equality holds in $T_+(G) \subseteq W_+(G)$

is true for some G, in which case it is the ideal statement. For general G, we seek weaker statements in the same spirit. Accordingly, consider the orthogonal projection $a \mapsto a^V$ from $\mathbb{R}(G^\sharp)$ onto an arbitrary subspace $V \subseteq \mathbb{R}(G^\sharp)$. Let $T_+(G, V)$, $W_+(G, V)$, $\tilde{T}_+(G, V)$, and $\hat{T}(G, V)$ be the images of $T_+(G)$, $W_+(G)$, $\tilde{T}_+(G)$, and $\hat{T}(G)$, respectively.

Definition 4.1. Let G be a finite group and let $V \subseteq \mathbb{R}(G^\sharp)$ be a subspace. If equality holds in $T_+(G, V) \subseteq W_+(G, V)$, then we say the tame-wild principle holds for (G, V).

As V gets larger, the tame-wild principle for (G, V) becomes a stronger statement. If it holds when V is all of $\mathbb{R}(G^\sharp)$, then we say it holds universally for G.

An important aspect of our formalism is as follows. Given (G, V), consider inertial subgroups I of G. For each I, one has the subspace $V_I \subseteq \mathbb{R}(I^\sharp)$ consisting of pullbacks of functions in V under the natural map $I^\sharp \to G^\sharp$. Then the tame-wild principle holds for (G, V) if and only if it holds for all (I, V_I). In fact, while G typically arises as a global Galois group in our applications, whether or not the tame-wild principle holds for (G, V) is purely a question about local Galois extensions.

4B. ***Two proof methods.*** Projection turns the chain (3-8) into a chain of cones in V:

$$T_+(G, V) \subseteq W_+(G, V) \subseteq \tilde{T}_+(G, V) \subseteq \hat{T}_+(G, V). \tag{4-1}$$

As we will see, for many G all three inclusions are strict in the universal case $V = \mathbb{R}(G^\sharp)$. However strict inclusions can easily become equalities after projection, giving us elementary but quite effective proof techniques. Namely, the *broad method* for proving that the tame-wild principle holds for (G, V) is to show that equality holds in $T_+(G, V) \subseteq \hat{T}_+(G, V)$. The *inertial method* is to show that equality holds in $T_+(G, V) \subseteq \tilde{T}_+(G, V)$.

Applying the broad method gives the following simple result which we highlight because of its wide applicability:

Theorem 4.2. *Let G be a finite group and let V be a subspace of $\mathbb{R}(G^\sharp)$. Suppose that the broad cone $\hat{T}_+(G, V)$ is generated by the $\hat{a}_\tau^V$ with τ of prime order. Then $T_+(G, V) = \hat{T}_+(G, V)$ and the tame-wild principle holds for (G, V).*

Proof. For τ of prime order one has

$$\hat{a}_\tau = \frac{\bar{\tau}}{\bar{\tau} - 1} a_\tau,$$

as noted after (3-4). Thus $\hat{T}_+(G, V)$ is contained in $T(G, V)$ and so all four sets in (4-1) are the same. □

In general, the broad method is very easy to apply, while the harder inertial method can work when the broad method does not.

4C. ***Calculations with permutation characters.*** Let $\phi_1, \ldots, \phi_r$ be permutation characters spanning V. Then we are exactly in the situation described in the introduction, and in this subsection we describe how one approaches the tame-wild principle in this particular coordinatization. We incorporate the ϕ_i into our notation in straightforward ways, for example by writing $(G, \phi_1, \ldots, \phi_r)$ rather than (G, V).

Throughout this subsection, we illustrate the generalities by returning to the introductory example with $G = S_5$ and $V = \langle \phi_5, \phi_6 \rangle$. The very simple two-dimensional picture of V in Figure 1 serves as an adequate model for mental images of the general situation. In particular, we always think of the a_τ^V, $a_{L/F,\mathfrak{p}}$, a_I^V, and $\hat{a}_\tau$ as in the drawn V. We think of our various cones in the drawn V as well. On the other hand, it is not useful to draw the ϕ_i on these pictures. Rather, via the identification of V with its dual by the inner product, we think of the ϕ_i as coordinate functions on the drawn V.

Conductor vectors. The space V is identified with a subspace of $\mathbb{R}^r$, viewed but not always written as column vectors, via $v \mapsto (c_1, \ldots, c_r)$ with $c_i = (v, \phi_i)$. For example, an Artin character $a_{L/F,\mathfrak{p}}^V$ becomes a vector of conductors as in the introduction:

$$c_{L/F,\mathfrak{p}} = (c_{\mathfrak{p}}(K_1), \ldots, c_{\mathfrak{p}}(K_r)).$$

The main case is when the ϕ_i are linearly independent, so that V is all of $\mathbb{R}^r$. One can always work in this main case by picking a basis from among the ϕ_i.

Various matrices. Our approach to calculations centers on matrices. The $r \times G^{\sharp 0}$ *partition matrix* $P(G, \phi_1, \ldots, \phi_r)$ has i-τ entry the cycle type $\lambda_\tau(\phi_i)$ of $\rho_i(g)$, where ρ_i is a permutation representation with character ϕ_i and $g \in G$ represents τ. Thus,

$$P(S_5, \phi_5, \phi_6) = \begin{pmatrix} 2111 & 221 & 311 & 41 & 5 & 32 \\ 222 & 2211 & 33 & 411 & 51 & 6 \end{pmatrix}. \tag{4-2}$$

Partition matrices are purely group-theoretic objects, but one can use fields in a standard way to help construct them. For example, the columns from left to right are the partitions obtained by factoring the pair $(f_5(x), f_6(x))$ from (2-1) and (2-2) modulo the primes 67, 211, 31, 13, 11, and 7 respectively.

One passes to the *tame matrix* $T(G, \phi_1, \ldots, \phi_r)$ by replacing each partition $\lambda_\tau(\phi_i)$ by its conductor $c_\tau(\phi_i) = (a_\tau, \phi_i)$ — its degree minus its number of parts. Thus

$$T(S_5, \phi_5, \phi_6) = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 3 \\ 3 & 2 & 4 & 3 & 4 & 5 \end{pmatrix}. \tag{4-3}$$

The *broad matrix* $\hat{T}(G, \phi_1, \ldots, \phi_r)$ consists of what we call *preconductors*, the preconductor $\hat{c}_\tau(\phi_i) = (\hat{a}_\tau, \phi_i)$ being the degree of $\lambda_\tau(\phi_i)$ minus its number of ones.

Thus

$$\hat{T}(S_5, \phi_5, \phi_6) = \begin{pmatrix} 2 & 4 & 3 & 4 & 5 & 5 \\ 6 & 4 & 6 & 4 & 5 & 6 \end{pmatrix}. \tag{4-4}$$

Inertial matrices $\tilde{T}(G, \phi_1, \ldots, \phi_r)$ typically have more columns, because columns are indexed by conjugacy classes of inertial subgroups I. But an entry is just the *formal conductor* $c_I(\phi_i) = (a_I, \phi_i)$, this being the degree of ρ_i minus the number of orbits of $\rho_i(I)$, just as in the cyclic case. The cones $T_+ \subseteq \tilde{T}_+ \subseteq \hat{T}_+ \subset \mathbb{R}^r$ are then generated by the columns of the corresponding matrices T, $\tilde{T}$, and $\hat{T}$.

Inclusions $a^V \in T_+(G, V)$ *in matrix terms.* By dropping rows, we can assume that $\phi_1, \ldots, \phi_r$ span V and so $T = T(G, \phi_1, \ldots, \phi_r)$ has full rank r, as discussed above. In general, let $c \in \mathbb{R}^r$ be a column r-vector. For each r-element subset $J \subseteq G^{\sharp 0}$ for which the corresponding minor $T(J)$ is invertible, let $u(J) = (u(J)_\tau)_{\tau \in J}$ be the vector $T(J)^{-1}c$. Then $c = \sum_{\tau \in J} u(J)_\tau T_\tau$, with T_τ the τ-th column of T. Then c is in the tame cone T_+ if and only if there exists such a J with $u(J)_\tau \geq 0$ for all $\tau \in J$.

To prove that the tame-wild principle holds for $(G, \phi_1, \ldots, \phi_r)$ directly, one would have to show this positivity condition holds for all conductor vectors $c_{L/F,\mathfrak{p}}$. To show it via the inertial method, one has to show that it holds for all formal conductor vectors c_I. To show it holds via the broad method, one has to show that it holds for all preconductor vectors $\hat{c}_\tau$.

Projectivization. In the introductory example, we emphasized taking ratios of conductors, thereby removing the phenomenon that wild conductors are typically much larger than tame conductors, but keeping the phenomenon we are interested in. We can do this in the general case as well, assuming without loss of generality that ϕ_r comes from a faithful permutation representation so that the conductors $c_\tau(\phi_r)$ are strictly positive for all $\tau \in G^{\sharp 0}$. We projectivize $c = (c_1, \ldots, c_r)$ to $c' = (c'_1, \ldots, c'_{r-1})$ with $c'_i = c_i/c_r$.

Applying this projectivization process to columns gives the projective tame, inertial, and broad matrices respectively, each notationally indicated by a $'$. In our continuing introductory example, one has, very simply,

$$T'(S_5, \phi_5, \phi_6) = \left(\tfrac{1}{3} \;\; 1 \;\; \tfrac{1}{2} \;\; 1 \;\; 1 \;\; \tfrac{3}{5}\right), \tag{4-5}$$

$$\hat{T}'(S_5, \phi_5, \phi_6) = \left(\tfrac{1}{3} \;\; 1 \;\; \tfrac{1}{2} \;\; 1 \;\; 1 \;\; \tfrac{5}{6}\right). \tag{4-6}$$

In general, the τ-columns of $T'(G, \phi_1, \ldots, \phi_r)$ and $\hat{T}'(G, \phi_1, \ldots, \phi_r)$ agree if τ has prime order. Here they disagree only in the last column corresponding to the composite order 6.

Let $T'_+(G, \phi_1, \ldots, \phi_r)$ be the convex hull of the columns of $T'(G, \phi_1, \ldots, \phi_r)$ and define $W'_+(G, \phi_1, \ldots, \phi_r)$, $\tilde{T}'_+(G, \phi_1, \ldots, \phi_r)$ and $\hat{T}'_+(G, \phi_1, \ldots, \phi_r)$ to be the

analogous hulls. Then (4-1) has its obvious analog at the level of hulls, and one can think about the broad method and the inertial method at this level. In the introductory example, (4-5) and (4-6) say that $T'_+(S_5, \phi_5, \phi_6) \subseteq \hat{T}'_+(S_5, \phi_5, \phi_6)$ is an equality because both sides are $\left[\frac{1}{3}, 1\right]$. Thus the tame-wild principle holds for (S_5, ϕ_5, ϕ_6).

The drop in dimension from r to $r-1$ has a number of advantages. As illustrated already by (4-5) and (4-6), it renders the $r=2$ case extremely concrete. As we will illustrate in Section 8A, it renders the $r=3$ case highly visible. In general, it lets one determine whether a^V is in $T'_+(G, \phi_1, \ldots, \phi_r)$ by computation with $(r-1)\times(r-1)$ minors rather than $r \times r$ minors.

4D. ***Alternative approaches.*** Our abstract formulation of the tame-wild principle is designed to be very flexible. For example, say that a vector $v \in \mathbb{R}(G^\sharp)$ is *bad* if $(a_\tau, v) \geq 0$ for all $\tau \in G^{\sharp 0}$ but $(a_{L/F,\mathfrak{p}}, v) < 0$ for some Artin character $a_{L/F,\mathfrak{p}}$. The bad vectors form a union of cones in $\mathbb{R}(G^\sharp)$ and the tame-wild principle holds for (G, V) if and only if V misses all these cones. In this sense, the one-dimensional V spanned by bad v are essential cases, but these V are never spanned by permutation characters.

Sections 5 and 6 are in the universal setting $V = \mathbb{R}(G^\sharp)$, and we do not use ϕ_i at all. Sections 7 and 8 return to the permutation character setting described in Section 4C. In general, the systematic study of the tame-wild principle for a fixed G and varying V would be facilitated by the canonical basis of $\mathbb{Q}(G^\sharp)$ given by irreducible rational characters.

5. The universal tame-wild principle holds for U-groups

In Section 5A, we present a diagrammatic way of understanding class sets $G^\sharp$. Making use of this viewpoint, Section 5B gives the canonical expansion of a formal Artin character a_I as a sum of tame characters a_τ. Next, Section 5C introduces the notion of U-group and proves that the universal tame-wild principle holds for U-groups. However the class of U-groups is quite small, as discussed in Section 5D.

5A. ***Divisibility posets.*** For G a finite group, the set $G^\sharp$ is naturally a partially ordered set via the divisibility relation. We draw this divisibility poset in the standard way with an edge from σ down to τ of vertical length one if $\sigma^p = \tau$ for some prime p. With notation as in Section 3A, the natural weight $d(\tau, \sigma) = [\sigma]/[\tau]$ plays an important role, and we write it next to the edge whenever it is different from 1, considering this data as part of the divisibility poset.

The product of the edge weights from any vertex σ down to another τ is path-independent, being in fact just $d(\tau, \sigma) = [\sigma]/[\tau]$. Define integers $u_{G,\sigma}$ via

$$\sum_{\tau|\sigma} d(\tau, \sigma) u_{G,\sigma} = 1. \tag{5-1}$$

Thus $u_{G,\tau} = 1$ for maximal τ, and all the integers $u_{G,\tau}$ can be computed by downwards induction on the divisibility poset $G^\sharp$.

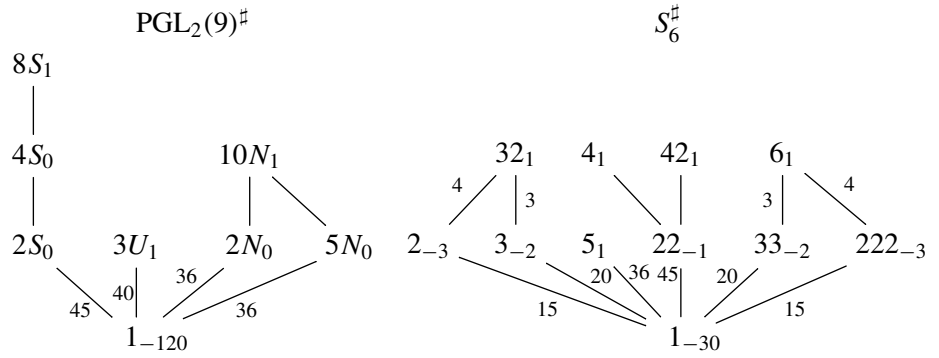


Figure 2. Two divisibility posets $G^\sharp$ with $u_{G,\tau}$ subscripted on $\tau \in G^\sharp$.

Figure 2 draws the divisibility posets $\mathrm{PGL}_2(9)^\sharp$ and $S_6^\sharp$, with each τ subscripted by its $u_{G,\tau}$. The case $\mathrm{PGL}_2(9)$ represents the general case $\mathrm{PGL}_2(p^f)$, with split-torus classes indexed by nonunital divisors of $p^f - 1$, a unipotent class pU, nonsplit-torus classes indexed by nonunital divisors of $p^f + 1$, and finally the identity class 1. The case S_6 represents the general case S_n, where classes are indexed by partitions of n, with 1s usually left unprinted.

In general, the largest edge weights on divisibility posets $G^\sharp$ tend to be on edges incident on the identity class. These edges do not play an important role for us and in the sequel we work instead with the divisibility poset associated to $G^{\sharp 0}$.

5B. ***Expansion of formal Artin characters.*** Divisibility posets for inertial groups I and the associated integers $u_{I,\sigma}$ are important to us because of the role they play in the following lemma.

Lemma 5.1. *Let G be a finite group, let I be a subgroup, and let $i : I^\sharp \to G^\sharp$ be the induced map. Then the expansion of the formal Artin character $a_I \in \mathbb{Q}(G^\sharp)^0$ in the basis $\{a_\tau\}_{\tau \in G^{\sharp 0}}$ can be read off from the divisibility poset $I^{\sharp 0}$ via the formula*

$$a_I = \frac{1}{|I|} \sum_{\sigma \in I^{\sharp 0}} u_{I,\sigma} [\sigma] \bar{\sigma} a_{i(\sigma)}. \tag{5-2}$$

Before proving the lemma, we explain the roles that various parts of (5-2) play in the sequel. The positive integer $[\sigma]\bar{\sigma}$ plays a very passive role: only the positivity of $[\sigma]\bar{\sigma}$ is used in the proof of Theorem 5.3; moreover, $[\sigma]\bar{\sigma}$ factors out in Section 6A and accordingly does not enter into Section 6B-Section 6E. The factor $|I|^{-1}$ is more important: while only its positivity enters into the proof of Theorem 5.3,

it contributes to the index factor in (6-1) which enters significantly into the rest of Section 6. The part with the most important role is $u_{I,\sigma}$, as it is the possible negativity of $u_{I,\sigma}$ that can lead to failures of the tame-wild principle. Our use of the function i relegates the difference between I and G to the background, but one should note that for $\tau \in G^{\sharp 0}$ the actual coefficient of a_τ in (5-2) has $|i^{-1}(\tau)|$ terms.

Proof. First consider the case $I = G$. Then both sides of (5-2) are in $\mathbb{Q}(I^\sharp)^0$. The left side takes the value $a_I(\tau) = -1$ for all $\tau \in I^{\sharp 0}$. We thus need to evaluate the right side on an arbitrary $\tau \in I^{\sharp 0}$ and see that it simplifies to -1:

$$\begin{aligned}
\frac{1}{|I|}\sum_{\sigma \in I^{\sharp 0}} u_{I,\sigma}[\sigma]\bar{\sigma}a_\sigma(\tau) &= \frac{1}{|I|}\sum_{\tau|\sigma} u_{I,\sigma}[\sigma]\bar{\sigma}a_\sigma(\tau) \\
&= \frac{1}{|I|}\sum_{\tau|\sigma} u_{I,\sigma}[\sigma]\bar{\sigma}\left(-\frac{\phi(\bar{\tau})}{\bar{\sigma}}\frac{|I|}{|C_\tau|}\right) \\
&= -\sum_{\tau|\sigma} u_{I,\sigma}[\sigma]\frac{\phi(\bar{\tau})}{|C_\tau|} \\
&= -\sum_{\tau|\sigma} \frac{u_{I,\sigma}[\sigma]}{[\tau]} \\
&= -\sum_{\tau|\sigma} d(\tau,\sigma)u_{I,\sigma} = -1.
\end{aligned}$$

Here we have used formulas from Section 3A and Section 3B as well as the definition of $d(\tau, \sigma)$ and the defining property of the $u_{I,\sigma}$ from Section 5A. Finally the case of general G follows, by induction of both sides from I to G. □

5C. ***Applying the inertial method.*** Say that a class $\tau \in G^\sharp$ is a *U-class* if it divides exactly one maximal element σ of $G^\sharp$ and $d(\tau, \sigma) = 1$. Otherwise, say it is an *N-class*. Here U stands for unique and N for nonunique. The following three facts are immediate from the definition. First, a maximal class τ is always a U-class with $u_{G,\tau} = 1$. Second, other U-classes τ have $u_{G,\tau} = 0$. Third, a maximal N-class τ always has $u_{G,\tau} < 0$.

We divide all finite groups into two types, as follows.

Definition 5.2. A finite group is a *U-group* if every nonidentity element is contained in exactly one maximal cyclic subgroup. Otherwise it is an *N-group*.

It is immediate that a group G is a U-group if and only if all classes $\tau \in G^{\sharp 0}$ are U-classes. Thus from Figure 2, $\mathrm{PGL}_2(9)$ is a U-group while S_6 is an N-group. It follows easily from the definition that any subgroup of a U-group is itself a U-group. Similarly, any quotient of a U-group is a U-group [Suzuki 1950].

Via (3-4), the chain (3-8) completely collapses to the equality $T_+(G) = \hat{T}_+(G)$ if and only if all nonidentity elements in G have prime order. The following theorem is a subtler version of this idea.

Theorem 5.3. *Suppose G is a group such that all inertial subgroups of G are U-groups. Then one has $T_+(G) = \tilde{T}_+(G)$, and so the tame-wild principle holds universally for G.*

Proof. Let I be an arbitrary inertial subgroup. Since I is assumed to be a U-group, the associated integers $u_{I,\sigma}$ are nonnegative for all $\sigma \in I^{\sharp 0}$. For any $\tau \in G^{\sharp 0}$, the terms $|I|^{-1} u_{I,\sigma}[\sigma]\bar{\sigma}$ contributing to the coefficient of a_τ in Lemma 5.1 are all nonnegative. Hence the coefficient itself is nonnegative and so a_I is in the tame cone $T_+(G)$. Since the a_I generate the inertial cone $\tilde{T}_+(G)$, equality holds in $T_+(G) \subseteq \tilde{T}_+(G)$. □

In particular, the tame-wild principle holds for all U-groups. This is the main import of Theorem 5.3, as we are not aware of any group satisfying the hypothesis of Theorem 5.3 which is not itself a U-group.

5D. ***Classification of U-groups.*** Given Theorem 5.3, it is of interest to classify U-groups. This problem has been addressed in the literature, with Kontorovich [1939; 1940] referring to U-groups as completely decomposable groups, and Suzuki [1950] calling them groups with a complete partition. We give a summary of the classification situation here.

The condition to be a U-group is very restrictive, but it is easy to check that it includes many groups of small order. In particular, the following groups are U-groups: cyclic groups, dihedral groups, groups of prime exponent, and the Frobenius groups $F_p = C_p : C_{p-1}$. The last class is particularly important in our context, since an extension of a p-adic field of degree p has normal closure with Galois group a subgroup of F_p. If q is a prime power, the linear groups $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(q)$ are U-groups, so that in particular $S_4 \cong \mathrm{PGL}_2(3)$, $S_5 \cong \mathrm{PGL}_2(5)$ and $A_6 \cong \mathrm{PSL}_2(9)$ are all U-groups. There are more U-groups than those listed here, most of them being more general types of Frobenius groups.

The following observation is useful in understanding the nature of U-groups. In two settings, the extreme members of a class of groups are exactly the U-groups as follows. First, consider abelian p-groups of order p^n. Up to isomorphism, they correspond to partitions of n. The groups which are U-groups are the two extreme ones $(C_p)^n$ and C_{p^n}. Second, consider semidirect products $C_a :_\gamma C_b$ with a and b being relatively prime and $\gamma : C_b \to \mathrm{Aut}(C_a)$. If γ is trivial, then $C_a :_\gamma C_b \cong C_{ab}$ is a U-group. If γ is injective, then $C_a :_\gamma C_b$ is again a U-group, being of a nature similar to F_p above. Again, it is the intermediate cases which are N-groups: if γ is neither trivial nor injective then nontrivial elements in the kernel of γ are in more than one maximal cyclic subgroup.

6. The universal tame-wild principle usually fails for N-groups

In this section, we study the universal tame-wild principle for N-groups. In Section 6A, we give the canonical expansion of a general Artin character $a_{L/F,\mathfrak{p}}$ in terms of tame characters a_τ. In Section 6B, we list out N-groups of order pqr where p, q, and r are not necessarily distinct primes, finding six series. We show in Section 6C that the universal tame-wild principle generally fails for groups in the first four series. In Section 6D we take a close look at the quaternion group Q_8, which is the first group of the fifth series, finding failure again. On the other hand we show in Section 6E that the universal tame-wild principle holds for all groups in the sixth series. Finally, Section 6F explains how the negative results for small groups support the principle that most N-groups do not satisfy the universal tame-wild principle.

6A. ***Expansion of general Artin characters.*** Let $a_{L/F,\mathfrak{p}} \in \mathbb{Q}(G^\sharp)^0$ be an Artin character coming from an inertial subgroup $I \subseteq G$. Equation (3-7) expands $a_{L/F,\mathfrak{p}}$ in terms of formal Artin characters $a_{I^{s_i}}$ and Lemma 5.1 in turn expands each $a_{I^{s_i}}$ in terms of tame characters. Putting these two expansions together and replacing the divisibility posets $(I^{s_i})^{\sharp 0}$ with their images in $I^{\sharp 0}$ gives the following lemma.

Lemma 6.1. *Let G be a group, and let $a_{L/F,\mathfrak{p}} \in \mathbb{Q}(G^\sharp)^0$ be an Artin character with inertia group $I = I^{s_1} \supset I^{s_2} \supset \cdots$ as in* (3-6)*. Let $i : I^\sharp \to G^\sharp$ be the induced map. Then one has the expansion*

$$a_{L/F,\mathfrak{p}} = \frac{1}{|I|} \sum_{\sigma \in I^{\sharp 0}} w_{L/F,\mathfrak{p},\sigma}[\sigma]\bar{\sigma} a_{i(\sigma)},$$

where

$$w_{L/F,\mathfrak{p},\sigma} = \sum_{i=1}^{k} (s_i - s_{i-1})[I : I^{s_i}] u_{I^{s_i},\sigma}. \tag{6-1}$$

While the lemma applies to the general situation, our focus in Sections 6B–6E is on the case $I = G$. Here $a_{L/F,\mathfrak{p}}$ is in the tame cone $T_+(G)$ if and only if $w_{L/F,\mathfrak{p},\tau} \geq 0$ for all $\tau \in G^{\sharp 0}$.

6B. ***Inertial N-groups of order pqr.*** Groups of order p or pq are U-groups. In the complete list of inertial groups of order pqr, in a rough sense about half of them are U-groups and the other half N-groups. For example, for a given prime p, there are two nonabelian groups of order p^3, the extra-special groups often denoted p_+^{1+2} and p_-^{1+2}. For p odd, p_+^{1+2} has exponent p and so is a U-group, while p_-^{1+2} is an N-group. Similarly the dihedral group $D_4 = 2_+^{1+2}$ is a U-group while the quaternion group $Q_8 = 2_-^{1+2}$ is an N-group.

In fact, it is easy to see that the inertial N-groups are as follows. Now p, q, and r are required to be different primes, with $q \mid p-1$ whenever $F_{p,q} = C_p : C_q$ is present:

1: The product $F_{p,q} \times C_r$.

2: The semidirect product $C_p : C_{q^2} \cong F_{p,q} *_{C_q} C_{q^2}$.

3: The abelian group $C_{pq} \times C_p \cong C_p \times C_p \times C_q$.

4: The product $F_{p,q} \times C_p$.

5: The extra-special group p_-^{1+2}.

6: The abelian group $C_{p^2} \times C_p$.

These groups I are all p-inertial groups, but not inertial groups for any other primes. Moreover, since all proper subgroups are U-groups, the universal tame-wild principle fails for I if and only if there exists a totally ramified local Galois extension L/F having $\mathrm{Gal}(L/F) \cong I$ with associated Artin character $a_{L/F,\mathfrak{p}}$ having a negative coefficient $w_{L/F,\mathfrak{p},\tau}$. Furthermore, in each case it turns out that there is exactly one N-class $\tau \in I^{\sharp 0}$. Only for this class τ could $w_{L/F,\mathfrak{p},\tau}$ possibly be negative, and this N-class is boxed in the displayed divisibility posets below.

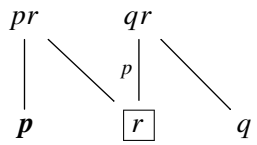
In general, let I be a p-inertial group. Then it is known that there indeed exists a totally ramified Galois extension of p-adic fields L/F with $\mathrm{Gal}(L/F)$ isomorphic to I. This fact for our particular I is essential to our proofs that the universal tame-wild principle does not hold. However it is easy to prove this fact for all the above I by direct exhibition of L/F. We will go into this level of detail only for the groups in Series 4 and Q_8 from Series 5, as here we need particular fields satisfying conditions on their wild ramification.

6C. ***Negative results for four series.*** Our first result concerns Series 1-4 and is negative:

Theorem 6.2. *$F_{3,2} \times C_3 \cong S_3 \times C_3$ satisfies the universal tame-wild principle, but otherwise the groups $F_{p,q} \times C_r$, $C_p : C_{q^2}$, $C_{pq} \times C_p$, and $F_{p,q} \times C_p$, do not.*

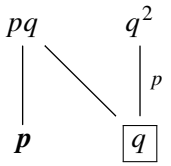
Proof. In the divisibility posets below, the wild classes, meaning the classes of p-power order, are put in boldface for further emphasis. For the first three series, the unique N-class τ has prime-to-p order and so we do not need to enter into an examination of wild slopes. In Series 4, τ has order p and bounds on wild slopes lead to the exception.

1. For the group $I = F_{p,q} \times C_r$, power-conjugacy classes are determined by their orders, and the divisibility poset $I^{\sharp 0}$ is



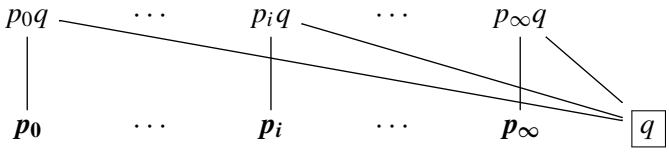
Equation (6-1) becomes $w_r = u_{I,r} = -p < 0$. So by the existence of totally ramified I-extensions as discussed in the previous subsection, the universal tame-wild principle does not hold for $I = F_{p,q} \times C_r$.

2. The group $I = C_p : C_{q^2}$ behaves very similarly. Again power-conjugacy classes are determined by their orders:



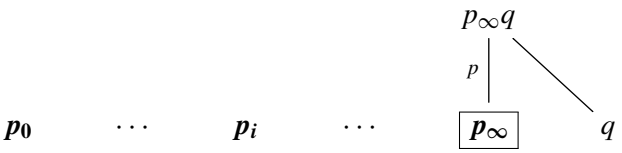
The key quantity $w_q = u_{I,q} = -p$ is again negative, so the universal tame-wild principle fails for $C_p : C_{q^2}$.

3. The group $I = C_p^2 \times C_q$ has a more complicated divisibility poset $I^{\sharp 0}$ but the behavior is otherwise similar. The classes of order p and the classes of order pq have the structure of projective lines over $\mathbb{F}_p$ in bijection with one another:



Once again $w_q = u_{I,q} = -p$ and so the universal tame-wild principle fails for $C_p^2 \times C_q$.

4. For $I = F_{p,q} \times C_p$ the divisibility poset $I^{\sharp 0}$ is disconnected:



Here p_0 and q lie in the factor $F_{p,q}$ while p_∞ lies in the factor C_p. The first term in (6-1) is $u_{I,p_\infty} = 1 - p$. However, now we must take into account how wild ramification contributes to the remaining terms. Let $s > 1$ be the slope associated to $F_{p,q}$ and let $c > 1$ be the slope associated to C_p. Since C_p is abelian, c must be integral and hence $c \geq 2$. On the other hand, s must have exact denominator q. Let

$m = \min(c, s)$, so that $I^m = C_p^2$ is the wild inertia group and $I^{\max(c,s)} \cong C_p$ is a higher inertia group. If $c > s$ then $(I^c)^{\sharp 0} = \{p_\infty\}$, while if $s > c$ then $(I^s)^{\sharp 0} = \{p_0\}$. Equation (6-1) becomes

$$w_{p_\infty} = \begin{cases} (1-p) + q(s-1) + qp(c-s) & \text{if } c > s, \\ (1-p) + q(c-1) & \text{if } s > c. \end{cases}$$

For $(p, q) = (3, 2)$, the general formula simplifies to

$$w_{3_\infty} = \begin{cases} 6c - 4s - 4 & \text{if } c > s, \\ 2c - 4 & \text{if } s > c. \end{cases}$$

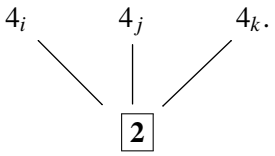
Thus, using $c \geq 2$, one has $w_{3_\infty} \geq 0$ and so the universal tame-wild principle holds for $F_{3,2} \times C_3$.

There are many ways to produce an explicit instance with $w_{p_\infty} < 0$ for the remaining (p, q). We will present one in the setting $s > c = 2$ in which case $w_{p_\infty} = 1 + q - p$ is indeed negative. To get an $F_{p,q}$ extension, start with $x^p - p$, which gives a totally ramified $F_{p,p-1}$ extension of $\mathbb{Q}_p$ with wild slope best written in the form $1 + p/(p-1)$. Write $e = (p-1)/q$ and extend the ground field from $\mathbb{Q}_p$ to $F_e = \mathbb{Q}_p[\pi]/(\pi^e - p)$. Then $x^p - p$ has Galois group $F_{p,q}$ over F_e, with wild slope $1 + ep/(p-1)$, as tame base-change always scales slopes this way. But now $x^p - \pi x^{p-1} + \pi$ has wild slope 2 and, after perhaps replacing F_e by an unramified extension F, Galois group C_p [Amano 1971]. The splitting field of $(x^p - p)(x^p - \pi x^{p-1} + \pi)$ gives the desired extension L/F, showing that the universal tame-wild principle does not hold for $F_{p,q} \times C_p$. □

6D. ***Negative result for Q_8.*** The fifth series, consisting of groups of the form p_-^{1+2}, is the most complicated. Here we treat only $2_-^{1+2} = Q_8$, getting a negative result.

Proposition 6.3. *The universal tame-wild principle fails for the quaternion group.*

Proof. The divisibility poset $Q_8^{\sharp 0}$, with unique N-class boxed as always, is



The generic case has three distinct slopes. We seek only counterexamples and so we focus on the special case with two slopes $s_1 < s_2$, with s_1 occurring with multiplicity two. The key quantity (6-1) here becomes $w_2 = -2s_1 + 4(s_2 - s_1) = 4s_2 - 6s_1$. Thus one gets a counterexample to the universal tame-wild principle if and only if $s_2 < 1.5s_1$.

The table of octic 2-adic fields [Jones and Roberts 2008] available from the website of [Jones and Roberts 2006] then give four types of counterexamples in this

context, after tame base-change from $\mathbb{Q}_2$ to its Galois extension F with ramification index t and residual field degree u:

#	c	slope content	I	D	s_1	s_2
2	10	$[1.\overline{3}, 1.\overline{3}, 1.5]_3^2$	$\mathrm{SL}_2(3)$	$\mathrm{GL}_2(3)$	2	2.5
2	16	$[2, 2, 2.5]^2$	Q_8	$\hat{Q}_8$	2	2.5
4	16	$[2, 2, 2.5]^4$	Q_8	8T17	2	2.5
8	22	$[2.\overline{6}, 2.\overline{6}, 3.5]_3^2$	$\mathrm{SL}_2(3)$	$\mathrm{GL}_2(3)$	6	8.5

(6-2)

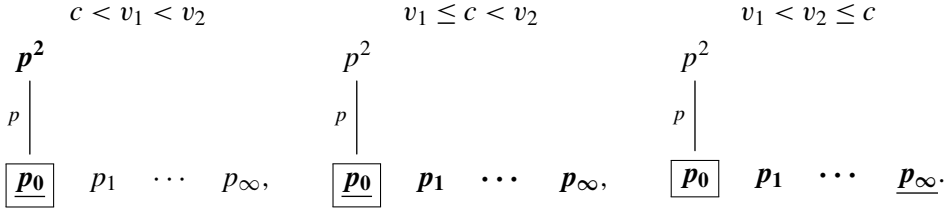
Here in the first and last cases, we use the general conversion from slope content $[\dots \sigma_i \dots]_t^u$ over $\mathbb{Q}_2$ to slope content $[\dots s_i \dots]_1^1$ over F given by $s_i = 1 + t(\sigma_i - 1)$. A full treatment of the range of possible counterexamples could have Proposition 4.4 of [Fontaine 1971] as its starting point. □

Our counterexamples in Section 7D and Section 8B will be built from one of the two fields with slope content $[2, 2, 2.5]^2$. A point to note here is that $\mathbb{Q}_2$ does have totally ramified quaternionic extensions, in fact four of them, all with slope content $[2, 3, 4]$ [Jones and Roberts 2008]. However, these extensions do not give counterexamples to the universal tame-wild principle for Q_8. The fact that the first local counterexamples come from $\hat{Q}_8 = 8\mathrm{T}8$ extensions of $\mathbb{Q}_2$ plays a prominent role in our later global counterexamples.

6E. ***Positive results for $C_{p^2} \times C_p$.*** Here we prove that the N-groups in Series 6 always satisfy the universal tame-wild principle. Unlike most of our previous positive results, but like the exception $S_3 \times C_3$ of Section 6C, this result is not purely group-theoretic. Rather it depends on a close analysis of the possibilities for wild slopes. Said in a different way, the situation for these I is $T_+(I) = W_+(I) \subset \tilde{T}_+(I)$, so that the universal tame-wild principle holds, even though it is not provable by the inertial method.

Theorem 6.4. *The groups $C_{p^2} \times C_p$ satisfy the universal tame-wild principle.*

Proof. Let K_{p^2}/F be a cyclic extension of degree p^2 and slopes $s_1 < s_2$. Let K_p/F be a cyclic extension of degree p and slope t. Switch to the indexing scheme of [Serre 1979] via $s_i = 1 + v_i$ and $t = 1 + c$, so as to better align also with our reference [Fontaine 1971] and in particular make (6-5) below as simple as possible. There are three possibilities for how the slope filtration goes through the group:



Here, assuming all inequalities are strict, classes in the higher inertia group of order p^2 are put in bold and the classes in the higher inertia group of order p are furthermore underlined. If one has equality, the formulas below still apply.

As in the previous two subsections, only one w_τ from Lemma 6.1 could possibly be negative, and in this case it is $w = w_{p_0}$. Equation (6-1) becomes

$$w = \begin{cases} (c+1)(1-p) + (v_2 - v_1)p^2 & \text{if } c < v_1 < v_2, \\ (v_1+1)(1-p) + (c - v_1)p + (v_2 - c)p^2 & \text{if } v_1 \le c < v_2, \\ (v_1+1)(1-p) + (v_2 - v_1)p & \text{if } v_1 < v_2 \le c. \end{cases} \tag{6-3}$$

Let e be the ramification index of $F/\mathbb{Q}_2$ and put $B = e/(p-1)$. From the known behavior of cyclic degree p extensions, one has

$$1 \le c \le pB \quad \text{and} \quad 1 \le v_1 \le pB. \tag{6-4}$$

There are two regimes to consider: the geometric regime, where $v_1 < B$, and the arithmetic regime, where $v_1 \ge B$. One has

$$v_2 \ge pv_1 \qquad \text{in the geometric regime,} \tag{6-5}$$

$$v_2 = v_1 + e \quad \text{in the arithmetic regime.} \tag{6-6}$$

These last two facts and other related information dating back to [Maus 1965] are conveniently available in [Fontaine 1971, Proposition 4.3].

The quantity e does not enter into the geometric inequality (6-5), and since we need to deal with arbitrary e the upper bounds in (6-4) are not available to us. This fact is the source of our terminology because the geometric case is now identified with the case where p-adic fields have been replaced by $\mathbb{F}_{p^f}((t))$, which have $e = \infty$. The worst case is always when $v_2 = pv_1$ and, in the second case, when c takes on its limiting bound v_2 as well. Substituting these worst cases into (6-3) and simplifying, one has

$$w \ge \begin{cases} (v_1+1)(1-p) + (pv_1 - v_1)p^2 & \text{if } c < v_1 < v_2, \\ (v_1+1)(1-p) + (pv_1 - v_1)p & \text{if } v_1 \le c < v_2, \\ (v_1+1)(1-p) + (pv_1 - v_1)p & \text{if } v_1 < v_2 \le c. \end{cases}$$

With m equal to p^2, p, p in the three cases, one further simplifies by

$$w \ge (p-1)(-v_1 - 1 + v_1 m) = (p-1)((m-1)v_1 - 1) \ge 0.$$

Thus, in the geometric regime, w is never negative.

In the arithmetic regime the substitute (6-6) for (6-5) is simpler in that it is an equality, but now the upper bounds in (6-4) will need to be used. The substitution $v_2 = v_1 + e = v_1 + B(p-1)$ into (6-3) makes w factor, and we divide by the positive

quantity $p-1$:

$$\frac{w}{p-1}=\begin{cases}Bp^2-1-c & \text{if } c<v_1<v_1+e,\\ Bp^2-v_1-p(c-v_1)-1 & \text{if } v_1\le c<v_1+e,\\ Bp-v_1-1 & \text{if } v_1<v_1+e\le c.\end{cases}$$

Using the bounds (6-4) one has

$$\frac{w}{p-1}\ge\begin{cases}Bp^2-1-Bp=Bp(p-1)-1=ep-1\ge 1,\\ Bp^2-Bp-p(e-1)-1=ep-p(e-1)-1=p-1\ge 1,\\ Bp-v_1-1\le Bp-(Bp-e)-1=e-1\ge 0.\end{cases}$$

in the three cases. Thus here too $w\ge 0$. □

6F. ***From smaller to larger groups.*** Our final topic in this section is to promote our counterexamples from the small groups I to larger groups G that contain them. In general, let $I\subseteq G$ be an inclusion of groups and consider the induced map $i: I^\sharp\to G^\sharp$. Then the lack of injectivity of i can obstruct the promotion process. For example, consider Series 4 groups $I=F_{p,q}\times C_p$ and their product embedding into $G=S_{p^2}$. Then all $p+1$ classes in $I^\sharp$ of order p go to the single class in $S^\sharp_{p^2}$ indexed by the partition $p^p=p\cdots p$. To get the coefficient of a_{p^p} of the pushed-forward formal Artin conductor $a_I\in\mathbb{Q}(S^\sharp_{p^2})^0$ one has to add the contributions of the fiber, as in Lemma 5.1. There are p contributions of $1/pq$ and one contribution of $(2-p)/pq$ for a total of $2/pq$. Equation (6-1) says that wild ramification can only increase this $2/pq$ to larger positive numbers, and so all pushed-forward Artin characters $a_{L/F,\mathfrak{p}}$ from I are in the tame cone $T_+(S_{p^2})$.

For Series 1–3 and also for Q_8, this complication does not arise because the unique N-class in $I^{\sharp 0}$ is the only class of its order. Hence the promotion process works:

Corollary 6.5. *Let G be a group containing a subgroup I of the form $F_{p,q}\times C_r$, $C_p:C_{q^2}$, $C_{pq}\times C_p$ or Q_8. Then G does not satisfy the universal tame-wild principle.*

Since there are so many possibilities for I, the hypothesis holds for many G. Moreover the fact that it holds for a given G is often easily verified. For example, when studying G one commonly has a list of maximal subgroups H, and one can often easily see that at least one I is in at least one of the H. As another example, the presence of $C_{pq}\times C_p$ can often be read off from the divisibility poset: suppose one has a class $\tau\in G^\sharp$ of order $\bar\tau=pq$ not dividing a class of order p^2q but such that p^2 divides the numerator of $|G|/|C_\tau|$. Then any representative g of τ lies in a group of type $C_{pq}\times C_p$. This criterion is satisfied particularly often for $p=2$ and some odd prime q.

7. Comparing an algebra with its splitting field

In this section we return to a very concrete setting, considering types (G, ϕ_i, ϕ_r) where ϕ_i comes from a faithful permutation representation $i : G \subseteq S_n$ and ϕ_r is the regular character. Thus we are considering algebras $K = K_i$ of a specified Galois type compared with their splitting fields $K^{\mathrm{gal}} = K_r$.

In Section 7A we introduce explicit notation for comparing two algebras and in Section 7B we explain how it is sometimes best to highlight root discriminants $\mathfrak{d}$ rather than discriminants $\mathfrak{D}$. The tame-wild principle in the notation set up then takes the following form:

$$\mathfrak{d}_{K^{\mathrm{gal}}/F}^{\underline{\alpha}(G,\phi_i,\phi_r)} \mid \mathfrak{d}_{K/F} \mid \mathfrak{d}_{K^{\mathrm{gal}}/F}^{\underline{\omega}(G,\phi_i,\phi_r)}.$$

We observe in Section 7C that the right divisibility often trivially holds. In Section 7D, we give four examples where it holds nontrivially and one where it fails to hold. In Section 7E we show that the left divisibility always holds, and discuss applications to number field tabulation.

7A. ***Generalities.*** The case $r = 2$ of just two algebras deserves special attention for at least three reasons. First, hulls $T_+'(G, \phi_1, \phi_2) \subset \mathbb{R}^1$ are intervals while hulls for larger r can have up to $|G^{\natural 0}|$ vertices. Second, the inequality for each face of any $T_+'(G, \phi_1, \ldots, \phi_r)$ also comes from some $T_+'(G, \psi_1, \psi_2)$ with the new characters ψ_j being certain sums of the old characters ϕ_i. Third, it is the case which applies most directly to number field tabulation.

To present results coming from $r = 2$ as explicitly as possible, we let $\alpha = \alpha(G, \phi_1, \phi_2)$ and $\omega = \omega(G, \phi_1, \phi_2)$ be the left and right endpoints of the interval $T_+'(G, \phi_1, \phi_2)$. The tame-wild principle says that all local exponents satisfy

$$\alpha c_{\mathfrak{p}}(K_2) \le c_{\mathfrak{p}}(K_1) \le \omega c_{\mathfrak{p}}(K_2). \tag{7-1}$$

In this $r = 2$ setting, the tame-wild principle breaks cleanly into two parts: the left and right tame-wild principles respectively say that the left and right inequalities in (7-1) always hold. Similarly, one has the perhaps larger inertial interval $[\tilde{\alpha}, \tilde{\omega}]$ and the perhaps even larger broad interval $[\hat{\alpha}, \hat{\omega}]$.

To transfer the additive inequalities (7-1) into the multiplicative language of divisibility, we make use of the following formalism. Note that the torsion-free group $\mathcal{I}$ of fractional ideals of a local or global number field F embeds into its tensor product over $\mathbb{Z}$ with $\mathbb{Q}$, a group we write as $\mathcal{I}^{\mathbb{Q}}$ to account for the fact that $\mathcal{I}$ is written multiplicatively. In $\mathcal{I}^{\mathbb{Q}}$, as our notation indicates, general rational exponents on ideals are allowed. Then (7-1) corresponds to

$$\mathfrak{D}_{K_2/F}^{\alpha} \mid \mathfrak{D}_{K_1/F} \mid \mathfrak{D}_{K_2/F}^{\omega}, \tag{7-2}$$

which makes sense for both local and global number fields. In this formalism, the relations of the introductory example take the form $\mathfrak{D}_{K_6/F}^{1/3} \mid \mathfrak{D}_{K_5/F} \mid \mathfrak{D}_{K_6/F}$.

7B. ***Mean-root normalization and the comparison interval.*** It is sometimes insightful to switch to a slightly different normalization. We call this normalization mean-root normalization, with "mean" capturing how additive quantities are renormalized and "root" capturing how multiplicative quantities are renormalized.

If K/F has degree n and discriminant $\mathfrak{D}_{K/F}$ then its *root discriminant* is by definition $\mathfrak{d}_{K/F} = \mathfrak{D}_{K/F}^{1/n}$. To make this shift in our formalism, we simply replace all permutation characters ϕ_i by the scaled-down quantities $\underline{\phi}_i = \phi_i/\phi_i(1)$. One has mean tame conductors $\underline{c}_\tau(\phi_i) = (a_\tau, \underline{\phi}_i)$ as well as their analogs $\underline{c}_I(\phi_i) = (a_I, \underline{\phi}_i)$ and $\underline{\hat{c}}_\tau(\phi_i) = (\hat{a}_\tau, \underline{\phi}_i)$. We always indicate this alternative convention by underlining. Thus the mean-root normalized tame hull for two characters indexed by dimension is $\underline{T}'_+(G, \phi_n, \phi_m) = [\underline{\alpha}, \underline{\omega}]$ where $\underline{\alpha} = m\alpha/n$ and $\underline{\omega} = m\omega/n$. The divisibility relation (7-2) becomes $\mathfrak{d}_{K_m/F}^{\underline{\alpha}} \mid \mathfrak{d}_{K_n/F} \mid \mathfrak{d}_{K_m/F}^{\underline{\omega}}$.

The *comparison interval* $[\underline{\alpha}, \underline{\omega}]$ just introduced supports an intuitive understanding of how ramification in K_n/F and K_m/F relate to each other. Suppose, for example, that $K_m/K_n/F$ is a tower of fields so that one has the standard divisibility relation

$$\mathfrak{d}_{K_n/F} \mid \mathfrak{d}_{K_m/F}. \tag{7-3}$$

Then, assuming K_m/F is actually ramified, the ratio

$$\frac{\log|\mathfrak{d}_{K_n/F}|}{\log|\mathfrak{d}_{K_m/F}|} \in [0, 1]$$

can be understood as the fraction of ramification in K_m/F which is seen already in K_n/F. If the corresponding tame-wild principle holds, then this quantity is guaranteed to be in $[\underline{\alpha}, \underline{\omega}]$.

The mean-root normalization introduces a sense of absolute scale, with the number one playing a prominent role, as illustrated by the preceding paragraph and the next three subsections. Assuming $\phi_n - \phi_m$ is not a constant, one always has strict inequality $\underline{\alpha} < \underline{\omega}$. The failure of resolvent constructions from (G, ϕ_n)-fields to (G, ϕ_m)-fields to preserve ordering by absolute discriminants is, roughly speaking, measured by the length of $[\underline{\alpha}, \underline{\omega}]$. For ϕ_n and ϕ_m coming from faithful transitive permutation representations, a very common situation is $\underline{\alpha} \le 1 \le \underline{\omega}$. This tendency gets stronger as n and m increase to $|G|$. For example, for $(A_5, \phi_{20}, \phi_{30})$ the partition matrix is

$$\begin{pmatrix} 2^{10} & 3^6 1^2 & 5^4 \\ 2^{14} 1^2 & 3^{10} & 5^6 \end{pmatrix},$$

and the comparison interval works out to $\left[\frac{9}{10}, \frac{15}{14}\right] \approx [0.90, 1.07]$.

7C. ***The right tame-wild principle often holds for*** (G, ϕ_i, ϕ_r)**.** Applying (7-3) in our setting gives

$$\mathfrak{d}_{K/F} \mid \mathfrak{d}_{K^{\mathrm{gal}}/F} \tag{7-4}$$

when K is a field. This relation holds also when K is an algebra, as can be seen by expressing K as a product of fields and comparing each factor to the field K^{gal}.

The critical quantity is simply expressed as

$$\underline{\omega}(G, \phi_i, \phi_r) = \max_{\tau \in G^{\sharp 0}} \frac{\underline{c}_\tau(\phi_i)}{\underline{c}_\tau(\phi_r)}. \tag{7-5}$$

The denominator depends only on the order $\bar{\tau}$ of τ via $\underline{c}_\tau(\phi_r) = (\bar{\tau} - 1)/\bar{\tau}$. For the more complicated numerator, one has $\underline{c}_\tau(\phi_i) \le (\bar{\tau} - 1)/\bar{\tau}$, with equality if and only if the partition $\lambda_\tau(\phi_i)$ has the form $\bar{\tau}^{n/\bar{\tau}} = \bar{\tau} \dots \bar{\tau}$. A permutation is *semiregular* if all cycles have the same length. Therefore $\underline{\omega}(G, \phi_r, \phi_i) \le 1$, with equality if and only if G contains a nonidentity element which is semiregular. Summarizing:

Proposition 7.1. *Let $G \subseteq S_n$ be a permutation group containing a nonidentity semiregular element, ϕ_i the given permutation character, and ϕ_r the regular character. Then the right tame-wild principle holds for (G, ϕ_i, ϕ_r) with*

$$\underline{\omega}(G, \phi_i, \phi_r) = 1.$$

However, this principle is nothing more than the classical statement that for any (K, K^{gal}) of type (G, ϕ_i, ϕ_r), one has $\mathfrak{d}_{K/F} \mid \mathfrak{d}_{K^{\mathrm{gal}}/F}$.

7D. ***Elusive groups.*** In the global setting, we are mainly interested in the case when K is a field and thus G is transitive. A transitive permutation group which does not contain a nonidentity semiregular element is called an *elusive group* [Cameron et al. 2002]. So Proposition 7.1 is the best statement for nonelusive transitive groups, but the situation needs to be investigated further for elusive groups.

Elusive groups are aptly named in that they are relatively rare. The smallest n for which S_n contains an elusive group is $n = 12$. There are five elusive groups in S_{12} up to permutation equivalence, listed in Table 1, all subgroups of the Mathieu group M_{11} in its transitive degree twelve realization 12T272. Here and in the sequel we use the T-notation for transitive permutation groups introduced in [Conway et al. 1998] and available online in several places, including [LMFDB 2013].

The following proposition treats these five groups.

Proposition 7.2. *The right tame-wild principle for (G, ϕ_i, ϕ_r) holds for the elusive groups* 12T46, 12T84, 12T181, *and* 12T272 *with $\underline{\omega}(G, \phi_i, \phi_r) = \frac{20}{21}$. Thus the strengthening $\mathfrak{d}_{K/F} \mid \mathfrak{d}^{20/21}_{K^{\mathrm{gal}}/F}$ of* (7-5) *holds for these groups. For* 12T47, *one has $\underline{\omega}(12\mathrm{T}47, \phi_i, \phi_r) = \frac{8}{9}$. Extensions (K_i, K_r) from* (7-6) *give an counterexample to the tame-wild principle over $\mathbb{Q}(\sqrt{-3})$, but there is no counterexample over $\mathbb{Q}$.*

12T46 $\cong C_3^2 : Q_8$	√	√	√			√			
12T47 $\cong M_9$	√	√	√						
12T84 $\cong C_3^2 : \hat{Q}_8$	√	√	√		√	√			
12T181 $\cong M_{10}$	√	√	√	√		√			
12T272 $\cong M_{11}$	√	√	√	√	√	√	√		
τ	2A	3A	4A	5A	6A	8A	11A	Q_8	I
$\lambda_\tau(\phi_i)$	$2^4 1^4$	$3^3 1^3$	$4^2 2^2$	$5^2 1^2$	6321	84	(11)1	84	
$c_\tau(\phi_i)$	4	6	8	8	8	10	10	10	≤ 10
$\underline{c}_\tau(\phi_r)$	1/2	2/3	3/4	4/5	5/6	7/8	10/11	7/8	$(\|I\|-1)/\|I\|$
$\underline{c}'_\tau$	2/3	3/4	8/9	5/6	4/5	20/21	11/12	20/21	$\leq 20/21$

Table 1. Information used in the proof of Proposition 7.2.

Proof. The part below the line of Table 1 supports applying the broad and inertial methods for 12T272 $\cong M_{11}$. Thus the line labeled τ lists out the seven elements of 12T272$^{\sharp 0}$. The next two lines gives the corresponding dodecic partitions $\lambda_\tau(\phi_i)$ and conductors $c_\tau(\phi_i) = 12\underline{c}_\tau(\phi_i)$ respectively. The next lines give

$$\underline{c}_\tau(\phi_r) = \frac{c_\tau(\phi_r)}{|M_{11}|} \quad \text{and} \quad \underline{c}'_\tau = \frac{\underline{c}_\tau(\phi_i)}{\underline{c}_\tau(\phi_r)}.$$

Thus the comparison interval is $[\underline{\alpha}, \underline{\omega}] = \left[\frac{2}{3}, \frac{20}{21}\right]$.

One inertial subgroup of 12T272 is Q_8, which has orbit partition 84. As indicated in the second-to-last column of Table 1, its associated quantity is $\underline{c}'_{Q_8} = \frac{20}{21}$, which is the right endpoint of $\left[\frac{2}{3}, \frac{20}{21}\right]$. In general, the difficulty with the inertial method is that there can be many inertial N-subgroups I to inspect. However here we can treat them all at once as follows. Since none of the elusive groups from the complete list are themselves inertial groups, I must act intransitively and so $c_I(\phi_i) \leq 10$. Also $\underline{c}_I(\phi_r) = (|I|-1)/|I| \geq \frac{7}{8}$, since N-groups have order at least 8. So, as indicated by the table, $\underline{c}'_I \leq \frac{20}{21}$. Thus our initial case Q_8 was in fact the worst case, and the right tame-wild principle holds for (M_{11}, ϕ_i, ϕ_r).

The part of the table above the line gives the partitions which arise for all the G, as a subset of those that we have listed for 11T272. The smaller groups 12T46, 12T84, and 12T181 still have elements of cycle type 84, and so the same argument goes through for them, proving the tame-wild principle for (G, ϕ_i, ϕ_r) in these cases. Note that our uniform treatment of all the G uses that $\underline{c}_\tau(\phi_r)$ is independent of G; in contrast, the unnormalized quantity $c_\tau(\phi_r)$ depends on G.

A counterexample seems likely for $G =$ 12T47 because it contains Q_8 and its comparison interval is only $[\underline{\alpha}, \underline{\omega}] = \left[\frac{2}{3}, \frac{8}{9}\right]$, with $\frac{8}{9}$ being considerably less than $\frac{20}{21}$. However, as discussed in reference to (6-2), there are no quaternionic extensions

of $\mathbb{Q}_2$ giving counterexamples to the universal tame-wild principle for Q_8. Other candidate I do not work either, and we are forced to leave $\mathbb{Q}$ as a ground field.

Our counterexample comes from fields $K_n = \mathbb{Q}[x]/f_n(x)$ with discriminants $D_n \in \mathbb{Z}$ and Galois groups $G_n = \mathrm{Gal}(K_n^{\mathrm{gal}}/\mathbb{Q})$ as follows:

n	$f_n(x)$	D_n	G_n
8	$x^8 + 6x^4 - 3$	$-2^{16}3^7$	$\hat{Q}_8$
9	$x^9 - 3x^8 + 18x^5 + 18x^4 - 27x + 9$	$-2^{16}3^{15}$	9T19
12	$x^{12} - 6x^{10} - 4x^9 + 12x^7 - 36x^5 + 30x^4 + 8x^3 - 8$	$2^{22}3^{18}$	12T84

(7-6)

The overgroup 12T84 $\cong C_3^2 : \hat{Q}_8 \supset$ 12T47 was chosen because it contains not just Q_8 but also $\hat{Q}_8$. The nonic group 9T19 is a lower degree realization of 12T84, where the isomorphism with $C_3^2 : \hat{Q}_8$ is naturally realized. The field K_8 was chosen as a strong candidate from which to build a counterexample, because $\mathrm{Gal}(K_8^{\mathrm{gal}}/\mathbb{Q})$ is its own decomposition group with slope content $[2, 2, 2.5]^2$ as in (6-2). The field K_9 was extracted from the database [Jones and Roberts $\geq$ 2014] as a 9T19 field with K_8 as a resolvent, and then K_{12} was obtained from K_9 by resolvent calculations.

The splitting field K_{12}^{gal} contains $\mathbb{Q}(\sqrt{-3})$ with $\mathrm{Gal}(K_{12}^{\mathrm{gal}}/\mathbb{Q}(\sqrt{-3})) = 12\mathrm{T}47$ by construction. The root discriminant of K_{12}^{gal} is $2^2 3^{127/72}$, as computed by using the website of [Jones and Roberts 2006] to analyze ramification in $K_9/\mathbb{Q}$. Here the exponent 2 can be confirmed from a standard computation associated with the slope content $[2, 2, 2.5]^2$, namely $\frac{2}{8} + \frac{2}{4} + \frac{2.5}{2} = 2$. On the other hand K_{12} has root discriminant $2^{22/12}3^{18/12}$. The quotient $\frac{22}{12}/2 = \frac{11}{12}$ is to the right of the root-normalized tame hull $[\underline{\alpha}, \underline{\omega}] = \left[\frac{2}{3}, \frac{8}{9}\right]$, giving a counterexample to the right tame-wild principle for $(12\mathrm{T}47, \phi_i, \phi_r)$ over $\mathbb{Q}(\sqrt{-3})$. □

7E. ***The left tame-wild principle always holds for*** (G, ϕ_i, ϕ_r)**.** The following theorem shows that an important part of the tame-wild principle holds for all finite groups G.

Theorem 7.3. *Let $G \subseteq S_n$ be any permutation group, ϕ_i the given permutation character, and ϕ_r the regular character. Let $\mathcal{F}(G)$ be the maximal number of fixed points of a nonidentity element of G. Then the left tame-wild principle holds for (G, ϕ_i, ϕ_r) with*

$$\underline{\alpha}(G, \phi_i, \phi_r) = 1 - \frac{\mathcal{F}(G)}{n}.$$

Thus for any (K, K^{gal}) of type (G, ϕ_i, ϕ_r), one has $\mathfrak{d}_{K^{\mathrm{gal}}/F}^{1-\mathcal{F}(G)/n} \mid \mathfrak{d}_{K/F}$.

Proof. We apply the broad method. Let τ be an arbitrary element of $G^{\sharp 0}$ and call its order t. Consider the corresponding column $\binom{\lambda}{\Lambda}$ of the partition matrix $P(G, \phi_i, \phi_r)$. Then $\lambda = \prod_{k=1}^{t} k^{m_k}$ is some partition of n and $\Lambda = t^{|G|/t}$ is the corresponding partition of $|G|$.

The projective matrices $T'(G, \phi_i, \phi_r)$ and $\hat{T}'(G, \phi_i, \phi_r)$ have just one row each. The entries in the τ column are respectively $c(\lambda)/c(\Lambda)$ and $\hat{c}(\lambda)/\hat{c}(\Lambda)$. Their difference is positive:

$$\begin{aligned}\frac{\hat{c}(\lambda)}{\hat{c}(\Lambda)} - \frac{c(\lambda)}{c(\Lambda)} &= \frac{\sum_{k=2}^{t} m_k k}{|G|} - \frac{\sum_{k=2}^{t} m_k (k-1)}{|G|(t-1)/t} \\ &= \frac{1}{|G|}\sum_{k=2}^{t} m_k \frac{k(t-1)}{t-1} - \frac{1}{|G|}\sum_{k=2}^{t} m_k \frac{t(k-1)}{t-1} \\ &= \frac{1}{|G|}\sum_{k=2}^{t} m_k \frac{(kt-k)-(kt-t)}{t-1} \\ &= \frac{1}{|G|}\sum_{k=2}^{t} m_k \frac{t-k}{t-1} \geq 0.\end{aligned}$$

Thus $\hat{c}(\lambda)/\hat{c}(\Lambda) \geq c(\lambda)/c(\Lambda)$, and so certainly all the $\hat{c}(\lambda)/\hat{c}(\Lambda)$ are at least

$$\alpha(G, \phi_i, \phi_r) = \min_{\tau \in G^{\sharp 0}} \frac{c_\tau(\lambda)}{c_\tau(\Lambda)}.$$

Thus the left tame-wild principle holds for (G, ϕ_i, ϕ_r), and moreover we can compute $\alpha(G, \phi_i, \phi_r)$ using $\hat{c}_\tau$ rather than c_τ, giving

$$\alpha(G, \phi_i, \phi_r) = \min_{\tau \in G^{\sharp 0}} \frac{\hat{c}_\tau(\lambda)}{\hat{c}_\tau(\Lambda)} = \min_{\tau \in G^{\sharp 0}} \frac{n - m_1}{|G|} = \frac{n - \mathscr{F}(G)}{|G|}.$$

Switching to the mean-root normalization gives $\underline{\alpha}(G, \phi_i, \phi_r) = 1 - \mathscr{F}(G)/n$. □

Number field tabulation. For certain solvable transitive groups $G \subset S_n$, the techniques of [Jones and Wallington 2012] let one compute all degree n fields K of type G where $|\mathfrak{d}_{K^{\mathrm{gal}}/\mathbb{Q}}|$ is at most some constant β. Then the theorem just proved can be applied through its corollary $|\mathfrak{d}_{K^{\mathrm{gal}}/\mathbb{Q}}^{1-\mathscr{F}(G)/n}| \leq |\mathfrak{d}_{K/\mathbb{Q}}|$ to obtain all K with $|\mathfrak{d}_{K/\mathbb{Q}}|$ at most $B = \beta^{1-\mathscr{F}(G)/n}$. This computation is carried out in [Jones 2013] for the primitive nonic groups 9T9, 9T14, 9T15, 9T16, 9T23, and 9T27 to obtain the corresponding nonic fields with smallest absolute discriminant. This particular application served as the catalyst for the present paper.

8. Examples and counterexamples

The positive and negative results of the previous sections give one a good idea of the extent to which the tame-wild principle holds and how it can be applied. We now refine this picture, by considering various $(G, \phi_1, \ldots, \phi_r)$ of interest and determining whether the tame-wild principle holds. In Section 8A, we give examples illustrating the broad method and the inertial method. In Section 8B, we conclude

by arguing that counterexamples to the tame-wild principle from pairs $(K_1, K_2)/\mathbb{Q}$ of number fields are not easily found, but present one such counterexample with Galois group 12T112 of order 192.

8A. ***The broad and inertial methods.*** We illustrate the two methods of Section 4B with positive results for three N-groups.

The broad method for $(\mathrm{Aff}_3(\mathbb{F}_2), \phi_7, \phi_8, \phi_{8a}, \phi_{8b})$. The group $\mathrm{Aff}_3(\mathbb{F}_2)$ provides a simple illustration of the broad method in the setting $r = 4$. It has five nontrivial small permutation representations $\rho_{7a}, \rho_{7b}, \rho_8, \rho_{8a}, \rho_{8b}$, with images the permutation groups 7T5, 7T5, 8T37, 8T48, 8T48. The first three representations are through the quotient $\mathrm{GL}_3(\mathbb{F}_2) \cong \mathrm{PGL}_2(\mathbb{F}_7)$ while the last two are faithful. The representations ρ_{7a} and ρ_{7b} share a common character ϕ_7. They are thus arithmetically equivalent and we call them identical twins. The representations ρ_{8a} and ρ_{8b} have different characters ϕ_{8a} and ϕ_{8b} and so we call them fraternal twins. The four characters ϕ_7, ϕ_8, ϕ_{8a}, and ϕ_{8b} are linearly independent.

Figure 3 first presents the partition matrix $P = P(\mathrm{Aff}_3(\mathbb{F}_2), \phi_7, \phi_8, \phi_{8a}, \phi_{8b})$ and the broad and tame matrices derived from it. For visualization purposes, it then drops consideration of ϕ_8. After this projection, it plots the columns of $\hat{T}'$ as +s and those of T' as •s. Since the +s are in the hull T'_+ of the •s, the tame-wild principle holds for $(\mathrm{Aff}_3(\mathbb{F}_2), \phi_7, \phi_{8a}, \phi_{8b})$. Working more algebraically, as described in Section 4C, one can verify the analogous convexity assertion in the presence of ϕ_8, giving the first sentence of the following result.

Proposition 8.1. *The tame-wild principle holds for* $(\mathrm{Aff}_3(\mathbb{F}_2), \phi_7, \phi_8, \phi_{8a}, \phi_{8b})$. *In particular, to find all* 8T48 *extensions with* $|\mathfrak{D}_{K_{8a}/F}| \leq B$, *one need look only at* 7T5 *extensions with* $|\mathfrak{D}_{K_7/F}| \leq B$ *and select from among the octic resolvents of their* 14T34 *quadratic overfields.*

The second sentence comes from an understanding of the algebraic meaning of Figure 3. Associate variables u, a, and b to ϕ_7, ϕ_{8a}, and ϕ_{8b} respectively. The four sides of the trapezoid $T'_+(\mathrm{Aff}_3(\mathbb{F}_2), \phi_7, \phi_{8a}, \phi_{8b})$ in the drawn $(u/b, a/b)$ plane correspond to the four faces of the cone $T_+(\mathrm{Aff}_3(\mathbb{F}_2), \phi_7, \phi_{8a}, \phi_{8b})$ in (u, a, b)-space. These four faces correspond to the four inequalities on local exponents on the left and they translate into divisibility relations among either local or global discriminants on the right:

$$u \leq a \leq u + b, \qquad \mathfrak{D}_{K_7/F} \mid \mathfrak{D}_{K_{8a}/F} \mid \mathfrak{D}_{K_7/F}\mathfrak{D}_{K_{8b}/F},$$
$$u \leq b \leq u + a, \qquad \mathfrak{D}_{K_7/F} \mid \mathfrak{D}_{K_{8b}/F} \mid \mathfrak{D}_{K_7/F}\mathfrak{D}_{K_{8a}/F}.$$

For tabulations of all extensions K_{8a}/F with $|\mathfrak{D}_{K_{8a}/F}|$ at most some bound B, the procedure referred to by the proposition is to look for all K_7/F with $|\mathfrak{D}_{K_7/F}| \leq B$,

τ	2A	2B	2C	3A	4A	4B	4C	6A	7A
	1^7	22111	22111	331	22111	421	421	331	7
P	1^8	2222	2222	3311	2222	44	44	3311	71
	2222	221111	2222	3311	44	4211	44	62	71
	2222	2222	221111	3311	44	44	4211	62	71
	0	4	4	6	4	6	6	6	7
$\hat{T}$	0	8	8	6	8	8	8	6	7
$+$	8	4	8	6	8	6	8	8	7
	8	8	4	6	8	8	6	8	7
	0	2	2	4	2	4	4	4	6
T	0	4	4	4	4	6	6	4	6
$\bullet$	4	2	4	4	6	4	6	6	6
	4	4	2	4	6	6	4	6	6

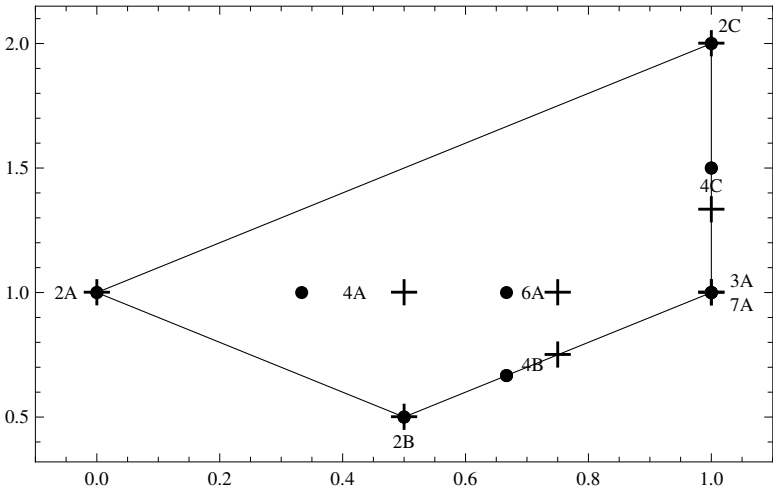


Figure 3. Top: The partition matrix, broad matrix, and tame matrix for $(\mathrm{Aff}_3(\mathbb{F}_2), \phi_7, \phi_8, \phi_{8a}, \phi_{8b})$. Bottom: the broad hull and tame hulls coinciding after removing ϕ_8 from consideration, proving the tame-wild principle for $(\mathrm{Aff}_3(\mathbb{F}_2), \phi_7, \phi_{8a}, \phi_{8b})$.

take suitable square roots to pass from 7T5 fields to 14T34 fields, and then use resolvents to obtain the desired 8T48 fields.

The inertial method for $(S_6, \phi_{6a}, \phi_{6b}, \phi_{10})$. The group S_6 has three faithful permutation representations of degree at most ten: two sextic ones ρ_{6a} and ρ_{6b} interchanged by the outer automorphism of S_6, and a decic one ρ_{10} coming from the exceptional isomorphism $S_6 \cong \mathrm{PSL}_2(\mathbb{F}_9)$. $\mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3) = 10\mathrm{T}32 \subset S_{10}$.

	2A	2B	2C	3A	3B	4A	4B	5A	6A	6B
P	222	21^4	2211	33	3111	42	411	51	6	321
	21^4	222	2211	3111	33	42	411	51	321	6
	$2^3 1^4$	$2^3 1^4$	$2^4 1^2$	3331	3331	4411	442	55	631	631
$\hat{T}$	6	2	4	6	3	6	4	5	6	5
+	2	6	4	3	6	6	4	5	5	6
	6	6	8	9	9	8	10	10	9	9
T	3	1	2	4	2	4	3	4	5	3
•	1	3	2	2	4	4	3	4	3	5
	3	3	4	6	6	6	7	8	7	7

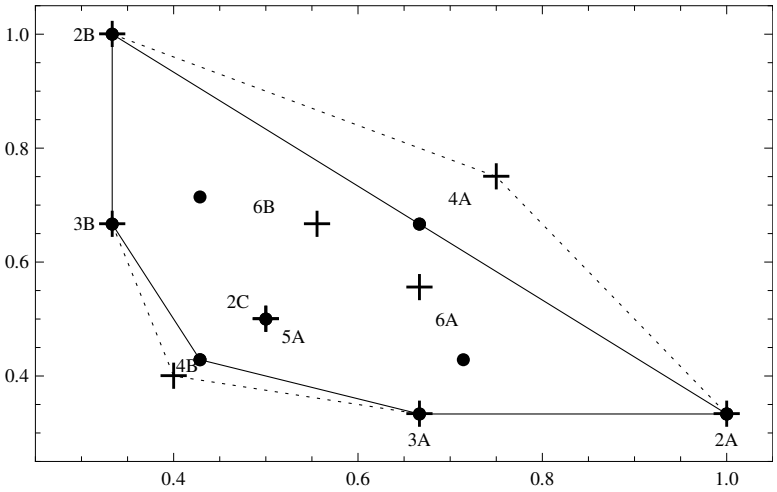


Figure 4. Top: The partition matrix, broad matrix, and tame matrix for $(S_6, \phi_{6a}, \phi_{6b}, \phi_{10})$. Bottom: the broad hull strictly containing the tame hull, showing that the broad method does not suffice to prove the tame-wild principle for $(S_6, \phi_{6a}, \phi_{6b}, \phi_{10})$.

Figure 4 presents our standard analysis of the situation. Since some +s are outside of the tame hull $T'_+(S_6, \phi_{6a}, \phi_{6b}, \phi_{10})$, the broad method does not suffice for $(S_6, \phi_{6a}, \phi_{6b}, \phi_{10})$. However after projection to the horizontal axis, the +s are indeed in the convex hull of the •s, so that the broad method establishes the tame-wild principle for $(S_6, \phi_{6a}, \phi_{10})$. Also the ratios a/b for the + points (a, b) are within the interval $\left[\frac{1}{3}, 3\right]$ formed by the ratios for the • points, proving the tame-wild principle for $(S_6, \phi_{6a}, \phi_{6b})$.

In fact, the tame-wild principle is true for $(S_6, \phi_{6a}, \phi_{6b}, \phi_{10})$ as follows. The only inertial subgroups not covered by previous considerations are $I_1 = D_4 \times C_2$

and the twin pair $(I_2, I_3) = (A_4 \times C_2, 6\text{T}6)$. The orbit partitions in the three cases are (42, 42, 442), (42, 6, 64), and (6, 42, 64). The associated conductor vectors are then (4, 4, 7), (4, 5, 8), and (5, 4, 8). Their projectivized versions are $(\frac{4}{7}, \frac{4}{7})$, $(\frac{1}{2}, \frac{5}{8})$, and $(\frac{5}{8}, \frac{1}{2})$. Since these points are visibly in $T'_+(S_6, \phi_{6a}, \phi_{6b}, \phi_{10})$, the tame-wild principle holds. We have given this argument to illustrate how the inertial method typically applies. However in this case the inertial groups I_2 and I_3 could also have been treated by using the techniques from Section 6, as in fact the universal tame-wild principle holds for $A_4 \times C_2$.

Summarizing, we have proved the first sentence:

Proposition 8.2. *The tame-wild principle holds for* $(S_6, \phi_{6a}, \phi_{6b}, \phi_{10})$. *In particular to find all decic* S_6*-extensions with* $|\mathfrak{D}_{K_{10}/F}| \le B$, *one need only look at sextic* S_6*-extensions with* $|\mathfrak{D}_{K_{6a}/F}| \le B^{2/3}$ *and select from among their decic resolvents.*

For the second sentence, note first that the locations of the rightmost and highest points of the tame hull in Figure 4 respectively correspond to the equivalent statements $\mathfrak{D}_{K_{6a}/F} \mid \mathfrak{D}_{K_{10}/F}$ and $\mathfrak{D}_{K_{6b}/F} \mid \mathfrak{D}_{K_{10}/F}$. Each of these says that to find all decics with absolute discriminant $\le B$, it suffices to look at all sextics up to that bound. A considerable improvement is to see that the long diagonal boundary between them corresponds to $\mathfrak{D}_{K_{6a}/F}\mathfrak{D}_{K_{6b}/F} \mid \mathfrak{D}_{K_{10}/F}^{4/3}$ which implies the statement.

The broad method for $(W(E_6), \phi_{27}, \phi_{36}, \phi_{40a}, \phi_{40b}, \phi_{45})$. As we have seen in Section 7E and by the earlier examples of this subsection, the broad method works well in the setting $r = 2$. As r increases, the difference between a_τ and $\hat{a}_\tau$ becomes more visible, and the broad method often fails even when the tame-wild principle is true, as we just saw for $(S_6, \phi_{6a}, \phi_{6b}, \phi_{10})$.

A clear illustration of the effectiveness of the broad method and its decay with increasing r comes from the Weyl group $W(E_6)$ of order $51840 = 2^6 3^4 5$ and the permutation characters $\phi_{27}, \phi_{36}, \phi_{40a}, \phi_{40b}, \phi_{45}$ corresponding to five maximal subgroups [Conway et al. 1985]. The broad method immediately shows that the tame-wild principle for $(W(E_6), \phi_u, \phi_v)$ holds for all ten possibilities for $\{u, v\}$. From ten pictures like Figures 3 and 4, now quite involved since $|W(E_6)^{\sharp 0}| = 24$, the broad method establishes the tame-wild principle in exactly four of the ten cases $(W(E_6), \phi_u, \phi_v, \phi_w)$ as follows.

Proposition 8.3. *For* $\{u, v, w\} = \{27, 36, 40a\}, \{27, 40a, 40b\}, \{36, 40a, 40b\}$, *and* $\{36, 40b, 45\}$, *the tame-wild principle holds for* $(W(E_6), \phi_u, \phi_v, \phi_w)$.

Pursuing this situation further with the inertial method would be harder, because $W(E_6)$ has many 2-inertial and 3-inertial subgroups.

8B. ***Best counterexamples.*** Let G be a group for which the universal tame-wild principle fails. Then there exists a vector $v \in \mathbb{Q}(G^\sharp)$ for which the tame-wild principle fails for $(G, \langle v \rangle)$. There are infinitely many solutions to $\phi_1 - \phi_2 \in \langle v \rangle$

with the ϕ_i permutation characters. So any failure of the universal tame-wild principle can be converted to a failure in the setting (G, ϕ_1, ϕ_2) of the introduction. By switching ϕ_1 and ϕ_2 if necessary, it can be converted to a failure of the left tame-wild principle for (G, ϕ_1, ϕ_2).

However these counterexamples are not guaranteed to have immediate bearing on our applications to tabulating number fields. All that is asserted by the failure of the principle for (G, ϕ_1, ϕ_2) is that there exists a pair of local extensions $(K_1, K_2)/F$ of the given type with

$$\mathfrak{D}_{K_2/F}^{\alpha(G,\phi_1,\phi_2)} \mid \mathfrak{D}_{K_1/F} \tag{8-1}$$

not holding. More directly relevant would be global counterexamples with the ϕ_i both coming from faithful transitive permutation representations and the extensions K_i/F full in the sense of each having Galois group $\mathrm{Gal}(K^{\mathrm{gal}}/F)$ all of G. More demanding still is to ask for counterexamples of this sort with $F = \mathbb{Q}$. Finally, one can seek examples for which even the weaker numerical statement

$$|\mathfrak{D}_{K_2/\mathbb{Q}}|^{\alpha(G,\phi_1,\phi_2)} \leq |\mathfrak{D}_{K_1/\mathbb{Q}}| \tag{8-2}$$

fails. Examples of this explicit nature often do not exist for a given G, and even when they exist they can be hard to find. The rest of this subsection discusses the construction of global counterexamples built from one of the two local counterexamples with $I = Q_8$ with slope-content $[2, 2, \frac{5}{2}]^2$ from (6-2). There are several points of contact with Section 7D, but here we find counterexamples to (8-1) over $\mathbb{Q}$.

Inadequacy of $G = \hat{Q}_8$ as a source of global counterexamples. The group $\hat{Q}_8$ itself is not a source of global counterexamples of the sort we seek because it has only two transitive faithful permutation characters and the tame-wild principle holds for the corresponding type $(\hat{Q}_8, \phi_8, \phi_{16})$. To illustrate the best that can be done with this group, take

n	$f_n(x)$	D_n	G_n	$\lvert G_n\rvert$
8	$x^8 + 6x^4 - 3$	$-2^{16}3^7$	$\hat{Q}_8$	16
4	$x^4 + 6x^2 - 3$	$-2^6 3^3$	D_4	8
2	$x^2 + 3$	-3	C_2	2

(8-3)

The global and 2-adic Galois groups of $f_8(x)$ agree, and so one has this agreement for the resolvents $f_4(x)$ and $f_2(x)$ as well. The Galois groups G_n and the field discriminants D_n are as indicated. The fields $K_n = \mathbb{Q}[x]/f_n(x)$ belong to transitive characters ϕ_8, ϕ_4, and ϕ_2 of $\hat{Q}_8$.

Figure 5 is an analog of Figure 1, but now for $(\hat{Q}_8, \phi_8+\phi_2, \phi_8+\phi_4)$. The algebra pair $(K_8 \times K_2, K_8 \times K_4)/\mathbb{Q}$ yields the exponent pair $(a_2, b_2) = (16, 22)$ which is just outside the tame cone. So this pair of algebras indeed contradicts (8-1), but we are seeking counterexamples among pairs of fields.

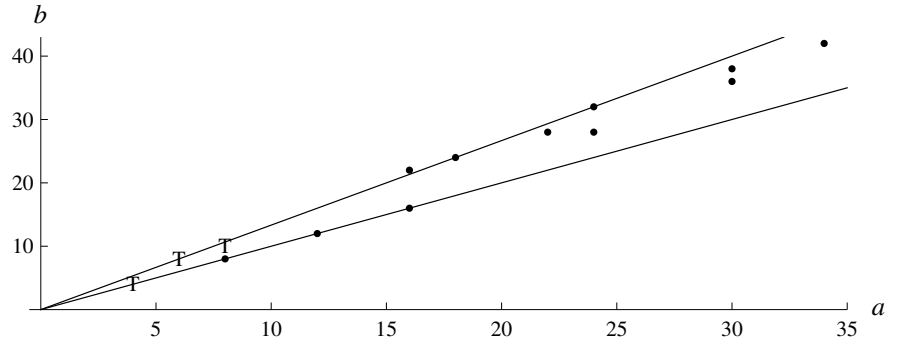


Figure 5. An analog of the introductory Figure 1 for the type $(\hat{Q}_8, \phi_8+\phi_2, \phi_8+\phi_4)$. The points are exactly all the possibilities for exponent pairs (a_2, b_2) from wild 2-adic ramification over $\mathbb{Q}_2$, and (16, 22) is just outside the tame cone.

Failure of the inertial method for $(M_{12}, \phi_{12a}, \phi_{12b})$. To get a better global counterexample corresponding to the same local counterexample, we need to replace $\hat{Q}_8$ by larger groups G containing it. An initial key observation is that the quaternion group Q_8 is the four-point stabilizer of the Mathieu group $M_{12} \subset S_{12}$ of order $12\cdot 11\cdot 10\cdot 9\cdot 8$ in its natural action, and also one has $Q_8 \subset \hat{Q}_8 \subset M_{12}$. On the one hand, the given character ϕ_{12a} of the Mathieu group has decomposition $\phi_8+\phi_2+2$ when restricted to $\hat{Q}_8$. On the other hand, there is a twin dodecic character ϕ_{12b} coming from the outer involution of M_{12}; its restriction to $\hat{Q}_8$ decomposes as $\phi_8+\phi_4$.

Further group-theoretic facts are necessary for this situation to give number fields as desired. First, the partition matrix and projective tame matrix of $(M_{12}, \phi_{12a}, \phi_{12b})$ are as follows:

τ	2A	2B	3A	3B	4A	4B	5A	6A	6B	8A	8B	10A	11AB	Q_8
$\lambda_\tau(\phi_{12a})$	2^6	$2^4 1^4$	$3^3 1^3$	3^4	$4^2 2^2$	$4^2 1^4$	$5^2 1^2$	6^2	6321	84	821^2	(10)2	(11)1	81^4
$\lambda_\tau(\phi_{12b})$	2^6	$2^4 1^4$	$3^3 1^3$	3^4	$4^2 1^4$	$4^2 2^2$	$5^2 1^2$	6^2	6321	821^2	84	(10)2	(11)1	84
c'_τ	1	1	1	1	$\frac{8}{6}$	$\frac{6}{8}$	1	1	1	$\frac{10}{8}$	$\frac{8}{10}$	1	1	$\frac{7}{10}$

Thus, in the language introduced in Section 8A, extensions $(K_{12a}, K_{12b})/F$ of full type $(M_{12}, \phi_{12a}, \phi_{12b})$ are fraternal twins, this being necessary for our purposes. But they are near-identical in the sense that the interval $[\alpha, \omega]$ is small, being $\left[\frac{3}{4}, \frac{4}{3}\right] = [0.75, 1.\overline{33}]$ here, rather than the intervals $\left[\frac{1}{2}, 2\right]$ and $\left[\frac{1}{3}, 3\right]$ seen in Section 8A for $\mathrm{Aff}_3(\mathbb{F}_2)$-twins and S_6-twins respectively. The orbit partitions of Q_8 are as indicated above, yielding $c_{Q_8}(\phi_{12a})/c_{Q_8}(\phi_{12b}) = \frac{7}{10} = 0.70$ which is outside the interval $[0.75, 1.\overline{33}]$. Thus the inertial method for proving the tame-wild principle fails here.

Failure of the tame-wild principle for $(M_{12}, \phi_{12a}, \phi_{12b})$. Computing with the slopes $[2, 2, 2.5]^2$, the 8s in the last column above give discriminant exponent $3\cdot 2+4\cdot 2.5 = 16$ while the 4 gives the discriminant exponent $3 \cdot 2 = 6$. So the ratio of wild conductors is $\frac{16}{22} = 0.\overline{72}$, which is still outside the interval $[0.75, 1.\overline{33}]$. Thus the tame-wild principle itself fails for $(M_{12}, \phi_{12a}, \phi_{12b})$.

Smaller groups. We have looked in several places, including the two-parameter family of [Malle 2000], for twin pairs (K_{12a}, K_{12b}) of M_{12} fields with the needed quaternionic 2-adic behavior. We did not find any, and so we consider smaller groups as follows as potential sources of counterexamples:

$$\begin{array}{ccccccccccccc}
 & & & & \mathrm{Aff}_2(\mathbb{F}_3) & & P\Gamma L_2(\mathbb{F}_9) & & & & & & \\
 & & & & \| & & \| & & & & & & \\
\hat{Q}_8 & \subset & & & M_9.2 & \subset & M_{10}.2 & & & & & & \\
\cup & & & & \cup & & \cup & & & & & & \\
Q_8 & \subset & & & M_9 & \subset & \boxed{M_{10}} & \subset & \boxed{M_{11}} & \subset & \boxed{M_{12}} & \supset & \boxed{T} \supset P.
\end{array}$$

The four groups in the middle are boxed to stress that they appear in Proposition 8.4 below.

Proceeding from M_{12} to the left, the groups M_{11} and M_{10} contain $\hat{Q}_8$, since $\hat{Q}_8$ has orbit partition 8211. Thus, using $0.\overline{72} \notin [0.75, 1.\overline{33}]$ exactly as above, the tame-wild principle fails also for $(M_{11}, \phi_{11}, \phi_{12b})$ and $(M_{10}, \phi_{10}, \phi_{12b})$. Here the transitive permutation groups in question are respectively (11T6, 12T272) and (10T31, 12T181). The analog of Figure 5 for M_{10} and M_{11} has the same tame cone, but more dots. For M_{12} there are many more dots, and a symmetry appears with the cone doubling so that its bounding lines have slope $\frac{3}{4}$ and $\frac{4}{3}$ rather than 1 and $\frac{4}{3}$.

Moving further leftward to M_9 and Q_8 relates our current discussion to our earlier counterexamples. For M_9, the transitive groups are (9T14, 12T47). However now $\hat{Q}_8$ is not contained in M_9 and so we do not have counterexamples over $\mathbb{Q}$. However the counterexample for $(M_9, \phi_{12b}, \phi_{72})$ over $\mathbb{Q}(\sqrt{-3})$ from Section 7D also gives a counterexample for $(M_9, \phi_9, \phi_{12b})$, as always because the projectivized wild Artin conductor $0.\overline{72}$ is not in the tame hull $[0.75, 1.\overline{33}]$. Finally for $\hat{Q}_8$ itself we recover (8-3), now interpreted as an intransitive counterexample for $(Q_8, \phi_8, \phi_8 + \phi_4)$ over $\mathbb{Q}(\sqrt{-3})$.

The extended groups $M_9.2$ and $M_{10}.2$ corresponding to the pairs (9T19, 12T84) and (10T35, 12T220) are natural candidates to support examples over $\mathbb{Q}$ because they contain $\hat{Q}_8$. However they have orbit partitions 921 and (10)2 as subgroups of M_{12}. Computation in the column headed by 8B then has to be adjusted, with the 2 in 821^2 removed. The conductor ratio is then $\frac{7}{10}$ rather than $\frac{8}{10}$ and in fact the inertial method above works to prove the tame-wild principle for $(M_9.2, \phi_9, \phi_{12b})$ and $(M_{10}.2, \phi_{10}, \phi_{12b})$. This phenomenon illustrates the fundamental difficulty in

promoting local nontransitive counterexamples to global transitive ones with a larger group. While wild Artin conductor ratios, here $0.\overline{72}$ stay the same, tame hulls increase, here from [0.75, 1] for $\hat{Q}_8$ itself to $\left[\frac{7}{10}, \frac{7}{6}\right] = [0.70, 1.1\overline{6}]$ for $M_{10}.2$.

There are other good candidates for global Galois groups. The 2-Sylow subgroup P of M_{12} of order 2^6 is not good for us, because neither the given orbit decomposition nor its twin is transitive, both having orbit partition 84. However an overgroup T of order $2^6 \cdot 3 = 192$ is good, with (ϕ_{12a}, ϕ_{12b}) remaining a fraternal pair of type (12T112, 12T112). Our computations have shown that the tame-wild principle fails for (12T112, ϕ_{12a}, ϕ_{12b}).

Number fields. Constructing number fields with nonsolvable Galois groups and prescribed ramification remains a difficult problem despite the increasing attention it has been receiving recently. Just as we have not found M_{12} fields with the appropriate quaternionic ramification, we have also not found M_{11} or M_{10} fields.

In contrast, it is relatively easy to build solvable fields step by step, and we have found many explicit pairs $(K_{12a}, K_{12b})/\mathbb{Q}$ providing counterexamples to the tame-wild principle for (12T112, ϕ_{12a}, ϕ_{12b}). One such, with tame ramification at the prime number $q = 277$, is

$$f_{12a}(x) = x^{12} + 223x^{10} + 14856x^8 + 1784qx^6 + 38160qx^4 + 1712q^2x^2 + 9216q^2,$$
$$f_{12b}(x) = x^{12} + 202x^8 + 49qx^4 + 4q^2.$$

The discriminants are $D_{12a} = 2^{16}277^8$ and $D_{12b} = 2^{22}277^6$, with the tame prime 277 having ramification partitions $\mu_{12a} = 4^2 2^2$ and $\mu_{12b} = 4^2 1^4$.

By design, $\mathfrak{D}_{K_{12b}}^{0.75} \nmid \mathfrak{D}_{K_{12a}}$. However the tame ramification at 277 completely overwhelms the wild ramification at 2 in terms of magnitudes, and easily $|\mathfrak{D}_{K_{12b}}|^{.75} \leq |\mathfrak{D}_{K_{12a}}|$. Indeed $|\mathfrak{D}_{K_{12b}}|^{1.15} \approx |\mathfrak{D}_{K_{12a}}|$. To improve upon the counterexample (f_{12a}, f_{12b}), one would like examples with $D_{12a} = 2^{16}(p_1 \dots p_k)^6$ and $D_{12b} = 2^{22}(p_1 \dots p_k)^8$ so that (8-2) is contradicted as well. However no such counterexamples exist with $G = 12T112$, as the subgroup Q_8 together with all elements of type $4^2 1^4$ generate an index two subgroup of type 12T63 and this subgroup does not contain $\hat{Q}_8$. Partially summarizing:

Proposition 8.4. *The tame-wild principle for* $(G, \phi_{12a}, \phi_{12b})$ *fails for the groups* $G = M_{12}$, M_{11}, M_{10} *and* 12T112. *For* $G =$ 12T112 *the pair of number fields* (K_{12a}, K_{12b}) *contradicts the divisibility statement* (8-1), *but no pair with* $G =$ 12T112 *contradicts the numerical statement* (8-2).

The group-theoretic argument for 12T112 does not apply to the three M_n and we expect that there exist pairs (K_{12a}, K_{12b}) for them contradicting not only (8-1) but also (8-2). In general, a closer analysis of the exact range of applicability of the tame-wild principle would be interesting.

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