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Moritz Kerz and Shuji Saito

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Lefschetz theorem for abelian fundamental group with modulus

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We prove a Lefschetz hypersurface theorem for abelian fundamental groups allowing wild ramification along some divisor. In fact, we show that isomorphism holds if the degree of the hypersurface is large relative to the ramification along the divisor.

1. Statement of main results

Lefschetz hyperplane theorems represent an important technique in the study of Grothendieck's fundamental group $\pi_1(X)$ of an algebraic variety X (we omit base points for simplicity). Roughly speaking, one gets an isomorphism of the form

$$\iota_{Y/X} : \pi_1(Y) \xrightarrow{\sim} \pi_1(X)$$

for a suitable hypersurface section $Y \to X$ if $\dim(X) \geq 3$. Purely algebraic Lefschetz theorems for projective varieties satisfying certain regularity assumptions were developed in [SGA 2 1968]. The case of nonproper varieties X and Y is more intricate because one needs a precise control of the ramification at the infinite locus. We show in the present note that for the abelian quotient of the fundamental group a Lefschetz hyperplane theorem does in fact hold. Our basic technical ingredient is the higher-dimensional ramification theory of Brylinski, Kato and Matsuda, which is recalled in Section 2. We expect that there is a noncommutative analog of our Lefschetz theorem, which should have applications to ℓ-adic representations of fundamental groups, especially over finite fields as studied in [Esnault and Kerz 2012].

To formulate our main result, let X be a normal variety over a perfect field k, and let $U \subset X$ be an open subset such that $X \setminus U$ is the support of an effective Cartier divisor on X. Let D be an effective Cartier divisor on X with support in $X \setminus U$. We introduce the abelian fundamental group $\pi_1^{\mathrm{ab}}(X, D)$ as a quotient of $\pi_1^{\mathrm{ab}}(U)$ classifying abelian étale coverings of U with ramification bounded by D. More precisely, for an integral curve $Z \subset U$, let Z^N be the normalization of the closure of Z in X with $\phi_Z : Z^N \to X$, the natural map. Let $Z_\infty \subset Z^N$ be the finite set

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of points x such that $\phi_Z(x) \notin U$. Then $\pi_1^{\mathrm{ab}}(X, D)$ is defined as the Pontryagin dual of the group $\mathrm{fil}_D\, H^1(U)$ of continuous characters $\chi : \pi_1^{\mathrm{ab}}(U) \to \mathbb{Q}/\mathbb{Z}$ such that, for any integral curve $Z \subset U$, its restriction $\chi|_Z : \pi_1^{\mathrm{ab}}(Z) \to \mathbb{Q}/\mathbb{Z}$ satisfies the following inequality of Cartier divisors on Z^N:

$$\sum_{y \in Z_\infty} \mathrm{art}_y(\chi|_Z)[y] \le \phi_Z^* D,$$

where $\mathrm{art}_y(\chi|_Z) \in \mathbb{Z}_{\ge 0}$ is the Artin conductor of $\chi|_Z$ at $y \in Z_\infty$ and $\phi_Z^* D$ is the pullback of D by the natural map $\phi_Z : Z^N \to X$.

Such a global measure of ramification in terms of curves has been first considered by Deligne and Laumon; see [Laumon 1981].

Now assume that X is smooth projective over k (we fix a projective embedding) and that $C = X \setminus U$ is a simple normal crossing divisor. Let Y be a smooth hypersurface section such that $Y \times_X C$ is a reduced simple normal crossing divisor on Y, and write $\deg(Y)$ for the degree of Y with respect to the fixed projective embedding of X. Set $E = Y \times_X D$. Then one sees from the definition that the map $Y \cap U \to U$ induces a natural map

$$\iota_{Y/X} : \pi_1^{\mathrm{ab}}(Y, E) \to \pi_1^{\mathrm{ab}}(X, D).$$

Our main theorem says:

Theorem 1.1. *Assume that Y is* sufficiently ample *for (X, D) (see Definition 3.1). If $d := \dim(X) \ge 3$, $\iota_{Y/X}$ is an isomorphism. If $d = 2$, $\iota_{Y/X}$ is surjective.*

The prime-to-p part of the theorem is due to [Schmidt and Spieß 2000], where $p = \mathrm{ch}(k)$. Below we see that Y is sufficiently ample if $\deg(Y) \gg 0$.

Corollary 1.2. *Let X be a normal proper variety over a finite field k. Then $\pi_1^{\mathrm{ab}}(X, D)^0$ is finite, where*

$$\pi_1^{\mathrm{ab}}(X, D)^0 = \mathrm{Ker}\big(\pi_1^{\mathrm{ab}}(X, D) \to \pi_1^{\mathrm{ab}}(\mathrm{Spec}(k))\big).$$

Proof. In case X and $X \setminus U$ satisfy the assumption of Theorem 1.1, the corollary follows from the corresponding statement for curves. The finiteness in the curves case is a consequence of class field theory. For the general case, one can take by [de Jong 1996] an alteration $f : X' \to X$ such that X' and $X' \setminus U'$ with $U' = f^{-1}(U)$ satisfy the assumption of Theorem 1.1. Then the assertion follows from the fact that the map $f_* : \pi_1^{\mathrm{ab}}(U') \to \pi_1^{\mathrm{ab}}(U)$ has a finite cokernel. □

Corollary 1.2 can also be deduced from [Raskind 1995, Theorem 6.2]. It has recently been generalized to the noncommutative setting by Deligne; see [Esnault and Kerz 2012].

Theorem 1.1 is a central ingredient in our paper [Kerz and Saito 2013]. There we use it to construct a reciprocity isomorphism between a Chow group of zero cycles with modulus and the abelian fundamental group with bounded ramification. In fact, Theorem 1.1 allows us to restrict to surfaces in this construction.

2. Review of ramification theory

First we review local ramification theory. Let K denote a henselian discrete valuation field of $\mathrm{ch}(K) = p > 0$ with the ring $\mathbb{O}_K$ of integers and residue field κ. Let π be a prime element of $\mathbb{O}_K$ and $\mathfrak{m}_K = (\pi) \subset \mathbb{O}_K$ the maximal ideal. By the Artin–Schreier–Witt theory, we have a natural isomorphism for $s \in \mathbb{Z}_{\geq 1}$,

$$\delta_s : W_s(K)/(1-F)W_s(K) \xrightarrow{\sim} H^1(K, \mathbb{Z}/p^s\mathbb{Z}), \tag{2-1}$$

where $W_s(K)$ is the ring of Witt vectors of length s and F is the Frobenius. We have the Brylinski–Kato filtration indexed by integers $m \geq 0$

$$\mathrm{fil}_m^{\log} W_s(K) = \{(a_{s-1}, \dots, a_1, a_0) \in W_s(K) \mid p^i v_K(a_i) \geq -m\},$$

where v_K is the normalized valuation of K. In this paper, we use its nonlog version introduced by Matsuda [1997]:

$$\mathrm{fil}_m W_s(K) = \mathrm{fil}_{m-1}^{\log} W_s(K) + V^{s-s'} \mathrm{fil}_m^{\log} W_{s'}(K),$$

where $s' = \min\{s, \mathrm{ord}_p(m)\}$. We define ramification filtrations on $H^1(K) := H^1(K, \mathbb{Q}/\mathbb{Z})$ as

$$\mathrm{fil}_m^{\log} H^1(K) = H^1(K)\{p'\} \oplus \bigcup_{s \geq 1} \delta_s(\mathrm{fil}_m^{\log} W_s(K)) \quad (m \geq 0),$$

$$\mathrm{fil}_m H^1(K) = H^1(K)\{p'\} \oplus \bigcup_{s \geq 1} \delta_s(\mathrm{fil}_m W_s(K)) \quad (m \geq 1),$$

where $H^1(K)\{p'\}$ is the prime-to-p part of $H^1(K)$. We note that this filtration is shifted by one from the filtration of Matsuda [1997, Definition 3.1.1]. We also let $\mathrm{fil}_0 H^1(K)$ be the subgroup of all unramified characters.

Definition 2.1. For $\chi \in H^1(K)$, we denote the minimal m with $\chi \in \mathrm{fil}_m H^1(K)$ by $\mathrm{art}_K(\chi)$ and call it the Artin conductor of χ.

We have the following facts (cf. [Kato 1989; Matsuda 1997]):

Lemma 2.2. (1) $\mathrm{fil}_1 H^1(K)$ *is the subgroup of tamely ramified characters.*

(2) $\mathrm{fil}_m H^1(K) \subset \mathrm{fil}_m^{\log} H^1(K) \subset \mathrm{fil}_{m+1} H^1(K)$.

(3) $\mathrm{fil}_m H^1(K) = \mathrm{fil}_{m-1}^{\log} H^1(K)$ *if* $(m, p) = 1$.

The structure of graded quotients

$$\mathrm{gr}_m H^1(K) = \mathrm{fil}_m H^1(K)/ \mathrm{fil}_{m-1} H^1(K) \quad (m > 1)$$

is described as follows. Let Ω^1_K be the absolute Kähler differential module, and put

$$\mathrm{fil}_m\,\Omega^1_K = \mathfrak{m}_K^{-m} \otimes_{\mathbb{O}_K} \Omega^1_{\mathbb{O}_K}.$$

We have an isomorphism

$$\mathrm{gr}_m\,\Omega^1_K = \mathrm{fil}_m\,\Omega^1_K / \mathrm{fil}_{m-1}\,\Omega^1_K \simeq \mathfrak{m}_K^{-m}\Omega^1_{\mathbb{O}_K} \otimes_{\mathbb{O}_K} \kappa. \tag{2-2}$$

We have the maps

$$F^s d : W_s(K) \to \Omega^1_K, \quad (a_{s-1}, \dots, a_1, a_0) \mapsto \sum_{i=0}^{s-1} a_i^{p^i-1} da_i,$$

and one can check $F^s d(\mathrm{fil}_n\, W_s(K)) \subset \mathrm{fil}_n\,\Omega^1_K$.

Theorem 2.3 [Matsuda 1997]. *The maps $F^s d$ factor through δ_s and induce a natural map*

$$\mathrm{fil}_n\, H^1(K) \to \mathrm{fil}_n\,\Omega^1_K,$$

which induces for $m > 1$ an injective map (called the refined Artin conductor for K)

$$\mathrm{art}_K : \mathrm{gr}_n\, H^1(K) \hookrightarrow \mathrm{gr}_n\,\Omega^1_K. \tag{2-3}$$

Next we review global ramification theory. Let X and C be as in the introduction, and fix a Cartier divisor D with $|D| \subset C$. We recall the definition of $\pi_1^{\mathrm{ab}}(X, D)$. We write $H^1(U)$ for the étale cohomology group $H^1(U, \mathbb{Q}/\mathbb{Z})$, which is identified with the group of continuous characters $\pi_1^{\mathrm{ab}}(U) \to \mathbb{Q}/\mathbb{Z}$.

Definition 2.4. We define $\mathrm{fil}_D\, H^1(U)$ to be the subgroup of $\chi \in H^1(U)$ satisfying this condition. For all integral curves $Z \subset X$ not contained in C, its restriction $\chi|_Z : \pi_1^{\mathrm{ab}}(Z) \to \mathbb{Q}/\mathbb{Z}$ satisfies the following inequality of Cartier divisors on Z^N:

$$\sum_{y \in Z_\infty} \mathrm{art}_y(\chi|_Z)[y] \le \phi_Z^* D,$$

where $\mathrm{art}_y(\chi|_Z) \in \mathbb{Z}_{\ge 0}$ is the Artin conductor of $\chi|_Z$ at $y \in Z_\infty$ and $\phi_Z^* D$ is the pullback of D by the natural map $\phi_Z : Z^N \to X$. Define

$$\pi_1^{\mathrm{ab}}(X, D) = \mathrm{Hom}(\mathrm{fil}_D\, H^1(U), \mathbb{Q}/\mathbb{Z}), \tag{2-4}$$

endowed with the usual profinite topology of the dual.

For the rest of this section, we assume that X is smooth and C is a simple normal crossing. Let I be the set of generic points of C, and let $C_\lambda = \overline{\{\lambda\}}$ for $\lambda \in I$. Write

$$D = \sum_{\lambda \in I} m_\lambda C_\lambda. \tag{2-5}$$

For $\lambda \in I$, let K_λ be the henselization of $K = k(X)$ at λ. Note that K_λ is a henselian discrete valuation field with residue field $k(C_\lambda)$.

Proposition 2.5. *We have*

$$\mathrm{fil}_D\, H^1(U) = \mathrm{Ker}\Bigl(H^1(U) \to \bigoplus_{\lambda \in I} H^1(K_\lambda)/\,\mathrm{fil}_{m_\lambda} H^1(K_\lambda)\Bigr).$$

Proof. This is a consequence of ramification theory developed in [Kato 1989; Matsuda 1997]. See [Kerz and Saito 2013, Corollary 2.7] for a proof. □

Proposition 2.6. *Fix $\lambda \in I$ such that $m_\lambda > 1$ in* (2-5). *The refined Artin conductor* art_{K_λ} (*cf. Theorem 2.3*) *induces a natural injective map*

$$\mathrm{art}_{C_\lambda} : \mathrm{fil}_D\, H^1(U)/\,\mathrm{fil}_{D-C_\lambda} H^1(U) \hookrightarrow H^0(C_\lambda, \Omega^1_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_{C_\lambda}),$$

which is compatible with pullback along maps $f : X' \to X$ of smooth varieties with the property that $f^{-1}(C)$ is a reduced simple normal crossing divisor.

Proof. This follows from the integrality result [Matsuda 1997, Corollary 4.2.2] of the refined Artin conductor. □

Proposition 2.6 motivates us to introduce the following log-variant of $\mathrm{fil}_D\, H^1(U)$:

Definition 2.7. We define $\mathrm{fil}^{\log}_D H^1(U)$ as

$$\mathrm{fil}^{\log}_D H^1(U) = \mathrm{Ker}\Bigl(H^1(U) \to \bigoplus_{\lambda \in I} H^1(K_\lambda)/\,\mathrm{fil}^{\log}_{m_\lambda} H^1(K_\lambda)\Bigr).$$

Lemma 2.8. (1) $\mathrm{fil}_C\, H^1(U)$ *is the subgroup of tamely ramified characters.*

(2) $\mathrm{fil}_D\, H^1(U) \subset \mathrm{fil}^{\log}_D H^1(U) \subset \mathrm{fil}_{D+C}\, H^1(U)$.

(3) $\mathrm{fil}_D\, H^1(U) = \mathrm{fil}^{\log}_{D-C} H^1(U)$ *if* $(m_\lambda, p) = 1$ *for all* $\lambda \in I$.

Proof. This is a direct consequence of Lemma 2.2. □

3. Proof of the main theorem

Let X be a smooth projective variety over a perfect field of characteristic $p > 0$ and $C \subset X$ a reduced simple normal crossing divisor on X. Let $j : U = X \setminus C \subset X$ be the open immersion. We use the same notation as in the last part of the previous section. Take an effective Cartier divisor

$$D = \sum_{\lambda \in I} m_\lambda C_\lambda \quad \text{with } m_\lambda \geq 0.$$

Let $I' = \bigl\{\lambda \in I \bigm| p \mid m_\lambda\bigr\}$, and put

$$D' = \sum_{\lambda \in I'} (m_\lambda + 1) C_\lambda + \sum_{\lambda \in I \setminus I'} m_\lambda C_\lambda.$$

Let Y be a smooth hypersurface section such that $Y \times_X C$ is a reduced simple normal crossing divisor on Y.

Definition 3.1. (1) Assuming $\dim(X) \geq 3$, we say that Y is sufficiently ample for (X, D) if the following conditions hold:

(A1) $H^i(X, \Omega_X^d(-\Xi + Y)) = 0$ for any effective Cartier divisor $\Xi \leq D$ and for $i = d, d-1, d-2$.

(A2) For any $\lambda \in I'$, we have

$$H^0(C_\lambda, \Omega_X^1(D' - Y) \otimes \mathcal{O}_{C_\lambda}) = H^0(C_\lambda, \mathcal{O}_{C_\lambda}(D' - Y)) = H^1(C_\lambda, \mathcal{O}_{C_\lambda}(D' - 2Y)) = 0.$$

(2) Assuming $\dim(X) = 2$, we say that Y is sufficiently ample for (X, D) if the following condition holds:

(B) $H^i(X, \Omega_X^d(-\Xi + Y)) = 0$ for any effective Cartier divisor $\Xi \leq D$ and for $i = 1, 2$.

We remark that there is an integer N such that any smooth Y of degree $\geq N$ is sufficiently ample for (X, D).

Theorem 1.1 is a direct consequence of the following:

Theorem 3.2. *Let Y be sufficiently ample for (X, D). Write $E = Y \times_X D$.*

(1) *Assuming $d := \dim(X) \geq 3$, we have isomorphisms*

$$\mathrm{fil}_D H^1(U) \xrightarrow{\sim} \mathrm{fil}_E H^1(U \cap Y) \quad \textit{and} \quad \mathrm{fil}_D^{\log} H^1(U) \xrightarrow{\sim} \mathrm{fil}_E^{\log} H^1(U \cap Y).$$

(2) *Assuming $d = 2$, we have injections*

$$\mathrm{fil}_D H^1(U) \hookrightarrow \mathrm{fil}_E H^1(U \cap Y) \quad \textit{and} \quad \mathrm{fil}_D^{\log} H^1(U) \hookrightarrow \mathrm{fil}_E^{\log} H^1(U \cap Y).$$

For an abelian group M, we let $M\{p'\}$ denote the prime-to-p torsion part of M.

Lemma 3.3. (1) *Assuming $d := \dim(X) \geq 3$, we have an isomorphism*

$$\mathrm{fil}_D H^1(U)\{p'\} \xrightarrow{\sim} \mathrm{fil}_E H^1(U \cap Y)\{p'\}$$

and the same isomorphism for $\mathrm{fil}_D^{\log}$.

(2) *Assuming $d = 2$, we have an injection*

$$\mathrm{fil}_D H^1(U)\{p'\} \hookrightarrow \mathrm{fil}_E H^1(U \cap Y)\{p'\}$$

and the same injection for $\mathrm{fil}_D^{\log}$.

Proof. Noting

$$\mathrm{fil}_D H^1(U)\{p'\} = \mathrm{fil}_C H^1(U)\{p'\} = \mathrm{fil}_C^{\log} H^1(U)\{p'\} = \mathrm{fil}_D^{\log} H^1(U)\{p'\},$$

this follows from the tame case of Theorem 1.1 due to [Schmidt and Spieß 2000]. □

By the above lemma, Theorem 3.2 is reduced to the following:

Theorem 3.4. *Let the assumption be as in Theorem 3.2. Take an integer $n > 0$.*

(1) *Assuming $d := \dim(X) \geq 3$, we have isomorphisms*

$$\mathrm{fil}_D\, H^1(U)[p^n] \xrightarrow{\sim} \mathrm{fil}_E\, H^1(U \cap Y)[p^n]$$

and the same isomorphism for $\mathrm{fil}_D^{\log}$.

(2) *Assuming $d = 2$, we have an injection*

$$\mathrm{fil}_D\, H^1(U)[p^n] \hookrightarrow \mathrm{fil}_E\, H^1(U \cap Y)[p^n]$$

and the same injection for $\mathrm{fil}_D^{\log}$.

In what follows, we consider an effective Cartier divisor with $\mathbb{Z}[1/p]$-coefficient:

$$D = \sum_{\lambda \in I} m_\lambda C_\lambda, \quad m_\lambda \in \mathbb{Z}[1/p]_{\geq 0}.$$

We put

$$[D] = \sum_{\lambda \in I} [m_\lambda] C_\lambda \quad \text{with } [m_\lambda] = \max\{i \in \mathbb{Z} \mid i \leq m_\lambda\}$$

and $\mathscr{F}(\pm D) = \mathscr{F} \otimes_{\mathbb{O}_X} \mathbb{O}_X(\pm[D])$ for an $\mathbb{O}_X$-module. For D as above, let $\mathrm{fil}_D^{\log} W_n \mathbb{O}_X$ be the subsheaf of $j_* W_n \mathbb{O}_U$ of local sections

$$\underline{a} \in W_n \mathbb{O}_U \quad \text{such that } \underline{a} \in \mathrm{fil}_{m_\lambda}^{\log} W_n(K_\lambda) \text{ for any } \lambda \in I,$$

where $\mathrm{fil}_{m_\lambda}^{\log} W_n(K_\lambda) := \mathrm{fil}_{[m_\lambda]}^{\log} W_n(K_\lambda)$ is defined in Section 2 for the henselization K_λ of $K = k(X)$ at λ. We note

$$\mathbb{O}_X(D) = \mathrm{fil}_D^{\log} W_n \mathbb{O}_X \quad \text{for } n = 1.$$

The following facts are easily checked:

- The Frobenius F induces $F : \mathrm{fil}_{D/p}^{\log} W_n \mathbb{O}_X \to \mathrm{fil}_D^{\log} W_n \mathbb{O}_X$.
- The Verschiebung V induces $V : \mathrm{fil}_D^{\log} W_{n-1} \mathbb{O}_X \to \mathrm{fil}_D^{\log} W_n \mathbb{O}_X$.
- The restriction R induces $R : \mathrm{fil}_D^{\log} W_n \mathbb{O}_X \to \mathrm{fil}_{D/p}^{\log} W_{n-1} \mathbb{O}_X$.
- The following sequence is exact:

$$0 \to \mathbb{O}_X(D) \xrightarrow{V^{n-1}} \mathrm{fil}_D^{\log} W_n \mathbb{O}_X \xrightarrow{R} \mathrm{fil}_{D/p}^{\log} W_{n-1} \mathbb{O}_X \to 0. \tag{3-1}$$

We define an object $(\mathbb{Z}/p^n\mathbb{Z})_{X|D}$ of the derived category $D^b(X)$ of bounded complexes of étale sheaves on X:

$$(\mathbb{Z}/p^n\mathbb{Z})_{X|D} = \mathrm{Cone}\big(\mathrm{fil}_{D/p}^{\log} W_n \mathbb{O}_X \xrightarrow{1-F} \mathrm{fil}_D^{\log} W_n \mathbb{O}_X\big)[-1].$$

We have a distinguished triangle in $D^b(X)$:

$$(\mathbb{Z}/p^n\mathbb{Z})_{X|D} \to \mathrm{fil}^{\log}_{D/p} W_n\mathbb{O}_X \xrightarrow{1-F} \mathrm{fil}^{\log}_{D} W_n\mathbb{O}_X \xrightarrow{+} . \tag{3-2}$$

Lemma 3.5. *There is a distinguished triangle*

$$(\mathbb{Z}/p\mathbb{Z})_{X|D} \to (\mathbb{Z}/p^n\mathbb{Z})_{X|D} \to (\mathbb{Z}/p^{n-1}\mathbb{Z})_{X|D/p} \xrightarrow{+} .$$

Proof. The lemma follows from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{O}_X(D/p) \arrow[r, "V^{n-1}"] \arrow[d, "1-F"] & \mathrm{fil}^{\log}_{D/p} W_n\mathbb{O}_X \arrow[r, "R"] \arrow[d, "1-F"] & \mathrm{fil}^{\log}_{D/p^2} W_{n-1}\mathbb{O}_X \arrow[r] \arrow[d, "1-F"] & 0 \\
0 \arrow[r] & \mathbb{O}_X(D) \arrow[r, "V^{n-1}"] & \mathrm{fil}^{\log}_{D} W_n\mathbb{O}_X \arrow[r, "R"] & \mathrm{fil}^{\log}_{D/p} W_{n-1}\mathbb{O}_X \arrow[r] & 0
\end{tikzcd}$$

□

Lemma 3.6. *There is a canonical isomorphism*

$$\mathrm{fil}^{\log}_D H^1(U)[p^n] \simeq H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}).$$

Proof. Noting that the restriction of $(\mathbb{Z}/p^n\mathbb{Z})_{X|D}$ to U is $\mathbb{Z}/p^n\mathbb{Z}$ on U, we have the localization exact sequence

$$H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \to H^1(U, \mathbb{Z}/p^n\mathbb{Z}) \to H^2_C(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}). \tag{3-3}$$

For the generic point λ of C_λ, (3-2) gives us an exact sequence

$$H^1_\lambda(X, \mathrm{fil}^{\log}_{D/p} W_n\mathbb{O}_X) \xrightarrow{1-F} H^1_\lambda(X, \mathrm{fil}^{\log}_{D} W_n\mathbb{O}_X) \\ \to H^2_\lambda(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \to H^2_\lambda(X, \mathrm{fil}^{\log}_{D/p} W_n\mathbb{O}_X).$$

By [Grothendieck 1967, Corollary 3.10] and (3-1), we have

$$H^i_\lambda(X, \mathrm{fil}^{\log}_{D/p} W_n\mathbb{O}_X) = H^i_\lambda(X, \mathrm{fil}^{\log}_{D} W_n\mathbb{O}_X) = 0 \quad \text{for } i \geq 2$$

and

$$H^1_\lambda(X, \mathrm{fil}^{\log}_{D/p} W_n\mathbb{O}_X) \simeq W_n(K_\lambda)/\mathrm{fil}^{\log}_{m_\lambda/p} W_n(K_\lambda),$$
$$H^1_\lambda(X, \mathrm{fil}^{\log}_{D} W_n\mathbb{O}_X) \simeq W_n(K_\lambda)/\mathrm{fil}^{\log}_{m_\lambda} W_n(K_\lambda).$$

Thus, we get

$$H^2_\lambda(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \simeq H^1(K_\lambda)[p^n]/\mathrm{fil}^{\log}_{m_\lambda} H^1(K_\lambda)[p^n].$$

Hence, Lemma 3.6 follows from (3-3) and the injectivity of

$$H^2_C(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \to \bigoplus_{\lambda\in I} H^2_\lambda(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}).$$

This injectivity is a consequence of:

Claim 3.7. *For $x \in C$ with* $\dim(\mathcal{O}_{X,x}) \geq 2$, *we have*

$$H^2_x(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) = 0.$$

By Lemma 3.5, it suffices to show Claim 3.7 in case $n = 1$. Triangle (3-2) gives us an exact sequence

$$H^1_x(X, \mathcal{O}_X(D)) \to H^2_x(X, (\mathbb{Z}/p\mathbb{Z})_{X|D}) \to H^2_x(X, \mathcal{O}_X(D/p)) \xrightarrow{1-F} H^2_x(X, \mathcal{O}_X(D)).$$

If $\dim(\mathcal{O}_{X,x}) > 2$, $H^1_x(X, \mathcal{O}_X(D)) = 0$ and $H^2_x(X, \mathcal{O}_X(D/p)) = 0$ by [Grothendieck 1967, Corollary 3.10], which implies $H^2_x(X, (\mathbb{Z}/p\mathbb{Z})_{X|D}) = 0$ as desired.

We now assume $\dim(\mathcal{O}_{X,x}) = 2$. Let $(\mathbb{Z}/p\mathbb{Z})_X$ denote the constant sheaf $\mathbb{Z}/p\mathbb{Z}$ on X, and put

$$\mathcal{F}_{X|D} = \operatorname{Coker}\bigl(\mathcal{O}_X(D/p) \xrightarrow{1-F} \mathcal{O}_X(D)\bigr).$$

Note that $\mathcal{F}_{X|D} = 0$ for $D = 0$. By definition, we have a distinguished triangle

$$(\mathbb{Z}/p\mathbb{Z})_X \to (\mathbb{Z}/p\mathbb{Z})_{X|D} \to \mathcal{F}_{X|D} \xrightarrow{+}.$$

By [SGA 1 1971, Exposé X, Théorèm 3.1], we have $H^2_x(X, (\mathbb{Z}/p\mathbb{Z})_X) = 0$. Hence, we are reduced to showing

$$H^2_x(X, \mathcal{F}_{X|D}) = 0. \tag{3-4}$$

Without loss of generality, we can assume that D has integral coefficients. We prove (3-4) by induction on multiplicities of D reducing to the case $D = 0$. Fix an irreducible component C_λ of C with the multiplicity $m_\lambda \geq 1$ in D, and put $D' = D - C_\lambda$. We have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & (\mathbb{Z}/p\mathbb{Z})_X \arrow[d] & (\mathbb{Z}/p\mathbb{Z})_X \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_X(D'/p) \arrow[r] \arrow[d, "1-F"] & \mathcal{O}_X(D/p) \arrow[r] \arrow[d, "1-F"] & \mathcal{L} \arrow[r] \arrow[d, "F"] & 0 \\
0 \arrow[r] & \mathcal{O}_X(D') \arrow[r] & \mathcal{O}_X(D) \arrow[r] & \mathcal{O}_{C_\lambda}(D) \arrow[r] & 0
\end{tikzcd}$$

Here $\mathcal{O}_{C_\lambda}(D) = \mathcal{O}_X(D) \otimes \mathcal{O}_{C_\lambda}$, and $\mathcal{L} = \mathcal{O}_{C_\lambda}(D/p)$ if $p \mid m_\lambda$, and $\mathcal{L} = 0$ otherwise. Thus, we get short exact sequences

$$\begin{aligned}
&0 \to \mathcal{F}_{X|D'} \to \mathcal{F}_{X|D} \to \mathcal{O}_{C_\lambda}(D) \to 0 && \text{if } p \nmid m_\lambda,\\
&0 \to \mathcal{F}_{X|D'} \to \mathcal{F}_{X|D} \to \mathcal{O}_{C_\lambda}(D)/\mathcal{O}_{C_\lambda}(D/p)^p \to 0 && \text{if } p \mid m_\lambda.
\end{aligned}$$

We may assume $H^2_x(X, \mathcal{F}_{X|D'}) = 0$ by the induction hypothesis. Hence, (3-4) follows from

$$H^2_x(C_\lambda, \mathbb{O}_{C_\lambda}(D)) = 0, \tag{3-5}$$

$$H^2_x(C_\lambda, \mathbb{O}_{C_\lambda}(D)/\mathbb{O}_{C_\lambda}(E)^p) = 0, \tag{3-6}$$

where we put $E = [D/p]$. We may assume $x \in C_\lambda$ so that $\dim(\mathbb{O}_{C_\lambda,x}) = 1$ by the assumption $\dim(\mathbb{O}_{X,x}) = 2$. Equation (3-5) is a consequence of [Grothendieck 1967, Corollary 3.10]. In view of an exact sequence

$$0 \to \mathbb{O}_{C_\lambda}(pE)/\mathbb{O}_{C_\lambda}(E)^p \to \mathbb{O}_{C_\lambda}(D)/\mathbb{O}_{C_\lambda}(E)^p \to \mathbb{O}_{C_\lambda}(D)/\mathbb{O}_{C_\lambda}(pE) \to 0,$$

(3-6) follows from

$$H^2_x(C_\lambda, \mathbb{O}_{C_\lambda}(pE)/\mathbb{O}_{C_\lambda}(E)^p) = 0 \quad \text{and} \quad H^2_x(C_\lambda, \mathbb{O}_{C_\lambda}(D)/\mathbb{O}_{C_\lambda}(pE)) = 0.$$

The first assertion follows from [Grothendieck 1967, Corollary 3.10] noting that $\mathbb{O}_{C_\lambda}(pE)/\mathbb{O}_{C_\lambda}(E)^p$ is a locally free $\mathbb{O}^p_{C_\lambda}$-module. The second assertion holds since $\mathbb{O}_{C_\lambda}(D)/\mathbb{O}_{C_\lambda}(pE)$ is supported in a proper closed subscheme T of C_λ and x is a generic point of T if $x \in T$. This completes the proof of Lemma 3.6. □

Proof of Theorem 3.4. In view of the above results, the assertions for $\mathrm{fil}^{\log}_D$ of Theorem 3.4(1) and (2) follow from the following:

Theorem 3.8. *Let the assumption be as in Theorem 3.2. The natural map*

$$H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \to H^1(Y, (\mathbb{Z}/p^n\mathbb{Z})_{Y|D})$$

is an isomorphism for $d := \dim(X) \geq 3$, and it is injective for $d = 2$.

Proof. By Lemma 3.5, we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
H^1(X, (\mathbb{Z}/p\mathbb{Z})_{X|D}) \arrow[r] \arrow[d] & H^1(Y, (\mathbb{Z}/p\mathbb{Z})_{Y|E}) \arrow[d] \\
H^1(X, (\mathbb{Z}/p^n\mathbb{Z})_{X|D}) \arrow[r] \arrow[d] & H^1(Y, (\mathbb{Z}/p^n\mathbb{Z})_{Y|D}) \arrow[d] \\
H^1(X, (\mathbb{Z}/p^{n-1}\mathbb{Z})_{X|D/p}) \arrow[r] \arrow[d] & H^1(Y, (\mathbb{Z}/p^{n-1}\mathbb{Z})_{Y|E/p}) \arrow[d] \\
H^2(X, (\mathbb{Z}/p\mathbb{Z})_{X|D}) \arrow[r] & H^2(Y, (\mathbb{Z}/p\mathbb{Z})_{Y|E})
\end{tikzcd}$$

The theorem follows by the induction on n from the following lemma. □

Lemma 3.9. *Let the assumption be as in Theorem 3.2.*

(1) *Assuming $d \geq 3$, the natural map*

$$H^i(X, (\mathbb{Z}/p\mathbb{Z})_{X|D}) \to H^i(Y, (\mathbb{Z}/p\mathbb{Z})_{Y|E})$$

is an isomorphism for $i = 1$ and injective for $i = 2$.

(2) *Assuming $d = 2$, the natural map*

$$H^1(X, (\mathbb{Z}/p\mathbb{Z})_{X|D}) \to H^1(Y, (\mathbb{Z}/p\mathbb{Z})_{Y|E})$$

is injective.

Proof. We define an object $\mathcal{K}$ of $D^b(X)$:

$$\mathcal{K} = \mathrm{Cone}\big(\mathcal{O}_X(D/p - Y) \xrightarrow{1-F} \mathcal{O}_X(D - Y)\big)[-1].$$

By the commutative diagram with exact horizontal sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(D/p - Y) \arrow[r] \arrow[d, "1-F"] & \mathcal{O}_X(D/p) \arrow[r] \arrow[d, "1-F"] & \mathcal{O}_Y(E/p) \arrow[r] \arrow[d, "1-F"] & 0 \\
0 \arrow[r] & \mathcal{O}_X(D - Y) \arrow[r] & \mathcal{O}_X(D) \arrow[r] & \mathcal{O}_Y(E) \arrow[r] & 0
\end{tikzcd}$$

we have a distinguished triangle in $D^b(X)$:

$$\mathcal{K} \to (\mathbb{Z}/p\mathbb{Z})_{X|D} \to (\mathbb{Z}/p\mathbb{Z})_{Y|E} \xrightarrow{+} .$$

Hence, it suffices to show $H^i(X, \mathcal{K}) = 0$ for $i = 1, 2$ in case $d \geq 3$ and $H^1(X, \mathcal{K}) = 0$ in case $d = 2$. We have an exact sequence

$$\begin{aligned} H^0(\mathcal{O}_X(D - Y)) \to H^1(X, \mathcal{K}) \to H^1(\mathcal{O}_X(D/p - Y)) \\ \to H^1(\mathcal{O}_X(D - Y)) \to H^2(X, \mathcal{K}) \to H^2(\mathcal{O}_X(D/p - Y)). \end{aligned}$$

By Serre duality, for a divisor Ξ on X, we have

$$H^i(X, \mathcal{O}_X(\Xi - Y)) = H^{d-i}(X, \Omega^d_X(-\Xi + Y))^\vee.$$

Thus, the desired assertion follows from Definition 3.1(A1) and (B). □

It remains to deduce the assertions for fil_D of Theorem 3.4(1) and (2) from those for $\mathrm{fil}^{\log}_D$. Let D' be as in the beginning of this section and $E' = D' \times_X Y$. Noting that the multiplicities of D' are prime to p, we have by Lemma 2.8(3)

$$\mathrm{fil}_{D'} H^1(U) = \mathrm{fil}^{\log}_{D'-C} H^1(U) \quad \text{and} \quad \mathrm{fil}_{E'} H^1(U \cap Y) = \mathrm{fil}^{\log}_{E'-C\cap Y} H^1(U \cap Y).$$

Thus, the assertions for $\mathrm{fil}^{\log}_{D'-C}$ of Theorem 3.4 imply that for $\mathrm{fil}_{D'}$. Since $\mathrm{fil}_D \subset \mathrm{fil}_{D'}$, it immediately implies the injectivity of

$$\mathrm{fil}_D\, H^1(U) \to \mathrm{fil}_E\, H^1(U \cap Y).$$

It remains to deduce its surjectivity from that of

$$\mathrm{fil}_{D'}\, H^1(U) \to \mathrm{fil}_{E'}\, H^1(U \cap Y)$$

assuming $d \geq 3$. For this it, suffices to show the injectivity of

$$\mathrm{fil}_{D'}\, H^1(U)/\,\mathrm{fil}_D\, H^1(U) \to \mathrm{fil}_{E'}\, H^1(U \cap Y)/\,\mathrm{fil}_E\, H^1(U \cap Y).$$

By Proposition 2.6, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{fil}_{D'}\, H^1(U)/\,\mathrm{fil}_D\, H^1(U) \arrow[r, hook] \arrow[d] & \bigoplus_{\lambda \in I'} H^0(C_\lambda, \Omega^1_X(D') \otimes_{\mathcal{O}_X} \mathcal{O}_{C_\lambda}) \arrow[d] \\
\mathrm{fil}_{E'}\, H^1(U \cap Y)/\,\mathrm{fil}_E\, H^1(U \cap Y) \arrow[r, hook] & \bigoplus_{\lambda \in I'} H^0(C_\lambda \cap Y, \Omega^1_Y(D') \otimes_{\mathcal{O}_Y} \mathcal{O}_{C_\lambda \cap Y})
\end{tikzcd}$$

Thus, we are reduced to showing the injectivity of the right vertical map. Putting $\mathcal{L} = \mathrm{Ker}(\Omega^1_X \to i_*\Omega^1_Y)$ where $i : Y \subset X$, the assertion follows from

$$H^0(C_\lambda, \mathcal{L}(D') \otimes_{\mathcal{O}_X} \mathcal{O}_{C_\lambda}) = 0.$$

Note that we used the fact that Y and C_λ intersect transversally. We have an exact sequence

$$0 \to \Omega^1_X(-Y) \to \mathcal{L} \to \mathcal{O}_X(-Y) \otimes \mathcal{O}_Y \to 0.$$

From this, we get an exact sequence

$$0 \to \Omega^1_X(D' - Y) \otimes_{\mathcal{O}_X} \mathcal{O}_{C_\lambda} \to \mathcal{L}(D') \otimes_{\mathcal{O}_X} \mathcal{O}_{C_\lambda} \to \mathcal{O}_{C_\lambda}(D' - Y) \otimes \mathcal{O}_{C_\lambda \cap Y} \to 0.$$

We also have an exact sequence

$$0 \to \mathcal{O}_{C_\lambda}(D' - 2Y) \to \mathcal{O}_{C_\lambda}(D' - Y) \to \mathcal{O}_{C_\lambda}(D' - Y) \otimes \mathcal{O}_{C_\lambda \cap Y} \to 0.$$

Therefore, the desired assertion follows from Definition 3.1(A2). This completes the proof of Theorem 3.4. □

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moritz.kerz@mathematik.uni-regensburg.de
Naturwissenschaftliche Fakultät I – Mathematik, Universität Regensburg, 93040 Regensburg, Germany

sshuji@msb.biglobe.ne.jp
Interactive Research Center of Science, Graduate School of Science and Engineering, Tokyo Institute of Technology, Ookayama, Meguro, Tokyo 152-8551, Japan

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