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Localization of spherical varieties

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We prove some fundamental structural results for spherical varieties in arbitrary characteristic. In particular, we study Luna's two types of localization and use them to analyze spherical roots, colors, and their interrelation. At the end, we propose a preliminary definition of a p-spherical system.

1. Introduction

Let G be a connected reductive group defined over an algebraically closed ground field k of arbitrary characteristic p. A normal G-variety X is called *spherical* if a Borel subgroup B of G has an open orbit in X. In characteristic zero, there exists by now an extensive body of research on spherical varieties culminating in a complete classification [Luna and Vust 1983; Luna 2001; Losev 2009; Cupit-Foutou 2010; Bravi and Pezzini 2011a; 2011b; 2011c].

In positive characteristic, much less work has been done. Most papers dealing with spherical varieties in positive characteristic are restricted to particular examples (like flag or symmetric varieties) or other special classes of spherical varieties (like varieties obtained by reduction mod p).

This paper is part of a program to develop a general theory of spherical varieties in arbitrary characteristic, possibly also leading to a classification. In this sense, the old paper [Knop 1991] on a characteristic-free approach to spherical embeddings is already part of the program.

A crucial portion of Luna's theory of spherical varieties depends on Akhiezer's classification [1983] of spherical varieties of rank 1. In this paper, we present results which are independent of that classification. On the other hand, in the companion paper [Knop 2013], we determine all spherical varieties of rank 1 in arbitrary characteristic and present results whose proofs depend (so far) on it.

More precisely, in this paper we recover most of Luna's results [1997] on the "big cell". We start by generalizing Luna's fundamental relations for the colors of a spherical variety. At this point, we introduce additional data needed to describe a spherical variety in positive characteristic. These are certain p-powers $q_{D,\alpha}$, where

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α is a simple root and D is a color "moved by α". Our exposition of this part is different from (and, we think, simpler than) Luna's, and seems to be new even in characteristic zero.

Next, we define the notion of spherical roots as properly normalized normal vectors to the valuation cone. Luna's method of viewing them as weights attached to a wonderful variety does not generalize.

Next, we consider Luna's construction [1997], called *localization at S*. Basically, it consists of analyzing the open Białynicki-Birula cell with respect to a dominant 1-parameter subgroup of G. Our results are more general than Luna's, even in characteristic zero, since Luna restricts his attention to wonderful varieties, while we formulate everything for so-called toroidal varieties. From this, we derive Luna's important result that the colors are, to a large extent, already determined by the spherical roots.

Then we consider a construction called *localization at Σ*. This procedure amounts to analyzing G-orbits of a toroidal variety. Since this technique is mostly classical, only the proof for the behavior of type-(a) colors is new. Unfortunately, our results remain somewhat incomplete, since it is unknown whether orbit closures in toroidal spherical embeddings are normal or not.

We use localization at Σ to prove the important nonpositivity result Corollary 6.6. Unlike Luna's proof, which uses Wasserman's tables [1996] of rank-2 varieties, our proof is conceptual.

Finally, in Section 7 we attempt to generalize Luna's notion of a spherical system to positive characteristic. This is a combinatorial structure describing the roots and the colors of a spherical variety. As additional data we propose the p-powers $q_{\alpha,D}$ mentioned above, and we hope that, at least for $p \neq 2, 3$, these data are enough to describe a spherical variety. As for the axioms, we restrict ourselves to those which immediately generalize axioms in characteristic zero. Conditions which are only meaningful in positive characteristic (like bounding the denominators of the pairings $\delta_D(\alpha)$) are deferred to future work.

So additional axioms will have to be added on at a later stage.

Notation. In the entire paper, the ground field k is algebraically closed. Its characteristic exponent is denoted by p, that is, $p = 1$ if $\operatorname{char} k = 0$ and $p = \operatorname{char} k$ otherwise. The group G is connected reductive, $B \subseteq G$ is a Borel subgroup, and $T \subseteq B$ is a maximal torus. Let $\Xi(T) = \Xi(B)$ be its character group. The set of simple roots with respect to B is denoted by $S \subset \Xi(T)$.

A rational function f on X is B-semiinvariant if there is a character $\chi_f \in \mathcal{X}(B) = \mathcal{X}(T)$ such that $f(b^{-1}x) = \chi_f(b)f(x)$ for all $b \in B$ and generic $x \in X$. If X is spherical, the character χ_f determines f up to a nonzero scalar. Let $\Xi(X) \subseteq \Xi(T)$ be the set of characters of the form χ_f. It is a finitely generated abelian group whose rank is called the rank of X. We also use $\Xi_{\mathbb{Q}}(X) := \Xi(X)\otimes\mathbb{Q}$ and $\Xi_p(X) := \Xi(X)\otimes\mathbb{Z}_p$ with $\mathbb{Z}_p := \mathbb{Z}[1/p]$.

2. Colors

Many properties of a spherical variety are determined by two sets of data and their interrelation: colors and valuations. We start with colors. Our results generalize those of [1997] in characteristic zero, but the approach is different. We do not use compactifications, but use the completeness of flag varieties instead.

Let X be a spherical G-variety with group of characters $\Xi(X)$, and let

$$N_{\mathbb{Q}}(X) = \operatorname{Hom}(\Xi(X), \mathbb{Q}). \tag{2-1}$$

A *color* of X is an irreducible divisor which is B- but not G-invariant. Every color D produces an element $\delta_D \in N_{\mathbb{Q}}(X)$ by

$$\delta_D(\chi_f) := v_D(f) \tag{2-2}$$

for all B-semiinvariants f. Here v_D is the valuation of $k(X)$ attached to D. The color D is, in general, not uniquely determined by δ_D.

Since X is spherical, we can choose a point $x_0 \in X$ such that Bx_0 is open and dense in X. Let $\Delta(X)$ be the set of colors of X. Since every color intersects the open G-orbit Gx_0, we have $\Delta(X) = \Delta(Gx_0) = \Delta(G/H)$, where $H = G_{x_0}$ is the isotropy subgroup scheme of x_0. We start by recalling a well known formula for the number of colors.

Proposition 2.1. *Let G be a semisimple group and $H \subseteq G$ a spherical subgroup. Then*

$$\#\Delta(G/H) = \operatorname{rk} G/H + \operatorname{rk} \Xi(H). \tag{2-3}$$

Proof. We compute the Picard group in two different ways. Set $X = G/H$. First, we have an exact sequence

$$\Xi(G) \to \operatorname{Pic}^G X \to \operatorname{Pic} X. \tag{2-4}$$

The group on the left is trivial since G is semisimple. The cokernel of the homomorphism on the right is torsion by [Sumihiro 1974]. On the other hand, $\operatorname{Pic}^G X = \Xi(H)$. Thus $\operatorname{rk}\operatorname{Pic} X = \operatorname{rk} \Xi(H)$. Now let $X_0 = Bx_0 \subseteq X$ be the open B-orbit. Then the colors are the irreducible components of $X \setminus X_0$. Thus we have an exact sequence

$$k^\times = \mathbb{O}(X)^\times \to \mathbb{O}(X_0)^\times \to \mathbb{Z}^a \to \operatorname{Pic} X \to \operatorname{Pic} X_0 = 0, \tag{2-5}$$

where $a = \#\Delta(X)$. By definition, $\operatorname{rk} \mathbb{O}(X_0)^\times/k^\times = \operatorname{rk} X$. Thus $\operatorname{rk}\operatorname{Pic} X = a - \operatorname{rk} X$. □

Given a simple root $\alpha \in S$, one can construct colors as follows: let $P_\alpha \subseteq G$ be the minimal parabolic subgroup corresponding to α. Then $P_\alpha x_0$ is an open B-stable subset of X which, according to [Knop 1995b, Lemma 3.2], decomposes into at most three B-orbits. One of them is the open B-orbit Bx_0; the others are of codimension 1 in X, and hence their closures are colors. We say that these colors are *moved by* α. Clearly, this just means that $P_\alpha D \neq D$. In particular, every color is moved by some (not necessarily unique) simple root.

A more precise description is as follows. Let $H_\alpha := (P_\alpha)_{x_0} = H \cap P_\alpha$ such that $P_\alpha x_0 = P_\alpha / H_\alpha$. Then the B-orbits in $P_\alpha x_0$ correspond to H_α^{red}-orbits in $B \backslash P_\alpha \cong \boldsymbol{P}^1$. Let $\overline{H}_\alpha$ denote the image of H_α^{red} in Aut $\boldsymbol{P}^1 \cong \mathrm{PGL}(2)$. Then, up to conjugation, there are four possibilities for $\overline{H}_\alpha$:

Type of α	$\overline{H}_\alpha$	colors
(p)	G_0	—
(b)	$S_0 U_0$	D
(a)	T_0	D, D'
$(2a)$	N_0	D

(2-6)

Here $G_0 = \mathrm{PGL}(2)$. The subgroups B_0, U_0, and T_0 of G_0 are a Borel subgroup, a maximal unipotent subgroup, and a maximal torus, respectively. Moreover, $S_0 \subseteq T_0$ and $N_0 = N_{G_0}(T_0)$. Thus, the set of simple roots decomposes as a disjoint union according to their type:

$$S = S^{(p)} \cup S^{(b)} \cup S^{(a)} \cup S^{(2a)}. \tag{2-7}$$

Observe that $\alpha \in S^{(p)}$ if and only if the open B-orbit Bx_0 is P_α-invariant. Thus, $S^{(p)}$ is the set of simple roots of the parabolic P_X, the stabilizer of the open B-orbit.

Let D be a color moved by α. Then the morphism

$$\varphi_{D,\alpha} : P_\alpha \times^B D \to X \tag{2-8}$$

is generically finite. Its separable degree is 1, that is, $\varphi_{D,\alpha}$ is bijective if and only if α is of type (b) or (a). It is 2 for α of type $(2a)$. The inseparable degree of $\varphi_{D,\alpha}$ will be denoted by $q_{D,\alpha} \in p^{\mathbb{N}}$.

Example. Assume $p > 3$ and let $P \subseteq G$ be a subgroup scheme which contains ${}^-B$, the Borel subgroup which is opposite to B. Wenzel [1993; 1994] showed that such subgroup schemes are classified by functions $f : S \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$, where $f(\alpha)$ can be defined as the supremum of the set of all $n \in \mathbb{Z}_{\geq 0}$ such that P contains the n-th Frobenius kernel of P_α. See also [Haboush and Lauritzen 1993] for a simplified account. Now let $X = G/P$, a complete homogeneous G-variety. Then the following is easy to see: A simple root $\alpha \in S$ is of type (p) if and only if $f(\alpha) = \infty$. All other simple roots are of type (b) and they all move a different divisor D_α. Moreover, $q_{D_\alpha,\alpha} = p^{f(\alpha)}$. In particular, this shows that in this example the numbers $q_{D,\alpha}$ may be arbitrary p-powers.

To formulate the following permanence property we renormalize δ_D as follows:

$$\delta_D^{(\alpha)} := q_{D,\alpha}\delta_D. \tag{2-9}$$

Lemma 2.2. *Let $\pi : X_1 \to X_0$ be a finite surjective equivariant morphism between spherical G-varieties, let E be a color of X_1, and let $D = \pi(E)$ be its image in X_0. Let, moreover, $\alpha \in S$ be a simple root moving E (and D). Then $\delta_D^{(\alpha)} = \delta_E^{(\alpha)}$.*

Proof. We consider first the case that α is of type (a) or (b) for X_0. Then its type for X_1 is the same. Moreover, both $\varphi_{D,\alpha}$ and $\varphi_{E,\alpha}$ are bijective and, as an equality of divisors, $\pi^{-1}(D) = qE$, where q is some p-power. Thus $\delta_E = q\delta_D$. Now consider the diagram

$$\begin{array}{ccc} P_\alpha \times^B \pi^{-1}(D) & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ P_\alpha \times^B D & \xrightarrow{\varphi_{D,\alpha}} & X_0 \end{array} \tag{2-10}$$

It is cartesian, and hence both horizontal arrows have the same (inseparable) degree, namely $q_{D,\alpha}$. On the other hand, the top arrow has degree $q\, q_{E,\alpha}$. Hence

$$\delta_D^{(\alpha)} = q_{D,\alpha}\delta_D = qq_{E,\alpha}\delta_D = q_{E,\alpha}\delta_E = \delta_E^{(\alpha)}. \tag{2-11}$$

Now assume that α is of type ($2a$) for X_0. Then there are two cases. If $\pi^{-1}(D)^{\text{red}}$ is irreducible then α is of type ($2a$) for X_1, as well. Moreover, the degree of both horizontal arrows is now $2q_{D,\alpha}$. From here one argues as above.

The second case is when α is of type (a) for X_0. Then $\pi^{-1}(D)^{\text{red}} = E_1 \cup E_2$ has two components. As divisors, we have $\pi^{-1}(D) = q_1E_1 + q_2E_2$. Thus $\delta_{E_1} = q_1\delta_D$ and $\delta_{E_2} = q_2\delta_D$. Moreover, as above, we get

$$2\delta_D^{(\alpha)} = 2q_{D,\alpha}\delta_D = q_1q_{E_1,\alpha}\delta_D + q_2q_{E_2,\alpha}\delta_D = \delta_{E_1}^{(\alpha)} + \delta_{E_2}^{(\alpha)}. \tag{2-12}$$

Now we claim that actually $\delta_{E_1}^{(\alpha)} = \delta_{E_2}^{(\alpha)}$, which would prove our assertion.

To prove the claim, we may assume that $X_0 = G/H_0$ and $X_1 = G/H_1$ are homogeneous. Moreover, the cases proved above allow replacement of H_0 and H_1 by H_0^{red} and H_1^{red}, respectively. We can even replace H_1 by its connected component of unity since E cannot split any further (otherwise D would split into more than two components). Then H_1 is normal in H_0 and π is the quotient by the finite group $\Gamma := H_0/H_1$. Since Γ acts transitively on the connected components of the fibers of π, there is an element $g \in \Gamma$ which maps E_1 to E_2, which proves the claim. □

For any simple root α, let $\alpha^r \in N_{\mathbb{Q}}(X)$ be the restriction of $\alpha^\vee$ to $\Xi_{\mathbb{Q}}(X)$:

$$\alpha^r(\chi) = \langle \chi, \alpha^\vee \rangle \quad \text{for all } \chi \in \Xi_{\mathbb{Q}}(X). \tag{2-13}$$

Proposition 2.3. *Fix a simple root $\alpha \in S$. Then the following relations hold*:

$$
\begin{array}{ll}
\textit{Type of } \alpha & \\
\hline
(p) & \alpha^r = 0 \\
(b) & \delta_D^{(\alpha)} = \alpha^r \\
(a) & \delta_D^{(\alpha)} + \delta_{D'}^{(\alpha)} = \alpha^r \\
 & \alpha \in \Xi_p(X),\, \delta_D^{(\alpha)}(\alpha) = \delta_{D'}^{(\alpha)}(\alpha) = 1 \\
(2a) & \delta_D^{(\alpha)} = \tfrac{1}{2}\alpha^r
\end{array}
\tag{2-14}
$$

Proof. Let $X = G/H$ be homogeneous and put $X_1 = G/H^{0,\mathrm{red}}$. We claim that it suffices to prove the assertions for X_1. Indeed, if α is of type (p) for X, then the same holds for X_1 and the claim follows from $\Xi_{\mathbb{Q}}(X) = \Xi_{\mathbb{Q}}(X_1)$. If α is not of type (p) for X and X_1, then the claim follows immediately from Lemma 2.2 if α has the same type for X and X_1. Otherwise, α is of type $(2a)$ for X (moving one color D) and of type (a) for X_1 (moving two colors E_1, E_2). But then Lemma 2.2 implies

$$\delta_D^{(\alpha)} = \tfrac{1}{2}\big(\delta_{E_1}^{(\alpha)} + \delta_{E_2}^{(\alpha)}\big) = \tfrac{1}{2}\alpha^r, \tag{2-15}$$

proving the claim.

Thus we may assume that H is connected and reduced. Then consider the diagram

$$
\begin{array}{ccc}
 & G & \\
 {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\
X = G/H & & B\backslash G =: \mathscr{F}
\end{array}
\tag{2-16}
$$

Both morphisms p_1 and p_2 are smooth with connected fibers. Therefore, an irreducible B-stable divisor $D \subset X$ corresponds to an irreducible H-stable divisor $E \subset \mathscr{F}$. Moreover, any B-semiinvariant rational function f on X corresponds to an H-invariant rational section s of the homogeneous line bundle $\mathscr{L}_\chi$ (with $\chi = \chi_f$) on $\mathscr{F}$. Furthermore, (D, f) is related to (E, s) by

$$v_E(s) = v_D(f) = \delta_D(\chi). \tag{2-17}$$

Now consider the $\boldsymbol{P}^1$-fibration $\pi : \mathscr{F} = B\backslash G \to \mathscr{F}_\alpha := P_\alpha\backslash G$. Moreover, let $y \in \mathscr{F}$ be in the open H-orbit and let $F \cong \boldsymbol{P}^1$ be the fiber through y.

Assume first that α is of type (p). Then the open B-orbit in X is P_α-stable, which translates into the open H-orbit in $\mathscr{F}$ being the preimage of an open set in $P_\alpha\backslash G$. But then $\mathscr{L}_\chi$ is a pull-back from $P_\alpha\backslash G$, which implies $\langle \chi_f, \alpha^\vee \rangle = 0$.

Now assume that $\alpha \in S^{(b)}$. Then E is the only H-invariant divisor mapping dominantly onto $\mathscr{F}_\alpha$. Moreover, since $E \cap F$ consists of a single point, the map $E \to \mathscr{F}_\alpha$ is generically bijective, and hence purely inseparable. Its degree is $q_{D,\alpha}$.

Thus we get

$$\langle \chi, \alpha^\vee \rangle = \deg \mathcal{L}_\chi|_F = (s) \cdot F = v_E(s) E \cdot F = q_{D,\alpha} \delta_D(\chi) = \delta_D^{(\alpha)}(\chi), \quad (2\text{-}18)$$

proving the assertion.

If $\alpha \in S^{(a)}$, then there are two divisors E, E' mapping generically injectively to $\mathcal{F}_\alpha$ with degree $q_{D,\alpha}$ and $q_{D',\alpha}$, respectively. Then

$$\langle \chi, \alpha^\vee \rangle = (s) \cdot F = \big(v_E(s) E + v_{E'}(s) E'\big) \cdot F = q_{D,\alpha} \delta_D(\chi) + q_{D',\alpha} \delta_{D'}(\chi). \quad (2\text{-}19)$$

Now we prove $\alpha \in \Xi_p(X)$. By construction, there is an equivariant morphism $P_\alpha x_0 \twoheadrightarrow \mathrm{PGL}(2)/\tilde{H}$ with $\tilde{H}^{\mathrm{red}} = T_0$. Thus the pull-back of any nonconstant B_0-semiinvariant is a B-semiinvariant with character $q_0 \alpha$ for some p-power q_0. The H_α-linearization of $\mathcal{L}_{q_0\alpha}|_F$ factors through a PGL(2)-linearization. One reason is, for example, that $\mathcal{L}_{-\alpha}$ is the relative canonical bundle of the fibration π. This implies that $s|_F$ has two zeroes of the same multiplicity on F. Hence $\delta_D^{(\alpha)}(\alpha) = \delta_{D'}^{(\alpha)}(\alpha)$, and therefore both are equal to 1.

Finally, assume that $\alpha \in S^{(2a)}$. Then there is one divisor E mapping generically 2 : 1 to $\mathcal{F}_\alpha$. The degree of inseparability of this map is $q_{D,\alpha}$. Then $E \cdot F = 2q_{D,\alpha}$, and therefore

$$\langle \chi, \alpha^\vee \rangle = (s) \cdot F = v_E(s) E \cdot F = 2q_{D,\alpha} \delta_D(\chi) = 2\delta_D^{(\alpha)}(\chi), \quad (2\text{-}20)$$

as claimed. □

We note the following consequence:

Corollary 2.4. *Let $p \neq 2$ and let $\alpha \in S$ be of type* (2*a*). *Then $\alpha \notin \Xi_p(X)$ and $\langle \chi, \alpha^\vee \rangle$ is even for all $\chi \in \Xi(X)$.*

Proof. We keep the notation of the proof of Proposition 2.3. Let $N_0 = \langle s_0 \rangle T_0$ and let $n \in \mathbb{Z}$ with $n\alpha \in \Xi(X)$. Then s_0 acts on the T_0-invariant section of $\mathcal{L}_{q_0\alpha}|_F$ by multiplication with $(-1)^n$. Hence n is even and $\alpha \notin \Xi_p(X)$. The rest follows directly from Proposition 2.3. □

Now we analyze the case where a color is moved by more than one simple root.

Lemma 2.5. *Let D be a color which is moved by two distinct simple roots α_1 and α_2. Then either $\alpha_1, \alpha_2 \in S^{(b)}$ or $\alpha_1, \alpha_2 \in S^{(a)}$. In the latter case, let D' and D'' be the second color moved by α_1 and α_2, respectively. Then $D' \neq D''$.*

Proof. Clearly, neither α_1 nor α_2 is of type (p). Recall from [Knop 1995b, §2] that any B-orbit on X has a rank attached to it. Moreover, if $\alpha \in S$ moves the color D, then rank D = rank X if α is of type (b), and rank D = rank $X - 1$ in case α is of type (a) or (2a) [Knop 1995b, §2 and Lemma 3.2]. This entails that α_1, α_2 are either both of type (b) or both of type (a) or (2a).

Suppose they are both of type $(2a)$. Then, since $\alpha_1 \in \Xi_{\mathbb{Q}}(X)$,

$$0 < q_{D,\alpha_1}^{-1} = q_{D,\alpha_1}^{-1}\,\tfrac{1}{2}\alpha_1^r(\alpha_1) = \delta_D(\alpha_1) = q_{D,\alpha_2}^{-1}\,\tfrac{1}{2}\alpha_2^r(\alpha_1) \leq 0. \tag{2-21}$$

Similarly, suppose α_1 is of type (a) and α_2 is of type $(2a)$. Then

$$0 < q_{D,\alpha_1}^{-1} = \delta_D(\alpha_1) = q_{D,\alpha_2}^{-1}\tfrac{1}{2}\alpha_2^r(\alpha_1) \leq 0. \tag{2-22}$$

This finishes the proof of the first part.

Now let both α_1 and α_2 be of type (a) and suppose $D' = D''$. Then

$$0 < \frac{q_{D,\alpha_2}}{q_{D,\alpha_1}} + \frac{q_{D',\alpha_2}}{q_{D',\alpha_1}} = q_{D,\alpha_2}\delta_D(\alpha_1) + q_{D',\alpha_2}\delta_{D'}(\alpha_1) = \alpha_2^r(\alpha_1) \leq 0. \qquad \square$$

Examples. 1. Let $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$ and $H = \mathrm{SL}(2)$ embedded into G via $\mathrm{id} \times F_q$, where F_q is a Frobenius morphism. Then $X := G/H$ has only one color D. Moreover, both simple roots α_1, α_2 are of type (b), and D is moved by both of them. Furthermore, $q_{D,\alpha_1} = 1$ and $q_{D,\alpha_2} = q$, which shows that $q_{D,\alpha}$ may depend on α.

2. Let $G = \mathrm{SL}(3)$, let q be a p-power, and let H be the subgroup consisting of the matrices

$$\begin{pmatrix} t^{q+2} & & \\ & t^{q-1} & \\ & & t^{-2q-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & x & y \\ & 1 & x^q \\ & & 1 \end{pmatrix} \quad \text{with } t \in \boldsymbol{G}_m \text{ and } x, y \in \boldsymbol{G}_a. \tag{2-23}$$

Then both simple roots are of type (a) and there are three colors D_0, D_1, D_2 where α_i moves D_0 and D_i. Furthermore, $q_{D_1,\alpha_1} = q_{D_0,\alpha_2} = q_{D_2,\alpha_2} = 1$, while $q_{D_0,\alpha_1} = q$. The values $\delta_D(\alpha_i)$ are given by the following table:

	δ_{D_0}	δ_{D_1}	δ_{D_2}
α_1	q^{-1}	1	$-1-q^{-1}$
α_2	1	$-q-1$	1

(2-24)

So indeed $q\delta_{D_0} + \delta_{D_1} = \alpha_1^r$ and $\delta_{D_0} + \delta_{D_2} = \alpha_2^r$.

Both examples show that G contains for $p \geq 2$ infinitely many conjugacy classes of self-normalizing spherical subgroups, a phenomenon which does not occur in characteristic zero.

Remark. The Lemma shows, in particular, that one can assign unambiguously a type to any color. Thereby, one gets a decomposition

$$\Delta(X) = \Delta^{(b)}(X) \cup \Delta^{(a)}(X) \cup \Delta^{(2a)}(X). \tag{2-25}$$

3. Spherical roots

For a spherical variety X, let $\mathcal{V}(X)$ be the set of G-invariant $\mathbb{Q}$-valued valuations of the field $k(X)$. The map

$$\mathcal{V}(X) \to N_{\mathbb{Q}}(X) = \mathrm{Hom}(\Xi(X), \mathbb{Q}) : v \mapsto (\chi_f \mapsto v(f)) \tag{3-1}$$

is injective [Knop 1991, Corollary 1.8]. According to [ibid., Corollary 5.3], it identifies $\mathcal{V}(X)$ with a finitely generated convex rational cone inside $N_{\mathbb{Q}}(X)$ which contains the image of the antidominant Weyl chamber under the projection $\mathrm{Hom}(\Xi(T), \mathbb{Q}) \twoheadrightarrow N_{\mathbb{Q}}(X)$. This can be phrased in terms of the dual cone $\mathcal{V}(X)^\vee$ of $\mathcal{V}(X)$: it is a finitely generated rational convex cone in $\Xi_{\mathbb{Q}}(X)$ with $\mathcal{V}(X) = (\mathcal{V}(X)^\vee)^\vee$ and $-\mathcal{V}(X)^\vee \subseteq \mathbb{Q}_{\geq 0}S$, where $\mathbb{Q}_{\geq 0}S$ is the cone generated by the simple roots of G. In particular, $-\mathcal{V}(X)^\vee$ is a pointed cone. Thus, it has a canonical set of generators:

Definition 3.1. An element $\sigma \in \Xi_{\mathbb{Q}}(X)$ is called a *spherical root of X* if

- $\mathbb{Q}_{\geq 0}\sigma$ is an extremal ray of $-\mathcal{V}(X)^\vee$ (thus $\sigma \in \mathbb{Q}S$) and
- σ is a primitive element of $\mathbb{Z}S \cap \Xi_p(X)$.

The set of spherical roots is denoted by $\Sigma(X)$.

Clearly, each extremal ray of $-\mathcal{V}(X)^\vee$ contains a unique spherical root. Moreover, the spherical roots determine the valuation cone via

$$\mathcal{V}(X) = \{v \in N_{\mathbb{Q}}(X) : \sigma(v) \leq 0 \text{ for all } \sigma \in \Sigma(X)\}, \tag{3-2}$$

and are in bijection with faces of codimension 1 of $\mathcal{V}(X)$.

The normalization for a spherical root is chosen such that the following statement holds:

Lemma 3.2. *Let $\varphi : X_1 \to X_2$ be a morphism of spherical varieties which is either purely inseparable or a quotient by a central subgroup scheme of G. Then $\Sigma(X_1) = \Sigma(X_2)$.*

Proof. If φ is purely inseparable, then clearly $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ and $\Xi_p(X_1) = \Xi_p(X_2)$. Hence $\Sigma(X_1) = \Sigma(X_2)$.

Now let φ be the quotient by $A \subseteq Z(G)$ (which might be positive dimensional). Then

$$\Xi_p(X_2) = \{\chi \in \Xi_p(X_1) : \chi|_A = 1\}. \tag{3-3}$$

Since roots of G are trivial on A, this implies

$$\mathbb{Z}S \cap \Xi_p(X_2) = \mathbb{Z}S \cap \Xi_p(X_1). \tag{3-4}$$

Also, $N_{\mathbb{Q}}(X_2)$ is a quotient of $N_{\mathbb{Q}}(X_1)$ and $\mathcal{V}(X_1)$ is the preimage of $\mathcal{V}(X_2)$ (this follows, for example, from [Knop 1991, Theorem 6.1]). Then $\Sigma(X_2) = \Sigma(X_1)$. □

4. Localization at S

There are two types of constructions, called localization at S and at Σ, respectively, which both allow reduction of a spherical variety to a simpler one. In this section we describe localization at S, which, in characteristic zero, was first introduced by Luna [1997]. To this end, we first recall and prove some properties of the Białynicki-Birula decomposition [1976] of a $\boldsymbol{G}_m$-variety.

Let X be a complete normal $\boldsymbol{G}_m$-variety. Then for any $x \in X$, the limit

$$\pi(x) := \lim_{t \to 0} t \cdot x \in X \tag{4-1}$$

exists and is a $\boldsymbol{G}_m$-fixed point. Thus, letting F be the set of connected components of the fixed point set $X^{\boldsymbol{G}}_m$, we get a set partition of X by putting

$$X_Z := \{x \in X : \pi(x) \in Z\}. \tag{4-2}$$

These are the Białynicki-Birula cells which are indexed by F. Except when X is smooth or projective, they are, in general, not very well behaved. One cell is always good, though:

Proposition 4.1. *Let X be a complete normal $\boldsymbol{G}_m$-variety. Then there is a unique connected component S of $X^{\boldsymbol{G}}_m$ (the* source *of X) such that X_S is open. Moreover, the map $\pi_S := \pi_S : X_S \to S$ is affine and a categorical quotient by $\boldsymbol{G}_m$. In particular, the source S is irreducible and normal.*

Proof. The statement is well known. For example, it follows from the theory in [Białynicki-Birula and Święcicka 1982]: Let S be the source, that is, the connected component of $X^{\boldsymbol{G}}_m$ such that $\pi(x) \in S$ for $x \in X$ generic. By [Białynicki-Birula and Święcicka 1982, Proposition 2.3], the set $A^+ = \{S\}$ defines a sectional set. Now the assertion is [Białynicki-Birula and Święcicka 1982, Theorem 1.5]. □

Lemma 4.2. *Let X be as above. Then the general fibers of π_S are irreducible and generically reduced.*

Proof. Since X_S is normal, the generic fiber of π_S is geometrically unibranch [Grothendieck 1965, 6.15.6]. Since all irreducible components contain the $\boldsymbol{G}_m$-fixed point, it follows that the generic fiber is geometrically irreducible. Thus, there is an open subset of S over which all fibers are also geometrically irreducible [Jouanolou 1983, Theorem 4.10].

The second property follows from the fact that π_S is a categorical quotient. This entails that $k(S)$ is separable inside $k(X_S)$. Therefore, π_S is smooth generically on X_S. □

There is a second well-behaved cell. For this, let $X^- := X$ but with the opposite $\boldsymbol{G}_m$-action: $t * x := t^{-1} \cdot x$. Then the source T of X^- is called the *sink of X*.

It is characterized by the fact that $\pi(x) \in T$ implies $x \in T$. Thus, $T = X_T$ is a Białynicki-Birula cell by itself. Symmetrically, S is the sink of X^- and therefore $X^-_S = S$.

Now assume that X is a G-variety for some connected reductive group G and that the $\boldsymbol{G}_m$-action is induced by a 1-parameter subgroup $\lambda : \boldsymbol{G}_m \to G$. Then we put $X^\lambda := S$ and $X_\lambda := X_S$ and $\pi_\lambda = \pi_S$. Observe that X^λ is not the entire fixed point set of $\lambda(\boldsymbol{G}_m)$ but only a very special component.

Let $G^\lambda := C_G(\lambda(\boldsymbol{G}_m))$ be the fixed point set under the conjugation action and let

$$G_\lambda := \{g \in G : \pi_G(g) := \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}. \tag{4-3}$$

Then G_λ is a parabolic subgroup with Levi complement G^λ and the map

$$\pi_G : G_\lambda \to G^\lambda$$

is the natural homomorphism with kernel G^u_λ, the unipotent radical of G_λ. The following lemma is well known; for example, the proof given in [Luna 1997] carries over verbatim to positive characteristic.

Lemma 4.3. *The open cell X_λ is G_λ-invariant and $\pi_\lambda : X_\lambda \to X^\lambda$ is G_λ-equivariant where G_λ acts on X^λ via $\pi_G : G_\lambda \to G^\lambda$, that is, $\pi_\lambda(gx) = \pi_G(g)\pi_\lambda(x)$ for all $g \in G_\lambda$ and $x \in X_\lambda$. Moreover, X^λ consists of fixed points for the opposite unipotent radical ${}^-G^u_\lambda$.*

We next provide a link between closed orbits in X and closed orbits in X^λ:

Lemma 4.4. *Let X be a complete normal G-variety and let $Z \subseteq X^\lambda$ be a closed G^λ-orbit. Then $GZ \subseteq X$ is a closed G-orbit with $GZ \cap X^\lambda = Z$.*

Proof. Since Z is complete and homogeneous, it contains a unique fixed point z for ${}^-B^\lambda$, the Borel subgroup opposite to B^λ. Since z is in the source, it is a ${}^-G^u_\lambda$-fixed point. So z is fixed by ${}^-B = {}^-B^\lambda\, {}^-G^u_\lambda$, which implies that $Gz = GZ$ is a complete, and hence closed, G-orbit of X. Moreover, since λ is dominant, we have $z \in (GZ)^\lambda$ and therefore $GZ \cap X^\lambda = (Gz)^\lambda = G^\lambda z = Z$. □

Recall that a complete spherical G-variety X is called *toroidal* if no color of X contains a G-orbit.

Proposition 4.5. *Let λ be a dominant* 1*-parameter subgroup and let X be a complete toroidal spherical variety. Then X^λ is a complete toroidal spherical G^λ-variety.*

Proof. This is basically [Luna 1997, Proposition 1.4]. We recall the proof and check that it is characteristic-free.

First of all, completeness of X^λ is clear, while irreducibility and normality were obtained in Proposition 4.1. Moreover, the dominance of λ implies $B \subseteq G_\lambda$. Hence the open B-orbit of X is contained in X_λ. Its image in X^λ is an open B^λ-orbit. Thus, X^λ is spherical as well.

Now let $D \subseteq X^\lambda$ be a color containing the closed G^λ-orbit Z. Then $\pi_\lambda^{-1}(D)$ contains a unique component E' which maps onto D (by Lemma 4.2 and the fact that D meets the open G^λ-orbit). Then E, the closure of D in X, is a color which contains Z. Since GZ is a closed G-orbit by Lemma 4.4, we have $GZ = \overline{BZ}$, and therefore $GZ \subseteq E$. It follows that E is G-invariant since X is toroidal. From this we get that $D = \pi_\lambda(E)$ is G^λ-invariant, in contradiction to D being a color. □

A toroidal spherical variety X determines a (pointed) fan $\mathcal{F} = \mathcal{F}(X)$ in $N_\mathbb{Q}(X)$ whose support is $\mathcal{V}(X)$. More precisely, for any G-orbit $Y \subseteq X$, the invariant valuations whose center in X contain Y form a cone $\mathcal{C}_Y \subseteq \mathcal{V}(X)$. Then $\mathcal{F}$ is the collection of cones of the form $\mathcal{C}_Y$.

The fan $\mathcal{F}$ is precisely the piece of information needed to reconstruct X from its open G-orbit X_0. In fact, X is the compactification of X_0 corresponding to the colored fan $(\mathcal{F}, \varnothing)$. See [Knop 1991] for more details.

Let λ be a dominant 1-parameter subgroup and let X be a complete toroidal spherical variety with associated fan $\mathcal{F}$. Then λ induces, via restriction to $\Xi_\mathbb{Q}(X)$, an element $\lambda^r \in N_\mathbb{Q}(X)$. The dominance of λ implies $-\lambda^r \in \mathcal{V}(X)$. Thus we can consider the fan

$$\tilde{\mathcal{F}}^\lambda := \{\mathcal{C} + \mathbb{Q}_{\geq 0}\lambda : -\lambda^r \in \mathcal{C} \in \mathcal{F}\}. \tag{4-4}$$

One may think of $\tilde{\mathcal{F}}^\lambda$ as the restriction of $\mathcal{F}$ to a neighborhood of $-\lambda^r$. Visibly, this fan is not pointed, since all its members contain the line $\mathbb{Q}\lambda^r$. More precisely, let $\mathcal{C}(\lambda)$ be the unique cone in $\mathcal{F}$ such that $-\lambda^r$ is contained in its relative interior. Then $V(\lambda) := \langle \mathcal{C}(\lambda) \rangle_\mathbb{Q} = \mathcal{C}(\lambda) + \mathbb{Q}_{\geq 0}\lambda^r$ is the unique element of $\tilde{\mathcal{F}}^\lambda$ which is a subspace. Thus,

$$\mathcal{F}^\lambda := \{\mathcal{C}/V(\lambda) : \mathcal{C} \in \tilde{\mathcal{F}}^\lambda\}. \tag{4-5}$$

is a pointed fan which lives in the vector space $N_\mathbb{Q}(X)/V(\lambda)$ and is called the *localization of* $\mathcal{F}$ *at* λ.

Theorem 4.6. *Let λ and X as in Proposition 4.5. Then $\Xi(X^\lambda) = \Xi(X) \cap V(\lambda)^\perp$, $N_\mathbb{Q}(X^\lambda) = N_\mathbb{Q}(X)/V(\lambda)$, and $\mathcal{F}(X^\lambda) = \mathcal{F}(X)^\lambda$.*

Proof. Let z be an arbitrary ${}^-B$-fixed point in X. Then z corresponds to the complete orbit Gz and therefore to a cone $\mathcal{C}_{Gz}$ of maximal dimension in $\mathcal{F}(X)$. Let P be the parabolic which is opposite to the reduced stabilizer G_z^{red}. Then the local structure theorem [Knop 1993, Satz 1.2] asserts the existence of a normal affine T-variety $\overline{A}$ and a T-equivariant morphism $\varphi_0 : \overline{A} \to X$ such that the morphism

$$\varphi : P_u \times \overline{A} \to X : (u, a) \mapsto u\varphi_0(a) \tag{4-6}$$

is finite onto an open neighborhood of z in X. Moreover, the torus T has an open orbit A in $\overline{A}$ such that $\Xi_{\mathbb{Q}}(A) = \Xi_{\mathbb{Q}}(X)$. The embedding $A \hookrightarrow \overline{A}$ corresponds to the cone $\mathscr{C}_{Gz}$. In particular, $\overline{A}$ contains a unique T-fixed point, denoted by 0, such that $\varphi_0(0) = z$.

From this we see that z lies in the source of λ on X if and only if λ has a source in $P_u \times \overline{A}$. This is automatically the case for P_u since λ is dominant and acts by conjugation. The fixed point set is $P_u^\lambda = P_u \cap G^\lambda$. On the other hand, we have

$$f_\chi(\lambda(t)a) = t^{-\lambda^r(\chi)} f_\chi(a). \tag{4-7}$$

For the limit when $t \to 0$ to exist for all $a \in \overline{A}$, it is necessary and sufficient that $-\lambda^r(\chi) \geq 0$ for all $\chi \in \Xi_{\mathbb{Q}}(\overline{A})$ with $f_\chi \in \mathbb{O}(\overline{A})$. This condition boils down to $-\lambda^r \in \mathscr{C}_{Gz}$. In that case, one readily checks that the fixed point set $\overline{A}^\lambda$ is the closure of the orbit corresponding to the face $\mathscr{C}(\lambda)$ of $\mathscr{C}_{Gz}$. The restricted morphism

$$\varphi^\lambda : P_u^\lambda \times \overline{A}^\lambda \to X^\lambda \tag{4-8}$$

describes the local structure of X^λ in a neighborhood of z.

From this we already infer that $\Xi_{\mathbb{Q}}(X^\lambda) = \Xi_{\mathbb{Q}}(\overline{A}^\lambda) = V(\lambda)^\perp$. We claim that $\Xi(X^\lambda) = \Xi_{\mathbb{Q}}(X^\lambda) \cap \Xi(X)$, which then proves the assertion $\Xi(X^\lambda) = \Xi(X) \cap V(\lambda)^\perp$. In fact, only "$\supseteq$" is an issue. To prove it, let $\chi \in \Xi_{\mathbb{Q}}(X^\lambda) \cap \Xi(X)$. Then there are $n \in \mathbb{Z}_{>0}$ and rational semiinvariants f_χ on X and $f_{n\chi}$ on X^λ such that $f_\chi^n = \pi_\lambda^* f_{n\chi}$. Let $X' \subseteq X^\lambda$ be the open subset on which $f_{n\chi}$ is regular. Then the normality of X implies that f_χ is regular on $\pi_\lambda^{-1}(X')$. Since f_χ is also λ-invariant, we conclude that f_χ pushes down to a rational function on X^λ, which shows $\chi \in \Xi(X^\lambda)$, as claimed. The equality $N_{\mathbb{Q}}(X^\lambda) = N_{\mathbb{Q}}(X)/V(\lambda)$ follows immediately.

Finally, we compute the fan $\mathscr{F}(X^\lambda)$. Clearly, it suffices to determine its cones $\mathscr{C}$ of maximal dimension corresponding to closed orbits. Lemma 4.4 and the discussion above show that the closed G^λ-orbits in X^λ correspond precisely to those closed G-orbits Gz in X such that $-\lambda^r \in \mathscr{C}_{Gz}$. In that case, it is easy to check that the toroidal embedding $A^\lambda \hookrightarrow \overline{A}^\lambda$ corresponds to the cone $(\mathscr{C}_{Gz} + \mathbb{Q}_{\geq 0}\lambda)/V(\lambda)$. But these are precisely the cones of maximal dimension in $\mathscr{F}^\lambda$, which shows $\mathscr{F}(X^\lambda) = \mathscr{F}(X)^\lambda$. □

Corollary 4.7. *Let λ and X be as above. Then*

$$\Sigma(X^\lambda) = \Sigma(X) \cap V(\lambda)^\perp = \Sigma(X) \cap \lambda^\perp. \tag{4-9}$$

Proof. The valuation cone $\mathscr{V}(X^\lambda)$ equals the support of $\mathscr{F}(X)^\lambda$. Its codimension-1 faces are, by construction, the codimension-1 faces of $\mathscr{V}(X)$ which contain $\mathscr{C}(\lambda)$. From $\Xi_p(X^\lambda) = \Xi_p(X) \cap V(\lambda)^\perp$ we get $\Sigma(X^\lambda) = \Sigma(X) \cap V(\lambda)^\perp$. The second equality follows from the fact that $\langle \sigma, \mathscr{V}(X) \rangle \geq 0$ for all $\sigma \in \Sigma(X)$. Hence $\langle \sigma, V(\lambda) \rangle = 0$ if and only if $\langle \sigma, \mathscr{C}(\lambda) \rangle = 0$ if and only if $\langle \sigma, \lambda \rangle = 0$. □

Proposition 4.8. *Let λ and X be as above. Then $GX^\lambda = \overline{Y}_0$, where $Y_0 \subseteq X$ is the G-orbit with $\mathscr{C}_{Y_0} = \mathscr{C}(\lambda)$. Moreover, for any G-orbit $Y \subseteq X$,*

$$\mathscr{C}(\lambda) \subseteq \mathscr{C}_Y \iff X^\lambda \cap Y \neq \varnothing, \tag{4-10}$$

$$\mathscr{C}(\lambda) \supseteq \mathscr{C}_Y \iff X^\lambda \subseteq \overline{Y}. \tag{4-11}$$

Proof. First note that X^λ is stable under ${}^-B = {}^-B^\lambda\,{}^-G^u_\lambda$. Hence GX^λ is closed, and hence an orbit closure $\overline{Y_0}$, in X. Choose a closed orbit Gz in $\overline{Y_0}$. The orbits of X which contain Gz in their closure correspond precisely to the T-orbits in the slice $\overline{A}$. It follows from (4-8) that in this way, $\overline{Y_0}$ corresponds to $\overline{A}^\lambda$, which shows $\mathscr{C}_{Y_0} = \mathscr{C}(\lambda)$, as claimed.

The two equivalences follow easily: we have $X^\lambda \cap Y \neq \varnothing$ if and only if $Y \subseteq GX^\lambda = \overline{Y_0}$ if and only if $\mathscr{C}_Y \supseteq \mathscr{C}(\lambda)$; and $X^\lambda \subseteq \overline{Y}$ if and only if $\overline{Y_0} = GX^\lambda \subseteq \overline{Y}$ if and only if $\mathscr{C}(\lambda) \supseteq \mathscr{C}_Y$. □

Next we compute the colors of X^λ. Let $D \subseteq X^\lambda$ be a color and let D_0 be its restriction to the open G^λ-orbit. Then $\pi_\lambda^{-1}(D_0)$ is irreducible by Lemma 4.2. Hence its closure $D^* \subset X$ is a B-stable prime divisor.

Proposition 4.9. *Let λ and X be as above.*

a) *Let $\alpha \in S^\lambda := S(G^\lambda) = S \cap \lambda^\perp$ and let D be a color of X^λ. Then*
   i) *α has the same type for X^λ as it has for X,*
   ii) *δ_D is the restriction of δ_{D^*} to $\Xi_{\mathbb{Q}}(X^\lambda) \subseteq \Xi_{\mathbb{Q}}(X)$, and*
   iii) *D is moved by α if and only if D^* is moved by α. In that case $q_{D,\alpha} = q_{D^*,\alpha}$.*

b) *The map $D \mapsto D^*$ is a bijection between the set of colors of X^λ and the set of colors of X which are moved by some $\alpha \in S^\lambda$.*

Proof. The first part of iii) follows from $P_\alpha \subseteq G_\lambda$ and the equivariance of π_λ. Then b) is an immediate consequence. For ii), recall that $\pi_\lambda^{-1}(D_0)$ is even a reduced divisor by Lemma 4.2. Thus, $v_{D^*}(\pi_\lambda^* f) = v_D(f)$ for all $f \in k(X^\lambda)$. Moreover, there is a commutative diagram

$$\begin{array}{ccc} P_\alpha \times^B D^* & \xrightarrow{\varphi_{D^*,\alpha}} & X \\ \uparrow & & \uparrow \\ P_\alpha \times^B \pi_\lambda^{-1}(D_0) & \longrightarrow & X_\lambda \\ \downarrow & & \downarrow{\scriptstyle \pi_\lambda} \\ P_\alpha \times^B D_0 & \longrightarrow & X^\lambda \\ \downarrow & & \| \\ P_\alpha^\lambda \times^{B^\lambda} D & \xrightarrow{\varphi^\lambda_{D,\alpha}} & X^\lambda \end{array} \tag{4-12}$$

where the middle square is cartesian and the injections are open embeddings. It follows that $\varphi^{\lambda}_{D,\alpha}$ and $\varphi_{D^*,\alpha}$ have the same inseparable degree, showing the second part of iii). Both morphisms also have the same separable degree (1 or 2). Thus, α is of type $(2a)$ for X^{λ} if and only if it is of type $(2a)$ for X, proving part of i). The other types (p), (b), and (a) are distinguished by the number of colors (0, 1, and 2, respectively) moved by α. Thus, the rest of i) follows from iii). □

5. The interrelation of roots and colors

In practice, the 1-parameter subgroup λ has to be chosen diligently.

Lemma 5.1. *Let X be a complete toroidal spherical G-variety and $S' \subseteq S$. Then there is a 1-parameter subgroup λ such that*

a) *$S(G^{\lambda}) = S'$ and*

b) *the connected center of G^{λ} acts trivially on X^{λ}.*

Proof. Let $F \subseteq N_{\mathbb{Q}}(T) := \mathrm{Hom}(\Xi(T), \mathbb{Q})$ be the open face of the Weyl chamber defined by $\alpha = 0$ for all $\alpha \in S'$ and $\alpha > 0$ for $\alpha \in S \setminus S'$. Then $\lambda \in F$ is equivalent to a) (such λ are called *adapted* to S').

Now consider the projection $\pi : N_{\mathbb{Q}}(T) \twoheadrightarrow N_{\mathbb{Q}}(X)$. Since $\pi(F) \subseteq \mathcal{V}(X)$, the fan $\mathcal{F}$ associated to X induces a complete fan $\mathcal{F}'$ on $\pi(F)$. Let $F^0 := F \setminus \bigcup \pi^{-1}(\mathcal{C})$, where $\mathcal{C}$ runs through all $\mathcal{C} \in \mathcal{F}'$ with $\dim \mathcal{C} < \dim \pi(F)$, which is obviously a dense open subset of F. We claim that $\lambda \in F^0$ ensures the second property b).

Indeed, $\pi(\lambda)$ lies by construction in the relative interior of a maximal dimensional cone $\mathcal{C}'$ of $\mathcal{F}'$. This implies that $\pi(F) \subseteq V'$, where V' is the span of $\mathcal{C}'$. Now let $\mathcal{C} \in \mathcal{F}$ be minimal with $\mathcal{C}' \subseteq \mathcal{C}$. Then $\pi(\lambda)$ is also in the relative interior of $\mathcal{C}$. Thus, the subspace V spanned by $\mathcal{C}$ contains $\pi(F)$. Hence $\langle F \rangle_{\mathbb{Q}} \subseteq \pi^{-1}(V)$, which implies b). □

If the fan is changed, one can do better.

Lemma 5.2. *Let X_0 be a homogeneous spherical G-variety and $S' \subseteq S$. Then there is a 1-parameter subgroup λ and a toroidal compactification X of X_0 such that*

a) *$S(G^{\lambda}) = S'$ and*

b) *$\Xi_{\mathbb{Q}}(X^{\lambda}) = \langle \Sigma(X^{\lambda}) \rangle_{\mathbb{Q}}$.*

Proof. Same construction as above, but this time we choose $\mathcal{F}$ such that $\dim \mathcal{C}$ is as large as possible, that is, the dimension of the smallest face of $\mathcal{V}$ which contains $\mathcal{C}'$. This means precisely b). □

Remark. A rather trivial application of the last lemma is when $S' = S$. Then X^{λ} has the same roots and colors as X_0 but $\Xi_{\mathbb{Q}}(X^{\lambda})$ is spanned by $\Sigma(X)$.

Following Luna [1997], an important application of this technique is:

Proposition 5.3. *Let $\alpha \in S$ be a simple root. If $p \neq 2$, then*:

a) $\alpha \in \Sigma(X)$ *if and only if α is of type (a). Thus* $S^{(a)} = S \cap \Sigma(X)$.

b) $2\alpha \in \Sigma(X)$ *if and only if α is of type $(2a)$. Thus* $S^{(2a)} = S \cap \frac{1}{2}\Sigma(X)$.

If $p = 2$, then:

c) $\alpha \in \Sigma(X)$ *if and only if α is of type (a) or $(2a)$. Thus* $S^{(a)} \cup S^{(2a)} = S \cap \Sigma(X)$.

Proof. Without loss of generality, we may replace X by a toroidal compactification of its open G-orbit. The corresponding fan is denoted by $\mathcal{F}$. Choose λ as in Lemma 5.1 with $S' := \{\alpha\}$. Then G^λ acts on X^λ only via a semisimple quotient G_0 of rank 1. Let G_0/H_0 be the open G_0-orbit in X^λ.

Assume first $p \neq 2$. Then

$$\alpha \in S^{(a)}(X) \iff \alpha \in S^{(a)}(X^\lambda) \iff H_0^{\mathrm{red}} \sim T_0 \iff \alpha \in \Sigma(X^\lambda) \iff \alpha \in \Sigma(X).$$

This proves a).

The argument for b) is analogous, with T_0 replaced by $N(T_0)$. Finally, for $p = 2$ we argue with $\Sigma(G_0/T_0) = \Sigma(G_0/N(T_0)) = \{\alpha\}$. □

Remarks. 1. The proposition shows that in case $p \neq 2$, the type of the simple roots can be recovered from $S^{(p)}$ and $\Sigma(X)$ as follows:

$$\alpha \in S \text{ is of type } \begin{cases} (p) & \text{if } \alpha \in S^{(p)}, \\ (a) & \text{if } \alpha \in \Sigma(X), \\ (2a) & \text{if } 2\alpha \in \Sigma(X), \\ (b) & \text{otherwise.} \end{cases} \tag{5-1}$$

This way, all colors can be recovered, but some might appear multiple times (see Lemma 2.5). For colors of type (b), that behavior is controlled by $\Sigma(X)$ as well (see Proposition 5.4 below).

2. For $p = 2$ and $\alpha \in S^{(2a)}$, it is tempting to define the corresponding spherical root to be 2α instead of α. This would make parts a) and b) of Proposition 5.3 work uniformly in all characteristics. We opted against this procedure. The main reason is that otherwise spherical roots would not be roots (possibly not simple) of some root system. Example: For $p = 2$ and $X = \mathrm{SL}(3)/\mathrm{SO}(3)$, the two roots are α_1 and $\alpha_1 + \alpha_2$ (see [Schalke 2011]), which are visibly contained in an A_2-root system. On the other hand, the root α_1 is of type $(2a)$ and the set $\{2\alpha_1, \alpha_1 + \alpha_2\}$ is not part of any root system.

Proposition 5.4. *For two distinct simple roots $\alpha_1, \alpha_2 \in S^{(b)}$, there is equivalence between*:

a) α_1 *and* α_2 *move the same color D.*

b) α_1 *and* α_2 *are orthogonal to each other and* $q_1\alpha_1 + q_2\alpha_2 \in \Sigma(X)$ *for two* p*-powers* q_1, q_2 (*one of which is necessarily equal to* 1).

If these conditions hold, *we have*

$$q_1^{-1}\alpha_1^r = q_2^{-1}\alpha_2^r, \tag{5-2}$$

and D *is not moved by any other simple root.*

Proof. Again replace X by a toroidal compactification and choose λ as in Lemma 5.1 with $S' = \{\alpha_1, \alpha_2\}$. Now G^λ acts on X^λ via a semisimple group G_0 of rank 2. The simple roots of G_0 are α_1 and α_2, and both are of type (b) with respect to X^λ.

Assume first b). Then G_0 is of type $\mathsf{A}_1\mathsf{A}_1$. An easy inspection of its subgroups shows that X^λ has a spherical root of the given form if and only if its open orbit is isogenous to $\mathrm{SL}(2) \times \mathrm{SL}(2)/(F_{q_1} \times F_{q_2})\,\mathrm{SL}(2)$ (with $F_q =$ Frobenius morphism of $\mathrm{SL}(2)$). In that case one checks that a) holds for X^λ and therefore for X.

Conversely assume a). Then X^λ has precisely one color D which is moved by both simple roots. Thus, (2-14) implies that a relation like (5-2) holds. In particular, the rank of X^λ is 1.

Now one could use the classification of spherical varieties of rank ≤ 1 in [Knop 2013] and conclude that G_0 is of type $\mathsf{A}_1\mathsf{A}_1$ having a root of the given form. A self-contained argument goes as follows. We may assume that $X = G/H$ is homogeneous, where H is reduced and connected. The color and the half-line $\mathcal{V}(X)$ lie opposite to each other. By [Knop 1991, Theorem 6.7], the variety X is affine, and thus H is reductive. Since there is only one color, (2-3) implies that H is semisimple. Moreover, the dimension formula [Knop 1991, Theorem 6.6] shows that $\dim H = 3, 4, 5, 7$ for $G = \mathsf{A}_1\mathsf{A}_1, \mathsf{A}_2, \mathsf{B}_2, \mathsf{G}_2$, respectively. This shows that G is isogenous to $\mathrm{SL}(2) \times \mathrm{SL}(2)$ and that $H \cong \mathrm{SL}(2)$ is embedded diagonally using the Frobenius morphisms. The assertion b) follows.

Formula (5-2) follows immediately from (2-14). Finally, assume $\alpha_3 \in S$ moves D as well. Then $\alpha_3 \in S^{(b)}$ (Lemma 2.5). Thus, by the above, α_3 would be orthogonal to α_1 and α_2. Moreover, $q_1'\alpha_1 + q_3\alpha_3 \in \Sigma(X)$ for some p-powers q_1', q_3. Now (5-2) implies the contradiction

$$2q_1'q_1^{-1} = \langle q_1'\alpha_1 + q_3\alpha_3, q_1^{-1}\alpha_1^\vee\rangle = \langle q_1'\alpha_1 + q_3\alpha_3, q_2^{-1}\alpha_2^\vee\rangle = 0. \qquad \square$$

6. Localization at Σ

Localization at S allows one to pass from S to a subset. There is a second, older, kind of localization which does the same thing with $\Sigma(X)$. Geometrically, it simply corresponds to looking at an orbit in a carefully chosen toroidal embedding. The next result summarizes what was already known about localization at Σ:

Proposition 6.1. *Let X be a toroidal spherical variety and let $Y \subseteq X$ be an orbit. Put $V := \langle \mathscr{C}_Y \rangle^\perp \subseteq \Xi_\mathbb{Q}(X)$. Then*:

a) $\Xi_p(Y) = \Xi_p(X) \cap V$.

b) $\Sigma(Y) = \Sigma(X) \cap V$.

c) $S^{(p)}(Y) = S^{(p)}(X)$.

Proof. Part a) follows, for example, from [Knop 1991, Theorem 1.3]. Moreover, $\mathscr{V}(Y) = (\mathscr{V}(X) + V)/V$, where $V = \langle \mathscr{C} \rangle_\mathbb{Q}$ (this follows from [Knop 1993, Satz 7.4]), which implies b). Part c) follows, for example, from the fact that all closed orbits in any toroidal compactification of X are of the form G/Q with $Q^{\text{red}} = {}^-P$, where P is the parabolic corresponding to $S^{(p)}$. □

If $p \neq 2$, then the remark after Proposition 5.3 allows us now to determine the type of a simple root for Y.

$$S^{(p)}(Y) = S^{(p)}(X), \tag{6-1}$$

$$S^{(a)}(Y) = S^{(a)}(X) \cap V, \tag{6-2}$$

$$S^{(2a)}(Y) = S^{(2a)}(X) \cap V, \tag{6-3}$$

$$S^{(b)}(Y) = S^{(b)}(X) \cup \big(S^{(a)}(X) \setminus V\big) \cup \big(S^{(2a)}(X) \setminus V\big). \tag{6-4}$$

For $p = 2$, equations (6-2) and (6-3) have to be replaced by the weaker equality

$$S^{(a)}(Y) \cup S^{(2a)}(Y) = \big(S^{(a)}(X) \cap V\big) \cup \big(S^{(2a)}(X) \cap V\big). \tag{6-5}$$

The next lemma shows (in particular) that moreover

$$S^{(2a)}(Y) \subseteq S^{(2a)}(X) \cap V. \tag{6-6}$$

Lemma 6.2. *Let X and Y be as above and let $\alpha \in S \cap V$ be of type* (*a*) *for X. Then α is also of type* (*a*) *for Y. Moreover, let D be a color of X which is moved by α. Then $E = (D \cap Y)^{\text{red}}$ is a color of Y which is moved by α and there is a p-power q such that δ_E is the restriction of $q\delta_D$ to $\Xi_p(Y)$.*

Proof. We plan to use localization at S but face the problem that $\mathscr{C}_Y$ may not meet $N_\mathbb{Q}^-(X)$, the image of the antidominant Weyl chamber in $N_\mathbb{Q}(X)$. To bypass this problem, we go one dimension up: the group $\overline{G} := G \times \boldsymbol{G}_m$ acts on $\overline{X}^0 := X \times \boldsymbol{G}_m$. Then $N_\mathbb{Q}(\overline{X}^0) = N_\mathbb{Q}(X) \oplus \mathbb{Q}$ and $\mathscr{V}(\overline{X}^0) = \mathscr{V}(X) \times \mathbb{Q}$. Now choose any v_0 in the relative interior of $N^-(X) \cap \{\alpha = 0\} \subseteq \mathscr{V}(X)$ and put $\overline{\mathscr{C}} := (\mathscr{C} \times 0) + \mathbb{Q}_{\geq 0}(v_0, 1)$. Choose any fan $\mathscr{F}$ whose support is $\mathscr{V}(\overline{X}^0)$ and which contains $\overline{\mathscr{C}}$. This gives rise to a toroidal $\overline{G}$-variety $\overline{X}$. Moreover, $\overline{X}$ contains $Y \times \boldsymbol{G}_m$ since $\mathscr{C}_Y$ is a face of $\overline{\mathscr{C}}$. Its closure is denoted by $\overline{Y}$.

Now choose any $v \in \mathscr{C}^0$ small enough that $v + v_0$ is still in the relative interior of $N^-(X) \cap \{\alpha = 0\}$, and choose $a \in \mathbb{Z}_{>0}$ such that $v_1 := a(v + v_0, 1)$ is the image of a 1-parameter subgroup λ of $\overline{G}$. Then, by construction, $S^\lambda = \{\alpha\}$ and $\overline{G}^\lambda$ is of semisimple rank 1. Moreover, $\overline{X}^\lambda$ is contained in $\overline{Y}$ by Proposition 4.8. Thus, we get a diagram

$$\begin{array}{ccccc} \tilde{Y}_\lambda & \twoheadrightarrow & \overline{Y}_\lambda & \hookrightarrow & \overline{X}_\lambda \\ \downarrow{\scriptstyle \tilde{\pi}_\lambda} & & \downarrow{\scriptstyle \pi_\lambda} & & \downarrow{\scriptstyle \pi_\lambda} \\ \tilde{Y}^\lambda & \overset{\nu}{\twoheadrightarrow} & \overline{X}^\lambda & = & \overline{X}^\lambda \end{array} \tag{6-7}$$

where $\tilde{Y}$ is the normalization of $\overline{Y}$ and where the vertical arrows represent Białynicki-Birula contractions on the open cell. By Proposition 4.9, the type of α on $\overline{X}^\lambda$ is (a). This means that the open $\overline{G}^\lambda$-orbit in $\overline{X}^\lambda$ is of the form $\overline{G}^\lambda/H_0$, where H_0^{red} is diagonalizable.

Now we argue that ν is purely inseparable. Indeed, the open orbit in $\tilde{Y}^\lambda$ is $\overline{G}^\lambda/H_1$, with $H_1^{\mathrm{red}} \subseteq H_0^{\mathrm{red}} \subseteq T$. This already shows that α is of type (a) for $\tilde{Y}$, and hence for $\overline{Y}$ and Y. Since both H_1^{red} and H_0^{red} are linearly reductive abelian groups, we have

$$\big[H_0^{\mathrm{red}} : H_1^{\mathrm{red}}\big] = \big[\Xi_p(\tilde{Y}^\lambda) : \Xi_p(\overline{X}^\lambda)\big]. \tag{6-8}$$

On the other hand,

$$\Xi_p(\tilde{Y}^\lambda) = \Xi_p(\tilde{Y}) = \Xi_p(\overline{Y}) = \Xi_p(\overline{X}) \cap \overline{V} = \Xi_p(\overline{X}^\lambda), \tag{6-9}$$

where $\overline{V} := \langle \overline{\mathscr{C}} \rangle$. This shows that ν is generically injective and therefore purely inseparable.

The color D gives rise to the color $D \times \boldsymbol{G}_m$ of $\overline{X}$. Its image D_0 in $\overline{X}^\lambda$ is a color of $\overline{X}^\lambda$. Now $E_0 = \nu^{-1}(D_0)^{\mathrm{red}}$ is a color of $\tilde{Y}^\lambda$. Clearly $\nu^{-1}(D_0) = qE_0$ for some p-power q. Finally, the closure of $(\tilde{\pi}_\lambda)^{-1}(E_0)$ is a color of $\overline{Y}$ which is of the form $E \times \boldsymbol{G}_m$, where $E = (D \cap Y)^{\mathrm{red}}$.

Recall $V = \langle \mathscr{C} \rangle_{\mathbb{Q}}^\perp$. Then

$$\overline{V} = \{(\chi, -v_0(\chi)) : \chi \in V\} \cong V \tag{6-10}$$

and $\Xi_{\mathbb{Q}}(\overline{X}) = \Xi_{\mathbb{Q}}(X) \oplus \mathbb{Q}$, $\Xi_{\mathbb{Q}}(\overline{Y}) = V \oplus \mathbb{Q}$, and $\Xi_{\mathbb{Q}}(\overline{X}^\lambda)$. Thus, for $\chi \in \Xi_{\mathbb{Q}}(X)$,

$$\delta_D(\chi) = \delta_{D\times \boldsymbol{G}_m}(\chi, 0) = \delta_{D\times \boldsymbol{G}_m}(\chi, -v_0(\chi)) = \delta_{D_0}(\chi, -v_0(\chi)); \tag{6-11}$$

and similarly, $\delta_E(\chi) = \delta_{E_0}(\chi, -v_0(\chi))$ for all $\chi \in V$. Thus, for $\chi \in V$,

$$\delta_D(\chi) = \delta_{D_0}(\chi, -v_0(\chi)) = q\delta_{E_0}(\chi, -v_0(\chi)) = q\delta_E(\chi). \qquad \square$$

Remark. With these results it is possible to recover all colors of Y and which color is being moved by which root. In characteristic zero this is good enough to compute the entire spherical system of Y. In positive characteristic we are missing information on the degrees $q_{D,\alpha}$ of Y, though. We plan to return to this question in the future.

Localization at Σ is, a priori, not possible for all subsets of $\Sigma(X)$. Therefore, we define:

Definition 6.3. A subset Σ' of $\Sigma(X)$ is called a *set of neighbors* if there is $v \in \mathcal{V}(X)$ such that

$$\Sigma' = \{\sigma \in \Sigma(X) : v(\sigma) = 0\}. \tag{6-12}$$

Equivalently, Σ' is a set of neighbors if $\mathbb{Q}_{\geq 0}\Sigma'$ is a face of $\mathbb{Q}_{\geq 0}\Sigma(X)$. Two spherical roots α and β are called *neighbors* if they are distinct and if $\{\alpha, \beta\}$ is a set of neighbors.

Clearly, if $\Sigma(X)$ is linearly independent, then all subsets are sets of neighbors. This is always the case if $p \neq 2$ (see [Brion 1990] for $\operatorname{char} k = 0$ and [Knop 2013, Corollary 4.8] for the general case). For $p = 2$ and $X = \mathrm{SL}(4)/\mathrm{SO}(4)$, Schalke has shown (unpublished) that $\Sigma(X) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3\}$. Since $\alpha_1 + (\alpha_2 + \alpha_3) = (\alpha_1 + \alpha_2) + \alpha_3$, the pairs $(\alpha_1, \alpha_2 + \alpha_3)$ and $(\alpha_1 + \alpha_2, \alpha_3)$ are not neighbors. In fact, $\mathcal{V}(X)$ is the cone over a quadrangle and the given pairs correspond to opposite faces.

Lemma 6.4. *If $\alpha, \beta \in \Sigma(X)$ are multiples of simple roots, then they are neighbors.*

Proof. The set $\mathbb{Q}_{\geq 0}\alpha + \mathbb{Q}_{\geq 0}\beta$ is already a face of $\mathbb{Q}_{\geq 0}S$ and therefore also of the smaller cone $\mathbb{Q}_{\geq 0}\Sigma(X)$. □

The following statement can be used to exclude certain configurations of colors and roots (see [Knop 2013, Proof of Theorem 4.5]).

Proposition 6.5. *Let $D \in \Delta(X)^{(a)}$ be moved by $\alpha \in S^{(a)}$ and let $\sigma \in \Sigma(X)$ be a neighbor of α with $\delta_D(\sigma) > 0$. Then $\sigma \in S^{(a)}$ and D is moved by σ, as well.*

Proof. We first reduce to the case that X is of rank 2 with $\Sigma = \{\alpha, \sigma\}$. Because α and σ are neighbors, one can choose a pointed cone $\mathcal{C}$ inside $\mathcal{V}(X) \cap \{\alpha = \sigma = 0\}$ which is of codimension 2 in $N_{\mathbb{Q}}(X)$. Let $\overline{X}$ be the simple embedding corresponding to $\mathcal{C}$ and let Y be its closed orbit. Then Proposition 6.1 implies $\Sigma(Y) = \{\alpha, \sigma\}$. Moreover, using the remark after Lemma 5.2, there is a spherical variety Y' with $\Sigma(Y') = \{\alpha, \sigma\}$ and $\operatorname{rk} Y' = 2$. Moreover, Lemma 6.2 implies that Y' still has a color E moved by α with $\delta_E(\sigma) = q\delta_D(\sigma) > 0$. Let us assume that we can prove that $\sigma \in S^{(a)}(Y')$ and that E is moved by σ. Clearly, D is moved by σ in X as well. Moreover, since $\alpha \in \Sigma(X)$, either $\alpha \in S^{(a)}(X)$ or $p = 2$ and $\alpha \in S^{(2a)}(X)$. But the latter case cannot happen, since then D could not be moved by any other simple root (Lemma 2.5). So $\sigma \in S^{(a)}(X)$, which finishes the reduction step.

From now on we assume that $\operatorname{rk} X = 2$ and, without loss of generality, that $X = G/H$ where H is reduced. Since $\delta_D(\sigma) > 0$ by assumption and $\delta_D(\alpha) = q_{D,\alpha} > 0$, the cone generated by $\mathcal{V}(X)$ and δ_D is the entire space $N := N_{\mathbb{Q}}(X)$. From that we get a morphism $X = G/H \to G/P$ with $\operatorname{rank} G/P = \dim N/N = 0$ [Knop 1991, Corollary 4.6].[1] Hence P is a parabolic subgroup with an identification $\Delta(G/P) = \Delta(G/H) \setminus \{D\}$. We may choose P in such a way that it is *opposite* to B.

Every $\beta \in S \setminus S^{(p)}$ moves at least one color and α moves even two. Assume first that these colors are *not* all different. Then, according to Lemma 2.5 and Proposition 5.4, there are two possibilities:

a) $\sigma = \alpha_1 + q\alpha_2$ with $\alpha_1, \alpha_2 \in S$ orthogonal. But then α_1 and α_2 would move the same color in G/P, which is impossible.

b) $S^{(a)}$ contains another element β besides α. But then $\beta \in \Sigma(X)$ (Proposition 5.3), and hence $\sigma = \beta \in S^{(a)}$. Moreover, there is a color D' moved by both α and σ. We claim that $D' = D$. Suppose not. Then D and D' are the two colors moved by α, and Proposition 2.3 implies the contradiction

$$\delta_D^{(\alpha)}(\sigma) = \langle \sigma, \alpha^\vee \rangle - \delta_{D'}^{(\alpha)}(\sigma) = \langle \sigma, \alpha^\vee \rangle - \frac{q_{D',\alpha}}{q_{D',\sigma}} \delta_{D'}^{(\sigma)}(\sigma) = \langle \sigma, \alpha^\vee \rangle - \frac{q_{D',\alpha}}{q_{D',\sigma}} < 0. \quad (6\text{-}13)$$

Thus, we are exactly in the asserted situation, that is, $\sigma \in S^{(a)}$ moving D.

So, assume from now on that the colors moved by all the $\beta \in S \setminus S^{(p)}$ are different. Then

$$\#\Delta(G/P) \geq \#S \setminus S^{(p)}. \quad (6\text{-}14)$$

Consider a toroidal completion $\overline{X}$ of X. Then the morphism $X \to G/P$ extends to $\overline{X}$ [Knop 1991, Theorem 4.1]. Every closed orbit in $\overline{X}$ is isogenous to $G/{}^{-}P_X$, where P_X is the parabolic attached to $S^{(p)}$. Hence $P_X \subseteq P^{\mathrm{red}}$. Thus (6-14) implies $P = P_X$.

Let $Y = G/H_1 \subset \overline{X}$ be the rank-1 orbit corresponding to the half-line $\mathcal{V}(X) \cap \{\sigma = 0\}$. Then $\Sigma(Y) = \{\sigma\}$. Because of $P = P_X$, the fiber P_X/H_1^{red} is one-dimensional and therefore isomorphic to $\boldsymbol{P}^1$, $\boldsymbol{G}_m$, or $\boldsymbol{A}^1$. The first case is impossible since H_1 is not parabolic. The second case is excluded since H_1 is not horospherical. Thus $P_X/H_1^{\mathrm{red}} \cong \boldsymbol{A}^1$.

This means in particular that (a conjugate of) H_1 contains the maximal torus T of G and that H_1 contains all root subgroups U_β which are contained in P_X except for one, which is denoted by γ, and which lies in P_X^u. The U_β corresponding to $\beta \in S$ generate the maximal unipotent subgroup of G. This implies $\gamma \in S$. Moreover, U_γ is a 1-dimensional module for the Levi part of P_X. This shows that H_1^{red} is, in fact, induced from $\mathrm{PGL}(2)/\boldsymbol{G}_m$ (on induction in arbitrary characteristic, see [Knop 2013, §2, in particular Lemma 2.1]). Hence $\sigma \in S^{(a)}$. But in that case, (6-14) would be a strict inequality, which is not true because of $P = P_X$. □

[1] The idea for this construction is due to Guido Pezzini.

Proposition 6.5 can be used to give bounds for $\delta_D(\sigma)$:

Corollary 6.6. *With α and σ as above, assume that $\sigma \notin S$ or that $\sigma \in S$ but does not move either color moved by α. Then*

$$q_{D,\alpha}^{-1}\langle \sigma, \alpha^\vee \rangle \le \delta_D(\sigma) \le 0. \tag{6-15}$$

Proof. The right-hand inequality follows directly from Proposition 6.5. For the left-hand inequality, apply Proposition 6.5 to the other color D' moved by α and observe that $\delta_{D'}^{(\alpha)} = \alpha^r - \delta_D^{(\alpha)}$ (Proposition 2.3). □

In positive characteristic, these bounds are less valuable since we didn't derive a bound on the denominator of $\delta_D(\sigma)$. Such a bound exists (in terms of the $q_{D,\alpha}$'s) and will be included in a future paper. Then (6-15) leaves only finitely many possibilities for $\delta_D(\sigma)$.

7. The p-spherical system

We summarize what we have proved so far in terms of a combinatorial structure which generalizes Luna's spherical systems. But first we need some more terminology:

Definition 7.1. Let G be a connected reductive group.

a) An element $\sigma \in \Xi_{\mathbb{Q}}(T)$ is called a *spherical root for G* if there is a spherical G-variety X such that σ is a spherical root for X. The set of spherical roots for G is denoted by $\Sigma(G)$.

b) A spherical root $\sigma \in \Sigma(G)$ is *compatible* with a subset $S^{(p)} \subseteq S$ if there is a spherical G-variety X with $\sigma \in \Sigma(X)$ and $S^{(p)} = S^{(p)}(X)$.

Remarks. 1. Proposition 6.1 shows that in the definition above, one may assume, without loss of generality, $\operatorname{rk} X = 1$.

2. Spherical varieties of rank 1 have been classified by Akhiezer [1983] in characteristic zero and Knop [2013] in general. In particular, for every G, there is a complete description of $\Sigma(G)$ (see [Knop 2013, §2 and §7]).

3. One result of that classification is that $\Sigma(G)$ is infinite unless $\operatorname{char} k = 0$ or G is simple of rank ≤ 2.

For the following, recall that $\mathbb{Z}_p = \mathbb{Z}[1/p]$ and $\Xi_p := \Xi \otimes \mathbb{Z}_p$ for any abelian group Ξ.

Definition 7.2. Let $p \neq 2$. Then a p-spherical system for G consists of

- a subgroup $\Xi \subseteq \Xi(T)$,
- a subset $\Sigma \subseteq \Xi_p \cap \Sigma(G)$,
- a subset $S^{(p)} \subseteq S$,

- a finite set $\Delta^{(a)}$,
- a map $\delta : \Delta^{(a)} \to \mathrm{Hom}(\Xi, \mathbb{Z}) : D \mapsto \delta_D$, and
- a map $S \setminus (S^{(p)} \cup S^{(a)}) \to p^{\mathbb{N}} : \alpha \mapsto q_\alpha$, where $S^{(a)} := S \cap \Sigma$.

Of course, these data are subject to some conditions. Here, we list only those which are straightforward generalizations of Luna's axioms. It is safe to say that more axioms have to be imposed which deal specifically with issues of positive characteristic. We keep the notation that α^r denotes the restriction of $\alpha^\vee$ to Ξ.

A1 All $\sigma \in \Sigma$ are primitive vectors of $\mathbb{Z}S \cap \Xi_p$.

A2 $\alpha^r = 0$ for all $\alpha \in S^{(p)}$.

A3 Every $\sigma \in \Sigma$ is compatible with $S^{(p)}$.

A4 For all $D \in \Delta^{(a)}$ and $\sigma \in \Sigma \setminus S^{(a)}$, we have $\delta_D(\sigma) \le 0$.

A5 For every $\alpha \in S^{(a)}$, there are exactly two $D \in \Delta^{(a)}$ with $\delta_D(\alpha) > 0$. Conversely, for every $D \in \Delta^{(a)}$, there is at least one $\alpha \in S^{(a)}$ with $\delta_D(\alpha) > 0$.

A6 For $\alpha \in S^{(a)}$, let $D^+ \ne D^- \in \Delta^{(a)}$ with $\delta_{D^\pm}(\alpha) > 0$. Then $q_{\alpha,D^\pm} := \delta_{D^\pm}(\alpha)^{-1} \in p^{\mathbb{N}}$ and $q_{\alpha,D^+}\delta_{D^+} + q_{\alpha,D^-}\delta_{D^-} = \alpha^r$.

A7 Let $\alpha \in S$ with $2\alpha \in \Sigma$. Then $\alpha \notin \Xi_p$ and $\frac{1}{2}\alpha^r(\Xi_p) \subseteq \mathbb{Z}_p$. Moreover, $\alpha^r(\sigma) \le 0$ for all $\sigma \in \Sigma \setminus \{2\alpha\}$.

A8 Let $q\alpha_1 + \alpha_2 \in \Sigma$ with $\alpha_1 \perp \alpha_2$. Then $q^{-1}\alpha_1^r = \alpha_2^r$ and $q^{-1}q_{\alpha_1} = q_{\alpha_2}$.

The point is, of course, that for $p \ne 2$ every homogeneous spherical variety, X gives rise to a p-spherical system. More specifically, we put

$$\begin{aligned} \Xi &:= \Xi(X), \qquad \Sigma := \Sigma(X), \qquad S^{(p)} := S^{(p)}(X), \\ \Delta^{(a)} &:= \Delta^{(a)}(X), \qquad \delta_D := \delta_D^X. \end{aligned} \tag{7-1}$$

The only new constituents are the p-powers. For $\alpha \in S \setminus (S^{(p)} \cup S^{(a)}) = S^{(b)} \cup S^{(2a)}$, we define q_α as $q_{\alpha,D}$ from Proposition 2.3, where D is the unique color moved by α.

Now we verify all axioms.

A1 Holds by definition of $\Sigma(X)$.

A2 See Proposition 2.3.

A3 Follows from the definition of "compatibility".

A4 This is Corollary 6.6 in conjunction with [Knop 2013, Corollary 4.8], which implies that for $p \ne 2$, any two spherical roots are neighbors.

A5 The first part follows also from Corollary 6.6 and Proposition 2.3. The second part holds by definition of $\Delta^{(a)}(X)$.

A6 This follows from Proposition 2.3.

A7 The first part is Proposition 5.3b) and Corollary 2.4. The second follows from [Knop 2013, Theorem 4.5].

A8 This is Propositions 5.4 and 2.3.

Remarks. 1. In characteristic 0, Luna [2001, 5.1] used Wasserman's tables [1996] of spherical rank-2 varieties to verify the axioms. So our approach is more conceptual in that it uses only the classification of rank-1 but not of rank-2 varieties.[2]

2. The case $p = 2$ requires some modifications. To distinguish simple roots of type (a) and $(2a)$, we redefine $S^{(a)}$ as

$$S^{(a)} := \{\alpha \in S \cap \Sigma : \delta_D(\alpha) > 0 \text{ for some } D \in \Delta^{(a)}\}. \tag{7-2}$$

This works indeed for spherical systems coming from spherical varieties: Suppose there are $\alpha \in S^{(2a)}(X)$ and $D \in \Delta^{(a)}$ with $\delta_D(\alpha) > 0$. Then D is moved by some $\beta \in S^{(a)}$. Since α and β are neighbors (Lemma 6.4), we get a contradiction to Proposition 6.5.

With this change, all axioms hold for $p = 2$ except for one: in A4, one has to require that σ and α are neighbors. Observe that A7 is vacuously satisfied.

3. It is a natural question whether spherical varieties are classified by their p-spherical system. In characteristic zero, the answer is "yes" according to work by Luna [2001], Losev [2009], Cupit-Foutou [2010], and Bravi and Pezzini [2011a; 2011b; 2011c]. For $p \neq 2$ or 3, it might be possible that the p-spherical system determines the variety uniquely. For example, all complete homogeneous varieties are classified by p-spherical systems with $\Xi = 0$ (see the example before Lemma 2.2). Furthermore, the author convinced himself that this also holds for spherical varieties of rank 1. If $p = 2$ or $p = 3$, then uniqueness does not even hold for complete homogeneous varieties (see [Wenzel 1994, Proposition 4]) due to exceptional isogenies. If $p = 2$, then uniqueness is wrong already for $G = \mathrm{SL}(2)$, as then G contains nonstandard horospherical subgroup schemes (see [Knop 1995a]).

4. The above list of axioms A1–A8 is definitely only preliminary. Even in the rank-1 case, they do not suffice. For example, there is no axiom bounding the lattice Ξ from below. We plan to return to this problem in the future.

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[2]Note that the tables in [Wasserman 1996] are slightly incomplete: of the series $G = \mathrm{Sp}(2n) \times \mathrm{Sp}(2)$, $H = \mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$ with $n \geq 2$, only the first case, $n = 2$, is stated. I would like to thank Guido Pezzini for pointing that out to me.

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