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Lefschetz operator and local Langlands modulo ℓ: the limit case

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Lefschetz operator and local Langlands modulo ℓ: the limit case

Jean-François Dat

Let K be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and let ℓ be a prime such that $q \equiv 1 (\mathrm{mod}\ \ell)$. We investigate the cohomology of the Lubin–Tate towers of K with coefficients in $\overline{\mathbb{F}}_\ell$, and we show how it encodes Vignéras' Langlands correspondence for unipotent $\overline{\mathbb{F}}_\ell$-representations.

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1. Main results

This paper is part of a project, outlined in [Dat 2012b], that aims at providing a geometric interpretation of the Vignéras correspondence for modulo-ℓ representations of p-adic linear groups.

1.1. *Motivation.*

1.1.1. *The problem.* Let K be a local p-adic field, ℓ a prime distinct from p, and $d \geqslant 1$ an integer. Vignéras [2001] established a bijection between (classes of) irreducible smooth $\overline{\mathbb{F}}_\ell$-representations of $\mathrm{GL}_d(K)$ and (classes of) d-dimensional Weil–Deligne $\overline{\mathbb{F}}_\ell$-representations for K. On the one hand we have fairly natural "automorphic objects", but on the other hand we get fairly *unnatural* "Galois objects". Indeed, the nilpotent part of a Weil–Deligne $\overline{\mathbb{F}}_\ell$-representation has no obvious Galois interpretation, in contrast with $\overline{\mathbb{Q}}_\ell$-representations, where it is related to the infinitesimal action of the tame inertia subgroup on some associated continuous $\overline{\mathbb{Q}}_\ell$ representation of the Weil group. Therefore in the $\overline{\mathbb{F}}_\ell$ case, this nilpotent part appears as an "extra datum", from the arithmetic point of view. In

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fact, Vignéras' correspondence was obtained by purely representation-theoretic arguments (a classification theorem à la Zelevinsky), and our aim is to find a geometric interpretation for it.

1.1.2. *The project.* Since Carayol's formulation of "nonabelian Lubin–Tate theory", the cohomology of the Lubin–Tate tower $(\mathcal{M}_{\mathrm{LT},n})_{n\in\mathbb{N}}$ of height d is a natural place to look for a realization of any Langlands-type local correspondence. Let us fix a completed algebraic closure K^{ca} of K and denote base changes to K^{ca} by adding the exponent "ca". It has long been noticed by the author that in order to get something correct for nonsupercuspidal representations, one should use the full "cohomology complex" $R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)$ as an object in the derived category $D^b(\mathrm{Rep}^\infty_{\mathbb{Z}_\ell}(\mathrm{GL}_d(K)))$ endowed with an action of the Weil group W_K and of the unit group $D^\times$ of the division algebra with center K and invariant $1/d$. Then for an irreducible representation π (over $\overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$), one considers the "derived π-coisotypical component"

$$R^*_\pi := \mathcal{H}^*\big(R\,\mathrm{Hom}_{\mathbb{Z}_\ell\,\mathrm{GL}_d(K)}(R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell), \pi)\big)[1-d],$$

which is a finite-dimensional graded smooth/continuous $D^\times \times W_K$-module supported in the range $[1-d, d-1]$. In [Dat 2012c], we proved the equality

$$[R^*_\pi] = \mathrm{LJ}(\pi) \otimes \sigma^{\mathrm{ss}}(\pi) \tag{1.1.3}$$

in the Grothendieck group of smooth/continuous $D^\times \times W_K$-modules. Here LJ stands for the so-called Langlands–Jacquet transfer of [Dat 2012d] and $\sigma^{\mathrm{ss}}(\pi)$ is the Weil part (that is, the semisimple part), of the Weil–Deligne representation $\sigma(\pi) = (\sigma^{\mathrm{ss}}(\pi), N_\pi)$ attached to π.

We want to enrich R^*_π with a nilpotent operator so that a similar formula holds in a suitable Grothendieck group of Weil–Deligne representations. The inspiration for this is Arthur's second SL_2 factor in the theory of automorphic forms. An hint that this may be useful for our purpose is the relation between switching the two SL_2-factors of a local A-parameter and the Zelevinsky involution. Indeed, Vignéras' correspondence is more a $\overline{\mathbb{F}}_\ell$-analog of the "Zelevinsky correspondence" than of the Langlands correspondence for $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{GL}_d(K)$.[1]

The origin of Arthur's SL_2 lies in the Lefschetz decomposition of the intersection cohomology of Shimura varieties. Building on this analogy, we defined in [Dat 2012b] a "Lefschetz operator"

$$L : R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell) \longrightarrow R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)[2](1)$$

[1] Recall that both correspondences are intertwined by the Zelevinsky involution.

as the cup product by the Chern class of a natural equivariant line bundle on the Lubin–Tate tower. In turn, this operator induces a graded equivariant map

$$L^*_\pi : R^*_\pi \longrightarrow R^*_\pi[2](1).$$

In its roughest formulation, our hope is that the pair (R^*_π, L^*_π) encodes the Weil–Deligne representation associated to π by Vignéras' correspondence. More precisely, one can define the class $[R^*_\pi, L^*_\pi] = [R^{\text{even}}_\pi, L^*_\pi] - [R^{\text{odd}}_\pi, L^*_\pi]$ of (R^*_π, L^*_π) in the Grothendieck group of Weil–Deligne $\overline{\mathbb{F}}_\ell D^\times$-representations, as in [Dat 2012a, 2.2.2]. The best we can hope for is then the equality

$$[R^*_\pi, L^*_\pi] = \mathrm{LJ}(\pi) \otimes [\sigma^{\mathrm{ss}}(\pi), N_\pi]. \tag{1.1.4}$$

1.1.5. *Previous results.* We proved such an equality in the following cases:

- When π is any irreducible $\overline{\mathbb{Q}}_\ell$-representation (in this case, by "Vignéras correspondence", we just mean the Zelevinsky correspondence) [Dat 2012b].
- When π is any *unipotent* irreducible $\overline{\mathbb{F}}_\ell$-representation and the order of q in $\mathbb{F}_\ell^\times$ is at least d [Dat 2012a].

In Vignéras' terminology, a representation is "unipotent" if it belongs to the principal block of the category $\mathrm{Rep}^\infty_{\overline{\mathbb{F}}_\ell}(G)$, that is, the unique block that contains the trivial representation. For an irreducible π, this is equivalent to π occurring as a subquotient of some $\mathrm{Ind}_B^G(\chi)$ for χ an unramified character of a Borel subgroup B. We note that, in the second case above, R^*_π can be computed in greater generality,[2] but the author is still unable to control L^*_π when π is not unipotent. Also, the arguments are much more difficult in the "Coxeter congruence" case (when the order of q in $\mathbb{F}_\ell^\times$ is d) than in the "banal" case (when this order is greater than d), due to some representation-theoretic complications. However, a common feature of these two cases is that one can still use enough of the theory of weights (on the Galois side) and exponents (on the GL_d side) so as to split the complex and compute explicitly Yoneda extensions. Moreover, in all the cases above, (1.1.3) and (1.1.4) involve no cancellation because R^*_π turns out to be either oddly or evenly graded. In fact, the "énoncé optimiste" from [Dat 2012b, 1.3.3] holds true in these cases.

1.2. ***This paper.*** Here we study the case when $q \equiv 1 (\mathrm{mod}\ \ell)$ and $\ell > d$. This is called the "limit case" in [Clozel et al. 2008, §5]. From the point of view of weights or exponents, this is the most degenerate case, due to the congruence on q. For example, in this situation, the unipotent summand of the complex $R\Gamma_c(\mathcal{M}^{\mathrm{ca}}_{\mathrm{LT}}, \mathbb{Z}_\ell)$ can be shown to be indecomposable. Moreover, as we have noted in [Dat 2012c, 2.2.7],

[2] And, in fact, for any irreducible π, if one admits that the $\mathbb{Z}_\ell$-cohomology of the Lubin–Tate tower is torsion free, as recently announced by Boyer.

R^*_π is generally not evenly nor oddly graded. Despite this bad news, computations are still feasible because of the additional assumption that $\ell > d$, which simplifies significantly the representation theory, as indicated by Vignéras' appendix to [Clozel et al. 2008].

1.2.1. *The result of the computation.* We explain in Corollary 3.1.3 that for π *unipotent*, R^*_π vanishes unless π is a subquotient of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$, or equivalently, unless $\sigma^{\mathrm{ss}}(\pi)$ is the trivial representation of W_K of dimension d. Then $\sigma(\pi) = (\sigma^{ss}(\pi), N)$ is given by a nilpotent $d \times d$ matrix N, whose Jordan form has shape λ for some partition λ of d. Through Vignéras' correspondence we thus get a parametrization $\lambda \in \mathcal{P}(d) \mapsto \pi_\lambda$ of all π's that occur as a subquotient of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$. We explain how to construct π_λ explicitly in Section 2.2.5. We now describe algebraically (R^*_π, L^*_π) and provide geometric intuition for the result obtained.

For a finite-dimensional $\overline{\mathbb{F}}_\ell$-vector space V with dual $\check{V}$, consider the graded space $H^* := \bigwedge^* \check{V} \otimes \bigwedge^* V$ endowed with the operator L^* of degree 2 which on the (p, q) part is given by

$$L : \textstyle\bigwedge^p \check{V} \otimes \bigwedge^q V \xrightarrow{\mathrm{Id}\otimes \mathrm{Id}_V \otimes \mathrm{Id}} \bigwedge^p \check{V} \otimes \check{V} \otimes V \otimes \bigwedge^q V \xrightarrow{\wedge\otimes\wedge} \bigwedge^{p+1} \check{V} \otimes \bigwedge^{q+1} V.$$

When the dimension of V is less than ℓ, this satisfies the hard Lefschetz property; see Section A.1.4. In particular, this holds for V the *standard representation* of $\mathfrak{S}_d$ (which has dimension $d-1$). In this case, H^* also carries an action of $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$ that commutes with L^*, so we may decompose it as $(H^*, L^*) = \sum_{\lambda\in\mathcal{P}(d)} (H^*_\lambda, L^*_\lambda)$ by applying central primitive idempotents associated to partitions.

Theorem. *For $\lambda \in \mathcal{P}(d)$, the action of $D^\times$ on $R^*_{\pi_\lambda}$ is trivial and that of W_K is unipotent. Moreover, there is an isomorphism $(R^*_{\pi_\lambda}, L^*_{\pi_\lambda})^{\oplus d_\lambda} \simeq (H^*_\lambda, L^*_\lambda)$ of graded $\overline{\mathbb{F}}_\ell$-vector spaces compatible with L operators.*

Here d_λ is the dimension of the simple $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$-module associated to λ.

There is geometric intuition behind this result. Consider the diagonal torus T in PGL_d and the discrete cocompact subgroup ϖ^X of T^{an} obtained by evaluating cocharacters at a fixed uniformizer ϖ of $\mathcal{O}_K$, and let A be the abelian variety T^{an}/ϖ^X, which has an action by the Weyl group $\mathfrak{S}_d$. Its cohomology is equivariantly isomorphic to H^* (note that $V = X_*(T) \otimes \overline{\mathbb{F}}_\ell$ is the standard representation of $\mathfrak{S}_d$), and there's a natural choice of a $\mathfrak{S}_d$-equivariant ample invertible sheaf on A whose associated Chern class can be put in the form described above. Now the special fiber (analytic reduction) of Mumford's formal model of A turns out to be isomorphic to the quotient of the special fiber of Deligne's formal model of Drinfeld's symmetric space by the action of B. This suggests a relation between the cohomology of A and $R^*_{i_B}$ with $i_B = \mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell) = \bigoplus_\lambda \pi_\lambda^{\oplus d_\lambda}$. In general, however, there is no such a relation because some multiplicities appear when one tries to compare vanishing cycles on both sides, but somehow these multiplicities disappear when $q = 1$ in the coefficients.

By representation theory of $\mathfrak{S}_d$ the theorem implies that $R^*_{\pi_\lambda}$ vanishes unless the Young tableau of λ is a hook or a double-hook. We explain in Section 2.2.7 that λ is a hook if and only if π_λ is *elliptic*, that is, $\mathrm{LJ}(\pi_\lambda) \neq 0$. In this case we can make the theorem more explicit.

Corollary. *Assume $\lambda = (d - j, 1^{(j)})$ for some $j \in \{0, \ldots, d-1\}$ and put $j' = d-1-j$. Then we have $(R^*_{\pi_\lambda}, L^*_{\pi_\lambda}) \simeq (\boldsymbol{P}_{j'})^{\oplus j+1} \oplus (\boldsymbol{P}_{j'-1})^{\oplus j}$ where $\boldsymbol{P}_k$ denotes the cohomology of a projective space of dimension k, shifted by $-k$, and with its tautological Lefschetz operator.*

In particular, the space $R^*_{\pi_\lambda}$ has total dimension $2jj' + d$. When $jj' \neq 0$, the pair $(R^*_{\pi_\lambda}, L^*_{\pi_\lambda})$ does not have the right dimension, and what is worse, it does not seem related to the Vignéras pair $(\sigma^{\mathrm{ss}}(\pi_\lambda), N_\lambda)$ in any reasonable Grothendieck group of Weil–Deligne representations. In other words, (1.1.4) fails in this case.

However, it is still true that it encodes the Vignéras pair, provided one uses extra structure.

1.2.2. *Main result.* Observe that R^*_π has the structure of a graded right module over the derived endomorphism algebra $\mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell G}(\pi, \pi)$. Consider the subalgebra $\mathscr{E}^*_\pi$ generated by extensions that "come from the boundary", namely by the kernel of the map $\mathrm{Ext}^1_{\overline{\mathbb{F}}_\ell G}(\pi, \pi) \longrightarrow \mathrm{Ext}^1_{\overline{\mathbb{F}}_\ell \overline{G}}(\bar{\pi}, \bar{\pi})$, where $\overline{G} = \mathrm{GL}_d(\mathbb{F}_q)$ and $\bar{\pi} = \pi^{1+\varpi M_d(\mathcal{O})}$. For a unipotent π, we'll see that $\mathscr{E}^*_\pi$ is also the image of a natural map $\mathrm{Ext}^*_{\mathscr{H}}(\pi^I, \pi^I) \longrightarrow \mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell G}(\pi, \pi)$, where I is an Iwahori subgroup and $\mathscr{H}$ is the corresponding Hecke algebra. This is a local graded algebra and we denote by $\mathscr{E}^+_\pi$ its maximal ideal. In the cases when (1.1.4) has been established, one also observes that either $\mathscr{E}^+_\pi = 0$ or at least its action on R^*_π vanishes. In contrast, in the limit case under study here, this action is nonzero and is somehow responsible for R^*_π being "too big". So, define $R^{\mathrm{red}}_\pi := R^*_\pi / R^*_\pi \mathscr{E}^+_\pi$. This is still a graded $\overline{\mathbb{F}}_\ell$-representation of $W_K \times D^\times$, and L^*_π induces an operator $L^{\mathrm{red}}_\pi : R^{\mathrm{red}}_\pi \longrightarrow R^{\mathrm{red}}_\pi[2](1)$. Let us finally denote by $[R^{\mathrm{red}}_\pi, L^{\mathrm{red}}_\pi]$ the image of the pair $(R^{\mathrm{red}}_\pi, L^{\mathrm{red}}_\pi)$ in the Grothendieck group of Weil–Deligne representations.

Theorem. *Let π be an elliptic unipotent irreducible $\overline{\mathbb{F}}_\ell$-representation of $\mathrm{GL}_d(K)$. As above, let $\pi = \pi_\lambda$ for $\lambda = (d - j, 1^{(j)})$. Then we have a $\overline{\mathbb{F}}_\ell$-linear isomorphism $(R^{\mathrm{red}}_\pi, L^{\mathrm{red}}_\pi) \simeq \boldsymbol{P}_{j'} \oplus (\boldsymbol{P}_0)^{\oplus j}[j']$. Hence in the Grothendieck group of Weil–Deligne $\overline{\mathbb{F}}_\ell D^\times$-representations we get*

$$[R^{\mathrm{red}}_\pi, L^{\mathrm{red}}_\pi] = \mathrm{LJ}(\pi) \otimes [\sigma^{\mathrm{ss}}(\pi), N_\pi].$$

When π is not elliptic but has a nonzero R^*_π (that is, π is associated to a double hook) we expect that $[R^{\mathrm{red}}_\pi, L^{\mathrm{red}}_\pi]$ vanishes, but we don't prove this here.

1.2.3. *A sketch of the argument.* Representation theoretic considerations tell us that, under our assumption on ℓ, the graded space R^*_π for π unipotent is the abutment of a spectral sequence whose E_2^{pq} term is $\mathrm{Ext}^p_{\mathcal{H}}(H^q_c(\mathcal{M}^{\mathrm{ca}}_{\mathrm{LT},I}, \overline{\mathbb{F}}_\ell), \pi^I)$, where $\mathcal{M}_{\mathrm{LT},I}$ is the Lubin–Tate space at Iwahori level, and $\mathcal{H}$ is the Hecke–Iwahori algebra as above. The main point of the paper is to compute this E_2 term and show that the spectral sequence degenerates at E_2. There are three ingredients coming into this computation.

- We can compute the $\overline{\mathbb{F}}_\ell$-cohomology of $\mathcal{M}_{\mathrm{LT},I}$ because the $\overline{\mathbb{Q}}_\ell$-cohomology is known and the simple geometry of $\mathcal{M}_{\mathrm{LT},I}$ shows the $\mathbb{Z}_\ell$-cohomology has no torsion.
- Representation theory of p-adic groups, under our assumption on ℓ, reduces the computation of the E_2 term to that of known multiplicities in certain tensor product representations of a symmetric group.
- Some numerical coincidences appear, that force degeneration of the spectral sequence.

Then comes the computation of L^*_π. Here we have to understand cup-products between the extensions mentioned above, and this also boils down to a problem in representation theory of the symmetric group that we solve in the Appendix. Once cup-products are understood, we need a handle on the Lefschetz operator (after all, it could be trivial!). This is provided by the explicit form of L^*_π when π is the trivial representation, which itself comes from the very definition of the Lefschetz operator as the Chern class of a bundle that is lifted from the crystalline (or Gross–Hopkins) period space. It turns out that knowing this particular L^*_π is enough to compute L^*_π for all unipotent elliptic π. Eventually, our computation of cup-products also allows going from the pair (R^*_π, L^*_π) to $(R^{\mathrm{red}}_\pi, L^{\mathrm{red}}_\pi)$.

Remark. Part of the above arguments can be generalized to approach the computation of R^*_π for any irreducible representation π (under the same hypothesis $q \equiv 1 \pmod{\ell}$ and $\ell > d$). Indeed, Boyer's announcement that the $\mathbb{Z}_\ell$-cohomology of the Lubin–Tate tower is torsion free implies that the only π's for which R^*_π is nonzero contain a simple type (J, τ). Then, in the above spectral sequence, one should replace $\mathcal{H}$ by the Hecke ring of (J, τ), π^I by $\mathrm{Hom}_J(\tau, \pi)$, and $H^q_c(\mathcal{M}^{\mathrm{ca}}_{\mathrm{LT},I}, \overline{\mathbb{F}}_\ell)$ by $R^q\,\mathrm{Hom}_J(\tau, R\Gamma_c(\mathcal{M}^{\mathrm{ca}}_{\mathrm{LT}}, \overline{\mathbb{F}}_\ell))$. The main problem may then be to show that the latter is torsion free. Granted this, and since the Hecke ring of (J, τ) is known to be isomorphic to a Hecke–Iwahori algebra, all the combinatorics should be the same as in this paper. *However*, as in the regular case, at the moment we don't see how to get any handle on L^*_π when π is not unipotent.

1.2.4. *Organization of the paper.* Section 2 deals with most of the representation-theoretic prerequisites. We recall and expand on Vignéras' appendix to [Clozel et al.

2008] to describe the unipotent block and the elliptic unipotent representations in our context. We then compute extensions of Iwahori-invariants of these representations. This involves representation theory of the symmetric group, and in particular some knowledge of the decomposition of tensor products. We postpone to the Appendix a delicate computation of cup-products in this context, which we use in the study of R_π^{red}. Section 3 deals with the cohomological study. The main point is to show that some spectral sequences degenerate, which implies that the cohomology complex at Iwahori level is split. With this splitting property and our knowledge of extensions and cup-products from Section 2 in hand, the results listed above are quite easy computations. The theorem we gave in Section 1.2.1 is proved in Section 3.3 and our main theorem in Section 3.4.

2. Representation theory

2.1. *The unipotent block.* We put $G := \mathrm{GL}_d(K)$ and denote by $\mathrm{Rep}_R^\infty(G)$ the abelian category of smooth representations of G with coefficients in the commutative ring R. Let $\boldsymbol{b}$ be the unique primitive idempotent of the center $\mathfrak{Z}_{\mathbb{Z}_\ell}(G)$ of the category $\mathrm{Rep}_{\mathbb{Z}_\ell}^\infty(G)$ which is nonzero on the trivial representation. Denote by $\mathrm{Rep}_{\boldsymbol{b}}^\infty(G)$ the full subcategory of all objects on which $\boldsymbol{b}$ acts by the identity. This is a Serre subcategory, called the *unipotent block of* $\mathrm{Rep}_{\mathbb{Z}_\ell}^\infty(G)$.

Let I be the standard Iwahori subgroup of G and let I^ℓ be the maximal prime-to-ℓ subgroup of I. This is a distinguished open subgroup of I and the quotient I/I^ℓ is isomorphic to the ℓ-Sylow subgroup $\mathrm{Syl}_\ell(\mathbb{F}_q^\times)^d$ of $(\mathbb{F}_q^\times)^d$.

2.1.1. Proposition. *The unipotent block is generated by the projective representation* $\mathbb{Z}_\ell[G/I^\ell]$.

Proof. When $\mathbb{Z}_\ell$ is replaced by $\overline{\mathbb{F}}_\ell$, this is explained in [Clozel et al. 2008, Appendix 1], and we could probably reduce our claim to this reference. Here is another argument relying on our description of the unipotent block in [Dat 2012a, §3.1]. Indeed, by [Dat 2012a, Proposition 3.1.2], our claim reduces to a claim about the finite group $\overline{G} := \mathrm{GL}_d(\mathbb{F}_q)$. Namely, let $\overline{B}^\ell$ be the maximal prime-to-ℓ subgroup of the standard Borel subgroup $\overline{B}$ of $\overline{G}$, and let $\boldsymbol{b}_{\overline{G}}$ be the central idempotent in $\mathbb{Z}_\ell[\overline{G}]$ corresponding to the sum of all blocks that contain a unipotent rational series. Explicitly, $\boldsymbol{b}_{\overline{G}}$ is the sum in $\overline{\mathbb{Q}}_\ell[\overline{G}]$ of all central primitive idempotents e_π corresponding to irreducible $\overline{\mathbb{Q}}_\ell\overline{G}$-representations π whose semisimple part s_π in Lusztig's classification is an ℓ-element. Then the claim is that $\overline{P} := \mathbb{Z}_\ell[\overline{G}/\overline{B}^\ell]$ *is a projective generator of the category* $\boldsymbol{b}_{\overline{G}}\,\mathrm{Rep}_{\mathbb{Z}_\ell}(\overline{G})$.

It is indeed clear that $\overline{P}$ is projective. Moreover, the Jordan–Hölder constituents of $\overline{P} \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}}_\ell$ are all the irreducible representations whose semisimple part in Lusztig's parametrization is an ℓ-element of a *split* torus of the dual group $\overline{G}^*$. But because $q \equiv 1 (\mathrm{mod}\ \ell)$, every semisimple ℓ-element of $\overline{G}^*$ lies in a split torus. $\square$

If H is an open compact subgroup of G, we denote by $\mathcal{H}_R(G, H)$ the Hecke algebra of left and right H-invariant measures on G with coefficients in R.

2.1.2. Corollary. *The functor $V \mapsto V^{I^\ell}$ induces an equivalence of categories*

$$\mathrm{Rep}^\infty_{\boldsymbol{b}}(G) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{H}_{\mathbb{Z}_\ell}(G, I^\ell)),$$

a quasi-inverse of which is the functor $M \mapsto \mathbb{Z}_\ell[G/I^\ell] \otimes_{\mathcal{H}_{\mathbb{Z}_\ell}(G,I^\ell)} M$.

The "intersection" $\mathrm{Rep}^\infty_{\overline{\mathbb{F}}_\ell}(G) \cap \mathrm{Rep}^\infty_{\boldsymbol{b}}(G)$ is Vignéras' unipotent block. In particular, the irreducible $\overline{\mathbb{F}}_\ell$-objects in $\mathrm{Rep}^\infty_{\boldsymbol{b}}(G)$ are the irreducible $\overline{\mathbb{F}}_\ell$-representations which appear as subquotients of some representation $\mathrm{Ind}_B^G(\chi)$, induced from an unramified $\overline{\mathbb{F}}_\ell$-character χ of a Borel subgroup B; see [Dat 2012a, Proposition 3.1.3]. Via the functor of the above corollary, these irreducible objects are in bijection with simple $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I^\ell)$-modules.

Let $\mathfrak{m}_I$ be the maximal ideal of the local subalgebra $\mathbb{Z}_\ell[I/I^\ell]$ of $\mathcal{H}_{\mathbb{Z}_\ell}(G, I^\ell)$.

2.1.3. Proposition. *The ideal $\mathfrak{m} := \mathfrak{m}_I \mathcal{H}_{\mathbb{Z}_\ell}(G, I^\ell)$ is two sided and is equal to $\mathcal{H}_{\mathbb{Z}_\ell}(G, I^\ell)\mathfrak{m}_I$. The map $[I^\ell g I^\ell] \mapsto [IgI]$ induces an isomorphism of $\mathbb{F}_\ell$-algebras*

$$\mathcal{H}_{\mathbb{Z}_\ell}(G, I^\ell)/\mathfrak{m} \xrightarrow{\sim} \mathcal{H}_{\mathbb{F}_\ell}(G, I).$$

Before proving the proposition, we introduce some more notation. Let T be the diagonal torus in G and let $N := N_G(T)$ be its normalizer. We denote by T^0 the maximal compact subgroup of T, and by T^ℓ the maximal prime-to-ℓ subgroup of T^0. Both are normal subgroups of N.

Proof. Since $T^\ell \subset I^\ell$, any element $w \in N/T^\ell$ gives rise to a well-defined Hecke operator $[I^\ell w I^\ell]$. By the Iwahori decomposition, the Hecke operators $[I^\ell w I^\ell]$, with w running on N/T^ℓ, form a $\mathbb{Z}_\ell$-basis of $\mathcal{H}_{\mathbb{Z}_\ell}(G, I^\ell)$. Among them, the operators $[I^\ell t I^\ell]$ with $t \in T^0/T^\ell$ form a basis of the subalgebra $\mathbb{Z}_\ell[I/I^\ell]$. Then, the formula

$$[I^\ell w I^\ell] * [I^\ell t I^\ell] = [I^\ell w t w^{-1} I^\ell] * [I^\ell w I^\ell]$$

shows that $\mathfrak{m}_I \mathcal{H}_{\mathbb{Z}_\ell}(G, I^\ell) = \mathcal{H}_{\mathbb{Z}_\ell}(G, I^\ell)\mathfrak{m}_I$ is a two-sided ideal, since $\mathfrak{m}_I$ is generated by elements $1 - [I^\ell t I^\ell]$, $t \in T/T^\ell$. The same formula shows that the map of the proposition is an isomorphism of $\mathbb{F}_\ell$-vector spaces. That it is a morphism of algebras follows from the definition of convolution products on both sides, and the fact that for an element $w \in N$ the obvious map

$$(I^\ell \cap w I^\ell w^{-1}) \backslash I^\ell \longrightarrow (I \cap w I w^{-1}) \backslash I$$

is a bijection. □

We note that if M is an $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I^\ell)$-module, then $M/\mathfrak{m}M$ identifies with the I-coinvariants M_I, where I acts through I/I^ℓ.

2.1.4. Corollary. *Any simple $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I^\ell)$-module is killed by $\mathfrak{m}$, and the map of the previous proposition induces a bijection between simple $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I)$-modules and simple $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I^\ell)$-modules. Equivalently, for any irreducible $\overline{\mathbb{F}}_\ell$-representations V of G, we have $V^{I^\ell} = V^I \xrightarrow{\sim} V_I$ and the functor $V \mapsto V^I$ induces a bijection between irreducible $\overline{\mathbb{F}}_\ell$-representations in $\mathrm{Rep}^\infty_{\boldsymbol{b}}(G)$ and simple $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I)$-modules.*

2.2. *Elliptic unipotent representations.* We first recall the structure of the Iwahori–Hecke algebra, taking into account the fact that $q = 1$ in $\overline{\mathbb{F}}_\ell$.

2.2.1. Fact. *The map $w \mapsto [IwI]$ is an isomorphism of $\overline{\mathbb{F}}_\ell$-algebras*

$$\overline{\mathbb{F}}_\ell[N/T^0] \xrightarrow{\sim} \mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I).$$

Proof. In general, this map induces an isomorphism of algebras $T_w \mapsto [IwI]$ from the Iwahori–Hecke algebra $\mathcal{H}_q(\tilde{W}) \otimes \overline{\mathbb{F}}_\ell$ with parameter q of the extended Weyl group $\tilde{W} = N/T^0$ of (G, T) to $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I)$. The multiplication in $\mathcal{H}_q(\tilde{W})$ is determined by the braid relation and the formula $T_s^2 = (q-1)T_s + q$ for each simple reflection s associated to I. Specializing at $q = 1$ therefore gives $\mathcal{H}_q(\tilde{W}) \otimes \overline{\mathbb{F}}_\ell = \overline{\mathbb{F}}_\ell[\tilde{W}]$. $\square$

We now revisit the classical relation between parabolic induction and compact induction in the context where $q = 1$ in $\overline{\mathbb{F}}_\ell$. Let $\chi : T/T^0 \longrightarrow \overline{\mathbb{F}}_\ell^\times$ be an unramified character of T and consider the (unnormalized) induction $\mathrm{Ind}_B^G(\chi)$. Let us also identify the symmetric group $\mathfrak{S}_d$ with the subgroup of permutation matrices of G in the usual way. Because of the double cosets decomposition $G = \bigsqcup_{w\in\mathfrak{S}_d} IwB = \bigsqcup_{w\in\mathfrak{S}_d} I^\ell wB$, we see that

$$\mathrm{Ind}_B^G(\chi)^{I^\ell} = \mathrm{Ind}_B^G(\chi)^I. \tag{2.2.2}$$

In particular, the action of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I^\ell)$ on $\mathrm{Ind}_B^G(\chi)^{I^\ell}$ factors through $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I)$. Through the previous isomorphism $\overline{\mathbb{F}}_\ell[N/T^0] \xrightarrow{\sim} \mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I)$, this action is given as follows.

2.2.3. Fact. *For $w \in \mathfrak{S}_d$, let $[IwB]_\chi$ be the unique element of $\mathrm{Ind}_B^G(\chi)^I$ that is supported on IwB and takes value 1 on w. Similarly, let $[wT]_\chi$ be the unique element of $\mathrm{ind}_T^N(\chi)$ that is supported on wT and takes value 1 on w. Then the map $[wT]_\chi \mapsto [IwB]_\chi$ is an isomorphism of $\mathbb{F}_\ell[N/T^0]$-modules*

$$\mathrm{ind}_T^N(\chi) \xrightarrow{\sim} \mathrm{Ind}_B^G(\chi)^I.$$

Proof. The mixed Bruhat decomposition shows that $([IwB]_\chi)_{w\in\mathfrak{S}_d}$ is a basis of $\mathrm{ind}_B^G(\chi)^I$ over $\overline{\mathbb{F}}_\ell$, therefore the map is an isomorphism of $\overline{\mathbb{F}}_\ell$-vector spaces. It is elementary to check that $[IwI] * [IB]_\chi = [IwB]_\chi$ for all $w \in \mathfrak{S}_d$, showing that the map is $\mathfrak{S}_d$-equivariant. Moreover if $t \in T$ dilates the unipotent radical of B, we see that $[ItI] * [IB]_\chi = \chi^{-1}(t)[IB]_\chi$. Since the semigroup T^+ of all elements that dilate the radical of B generates the group T, this equality is true for all $t \in T$. The T-equivariance of the map follows. $\square$

As a particular case, we get $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)^{I^\ell} \simeq \overline{\mathbb{F}}_\ell[N/T] = \overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$. Because of our assumption that $\ell > d$, the right-hand side is a semisimple $\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G, I^\ell)$-module. We summarize this as follows.

2.2.4. Corollary. $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ *is a semisimple representation of G and the functor $V \mapsto V^I$ induces an isomorphism between the poset of subrepresentations of* $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ *and that of subrepresentations of the regular $\overline{\mathbb{F}}_\ell$-representation of the symmetric group $\mathfrak{S}_d$.*

2.2.5. *More notation.* We put $S := \{1, \dots, d-1\}$ and we think of S as the set of simple roots of T in the upper triangular matrices, numbered by rows. To each subset $J \subset S$ is associated a unique standard parabolic subgroup P_J which contains B and such that J is the set of simple roots of T in the upper triangular matrices of the Levi component L_J of P_J. Denote the Weyl group of L_J by $\mathfrak{S}_J$, a parabolic subgroup of the Weyl group $\mathfrak{S}_S = \mathfrak{S}_d$ of G. Then we have an isomorphism of $\overline{\mathbb{F}}_\ell[N/T^0]$-modules

$$\mathrm{Ind}_{P_J}^G(\overline{\mathbb{F}}_\ell)^{I^\ell} = \mathrm{Ind}_{P_J}^G(\overline{\mathbb{F}}_\ell)^I \simeq \overline{\mathbb{F}}_\ell[\mathfrak{S}_d/\mathfrak{S}_J].$$

In fact, the image of the submodule $\mathrm{Ind}_{P_J}^G(\overline{\mathbb{F}}_\ell)^I$ of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)^I$ by the map of Corollary 2.2.4 is the submodule $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d/\mathfrak{S}_J]$ of $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$. As usual in this context, for any ring R we put

$$i_J(R) := \mathrm{Ind}_{P_J}^G(R) \quad \text{and} \quad v_J(R) := i_J(R) \Big/ \sum_{K \supset J} i_K(R).$$

Recall that $\mathrm{Ind}_B^G(\overline{\mathbb{Q}}_\ell)$ is multiplicity free, with pairwise distinct irreducible subquotients all $v_J(\overline{\mathbb{Q}}_\ell)$, $J \subset S$. In contrast, $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ is not multiplicity free, and $v_J(\overline{\mathbb{F}}_\ell)$ need not be irreducible.

2.2.6. *Notation again.* We denote by $P(d)$ the set of partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots)$ of d. To such a partition are associated the parabolic subgroup $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots$ of $\mathfrak{S}_d$, the permutation module $M_\lambda := \overline{\mathbb{F}}_\ell[\mathfrak{S}_d/\mathfrak{S}_\lambda]$, and the simple $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$-module S_λ. The latter appears with multiplicity one in M_λ and may be inductively characterized by equalities $M_\lambda = S_\lambda + \sum_{\mu > \lambda} m_{\lambda,\mu} S_\mu$ in the Grothendieck group of $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$-modules. We will denote by π_λ the unique irreducible $\overline{\mathbb{F}}_\ell$-representation of G such that $(\pi_\lambda)^I \simeq S_\lambda$.

To a subset $J \subset S$ we associate the unique partition λ_J such that $\mathfrak{S}_{\lambda_J}$ is conjugate to $\mathfrak{S}_J$. We then have $i_J(\overline{\mathbb{F}}_\ell)^I \simeq M_{\lambda_J}$, so that

$$i_J(\overline{\mathbb{F}}_\ell) = \pi_{\lambda_J} + \sum_{\mu > \lambda_J} m_{\lambda_J,\mu} \pi_\mu$$

in the Grothendieck group of finite-length $\overline{\mathbb{F}}_\ell$-representations of G. We can also write

$$v_J(\overline{\mathbb{F}}_\ell) = \pi_{\lambda_J} + \sum_{\mu > \lambda_J} m'_{\lambda_J,\mu} \pi_\mu,$$

but in general $m'_{\lambda_J,\mu}$ need not vanish.

2.2.7. *Elliptic unipotent representations.* An irreducible representation of G is called *elliptic* if it is not a virtual sum of parabolically induced representations. We know from [Dat 2012d, lemme 3.2.1] that up to unramified twist, an elliptic unipotent representation occurs as a subquotient of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$. However, in contrast with the regular case, not all such subquotients are elliptic.

Proposition. *The representation π_λ is elliptic if and only if λ is hook-shaped, that is, if $\lambda = (i, 1^{(d-i)})$ for some $i \in \{1, \dots, d\}$.*

Proof. The set $[M_\mu]_{\mu \in P(d)}$ is a basis of the Grothendieck group of $\overline{\mathbb{F}}_\ell$-representations of $\mathfrak{S}_d$ (recall that $\ell > d$). Write $[S_\lambda] = \sum_{\mu \geqslant \lambda} a_{\lambda,\mu}[M_\mu]$. By the foregoing, π_λ is elliptic if and only if $a_{\lambda,(d)} \neq 0$. It is proved in [James and Kerber 1981, 2.3.17] that this is equivalent to λ being a hook. □

Therefore, there are only d elliptic constituents in $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$, in high contrast with the ℓ-adic or banal case (2^{d-1} of them) or the regular nonbanal case ($2^d - 1$ of them).

There is a convenient realization of the modules $S_{(i,1^{(n-i)})}$. Denote by Std the standard $(d-1)$-dimensional $\overline{\mathbb{F}}_\ell$-representation of $\mathfrak{S}_d$. This is the subrepresentation of the permutation representation on $\overline{\mathbb{F}}_\ell^d$ on the subspace of vectors whose sum of coordinates vanish.

Fact. *For $i = 1, \dots d$, we have $S_{(i,1^{(d-i)})} = \bigwedge^{d-i} \mathrm{Std}$. In particular, $S_{(d)}$ is the trivial representation, and $S_{1^{(d)}}$ is the sign representation.*

The next fact will be an important technical tool in the study of the unipotent part of the cohomology complex of the Lubin–Tate tower.

2.2.8. Proposition. *For $i \in \{1, \dots, d\}$, we have*

$$v_{\{1,\dots,i-1\}}(\overline{\mathbb{F}}_\ell) \simeq v_{\{d-i+1,\dots,d-1\}}(\overline{\mathbb{F}}_\ell) \simeq \pi_{(i,1^{(d-i)})}.$$

Proof. Because $\pi_{(i,1^{(d-i)})}$ is a Jordan–Hölder factor of both $v_{\{1,\dots,i-1\}}(\overline{\mathbb{F}}_\ell)$ and $v_{\{d-i+1,\dots,d-1\}}(\overline{\mathbb{F}}_\ell)$, it suffices to prove the following equalities of dimensions:

$$\dim_{\overline{\mathbb{F}}_\ell} v_{\{1,\dots,i-1\}}(\overline{\mathbb{F}}_\ell)^I = \dim_{\overline{\mathbb{F}}_\ell} v_{\{d-i+1,\dots,d-1\}}(\overline{\mathbb{F}}_\ell)^I = \dim_{\overline{\mathbb{F}}_\ell} S_{(i,1^{(d-i)})}.$$

From the previous fact or from the hook-length formula, we have

$$\dim_{\overline{\mathbb{F}}_\ell} S_{(i,1^{(d-i)})} = \binom{d-1}{d-i}.$$

On the other hand, for any subset $J \subset S = \{1, \dots, d-1\}$, we have by definition $v_J(\overline{\mathbb{F}}_\ell) = v_J(\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{F}}_\ell$ and we know from [Schneider and Stuhler 1991, Corollary 4.5] that $v_J(\mathbb{Z}_\ell)$ is free over $\mathbb{Z}_\ell$. Therefore we have

$$\dim_{\overline{\mathbb{F}}_\ell} v_J(\overline{\mathbb{F}}_\ell)^I = \dim_{\overline{\mathbb{F}}_\ell} v_J(\overline{\mathbb{F}}_\ell)^{I^\ell} = \dim_{\overline{\mathbb{Q}}_\ell} v_J(\overline{\mathbb{Q}}_\ell)^{I^\ell} = \dim_{\overline{\mathbb{Q}}_\ell} v_J(\overline{\mathbb{Q}}_\ell)^I.$$

Denote by r_B the normalized Jacquet functor along B. Then Borel's theorem on principal series representations tells us that for any $J \subset S$, we have

$$\dim_{\overline{\mathbb{Q}}_\ell} v_J(\overline{\mathbb{Q}}_\ell)^I = \dim_{\overline{\mathbb{Q}}_\ell} r_B(v_J(\overline{\mathbb{Q}}_\ell)).$$

Now, for $w \in \mathfrak{S}_d$, put $J(w) := \{j \in \{1, \dots, d-1\}, w(j) < w(j+1)\}$. By [Dat 2012a, Fact 2.1.1 and subsequent paragraph], we have

$$\dim_{\overline{\mathbb{Q}}_\ell} r_B(v_J(\overline{\mathbb{Q}}_\ell)) = \#\{w \in \mathfrak{S}_d,\, J(w) = J\}.$$

Observe that in the cases where $J = \{1, \dots, i-1\}$ or $J = \{d-i, \dots, d-1\}$, the map $w \mapsto w_{|J}$ induces a bijection from $\{w \in \mathfrak{S}_d, J(w) = J\}$ to the set of nondecreasing maps $J \to \{1, \dots, d-1\}$. Therefore, in the same cases, the map $w \mapsto w(J)$ induces a bijection from $\{w \in \mathfrak{S}_d, J(w) = J\}$ to the set of subsets of size $|J| = i-1$ in S, whence the desired equalities

$$\dim_{\overline{\mathbb{F}}_\ell} v_{\{1,\dots,i-1\}}(\overline{\mathbb{F}}_\ell)^I = \dim_{\overline{\mathbb{F}}_\ell} v_{\{d-i+1,\dots,d-1\}}(\overline{\mathbb{F}}_\ell)^I = \binom{d-1}{i-1}. \qquad \square$$

2.3. ***Extensions between some simple $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, I)$-modules.***

2.3.1. Let G^0 be the subgroup of G generated by compact elements. We have $G^0 = \ker(|\det|_K : G \longrightarrow \mathbb{R}_+^\times)$. The isomorphism of Fact 2.2.1 restricts to an isomorphism

$$\overline{\mathbb{F}}_\ell[(N \cap G^0)/T^0] \xrightarrow{\sim} \mathcal{H}_{\overline{\mathbb{F}}_\ell}(G^0, I).$$

The group $(N \cap G^0)/T^0$ is an extension

$$(T \cap G^0)/T^0 \hookrightarrow (N \cap G^0)/T^0 \twoheadrightarrow \mathfrak{S}_d,$$

where $(T \cap G^0)/T^0$ is a free abelian group of rank $d-1$ on which the conjugation action of $N \cap G^0$ factors through $\mathfrak{S}_d$ and is the standard representation, namely

$$\overline{\mathbb{F}}_\ell \otimes_{\mathbb{Z}} ((T \cap G^0)/T^0) \simeq_{\mathfrak{S}_d} \mathrm{Std}.$$

2.3.2. Proposition. *Let A and B be two $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$-modules, that we may see as $\overline{\mathbb{F}}_\ell[N/T^0]$-modules via the projection $N/T^0 \twoheadrightarrow \mathfrak{S}_d$.*

(i) *There is a natural isomorphism*

$$\mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]}(A, B \otimes {\textstyle\bigwedge^*} \mathrm{Std})$$

functorial in A and B.

(ii) *If C is another $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$-module, cup-products are given by the following compositions*:

$$\begin{aligned}
&\mathrm{Ext}^k_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(A,B)\otimes \mathrm{Ext}^l_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(B,C)\\
\xrightarrow{\sim}\ &\mathrm{Hom}_{\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]}\big(A, B\otimes {\textstyle\bigwedge^k}\mathrm{Std}\big)\otimes \mathrm{Hom}_{\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]}\big(B, C\otimes {\textstyle\bigwedge^l}\mathrm{Std}\big)\\
\longrightarrow\ &\mathrm{Hom}_{\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]}\big(A, C\otimes {\textstyle\bigwedge^l}\mathrm{Std}\otimes {\textstyle\bigwedge^k}\mathrm{Std}\big)\\
\longrightarrow\ &\mathrm{Hom}_{\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]}\big(A, C\otimes {\textstyle\bigwedge^{k+l}}\mathrm{Std}\big)\\
\xrightarrow{\sim}\ &\mathrm{Ext}^{k+l}_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(A,C),
\end{aligned}$$

where the second map is composition and the third is induced by the exterior product.

Proof. As with any free abelian group of finite rank, there is a natural isomorphism of graded algebras

$$\mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell[(T\cap G^0)/T^0]}(\overline{\mathbb{F}}_\ell,\overline{\mathbb{F}}_\ell)\simeq {\textstyle\bigwedge^*}\big(\overline{\mathbb{F}}_\ell\otimes_{\mathbb{Z}}((T\cap G^0)/T^0)\big).$$

This isomorphism is compatible with automorphisms of the group $(T\cap G^0)/T^0$, and in particular with the action of $\mathfrak{S}_d$. As already noted above the proposition, the right-hand side with its $\mathfrak{S}_d$ action is $\bigwedge^*\mathrm{Std}$. With A and B as in the proposition, we thus get

$$\begin{aligned}
\mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(A,B) &\simeq \mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell[(T\cap G^0)/T^0]}(A,B)^{\mathfrak{S}_d}\\
&\simeq \big(\mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(A,B)\otimes {\textstyle\bigwedge^*}\mathrm{Std}\big)^{\mathfrak{S}_d}\\
&\simeq \mathrm{Hom}_{\overline{\mathbb{F}}_\ell\mathfrak{S}_d}\big(A, B\otimes {\textstyle\bigwedge^*}\mathrm{Std}\big).
\end{aligned}$$

Here in the first line we have used that $\ell > d$, so that $\mathfrak{S}_d$ has no higher cohomology on $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$-modules. This also shows (ii) on the cup-products, since the algebra structure on $\mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell[(T\cap G^0)/T^0]}(\overline{\mathbb{F}}_\ell,\overline{\mathbb{F}}_\ell)$ is given by the exterior product. □

We see in particular that the dimension of $\mathrm{Ext}^k_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(S_\lambda,S_\mu)$ equals the multiplicity of S_λ as a constituent of $S_\mu\otimes S_{(n-k,1^{(k)})}$. Computing such multiplicities is a notoriously difficult problem and remains largely open. Fortunately, enough is known for our purposes in this paper.

2.3.3. Corollary. *For $i,j,k\in\{0,\dots,d-1\}$, the dimension over $\overline{\mathbb{F}}_\ell$ of the extension space $\mathrm{Ext}^k_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}\big(\bigwedge^i\mathrm{Std},\bigwedge^j\mathrm{Std}\big)$ is either 0 or 1. It is 1 if and only if the following inequalities hold*:

$$i+j\geqslant k,\quad j+k\geqslant i,\quad k+i\geqslant j,\quad i+j+k\leqslant 2d-2.$$

Remark. The symmetry of the above conditions should not be surprising since the dimension we are interested in is that of $\big(\bigwedge^i \mathrm{Std}\otimes\bigwedge^j \mathrm{Std}\otimes\bigwedge^k \mathrm{Std}\big)^{\mathfrak{S}_d}$ by the last proposition and the self-duality of irreducible representations of $\mathfrak{S}_d$.

The above conditions are also invariant under the transformation

$$(i,j)\mapsto(i',j'):=(d-1-i,d-1-j).$$

This corresponds to the fact that $\bigwedge^{i'}\mathrm{Std}\simeq\bigwedge^i \mathrm{Std}\otimes\bigwedge^{d-1}\mathrm{Std}$.

A less symmetric formulation of the inequalities of the corollary, which is sometimes more convenient, is:

$$|j-i|\leqslant k\leqslant \min(i+j,i'+j'). \tag{2.3.4}$$

Proof of the Corollary. This is Theorem 2.1 of [Remmel 1989], formulated in a more symmetric way, and corrected. More precisely, assume, as we may from the above remark, that $i\leqslant j$ and $i+j\leqslant d-1$. Then our claim is that we have nonvanishing (and multiplicity one) if and only if $j-i\leqslant k\leqslant i+j$. To match the notation of [Remmel 1989], we put $n:=d$, $r:=n-i$, and $s:=n-j$. Then Remmel asserts that nonvanishing (and multiplicity one) holds if and only if $j-i\leqslant k\leqslant i+j+1$, which seems incompatible with our claim, and which is obviously false when $i=j=0$. However, there is a slight mistake in the proof, located in the third line of [Remmel 1989, p. 113], where it is asserted that "there are two possibilities for the positions of the remaining green cells [...]". Indeed, in the case that there are no green cells at all (that is, when $x=n-i-j$) there is only *one* possibility. Once corrected, we get our claim. □

In our study of the cohomology complex of the Lubin–Tate tower we will need some cup-products between some nonvanishing Ext spaces of the above corollary. In order to simplify the notation a bit, we will abbreviate $\Lambda^i:=\bigwedge^i\mathrm{Std}$ and write $\mathrm{Ext}^k(\Lambda^i,\Lambda^j)$ instead of $\mathrm{Ext}^k_{\overline{\mathbb{F}}_\ell[(G^0\cap T)/T^0]}\big(\bigwedge^i\mathrm{Std},\bigwedge^j\mathrm{Std}\big)$.

2.3.5. Theorem. *For $i=1,\dots,d-2$, we fix a generator $\zeta^1_{i,i}$ of $\mathrm{Ext}^1(\Lambda^i,\Lambda^i)$. Let i,j,k be integers such that $\mathrm{Ext}^k(\Lambda^i,\Lambda^j)$ is nonzero and $\mathrm{Ext}^{k+1}(\Lambda^i,\Lambda^j)$ is also nonzero. Then, both the cup-product maps from $\mathrm{Ext}^k(\Lambda^i,\Lambda^j)$ to $\mathrm{Ext}^{k+1}(\Lambda^i,\Lambda^j)$,*

$$\xi^1_{i,i}\cup- \quad \textit{and} \quad -\cup\xi^1_{j,j},$$

are isomorphisms.

We postpone the proof of this theorem to the Appendix, in order to lighten the exposition a bit. Let us mention the following corollary, in which we use the notation $i'=d-1-i$ introduced above.

2.3.6. Corollary. *For $i\in\{1,\dots,d-1\}$, the self-extension algebra $\mathrm{Ext}^*(\Lambda^i,\Lambda^i)$ is (graded) isomorphic to $\overline{\mathbb{F}}_\ell[X]/(X^{2\min(i,i')})$ via $X\mapsto\xi^1_{i,i}$. Moreover, for any other*

$j \in \{1, \dots, d-1\}$, *the graded space* $\mathrm{Ext}^*(\Lambda^i, \Lambda^j)$ *is a (shifted) cyclic module over* $\mathrm{Ext}^*(\Lambda^i, \Lambda^i)$.

The proof of the next result is also postponed to the Appendix, Section A.1.4.

2.3.7. Proposition. *Let* $i \leqslant j \in \{0, \dots, d-1\}$ *and let* $\xi_{j,i}^{j-i}$ *be a fixed generator of* $\mathrm{Ext}^{j-i}(\Lambda^j, \Lambda^i)$. *For any simple module* S *over* $\bar{\mathbb{F}}_\ell[\mathfrak{S}_d]$, *the cup-product*

$$\xi_{j,i}^{j-i} \cup - : \mathrm{Ext}^{j'}(\Lambda^i, S) \longrightarrow \mathrm{Ext}^{i'}(\Lambda^j, S)$$

is an isomorphism.

3. Cohomology and the Lefschetz operator

As in Section 1, we denote by $R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)$ the cohomology complex of the Lubin–Tate tower of height d of the field K. This is an object of the derived category $D^b(\mathrm{Rep}_{\mathbb{Z}_\ell}^\infty(G))$ with an action of the Weil group W_K and of the unit group $D^\times$ of the division algebra with invariant $1/d$ over K. We refer to [Dat 2007, §3.2] for a precise definition of this object.

We want to compute the graded $\bar{\mathbb{F}}_\ell[D^\times \times W_K]$-module

$$R_\pi^* := \mathcal{H}^*(R\,\mathrm{Hom}_{\mathbb{Z}_\ell G}(R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell), \pi))[1-d]$$

for π a unipotent irreducible $\bar{\mathbb{F}}_\ell$-representation of G. Notice the shift by $1-d$, which is here for convenience. Indeed, by [Dat 2012c, Proposition 2.1.3], the graded space R_π^* is supported in the range $[1-d, d-1]$.

3.1. ***The unipotent part of the cohomology complex.*** Thanks to the equivalence of categories of Corollary 2.1.2 we have

$$R_\pi^* \simeq R^*\,\mathrm{Hom}_{\mathcal{H}_{\mathbb{Z}_\ell}(G, I^\ell)}(R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)^{I^\ell}, \pi^{I^\ell})[1-d].$$

Moreover, since the action of I on π^{I^ℓ} is trivial, Proposition 2.1.3 implies

$$\begin{aligned} R_\pi^* &\simeq \mathcal{H}^*\big(R\,\mathrm{Hom}_{\mathcal{H}_{\mathbb{F}_\ell}(G,I)}(R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)^{I^\ell} \otimes^L_{\mathcal{H}_{\mathbb{Z}_\ell}(G,I^\ell)} \mathcal{H}_{\mathbb{F}_\ell}(G, I), \pi^I)\big)[1-d] \\ &\simeq \mathcal{H}^*\big(R\,\mathrm{Hom}_{\mathcal{H}_{\mathbb{Z}_\ell}(G,I)}(L_I(R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)), \pi^I)\big)[1-d], \end{aligned}$$

where L_I denotes the left-derived functor of the I-coinvariant functor. Let $\mathcal{M}_{\mathrm{LT},I}$ denote the Iwahori level of the Lubin–Tate tower (a quotient of the tame level $\mathcal{M}_{\mathrm{LT},1}$). We have $L_I(R\Gamma_c(\mathcal{M}_{\mathrm{LT}}^{\mathrm{ca}}, \mathbb{Z}_\ell)) \simeq R\Gamma_c(\mathcal{M}_{\mathrm{LT},I}^{\mathrm{ca}}, \mathbb{Z}_\ell)$. Now recall that the G-tower $\mathcal{M}_{\mathrm{LT}}$ is induced from a G^0-tower $\mathcal{M}_{\mathrm{LT}}^{(0)}$ (the notation we use is that of [Dat 2007, 3.5.1]). The analytic space $\mathcal{M}_{\mathrm{LT},I}^{(0)}$ is the deformation space with Iwahori level structure of a formal $\mathcal{O}_K$-module of height d over $\bar{k}$. Finally we have obtained

$$R_\pi^* \simeq \mathcal{H}^*\big(R\,\mathrm{Hom}_{\mathcal{H}_{\bar{\mathbb{F}}_\ell}(G^0,I)}(R\Gamma_c(\mathcal{M}_{\mathrm{LT},I}^{\mathrm{ca},(0)}, \bar{\mathbb{F}}_\ell), \pi^I)\big)[1-d]. \tag{3.1.1}$$

Let us abbreviate

$$\mathscr{C}_I := R\Gamma_c(\mathcal{M}^{\mathrm{ca},(0)}_{\mathrm{LT},I}, \overline{\mathbb{F}}_\ell)[d-1] \in D^b(\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0, I)).$$

Its cohomology is quite easy to describe, although the author does not know any elementary proof. Recall that any $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$-module inflates to a $\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0, I)$-module via the isomorphism $\overline{\mathbb{F}}_\ell[(N \cap G^0)/T^0] \xrightarrow{\sim} \mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0, I)$ and the projection

$$(N \cap G^0)/T^0 \twoheadrightarrow \mathfrak{S}_d.$$

3.1.2. Proposition. *The cohomology of $\mathscr{C}_I$ vanishes outside the range $\{0, \dots, d-1\}$. For $i \in \{0, \dots, d-1\}$, we have $\mathscr{H}^i(\mathscr{C}_I) \simeq \bigwedge^{i'} \mathrm{Std}$, where $i' := d-1-i$.*

Proof. The deformation ring $\mathscr{R}_I$ with Iwahori level structure of a formal $\mathscr{O}_K$-module of height d is known to be isomorphic to $\breve{\mathscr{O}}[\![X_1, \dots, X_d]\!]/(X_1X_2\cdots X_d - \varpi)$, where $\breve{\mathscr{O}}$ is the completed maximal unramified extension of $\mathscr{O}_K$ and ϖ is a uniformizer of $\mathscr{O}_K$. A reference is [Taylor and Yoshida 2007, top of p. 483]. It follows that the vanishing cycles $\Psi^i(\mathscr{R}_I, \mathbb{Z}_\ell)$ are isomorphic, as $\mathbb{Z}_\ell$-modules, to $\bigwedge^i(\mathbb{Z}_\ell^{d-1})$. Since $\mathscr{H}^i(\mathscr{C}_I) = H_c^{d-1+i}(\mathcal{M}^{\mathrm{ca},(0)}_{\mathrm{LT},I}, \overline{\mathbb{F}}_\ell) = \Psi^{d-1-i}(\mathscr{R}_I, \overline{\mathbb{F}}_\ell)^\vee$, we get at least the expected dimension for these cohomology spaces.

Unfortunately, computing the action of the Hecke algebra is not so easy. However, here we observe that $H_c^{d-1+i}(\mathcal{M}^{\mathrm{ca},(0)}_{\mathrm{LT},I}, \mathbb{Z}_\ell)$ is torsion free, so that

$$H_c^{d-1+i}(\mathcal{M}^{\mathrm{ca},(0)}_{\mathrm{LT},I}, \overline{\mathbb{F}}_\ell) = H_c^{d-1+i}(\mathcal{M}^{\mathrm{ca},(0)}_{\mathrm{LT},I}, \mathbb{Z}_\ell) \otimes \overline{\mathbb{F}}_\ell,$$

and we may hope to deduce $H_c^{d-1+i}(\mathcal{M}^{\mathrm{ca},(0)}_{\mathrm{LT},I}, \overline{\mathbb{F}}_\ell)$ by reduction modulo ℓ of

$$H_c^{d-1+i}(\mathcal{M}^{\mathrm{ca},(0)}_{\mathrm{LT},I}, \overline{\mathbb{Q}}_\ell).$$

Indeed, by Proposition 2.2.8, it suffices to know that

$$H_c^{d-1+i}(\mathcal{M}^{\mathrm{ca},(0)}_{\mathrm{LT},I}, \overline{\mathbb{Q}}_\ell) \simeq v_{\{1,\dots,i\}}(\overline{\mathbb{Q}}_\ell)^I.$$

There are two ways to infer such an isomorphism. It follows from Boyer's local theorem in [Boyer 2009], which uses global arguments, but remains "on the Lubin–Tate side". It also follows by purely local arguments, from the Faltings–Fargues theorem [Fargues 2008] that the cohomology of the Lubin–Tate tower coincides with that of the Drinfeld tower, and the Schneider–Stuhler computation [Schneider and Stuhler 1991] of the cohomology of the Drinfeld symmetric space. □

3.1.3. Corollary. *We have $R_\pi^* = 0$ unless π is a subquotient of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ and $\pi^I = \pi_\lambda$ with λ a hook or double-hook partition. Moreover, the action of $D^\times$ is trivial and that of W_K is unipotent.*

Proof. By the proposition we have a spectral sequence

$$E_2^{p,q} := \mathrm{Ext}^p_{\mathscr{H}(G^0,I)}\big(\textstyle\bigwedge^{q'} \mathrm{Std}, \pi^I\big) \Rightarrow R_\pi^{p-q}.$$

So R^*_π vanishes unless $\mathrm{Ext}^p_{\mathcal{H}(G^0,I)}\left(\bigwedge^{q'}\mathrm{Std}, \pi^I\right) \neq 0$ for some p and q. In this case, π^I has to be trivial on $(T\cap G^0)/T^0 \subset \mathcal{H}(G^0, I)$, so that π is a subquotient of $\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ and π^I comes from a simple $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$-module. Then, by Proposition 2.3.2 this simple module occurs in $\bigwedge^{q'}\mathrm{Std}\otimes\bigwedge^p\mathrm{Std}$. It follows from Remmel's theorem [Remmel 1989] that this simple module is associated to a double-hook or hook partition.

Let us turn to the actions of W_K and $D^\times$. We know that W_K acts trivially on the cohomology of $\mathcal{C}_I$ (because of $q = 1$ in $\overline{\mathbb{F}}_\ell$), therefore W_K acts unipotently on $\mathcal{C}_I$ hence also on R^*_π. For the same reason, the action of $D^\times$ on $\mathcal{C}_I$ has to be unipotent. However, the center $F^\times$ of $D^\times$ acts on R^*_π by the same character as $F^\times$ acts on π, that is, the trivial character. Since ℓ does not divide the pro-order of $D^\times/F^\times$, we deduce that $D^\times$ acts trivially. □

One consequence of the next section will be the following theorem.

3.1.4. Theorem. *The complex $\mathcal{C}_I$ is split in $D^b(\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G^0, I))$. Namely, we have (noncanonically) $\mathcal{C}_I \simeq \bigoplus_{i=0}^{n-1}\bigwedge^{i'}\mathrm{Std}[-i]$ in $D^b(\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G^0, I))$.*

Proof. In the proof of Theorem 3.2.1 below, we get the following property on $\mathcal{C}_I$. For all $i = 0, \dots, d-1$, the spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_{\mathcal{H}(G^0,I)}(H^q(\mathcal{C}_I), H^i(\mathcal{C}_I)) \Rightarrow \mathrm{Hom}_{D^b(\mathcal{H}(G^0,I))}(\mathcal{C}_I[q-p], H^i(\mathcal{C}_I))$$

degenerates at E_2. But then, it follows from the proof of the implication (i) ⇒ (ii) of [Deligne 1968, Proposition (1.2)] (or rather a dual version of it, as in [ibid., Remark (1.4)]) that the complex $\mathcal{C}_I$ is split. □

Remark. In contrast, the complex $\boldsymbol{b}_G R\Gamma_c(\mathcal{M}_{\mathrm{LT}}, \overline{\mathbb{F}}_\ell)$ is certainly *not split* in $D^b_{\overline{\mathbb{F}}_\ell}(G)$. Equivalently, $\mathcal{C} := R\Gamma_c(\mathcal{M}_{\mathrm{LT}}, \overline{\mathbb{F}}_\ell)^{I^\ell}$ is not split in $D^b(\mathcal{H}(G, I^\ell))$. Indeed, it is a perfect complex of $\overline{\mathbb{F}}_\ell[I/I^\ell]$-modules whose cohomology spaces are not of finite projective dimension since I acts trivially on them.

3.2. *The graded dimension of R^*_π when π is elliptic unipotent.* For $j = 0, \dots, d-1$, we put $\pi_j := \pi_{(d-j,1^{(j)})}$, so that $(\pi_j)^I \simeq \bigwedge^j\mathrm{Std}$. As in Proposition 3.1.2, we put

$$j' := d-1-j.$$

3.2.1. Theorem. *The graded vector space $R^*_{\pi_j}$ is supported in the range $[-j', j']$. For $k \in [-j', j']$ we have*

$$\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\pi_j}) = \begin{cases} j+1, & \text{if } k-j' \text{ is even,} \\ j, & \text{if } k-j' \text{ is odd.}\end{cases}$$

Proof. We prove equality of dimensions by proving inequalities in both directions.

In order to bound above $\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\pi_j})$, we use the spectral sequence

$$E_2^{p,q} := \mathrm{Ext}^p_{\mathcal{H}(G^0,I)}(\mathcal{H}^q(\mathcal{C}_I), \pi_j^I) \Rightarrow R^{p-q}_{\pi_j}. \tag{3.2.2}$$

Proposition 3.1.2 tells us that $E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{H}(G^0,I)}\big(\bigwedge^{q'}\mathrm{Std}, \bigwedge^j \mathrm{Std}\big)$ and Corollary 2.3.3 then ensures that $\dim_{\overline{\mathbb{F}}_\ell}(E_2^{p,q}) \leqslant 1$ for all p and q and

$$\begin{aligned}\dim_{\overline{\mathbb{F}}_\ell}(E_2^{p,q}) = 1 &\iff (-j' \leqslant p-q \leqslant j' \text{ and } -j \leqslant p+q-(d-1) \leqslant j)\\ &\iff (p,q) \text{ lies in the rectangle } (0,j'), (j',0), (d-1,j), (j,d-1).\end{aligned}$$

This rectangle is contained in the square $[0,d-1]\times[0,d-1]$ and its faces have slopes ± 1. Since this spectral sequence has finite support, it converges and we have

$$\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\pi_j}) = \sum_{i=0}^{d-1} \dim(E_\infty^{k+i,i}) \leqslant \sum_{i=0}^{d-1} \dim(E_2^{k+i,i}).$$

In particular, we see that $R^k_{\pi_j}$ vanishes unless $-j' \leqslant k \leqslant j'$, in which case we get

$$\begin{aligned}\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\pi_j}) &\leqslant \#\{i \in \{0,\dots,d-1\}, -j \leqslant k+2i-(d-1) \leqslant j\}\\ &= \#\{i \in \{0,\dots,d-1\}, -j+k' \leqslant 2i \leqslant j-k'\}\\ &= \#\{\text{even integers in the range } [-j+k', j+k']\}.\end{aligned}$$

For the last equality, we use that $[-j+k', j+k'] \subset [0, 2d-2]$, which is indeed equivalent to $-j' \leqslant k \leqslant j'$. Now the last expression in the right-hand side above is $j+1$ if $-j+k' = j'-k$ is even, and is j otherwise.

We now look for lower bounds on $\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\pi_j})$. We will use the fact that, by Proposition 2.2.8, we have

$$\pi_j \simeq v_{\{1,\dots,j'\}}(\overline{\mathbb{F}}_\ell) \simeq v_{\{j+1,\dots,d-1\}}(\overline{\mathbb{F}}_\ell).$$

Denote by $\breve{\mathbb{Z}}_\ell$ the Witt vectors of $\overline{\mathbb{F}}_\ell$. We put

$$\omega_j^+ := v_{\{1,\dots,j'\}}(\breve{\mathbb{Z}}_\ell) \quad \text{and} \quad \omega_j^- := v_{\{j+1,\dots,d-1\}}(\breve{\mathbb{Z}}_\ell).$$

As recalled in the proof of Proposition 2.2.8, these are liftings of π_j over $\breve{\mathbb{Z}}_\ell$, that is, admissible free $\breve{\mathbb{Z}}_\ell$-representations of G such that

$$\omega_j^\pm \otimes_{\breve{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell \simeq \pi_j.$$

Therefore we have universal coefficients exact sequences

$$R^k_{\omega_j^\pm} \otimes_{\breve{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell \hookrightarrow R^k_{\pi_j} \twoheadrightarrow R^{k+1}_{\omega_j^\pm}[\ell],$$

for all $k \in \mathbb{Z}$, and where the $[\ell]$ denotes ℓ-torsion (kernel of multiplication by ℓ).

Since the $R^k_{\omega_j^\pm}$ are finitely generated $\breve{\mathbb{Z}}_\ell$-modules, we have equalities

$$\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^\pm}\otimes_{\breve{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell)=\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^\pm}[\ell])+\dim_{\breve{\mathbb{Q}}_\ell}(R^k_{\omega_j^\pm}[1/\ell]),$$

where $\breve{\mathbb{Q}}_\ell=\breve{\mathbb{Z}}_\ell[1/\ell]$ is the fraction field of $\breve{\mathbb{Z}}_\ell$. Therefore we get

$$\text{(3.2.3)}\quad \dim_{\overline{\mathbb{F}}_\ell}(R^k_{\pi_j})=\dim_{\overline{\mathbb{F}}_\ell}(R^{k+1}_{\omega_j^\mp}[\ell])+\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^\pm}[\ell])+\dim_{\breve{\mathbb{Q}}_\ell}(R^k_{\omega_j^\pm}[1/\ell]).$$

Since $\omega_j^+[1/\ell]=v_{\{1,\dots,j'\}}(\breve{\mathbb{Q}}_\ell)$ and $\omega_j^-[1/\ell]=v_{\{j+1,\dots,n-1\}}(\breve{\mathbb{Q}}_\ell)$, we have already computed the last summand of the right-hand side in [Dat 2006] (see, more precisely, the display below [Dat 2006, Lemma 4.4.1]). This gives

$$\text{(3.2.5)}\qquad \dim_{\breve{\mathbb{Q}}_\ell}(R^k_{\omega_j^\pm}[1/\ell])=1\quad \text{if } -j'<k<j' \text{ and } j'-k \text{ is even},$$

$$\text{(3.2.6)}\qquad \dim_{\breve{\mathbb{Q}}_\ell}(R^{-j'}_{\omega_j^+}[1/\ell])=j+1,\quad \dim_{\breve{\mathbb{Q}}_\ell}(R^{j'}_{\omega_j^+}[1/\ell])=1,$$

$$\text{(3.2.7)}\qquad \dim_{\breve{\mathbb{Q}}_\ell}(R^{-j'}_{\omega_j^-}[1/\ell])=1,\quad \dim_{\breve{\mathbb{Q}}_\ell}(R^{j'}_{\omega_j^-}[1/\ell])=j+1,$$

$$\text{(3.2.8)}\qquad \dim_{\breve{\mathbb{Q}}_\ell}(R^k_{\omega_j^\pm}[1/\ell])=0\quad \text{in all other cases.}$$

Case $k=-j'$. In this case, the equality $\dim_{\overline{\mathbb{F}}_\ell}(R^{-j'}_{\pi_j})=j+1$ follows from (3.2.3) applied to ω_j^+ in degree $-j'$, (3.2.6), and our previously obtained upper bound.

Case $k=j'$. Similarly, the equality $\dim_{\overline{\mathbb{F}}_\ell}(R^{j'}_{\pi_j})=j+1$ follows from (3.2.3) applied to ω_j^- in degree j', (3.2.7), and our previously obtained upper bound.

Case $-j'<k<j'$. For k in this range, we are going to prove that

$$\text{(3.2.8)}\qquad \begin{aligned}&\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^+}[\ell])=j \text{ and } \dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^-}[\ell])=0 \quad \text{if } j'-k \text{ is even},\\ &\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^+}[\ell])=0 \text{ and } \dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^-}[\ell])=j \quad \text{if } j'-k \text{ is odd}.\end{aligned}$$

Because of (3.2.3) and (3.2.5), this implies our desired equalities:

$$\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\pi_j})=\begin{cases} j+1 & \text{if } j'-k \text{ is even},\\ j & \text{if } j'-k \text{ is odd}.\end{cases}$$

We will prove (3.2.8) by induction on k. The first case is $k=-j'+1$. When (3.2.3) is applied to ω_j^- in degree $-j'$ it reads

$$j+1=\dim_{\overline{\mathbb{F}}_\ell}(R^{-j'+1}_{\omega_j^-}[\ell])+\dim_{\overline{\mathbb{F}}_\ell}(R^{-j'}_{\omega_j^-}[\ell])+1.$$

The same equation in degree $-j'-1$ tells us that $R^{-j'}_{\omega_j^-}[\ell]=0$, whence the desired equality $\dim_{\overline{\mathbb{F}}_\ell}(R^{-j'+1}_{\omega_j^-}[\ell])=j$. On the other hand, (3.2.3) applied to ω_j^+ in degree $-j'$ immediately implies that $\dim_{\overline{\mathbb{F}}_\ell}(R^{-j'+1}_{\omega_j^+}[\ell])=0$.

We now assume that (3.2.8) has been proved up to $k-1$ and we want to prove it for k. We distinguish two cases.

Suppose first that $j'-k$ is even. Then our induction hypothesis tells us that $\dim_{\overline{\mathbb{F}}_\ell}(R^{k-1}_{\omega_j^-}[\ell])=j$ so that the upper bound already obtained and (3.2.3) for ω_j^- in degree $k-1$ imply that $\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^-}[\ell])=0$ and also that $\dim_{\overline{\mathbb{F}}_\ell}(R^{k-1}_{\pi_j})=j$. Then, (3.2.3) for ω_j^+ in degree $k-1$ together with the vanishing of $R^{k-1}_{\omega_j^+}[\ell]$ (induction hypothesis) and (3.2.8) tell us that $\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^+}[\ell])=j$, as desired.

Next, suppose that $j'-k$ is odd, and apply (3.2.3) to ω_j^+ in degree $k-1$. By the induction hypothesis, the upper bound, and (3.2.5), we get that $\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^+}[\ell])=0$ and also that $\dim_{\overline{\mathbb{F}}_\ell}(R^{k-1}_{\pi_j})=j+1$. Apply then (3.2.3) to ω_j^- in degree $k-1$. Again the induction hypothesis and (3.2.5) tell us that $\dim_{\overline{\mathbb{F}}_\ell}(R^k_{\omega_j^-}[\ell])=j$, as desired. □

Corollary (of the proof). *The spectral sequence (3.2.2) degenerates at E_2.*

In particular, Theorem 3.1.4 is now proved. We may use it to recast the foregoing result in the following way.

3.2.9. Corollary. *Any splitting $\bigoplus_{q=0}^{d-1}\bigwedge^{q'}\mathrm{Std}[-q]\xrightarrow{\sim}\mathscr{C}_I$ as in Theorem 3.1.4 induces a graded isomorphism*

$$R^*_{\pi_j}\xrightarrow{\sim}\bigoplus_{\substack{-j'\leqslant p-q\leqslant j'\\ -j\leqslant p+q+1-d\leqslant j}}\mathrm{Ext}^p_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}\big(\textstyle\bigwedge^{q'}\mathrm{Std},\bigwedge^{j}\mathrm{Std}\big)[q-p].$$

Moreover, each term of the above sum has dimension 1.

3.3. *The description of the pair (R^*_π, L^*_π).*

In this section we prove the theorem from Section 1.2.1 and its corollary.

3.3.1. Let us write $i_B:=\mathrm{Ind}_B^G(\overline{\mathbb{F}}_\ell)$ and consider the graded $\overline{\mathbb{F}}_\ell$-vector space $R^*_{i_B}$. By (3.1.1) we have

$$R^*_{i_B}\simeq\mathscr{H}^*\big(R\,\mathrm{Hom}_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}(R\Gamma_c(\mathscr{M}^{\mathrm{ca},(0)}_{\mathrm{LT},I},\overline{\mathbb{F}}_\ell),(i_B)^I)\big)[1-d].$$

On the other hand we have on the left $\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)$-module $(i_B)^I=\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$ a right module structure over $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$ which induces a left module structure on $R^*_{i_B}$. Now let $\lambda\in\mathscr{P}(d)$ and denote by ε_λ the central idempotent corresponding to the simple module S_λ, as well as $d_\lambda:=\dim_{\overline{\mathbb{F}}_\ell}S_\lambda$. We then may recover $R^*_{\pi_\lambda}$ by applying ε_λ:

$$(R^*_{\pi_\lambda})^{\oplus d_\lambda}=\varepsilon_\lambda R^*_{i_B}.$$

As in Corollary 3.2.9, the splitting property of $\mathscr{C}_I$ shows that for $k=0,\dots,2d-2$ we have

$$R^{k+1-d}_{i_B}\simeq\bigoplus_{p-q=k+1-d}\mathrm{Ext}^p_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}\big(\textstyle\bigwedge^{q'}\mathrm{Std},\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]\big).$$

Inserting Proposition 2.3.2 we get

$$R^{k+1-d}_{i_B}\simeq\bigoplus_{p+q'=k}\mathrm{Hom}_{\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]}\big(\textstyle\bigwedge^{q'}\mathrm{Std},\bigwedge^{p}\mathrm{Std}\otimes\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]\big).$$

By Frobenius reciprocity and self-duality we finally get a $\mathfrak{S}_d$-equivariant isomorphism

$$R^{k+1-d}_{i_B} \simeq \bigoplus_{p+q'=k} \bigwedge^{q'} \mathrm{Std} \otimes \bigwedge^{p} \mathrm{Std},$$

which shows that, as a graded vector space $R^*_{i_B}[d-1]$ is $\mathfrak{S}_d$-equivariantly isomorphic to the graded space H^* considered in the theorem given in Section 1.2.1. So we have obtained half of this theorem and we now have to study compatibility with Lefschetz operators.

3.3.2. We refer to [Dat 2012b] for the precise definition of the Lefschetz operator

$$L : R\Gamma_c(\mathcal{M}^{\mathrm{ca}}_{\mathrm{LT}}, \overline{\mathbb{F}}_\ell) \longrightarrow R\Gamma_c(\mathcal{M}^{\mathrm{ca}}_{\mathrm{LT}}, \overline{\mathbb{F}}_\ell)[2](1)$$

on the cohomology complex of the Lubin–Tate tower. By functoriality, the latter induces a graded equivariant map $L^*_\pi : R^*_\pi \longrightarrow R^*_\pi[2](1)$ for any smooth $\overline{\mathbb{F}}_\ell$-representation of G. For our purposes here, the most useful feature of L is that it is lifted from the Chern class of the tautological bundle on the crystalline period space $\mathbb{P}^{d-1}$ of the Lubin–Tate space. This explains the following description of the pair $(R^*_{\pi_0}, L^*_{\pi_0})$ where π_0 denotes the unit representation of G over $\overline{\mathbb{F}}_\ell$ (see the first paragraph of the proof of [Dat 2012a, Theorem 4.2.2]).

Fact. *We have an isomorphism* $R^*_{\pi_0} \simeq H^*(\mathbb{P}^{d-1,\mathrm{ca}}, \overline{\mathbb{F}}_\ell)[d-1] = \bigoplus_{i=0}^{d-1} \overline{\mathbb{F}}_\ell[d-1-2i]$ *with* $L^*_{\pi_0}$ *corresponding to the Chern class of the tautological sheaf on* $\mathbb{P}^{d-1}$.

Because of our assumption that $q \equiv 1[\ell]$, we may and will forget all Tate twists in the sequel. We will also denote by

$$L_I \in \mathrm{Hom}_{D^b(\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G^0,I))}(\mathcal{C}_I, \mathcal{C}_I[2])$$

the morphism induced by L on the complex at Iwahori level. When π is in the unipotent block and $\pi^I = \pi^{I^\ell}$, the morphism L^*_π is induced by L_I through the identification (3.1.1). We also denote by $L^{(k)} := L[2k-2]\circ\cdots\circ L : R\Gamma_c \longrightarrow R\Gamma_c[2k]$ the k-th iterate of L, and similarly for $(L_I)^{(k)}$ or $(L^*_\pi)^{(k)}$.

3.3.3. Theorem. *Let π be any unipotent irreducible $\overline{\mathbb{F}}_\ell$-representation of G. Then $(L^*_\pi)^{(k)}$ induces an isomorphism $R^{-k}_\pi \xrightarrow{\sim} R^k_\pi$ for any $k \geqslant 0$.*

Proof. We know from Theorem 3.1.4 that $\mathcal{C}_I$ is a split complex. Let us choose a splitting $\mathcal{C}_I \xrightarrow{\sim} \bigoplus_{i=0}^{d-1} \mathcal{H}^i(\mathcal{C}_I)[-i]$. As in Corollary 3.2.9, this induces an isomorphism from R^*_π to the graded space associated to the bigraded space

$$(p,q) \mapsto E^{p,q}_\pi := \mathrm{Ext}^p_{\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}(\Lambda^{q'}, \pi^I).$$

This also induces an isomorphism

$$\mathrm{Hom}_{D^b(\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G^0,I))}(\mathcal{C}_I, \mathcal{C}_I[2]) \simeq \bigoplus_{i,j=0}^{d-1} \mathrm{Ext}^{i-j+2}(\mathcal{H}^i(\mathcal{C}_I), \mathcal{H}^j(\mathcal{C}_I))$$

according to which we have a decomposition $L_I = \sum_{i,j=0}^{d-1} L_I^{i,j}$. By Proposition 3.1.2 we have

$$\mathrm{Ext}^{i-j+2}(\mathcal{H}^i(\mathcal{C}_I), \mathcal{H}^j(\mathcal{C}_I)) \simeq \mathrm{Ext}^{i-j+2}(\Lambda^{i'}, \Lambda^{j'})$$

and by Corollary 2.3.3 (see (2.3.4)), the latter has dimension 1 if $|i-j| \leqslant i-j+2 \leqslant \min(i+j, i'+j')$ and vanishes otherwise. In particular, when it does not vanish, we have $i-j+2 \geqslant 1$ with equality if and only if $j = i+1$. Now, each $L_I^{i,j}$ acts on the bigraded space E_π by a map of degree $(i-j+2, i-j)$. Since $(i-j+2)+(i-j) \geqslant 0$, it follows that L_I preserves the decreasing filtration on E_π defined by $\mathrm{Fil}_r\, E_\pi := \bigoplus_{p+q \geqslant r} E_\pi^{p,q}$, hence that L_π^* preserves the filtration induced on R_π^*. In particular, we may check the expected property of $(L_\pi^*)^k$ on the associated graded space. Concretely, this means that it suffices to prove that for all $0 \leqslant p \leqslant q \leqslant d-1$, the map

$$E_\pi^{p,q} \hookrightarrow E_\pi \xrightarrow{(L_I)^{(q-p)}} E_\pi \twoheadrightarrow E_\pi^{q,p}$$

is an isomorphism. This map is the composition

$$E^{p,q} \xrightarrow{L_I^{p,p+1}} E^{p+1,q-1} \xrightarrow{L_I^{p+1,p+2}} \cdots \xrightarrow{L_I^{q-1,q}} E^{q,p}$$

and is given by cup-product:

$$(L_I^{p,p+1} \cup \cdots \cup L_I^{q-1,q}) \cup - : \mathrm{Ext}^p(\Lambda^{q'}, \pi^I) \longrightarrow \mathrm{Ext}^q(\Lambda^{p'}, \pi^I).$$

But Proposition 2.3.7 and the next lemma imply that the latter map is an isomorphism. $\square$

Lemma. *The element* $L_I^{p,p+1} \cup \cdots \cup L_I^{q-1,q}$ *is nonzero in* $\mathrm{Ext}^{q-p}(\Lambda^{p'}, \Lambda^{q'})$.

Proof. Clearly, it suffices to prove this for $p=0$ and $q=d-1$. Let us consider the space $\mathrm{Hom}_{D^b(\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G^0,I))}(\mathcal{C}_I, \mathcal{C}_I[2d-2])$. It is isomorphic to

$$\bigoplus_{i,j=0}^{d-1} \mathrm{Ext}^{i-j+2d-2}(\mathcal{H}^i(\mathcal{C}_I), \mathcal{H}^j(\mathcal{C}_I))$$

via our splitting of $\mathcal{C}_I$. But for $i, j \in \{0, \ldots, d-1\}$ the space $\mathrm{Ext}^{i-j+2d-2}(\Lambda^{i'}, \Lambda^{j'})$ vanishes unless $i-j+2d-2 \leqslant d-1$, which happens only when $i=0$ and $j=d-1$. In other words, we have

$$\begin{aligned}\mathrm{Hom}_{D^b(\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G^0,I))}&(\mathcal{C}_I, \mathcal{C}_I[2d-2]) \\ &\simeq \mathrm{Ext}^{d-1}(\mathcal{H}^0(\mathcal{C}_I), \mathcal{H}^{d-1}(\mathcal{C}_I)) = \mathrm{Ext}^{d-1}(\Lambda^{d-1}, \Lambda^0).\end{aligned}$$

Moreover, through this identification we have

$$(L_I)^{(d-1)} = L_I^{0,1} \cup L_I^{1,2} \cup \cdots \cup L_I^{d-2,d-1}.$$

So we are left to show that $(L_I)^{(d-1)}$ is nonzero.

Now, in the case that $\pi = \pi_0$ is the unit representation of G over $\overline{\mathbb{F}}_\ell$, the explicit description recalled in the fact given in Section 3.3.2 shows that $(L^*_{\pi_0})^{(d-1)}$ is nonzero, and hence so is $(L_I)^{(d-1)}$. □

3.3.4. Theorem 3.3.3 tells us that $(R^*_{i_B}, L^*_{i_B})$ satisfies the "hard Lefschetz theorem". Therefore, by the discussion in Section 3.3.1 it is $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$-equivariantly isomorphic to the pair (H^*, L^*) shifted by $1-d$ of the theorem given in Section 1.2.1, which finishes the proof of the latter theorem.

3.3.5. For π elliptic, we will now describe precisely the isomorphism class of the pair (R^*_π, L^*_π) in the category of finite-dimensional graded vector spaces endowed with a degree-2 endomorphism. This category is abelian artinian and its indecomposable objects are isomorphic, up to shift, to some

$$\boldsymbol{P}_k := \left(\bigoplus_{i=0}^{k} \overline{\mathbb{F}}_\ell[k-2i], L_k \right) \quad \text{with } L_k \text{ the unique map of degree 2 and rank } k.$$

Note also that

$$\boldsymbol{P}_k \simeq (H^*(\mathbb{P}^k, \overline{\mathbb{F}}_\ell)[k], c(\mathbb{O}_{\mathbb{P}^k(1)})),$$

the shifted cohomology of a projective space of dimension k with its tautological Lefschetz operator.

Corollary. *For $j = 0, \dots, d-1$, we have* $(R^*_{\pi_j}, L^*_{\pi_j}) \simeq (\boldsymbol{P}_{j'})^{\oplus j+1} \oplus (\boldsymbol{P}_{j'-1})^{\oplus j}$.

Proof. By Theorem 3.2.1, the graded pieces $R^{-j'}_{\pi_j}, R^{-j'+2}_{\pi_j}, \dots, R^{j'}_{\pi_j}$ all have dimension $j+1$. By Theorem 3.3.3 we see that $L^*_{\pi_j}$ induces isomorphisms

$$R^{-j'}_{\pi_j} \xrightarrow{\sim} R^{-j'+2}_{\pi_j} \xrightarrow{\sim} \cdots \xrightarrow{\sim} R^{j'}_{\pi_j},$$

whence the summand $(\boldsymbol{P}_{j'})^{\oplus j+1}$. Similarly, $L^*_{\pi_j}$ induces isomorphisms

$$R^{-j'+1}_{\pi_j} \xrightarrow{\sim} R^{-j'+3}_{\pi_j} \xrightarrow{\sim} \cdots \xrightarrow{\sim} R^{j'-1}_{\pi_j}$$

between all these j-dimensional spaces, whence the summand $(\boldsymbol{P}_{j'-1})^{\oplus j}$. □

3.4. ***Computation of*** $(\boldsymbol{R}^{\mathrm{red},*}_{\boldsymbol{\pi}}, \boldsymbol{L}^{\mathrm{red}}_{\boldsymbol{\pi}})$**.** We now assume that the unipotent representation π is elliptic. So it is of the form $\pi = \pi_j$ for some $j \in \{0, \dots, d-1\}$. The particular case $j = 0$ corresponds to the trivial representation $\pi_j \simeq \overline{\mathbb{F}}_\ell$. In this case we find that $R^*_{\pi_0} = \bigoplus_{i=0}^{d-1} \overline{\mathbb{F}}_\ell[-2i+d-1]$ so that the total dimension of $R^*_{\pi_0}$ is d. In the other extreme case, when $j = d-1$ so that π_j is the Steinberg representation, we get that $R^*_{\pi_{d-1}}$ is concentrated in degree 0 and has dimension d. These are, however, the only cases where the total dimension is d, in contrast with the other situations we have studied in previous papers (ℓ-adic, banal, and regular cases).

3.4.1. In order to recover a d-dimensional vector space, we consider the following "reduced" version of R^*_π. We put

$$\mathscr{E}^*_{\pi^I} := R^* \operatorname{End}_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}(\pi^I) = \bigoplus_{k\geqslant 0} \operatorname{Ext}^k_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}(\pi^I, \pi^I),$$

the self-extension algebra of the $\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0, I)$-module π^I. This is a positively graded algebra and we denote by $\mathscr{E}^+_{\pi^I} := \bigoplus_{k>0} \operatorname{Ext}^k_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}(\pi^I, \pi^I)$ its augmentation ideal. Via the isomorphism (3.1.1), the graded $\overline{\mathbb{F}}_\ell$-vector space R^*_π carries a graded right-module structure over $\mathscr{E}^*_{\pi^I}$. We then put

$$R^{\mathrm{red},*}_\pi := R^*_\pi / R^*_\pi \mathscr{E}^+_{\pi^I}.$$

For π elliptic, we will show that this graded vector space has total dimension d, as desired.

3.4.2. Theorem. *Choose a splitting* $\bigoplus_{q=0}^{d-1} \bigwedge^{q'} \mathrm{Std}[-q] \xrightarrow{\sim} \mathscr{E}_I$ *as in Theorem 3.1.4. Then, through the isomorphism of Corollary 3.2.9, we have*

$$R^*_{\pi_j} \mathscr{E}^+_{\pi_j^I} \xrightarrow{\sim} \bigoplus_{\substack{-j'<p-q\leqslant j' \\ -j<p+q+1-d\leqslant j}} \operatorname{Ext}^p_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}\big(\textstyle\bigwedge^{q'} \mathrm{Std}, \bigwedge^j \mathrm{Std}\big)[q-p].$$

Therefore we also get an isomorphism

$$R^{\mathrm{red}}_{\pi_j} \simeq \bigoplus_{q=0}^{j'} \operatorname{Ext}^{j'-q}_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}(\Lambda^{q'}, \Lambda^j)[-j'+2q] \oplus \bigoplus_{q=j'+1}^{d-1} \operatorname{Ext}^{q-j'}_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}(\Lambda^{q'}, \Lambda^j)[j'].$$

In particular $R^{\mathrm{red}}_{\pi_j}$ *has total dimension* d *and its graded dimension is given by*

$$\dim_{\overline{\mathbb{F}}_\ell} R^{\mathrm{red},k}_{\pi_j} = \begin{cases} j+1 & \textit{if } k = -j', \\ 1 & \textit{if } k = -j'+2i \textit{ with } 0 < i \leqslant j', \\ 0 & \textit{else.} \end{cases}$$

Proof. Let D be the square defined by the inequalities $-j' \leqslant p-q \leqslant j'$ and $-j \leqslant p+q-d+1 \leqslant j$ and let D^+ be the complement in D of the left-side edges defined by $p-q=-j'$ and $p+q=-j+d-1$. Corollary 3.2.9 expresses $R^*_{\pi_j}$ as the graded vector space associated to the double graded vector space $(p,q) \mapsto E^{pq} := \operatorname{Ext}^p(\Lambda^{q'}, \Lambda^j)$, whose support is D. For $l > 0$, an element of $\mathscr{E}^l_{\pi^I} = \operatorname{Ext}^l(\Lambda^j, \Lambda^j)$ acts through a bigraded map of degree $(l, 0)$. Therefore, $R^*_{\pi_j} \mathscr{E}^+_{\pi_j^I}$ is the graded space associated to a bigraded subspace of $E^{*\bullet}$ whose support is contained in D^+. In other words, the isomorphism of Corollary 3.2.9 takes $R_{\pi_j} \mathscr{E}^+_{\pi_j^I}$ in the right-hand side of our statement.

To get the other inclusion, we need to understand the action of a generator $\xi^1_{j,j}$ of $\mathscr{E}^1_{\pi_j} = \mathrm{Ext}^1(\Lambda^j, \Lambda^j)$, which is given by the cup-product

$$E^{p,q} = \mathrm{Ext}^p_{\mathscr{H}(G^0,I)}(\Lambda^{q'}, \Lambda^j) \xrightarrow{-\cup\xi^1_{j,j}} E^{p+1,q} = \mathrm{Ext}^{p+1}_{\mathscr{H}(G^0,I)}(\Lambda^{q'}, \Lambda^j).$$

But Theorem 2.3.5 tells us that this is an isomorphism, whenever both sides are nonzero. It follows that $E^{p+1,q} \subset R^*_{\pi_j}\mathscr{E}^+_{\pi_j^I}$, and finally that the isomorphism of Corollary 3.2.9 induces the claimed isomorphism. The claim on $R^{\mathrm{red}}_{\pi_j}$ is a direct a consequence of the latter. □

By definition the graded map L^*_π commutes with $\mathscr{E}^+_{\pi^I}$, so L^*_π induces a graded linear map $L^{\mathrm{red}}_\pi : R^{\mathrm{red}}_\pi \longrightarrow R^{\mathrm{red}}_\pi[2]$.

3.4.3. Corollary. *For $j = 0, \dots, d-1$, we have $(R^{\mathrm{red}}_{\pi_j}, L^{\mathrm{red}}_{\pi_j}) \simeq \boldsymbol{P}_{j'} \oplus (\boldsymbol{P}_0)^{\oplus j}[j']$. In particular we have the equality*

$$[R^{\mathrm{red}}_{\pi_j}, L^{\mathrm{red}}_{\pi_j}] = \mathrm{LJ}(\pi_j) \otimes (\sigma^{\mathrm{ss}}(\pi_j), N_{\pi_j})$$

in the Grothendieck group of Weil–Deligne $\overline{\mathbb{F}}_\ell D^\times$-representations.

Proof. By the corollary in Section 3.3.5 (and its proof) and the very definition of $R^{\mathrm{red}}_{\pi_j}$ as a quotient of $R^*_{\pi_j}$, we see that $L^{\mathrm{red}}_{\pi_j}$ induces a *surjective* map $R^{\mathrm{red},k}_{\pi_j} \twoheadrightarrow R^{\mathrm{red},k+2}_{\pi_j}$ for all $k \geqslant -j'$. So our first claim follows from the description of the graded dimension of $R^*_{\pi_j}$ in the above theorem. The second claim follows from the equalities

- $\mathrm{LJ}(\pi_j) = (-1)^j \,\mathrm{LJ}(\pi_0) = (-1)^{j'} \,\mathrm{LJ}(\pi_{d-1}) = (-1)^{j'}[1_{D^\times}]$ and
- $\sigma^{\mathrm{ss}}(\pi_j) = (1_{W_K})^{\oplus d}$ and N_{π_j} has Jordan type $\lambda = (d-j, 1^{(j)})$.

The latter equalities are seen, for example, by reducing modulo ℓ the corresponding equalities for $v_{\{1,\dots,j'\}}(\overline{\mathbb{Q}}_\ell)$, since we have $\pi_j = v_{\{1,\dots,j'\}}(\overline{\mathbb{F}}_\ell)$. □

3.4.4. Remark. There are dual versions of the theorem and corollary. Instead of considering $R^{\mathrm{red},*}_\pi$, one could consider

$$R^{\mathrm{cored},*}_\pi := R^*_\pi[\mathscr{E}^+_{\pi^I}]$$

the graded subvector space of R^*_π which is annihilated by the augmentation ideal $\mathscr{E}^+_{\pi^I}$. Then a very similar argument to the proof of the above theorem shows that the isomorphism of Corollary 3.2.9 induces the following one:

$$R^{\mathrm{cored}}_{\pi_j} \simeq \bigoplus_{q=0}^{j-1} \mathrm{Ext}^{j'+q}_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}(\Lambda^{q'}, \Lambda^j)[j'] \oplus \bigoplus_{q=j}^{d-1} \mathrm{Ext}^{j+q'}_{\mathscr{H}_{\overline{\mathbb{F}}_\ell}(G^0,I)}(\Lambda^{q'}, \Lambda^j)[j'-2q].$$

In particular $R^{\mathrm{cored},*}_{\pi_j}$ has total dimension d and graded dimension given by

$$\dim_{\overline{\mathbb{F}}_\ell} R^{\mathrm{red},k}_{\pi_j} = \begin{cases} j+1 & \text{if } k = j', \\ 1 & \text{if } k = j'-2i \text{ with } 0 < i \leqslant j', \\ 0 & \text{else.} \end{cases}$$

Similar arguments as in the last corollary show that L_π^* takes R_π^{cored} to itself, and that for $j=0,\ldots,d-1$, we have $(R_{\pi_j}^{\text{cored}}, L_{\pi_j}^{\text{cored}}) \simeq \boldsymbol{P}_{j'} \oplus (\boldsymbol{P}_0)^{\oplus j}[-j']$, leading to the same conclusion that $[R_{\pi_j}^{\text{cored}}, L_{\pi_j}^{\text{cored}}] = \mathrm{LJ}(\pi_j) \otimes (\sigma^{\text{ss}}(\pi_j), N_{\pi_j})$.

3.4.5. We now give the relation between the algebra $\mathscr{E}_{\pi^I}^*$ used here and that used in the statement of the main theorem in Section 1.2.2. The inflation functor from $\mathscr{H}_{\bar{\mathbb{F}}_\ell}(G^0, I)$-modules to $\mathscr{H}_{\bar{\mathbb{F}}_\ell}(G^0, I^\ell)$-modules along the map of Proposition 2.1.3 yields a morphism of graded algebras

$$\mathscr{E}_{\pi^I}^* \longrightarrow \mathrm{Ext}^*_{\bar{\mathbb{F}}_\ell G^0}(\pi, \pi)$$

and the right action of $\mathscr{E}_{\pi^I}^*$ on R_π^* that one deduces from this morphism coincides with the right-action used above, which was obtained from the expression (3.1.1). We claim that the image of the above map is the algebra $\mathscr{E}_\pi^*$ defined in Section 1.2.2. Since $\mathscr{E}_{\pi^I}^*$ is generated in degree 1, we only need to see that sequence

$$\mathrm{Ext}^1_{\mathscr{H}_{\bar{\mathbb{F}}_\ell}(G^0,I)}(\pi^I, \pi^I) \longrightarrow \mathrm{Ext}^1_{\bar{\mathbb{F}}_\ell G^0}(\pi, \pi) \longrightarrow \mathrm{Ext}^1_{\bar{\mathbb{F}}_\ell \bar{G}}(\bar{\pi}, \bar{\pi})$$

is exact. But through the equivalence of Corollary 2.1.2 this sequence reads

$$\mathrm{Ext}^1_{\mathscr{H}_{\bar{\mathbb{F}}_\ell}(G^0,I)}(\pi^I, \pi^I) \longrightarrow \mathrm{Ext}^1_{\mathscr{H}_{\bar{\mathbb{F}}_\ell}(G^0,I^\ell)}(\pi^I, \pi^I) \longrightarrow \mathrm{Ext}^1_{\mathscr{H}_{\bar{\mathbb{F}}_\ell}(\mathrm{GL}_d(\mathbb{O}),I^\ell)}(\pi^I, \pi^I).$$

On the other hand, since $\ell > d$, the right-hand side coincides with $\mathrm{Ext}^1_{\bar{\mathbb{F}}_\ell[I^\ell/I]}(\pi^I, \pi^I)$. Therefore the exactness is equivalent to the fact that any extension E of π^I by π^I as $\mathscr{H}(G^0, I^\ell)$-modules is a $\mathscr{H}(G^0, I)$-module if and only if it splits as an extension of I^ℓ/I-modules. But this is clear.

Appendix: Proof of Theorem 2.3.5 and Proposition 2.3.7

In order to compute enough cup-products between the nonvanishing Ext spaces of Corollary 2.3.3, we need to exhibit explicit generators of some of these Ext spaces.

A.1. ***Construction of extensions.*** Recall that we have abbreviated $\Lambda^i := \bigwedge^i \mathrm{Std}$ and $\mathrm{Ext}^k_{\bar{\mathbb{F}}_\ell[(G^0\cap T)/T^0]}\big(\bigwedge^i \mathrm{Std}, \bigwedge^j \mathrm{Std}\big)$ by $\mathrm{Ext}^k(\Lambda^i, \Lambda^j)$.

A.1.1. *Further notation.* The standard representation $\Lambda^1 = \mathrm{Std}$ can be presented as the quotient of the permutation module $\bar{\mathbb{F}}_\ell^d$ by the line generated by $(1, 1, \ldots, 1)$. Let $e_1, \ldots, e_d$ be the image of the canonical basis of $\bar{\mathbb{F}}_\ell^d$ in Λ^1. This set of vectors enjoys the following properties:

- it is a generating set with "only" linear relation $\sum_{r=1}^d e_r = 0$ and
- the action of $\mathfrak{S}_d$ is given by $\sigma(e_r) = e_{\sigma(r)}$, where we see $\mathfrak{S}_d$ as the permutation group of $\{1, \ldots, d\}$.

More generally, if I is a subset of $\{1,\dots,d\}$ of size $|I|=i$ we put

$$e_I := e_{r_1}\wedge\cdots\wedge e_{r_i}\in\Lambda^i,$$

where $I=\{r_1,r_2,\dots,r_i\}$ and $r_1<r_2<\cdots<r_i$. For a collection $I_1,\dots,I_n$ of subsets of $\{1,\dots,d\}$ we define $\varepsilon(I_1,\dots,I_n)$ by the equality

$$e_{I_1}\wedge\cdots\wedge e_{I_n}=\varepsilon(I_1,\dots,I_n)e_{I_1\cup\cdots\cup I_n}.$$

Thus $\varepsilon(I_1,\dots,I_n)$ is 0 unless all I_t's are pairwise disjoint, in which case it is a sign. Now, if we fix i, the set of vectors $\{e_I, I\subset\{1,\dots,d\}, |I|=i\}$ enjoys the following properties:

- It is a generating set whose space of linear relations is generated by the following ones: $\sum_{r=1}^d \varepsilon(J,\{r\})e_{J\cup\{r\}}=0$ for each $J\subset\{1,\dots,d\}$ of size $|J|=i-1$. In particular, the subset of all e_I's for I contained in $\{1,\dots,d-1\}$ is a basis of Λ^i.
- The action of $\mathfrak{S}_d$ is given by $\sigma(e_I)=\mathrm{sgn}(\sigma_{|I})e_{\sigma(I)}$, where $\sigma_{|I}$ is considered as a permutation of $\{1,\dots,i\}$ via the orderings on I and $\sigma(I)$ inherited from that of $\{1,\dots,d\}$.

When $i=i_1+\cdots+i_n$ is a composition series of i, we have a canonical map (exterior product) $\Lambda^{i_1}\otimes\cdots\otimes\Lambda^{i_n}\xrightarrow{\mathrm{prod}}\Lambda^i$. We will later need the quasisection $\Lambda^i\xrightarrow{\mathrm{can}}\Lambda^{i_1}\otimes\cdots\otimes\Lambda^{i_n}$ of this map defined by

$$\begin{aligned}&\mathrm{can}(v_1\wedge\cdots\wedge v_i)\\ &\quad=\sum_{\tau\in[\mathfrak{S}_i/\mathfrak{S}_{i_1}\times\cdots\times\mathfrak{S}_{i_n}]}\mathrm{sgn}(\tau)(v_{\tau(1)}\wedge\cdots\wedge v_{\tau(i_1)})\otimes\cdots\otimes(v_{\tau(i-i_n+1)}\wedge\cdots\wedge v_{\tau(i)}),\end{aligned}$$

where the index set stands for permutations τ that are increasing on each $\left[\sum_{s<t} i_s+1, \sum_{s\leqslant t} i_s\right]$. Note that prod $\circ$ can is the multiplication by $i!/(i_1!\cdots i_n!)$, which is nonzero in $\overline{\mathbb{F}}_\ell$, due to our assumption that $\ell>d$. Note also that can is $\mathfrak{S}_d$-equivariant and the evaluation on e_I is given by

$$\mathrm{can}(e_I)=\sum_{I_1\sqcup\cdots\sqcup I_n=I,|I_t|=i_t}\varepsilon(I_1,\dots,I_n)e_{I_1}\otimes\cdots\otimes e_{I_n}. \tag{A.1.2}$$

A.1.3. Definition. *For $d-1\geqslant i\geqslant j\geqslant 0$, we denote by $\xi_{i,j}^{i-j}$ the generator of $\mathrm{Ext}^{i-j}(\Lambda^i,\Lambda^j)$ provided by the nonzero $\mathfrak{S}_d$-equivariant map $\Lambda^i\xrightarrow{\mathrm{can}}\Lambda^j\otimes\Lambda^{i-j}$ via the isomorphism of Proposition 2.3.2(i).*

A.1.4. *Proof of Proposition 2.3.7.* Since the regular representation $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]$ is semi-simple, it is sufficient to prove that for all $i\leqslant j$ the cup-product

$$\xi_{j,i}^{j-i}\cup - : \mathrm{Ext}^{j'}(\Lambda^i,\overline{\mathbb{F}}_\ell[\mathfrak{S}_d])\longrightarrow\mathrm{Ext}^{i'}(\Lambda^j,\overline{\mathbb{F}}_\ell[\mathfrak{S}_d]) \tag{A.1.5}$$

is an isomorphism. Note that, as a $\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]$-module, we have $\overline{\mathbb{F}}_\ell[\mathfrak{S}_d] = \mathrm{ind}_{T\cap G^0}^{N\cap G^0}(\overline{\mathbb{F}}_\ell)$, so Frobenius reciprocity tells us that

$$\mathrm{Ext}^{j'}_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(\Lambda^i, \overline{\mathbb{F}}_\ell[\mathfrak{S}_d]) \simeq \mathrm{Ext}^{j'}_{\overline{\mathbb{F}}_\ell[(T\cap G^0)/T^0]}(\Lambda^i, \overline{\mathbb{F}}_\ell) \simeq \mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^i, \Lambda^{j'}).$$

Now, by the formula for cup-products in Proposition 2.3.2(ii) and the expression of $\xi^{j-i}_{j,i}$ in Definition A.1.3, we see that the map (A.1.5) is isomorphic to the map

$$\text{(A.1.6)}\quad \begin{aligned} &\mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^i, \Lambda^{j'}) \to \mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^j, \Lambda^{i'}) \\ &\qquad \varphi \mapsto \psi : (\Lambda^j \xrightarrow{\text{can}} \Lambda^i \otimes \Lambda^{j-i} \xrightarrow{\varphi\otimes\mathrm{Id}} \Lambda^{j'} \otimes \Lambda^{i'-j'} \xrightarrow{\wedge} \Lambda^{i'}). \end{aligned}$$

That the latter map is an isomorphism is an avatar of the Lefschetz decomposition of the exterior algebra of an hermitian space (see, for example, [Griffiths and Harris 1978, Chapter 0.7]). Since our field of coefficients $\overline{\mathbb{F}}_\ell$ has positive characteristic we briefly review the argument in order to ensure that it is still valid in our context.

Let us consider the graded space $H^* := \mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^*, \Lambda^*) = \check{\Lambda}^* \otimes \Lambda^*$, where $\check{\Lambda}$ is the $\overline{\mathbb{F}}_\ell$-linear dual space of Λ. We endow it with an operator of degree 2 which on the (p, q) part is given by

$$\begin{aligned} L :\quad &\mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^p, \Lambda^q) \to \mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^{p+1}, \Lambda^{q+1}) \\ &\qquad \varphi \mapsto L\varphi : (\Lambda^{p+1} \xrightarrow{\text{can}} \Lambda^p \otimes \Lambda \xrightarrow{\varphi\otimes\mathrm{Id}} \Lambda^q \otimes \Lambda \xrightarrow{\wedge} \Lambda^{q+1}), \end{aligned}$$

and an operator of degree -2 which on the (p, q)-part is given by

$$\begin{aligned} {}^tL :\quad &\mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^p, \Lambda^q) \to \mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^{p-1}, \Lambda^{q-1}) \\ &\qquad \varphi \mapsto {}^tL\varphi : (\Lambda^{p-1} \xrightarrow{\wedge} \Lambda^p \otimes \check{\Lambda} \xrightarrow{\varphi\otimes\mathrm{Id}} \Lambda^q \otimes \check{\Lambda} \xrightarrow{\text{can}} \Lambda^{q-1}). \end{aligned}$$

Here the last map is the composition of can with the evaluation map $\Lambda \otimes \check{\Lambda} \longrightarrow \overline{\mathbb{F}}_\ell$. Explicitly, denoting by $e_1^*, \dots, e_{d-1}^*$ the dual basis of $e_1, \dots, e_{d-1}$, we have $L\varphi = \varphi \wedge \left(\sum_i e_i^* \otimes e_i\right)$, or, even more explicitly, we have for $I, J \subset \{1, \dots, d-1\}$ with $|I| = p$ and $|J| = q$, denoting by I^c the complementary subset,

$$L(e_I^* \otimes e_J) = \sum_{k\in I^c\cap J^c} \varepsilon(I, k)\varepsilon(J, k) e^*_{I\cup\{k\}} \otimes e_{J\cup\{k\}},$$

while

$${}^tL(e_I^* \otimes e_J) = \sum_{k\in I\cap J} \varepsilon(I \setminus \{k\}, k)\varepsilon(J \setminus \{k\}, k) e^*_{I\setminus\{k\}} \otimes e_{J\setminus\{k\}}.$$

Then a simple computation shows that commutator $[L, {}^tL]$ acts on $\mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^p, \Lambda^q)$ by multiplication by $p + q - d + 1$, hence it acts on H^k by multiplication by $k - d + 1$. It follows that the triple $(L, [L, {}^tL], {}^tL)$ is a $\mathfrak{sl}_2$-triple, that is, is the image of the canonical basis $(E_{12}, E_{11} - E_{22}, E_{21})$ of $\mathfrak{sl}_2(\overline{\mathbb{F}}_\ell)$ by a unique structure of $\mathfrak{sl}_2(\overline{\mathbb{F}}_\ell)$-modules on H^*. Moreover, the weights are $1-d, 2-d, \dots, d-1$ and the eigenspace of weight $1 - d + k$ is H^k. Now, we use our assumption that $\ell > d$.

This means that the simple $\mathfrak{sl}_2(\overline{\mathbb{F}}_\ell)$-modules with weights in the above range are constructed in the same way as for $\mathfrak{sl}_2(\mathbb{C})$. In particular, for each $\lambda \in \{0, \dots, d-1\}$ there is a unique simple module (V_λ, r_λ) of highest weight λ, its weights are $-\lambda, \lambda+2, \dots, \lambda$, and $r_\lambda(E_{12})$ induces an isomorphism between eigenspaces $V_\lambda^k \xrightarrow{\sim} V_\lambda^{k+2}$ for $-\lambda \leqslant k < \lambda$. Now, taking a filtration of H^* with simple subquotients, we see that L^k induces an isomorphism $H^{d-1-k} \xrightarrow{\sim} H^{d-1+k}$ for all $k = 0, \dots, d-1$, and consequently an isomorphism

$$L^k : \mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^p, \Lambda^q) \xrightarrow{\sim} \mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(\Lambda^{p+k}, \Lambda^{q+k})$$

whenever $p+q = d-1-k$. But we have $L^k\varphi = \varphi \wedge \big(k! \sum_{|K|=k} e_K^* \otimes e_K\big)$, so that L^k is also given by

$$L^k\varphi = k!(\Lambda^{p+k} \xrightarrow{\text{can}} \Lambda^p \otimes \Lambda^k \xrightarrow{\varphi\otimes\mathrm{Id}} \Lambda^q \otimes \Lambda^k \xrightarrow{\wedge} \Lambda^{q+k}).$$

Now, taking $p = i$, $k = j - i = i' - j'$, and $q = j'$ we get that (A.1.6) is an isomorphism, as desired. □

A.1.7. Lemma. *For $i = 1, \dots, d-1$, we define inductively an element $F_i \in \Lambda^i \otimes \Lambda^i$ by setting*

$$\text{(A.1.8)} \qquad F_1 := \frac{1}{d}\sum_{r=1}^{d} e_r \otimes e_r, \quad \textit{and} \quad F_i = \frac{1}{d}\sum_{r=1}^{d} e_r \wedge F_{i-1} \wedge e_r, \ \textit{for } i > 1.$$

Then, F_i is a generator of $(\Lambda^i \otimes \Lambda^i)^{\mathfrak{S}_d}$ and we have the formula

$$\text{(A.1.9)} \qquad F_i = (-1)^{i(i-1)/2}\frac{i!}{d^i} \sum_{I\subset\{1,\dots,d\}, |I|=i} e_I \otimes e_I.$$

Proof. Formula (A.1.9) is easily checked by induction, using the fact that

$$e_r \wedge e_I \otimes e_I \wedge e_r = \varepsilon(\{r\}, I)\varepsilon(I, \{r\})e_{I\cup\{r\}} \otimes e_{I\cup\{r\}}$$

is 0 unless $r \in \{1, \dots, d\} \setminus I$, in which case we have $\varepsilon(\{r\}, I)\varepsilon(I, \{r\}) = (-1)^{|I|}$. Now, F_i is clearly $\mathfrak{S}_d$-invariant, so it only remains to check it is nonzero. Consider the element

$$E_i := e_{\{i+1,\dots,d-1\}} \wedge F_i \wedge e_{\{i+2,\dots,d\}} \in \Lambda^{d-1} \otimes \Lambda^{d-1} \simeq \overline{\mathbb{F}}_\ell.$$

In our formula for F_i, the only subset I that will contribute to E_i is $I = \{1, \dots, i\}$, so that we get

$$E_i = c_i \cdot e_{\{1,\dots,d\}\setminus\{d-1\}} \otimes e_{\{1,\dots,d\}\setminus\{i+1\}},$$

with $c_i = \pm i!/d^i$ nonzero. Hence $E_i \neq 0$ and therefore $F_i \neq 0$. □

The element F_i defines a nonzero morphism $\xi^i_{0,i} : \Lambda^0 = \overline{\mathbb{F}}_\ell \longrightarrow \Lambda^i \otimes \Lambda^i$, and therefore provides a generator $\xi^i_{0,i}$ for $\mathrm{Ext}^i(\Lambda^0, \Lambda^i)$. More generally, for $i \leqslant j$, the map $\Lambda^i \longrightarrow \Lambda^j \otimes \Lambda^{j-i}$, $v \mapsto v \wedge F_{j-i}$ is a $\mathfrak{S}_d$-equivariant map and the above proof shows that it takes nonzero value on $v = e_{\{d-i,\dots,d-1\}}$, for example.

A.1.10. Definition. *For $0 \leqslant i \leqslant j \leqslant d-1$, we denote by $\xi^{j-i}_{i,j}$ the generator of $\mathrm{Ext}^{j-i}(\Lambda^i, \Lambda^j)$ provided by the nonzero $\mathfrak{S}_d$-equivariant map $v \mapsto v \wedge F_{j-i}$, $\Lambda^i \xrightarrow{\text{can}} \Lambda^j \otimes \Lambda^{j-i}$ via the isomorphism of Proposition 2.3.2(i).*

A.1.11. Lemma. *For $0 < i < d-1$, the mapping $e_r \mapsto e_r \wedge F_{i-1} \wedge e_r - F_i$ extends uniquely to a $\mathfrak{S}_d$-equivariant $\overline{\mathbb{F}}_\ell$-linear nonzero map*

$$\xi^i_{1,i} : \Lambda^1 \longrightarrow \Lambda^i \otimes \Lambda^i.$$

Proof. Since the e_r's generate Λ^1, the extension is unique. For its existence we need to check compatibility with the linear relation $\sum_r e_r = 0$. But this follows from the recursive definition (A.1.8). Equivariance is clear, given the invariance of F_i and F_{i-1}. It remains to check nonvanishing. But we have

$$\begin{aligned} e_{\{i+1,\dots,d-1\}} \wedge \xi^i_{1,i}(e_d) \wedge e_{\{i+2,\dots,d\}} &= e_{\{i+1,\dots,d-1\}} \wedge (e_d \wedge F_{i-1} \wedge e_d - F_i) \wedge e_{\{i+2,\dots,d\}} \\ &= -\, e_{\{i+1,\dots,d-1\}} \wedge F_i \wedge e_{\{i+2,\dots,d\}}, \end{aligned}$$

which was shown to be nonzero in the previous proof. $\square$

The homomorphism $\xi^i_{1,i}$ therefore provides a generator of $\mathrm{Ext}^i(\Lambda, \Lambda^i)$. We will now construct generators of some other nonzero Ext spaces.

A.1.12. *Explicit basis of* Ext *spaces.* In this section we fix integers $i, j, k \in \{0, \dots, d-1\}$ and we assume that $i \leqslant j$ and $i + j \leqslant d-1$. Under these assumptions, Corollary 2.3.3 tells us that $\mathrm{Ext}^k(\Lambda^i, \Lambda^j)$ has dimension 1 exactly when $j - i \leqslant k \leqslant j + i$.

Definition. *In this context, if k is of the form $k = j - i + 2l$, we define*

$$\begin{aligned} \xi^k_{i,j} : \quad \Lambda^i \xrightarrow{\text{can}} \Lambda^{i-l} \otimes \Lambda^l &\to \Lambda^j \otimes \Lambda^k \\ v \otimes w &\mapsto v \wedge F_{k-l} \wedge w. \end{aligned}$$

If k is of the form $k = j - i + 2l + 1$, we define

$$\begin{aligned} \xi^k_{i,j} : \quad \Lambda^i \xrightarrow{\text{can}} \Lambda^{i-l-1} \otimes \Lambda^1 \otimes \Lambda^l &\to \Lambda^j \otimes \Lambda^k \\ v \otimes u \otimes w &\mapsto v \wedge \xi^{k-l}_{1,k-l}(u) \wedge w. \end{aligned}$$

By construction, the map $\xi^k_{i,j}$ is $\mathfrak{S}_d$-equivariant, and we denote by the same symbol the corresponding element of $\mathrm{Ext}^k(\Lambda^i, \Lambda^j)$ via Proposition 2.3.2(i).

Remark. This definition is consistent with Definition A.1.10. Moreover, by Proposition 2.3.2(ii), we see that by construction we have a factorization $\xi_{i,j}^k = \xi_{i,i-l}^l \cup \xi_{i-l,j}^{k-l}$ with l as in the definition above.

The following proposition is the key to Theorem 2.3.5.

Proposition. *For all k such that $j-i \leqslant k \leqslant j+i$, the element $\xi_{i,j}^k$ is a generator of* $\mathrm{Ext}^k(\Lambda^i, \Lambda^j)$. *Moreover, if $k < j+i$, we have*

(i) $\xi_{i,i}^1 \cup \xi_{i,j}^k \in \overline{\mathbb{F}}_\ell^\times \cdot \xi_{i,j}^{k+1}$ *and*

(ii) $\xi_{i,j}^k \cup \xi_{j,j}^1 \in \overline{\mathbb{F}}_\ell^\times \cdot \xi_{i,j}^{k+1}$.

Proof. The proofs of (i) and (ii) are somewhat lengthy and complicated computations, and are postponed to Section A.2 in order to lighten a bit this section.

Let us deduce that $\xi_{i,j}^k$ is a generator of $\mathrm{Ext}^k(\Lambda^i, \Lambda^j)$. We know that this is equivalent to $\xi_{i,j}^k$ being nonzero. By either (i) and (ii), it is enough to prove that $\xi_{i,j}^{i+j}$ is nonzero. By definition, the homomorphism $\xi_{i,j}^{i+j} : \Lambda^i \longrightarrow \Lambda^j \otimes \Lambda^{i+j}$ is given by $v \mapsto F_j \wedge v$. But the proof of Lemma (A.1.9) shows that, for example, $F_j \wedge e_{\{d-i+1,\dots,d\}}$ is nonzero (note that $j+2 \leqslant d-i+1$). $\square$

A.1.13. *Proof of Theorem 2.3.5.* The last proposition implies the claim in the case where $i \leqslant j$ and $i+j \leqslant d-1$. Using duality we get the case where $i \geqslant j$ and $i+j \leqslant d-1$. Here by duality we mean contragredient $A \mapsto A^* = \mathrm{Hom}_{\overline{\mathbb{F}}_\ell}(A, \overline{\mathbb{F}}_\ell)$ of $\overline{\mathbb{F}}_\ell$-representations of $\mathfrak{S}_d$. Indeed, for such representations we have functorial isomorphisms

$$\mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(A, B) \xrightarrow{\sim} \mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(B^*, A^*)$$

that are compatible with cup-products in the obvious sense, and, on the other hand, we have $(\Lambda^i)^* \simeq \Lambda^i$ and $(\Lambda^j)^* \simeq \Lambda^j$. So the claim is now proved for $i+j \leqslant d-1$.

In order to get the case $i+j \geqslant d-1$ we use the endoequivalence of the category of $\overline{\mathbb{F}}_\ell$-representations of $\mathfrak{S}_d$ given by $A \mapsto A \otimes \Lambda^{d-1}$. Again we have functorial isomorphisms

$$\mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(A, B) \xrightarrow{\sim} \mathrm{Ext}^*_{\overline{\mathbb{F}}_\ell[(N\cap G^0)/T^0]}(A \otimes \Lambda^{d-1}, B \otimes \Lambda^{d-1})$$

that are compatible with cup-products. On the other hand, we have

$$\Lambda^i \otimes \Lambda^{d-1} \simeq \Lambda^{d-1-i} \quad \text{and} \quad \Lambda^j \otimes \Lambda^{d-1} \simeq \Lambda^{d-1-j},$$

and $(d-1-i)+(d-1-j) \leqslant d-1$ whenever $i+j \geqslant d-1$.

A.2. ***Proof of the preceding proposition.*** We now prove items (i) and (ii) of the proposition appearing on this page. We will evaluate both sides of the claimed equalities on elements e_I for $I \subset \{1, \dots, d\}$ of size i. Inserting (A.1.2) in our definitions, we get

$$\begin{aligned}\xi^1_{i,i}(e_I) &= \sum_{r\in I} \varepsilon(I \setminus \{r\}, \{r\})\, e_{I\setminus\{r\}} \wedge (e_r \otimes e_r - F_1)\\ &= \sum_{r\in I} e_I \otimes e_r - \sum_{r\in I} \varepsilon(I \setminus \{r\}, \{r\})\, e_{I\setminus\{r\}} \wedge F_1.\end{aligned}$$

Similarly, when $k = j - i + 2l$ we get

$$\xi^k_{i,j}(e_I) = \sum_{I=I_{i-l}\sqcup I_l} \varepsilon(I_{i-l}, I_l)\, e_{I_{i-l}} \wedge F_{k-l} \wedge e_{I_l},$$

and when $k = j - i + 2l + 1$ we get

$$\xi^k_{i,j}(e_I) = \sum_{I=I_{i-l-1}\sqcup\{r\}\sqcup I_l} \varepsilon(I_{i-l-1}, \{r\}, I_l)\, e_{I_{i-l-1}} \wedge (e_r \wedge F_{k-l-1} \wedge e_r - F_{k-l}) \wedge e_{I_l}.$$

A.2.1. *Proof of* (i) *in the case $k = j - i + 2l$.* We assume that $k = j - i + 2l$. From the two expressions above and from the dictionary for cup-products in Proposition 2.3.2(ii), we get

$$(\xi^1_{i,i} \cup \xi^k_{i,j})(e_I) = A - B,$$

with

$$A = \sum_{r\in I} \xi^k_{i,j}(e_I) \wedge e_r = \sum_{r\in I} \sum_{I=I_{i-l}\sqcup I_l} \varepsilon(I_{i-l}, I_l)\, e_{I_{i-l}} \wedge F_{k-l} \wedge e_{I_l} \wedge e_r$$

and

$$B = \frac{1}{d} \sum_{r\in I} \varepsilon(I \setminus \{r\}, \{r\}) \sum_{s=1}^{d} \xi^k_{i,j}(e_{I\setminus\{r\}} \wedge e_s) \wedge e_s.$$

Now in the expression A, each summand is 0 unless $r \in I_{i-l}$, in which case we have

$$\begin{aligned}&\varepsilon(I_{i-l}, I_l)\, e_{I_{i-l}} \wedge F_{k-l} \wedge e_{I_l} \wedge e_r\\ &\qquad = (-1)^l \varepsilon(I_{i-l}, I_l)\varepsilon(I_{i-l} \setminus \{r\}, \{r\})\, e_{I_{i-l}\setminus\{r\}} \wedge (e_r \wedge F_{k-l} \wedge e_r) \wedge e_{I_l}\\ &\qquad = (-1)^l \varepsilon(I_{i-l} \setminus \{r\}, \{r\}, I_l)\, e_{I_{i-l}\setminus\{r\}} \wedge (e_r \wedge F_{k-l} \wedge e_r) \wedge e_{I_l},\end{aligned}$$

so that eventually we may rewrite

$$A = (-1)^l \sum_{I=I_{i-l-1}\sqcup\{r\}\sqcup I_l} \varepsilon(I_{i-l-1}, \{r\}, I_l)\, e_{I_{i-l-1}} \wedge (e_r \wedge F_{k-l} \wedge e_r) \wedge e_{I_l}$$

and thus recognize one summand of the expression of $\xi^{k+1}_{i,j}(e_I)$. Moreover, in expression B each summand is 0 unless s is outside $I \setminus \{r\}$, in which case, putting

$I^{-r+s} := I \setminus \{r\} \sqcup \{s\}$, we may write

$$\begin{aligned}
&\xi^k_{i,j}(e_{I\setminus\{r\}} \wedge e_s) \wedge e_s \\
&\quad = \varepsilon(I \setminus \{r\}, \{s\}) \xi^k_{i,j}(e_{I^{-r+s}}) \wedge e_s \\
&\quad = \varepsilon(I \setminus \{r\}, \{s\}) \sum_{I^{-r+s}=I^{-r+s}_{i-l} \sqcup I^{-r+s}_l} \varepsilon(I^{-r+s}_{i-l}, I^{-r+s}_l) e_{I^{-r+s}_{i-l}} \wedge F_{k-l} \wedge e_{I^{-r+s}_l} \wedge e_s.
\end{aligned}$$

Now in the last expression, each summand vanishes unless $s \in I^{-r+s}_{i-l}$ and, as above, we have

$$\begin{aligned}
&\varepsilon(I^{-r+s}_{i-l}, I^{-r+s}_l) e_{I^{-r+s}_{i-l}} \wedge F_{k-l} \wedge e_{I^{-r+s}_l} \wedge e_s \\
&\qquad = (-1)^l \varepsilon(I^{-r+s}_{i-l} \setminus \{s\}, \{s\}, I^{-r+s}_l) e_{I^{-r+s}_{i-l}\setminus\{s\}} \wedge (e_s \wedge F_{k-l} \wedge e_s) \wedge e_{I^{-r+s}_l}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&\xi^k_{i,j}(e_{I\setminus\{r\}} \wedge e_s) \wedge e_s \\
&= (-1)^l \varepsilon(I \setminus \{r\}, \{s\}) \sum_{I\setminus\{r\}=I_{i-l-1}\sqcup I_l} \varepsilon(I_{i-l-1}, \{s\}, I_l) e_{I_{i-l-1}} \wedge (e_s \wedge F_{k-l} \wedge e_s) \wedge e_{I_l}.
\end{aligned}$$

Because of the equality

$$\varepsilon(I_{i-l-1}, \{s\}, I_l) = (-1)^l \varepsilon(I_{i-l-1}, I_l, \{s\}) = (-1)^l \varepsilon(I_{i-l-1}, I_l) \varepsilon(I \setminus \{r\}, \{s\}),$$

we have

$$\varepsilon(I \setminus \{r\}, \{s\}) \varepsilon(I_{i-l-1}, \{s\}, I_l) = \varepsilon(I \setminus \{r\}, \{r\}) \varepsilon(I_{i-l-1}, \{r\}, I_l),$$

so that eventually

$$\begin{aligned}
B &= \frac{(-1)^l}{d} \sum_{s=1}^{d} \sum_{I=I_{i-l-1}\sqcup\{r\}\sqcup I_l} \varepsilon(I_{i-l-1}, \{r\}, I_l) e_{I_{i-l-1}} \wedge (e_s \wedge F_{k-l} \wedge e_s) \wedge e_{I_l} \\
&= (-1)^l \sum_{I=I_{i-l-1}\sqcup\{r\}\sqcup I_l} \varepsilon(I_{i-l-1}, \{r\}, I_l) e_{I_{i-l-1}} \wedge F_{k-l+1} \wedge e_{I_l},
\end{aligned}$$

which gives the second summand in the expression of $\xi^{k+1}_{i,j}(e_I)$. Namely, we have shown that $\xi^1_{i,i} \cup \xi^k_{i,j} = (-1)^l \xi^{k+1}_{i,j}$ in this case.

A.2.2. *Proof of* (i) *in the case* $k=j-i+2l+1$. We now assume that $k=j-i+2l+1$. Again we can write

$$(\xi^1_{i,i} \cup \xi^k_{i,j})(e_I) = A - B,$$

with

$$A = \sum_{r\in I} \xi^k_{i,j}(e_I) \wedge e_r$$

and

$$B = \frac{1}{d} \sum_{r \in I} \varepsilon(I \setminus \{r\}, \{r\}) \sum_{s=1}^{d} \xi_{i,j}^{k}(e_{I \setminus \{r\}} \wedge e_s) \wedge e_s.$$

Inserting the expression of $\xi_{i,j}^{k}$ in A we may decompose $A = A_1 + A_2$ with

$$A_1 = \sum_{r \in I} \sum_{I = I_{i-l-1} \sqcup \{t\} \sqcup I_l} \varepsilon(I_{i-l-1}, t, I_l) e_{I_{i-l-1}} \wedge e_t \wedge F_{k-l-1} \wedge e_t \wedge e_{I_l} \wedge e_r$$

and

$$A_2 = -\sum_{r \in I} \sum_{I = I_{i-l-1} \sqcup \{t\} \sqcup I_l} \varepsilon(I_{i-l-1}, t, I_l) e_{I_{i-l-1}} \wedge F_{k-l} \wedge e_{I_l} \wedge e_r.$$

Each summand of A_1 vanishes unless $r \in I_{i-l-1}$. This remark allows us to rewrite the sum in the following way:

$$A_1 = \sum_{I = I_{i-l} \sqcup I_l} \varepsilon(I_{i-l}, I_l) \sum_{r,t \in I_{i-l}} e_{I_{i-l}} \wedge F_{k-l-1} \wedge e_t \wedge e_{I_l} \wedge e_r = 0.$$

Indeed, the sum vanishes because the summands are antisymmetric in (r, t). As for A_2, we split it further as $A_{21} + A_{22}$, according to whether $r = t$ or $r \neq t$:

$$A_{21} = - \sum_{I = I_{i-l-1} \sqcup \{t\} \sqcup I_l} \varepsilon(I_{i-l-1}, t, I_l) e_{I_{i-l-1}} \wedge F_{k-l} \wedge e_{I_l} \wedge e_t,$$

$$A_{22} = - \sum_{I = I_{i-l-1} \sqcup \{t\} \sqcup I_l} \sum_{r \neq t} \varepsilon(I_{i-l-1}, t, I_l) e_{I_{i-l-1}} \wedge F_{k-l} \wedge e_{I_l} \wedge e_r.$$

We are then pleased to see that

$$\begin{aligned} A_{21} &= -(-1)^l \sum_{I = I_{i-l-1} \sqcup \{t\} \sqcup I_l} \varepsilon(I_{i-l-1}, I_l \cup \{t\}) e_{I_{i-l-1}} \wedge F_{k-l} \wedge e_{I_l \cup \{t\}} \\ &= (-1)^{l+1}(l+1) \sum_{I = I_{i-l-1} \sqcup I_{l+1}} \varepsilon(I_{i-l-1}, I_{l+1}) e_{I_{i-l-1}} \wedge F_{k-l} \wedge e_{I_{l+1}} \\ &= (-1)^{l+1}(l+1) \xi_{i,j}^{k+1}(e_I). \end{aligned}$$

The term A_{22} will be canceled by a term occurring in B, so we now turn to B, and begin to decompose it as $B_1 + B_2$, with

$$\begin{aligned} B_1 = \sum_{r \in I} \varepsilon(I \setminus r, r) \Bigg(\frac{1}{d} \sum_{s=1}^{d} \varepsilon(I \setminus r, s) \\ \times \sum_{I^s = I^s_{i-l-1} \sqcup \{t\} \sqcup I^s_l} \varepsilon(I^s_{i-l-1}, \{t\}, I^s_l) e_{I^s_{i-l-1}} \wedge (e_t \wedge F_{k-l-1} \wedge e_t) \wedge e_{I^s_l} \wedge e_s \Bigg) \end{aligned}$$

and

$$B_2 = -\sum_{r\in I}\varepsilon(I\setminus r, r)\Bigg(\frac{1}{d}\sum_{s=1}^{d}\varepsilon(I\setminus r, s) \times \sum_{I^s=I^s_{i-l-1}\sqcup\{t\}\sqcup I^s_l}\varepsilon(I^s_{i-l-1}, \{t\}, I^s_l)e_{I^s_{i-l-1}}\wedge F_{k-l}\wedge e_{I^s_l}\wedge e_s,\Bigg)$$

where we have written $I^s := I\setminus\{r\}\cup\{s\}$. We may split further, $B_2 = B_{21} + B_{22}$, according to whether $t = s$ or $t \neq s$. Then we have

$$\begin{aligned} B_{21} &= -\sum_{r\in I}\varepsilon(I\setminus r, r)\Bigg(\frac{1}{d}\sum_{s=1}^{d}\varepsilon(I\setminus r, s) \times \sum_{I\setminus\{r\}=I_{i-l-1}\sqcup I_l}\varepsilon(I_{i-l-1}, \{s\}, I_l)e_{I_{i-l-1}}\wedge F_{k-l}\wedge e_{I_l}\wedge e_s\Bigg)\\ &= -\sum_{r\in I}\varepsilon(I\setminus r, r)\frac{1}{d}\sum_{s=1}^{d}\sum_{I\setminus\{r\}=I_{i-l-1}\sqcup I_l}(-1)^l\varepsilon(I_{i-l-1}, I_l)e_{I_{i-l-1}}\wedge F_{k-l}\wedge e_{I_l}\wedge e_s\\ &= -\sum_{r\in I}\varepsilon(I\setminus r, r)\sum_{I\setminus\{r\}=I_{i-l-1}\sqcup I_l}(-1)^l\varepsilon(I_{i-l-1}, I_l)e_{I_{i-l-1}}\wedge F_{k-l}\wedge e_{I_l}\wedge\Bigg(\frac{1}{d}\sum_{s=1}^{d}e_s\Bigg)\\ &= 0 \quad \Bigg(\text{since } \sum_{s=1}^{d}e_s = 0\Bigg). \end{aligned}$$

As for B_{22}, we note that each summand vanishes unless $s \in I^s_{i-l-1}$. In this case, we may write $e_{I^s_{i-l-1}} = \varepsilon(I_{i-l-2}, s)e_{I_{i-l-2}}\wedge e_s$ for I_{i-l-2} a subset of $I\setminus\{r, t\}$, and we note that I^s_l is also a subset of $I\setminus\{r, s\}$, so we simply denote it by I_l. Therefore we may rearrange the sum in the following way:

$$B_{22} = -\sum_{r,t\in I, r\neq t}\;\sum_{I\setminus\{r,t\}=I_{i-l-2}\sqcup I_l}\frac{1}{d}\sum_{s=1}^{d}\operatorname{sign}(r, s, t, I_{i-l-2}, I_l)e_{I_{i-l-2}}\wedge e_s\wedge F_{k-l}\wedge e_{I_l}\wedge e_s,$$

where

$$\begin{aligned} \operatorname{sign}(r, s, t, I_{i-l-2}, I_l) &= \varepsilon(I\setminus r, r)\varepsilon(I\setminus r, s)\varepsilon(I_{i-l-2}\cup\{s\}, \{t\}, I_l)\varepsilon(I_{i-l-2}, \{s\})\\ &= \varepsilon(I\setminus r, r)\varepsilon(I\setminus r, s)\varepsilon(I_{i-l-2}, \{s\}, \{t\}, I_l)\\ &= (-1)^{l+1}\varepsilon(I\setminus r, r)\varepsilon(I\setminus r, s)\varepsilon(I_{i-l-2}, \{t\}, I_l, \{s\})\\ &= (-1)^{l+1}\varepsilon(I\setminus r, r)\varepsilon(I_{i-l-2}, \{t\}, I_l)\\ &= (-1)^{l+1}\varepsilon(I_{i-l-2}, \{t\}, I_l, \{r\})\\ &= \varepsilon(I_{i-l-2}, \{r\}, \{t\}, I_l). \end{aligned}$$

But since $\varepsilon(I_{i-l-2}, \{r\}, \{t\}, I_l) = -\varepsilon(I_{i-l-2}, \{t\}, \{r\}, I_l)$, we see that $B_{22} = 0$. It remains to deal with B_1. As in the case of B_{22}, we see that each summand vanishes unless $s \in I^s_{i-l-1}$, so that we may rearrange the sum

$$\begin{aligned}
B_1 &= \sum_{r,t\in I, r\neq t} \sum_{I\setminus\{r,t\}=I_{i-l-2}\sqcup I_l} \frac{1}{d}\sum_{s=1}^{d} \varepsilon(I_{i-l-2}, \{r\}, \{t\}, I_l) \\
&\qquad\qquad \times e_{I_{i-l-2}} \wedge e_s \wedge (e_t \wedge F_{k-l-1} \wedge e_t) \wedge e_{I_l} \wedge e_s \\
&= (-1)^l \sum_{r,t\in I, r\neq t} \sum_{I\setminus\{r,t\}=I_{i-l-2}\sqcup I_l} \varepsilon(I_{i-l-2}, \{r\}, \{t\}, I_l) e_{I_{i-l-2}} \wedge (e_t \wedge F_{k-l} \wedge e_t) \wedge e_{I_l} \\
&= -\sum_{r,t\in I, r\neq t} \sum_{I\setminus\{r,t\}=I_{i-l-2}\sqcup I_l} \varepsilon(I_{i-l-2}\cup\{t\}, \{r\}, I_l) e_{I_{i-l-2}\cup\{t\}} \wedge F_{k-l} \wedge e_{I_l} \wedge e_t \\
&= -\sum_{I=I_{i-l-1}\sqcup\{r\}\sqcup I_l} \sum_{t\neq r} \varepsilon(I_{i-l-1}, r, I_l) e_{I_{i-l-1}} \wedge F_{k-l} \wedge e_{I_l} \wedge e_t \\
&= A_{22}.
\end{aligned}$$

Finally we have proved that in the case $k = j - i + 2l + 1$,

$$(\xi^1_{i,i} \cup \xi^k_{i,j})(e_I) = A_1 + A_{21} + A_{22} - B_1 - B_{21} - B_{22} = A_{21} = (-1)^{l+1}(l+1)\xi^{k+1}_{i,j}(e_I).$$

A.2.3. *Proof of* (ii). It is certainly possible to do a direct computation as above, but it seems easier to use case (i) and prove (ii) by induction on k. Indeed, if we assume that $\xi^{k-1}_{i,j} \cup \xi^1_{j,j} \in \bar{\mathbb{F}}^\times_\ell \cdot \xi^k_{i,j}$, then we get, thanks to (i),

$$\xi^k_{i,j} \cup \xi^1_{j,j} \in \bar{\mathbb{F}}^\times_\ell \cdot \xi^1_{i,i} \cup \xi^{k-1}_{i,j} \cup \xi^1_{j,j} \in \bar{\mathbb{F}}^\times_\ell \cdot \xi^1_{i,i} \cup \xi^k_{i,j} \in \bar{\mathbb{F}}^\times_\ell \cdot \xi^{k+1}_{i,j}.$$

Therefore all we have to do is initialize the induction by proving that $\xi^{j-i}_{i,j} \cup \xi^1_{j,j} \in \bar{\mathbb{F}}^\times_\ell \cdot \xi^{j-i+1}_{i,j}$. But since we already know that $\xi^1_{j,j} \cup \xi^1_{j,j} \in \bar{\mathbb{F}}^\times_\ell \xi^2_{j,j}$, it will suffice to check that $\xi^{j-i}_{i,j} \cup \xi^2_{j,j} \in \bar{\mathbb{F}}^\times_\ell \cdot \xi^{j-i+2}_{i,j}$ (this involves less computation). Again we evaluate both sides on elements e_I. Let us write

$$\xi^{j-i}_{i,j}(e_I) = e_I \wedge F_{j-i} = C_{j-i} \sum_{|K|=j-i} e_I \wedge e_K \otimes e_K,$$

with $C_{j-i} = (-1)^{(j-i)(j-i-1)/2}(j-i)!/(d^{j-i})$. We then have

$$(\xi^{j-i}_{i,j} \cup \xi^2_{j,j})(e_I) = C_{j-i} \sum_{|K|=j-i} \varepsilon(I, K)\xi^2_{j,j}(e_{I\cup K}) \wedge e_K.$$

Inserting the expression

$$\xi^2_{j,j}(e_{I\cup K}) = \sum_{r\in I\cup K} \varepsilon(I\cup K \setminus r, r) e_{I\cup K\setminus r} \wedge F_1 \wedge e_r,$$

which is valid for I and K disjoint, we get

$$(\xi_{i,j}^{j-i}\cup\xi_{j,j}^{2})(e_I)=C_{j-i}\sum_{|K|=j-i}\sum_{r\in I\cup K}\varepsilon(I,K)\varepsilon(I\cup K\setminus r,r)e_{I\cup K\setminus r}\wedge F_1\wedge e_r\wedge e_K.$$

We see that those summands where $r\in K$ vanish. Hence we may restrict the second sum to I:

$$\begin{aligned}(\xi_{i,j}^{j-i}\cup\xi_{j,j}^{2})(e_I)&=C_{j-i}\sum_{|K|=j-i}\sum_{r\in I}\varepsilon(I,K)\varepsilon(I\setminus r\cup K,r)e_{I\setminus r\cup K}\wedge F_1\wedge e_r\wedge e_K\\&=C_{j-i}\sum_{|K|=j-i}\sum_{r\in I}\mu(I,K,r)e_{I\setminus r}\wedge e_K\wedge F_1\wedge e_K\wedge e_r,\end{aligned}$$

with

$$\begin{aligned}\mu(I,K,r)&=\varepsilon(I,K)\varepsilon(I\setminus r\cup K,r)\varepsilon(I\setminus r,K)(-1)^{j-i}\\&=\varepsilon(I,K)\varepsilon(I\setminus r,K,r)(-1)^{j-i}=\varepsilon(I,K)\varepsilon(I\setminus r,r,K)=\varepsilon(I\setminus r,r).\end{aligned}$$

Eventually we get

$$\begin{aligned}(\xi_{i,j}^{j-i}\cup\xi_{j,j}^{2})(e_I)&=C_{j-i}\sum_{|K|=j-i}\sum_{r\in I}\varepsilon(I\setminus r,r)e_{I\setminus r}\wedge e_K\wedge F_1\wedge e_K\wedge e_r\\&=\sum_{r\in I}\varepsilon(I\setminus r,r)e_{I\setminus r}\wedge\Big(C_{j-i}\sum_{|K|=j-i}e_K\wedge F_1\wedge e_K\Big)\wedge e_r\\&=\sum_{r\in I}\varepsilon(I\setminus r,r)e_{I\setminus r}\wedge F_{j-i+1}\wedge e_r\\&=\xi_{i,j}^{j-i+2}(e_I).\end{aligned}$$

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