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Compatibility between Satake and Bernstein isomorphisms in characteristic p

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We study the center of the pro-p Iwahori–Hecke ring $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ of a connected split p-adic reductive group G. For k an algebraically closed field of characteristic p, we prove that the center of the k-algebra $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ contains an affine semigroup algebra which is naturally isomorphic to the Hecke k-algebra $\mathscr{H}(\mathrm{G}, \rho)$ attached to an irreducible smooth k-representation ρ of a given hyperspecial maximal compact subgroup of G. This isomorphism is obtained using the inverse Satake isomorphism defined in our previous work.

We apply this to classify the simple supersingular $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$-modules, study the supersingular block in the category of finite-length $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$-modules, and relate the latter to supersingular representations of G.

1. Introduction

The Iwahori–Hecke ring of a split p-adic reductive group G is the convolution ring of $\mathbb{Z}$-valued functions with compact support in $\mathrm{I}\backslash\mathrm{G}/\mathrm{I}$, where I denotes an Iwahori subgroup of G. It is isomorphic to the quotient of the extended braid group ring associated to G by quadratic relations in the standard generators. If one replaces I by its pro-p Sylow subgroup $\tilde{\mathrm{I}}$, then one obtains the pro-p Iwahori–Hecke ring $\widetilde{\mathrm{H}}_{\mathbb{Z}}$. In this article we study the center of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$. We are motivated by the smooth representation theory of G over an algebraically closed field k with

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characteristic p and subsequently will be interested in the k-algebra $\widetilde{\mathrm{H}}_k := \widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. We construct an isomorphism of k-algebras between a subring of the center of $\widetilde{\mathrm{H}}_k$ and (generalizations of) spherical Hecke k-algebras by means of the inverse mod p Satake isomorphism defined in [Ollivier 2012]. This result is the *compatibility between Bernstein and Satake isomorphisms* referred to in the title of this article. We then explore some consequences of this compatibility. In particular, we study and relate the notions of supersingularity for Hecke modules and k-representations of G.

1A. ***Framework and results.*** Let $\mathfrak{F}$ be a nonarchimedean locally compact field with residue characteristic p and k an algebraic closure of the residue field. We choose a uniformizer ϖ. Let $\mathrm{G} := \mathbf{G}(\mathfrak{F})$ be the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ over $\mathfrak{F}$, which we assume to be $\mathfrak{F}$-split. In the semisimple building $\mathscr{X}$ of G, we choose and fix a chamber C, which amounts to choosing an Iwahori subgroup I in G, and we denote by $\tilde{\mathrm{I}}$ the pro-p Sylow subgroup of I. The choice of C is unique up to conjugacy by an element of G. We consider the associated pro-p Iwahori–Hecke ring $\widetilde{\mathrm{H}}_{\mathbb{Z}} := \mathbb{Z}[\tilde{\mathrm{I}}\backslash\mathrm{G}/\tilde{\mathrm{I}}]$ of $\mathbb{Z}$-valued functions with compact support in $\tilde{\mathrm{I}}\backslash\mathrm{G}/\tilde{\mathrm{I}}$ under convolution.

Since G is split, C has at least one hyperspecial vertex x_0, and we denote by K the associated maximal compact subgroup of G. Fix a maximal $\mathfrak{F}$-split torus T in G such that the corresponding apartment $\mathscr{A}$ in $\mathscr{X}$ contains C. The set $\mathrm{X}_*(\mathrm{T})$ of cocharacters of T is naturally equipped with an action of the finite Weyl group $\mathfrak{W}$. The choice of x_0 and C induces a natural choice of a positive Weyl chamber of $\mathscr{A}$, that is to say, of a semigroup $\mathrm{X}_*^+(\mathrm{T})$ of dominant cocharacters of T.

1A1. *The complex case.* The structure of the spherical algebra $\mathbb{C}[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}]$ of complex functions compactly supported on $\mathrm{K}\backslash\mathrm{G}/\mathrm{K}$ is understood thanks to the classical Satake isomorphism [1963] (see also [Gross 1998; Haines 2001])

$$s : \mathbb{C}[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}] \xrightarrow{\sim} (\mathbb{C}[\mathrm{X}_*(\mathrm{T})])^{\mathfrak{W}}.$$

On the other hand, the complex Iwahori–Hecke algebra $\mathrm{H}_{\mathbb{C}} := \mathbb{C}[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ of complex functions compactly supported on $\mathrm{I}\backslash\mathrm{G}/\mathrm{I}$ contains a large commutative subalgebra $\mathcal{A}_{\mathbb{C}}$ defined as the image of the *Bernstein map* $\theta : \mathbb{C}[\mathrm{X}_*(\mathrm{T})] \hookrightarrow \mathrm{H}_{\mathbb{C}}$, which depends on the choice of the dominant Weyl chamber (see [Lusztig 1989, Section 3.2]). The algebra $\mathrm{H}_{\mathbb{C}}$ is free of finite rank over $\mathcal{A}_{\mathbb{C}}$ and its center $\mathscr{Z}(\mathrm{H}_{\mathbb{C}})$ is contained in $\mathcal{A}_{\mathbb{C}}$. Furthermore, the map θ yields an isomorphism

$$b : (\mathbb{C}[\mathrm{X}_*(\mathrm{T})])^{\mathfrak{W}} \xrightarrow{\sim} \mathscr{Z}(\mathrm{H}_{\mathbb{C}}).$$

This was proved by Bernstein ([Lusztig 1989, Section 3.5]; see also [Haines 2001, Theorem 2.3]). By [Dat 1999, Corollary 3.1] and [Haines 2001, Proposition 10.1],

the *Bernstein isomorphism* b is compatible with s, in the sense that the composition $(e_{\mathrm{K}} \star \cdot)b$ is an inverse for s, where $(e_{\mathrm{K}} \star \cdot)$ denotes the convolution by the characteristic function of K.

1A2. *Bernstein and Satake isomorphisms in characteristic p.* After defining an integral version of the complex Bernstein map, Vignéras [2005] gave a basis for the center of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ and proved that $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is noetherian and finitely generated over its center. In the first section of this article, we define a subring $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ of the center of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ over which $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is still finitely generated. In Proposition 2.8 we prove that $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ is not affected by the choice of another apartment containing C and of another hyperspecial vertex of C, as long as it is conjugate to x_0. In particular, if $\mathbf{G}$ is of adjoint type or $\mathbf{G} = \mathrm{GL}_n$, then $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ depends only on the choice of the uniformizer ϖ.

The natural image of $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ in $\widetilde{\mathrm{H}}_k = \widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ is denoted by $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_k)$, and we prove that it has an affine semigroup algebra structure. More precisely, we have an isomorphism of k-algebras (Proposition 2.10)

$$k[\mathrm{X}_*^+(\mathrm{T})] \xrightarrow{\sim} \mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_k) \subseteq \widetilde{\mathrm{H}}_k. \tag{1-1}$$

By the main theorem in [Herzig 2011b] (and in [Ollivier 2012]), this makes $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_k)$ isomorphic to the algebra $\mathscr{H}(\mathrm{G}, \rho)$ of any irreducible smooth k-representation ρ of K. Note that when ρ is the k-valued trivial representation $\mathbf{1}_{\mathrm{K}}$ of K, one retrieves the convolution algebra $k[\mathrm{K}\backslash\mathrm{G}/\mathrm{K}] = \mathscr{H}(\mathrm{G}, \mathbf{1}_{\mathrm{K}})$.

In [Ollivier 2012], we constructed an isomorphism

$$\mathcal{T} : k[\mathrm{X}_*^+(\mathrm{T})] \xrightarrow{\sim} \mathscr{H}(\mathrm{G}, \rho). \tag{1-2}$$

Here we prove the following theorem:

Theorem 4.3. *We have a commutative diagram of isomorphisms of k-algebras*

$$\begin{array}{ccc} k[\mathrm{X}_*^+(\mathrm{T})] & \xrightarrow{(1\text{-}1)} & \mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_k) \\ \| & & \downarrow \\ k[\mathrm{X}_*^+(\mathrm{T})] & \xrightarrow{\mathcal{T}} & \mathscr{H}(\mathrm{G}, \rho) \end{array} \tag{1-3}$$

where the vertical arrow on the right-hand side is the natural morphism of k-algebras (4-3) *described in Section 4.*

The isomorphism $\mathcal{T}$ was constructed in [Ollivier 2012] by means of generalized integral Bernstein maps, as are the subring $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_k)$ and the map (1-1) in the current article. By analogy with the complex case, we can see the map (1-1) as an isomorphism à la Bernstein in characteristic p. The above commutative diagram can then be interpreted as a statement of compatibility between Satake and Bernstein

isomorphisms in characteristic p. Note that under the hypothesis that the derived subgroup of $\mathbf{G}$ is simply connected, it is proved in [Ollivier 2012] that $\mathcal{T}$ is the inverse of the mod p Satake isomorphism defined in [Herzig 2011b]. (The extra hypothesis on $\mathbf{G}$ is probably not necessary).

If we worked with the Iwahori–Hecke algebra $k[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$, the analog of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ would actually be the whole center of $k[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$. We prove:

Theorem 2.14. *The center of the Iwahori–Hecke k-algebra $k[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ is isomorphic to $k[\mathrm{X}_*^+(\mathrm{T})]$.*

1A3. *Generalized integral Bernstein maps.* One ingredient of the construction of $\mathcal{T}$ in [Ollivier 2012] and of the proof of Theorem 4.3 is the definition of $\mathbb{Z}$-linear injective maps

$$\mathcal{B}_F^\sigma : \mathbb{Z}[\widetilde{\mathrm{X}}_*(\mathrm{T})] \to \widetilde{\mathrm{H}}_{\mathbb{Z}}$$

defined on the group ring of the (extended) cocharacters $\widetilde{\mathrm{X}}_*(\mathrm{T})$, which are multiplicative when restricted to the semigroup ring of any chosen Weyl chamber of $\widetilde{\mathrm{X}}_*(\mathrm{T})$ (see Section 1B5 for the definition of $\widetilde{\mathrm{X}}_*(\mathrm{T})$). The image of $\mathcal{B}_F^\sigma$ happens to be a commutative subring of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$, which we denote by $\mathcal{A}_F^\sigma$. The parameter σ is a sign and F is a standard facet (a facet of C containing x_0 in its closure). The choice of F corresponds to the choice of a Weyl chamber in $\mathscr{A}$: for example, if $F = C$ (resp. x_0), then the corresponding Weyl chamber is the dominant (resp. antidominant) one.

The maps $\mathcal{B}_F^\sigma$ are called *integral Bernstein maps* because they are generalizations of the Bernstein map θ mentioned in Section 1A1. In the complex case, it is customary to consider either θ which is constructed using the dominant chamber, or θ^- which is constructed using the antidominant chamber (see the discussion in the introduction of [Haines and Pettet 2002] for example). By a result by Bernstein [Lusztig 1983], a basis for the center of $\mathrm{H}_{\mathbb{C}}$ is given by the central Bernstein functions

$$\sum_{\lambda' \in \mathbb{O}} \theta(\lambda'),$$

where $\mathbb{O}$ ranges over the $\mathfrak{W}$-orbits in $\mathrm{X}_*(\mathrm{T})$. We refer to [Haines 2001] for the geometric interpretation of these functions. It is natural to ask whether using θ^- instead of θ in the previous formula yields the same central element in $\mathrm{H}_{\mathbb{C}}$. The answer is yes (see [Haines and Pettet 2002, Section 2.2.2]). The proof is based on [Lusztig 1983, Corollary 8.8] and relies on the combinatorics of the Kazhdan–Lusztig polynomials. Note that there is no theory of Kazhdan–Lusztig polynomials for the complex pro-p Iwahori–Hecke algebra.

Integral (and pro-p) versions of θ and θ^- for the ring $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ were defined in [Vignéras 2005]. In our language they correspond respectively to $\mathcal{B}_C^+ = \mathcal{B}_{x_0}^-$ and

$\mathcal{B}^+_{x_0} = \mathcal{B}^-_{\boldsymbol{C}}$. It is also proved there that a $\mathbb{Z}$-basis for the center of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is given by

$$\sum_{\lambda' \in \mathbb{O}} \mathcal{B}^+_{\boldsymbol{C}}(\lambda'), \tag{1-4}$$

where $\mathbb{O}$ ranges over the $\mathfrak{W}$-orbits in $\widetilde{\mathrm{X}}_*(\mathrm{T})$. It is now natural to ask whether the element (1-4) is the same if (a) we use $-$ instead of $+$, and if, more generally, (b) we use any standard facet F instead of C, and any sign σ. We prove:

Lemma 3.4. *The element*

$$\sum_{\lambda' \in \mathbb{O}} \mathcal{B}^{\sigma}_{F}(\lambda')$$

in $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ *does not depend on the choice of the standard facet* F *and of the sign* σ.

To prove the lemma, we first answer positively question (a) above; we then study and exploit the behavior of the integral Bernstein maps upon a process of parabolic induction. In passing we also consider question (a) in the k-algebra $\widetilde{\mathrm{H}}_k$ in the case when G is semisimple, and we suggest a link between such questions and the duality for finite-length $\widetilde{\mathrm{H}}_k$-modules defined in [Ollivier and Schneider 2012] (see Proposition 3.3).

1A4. In Section 5, we define and study a natural topology on $\widetilde{\mathrm{H}}_k$ which depends only on the conjugacy class of x_0. It is the $\mathfrak{I}$-adic topology, where $\mathfrak{I}$ is a natural monomial ideal of the affine semigroup algebra $\mathfrak{X}^\circ(\widetilde{\mathrm{H}}_k)$.

We define the supersingular block of the category of finite length $\widetilde{\mathrm{H}}_k$-modules to be the full subcategory of the modules that are continuous for the $\mathfrak{I}$-adic topology on $\widetilde{\mathrm{H}}_k$ (Proposition-Definition 5.10). A finite length $\widetilde{\mathrm{H}}_k$-module then turns out to be in the supersingular block if and only if all its irreducible constituents are supersingular in the sense of [Vignéras 2005].

In the case when the root system of G is irreducible, we establish the following results. We classify the simple supersingular $\widetilde{\mathrm{H}}_k$-modules (Theorem 5.14 and subsequent corollary). (For example, when G is semisimple simply connected, the simple supersingular modules all have dimension 1.) We prove in passing that even if the ideal $\mathfrak{I}$ does depend on the choices made, the supersingular block is independent of all the choices.

Theorem 5.14 extends Theorem 5 of [Vignéras 2005] and Theorem 7.3 of [Ollivier 2010], which dealt with the case of GL_n and relied on explicit *minimal expressions* for certain Bernstein functions associated to the minuscule coweights. The results of those two papers together proved a "numerical Langlands correspondence for Hecke modules" of $\mathrm{GL}_n(\mathfrak{F})$: there is a bijection between the finite set of all simple n-dimensional supersingular $\widetilde{\mathrm{H}}_k$-modules and the finite set of all irreducible n-dimensional smooth k-representations of the absolute Galois group of $\mathfrak{F}$, where

the action of the uniformizer ϖ on the Hecke modules and the determinant of the Frobenius on the Galois representations are fixed. Recently, Große-Klönne constructed a functor from the category of finite-length $\widetilde{\mathrm{H}}_k$-modules for $\mathrm{GL}_n(\mathbb{Q}_p)$ to the category of étale (φ, Γ)-modules. This functor induces a bijection between the two finite sets above, turning the "numerical" correspondence into a natural and explicit correspondence in the case of $\mathrm{GL}_n(\mathbb{Q}_p)$. In fact, Große-Klönne [2013a] has constructed such a functor (with values in a category of modified étale (φ, Γ)-modules) in the case of a general split group over $\mathbb{Q}_p$. In the case of $\mathrm{SL}_n(\mathfrak{F})$, Koziol [2013] has defined packets of simple supersingular $\widetilde{\mathrm{H}}_k$-modules and built a bijection between the set of packets and a certain set of projective k-representations of the absolute Galois group of $\mathfrak{F}$; if $\mathfrak{F} = \mathbb{Q}_p$, this bijection is proved to be compatible with Große-Klönne's functor and therefore with the explicit Langlands-type correspondence for Hecke modules of $\mathrm{GL}_n(\mathbb{Q}_p)$. This result is a first step towards a mod p principle of functoriality for Hecke modules.

The current article provides, in the case of a general split group, a classification of the objects that one wants to apply Große-Klönne's functor to, in order to investigate the possibility of a Langlands-type correspondence for Hecke modules in general.

1A5. In Section 5F we consider an admissible irreducible smooth k-representation π of G. In the case where the derived subgroup of $\mathbf{G}$ is simply connected, we use the fact that (1-2) is the inverse of the mod p Satake isomorphism to prove that if π is supersingular, then

$$\pi \text{ is a quotient of } \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1 \,/\, \mathfrak{I}\, \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1. \tag{1-5}$$

The condition (1-5) is equivalent to saying that $\pi^{\tilde{\mathrm{I}}}$ contains an irreducible supersingular $\widetilde{\mathrm{H}}_k$-module.

When $\mathrm{G} = \mathrm{GL}_n(\mathfrak{F})$ and $\mathfrak{F}$ is a finite extension of $\mathbb{Q}_p$, we use the classification of the nonsupersingular representations obtained in [Herzig 2011a], the work on generalized special representations in [Große-Klönne 2013b], and our Lemma 3.4 to prove that the condition (1-5) is in fact a characterization of the supersingular representations (Theorem 5.27).

Finally, we comment in Section 5F on the generalization of this characterization to the case of a split group (with simply connected derived subgroup), and on the independence of the characterization of the choices made.

We raise the question of the possibility of a direct proof of this characterization that does not use the classification of the nonsupersingular representations.

1B. ***Notation and preliminaries.*** We choose the valuation $\mathrm{val}_{\mathfrak{F}}$ on $\mathfrak{F}$ normalized by $\mathrm{val}_{\mathfrak{F}}(\varpi) = 1$, where ϖ is the chosen uniformizer. The ring of integers of $\mathfrak{F}$ is denoted by $\mathfrak{O}$ and its residue field by $\mathbb{F}_q$, where q is a power of the prime number p. Recall that k denotes an algebraic closure of $\mathbb{F}_q$. Let $\mathbf{G}_{x_0}$ and $\mathbf{G}_{\boldsymbol{C}}$

denote the Bruhat–Tits group schemes over $\mathfrak{O}$ whose $\mathfrak{O}$-valued points are K and I respectively. Their reductions over the residue field $\mathbb{F}_q$ are denoted by $\overline{\mathbf{G}}_{x_0}$ and $\overline{\mathbf{G}}_C$. Note that $\mathrm{G} = \mathbf{G}_{x_0}(\mathfrak{F}) = \mathbf{G}_C(\mathfrak{F})$. By [Tits 1979, 3.4.2, 3.7 and 3.8], $\overline{\mathbf{G}}_{x_0}$ is connected reductive and $\mathbb{F}_q$-split. Therefore we have $\mathbf{G}^\circ_C(\mathfrak{O}) = \mathbf{G}_C(\mathfrak{O}) = \mathrm{I}$ and $\mathbf{G}^\circ_{x_0}(\mathfrak{O}) = \mathbf{G}_{x_0}(\mathfrak{O}) = \mathrm{K}$. Denote by K_1 the prounipotent radical of K. The quotient K/K_1 is isomorphic to $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$. The Iwahori subgroup I is the preimage in K of the $\mathbb{F}_q$-rational points of a Borel subgroup $\overline{\mathbf{B}}$ with Levi decomposition $\overline{\mathbf{B}} = \overline{\mathbf{T}}\overline{\mathbf{N}}$. The pro-$p$ Iwahori subgroup $\tilde{\mathrm{I}}$ is the preimage in I of $\overline{\mathbf{N}}(\mathbb{F}_q)$. The preimage of $\overline{\mathbf{T}}(\mathbb{F}_q)$ is the maximal compact subgroup T^0 of T. Note that $\mathrm{T}^0/\mathrm{T}^1 = \mathrm{I}/\tilde{\mathrm{I}} = \overline{\mathbf{T}}(\mathbb{F}_q)$, where $\mathrm{T}^1 := \mathrm{T}^0 \cap \tilde{\mathrm{I}}$.

1B1. *Affine root datum.* To the choice of T is attached the root datum

$$(\Phi, \mathrm{X}^*(\mathrm{T}), \check{\Phi}, \mathrm{X}_*(\mathrm{T})).$$

This root system is reduced because the group $\mathbf{G}$ is $\mathfrak{F}$-split. We denote by $\mathfrak{W}$ the finite Weyl group $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$, the quotient by T of the normalizer of T. Recall that $\mathscr{A}$ denotes the apartment of the semisimple building attached to T (see [Tits 1979; Schneider and Stuhler 1997, Section I.1], and we follow the notation of [Ollivier 2012, Section 2.2]). We denote by $\langle \cdot, \cdot \rangle$ the perfect pairing $\mathrm{X}_*(\mathrm{T}) \times \mathrm{X}^*(\mathrm{T}) \to \mathbb{Z}$. The elements in $\mathrm{X}_*(\mathrm{T})$ will be called coweights. We identify $\mathrm{X}_*(\mathrm{T})$ with the subgroup T/T^0 of the extended Weyl group $\mathrm{W} = N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}^0$ as in [Tits 1979, I.1] and [Schneider and Stuhler 1997, Section I.1]: to an element $g \in \mathrm{T}$ corresponds the vector $\nu(g) \in \mathbb{R} \otimes_{\mathbb{Z}} \mathrm{X}_*(\mathrm{T})$ defined by

$$\langle \nu(g), \chi \rangle = -\operatorname{val}_{\mathfrak{F}}(\chi(g)) \quad \text{for any } \chi \in \mathrm{X}^*(\mathrm{T}), \tag{1-6}$$

and ν induces the required isomorphism $\mathrm{T}/\mathrm{T}^0 \cong \mathrm{X}_*(\mathrm{T})$. The group T/T^0 acts by translation on $\mathscr{A}$ via ν. The actions of $\mathfrak{W}$ and T/T^0 combine into an action of W on $\mathscr{A}$ as recalled in [Schneider and Stuhler 1997, p. 102]. Since x_0 is a special vertex of the building, W is isomorphic to the semidirect product $\mathfrak{W} \ltimes \mathrm{X}_*(\mathrm{T})$, where we see $\mathfrak{W}$ as the fixator in W of any lift of x_0 in the extended apartment [Tits 1979, 1.9]. A coweight λ will sometimes be denoted by e^λ to underline that we see it as an element in W, meaning as a translation on $\mathscr{A}$.

Denote by Φ_{aff} the set of affine roots. The choice of the chamber C implies in particular the choice of the positive affine roots Φ^+_{aff} taking nonnegative values on C. The choice of x_0 as an origin of $\mathcal{A}$ implies that we identify the affine roots taking value zero at x_0 with Φ. We set $\Phi^+ := \Phi^+_{\mathrm{aff}} \cap \Phi$ and $\Phi^- = -\Phi^+$. The affine roots can be described the following way: $\Phi_{\mathrm{aff}} = \Phi \times \mathbb{Z} = \Phi^+_{\mathrm{aff}} \sqcup \Phi^-_{\mathrm{aff}}$, where

$$\Phi^+_{\mathrm{aff}} := \{(\alpha, r) : \alpha \in \Phi, r > 0\} \cup \{(\alpha, 0) : \alpha \in \Phi^+\}.$$

Let Π be the basis for Φ^+ consisting of the set of simple roots. The finite Weyl

group $\mathfrak{W}$ is a Coxeter system with generating set $S := \{s_\alpha : \alpha \in \Pi\}$, where s_α denotes the (simple) reflection at the hyperplane $\langle \cdot, \alpha \rangle = 0$. Denote by $\preceq$ the partial ordering on $\mathrm{X}_*^+(\mathrm{T})$ associated to Π. Let Π_m be the set of roots in Φ that are minimal elements for $\preceq$. Define the set of simple affine roots by $\Pi_{\mathrm{aff}} := \{(\alpha, 0) : \alpha \in \Pi\} \cup \{(\alpha, 1) : \alpha \in \Pi_m\}$. Identifying α with $(\alpha, 0)$, we consider Π a subset of Π_{aff}. For $A \in \Pi_{\mathrm{aff}}$, denote by s_A the following associated reflection: $s_A = s_\alpha$ if $A = (\alpha, 0)$ and $s_A = s_\alpha e^{\check{\alpha}}$ if $A = (\alpha, 1)$. The action of W on the coweights induces an action on the set of affine roots: W acts on Φ_{aff} by $we^\lambda : (\alpha, r) \mapsto (w\alpha, r - \langle \lambda, \alpha \rangle)$, where we denote by $(w, \alpha) \mapsto w\alpha$ the natural action of $\mathfrak{W}$ on Φ. The length on the Coxeter system $(\mathfrak{W}, S)$ extends to W in such a way that the length $\ell(w)$ of $w \in \mathrm{W}$ is the number of affine roots $A \in \Phi_{\mathrm{aff}}^+$ such that $w(A) \in \Phi_{\mathrm{aff}}^-$. It satisfies the following formula, for $A \in \Pi_{\mathrm{aff}}$ and $w \in \mathrm{W}$:

$$\ell(ws_A) = \begin{cases} \ell(w) + 1 & \text{if } w(A) \in \Phi_{\mathrm{aff}}^+, \\ \ell(w) - 1 & \text{if } w(A) \in \Phi_{\mathrm{aff}}^-. \end{cases} \tag{1-7}$$

The affine Weyl group is defined as the subgroup $\mathrm{W}_{\mathrm{aff}}$ of W generated by $S_{\mathrm{aff}} := \{s_A : A \in \Pi_{\mathrm{aff}}\}$. The length function ℓ restricted to $\mathrm{W}_{\mathrm{aff}}$ coincides with the length function of the Coxeter system $(\mathrm{W}_{\mathrm{aff}}, S_{\mathrm{aff}})$ [Bourbaki 1968, V.3.2, Théorème 1(i)]. Recall from [Lusztig 1989, Section 1.5] that $\mathrm{W}_{\mathrm{aff}}$ is a normal subgroup of W: the set Ω of elements with length zero is an abelian subgroup of W and W is the semidirect product $\mathrm{W} = \Omega \ltimes \mathrm{W}_{\mathrm{aff}}$. The length ℓ is constant on the double cosets of W mod Ω. In particular, Ω normalizes S_{aff}.

The extended Weyl group W is equipped with a partial order $\leq$ that extends the Bruhat order on $\mathrm{W}_{\mathrm{aff}}$. By definition, given $w = \omega w_{\mathrm{aff}}$, $w = \omega' w'_{\mathrm{aff}} \in \Omega \ltimes \mathrm{W}_{\mathrm{aff}}$, we have $w \leq w'$ if $\omega = \omega'$ and $w_{\mathrm{aff}} \leq w'_{\mathrm{aff}}$ in the Bruhat order on $\mathrm{W}_{\mathrm{aff}}$ (see for example [Haines 2001, Section 2.1]).

We fix a lift $\hat{w} \in N_{\mathrm{G}}(\mathrm{T})$ for any $w \in \mathrm{W}$. By Bruhat decomposition, G is the disjoint union of all $\mathrm{I}\hat{w}\mathrm{I}$ for $w \in \mathrm{W}$.

1B2. *Orientation character.* The stabilizer of the chamber C in W is Ω. We define as in [Ollivier and Schneider 2012, Section 3.1] the orientation character $\epsilon_C : \Omega \to \{\pm 1\}$ of C by setting $\epsilon_C(\omega) = +1$ (resp. -1) if ω preserves (resp. reverses) a given orientation of C. Since $\mathrm{W}/\mathrm{W}_{\mathrm{aff}} = \Omega$, we can see ϵ_C as a character of W trivial on $\mathrm{W}_{\mathrm{aff}}$. By definition of the Bruhat order on W, we have $\epsilon_C(w) = \epsilon_C(w')$ for $w, w' \in \mathrm{W}$ satisfying $w \leq w'$.

On the other hand, the extended Weyl group acts by affine isometries on the Euclidean space $\mathscr{A}$. We therefore have a determinant map $\det : \mathrm{W} \to \{\pm 1\}$ which is trivial on $\mathrm{X}_*(\mathrm{T})$. An orientation of C is a choice of a cyclic ordering of its set of vertices (in the geometric realization of $\mathscr{A}$). Therefore, $\det(\omega)$ is the signature of the permutation of the vertices of C induced by $\omega \in \Omega$, and $\det(\omega) = \epsilon_C(\omega)$.

Lemma 1.4. (i) *For $w \in \mathrm{W}_{\mathrm{aff}}$, we have* $\det(w) = (-1)^{\ell(w)}$.

(ii) *For $\lambda \in \mathrm{X}_*(\mathrm{T})$, we have $\epsilon_C(w) = (-1)^{\ell(e^\lambda)}$ for any $w \in \mathrm{W}$ such that $w \leq e^\lambda$.*

Proof. Part (i) comes from the fact that $\det s = -1$ for $s \in S_{\mathrm{aff}}$. For (ii), by definition of the Bruhat order it is enough to prove that $\epsilon_C(e^\lambda) = (-1)^{\ell(e^\lambda)}$ for $\lambda \in \mathrm{X}_*(\mathrm{T})$. Decompose $e^\lambda = \omega w_{\mathrm{aff}}$ with $w \in \mathrm{W}_{\mathrm{aff}}$ and $\omega \in \Omega$. Recall that ω has length zero. Since ϵ_C is trivial on $\mathrm{W}_{\mathrm{aff}}$, we have $\epsilon_C(e^\lambda) = \epsilon_C(\omega) = \det \omega$. Since e^λ has unit determinant, we get $\det \omega = \det w_{\mathrm{aff}} = (-1)^{\ell(w_{\mathrm{aff}})} = (-1)^{\ell(e^\lambda)}$. □

1B3. *Distinguished cosets representatives.*

Proposition 1.5. (i) *The set $\mathcal{D}$ of all elements $d \in \mathrm{W}$ satisfying $d^{-1}(\Phi^+) \subset \Phi^+_{\mathrm{aff}}$ is a system of representatives of the right cosets $\mathfrak{W}\backslash\mathrm{W}$. It satisfies*

$$\ell(wd) = \ell(w) + \ell(d) \quad \text{for any } w \in \mathfrak{W} \text{ and } d \in \mathcal{D}. \tag{1-8}$$

In particular, d is the unique element with minimal length in $\mathfrak{W}d$.

(ii) *An element $d \in \mathcal{D}$ can be written uniquely as $d = e^\lambda w$, with $\lambda \in \mathrm{X}^+_*(\mathrm{T})$ and $w \in \mathfrak{W}$. We then have $\ell(e^\lambda) = \ell(d) + \ell(w^{-1}) = \ell(d) + \ell(w)$.*

(iii) *For $s \in S_{\mathrm{aff}}$ and $d \in \mathcal{D}$, we are in one of the following situations*:

- $\ell(ds) = \ell(d) - 1$, *in which case* $ds \in \mathcal{D}$.
- $\ell(ds) = \ell(d) + 1$, *in which case either* $ds \in \mathcal{D}$ *or* $ds \in \mathfrak{W}d$.

Proof. This proposition is proved in [Ollivier 2010, Lemma 2.6, Proposition 2.7] in the case of $\mathrm{G} = \mathrm{GL}_n(\mathfrak{F})$. It is checked in [Ollivier and Schneider 2012, Proposition 4.6] that it remains valid for a general split reductive group (see also [Ollivier 2012, Proposition 2.2] for (ii)), except for point (iii) when $s \in S_{\mathrm{aff}} - S$. We check here that the argument goes through. Let $s \in S_{\mathrm{aff}}$ and A be the corresponding affine root. Let $d \in \mathcal{D}$ and suppose that $ds \notin \mathcal{D}$; then there is $\beta \in \Pi$ such that $(ds)^{-1}\beta \in \Phi^-_{\mathrm{aff}}$ while $d^{-1}\beta \in \Phi^+_{\mathrm{aff}}$. This implies that $d^{-1}\beta = A$, which in particular ensures that $dA \in \Phi^+_{\mathrm{aff}}$ and therefore $\ell(ds) = \ell(d) + 1$. Furthermore, $dsd^{-1} = s_{dA} = s_\beta \in \mathfrak{W}$. □

There is an action of the group G on the semisimple building $\mathscr{X}$ recalled in [Schneider and Stuhler 1997, p. 104] that extends the action of $N_{\mathrm{G}}(\mathrm{T})$ on the standard apartment. For F a standard facet, we denote by $\mathcal{P}^\dagger_F$ the stabilizer of F in G.

Proposition 1.6. (i) *The Iwahori subgroup* I *acts transitively on the apartments of $\mathscr{X}$ containing C.*

(ii) *The stabilizer $\mathcal{P}^\dagger_{x_0}$ of x_0 acts transitively on the chambers of $\mathscr{X}$ containing x_0 in their closure.*

(iii) *A* G*-conjugate of x_0 in the closure of C is a $\mathcal{P}^\dagger_C$-conjugate of x_0.*

Proof. Part (i) is [Bruhat and Tits 1984, 4.6.28]. For (ii), we first consider C' a chamber of $\mathscr{A}$ containing x_0 in its closure. Since the group W acts transitively on the chambers of $\mathscr{A}$, there is $d \in \mathcal{D}$ and $w_0 \in \mathfrak{W}$ such that $C' = w_0 dC$ and C contains $d^{-1}x_0$ in its closure. By [Ollivier and Schneider 2012, Proposition 4.13i.], this implies that $d^{-1}C = C$, and therefore $C' = w_0 C$ or, when considering the action of G on the building, $C' = \hat{w}_0 C$, where $\hat{w}_0 \in \mathrm{K} \cap N_G(\mathrm{T})$ denotes a lift for w_0. Now, let C'' be a chamber of $\mathscr{X}$ containing x_0 in its closure. By [Bruhat and Tits 1972, Corollaire 2.2.6], there is $k \in \mathcal{P}^\dagger_{x_0}$ such that kC'' is in $\mathscr{A}$. Applying the previous observation, C'' is a $\mathcal{P}^\dagger_{x_0}$-conjugate of C. Lastly, let gx_0 (with $g \in \mathrm{G}$) be a conjugate of x_0 in the closure of C. By (ii), the chamber $g^{-1}C$ is of the form kC for $k \in \mathcal{P}^\dagger_{x_0}$, which implies that $gk \in \mathcal{P}^\dagger_C$ and gx_0 is a $\mathcal{P}^\dagger_C$-conjugate of x_0. □

Remark 1.7. By [Ollivier and Schneider 2012, Lemma 4.9], $\mathcal{P}^\dagger_C$ is the disjoint union of all $\mathrm{I}\hat{\omega}\mathrm{I} = \hat{\omega}\mathrm{I}$ for $\omega \in \Omega$. Therefore, a G-conjugate of x_0 in the closure of C is a $\mathcal{P}^\dagger_C \cap N_{\mathrm{G}}(\mathrm{T})$-conjugate of x_0.

1B4. *Weyl chambers.* The set of dominant coweights $\mathrm{X}_*^+(\mathrm{T})$ is the set of all $\lambda \in \mathrm{X}_*(\mathrm{T})$ such that $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$. It is called the dominant chamber. Its opposite is the antidominant chamber. A coweight λ such that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Phi^+$ is called strongly dominant. By [Bushnell and Kutzko 1998, Lemma 6.14], strongly dominant elements do exist.

We call a facet F of $\mathscr{A}$ standard if it is a facet of C containing x_0 in its closure. Attached to a standard facet F is the subset Φ_F of all roots in Φ taking value zero on F and the subgroup $\mathfrak{W}_F$ of $\mathfrak{W}$ generated by the simple reflections stabilizing F. Let $\Phi_F^+ := \Phi^+ \cap \Phi_F$ and $\Phi_F^- := \Phi^- \cap \Phi_F$. Define the following Weyl chambers in $\mathrm{X}_*(\mathrm{T})$ as in [Ollivier 2012, Section 4.1.1]:

$$\mathscr{C}^+(F) = \{\lambda \in \mathrm{X}_*(\mathrm{T}) \text{ such that } \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in (\Phi^+ - \Phi_F^+) \cup \Phi_F^-\}$$

and its opposite $\mathscr{C}^-(F) = -\mathscr{C}^+(F)$. They are respectively the images of the dominant and antidominant chambers under the longest element w_F in $\mathfrak{W}_F$.

By Gordan's lemma [Kempf et al. 1973, p. 7], a Weyl chamber is finitely generated as a semigroup.

1B5. We follow the notations of [Ollivier 2012, Sections 2.2.2, 2.2.3]. Recall that T^1 is the pro-p Sylow subgroup of T^0. We denote by $\widetilde{\mathrm{W}}$ the quotient of $N_{\mathrm{G}}(\mathrm{T})$ by T^1, and obtain the exact sequence

$$0 \longrightarrow \mathrm{T}^0/\mathrm{T}^1 \longrightarrow \widetilde{\mathrm{W}} \longrightarrow \mathrm{W} \longrightarrow 0.$$

The group $\widetilde{\mathrm{W}}$ parametrizes the double cosets of G modulo $\tilde{\mathrm{I}}$. We fix a lift $\hat{w} \in N_{\mathrm{G}}(\mathrm{T})$ for any $w \in \widetilde{\mathrm{W}}$ and denote by τ_w the characteristic function of the double coset $\tilde{\mathrm{I}}\hat{w}\tilde{\mathrm{I}}$. The set of all $(\tau_w)_{w \in \widetilde{\mathrm{W}}}$ is a $\mathbb{Z}$-basis for $\widetilde{\mathrm{H}}_{\mathbb{Z}}$, which was defined in the introduction

to be the convolution ring of $\mathbb{Z}$-valued functions with compact support in $\tilde{\mathrm{I}}\backslash\mathrm{G}/\tilde{\mathrm{I}}$. For $g \in \mathrm{G}$, we will also use the notation τ_g for the characteristic function of the double coset $\tilde{\mathrm{I}}g\tilde{\mathrm{I}}$.

For Y a subset of W, we denote by $\widetilde{Y}$ its preimage in $\widetilde{\mathrm{W}}$. In particular, we have the preimage $\widetilde{\mathrm{X}}_*(\mathrm{T})$ of $\mathrm{X}_*(\mathrm{T})$. Similarly to those of $\mathrm{X}_*(\mathrm{T})$, its elements will be denoted by λ or e^λ and called coweights. For $\alpha \in \Phi$, we inflate the function $\langle \cdot, \alpha \rangle$ defined on $\mathrm{X}_*(\mathrm{T})$ to $\widetilde{\mathrm{X}}_*(\mathrm{T})$. We still call the elements in the preimage $\widetilde{\mathrm{X}}_*^+(\mathrm{T})$ of $\mathrm{X}_*^+(\mathrm{T})$ *dominant coweights*. For σ a sign and F a standard facet, we consider the preimage of $\mathscr{C}^\sigma(F)$ in $\widetilde{\mathrm{X}}_*(\mathrm{T})$, and we still denote it by $\mathscr{C}^\sigma(F)$.

The length function ℓ on W pulls back to a length function ℓ on $\widetilde{\mathrm{W}}$ [Vignéras 2005, Proposition 1]. For $u, v \in \widetilde{\mathrm{W}}$ we write $u \le v$ (resp. $u < v$) if their projections $\bar{u}$ and $\bar{v}$ in W satisfy $\bar{u} \le \bar{v}$ (resp. $\bar{u} < \bar{v}$).

1B6. We emphasize the following remark which will be important for the definition of the subring $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_\mathbb{Z})$ of the center of $\widetilde{\mathrm{H}}_\mathbb{Z}$ in Section 2B.

For $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, the element $\lambda(\varpi^{-1}) \in N_\mathrm{G}(\mathrm{T})$ is a lift for e^λ, viewed in W by our convention (1-6). The map

$$\lambda \in \mathrm{X}_*(\mathrm{T}) \to [\lambda(\varpi^{-1}) \bmod \mathrm{T}^1] \in \widetilde{\mathrm{X}}_*(\mathrm{T}) \tag{1-9}$$

is a $\mathfrak{W}$-equivariant splitting for the exact sequence of abelian groups

$$0 \longrightarrow \mathrm{T}^0/\mathrm{T}^1 \longrightarrow \widetilde{\mathrm{X}}_*(\mathrm{T}) \longrightarrow \mathrm{X}_*(\mathrm{T}) \longrightarrow 0. \tag{1-10}$$

We will identify $\mathrm{X}_*(\mathrm{T})$ with its image in $\widetilde{\mathrm{X}}_*(\mathrm{T})$ via (1-9). Note that this identification depends on the choice of the uniformizer ϖ.

Remark 1.8. We have the decomposition of $\widetilde{\mathrm{W}}$ as the semidirect product $\widetilde{\mathrm{W}} = \widetilde{\mathfrak{W}} \ltimes \mathrm{X}_*(\mathrm{T})$, where $\widetilde{\mathfrak{W}}$ denotes the preimage of $\mathfrak{W}$ in $\widetilde{\mathrm{W}}$.

1B7. *Pro-p Hecke rings.* The product in the generic pro-p Iwahori–Hecke ring $\widetilde{\mathrm{H}}_\mathbb{Z}$ is described in [Vignéras 2005, Theorem 1]. It is given by *quadratic relations* and *braid relations*. Stating the quadratic relations in $\widetilde{\mathrm{H}}_\mathbb{Z}$ requires some more notation. We are only going to use them in $\widetilde{\mathrm{H}}_k$ where they have a simpler form, and we postpone their description to Section 1B8. We recall here the braid relations

$$\tau_{ww'} = \tau_w \tau_{w'} \text{ for } w, w' \in \widetilde{\mathrm{W}} \text{ satisfying } \ell(ww') = \ell(w) + \ell(w'). \tag{1-11}$$

The functions in $\widetilde{\mathrm{H}}_\mathbb{Z}$ with support in the subgroup of G generated by all parahoric subgroups form a subring $\widetilde{\mathrm{H}}_\mathbb{Z}^{\mathrm{aff}}$ called the affine subring. It has $\mathbb{Z}$-basis the set of all τ_w for w in the preimage $\widetilde{\mathrm{W}}_{\mathrm{aff}}$ of $\mathrm{W}_{\mathrm{aff}}$ in $\widetilde{\mathrm{W}}$ (see for example [Ollivier and Schneider 2012, Section 4.5]). It is generated by all τ_s for s in the preimage $\widetilde{S}_{\mathrm{aff}}$ of S_{aff} and all τ_t for $t \in \mathrm{T}^0/\mathrm{T}^1$.

There is an involutive automorphism defined on $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ by

$$\iota : \tau_w \mapsto (-q)^{\ell(w)} \tau_{w^{-1}}^{-1} \tag{1-12}$$

[Vignéras 2005, Corollary 2], and it actually yields an involution on $\widetilde{\mathrm{H}}_{\mathbb{Z}}$. Inflating the character $\epsilon_C : \mathrm{W} \to \{\pm 1\}$ defined in Section 1B2 to a character of $\widetilde{\mathrm{W}}$, we define a $\mathbb{Z}$-linear involution υ_C of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ by

$$\upsilon_C(\tau_w) = \epsilon_C(w)\tau_w \quad \text{for any } w \in \widetilde{\mathrm{W}}.$$

It is the identity on the affine subring $\widetilde{\mathrm{H}}_{\mathbb{Z}}^{\mathrm{aff}}$. We will consider the following $\mathbb{Z}$-linear involution on $\widetilde{\mathrm{H}}_{\mathbb{Z}}$:

$$\iota_C = \iota \circ \upsilon_C. \tag{1-13}$$

Remark 1.9. The involution ι fixes all τ_w for $w \in \widetilde{\mathrm{W}}$ with length zero. The involution ι_C fixes all τ_{e^λ} for $\lambda \in \widetilde{\mathrm{X}}_*(\mathrm{T})$ with length zero.

1B8. Let R be a ring with unit 1_{R}, containing an inverse for $(q1_{\mathrm{R}} - 1)$ and a primitive $(q-1)$-th root of 1_{R}. The group of characters of $\mathrm{T}^0/\mathrm{T}^1 = \overline{\mathbf{T}}(\mathbb{F}_q)$ with values in $\mathrm{R}^\times$ is isomorphic to the group of characters of $\overline{\mathbf{T}}(\mathbb{F}_q)$ with values in $\mathbb{F}_q^\times$, which we denote by $\widehat{\overline{\mathbf{T}}}(\mathbb{F}_q)$. To $\xi \in \widehat{\overline{\mathbf{T}}}(\mathbb{F}_q)$ we attach the idempotent element $\epsilon_\xi \in \widetilde{\mathrm{H}}_{\mathrm{R}}$ as in [Vignéras 2005] (definition recalled in [Ollivier 2012, Section 2.4.3]). For $t \in \mathrm{T}^0$ we have $\epsilon_\xi \tau_t = \tau_t \epsilon_\xi = \xi(t)\epsilon_\xi$. The idempotent elements ϵ_ξ, $\xi \in \widehat{\overline{\mathbf{T}}}(\mathbb{F}_q)$ are pairwise orthogonal and their sum is the identity in $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{R}$.

For $A \in \Pi_{\mathrm{aff}}$, choose the lift $n_A \in \mathrm{G}$ for s_A defined after fixing an épinglage for G as in [Vignéras 2005, Section 1.2]. We refer to [Ollivier 2012, Section 2.2.5] for the definition of the associated subgroup T_A of T^0, which is identified with a subgroup of $\mathrm{T}^0/\mathrm{T}^1$.

For $\xi \in \widehat{\overline{\mathbf{T}}}(\mathbb{F}_q)$, we have in $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{R}$

$$\epsilon_\xi \tau_{n_A}^2 = \begin{cases} \epsilon_\xi((q1_{\mathrm{R}} - 1)\tau_{n_A} + q1_{\mathrm{R}}) & \text{if } \xi \text{ is trivial on } \mathrm{T}_A, \\ \text{an element of } q\mathrm{R}^\times \epsilon_\xi & \text{otherwise.} \end{cases} \tag{1-14}$$

The field k is an example of ring R as above. In $\widetilde{\mathrm{H}}_k$ we have

$$\epsilon_\xi \tau_{n_A}^2 = \begin{cases} -\epsilon_\xi \tau_{n_A} & \text{if } \xi \text{ is trivial on } \mathrm{T}_A, \\ 0 & \text{otherwise.} \end{cases} \tag{1-15}$$

Remark 1.10. In $\widetilde{\mathrm{H}}_k$ we have $\tau_{n_A}\iota(\tau_{n_A}) = 0$ for all $A \in S_{\mathrm{aff}}$. Furthermore, $\iota(\tau_{n_A}) + \tau_{n_A}$ lies in the subalgebra of $\widetilde{\mathrm{H}}_k$ generated by all τ_t, $t \in \mathrm{T}^0/\mathrm{T}^1$, or equivalently by all ϵ_ξ, $\xi \in \widehat{\overline{\mathbf{T}}}(\mathbb{F}_q)$. This can be seen using for example [Ollivier 2012, Remark 2.10], which also implies the following:

- If ξ is trivial on T_A, then $\iota(\epsilon_\xi \tau_{n_A}) = \epsilon_\xi \iota(\tau_{n_A}) = -\epsilon_\xi(\tau_{n_A} + 1)$.

- If ξ is not trivial on T_A, then $\iota(\epsilon_\xi \tau_{n_A}) = -\epsilon_\xi \tau_{n_A}$.

1B9. *Parametrization of the weights.* The functions in $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ with support in K form a subring $\widetilde{\mathfrak{H}}_{\mathbb{Z}}$. It has $\mathbb{Z}$-basis the set of all τ_w for $w \in \widetilde{\mathfrak{W}}$. Denote by $\widetilde{\mathfrak{H}}_k$ the k-algebra $\widetilde{\mathfrak{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$. The simple modules of $\widetilde{\mathfrak{H}}_k$ are one-dimensional [Sawada 1977, (2.11)].

An irreducible smooth k-representation ρ of K will be called a weight. By [Carter and Lusztig 1976, Corollary 7.5], the weights are in one-to-one correspondence with the characters of $\widetilde{\mathfrak{H}}_k$ via $\rho \mapsto \rho^{\mathrm{I}}$. To a character $\chi : \widetilde{\mathfrak{H}}_k \to k$ is attached the morphism $\bar{\chi} : \mathrm{T}^0/\mathrm{T}^1 \to k^\times$ such that $\bar{\chi}(t) = \chi(\tau_t)$ for all $t \in \mathrm{T}^0/\mathrm{T}^1$ and the set $\Pi_{\bar{\chi}}$ of all simple roots $\alpha \in \Pi$ such that $\bar{\chi}$ is trivial on T_α. We then have $\chi(\tau_{\tilde{s}_\alpha}) = 0$ for all $\alpha \in \Pi - \Pi_{\bar{\chi}}$, where $\tilde{s}_\alpha \in \widetilde{\mathrm{W}}$ is any lift for $s_\alpha \in \mathrm{W}$. We denote by Π_χ the subset of all $\alpha \in \Pi_{\bar{\chi}}$ such that $\chi(\tau_{\tilde{s}_\alpha}) = 0$. The character χ is determined by the data of $\bar{\chi}$ and Π_χ (see also [Ollivier 2012, Section 3.4]).

Remark 1.11. Choosing a standard facet F is equivalent to choosing the subset Π_F of Π of the simple roots taking value zero on F. The standard facet corresponding to Π_χ in the previous discussion will be denoted by F_χ.

2. On the center of the pro-p Iwahori–Hecke algebra in characteristic p

2A. ***Commutative subrings of the pro-p Iwahori–Hecke ring.*** Let σ be a sign and F a standard facet.

2A1. As in [Ollivier 2012, Section 4.1.1], we introduce the multiplicative injective map

$$\Theta_F^\sigma : \widetilde{\mathrm{X}}_*(\mathrm{T}) \longrightarrow \widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$$

and the elements $\mathcal{B}_F^\sigma(\lambda) := q^{\ell(e^\lambda)/2}\Theta_F^\sigma(\lambda)$ for all $\lambda \in \widetilde{\mathrm{X}}_*(\mathrm{T})$. Recall that $\mathcal{B}_F^\sigma(\lambda) = \tau_{e^\lambda}$ if $\lambda \in \mathscr{C}^\sigma(F)$.

The map $\mathcal{B}_F^\sigma$ does not respect the product in general, but it is multiplicative when restricted to any Weyl chamber (see [ibid., Remark 4.3]). For any coweight $\lambda \in \widetilde{\mathrm{X}}_*(\mathrm{T})$, the element $\mathcal{B}_F^\sigma(\lambda)$ lies in $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ (see Lemma 2.3 below). Furthermore, combining Lemmas 1.4(ii), 2.3 and [ibid., Lemma 4.4],

$$\iota_C(\mathcal{B}_F^+(\lambda)) = \mathcal{B}_F^-(\lambda). \tag{2-1}$$

Extend Θ_F^σ linearly to an injective morphism of $\mathbb{Z}[q^{\pm 1/2}]$-algebras

$$\mathbb{Z}[q^{\pm 1/2}][\widetilde{\mathrm{X}}_*(\mathrm{T})] \longrightarrow \widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}].$$

We consider the commutative subring $\mathcal{A}_F^\sigma := \widetilde{\mathrm{H}}_{\mathbb{Z}} \cap \mathrm{Im}(\Theta_F^\sigma)$. By [ibid., Proposition 4.5], it is a free $\mathbb{Z}$-module with basis the set of all $\mathcal{B}_F^\sigma(\lambda)$ for $\lambda \in \widetilde{\mathrm{X}}_*(\mathrm{T})$. Since the Weyl chambers (in $\widetilde{\mathrm{X}}_*(\mathrm{T})$) are finitely generated semigroups, $\mathcal{A}_F^\sigma$ is finitely generated as a ring.

Remark 2.1. Note that $\mathcal{B}_C^+ = \mathcal{B}_{x_0}^-$ (resp. $\mathcal{B}_C^- = \mathcal{B}_{x_0}^+$) coincides with the integral Bernstein map E^+ (resp. E) introduced in [Vignéras 2005] and $\mathcal{A}_C^+$ (resp. $\mathcal{A}_C^-$) with the commutative ring denoted by $\mathcal{A}^{+,(1)}$ (resp. $\mathcal{A}^{(1)}$) in Theorem 2 of the same paper.

Identify $\mathrm{X}_*(\mathrm{T})$ with its image in $\widetilde{\mathrm{X}}_*(\mathrm{T})$ via (1-9). We denote by $(\mathcal{A}_F^\sigma)^\circ$ the intersection

$$(\mathcal{A}_F^\sigma)^\circ := \widetilde{\mathrm{H}}_\mathbb{Z} \cap \Theta_F^\sigma(\mathbb{Z}[\mathrm{X}_*(\mathrm{T})]) \subseteq \mathcal{A}_F^\sigma.$$

A $\mathbb{Z}$-basis for $(\mathcal{A}_F^\sigma)^\circ$ is given by all $\mathcal{B}_F^\sigma(\lambda)$ for $\lambda \in \mathrm{X}_*(\mathrm{T})$. It is finitely generated as a ring.

Proposition 2.2. *The commutative $\mathbb{Z}$-algebra $\mathcal{A}_F^\sigma$ is isomorphic to the tensor product of the $\mathbb{Z}$-algebras $\mathbb{Z}[\mathrm{T}^0/\mathrm{T}^1]$ and $(\mathcal{A}_F^\sigma)^\circ$. In particular, $(\mathcal{A}_F^\sigma)^\circ$ is a direct summand of $\mathcal{A}_F^\sigma$ as a $\mathbb{Z}$-module.*

Proof. Since the exact sequence (1-10) splits, $\mathcal{A}_F^\sigma$ is a free $(\mathcal{A}_F^\sigma)^\circ$-module with basis the set of all τ_t for $t \in \mathrm{T}^0/\mathrm{T}^1$. Indeed, recall that

$$\mathcal{B}_F^\sigma(\lambda + t) = \mathcal{B}_F^\sigma(\lambda)\tau_t = \tau_t \mathcal{B}_F^\sigma(\lambda)$$

for all $\lambda \in \widetilde{\mathrm{X}}_*(\mathrm{T})$ and $t \in \mathrm{T}^0/\mathrm{T}^1$. □

2A2. The following is a direct consequence of the lemma proved in [Haines 2001, §5] and adapted to the pro-p Iwahori–Hecke algebra in [Vignéras 2005, Lemma 13] (see also [Vignéras 2006, Sections 1.2 and 1.5]).

Lemma 2.3. *Let F be a standard facet and σ a sign. For any $\lambda \in \widetilde{\mathrm{X}}_*(\mathrm{T})$, we have*

$$\mathcal{B}_F^\sigma(\lambda) = \tau_{e^\lambda} + \sum_{w < e^\lambda} a_w \tau_w,$$

where $(a_w)_w$ is a family of elements in $\mathbb{Z}$ (depending on σ, F and λ) indexed by the set of $w \in \widetilde{\mathrm{W}}$ such that $w < e^\lambda$. For those w, we have in particular $\ell(w) < \ell(e^\lambda)$.

2A3. In this subsection, we suppose that the root system of G is irreducible. This implies in particular that there is a unique element in Π_m. It can be written $-\alpha_0$, where $\alpha_0 \in \Phi^+$ is the highest root; we have $\beta \preceq \alpha_0$ for all $\beta \in \Phi$ [Bourbaki 1968, VI.1.8]. For any standard facet $F \neq x_0$, we have $\alpha_0 \notin \Phi_F$. Denote by $s_0 \in S_{\mathrm{aff}}$ the simple reflection associated to $(-\alpha_0, 1) \in \Pi_{\mathrm{aff}}$ and $n_0 := n_{(-\alpha_0,1)} \in \mathrm{G}$ the lift for s_0 as chosen in Section 1B8.

Lemma 2.4. *Suppose that $F \neq x_0$ and let $\lambda \in \widetilde{\mathrm{X}}_*^+(\mathrm{T})$ be such that $\ell(e^\lambda) \neq 0$. We have*

$$\mathcal{B}_F^+(\lambda) \in \tau_{n_0} \widetilde{\mathrm{H}}_\mathbb{Z}.$$

Proof. It suffices to check the claim for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$. Let $\mu, \nu \in \mathrm{X}_*(\mathrm{T})$, such that $\lambda = \mu - \nu$ and $w_F\mu, w_F\nu \in \mathrm{X}_*^+(\mathrm{T})$, where w_F denotes the longest element in $\mathfrak{W}_F$. Note that $w_F\alpha_0 \in \Phi^+$ because $F \neq x_0$. Furthermore, $\langle\lambda, \alpha_0\rangle \geq 1$ because there is $\beta \in \Pi$ such that $\langle\lambda, \beta\rangle \geq 1$ and $\beta \preceq \alpha_0$.

We have $e^\nu(-\alpha_0, 1) = (-\alpha_0, 1 + \langle\nu, \alpha_0\rangle) = (-\alpha_0, 1 + \langle w_F\nu, w_F\alpha_0\rangle) \in \Phi^+_{\mathrm{aff}}$. Therefore $\ell(e^\nu n_0) = \ell(e^\nu) + 1$ and $\tau_{e^\nu}\tau_{n_0} = \tau_{e^\nu n_0}$ in $\widetilde{\mathrm{H}}_{\mathbb{Z}}$. On the other hand, $e^{-\lambda}(-\alpha_0, 1) = (-\alpha_0, 1 - \langle\lambda, \alpha_0\rangle) \in \Phi^-_{\mathrm{aff}}$, and therefore $\ell(n_0 e^\lambda) = \ell(e^\lambda) - 1$.

We perform the computations in $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$, where, by definition, $\mathcal{B}_F^+(\lambda) = q^{\frac{1}{2}(\ell(e^\lambda)+\ell(e^\nu)-\ell(e^\mu))}\tau_{e^\nu}^{-1}\tau_{e^\mu}$. By the previous remarks,

$$\mathcal{B}_F^+(\lambda) = \tau_{n_0} q^{\frac{1}{2}(\ell(n_0 e^\lambda)+\ell(e^\nu n_0)-\ell(e^\mu))}\tau_{e^\nu n_0}^{-1}\tau_{e^\mu},$$

which, by the lemma evoked in Section 2A2, lies in $\tau_{n_0}\widetilde{\mathrm{H}}_{\mathbb{Z}}$. □

2B. *On the center of the pro-p Iwahori–Hecke ring.*

2B1. The ring $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is finitely generated as a module over its center $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathbb{Z}}) = (\mathcal{A}_C^+)^{\mathfrak{W}}$, and the latter has $\mathbb{Z}$-basis the set of all

$$\sum_{\lambda' \in \mathbb{O}} \mathcal{B}_C^+(\lambda'), \tag{2-2}$$

where $\mathbb{O}$ ranges over the $\mathfrak{W}$-orbits in $\widetilde{\mathrm{X}}_*(\mathrm{T})$. Moreover, $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ is a finitely generated $\mathbb{Z}$-algebra. Those results are proved in [Vignéras 2005, Theorem 4] (the hypothesis of irreducibility of the root system of G made there is not necessary for the statements about the center). One can also find a proof in [Schmidt 2009].

2B2. We denote by $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ the intersection of $(\mathcal{A}_C^+)^\circ$ with $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$. We have $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_{\mathbb{Z}}) = ((\mathcal{A}_C^+)^\circ)^{\mathfrak{W}}$. It has $\mathbb{Z}$-basis the set of all

$$z_\lambda := \sum_{\lambda' \in \mathbb{O}(\lambda)} \mathcal{B}_C^+(\lambda') \quad \text{for } \lambda \in \mathrm{X}_*^+(\mathrm{T}), \tag{2-3}$$

where we denote by $\mathbb{O}(\lambda)$ the $\mathfrak{W}$-orbit of λ.

Proposition 2.5. (i) *The left and right $(\mathcal{A}_C^+)^\circ$-modules $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ are finitely generated.*

(ii) *As a $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_{\mathbb{Z}})$-module, $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is finitely generated.*

(iii) *$\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ is a finitely generated $\mathbb{Z}$-algebra.*

(iv) *As $\mathbb{Z}$-modules, $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$, $\mathcal{A}_C^+$, $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ and $(\mathcal{A}_C^+)^\circ$ are direct summands of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$.*

Proof. Using Proposition 2.2 and [Vignéras 2005, Theorems 3 and 4], which state that $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is finitely generated over $\mathcal{A}_C^+$ (see Remark 2.1), we see that $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is finitely generated over $(\mathcal{A}_C^+)^\circ$. Statements (ii) and (iii) follow from [Bourbaki 1964, V.1.9, Théorème 2] because $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ is the ring of $\mathfrak{W}$-invariants of $(\mathcal{A}_C^+)^\circ$ and $\mathbb{Z}$ is

noetherian. For (iv), we first remark that the $\mathbb{Z}$-module $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ (resp. $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$) is a direct summand of $\mathcal{A}_C^+$ (resp. $(\mathcal{A}_C^+)^{\circ}$) since $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathbb{Z}}) = (\mathcal{A}_C^+)^{\mathfrak{W}}$ (resp. $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}}) = ((\mathcal{A}_C^+)^{\circ})^{\mathfrak{W}}$). The $\mathbb{Z}$-module $(\mathcal{A}_C^+)^{\circ}$ is a direct summand of $\mathcal{A}_C^+$ by Proposition 2.2. It remains to show that $\mathcal{A}_C^+$ is a direct summand of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$, which can be done by considering the integral Bernstein basis for the whole Hecke ring $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ introduced in [Vignéras 2005]. We recall it later in Section 5A and finish the proof of (iv) in Remark 5.1. □

2B3. Given a ring R with unit 1_{R}, we denote by $\widetilde{\mathrm{H}}_{\mathrm{R}}$ the R-algebra $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathrm{R}$; we identify q with its image in R. By Proposition 2.5(iv), the R-algebras $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathrm{R}$, $\mathcal{A}_C^+ \otimes_{\mathbb{Z}} \mathrm{R}$, $(\mathcal{A}_C^+)^{\circ} \otimes_{\mathbb{Z}} \mathrm{R}$ and $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathrm{R}$ are identified with subalgebras of $\widetilde{\mathrm{H}}_{\mathrm{R}}$, which we denote by $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathrm{R}})$ $(\mathcal{A}_C^+)_R$, $(\mathcal{A}_C^+)_R^{\circ}$ and $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathrm{R}})$, respectively. By [Schmidt 2009], $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathrm{R}})$ is not only contained in but is equal to the center of $\widetilde{\mathrm{H}}_{\mathrm{R}}$.

Remark 2.6. Proposition 2.5 remains valid with x_0 instead of C (use the involution ι_C and (2-1)). We introduce the subalgebras $(\mathcal{A}_{x_0}^+)_R$ and $(\mathcal{A}_{x_0}^+)_R^{\circ}$ of $\widetilde{\mathrm{H}}_{\mathrm{R}}$ with the obvious definitions.

For $\lambda \in \widetilde{\mathrm{X}}_*(\mathrm{T})$ (resp. $w \in \widetilde{\mathrm{W}}$), we still denote by $\mathcal{B}_F^{\sigma}(\lambda)$ (resp. τ_w) its natural image $\mathcal{B}_F^{\sigma}(\lambda) \otimes 1$ (resp. $\tau_w \otimes 1$) in $\widetilde{\mathrm{H}}_{\mathrm{R}}$. An R-basis for $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathrm{R}})$ is given by the set of all z_λ for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, where again we identify the element z_λ with its image in $\widetilde{\mathrm{H}}_{\mathrm{R}}$.

From Proposition 2.5 we deduce:

Proposition 2.7. *Let* R *be a field. A morphism of* R*-algebras* $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathrm{R}}) \to \mathrm{R}$ *can be extended to a morphism of* R*-algebras* $\mathscr{Z}(\widetilde{\mathrm{H}}_{\mathrm{R}}) \to \mathrm{R}$.

2B4. In the process of constructing $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$, we first fixed a hyperspecial vertex x_0 of C and then an apartment $\mathscr{A}$ containing C.

Proposition 2.8. *The ring* $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ *is not affected by*

- *the choice of another apartment* $\mathscr{A}'$ *containing* C,
- *the choice of another vertex* x_0' *of* C, *provided it is* G*-conjugate to* x_0.

Proof. Let g be in the stabilizer $\mathcal{P}_C^{\dagger}$ of C in G. Let $\mathrm{T}' := g\mathrm{T}g^{-1}$ and $x_0' = gx_0g^{-1}$. The apartment $\mathscr{A}'$ corresponding to T' contains C and x_0' is a hyperspecial vertex of C. Starting from T' and x_0' we proceed to the construction of the corresponding commutative subring $\mathscr{Z}^{\circ}(\widetilde{\mathrm{H}}_{\mathbb{Z}})'$ of the center of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$. Since $g \in \mathcal{P}_C^{\dagger}$, we have $\tilde{\mathrm{I}}g\tilde{\mathrm{I}} = \tilde{\mathrm{I}}\hat{\omega}\tilde{\mathrm{I}} = \tilde{\mathrm{I}}\hat{\omega}$ for some $\omega \in \widetilde{\Omega}$. Since this element ω has length zero, for $\lambda \in \mathrm{X}_*(\mathrm{T})$ the characteristic function of $\tilde{\mathrm{I}}g\lambda(\varpi)g^{-1}\tilde{\mathrm{I}}$ is equal to the product $\tau_g\tau_{\lambda(\varpi)}\tau_g^{-1}$. Therefore, the restriction to $\mathrm{X}_*(\mathrm{T})$ of the new map $(\mathcal{B}_C^+)'$ corresponding to the choice of x_0' and T' is defined by

$$\mathrm{X}_*(\mathrm{T}') \longrightarrow \widetilde{\mathrm{H}}_{\mathbb{Z}}, \quad \lambda \mapsto \tau_g \mathcal{B}_C^+(g^{-1}\lambda g)\tau_g^{-1}.$$

The element $z'_\lambda \in \mathscr{Z}^\circ(\widetilde{\mathrm{H}}_\mathbb{Z})'$ corresponding to the choice of $\lambda \in \mathrm{X}_*^+(\mathrm{T}') = g\mathrm{X}_*^+(\mathrm{T})g^{-1}$ is therefore $\tau_g z_{g^{-1}\lambda g}\tau_g^{-1} = z_\lambda$. We have proved that $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_\mathbb{Z})' = \mathscr{Z}^\circ(\widetilde{\mathrm{H}}_\mathbb{Z})$.

By Proposition 1.6(i) and Remark 1.7

- changing $\mathscr{A}$ into another apartment $\mathscr{A}'$ containing C, and
- changing x_0 into another vertex x'_0 of C which is G-conjugate to x_0

can be done independently of each other by conjugating by an element of I and of $\mathcal{P}_C^\dagger \cap N_\mathrm{G}(\mathrm{T})$ respectively. We have checked that these changes do not affect $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_\mathbb{Z})$. □

If $\mathbf{G}$ is of adjoint type or $\mathbf{G} = \mathrm{GL}_n$, then all hyperspecial vertices are conjugate:

Corollary 2.9 [Tits 1979, Section 2.5]. *If $\mathbf{G}$ is of adjoint type or $\mathbf{G} = \mathrm{GL}_n$, then $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_\mathbb{Z})$ depends only on the choice of the uniformizer ϖ.*

2C. ***An affine semigroup algebra in the center of the pro-p Iwahori–Hecke algebra in characteristic p.*** We will use the following observation several times in this subsection: Let F be a standard facet and σ a sign. For $\mu_1, \mu_2 \in \mathrm{X}_*(\mathrm{T})$, we have in $\widetilde{\mathrm{H}}_k$

$$\mathcal{B}_F^\sigma(\mu_1)\mathcal{B}_F^\sigma(\mu_2) = \begin{cases} \mathcal{B}_F^\sigma(\mu_1+\mu_2) & \text{if } \mu_1 \text{ and } \mu_2 \text{ lie in a common Weyl chamber,} \\ 0 & \text{otherwise.} \end{cases} \tag{2-4}$$

In $\widetilde{\mathrm{H}}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}[q^{\pm 1/2}]$ we have indeed

$$\mathcal{B}_F^\sigma(\mu_1)\mathcal{B}_F^\sigma(\mu_2) = q^{(\ell(e^{\mu_1})+\ell(e^{\mu_2})-\ell(e^{\mu_1+\mu_2}))/2}\mathcal{B}_F^\sigma(\mu_1+\mu_2).$$

If μ_1 and μ_2 lie in a common Weyl chamber, then $\ell(e^{\mu_1}) + \ell(e^{\mu_2}) - \ell(e^{\mu_1+\mu_2})$ is zero; otherwise, there is $\alpha \in \Pi$ satisfying $\langle \mu_1, \alpha\rangle\langle \mu_2, \alpha\rangle < 0$, which implies that this quantity is ≥ 2. This gives the required equality in $\widetilde{\mathrm{H}}_k$.

2C1. *The structure of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$.*

Proposition 2.10. *The map*

$$k[\mathrm{X}_*^+(\mathrm{T})] \longrightarrow \mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k), \quad \lambda \longmapsto z_\lambda, \tag{2-5}$$

is an isomorphism of k-algebras.

Proof. We already know that (2-5) maps a k-basis for $k[\mathrm{X}_*^+(\mathrm{T})]$ onto a k-basis for $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$. We have to check that it respects the product. Let $\lambda_1, \lambda_2 \in \mathrm{X}_*^+(\mathrm{T})$, with respective $\mathfrak{W}$-orbits $\mathbb{O}(\lambda_1)$ and $\mathbb{O}(\lambda_2)$. We consider the product

$$z_{\lambda_1} z_{\lambda_2} = \sum_{\substack{\mu_1 \in \mathbb{O}(\lambda_1), \\ \mu_2 \in \mathbb{O}(\lambda_2)}} \mathcal{B}_F^\sigma(\mu_1)\mathcal{B}_F^\sigma(\mu_2) \in \widetilde{\mathrm{H}}_k.$$

A Weyl chamber in $X_*(T)$ is a $\mathfrak{W}$-conjugate of $X_*^+(T)$. Given a Weyl chamber and a coweight (in $X_*(T)$), there is a unique $\mathfrak{W}$-conjugate of the coweight in the chosen Weyl chamber. The map $(\mu_1, \mu_2) \mapsto \mu_1 + \mu_2$ yields a bijection between the set of all $(\mu_1, \mu_2) \in \mathbb{O}(\lambda_1) \times \mathbb{O}(\lambda_2)$ such that μ_1 and μ_2 lie in the same Weyl chamber and the $\mathfrak{W}$-orbit $\mathbb{O}(\lambda_1 + \lambda_2)$ of $\lambda_1 + \lambda_2$: it is indeed surjective, and one checks that the two sets in question have the same size because, λ_1 and λ_2 being both dominant, the stabilizer in $\mathfrak{W}$ of $\lambda_1 + \lambda_2$ is the intersection of the stabilizers of λ_1 and of λ_2. Together with (2-4), this proves that $z_{\lambda_1+\lambda_2} = z_{\lambda_1} z_{\lambda_2}$. □

For a different proof of this proposition, see the remark after Theorem 4.3.

2C2. Since $X_*(T)$ is a free abelian group (of rank $\dim(T)$), the k-algebra $k[X_*(T)]$ is isomorphic to an algebra of Laurent polynomials and has a trivial nilradical. By Gordan's lemma, $X_*^+(T)$ is finitely generated as a semigroup. So, $k[X_*^+(T)]$ is a finitely generated k-algebra and its Jacobson radical coincides with its nilradical.

The Jacobson radical of $\mathcal{Z}^\circ(\widetilde{H}_k)$ is therefore trivial.

Proposition 2.11. *The Jacobson radical of $\mathcal{Z}(\widetilde{H}_k)$ is trivial.*

Proof. Since $\mathcal{Z}(\widetilde{H}_k)$ is a finitely generated k-algebra contained in $(\mathcal{A}_C^+)_k$, it is enough to prove that the nilradical of $(\mathcal{A}_C^+)_k$ is trivial. Using the notation of Section 1B8, it is enough to prove that, for any $\xi \in \widehat{\widetilde{\mathbf{T}}}(\mathbb{F}_q)$, the nilradical of the k-algebra $\epsilon_\xi(\mathcal{A}_C^+)_k$ with unit ϵ_ξ is trivial. By Proposition 2.2, the latter algebra is isomorphic to $(\mathcal{A}_C^+)_k^\circ$. It is therefore enough to prove that the nilradical of $(\mathcal{A}_C^+)_k^\circ$ is trivial.

By definition (see the convention in Section 2B3), the image of the k-linear injective map

$$\mathcal{B}_C^+ : k[X_*(T)] \longrightarrow \widetilde{H}_k$$

coincides with $(\mathcal{A}_C^+)_k^\circ$.

Fact i. *Let $\lambda_0 \in X_*^+(T)$ be a strongly dominant coweight. The ideal of $(\mathcal{A}_C^+)_k^\circ$ generated by $\mathcal{B}_C^+(\lambda_0)$ does not contain any nontrivial nilpotent element.*

An element $a \in (\mathcal{A}_C^+)_k^\circ$ is a k-linear combination of elements $\mathcal{B}_C^+(\lambda)$ for $\lambda \in X_*(T)$, and we say that $\lambda \in X_*(T)$ is in the support of a if the coefficient of $\mathcal{B}_C^+(\lambda)$ is nonzero. Suppose that a is nilpotent and nontrivial. After conjugating by an element of $\mathfrak{W}$, we can suppose that there is an element of $X_*^+(T)$ in the support of a. Then, let $\lambda_0 \in X_*^+(T)$ be strongly dominant. The element $a\mathcal{B}_C^+(\lambda_0)$ is nilpotent and by (2-4) it is nontrivial. By Fact i, we have a contradiction. □

Proof of the fact. The restriction of $\mathcal{B}_C^+$ to $k[X_*^+(T)]$ induces an isomorphism of k-algebras $k[X_*^+(T)] \cong \mathcal{B}_C^+(k[X_*^+(T)])$. By (2-4), the ideal $\mathfrak{A}$ of $(\mathcal{A}_C^+)^\circ$ generated by $\mathcal{B}_C^+(\lambda_0)$ coincides with the ideal of $\mathcal{B}_C^+(k[X_*^+(T)])$ generated by $\mathcal{B}_C^+(\lambda_0)$. Since

the k-algebra $k[\mathrm{X}_*^+(\mathrm{T})]$ does not contain any nontrivial nilpotent element, neither does $\mathfrak{A}$. □

Since k is algebraically closed, we have:

Corollary 2.12. *Let $z \in \mathscr{Z}(\tilde{\mathrm{H}}_k)$. If $\zeta(z) = 0$ for all characters $\zeta : \mathscr{Z}(\tilde{\mathrm{H}}_k) \to k$, then $z = 0$.*

2C3. *The center of the Iwahori–Hecke algebra in characteristic p.* Let R be a ring containing an inverse for $(q1_\mathrm{R} - 1)$ and a primitive $(q-1)$-th root of 1_R. We can apply the observations of Section 1B8 and consider the algebra

$$\tilde{\mathrm{H}}_\mathrm{R}(\xi) := \epsilon_\xi \tilde{\mathrm{H}}_\mathrm{R} \epsilon_\xi.$$

It can be seen as the algebra $\mathscr{H}(\mathrm{G}, \mathrm{I}, \xi^{-1})$ of G-endomorphisms of the representation $\epsilon_\xi \operatorname{ind}_\mathrm{I}^\mathrm{G} \mathbf{1}_\mathrm{R}$, which is isomorphic to the compact induction $\operatorname{ind}_\mathrm{I}^\mathrm{G} \xi^{-1}$ of ξ^{-1} seen as an R-character of I trivial on $\tilde{\mathrm{I}}$: denote by $1_{\mathrm{I},\xi^{-1}} \in \operatorname{ind}_\mathrm{I}^\mathrm{G} \xi^{-1}$ the unique function with support in I and value 1_R at 1_G, and then the map

$$\tilde{\mathrm{H}}_\mathrm{R}(\xi) \to \mathscr{H}(\mathrm{G}, \mathrm{I}, \xi^{-1}), \quad h \mapsto [1_{\mathrm{I},\xi^{-1}} \mapsto 1_{\mathrm{I},\xi^{-1}} h] \tag{2-6}$$

gives the identification. In particular, when $\xi = \mathbf{1}$ is the trivial character, then the algebra $\tilde{\mathrm{H}}_\mathrm{R}(\mathbf{1})$ identifies with the usual Iwahori–Hecke algebra $\mathrm{H}_\mathrm{R} = \mathrm{R}[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ with coefficients in R.

Remark 2.13. Let $\xi \in \hat{\tilde{\mathbf{T}}}(\mathbb{F}_q)$. We have inclusions

$$\epsilon_\xi \mathscr{Z}^\circ(\tilde{\mathrm{H}}_\mathrm{R}) \subseteq \epsilon_\xi \mathscr{Z}(\tilde{\mathrm{H}}_\mathrm{R}) \subseteq \mathscr{Z}(\tilde{\mathrm{H}}_\mathrm{R}(\xi)),$$

where the latter space is the center of $\tilde{\mathrm{H}}_\mathrm{R}(\xi)$. The inclusion $\epsilon_\xi \mathscr{Z}^\circ(\tilde{\mathrm{H}}_\mathrm{R}) \subseteq \mathscr{Z}(\tilde{\mathrm{H}}_\mathrm{R}(\epsilon_\xi))$ is strict in general. For example if $\mathrm{G} = \mathrm{GL}_2(\mathfrak{F})$, $\mathrm{R} = k$, and ξ is not fixed by the nontrivial element of $\mathfrak{W}$, then $\tilde{\mathrm{H}}_k(\xi)$ is commutative with a k-basis indexed by the elements in $\mathrm{X}_*(\mathrm{T})$ and contains zero divisors [Barthel and Livné 1994, Proposition 13] while the k-algebra $\epsilon_\xi \mathscr{Z}^\circ(\tilde{\mathrm{H}}_k)$ is isomorphic to $k[\mathrm{X}_*^+(\mathrm{T})]$.

If $\xi = \mathbf{1}$ however, these inclusions are equalities: one easily checks by direct comparison of the basis elements (2-2) and (2-3) that the first inclusion is an equality. The second one comes from the fact that $\epsilon_\mathbf{1}$ is a central idempotent in $\tilde{\mathrm{H}}_\mathrm{R}$. In particular we have:

Theorem 2.14. *The center of the Iwahori–Hecke k-algebra $k[\mathrm{I}\backslash\mathrm{G}/\mathrm{I}]$ is isomorphic to $k[\mathrm{X}_*^+(\mathrm{T})]$.*

Proof. The map

$$k[\mathrm{X}_*^+(\mathrm{T})] \longrightarrow \epsilon_\mathbf{1} \mathscr{Z}(\tilde{\mathrm{H}}_k), \quad \lambda \longmapsto \epsilon_\mathbf{1} z_\lambda$$

is surjective by the previous discussion. It is easily checked to be injective using Lemma 2.3 (compare with [Vignéras 2006, (1.6.5)]). □

3. The central Bernstein functions in the pro-p Iwahori–Hecke ring

Let $\mathbb{O}$ be a $\mathfrak{W}$-orbit in $\widetilde{\mathrm{X}}_*(\mathrm{T})$. We call the central element of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$

$$z_{\mathbb{O}} := \sum_{\lambda' \in \mathbb{O}} \mathcal{B}_C^+(\lambda') \tag{2-2}$$

the associated central Bernstein function.

3A. ***The support of the central Bernstein functions.*** For $h \in \widetilde{\mathrm{H}}_{\mathbb{Z}}$, the set of all $w \in \widetilde{\mathrm{W}}$ such that $h(\hat{w}) \neq 0$ is called the *support* of h. For $\mathbb{O}$ a $\mathfrak{W}$-orbit in $\widetilde{\mathrm{X}}_*(\mathrm{T})$, we denote by $\ell_{\mathbb{O}}$ the common length of all the coweights in $\mathbb{O}$.

Lemma 3.1. *Let $\mathbb{O}$ be a $\mathfrak{W}$-orbit in $\widetilde{\mathrm{X}}_*(\mathrm{T})$. The support of $z_{\mathbb{O}}$ contains the set of all e^{μ} for $\mu \in \mathbb{O}$: more precisely, the coefficient of $\tau_{e^{\mu}}$ in the decomposition of $z_{\mathbb{O}}$ is equal to* 1. *Any other element in the support of $z_{\mathbb{O}}$ has length $< \ell_{\mathbb{O}}$. The same is true with $\iota_C(z_{\mathbb{O}})$ instead of $z_{\mathbb{O}}$.*

Proof. This is a consequence of Lemma 2.3 (and of (2-1)). □

Proposition 3.2. *The involution ι_C fixes the elements in the center $\mathfrak{Z}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$. In particular, for $\mathbb{O}$ a $\mathfrak{W}$-orbit in $\widetilde{\mathrm{X}}_*(\mathrm{T})$, the element $\sum_{\lambda' \in \mathbb{O}} \mathcal{B}_C^{\sigma}(\lambda') \in \widetilde{\mathrm{H}}_{\mathbb{Z}}$ does not depend on the sign σ.*

Proof. We prove that ι_C fixes $z_{\mathbb{O}}$ by induction on $\ell_{\mathbb{O}}$.

If $\ell_{\mathbb{O}} = 0$, we conclude using Remark 1.9. Let $\mathbb{O}$ be a $\mathfrak{W}$-orbit in $\widetilde{\mathrm{X}}_*(\mathrm{T})$ such that $\ell_{\mathbb{O}} > 0$. The element $\iota_C(z_{\mathbb{O}})$ is central in $\widetilde{\mathrm{H}}_{\mathbb{Z}}$. Recall that a $\mathbb{Z}$-basis for $\mathfrak{Z}(\widetilde{\mathrm{H}}_{\mathbb{Z}})$ is given by the central Bernstein functions $z_{\mathbb{O}}$, where $\mathbb{O}$ ranges over the $\mathfrak{W}$-orbits in $\widetilde{\mathrm{X}}_*(\mathrm{T})$. Lemma 3.1 implies that $\iota_C(z_{\mathbb{O}})$ decomposes as a sum

$$\iota_C(z_{\mathbb{O}}) = z_{\mathbb{O}} + \sum_{\mathbb{O}'} a_{\mathbb{O}'} z_{\mathbb{O}'},$$

where $\mathbb{O}'$ ranges over a finite set of $\mathfrak{W}$-orbits in $\widetilde{\mathrm{X}}_*(\mathrm{T})$ such that $\ell_{\mathbb{O}'} < \ell_{\mathbb{O}}$ and $a_{\mathbb{O}'} \in \mathbb{Z}$. By induction and applying the involution ι_C, we get

$$z_{\mathbb{O}} = \iota_C(z_{\mathbb{O}}) + \sum_{\mathbb{O}'} a_{\mathbb{O}'} z_{\mathbb{O}'}$$

and $2(\iota(z_{\mathbb{O}}) - z_{\mathbb{O}}) = 0$. Since $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ has no $\mathbb{Z}$-torsion, $\iota(z_{\mathbb{O}}) = z_{\mathbb{O}}$. The second statement follows from (2-1). □

If G is semisimple, the projection in $\widetilde{\mathrm{H}}_k$ of the equality proved in Proposition 3.2 can be obtained independently, using the duality for finite-length $\widetilde{\mathrm{H}}_k$-modules defined in [Ollivier and Schneider 2012]:

Proposition 3.3. *Suppose that* G *is semisimple. The element $\sum_{\lambda' \in \mathbb{O}} \mathcal{B}_C^{\sigma}(\lambda') \in \widetilde{\mathrm{H}}_k$ is fixed by the involution ι_C and therefore does not depend on the sign σ.*

Proof. Suppose that G is semisimple. Let $\mathbb{O}$ be a $\mathfrak{W}$-orbit in $\widetilde{\mathrm{X}}_*(\mathrm{T})$. We want to prove, without using Proposition 3.2, that in $\widetilde{\mathrm{H}}_k$ we have $z_{\mathbb{O}} = \iota_C(z_{\mathbb{O}})$.

Let $\zeta : \mathscr{Z}(\widetilde{\mathrm{H}}_k) \to k$ be a character and $M = \widetilde{\mathrm{H}}_k \otimes_{\mathscr{Z}(\widetilde{\mathrm{H}}_k)} \zeta$ the induced $\widetilde{\mathrm{H}}_k$-module. It is finite dimensional over k and therefore by [Ollivier and Schneider 2012, Corollary 6.12] we have an isomorphism of right $\widetilde{\mathrm{H}}_k$-modules

$$\mathrm{Ext}^d_{\widetilde{\mathrm{H}}_k}(M, \widetilde{\mathrm{H}}_k) = \mathrm{Hom}_k(\iota_C^* M, k),$$

where d is the semisimple rank of G and $\iota_C^* M$ denotes the left $\widetilde{\mathrm{H}}_k$-module M with action twisted by the involution ι_C defined by (1-13). The category of left $\widetilde{\mathrm{H}}_k$-modules is naturally a $\mathscr{Z}(\widetilde{\mathrm{H}}_k)$-linear category, and therefore, for X and Y two left $\widetilde{\mathrm{H}}_k$-modules, $\mathrm{Ext}^d_{\widetilde{\mathrm{H}}_k}(X, Y)$ inherits the structure of a central $\mathscr{Z}(\widetilde{\mathrm{H}}_k)$-bimodule. Hence, the right $\widetilde{\mathrm{H}}_k$-module $\mathrm{Ext}^d_{\widetilde{\mathrm{H}}_k}(M, \widetilde{\mathrm{H}}_k)$ has a central character equal to ζ. On the other hand, $\mathrm{Hom}_k(\iota_C^* M, k)$ has $\zeta \circ \iota_C$ as a central character. Therefore, $\zeta(z_{\mathbb{O}}) = \zeta \circ \iota_C(z_{\mathbb{O}})$. By Corollary 2.12, we have the required equality $z_{\mathbb{O}} = \iota_C(z_{\mathbb{O}})$. □

3B. ***Independence lemma.*** The following lemma will be proved in Section 3C3.

Lemma 3.4. *For $\mathbb{O}$ a $\mathfrak{W}$-orbit in $\widetilde{\mathrm{X}}_*(\mathrm{T})$, the element*

$$\sum_{\lambda \in \mathbb{O}} \mathcal{B}_F^\sigma(\lambda)$$

in $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ does not depend on the choice of the standard facet F and of the sign σ.

Corollary 3.5. *The center of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is contained in the intersection of all the commutative rings $\mathcal{A}_F^\sigma$ for F a standard facet and σ a sign.*

3C. ***Inducing the generalized integral Bernstein functions.*** We study the behavior of the integral Bernstein maps upon parabolic induction and subsequently prove Lemma 3.4.

3C1. Let F be a standard facet, Π_F the associated set of simple roots and P_F the corresponding standard parabolic subgroup, with Levi decomposition $\mathrm{P}_F = \mathrm{M}_F \mathrm{N}_F$. The root datum attached to the choice of the split torus T in M_F is $(\Phi_F, \mathrm{X}^*(\mathrm{T}), \check{\Phi}_F, \mathrm{X}_*(\mathrm{T}))$ (notation in Section 1B4). The extended Weyl group of M_F is $\mathrm{W}_F = (N_{\mathrm{G}}(\mathrm{T}) \cap \mathrm{M}_F)/\mathrm{T}^0$. It is isomorphic to the semidirect product $\mathfrak{W}_F \ltimes \mathrm{X}_*(\mathrm{T})$, where $\mathfrak{W}_F$ is the finite Weyl group $(N_{\mathrm{G}}(\mathrm{T}) \cap \mathrm{M}_F)/\mathrm{T}$ (also defined in Section 1B4). We denote by ℓ_F its length function and by $\leq_F$ the Bruhat order on W_F.

Set $\widetilde{\mathrm{W}}_F = (N_{\mathrm{G}}(\mathrm{T}) \cap \mathrm{M}_F)/\mathrm{T}^1$. It is a subgroup of $\widetilde{\mathrm{W}}$. The double cosets of M_F modulo its pro-p Iwahori subgroup $\tilde{\mathrm{I}} \cap \mathrm{M}_F$ are indexed by the elements in $\widetilde{\mathrm{W}}_F$. For $w \in \mathrm{W}_F$, we denote by τ_w^F the characteristic function of the double coset containing the lift $\hat{w}$ for w (which lies in $N_{\mathrm{G}}(\mathrm{T}) \cap \mathrm{M}_F$). The set of all τ_w^F

for $w \in \mathrm{W}_F$ is a basis for the pro-p Iwahori–Hecke ring $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ of $\mathbb{Z}$-valued functions with compact support in $(\tilde{\mathrm{I}} \cap \mathrm{M}_F)\backslash \mathrm{M}_F / (\tilde{\mathrm{I}} \cap \mathrm{M}_F)$. The ring $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ does not inject in $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ in general.

An element in $w \in \mathrm{W}_F$ is called F-positive if $w^{-1}(\Phi^+ - \Phi_F^+) \subset \Phi_{\mathrm{aff}}^+$. For example, for $\lambda \in \mathrm{X}_*(\mathrm{T})$, the element e^λ is F-positive if and only if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+ - \Phi_F^+$. In this case, we will say that the coweight λ itself is F-positive. If furthermore $\langle \lambda, \alpha \rangle > 0$ for $\alpha \in \Phi^+ - \Phi_F^+$ and $\langle \lambda, \alpha \rangle = 0$ for $\alpha \in \Phi_F^+$, then it is called strongly F-positive. The F-positive coweights are the $\mathfrak{W}_F$-conjugates of the dominant coweights. The C-positive (resp. strongly C-positive) coweights are the dominant (resp. strongly dominant) coweights. An element in W_F is F-positive if and only if it belongs to $e^\lambda \mathfrak{W}_F$ for some F-positive coweight $\lambda \in \mathrm{X}_*(\mathrm{T})$. If μ and $\nu \in \mathrm{X}_*(\mathrm{T})$ are F-positive coweights such that $\mu - \nu$ is also F-positive, then we have the equality (see [Ollivier 2012, Section 1.2] for example)

$$\ell(e^{\mu-\nu}) + \ell(e^\nu) - \ell(e^\mu) = \ell_F(e^{\mu-\nu}) + \ell_F(e^\nu) - \ell_F(e^\mu). \tag{3-1}$$

An element in $\widetilde{\mathrm{W}}_F$ will be called F-positive if its projection in W_F is F-positive.

The subspace of $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ generated over $\mathbb{Z}$ by all τ_w^F for F-positive $w \in \widetilde{\mathrm{W}}_F$ is denoted by $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^+$. It is in fact a ring, and there is an injection of rings

$$j_F^+ : \widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^+ \longrightarrow \widetilde{\mathrm{H}}_{\mathbb{Z}}, \quad \tau_w^F \longmapsto \tau_w$$

which extends to an injection of $\mathbb{Z}[q^{\pm 1/2}]$-algebras

$$j_F : \widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \to \widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}].$$

This is a classical result for complex Hecke algebras [Bushnell and Kutzko 1998, (6.12)]. The argument is valid over $\mathbb{Z}[q^{\pm 1/2}]$.

Remark 3.6. An element $w \in \widetilde{\mathrm{W}}_F$ is called F-negative (resp. strongly F-negative) if w^{-1} is F-positive (resp. strongly F-positive), and, as before, $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ contains as a subring the space $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^-$ generated over $\mathbb{Z}$ by all τ_w^F for F-negative $w \in \widetilde{\mathrm{W}}_F$. There is an injection of rings $j_F^- : \widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^- \longrightarrow \widetilde{\mathrm{H}}_{\mathbb{Z}}, \tau_w^F \longmapsto \tau_w$.

Fact ii. *Let $v \in \mathrm{W}_F$, such that $v \leq_F e^\lambda$ for $\lambda \in \mathrm{X}_*(\mathrm{T})$ an F-positive coweight. Then v is F-positive.*

Proof. Suppose first that λ is dominant. Then the claim is Lemma 2.9(ii) of [Ollivier 2012]. In general, λ is a $\mathfrak{W}_F$-conjugate of a dominant coweight λ_0: there is $u \in \mathfrak{W}_F$ such that $e^\lambda = u e^{\lambda_0} u^{-1}$. We argue by induction on $\ell_F(u)$. Let s be a simple reflection in $\mathfrak{W}_F$ such that $\ell_F(su) = \ell_F(u) - 1$. By the properties of the Bruhat order (see [Haines 2001, Lemma 4.3] for example), one of v, vs, sv, svs is $\leq_F se^\lambda s$, and by induction this element is F-positive, which implies that v is F-positive. □

3C2. Let $F' \subseteq \overline{C}$ be another facet containing x_0 in its closure, such that $F \subseteq \overline{F}'$. This implies that $\Phi_{F'} \subseteq \Phi_F$ and $\Phi^+_{F'} \subseteq \Phi^+_F$. Let ${}_F\Theta^+_{F'}$ be the map constructed as in Section 2A with respect to the root data attached to M_F:

$$ {}_F\Theta^+_{F'} : \mathbb{Z}[q^{\pm 1/2}][\widetilde{\mathrm{X}}_*(\mathrm{T})] \longrightarrow \widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]. $$

The corresponding $\mathbb{Z}$-linear integral map is denoted by ${}_F\mathcal{B}^+_{F'} : \mathbb{Z}[\widetilde{\mathrm{X}}_*(\mathrm{T})] \longrightarrow \widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ and defined by ${}_F\mathcal{B}^+_{F'}(\lambda) = q^{\ell_F(e^\lambda)/2}\, {}_F\Theta^+_{F'}(\lambda)$ for all $\lambda \in \widetilde{\mathrm{X}}_*(\mathrm{T})$. It satisfies ${}_F\mathcal{B}^+_{F'}(\lambda) = \tau^F_{e^\lambda}$ if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in (\Phi^+_F - \Phi^+_{F'}) \cup \Phi^-_{F'}$.

Remark 3.7. If $F = x_0$ then ${}_{x_0}\mathcal{B}^+_{F'} = \mathcal{B}^+_{F'}$.

Lemma 3.8. *Let $\lambda \in \widetilde{\mathrm{X}}_*(\mathrm{T})$ be an F-positive coweight. Then ${}_F\mathcal{B}^+_{F'}(\lambda)$ lies in $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)^+$ and*

$$ j^+_F({}_F\mathcal{B}^+_{F'}(\lambda)) = \mathcal{B}^+_{F'}(\lambda). \tag{3-2} $$

Proof. Decompose $\lambda = \mu - \nu$ with $\mu, \nu \in \mathcal{C}^+(F')$. Then in $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ we have ${}_F\mathcal{B}^+_{F'}(\lambda) = q^{(\ell_F(e^\lambda) + \ell_F(e^\nu) - \ell_F(e^\mu))/2}\, \tau^F_{e^\mu}(\tau^F_{e^\nu})^{-1}$. By Lemma 2.3 applied to the pro-p Iwahori–Hecke algebra $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$, this element decomposes in $\widetilde{\mathrm{H}}_{\mathbb{Z}}(\mathrm{M}_F)$ into a linear combination of $\tau^F_{\tilde{w}}$ for $\tilde{w} \in \widetilde{\mathrm{W}}_F$, where the projection w of $\tilde{w}$ in W_F satisfies $w \leq_F e^\lambda$. Fact ii ensures that these w (and $\tilde{w}$) are F-positive. Now, j_F respects the product and

$$ j^+_F({}_F\mathcal{B}^+_{F'}(\lambda)) = j_F({}_F\mathcal{B}^+_{F'}(\lambda)) = q^{(\ell_F(e^\lambda) + \ell_F(e^\nu) - \ell_F(e^\mu))/2}\, \tau_{e^\mu}(\tau_{e^\nu})^{-1} $$

because μ and ν are in particular F-positive. Apply (3-1) to finish the proof. □

3C3. We prove Lemma 3.4. Let $\mathbb{O}$ be a $\mathfrak{W}$-orbit in $\widetilde{\mathrm{X}}_*(\mathrm{T})$. Since $\mathcal{B}^+_{x_0} = \mathcal{B}^-_C$, and using (2-1), it is enough to prove

$$ \sum_{\lambda \in \mathbb{O}} \mathcal{B}^+_F(\lambda) = \sum_{\lambda \in \mathbb{O}} \mathcal{B}^+_C(\lambda) \tag{3-3} $$

for any standard facet F. If $F = x_0$ then the result is given by Proposition 3.2. Let F be a standard facet, such that $F \neq x_0$.

(1) Let $\mu \in \widetilde{\mathrm{X}}_*(\mathrm{T})$ be an F-positive coweight with $\mathfrak{W}_F$-orbit $\mathbb{O}_F$. We have the following identity:

$$ \sum_{\mu' \in \mathbb{O}_F} \mathcal{B}^+_F(\mu') = \sum_{\mu' \in \mathbb{O}_F} j^+_F({}_F\mathcal{B}^+_F(\mu')) = \sum_{\mu' \in \mathbb{O}_F} j^+_F({}_F\mathcal{B}^+_C(\mu')) = \sum_{\mu' \in \mathbb{O}_F} \mathcal{B}^+_C(\mu'), $$

where the first and third equalities come from (3-2) and the second one from Proposition 3.2 applied to M_F.

(2) Choose ν a strongly F-positive coweight such that $\lambda + \nu$ is F-positive for all $\lambda \in \mathbb{O}$. Decompose the $\mathfrak{W}$-orbit $\mathbb{O}$ into the disjoint union of $\mathfrak{W}_F$-orbits $\mathbb{O}^i_F$

for $i \in \{1, \dots, r\}$. Since ν lies in both $\widetilde{\mathrm{X}}_*^+(\mathrm{T})$ and $\mathscr{C}^+(F)$, we have $\mathcal{B}_F^+(-\nu) = \mathcal{B}_C^+(-\nu) = \iota_C(\tau_{e^{-\nu}})$.

Let $i \in \{1, \dots, r\}$ and $\lambda \in \mathbb{O}_F^i$. We have in $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ that

$$\mathcal{B}_F^+(\lambda) = q^{\frac{1}{2}(\ell(e^\lambda) - \ell(e^{\lambda+\nu}) - \ell(e^\nu))} \mathcal{B}_F^+(\lambda+\nu)\mathcal{B}_F^+(-\nu).$$

Note that $\ell(e^\lambda) - \ell(e^{\lambda+\nu}) - \ell(e^\nu)$ does not depend on $\lambda \in \mathbb{O}_F^i$: since $\langle \nu, \alpha \rangle = 0$ for all $\alpha \in \Phi_F^+$, this quantity is equal to $\sum_{\alpha \in \Phi^+ - \Phi_F^+} |\langle \lambda, \alpha \rangle| - |\langle \lambda + \nu, \alpha \rangle| - |\langle \nu, \alpha \rangle|$, which does not depend on the choice of $\lambda \in \mathbb{O}_F^i$ because $\Phi^+ - \Phi_F^+$ is invariant under the action of $\mathfrak{W}_F$. Therefore, if we pick a representative $\lambda_i \in \mathbb{O}_F^i$, we have

$$\begin{aligned}\sum_{\lambda \in \mathbb{O}_F^i} \mathcal{B}_F^+(\lambda) &= q^{\frac{1}{2}(\ell(e^{\lambda_i}) - \ell(e^{\lambda_i+\nu}) - \ell(e^\nu))} \sum_{\lambda \in \mathbb{O}_F^i} \mathcal{B}_F^+(\lambda+\nu)\mathcal{B}_C^+(-\nu). \\ &= q^{\frac{1}{2}(\ell(e^{\lambda_i}) - \ell(e^{\lambda_i+\nu}) - \ell(e^\nu))} \sum_{\lambda \in \mathbb{O}_F^i} \mathcal{B}_C^+(\lambda+\nu)\mathcal{B}_C^+(-\nu) = \sum_{\lambda \in \mathbb{O}_F^i} \mathcal{B}_C^+(\lambda)\end{aligned}$$

(where the second equality follows from (1) applied to the $\mathfrak{W}_F$-orbit of $\lambda + \nu$), which proves that $\sum_{\lambda \in \mathbb{O}} \mathcal{B}_F^+(\lambda) = \sum_{\lambda \in \mathbb{O}} \mathcal{B}_C^+(\lambda)$.

4. Compatibility between Satake and Bernstein isomorphisms in characteristic p

In this section all the algebras have coefficients in k.

Let (ρ, V) be a weight and v a chosen nonzero $\tilde{\mathrm{I}}$-fixed vector. Let $\chi : \widetilde{\mathfrak{H}}_k \to k$ be the associated character and F_χ the corresponding standard facet (Remark 1.11). We consider the compact induction $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho$ and its k-algebra of G-endomorphisms $\mathscr{H}(\mathrm{G}, \rho)$. The $\tilde{\mathrm{I}}$-invariant subspace $(\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho)^{\tilde{\mathrm{I}}}$ is naturally a right $\widetilde{\mathrm{H}}_k$-module. Let $\mathbf{1}_{\mathrm{K},v} \in \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho$ be the ($\tilde{\mathrm{I}}$-invariant) function with support K and value v at 1.

The map

$$\mathscr{Z}(\widetilde{\mathrm{H}}_k) \longrightarrow \mathrm{Hom}_{\widetilde{\mathrm{H}}_k}((\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho)^{\tilde{\mathrm{I}}}, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho)^{\tilde{\mathrm{I}}}), \quad z \longmapsto [f \mapsto fz], \tag{4-1}$$

defines a morphism of k-algebras. On the other hand, by [Ollivier 2012, Corollary 3.14], passing to $\tilde{\mathrm{I}}$-invariants yields an isomorphism of k-algebras

$$\mathscr{H}(\mathrm{G}, \rho) = \mathrm{Hom}_{\mathrm{G}}(\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho, \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho) \xrightarrow{\sim} \mathrm{Hom}_{\widetilde{\mathrm{H}}_k}((\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho)^{\tilde{\mathrm{I}}}, (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho)^{\tilde{\mathrm{I}}}). \tag{4-2}$$

Composing (4-1) with the inverse of (4-2) therefore gives a morphism of k-algebras $\mathscr{Z}(\widetilde{\mathrm{H}}_k) \to \mathscr{H}(\mathrm{G}, \rho)$, and we consider its restriction to $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$:

$$\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k) \longrightarrow \mathscr{H}(\mathrm{G}, \rho), \quad z \longmapsto [\mathbf{1}_{\mathrm{K},v} \mapsto \mathbf{1}_{\mathrm{K},v} z]. \tag{4-3}$$

For $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, we denote by $\mathcal{T}'_\lambda \in \mathscr{H}(\mathrm{G}, \rho)$ the image under (4-3) of the central Bernstein function z_λ defined by (2-3).

On the other hand, recall that we have the isomorphism of k-algebras [Ollivier 2012, Theorem 4.11]

$$\mathcal{T} : k[\mathrm{X}_*^+(\mathrm{T})] \xrightarrow{\sim} \mathscr{H}(\mathrm{G}, \rho), \tag{4-4}$$

where $\mathcal{T}_\lambda$ for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ is defined by

$$\mathcal{T}_\lambda : \mathbf{1}_{\mathrm{K},v} \mapsto \mathbf{1}_{\mathrm{K},v} \mathcal{B}^+_{F_\chi}(\lambda). \tag{4-5}$$

Proposition 4.1. *We have $\mathcal{T}'_\lambda = \mathcal{T}_\lambda$ for all $\lambda \in \mathrm{X}_*^+(\mathrm{T})$.*

Proof. It is enough to check that these operators coincide on $\mathbf{1}_{\mathrm{K},v}$. If λ has length zero, then $\mathcal{B}^+_{F_\chi}(\lambda) = z_\lambda = \tau_{e^\lambda}$ and the claim is true. Otherwise λ has length > 0; recall that $\mathbb{O}(\lambda)$ denotes the $\mathfrak{W}$-orbit of λ.

(a) Let $\lambda' \in \mathbb{O}(\lambda)$ and suppose that $\lambda' \neq \lambda$. By (2-4), we have $\mathcal{B}^+_{F_\chi}(\lambda')\mathcal{B}^+_{F_\chi}(\lambda) = \mathcal{B}^+_{F_\chi}(\lambda)\mathcal{B}^+_{F_\chi}(\lambda') = 0$ in $\widetilde{\mathrm{H}}_k$. This implies that $\mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda')) = 0$ and therefore that $\mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda') = 0$ by [Herzig 2011a, Corollary 6.5], which claims that $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho$ is a torsion-free $\mathscr{H}(\mathrm{G}, \rho)$-module.

(b) By Lemma 3.4, we have

$$\begin{aligned}\mathcal{T}'_\lambda(\mathbf{1}_{\mathrm{K},v}) &= \mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda) + \sum_{\substack{\lambda' \in \mathbb{O}(\lambda),\\ \lambda' \neq \lambda}} \mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda') = \mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v}) + \sum_{\substack{\lambda' \in \mathbb{O}(\lambda),\\ \lambda' \neq \lambda}} \mathbf{1}_{\mathrm{K},v}\mathcal{B}^+_{F_\chi}(\lambda') \\ &= \mathcal{T}_\lambda(\mathbf{1}_{\mathrm{K},v}),\end{aligned}$$

where the last equality follows from (a). □

Remark 4.2. By [Ollivier 2012, Lemma 3.6], the map

$$\chi \otimes_{\widetilde{\mathfrak{H}}_k} \widetilde{\mathrm{H}}_k \cong (\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho)^{\mathrm{I}}, \quad 1 \otimes 1 \mapsto \mathbf{1}_{\mathrm{K},v}, \tag{4-6}$$

induces an $\widetilde{\mathrm{H}}_k$-equivariant isomorphism. Proposition 4.1, combined with (4-6), proves that for $\lambda \in \mathrm{X}_*^+(\mathrm{T})$, the right actions of z_λ and of $\mathcal{B}^+_{F_\chi}(\lambda)$ on $1 \otimes 1 \in \chi \otimes_{\widetilde{\mathfrak{H}}_k} \widetilde{\mathrm{H}}_k$ coincide. This remark will be important for the classification of the simple supersingular $\widetilde{\mathrm{H}}_k$-modules in Section 5D.

Proposition 4.1 implies:

Theorem 4.3. *The diagram*

$$\begin{array}{ccc} k[\mathrm{X}_*^+(\mathrm{T})] & \xrightarrow{(2\text{-}5)} & \mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k) \\ \| & & \downarrow{\scriptstyle (4\text{-}3)} \\ k[\mathrm{X}_*^+(\mathrm{T})] & \xrightarrow{\mathcal{T}} & \mathscr{H}(\mathrm{G}, \rho) \end{array} \tag{4-7}$$

is a commutative diagram of isomorphisms of k-algebras.

We remark that we have not used the fact that (2-5) is multiplicative. We proved this fact beforehand in Proposition 2.10, but it can also be seen as a consequence of the commutativity of the diagram.

5. Supersingularity

We turn to the study of the $\widetilde{\mathrm{H}}_k$-modules with finite length. We consider right modules unless otherwise specified. Recall that k is algebraically closed with characteristic p.

5A. ***A basis for the pro-p Iwahori–Hecke ring.*** We recall the $\mathbb{Z}$-basis for $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ defined in [Vignéras 2005]. It is indexed by $w \in \widetilde{\mathrm{W}}$ and is denoted by $(E_w)_{w\in\widetilde{\mathrm{W}}}$ there. We will call it $(\mathcal{B}^+_{x_0}(w))_{w\in\widetilde{\mathrm{W}}}$ because it coincides on $\widetilde{\mathrm{X}}_*(\mathrm{T})$ with the definition introduced in Section 2A (see also Remark 2.1). Recall that we have a decomposition of $\widetilde{\mathrm{W}}$ as the semidirect product

$$\widetilde{\mathrm{W}} = \mathrm{X}_*(\mathrm{T}) \rtimes \widetilde{\mathfrak{W}}.$$

For $w_0 \in \widetilde{\mathfrak{W}}$, set $\mathcal{B}^+_{x_0}(w_0) = \tau_{w_0}$ and for $w = e^\lambda w_0 \in \mathrm{X}_*(\mathrm{T}) \rtimes \widetilde{\mathfrak{W}}$, define in $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$

$$\mathcal{B}^+_{x_0}(w) = q^{(\ell(w)-\ell(w_0)-\ell(e^\lambda))/2}\mathcal{B}^+_{x_0}(\lambda)\mathcal{B}^+_{x_0}(w_0) = q^{(\ell(w)-\ell(w_0))/2}\Theta^+_{x_0}(\lambda)\tau_{w_0}.$$

By [Vignéras 2005, Theorem 2 and Proposition 8], this element lies in $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ and the set of all $(\mathcal{B}^+_{x_0}(w))_{w\in\widetilde{\mathrm{W}}}$ is a $\mathbb{Z}$-basis for $\widetilde{\mathrm{H}}_{\mathbb{Z}}$.

Remark 5.1. As a $\mathbb{Z}$-module, $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is the direct sum of $\mathcal{A}^+_{x_0}$ and of the $\mathbb{Z}$-module with basis $(\mathcal{B}^+_{x_0}(e^\lambda w_0))$, where λ ranges over $\mathrm{X}_*(\mathrm{T})$ and w_0 over the set of elements in $\widetilde{\mathfrak{W}}$ the projection of which in $\mathfrak{W}$ is nontrivial. Applying (2-1), we obtain that the $\mathbb{Z}$-module $\mathcal{A}^+_C$ is a direct summand of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ as well.

Remark 5.2. Let $d \in \mathcal{D}$ and $\tilde{d} \in \widetilde{\mathrm{W}}$ be a lift for d. Write $\tilde{d} = e^\lambda w_0$ with $w_0 \in \widetilde{\mathfrak{W}}$, $\lambda \in \mathrm{X}^+_*(\mathrm{T})$ and $\ell(e^\lambda) = \ell(d) + \ell(w_0)$ (Proposition 1.5).

Then in $\widetilde{\mathrm{H}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$, we have

$$\mathcal{B}^+_{x_0}(\tilde{d}) = q^{(\ell(\tilde{d})-\ell(w_0)+\ell(e^\lambda))/2}\tau^{-1}_{e^{-\lambda}}\tau_{w_0} = q^{\ell(\tilde{d})}\tau^{-1}_{\tilde{d}^{-1}} = (-1)^{\ell(d)}\iota(\tau_{\tilde{d}}). \quad (5\text{-}1)$$

5B. ***Topology on the pro-p Iwahori–Hecke algebra in characteristic p.*** We consider the (finitely generated) ideal $\mathfrak{I}$ of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ generated by all z_λ for $\lambda \in \mathrm{X}^+_*(\mathrm{T})$ such that $\ell(e^\lambda) > 0$, and the associated ring filtration of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$. A $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$-module M can be endowed with the $\mathfrak{I}$-adic topology induced by the filtration

$$M \supseteq M\mathfrak{I} \supseteq M\mathfrak{I}^2 \supseteq \cdots.$$

An example of such a module is $\widetilde{\mathrm{H}}_k$ itself. We define on $\widetilde{\mathrm{H}}_k$ another decreasing filtration $(F_n\widetilde{\mathrm{H}}_k)_{n\in\mathbb{N}}$ by k-vector spaces, where

$$F_n\widetilde{\mathrm{H}}_k := \text{the } k\text{-vector space generated by } \mathcal{B}^+_{x_0}(w),\ w\in\widetilde{\mathrm{W}} \text{ with } \ell(w)\geq n. \quad (5\text{-}2)$$

Lemma 5.3. *The filtration* (5-2) *is a filtration of* $\widetilde{\mathrm{H}}_k$ *as a left* $\mathcal{A}^+_{x_0}$*-module. In particular, it is a filtration of* $\widetilde{\mathrm{H}}_k$ *as a (left and right)* $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$*-module. It is compatible with the* $\mathfrak{I}$*-filtration: for all* $n\in\mathbb{N}$*, we have*

$$(F_n\widetilde{\mathrm{H}}_k)\mathfrak{I} = \mathfrak{I}\,(F_n\widetilde{\mathrm{H}}_k)\ \subseteq F_{n+1}\widetilde{\mathrm{H}}_k.$$

Proof. Let $\lambda\in\widetilde{\mathrm{X}}_*(\mathrm{T})$ and $w\in\widetilde{\mathrm{W}}$. From the definition of $\mathcal{B}^+_{x_0}$, we see that

$$\mathcal{B}^+_{x_0}(\lambda)\mathcal{B}^+_{x_0}(w) = q^{(\ell(e^\lambda)+\ell(w)-\ell(e^\lambda w))/2}\mathcal{B}^+_{x_0}(e^\lambda w)$$

and therefore in $\widetilde{\mathrm{H}}_k$ we have $\mathcal{B}^+_{x_0}(\lambda)\mathcal{B}^+_{x_0}(w)=0$ if $\ell(e^\lambda)+\ell(w)>\ell(e^\lambda w)$ and $\mathcal{B}^+_{x_0}(\lambda)\mathcal{B}^+_{x_0}(w)=\mathcal{B}^+_{x_0}(e^\lambda w)$ if $\ell(w)+\ell(e^\lambda)=\ell(e^\lambda w)$. This proves the claims. □

Proposition 5.4. *The* $\mathfrak{I}$*-adic topology on* $\widetilde{\mathrm{H}}_k$ *is equivalent to the topology on* $\widetilde{\mathrm{H}}_k$ *induced by the filtration* $(F_n\widetilde{\mathrm{H}}_k)_{n\in\mathbb{N}}$*. In particular, it is independent of the choice of the uniformizer* ϖ*.*

Proof. We have to prove that given $m\in\mathbb{N}$, $m\geq 1$, there is $n\in\mathbb{N}$ such that $F_n\widetilde{\mathrm{H}}_k\subseteq\mathfrak{I}^m\widetilde{\mathrm{H}}_k$.

Fact iii. *For* $\lambda\in\mathrm{X}_*(\mathrm{T})$ *such that* $\ell(e^\lambda)>0$ *and* $m\geq 1$*, we have* $\mathcal{B}^+_{x_0}((m+1)\lambda)\in\mathfrak{I}^m\widetilde{\mathrm{H}}_k$.

Proof. We check that for $m\in\mathbb{N}$ we have $\mathcal{B}^+_{x_0}((m+1)\lambda)=z_\lambda^m\mathcal{B}^+_{x_0}(\lambda)$. Notice that $\mathcal{B}^+_{x_0}(2\lambda)=\mathcal{B}^+_{x_0}(\lambda)\mathcal{B}^+_{x_0}(\lambda)=z_\lambda\mathcal{B}^+_{x_0}(\lambda)$ by (2-4) and Lemma 3.4. Now let $m\geq 2$. We have $\mathcal{B}^+_{x_0}((m+1)\lambda)=\mathcal{B}^+_{x_0}(m\lambda)\mathcal{B}^+_{x_0}(\lambda)=z_\lambda^m\mathcal{B}^+_{x_0}(\lambda)$ by induction. □

Fact iv. *Let* $m\geq 1$*. There is* $A_m\in\mathbb{N}$ *such that for any* $\lambda\in\mathrm{X}_*(\mathrm{T})$*, if* $\ell(e^\lambda)>A_m$ *then* $\mathcal{B}^+_{x_0}(\lambda)\in\mathfrak{I}^m\widetilde{\mathrm{H}}_k$.

Proof. Let $\{z_{\lambda_1},\dots,z_{\lambda_r}\}$ be a system of generators of $\mathfrak{I}$ with $\lambda_1,\dots,\lambda_r\in\mathrm{X}_*^+(\mathrm{T})$. Let $A_m:=m\sum_{i=1}^r\ell(e^{\lambda_i})$. Let $\lambda\in\mathrm{X}_*(\mathrm{T})$ such that $\ell(e^\lambda)>0$. This is $\mathfrak{W}$-conjugate to an element $\lambda_0\in\mathrm{X}_*^+(\mathrm{T})$, and one can write $\lambda=w_0.\lambda_0$ with $w_0\in\mathfrak{W}$ and $\lambda_0=\sum_{i=1}^r a_i\lambda_i$ with $a_i\in\mathbb{N}$ (not all equal to zero). If $\ell(e^\lambda)=\ell(e^{\lambda_0})>A_m$, then there is $i_0\in\{1,\dots,r\}$ such that $a_{i_0}>m$ and $\mathcal{B}^+_{x_0}(\lambda)=\prod_{i=1}^r\mathcal{B}^+_{x_0}(a_i(w_0.\lambda_i))\in\mathcal{B}^+_{x_0}((m+1)(w_0.\lambda_{i_0}))\widetilde{\mathrm{H}}_k\subseteq\mathfrak{I}^m\widetilde{\mathrm{H}}_k$ by Fact iii. □

We now turn to the proof of the proposition. Let $m\geq 1$. To any $w_0\in\mathfrak{W}$ corresponds, by [Vignéras 2006, (1.6.3)], a finite set $\mathrm{X}(w_0)$ of elements in $\mathrm{X}_*(\mathrm{T})$ such that

$$\text{for all } \lambda\in\mathrm{X}_*(\mathrm{T}) \text{ there is } \mu\in\mathrm{X}(w_0) \text{ such that } \ell(e^\lambda w_0)=\ell(e^{\lambda-\mu})+\ell(e^\mu w_0).$$

Let $\tilde{w} \in \widetilde{\mathrm{W}}$ with image w_0 under the projection $\widetilde{\mathrm{W}} \to \mathfrak{W}$. Its image w under $\widetilde{\mathrm{W}} \to \mathrm{W}$ has the form $w = e^{\lambda} w_0 \in \mathrm{X}_*(\mathrm{T}) \rtimes \mathfrak{W}$, and there is $\mu \in \mathrm{X}(w_0)$ such that $\ell(w) = \ell(e^{\lambda-\mu}) + \ell(e^{\mu} w_0)$. Choose lifts $\widetilde{e}^{\mu} w_0$ and $\widetilde{e}^{\lambda-\mu}$ in $\widetilde{\mathrm{W}}$ for $e^{\mu} w_0$ and $e^{\lambda-\mu}$. The product $\widetilde{e}^{\lambda-\mu}\widetilde{e}^{\mu} w_0$ differs from $\tilde{w}$ by an element in $\mathrm{T}^0/\mathrm{T}^1$ (which has length zero). Therefore, $\mathcal{B}^+_{x_0}(\tilde{w}) \in \mathcal{B}^+_{x_0}(\lambda-\mu)\widetilde{\mathrm{H}}_k$ (see the proof of Lemma 5.3, for example). If $\ell(\tilde{w}) > A_m(w_0) := A_m + \max\{\ell(e^{\mu'} w_0),\ \mu' \in \mathrm{X}(w_0)\}$ then $\ell(e^{\lambda-\mu}) > A_m$ and $\mathcal{B}^+_{x_0}(\tilde{w}) \in \mathfrak{I}^m \widetilde{\mathrm{H}}_k$ by Fact iv.

We have proved that $n > \max\{A_m(w_0),\ w_0 \in \mathfrak{W}\}$ implies $F_n \widetilde{\mathrm{H}}_k \subseteq \mathfrak{I}^m \widetilde{\mathrm{H}}_k$. □

5C. ***The category of modules of finite length over the pro-p Iwahori–Hecke algebra in characteristic p.*** We consider the abelian category $\mathrm{Mod}_{fg}(\widetilde{\mathrm{H}}_k)$ of all $\widetilde{\mathrm{H}}_k$-modules with finite length.

For an $\widetilde{\mathrm{H}}_k$-module, having finite length is equivalent to being finite-dimensional as a k-vector space (see [Vignéras 2007, Section 5.3] or [Ollivier and Schneider 2012, Lemma 6.9]). Therefore, any irreducible $\widetilde{\mathrm{H}}_k$-module is finite dimensional and has a central character, and any module in $\mathrm{Mod}_{fg}(\widetilde{\mathrm{H}}_k)$ decomposes uniquely into a direct sum of indecomposable modules.

5C1. *The category of finite-dimensional $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$-modules.* Let $\mathrm{Mod}_{fd}(\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k))$ denote the category of finite-dimensional $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$-modules. For $\mathfrak{M}$ a maximal ideal of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$, we consider the full subcategory

$$\mathfrak{M}\text{-}\,\mathrm{Mod}_{fd}(\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k))$$

of modules M of $\mathfrak{M}$-torsion, that is, such that there is $e \in \mathbb{N}$ satisfying $M\mathfrak{M}^e = 0$. The category $\mathrm{Mod}_{fd}(\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k))$ decomposes into the direct sum

$$\bigoplus_{\mathfrak{M}} \mathfrak{M}\text{-}\,\mathrm{Mod}_{fd}(\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)),$$

where $\mathfrak{M}$ ranges over the maximal ideals of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$.

5C2. *Blocks of $\widetilde{\mathrm{H}}_k$-modules with finite length.* For $\mathfrak{M}$ a maximal ideal of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$, we say that an $\widetilde{\mathrm{H}}_k$-module with finite length is an $\mathfrak{M}$-torsion module if its restriction to a $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$-module lies in the subcategory $\mathfrak{M}\text{-}\,\mathrm{Mod}_{fd}(\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k))$. We denote by

$$\mathfrak{M}\text{-}\,\mathrm{Mod}_{fg}(\widetilde{\mathrm{H}}_k) \tag{5-3}$$

the full subcategory of $\mathrm{Mod}_{fg}(\widetilde{\mathrm{H}}_k)$ whose objects are the $\mathfrak{M}$-torsion modules.

Lemma 5.5. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be two maximal ideals of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$. If there is a nonzero $\mathfrak{M}$-torsion module M and a nonzero $\mathfrak{N}$-torsion module N such that $\mathrm{Ext}^r_{\widetilde{\mathrm{H}}_k}(M, N) \neq 0$ for some $r \geq 0$, then $\mathfrak{M} = \mathfrak{N}$.*

Proof. For any $\widetilde{\mathrm{H}}_k$-modules X and Y, the natural morphisms of algebras $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k) \to \mathrm{End}_{\widetilde{\mathrm{H}}_k}(X)$ and $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k) \to \mathrm{End}_{\widetilde{\mathrm{H}}_k}(Y)$ equip $\mathrm{Hom}_{\widetilde{\mathrm{H}}_k}(X, Y)$ with the structure of a

central $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$-bimodule. The space $\mathrm{Ext}^r_{\widetilde{\mathrm{H}}_k}(M, N)$ is therefore naturally a central $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$-bimodule. It is an $\mathfrak{M}$-torsion module and an $\mathfrak{N}$-torsion module; it is zero unless $\mathfrak{M} = \mathfrak{N}$. □

Since $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ is a central finitely generated subalgebra of $\widetilde{\mathrm{H}}_k$, an indecomposable $\widetilde{\mathrm{H}}_k$-module with finite length is an $\mathfrak{M}$-torsion module for some maximal ideal $\mathfrak{M}$ of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$.

Remark 5.6. An $\widetilde{\mathrm{H}}_k$-module with finite length M lies in the block corresponding to some maximal ideal $\mathfrak{M}$ if and only if all the characters of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ contained in M have kernel $\mathfrak{M}$.

Remark 5.7. The blocks (5-3) are not indecomposable. They can for example be further decomposed via the idempotents introduced in Section 1B8.

5C3. *The supersingular block.*

Definition 5.8. We call a maximal ideal $\mathfrak{M}$ of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ supersingular if it contains the ideal $\mathfrak{I}$ defined in Section 5B. A character of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ is called supersingular if its kernel is a supersingular maximal ideal of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$.

Given a character ω of the connected center Z of G, there is a unique supersingular character ζ_ω of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ satisfying $\zeta_\omega(z_\lambda) = \omega(\lambda(\varpi))$ for any $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ with length zero. A character of the center of $\widetilde{\mathrm{H}}_k$ is called "null" in [Vignéras 2005] if it takes value zero at all central elements (2-2) for all $\mathfrak{W}$-orbits $\mathbb{O}$ in $\widetilde{\mathrm{X}}_*(\mathrm{T})$ containing a coweight with nonzero length.

Lemma 5.9. *A character $\mathscr{Z}(\widetilde{\mathrm{H}}_k) \to k$ is null if and only if its restriction to $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ is a supersingular character in the sense of Definition 5.8.*

Proof. Consider a character $\zeta : \mathscr{Z}(\widetilde{\mathrm{H}}_k) \to k$ whose restriction to $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ is supersingular. We want to prove that ζ is null. Since the $\widetilde{\mathrm{H}}_k$-module $\widetilde{\mathrm{H}}_k \otimes_{\mathscr{Z}(\widetilde{\mathrm{H}}_k)} \zeta$ is finite dimensional, it contains a character $\hat{\zeta}$ for the commutative finitely generated k-algebra $(\mathcal{A}^+_{x_0})_k$ and the restriction of $\hat{\zeta}$ to $\mathscr{Z}(\widetilde{\mathrm{H}}_k)$ coincides with ζ.

Let $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ with $\ell(e^\lambda) \neq 0$; by (2-4), there is at most one $\mathfrak{W}$-conjugate λ' of λ such that $\hat{\zeta}(\mathcal{B}^+_{x_0}(\lambda')) \neq 0$, and if there exists such a λ', then $\hat{\zeta}(z_\lambda) = \zeta(z_\lambda) \neq 0$, which is a contradiction; we have proved that $\hat{\zeta}(\mathcal{B}^+_{x_0}(\lambda')) = 0$ for all $\lambda' \in \mathrm{X}_*(\mathrm{T})$ with $\ell(e^{\lambda'}) \neq 0$, which implies that this is also the case for $\lambda' \in \widetilde{\mathrm{X}}_*(\mathrm{T})$ with $\ell(e^{\lambda'}) \neq 0$. Therefore, ζ is null. □

A finite-dimensional $\widetilde{\mathrm{H}}_k$-module M with central character is called supersingular in [Vignéras 2005] if this central character is null. We extend this definition:

Proposition-Definition 5.10. *A finite-length $\widetilde{\mathrm{H}}_k$-module is in the supersingular block and is called* supersingular *if and only if, equipped with the discrete topology, it is a continuous module for the $\mathfrak{I}$-adic topology on $\widetilde{\mathrm{H}}_k$ or, equivalently, for the topology induced by the filtration* (5-2).

Proof. An indecomposable $\widetilde{\mathrm{H}}_k$-module M with finite length is in the supersingular block if and only if there is $m \geq 1$ such that $M\mathfrak{I}^m = \{0\}$. Then use Proposition 5.4. □

5D. ***Classification of the simple supersingular modules over the pro-p Iwahori–Hecke algebra in characteristic p.*** We establish this classification in the case where the root system of G is irreducible, which we will suppose in Section 5D4. Until then the results are valid without further assumption on the root system.

5D1. Denote by $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ the natural image in $\widetilde{\mathrm{H}}_k$ of the affine Hecke subring $\widetilde{\mathrm{H}}_{\mathbb{Z}}^{\mathrm{aff}}$ of $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ defined in Section 1B7. We generalize [Ollivier 2010, Theorem 7.3]:

Proposition 5.11. *A finite-length $\widetilde{\mathrm{H}}_k$-module in the supersingular block contains a character for the affine Hecke subalgebra $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$.*

Proof. Let M be an $\widetilde{\mathrm{H}}_k$-module with finite length in the supersingular block. By Proposition-Definition 5.10, there is $n \in \mathbb{N}$ such that for any $w \in \widetilde{\mathrm{W}}$, if $\ell(w) > n$ then $M\mathcal{B}_{x_0}^+(w) = 0$. Let $x \in M$, and suppose that it supports a character for $\widetilde{\mathfrak{H}}_k$ (see Section 1B9) and let $d \in \mathcal{D}$ with maximal length such that $x\mathcal{B}_{x_0}^+(\tilde{d}) \neq 0$, where $\tilde{d} \in \widetilde{\mathrm{W}}$ denotes a lift for d (the property $x\mathcal{B}_{x_0}^+(\tilde{d}) \neq 0$ does not depend on the choice of the lift $\tilde{d}$). As in the proof of [Ollivier 2010, Theorem 7.3], we prove that $x' := x\mathcal{B}_{x_0}^+(\tilde{d})$ supports a character for $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ which is the k-algebra generated by all τ_t and all $\tau_{\tilde{s}}$ for $t \in \mathrm{T}^0/\mathrm{T}^1$ and $s \in S_{\mathrm{aff}}$ with chosen lift $\tilde{s} \in \widetilde{\mathrm{W}}$ (see paragraph Section 1B7). From the relations (1-11) we get that $x'\tau_t = x\tau_{dtd^{-1}}\mathcal{B}_{x_0}^+(\tilde{d})$ is proportional to x'. Now let $s \in S_{\mathrm{aff}}$. If $\ell(ds) = \ell(d) - 1$, then $ds \in \mathcal{D}$ by Proposition 1.5 and, by (5-1), the element x' is equal to $x\iota(\tau_{\tilde{d}\tilde{s}})\iota(\tau_{\tilde{s}})$ (up to an invertible element in k), so $x'\tau_{\tilde{s}} = 0$ by Remark 1.10. If $\ell(ds) = \ell(d) + 1$ and $ds \in \mathcal{D}$, then $x\mathcal{B}_{x_0}^+(\tilde{d}\tilde{s})$ is equal to zero on one side and, by (5-1), to $x'\iota(\tau_{\tilde{s}})$ (up to an invertible element in k) on the other side. This proves that $x'\tau_{\tilde{s}}$ is proportional to x' by Remark 1.10. If $\ell(ds) = \ell(d) + 1$ and $ds \notin \mathcal{D}$ then there is $s' \in S$ such that $ds = s'd$ by Proposition 1.5, and $x'\iota(\tau_{\tilde{s}})$ is proportional to $x\iota(\tau_{\tilde{s}'})\mathcal{B}_{x_0}^+(\tilde{d})$ and therefore to x' because $\iota(\tau_{\tilde{s}'}) \in \widetilde{\mathfrak{H}}_k$. We conclude that $x'\tau_{\tilde{s}}$ is proportional to x' by Remark 1.10. □

5D2. *Characters of* $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$. We call a morphism of k-algebras $\widetilde{\mathrm{H}}_k^{\mathrm{aff}} \to k$ a character of $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$. A character $\mathcal{X}$ of $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ is completely determined by:

- The unique $\xi \in \widehat{\overline{\mathbf{T}}}(\mathbb{F}_q)$ such that $\mathcal{X}(\epsilon_\xi) = 1$ (see notation in Section 1B8). This ξ is defined by $\xi(t) = \mathcal{X}(\tau_t)$, where $t \in \mathrm{T}^0/\mathrm{T}^1 = \overline{\mathbf{T}}(\mathbb{F}_q)$, and we call ξ the restriction of $\mathcal{X}$ to $k[\mathrm{T}^0/\mathrm{T}^1]$.
- The values $\mathcal{X}(\tau_{n_A})$ for all $A \in S_{\mathrm{aff}}$, which, by the quadratic relations (1-15) satisfy $\mathcal{X}(\tau_{n_A}) \in \{0, -1\}$, if ξ is trivial on T_A, and $\mathcal{X}(\tau_{n_A}) = 0$ otherwise.

Conversely, one checks that any such datum of $\xi \in \hat{\widetilde{\mathbf{T}}}(\mathbb{F}_q)$ and values $\mathcal{X}(\tau_{n_A})$ for all $A \in S_{\mathrm{aff}}$ satisfying the above conditions defines a character $\mathcal{X}$ of $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$.

Example. The pro-p Iwahori–Hecke ring $\widetilde{\mathrm{H}}_{\mathbb{Z}}$ is endowed with two natural morphisms of rings $\widetilde{\mathrm{H}}_{\mathbb{Z}} \to \mathbb{Z}$ defined by

$$\tau_w \mapsto q^{\ell(w)} \quad \text{and} \quad \tau_w \mapsto (-1)^{\ell(w)}.$$

We denote by $\mathcal{X}_{\mathrm{triv}}$ and $\mathcal{X}_{\mathrm{sign}}$ the characters of $\widetilde{\mathrm{H}}_k$ that they respectively induce, as well as their restrictions to characters of $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$. The former can be described by $\xi = \mathbf{1}$ and $\mathcal{X}_{\mathrm{triv}}(\tau_{n_A}) = 0$ for all $A \in S_{\mathrm{aff}}$, the latter by $\xi = \mathbf{1}$ and $\mathcal{X}_{\mathrm{sign}}(\tau_{n_A}) = -1$ for all $A \in S_{\mathrm{aff}}$.

Let $\mathcal{X}$ be a character of $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ and ξ the corresponding element in $\hat{\widetilde{\mathbf{T}}}(\mathbb{F}_q)$.

- Let $\xi_0 \in \hat{\widetilde{\mathbf{T}}}(\mathbb{F}_q)$, and suppose that ξ_0 is trivial on T_α for all $\alpha \in \Pi$. Then one can consider the twist $(\xi_0)\mathcal{X}$ of $\mathcal{X}$ by ξ_0 in the obvious way. The restriction of $(\xi_0)\mathcal{X}$ to $k[\mathrm{T}^0/\mathrm{T}^1]$ is the product $\xi_0\xi$, and $(\xi_0)\mathcal{X}$ coincides with $\mathcal{X}$ on the elements of type τ_{n_A} for $A \in S_{\mathrm{aff}}$. By a *twist of the character* $\mathcal{X}$, we mean from now on a twist of $\mathcal{X}$ by an element in $\hat{\widetilde{\mathbf{T}}}(\mathbb{F}_q)$ that is trivial on T_α for all $\alpha \in \Pi$.
- The involution ι_C extends to an involution of the k-algebra $\widetilde{\mathrm{H}}_k$. The composition $\mathcal{X} \circ \iota_C$ is then also a character for $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$. Note that $\mathcal{X}$ and $\mathcal{X} \circ \iota_C$ have the same restriction to $k[\mathrm{T}^0/\mathrm{T}^1]$ (Remark 1.9). Furthermore, if $\mathcal{X}(\tau_{n_A}) = -1$ for some $A \in S_{\mathrm{aff}}$, then $\mathcal{X} \circ \iota_C(\tau_{n_A}) = 0$ (use Remark 1.10). For example, $\mathcal{X}_{\mathrm{triv}} = \mathcal{X}_{\mathrm{sign}} \circ \iota_C$.
- There is an action of $\widetilde{\Omega}$ by conjugacy on $\widetilde{\mathrm{W}}_{\mathrm{aff}}$. Since the elements in $\widetilde{\Omega}$ have length zero, this yields an action of $\widetilde{\Omega}$ on $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ and its characters. For $\omega \in \widetilde{\Omega}$, we denote by $\omega.\mathcal{X}$ the character $\mathcal{X}(\tau_{\omega^{-1}} \,.\, \tau_\omega)$.

Lemma 5.12. *A simple $\widetilde{\mathrm{H}}_k$-module containing a twist of the character $\mathcal{X}_{\mathrm{triv}}$ or of the character $\mathcal{X}_{\mathrm{sign}}$ of $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ is not supersingular.*

Proof. Let M be a simple $\widetilde{\mathrm{H}}_k$-module. Suppose that it contains a twist of the character $\mathcal{X}_{\mathrm{sign}}$ supported by the nonzero vector $m \in M$. In particular, m supports the character of $\widetilde{\mathfrak{H}}_k$ parametrized by (a twist of) the trivial character of $\hat{\widetilde{\mathbf{T}}}(\mathbb{F}_q)$ and by the facet C (see Section 1B9). By Remark 4.2, we have

$$m z_\lambda = m\mathcal{B}_C^+(\lambda)$$

for all $\lambda \in X_*^+(\mathrm{T})$. There are $\omega \in \widetilde{\Omega}$ and $w \in \widetilde{\mathrm{W}}_{\mathrm{aff}}$ such that $\lambda(\varpi^{-1}) \mod \mathrm{T}^1$ corresponds to $w\omega \in \widetilde{\mathrm{W}}$. Since $\mathcal{B}_C^+(\lambda) = \tau_{\lambda(\varpi^{-1})}$, the element $m\mathcal{B}_C^+(\lambda)$ is equal to $(-1)^{\ell(w)} m\tau_\omega$ (up to multiplication by an element in $k^\times$), and we recall that τ_ω is invertible in $\widetilde{\mathrm{H}}_k$. We have proved that $m.z_\lambda \neq 0$ and M is not supersingular.

Now if M contains a twist of the character $\mathcal{X}_{\text{triv}}$, then $\iota_C^* M$ contains a twist of the character $\mathcal{X}_{\text{sign}}$ and is not supersingular (notation in the proof of Proposition 3.3). By Proposition 3.2, this implies that M is not supersingular either. □

5D3. Consider the image of $\widetilde{\Omega}$ in $\widetilde{\mathrm{H}}_k$ via $\omega \mapsto \tau_\omega$. For $\mathcal{X}$ a character of $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$, denote by $\widetilde{\Omega}_{\mathcal{X}}$ its fixator under the action of $\widetilde{\Omega}$; obviously $\widetilde{\Omega}_{\mathcal{X}}$ contains $\mathrm{T}^0/\mathrm{T}^1$ as a subgroup. We consider the set $\mathcal{P}$ of pairs $(\mathcal{X}, \sigma)$ where $\mathcal{X}$ is a character of $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ and $(\sigma, \mathrm{V}_\sigma)$ an irreducible finite-dimensional k-representation of $\widetilde{\Omega}_{\mathcal{X}}$ (up to isomorphism) whose restriction to $\mathrm{T}^0/\mathrm{T}^1$ coincides with the inverse of the restriction of $\mathcal{X}$; for any $t \in \mathrm{T}^0/\mathrm{T}^1$ and $v \in \mathrm{V}_\sigma$, we have $\sigma(t)v = \mathcal{X}(\tau_{t^{-1}})v$.

The set $\mathcal{P}$ is naturally endowed with an action of $\widetilde{\Omega}$: for $(\mathcal{X}, \sigma) \in \mathcal{P}$ and $\omega \in \widetilde{\Omega}$, denote by $\omega.\sigma$ the representation of $\widetilde{\Omega}_{\omega.\mathcal{X}} = \omega \widetilde{\Omega}_{\mathcal{X}} \omega^{-1}$ naturally obtained by conjugating σ; then $\omega.(\mathcal{X}, \sigma) := (\omega.\mathcal{X}, \omega.\sigma) \in \mathcal{P}$.

Let $(\mathcal{X}, \sigma) \in \mathcal{P}$. Consider the subalgebra $\widetilde{\mathrm{H}}_k(\mathcal{X})$ of $\widetilde{\mathrm{H}}_k$ generated by $k[\widetilde{\Omega}_{\mathcal{X}}]$ and $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$. It is isomorphic to the twisted tensor product of algebras

$$\widetilde{\mathrm{H}}_k(\mathcal{X}) \simeq k[\widetilde{\Omega}_{\mathcal{X}}] \otimes_{k[\mathrm{T}^0/\mathrm{T}^1]} \widetilde{\mathrm{H}}_k^{\mathrm{aff}},$$

where the product is given by $(\omega \otimes h)(\omega' \otimes h') = \omega\omega' \otimes \tau_{\omega'}^{-1} h \tau_{\omega'} h'$. As a left $\widetilde{\mathrm{H}}_k(\mathcal{X})$-module, $\widetilde{\mathrm{H}}_k$ is free with basis the set of all τ_ω, where ω ranges over a set of representatives of the right cosets $\widetilde{\Omega}_{\mathcal{X}} \backslash \widetilde{\Omega}$. The tensor product $\sigma \otimes \mathcal{X}$ is naturally a right $\widetilde{\mathrm{H}}_k(\mathcal{X})$-module: the right action of $\omega \otimes h$ on $v \in \mathrm{V}_\sigma$ is given by $\mathcal{X}(h)\sigma(\omega^{-1})v$. The right $\widetilde{\mathrm{H}}_k(\mathcal{X})$-module $\sigma \otimes \mathcal{X}$ is irreducible. As an $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$-module, it is isomorphic to a direct sum of copies of $\mathcal{X}$.

Lemma 5.13. *The isomorphism classes of the simple $\widetilde{\mathrm{H}}_k$-modules containing a character for $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ are represented by the induced modules*

$$\mathfrak{m}(\mathcal{X}, \sigma) := (\sigma \otimes \mathcal{X}) \otimes_{\widetilde{\mathrm{H}}_k(\mathcal{X})} \widetilde{\mathrm{H}}_k,$$

where $(\mathcal{X}, \sigma)$ ranges over the set of orbits in $\mathcal{P}$ under the action of $\widetilde{\Omega}$.

Proof. First note that for any $\omega \in \widetilde{\Omega}$, the $(\widetilde{\mathrm{H}}_k^{\mathrm{aff}}, \omega.\mathcal{X})$-isotypic component of $\mathfrak{m}(\mathcal{X}, \sigma)$ is isomorphic to $\omega.\sigma \otimes \omega\mathcal{X}$ as a right $\widetilde{\mathrm{H}}_k(\omega.\mathcal{X})$-module.

(1) We check that an $\widetilde{\mathrm{H}}_k$-module of the form $\mathfrak{m}(\mathcal{X}, \sigma)$ is irreducible. Restricted to $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$, it is semisimple and isomorphic to a direct sum of $\mathcal{X}$ and of its conjugates. Therefore, a submodule $\mathfrak{m}$ of $\mathfrak{m}(\mathcal{X}, \sigma)$ contains a nonzero $(\widetilde{\mathrm{H}}_k^{\mathrm{aff}}, \omega.\mathcal{X})$-isotypic vector for some $\omega \in \widetilde{\Omega}$, and, after translating by $\tau_{\omega^{-1}}$, we see that $\mathfrak{m}$ contains a nonzero $(\widetilde{\mathrm{H}}_k^{\mathrm{aff}}, \mathcal{X})$-isotypic vector. But the $(\widetilde{\mathrm{H}}_k^{\mathrm{aff}}, \mathcal{X})$-isotypic component in $\mathfrak{m}(\mathcal{X}, \sigma)$ supports the irreducible representation σ of $k[\widetilde{\Omega}_{\mathcal{X}}]$. Therefore $\mathfrak{m} = \mathfrak{m}(\mathcal{X}, \sigma)$.

(2) Let $\mathfrak{m}$ be a simple $\widetilde{\mathrm{H}}_k$-module containing the character $\mathcal{X}$ of $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$. Its $(\widetilde{\mathrm{H}}_k^{\mathrm{aff}}, \mathcal{X})$-isotypic component contains an irreducible (finite-dimensional) representation σ of

$k[\widetilde{\Omega}_{\mathcal{X}}]$ which coincides with the inverse of $\mathcal{X}$ on $k[\mathrm{T}^0/\mathrm{T}^1]$. Therefore, using (1), $\mathfrak{m} \simeq (\sigma \otimes \mathcal{X}) \otimes_{\widetilde{\mathrm{H}}_k(\mathcal{X})} \widetilde{\mathrm{H}}_k$.

(3) Let $\omega \in \widetilde{\Omega}$ and $(\mathcal{X}, \sigma) \in \mathcal{P}$. The $(\widetilde{\mathrm{H}}_k^{\mathrm{aff}}, \mathcal{X})$-isotypic component of $\mathfrak{m}(\omega.(\mathcal{X}, \sigma))$ contains the representation σ of $k[\widetilde{\Omega}_{\mathcal{X}}]$. The simple $\widetilde{\mathrm{H}}_k$-module $\mathfrak{m}(\omega.(\mathcal{X}, \sigma))$ is therefore isomorphic to $\mathfrak{m}(\mathcal{X}, \sigma)$ by (2).

(4) Let $(\mathcal{X}, \sigma)$ and $(\mathcal{X}', \sigma')$ be in $\mathcal{P}$ and suppose that they induce isomorphic $\widetilde{\mathrm{H}}_k$-modules. Looking at the restriction of the latter to $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$, we see that there is $\omega \in \widetilde{\Omega}$ such that $\mathcal{X}' = \omega.\mathcal{X}$.

Therefore, by (3), $\mathfrak{m}(\mathcal{X}, \omega^{-1}\sigma')$ and $\mathfrak{m}(\mathcal{X}, \sigma)$ are isomorphic, and looking at the restriction to the $(\widetilde{\mathrm{H}}_k^{\mathrm{aff}}, \mathcal{X})$-isotypic component shows that $\sigma' \simeq \omega.\sigma$. Therefore, $(\mathcal{X}', \sigma')$ and $(\mathcal{X}, \sigma)$ are conjugate. □

5D4. *Classification of the simple supersingular $\widetilde{\mathrm{H}}_k$-modules when the root system of* G *is irreducible.* We generalize [Vignéras 2005, Theorem 5(1)] and [Ollivier 2010, Theorem 7.3].

Theorem 5.14. *Suppose that the root system of* G *is irreducible. A simple $\widetilde{\mathrm{H}}_k$-module is supersingular if and only if it contains a character for $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ that is different from a twist of $\mathcal{X}_{\mathrm{triv}}$ or $\mathcal{X}_{\mathrm{sign}}$.*

Remark 5.15. This proves in particular (if the root system of G is irreducible) that the notion of supersingularity for Hecke modules does not depend on any of the choices made.

Proof of Theorem 5.14. We already proved in Proposition 5.11 (without restriction on the root system of G) that a simple supersingular module contains a character for $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$, and by Lemma 5.12 we know that this character is not a twist of $\mathcal{X}_{\mathrm{triv}}$ or $\mathcal{X}_{\mathrm{sign}}$.

Conversely, let $\mathfrak{m}$ be a simple $\widetilde{\mathrm{H}}_k$-module containing the character $\mathcal{X}$ for $\widetilde{\mathrm{H}}_k^{\mathrm{aff}}$ and suppose that $\mathcal{X}$ is not a twist of $\mathcal{X}_{\mathrm{triv}}$ or $\mathcal{X}_{\mathrm{sign}}$. We want to prove that $\mathfrak{m}$ is supersingular. Since, by Proposition 3.2, this is equivalent to showing that $\iota_C^*\mathfrak{m}$ is supersingular (notation in the proof of Proposition 3.3), we can suppose (see the discussion before Lemma 5.12) that $\mathcal{X}(\tau_{n_0}) = 0$, where n_0 was introduced in Section 2A3.

Let $m \in \mathfrak{m}$ be a nonzero vector supporting $\mathcal{X}$. Let χ be the restriction of $\mathcal{X}$ to $\widetilde{\mathfrak{I}}_k$ and F_χ the associated standard facet. Suppose that $F_\chi = x_0$; then $\Pi_{\bar{\chi}} = \Pi_\chi = \Pi$ (notation in Section 1B9) and $\mathcal{X}(\tau_{n_\alpha}) = 0$ for all $\alpha \in \Pi$. Since, by hypothesis, we also have $\mathcal{X}(\tau_{n_0}) = 0$, the character $\mathcal{X}$ is equal to $\mathcal{X}_{\mathrm{triv}}$ up to twist. Therefore, $F_\chi \neq x_0$. Let $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ with $\ell(e^\lambda) > 0$. By Remark 4.2,

$$m.z_\lambda = m.\mathcal{B}^+_{F_\chi}(\lambda),$$

and, since $F_\chi \neq x_0$, we have $m.z_\lambda = 0$ by Lemma 2.4. We have proved that $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ acts on m and therefore on $\mathfrak{m}$ by a supersingular character. □

Let $\mathcal{P}^*$ denote the subsets of pairs $(\mathcal{X}, \sigma)$ in $\mathcal{P}$ such that $\mathcal{X}$ is different from a twist of $\mathcal{X}_{\mathrm{triv}}$ or $\mathcal{X}_{\mathrm{sign}}$. It is stable under the action of $\widetilde{\Omega}$. Lemma 5.13 and Theorem 5.14 together give the following:

Corollary 5.16. *Suppose that the root system of* G *is irreducible. The map*

$$(\mathcal{X}, \sigma) \mapsto \mathfrak{m}(\mathcal{X}, \sigma)$$

induces a bijection between the $\widetilde{\Omega}$*-orbits of pairs* $(\mathcal{X}, \sigma) \in \mathcal{P}^*$ *and a system of representatives of the isomorphism classes of the simple supersingular* $\widetilde{\mathrm{H}}_k$*-modules.*

5E. *Pro-p Iwahori invariants of parabolic inductions and of special representations.*

5E1. In this section, $\mathbf{k}$ is an arbitrary field. Let F be a standard facet, Π_F the associated set of simple roots and P_F the group of $\mathfrak{F}$-points of the corresponding standard parabolic subgroup, with Levi decomposition $\mathrm{P}_F = \mathrm{M}_F \mathrm{N}_F$. We use the same notation as in Section 3C1. The unipotent subgroup N_F is generated by all the root subgroups $\mathcal{U}_\alpha$ for $\alpha \in \Phi^+ - \Phi_F^+$. Let N_F^- denote the opposite unipotent subgroup of G. The pro-p Iwahori subgroup $\tilde{\mathrm{I}}$ has the decomposition

$$\tilde{\mathrm{I}} = \tilde{\mathrm{I}}_F^+ \, \tilde{\mathrm{I}}_F^0 \, \tilde{\mathrm{I}}_F^-,$$

where

$$\tilde{\mathrm{I}}_F^+ := \tilde{\mathrm{I}} \cap \mathrm{N}_F, \quad \tilde{\mathrm{I}}_F^0 := \tilde{\mathrm{I}} \cap \mathrm{M}_F, \quad \tilde{\mathrm{I}}_F^- := \tilde{\mathrm{I}} \cap \mathrm{N}_F^-.$$

By Remark 3.6, the subspace $\widetilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)^-$ of $\widetilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)$ generated over $\mathbf{k}$ by τ_w^F for F-negative $w \in \widetilde{\mathrm{W}}_F$ is identified with a sub-$\mathbf{k}$-algebra of $\widetilde{\mathrm{H}}_{\mathbf{k}}$ via the injection

$$j_F^- : \widetilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)^- \longrightarrow \widetilde{\mathrm{H}}_{\mathbf{k}}, \quad \tau_w^F \longmapsto \tau_w.$$

This endows $\widetilde{\mathrm{H}}_k$ with the structure of left module over $\widetilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)^-$.

Proposition 5.17. *Let* $(\sigma, \mathrm{V}_\sigma)$ *be a smooth* $\mathbf{k}$*-representation of* M_F*. Consider the parabolic induction* $\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma$ *and its* $\tilde{\mathrm{I}}$*-invariant subspace* $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma)^{\tilde{\mathrm{I}}}$*. There is a surjective morphism of right* $\widetilde{\mathrm{H}}_{\mathbf{k}}$*-modules*

$$\sigma^{\tilde{\mathrm{I}}_F^0} \otimes_{\widetilde{\mathrm{H}}_{\mathbf{k}}(\mathrm{M}_F)^-} \widetilde{\mathrm{H}}_{\mathbf{k}} \longrightarrow (\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma)^{\tilde{\mathrm{I}}} \tag{5-4}$$

sending $v \otimes 1$ *to the unique* $\tilde{\mathrm{I}}$*-invariant function with support in* $\mathrm{P}_F \tilde{\mathrm{I}}$ *and value* v *at* 1_{G}.

Remark 5.18. In the cases $\mathbf{G} = \mathrm{PGL}_n$ or GL_n, Proposition 5.2 in [Ollivier 2010] implies that (5-4) is an isomorphism. This result should be true for a general (split) $\mathbf{G}$, but we will only use the surjectivity here.

The proposition follows from the discussion below. All the lemmas are proved in the next section.

Lemma 5.19. *Let* $\mathcal{D}_F = \{d \in \mathfrak{W} : d^{-1}\Phi_F^+ \subseteq \Phi^+\}$.

(i) *For* $d \in \mathcal{D}_F$, *we have* $\mathrm{P}_F \tilde{\mathrm{I}}\, \hat{d}\, \tilde{\mathrm{I}} = \mathrm{P}_F \hat{d}\, \tilde{\mathrm{I}}$.

(ii) *The set of all* $\hat{d} \in \mathrm{G}$ *for* $d \in \mathcal{D}_F$ *is a system of representatives of the double cosets* $\mathrm{P}_F \backslash \mathrm{G} / \tilde{\mathrm{I}}$.

(iii) *For* $d \in \mathcal{D}_F$, *let* $\tilde{\mathrm{I}}\, \hat{d}\, \tilde{\mathrm{I}} = \bigsqcup_y \tilde{\mathrm{I}}\, \hat{d}\, y$ *be a decomposition into right cosets. Then*

$$\mathrm{P}_F \hat{d}\, \tilde{\mathrm{I}} = \bigsqcup_y \mathrm{P}_F \tilde{\mathrm{I}}\, \hat{d}\, y.$$

(iv) *Let* $d \in \mathcal{D}_F$. *Under the projection* $\mathrm{P}_F \twoheadrightarrow \mathrm{M}_F$, *the image of* $\mathrm{P}_F \cap \hat{d}\tilde{\mathrm{I}}\hat{d}^{-1}$ *is* $\tilde{\mathrm{I}}_F^0$.

An element $m \in \mathrm{M}_F$ contracts $\tilde{\mathrm{I}}_F^+$ and dilates $\tilde{\mathrm{I}}_F^-$ if it satisfies the conditions

$$m\tilde{\mathrm{I}}_F^+ m^{-1} \subseteq \tilde{\mathrm{I}}_F^+, \quad m^{-1}\tilde{\mathrm{I}}_F^- m \subseteq \tilde{\mathrm{I}}_F^- \tag{5-5}$$

(see [Bushnell and Kutzko 1998, (6.5)]).

Remark 5.20. This property of an element $m \in \mathrm{M}_F$ only depends on the double coset $\tilde{\mathrm{I}}_F^0 m \tilde{\mathrm{I}}_F^0$. Furthermore, if $m \in \mathrm{K} \cap \mathrm{M}_F$ then $m\tilde{\mathrm{I}}_F^+ m^{-1} = \tilde{\mathrm{I}}_F^+$ and $m^{-1}\tilde{\mathrm{I}}_F^- m = \tilde{\mathrm{I}}_F^-$.

Lemma 5.21. *Let* $w \in \widetilde{\mathrm{W}}_F$. *The element* $\hat{w}$ *satisfies* (5-5) *if and only if* w *is* F*-negative.*

Let $(\sigma, \mathrm{V}_\sigma)$ be as in the proposition. Let $v \in \mathrm{V}_\sigma^{\tilde{\mathrm{I}}_F^0}$ and $d \in \mathcal{D}_F$. By (ii) and (iv) of Lemma 5.19, the $\tilde{\mathrm{I}}$-invariant function

$$f_{d,v} \in (\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma)^{\tilde{\mathrm{I}}}$$

with support in $\mathrm{P}_F \hat{d}\, \tilde{\mathrm{I}}$ and value v at $\hat{d}$ is well defined, and the set of all $f_{d,v}$ form a basis of $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma)^{\tilde{\mathrm{I}}}$, where d ranges over $\mathcal{D}_F$ and v over a basis of $\mathrm{V}_\sigma^{\tilde{\mathrm{I}}_F^0}$.

Lemma 5.22. (i) *If* w *is an* F*-negative element in* $\widetilde{\mathrm{W}}_F$, *then* $f_{1,v}.\tau_w = f_{1,v.\tau_w^F}$.

(ii) *We have* $f_{1,v}.\tau_{\hat{d}} = f_{d,v}$.

5E2. *Proof of the lemmas.* Recall that given $\alpha \in \Phi$, the root subgroup $\mathcal{U}_\alpha$ is endowed with a filtration $\mathcal{U}_{(\alpha,k)}$ for $k \in \mathbb{Z}$ (see for example [Schneider and Stuhler 1997, Section I.1] or [Ollivier and Schneider 2012, Section 4.2]) and that the product map

$$\prod_{\alpha \in \Phi^-} \mathcal{U}_{(\alpha,1)} \times \mathrm{T}^1 \times \prod_{\alpha \in \Phi^+} \mathcal{U}_{(\alpha,0)} \xrightarrow{\sim} \tilde{\mathrm{I}} \tag{5-6}$$

induces a bijection, where the products on the left side are ordered in some arbitrary

chosen way [Schneider and Stuhler 1997, Proposition I.2.2]. The subgroup $\tilde{\mathrm{I}}_F^+$ of $\tilde{\mathrm{I}}$ is generated by the image of $\prod_{\alpha\in\Phi^+-\Phi_F^+}\mathcal{U}_{(\alpha,0)}$, while $\tilde{\mathrm{I}}_F^-$ is generated by that of $\prod_{\alpha\in\Phi^--\Phi_F^-}\mathcal{U}_{(\alpha,1)}$. The subgroup $\tilde{\mathrm{I}}_F^0$ is generated by the image of

$$\prod_{\alpha\in\Phi_F^-}\mathcal{U}_{(\alpha,1)}\times\mathrm{T}^1\times\prod_{\alpha\in\Phi_F^+}\mathcal{U}_{(\alpha,0)}.$$

Proof of Lemma 5.19. (i) We have $\mathrm{P}_F\tilde{\mathrm{I}}\hat{d}\tilde{\mathrm{I}}=\mathrm{P}_F\tilde{\mathrm{I}}_F^-\hat{d}\tilde{\mathrm{I}}$. But for $\alpha\in\Phi^+$, we have $\hat{d}^{-1}\mathcal{U}_{(-\alpha,1)}\hat{d}=\mathcal{U}_{(-d^{-1}\alpha,1)}\subseteq\tilde{\mathrm{I}}$, so $\tilde{\mathrm{I}}_F^-\hat{d}\subseteq\hat{d}\tilde{\mathrm{I}}$ and $\mathrm{P}_F\tilde{\mathrm{I}}\hat{d}\tilde{\mathrm{I}}=\mathrm{P}_F\hat{d}\tilde{\mathrm{I}}$. Point (ii) follows by Bruhat decomposition for K and Iwasawa decomposition for G. For (iii), we first recall that the image of $\mathrm{P}_F\cap\mathrm{K}$ under the reduction $\mathrm{red}\,\mathrm{K}\to\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$ modulo K_1 is a parabolic subgroup $\overline{\mathbf{P}}_F(\mathbb{F}_q)$ containing $\overline{\mathbf{B}}(\mathbb{F}_q)$ (notation in Section 1B).

Recall that the Weyl group of $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$ is $\mathfrak{W}$; for $w\in\mathfrak{W}$ we will still denote by w a chosen lift in $\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)$. The set $\mathcal{D}_F$ is a system of representatives of $\overline{\mathbf{P}}_F(\mathbb{F}_q)\backslash\overline{\mathbf{G}}_{x_0}(\mathbb{F}_q)/\overline{\mathbf{N}}(\mathbb{F}_q)$. For $d\in\mathcal{D}_F$ we have, using [Carter 1985, 2.5.12],

$$\overline{\mathbf{P}}_F(\mathbb{F}_q)\cap d\overline{\mathbf{N}}(\mathbb{F}_q)d^{-1}\subset\overline{\mathbf{N}}(\mathbb{F}_q).$$

We deduce that the image of $\mathrm{P}_F\cap\tilde{\mathrm{I}}_F^-\hat{d}\tilde{\mathrm{I}}\hat{d}^{-1}$ by red is contained in $\overline{\mathbf{N}}(\mathbb{F}_q)$ and therefore $\mathrm{P}_F\cap\tilde{\mathrm{I}}_F^-\hat{d}\tilde{\mathrm{I}}\hat{d}^{-1}$ is contained in $\tilde{\mathrm{I}}$.

Now let $d\in\mathcal{D}_F$ and $y\in\tilde{\mathrm{I}}$. By the previous observations, $\hat{d}\in\mathrm{P}_F\tilde{\mathrm{I}}\hat{d}\,y=\mathrm{P}_F\tilde{\mathrm{I}}_F^-\hat{d}\,y$ implies $\hat{d}\in\tilde{\mathrm{I}}\hat{d}\,y$. This proves (iii). In passing we proved that $\mathrm{P}_F\cap\hat{d}\tilde{\mathrm{I}}\hat{d}^{-1}$ is contained in $\mathrm{P}_F\cap\tilde{\mathrm{I}}=\tilde{\mathrm{I}}_F^0\tilde{\mathrm{I}}_F^+$. Since $\tilde{\mathrm{I}}_F^0$ is contained in $\mathrm{P}_F\cap\hat{d}\tilde{\mathrm{I}}\hat{d}^{-1}$ by definition of $\mathcal{D}_F$, this proves (iv). □

Proof of Lemma 5.21. By Remark 5.20, it is enough to prove the result for $w=e^\lambda\in\mathrm{X}_*(\mathrm{T})$. A lift for e^λ is given by $\lambda(\varpi^{-1})$. The element $\lambda(\varpi^{-1})$ satisfies (5-5) if

$$\begin{aligned}\text{for all }\alpha\in\Phi^+-\Phi_F^+\text{ we have }&\lambda(\varpi^{-1})\,\mathcal{U}_{(\alpha,0)}\lambda(\varpi)\subseteq\tilde{\mathrm{I}}_F^+\\ &\text{and }\lambda(\varpi)\mathcal{U}_{(-\alpha,1)}\lambda(\varpi^{-1})\subseteq\tilde{\mathrm{I}}_F^-.\end{aligned}\tag{5-7}$$

By [Ollivier and Schneider 2012, Remark 4.1(1)] for example,

$$\lambda(\varpi^{-1})\mathcal{U}_{(\alpha,0)}\lambda(\varpi)=\mathcal{U}_{(\alpha,-\langle\alpha,\lambda\rangle)}\quad\text{and}\quad\lambda(\varpi)\mathcal{U}_{(-\alpha,1)}\lambda(\varpi^{-1})=\mathcal{U}_{(-\alpha,1-\langle\alpha,\lambda\rangle)}.$$

Condition (5-7) is satisfied if and only if λ is F-negative (definition in Section 3C1). □

Proof of Lemma 5.22. (i) Let w be an F-negative element in $\widetilde{\mathrm{W}}_F$. The function $f_{1,v}.\tau_w$ has support in $\mathrm{P}_F\tilde{\mathrm{I}}_F^-\hat{w}\tilde{\mathrm{I}}$. Since $\hat{w}$ satisfies (5-5), we have $\mathrm{P}_F\tilde{\mathrm{I}}_F^-\hat{w}\tilde{\mathrm{I}}=\mathrm{P}_F\hat{w}\tilde{\mathrm{I}}=\mathrm{P}_F\tilde{\mathrm{I}}$. It remains to compute the value of $f_{1,v}.\tau_w$ at 1_G (we choose the unit element 1_G of G as a lift for $1\in\mathcal{D}_F$). The proof goes through exactly as in [Ollivier 2010, Section 6A.3], where it is written up in the case of $\mathbf{G}=\mathrm{GL}_n$.

(ii) Let $d \in \mathcal{D}_F$. By Lemma 5.19(i), the $\tilde{\mathrm{I}}$-invariant function $f_{1,v} \,.\, \tau_d$ has support in $\mathrm{P}_F \hat{d} \tilde{\mathrm{I}}$, and it follows from Lemma 5.19(iii) that it takes value v at $\hat{d}$. □

5E3. Here we consider again representations with coefficients in an algebraically closed field k with characteristic p. We draw corollaries from Proposition 5.17.

Corollary 5.23. *Let $F \neq x_0$ be a standard facet. If σ is an admissible k-representation of M_F with a central character, then $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma)^{\tilde{\mathrm{I}}}$ is a finite-dimensional $\tilde{\mathrm{H}}_k$-module whose irreducible subquotients are not supersingular.*

Proof. That $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma)^{\tilde{\mathrm{I}}}$ is finite-dimensional is a consequence of the admissibility of σ. Let $\lambda \in \mathrm{X}_*(\mathrm{T})$ be a strongly F-negative coweight (see Remark 3.6) and $\lambda_0 \in \mathrm{X}_*^+(\mathrm{T})$ the unique dominant coweight in its $\mathfrak{W}$-orbit $\mathbb{O}(\lambda)$. By Lemma 3.4,

$$z_{\lambda_0} = \sum_{\lambda' \in \mathbb{O}(\lambda)} \mathcal{B}_F^-(\lambda').$$

We compute the action of z_{λ_0} on an element of the form $v \otimes 1 \in \sigma^{\tilde{\mathrm{I}}_F^0} \otimes_{\tilde{\mathrm{H}}_k(\mathrm{M}_F)^-} \tilde{\mathrm{H}}_k$. We have $\mathcal{B}_F^-(\lambda) = \tau_{e^\lambda}$ and therefore

$$(v \otimes 1)\mathcal{B}_F^-(\lambda) = v \otimes \tau_{e^\lambda} = v \otimes j_F^-(\tau_{e^\lambda}^F) = (v\tau_{e^\lambda}^F) \otimes 1.$$

Recall that $\tau_{e^\lambda}^F = \tau_{\lambda(\varpi^{-1})}^F$ and that $\lambda(\varpi^{-1})$ is a central element in M_F. Therefore, $v\tau_{e^\lambda}^F = \omega(\lambda(\varpi))v$, where ω denotes the central character of σ. By (2-4), this implies in particular that $(v \otimes 1)\mathcal{B}_F^-(\lambda') = 0$ for $\lambda' \in \mathbb{O}(\lambda)$ distinct from λ. We have proved that z_{λ_0} acts by multiplication by $\omega(\lambda(\varpi)) \neq 0$ on $\sigma^{\tilde{\mathrm{I}}_F^0} \otimes_{\tilde{\mathrm{H}}_k(\mathrm{M}_F)^-} \tilde{\mathrm{H}}_k$, and therefore on $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} \sigma)^{\tilde{\mathrm{I}}}$ by Proposition 5.17. This proves the claim. □

Corollary 5.24. *Let F be a standard facet. Let Sp_F be the generalized special k-representation of* G

$$\mathrm{Sp}_F = \frac{\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} 1}{\sum_{F' \neq F \subset \overline{F}} \mathrm{Ind}_{\mathrm{P}_{F'}}^{\mathrm{G}} 1},$$

where F' ranges over the set of standard facets $\neq F$ contained in the closure of F. The $\tilde{\mathrm{I}}$-invariant subspace of Sp_F is a finite-dimensional $\tilde{\mathrm{H}}_k$-module whose irreducible subquotients are not supersingular.

Proof. Suppose first that $F \neq x_0$. By [Große-Klönne 2013b, (18)] (which is valid with no restriction on the split group G), $(\mathrm{Sp}_F)^{\tilde{\mathrm{I}}}$ is a quotient of $(\mathrm{Ind}_{\mathrm{P}_F}^{\mathrm{G}} 1)^{\tilde{\mathrm{I}}}$. Apply Corollary 5.23. If $F = x_0$, then the special representation in question is the trivial character of G, whose $\tilde{\mathrm{I}}$-invariant subspace is isomorphic to the trivial character of $\tilde{\mathrm{H}}_k$ and is not supersingular (see the example in Section 5D2 and Lemma 5.12). □

5F. ***On supersingular representations.*** Let ρ be a weight of K. By (4-7), there is a correspondence between the k-characters of $\mathscr{H}(\mathrm{G},\rho)$ and the k-characters of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$, and we will use the letter ζ for each of the two characters paired up by (4-7). With this notation, by the work in Section 4 we have a surjective morphism of representations of G:

$$\zeta \otimes_{\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)} \mathrm{ind}_{\widetilde{\mathrm{I}}}^{\mathrm{G}} 1 \longrightarrow \zeta \otimes_{\mathscr{H}(\mathrm{G},\rho)} \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho. \tag{5-8}$$

For ω a character of the connected center of G, let ζ_ω the supersingular character of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ as in Section 5C3. We remark that the representation $\zeta_\omega \otimes_{\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)} \mathrm{ind}_{\widetilde{\mathrm{I}}}^{\mathrm{G}} 1$ of G has central character ω.

From now on we suppose that the derived group of G is simply connected and that $\mathfrak{F}$ is a finite extension of $\mathbb{Q}_p$.

Lemma 5.25. *A character $\mathscr{H}(\mathrm{G},\rho) \to k$ is parametrized by the pair (G,ω) in the sense of* [Herzig 2011a, Proposition 4.1] *if and only if it corresponds to the supersingular character ζ_ω of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$ via* (4-7)*.*

Proof. In this proof we denote by $\psi : \mathscr{H}(\mathrm{G},\rho) \to k$ and $\zeta : \mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k) \to k$ a pair of characters corresponding to each other by (4-7). Recall that $\mathcal{T}$ denotes the inverse Satake isomorphism (4-4). By [ibid., Corollary 4.2] (see also Corollary 2.19 there), the character $\psi : \mathscr{H}(\mathrm{G},\rho) \to k$ is parametrized by the pair (G,ω) if and only if $\psi \circ \mathcal{T}(\lambda) = 0$ for all $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ such that $\ell(e^\lambda) \neq 0$ and if $\psi \otimes_{\mathscr{H}(\mathrm{G},\rho)} \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho$ has central character equal to ω (see Lemma 4.4 and its proof there). Since for all $\lambda \in \mathrm{X}_*^+(\mathrm{T})$ we have $\zeta(z_\lambda) = \psi \circ \mathcal{T}(\lambda)$ and since $\psi \otimes_{\mathscr{H}(\mathrm{G},\rho)} \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho$ is a quotient of $\zeta \otimes_{\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)} \mathrm{ind}_{\widetilde{\mathrm{I}}}^{\mathrm{G}} 1$, we have proved (using the remark before the statement of this lemma) that ψ is parametrized by the pair (G,ω) if and only if $\zeta = \zeta_\omega$. □

A smooth irreducible admissible k-representation of G has a central character. A smooth irreducible admissible k-representation π with central character $\omega : Z \to k^\times$ is called supersingular with respect to $(\mathrm{K},\mathrm{T},\mathrm{B})$ [ibid., Definition 4.7] if for all weights ρ of K, any map $\mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho \to \pi$ factors through

$$\zeta_\omega \otimes_{\mathscr{H}(\mathrm{G},\rho)} \mathrm{ind}_{\mathrm{K}}^{\mathrm{G}} \rho \longrightarrow \pi.$$

Note that if the first map is zero, then the condition is trivial. By (5-8), a supersingular representation with central character $\omega : Z \to k^\times$ is therefore a quotient of $\zeta_\omega \otimes_{\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)} \mathrm{ind}_{\widetilde{\mathrm{I}}}^{\mathrm{G}} 1$ and, by Definition 5.8, of

$$\mathrm{ind}_{\widetilde{\mathrm{I}}}^{\mathrm{G}} 1 \,/\, \mathcal{I} \,\mathrm{ind}_{\widetilde{\mathrm{I}}}^{\mathrm{G}} 1.$$

Remark 5.26. (i) The representation $\mathrm{ind}_{\widetilde{\mathrm{I}}}^{\mathrm{G}} 1 \,/\, \mathcal{I} \,\mathrm{ind}_{\widetilde{\mathrm{I}}}^{\mathrm{G}} 1$ depends only on the conjugacy class of x_0. It is independent of any choices if $\mathbf{G}$ is of adjoint type or $\mathbf{G} = \mathrm{GL}_n$.

(ii) An irreducible admissible representation π of G is a quotient of $\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1 / \mathfrak{I}\, \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1$ if and only if $\pi^{\tilde{\mathrm{I}}}$ contains a supersingular $\widetilde{\mathrm{H}}_k$-module. Recall that when the root system of G is irreducible, we have proved that the notion of supersingularity for $\widetilde{\mathrm{H}}_k$-modules is independent of all the choices made.

Theorem 5.27. *If* $\mathrm{G} = \mathrm{GL}_n(\mathfrak{F})$ *or* $\mathrm{PGL}_n(\mathfrak{F})$, *a smooth irreducible admissible* k*-representation* π *is supersingular if and only if* $\pi^{\tilde{\mathrm{I}}}$ *contains a supersingular* $\widetilde{\mathrm{H}}_k$*-module*; *that is to say*, *if and only if* π *is a quotient of*

$$\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1 / \mathfrak{I}\, \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1. \tag{5-9}$$

Proof. Let π be a smooth irreducible admissible k-representation of G with central character ω. If it is a quotient of $\mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1 / \mathfrak{I}\, \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1$, then it is a quotient of $\zeta_\omega \otimes_{\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)} \mathrm{ind}_{\tilde{\mathrm{I}}}^{\mathrm{G}} 1$, and $\pi^{\tilde{\mathrm{I}}}$ contains the supersingular character ζ_ω of $\mathscr{Z}^\circ(\widetilde{\mathrm{H}}_k)$. Therefore it contains a supersingular $\widetilde{\mathrm{H}}_k$-module. By Corollaries 5.23 and 5.24, this implies that π is neither a representation induced from a strict parabolic subgroup of G nor (a twist by a character of G of) a generalized special representation. By [Herzig 2011a, Theorem 1.1], which classifies all smooth irreducible admissible k-representation of G, we conclude by elimination that the representation π is supersingular. □

The results of [Herzig 2011a] have been generalized to the case of an $\mathfrak{F}$-split connected reductive group G in [Abe 2013]: the classification of the smooth irreducible admissible representations of G is quite similar to the case of $\mathrm{GL}_n(\mathfrak{F})$ (expect for a certain subtlety when the root system of G is not irreducible). Based on this classification and on Corollaries 5.23 and 5.24, N. Abe confirmed that the space of $\tilde{\mathrm{I}}$-invariant vectors of a nonsupersingular representation does not contain any supersingular $\widetilde{\mathrm{H}}_k$-module. Therefore, Theorem 5.27 is true for a general split group with simply connected derived subgroup.

References

[Abe 2013] N. Abe, "On a classification of irreducible admissible modulo p representations of a p-adic split reductive group", *Compos. Math.* **149**:12 (2013), 2139–2168. MR 3143708 Zbl 06250165 arXiv 1103.2525

[Barthel and Livné 1994] L. Barthel and R. Livné, "Irreducible modular representations of GL_2 of a local field", *Duke Math. J.* **75**:2 (1994), 261–292. MR 95g:22030 Zbl 0826.22019

[Bourbaki 1964] N. Bourbaki, Actualités Scientifiques et Industrielles **1308**, Hermann, Paris, 1964. Translated in *Commutative algebra: chapters* 1–7, Springer, 1989. MR 33 #2660 Zbl 0205.34302

[Bourbaki 1968] N. Bourbaki, *Groupes et algébres de Lie: chapitres 4 à 6*, Actualités Scientifiques et Industrielles **1337**, Hermann, Paris, 1968. Translated as *Lie groups and Lie algebras, chapters 4–6*, Springer, Berlin, 2002. French original reprinted by Springer, Berlin, 2007. MR 39 #1590 Zbl 0186.33001

[Bruhat and Tits 1972] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local, I: Données radicielles valuées", *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 5–251. MR 48 #6265 Zbl 0254.14017

[Bruhat and Tits 1984] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local, II: Schémas en groupes. Existence d'une donnée radicielle valuée", *Inst. Hautes Études Sci. Publ. Math.* **60** (1984), 5–184. MR 86c:20042 Zbl 0597.14041

[Bushnell and Kutzko 1998] C. J. Bushnell and P. C. Kutzko, "Smooth representations of reductive p-adic groups: structure theory via types", *Proc. London Math. Soc.* (3) **77**:3 (1998), 582–634. MR 2000c:22014 Zbl 0911.22014

[Carter 1985] R. W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, Wiley, New York, 1985. MR 87d:20060 Zbl 0567.20023

[Carter and Lusztig 1976] R. W. Carter and G. Lusztig, "Modular representations of finite groups of Lie type", *Proc. London Math. Soc.* (3) **32**:2 (1976), 347–384. MR 53 #592 Zbl 0338.20013

[Dat 1999] J.-F. Dat, "Caractères à valeurs dans le centre de Bernstein", *J. Reine Angew. Math.* **508** (1999), 61–83. MR 2000f:22021 Zbl 0938.22016

[Gross 1998] B. H. Gross, "On the Satake isomorphism", pp. 223–238 in *Galois representations in arithmetic algebraic geometry* (Durham, 1996), edited by A. J. Scholl and R. L. Taylor, London Math. Soc. Lecture Note Ser. **254**, Cambridge University Press, 1998. MR 2000e:22008 Zbl 0996.11038

[Große-Klönne 2013a] E. Große-Klönne, "From pro-p Iwahori–Hecke modules to (φ, Γ)-modules, I", preprint, 2013, http://www.math.hu-berlin.de/~zyska/Grosse-Kloenne/iwahecgal.pdf.

[Große-Klönne 2013b] E. Große-Klönne, "On special representations of p-adic reductive groups", preprint, 2013, http://www.math.hu-berlin.de/~zyska/Grosse-Kloenne/spec.pdf.

[Haines 2001] T. J. Haines, "The combinatorics of Bernstein functions", *Trans. Amer. Math. Soc.* **353**:3 (2001), 1251–1278. MR 2002j:20012 Zbl 0962.14018

[Haines and Pettet 2002] T. J. Haines and A. Pettet, "Formulae relating the Bernstein and Iwahori–Matsumoto presentations of an affine Hecke algebra", *J. Algebra* **252**:1 (2002), 127–149. MR 2003f:20012 Zbl 1056.20003

[Herzig 2011a] F. Herzig, "The classification of irreducible admissible mod p representations of a p-adic GL_n", *Invent. Math.* **186**:2 (2011), 373–434. MR 2845621 Zbl 1235.22030

[Herzig 2011b] F. Herzig, "A Satake isomorphism in characteristic p", *Compos. Math.* **147**:1 (2011), 263–283. MR 2012c:22020 Zbl 1214.22004

[Kempf et al. 1973] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings, I*, Lecture Notes in Math. **339**, Springer, Berlin, 1973. MR 49 #299 Zbl 0271.14017

[Koziol 2013] K. Koziol, "Restriction of pro-p-Iwahori–Hecke modules", preprint, 2013. arXiv 1308.6239

[Lusztig 1983] G. Lusztig, "Singularities, character formulas, and a q-analog of weight multiplicities", pp. 208–229 in *Analyse et topologie sur les espaces singuliers, II, III* (Luminy, 1981), edited by A. A. Beilinson et al., Astérisque **101**, Soc. Math. France, Paris, 1983. MR 85m:17005 Zbl 0561.22013

[Lusztig 1989] G. Lusztig, "Affine Hecke algebras and their graded version", *J. Amer. Math. Soc.* **2**:3 (1989), 599–635. MR 90e:16049 Zbl 0715.22020

[Ollivier 2010] R. Ollivier, "Parabolic induction and Hecke modules in characteristic p for p-adic GL_n", *Algebra Number Theory* **4**:6 (2010), 701–742. MR 2012c:20007 Zbl 1243.22017

[Ollivier 2012] R. Ollivier, "An inverse Satake isomorphism in characteristic p", preprint, 2012. arXiv 1207.5557

[Ollivier and Schneider 2012] R. Ollivier and P. Schneider, "Pro-p Iwahori–Hecke algebras are Gorenstein", preprint, 2012. arXiv 1207.3769

[Satake 1963] I. Satake, "Theory of spherical functions on reductive algebraic groups over p-adic fields", *Inst. Hautes Études Sci. Publ. Math.* **18** (1963), 5–69. MR 33 #4059 Zbl 0122.28501

[Sawada 1977] H. Sawada, "A characterization of the modular representations of finite groups with split (B, N)-pairs", *Math. Z.* **155**:1 (1977), 29–41. MR 56 #8679 Zbl 0345.20009

[Schmidt 2009] N. A. Schmidt, *Generische pro-p Hecke-Algebren*, thesis, Humboldt University, Berlin, 2009, http://www2.mathematik.hu-berlin.de/~schmidtn/hecke.pdf.

[Schneider and Stuhler 1997] P. Schneider and U. Stuhler, "Representation theory and sheaves on the Bruhat–Tits building", *Inst. Hautes Études Sci. Publ. Math.* **85** (1997), 97–191. MR 98m:22023 Zbl 0892.22012

[Tits 1979] J. Tits, "Reductive groups over local fields", pp. 29–69 in *Automorphic forms, representations and L-functions, Part 1* (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979. MR 80h:20064 Zbl 0415.20035

[Vignéras 2005] M.-F. Vignéras, "Pro-p-Iwahori Hecke ring and supersingular $\overline{\mathbf{F}}_p$-representations", *Math. Ann.* **331**:3 (2005), 523–556. [Erratum at ibid. **333**:3 (2005), 699–701]. MR 2005m:22020 Zbl 1107.22011

[Vignéras 2006] M.-F. Vignéras, "Algèbres de Hecke affines génériques", *Represent. Theory* **10** (2006), 1–20. MR 2006i:20005 Zbl 1134.22014

[Vignéras 2007] M.-F. Vignéras, "Représentations irréductibles de GL(2, F) modulo p", pp. 548–563 in *L-functions and Galois representations*, edited by D. Burns et al., London Math. Soc. Lecture Note Ser. **320**, Cambridge University Press, 2007. MR 2009h:11084 Zbl 1172.11017

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