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We strengthen the compatibility between local and global Langlands correspondences for GL_n when n is even and $l = p$. Let L be a CM field and Π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ which is conjugate self-dual and regular algebraic. In this case, there is an l-adic Galois representation associated to Π, which is known to be compatible with local Langlands in almost all cases when $l = p$ by recent work of Barnet-Lamb, Gee, Geraghty and Taylor. The compatibility was proved only up to semisimplification unless Π has Shin-regular weight. We extend the compatibility to Frobenius semisimplification in all cases by identifying the monodromy operator on the global side. To achieve this, we derive a generalization of Mokrane's weight spectral sequence for log crystalline cohomology.

1. Introduction

This paper is a continuation of [Caraiani 2012]. Here we extend our local-global compatibility result to the case $l = p$.

Theorem 1.1. *Let $n \in \mathbb{Z}_{\geq 2}$ be an integer and L be a CM field with complex conjugation c. Let l be a prime of $\mathbb{Q}$ and $\iota_l : \bar{\mathbb{Q}}_l \to \mathbb{C}$ be an isomorphism. Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$ satisfying*

- $\Pi^\vee \simeq \Pi \circ c$,

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- Π *is cohomological for some irreducible algebraic representation* Ξ *of* $\mathrm{GL}_n(L \otimes_{\mathbb{Q}} \mathbb{C})$.

Let

$$R_l(\Pi) : \mathrm{Gal}(\bar{L}/L) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_l)$$

be the Galois representation associated to Π *by* [Shin 2011; Chenevier and Harris 2013]. *Let* y *be a place of* L *above* l. *Then we have the isomorphism of Weil–Deligne representations*

$$WD(R_l(\Pi)|_{\mathrm{Gal}(\bar{L}_y/L_y)})^{\text{F-ss}} \simeq \iota_l^{-1}\mathscr{L}_{n,L_y}(\Pi_y).$$

Here $\mathscr{L}_{n,L_y}(\Pi_y)$ is the image of Π_y under the local Langlands correspondence, using the geometric normalization; i.e., $\mathscr{L}_{n,L_y}(\Pi_y) := \mathrm{rec}\big(\Pi_y^\vee \otimes |\det|^{\frac{1-n}{2}}\big)$, where rec is the local Langlands correspondence compatible with L- and ϵ-factors (see the introduction to [Harris and Taylor 2001] for more details). $WD(r)$ is the Weil–Deligne representation attached to a de Rham l-adic representation r of the absolute Galois group of an l-adic field; F-ss denotes Frobenius semisimplification. Note that we are assuming throughout that $n \geq 2$. The local-global compatibility of Langlands correspondences for GL_1 follows from the compatibility between local and global class field theory.

This theorem is proved in [Barnet-Lamb et al. 2012; Barnet-Lamb et al. 2011] in the case when Π has Shin-regular weight (either n is odd or if n is even then Π satisfies an additional regularity condition) and in general up to semisimplification. The strategy for obtaining the local-global compatibility of monodromy operators in these cases is to make use of the fact that the l-adic Galois representation associated to Π occurs in the cohomology of certain very special unitary Shimura varieties. These are associated to unitary similitude groups with signature $(1, n-1)$ (respectively, $(1, n)$ if n is even) at exactly one infinite place and signature $(0, n)$ (respectively, $(0, n+1)$) at all the other infinite places. The problem can be reduced to the case when Π_y has an Iwahori-fixed vector, in which case one has to compute the crystalline cohomology of a compact Shimura variety which is strictly semistable. This computation makes use of the weight spectral sequence for crystalline cohomology due to Mokrane [1993], which is shown to degenerate at the first page. We remark that the l-adic Galois representation associated to Π is only known to occur in the cohomology of a proper, smooth variety in the case when Π has Shin-regular weight.

Our goal in this paper is to match up the monodromy operators in the case when n is even and Π does not necessarily have Shin-regular weight. Following the conventions of [Taylor and Yoshida 2007], we call a Weil–Deligne representation *pure of weight* k if it admits a weight filtration, with all the weights in $k + \mathbb{Z}$, such that the (iterated) monodromy operator induces an isomorphism of the $(k+i)$-th

and $(k-i)$-th graded pieces for all positive integers i. By Lemma 1.4 (4) of [Taylor and Yoshida 2007], given a semisimple representation of the Weil group of some l-adic field, there is at most one way to choose the monodromy operator such that the resulting Weil–Deligne representation is pure of some weight. By Theorem 1.2 of [Caraiani 2012], Π_y is tempered, so we know that $\iota_l^{-1}\mathcal{L}_{n,L_y}(\Pi_y)$ is pure of some weight.

By Theorem A of [Barnet-Lamb et al. 2011], we also know that we have an isomorphism up to semisimplification:

$$WD(R_l(\Pi)|_{\mathrm{Gal}(\bar{L}_y/L_y)})^{\mathrm{ss}} \simeq \iota_l^{-1}\mathcal{L}_{n,L_y}(\Pi_y)^{\mathrm{ss}}.$$

We note that Theorem A of [Barnet-Lamb et al. 2011] is stated for an imaginary CM field F. For our CM field L we proceed as on pages 230–231 of [Harris and Taylor 2001] to find a quadratic extension F/L which is an imaginary CM field, in which $y = y'y''$ splits, such that

$$[R_l(\Pi)|_{\mathrm{Gal}(\bar{L}/F)}] = [R_l(BC_{F/L}(\Pi))].$$

This together with Theorem A of [Barnet-Lamb et al. 2011] gives the compatibility up to semisimplification for the place y of L. Therefore, in order to complete the proof of Theorem 1.1, it suffices to show that $W := WD(R_l(\Pi)_{\mathrm{Gal}(\bar{L}_y/L_y)})^{\mathrm{F\text{-}ss}}$ is pure of some weight when n is even. From now on we will let $n \in \mathbb{Z}_{\geq 2}$ be an even integer.

Our argument will follow the same general lines as that of [Taylor and Yoshida 2007], which is also the strategy followed by [Barnet-Lamb et al. 2012; Barnet-Lamb et al. 2011]. We reduce the problem to the case when Π_y has an Iwahori-fixed vector. In this case, we find not W itself, but rather the tensor square of W in the log crystalline cohomology of a compact Shimura variety with Iwahori-level structure, and finally compute a part of this cohomology explicitly. For the last step, however, we can not make use of the Mokrane spectral sequence, since our Iwahori-level Shimura variety is no longer semistable, but rather Zariski-locally étale over a product of strictly semistable schemes. Therefore, we need to derive a formula for the log crystalline cohomology of the special fiber of this Shimura variety in terms of the crystalline cohomology of closed Newton polygon strata in the special fiber. Deriving this formula constitutes the heart of this paper; we obtain it in the form of a generalization of the Mokrane spectral sequence or as a crystalline analogue of Corollary 4.28 of [Caraiani 2012].

We briefly outline the structure of our paper. In Section 2 we reduce to the case where Π has an Iwahori-fixed vector, we define an inverse system of compact Shimura varieties associated to a unitary group and we show that the crystalline cohomology of the Iwahori-level Shimura variety realizes the tensor square of W. The Shimura varieties we work with are the same as those studied in [Caraiani 2012],

so in Section 2 we also recall the main results from [Caraiani 2012] concerning them. In Section 3 we recall and adapt to our situation some standard results from the theory of log crystalline cohomology and the de Rham–Witt complex; we define and study some slight generalizations of the logarithmic de Rham–Witt complex. In Section 4 we generalize the Mokrane spectral sequence to our geometric setting. The main technical result is Theorem 4.6. In Section 5 we prove Theorem 1.1.

2. Shimura varieties

Let L, Π, $R_l(\Pi)$ and y be as described in the introduction. Below, we show that we can understand the Weil–Deligne representation $W = WD(R_l(\Pi)_{\mathrm{Gal}(\bar{L}_y/L_y)})^{\mathrm{F\text{-}ss}}$ by computing a part of the crystalline cohomology of an inverse system of Shimura varieties. In the first part we closely follow Sections 2 and 7 of [Caraiani 2012] and afterwards we use some results from Section 5 of the same work.

We claim first that we can reduce the problem to the case when Π has Iwahori-fixed vectors at y, and we can also put ourselves in a situation where the base change from unitary groups to GL_n is well understood. This means that we can reduce the problem to understanding the cohomology of certain Iwahori-level unitary Shimura varieties. More precisely, we can find a CM field extension F' of L such that:

- $F' = EF_1$, where E is an imaginary quadratic field in which l splits and $F_1 = (F')^{c=1}$ has $[F_1 : \mathbb{Q}] \geq 2$;
- F' is soluble and Galois over L;
- $\Pi^0_{F'} := BC_{F'/L}(\Pi)$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F'})$; and
- there is a place $\mathfrak{p}$ above the place y of L such that $\Pi^0_{F',\mathfrak{p}}$ has a nonzero Iwahori-fixed vector;

and a CM field F which is a quadratic extension of F', such that:

- $\mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2$ splits in F;
- $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \subset \mathrm{Spl}_{F/F_2,\mathbb{Q}}$, where $F_2 := (F)^{c=1}$; and
- $\Pi^0_F = BC_{F/F'}(\Pi^0_{F'})$ is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$.

We can find F and F' as in the proof of Corollary 5.9 of [Caraiani 2012]. Since purity is preserved under finite extensions by Lemma 1.4 of [Taylor and Yoshida 2007], to show that W is pure it suffices to show that

$$W_{F'} := WD\big(R_l(\Pi^0_{F'})|_{\mathrm{Gal}(\overline{F'_{\mathfrak{p}}}/F'_{\mathfrak{p}})}\big)^{\mathrm{F\text{-}ss}}$$

is pure. Note that in this new situation $\Pi^0_{F',\mathfrak{p}}$ has a nonzero Iwahori-fixed vector.

We can define an algebraic group G over $\mathbb{Q}$ and an inverse system of Shimura varieties over F' corresponding to a PEL Shimura datum $(F, *, V, \langle\cdot,\cdot\rangle, h)$. Here

F is the CM field defined above and $* = c$ is the involution corresponding to complex conjugation. We take V to be the F-vector space F^n. The pairing

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{Q}$$

is a nondegenerate Hermitian pairing such that $\langle fv_1, v_2 \rangle = \langle v_1, f^* v_2 \rangle$ for all $f \in F$ and $v_1, v_2 \in V$. The last element we need is an $\mathbb{R}$-algebra homomorphism $h : \mathbb{C} \to \mathrm{End}_F(V) \otimes_{\mathbb{Q}} \mathbb{R}$ such that the bilinear pairing

$$(v_1, v_2) \to \langle v_1, h(i) v_2 \rangle$$

is symmetric and positive definite. We define the algebraic group G over $\mathbb{Q}$ by

$$G(R) = \left\{ (g, \lambda) \in \mathrm{End}_{F \otimes_{\mathbb{Q}} \mathbb{R}}(V \otimes_{\mathbb{Q}} R)^\times \times R^\times \mid \langle gv_1, gv_2 \rangle = \lambda \langle v_1, v_2 \rangle \right\}$$

for any $\mathbb{Q}$-algebra R.

We choose embeddings $\tau_i : F \hookrightarrow \mathbb{C}$ such that $\tau_2 = \tau_1 \circ \sigma$, where σ is an element of $\mathrm{Gal}(F/F')$ which takes $\mathfrak{p}_1$ to $\mathfrak{p}_2$. For $\sigma \in \mathrm{Hom}_{E, \tau_E}(F, \mathbb{C})$ we let (p_σ, q_σ) be the signature at σ of the pairing $\langle \cdot, \cdot \rangle$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$. In particular, $\tau_E := \tau_1|_E = \tau_2|_E$ is well-defined. We claim that it is possible to choose a PEL datum as above such that $(p_\tau, q_\tau) = (1, n-1)$ for $\tau = \tau_1$ or τ_2 and $(p_\tau, q_\tau) = (0, n)$ otherwise and such that $G_{\mathbb{Q}_v}$ is quasisplit at every finite place v of $\mathbb{Q}$. This follows from Lemma 2.1 of [Caraiani 2012] and the discussion following it, and it depends crucially on the fact that n is even. We choose such a PEL datum and we let G be the corresponding algebraic group over $\mathbb{Q}$ with the prescribed signature at infinity and quasisplit at all the finite places.

Let $\Xi^0_F := BC_{F/L}(\Xi)$ and $F_2 = F^{c=1}$. Lemma 7.2 of [Shin 2011] says that we can find a character $\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ and an algebraic representation $\xi_{\mathbb{C}}$ of G over $\mathbb{C}$ satisfying the following conditions:

- $\psi_{\Pi^0_F} = \psi^c / \psi$.
- Ξ^0_F is isomorphic to the restriction of Ξ' to $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n) \times_{\mathbb{Q}} \mathbb{C}$, where Ξ' is obtained from $\xi_{\mathbb{C}}$ by base change from G to $\mathbb{G}_n := \mathrm{Res}_{E/\mathbb{Q}}(G \times_{\mathbb{Q}} E)$.
- $\xi_{\mathbb{C}}|^{-1}_{E^\times_\infty} = \psi^c_\infty$.
- $\mathrm{Ram}_{\mathbb{Q}}(\psi) \subset \mathrm{Spl}_{F/F_2, \mathbb{Q}}$.
- $\psi|_{\mathcal{O}_{E_u^\times}} = 1$, where u is the place above l induced by $\iota_l^{-1} \tau_E$.

Define $\xi := \iota_l \xi_{\mathbb{C}}$, and define $\Pi^1 := \psi \otimes \Pi^0_F$, which is a cuspidal automorphic representation of $\mathrm{GL}_1(\mathbb{A}_E) \times \mathrm{GL}_n(\mathbb{A}_F)$.

Corresponding to the PEL datum $(F, *, V, \langle \cdot, \cdot \rangle, h)$, we have a PEL-type moduli problem of abelian varieties. This moduli problem is defined in Section 2.1 of [Caraiani 2012], and here we recall some facts about it. Since the reflex field of the PEL datum is F', the moduli problem for an open compact subgroup $U \subset G(\mathbb{A}^\infty)$ is

representable by a Shimura variety X_U/F', which is a smooth and quasiprojective scheme of dimension $2n-2$. The inverse system of Shimura varieties X_U as U varies has an action of $G(\mathbb{A}^\infty)$. As in Section III.2 of [Harris and Taylor 2001], starting with ξ, which is an irreducible algebraic representation of G over $\overline{\mathbb{Q}}_l$, we can define a lisse $\overline{\mathbb{Q}}_l$-sheaf $\mathscr{L}_\xi$ over each X_U, and the action of $G(\mathbb{A}^\infty)$ extends to the inverse system of sheaves. The direct limit

$$H^i(X, \mathscr{L}_\xi) := \varinjlim H^i(X_U \times_{F'} \overline{F'}, \mathscr{L}_\xi)$$

is a semisimple admissible representation of $G(\mathbb{A}^\infty)$ with a continuous action of $\mathrm{Gal}(\overline{F'}/F')$. It can be decomposed as

$$H^i(X, \mathscr{L}_\xi) = \bigoplus_\pi \pi \otimes R^i_{\xi,l}(\pi),$$

where the sum runs over irreducible admissible representations π of $G(\mathbb{A}^\infty)$ over $\overline{\mathbb{Q}}_l$. The $R^i_{\xi,l}(\pi)$are finite-dimensional continuous representations of $\mathrm{Gal}(\overline{F'}/F')$ over $\overline{\mathbb{Q}}_l$. Let $\mathscr{A}_U$ be the universal abelian variety over X_U, to the inverse system of which the action of $G(\mathbb{A}^\infty)$ extends. To the irreducible representation ξ of G we can associate as in Section III.2 of [Harris and Taylor 2001] nonnegative integers m_ξ and t_ξ as well as an idempotent a_ξ of $H^*(\mathscr{A}_U^{m_\xi} \times_{F'} \overline{F'}, \overline{\mathbb{Q}}_l(t_\xi))$. (Here $\mathscr{A}_U^{m_\xi}$ denotes the m_ξ-fold product of $\mathscr{A}_U$ with itself over X_U and $\overline{\mathbb{Q}}_l(t_\xi)$ is a Tate twist.) We have an isomorphism

$$H^i(X_U \times_{F'} \overline{F'}, \mathscr{L}_\xi) \simeq a_\xi H^{i+m_\xi}(\mathscr{A}_U^{m_\xi} \times_{F'} \overline{F'}, \overline{\mathbb{Q}}_l(t_\xi)),$$

which commutes with the $G(\mathbb{A}^\infty)$-action.

For every finite place v of $\mathbb{Q}$ we can define a base-change morphism taking certain admissible $G(\mathbb{Q}_v)$-representations to admissible $\mathbb{G}(\mathbb{Q}_v)$-representations, as in Section 4.2 of [Shin 2011]. Recall that $\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}\, \Pi^1 \subset \mathrm{Spl}_{F/F_2,\mathbb{Q}}$. If $v \notin \mathrm{Spl}_{F/F_2,\mathbb{Q}}$ then we can define the morphism

$$BC : \mathrm{Irr}^{\mathrm{ur}}_{(l)}(G(\mathbb{Q}_v)) \to \mathrm{Irr}^{\mathrm{ur},\theta\text{-st}}_{(l)}(\mathbb{G}(\mathbb{Q}_v)),$$

taking unramified representations of $G(\mathbb{Q}_v)$ to unramified, θ-stable representations of $\mathbb{G}(\mathbb{Q}_v)$. If $v \in \mathrm{Spl}_{F/F_2,\mathbb{Q}}$ then the morphism

$$BC : \mathrm{Irr}_{(l)}(G(\mathbb{Q}_v)) \to \mathrm{Irr}^{\theta\text{-st}}_{(l)}(\mathbb{G}(\mathbb{Q}_v))$$

can be defined explicitly since $G(\mathbb{Q}_v)$ is split. Putting these maps together, we get, for any finite set of primes $\mathfrak{S}_{\mathrm{fin}}$ such that

$$\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \subset \mathfrak{S}_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F_2,\mathbb{Q}},$$

a base-change morphism

$$BC : \mathrm{Irr}^{\mathrm{ur}}_{(l)}\big(G(\mathbb{A}^{\mathfrak{S}_{\mathrm{fin}}\cup\{\infty\}})\big) \otimes \mathrm{Irr}_{(l)}\big(G(\mathbb{A}_{\mathfrak{S}_{\mathrm{fin}}})\big) \to \mathrm{Irr}^{\mathrm{ur},\theta\text{-st}}_{(l)}\big(\mathbb{G}(\mathbb{A}^{\mathfrak{S}_{\mathrm{fin}}\cup\{\infty\}})\big) \otimes \mathrm{Irr}^{\theta\text{-st}}_{(l)}\big(\mathbb{G}(\mathbb{A}_{\mathfrak{S}_{\mathrm{fin}}})\big).$$

Let p be a prime of $\mathbb{Q}$ which splits in E and such that there is a place of F' above p which splits in F. Let $\mathfrak{S}_{\mathrm{fin}}$ be a finite set of primes such that

$$\mathrm{Ram}_{F/\mathbb{Q}} \cup \mathrm{Ram}_{\mathbb{Q}}(\Pi) \cup \{p\} \subset \mathfrak{S}_{\mathrm{fin}} \subset \mathrm{Spl}_{F/F_2,\mathbb{Q}},$$

and set $\mathfrak{S} := \mathfrak{S}_{\mathrm{fin}} \cup \{\infty\}$. For any $R \in \mathrm{Groth}(G(\mathbb{A}^{\mathfrak{S}}) \times G(\mathbb{A}_{\mathfrak{S}_{\mathrm{fin}}}) \times \mathrm{Gal}(\overline{F'}/F'))$ (over $\overline{\mathbb{Q}}_l$) and $\pi^{\mathfrak{S}} \in \mathrm{Irr}^{\mathrm{ur}}(G(\mathbb{A}^{\mathfrak{S}}))$ define the $\pi^{\mathfrak{S}}$-isotypic part of R to be

$$R\{\pi^{\mathfrak{S}}\} := \sum_{\rho} n(\pi^{\mathfrak{S}} \otimes \rho)[\pi^{\mathfrak{S}}][\rho],$$

where ρ runs over $\mathrm{Irr}_l(G(\mathbb{A}_{\mathfrak{S}}) \times \mathrm{Gal}(\overline{F'}/F'))$. Also define

$$R[\Pi^{1,\mathfrak{S}}] := \sum_{\pi^{\mathfrak{S}}} R[\pi^{\mathfrak{S}}],$$

where each sum runs over $\pi^{\mathfrak{S}} \in \mathrm{Irr}^{\mathrm{ur}}_l(G(\mathbb{A}^{\mathfrak{S}}))$ such that $BC(\iota_l \pi^{\mathfrak{S}}) \simeq \Pi^{1,\mathfrak{S}}$.

Proposition 2.1. *Let $\mathfrak{S} = \mathfrak{S}_{\mathrm{fin}} \cup \{\infty\}$ be as above. We have the equality*

$$BC(H^{2n-2}(X, \mathscr{L}_\xi)[\Pi^{1,\mathfrak{S}}]) \simeq C_G[\iota_l^{-1}\Pi^{1,\infty}][R_l(\Pi^0_{F'})^{\otimes 2} \otimes \mathrm{rec}_{l,\iota_l}(\psi)]$$

of elements of $\mathrm{Groth}(G(\mathbb{A}^{\infty}) \times \mathrm{Gal}(\overline{F'}/F))$. *Here C_G is a positive integer and $\mathrm{rec}_{l,\iota_l}(\psi)$ is the continuous l-adic character $\mathrm{Gal}(\bar{E}/E) \to \overline{\mathbb{Q}}_l^{\times}$ associated to ψ by global class field theory, normalized so that it matches uniformizers with geometric Frobenius elements.*

Remark. Unlike in the classical situation of modular forms or in the case of Harris–Taylor-type Shimura varieties [Harris and Taylor 2001; Shin 2011], the cohomology of our inverse system of Shimura varieties realizes a twist of the tensor square of the l-adic Galois representation associated to Π, because we have chosen our unitary similitude groups to have signature $(1, n-1)$ two infinite places. One could use Matsushima's formula and $(\mathfrak{g}, K)$-cohomology to check that the dimension of the Galois representation seen by this cohomology is n^2, as predicted by the statement.

Proof. Let $p \in \mathfrak{S}_{\mathrm{fin}}$ be a prime which splits in E such that there is a place w of F' above the place induced by τ_E over p which splits in F, $w = w_1 w_2$. We start by recalling some constructions and results from Sections 2 and 5 of [Caraiani 2012]. It is possible to define an integral model of each X_U over the ring of integers $\mathbb{O}_K$ in $K := F_{w_1} \simeq F_{w_2}$, which itself represents a moduli problem of abelian varieties and to which the sheaf $\mathscr{L}_\xi$ extends. The special fiber Y_U of this integral model has a stratification by open Newton polygon strata $Y^{\circ}_{U,S,T}$, according to the formal

(or étale) height of the p-divisible group of the abelian variety at w_1 and w_2. Each open Newton polygon stratum is covered by a tower of Igusa varieties $\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}}$, where $0 \leq h_1, h_2 \leq n-1$ represent the étale heights of the p-divisible groups at w_1 and w_2, and $\vec{m}$ is a tuple of positive integers describing the level structure at p.

Define

$$J^{(h_1,h_2)}(\mathbb{Q}_p) := \mathbb{Q}_p^\times \times D_{K,n-h_1}^\times \times \mathrm{GL}_{h_1}(K) \times D_{K,n-h_2}^\times \times \mathrm{GL}_{h_2}(K) \times \prod_w \mathrm{GL}_n(F_w),$$

where $D_{K,n-h}$ is the division algebra over K of invariant $1/(n-h)$ and w runs over places of F above τ_E other than w_1 and w_2. The group $J^{(h_1,h_2)}(\mathbb{Q}_p)$ acts on the directed system of $H^j_c(\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}}, \mathscr{L}_\xi)$, as U^p and $\vec{m}$ vary. Let

$$H_c(\mathrm{Ig}^{(h_1,h_2)}, \mathscr{L}_\xi) \in \mathrm{Groth}\big(G(\mathbb{A}^{\infty,p}) \times J^{(h_1,h_2)}\big)$$

be the alternating sum of the direct limit of $H^j_c(\mathrm{Ig}^{(h_1,h_2)}_{U^p,\vec{m}}, \mathscr{L}_\xi)$ as in Section 5.1 of [Caraiani 2012]. Let $\pi_p \in \mathrm{Irr}_l(G(\mathbb{Q}_p))$ be a representation such that $BC(\pi_p) \simeq \iota_l^{-1}\Pi^1_p$ (such a π_p is unique up to isomorphism since p splits in E). Theorem 5.6 of [Caraiani 2012] gives a formula for computing the cohomology of Igusa varieties, as elements of $\mathrm{Groth}\big(\mathbb{G}(\mathbb{A}^{\mathfrak{S}}) \times \mathbb{G}(\mathbb{A}_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}) \times J^{(h_1,h_2)}(\mathbb{Q}_p)\big)$:

$$\begin{aligned} BC^p\big(H_c(\mathrm{Ig}^{(h_1,h_2)}, \mathscr{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big) \\ = e_0(-1)^{h_1+h_2} C_G [\iota_l^{-1}\Pi^{1,\mathfrak{S}}][\iota_l^{-1}\Pi^1_{\mathfrak{S}_{\mathrm{fin}}\setminus\{p\}}][\mathrm{Red}_n^{(h_1,h_2)}(\pi_p)] \end{aligned} \quad (2.1)$$

Here $e_0 = \pm 1$ independently of h_1, h_2 and $\mathrm{Red}_n^{(h_1,h_2)}$ is a group morphism from $\mathrm{Groth}(G(\mathbb{Q}_p))$ to $\mathrm{Groth}(J^{(h_1,h_2)}(\mathbb{Q}_p))$, defined explicitly above Theorem 5.6 of [Caraiani 2012].

We can combine the above formula with Mantovan's formula for the cohomology of Shimura varieties. This is the equality

$$H(X, \mathscr{L}_\xi) = \sum_{0 \leq h_1, h_2 \leq n-1} (-1)^{h_1+h_2} \mathrm{Mant}_{(h_1,h_2)}\big(H_c(\mathrm{Ig}^{(h_1,h_2)}, \mathscr{L}_\xi)\big) \quad (2.2)$$

of elements of $\mathrm{Groth}(G(\mathbb{A}^\infty) \times W_K)$. Here $H(X, \mathscr{L}_\xi)$ is the alternating sum of the direct limit of the cohomology of the Shimura fibers (generic fibers) and

$$\mathrm{Mant}_{(h_1,h_2)} : \mathrm{Groth}(J^{(h_1,h_2)}(\mathbb{Q}_p)) \to \mathrm{Groth}(G(\mathbb{Q}_p) \times W_K)$$

is the functor defined in [Mantovan 2005]. The formula (2.2) is what Theorem 22 of [Mantovan 2005] amounts to in our situation, where h_1 and h_2 are the parameters for the Newton stratification. The extra term $(-1)^{h_1+h_2}$ occurs on the right-hand side because we use the same convention for the alternating sum of cohomology as in [Caraiani 2012], which differs by a sign from the conventions used in [Mantovan 2005] and [Shin 2011].

By combining formulas (2.1) and (2.2) we get

$$BC^p\big(H(X, \mathscr{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big) = e_0 C_G[\iota_l^{-1}\Pi^{1,\infty,p}]\Bigg(\sum_{0\le h_1,h_2\le n-1}[\mathrm{Mant}_{(h_1,h_2)}(\mathrm{Red}_n^{(h_1,h_2)}(\pi_p))]\Bigg)$$

in $\mathrm{Groth}(\mathbb{G}(\mathbb{A}^{\infty,p}) \times G(\mathbb{Q}_p) \times W_K)$. We claim that

$$\sum_{0\le h_1,h_2\le n-1}[\mathrm{Mant}_{(h_1,h_2)}(\mathrm{Red}_n^{(h_1,h_2)}(\pi_p))] = [\pi_p]\big[(\pi_{p,0}\circ \mathrm{Art}_{\mathbb{Q}_p}^{-1})|_{W_K} \otimes \iota_l^{-1}\mathscr{L}_{K,n}(\Pi^0_{F',w})\big]. \quad (2.3)$$

By its definition above Theorem 5.6 of [Caraiani 2012], the morphism $\mathrm{Red}_n^{(h_1,h_2)}(\pi_p)$ breaks down as a product

$$(-1)^{h_1+h_2}\pi_{p,0} \otimes \mathrm{Red}^{n-h_1,h_1}(\pi_{w_1}) \otimes \mathrm{Red}^{n-h_2,h_2}(\pi_{w_2}) \otimes \bigotimes_{w\ne w_1,w_2} \pi_w,$$

where w runs over places above the place of p induced by τ_E other than w_1 and w_2. The morphism

$$\mathrm{Red}^{n-h,h} : \mathrm{Groth}(\mathrm{GL}_n(K)) \to \mathrm{Groth}(D^\times_{K,n-h} \times \mathrm{GL}_h(K))$$

is also defined above Theorem 5.6 of [Caraiani 2012]. On the other hand, the functor $\mathrm{Mant}_{(h_1,h_2)}$ also decomposes as a product (see [Shin 2011, Formula 5.6]), into

$$\mathrm{Mant}_{(h_1,h_2)}(\rho) = \mathrm{Mant}_{1,0}(\rho_0) \otimes \mathrm{Mant}_{n-h_1,h_1}(\rho_{w_1}) \otimes \mathrm{Mant}_{n-h_2,h_2}(\rho_{w_2}) \otimes \bigotimes_{w\ne w_1,w_2} \mathrm{Mant}_{0,m}(\rho_w),$$

where w again runs over places above the place of p induced by τ_E other than w_1 and w_2. So

$$\sum_{0\le h_1,h_2\le n-1}[\mathrm{Mant}_{(h_1,h_2)}(\mathrm{Red}_n^{(h_1,h_2)}(\pi_p))] = [\mathrm{Mant}_{1,0}(\pi_{p,0})] \otimes \sum_{h_1=0}^{n-1}(-1)^{h_1}[\mathrm{Mant}_{n-h_1,h_1}(\mathrm{Red}^{n-h_1,h_1}(\pi_{w_1}))] \otimes \sum_{h_2=0}^{n-1}(-1)^{h_2}[\mathrm{Mant}_{n-h_2,h_2}(\mathrm{Red}^{n-h_2,h_2}(\pi_{w_2}))] \otimes \bigotimes_{w\ne w_1,w_2}[\pi_w].$$

Now by applying Propositions 2.2(i) and 2.3 of [Shin 2011] we get the desired result (note that the normalization used in their statements is slightly different than

ours, but the relation between the two different normalizations is explained above the statement of Proposition 2.3).

Applying Equation (2.3), we first see that

$$BC\big(H(X,\mathscr{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big) = e_0 C_G[\iota_l^{-1}\Pi^{1,\infty}]\big[(\pi_{p,0}\circ \mathrm{Art}_{\mathbb{Q}_p}^{-1})|_{W_K}\otimes \iota_l^{-1}\mathscr{L}_{K,n}(\Pi^0_{F',w})\big] \quad (2.4)$$

in $\mathrm{Groth}(\mathbb{G}(\mathbb{A}^\infty)\times W_K)$, which means that

$$BC\big(H(X,\mathscr{L}_\xi)[\Pi^{1,\mathfrak{S}}]\big) = e_0[\iota_l^{-1}\Pi^{1,\infty}][R'(\Pi^1)],$$

for some $[R'(\Pi^1)]\in \mathrm{Groth}(\mathrm{Gal}(\overline{F}'/F))$. We show now that

$$[R'(\Pi^1)] = C_G[R(\Pi^0_{F'})^{\otimes 2}\otimes \mathrm{rec}_{l,\iota_l}(\psi)]$$

in $\mathrm{Groth}(\mathrm{Gal}(\overline{F}/F'))$, using the Cebotarev density theorem. Note first that $R'(\Pi^1)$ is simply the sum of (the alternating sum of) $R^k_{\xi,l}(\pi^\infty)$, where π^∞ runs over $\mathrm{Irr}_l(G(\mathbb{A}^\infty))$ such that

- $BC(\iota_l\pi^{\mathfrak{S}})\simeq \Pi^{1,\mathfrak{S}}$,
- $BC(\iota_l\pi_{\mathfrak{S}_{\mathrm{fin}}})\simeq \Pi_{\mathfrak{S}_{\mathrm{fin}}}$,
- $R^k_{\xi,l}(\pi^\infty)\neq 0$ for some k.

The set of such π doesn't depend on $\mathfrak{S}$ if $\mathfrak{S}$ is chosen as described above this proposition, so the Galois representation $R'(\Pi^1)$ is also independent of $\mathfrak{S}$. Therefore, for any prime w_1 of F where Π^1 is unramified and which is above a prime w of F' which splits in F and above a prime $p\neq l$ of $\mathbb{Q}$ which splits in E, we can choose a finite set of places $\mathfrak{S}$ containing p such that we get from Equation (2.4)

$$[R'(\Pi^1)|_{W_{F_{w_1}}}] = C_G\big[(R(\Pi^0_{F'})^{\otimes 2}\otimes \mathrm{rec}_{l,\iota_l}(\psi))_{W_{F_{w_1}}}\big].$$

By the Cebotarev density theorem (which tells us the Frobenius elements of primes w_1 are dense in $\mathrm{Gal}(\overline{F}'/F)$) we conclude that

$$[R'(\Pi^1)] = C_G[R(\Pi^0_{F'})^{\otimes 2}\otimes \mathrm{rec}_{l,\iota_l}(\psi)]$$

in $\mathrm{Groth}(\mathrm{Gal}(\overline{F}/F'))$.

It remains to see that $e_0=1$ and that $H^k(X,\mathscr{L}_\xi)[\Pi^{1,\mathfrak{S}}]=0$ unless $k=2n-2$. In fact, it suffices to show the latter, since then $H(X,\mathscr{L}_\xi)[\Pi^{1,\mathfrak{S}}]$ will have to be an actual representation, so that would force $e_0=1$. The fact that $H^k(X,\mathscr{L}_\xi)[\Pi^{1,\mathfrak{S}}]=0$ for $k\neq 2n-2$ can be seen, as in the proof of Corollary 7.3 of [Caraiani 2012], by choosing a prime $p\neq l$ to work with and applying the spectral sequences in Proposition 7.2 of [ibid.], and noting that the terms of those spectral sequence are 0 outside the diagonal corresponding to $k=2n-2$. □

Corollary 2.2. *By Lemmas* 1.4 *and* 1.7 *of* [Taylor and Yoshida 2007] *and by the same argument as in the proof of Theorem* 7.4 *of* [Caraiani 2012], *in order to show that*

$$WD\big(R_l(\Pi^0_{F'})|_{\mathrm{Gal}(\overline{F'_{\mathfrak{p}}}/F'_{\mathfrak{p}})}\big)^{\mathrm{F\text{-}ss}}$$

is pure, it suffices to show that

$$WD\big(BC(H^{2n-2}(X,\mathscr{L}_\xi)[\Pi^{\mathfrak{S}}])|_{\mathrm{Gal}(\overline{F'_{\mathfrak{p}}}/F'_{\mathfrak{p}})}\big)^{\mathrm{F\text{-}ss}}$$

is pure, where $\mathfrak{S}$ *is chosen such that it contains* l.

At this point, we've reduced the question of proving the local-global compatibility of monodromy operators when $l = p$ to proving that the $\Pi^{\mathfrak{S}}$-part of the cohomology of a system of proper, smooth Shimura varieties over F' gives rise to a pure Weil–Deligne representation. In the rest of this section, we shall describe integral models of these Shimura varieties which are no longer smooth but are log smooth and of Cartier type. We shall relate their log crystalline cohomology to the Weil–Deligne representation we are interested in. The upshot is that we reduce the question of local-global compatibility to proving the purity of (the $\Pi^{\mathfrak{S}}$-part of) certain log crystalline cohomology groups. This statement is made precise in Corollary 2.3 below.

Recall that $\mathfrak{p}$ is a place of F' above l such that $\mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2$. From now on, set $K := F_{\mathfrak{p}_1} \simeq F_{\mathfrak{p}_2}$, where the isomorphism is via σ. Let $\mathbb{O}_K$ be the ring of integers in K with uniformizer ϖ and residue field k. For $i = 1, 2$ let $\mathrm{Iw}_{n,\mathfrak{p}_i}$ be the subgroup of matrices in $\mathrm{GL}_n(\mathbb{O}_K)$ which reduce modulo $\mathfrak{p}_i$ to the Borel subgroup $B_n(k)$. Now we set

$$U_{\mathrm{Iw}} = U^l \times U_l^{\mathfrak{p}_1,\mathfrak{p}_2}(m) \times \mathrm{Iw}_{n,\mathfrak{p}_1} \times \mathrm{Iw}_{n,\mathfrak{p}_2} \subset G(\mathbb{A}^\infty),$$

for some $U^l \subset G(\mathbb{A}^\infty)$ compact open and $U_l^{\mathfrak{p}_1,\mathfrak{p}_2}$ a congruence subgroup at l away from $\mathfrak{p}_1$ and $\mathfrak{p}_2$. In Section 2.2 of [Caraiani 2012], an integral model for $X_{U_{\mathrm{Iw}}}/\mathbb{O}_K$ is defined. This is a proper scheme of dimension $2n-1$ with smooth generic fiber. The special fiber $Y_{U_{\mathrm{Iw}}}$ has a stratification by closed Newton polygon strata $Y_{U_{\mathrm{Iw}},S,T}$ with $S, T \subseteq \{1,\dots,n\}$ nonempty subsets. These strata are proper, smooth schemes over k of dimension $2n - \#S - \#T$. In fact,

$$Y_{U_{\mathrm{Iw}},S,T} = \Big(\bigcap_{i\in S} Y_{1,i}\Big) \cap \Big(\bigcap_{j\in T} Y_{2,j}\Big),$$

where each $Y_{i,j}$ for $i = 1, 2$ and $j = 1,\dots,n$ is cut out by one local equation. We can also define

$$Y^{(l_1,l_2)}_{U_{\mathrm{Iw}}} = \bigsqcup_{\substack{S,T\subseteq\{1,\dots,n\}\\ \#S=l_1\\ \#T=l_2}} Y_{U_{\mathrm{Iw}},S,T}.$$

By Proposition 2.8 of [Caraiani 2012], the completed local rings of $X_{U_{\mathrm{Iw}}}$ at closed geometric points s of $X_{U_{\mathrm{Iw}}}$ are isomorphic to

$$\mathbb{O}^{\wedge}_{X_{U_{\mathrm{Iw}}},s} \simeq W_{(K)}[\![X_1,\dots,X_n,Y_1,\dots,Y_n]\!]/(X_{i_1}\cdots X_{i_r}-\varpi, Y_{j_1}\cdots Y_{j_s}-\varpi),$$

where $\{i_1,\dots,i_r\}\subseteq\{1,\dots,n\}$, $\{j_1,\dots,j_r\}\subseteq\{1,\dots,n\}$ and $W_{(K)}$ is the ring of integers in the completion of the maximal unramified extension of K. The closed subscheme Y_{1,i_l} is cut out in $\mathbb{O}^{\wedge}_{X_{U_{\mathrm{Iw}}},s}$ by $X_{i_l}=0$ and Y_{2,j_l} is cut out by $Y_{j_l}=0$.

The action of $G(\mathbb{A}^{\infty,p})$ extends to the inverse system $X_{U_{\mathrm{Iw}}}/\mathbb{O}_K$. There is a universal abelian variety $\mathscr{A}_{U_{\mathrm{Iw}}}/\mathbb{O}_K$ and the actions of $G(\mathbb{A}^{\infty})$ and a_ξ extend to it. We can define a stratification of the special fiber of $\mathscr{A}_{U_{\mathrm{Iw}}}$ by

$$\mathscr{A}_{U_{\mathrm{Iw}},S,T} = \mathscr{A}_{U_{\mathrm{Iw}}}\times_{X_{U_{\mathrm{Iw}}}} X_{U_{\mathrm{Iw}},S,T}.$$

Moreover, $\mathscr{A}_{U_{\mathrm{Iw}}}$ satisfies the same geometric properties as $X_{U_{\mathrm{Iw}}}$ with respect to the above stratification of its special fiber and the analogous statement holds for $\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}$. In particular, we shall see in the next section (or it follows from Section 3 of [Caraiani 2012]) that $\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}$ can be endowed with a vertical logarithmic structure M such that

$$(\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}, M)\to(\operatorname{Spec}\mathbb{O}_K,\mathbb{N})$$

is log smooth, where $(\operatorname{Spec}\mathbb{O}_K,\mathbb{N})$ is the canonical log structure associated to the closed point. Also, we will see that its special fiber is of Cartier type. This means that we can define the log crystalline cohomology of $(\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}, M)$. Indeed, if $W=W(k)$ is the ring of Witt vectors of k, then we let

$$H^*_{\mathrm{cris}}(\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}/W)$$

be the log crystalline cohomology of $(\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}\times_{\mathbb{O}_K}k, M)$ (here we suppress M from the notation). This also has an action of a_ξ as an idempotent and of $G(\mathbb{A}^{\mathfrak{S}})$.

From the isomorphism

$$H^{2n-2}(X,\mathscr{L}_\xi)\simeq a_\xi H^{2n-2+m_\xi}(\mathscr{A}^{m_\xi},\overline{\mathbb{Q}}_l(t_\xi))$$

and Corollary 2.2, we see that it is enough to show that

$$a_\xi WD\big(H^{2n-2+m_\xi}(\mathscr{A}^{m_\xi},\overline{\mathbb{Q}}_l(t_\xi)|_{\mathrm{Gal}(\overline{K}/K)})[\Pi^{1,\mathfrak{S}}]\big)$$

is pure. Let $\tau_0: W\hookrightarrow\overline{\mathbb{Q}}_l$ be an embedding over $\mathbb{Z}_l$. By the semistable comparison theorem of [Nizioł 2008], we have

$$\begin{aligned}\varinjlim_{U_{\mathrm{Iw}}} a_\xi\big(H^{2n-2+m_\xi}_{\mathrm{cris}}(\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}\times_{\mathbb{O}_K}k/W)\otimes_{W,\tau_0}\overline{\mathbb{Q}}_l(t_\xi)\big)[\Pi^{1,\mathfrak{S}}]&\\ \simeq \varinjlim_{U_{\mathrm{Iw}}} a_\xi WD\big(H^{2n-2+m_\xi}(\mathscr{A}^{m_\xi}\times_{\mathbb{O}_K}\overline{K},\overline{\mathbb{Q}}_l(t_\xi)|_{\mathrm{Gal}(\overline{K}/K)})[\Pi^{1,\mathfrak{S}}]\big),&\end{aligned}$$

where the crystalline cohomology on the left-hand side as constructed in [Hyodo and Kato 1994] has *a priori* the structure of a (φ, N)-module over W, but which gives rise to a Weil–Deligne representation (r, N) of W_K by setting $r(\sigma) := \varphi^{n[k:\mathbb{F}_p]}$ whenever $\sigma \in W_K$ is a lift of Frob_k^n. Therefore, it suffices to understand the (direct limit of the) log crystalline cohomology of the special fiber of $\mathscr{A}_{U_{\mathrm{Iw}}}^{m_\xi}$. Note that the semistable comparison theorem was first proved by Kato [1994a] and Tsuji [1999] for proper, vertical log schemes with semistable reduction; the reason for citing Niziol's work is that her main theorem applies to a general fine and saturated, log-smooth, proper, vertical $(\mathrm{Spec}\,\mathcal{O}_K, \mathbb{N})$-scheme with special fiber of Cartier type. The fact that $(\mathscr{A}_{U_{\mathrm{Iw}}}^{m_\xi}, M)$ satisfies all these properties follows immediately from the explicit description of the log structure M in Section 3.

We summarize the above discussion in the following corollary:

Corollary 2.3. *The Weil–Deligne representation*

$$WD(R_l(\Pi^0_{F'})|_{\mathrm{Gal}(\overline{F'_{\mathfrak{p}}}/F'_{\mathfrak{p}})})^{\mathrm{F\text{-}ss}}$$

is pure if

$$\varinjlim_{U_{\mathrm{Iw}}} a_\xi \big(H^{2n-2+m_\xi}_{\mathrm{cris}}(\mathscr{A}_{U_{\mathrm{Iw}}}^{m_\xi} \times_{\mathcal{O}_K} k/W) \otimes_{W,\tau_0} \overline{\mathbb{Q}}_l(t_\xi)\big)[\Pi^{1,\mathfrak{S}}]$$

is pure, where $\mathfrak{S}$ is chosen such that it contains l.

3. Log crystalline cohomology

3A. ***Log structures.*** Let $\mathcal{O}_K$ be the ring of integers in a finite extension K of $\mathbb{Q}_p$ with uniformizer ϖ and residue field k. (Here, p is some prime number, which will be taken to equal l for our applications to local-global compatibility.) Let $W = W(k)$ be the ring of Witt vectors of k, with $W_n = W_n(k)$ referring to the Witt vectors of length n over k. Let $W_{(K)}$ be the ring of integers in the completion of the maximal unramified extension of K.

Let $X/\mathcal{O}_K$ be a scheme locally of finite type such that the completions of the strict henselizations $\mathcal{O}^\wedge_{X,s}$ at closed geometric points s of X are isomorphic to

$$W_{(K)}[\![X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z_1, \ldots, Z_m]\!]/(X_{i_1} \cdots X_{i_r} - \varpi, Y_{j_1} \cdots Y_{j_s} - \varpi)$$

for some indices $i_1, \ldots, i_r, j_1, \ldots, j_s \in \{1, \ldots, n\}$ and some $1 \le r, s \le n$. Also assume that the special fiber Y is a union of closed subschemes $Y_{1,j}$ with $j \in \{1, \ldots, n\}$ which are cut out by one local equation in $\mathcal{O}_X$, such that if s is a closed geometric point of $Y_{1,j}$, then $j \in \{i_1, \ldots, i_r\}$ and $Y_{1,j}$ is cut out in $\mathcal{O}^\wedge_{X,s}$ by the equation $X_j = 0$. Similarly, assume that Y is a union of closed subschemes $Y_{2,j}$ with $j \in \{1, \ldots, n\}$, which are cut out by one local equation in $\mathcal{O}_X$ such that if s is a closed geometric point of $Y_{2,j}$ then $j \in \{j_1, \ldots, j_r\}$ and $Y_{2,j}$ is cut out in $\mathcal{O}^\wedge_{X',s}$ by

the equation $Y_j = 0$. Then, by Lemma 2.9 of [Caraiani 2012], X is Zariski-locally étale over

$$X_{r,s,m} = \operatorname{Spec} \mathcal{O}_K[X_1, \dots, X_n, Y_1, \dots, Y_n, Z_1, \dots, Z_m]/(X_1 \cdots \varpi, Y_1 \cdots Y_s - \varpi).$$

The closed subschemes $Y_{i,j}$ for $i = 1, 2$ and $j = 1, \dots, n$ are Cartier divisors, which in the local model $X_{r,s,m}$ correspond to the divisors $X_j = 0$ or $Y_j = 0$.

Let Y/k be the special fiber of X. For $1 \leq i, j \leq n$ we define $Y^{(i,j)}$ to be the disjoint union of the closed subschemes of Y

$$(Y_{1,l_1} \cap \cdots \cap Y_{1,l_i}) \bigcap (Y_{2,m_1} \cap \cdots \cap Y_{2,m_j}),$$

as $\{l_1, \dots, l_i\}$ (resp. $\{m_1, \dots, m_j\}$) range over subsets of $\{1, \dots, n\}$ of cardinality i (resp. j). Each $Y^{(i,j)}$ is a proper smooth scheme over k of dimension $2n - i - j$.

Remark 3.1. Even though this section is general, we will only apply the results of this section in the case when X is $\mathscr{A}_{U_{\mathrm{Iw}}}$ for some compact open subgroup $U_{\mathrm{Iw}} \subset G(\mathbb{A}^\infty)$ with Iwahori-level structure at $\mathfrak{p}_1$ and $\mathfrak{p}_2$. $X_{U_{\mathrm{Iw}}}$ (and therefore $\mathscr{A}_{U_{\mathrm{Iw}}}$ as well) satisfies the above conditions by Proposition 2.8 of [Caraiani 2012]. The prime p is meant to be identified with l.

Let $(\operatorname{Spec} \mathcal{O}_K, \mathbb{N})$ be the log scheme corresponding to $\operatorname{Spec} \mathcal{O}_K$ endowed with the canonical log structure associated to the special fiber. This is given by the map $1 \in \mathbb{N} \mapsto \varpi \in \mathcal{O}_K$. We endow X with the log structure M associated to the special fiber Y. Let $j : X_K \to X$ be the open immersion and $i : Y \to X$ be the closed immersion. This log structure is defined by

$$M = j_*(\mathcal{O}_{X_K}^\times) \cap \mathcal{O}_X \to \mathcal{O}_X.$$

We have a map of log schemes $(X, M) \to (\operatorname{Spec} \mathcal{O}_K, \mathbb{N})$, given by sending $1 \in \mathbb{N}$ to $\varpi \in M$. Locally, we have a chart for this map, given by

$$\mathbb{N} \to \mathbb{N}^r \oplus \mathbb{N}^s/(1, \dots, 1, 0, \dots, 0) = (0, \dots, 0, 1, \dots, 1),$$
$$1 \mapsto (1, \dots, 1, 0, \dots, 0) = (0, \dots, 0, 1, \dots, 1).$$

It is easy to see from this that $(X, M)/(\operatorname{Spec} \mathcal{O}_K, \mathbb{N})$ is log smooth and that the log structure M on X is fine, saturated and vertical. We can pull back M to a log structure on Y, which we still denote M and then we get a log-smooth map of log schemes

$$(Y, M) \to (\operatorname{Spec} k, \mathbb{N}).$$

(Here we have the canonical log structure on k associated to $1 \in \mathbb{N} \mapsto 0 \in k$, which is the same as the pullback of the canonical log structure on $\operatorname{Spec} \mathcal{O}_K$.) Note that, since (X, M) is saturated over $(\operatorname{Spec} \mathcal{O}_K, \mathbb{N})$, its special fiber is of Cartier type (see [Tsuji 1997]).

We can also endow X with log structures $\widetilde{M}_1$, $\widetilde{M}_2$ and $\widetilde{M}$. Let $U_{i,j}$ be the complement of $Y_{i,,j}$ in X for $i = 1, 2$ and $j = 1, \ldots, n$. Let

$$j_{i,j} : U_{i,j} \to X$$

denote the open immersion. We define $\widetilde{M}_1$, $\widetilde{M}_2$ and $\widetilde{M}$ as follows

$$\widetilde{M}_1 = \left(\bigoplus_{j=1}^{n} (j_{1,j*}(\mathcal{O}^{\times}_{U_{1,j}}) \cap \mathcal{O}_X) \right) / \sim,$$

$$\widetilde{M}_2 = \left(\bigoplus_{j=1}^{n} (j_{1,j*}(\mathcal{O}^{\times}_{U_{1,j}}) \cap \mathcal{O}_X) \right) / \sim,$$

$$\widetilde{M} = \left(\bigoplus_{j=1}^{n} (j_{1,j*}(\mathcal{O}^{\times}_{U_{1,j}}) \cap \mathcal{O}_X) \oplus \bigoplus_{j=1}^{n} (j_{2,j*}(\mathcal{O}^{\times}_{U_{2,j}}) \cap \mathcal{O}_X) \right) / \sim,$$

where $\sim$ signifies that we have identified the image of $\mathcal{O}^{\times}_X$ in all the terms of the direct sums (in other words, we are taking an amalgamated sum of the log structures associated to each of the $Y_{i,j}$). We have a map $\widetilde{M} \to M$ given by inclusion on each $\mathcal{O}^{\times}_{U_{i,j}}$.

Lemma 3.2. *Locally on X, we have a chart for $\widetilde{M}$ given by*

$$X \to \operatorname{Spec} \mathcal{O}_k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z_1, \ldots, Z_m]/(X_1 \cdots X_r - \varpi, Y_1 \cdots Y_r - \varpi)$$
$$\to \operatorname{Spec} \mathbb{Z}[\mathbb{N}^r \oplus \mathbb{N}^s],$$

where $(0, \ldots, 0, 1, 0, \ldots, 0) \mapsto X_i$ if the 1 is in the i-th position and $1 \leq i \leq r$ and $(0, \ldots, 0, 1, 0, \ldots, 0) \mapsto Y_{i-r}$ if the 1 is in the i-th position and $r + 1 \leq i \leq r + s$.

Proof. We shall make use of Kato and Niziol's results on log smoothness and log regularity, namely:

- If $f : T \to S$ is a log smooth morphism of fs log schemes with S log regular then T is log regular (see 8.2 of [Kato 1994b]).
- If T is log regular, then $M_T = j_* \mathcal{O}^{\times}_U \cap \mathcal{O}_X$, where $j : U \hookrightarrow T$ is the inclusion of the open subset of triviality of T (see 8.6 of [Nizioł 2006]).

Let us define the following log schemes over $(\operatorname{Spec} \mathcal{O}_K, \text{triv})$:

$$\widetilde{U} := \operatorname{Spec} \mathcal{O}_K[X_1, \ldots, X_n, \sigma]/(X_1 \cdots X_r - \sigma),$$
$$\widetilde{V} := \operatorname{Spec} \mathcal{O}_K[Y_1, \ldots, Y_n, \tau]/(Y_1 \cdots Y_s - \tau),$$
$$W := \operatorname{Spec} \mathcal{O}_K[Z_1, \ldots, Z_m],$$
$$Z := \widetilde{U} \times_{(\operatorname{Spec} \mathcal{O}_K, \text{triv})} \widetilde{V} \times_{(\operatorname{Spec} \mathcal{O}_K, \text{triv})} W.$$

Then Z, equipped with the product log structure L, is smooth over $\mathbb{O}_K$ and log smooth over $(\operatorname{Spec}\mathbb{O}_K[\sigma,\tau],\text{triv})$. Therefore, Z is regular. The log structure L is given by the simple normal crossings divisor

$$D := \left(\bigcup_{j=1}^{r}(X_j = 0)\right) \cup \left(\bigcup_{j=1}^{s}(Y_j = 0)\right).$$

Since Z is regular, the log structure L is the same as the amalgamation of the log structures defined by the smooth divisors $(X_j = 0)$, $(Y_j = 0)$. Locally on X, we have a commutative diagram of schemes with a cartesian square

$$\begin{array}{ccccc} X & \longrightarrow & X_{r,s,m} & \longrightarrow & Z \\ & & \downarrow & & \downarrow \\ & & \operatorname{Spec}\mathbb{O}_K & \longrightarrow & \operatorname{Spec}\mathbb{O}_K[\tau,\sigma] \end{array} \tag{3A.1}$$

where the inverse image of $(X_j = 0)$ in X is Y_j^1 and the inverse image of $(Y_j = 0)$ in X is Y_j^2. Therefore, the log structure on X induced by that of Z coincides with the log structure $\widetilde{M}$ defined as the amalgamated sum of the log structures induced by the Y_j^1 and Y_j^2. □

If we endow $\operatorname{Spec}\mathbb{O}_K$ with the log structure $\mathbb{N}^2$ associated to $(a,b) \in \mathbb{N}^2 \mapsto \pi^{a+b} \in \mathbb{O}_K$, then we claim that we have a log-smooth map of log schemes

$$(X, \widetilde{M}) \to (\operatorname{Spec}\mathbb{O}_K, \mathbb{N}^2) \tag{3A.2}$$

whose chart is given locally by

$$(a,b) \in \mathbb{N}^2 \mapsto (a,\dots,a,b,\dots b) \in \mathbb{N}^r \oplus \mathbb{N}^s.$$

By definition, $\widetilde{M}$ is the amalgamated sum of $\widetilde{M}_1$ and $\widetilde{M}_2$ as log structures on X (or, in other words, $\widetilde{M}$ is the log structure associated to the prelog structure $\widetilde{M}_1 \oplus \widetilde{M}_2 \to \mathbb{O}_X$). Therefore, it suffices to prove the following lemma:

Lemma 3.3. *We can define a global map of log schemes* $(X, \widetilde{M}_1) \to (\operatorname{Spec}\mathbb{O}_K, \mathbb{N})$ *which locally admits the chart given by the diagonal embedding* $\mathbb{N} \to \mathbb{N}^r$.

Proof. It suffices to show that ϖ is a global section of $\widetilde{M}_1$, since then we can simply map $1 \in \mathbb{N}$ to $\varpi \in \widetilde{M}_1$. For this, note that we have a natural map of log structures on X

$$\widetilde{M}_1 \to M,$$

since the open subset of triviality of $\widetilde{M}_1$ is the generic fiber of X and M is the log structure defined by the inclusion of the generic fiber. Moreover, we can check locally that this map is injective, since it can be described by the chart $\mathbb{N}^r \to \mathbb{N}^r \oplus \mathbb{N}^s \to (\mathbb{N}^r \oplus \mathbb{N}^s)/\mathbb{N}$ for $r, s \geq 1$, where the first map is the identity on

the first factor. Now, locally on X we have the equation $X_1 \cdots X_r = \varpi$, where X_i are local equations defining the closed subschemes Y_i^1 of X. By definition, the X_i are local sections of $\widetilde{M}_1$, so ϖ is a local section of $\widetilde{M}_1$. But ϖ is also a global section of M and $\widetilde{M}_1 \hookrightarrow M$, so ϖ is a global section of $\widetilde{M}_1$. □

Lemma 3.4. *We have a cartesian diagram of maps of log schemes*

$$\begin{array}{ccc} (X, M) & \longrightarrow & (X, \widetilde{M}) \\ \downarrow & & \downarrow \\ (\operatorname{Spec} \mathcal{O}_K, \mathbb{N}) & \longrightarrow & (\operatorname{Spec} \mathcal{O}_K, \mathbb{N}^2) \end{array}$$

where the bottom horizontal arrow is the identity on the underlying schemes and maps $(a, b) \in \mathbb{N}^2$ *to* $a + b \in \mathbb{N}$.

Proof. We go back to the notation used in the proof of Lemma 3.2. Locally on X, we have the following commutative diagram of log schemes

$$\begin{array}{ccccc} (X, M) & \longrightarrow & \widetilde{U} \times_{\operatorname{Spec} \mathcal{O}_K[u]} \widetilde{V} \times W & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ (\operatorname{Spec} \mathcal{O}_K, \mathbb{N}) & \longrightarrow & (\operatorname{Spec} \mathcal{O}_K[u], \mathbb{N}) & \longrightarrow & (\operatorname{Spec} \mathcal{O}_K[\tau, \sigma], \mathbb{N}^2) \end{array}$$

where in the bottom row both τ and σ are mapped to u, which is in turn mapped to 0. The second square is cartesian and the horizontal maps in it are closed, but not exact, immersions. The first bottom map is an exact closed immersion, while the first top map is the composition of an étale morphism with an exact closed immersion. The lemma follows from the commutative diagram (3A.1) and the above diagram. □

3B. ***Variations on the logarithmic de Rham–Witt complex.*** Define the prelog structure $\mathbb{N}^2 \to W_n[\tau, \sigma]$ given by $(a, b) \mapsto \tau^a \sigma^b$. By abuse of notation, we write $(\operatorname{Spec} W_n[\tau, \sigma], \mathbb{N}^2)$ for the log scheme endowed with the associated log structure. We have the composite map of log schemes

$$(Y, \widetilde{M}) \to (\operatorname{Spec} k, \mathbb{N}^2) \to (\operatorname{Spec} W_n[\tau, \sigma], \mathbb{N}^2),$$

where $\mathbb{N}^2 \to \mathbb{N}^2$ is the obvious isomorphism. We shall call $(Z, \widetilde{N})$ a *lifting* for this morphism if $(Z, \widetilde{N})$ is a fine log scheme such that the composite map $(Y, \widetilde{M}) \to (\operatorname{Spec} W_n[\tau, \sigma], \mathbb{N}^2)$ factors through $f : (Y, \widetilde{M}) \to (Z, \widetilde{N})$ which is a closed immersion, and a map $(Z, \widetilde{N}) \to (\operatorname{Spec} W_n[\tau, \sigma], \mathbb{N}^2)$ which is log smooth. Such liftings always exists locally on Y and give rise to embedding systems as defined in Paragraph 2.18 of [Hyodo and Kato 1994]. If $(U, \widetilde{M}_U) \to (Y, \widetilde{M})$ is a covering and $(Z, \widetilde{N})$ is a lifting for $(U, \widetilde{M}_U) \to (\operatorname{Spec} W_n[\tau, \sigma]), \mathbb{N}^2)$, then we may define

an embedding system $((U^i, \widetilde{M}^i_U), (Z^i, \widetilde{N}^i))$ for $(Y, \widetilde{M}) \to (\mathrm{Spec}\, W_n[\tau,\sigma], \mathbb{N}^2)$ by taking the fiber product of $i+1$ copies of U over Y and of $i+1$ copies of $(Z, \widetilde{N})$ over $(\mathrm{Spec}\, W_n[\tau,\sigma], \mathbb{N}^2)$. Since $(Y, \widetilde{M})$ is an fs log scheme, we may assume the same for the local lifting $(Z, \widetilde{N})$.

Let $C^\bullet_{(Y,\widetilde{M})/(W_n,\mathrm{triv})}$ be the crystalline complex associated to the embedding system obtained from local liftings $(Z^\bullet, \widetilde{N}^\bullet)$, and define

$$\widetilde{\widetilde{C}}^\bullet_Y := C_{(Y,\widetilde{M})^\bullet/(W_n,\mathrm{triv})} \otimes_{W_n\langle\tau,\sigma\rangle} W_n.$$

Let $\mathrm{Spec}\, W_n[u]$ be endowed with the log structure associated to $1 \in \mathbb{N} \mapsto u \in W_n[u]$. Consider the map of log schemes $G : (\mathrm{Spec}\, W_n[u], \mathbb{N}) \to (\mathrm{Spec}\, W_n[\tau,\sigma], \mathbb{N}^2)$ given by $\tau, \sigma \mapsto u$ and $(a,b) \in \mathbb{N}^2 \mapsto a+b \in \mathbb{N}$. The pullback of $(Y, \widetilde{M})$ along G is just (Y, M). Let (Z', N') be the pullback of $(Z, \widetilde{N})$ along G, equipped with a map $f' : (Y', M') \to (Z', N')$ which is the pullback of f. Then (Z', N') is a (local) lifting for $(Y, M) \to (\mathrm{Spec}\, W_n[u], \mathbb{N})$, and gives rise to an embedding system for this morphism. Indeed, what we need to check is that $(Z', N') \to (\mathrm{Spec}\, W_n[u], \mathbb{N})$ is log smooth and that f' is a closed immersion of log schemes. For the first we note that log-smoothness is preserved under base change in the category of log schemes, and that

$$(Z', N') = \left(((Z, \widetilde{N}) \times_G (\mathrm{Spec}\, W_n[u], \mathbb{N}))^{\mathrm{int}}\right)^{\mathrm{sat}} \to (Z, \widetilde{N}) \times_G (\mathrm{Spec}\, W_n[u], \mathbb{N})$$

is log smooth. We also note that $g : Y \to (Z \times_{\mathrm{Spec}\, W_n[\tau,\sigma]} \mathrm{Spec}\, W_n[u])$ is a closed immersion, since $Y \to Z$ is a closed immersion. The morphism of schemes $Z' \to (Z \times_{\mathrm{Spec}\, W_n[\tau,\sigma]} \mathrm{Spec}\, W_n[u])$ is a composition of a finite morphism with a closed immersion, so $Y \to Z'$ is a closed immersion as well. Also, $g^*(\widetilde{N} \oplus_{\mathbb{N}^2} \mathbb{N}) \to M$ is surjective and factors through $(f')^*(N') \to M$, so $(f')^*(N') \to M$ is surjective as well.

We now follow the constructions in Section 3.6 of [Hyodo and Kato 1994] using the embedding system obtained from the liftings (Z', N'). Let $C^\bullet_{(Y,M)/(W_n,\mathrm{triv})}$ be the crystalline complex associated to the composite $(Z', N') \to (W_n, \mathrm{triv})$. Define

$$\widetilde{C}^\bullet_Y := C^\bullet_{(Y,M)/(W_n,\mathrm{triv})} \otimes_{W_n\langle u\rangle} W_n.$$

On the other hand, let $Z'' = Z' \times_{\mathrm{Spec}\, W_n[u]} \mathrm{Spec}\, W_n\langle u\rangle$ be endowed with N'' the inverse image of the log structure N'. Let $\mathscr{L}$ be the log structure on $\mathrm{Spec}\, W_n\langle u\rangle$ obtained by taking the inverse image of (the log structure associated to) $\mathbb{N}$ on $\mathrm{Spec}\, W_n[u]$. Then (Z'', N'') gives rise to an embedding system for

$$(Y, M) \to (\mathrm{Spec}\, W_n\langle u\rangle, \mathscr{L}),$$

with crystalline complex $C^\bullet_{(Y,M)/(\mathrm{Spec}\, W_n\langle u\rangle,\mathscr{L})}$. Define

$$C^\bullet_Y := C^\bullet_{(Y,M)/(\mathrm{Spec}\, W_n\langle u\rangle,\mathscr{L})} \otimes_{W_n\langle u\rangle} W_n.$$

Note that $C_Y^\bullet$ is the crystalline complex $C^\bullet_{(Y,M)/(W_n,\mathbb{N})}$ with respect to the embedding system obtained from $(Z' \times_{\mathrm{Spec}\, W_n[u]} \mathrm{Spec}\, W_n, N''')$. As in Section 3.6 of [Hyodo and Kato 1994], we have an exact sequence of complexes

$$0 \to C_Y^\bullet[-1] \to \widetilde{C}_Y^\bullet \to C_Y^\bullet \to 0, \tag{3B.1}$$

where the second arrow is $\wedge(du/u)$ and the third arrow is the canonical projection. The monodromy operator on the crystalline cohomology of (Y, M) is induced by the connecting homomorphism of this exact sequence.

Lemma 3.5. *Let $C_Z^\bullet$ be one of the complexes $\widetilde{\widetilde{C}}_Y^\bullet$, $\widetilde{C}_Y^\bullet$ or $C_Y^\bullet$ obtained with respect to a lifting $(Z, \widetilde{N})$ of some cover $U \to Y$. In the derived category, $C_Z^\bullet$ is independent of the choice of lifting $(Z, \widetilde{N})$.*

Proof. We may work étale locally on Y, in which case we have to show that for any two liftings $(Z_1, \widetilde{N}_1)$ and $(Z_2, \widetilde{N}_2)$ we have a canonical quasi-isomorphism between the corresponding complexes and moreover, that these quasi-isomorphisms satisfy the obvious cocycle condition for three different liftings.

First, we show that the complexes corresponding to $(Z_1, \widetilde{N}_1)$ and $(Z_2, \widetilde{N}_2)$ are quasi-isomorphic. We may assume that $i_i : (Y, \widetilde{M}) \to (Z_i, \widetilde{N}_i)$ is an exact closed immersion for $i = 1, 2$. Let $i_{12} : (Y, \widetilde{M}) \to (Z_1 \times_{W_n} Z_2, \widetilde{N}_{1\times 2})$ be the diagonal immersion of $(Y, \widetilde{M})$ into the fiber product of $(Z_1, \widetilde{N}_1)$ and $(Z_2, \widetilde{N}_2)$ as fs log schemes over (W_n, triv). Let $(Z_{12}, \widetilde{N}_{12})$ be a log scheme such that étale locally on Y we have a factorization of i_{12}

$$(Y, \widetilde{M}) \xrightarrow{f} (Z_{12}, \widetilde{N}_{12}) \xrightarrow{g} (Z_1 \times Z_2, \widetilde{N}_{1\times 2}),$$

with g log étale and f an exact closed immersion. This factorization is possible by Lemma 4.10 of [Kato 1989]. Let D_i be the PD-envelope of Y in Z_i (again, for $i = 1, 2$ or 12). (Since we have exact closed immersions, the logarithmic PD-envelope coincides with the usual PD-envelope in these cases.) It suffices to show that the canonical map

$$\omega^\bullet_{(Z_1,\widetilde{N}_1)/W_n,\mathrm{triv}} \otimes_{\mathcal{O}_{Z_1}} \mathcal{O}_{D_1} \to \omega^\bullet_{(Z_{12},\widetilde{N}_{12})/W_n,\mathrm{triv}} \otimes_{\mathcal{O}_{Z_{12}}} \mathcal{O}_{D_{12}} \tag{3B.2}$$

is a quasi-isomorphism. This follows from Paragraph 2.21 of [Hyodo and Kato 1994]. For completeness, we sketch the proof here. Let $p_1 : (Z_{12}, N_{12}) \to (Z_1, N_1)$ be the log-smooth map induced by projection onto the first factor. For any geometric point $\bar{y}$ of Y, the stalks at $\bar{y}$ of N_{12} and $p_1^* N_1$ coincide, so by replacing (Z_{12}, N_{12}) with an étale neighborhood of $\bar{y} \to Z_{12}$, we may assume that $N_{12} = p_1^* N_1$. Then the map $p_1 : Z_{12} \to Z_1$ is smooth in the usual sense. Since

the problem is étale local on Y, we may assume that $Z_{12} \simeq Z_1 \otimes_{W_n} W_n[t_1, \dots, t_r]$ for some positive integer r such that Y is contained in the closed subscheme of Z_{12} defined by $t_1 = \cdots = t_r = 0$. As in Proposition 6.5 of [Kato 1989], we also have $\mathcal{O}_{D_{12}} \simeq \mathcal{O}_{D_1}\langle t_1, \dots, t_r\rangle$, the PD-polynomial ring over $\mathcal{O}_{D_1}$ in r variables. The quasi-isomorphism (3B.2) is reduced then to the standard quasi-isomorphism

$$W_n \to \Omega_{W_n[t_1,\dots,t_r]} \otimes_{W_n[t_1,\dots,t_r]} W_n\langle t_1, \dots, t_r\rangle.$$

The quasi-isomorphism (3B.2) commutes with $\otimes_{W_n\langle\tau,\sigma\rangle} W_n$, so it induces a quasi-isomorphism

$$\tilde{\tilde{C}}^{\bullet}_{Z_1} \xrightarrow{\sim} \tilde{\tilde{C}}^{\bullet}_{Z_{12}}.$$

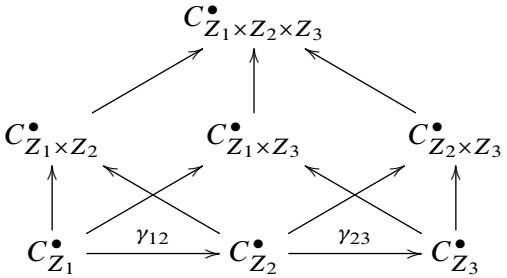
Now consider the morphism $Z'_{12} \to Z'_1$ obtained by pulling back $Z_{12} \to Z_1$ along G. We claim that the canonical morphisms $\tilde{C}^{\bullet}_{Z_{12}} \to \tilde{C}^{\bullet}_{Z_1}$ and $C^{\bullet}_{Z_{12}} \to C^{\bullet}_{Z_1}$ are quasi-isomorphisms as well. This is proved in the same way as in the case of $\tilde{\tilde{C}}^{\bullet}$ (for $C^{\bullet}_{Z_{12}} \to C^{\bullet}_{Z_1}$ this amounts to proving that the logarithmic de Rham–Witt complex is independent of the choice of embedding system). The quasi-isomorphisms are also compatible with the canonical maps $\tilde{\tilde{C}}^{\bullet}_Z \to \tilde{C}^{\bullet}_Z \to C^{\bullet}_Z$.

Note that the above result also implies that in the derived category, $C^{\bullet}$ commutes with étale base change. Indeed, if Y_2/Y_1 is étale and $(Z_1, \tilde{N}_1)$ is a lifting for $(Y_1, \tilde{M}) \to (\operatorname{Spec} W_n[\tau, \sigma], \mathbb{N}^2)$, then by [Grothendieck 1967, 18.1.1] we can find, Zariski locally on Y_2, an étale morphism $Z_2 \to Z_1$ such that the following diagram is cartesian

$$\begin{array}{ccc} Y_2 & \longrightarrow & Z_2 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Z_1 \end{array}$$

We take $\tilde{N}_2$ on Z_2 to be the inverse image of $\tilde{N}_1$. Then $(Z_2, \tilde{N}_2)$ is a lifting for $(Y_2, \tilde{M}) \to (\operatorname{Spec} W_n[\tau, \sigma], \mathbb{N}^2)$ and, since log differentials commute with étale base change [Kato 1989, Proposition 3.12], $C^{\bullet}_{(Z_2)}$ on Y_2 is just the pullback of $C^{\bullet}_{(Z_2)}$ on Y_1.

We are left with verifying the cocycle condition. The canonical quasi-isomorphism $\gamma_{12} : C^{\bullet}_{Z_1} \xrightarrow{\sim} C_{\bullet} Z_2^{\bullet}$ factors through $C^{\bullet}_{Z_1 \times Z_2}$, since by construction Z_{12} is log étale over $Z_1 \times Z_2$ and so we have a quasi-isomorphism $C^{\bullet}_{Z_1\times Z_2} \xrightarrow{\sim} C^{\bullet}_{Z_{12}}$. Let $(Z_3, \tilde{N}_3)$ be another lifting. Then we have the following commutative diagram of complexes:



where all the maps are quasi-isomorphisms. This proves the cocycle condition. □

Corollary 3.6. *The following sheaves on Y are well-defined and commute with étale base change*:

$$W_n\tilde{\tilde{\omega}}^q_Y := \mathcal{H}^q\big(\tilde{\tilde{C}}^\bullet_Y\big), \quad W_n\tilde{\omega}^q_Y := \mathcal{H}^q\big(\tilde{C}^\bullet_Y\big) \quad \text{and} \quad W_n\omega^q_Y := \mathcal{H}^q(C^\bullet_Y).$$

The sheaves $W_n\omega^q_Y$ make up the q-th terms of the log de Rham–Witt complex associated to (Y, M). We have canonical morphisms of sheaves on Y:

$$W_n\tilde{\tilde{\omega}}^q_Y \to W_n\tilde{\omega}^q_Y \to W_n\omega^q_Y.$$

In order to understand the monodromy N, we will study the short exact sequence of complexes

$$0 \to W_n\omega^\bullet_Y[-1] \to W_n\tilde{\omega}^\bullet_Y \to W_n\omega^\bullet_Y \to 0,$$

which we obtain below from the short exact sequence (3B.1). In Section 4 we will construct a resolution of this short exact sequence in terms of some subquotients of $W_n\tilde{\tilde{\omega}}^\bullet_Y$. For now, since these complexes are independent of the choice of lifting, we will fix some specific kinds of liftings of $(Y, \widetilde{M})$ over $(W[\tau,\sigma], \mathbb{N}^2)$, which we call *admissible liftings,* following the terminology used in [Hyodo 1991] and [Mokrane 1993]. Since Y is locally étale over

$$Y_{r,s,m} = \operatorname{Spec} k[X_1,\dots,X_n,Y_1,\dots,Y_n,Z_1,\dots Z_m]/(X_1\cdots X_r, Y_1\cdots Y_s),$$

we consider the lifting

$$Z_{r,s,m} = \operatorname{Spec} W[X_1,\dots,X_n,Y_1,\dots,Y_n,Z_1,\dots Z_m,\tau,\sigma]/(X_1\cdots X_r-\tau, Y_1\cdots Y_s-\sigma)$$

of $(Y_{r,s,m}, \mathbb{N}^r \oplus \mathbb{N}^s)/(W[\tau,\sigma], \mathbb{N}^2)$. The log structure on $Z_{r,s,m}$ is also induced from $\mathbb{N}^r \oplus \mathbb{N}^s$ (with the obvious structure map sending $\mathbb{N}^r$ to products of the X_i

and $\mathbb{N}^s$ to products of the Y_j). We let $Z/Z_{r,s,m}$ be étale and such that the diagram

$$\begin{array}{ccc} (Y,\widetilde{M}) & \longrightarrow & (Z,\widetilde{N}) \\ \downarrow & & \downarrow \\ (Y_{r,s,m},\mathbb{N}^r\oplus\mathbb{N}^s) & \longrightarrow & (Z_{r,s,m},\mathbb{N}^r\oplus\mathbb{N}^s) \end{array}$$

is cartesian, with the log structures on top obtained by pullback from the ones on the bottom. Then, locally on Y, the complexes $W_n\tilde{\tilde{\omega}}_Y^\bullet$, $W_n\tilde{\omega}_Y^\bullet$ and $W_n\omega_Y^\bullet$ are just pullbacks of the corresponding complexes on $Y_{r,s,m}$ with respect to the lifting $(Z_{r,s,m},\mathbb{N}^r\oplus\mathbb{N}^s)$. Note that admissible liftings exist locally on Y.

Now we will explain the relationships between $\widetilde{\widetilde{C}}_Y^\bullet$, $\widetilde{C}_Y^\bullet$ and $C_Y^\bullet$. First, note that we have the functoriality map $G^*\omega_{(Z,\widetilde{N})/(W_n,\mathrm{triv})}\to\omega_{(Z',N')/(W_n,\mathrm{triv})}$, which induces a canonical map

$$C^\bullet_{(Y,\widetilde{M})/(W_n,\mathrm{triv})}\otimes_{W_n\langle\tau,\sigma\rangle}W_n\langle u\rangle\to C^\bullet_{(Y,M)/(W_n,\mathrm{triv})},$$

which in turn induces a canonical map $\widetilde{\widetilde{C}}_Y^\bullet\to\widetilde{C}_Y^\bullet$. By composition, we also get a map $\widetilde{\widetilde{C}}_Y^\bullet\to C_Y^\bullet$. We claim that we can identify $\widetilde{C}_Y^\bullet$ with $\widetilde{\widetilde{C}}_Y^\bullet/(d\tau/\tau-d\sigma/\sigma)\wedge\widetilde{\widetilde{C}}_Y^\bullet$ and $C_Y^\bullet$ with $\widetilde{\widetilde{C}}_Y^\bullet/((d\tau/\tau)\wedge\widetilde{\widetilde{C}}_Y^\bullet+(d\sigma/\sigma)\wedge\widetilde{\widetilde{C}}_Y^\bullet)$. We explain this in the case of $\widetilde{C}_Y^\bullet$:

Lemma 3.7. *We have an isomorphism*

$$\widetilde{\widetilde{C}}_Y^\bullet\Big/\Big(\frac{d\tau}{\tau}-\frac{d\sigma}{\sigma}\Big)\wedge\widetilde{\widetilde{C}}_Y^{\bullet-1}\xrightarrow{\sim}\widetilde{C}_Y^\bullet.$$

Proof. Let $(Z,\widetilde{N})$ be an admissible lifting of $(Y,\widetilde{M})$ over $(\mathrm{Spec}\,W_n[\tau,\sigma],\mathbb{N}^2)$. Let $(D,\widetilde{M}_D)$ be the divided power envelope of $(Y,\widetilde{M})$ in $(Z,\widetilde{N})$. Note that the kernel of the map $\mathcal{O}_D\to\mathcal{O}_Y$ is generated by $\tau^{[n]}$ and $\sigma^{[n]}$. The divided power envelope (D',M'_D) of (Y,M) in (Z',N') satisfies the property

$$\mathcal{O}_{D'}\simeq\mathcal{O}_D\otimes_{W_n\langle\tau,\sigma\rangle}W_n\langle u\rangle,$$

where the map $W_n\langle\tau,\sigma\rangle\to W_n\langle u\rangle$ is given by $\tau^{[n]},\sigma^{[n]}\mapsto u^{[n]}$. The complexes $\widetilde{\widetilde{C}}_Y^\bullet$ and $\widetilde{C}_Y^\bullet$ are defined by

$$\begin{aligned}\widetilde{\widetilde{C}}_Y^\bullet&:=\big(\omega^\bullet_{(Z,\widetilde{N})/(W_n,\mathrm{triv})}\otimes_{\mathcal{O}_Z}\mathcal{O}_D\big)\otimes_{W_n\langle\tau,\sigma\rangle}W_n\langle u\rangle\otimes_{W_n\langle u\rangle}W_n\\&=\big(\omega^\bullet_{(Z,\widetilde{N})/(W_n,\mathrm{triv})}\otimes_{W_n[\tau,\sigma]}W_n[u]\big)\otimes_{\mathcal{O}_{Z'}}\mathcal{O}_{D'}\otimes_{W_n\langle u\rangle}W_n\end{aligned}$$

and

$$\widetilde{C}_Y^\bullet=(\omega^\bullet_{Z',N'/W_n,\mathrm{triv}})\otimes_{\mathcal{O}_{Z'}}\mathcal{O}_{D'}\otimes_{W_n\langle u\rangle}W_n.$$

Note that since we have chosen an admissible lifting, (Z',N') has $Z\times_{W_n[\tau,\sigma]}W_n[u]$ as its underlying scheme because $\widetilde{N}\oplus_{\mathbb{N}^2}\mathbb{N}$ is already fine and saturated. It is enough

to show that the sequence

$$\omega^{\bullet-1}_{(Z,\tilde{N})/(W_n,\mathrm{triv})} \otimes_{W_n[\tau,\sigma]} W_n[u] \to \omega^{\bullet}_{(Z,\tilde{N})/(W_n,\mathrm{triv})} \otimes_{W_n[\tau,\sigma]} W_n[u] \to \omega^{\bullet}_{(Z',N')/(W_n,\mathrm{triv})} \to 0 \quad (3\mathrm{B}.3)$$

is exact, where the first map is $\wedge(d\tau/\tau - d\sigma/\sigma)$ and the second map is induced by functoriality. We denote by G^* the pullback along $\operatorname{Spec} W_n[u] \to \operatorname{Spec} W_n[\tau,\sigma]$ or along $Z' \to Z$. By Proposition 3.12 of [Kato 1989], we have the following diagram of (vertical) exact sequences of sheaves on Z':

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
G^*\omega^1_{(\operatorname{Spec} W_n[\tau,\sigma],\mathbb{N}^2)/(W_n,\mathrm{triv})} \otimes_{W_n[u]} \mathcal{O}_{Z'} & \longrightarrow & \omega^1_{(\operatorname{Spec} W_n[u],\mathbb{N})/(W_n,\mathrm{triv})} \otimes_{W_n[u]} \mathcal{O}_{Z'} \\
\downarrow & & \downarrow \\
G^*\omega^1_{(Z,\tilde{N})/(W_n,\mathrm{triv})} & \longrightarrow & \omega^1_{(Z',N')/(W_n,\mathrm{triv})} \\
\downarrow & & \downarrow \\
G^*\omega^1_{(Z,\tilde{N})/(\operatorname{Spec} W_n[\tau,\sigma],\mathbb{N}^2)} & \longrightarrow & \omega^1_{(Z',N')/(\operatorname{Spec} W_n[u],\mathbb{N})} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

The bottom horizontal arrow is an isomorphism, since (Z', N') was obtained by pullback from $(Z, \tilde{N})$. In order to show that the middle horizontal arrow is a surjection, it is enough to check that du/u is in its image, but both $d\sigma/\sigma$ and $d\tau/\tau$ map to du/u. We also see similarly that the kernel of the middle horizontal arrow is generated by $d\tau/\tau - d\sigma/\sigma$. The exactness of (3B.3) follows. □

Corollary 3.8. *We have an isomorphism*

$$\tilde{\tilde{C}}^{\bullet}_Y \Big/ \Big(\frac{d\tau}{\tau} \wedge \tilde{\tilde{C}}^{\bullet-1}_Y + \frac{d\sigma}{\sigma} \wedge \tilde{\tilde{C}}^{\bullet-1}_Y \Big) \xrightarrow{\sim} C^{\bullet}_Y.$$

Proof. This follows from the exact sequence (3B.1) and Lemma 3.7. □

Lemma 3.9. *The sections $d\tau/\tau, d\sigma/\sigma \in W_n\tilde{\tilde{\omega}}^1_Y$ are global sections, independent of the choice of admissible lifting. The same holds for $du/u \in W_n\tilde{\omega}^1_Y$.*

Proof. We will explain the proof only for $d\tau/\tau$ since the same proof also works for $d\sigma/\sigma$ and du/u. We use basically the same argument as for part 3 of Lemma 3.4 of [Mokrane 1993]. We consider two admissible liftings of $(Y, \tilde{M})$, $(Z_1, \tilde{N}_1)$ and

$(Z_2, \widetilde{N}_2)$, and we let $(Z_{12}, \widetilde{N}_{12})$ be defined as in Lemma 3.5. It is enough to show that locally on Y

$$\frac{d\tau}{\tau} \in \omega^1_{(Z_1,\widetilde{N}_1)/(W_n,\mathrm{triv})} \otimes_{\mathbb{O}_{Z_1}} \mathbb{O}_{D_1}$$

and

$$\frac{d\tau'}{\tau'} \in \omega^1_{(Z_2,\widetilde{N}_2)/(W_n,\mathrm{triv})} \otimes_{\mathbb{O}_{Z_2}} \mathbb{O}_{D_2}$$

have the same image in $\mathcal{H}^1(\omega^\bullet_{(Z_{12},\widetilde{N}_{12})/(W_n,\mathrm{triv})} \otimes_{\mathbb{O}_{Z_{12}}} \mathbb{O}_{D_{12}})$.

Note that $d\tau/\tau \in \widetilde{N}_1$ and $d\tau'/d\tau' \in \widetilde{N}_2$ have the same image in $\widetilde{M}$. This is because locally on Y we have commutative diagrams

$$\begin{array}{ccc} (Y, \widetilde{M}) & \longrightarrow & (Z_i, \widetilde{N}_i) \\ \downarrow & & \downarrow \\ (k, \mathbb{N}^2) & \longrightarrow & (W_n[\tau,\sigma], \mathbb{N}^2) \end{array}$$

for $i = 1, 2$, so both $d\tau/\tau$ and $d\tau'/\tau'$ map to the image of $(1, 0) \in \mathbb{N}^2$ in $\widetilde{M}$. By the construction of $(Z_{12}, \widetilde{N}_{12})$ (see the proof of Proposition 4.10 of [Kato 1989]), we know that $d\tau/\tau - d\tau'/\tau' = m \in \widetilde{N}_{12}$. Moreover, if $\alpha_{12} : N_{12} \to \mathbb{O}_{Z_{12}}$ is the map defining the log structure of Z_{12} then m maps to $0 \in \widetilde{M}$, so $v = \alpha_{12}(m) \in \mathbb{O}^\times_{Z_{12}}$ maps to $1 \in \mathbb{O}_Y$. Therefore,

$$\frac{d\tau}{\tau} - \frac{d\tau'}{\tau'} = \frac{dv}{v}$$

for some $v \in \mathbb{O}_{D_{12}}$ for which $W_n\langle v-1\rangle \subseteq \mathbb{O}_{D_{12}}$. But then we see that $dv/v \in d(W_n\langle v-1\rangle)$, using the fact that the power series expansion of $\log(v)$ around 1 belongs to $W_n\langle v-1\rangle$. Therefore, $d\tau/\tau - d\tau'/\tau'$ is exact and the lemma follows. □

As in the classical case [Illusie and Raynaud 1983; Hyodo and Kato 1994], we can define operators $F : W_{n+1}\tilde{\omega}^q \to W_n\tilde{\omega}^q$, $V : W_n\tilde{\omega}^q \to W_{n+1}\tilde{\omega}^q$ and the differential $d : W_n\tilde{\omega}^q \to W_n\tilde{\omega}^{q+1}$, which satisfy

$$d^2 = 0, \quad FV = VF = p, \quad dF = pFd, \quad Vd = pdV \quad \text{and} \quad FdV = d.$$

Indeed, fix local liftings $(Z_n, \widetilde{N}_n)$ of $(Y, \widetilde{M}) \to (\mathrm{Spec}\, W_n[\tau,\sigma], \mathbb{N}^2)$ and denote the crystalline complex $\widetilde{C}^\bullet_{Z_n}$ by $\widetilde{C}^\bullet_n$. We can see that $\widetilde{C}^\bullet_n$ is flat over W_n in the same way as in Lemma 2.22 of [Hyodo and Kato 1994] (using an admissible lifting), and we have

$$\widetilde{C}^\bullet_n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\sim} \widetilde{C}^\bullet_m$$

for $m \le n$. We let $F : W_{n+1}\tilde{\omega}^\bullet \to W_n\tilde{\omega}^\bullet$ be the map induced by $\widetilde{C}^\bullet_{n+1} \to \widetilde{C}^\bullet_n$ and $V : W_n\tilde{\omega}^\bullet \to W_{n+1}\tilde{\omega}^\bullet$ be the map induced by $p : \widetilde{C}^\bullet_n \to \widetilde{C}^\bullet_{n+1}$. We define d to

be the connecting homomorphism in the exact sequence of cohomology sheaves associated to the exact sequence of crystalline complexes

$$0 \to \widetilde{\widetilde{C}}{}^{\bullet}_{n} \xrightarrow{p^n} \widetilde{\widetilde{C}}{}^{\bullet}_{2n} \to \widetilde{\widetilde{C}}{}^{\bullet}_{n} \to 0.$$

The same operators can be defined for $W_\bullet\tilde{\omega}^\bullet_Y$ and $W_\bullet\omega^\bullet_Y$.

Lemma 3.10. *Let $n = 1$. Locally, fix an admissible lifting $(Z, \tilde{N})$ as above. Let* Fr *be the relative Frobenius of Y/k. We have Cartier isomorphisms*

$$C^{-1} : \omega^q_Y \xrightarrow{\sim} \mathcal{H}^q(\mathrm{Fr}_*\,\omega^\bullet_Y),$$
$$\widetilde{C}^{-1} : \omega^q_{(Z',N')/(k,\mathrm{triv})} \otimes_{k[u]} k \xrightarrow{\sim} \mathcal{H}^q(\mathrm{Fr}_*(\omega^\bullet_{(Z',N')/k,\mathrm{triv}} \otimes_{k[u]} k)),$$
$$\widetilde{\widetilde{C}}{}^{-1} : \omega^q_{(Z,\tilde{N})/(k,\mathrm{triv})} \otimes_{k[t,s]} k \xrightarrow{\sim} \mathcal{H}^q(\mathrm{Fr}_*(\omega^\bullet_{(Z,\tilde{N})/k,\mathrm{triv}} \otimes_{k[t,s]} k)).$$

Proof. Note that $(Y, M)/(\mathrm{Spec}\,k, \mathbb{N})$ is log smooth of Cartier type. The Cartier isomorphism for $W_1\omega^q_Y$ is then defined in Section 2.12 of [Hyodo and Kato 1994]. Similarly, $(Z', N')/(\mathrm{Spec}\,k, \mathrm{triv})$ and $(Z, \tilde{N})/(\mathrm{Spec}\,k, \mathrm{triv})$ are log smooth and of Cartier type. Thus, the morphisms $\widetilde{C}^{-1}$ and $\widetilde{\widetilde{C}}{}^{-1}$ for $\widetilde{C}^q_Y$ and $\widetilde{\widetilde{C}}{}^q_Y$ are induced from the Cartier isomorphisms for these schemes.

Since we are working locally on Y, we may assume that $Y = Y_1 \times_k Y_2$ and that the lifting $Z = Z_1 \times Z_2$, where Z_1, Z_2 are smooth over k and Y_i is a reduced normal crossings divisor in Z_i. Let $\mathcal{I}_i$ be the ideal defining $Y_i \times_k Z_{3-i}$ in Z for $i = 1, 2$. Define $\omega^\bullet_{1,2} := \omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1 \oplus \omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_2$ To check that $\widetilde{\widetilde{C}}{}^{-1}$ is an isomorphism, we use the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
\omega^q_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1\mathcal{I}_2 & \longrightarrow & \omega^q_{1,2} & \longrightarrow & \omega^q_{(Z,\tilde{N})/k} & \longrightarrow & \widetilde{\widetilde{C}}{}^q_Y \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{H}^q(\mathrm{Fr}_*\,\omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1\mathcal{I}_2) & \longrightarrow & \omega^q_{1,2} & \longrightarrow & \mathcal{H}^q(\mathrm{Fr}_*\,\omega^\bullet_{(Z,\tilde{N})/k}) & \longrightarrow & \mathcal{H}^q(F_*\widetilde{\widetilde{C}}{}^\bullet_Y) \longrightarrow 0
\end{array}$$

The complex $\omega^\bullet_{(Z,\tilde{M})/k,\mathrm{triv}}$ is the same as $\Omega^\bullet_{Z_1/k}(\log Y_1) \otimes_k \Omega^\bullet_{Z_2/k}(\log Y_2)$, so it does satisfy a Cartier isomorphism, by [Deligne and Illusie 1987, 4.2.1.1]. Similarly, the complexes on its left are (sums of) products of complexes of the form $\Omega^\bullet_{Z_i/k}(\pm \log Y_i)$ for $i = 1, 2$, which also satisfy a Cartier isomorphism, by [Deligne and Illusie 1987, 4.2.1.3]. Therefore, the first three vertical arrows are isomorphisms. Once we know the exactness of the top and bottom sequence we can also deduce that the rightmost vertical arrow is an isomorphism. The exactness of the top row follows from the definition of $\widetilde{\widetilde{C}}{}^q_Y$.

The exactness of the bottom row follows from the cohomology long exact sequence associated to the short exact sequences obtained from the top row combined

with the Cartier isomorphisms for the first three vertical arrows, which tell us that the coboundary morphisms of these long exact sequences are all 0. Indeed, if we let $\bar{\omega}^\bullet_{(Z,\tilde{N})}$ be the complex obtained by completing the inclusion of complexes

$$\omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1\mathcal{I}_2 \to \omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1 \oplus \omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_2$$

to a distinguished triangle, then we get a long exact sequence

$$\cdots \to \mathcal{H}^q(\omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1\mathcal{I}_2) \to \mathcal{H}^q(\omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1) \oplus \mathcal{H}^q(\omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_2) \to \mathcal{H}^q(\bar{\omega}^\bullet_{(Z,\tilde{N})}) \to \cdots.$$

From the Cartier isomorphisms for $\omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1\mathcal{I}_2$ and $\omega^\bullet_{1,2}$, we deduce that

$$\mathcal{H}^q(\omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1\mathcal{I}_2) \hookrightarrow \mathcal{H}^q(\omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_1) \oplus \mathcal{H}^q(\omega^\bullet_{(Z,\tilde{N})/k} \otimes \mathcal{I}_2),$$

so the coboundaries of the long exact sequence are all 0. By continuing this argument, we deduce the exactness of the entire bottom row, and this proves that $\tilde{\tilde{C}}^{-1}$ is an isomorphism.

Now we prove that $\tilde{C}^{-1}$ is an isomorphism. We will show that $\tilde{C}^{-1}$ is an isomorphism in degree q as well. From the short exact sequence (3B.1), we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_Y^{q-1} & \longrightarrow & \tilde{C}_Y^q & \longrightarrow & C_Y^q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{H}^{q-1}(\mathrm{Fr}_*\, C_Y^\bullet) & \longrightarrow & \mathcal{H}^q(\mathrm{Fr}_*\, \tilde{C}_Y^\bullet) & \longrightarrow & \mathcal{H}^q(\mathrm{Fr}_*\, C_Y^\bullet) & \longrightarrow & 0
\end{array}$$

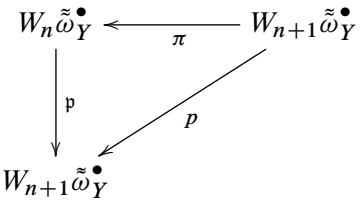
To see that the bottom row is exact, we have to check that in the long exact cohomology sequence associated to the top row the coboundaries are all 0, which is equivalent to showing surjectivity of $\mathcal{H}^q(\mathrm{Fr}_*\, \tilde{C}_Y^\bullet) \to \mathcal{H}^q(\mathrm{Fr}_*\, C_Y^\bullet)$. However, by the top row and the Cartier isomorphism C^{-1}, the composite

$$\tilde{C}_Y^q \to C_Y^q \to \mathcal{H}^q(\mathrm{Fr}_*\, C_Y^\bullet)$$

is surjective, so the desired map is surjective as well. Now we have a map of short exact sequences, where the left and right vertical maps are isomorphisms, so the middle one must be as well. □

We can define canonical projections $\pi : W_{n+1}\tilde{\tilde{\omega}}^\bullet_Y \to W_n\tilde{\tilde{\omega}}^\bullet_Y$ using the Cartier isomorphisms. The construction works in the same way for $W_n\tilde{\omega}^\bullet_Y$. The definition of π for $W_n\omega^\bullet_Y$ can be found in Section 1 of [Hyodo 1991] in the semistable case and in Section 4 of [Hyodo and Kato 1994] in general. The constructions in [Hyodo 1991] and in [Hyodo and Kato 1994] are the same, although they are formulated slightly differently. Our construction follows that in Section 1 of [Hyodo 1991],

by first defining a map $\mathfrak{p} : W_n\tilde{\tilde{\omega}}_Y^\bullet \to W_{n+1}\tilde{\tilde{\omega}}_Y^\bullet$ and then showing that $\mathfrak{p}$ is injective and its image coincides with the image of multiplication by p on $W_{n+1}\tilde{\tilde{\omega}}_Y^\bullet$. The projection π will then be the unique map which makes the following diagram commute:



The map $\mathfrak{p} : W_n\tilde{\tilde{\omega}}_Y^i \to W_{n+1}\tilde{\tilde{\omega}}_Y^i$ is induced from $p^{-i+1}\,\mathrm{Fr}^* : \tilde{\tilde{C}}_Y^i \to \tilde{\tilde{C}}_Y^i$, where $\mathrm{Fr} : (Z, \tilde{N}) \to (Z, \tilde{N})$ is a lifting of the Frobenius endomorphism of $(Z, \tilde{N}) \times_W k$ such that $\mathrm{Fr}^*(W[\tau, \sigma]) \subset W[\tau, \sigma]$. The injectivity of $\mathfrak{p}$ and the fact that its image coincides with that of multiplication by p are deduced as in Section 2 of [Hyodo 1991] (or as in Lemma 6.8 of [Nakkajima 2005]) from the Cartier isomorphism and from the fact that $\tilde{\tilde{C}}_Y^\bullet$ is W-torsion-free (when we take $\tilde{\tilde{C}}_Y^\bullet$ to be the crystalline complex associated to an embedding system for $(Y, \tilde{M})$ over W).

Now we will consider a different interpretation of the monodromy operator N. Taking the cohomology sheaves of the short exact sequence

$$0 \to C_Y^\bullet[-1] \to \tilde{C}_Y^\bullet \to C_Y^\bullet \to 0,$$

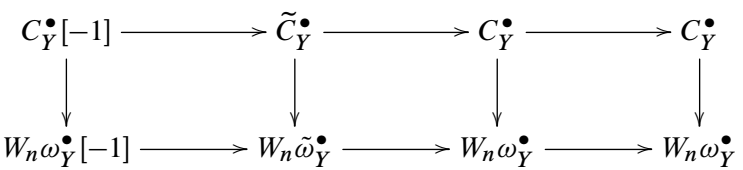
we get a long exact sequence of sheaves on Y

$$\cdots \to W_n\omega_Y^{q-1} \to W_n\tilde{\omega}_Y^q \to W_n\omega_Y^q \to \cdots$$

whose coboundaries are actually all 0. This can be checked as in Lemma 1.4.3 of [Hyodo 1991], since it suffices to see that the induced map on cocycles $Z^q(\tilde{C}_Y) \to Z^q(C_Y)$ modulo p^n is surjective, and we can use the Cartier isomorphisms in Lemma 3.10 to give an explicit formula for cocycles modulo p^n. So we have a short exact sequence of sheaves on Y

$$0 \to W_n\omega_Y^{q-1} \to W_n\tilde{\omega}_Y^q \to W_n\omega_Y^q \to 0, \tag{3B.4}$$

which is compatible with operators π, F, V and d. We have a morphism of distinguished triangles in the derived category $D(Y_{\text{ét}}, W)$ of sheaves of W-modules on Y:



The left and right vertical maps are defined in the proof of Theorem 4.19 of [Hyodo and Kato 1994], and the middle one can be defined in exactly the same way. Note that the definition of the maps in Theorem 4.19 has a gap which is corrected in Lemma 7.18 of [Nakkajima 2005], namely, checking that they commute with the transition morphisms $\pi : W_{n+1}\omega_Y^\bullet \to W_n\omega_Y^\bullet$. The fact that the middle map commutes with the transition morphisms $\pi : W_{n+1}\tilde{\omega}_Y^\bullet \to W_n\tilde{\omega}_Y^\bullet$ can be checked in the same way as in Lemma 7.18 of [Nakkajima 2005], using the corresponding Cartier isomorphism to check that the complexes $W_n\tilde{\omega}_Y^\bullet$ give rise to formal de Rham–Witt complexes as in Definition 6.1 of [ibid.] and thus applying Corollary 6.28(8). We also need to check that $\varprojlim W_n\tilde{\omega}_Y^1$ is torsion-free, but we can use the fact that this is known for $\varprojlim W_n\omega_Y^1$ and the exact sequence (3B.4). The first and third vertical maps are quasi-isomorphisms by Theorem 4.19 of [Hyodo and Kato 1994], so we get an isomorphism of distinguished triangles. Thus, the exact sequence (3B.4) induces the monodromy operator N on cohomology.

Assume that Y has an admissible lifting $\underline{Z}$ over $(W[t,s],\mathbb{N}^2)$, and set $Z = \underline{Z} \otimes_W k$. We consider a few more variations on the de Rham–Witt complex, which we will only define locally on Z. Let $W_n\Omega_Z^\bullet$ be the de Rham–Witt complex of Z. Let

$$Y^1 = \operatorname{Spec} k[X_1,\dots,X_n,Y_1,\dots,Y_n,Z_1,\dots,Z_m]/X_1\cdots X_r$$

and

$$Y^2 = \operatorname{Spec} k[X_1,\dots,X_n,Y_1,\dots,Y_n,Z_1,\dots,Z_m]/Y_1\cdots Y_s.$$

Each Y^i is a normal crossings divisor in $Z_{r,s,m}\times_W k$. Let $\mathcal{D}_n^i$ be the structure sheaf of the divided power envelope of Y^i in $Z_{r,s,m}$ and $\mathcal{J}\mathcal{D}_n^i = \ker(\mathcal{D}_n^i \to \mathcal{O}_{Y^i})$. For $i = 1,2$, let $W_n\Omega_Z^\bullet(-\log Y^i)$ be the (pullback to Z) of the "compact support" version of the de Rham–Witt complex of $Z_{r,s,m}$ with respect to Y^i. This complex was introduced by Hyodo [1991, Section 1] and is defined by

$$W_n\Omega^q_{Z_{r,s,m}}(-\log Y^i) = \mathcal{H}^q(\Omega^\bullet_{Z/W_n}(\log Y^i)\otimes_{\mathcal{O}_{Z_{r,s,m}}} \mathcal{J}\mathcal{D}_n^i).$$

Let $W_n\Omega_Z^\bullet(-\log Y^1 - \log Y^2)$ be the pullback from $Z_{r,s}$ to Z of the complex defined by

$$W_n\Omega^q_{Z_{r,s,m}}(-\log Y^1 - \log Y^2) := \mathcal{H}^q(\omega^\bullet_{Z_{r,s,m},\mathbb{N}^r\oplus\mathbb{N}^s/W_n}\otimes_{\mathcal{O}_Z}\mathcal{J}\mathcal{D}_1\mathcal{J}\mathcal{D}_2).$$

This third complex is meant to approximate a product of complexes of the form $W_n\Omega_Z(-\log Y)$. When $n=1$, consider $Z^1 = \operatorname{Spec} k[X_1,\dots,X_n,t]/(X_1\cdots X_r - t)$, $Z^2 = \operatorname{Spec} k[Y_1,\dots,Y_n,u]/(Y_1\cdots Y_s - u)$ and $Z^3 = \operatorname{Spec} k[Z_1,\dots,Z_m]$. Then

$$W_1\Omega^\bullet_{Z_{r,s,m}}(-\log Y^1 - \log Y^2) \simeq \Omega^\bullet_{Z^1/k}(-\log Y^1)\otimes_k \Omega^\bullet_{Z^2/k}(-\log Y^2)\otimes_k \Omega^\bullet_{Z^3/k}. \quad (3B.5)$$

All these also are endowed with operators F, V, differential d and projection π, and they also satisfy a Cartier isomorphism.

Let $\mathbb{R}$ be Raynaud's ring, introduced in [Illusie and Raynaud 1983], i.e., the graded W-algebra generated by F, V in degree 0 and d in degree 1, subject to the usual relations

$$d^2 = 0, \quad FV = VF = p, \quad dF = pFd, \quad Vd = pdV \quad \text{and} \quad FdV = d.$$

Let $\mathbb{R}_n$ be the right $\mathbb{R}$-module $\mathbb{R}/(V^n\mathbb{R} + dV^n\mathbb{R})$.

Lemma 3.11. *Let $W_n\Omega^\bullet$ be one of the complexes $W_n\Omega^\bullet_Z$, $W_n\Omega^\bullet_Z(-\log Y^i)$ for $i = 1, 2$ or $W_n\Omega^\bullet_Z(-\log Y^1 - \log Y^2)$. Let*

$$W\Omega^\bullet = \varprojlim W_n\Omega^\bullet.$$

Then $W\Omega^\bullet \otimes^L_{\mathbb{R}} \mathbb{R}_n = W_n\Omega^\bullet$.

Proof. For $n = 1$, and $W_n\Omega^\bullet_Z$ and $W_n\Omega^\bullet_Z(-\log Y^i)$, we have Cartier isomorphisms

$$W_1\Omega^i \xrightarrow{\sim} \mathscr{H}^i(F_*W_1\Omega^\bullet),$$

by [Deligne and Illusie 1987, Result 4.2.1.3]. For $W_n\Omega^\bullet_Z(-\log Y^1 - \log Y^2)$ the Cartier isomorphism follows from the product formula (3B.5) and from the Cartier isomorphisms above. Let $\mathscr{Z}_n = \underline{Z} \times_W W_n$. By abuse of notation, we write $\Omega^\bullet_{Z_n}$ for the complex of sheaves of W_n-modules such that

$$W_n\Omega^i = \mathscr{H}^i(\Omega^\bullet_{\mathscr{Z}_n}).$$

In fact, we have complexes $\Omega^\bullet_{\underline{Z}}$, $\Omega^\bullet_{\underline{Z}}(-\log Y^i)$ or $\Omega^\bullet_{\underline{Z}}(-\log Y^1 - \log Y^2)$ which give the corresponding complexes $\Omega^\bullet_{\mathscr{Z}_n}$, $\Omega^\bullet_{\mathscr{Z}_n}(-\log Y^i)$ or $\Omega^\bullet_{\mathscr{Z}_n}(-\log Y^1 - \log Y^2)$ when reduced modulo p^n. We also denote any of the initial complexes over W as $\Omega^\bullet_{\underline{Z}}$. Then there is an explicit description of cocycles modulo p^n given by

$$d^{-1}(p^n\Omega^{i+1}_{\underline{Z}}) = \sum_{k=0}^{n} p^k f^{n-k}\Omega^i_{\underline{Z}} + \sum_{k=0}^{n-1} f^k d\Omega^{i-1}_{\underline{Z}},$$

where $f : \Omega^i_{\underline{Z}} \to \Omega^i_{\underline{Z}}$ is defined by $f = \mathrm{Fr}/p^i$. This is the same as Formula A from Editorial Comment 11 in [Hyodo 1991] and is proven in the same way as in that paper and in the same way as in the classical crystalline cohomology case (see [Illusie 1979, 0.2.3.13]).

As in the case of $W_n\omega_Y$, $W_\bullet\Omega^\bullet$ (and $W\Omega^\bullet$) is endowed with a differential d, operators F, V satisfying the usual relations and a canonical projection $\pi_n : W_{n+1}\Omega^\bullet \to W_n\Omega^\bullet$ such that $\mathfrak{p} \circ \pi_n$ coincides with multiplication by p on $W_{n+1}\Omega^\bullet$.

We claim that the lemma follows from the Cartier isomorphism, from the description of cocycles modulo p^n in $\Omega^\bullet_{\underline{Z}}$ and from the formal properties of $W_n\Omega^\bullet$. The

proof is the same as for Lemma 1.3.3 of [Mokrane 1993]. We outline the argument in order to show that it applies to our case as well. To prove the desired result, we use the flat resolution of $\mathbb{R}_n$ as an $\mathbb{R}$-module given by

$$0 \to \mathbb{R} \xrightarrow{(F^n, -F^n d)} \mathbb{R} \oplus \mathbb{R} \xrightarrow{dV^n + V^n} \mathbb{R} \to \mathbb{R}_n \to 0,$$

and it suffices by Corollary 1.3.3 of [Illusie and Raynaud 1983] to prove that the sequence

$$0 \to W\Omega^{i-1} \xrightarrow{(F^n, -F^n d)} W\Omega^{i-1} \oplus W\Omega^i \xrightarrow{dV^n + V^n} W\Omega^i \to W_n\Omega^i \to 0$$

is exact. The last map is the canonical projection $\pi : W\Omega^i \to W_n\Omega^i$.

Exactness at the first term follows from the fact that multiplication by p (and hence also F) is injective on $W\Omega^\bullet$. Indeed, multiplication by p on $W_n\Omega^\bullet$ factors as $\mathfrak{p} \circ \pi_n$ and $\mathfrak{p}$ is injective by definition, so if $p(x_n) = 0$ for all n then $\pi_n(x_n) = x_{n-1} = 0$ for all n, so $x = (x_n) = 0$.

Exactness at the last term is the statement that π is surjective, which follows by construction, since $p = \mathfrak{p} \circ \pi$, $\mathfrak{p}$ is injective and the image of $\mathfrak{p} : W_n\Omega^\bullet \to W_{n+1}\Omega^\bullet$ coincides with the image of multiplication by p.

Now we check that $\ker \pi = dV^n W\Omega^\bullet + V^n W\Omega^\bullet$. Recall that $\pi_n : W_{n+1} \to W_n$ is the canonical projection. It is enough to show that $\ker \pi_n = dV^n W_1\Omega^\bullet + V^n W_1\Omega^\bullet$. First, if $x = V^n a + dV^n b \in W_{n+1}\Omega$, it suffices to check that $px = 0$ and indeed $px = FV^{n+1}a + dFV^{n+1}b = 0$. Now, let $[x]_{n+1} \in \ker \pi_n$, where x is an element of $\Omega^\bullet_{\underline{Z}}$ modulo p^{n+1}. Then $[px]_{n+1} = p[x]_{n+1} = 0$, so it must be the case that $px = p^{n+1}a + db$. We get $db = 0 \mod p$, so by the description of cocycles mod p we have $b = pb' + Fb'' + db''$, so that $db = pdb' + pFdb''$. Thus,

$$\begin{aligned}[x]_{n+1} &= [p^n a]_{n+1} + [db']_{n+1} + [Fdb'']_{n+1} \\ &= V^n[a]_{n+1} + d[p^n Fb'']_{n+1} = V^n[a] + dV^n[Fb''].\end{aligned}$$

Now we check exactness at the second term. First, note that the sequence

$$W_{2n}\Omega^{q-1} \xrightarrow{F^n} W_n\Omega^{q-1} \xrightarrow{d} W_n\Omega^q$$

is exact, which is proved in the same way as Lemma 1.3.4 of [Mokrane 1993], by taking the long exact sequence of cohomology sheaves of the short exact sequence

$$0 \to \Omega^\bullet_{\underline{Z}}/p^n\Omega^\bullet_{\underline{Z}} \xrightarrow{p^n} \Omega^\bullet_{\underline{Z}}/p^{2n}\Omega^\bullet_{\underline{Z}} \to \Omega^\bullet_{\underline{Z}}/p^n\Omega^\bullet_{\underline{Z}} \to 0.$$

We note that the proof of the analogous statement in the classical case in [Illusie 1979, I(3.21)] is wrong and corrected in [Illusie and Raynaud 1983, II(1.3)]. Nakkajima [2005, 6.28(6)] proves this statement for formal de Rham–Witt complexes, using the same argument as Lemma 1.3.4 of [Mokrane 1993].

We now claim that the projection

$$W\Omega^\bullet / p^n W\Omega^\bullet \to W_n\Omega^\bullet$$

is a quasi-isomorphism. This implies that

$$d^{-1}(p^n W\Omega^q) = F^n W\Omega^{q-1},$$

so if $dV^n x + V^n y = 0$, then $dx + p^n y = 0$, which in turn implies $x = F^n z$ and $y = -F^n dz$ for some $z \in W\Omega^{q-1}$. This checks exactness at the second term. Moreover, the fact that

$$W\Omega^\bullet / p^n W\Omega^\bullet \to W_n\Omega^\bullet$$

is a quasi-isomorphism follows in the same way as Corollary 3.17 of [Illusie 1979], boiling down to the Cartier isomorphism and to the description of $\ker \pi$ as $dV^n + V^n$. □

Remark 3.12. One can use the Cartier isomorphisms to check Properties 6.0.1–6.0.5 of [Nakkajima 2005] for $\Omega^\bullet_{\underline{Z}}, \Omega^\bullet_{\underline{Z}}(-\log Y^i)$ and $\Omega^\bullet_{\underline{Z}}(-\log Y^1 - \log Y^2)$, thus proving the analogue of Proposition 6.27 there for all three complexes. Then Theorem 6.24 of [Nakkajima 2005] also implies Lemma 3.11.

3C. ***The weight filtration.*** The goal of this section is to define a double filtration $P_{k,l}$ on $W\tilde{\tilde{\omega}}^\bullet_Y$, which will be an analogue of the weight filtration defined by Mokrane on $W_n\tilde{\omega}^\bullet_Y$ in the semistable case (see [1993, Section 3]).

Let $(Z, \widetilde{N})$ be an admissible lifting of $(Y, \widetilde{M})$ over $(W[\tau, \sigma], \mathbb{N}^2)$. We know that such liftings exist étale locally. Let $\mathfrak{Z}_n = Z \times_W W_n$. Let $\widetilde{N}_1$ be the log structure on Z (or $\mathfrak{Z}_n$) obtained by pulling back the log structure on $Z_{r,s,m}$ associated to

$$\begin{aligned} \mathbb{N}^r &\to W[X_1, \dots, X_n, Y_1, \dots, Y_n, Z_1, \dots, Z_m], \\ (0, \dots, 0, 1, 0, \dots, 0) &\mapsto X_i, \end{aligned}$$

where 1 is in the i-th position. Define $\widetilde{N}_2$ analogously. The pullback of $\widetilde{N}_i$ to Y is the same as $\widetilde{M}_i$. For $i = 1, 2$, we have maps of sheaves of monoids $\widetilde{N}_i \to \widetilde{N}$.

We define the following filtration on $\omega^\bullet_{(\mathfrak{Z}_n, \widetilde{N})/(W_n, \mathrm{triv})}$:

$$\begin{aligned} &P_{i,j}\omega^q_{(\mathfrak{Z}_n, \widetilde{N})/(W_n, \mathrm{triv})} \\ &\quad := \mathrm{Im}\big(\omega^i_{(\mathfrak{Z}_n, \widetilde{N}_1)/(W_n, \mathrm{triv})} \otimes \omega^j_{(\mathfrak{Z}_n, \widetilde{N}_2)/(W_n, \mathrm{triv})} \otimes \Omega^{q-i-j}_{\mathfrak{Z}_n/k} \to \omega^q_{(\mathfrak{Z}_n, \widetilde{N})/(W_n, \mathrm{triv})}\big) \end{aligned}$$

for $i, j \geq 0$ and $i + j \leq q$. This filtration respects the differential and induces a filtration $P_{i,j}\tilde{\tilde{C}}^\bullet_Y$ on $\tilde{\tilde{C}}^\bullet_Y$ (which can be thought of as a quotient of $\omega^\bullet_{(\mathfrak{Z}_n, \widetilde{M})/(W_n, \mathrm{triv})}$, as in the proof of Lemma 3.10). Note that if we let

$$P_k\omega^q_{(\mathfrak{Z}_n, \widetilde{N})/(W_n, \mathrm{triv})} = \mathrm{Im}\big(\omega^k_{(\mathfrak{Z}_n, \widetilde{N})/(W_n, \mathrm{triv})} \otimes \Omega^{q-k}_{\mathfrak{Z}_n/k} \to \omega^q_{(\mathfrak{Z}_n, \widetilde{N})/(W_n, \mathrm{triv})}\big),$$

then P_k is the weight filtration defined in [Mokrane 1993, 1.1.1], and

$$P_{i,j}\omega^\bullet_{(\mathfrak{X}_n,\tilde{N})/(W_n,\mathrm{triv})} \subset P_{i+j}\omega^\bullet_{(\mathfrak{X}_n,\tilde{N})/(W_n,\mathrm{triv})}.$$

For $i = 1,\dots,r$, let $D_{1,i}$ be the pullback to Z of the divisor of $Z_{r,s,m}$ obtained by setting $X_i = 0$. Similarly, for $i = 1,\dots,s$, let $D_{2,i}$ be the pullback to Z of the divisor of $Z_{r,s,m}$ obtained by setting $Y_i = 0$. For $i, j \geq 0$ let $D^{(i,j)}$ be the disjoint union of

$$D_{1,k_1} \times_Z \cdots \times_Z D_{1,k_i} \times_Z D_{2,l_1} \times_Z \cdots \times_Z D_{2,l_j},$$

over all $k_1,\dots,k_i \in \{1,\dots,r\}$ and $l_1,\dots,l_j \in \{1,\dots,s\}$. And let $\tau_{i,j} : D^{(i,j)} \to Z$ be the obvious morphism, with $\mathscr{D}_n^{(i,j)}$, $\tau_{i,j}$ the pullbacks to $\mathscr{X}_n$. Let

$$\begin{aligned}&\mathrm{Gr}_{i,j}\,\omega^q_{(\mathfrak{X}_n,\tilde{N})/(W_n,\mathrm{triv})}\\ &\quad := P_{i,j}\omega^q_{(\mathfrak{X}_n,\tilde{N})/(W_n,\mathrm{triv})}\Big/\big(P_{i-1,j}\omega^q_{(\mathfrak{X}_n,\tilde{N})/(W_n,\mathrm{triv})} + P_{i,j-1}\omega^q_{(\mathfrak{X}_n,\tilde{N})/(W_n,\mathrm{triv})}\big).\end{aligned}$$

For $i, j \geq 1$, we will define a morphism of sheaves

$$\mathrm{Res} : \mathrm{Gr}_{i,j}\,\omega^q_{(\mathfrak{X}_n,\tilde{N})/(W_n,\mathrm{triv})} \to (\tau_{i,j})_*\Omega^{q-i-j}_{\mathscr{D}_n^{(i,j)}/W_n},$$

which extends to a morphism of complexes. If

$$\omega = \alpha \wedge \frac{dX_{k_1}}{X_{k_1}} \wedge \cdots \wedge \frac{dX_{k_i}}{X_{k_i}} \wedge \frac{dY_{l_1}}{Y_{l_1}} \wedge \cdots \wedge \frac{dY_{k_j}}{Y_{k_j}}$$

is a local section of $P_{i,j}\omega^q_{(\mathfrak{X}_n,\tilde{N})/(W_n,\mathrm{triv})}$ with $k_1 < \cdots < k_i$ and $l_1 < \cdots < l_j$, then

$$\mathrm{Res}(\omega) := \alpha|_{D_{1,k_1}\times_Z\cdots\times_Z D_{1,k_i}\times_Z D_{2,l_1}\times_Z\cdots\times_Z D_{2,l_j}}.$$

This factors through $P_{i-1,j} + P_{i,j-1}$ and extends to a global map of sheaves.

Alternatively, we can follow the construction in Section 3 of Chapter II of [Deligne 1970]. Let $\mathscr{D}_n^k$ be the disjoint union of intersections of k divisors D_{j,k_i} with $j = 1, 2$ and $k_i \in \{1,\dots,n\}$. These intersections are in one-to-one correspondence with images of injections

$$f : \{1,\dots,k\} \to \{1,\dots,n\} \cup \{1,\dots,n\},$$

and so we denote one of these k intersections by $\mathscr{D}_n^f$ (even though it only really depends on $\mathrm{Im}\, f$). We have

$$\mathscr{D}_n^k = \bigsqcup_{\substack{i+j=k\\ i,j\geq 0}} \mathscr{D}_n^{i,j} = \bigsqcup_{\mathrm{Im}\, f} \mathscr{D}_n^f.$$

Let $\tau_f : \mathscr{D}_n^f \to \mathscr{Z}_n$ be the closed immersion. In [Deligne 1970, 3.5.2], a morphism

$$\rho_1 : (\tau_f)_* \Omega^{q-k}_{\mathscr{D}_n^f} \to P_k \omega^q_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} / P_{k-1}$$

(and then a morphism ρ_2, which depends on an ordering of $\{1, \dots, n\} \cup \{1, \dots, n\}$) is associated to each such injection, and the sum of ρ_2 over all injections f determines an isomorphism

$$\rho : (\tau_k)_* \Omega^{\bullet}_{\mathscr{D}_n^k / W_n}[-k] \xrightarrow{\sim} P_k \omega^q_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} / P_{k-1}$$

by Proposition 3.6 of Chapter II of [Deligne 1970].

We are only interested in injections $q_{i,j} : \{1, \dots, i+j\} \to \{1, \dots, n\} \cup \{1, \dots, n\}$ with image of cardinality i in the first $\{1, \dots, n\}$ term and cardinality j in the second $\{1, \dots, n\}$ term. We let Res^{-1} be the sum of the morphisms ρ_2 over all injections $q_{i,j}$. When we have an injection of type $q_{i,j}$, the image of the morphism ρ_2 defined by Deligne falls in

$$P_{i,j} \omega^q_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} / (P_{i-1,j} + P_{i,j-1}) \subset P_{i+j} \omega^q_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} / P_{i+j-1}.$$

For $k \geq 1$, we have the direct sum decompositions

$$P_k \omega^{\bullet}_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} / P_{k-1} = \bigoplus_{\substack{i+j=k \\ i,j \geq 0}} \mathrm{Gr}_{i,j}\, \omega^{\bullet}_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})},$$

$$(\tau_k)_* \Omega^{q-k}_{\mathscr{D}_n^{(k)} / W_n} = \bigoplus_{\substack{i+j=k \\ i,j \geq 0}} (\tau_{i,j})_* \Omega^{q-i-j}_{\mathscr{D}_n^{(i,j)} / W_n}.$$

It is easy to check that the isomorphism ρ matches up the (i, j) terms in each decomposition. Putting this discussion together, we get the following:

Lemma 3.13. *For $i, j \geq 1$, the map*

$$\mathrm{Res}^{-1} : (\tau_{i,j})_* \Omega^{q-i-j}_{\mathscr{D}_n^{(i,j)} / W_n} \to \mathrm{Gr}_{i,j}\, \omega^q_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})}$$

is an isomorphism.

We also have the following analogue of Lemma 1.2 of [Mokrane 1993].

Lemma 3.14. *We have an exact sequence of complexes*

$$\begin{aligned} 0 \to P_{i-1,j-1} \omega^{\bullet}_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} \\ \to P_{i-1,j} \omega^{\bullet}_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} \oplus P_{i,j-1} \omega^{\bullet}_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} \\ \to P_{i,j} \omega^{\bullet}_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} \to \mathrm{Gr}_{i,j}\, \omega^{\bullet}_{(\mathscr{Z}_n, \tilde{N})/(W_n, \mathrm{triv})} \to 0. \end{aligned}$$

The long exact cohomology sequence(s) associated to this have all coboundaries 0, *so we get the exact sequence*

$$0 \to \mathcal{H}^q(P_{i-1,j-1}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})}) \to \mathcal{H}^q(P_{i-1,j}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})}) \oplus \mathcal{H}^q(P_{i,j-1}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})}) \to \mathcal{H}^q(P_{i,j}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})}) \to \mathcal{H}^q(\Omega^\bullet_{\mathscr{D}_n^{(i,j)}/W_n}[-i-j]) \to 0.$$

Proof. The first assertion is clear. In order to show that the second sequence is exact, it suffices to show the following two statements about cocycles:

(1) $$ZP_{i,j}\omega^q_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})} \twoheadrightarrow Z\Omega^{q-i-j}_{\mathscr{D}_n^{(i,j)}/W_n}.$$

(2) $$ZP_{i-1,j}\omega^q_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})} \oplus ZP_{j,i-1}\omega^q_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})} \twoheadrightarrow Z\big(P_{i-1,j}\omega^q_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})} + P_{i,j-1}\omega^q_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})}\big).$$

The first statement is proved in the same way as the main step in Lemma 1.1.2 of [Mokrane 1993]. If α is a local section of $Z\Omega^{q-i-j}_{\mathscr{D}_n^{(i,j)}/W_n}$, assume that α is supported on some

$$D_{1,k_1} \times_Z \cdots \times_Z D_{1,k_i} \times_Z D_{2,l_1} \times_Z \cdots \times_Z D_{2,l_j},$$

for some $k_1, \ldots, k_i, l_1, \ldots, l_j \in \{1, \ldots, n\}$. Let

$$\rho : \mathscr{Z}_n \to D_{1,k_1} \times_Z \cdots \times_Z D_{1,k_i} \times_Z D_{2,l_1} \times_Z \cdots \times_Z D_{2,l_j}$$

be the retraction associated to the immersion

$$D_{1,k_1} \times_Z \cdots \times_Z D_{1,k_i} \times_Z D_{2,l_1} \times_Z \cdots \times_Z D_{2,l_j} \to \mathscr{Z}_n.$$

Then $\rho^*\alpha$ lifts α to a section of $Z\Omega^{q-i-j}_{\mathscr{Z}_n/W_n}$ and the section

$$\omega_\alpha = \rho^*\alpha \wedge \frac{dX_{k_1}}{X_{k_1}} \wedge \cdots \wedge \frac{dX_{k_i}}{X_{k_i}} \wedge \frac{dY_{l_1}}{Y_{l_1}} \wedge \cdots \wedge \frac{dY_{l_j}}{Y_{l_j}} \in P_{i,j}\omega^q_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})}$$

satisfies $d\omega = 0$ and $\mathrm{Res}(\omega) = \alpha$. From this, we know that the coboundaries of the long exact sequence associated to

$$0 \to P_{i-1,j}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})} + P_{i,j-1}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})} \to P_{i,j}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})} \to \mathrm{Gr}_{i,j}\,\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})} \to 0$$

are 0, so we also know that

$$\mathcal{H}^q\big(P_{i-1,j}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})} + P_{i,j-1}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})}\big) \hookrightarrow \mathcal{H}^q(P_{i,j}\omega^\bullet_{(\mathscr{Z}_n,\widetilde{N})/(W_n,\mathrm{triv})})$$

for every $i, j \geq 1$.

For the second statement, we have to prove that if $\alpha \in P_{i-1,j}\omega^q_{(\mathfrak{X}_n,\widetilde{N})/(W_n,\mathrm{triv})}$ and $\beta \in P_{i,j-1}\omega^q_{(\mathfrak{X}_n,\widetilde{N})/(W_n,\mathrm{triv})}$ satisfy $d(\alpha+\beta)=0$, then we can find $\alpha' \in ZP_{i-1,j}\omega^q_{(\mathfrak{X}_n,\widetilde{N})/(W_n,\mathrm{triv})}$ and $\beta' \in ZP_{i,j-1}\omega^q_{(\mathfrak{X}_n,\widetilde{N})/(W_n,\mathrm{triv})}$ such that $\alpha'+\beta' = \alpha+\beta$. If $\alpha \in P_{i-1,j-1}\omega^q_{(\mathfrak{X}_n,\widetilde{N})/(W_n,\mathrm{triv})}$, then we are done, since we can just take $\alpha'=0$, $\beta'=\alpha+\beta$. The same holds for β. Otherwise, we have $d\alpha \in P_{i-1,j-1}$, so by the injectivity proved in statement (1) for $(i-1,j)$, we know that $d\alpha = d\alpha_1 + d\alpha_2$ for some $\alpha_1 \in P_{i-1,j-1}$ and $\alpha_2 \in P_{i-2,j}$. Thus, we've reduced our problem from $(i-1,j)$ to $(i-2,j)$. Proceeding by induction, we may assume that $i=0$. In that case $d\alpha_{2i} \in P_{0,j-1}$. By (the same argument as in the proof of) Lemma 1.1.2 of [Mokrane 1993], we have an injection

$$\mathcal{H}^q(P_{0,j-1}\omega^\bullet_{(\mathfrak{X}_n,\widetilde{N})/(W_n,\mathrm{triv})}) \hookrightarrow \mathcal{H}^q(P_{0,j}\omega^\bullet_{(\mathfrak{X}_n,\widetilde{N})/(W_n,\mathrm{triv})}),$$

which implies $d\alpha_{2i} = d\alpha_{2i+1}$ for some $\alpha_{2i+1} \in P_{0,j-1}$. Then

$$\alpha' := \alpha - \sum_{i'=0}^{i} \alpha_{2i'+1} \in ZP_{i-1,j}, \quad \beta' := \beta + \sum_{i'=0}^{i} \alpha_{2i'+1} \in ZP_{i,j-1}$$

satisfy the desired relations. □

The double filtration $P_{i,j}$ on $\omega^\bullet_{(\mathfrak{X}_n,\widetilde{N})/(W_n,\mathrm{triv})}$ induces a double filtration $P_{i,j}$ on $\widetilde{\widetilde{C}}_{\mathfrak{X}_n}$, and, for $i,j \geq 1$, the residue morphism $\mathrm{Res} : P_{i,j}\omega^q_{(\mathfrak{X}_n,\widetilde{N})/(W_n,\mathrm{triv})} \to \Omega^{q-i-j}_{\mathfrak{D}_n^{(i,j)}/W_n}$ factors through $P_{i,j}\widetilde{\widetilde{C}}_{\mathfrak{X}_n}$.

Lemma 3.15. *For any two admissible liftings $(Z_1,\widetilde{N})$ and $(Z_2,\widetilde{N})$ of $(Y,\widetilde{M})$, we have a canonical isomorphism*

$$\alpha_{Z_1Z_2} : \mathcal{H}^q(P_{i,j}\widetilde{\widetilde{C}}^\bullet_{\mathfrak{X}_{1,n}}) \to \mathcal{H}^q(P_{i,j}\widetilde{\widetilde{C}}^\bullet_{\mathfrak{X}_{2,n}})$$

satisfying the cocycle condition for any three admissible liftings.

Moreover, the residue morphism

$$\mathrm{Res}_Z : \mathcal{H}^q(P_{i,j}\widetilde{\widetilde{C}}^\bullet_{\mathfrak{X}_n}) \to \mathcal{H}^{q-i-j}(\Omega^\bullet_{\mathfrak{D}_n^{(i,j)}/W_n}) \simeq W_n\Omega^q_{Y^{(i,j)}}$$

induced on cohomology satisfies the compatibility

$$\mathrm{Res}_{Z_1} = \mathrm{Res}_{Z_2} \circ \alpha_{Z_1Z_2}.$$

Proof. The proof of the first part is basically the same as the proof of Lemma 3.5. We take admissible lifts $(Z_1,\widetilde{N})$ and $(Z_2,\widetilde{N})$ (we denote the log structures on both simply by $\widetilde{N}$, as it will be understood from the context which is the underlying

scheme). As in the proof of Lemma 3.5, we form $(Z_{12}, \widetilde{N})$, which is smooth over $(Z_i, \widetilde{N})$, even though it is not quite an admissible lift. However, Z_{12} is étale over

$$\operatorname{Spec} W[X_1, \dots, X_n, Y_1, \dots, Y_n, X_1', \dots X_n', Y_1', \dots, Y_n', v_1^{\pm 1}, \dots, v_r^{\pm 1}, u_1^{\pm 1}, \dots u_s^{\pm 1}] / (X_i v_i - X_i'.Y_j v_j - Y_j').$$

So we can endow $\widetilde{\widetilde{C}}^{\bullet}_{\mathscr{X}_{12,n}}$ with a filtration $P_{i,j}\widetilde{\widetilde{C}}^{\bullet}_{\mathscr{X}_{12},n}$ defined as above, in terms of log structures $\widetilde{N}_1$ and $\widetilde{N}_2$ (which come from formally "inverting" the X_i and X_i' or the Y_i and Y_i'). Then the same argument used in the proof of Lemma 3.5 gives us quasi-isomorphisms

$$P_{i,j}\widetilde{\widetilde{C}}_{\mathscr{X}_{i,n}} \to P_{i,j}\widetilde{\widetilde{C}}_{\mathscr{X}_{12,n}}$$

for $i = 1, 2$, which satisfy the right compatibility condition for three admissible lifts.

For the second part, we follow the argument in Lemma 3.4(2) of [Mokrane 1993]. We let

$$\omega = \alpha \wedge \frac{dX_{k_1}}{X_{k_1}} \wedge \cdots \wedge \frac{dX_{k_i}}{X_{k_i}} \wedge \frac{dY_{l_1}}{Y_{l_1}} \wedge \cdots \wedge \frac{dY_{l_i}}{Y_{l_i}}$$

be a section of $P_{i,j}\omega^q_{(\mathscr{X}_{1,n},\widetilde{N})/(W_n,\mathrm{triv})}$ and

$$\omega' = \alpha' \wedge \frac{dX'_{k_1}}{X'_{k_1}} \wedge \cdots \wedge \frac{dX'_{k_i}}{X'_{k_i}} \wedge \frac{dY'_{l_1}}{Y'_{l_1}} \wedge \cdots \wedge \frac{dY'_{l_i}}{Y'_{l_i}}$$

be a section of $P_{i,j}\omega^q_{(\mathscr{X}_{2,n},\widetilde{N})/(W_n,\mathrm{triv})}$ such that $\omega = \omega'$ in $P_{i,j}\omega^q_{(\mathscr{X}_{12,n},\widetilde{N})/(W_n,\mathrm{triv})}$. We have to check that $\alpha|_{\mathscr{D}^{(i,j)}_{12,n}} = \alpha'|_{\mathscr{D}^{(i,j)}_{12,n}}$. But

$$\omega - \omega' = (\alpha - \alpha') \wedge \frac{dX_{k_1}}{X_{k_1}} \wedge \cdots \wedge \frac{dX_{k_i}}{X_{k_i}} \wedge \frac{dY_{l_1}}{Y_{l_1}} \wedge \cdots \wedge \frac{dY_{l_i}}{Y_{l_i}} + \Psi,$$

where $\Psi \in P_{i,j-1}\omega^q_{(\mathscr{X}_{2,n},\widetilde{N})/(W_n,\mathrm{triv})} + P_{i-1,j}\omega^q_{(\mathscr{X}_{2,n},\widetilde{N})/(W_n,\mathrm{triv})}$. This means that

$$(\alpha - \alpha') \wedge \frac{dX_{k_1}}{X_{k_1}} \wedge \cdots \wedge \frac{dX_{k_i}}{X_{k_i}} \wedge \frac{dY_{l_1}}{Y_{l_1}} \wedge \cdots \wedge \frac{dY_{l_i}}{Y_{l_i}}$$

is also a section of $P_{i,j-1}\omega^q_{(\mathscr{X}_{2,n},\widetilde{N})/(W_n,\mathrm{triv})} + P_{i-1,j}\omega^q_{(\mathscr{X}_{2,n},\widetilde{N})/(W_n,\mathrm{triv})}$, and so $(\alpha - \alpha')|_{\mathscr{D}^{(i,j)}_{12,n}} = 0$. □

Corollary 3.16. *We can define the sheaves*

$$P_{i,j}W_n\tilde{\omega}^q_Y := \mathscr{H}^q(P_{i,j}\widetilde{\widetilde{C}}^{\bullet}_Y).$$

The complexes $P_{i,j}W_n\tilde{\tilde{\omega}}_Y^\bullet$ form an increasing double filtration of $W_n\tilde{\tilde{\omega}}_Y^\bullet$ such that the graded pieces for $i, j \geq 1$

$$\mathrm{Gr}_{i,j}\, W_n\tilde{\tilde{\omega}}_Y^\bullet := P_{i,j}W_n\tilde{\tilde{\omega}}_Y^\bullet / P_{i,j-1} + P_{i-1,j}$$

are canonically isomorphic to the de Rham–Witt complexes of the smooth subschemes $Y^{(i,j)}$:

$$\mathrm{Res} : \mathrm{Gr}_{i,j}\, W_n\tilde{\tilde{\omega}}_Y^\bullet \xrightarrow{\sim} W_n\Omega^\bullet_{Y^{(i,j)}}[-i-j](-i-j).$$

Lemma 3.17. *The constructions in Section 3C are compatible with the transition morphisms π, in the following way*:

(1) *The following diagrams are commutative*:

$$\begin{array}{ccc} W_{n+1}\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\pi} & W_n\tilde{\tilde{\omega}}_Y^q \\ {\scriptstyle \wedge\frac{d\tau}{\tau}}\downarrow & & \downarrow{\scriptstyle \wedge\frac{d\tau}{\tau}} \\ W_{n+1}\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\pi} & W_n\tilde{\tilde{\omega}}_Y^q \end{array} \qquad \text{and} \qquad \begin{array}{ccc} W_{n+1}\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\pi} & W_n\tilde{\tilde{\omega}}_Y^q \\ {\scriptstyle \wedge\frac{d\sigma}{\sigma}}\downarrow & & \downarrow{\scriptstyle \wedge\frac{d\sigma}{\sigma}} \\ W_{n+1}\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\pi} & W_n\tilde{\tilde{\omega}}_Y^q \end{array}$$

(2) *The projection $\pi : W_{n+1}\tilde{\tilde{\omega}}_Y^q \to W_n\tilde{\tilde{\omega}}_Y^q$ preserves the weight filtration $P_{i,j}$ on $W_m\tilde{\tilde{\omega}}_Y^q$ for $m = n, n+1$.*

(3) *The morphism $\pi : P_{i,j}W_{n+1}\tilde{\tilde{\omega}}_Y^q \to P_{i,j}W_n\tilde{\tilde{\omega}}_Y^q$ is surjective.*

Proof. The first part follows in the same way as Proposition 8.1 of [Nakkajima 2005], by using a local admissible lifting $(Z, \tilde{N})$ of $(Y, \tilde{M})$ together with a lift of the Frobenius Φ. Then $\Phi^*(\tau) = \tau^p(1 + pu)$ for some

$$u \in \mathbb{O}_Z \otimes_{W[\tau,\sigma]} W_n\langle\tau, \sigma\rangle$$

and so $\Phi^*(d\log\tau)$ is equivalent to $pd\log\tau$ modulo an exact form. The same holds for σ.

The second part follows in the same way as Proposition 8.4 of [Nakkajima 2005]. The question is local, so we may assume that the admissible lift $(Z, \tilde{N})$ is étale over $\mathrm{Spec}\, W[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$, $\mathbb{N}^r \oplus \mathbb{N}^s$. First we see that, for a lift Φ of Frobenius we have that $\Phi^*(d\log X_i)$ is equivalent modulo an exact form to $pd\log X_i$ for $1 \leq i \leq r$ and that $\Phi^*(d\log Y_j)$ is equivalent modulo an exact form to $pd\log Y_j$ for $1 \leq j \leq s$. This implies that the map $\mathfrak{p} : W_n\tilde{\tilde{\omega}}_Y^q \to W_{n+1}\tilde{\tilde{\omega}}_Y^q$ preserves the weight filtration $P_{i,j}$.

In order to see that $\pi : W_{n+1}\tilde{\tilde{\omega}}_Y^q \to W_n\tilde{\tilde{\omega}}_Y^q$ also preserves $P_{i,j}$ we use a descending induction on (i, j) in lexicographic order. Note that $P_{r,s}W_n\tilde{\tilde{\omega}}_Y^q = W_n\tilde{\tilde{\omega}}_Y^q$, so there is nothing to prove in this case. We can prove the result for $(r, s-1)$ in the same way as Proposition 8.4(2) of [Nakkajima 2005], using the commutative diagrams

$$\begin{array}{ccc} P_{i,j}W_{n+1}\tilde{\tilde{\omega}}^q_Y & \xrightarrow{\text{Res}} & W_{n+1}\Omega^{q-i-j}_{Y^{(i,j)}} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ P_{i,j}W_n\tilde{\tilde{\omega}}^q_Y & \xrightarrow{\text{Res}} & W_n\Omega^{q-i-j}_{Y^{(i,j)}} \end{array}$$

for (i, j) successively equal to $(r, s), (r-1, s), \ldots, (1, s)$. At the last step we get a commutative diagram of (vertical) exact sequences

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ P_{r,s-1}W_{n+1}\tilde{\tilde{\omega}}^q_Y + P_{0,s}W_{n+1}\tilde{\tilde{\omega}}^q_Y & & P_{r,s-1}W_n\tilde{\tilde{\omega}}^q_Y + P_{0,s}W_n\tilde{\tilde{\omega}}^q_Y \\ \downarrow & & \downarrow \\ P_{r,s-1}W_{n+1}\tilde{\tilde{\omega}}^q_Y + P_{1,s}W_{n+1}\tilde{\tilde{\omega}}^q_Y & \xrightarrow{\pi} & P_{r,s-1}W_n\tilde{\tilde{\omega}}^q_Y + P_{1,s}W_n\tilde{\tilde{\omega}}^q_Y \\ \downarrow & & \downarrow \\ W_{n+1}\Omega^{q-s-1}_{Y^{(1,s)}} & \xrightarrow{\pi} & W_n\Omega^{q-s-1}_{Y^{(1,s)}} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

which means there is an induced morphism $\pi : P_{r,s-1}W_{n+1}\tilde{\tilde{\omega}}^q_Y + P_{0,s}W_{n+1}\tilde{\tilde{\omega}}^q_Y \to P_{r,s-1}W_n\tilde{\tilde{\omega}}^q_Y + P_{0,s}W_n\tilde{\tilde{\omega}}^q_Y$.

At this stage, we note that we can define

$$Y^{(0,s)} = \coprod_{\substack{T\subseteq\{1,\ldots,n\} \\ \#T=s}} \left(\bigcap_{i\in T} Y_i^2\right).$$

This will be a simple reduced normal crossings divisor over k, and we can endow it with the pullback of the log structure $\widetilde{M}_1$ so that $(Y, \widetilde{M})$ is a $(k, \mathbb{N})$-semistable log scheme, in the terminology of Section 2.4 of [Mokrane 1993]. There is a surjective residue morphism obtained via the restriction

$$P_{i,j}W_n\tilde{\tilde{\omega}}^q_Y \xrightarrow{\text{Res}} P_iW_n\tilde{\omega}^{q-j}_{Y^{(0,j)}},$$

which respects the weight filtrations. Just as the commutative diagram 8.4.3 of [Nakkajima 2005] is obtained, we can use the injectivity of $\mathfrak{p} : W_n\tilde{\omega}^q_{Y^{(0,s)}} \to W_{n+1}\tilde{\omega}^q_{Y^{(0,s)}}$ for $Y^{(0,s)}/k$ [Nakkajima 2005, Corollary 6.28(2)] to see that there is

a commutative diagram

$$\begin{array}{ccc} P_{0,s}W_{n+1}\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\text{Res}} & P_0W_{n+1}\tilde{\omega}_{Y(0,s)}^{q-s} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ P_{0,s}W_n\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\text{Res}} & P_0W_n\tilde{\omega}_{Y(0,s)}^{q-s} \end{array}$$

We therefore get a commutative diagram of (vertical) exact sequences:

$$\begin{array}{ccccc} 0 & & & & 0 \\ \downarrow & & & & \downarrow \\ P_{r,s-1}W_{n+1}\tilde{\tilde{\omega}}_Y^q & & & & P_{r,s-1}W_n\tilde{\tilde{\omega}}_Y^q \\ \downarrow & & & & \downarrow \\ P_{r,s-1}W_{n+1}\tilde{\tilde{\omega}}_Y^q + P_{0,s}W_{n+1}\tilde{\tilde{\omega}}_Y^q & & \xrightarrow[\pi]{} & & P_{r,s-1}W_n\tilde{\tilde{\omega}}_Y^q + P_{0,s}W_n\tilde{\tilde{\omega}}_Y^q \\ \downarrow & & & & \downarrow \\ P_0W_{n+1}\tilde{\omega}_{Y(0,s)}^{q-s} & & \xrightarrow[\pi]{} & & P_0W_n\tilde{\omega}_{Y(0,s)}^{q-s} \\ \downarrow & & & & \downarrow \\ 0 & & & & 0 \end{array}$$

so there is an induced morphism $\pi : P_{r,s-1}W_{n+1}\tilde{\tilde{\omega}}_Y^q \to P_{r,s-1}W_n\tilde{\tilde{\omega}}_Y^q$.

Finally, the third part follows in the same way as Corollary 8.6.4 of [Nakkajima 2005]. For an admissible lift $(Z, \tilde{N})$, let $Z_1 := Z \times_W k$. We have surjective morphisms $W_n\Omega_{Z_1}^q \to P_{0,0}W_n\tilde{\tilde{\omega}}_Y^q$, which commute with the transition morphisms π. So π is surjective for $P_{0,0}$. Using the exact sequences of the form

$$0 \to P_{0,j-1}W_n\tilde{\tilde{\omega}}_Y^q \to P_{0,j}W_n\tilde{\tilde{\omega}}_Y^q \to P_0W_n\tilde{\omega}_{Y(0,j)}^{q-j} \to 0$$

and the surjectivity of π on the third term, we prove by induction on j that π is surjective for $P_{0,j}$. The same statement holds for $P_{i,0}$. Then, we prove that π is surjective for a general $P_{i,j}$ by induction on $i + j$, using the exact sequences of the form

$$0 \to P_{i-1,j}W_n\tilde{\tilde{\omega}}_Y^q + P_{i,j-1}W_n\tilde{\tilde{\omega}}_Y^q \to P_{i,j}W_n\tilde{\tilde{\omega}}_Y^q \to W_n\Omega_{Y(i,j)}^{q-i-j} \to 0. \qquad \square$$

4. Generalizing the Mokrane spectral sequence

In this section, we derive a generalization of the Mokrane spectral sequence which will allow us to compute the log crystalline cohomology of the Shimura varieties we are interested in terms of the crystalline cohomology of certain proper smooth

Newton polygon strata in the special fiber. Mokrane's spectral sequence applies to the case of semistable reduction. Here we treat the case of a scheme whose singularities are locally those of a product of semistable schemes which is no longer semistable.

We define a double complex $W_n A^{\bullet\bullet}$ as follows. Its terms are

$$W_n A^{ij} := \bigoplus_{k=0}^{j} W_n \tilde{\tilde{\omega}}_Y^{i+j+2}/(P_{k,i+j+2} + P_{i+j+2,j-k}) \quad \text{for } i, j \geq 0,$$

and $W_n A^{ij} := 0$ otherwise. The operators d, π, F, V of $W_\bullet \tilde{\tilde{\omega}}^\bullet$ induce operators d', π, F, V of the procomplexes $W_\bullet A^{\bullet j}$. For x in the direct summand $W_n \tilde{\tilde{\omega}}_Y^{i+j+2}/(P_{k,i+j+2} + P_{i+j+2,j-k})$ of $W_n A^{ij}$, $d'x$ is the class of $(-1)^j d\tilde{x}$, where $\tilde{x}$ is a lift of x in $W_n \tilde{\tilde{\omega}}_Y^{i+j+2}$. We also have a differential $d'' : W_n A^{ij} \to W_n A^{ij+1}$ given by

$$d''x = (-1)^i \left(\frac{d\tau}{\tau} \wedge x + \frac{d\sigma}{\sigma} \wedge x \right),$$

where $d\tau/\tau$ and $d\sigma/\sigma$ are the global sections of $W_n \tilde{\tilde{\omega}}_Y^1$ defined in Lemma 3.9. We have $d'd'' = d''d'$, so we indeed get a double procomplex $(W_\bullet A^{\bullet\bullet}, d', d'')$. As in Lemma 3.9 of [Mokrane 1993], we can use dévissage by weights to see that the components of this procomplex are p-torsion-free. Let $W_\bullet A^\bullet$ be the simple procomplex associated to the double procomplex $W_\bullet A^{\bullet\bullet}$.

We define now an endomorphism ν of bidegree $(-1, 1)$ of $W_n A^{\bullet\bullet}$ which will induce the monodromy operator on cohomology. For each $k \in \{0, \ldots, j\}$ we have natural maps

$$\begin{aligned} W_n \tilde{\tilde{\omega}}_Y^{i+j+2}/(P_{k,i+j+2} + P_{i+j+2,j-k}) &\\ \to W_n \tilde{\tilde{\omega}}_Y^{i+j+2}/(P_{k,i+j+2} &+ P_{i+j+2,j+1-k}) \\ \oplus W_n \tilde{\tilde{\omega}}_Y^{i+j+2}/(P_{k+1,i+j+2} &+ P_{i+j+2,j-k}), \end{aligned}$$

which are sums of $(-1)^{i+j+1}$ proj on each factor. Summing over k we get maps $\nu : W_n A^{ij} \to W_n A^{i-1j+1}$, which induce an endomorphism ν of bidegree $(-1, 1)$.

The morphism of complexes $W_n \tilde{\tilde{\omega}}_Y^\bullet \to W_n A^{\bullet 0}$ given by

$$x \mapsto \frac{d\tau}{\tau} \wedge \frac{d\sigma}{\sigma} \wedge x$$

factors through $W_n \omega_Y^\bullet$. We get a morphism of complexes

$$\Theta : W_n \omega_Y^\bullet \to W_n A^\bullet.$$

The following lemma is analogous to Theorem 9.9 of [Nakkajima 2005]. It ensures that the resulting spectral sequence will be compatible with the Frobenius endomorphism (defined as an endomorphism of W_n-modules). We let $\Phi_n : W_n\omega_Y \to W_n\omega_Y$ be the Frobenius endomorphism induced by the absolute Frobenius endomorphism of (Y, M).

Lemma 4.1. *Let n be a positive integer. Then the following hold*:

(1) *There exists a unique endomorphism $\tilde{\tilde{\Phi}}_n^{\bullet,\bullet}$ of $W_nA^{\bullet\bullet}$ of double complexes, making the following diagram commutative*:

$$\begin{array}{ccc} W_{n+1}A^{qm} & \xrightarrow{\pi} & W_nA^{qm} \\ {\scriptstyle p^qF}\downarrow & & \downarrow{\scriptstyle \tilde{\tilde{\Phi}}_n^{qm}} \\ W_nA^{qm} & \xrightarrow{\mathrm{id}} & W_nA^{qm} \end{array}$$

(2) *The endomorphism $\tilde{\tilde{\Phi}}_n^{\bullet,\bullet}$ induces an endomorphism $\tilde{\tilde{\Phi}}_n$ of the complex $W_nA^{\bullet}$, fitting in a commutative diagram*

$$\begin{array}{ccc} W_n\omega_Y^{\bullet} & \xrightarrow{\Phi_n} & W_n\omega_Y^{\bullet} \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta} \\ W_nA^{\bullet} & \xrightarrow{\tilde{\tilde{\Phi}}_n} & W_nA^{\bullet} \end{array}$$

(3) *Finally, the Poincaré residue isomorphism* Res *fits in the following commutative diagrams for $i, j \geq 1$*:

$$\begin{array}{ccc} Gr_{i,j}W_n\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\mathrm{Res}} & W_n\Omega_{Y(i,j)}^{q-i-j} \\ {\scriptstyle\Psi_n}\downarrow & & \downarrow{\scriptstyle p^{i+j}\Phi_n} \\ Gr_{i,j}W_n\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\mathrm{Res}} & W_n\Omega_{Y(i,j)}^{q-i-j} \end{array}$$

where Ψ_n is an endomorphism of $W_n\tilde{\tilde{\omega}}_Y^{\bullet}$ which respects the weight filtration $P_{i,j}$ and which induces $\tilde{\tilde{\Phi}}_n^{\bullet,\bullet}$ on $W_nA^{\bullet,\bullet}$.

Proof. The proof is essentially the same as that of Theorem 9.9 of [Nakkajima 2005]. We emphasize only the key points. We can define a morphism

$$\Psi_n^{j,q} : W_n\tilde{\tilde{\omega}}_Y^q \to W_n\tilde{\tilde{\omega}}_Y^q$$

via the composition

$$W_n\tilde{\tilde{\omega}}_Y^q \xrightarrow{\mathfrak{p}} W_{n+1}\tilde{\tilde{\omega}}_Y^q \xrightarrow{p^{j-1}} W_{n+1}\tilde{\tilde{\omega}}_Y^q \xrightarrow{F} W_n\tilde{\tilde{\omega}}_Y^q.$$

The fact that these morphisms commute with the maps $(d\tau/\tau)\wedge$ and $(d\sigma/\sigma)\wedge$ follows from the proof of the first part of Lemma 3.17. This implies that the second diagram is commutative. The fact that the $\Psi_n^{\bullet\bullet}$ respect the weight filtration follows from the analogous statement for $\mathfrak{p}$, which is proved in Lemma 3.17 as well. This means that we can use Ψ_n^{jj+q+2} to define endomorphisms $\tilde{\tilde{\Phi}}_n^{jq}$ of W_nA^{jq}, at least for $j \geq 1$. For $j = 0$ we use the Frobenius endomorphism Φ_n of $W_n(\mathbb{O}_{Y^{(k+1,j-k+1)}})$ together with the residue isomorphisms to define $\tilde{\tilde{\Phi}}_n^{0q}$. The commutativity of the first diagram now follows from the definitions, from the commutative diagram

$$\begin{array}{ccc} W_{n+1}\tilde{\tilde{\omega}}_Y^{qm} & \xrightarrow{\pi} & W_n\tilde{\tilde{\omega}}_Y^{qm} \\ \downarrow{\scriptstyle p^qF} & & \downarrow{\scriptstyle \Psi_n^{qm}} \\ W_nA^{qm} & \xrightarrow{\mathrm{id}} & W_nA^{qm} \end{array}$$

(which is deduced from $\mathfrak{p}d = d\mathfrak{p}$ and $dF = pFd$) and from Diagram 9.2.2 of [Nakkajima 2005] in the case of a smooth morphism. The fact that the first diagram is commutative ensures the uniqueness of $\Phi_n^{q,m}$. Finally, the third commutative diagram follows from the surjectivity of π proved in Lemma 3.17, from Diagram 9.2.2 of [Nakkajima 2005] in the case of a smooth morphism and from the commutative diagrams

$$\begin{array}{ccc} P_{i,j}W_{n+1}\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\mathrm{Res}} & W_{n+1}\Omega_{Y^{(i,j)}}^{q-i-j} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ P_{i,j}W_n\tilde{\tilde{\omega}}_Y^q & \xrightarrow{\mathrm{Res}} & W_n\Omega_{Y^{(i,j)}}^{q-i-j} \end{array}$$

for $i, j \geq 1$. □

Proposition 4.2. *The sequence*

$$0 \to W_n\omega_Y^\bullet \xrightarrow{\Theta} W_nA^{\bullet 0} \xrightarrow{d''} W_nA^{\bullet 1} \xrightarrow{d''} \cdots$$

is exact.

Proof. We follow the proof of Proposition 3.15 of [Mokrane 1993]. Let $\theta : W_n\tilde{\tilde{\omega}}_Y^{i-1} \oplus W_n\tilde{\tilde{\omega}}_Y^{i-1} \to W_n\tilde{\tilde{\omega}}_Y^i$ be defined by

$$(x, y) \mapsto \frac{d\tau}{\tau} \wedge x + \frac{d\sigma}{\sigma} \wedge y.$$

It suffices to check that the sequence

$$\begin{aligned} W_n\tilde{\tilde{\omega}}_Y^{i-2} &\xrightarrow{(\frac{d\sigma}{\sigma}\wedge, \frac{d\tau}{\tau}\wedge)} W_n\tilde{\tilde{\omega}}_Y^{i-1} \oplus W_n\tilde{\tilde{\omega}}_Y^{i-1} \xrightarrow{\theta} W_n\tilde{\tilde{\omega}}_Y^{i} \\ &\xrightarrow{\frac{d\tau}{\tau}\wedge\frac{d\sigma}{\sigma}\wedge} W_n\tilde{\tilde{\omega}}_Y^{i+2}/(P_{0,i+2} + P_{i+2,0}) \\ &\xrightarrow{d''} W_n\tilde{\tilde{\omega}}_Y^{i+3}/(P_{1,i+3} + P_{i+3,0}) \oplus W_n\tilde{\tilde{\omega}}_Y^{i+3}/(P_{0,i+3} + P_{i+3,1}) \\ &\xrightarrow{d''} \cdots \end{aligned} \tag{4.1}$$

is exact. We do this by using first a dévissage by weights, reducing to the case $n = 1$ and then using the fact that the scheme Y is Zariski-locally étale over a product of (the special fibers of) strictly semistable schemes.

We let

$$\begin{aligned} K_{-4} &= W_n\tilde{\tilde{\omega}}_Y^{i-2}, \\ K_{-3} &= W_n\tilde{\tilde{\omega}}_Y^{i-1} \oplus W_n\tilde{\tilde{\omega}}_Y^{i-1}, \\ K_{-2} &= W_n\tilde{\tilde{\omega}}_Y^{i}, \\ K_j &= \bigoplus_{k=0}^{j} W_n\tilde{\tilde{\omega}}_Y^{i+j+2}/(P_{k,i+j+2} + P_{i+j+2,j-k}), \quad j \geq 0. \end{aligned}$$

For $j \geq -4$, $j \neq -1$ we define a double filtration of K_j as follows:

$$\begin{aligned} P_{l,m}K_{-4} &= P_{l-2,m-2}W_n\tilde{\tilde{\omega}}_Y^{i-2}, \\ P_{l,m}K_{-3} &= P_{l-2,m-1}W_n\tilde{\tilde{\omega}}_Y^{i-1} \oplus P_{l-1,m-2}W_n\tilde{\tilde{\omega}}_Y^{i-1}, \\ P_{l,m}K_{-2} &:= P_{l-1,m-1}W_n\tilde{\tilde{\omega}}_Y^{i}, \\ P_{l,m}K_j &:= \bigoplus_{k=0}^{j} P_{l+k,m+j-k}W_n\tilde{\tilde{\omega}}_Y^{i+j+2}/(P_{k,i+j+2} + P_{i+j+2,j-k}), \quad j \geq 0. \end{aligned}$$

Here we set the convention $P_{l,m}W_n\tilde{\tilde{\omega}}^{i} = 0$ if either $l < 0$ or $m < 0$. The sequence (4.1) is a filtered sequence and to prove exactness it suffices to prove exactness for each graded piece

$$\mathrm{Gr}_{l,m}\, K_j := P_{l,m}K_j/(P_{l,m-1}K_j + P_{l-1,m}K_j).$$

For $l, m \geq 0$ we can rewrite the sequences of graded pieces as:

$$\begin{aligned} \mathrm{Gr}_{l-2,m-2}\, W_n\tilde{\tilde{\omega}}_Y^{i-2} &\to \mathrm{Gr}_{l-2,m-1}\, W_n\tilde{\tilde{\omega}}_Y^{i-1} \oplus \mathrm{Gr}_{l-1,m-2}\, W_n\tilde{\tilde{\omega}}_Y^{i-1} \\ &\to \mathrm{Gr}_{l-1,m-1}\, W_n\tilde{\tilde{\omega}}_Y^{i} \to \mathrm{Gr}_{l,m}\, W_n\tilde{\tilde{\omega}}_Y^{i+2} \\ &\to \mathrm{Gr}_{l+1,m}\, W_n\tilde{\tilde{\omega}}_Y^{i+3} \oplus \mathrm{Gr}_{l,m+1}\, W_n\tilde{\tilde{\omega}}_Y^{i+3} \to \cdots. \end{aligned}$$

For $l < 0$ or $m < 0$ the sequence is trivial.

It suffices to show that the sequence of complexes

$$\begin{aligned}\mathrm{Gr}_{l-2,m-2}\, W_n\tilde{\tilde{\omega}}_Y^\bullet[-2] \to \mathrm{Gr}_{l-2,m-1}\, W_n\tilde{\tilde{\omega}}_Y^\bullet[-1] \oplus \mathrm{Gr}_{l-1,m-2}\, W_n\tilde{\tilde{\omega}}_Y^\bullet[-1]\\ \to \mathrm{Gr}_{l-1,m-1}\, W_n\tilde{\tilde{\omega}}_Y^\bullet \xrightarrow{\iota} \mathrm{Gr}_{l,m}\, W_n\tilde{\tilde{\omega}}_Y^\bullet[2]\\ \to \mathrm{Gr}_{l+1,m}\, W_n\tilde{\tilde{\omega}}_Y^\bullet[3] \oplus \mathrm{Gr}_{l,m+1}\, W_n\tilde{\tilde{\omega}}_Y^\bullet[3] \to \cdots \quad (4.2)\end{aligned}$$

is exact. Note that we can check this locally. When $l, m \geq 1$, we know by Corollary 3.16 that

$$\mathrm{Gr}_{l,m}\, W_n\tilde{\tilde{\omega}}_Y^\bullet \simeq W_n\Omega^\bullet_{Y_{(l,m)}}[-l-m](-l-m).$$

For $l = 0$ and $m \geq 1$ let $Y_{D^{0,m}}$ be the normal crossing divisor of $D^{0,m}$ corresponding to $s = 0$. In this case we have

$$\mathrm{Gr}_{l,m}\, W_n\tilde{\tilde{\omega}}_Y^\bullet \simeq [W_n\Omega_{D^{0,m}}(-\log Y_{D^{o,m}}) \to W_n\Omega_{D^{0,m}}],$$

and for $l = 0, m = 0$ we have the quasi-isomorphism

$$\begin{aligned}\mathrm{Gr}_{l,m}\, W_n\tilde{\tilde{\omega}}_Y^\bullet \simeq [W_n\Omega^\bullet_Z(-\log Y^1 - \log Y^2)\\ \to W_n\Omega^\bullet_Z(-\log Y^1) \oplus W_n\Omega^\bullet_Z(-\log Y^2) \to W_n\Omega^\bullet_Z],\end{aligned}$$

where $Z = \underline{Z} \otimes_W k$. In any case, $\mathrm{Gr}_{l,m}\, W_n\tilde{\tilde{\omega}}^\bullet$ satisfies the property

$$(\lim \mathrm{Gr}_{l,m}\, W_n\tilde{\tilde{\omega}}^\bullet) \otimes^L_{\mathbb{R}} \mathbb{R}_n \simeq \mathrm{Gr}_{l,m}\, W_n\tilde{\tilde{\omega}}^\bullet$$

by Lemma 1.3.3 of [Mokrane 1993] and Lemma 3.11. By Proposition 2.3.7 of [Illusie 1983], it suffices to check exactness of the sequence (4.2) for $n = 1$.

For $n = 1$ and working locally with our admissible lifts, we know that the exact sequence (4.2) is the pullback to Y of the corresponding exact sequence on $Y_1 \times_k Y_2$. We can assume that $Y = Y_1 \times_k Y_2$ and $Z = Z_1 \times_k Z_2$. Each Y_i for $i = 1, 2$ is a reduced normal crossings divisor in Z_i, for which we know that

$$\mathrm{Gr}_{l_i-2}\, W_1\tilde{\omega}^\bullet_{Y_i}[-1] \to \mathrm{Gr}_{l_i-1}\, W_1\tilde{\omega}_{Y_i} \to \mathrm{Gr}_l\, W_1\tilde{\omega}^\bullet_{Y_i}[1] \to \mathrm{Gr}_{l+1}\, W_1\tilde{\omega}^\bullet_{Y_i}[2] \to \cdots$$

is exact, by the proof of Proposition 3.15 of [Mokrane 1993]. In other words, for $i = 1, 2$ we have quasi-isomorphisms between the top row and the bottom row. Multiplying the quasi-isomorphisms for $i = 1, 2$ gives us exactly the quasi-isomorphism ι needed to prove the exactness of (4.2) in the case $n = 1$. Here, we use the Cartier isomorphisms for $W_1\tilde{\omega}_{Y_i}$ and for $W_1\tilde{\tilde{\omega}}_Y$ and the fact that

$$(\omega^\bullet_{(Z_1,\tilde{N}_1)/k} \otimes_{\mathbb{O}_{Z_1}} \mathbb{O}_{Y_1}) \otimes_k (\omega^\bullet_{(Z_2,\tilde{N}_2)/k} \otimes_{\mathbb{O}_{Z_2}} \mathbb{O}_{Y_2}) \simeq \omega^\bullet_{(Z,\tilde{N})} \otimes_{\mathbb{O}_Z} \mathbb{O}_Y,$$

where the two complexes on the left determine $W_1\tilde{\omega}^\bullet_{Y_i}$ for $i = 1, 2$ and the one on the right determines $W_1\tilde{\tilde{\omega}}^\bullet_Y$. □

Corollary 4.3. *The morphism of complexes $\Theta : W_n\omega_Y^\bullet \to W_n A^\bullet$ is a quasi-isomorphism. It induces a quasi-isomorphism $\Theta : W\omega_Y^\bullet \xrightarrow{\simeq} WA^\bullet$.*

Proposition 4.4. *The endomorphism v of $W_\bullet A^{\bullet\bullet}$ induces the monodromy operator N over $H^*_{\mathrm{cris}}((Y, M)/(W, \mathbb{N}))$.*

Proof. We define the double complex $B_n^{\bullet\bullet}$ as follows:

$$\begin{aligned} B_n^{\bullet\bullet} &= W_n A^{i-1,j} \oplus W_n A^{ij}, \quad i, j \geq 0, \\ d'(x_1, x_2) &= (d'x_1, d'x_2), \\ d''(x_1, x_2) &= (d''x_1 + v(x_2), d''x_2). \end{aligned}$$

We have a morphism of complexes $\Psi : W_n\tilde{\omega}_Y^\bullet \to B_n^\bullet$ defined as follows: for $x \in W_n\tilde{\omega}_Y^i$,

$$\begin{aligned} &\Psi(x) \\ &= \Big(\Big(\frac{d\sigma}{\sigma} - \frac{d\tau}{\tau}\Big) \wedge x \ (\mathrm{mod}\ P_{0,i+1} + P_{i+1,0}), \frac{d\tau}{\tau} \wedge \frac{d\sigma}{\sigma} \wedge x \ (\mathrm{mod}\ P_{0,i+2} + P_{i+2,0})\Big). \end{aligned}$$

Thus we have a commutative diagram of exact sequences of complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_n\omega_Y^\bullet[-1] & \longrightarrow & W_n\tilde{\omega}_Y^\bullet & \longrightarrow & W_n\omega_Y^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Theta[-1]} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Theta} & & \\ 0 & \longrightarrow & W_n A^\bullet[-1] & \longrightarrow & B_n^\bullet & \longrightarrow & W_n A^\bullet & \longrightarrow & 0 \end{array}$$

where the left and right downward arrows are quasi-isomorphisms. Thus, Ψ is also a quasi-isomorphism and the commutative diagram defines an isomorphism of distinguished triangles. Thus the monodromy operator N on cohomology is induced by the coboundary operator of the bottom exact sequence, which by construction is v. □

We can compute the monodromy filtration of the nilpotent operator N on cohomology from the monodromy filtration of v on $W_n A^\bullet$. We will exhibit a filtration $P_k(W_n A^\bullet) = \bigoplus_{i,j\geq 0} P_k(W_n A^{ij})$ which satisfies the following:

(1) $v(P_k(W_\bullet A^\bullet)) \subset P_{k-2}(W_\bullet A^\bullet)(-1)$.

(2) For $k \geq 0$, the induced map $v^k : \mathrm{Gr}_k(W_\bullet A^\bullet) \to \mathrm{Gr}_{-k}(W_\bullet A^\bullet)(-k)$ is an isomorphism.

A filtration satisfying these two properties must be the monodromy filtration of v.

Note 4.5. From now on, we will not work in the category $\mathfrak{C}$ of complexes of sheaves of W-modules but rather in $\mathbb{Q} \otimes \mathfrak{C}$, which is the category with the same set of objects as $\mathfrak{C}$, but with morphisms $\mathbb{Q} \otimes \mathrm{Hom}_{\mathfrak{C}}(A, B)$. We will in fact identify the monodromy filtration of v on $\mathbb{Q} \otimes W_n A^\bullet$, but for simplicity of notation we still denote an object A of $\mathfrak{C}$ as A when we regard it as an object of $\mathbb{Q} \otimes \mathfrak{C}$.

Define $P_l(W_nA^{\bullet\bullet}) := \bigoplus_{i,j\geq 0} P_l(W_nA^{ij})$ for $l \geq 0$, where $P_l(W_nA^{ij})$ is 0 if $l < 2n-2-j$ and

$$\bigoplus_{k=0}^{j}\Bigg(\sum_{m=0}^{l-2n+2+j} P_{k+m+1,2j-k+l-2n-m+3}W_n\tilde{\tilde{\omega}}_Y^{i+j+2}/(P_{k,i+j+2}+P_{j-k,i+j+2})\Bigg)$$

if $l \geq 2n-2-j$. It is easy to check that $\nu(P_l(W_nA^{ij})) \subset P_{l-2}W_nA^{i+1,j-1}$. Moreover, we can also compute the graded pieces $\mathrm{Gr}_l(W_nA^{\bullet\bullet}) = \bigoplus_{i,j\geq 0}\mathrm{Gr}_l(W_nA^{ij})$, where

$$\mathrm{Gr}_l(W_nA^{ij}) = \begin{cases} 0 & \text{if } l < 2n-2-j, \\ \bigoplus_{k=0}^{j}\bigoplus_{m=0}^{l-2n+2+j}\mathrm{Gr}_{k+m+1,2j-k+l-2n-m+3}\,W_n\tilde{\tilde{\omega}}_Y^{i+j+2} & \text{if } l \geq 2n-2-j. \end{cases}$$

For $l = 2n-2+h$, with $h > 0$, we claim that ν induces an injection $\mathrm{Gr}_l(W_nA^{ij}) \hookrightarrow \mathrm{Gr}_{l-1}(W_nA^{ij})$. This can be verified through a standard combinatorial argument. We have

$$\mathrm{Gr}_l(W_nA^{ij}) = \bigoplus_{k=0}^{j}\bigoplus_{m=0}^{h+j}\mathrm{Gr}_{(k+m)+1,2j+h+1-(k+m)}\,W_n\tilde{\tilde{\omega}}_Y^{i+j+2}$$

and

$$\mathrm{Gr}_{l-1}(W_nA^{ij}) = \bigoplus_{k=0}^{j+1}\bigoplus_{m=0}^{h+j-1}\mathrm{Gr}_{(k+m)+1,2j+h+1-(k+m)}\,W_n\tilde{\tilde{\omega}}_Y^{i+j+2}.$$

The map ν sends the term corresponding to a pair (k,m) to the direct sum of terms corresponding to (k,m) and to $(k+1,m-1)$. Therefore, it is easy to see that ν restricted to the direct sum of terms for which $k+m$ is constant is injective, so ν is injective. Moreover, we see that ν^h induces an isomorphism $\mathrm{Gr}_{2n-2+h}(W_nA^{ij}) \simeq \mathrm{Gr}_{2n-2-h}(W_nA^{i-h,j+h})$, since the terms on the right-hand side are of the form

$$\bigoplus_{k=0}^{j}\bigoplus_{m=0}^{h+j}\mathrm{Gr}_{(k+m)+1,2j+h+1-(k+m)}\,W_n\tilde{\tilde{\omega}}_Y^{i+j+2}$$

and the terms on the left-hand side are of the form

$$\bigoplus_{m=0}^{j}\bigoplus_{k=0}^{h+j}\mathrm{Gr}_{(k+m)+1,2j+h+1-(k+m)}\,W_n\tilde{\tilde{\omega}}_Y^{i+j+2},$$

so on either side we have the same number of terms corresponding to $k+m$. Since the filtration $P_l(W_nA^{\bullet\bullet})$ satisfies the two properties above, it must be the monodromy filtration of ν.

Note that the differentials d'' on $\mathrm{Gr}_l(W_\bullet A^{\bullet\bullet})$ are always 0. Using the isomorphisms in Corollary 3.16 we can rewrite

$$\mathrm{Gr}_{2n-2+h}(W_\bullet A^\bullet) \simeq \bigoplus_{j\geq 0, j\geq -h} \bigoplus_{k=0}^{j} \bigoplus_{m=0}^{j+h} (W_\bullet \Omega^\bullet_{Y^{(k+m+1,2j+h+1-(k+m))}})[-2j-h](-j-h).$$

This leads to the main geometric result of the paper. Recall from Section 3.1 that Y/k is the special fiber of $X/\mathbb{O}_K$, which is Zariski-locally étale over a product of strictly semistable schemes. Recall also that Y is globally the union of certain proper, smooth $(2n-2)$-dimensional subschemes $Y_{i,j}$ with $i = 1, 2$, $j = 1, \ldots, n$. Taking disjoint unions of intersections of these subschemes gives rise to schemes $Y^{(l_1,l_2)}/k$ for $1 \leq l_1, l_2 \leq n$, which cover closed strata in Y. Each $Y^{(l_1,l_2)}$ is proper, smooth and has dimension $2n - l_1 - l_2$.

Theorem 4.6. *There is a spectral sequence*

$$E_1^{-h,i+h} = \bigoplus_{j\geq 0, j\geq -h} \bigoplus_{k=0}^{j} \bigoplus_{m=0}^{j+h} H^{i-2j-h}_{\mathrm{cris}}(Y^{(k+m+1,2j+h+1-(k+m))}/W)(-j-h) \Rightarrow H^i_{\mathrm{cris}}(Y/W).$$

Remark 4.7. Note that the schemes $Y^{(l_1,l_2)}$ are proper and smooth so the $E^{-h,i+h}$ terms of the spectral sequence are strictly pure of weight $i+h$. If the above spectral sequence degenerates at the first page, then $H^i_{\mathrm{cris}}(Y/W)$ is pure of weight i.

5. Proof of the main theorem

In this section we prove the main theorem. By Corollary 2.3, its proof reduces to the following proposition.

Proposition 5.1. *Let $\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}$ be the universal abelian variety over $X^{m_\xi}_{U_{\mathrm{Iw}}}$. The direct limit of log crystalline cohomologies*

$$\varinjlim_{U_{\mathrm{Iw}}} a_\xi \left(H^{2n-2+m_\xi}_{\mathrm{cris}}(\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}} \times_{\mathbb{O}_K} k/W) \otimes_W \overline{\mathbb{Q}}_l(t_\xi) \right)[\Pi^{1,\mathfrak{S}}]$$

is pure of a certain weight.

Proof. Recall that we have chosen

$$U_{\mathrm{Iw}} = U^l \times U_l^{\mathfrak{p}_1,\mathfrak{p}_2}(m) \times \mathrm{Iw}_{n,\mathfrak{p}_1} \times \mathrm{Iw}_{n,\mathfrak{p}_2} \subset G(\mathbb{A}^\infty).$$

Pick m large enough such that $(\pi_l)^{U_l^{\mathfrak{p}_1,\mathfrak{p}_2}(m)\times \mathrm{Iw}_{n,\mathfrak{p}_1} \times \mathrm{Iw}_{n,\mathfrak{p}_2}}$ is nonzero, where $\pi_l \in \mathrm{Irr}_l(G(\mathbb{Q}_l))$ is such that $BC(\pi_l) = \iota_l^{-1}\Pi_l$. The results of Sections 3 and 4 apply to $\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}$. We have a stratification of its special fiber by closed Newton polygon

strata $\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}},S,T}$ with $S,T\subseteq\{1,\dots,n\}$ nonempty. For brevity, let $K_1:=k+m+1$ and $K_2:=2j+h+1-(k+m)$. By Theorem 4.6, we have a spectral sequence

$$E_1^{-h,i+h}=\bigoplus_{\substack{j\geq 0\\ j\geq -h}}\bigoplus_{k=0}^{j}\bigoplus_{m=0}^{j+h}\bigoplus_{\substack{\#S=K_1\\ \#T=K_2}} H^{i-2j-h}_{\mathrm{cris}}\big((\mathscr{A}^{m_{\xi,}}_{U_{\mathrm{Iw}},S,T}/W)(-j-h)\big)$$
$$\Rightarrow H^i_{\mathrm{cris}}(\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}\times_{\mathbb{O}_K}k/W).$$

We replace the cohomology degree i by $i+m_\xi$, tensor with $\bar{\mathbb{Q}}_l(t_\xi)$, apply a_ξ (which is obtained from a linear combination of étale morphisms); passing to a direct limit over U^l and taking the $\Pi^{1,\mathfrak{S}}$-isotypic components we get a spectral sequence

$$E_1^{-h,i+h}=\bigoplus_{\substack{j\geq 0\\ j\geq -h}}\bigoplus_{k=0}^{j}\bigoplus_{m=0}^{j+h}\bigoplus_{\substack{\#S=K_1\\ \#T=K_1}}\varinjlim_{U^l}\Big(a_\xi H^{i+m_\xi-2j-h}_{\mathrm{cris}}\big((\mathscr{A}^{m_{\xi,}}_{U_{\mathrm{Iw}},S,T}/W)(-j-h)\otimes_{W,\tau_0}\bar{\mathbb{Q}}_l(t_\xi)\big)[\Pi^{1,\mathfrak{S}}]\Big)$$
$$\Rightarrow\varinjlim_{U^l}\big(a_\xi H^{i+m_\xi}_{\mathrm{cris}}(\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}}}\times_{\mathbb{O}_K}k/W)\otimes_{W,\tau_0}\bar{\mathbb{Q}}_l(t_\xi)\big)[\Pi^{1,\mathfrak{S}}].$$

For any compact open subgroup $U^l\subset G(\mathbb{A}^{\infty,l})$ and any prime $p\neq l$ with isomorphism $\iota_p:\bar{\mathbb{Q}}_p\xrightarrow{\sim}\mathbb{C}$, set $\xi':=(\iota_p)^{-1}\iota_l\xi$ and $\Pi':=(\iota_p)^{-1}\Pi^1$.

We have

$$\dim_{\bar{\mathbb{Q}}_l}\big(\varinjlim_{U^l}a_\xi H^{i+m_\xi-2j-h}_{\mathrm{cris}}\big((\mathscr{A}^{m_{\xi,}}_{U_{\mathrm{Iw}},S,T}/W)(-j-h)\otimes_{W,\tau_0}\bar{\mathbb{Q}}_l\big)\big)[\Pi^{1,\mathfrak{S}}]^{U^l}$$
$$=\dim_{\bar{\mathbb{Q}}_p}\big(\varinjlim_{U^l}a_{\xi'}H^{i+m_{\xi'}-2j-h}(\mathscr{A}^{m_\xi}_{U_{\mathrm{Iw}},S,T},\bar{\mathbb{Q}}_p)\big)[(\Pi')^{\mathfrak{S}}]^{U^l}$$
$$=\dim_{\bar{\mathbb{Q}}_p}(\varinjlim_{U^l}H^{i-2j-h}\big(X_{U_{\mathrm{Iw}},S,T},\mathscr{L}_{\xi'}\big))[(\Pi')^{\mathfrak{S}}]^{U^l}.$$

The first equality is a consequence of the main theorem of [Gillet and Messing 1987] and of Theorem 2(2) of [Katz and Messing 1974]. The former proves that crystalline cohomology is a Weil cohomology theory in the strong sense. The latter is the statement that the characteristic polynomial on $H^i(X)$ of an integrally algebraic cycle on $X\times X$ of codimension n, for a projective smooth variety X/k of dimension n, is independent of the Weil cohomology theory H.

The dimension in the third row is equal to 0 unless $i=2n-2$ by Proposition 5.10 of [Caraiani 2012]. Therefore, $E_1^{-h,i+h}=0$ unless $i=2n-2$, so the E_1 page of the spectral sequence is concentrated on a diagonal. The spectral sequence degenerates at the E_1 page and the term corresponding to $E_1^{h,2n-2+h}$ is strictly pure of weight $h+2n-2+m_\xi-2t_\xi$, which shows that the abutment is pure. □

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