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outside the critical strip**

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# Zeros of $L$ -functions outside the critical strip

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For a wide class of Dirichlet series associated to automorphic forms, we show that those without Euler products must have zeros within the region of absolute convergence. For instance, we prove that if  $f \in S_k(\Gamma_1(N))$  is a classical holomorphic modular form whose  $L$ -function does not vanish for  $\Re(s) > (k+1)/2$ , then  $f$  is a Hecke eigenform. Our proof adapts and extends work of Saias and Weingartner, who proved a similar result for degree-1  $L$ -functions.

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## 1. Introduction

Saias and Weingartner [2009] showed that if  $L(s) = \sum_{m=1}^{\infty} \lambda(m)/m^s$  is a Dirichlet series with periodic coefficients, then either  $L(s) = 0$  for some  $s$  with real part  $> 1$ , or  $\lambda(m)$  is multiplicative at almost all primes (so that  $L(s) = D(s)L(s, \chi)$  for some primitive Dirichlet character  $\chi$  and finite Dirichlet series  $D$ ). Earlier work of Davenport and Heilbronn [1936a; 1936b] established this result for the special case of the Hurwitz zeta-function  $\zeta(s, \alpha)$  with rational parameter  $\alpha$ , and proved an analogue for the degree-2 Epstein zeta-functions. Also in degree 2, Conrey and Ghosh [1994] showed that the  $L$ -function associated to the square of Ramanujan's  $\Delta$  modular form has infinitely many zeros outside of its critical strip. In this paper, we generalize all of these results and study the extent to which, among all Dirichlet series associated to automorphic forms (appropriately defined), the existence of an Euler product is characterized by nonvanishing in the region of absolute convergence. For instance, for classical degree-2  $L$ -functions, we prove the following:

**Theorem 1.1.** *Let  $f \in S_k(\Gamma_1(N))$  be a holomorphic cuspform of arbitrary weight and level. If the associated complete  $L$ -function  $\Lambda_f(s) = \int_0^\infty f(iy)y^{s-1} dy$  does not vanish for  $\Re(s) > (k+1)/2$ , then  $f$  is an eigenfunction of the Hecke operators  $T_p$  for all primes  $p \nmid N$ .*

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Our method is sufficiently general to apply to  $L$ -functions of all degrees, and in fact we obtain Theorem 1.1 as a corollary of the following general result:

**Theorem 1.2.** *Fix a positive integer  $n$ . For  $j = 1, \dots, n$ , let  $r_j$  be a positive integer and  $\pi_j$  a unitary cuspidal automorphic representation of  $\mathrm{GL}_{r_j}(\mathbb{A}_{\mathbb{Q}})$  with  $L$ -series  $L(s, \pi_j) = \sum_{m=1}^{\infty} \lambda_j(m) m^{-s}$ . Assume that the  $\pi_j$  satisfy the generalized Ramanujan conjecture at all finite places (so that, in particular,  $|\lambda_j(p)| \leq r_j$  for all primes  $p$ ) and are pairwise nonisomorphic. Let*

$$R = \left\{ \sum_{m=1}^M \frac{a_m}{m^s} : M \in \mathbb{Z}_{\geq 0}, (a_1, \dots, a_M) \in \mathbb{C}^M \right\}$$

denote the ring of finite Dirichlet series, and let  $P \in R[x_1, \dots, x_n]$  be a polynomial with coefficients in  $R$ . Then either  $P(L(s, \pi_1), \dots, L(s, \pi_n))$  has a zero with real part  $> 1$  or  $P = D(s)x_1^{d_1} \cdots x_n^{d_n}$  for some  $D \in R, d_1, \dots, d_n \in \mathbb{Z}_{\geq 0}$ .

**Remarks.** (1) For  $\pi_j$  as in the statement of the theorem, it is known (see [Jacquet and Shalika 1976]) that  $L(s, \pi_j)$  does not vanish for  $\Re(s) \geq 1$ . Thus if  $P = D(s)x_1^{d_1} \cdots x_n^{d_n}$  is a monomial, the matter of whether  $P(L(s, \pi_1), \dots, L(s, \pi_n))$  vanishes for some  $s$  with  $\Re(s) > 1$  is determined entirely by the finite Dirichlet series  $D(s)$ . Further, the grand Riemann hypothesis (GRH) predicts that each  $L(s, \pi_j)$  does not vanish for  $\Re(s) > \frac{1}{2}$ . Theorem 1.2 demonstrates that the GRH, if it is true, is a very rigid phenomenon.

(2) By the almost-periodicity of Dirichlet series, if  $P(L(s, \pi_1), \dots, L(s, \pi_n))$  has at least one zero with real part  $> 1$  then it must have infinitely many such zeros. In fact, our proof shows that there is some number  $\eta = \eta(P; \pi_1, \dots, \pi_n) > 0$  such that for any  $\sigma_1, \sigma_2$  with  $1 < \sigma_1 < \sigma_2 \leq 1 + \eta$ , we have

$$\#\{s \in \mathbb{C} : \Re(s) \in [\sigma_1, \sigma_2], \Im(s) \in [-T, T], P(L(s, \pi_1), \dots, L(s, \pi_n)) = 0\} \gg T \quad (1-1)$$

for  $T$  sufficiently large (where both the implied constant and the meaning of “sufficiently large” depend on  $\sigma_1, \sigma_2$  as well as  $P$  and  $\pi_1, \dots, \pi_n$ ).

On the other hand, if we restrict to  $\mathbb{C}$ -linear combinations (that is, homogeneous degree-1 polynomials  $P \in \mathbb{C}[x_1, \dots, x_n]$ ) and  $\pi_1, \dots, \pi_n$  with a common conductor and archimedean component  $\pi_{1,\infty} \cong \cdots \cong \pi_{n,\infty}$ , Bombieri and Hejhal [1995] showed, subject to GRH and a weak form of the pair correlation conjecture for  $L(s, \pi_j)$ , that asymptotically 100% of the nontrivial zeros of  $P(L(s, \pi_1), \dots, L(s, \pi_n))$  have real part  $\frac{1}{2}$ .

(3) The assumption of the Ramanujan conjecture in Theorem 1.2 could be relaxed. For instance, it would suffice to have, for each fixed  $j$ :

- (i) Some mild control over the coefficients of the logarithmic derivative

$$\frac{L'}{L}(s, \pi_j) = \sum_{m=1}^{\infty} c_j(m) m^{-s}$$

at prime powers, namely  $\sum_p |c_j(p^k)|^2/p^k < \infty$  for any fixed  $k \geq 2$  (cf. [Rudnick and Sarnak 1996, Hypothesis H]).

- (ii) An average bound for
- $|\lambda_j(p)|^4$
- over arithmetic progressions of primes, namely

$$\limsup_{x \rightarrow \infty} \frac{\sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} |\lambda_j(p)|^4}{\sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} 1} \leq C_j$$

for all coprime  $a, q \in \mathbb{Z}_{>0}$ , where  $C_j > 0$  is independent of  $a, q$ .

Note that (i) is known to hold when  $r_j \leq 4$  (see [Rudnick and Sarnak 1996; Kim 2006]). Further, both estimates follow from the Rankin–Selberg method if, for instance, the tensor square  $\pi_j \otimes \pi_j$  is automorphic for each  $j$ . Since this is known when  $r_j = 2$  (see [Gelbart and Jacquet 1978]), Theorem 1.2 could be extended to include the  $L$ -functions associated to Maass forms.

(4) The main tool used in the proof is the quasi-orthogonality of the coefficients  $\lambda_j(p)$ , i.e., asymptotic estimates for sums of the form  $\sum_{p \leq x} \lambda_j(p) \overline{\lambda_k(p)}/p$  as  $x \rightarrow \infty$ . These follow from the Rankin–Selberg method, and were obtained in a precise form independently by Wu and Ye [2007, Theorem 3] and Avdispahić and Smajlović [2010, Theorem 2.2]. (We also make use of similar estimates for sums over  $p$  in an arithmetic progression — see Lemma 2.1 for the exact statement — though it is likely that this could be avoided at the expense of making the proof more complicated.)

Since quasi-orthogonality and the Ramanujan conjecture are essentially the only properties of automorphic  $L$ -functions that we require, one could instead take these as hypotheses and state the theorem for an axiomatically defined class of  $L$ -functions, such as the Selberg class. However, it has been conjectured that the Selberg class coincides with the class of automorphic  $L$ -functions, so this likely offers no greater generality.

(5) The conclusion of Theorem 1.2 is interesting even for  $n = 1$ . For instance, Nakamura and Pańkowski [2012] have shown for a wide class of  $L$ -functions  $L(s)$  that if  $P \in R[x]$  is not a monomial and  $\delta > 0$ , then  $P(L(s))$  necessarily has zeros in the half-plane  $\Re(s) > 1 - \delta$ . Our result strengthens this to  $\Re(s) > 1$ . (On the other hand, their results also yield the estimate (1-1) for any  $[\sigma_1, \sigma_2] \subseteq (\frac{1}{2}, 1)$ , which does not follow from our method.)

(6) Our results are related to universality results for zeta and  $L$ -functions. Voronin [1975] proved, for any compact set  $K$  with connected complement contained within the strip  $\Re(s) \in (\frac{1}{2}, 1)$  and any nonvanishing, continuous function  $f : K \rightarrow \mathbb{C}$  holomorphic on the interior of  $K$ , that  $f$  can be uniformly approximated by vertical translates of the zeta function.

Voronin's results were extended by a number of authors. One result similar to ours, due to Laurinćikas and Matsumoto [2004], states that given  $m$  functions  $f_1, \dots, f_m$  as above, and  $L$ -functions  $L_j(s, F)$  associated to twists of a Hecke newform  $F$  by pairwise inequivalent Dirichlet characters, that the  $f_j$  may be simultaneously approximated by a single vertical translate of the functions  $L_j(s, F)$ . This implies [loc. cit., Theorem 4] that nontrivial linear combinations of the  $L_j(s, F)$  must contain zeros inside the critical strip with  $\Re(s) > \frac{1}{2}$ .

References to many more works on universality can be found in [loc. cit.].

**Summary of the proof.** Our proof closely follows Saias and Weingartner in broad outline, but becomes more technical in some places. The reader may wish to read [Saias and Weingartner 2009].

The technical heart of our paper is Proposition 3.1, an extension of their Lemma 2. Given  $n$  nonzero complex numbers  $z_1, \dots, z_n$ , we would like to simultaneously solve the equations  $L(s, \pi_j) = z_j$ , leading to a quick proof of the main theorem. As a substitute, we solve equations of the form  $\prod_{p>y} L(\sigma + it_p, \pi_{j,p}) = z_j$ , where the ordinate of  $s$  is allowed to vary for each prime.

Given this, in Section 4 we prove our main theorem, following the proof of Theorem 2 in [Saias and Weingartner 2009]. As in that work, the main tools are Weyl's criterion, allowing us to simultaneously approximate all of the  $p^{-\sigma - it_p}$  by  $p^{-\sigma - it}$  for a single  $t$ , and Rouché's theorem, which states that actual zeros must exist near approximate zeros.

The proof of Proposition 3.1 follows those of Lemmas 1 and 2 of [Saias and Weingartner 2009]. However, in that work the Dirichlet coefficients  $\lambda(m)$  are all periodic to some fixed modulus, and this fact, combined with the prime number theorem for arithmetic progressions, allows for easy control of various partial sums that need to be estimated. Here, we must do without this periodicity.

To prove Proposition 3.1, we choose (in Proposition 3.3) a partition of the set of primes  $p > y$  into disjoint subsets  $S$ , and complex numbers  $\epsilon_p \in S^1$  for each  $p > y$ , so that the vectors of partial sums  $\sum_{p \in S} \epsilon_p \lambda_j(p) p^{-\sigma}$  are linearly independent in a precise quantitative sense. Our main tool is the Rankin–Selberg method (substituting for periodicity and orthogonality of Dirichlet characters); see Lemma 2.1.

We also rely on the rather technical Proposition 3.2, which says that for matrices  $g_1, \dots, g_m$ , we can continuously solve equations of the form  $\sum_{i=1}^m g_i f_i(z) = z$  for  $n$ -tuples of complex numbers  $z = (z_1, \dots, z_n)$ . The  $g_i$  are constructed from the sums

over  $p \in S$  considered in Proposition 3.3, but we are able to formulate Proposition 3.2 in a general manner, without reference to automorphic forms or primes.

The conclusion of Proposition 3.2 is guaranteed only for large  $m$ , so that the number of subsets  $S$  needed may be large. We choose these subsets to be arithmetic progressions, for which the Rankin–Selberg estimates presented in Lemma 2.1 are known to hold. If such estimates were unavailable, it seems likely that we could still obtain our result by constructing the  $S$  in a more ad hoc fashion instead. In any case, and in contrast to Saias–Weingartner, the modulus of the arithmetic progression has no particular arithmetic significance, and is chosen to be coprime to all the conductors of the  $\pi_j$ .

### 2. Preliminaries

**Automorphic  $L$ -functions.** Let  $\pi_j$  be as in the statement of Theorem 1.2. Each  $\pi_j$  can be written as a restricted tensor product  $\pi_{j,\infty} \otimes \bigotimes_p \pi_{j,p}$  of local representations, where  $p$  runs through all prime numbers. Then we have

$$L(s, \pi_j) = \prod_p L(s, \pi_{j,p}) \quad \text{for } \Re(s) > 1. \tag{2-1}$$

Here each local factor  $L(s, \pi_{j,p})$  is a rational function of  $p^{-s}$ , of the form

$$L(s, \pi_{j,p}) = \frac{1}{(1 - \alpha_{j,p,1} p^{-s}) \cdots (1 - \alpha_{j,p,r_j} p^{-s})} \tag{2-2}$$

for certain complex numbers  $\alpha_{j,p,\ell}$ . The generalized Ramanujan conjecture asserts that  $|\alpha_{j,p,\ell}| \leq 1$ , with equality holding for all  $p \nmid \text{cond}(\pi_j)$ , where  $\text{cond}(\pi_j) \in \mathbb{Z}_{>0}$  is the conductor of  $\pi_j$ . In particular,  $|\lambda_j(p)| = |\alpha_{j,p,1} + \cdots + \alpha_{j,p,r_j}| \leq r_j$ .

**Lemma 2.1.** *Let  $a$  and  $q$  be positive integers satisfying  $(q, a \prod_{j=1}^n \text{cond}(\pi_j)) = 1$ . Then*

$$\sum_{\substack{p > y \\ p \equiv a \pmod{q}}} \frac{|u_1 \lambda_1(p) + \cdots + u_n \lambda_n(p)|^2}{p^\sigma} = \left( \frac{1}{\phi(q)} + O(\sigma - 1) \right) \sum_{p > y} p^{-\sigma}$$

for all  $y > 0$ ,  $\sigma \in (1, 2]$  and all unit vectors  $(u_1, \dots, u_n)$ , where the implied constant depends only on  $\pi_1, \dots, \pi_n$  and  $q$ .

*Proof.* Let  $\chi \pmod{q}$  be a Dirichlet character, not necessarily primitive. We consider the sum

$$E_{jk\chi}(x) = \sum_{p \leq x} (\lambda_j(p) \overline{\lambda_k(p)} \chi(p) - \delta_{jk\chi}) \frac{\log p}{p},$$

running over primes  $p \leq x$ , where  $\delta_{jk\chi} = 1$  if  $j = k$  and  $\chi$  is the trivial character, and  $\delta_{jk\chi} = 0$  otherwise. Applying [Avdispahić and Smajlović 2010, (2) and (3)]

with  $(\pi, \pi') = (\pi_j \otimes \chi, \pi_k)$  and, if  $\chi$  is imprimitive, subtracting any contribution from the terms with  $p \mid q$ , we obtain the bound  $E_{jk\chi}(x) \ll_q 1$ .

Next, for any nonintegral  $y \geq \frac{3}{2}$  and any  $\sigma \in (1, 2]$ , we have

$$\sum_{p>y} \frac{\lambda_j(p)\overline{\lambda_k(p)}\chi(p) - \delta_{jk\chi}}{p^\sigma} = \int_y^\infty \frac{t^{1-\sigma}}{\log t} dE_{jk\chi}(t).$$

Integrating by parts and applying the above estimate for  $E_{jk\chi}$ , we see that this is  $\ll_q y^{1-\sigma} / \log y$ .

Now, expanding the square and using orthogonality of Dirichlet characters, we have

$$\begin{aligned} \sum_{\substack{p>y \\ p \equiv a \pmod{q}}} \frac{|u_1\lambda_1(p) + \dots + u_n\lambda_n(p)|^2}{p^\sigma} &= \frac{1}{\phi(q)} \sum_{j=1}^n \sum_{k=1}^n \sum_{\chi \pmod{q}} u_j \overline{u_k} \overline{\chi(a)} \sum_{p>y} \frac{\lambda_j(p)\overline{\lambda_k(p)}\chi(p)}{p^\sigma} \\ &= O_q\left(\frac{y^{1-\sigma}}{\log y}\right) + \frac{1}{\phi(q)} \sum_{p>y} p^{-\sigma}. \end{aligned}$$

Finally, by the prime number theorem we have  $\sum_{p>y} p^{-\sigma} \gg y^{1-\sigma} / ((\sigma - 1) \log y)$ , uniformly for  $y \geq \frac{3}{2}$  and  $\sigma \in (1, 2]$ . The lemma follows.  $\square$

**A few lemmas.** In the remainder of this section we discuss the topology of  $\text{GL}_n(\mathbb{C})$  and prove some simple lemmas, to be used in the more technical propositions which follow.

Let  $\text{Mat}_{n \times n}(\mathbb{C})$  denote the set of  $n \times n$  matrices with entries in  $\mathbb{C}$ . For  $A = (a_{ij}) \in \text{Mat}_{n \times n}(\mathbb{C})$ , the *Frobenius norm* is defined by

$$\|A\| = \sqrt{\text{tr}(\overline{A}^T A)} = \sqrt{\sum |a_{ij}|^2}.$$

Note that this agrees with the Euclidean norm under the identification of  $\text{Mat}_{n \times n}(\mathbb{C})$  with  $\mathbb{C}^{n^2}$ . By the Schwarz inequality, we have  $|Av| \leq \|A\| \cdot \|v\|$  for any  $A \in \text{Mat}_{n \times n}(\mathbb{C})$  and  $v \in \mathbb{C}^n$ .

We endow  $\text{GL}_n(\mathbb{C}) = \{g \in \text{Mat}_{n \times n}(\mathbb{C}) : \det g \neq 0\}$  with the subspace topology. In particular, it is easy to see that a set  $K \subseteq \text{GL}_n(\mathbb{C})$  is compact if and only if  $K$  is closed in  $\text{Mat}_{n \times n}(\mathbb{C})$  and there are positive real numbers  $c$  and  $C$  such that

$$\|g\| \leq C \text{ and } |\det g| \geq c \text{ for all } g \in K.$$

Since  $g^{-1}$  can be expressed in terms of  $1/\det g$  and the cofactor matrix of  $g$ , it follows that  $\|g^{-1}\|$  is bounded on  $K$  (and indeed the map  $g \mapsto g^{-1}$  is continuous, so that  $\text{GL}_n(\mathbb{C})$  is a topological group with this topology).



**Lemma 2.2.** *Suppose  $K$  is a compact subset of  $GL_n(\mathbb{C})$ ,  $g \in K$ , and  $U \subseteq \mathbb{C}^n$  contains an open  $\delta$ -neighborhood of some point. Then  $gU$  contains an  $\varepsilon$ -neighborhood, where  $\varepsilon > 0$  depends only on  $\delta$  and  $K$ .*

*Proof.* By linearity, we may assume without loss of generality that  $U$  contains the  $\delta$ -neighborhood of the origin,  $N_\delta$ . Since  $K$  is compact, there is a number  $C > 0$  such that  $\|g^{-1}\| \leq C$  for all  $g \in K$ . Put  $\varepsilon = C^{-1}\delta$ , and let  $N_\varepsilon$  be the  $\varepsilon$ -neighborhood of the origin. For any  $v \in N_\varepsilon$  we have  $|g^{-1}v| \leq \|g^{-1}\| \cdot |v| < C\varepsilon = \delta$ , so that  $v = g(g^{-1}v) \in gN_\delta$ . Since  $v$  was arbitrary,  $gN_\delta \supseteq N_\varepsilon$ .  $\square$

**Lemma 2.3.** *For any  $v_0, \dots, v_k \in \mathbb{C}^n$ , there exist  $\theta_0, \dots, \theta_k \in [0, 1]$  such that*

$$\left| \sum_{j=0}^k e(\theta_j)v_j \right| \leq \sqrt{\sum_{j=0}^k |v_j|^2}.$$

*Proof.* We have

$$\int_{[0,1]^k} \left| \sum_{j=0}^k e(\theta_j)v_j \right|^2 d\theta_1 \cdots d\theta_k = \sum_{j=0}^k |v_j|^2.$$

Thus, the average choice of  $(\theta_0, \dots, \theta_k)$  satisfies the conclusion.  $\square$

**Lemma 2.4.** *Let  $P \in \mathbb{C}[x_1, \dots, x_n]$ . Suppose that every solution to the equation  $P(x_1, \dots, x_n) = 0$  satisfies  $x_1 \cdots x_n = 0$ . Then  $P$  is a monomial; i.e.,  $P = cx_1^{d_1} \cdots x_n^{d_n}$  for some  $c \neq 0$  and nonnegative integers  $d_1, \dots, d_n$ .*

*Proof.* Let  $V = \{(x_1, \dots, x_n) \in \mathbb{C}^n : P(x_1, \dots, x_n) = 0\}$  be the vanishing set of  $P$ . By hypothesis, the polynomial  $x_1 \cdots x_n$  vanishes on  $V$ . Thus, since  $\mathbb{C}$  is algebraically closed, Hilbert’s Nullstellensatz implies that there is some  $d \in \mathbb{Z}_{\geq 0}$  such that  $(x_1 \cdots x_n)^d$  is contained in the ideal generated by  $P$ ; i.e.,  $P \mid (x_1 \cdots x_n)^d$ . Since  $\mathbb{C}[x_1, \dots, x_n]$  is a unique factorization domain, this is only possible if  $P$  is a monomial.  $\square$

**Lemma 2.5.** *Let  $P \in \mathbb{C}[x_1, \dots, x_n]$  and suppose that  $y \in \mathbb{C}^n$  is a zero of  $P$ . Then for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that any polynomial  $Q \in \mathbb{C}[x_1, \dots, x_n]$  obtained by changing any of the nonzero coefficients of  $P$  by at most  $\delta$  each has a zero  $z \in \mathbb{C}^n$  with  $|y - z| < \varepsilon$ .*

*Proof.* If  $P$  is identically 0 then so is  $Q$ , so we may take  $z = y$ . Otherwise, set

$$p(t) = P(y + tu) \quad \text{and} \quad q(t) = Q(y + tu)$$

for  $t \in \mathbb{C}$ , where  $u$  is any unit vector for which  $p(t)$  does not vanish for all  $t$ ; shrinking  $\varepsilon$  if necessary, assume that  $p(t)$  does not vanish on  $C_\varepsilon = \{t \in \mathbb{C} : |t| = \varepsilon\}$ , and let  $\gamma > 0$  be the minimum of  $|p(t)|$  on  $C_\varepsilon$ . For  $t \in C_\varepsilon$  we have

$$|q(t) - p(t)| < \delta N(1 + \varepsilon + |y|)^{\deg P},$$

where  $N$  is the number of nonzero coefficients of  $P$ . Choosing  $\delta$  so that the right side of this expression is bounded by  $\gamma$ , we have  $|q(t) - p(t)| < |p(t)|$  for  $t \in C_\varepsilon$ . By Rouché’s theorem  $q(t)$  has a zero  $t_0$  of modulus  $|t_0| < \varepsilon$ , and taking  $z = y + t_0u$  completes the proof.  $\square$

### 3. Simultaneous representations of $n$ -tuples of complex numbers

The technical heart of our work is the following analogue of Lemma 2 of [Saias and Weingartner 2009]:

**Proposition 3.1.** *For any real numbers  $y, R > 1$  there exists  $\eta > 0$  such that, for all  $\sigma \in (1, 1 + \eta]$ , we have*

$$\left\{ \left( \prod_{p>y} L(\sigma + it_p, \pi_{j,p}) \right)_{j=1,\dots,n} : t_p \in \mathbb{R} \text{ for each prime } p > y \right\} \\ \supseteq \{(z_1, \dots, z_n) \in \mathbb{C}^n : R^{-1} \leq |z_j| \leq R \text{ for all } j\}.$$

Loosely speaking, after simultaneously approximating the  $t_p$  by a common  $t$ , it will follow that we can make the  $L(s, \pi_j)$  independently approach any desired  $n$ -tuple of nonzero complex numbers, and this will allow us to find zeros in linear or polynomial combinations.

The proof relies on an analogue of Lemma 1 of [Saias and Weingartner 2009], whose adaptation is not especially straightforward. We carry out this work by proving two technical propositions; the first establishes the existence of solutions to a certain equation involving matrices in a fixed compact subset of  $GL_n(\mathbb{C})$ .

**Proposition 3.2.** *Let*

$$T = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1| = \dots = |z_n| = 1\}, \\ D = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|, \dots, |z_n| \leq 1\},$$

*and fix a compact set  $K \subseteq GL_n(\mathbb{C})$ . Then there is a number  $m_0 > 0$  such that for every  $m \geq m_0$  and all  $(g_1, \dots, g_m) \in K^m$ , there are continuous functions  $f_1, \dots, f_m : D \rightarrow T$  such that  $\sum_{i=1}^m g_i f_i(z) = z$  for all  $z \in D$ .*

We will carry out the proof in three steps:

- (1) We first show that there exist  $\varepsilon > 0$  and  $m_1$  such that for all  $m \geq m_1$  and all  $(g_1, \dots, g_m) \in K^m$ , the set  $\{\sum_{i=1}^m g_i t_i : t_1, \dots, t_m \in T\}$  contains an open  $\varepsilon$ -neighborhood of a point in  $\mathbb{C}^n$ .
- (2) “Fattening” the neighborhood constructed in the first step, we will then show that there exists  $m_2$  such that for  $m \geq m_2$  and all  $(g_1, \dots, g_m) \in K^m$ ,  $\{\sum_{i=1}^m g_i t_i : t_1, \dots, t_m \in T\}$  contains  $\{(z_1, \dots, z_n) : |z_1|^2 + \dots + |z_n|^2 \leq 4\}$ , the closed ball of radius 2.

- (3) Although the previous step yields a parametrization of a large closed set, it is not obviously continuous. By repeating the construction from step (1) using the added knowledge of step (2), we show that one can achieve a continuous parametrization of  $D$ .

*Proof.* We begin by showing (1). By compactness, there is an  $m_1$  such that for any  $m \geq m_1$  and any  $m$ -tuple  $(g_1, \dots, g_m)$ , there is a distinct pair of indices  $i, j$  such that  $\|g_i^{-1}g_j - I\| < 1/(3\sqrt{n})$ . Assume, without loss of generality, that  $(i, j) = (1, 2)$ , and put  $\Delta = g_1^{-1}g_2 - I$ . Then for any choice of  $t_1, t_2 \in T$ , we have

$$g_1t_1 + g_2t_2 = g_1(t_1 + (I + \Delta)t_2),$$

where  $\|\Delta\| < 1/(3\sqrt{n})$ .

We introduce some notation. First, define  $A = \{z \in \mathbb{C} : |z - 1| \leq \frac{1}{3}\}$  and  $B = \{z \in \mathbb{C} : |z - 1| \leq \frac{2}{3}\}$ . Next, let  $s_1, s_2 : B \rightarrow \mathbb{C}$  be the unique continuous functions satisfying  $z = s_1(z) + s_2(z)$ ,  $|s_1(z)| = |s_2(z)| = 1$  and  $\Im(s_1(z)/s_2(z)) > 0$  for all  $z \in B$ . For  $j = 1, 2$ , let  $t_j : B^n \rightarrow T$  be defined by  $t_j(z_1, \dots, z_n) = (s_j(z_1), \dots, s_j(z_n))$ .

Given an arbitrary element  $w \in A^n$ , we define a continuous function  $h_w : B^n \rightarrow \mathbb{C}^n$  by  $h_w(z) = w - \Delta t_2(z)$ . Since  $|t_2(z)| = \sqrt{n}$  and  $\|\Delta\| < 1/(3\sqrt{n})$ , we have  $|\Delta t_2(z)| < \frac{1}{3}$ . In particular, each entry of  $\Delta t_2(z)$  is bounded in magnitude by  $\frac{1}{3}$ , so by the triangle inequality, the image of  $h_w$  is contained in  $B^n$ . By the Brouwer fixed point theorem, there exists  $z \in B^n$  with  $h_w(z) = z$ , so that

$$t_1(z) + (I + \Delta)t_2(z) = z + \Delta t_2(z) = z + w - h_w(z) = w.$$

Therefore, all of  $A^n$  is in the image of the map  $z \mapsto t_1(z) + (I + \Delta)t_2(z)$ , so that in particular

$$A^n \subseteq \{t_1 + g_1^{-1}g_2t_2 : t_1, t_2 \in T\}.$$

Applying Lemma 2.2 with  $\delta = \frac{1}{3}$ , there is an  $\varepsilon > 0$  depending only on  $K$  such that  $\{g_1t_1 + g_2t_2 : t_1, t_2 \in T\}$  contains an  $\varepsilon$ -neighborhood of some point in  $\mathbb{C}^n$ . We conclude the same of the set  $\{g_1t_1 + \dots + g_mt_m : t_1, \dots, t_m \in T\}$  by choosing arbitrary fixed  $t_3, \dots, t_m \in T$ .

Proceeding to step (2), let  $k_1$  be a large integer to be determined later, set  $m_2 = m_1k_1$ , and for any  $m \geq m_2$  write  $m = km_1 + l$  with  $k \geq k_1$  and  $0 \leq l < m_1$ .

For each  $j$  with  $0 \leq j < k$ , applying step (1) to  $(g_{jm_1+1}, \dots, g_{j(m_1+m_1)})$ , we obtain an  $\varepsilon$ -neighborhood centered at some  $v_j \in \mathbb{C}^n$ . Further, we put  $v_k = g_{km_1+1}\vec{1} + \dots + g_{k(m_1+l)}\vec{1}$ , where  $\vec{1} = (1, \dots, 1) \in T$ . Since  $m_1$  is fixed and  $K$  is compact, we have  $|v_j| \leq C$  for  $0 \leq j \leq k$ , for some  $C$  independent of the individual  $g_i$ .

Let  $N_\varepsilon = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + \dots + |z_n|^2 < \varepsilon^2\}$  be the  $\varepsilon$ -neighborhood of the origin in  $\mathbb{C}^n$ . Then by the above observations, for any  $\theta_0, \dots, \theta_k \in [0, 1]$ ,

$\{\sum_{i=1}^m g_i t_i : t_1, \dots, t_m \in T\}$  contains the set

$$\sum_{j=0}^{k-1} e(\theta_j)(v_j + N_\varepsilon) + e(\theta_k)v_k = \sum_{j=0}^k e(\theta_j)v_j + kN_\varepsilon.$$

By Lemma 2.3, there is a choice of  $\theta_0, \dots, \theta_k$  for which  $|\sum_{j=0}^k e(\theta_j)v_j| \leq C\sqrt{k+1}$ . Now let  $k_1$  be the smallest positive integer satisfying  $k_1\varepsilon > C\sqrt{k_1+1} + 2$ . Then for  $k \geq k_1$ , we have shown that  $\{\sum_{i=1}^m g_i t_i : t_1, \dots, t_m \in T\}$  contains the closed ball of radius 2.

Proceeding to step (3), we put  $m_0 = 3nm_2$ . Suppose that  $m \geq m_0$  and  $(g_1, \dots, g_m)$  are given, and choose a partition of  $\{1, \dots, m\}$  into  $3n$  sets  $I_{j,\ell}$  (for  $1 \leq j \leq n$ ,  $1 \leq \ell \leq 3$ ), each of size at least  $m_2$ . For each  $j$  with  $1 \leq j \leq n$ , write

$$v_j = v_{j,1} = v_{j,2} = v_{j,3} = (0, \dots, 0, 2, 0, \dots, 0),$$

where the 2 is in the  $j$ -th position. For each  $j$  and  $\ell$  we use step (2) to express  $v_{j,\ell}$  in the form

$$v_{j,\ell} = \sum_{i \in I_{j,\ell}} g_i t_i \tag{3-1}$$

for some  $t_i \in T$ .

Next, note that the set  $\{2[(1, \dots, 1) + \alpha + \beta] : \alpha, \beta \in T\}$  contains  $D$ . As in step (1), we can choose continuous functions  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $\beta = (\beta_1, \dots, \beta_n) : D \rightarrow T$  such that  $z_j = 2[1 + \alpha_j(z) + \beta_j(z)]$  for every  $z = (z_1, \dots, z_n) \in D$ . Thus,

$$z = \sum_{j=1}^n [1 + \alpha_j(z) + \beta_j(z)]v_j = \sum_{j=1}^n [v_{j,1} + \alpha_j(z)v_{j,2} + \beta_j(z)v_{j,3}].$$

Finally, we use (3-1) to rewrite this as

$$z = \sum_{j=1}^n \left( \sum_{i \in I_{j,1}} g_i t_i + \sum_{i \in I_{j,2}} g_i (t_i \alpha_j(z)) + \sum_{i \in I_{j,3}} g_i (t_i \beta_j(z)) \right),$$

which is a decomposition of the type required. □

Next, we use the quasi-orthogonality of the coefficients  $\lambda_j(p)$  (Lemma 2.1) to show that, by choosing an arbitrary “twist”  $\epsilon_p \in S^1$  for each large prime  $p$ , we can make sums of the  $\epsilon_p \lambda_j(p)$  line up in linearly independent directions, as quantified in the following proposition.

Given a real parameter  $y > 0$ , we write

$$S(y) = \{p \text{ prime} : p > y\} \quad \text{and} \quad s(y, \sigma) = \sum_{p \in S(y)} p^{-\sigma}.$$

**Proposition 3.3.** *There is a compact set  $K \subseteq \text{GL}_n(\mathbb{C})$ , explicitly defined in (3-5), depending only on the degrees  $r_1, \dots, r_n$ , with the following property:*

*Let  $m$  be a positive integer. Then there is a real number  $\delta > 0$  (depending on the  $\pi_j$  and  $m$ ) such that for any  $y > 0$  and any  $\sigma \in (1, 1 + \delta]$ , there exists a partition of  $S(y)$  into  $mn$  pairwise disjoint subsets  $S_{ik}(y)$  ( $i = 1, \dots, m, k = 1, \dots, n$ ) and a choice of  $\epsilon_p \in S^1$  for each  $p \in S(y)$  such that the  $m$ -tuple of matrices  $(g_1, \dots, g_m)$  defined by*

$$g_i = \left( \frac{mn}{s(y, \sigma)} \sum_{p \in S_{ik}(y)} \frac{\epsilon_p \lambda_j(p)}{p^\sigma} \right)_{1 \leq j, k \leq n}, \quad i = 1, \dots, m \tag{3-2}$$

*lies in  $K^m$ .*

*Proof.* Let  $q$  be the smallest prime number satisfying  $q \equiv 1 \pmod{mn}$  and  $q \nmid \prod_{j=1}^n \text{cond}(\pi_j)$ . We put  $t = (q - 1)/mn$  and define  $S_{ik}^\circ(y)$  to be the union of residue classes

$$S_{ik}^\circ(y) = \bigcup_{\ell=1}^t \{p \in S(y) : p \equiv tn(i - 1) + t(k - 1) + \ell \pmod{q}\},$$

and

$$S_{ik}(y) = \begin{cases} S_{ik}^\circ(y) \cup \{q\} & \text{if } i = k = 1 \text{ and } y < q, \\ S_{ik}^\circ(y) & \text{otherwise.} \end{cases}$$

Then the  $S_{ik}(y)$  are pairwise disjoint and cover  $S(y)$ .

For a fixed choice of  $i$ , let  $v_k$  denote the  $k$ -th column of  $g_i$ , as defined in (3-2), with the  $\epsilon_p$  yet to be chosen. We will show by induction that there is a choice of the  $\epsilon_p$  such that

$$|v_\ell - \text{proj}_{\text{span}\{v_1, \dots, v_{\ell-1}\}} v_\ell| \geq \frac{1}{2r} \tag{3-3}$$

holds for every  $\ell = 1, \dots, n$ , where  $r = \sqrt{r_1^2 + \dots + r_n^2}$ . To that end, let  $k$  be given, and assume that (3-3) has been established for  $\ell = 1, \dots, k - 1$ . Choose a unit vector  $u = (u_1, \dots, u_n)$  orthogonal to  $v_1, \dots, v_{k-1}$ . By the Schwarz inequality and the Ramanujan bound  $|\lambda_j(p)| \leq r_j$ , for each prime  $p$  we have  $|\bar{u}_1 \lambda_1(p) + \dots + \bar{u}_n \lambda_n(p)| \leq r$ . Therefore

$$\begin{aligned} \frac{mn}{s(y, \sigma)} \sum_{p \in S_{ik}(y)} \frac{|\bar{u}_1 \lambda_1(p) + \dots + \bar{u}_n \lambda_n(p)|}{p^\sigma} &\geq \frac{mn}{rs(y, \sigma)} \sum_{p \in S_{ik}^\circ(y)} \frac{|\bar{u}_1 \lambda_1(p) + \dots + \bar{u}_n \lambda_n(p)|^2}{p^\sigma} \\ &= \frac{1 + O_{m,n}(\sigma - 1)}{r}, \end{aligned} \tag{3-4}$$

the latter equality following by Lemma 2.1. We choose  $\delta$  so that the  $O$  term above is bounded in modulus by  $\frac{1}{2}$ , and for each  $p \in S_{ik}(y)$  we choose  $\epsilon_p$  such that  $\epsilon_p(\bar{u}_1\lambda_1(p) + \dots + \bar{u}_n\lambda_n(p))$  is real and nonnegative. Then the left side of (3-4) equals

$$\langle u, v_k \rangle = \langle u, v_k - \text{proj}_{\text{span}\{v_1, \dots, v_{k-1}\}} v_k \rangle \leq |v_k - \text{proj}_{\text{span}\{v_1, \dots, v_{k-1}\}} v_k|,$$

so that (3-3) follows for  $\ell = k$ .

Applying Gram–Schmidt orthogonalization to  $v_1, \dots, v_n$ , it follows from (3-3) that  $|\det g_i| \geq (2r)^{-n}$ . Moreover, by the Schwarz inequality and Lemma 2.1 again, each entry of  $g_i$  is bounded above by  $1 + O_{m,n}(\sigma - 1)$ , so that  $\|g_i\| \leq 2n$  for a suitable choice of  $\delta$ . Thus,

$$K = \{g \in \text{GL}_n(\mathbb{C}) : \|g\| \leq 2n, |\det g| \geq (2r)^{-n}\} \tag{3-5}$$

has the desired properties. □

We are now ready to prove Proposition 3.1, largely following [Saias and Wein-gartner 2009].

*Proof of Proposition 3.1.* We use Propositions 3.3 and 3.2 to determine a compact set  $K \subseteq \text{GL}_n(\mathbb{C})$ , a positive integer  $m_0$ , and a real number  $\delta > 0$  with the properties described there. Taking  $m = m_0$ , the aforementioned propositions yield, for any  $\sigma \in (1, 1 + \delta]$ , an  $m$ -tuple of matrices  $(g_1, \dots, g_m) \in K^m$ , elements  $\epsilon_p \in S^1$  for each prime  $p > y$ , and continuous functions  $f_1, \dots, f_m : D \rightarrow T$  such that

$$\sum_{i=1}^m g_i f_i(z) = z \quad \text{for all } z \in D. \tag{3-6}$$

Now, let  $\mu = s(y, \sigma)/(mn)$ . For each prime  $p > y$ , we define a continuous function  $t_p : \mu D \rightarrow \mathbb{R}$  satisfying

$$p^{-it_p(z)} = \epsilon_p f_i(\mu^{-1}z)_k, \tag{3-7}$$

where  $(i, k)$  is the unique pair of indices for which  $p \in S_{ik}(y)$ , and  $f_i(\mu^{-1}z)_k$  denotes the  $k$ -th component of  $f_i(\mu^{-1}z)$ . (Note that the lift from  $S^1$  to  $\mathbb{R}$  is possible since  $D$  is simply connected.)

Define an error term  $E(z) = (E_1(z), \dots, E_n(z))$  by writing, for each  $j = 1, \dots, n$ ,

$$E_j(z) = \sum_{p>y} (\log L(\sigma + it_p(z), \pi_{j,p}) - \lambda_j(p) p^{-(\sigma+it_p(z))}).$$

By the Ramanujan bound, we have

$$\log L(s, \pi_{j,p}) - \lambda_j(p) p^{-s} = O(p^{-2})$$

uniformly for  $\Re(s) \geq 1$ . Since  $\sum_p p^{-2}$  converges, the continuity of  $E$  follows from

that of the individual  $t_p$ . Moreover, each component  $E_j(z)$  is bounded by a number  $C > 0$ , independent of  $j, z, y$ , or  $\sigma$ .

Set  $R' = \sqrt{\pi^2 + \log^2 R}$ . We take  $\eta \in (0, \delta]$  small enough that the condition  $\sigma \in (1, 1 + \eta]$  ensures that  $\mu \geq C + R'$ . By (3-6), (3-7), and Proposition 3.3 we have

$$\sum_{p>y} \lambda_j(p) p^{-(\sigma+it_p(z))} = \sum_{i=1}^m \sum_{k=1}^n \sum_{p \in S_{ik}(y)} \frac{\lambda_j(p) \epsilon_p f_i(\mu^{-1}z)_k}{p^\sigma} = z_j$$

for any  $z = (z_1, \dots, z_n) \in \mu D$ . Now, fix  $w \in R'D$  and define a function  $F_w : (C + R')D \rightarrow \mathbb{C}$  by  $F_w(z) = w - E(z)$ . By the estimate for  $E_j(z)$  above, the image of  $F_w$  is contained in  $(C + R')D$ . Thus, by the Brouwer fixed point theorem, there exists  $z \in (C + R')D$  with  $F_w(z) = z$ , so that

$$\left( \sum_{p>y} \log L(\sigma + it_p(z), \pi_{j,p}) \right)_{j=1, \dots, n} = z + E(z) = z + w - F_w(z) = w.$$

Taking exponentials yields the proposition. □

### 4. Proof of Theorem 1.2

The proof will be carried out in two steps:

- (1) Applying our previous results, we show that unless  $P$  is a monomial (as described in Theorem 1.2), for every  $\sigma > 1$  sufficiently close to 1 there are real numbers  $t_p$  (for each prime  $p$ ) and  $t_0$  such that  $P|_{s=\sigma+it_0}$  vanishes at  $(\prod_p L(\sigma + it_p, \pi_{1,p}), \dots, \prod_p L(\sigma + it_p, \pi_{n,p}))$ .
- (2) Simultaneously, approximating the  $p^{-it_p}$  by  $p^{-it}$  for a common value of  $t$ , we use Rouché's theorem to find a zero of  $P(L(s, \pi_1), \dots, L(s, \pi_n))$  close to  $\sigma + it$ .

Note that the second step is standard and is applied in [Saias and Weingartner 2009] in much the same way.

We begin with a polynomial  $P$  whose coefficients are finite Dirichlet series  $D(s) = \sum_{m=1}^M a_m m^{-s}$ , and let  $y$  be the largest value of  $M$  occurring in any of these coefficients. We rewrite each  $L(s, \pi_j)$  as  $L_{\leq y}(s, \pi_j)L_{> y}(s, \pi_j)$ , splitting each Euler product into products over primes  $p \leq y$  and  $p > y$  respectively. Setting

$$Q(x_1, \dots, x_n) = P(L_{\leq y}(s, \pi_1)x_1, \dots, L_{\leq y}(s, \pi_n)x_n),$$

we have  $P(L(s, \pi_1), \dots, L(s, \pi_n)) = Q(L_{> y}(s, \pi_1), \dots, L_{> y}(s, \pi_n))$ .

The coefficients of  $Q$  are rational functions of the  $p^{-s}$  for  $p \leq y$ . More precisely, for any monomial term  $D(s)x_1^{d_1} \dots x_n^{d_n}$  in the expansion of  $P$ , the corresponding term of  $Q$  is

$$D(s)L_{\leq y}(s, \pi_1)^{d_1} \dots L_{\leq y}(s, \pi_n)^{d_n} x_1^{d_1} \dots x_n^{d_n}.$$

Since the finite Euler products  $L_{\leq y}(s, \pi_j)$  are nonvanishing holomorphic functions on  $\{s \in \mathbb{C} : \Re(s) \geq 1\}$ , the corresponding terms of  $P$  and  $Q$  have the same zeros there.

Let  $D_1(s), \dots, D_m(s)$  run through the coefficients of  $P$  which do not vanish identically, and consider their product  $f(s) = D_1(s) \cdots D_m(s)$ . Then  $f$  is itself a finite Dirichlet series which does not vanish identically. By complex analysis,  $f$  cannot vanish at  $1 + it$  for every  $t \in \mathbb{R}$ , so there is some  $t_0$  for which  $D_1(1 + it_0), \dots, D_m(1 + it_0)$  are all nonzero, and the same holds for the corresponding terms of  $Q$ .

Next, we specialize the coefficients of  $Q$  to a fixed value of  $s$ , obtaining a polynomial  $h_s \in \mathbb{C}[x_1, \dots, x_n]$ . Considering  $s = 1 + it_0$ , Lemma 2.4 implies either that  $h_{1+it_0} = cx_1^{d_1} \cdots x_n^{d_n}$  for some  $c \in \mathbb{C}$  and  $d_1, \dots, d_n \in \mathbb{Z}_{\geq 0}$ , or that there are  $y_1, \dots, y_n \in \mathbb{C}$ , none zero, for which  $h_{1+it_0}(y_1, \dots, y_n) = 0$ . In the former case, it follows from our choice of  $t_0$  that  $P = D(s)x_1^{d_1} \cdots x_n^{d_n}$  is a monomial, as allowed in the conclusion of Theorem 1.2. Henceforth, we assume that we are in the latter case, and aim to show that  $P(L(s, \pi_1), \dots, L(s, \pi_n))$  has a zero with  $\Re(s) > 1$ .

We choose  $R > 1$  so that  $R^{-1/2} \leq |y_j| \leq R^{1/2}$  for every  $j$ . By Lemma 2.5, there is a number  $\varepsilon > 0$  such that for every  $\sigma \in (1, 1 + \varepsilon]$ , there exists  $(z_1(\sigma), \dots, z_n(\sigma)) \in \mathbb{C}^n$  satisfying  $h_{\sigma+it_0}(z_1(\sigma), \dots, z_n(\sigma)) = 0$  and  $R^{-1} \leq |z_j(\sigma)| \leq R$  for every  $j$ . We use Proposition 3.1 to determine  $\eta$  in terms of  $y$  and  $R$ , and assume that  $\eta \leq \varepsilon$  by shrinking  $\eta$  if necessary. Proposition 3.1 then guarantees that, for every  $\sigma \in (1, 1 + \eta]$ , we can solve the simultaneous system of equations

$$\prod_{p > y} L(\sigma + it_p, \pi_{j,p}) = z_j(\sigma), \quad j = 1, \dots, n,$$

in the  $t_p$  for  $p > y$ . For  $p \leq y$  we set  $t_p = t_0$ , thereby completing step (1).

Turning to step (2), let  $\sigma_1, \sigma_2 \in \mathbb{R}$  with  $1 < \sigma_1 < \sigma_2 \leq 1 + \eta$ , and put  $\sigma = (\sigma_1 + \sigma_2)/2$ . With the  $t_0$  and  $t_p$  resulting from step (1) for this choice of  $\sigma$ , let  $P_{it_0}$  denote the polynomial obtained from  $P$  by replacing  $s$  by  $s + it_0$  in all of its coefficients, and define

$$F(s) = P_{it_0} \left( \prod_p L(s + it_p, \pi_{1,p}), \dots, \prod_p L(s + it_p, \pi_{n,p}) \right). \quad (4-1)$$

Then  $F$  is holomorphic for  $|s - \sigma| < \sigma - 1$  and satisfies  $F(\sigma) = 0$  by construction. It follows that there is a number  $\rho \in (0, (\sigma_2 - \sigma_1)/2]$  such that  $F(s) \neq 0$  for all  $s \in C_\rho = \{s \in \mathbb{C} : |s - \sigma| = \rho\}$ . Write  $\gamma$  for the minimum of  $|F(s)|$  on  $C_\rho$ .

Next, by abuse of notation, we write  $P(s)$  to denote  $P(L(s, \pi_1), \dots, L(s, \pi_n))$ . As  $P(s) = \sum_{m=1}^\infty a_m m^{-s}$  converges absolutely as a Dirichlet series for  $\Re(s) > 1$ , there is an integer  $M > 0$  with  $\sum_{m=M}^\infty |a_m| m^{-\sigma_1} \leq \gamma/3$ . By (4-1) we have  $F(s) = \sum_{m=1}^\infty b_m m^{-s}$ , where  $b_m = a_m \prod_{p|m} p^{-it_p \text{ ord}_p(m)}$ , and by the joint uniform



distribution of  $p^{it}$  for primes  $p < M$ , it follows that the set of  $t \in \mathbb{R}$  satisfying

$$\sum_{m=1}^{M-1} \frac{|a_m m^{-it} - b_m|}{m^{\sigma_1}} < \frac{\gamma}{3}$$

has positive lower density. The triangle inequality yields  $|P(s + it) - F(s)| < \gamma$  for any such  $t$  and for all  $s$  with  $\Re(s) \geq \sigma_1$ , and in particular for all  $s \in C_\rho$ . By Rouché's theorem, it follows that  $P(s + it)$  has a zero  $s$  with  $|s - \sigma| < \rho$ . Thus,  $P(s)$  has zeros with real part in  $[\sigma_1, \sigma_2]$ , and indeed we have

$$\#\{s \in \mathbb{C} : \Re(s) \in [\sigma_1, \sigma_2], \Im(s) \in [-T, T], P(s) = 0\} \gg_{\sigma_1, \sigma_2} T$$

for all  $T \geq T_0(\sigma_1, \sigma_2)$ .

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
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