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Adequate groups of low degree

Robert Guralnick, Florian Herzig and Pham Huu Tiep

The notion of adequate subgroups was introduced by Jack Thorne. It is a weakening of the notion of big subgroups used in generalizations of the Taylor–Wiles method for proving the automorphy of certain Galois representations. Using this idea, Thorne was able to strengthen many automorphy lifting theorems. It was shown by Guralnick, Herzig, Taylor, and Thorne that if the dimension is small compared to the characteristic, then all absolutely irreducible representations are adequate. Here we extend that result by showing that, in almost all cases, absolutely irreducible kG-modules in characteristic p whose irreducible G^+-summands have dimension less than p (where G^+ denotes the subgroup of G generated by all p-elements of G) are adequate.

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1. Introduction

Throughout the paper, let k be a field of characteristic p and let V be a finite-dimensional vector space over k. Let $\rho : G \to \mathrm{GL}(V)$ be an absolutely irreducible representation. Thorne [2012] called (G, V) *adequate* if the following conditions hold (we rephrase the conditions slightly by combining two of the properties into one):

(i) p does not divide $\dim V$.

(ii) $\mathrm{Ext}^1_G(V, V) = 0$.

(iii) $\mathrm{End}(V)$ is spanned by the elements $\rho(g)$ with $\rho(g)$ semisimple.

We remark that recently Thorne has shown that one can relax condition (i) above (see [Thorne 2015, Corollary 7.3] and [Guralnick et al. 2014, §1]).

If G is a finite group of order prime to p, then it is well known that (G, V) is adequate. In this case, condition (iii) is often referred to as Burnside's lemma. It is a trivial consequence of the Artin–Wedderburn theorem. Also, (G, V) is adequate if G is a connected algebraic group over $k = \bar{k}$ and V is a rational irreducible kG-module; see [Guralnick 2012a, Theorem 1.2].

The adequacy conditions are a generalization to higher dimension of the conditions used by Wiles and Taylor in studying the automorphic lifts of certain 2-dimensional Galois representations, and they are a weakening of the previously introduced *bigness* condition [Clozel et al. 2008]. Thorne [2012] strengthened the existing automorphy lifting theorems for n-dimensional Galois representations assuming the weaker adequacy hypotheses. We refer the reader to [Thorne 2012] for more references and details.

The following theorem was proved in [Guralnick et al. 2012, Theorem 9]:

Theorem 1.1. *Let k be a field of characteristic p and G a finite group. Let V be an absolutely irreducible faithful kG-module. Let G^+ denote the subgroup generated by the p-elements of G. If $\dim W \le (p-3)/2$ for an absolutely irreducible kG^+-submodule W of V, then (G, V) is adequate.*

The example $G = \mathrm{SL}_2(p)$ with V irreducible of dimension $(p-1)/2$ shows that the previous theorem is the best possible. However, the counterexamples are rare. Our first goal is to prove a similar theorem under the assumption that $\dim W < p$. We show that almost always (G, V) is adequate; see Corollary 1.4. Indeed, we show that the spanning condition always holds under the weaker hypothesis. We show that there are only a handful of examples where $\mathrm{Ext}^1_G(V, V) \ne 0$. See Theorems 1.2 and 1.3 for more precise statements.

Theorem 1.1 was crucial in several recent applications of automorphy lifting theorems, such as [Barnet-Lamb et al. 2014; Calegari 2012; Dieulefait 2014]. In fact, the main two technical hypotheses in the most recent automorphy lifting theorems are potential diagonalizability (a condition in p-adic Hodge theory) and adequacy of the residual image [Dieulefait and Gee 2012]. Since some important applications of automorphy lifting theorems [Breuil et al. 2001; Khare and Wintenberger 2009; Dieulefait 2014] require working with primes p that are small relative to the dimension of the Galois representation, we expect that our results will be useful in obtaining further arithmetic applications of automorphy lifting theorems. (Note that adequacy of 2-dimensional linear groups has been analyzed in Appendix A of [Barnet-Lamb et al. 2013].)

An outgrowth of our results leads us to prove an analogue of [Guralnick 1999] and answer a question of Serre on complete reducibility of finite subgroups of orthogonal and symplectic groups of small degree. This is done in the sequel [Guralnick et al. 2014], where we essentially classify indecomposable modules in characteristic p of dimension less than $2p-2$. We also extend adequacy results to the case of linear groups of degree p and generalize the asymptotic result [Guralnick 2012a, Theorem 1.2] to disconnected algebraic groups $\mathcal{G}$ (with $p \nmid [\mathcal{G}:\mathcal{G}^0]$), allowing at the same time that p divides the dimension of the $\mathcal{G}$-module.

Note that if the kernel of ρ has order prime to p, then there is no harm in passing to the quotient. So we will generally assume that either ρ is faithful or more generally has kernel of order prime to p. Also, note that the dimensions of cohomology groups and the dimension of the span of the semisimple elements in G in $\mathrm{End}(V)$ do not change under extension of scalars. Hence, most of the time we will work over an algebraically closed field k.

Following [Guralnick 2012b], we say that the representation $\rho : G \to \mathrm{GL}(V)$, or the pair (G, V), is *weakly adequate* if $\mathrm{End}(V)$ is spanned by the elements $\rho(g)$ with $\rho(g)$ semisimple.

Our main results are the following:

Theorem 1.2. *Let k be a field of characteristic p and G a finite group. Let V be an absolutely irreducible faithful kG-module. Let G^+ denote the subgroup generated by the p-elements of G. If $p > \dim W$ for an irreducible kG^+-submodule W of V, then (G, V) is weakly adequate.*

Theorem 1.3. *Let $k = \bar{k}$ be a field of characteristic p and G a finite group. Let V be an irreducible faithful kG-module. Let G^+ denote the subgroup generated by the p-elements of G. Suppose that $p > d := \dim W$ for an irreducible kG^+-submodule W of V, and let $H < \mathrm{GL}(W)$ be induced by the action of G^+ on W. Then one of the following holds*:

(a) *p is a Fermat prime, $d = p-1$, G^+ is solvable (and so G is p-solvable), and $H/\boldsymbol{O}_{p'}(H) = C_p$.*

(b) *$H^1(G, k) = 0$. Furthermore, either $\mathrm{Ext}^1_G(V, V) = 0$, or one of the following holds*:

(i) $H = \mathrm{PSL}_2(p)$ *or* $\mathrm{SL}_2(p)$, *and* $d = (p \pm 1)/2$.
(ii) $H = \mathrm{SL}_2(p) \times \mathrm{SL}_2(p^a)$ (*modulo a central subgroup*), $d = p - 1$, *and* W *is a tensor product of a* $(p-1)/2$-*dimensional* $\mathrm{SL}_2(p)$-*module and a* 2-*dimensional* $\mathrm{SL}_2(p^a)$-*module.*
(iii) $p = (q+1)/2$, $d = p - 1$, *and* $H \cong \mathrm{SL}_2(q)$.
(iv) $p = 2^f + 1$ *is a Fermat prime*, $d = p - 2$, *and* $H \cong \mathrm{SL}_2(2^f)$.
(v) $(H, p, d) = (3\mathsf{A}_6, 5, 3)$ *and* $(2\mathsf{A}_7, 7, 4)$.
(vi) $(H, p, d) = (\mathrm{SL}_2(3^a), 3, 2)$ *and* $a \geq 2$.

Theorems 1.2 and 1.3 immediately imply:

Corollary 1.4. *Let* k *be a field of characteristic* p *and* G *a finite group. Let* V *be an absolutely irreducible faithful* kG-*module, and let* G^+ *denote the subgroup generated by the* p-*elements of* G. *Suppose that the dimension of any irreducible* kG^+-*submodule in* V *is less than* p. *Let* W *be an irreducible* $\bar{k}G^+$-*submodule of* $V \otimes_k \bar{k}$. *Then* (G, V) *is adequate, unless the group* $H < \mathrm{GL}(W)$ *induced by the action of* G^+ *on* W *is as described in one of the exceptional cases* (a), (b)(i)–(vi) *listed in Theorem 1.3.*

Corollary 1.5. *Let* k *be a field of characteristic* p *and* G *a finite group. Let* V *be an absolutely irreducible faithful* kG-*module, and let* G^+ *denote the subgroup generated by the* p-*elements of* G. *Suppose that the dimension* d *of any irreducible* kG^+-*submodule in* V *is less than* $p-3$. *Let* W *be an irreducible* $\bar{k}G^+$-*submodule of* $V \otimes_k \bar{k}$. *Then* (G, V) *is adequate, unless* $d = (p \pm 1)/2$ *and the group* $\overline{H} < \mathrm{PGL}(W)$ *induced by the action of* G^+ *on* W *is* $\mathrm{PSL}_2(p)$.

One should emphasize that, in all the aforementioned results, the dimension bound $\dim W < p$ is imposed only on an irreducible G^+-summand of V. In general, G/G^+ can be an arbitrary p'-group, and likewise, $\dim V / \dim W$ can be arbitrarily large. So a major portion of the proofs, especially for Theorem 1.2, is spent establishing the results under these more general hypotheses.

This paper is organized as follows. In Section 2, based on results of [Blau and Zhang 1993], we describe the structure of (non-p-solvable) finite linear groups $G < \mathrm{GL}(V)$ such that the dimension of irreducible G^+-summands in V is less than p; see Theorem 2.4. Sections 3 and 4 are devoted to establish weak adequacy for Chevalley groups in characteristic p. In Sections 5 and 6, we prove adequacy for the remaining families of finite groups occurring in Theorem 2.4 and complete the proof of Theorem 1.2. In Section 7, we collect various facts concerning extensions and self-extensions of simple modules. The main result of Section 8, Proposition 8.2, classifies self-dual indecomposable $\mathrm{SL}_2(q)$-modules for $p \mid q$. In Section 9, we describe the structure of finite groups G possessing a reducible indecomposable module of dimension $\leq 2p - 3$ (Proposition 9.7). Theorem 1.3 and Corollary 1.4 are proved in Section 10.

Notation. If V is a kG-module and $X \le G$ is a subgroup, then V_X denotes the restriction of V to X. The containments $X \subset Y$ (for sets) and $X < Y$ (for groups) are strict. Fix a prime p and an algebraically closed field k of characteristic p. Then for any finite group G, $\mathrm{IBr}_p(G)$ is the set of isomorphism classes of irreducible kG-representations (or their Brauer characters, depending on the context), $\mathfrak{d}_p(G)$ denotes the smallest degree of the nontrivial $\varphi \in \mathrm{IBr}_p(G)$, and $B_0(G)$ denotes the principal p-block of G. Sometimes we use $\mathbb{1}$ to denote the principal representation. $\boldsymbol{O}_p(G)$ is the largest normal p-subgroup of G, $\boldsymbol{O}^p(G)$ is the smallest normal subgroup N of G subject to G/N being a p-group, and similarly for $\boldsymbol{O}_{p'}(G)$ and $\boldsymbol{O}^{p'}(G) = G^+$. Furthermore, the *Fitting subgroup* $F(G)$ is the largest nilpotent normal subgroup of G, and $E(G)$ is the product of all subnormal quasisimple subgroups of G, so that $F^*(G) = F(G)E(G)$ is the *generalized Fitting subgroup* of G. Given a finite-dimensional kG-representation $\Phi : G \to \mathrm{GL}(V)$, we denote by $\mathcal{M}$ the k-span

$$\langle \Phi(g) : \Phi(g) \text{ semisimple} \rangle_k.$$

If M is a finite-length module over a ring R, then define $\mathrm{soc}_i(M)$ by $\mathrm{soc}_0(M) = 0$ and $\mathrm{soc}_j(M)/\mathrm{soc}_{j-1}(M) = \mathrm{soc}(M/\mathrm{soc}_{j-1}(M))$. If $M = \mathrm{soc}_j(M)$ with j minimal, we say that j is the *socle length* of M.

2. Linear groups of low degree

First we describe the structure of absolutely irreducible non-p-solvable linear groups of low degree, relying on the main result of [Blau and Zhang 1993]:

Theorem 2.1. *Let W be a faithful, absolutely irreducible kH-module for a finite group H with $\boldsymbol{O}^{p'}(H) = H$. Suppose that* $1 < \dim W < p$. *Then one of the following cases holds, where* $P \in \mathrm{Syl}_p(H)$:

(a) *p is a Fermat prime, $|P| = p$, $H = \boldsymbol{O}_{p'}(H)P$ is solvable*, $\dim W = p-1$, *and $\boldsymbol{O}_{p'}(H)$ is absolutely irreducible on W.*

(b) $|P| = p$, $\dim W = p-1$, *and one of the following conditions holds*:

- (b1) $(H, p) = (\mathrm{SU}_n(q), (q^n+1)/(q+1))$, $(\mathrm{Sp}_{2n}(q), (q^n+1)/2)$, $(2\mathsf{A}_7, 5)$, $(3J_3, 19)$, *or* $(2Ru, 29)$.
- (b2) $p = 7$ *and* $H = 6_1 \cdot \mathrm{PSL}_3(4)$, $6_1 \cdot \mathrm{PSU}_4(3)$, $2J_2$, $3\mathsf{A}_7$, *or* $6\mathsf{A}_7$.
- (b3) $p = 11$ *and* $H = M_{11}$, $2M_{12}$, *or* $2M_{22}$.
- (b4) $p = 13$ *and* $H = 6 \cdot Suz$ *or* $2\mathrm{G}_2(4)$.

(c) $|P| = p$, $\dim W = p-2$, *and* $(H, p) = (\mathrm{PSL}_n(q), (q^n-1)/(q-1))$, (A_p, p), $(3\mathsf{A}_6, 5)$, $(3\mathsf{A}_7, 5)$, $(M_{11}, 11)$, *or* $(M_{23}, 23)$.

(d) $(H, p, \dim W) = (2\mathsf{A}_7, 7, 4)$, $(J_1, 11, 7)$.

(e) *Extraspecial case*: $|P| = p = 2^n + 1 \geq 5$, $\dim W = 2^n$, $\boldsymbol{O}_{p'}(H) = R\boldsymbol{Z}(H)$, $R = [P, R]\boldsymbol{Z}(R) \in \mathrm{Syl}_2(\boldsymbol{O}_{p'}(H))$, $[P, R]$ *is an extraspecial* 2*-group of order* 2^{1+2n}, *and* $V_{[P,R]}$ *is absolutely irreducible. Furthermore*, $S := H/\boldsymbol{O}_{p'}(H)$ *is simple nonabelian, and either* $S = \mathrm{Sp}_{2a}(2^b)'$ *or* $\Omega^-_{2a}(2^b)'$ *with* $ab = n$ *or* $S = \mathrm{PSL}_2(17)$ *and* $p = 17$.

(f) $\mathrm{Lie}(p)$ *case*: $H/\boldsymbol{Z}(H)$ *is a direct product of simple groups of Lie type in characteristic* p.

Furthermore, *in the cases* (b)–(d), H *is quasisimple with* $\boldsymbol{Z}(H)$ *a* p'*-group.*

Proof. We apply Theorem A of [Blau and Zhang 1993] and arrive at one of the possibilities (a)–(j) listed there. Note that possibility (j) is restated as our case (f), and possibilities (f)–(i) do not occur since H is absolutely irreducible. Possibility (a) does not arise either since $\dim W > 1$, and possibility (b) is restated as our case (a). Next, in the case of possibility (c), either we are back to our case (a), or else we are in case (e), where the simplicity of S follows from the assumption that $H = \boldsymbol{O}^{p'}(H)$. (Also, $S \not\cong \Omega^+_{2a}(2^b)$ since $|S|_p = |P| = p$.)

In the remaining cases (d), (e), and (g) of [Blau and Zhang 1993, Theorem A], we have that $H/\boldsymbol{Z}(H) = S$ is a simple nonabelian group, and $\boldsymbol{Z}(H)$ is a cyclic p'-group by Schur's lemma. Hence, $H^{(\infty)}$ is a perfect normal subgroup of p'-index in $H = \boldsymbol{O}^{p'}(H)$. It follows that $H = H^{(\infty)}$ and so it is quasisimple. Also, the possibilities for $(S, \dim W, p)$ are listed. Using

- [Guralnick and Tiep 1999] if $S = \mathrm{PSL}_n(q)$,
- [Guralnick et al. 2002] if $S = \mathrm{PSU}_n(q)$ or $\mathrm{PSp}_{2n}(q)$,
- [Guralnick and Tiep 2005, Lemma 6.1] if $S = \mathsf{A}_p$ and $p \geq 17$, and
- [Jansen et al. 1995] for the other simple groups,

we arrive at cases (b)–(d). □

Next we prove some technical lemmas in the spirit of [Blau and Zhang 1993, Lemma 3.10].

Lemma 2.2. *Let* G *be a finite group with normal subgroups* K_1 *and* K_2 *such that* $K_1 \cap K_2 \leq \boldsymbol{O}_{p'}(G)$. *For any finite group* X, *let* $\overline{X}$ *denote* $X/\boldsymbol{O}_{p'}(X)$. *Suppose that* $\overline{G/K_1} \cong \prod_{i\in I} M_i$ *and* $\overline{G/K_2} \cong \prod_{j\in J} N_j$ *are direct products of simple nonabelian groups. Then there are some sets* $I' \subseteq I$ *and* $J' \subseteq J$ *such that*

$$\overline{G} \cong \prod_{i\in I'} M_i \times \prod_{j\in J'} N_j.$$

Proof. For $i = 1, 2$, let $K_i \leq H_i \lhd G$ be such that $H_i/K_i = \boldsymbol{O}_{p'}(G/K_i)$. Then

$$G/H_1 \cong \prod_{i\in I} M_i, \quad G/H_2 \cong \prod_{j\in J} N_j.$$

By [Blau and Zhang 1993, Lemma 3.9], there are sets $I' \subseteq I$ and $J' \subseteq J$ such that

$$G/(H_1 \cap H_2) \cong \prod_{i \in I'} M_i \times \prod_{j \in J'} N_j.$$

It remains to show that $H_1 \cap H_2 = \boldsymbol{O}_{p'}(G)$. Certainly, $H_1 \cap H_2 \geq \boldsymbol{O}_{p'}(G)$. Conversely,

$$(H_1 \cap K_2)/(K_1 \cap K_2) \hookrightarrow H_1/K_1, \quad (H_1 \cap H_2)/(H_1 \cap K_2) \hookrightarrow H_2/K_2,$$

and $K_1 \cap K_2 \leq \boldsymbol{O}_{p'}(G)$. It follows that $H_1 \cap H_2$ is a p'-group. □

Lemma 2.3. *Let G be a finite group with a faithful kG-module V. Suppose that $V = W_1 \oplus \cdots \oplus W_t$ is a direct sum of kG-submodules, and let $H_i \leq \mathrm{GL}(W_i)$ be the linear group induced by the action of G on W_i. Suppose that $S_i := H_i/\boldsymbol{O}_{p'}(H_i)$ is a simple nonabelian group for each i. Then there is a subset $J \subseteq \{1, 2, \ldots, t\}$ such that*

$$G/\boldsymbol{O}_{p'}(G) \cong \prod_{j \in J} S_j.$$

In particular, if $\boldsymbol{O}_{p'}(H_i) = 1$ for all i, then $G \cong \prod_{j \in J} S_j$.

Proof. We proceed by induction on t. The induction base $t = 1$ is obvious. For the induction step, let K_i denote the kernel of the action of G on W_i, so that $H_i = G/K_i$. The faithfulness of V implies that $\bigcap_{i=1}^{t} K_i = 1$. Adopt the bar notation $\overline{X}$ of Lemma 2.2. By the assumption, $\overline{G/K_1} \cong S_1$. Next, observe that $L := \bigcap_{i=2}^{t} K_i$ is the kernel of the action of G on $V' := W_2 \oplus \cdots \oplus W_t$, and the action of G/L on W_i induces H_i for all $i \geq 2$. Applying the induction hypothesis to G/L acting on V', we see that $\overline{G/L} \cong \prod_{j \in J'} S_j$ for some $J' \subseteq \{2, 3, \ldots, t\}$. Also, $K_1 \cap L = 1$. Hence we can apply Lemma 2.2 to get $\overline{G} \cong \prod_{j \in J} S_j$ for some $J \subseteq \{1, 2, 3, \ldots, t\}$.

Finally, if $\boldsymbol{O}_{p'}(H_i) = 1$ for all i, then the action of $\boldsymbol{O}_{p'}(G)$ on W_i induces a normal p'-subgroup of H_i for all i, whence $\boldsymbol{O}_{p'}(G) \leq \bigcap_{i=1}^{t} K_i = 1$, and we are done. □

Theorem 2.4. *Let V be a finite-dimensional vector space over an algebraically closed field k of characteristic p and $G < \mathrm{GL}(V)$ a finite irreducible subgroup. Suppose that an irreducible G^+-submodule W of V has dimension $< p$ and G^+ is not solvable. Then G^+ is perfect and has no composition factor isomorphic to C_p; in particular, $H^1(G, k) = 0$. Furthermore, if H is the image of G^+ in $\mathrm{GL}(W)$, then one of the following statements holds*:

(i) *One of the cases* (b)–(d) *of Theorem 2.1 holds for H, and $G^+/\boldsymbol{Z}(G^+) = S_1 \times \cdots \times S_n \cong S^n$ is a direct product of n copies of the simple nonabelian group $S = H/\boldsymbol{Z}(H)$. Here, G permutes these n direct factors $S_1, \ldots, S_n$ transitively. Furthermore, $G^+ = L_1 * \cdots * L_n$ is a central product of quasisimple groups L_i, each being a central cover of S, and the action of G^+ on each irreducible G^+-submodule W_i of W induces a quasisimple subgroup of $\mathrm{GL}(W_i)$. Finally, if H is* *the full covering group of S or if $H = S$, then*

$$G^+ = L_1 \times L_2 \times \cdots \times L_n \cong H^n.$$

(ii) *Case* (e) *of Theorem 2.1 holds for H. Furthermore, $\boldsymbol{O}_{p'}(G^+)$ is irreducible on any irreducible G^+-submodule W_i of V, and $G^+/\boldsymbol{O}_{p'}(G^+) \cong S^m$ is a direct product of $m \geq 1$ copies of the simple nonabelian group S listed in case* (e) *of Theorem 2.1.*

(iii) *Case* (f) *of Theorem 2.1 holds for H, and $G^+ = L_1 * \cdots * L_n$ is a central product of quasisimple groups L_i of Lie type in characteristic p with $\boldsymbol{Z}(L_i)$ a p'-group.*

Proof. (a) By Clifford's theorem, $V_{G^+} \cong e \sum_{i=1}^{t} W_i$ for some $e, t \geq 1$, and $\{W_1, \ldots, W_t\}$ is a full set of representatives of isomorphism classes of G-conjugates of $W \cong W_1$. Let $\Phi_i : G^+ \to \mathrm{GL}(W_i)$ denote the corresponding representation, and let $K_i := \mathrm{Ker}(\Phi_i)$, so that $G^+/K_i \cong H$ for all i, where we denote by H the subgroup of $\mathrm{GL}(W)$ induced by the action of G^+ on W. The faithfulness of the action of G on V implies that $\bigcap_{i=1}^{t} K_i = 1$. In particular, G^+ injects into $\prod_{i=1}^{t}(G^+/K_i) \cong H^t$. Hence case (a) of Theorem 2.1 is impossible since G^+ is not solvable. In case (f) of Theorem 2.1, an argument similar to the proof of Lemma 2.3 shows that $G^+/\boldsymbol{Z}(G^+) = S_1 \times \cdots \times S_n$ is a direct product of simple groups S_i of Lie type in characteristic p. Since $G^+ = \boldsymbol{O}^{p'}(G^+)$ and $\boldsymbol{O}_p(G^+) \leq \boldsymbol{O}_p(G) = 1$, it then follows that G^+ equals $L_1 * \cdots * L_n$, a central product of quasisimple groups L_i of Lie type in characteristic p with $\boldsymbol{Z}(L_i)$ a p'-group (just take L_i to be a perfect inverse image of S_i in G^+), i.e., (iii) holds. In the remaining cases (b)–(e) of Theorem 2.1, $H/\boldsymbol{O}_{p'}(H) \cong S$, where S is a nonabelian simple group described in Theorem 2.1(b)–(e). By Lemma 2.3, $G^+/\boldsymbol{O}_{p'}(G^+) \cong S^n$, a direct product of $n \geq 1$ copies of S. Thus in all cases, G^+ has no composition factor isomorphic to C_p and $\boldsymbol{Z}(G^+) \leq \boldsymbol{O}_{p'}(G^+)$. Furthermore, $G^+ = (G^+)^{(\infty)}\boldsymbol{O}_{p'}(G^+)$ and so $(G^+)^{(\infty)}$ is a normal subgroup of p'-index in $G^+ = \boldsymbol{O}^{p'}(G^+)$, whence G^+ is perfect. Thus the first claim of Theorem 2.4 holds in all cases.

(b) Suppose next that we are in the cases (b)–(d) of Theorem 2.1. Then H is quasisimple and $\boldsymbol{Z}(H)$ is a p'-group; in particular, $\boldsymbol{O}_{p'}(H) = \boldsymbol{Z}(H)$ and $H/\boldsymbol{Z}(H) = S$. Note that $\Phi_i(\boldsymbol{O}_{p'}(G^+))$ is a normal p'-subgroup of $H_i = \Phi_i(G^+) \cong H$, whence $\Phi_i(\boldsymbol{O}_{p'}(G^+)) \leq \boldsymbol{Z}(H_i)$. Thus, for any $z \in \boldsymbol{O}_{p'}(G^+)$ and any $g \in G^+$, $[z, g]$ acts trivially on each W_i and so $[z, g] \in \bigcap_{i=1}^{t} K_i = 1$, i.e., $z \in \boldsymbol{Z}(G^+)$. We have shown that $\boldsymbol{O}_{p'}(G^+) = \boldsymbol{Z}(G^+) =: Z$.

Now we can write $G^+/Z = S_1 \times \cdots \times S_n$ with $S_i \cong S$. Let M_i denote the full inverse image of S_i in G^+ and let $L_i := M_i^{(\infty)}$. Then $M_i = L_i Z$, $L_i/(L_i \cap Z) \cong M_i/Z \cong S$, and so L_i is quasisimple and a central cover of S. Next, for $i \neq j$ we have $[L_i, L_j] \leq Z$ and so, since L_i is perfect,

$$[L_i, L_j] = [[L_i, L_i], L_j] = 1$$

by the three subgroups lemma. It follows that $M := L_1L_2\cdots L_n$ is a central product of the L_i. But $G^+ = MZ$ and G^+ is perfect, so $G^+ = M$.

The remaining claims in (i) are obvious if $t = 1$, so we will now assume that $t > 1$. First we show that G acts transitively on $\{S_1, \dots, S_n\}$. Relabeling the W_i suitably we may assume that $K_1Z/Z \geq \prod_{i\neq 1} S_i$ and $K_2Z/Z \geq \prod_{i\neq 2} S_i$. But $G^+/K_j = \Phi_j(G^+)$ is quasisimple, so in fact $K_jZ/Z = \prod_{i\neq j} S_i$ for $j = 1, 2$. By Clifford's theorem, $W_2 = W_1^g$ for some $g \in G$. Now g sends K_1 to K_2, and so it sends S_1 to S_2, as desired. If furthermore $H = S$, then $\boldsymbol{O}_{p'}(H) = 1$, whence $G^+ = S_1 \times \cdots \times S_n \cong H^n$ by Lemma 2.3. Consider the opposite situation: H is the full covering group of S. Again relabeling the W_i suitably and arguing as above, we may assume that $K_1Z/Z = \prod_{i\neq 1} S_i$. In this case, $K_1Z \geq L_i$ for $i \geq 2$, whence $L_i = [L_i, L_i] \leq [K_1Z, K_1Z] \leq K_1$ and $K_1 \geq L_2L_3\cdots L_n$. It also follows that $G^+ = K_1L_1$ and so $L_1/(K_1 \cap L_1) \cong G^+/K_1 \cong H$. Recall that L_1 is perfect and $L_1/(L_1 \cap Z) \cong S$, i.e., L_1 is a central extension of the simple group S. But H is the full covering group of S, so $|L_1| \leq |H|$. It follows that $L_1 \cap K_1 = 1$ and $L_1 \cong H$; in particular, $L_1 \cap \prod_{j\neq 1} L_j = 1$. Similarly, $L_i \cong H$ and $L_i \cap \prod_{j\neq i} L_j = 1$ for all i. Thus $G^+ = L_1 \times \cdots \times L_n \cong H^n$.

(c) Assume now that we are in case (e) of Theorem 2.1. Then $P_i := \Phi_i(\boldsymbol{O}_{p'}(G^+))$ is again a normal p'-subgroup of H_i, and so $P_i \leq \boldsymbol{O}_{p'}(H_i)$. On the other hand, H_i/P_i is a quotient of $G^+/\boldsymbol{O}_{p'}(G^+) \cong S^n$, whence all composition factors of H_i/P_i are isomorphic to S. Since $H_i/\boldsymbol{O}_{p'}(H_i) \cong S$, we conclude that $P_i = \boldsymbol{O}_{p'}(H_i)$; in particular, $\boldsymbol{O}_{p'}(G^+)$ is irreducible on W_i. □

3. Weak adequacy for $\mathrm{SL}_2(\mathbb{F}_p)$

Proposition 3.1. *Any nontrivial irreducible representation V of $\mathrm{SL}_2(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$ is weakly adequate except when $\dim V = p$ and $p \leq 3$.*

Remark 3.2. When $p \leq 3$ the p'-elements of $\mathrm{SL}_2(\mathbb{F}_p)$ generate a normal subgroup of index p. If moreover $\dim V = p$ then this subgroup does not act irreducibly; hence V cannot be weakly adequate.

The rest of the section is devoted to proving Proposition 3.1. Note that $p > 2$. In the following we write $V = L(a)$ with $0 < a \leq p-1$. If $a \leq (p-3)/2$ then the argument of [Guralnick et al. 2012, Theorem 9] applies. (Let $\mathcal{T} \subset \mathrm{SL}_2$ denote the diagonal maximal torus. Then distinct weights of $\mathcal{T}_{/\overline{\mathbb{F}}_p}$ on $L(a)$ restrict distinctly to $\mathcal{T}(\mathbb{F}_p)$, and $\operatorname{End} V$ is semisimple by [Serre 1994] with p-restricted highest weights.) We will assume from now on that $a \geq (p-1)/2$.

Lemma 3.3. *Suppose that $(p-1)/2 \leq a \leq p-1$. Then*

$$\operatorname{head}_{\mathrm{SL}_2}(L(a)\otimes L(a)) \cong \bigoplus_{i=0}^{(p-1)/2} L(2i).$$

Moreover, if $a \neq p-1$, $\mathrm{head}_{\mathrm{SL}_2(\mathbb{F}_p)}(L(a)\otimes L(a)) = \mathrm{head}_{\mathrm{SL}_2}(L(a)\otimes L(a))$, *whereas if* $a = p-1$,

$$\mathrm{head}_{\mathrm{SL}_2(\mathbb{F}_p)}(L(a)\otimes L(a)) \cong \bigoplus_{i=0}^{(p-1)/2} L(2i) \oplus L(p-1).$$

Proof. By [Doty and Henke 2005, Lemmas 1.1, 1.3], we see that for SL_2,

$$L(a)\otimes L(a) \cong \bigoplus_{i=0}^{p-2-a} L(2i) \oplus \bigoplus_{i=p-1-a}^{(p-3)/2} T(2p-2-2i) \oplus L(p-1), \tag{3-1}$$

where the tilting module $T(2p-2-r)$ for $0 \le r \le p-2$ is uniserial of the form $(L(r) \mid L(2p-2-r) \mid L(r))$. This proves the first part of the lemma. As is pointed out in Lemma 1.1 of [Doty and Henke 2005], $T(2p-2-r) \cong Q(r)$ for $0 \le r \le p-2$, which implies that $T(2p-2-r)|_{\mathrm{SL}_2(\mathbb{F}_p)}$ is projective. See also [Jantzen 2003, §2.7].

Noting that $L(2p-2-r)|_{\mathrm{SL}_2(\mathbb{F}_p)} \cong L(p-1-r) \oplus L(p-3-r)$ and using that $L(p-1)$ is the only irreducible projective $\mathrm{SL}_2(\mathbb{F}_p)$-module, it follows that

$$T(2p-2-r)|_{\mathrm{SL}_2(\mathbb{F}_p)} \cong \begin{cases} U(r) & \text{if } 0 < r \le p-2, \\ U(0) \oplus L(p-1) & \text{if } r = 0, \end{cases} \tag{3-2}$$

where $U(i)$ denotes the projective cover of $L(i)$. The claim follows. □

In the following, we will think of $V \cong L(a)$ as the space of homogeneous polynomials in X, Y of degree a.

Lemma 3.4. $(\mathrm{End}\, V)^{\mathcal{U}} \cong \bigoplus_{k=0}^{a} \overline{\mathbb{F}}_p \cdot (X(\partial/\partial Y))^k$, *where* $\mathcal{U} = \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \subset \mathrm{SL}_2$.

Proof. The torus $\mathcal{T} = \left(\begin{smallmatrix} * & \\ & * \end{smallmatrix}\right) \subset \mathrm{SL}_2$ acts on $(\mathrm{End}\, V)^{\mathcal{U}}$, and, for $\lambda \in X(\mathcal{T})$,

$$\mathrm{Hom}_{\mathcal{T}}(\lambda, (\mathrm{End}\, V)^{\mathcal{U}}) \cong \mathrm{Hom}_{\mathrm{SL}_2}(V(\lambda), \mathrm{End}\, V). \tag{3-3}$$

So it follows from (3-1) that $\dim(\mathrm{End}\, V)^{\mathcal{U}} = a+1$. (Namely, $\lambda = 0, 2, \dots, 2a$ each work once.) A simple calculation shows that $X(\partial/\partial Y)$ is $\mathcal{U}$-invariant; hence, so are $(X(\partial/\partial Y))^k$, $(0 \le k \le a)$, which are clearly nonzero. Since $(X(\partial/\partial Y))^k$ has weight $2k$, they are linearly independent. □

By Lemma 3.4 and (3-1), for $0 \le k \le a$, the SL_2-representation generated by $(X(\partial/\partial Y))^k$ is $V(2k) \subset \mathrm{End}(V)$.

Lemma 3.5. *The weight-*0 *subspace in* $V(2k) \subset \mathrm{End}\, V$ *is the line spanned by*

$$\Delta_k := \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i}^2 X^i Y^{k-i} \left(\frac{\partial}{\partial X}\right)^i \left(\frac{\partial}{\partial Y}\right)^{k-i} \quad (0 \le k \le a).$$

Proof. We compute the weight-0 part of $\left(\begin{smallmatrix} 1 & \\ -1 & 1\end{smallmatrix}\right)\cdot(X(\partial/\partial Y))^k$. Take $f\in\overline{\mathbb{F}}_p[X,Y]$ homogeneous of degree a. Under $\left(\begin{smallmatrix} 1 & \\ -1 & 1\end{smallmatrix}\right)\cdot(X(\partial/\partial Y))^k$ the element f is sent to

$$\begin{aligned}\left(\begin{pmatrix} 1 & \\ -1 & 1\end{pmatrix}\cdot\Big(X\frac{\partial}{\partial Y}\Big)^k\right)f(X+Y,Y)\\ &=\begin{pmatrix} 1 & \\ -1 & 1\end{pmatrix}\left[X^k\sum_{i=0}^{k}\binom{k}{i}\Big(\Big(\frac{\partial}{\partial X}\Big)^i\Big(\frac{\partial}{\partial Y}\Big)^{k-i}f\Big)(X+Y,Y)\right]\\ &=(X-Y)^k\sum_{i=0}^{k}\binom{k}{i}\Big(\frac{\partial}{\partial X}\Big)^i\Big(\frac{\partial}{\partial Y}\Big)^{k-i}f.\end{aligned}$$

The weight-0 part is the part that does not change the monomial degree, so it is Δ_k. Finally, note that $\Delta_k\neq 0$ as $\Delta_k(X^a)\neq 0$. $\square$

Now suppose that $0\le k\le(p-1)/2$. By the SL_2-invariant trace pairing on $\operatorname{End}V$, the element $\Delta_k\in\operatorname{soc}_{\mathrm{SL}_2}(\operatorname{End}V)$ induces an element $\delta_k\in(\operatorname{head}_{\mathrm{SL}_2}(\operatorname{End}V))^*$ that is zero on all irreducible constituents of $\operatorname{head}_{\mathrm{SL}_2}(\operatorname{End}V)$ except for $L(2k)$. Let $\pi_\ell\in\operatorname{End}V$ $(0\le\ell\le a)$ denote the projection $X^iY^{a-i}\mapsto\delta_{i\ell}X^iY^{a-i}$.

Lemma 3.6. *If $0\le k\le(p-1)/2$, then $\delta_k(\pi_\ell)$ is a polynomial in ℓ of degree exactly k.*

Proof. Note that $\delta_k(\pi_\ell)=\operatorname{tr}(\pi_\ell\circ\Delta_k)$ is the eigenvalue of Δ_k on $X^\ell Y^{a-\ell}$, and hence equals

$$\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}^2\ell(\ell-1)\cdots(\ell-i+1)(a-\ell)(a-\ell-1)\cdots(a-\ell-k+i+1).$$

This is a polynomial in ℓ of degree at most k. The coefficient of ℓ^k is $\sum_{i=0}^{k}\binom{k}{i}^2=\binom{2k}{k}\not\equiv 0\pmod p$, as $k<p/2$. $\square$

Let us denote this polynomial by $p_k(z)\in\mathbb{F}_p[z]$.

Proof of Proposition 3.1. Recall that $(p-1)/2\le a\le p-1$. Let $\mathcal{M}$ denote the span of the image of the p'-elements in $\operatorname{End}V$, and let M denote the image of $\mathcal{M}$ in $\operatorname{head}_{\mathrm{SL}_2(\mathbb{F}_p)}(\operatorname{End}V)$. Since $\mathcal{M}$ is $\mathrm{SL}_2(\mathbb{F}_p)$-stable, it suffices to show that $M=\operatorname{head}_{\mathrm{SL}_2(\mathbb{F}_p)}(\operatorname{End}V)$.

(a) Suppose that $a<p-1$. By Lemma 3.3, $\operatorname{head}_{\mathrm{SL}_2(\mathbb{F}_p)}(\operatorname{End}V)\cong\bigoplus_{i=0}^{(p-1)/2}L(2i)$. Suppose that M does not contain $L(2k)$ for some $0\le k\le(p-1)/2$. Then δ_k annihilates the image of all p'-elements. The images of the diagonal elements of $\mathrm{SL}_2(\mathbb{F}_p)$ in $\operatorname{End}(V)$ span the subspace with basis

$$\pi_i\Big(a-\frac{p-3}{2}\le i\le\frac{p-3}{2}\Big)\quad \pi_i+\pi_{i+\frac{p-1}{2}}\Big(0\le i\le a-\frac{p-1}{2}\Big).$$

Hence

$$\begin{aligned} p_k(i) &= 0 \quad \left(a - \frac{p-3}{2} \le i \le \frac{p-3}{2}\right), \\ p_k(i) + p_k\left(i + \frac{p-1}{2}\right) &= 0 \quad \left(0 \le i \le a - \frac{p-1}{2}\right). \end{aligned} \tag{3-4}$$

Now repeat the same argument with a nonsplit Cartan subgroup. After a linear change of variables $(X, Y) \mapsto (X', Y')$ over $\mathbb{F}_{p^2}$, this subgroup acts as

$$\left\{ \begin{pmatrix} x & \\ & x^p \end{pmatrix} : x \in \mathbb{F}_{p^2}^\times,\ x^{p+1} = 1 \right\}.$$

In this new basis of V we have corresponding elements Δ'_k, δ'_k, π'_ℓ. However, p_k is unchanged, as it is given by the explicit formula in the proof of Lemma 3.6. We thus get

$$\begin{aligned} p_k(i) &= 0 \quad \left(a - \frac{p-1}{2} \le i \le \frac{p-1}{2}\right), \\ p_k(i) + p_k\left(i + \frac{p+1}{2}\right) &= 0 \quad \left(0 \le i \le a - \frac{p+1}{2}\right). \end{aligned} \tag{3-5}$$

From (3-4) and (3-5) we get that $p_k(\ell) = 0$ for all $0 \le \ell \le a$. This contradicts the fact that $\deg p_k = k \le (p-1)/2 \le a$.

(b) Suppose that $a = p - 1$, so that $p \ge 5$ by our assumption. By Lemma 3.3, $\mathrm{head}_{\mathrm{SL}_2(\mathbb{F}_p)}(\mathrm{End}\, V) \cong \bigoplus_{i=0}^{(p-1)/2} L(2i) \oplus L(p-1)$.

(b1) Suppose that M does not contain $L(2k)$ for some $k \le (p-3)/2$. Then δ_k and δ'_k annihilate the image of all p'-elements, so by an argument analogous to the one in (a) we get

$$\begin{aligned} p_k(i) + p_k\left(i + \frac{p-1}{2}\right) &= 0 \quad \left(0 < i < \frac{p-1}{2}\right), \\ p_k(0) + p_k\left(\frac{p-1}{2}\right) + p_k(p-1) &= 0; \end{aligned} \tag{3-6}$$

$$\begin{aligned} p_k(i) + p_k\left(i + \frac{p+1}{2}\right) &= 0 \quad \left(0 \le i \le \frac{p-3}{2}\right), \\ p_k\left(\frac{p-1}{2}\right) &= 0. \end{aligned} \tag{3-7}$$

Then $p_k(z+1) - p_k(z)$ is a polynomial of degree $k - 1 < (p-1)/2$ with zeroes at $z = 0, 1, \ldots, (p-5)/2$ and $z = (p+1)/2, (p+3)/2, \ldots, p-2$. As $p-3 \ge (p-1)/2$, it follows that $p_k(z+1) \equiv p_k(z)$; hence by (3-7) we get $p_k(\ell) = 0$ for all $0 \le \ell \le p-1$, contradicting the fact that p_k has degree $0 \le k < p$.

(b2) Suppose that M does not contain $L(p-1)^{\oplus 2}$. Note first that the second copy of $L(p-1) \subset \mathrm{End}(V)$ is contained in the Weyl module $V(2p-2) \hookrightarrow T(2p-2)$. Using (3-2) we have $V(2p-2)|_{\mathrm{SL}_2(\mathbb{F}_p)} \cong L(p-1) \oplus M$, where $0 \to L(0) \to M \to L(p-3) \to 0$ is nonsplit. It follows using (3-3) that $V(2p-2)^{\mathcal{U}(\mathbb{F}_p)} = V(2p-2)^{\mathcal{U}}$ (both are two-dimensional). Hence there is a $\mathcal{U}(\mathbb{F}_p)$-fixed vector in the second copy of $L(p-1)$ of the form $v := (X(\partial/\partial Y))^{p-1} + c$ for some $c \in \overline{\mathbb{F}}_p$. We first compute c. Note that if V is an $\mathrm{SL}_2(\mathbb{F}_p)$-representation over $\overline{\mathbb{F}}_p$ and $v \neq 0$ is fixed by the Borel subgroup $B := \left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right) \subset \mathrm{SL}_2(\mathbb{F}_p)$, then v generates the p-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{F}_p)$ if and only if

$$\sum_{\mathrm{SL}_2(\mathbb{F}_p)/\left(\begin{smallmatrix} * & * \\ & * \end{smallmatrix}\right)} gv = 0 \quad \Longleftrightarrow \quad \sum_{u \in \mathbb{F}_p} \begin{pmatrix} 1 & \\ -u & 1 \end{pmatrix} v + \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} v = 0.$$

As in Lemma 3.5,

$$\begin{pmatrix} 1 & \\ -u & 1 \end{pmatrix} \cdot \left(X \frac{\partial}{\partial Y}\right)^{p-1} = (X - uY)^{p-1} \sum_{i=0}^{p-1} (-u)^i \left(\frac{\partial}{\partial X}\right)^i \left(\frac{\partial}{\partial Y}\right)^{p-1-i};$$

hence

$$\sum_{u \in \mathbb{F}_p} \begin{pmatrix} 1 & \\ -u & 1 \end{pmatrix} \cdot \left[\left(X \frac{\partial}{\partial Y}\right)^{p-1} + c\right] = -\left[\Delta_{p-1} + Y^{p-1} \cdot \left(\frac{\partial}{\partial X}\right)^{p-1}\right].$$

Since

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \cdot \left[\left(X \frac{\partial}{\partial Y}\right)^{p-1} + c\right] = \left(Y \frac{\partial}{\partial X}\right)^{p-1} + c,$$

we deduce that $c = -1$.

Consider the annihilator $M^{\perp} \subset \mathrm{soc}_{\mathrm{SL}_2(\mathbb{F}_p)}(\mathrm{End}\, V)$ of M under the trace pairing. By assumption, $N := M^{\perp} \cap L(p-1)^{\oplus 2} \neq 0$. Let $\psi \in N^B - \{0\}$, so that by the previous paragraph we can write $\psi = \lambda (X(\partial/\partial Y))^{(p-1)/2} + \mu\big((X(\partial/\partial Y))^{p-1} - 1\big)$ for some $(\lambda, \mu) \in \overline{\mathbb{F}}_p^2 - \{0\}$. As $\psi \in M^{\perp}$, we get by a simple calculation that $0 = \mathrm{tr}\big(\left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right) \circ \psi\big) = -\mu$ for any $\alpha \in \mathbb{F}_p^{\times} - \{\pm 1\} \neq \varnothing$. Thus we may assume that $\psi = (X(\partial/\partial Y))^{(p-1)/2}$. As the $\mathrm{SL}_2(\mathbb{F}_p)$-subrepresentation of $\mathrm{End}(V)$ generated by ψ is the unique SL_2-subrepresentation $L(p-1) \subset \mathrm{End}(V)$, we see that N contains Δ_k and Δ'_k for $k = (p-1)/2$, so δ_k and δ'_k annihilate M. Now the argument of (b1) gives a contradiction. □

4. Weak adequacy for Chevalley groups

Lemma 4.1. *Suppose $(X, \Phi, X^{\vee}, \Phi^{\vee})$ is a reduced based root datum with Φ irreducible.*

(a) *If* Φ *is not of type* A_1, *then*

$$2\alpha_0^\vee \leq \sum_{\alpha\in\Phi_+} \alpha^\vee,$$

where $\alpha_0^\vee$ *is the highest coroot.*

(b) *If* Φ *is not of type* A_1, A_2, A_3, *or* B_2, *then*

$$4\beta_0^\vee \leq \sum_{\alpha\in\Phi_+} \alpha^\vee,$$

where $\beta_0^\vee$ *is the highest short coroot.*

Proof. (a) Let $\{\alpha_i : i = 1, \dots, r\}$ denote the simple roots. Then $\langle\alpha_0^\vee, \alpha_i\rangle \geq 0$ for all i and $\langle\alpha_0^\vee, \alpha_j\rangle > 0$ for some j. Since $\alpha_0^\vee \neq \alpha_j^\vee$ (as Φ is not of type A_1), $\beta^\vee := \alpha_0^\vee - \alpha_j^\vee \in \Phi^\vee$. Since $\alpha_0^\vee = \alpha_j^\vee + \beta^\vee$ it follows that $\beta^\vee > 0$. Also, $\alpha_j^\vee \neq \beta^\vee$, as Φ is reduced. Hence

$$2\alpha_0^\vee = \alpha_0^\vee + \alpha_j^\vee + \beta^\vee \leq \sum_{\alpha\in\Phi_+} \alpha^\vee.$$

(b) We pass to the dual root system to simplify notation. We want to show that

$$4\beta_0 \leq \sum_{\alpha\in\Phi_+} \alpha,$$

where β_0 is the highest short root. It suffices to express β_0 as sum of positive roots in three nontrivial ways that do not overlap (similarly as in the proof of (a)). If Φ is not simply laced, we only need two nontrivial ways because we can also use that $\beta_0 < \alpha_0$, where α_0 is the highest root.

In the following we use Bourbaki notation:

- Type A_{n-1} ($n \geq 5$):

$$\beta_0 = \varepsilon_1 - \varepsilon_n = (\varepsilon_1 - \varepsilon_i) + (\varepsilon_i - \varepsilon_n) \quad (1 < i < n).$$

- Type B_n ($n \geq 3$):

$$\beta_0 = \varepsilon_1 = (\varepsilon_1 - \varepsilon_i) + \varepsilon_i \quad (1 < i \leq n).$$

  If $n = 3$, we also use $\beta_0 < \alpha_0 = \varepsilon_1 + \varepsilon_2$.

- Type C_n ($n \geq 3$):

$$\beta_0 = \varepsilon_1 + \varepsilon_2 = (\varepsilon_1 - \varepsilon_i) + (\varepsilon_2 + \varepsilon_i) = (\varepsilon_1 + \varepsilon_i) + (\varepsilon_2 - \varepsilon_i) \quad (2 < i \leq n).$$

  If $n = 3$, we also use $\beta_0 < \alpha_0 = 2\varepsilon_1$.

- Type D_n $(n \geq 4)$:

$$\beta_0 = \varepsilon_1 + \varepsilon_2 = (\varepsilon_1 - \varepsilon_i) + (\varepsilon_2 + \varepsilon_i) = (\varepsilon_1 + \varepsilon_i) + (\varepsilon_2 - \varepsilon_i) \quad (2 < i \leq n).$$

- Type E_6:

$$\beta_0 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8).$$

Note that $\beta_0 - (\varepsilon_i + \varepsilon_j)$ and $\varepsilon_i + \varepsilon_j$ are positive $(1 \leq i < j \leq 5)$.

- Type E_7:

$$\beta_0 = \varepsilon_8 - \varepsilon_7 = \tfrac{1}{2}\Biggl(\varepsilon_8 - \varepsilon_7 + \varepsilon_6 + \sum_{i=1}^{5}(-1)^{v(i)}\varepsilon_i\Biggr) + \tfrac{1}{2}\Biggl(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 - \sum_{i=1}^{5}(-1)^{v(i)}\varepsilon_i\Biggr),$$

where $\sum_{i=1}^{5} v(i)$ is odd.

- Type E_8:

$$\beta_0 = \varepsilon_7 + \varepsilon_8 = (-\varepsilon_i + \varepsilon_7) + (\varepsilon_i + \varepsilon_8) \quad (1 \leq i < 7).$$

- Type F_4:

$$\beta_0 = \varepsilon_1 = (\varepsilon_1 - \varepsilon_i) + \varepsilon_i \quad (1 < i \leq 4).$$

- Type G_2:

$$\beta_0 = 2\alpha_1 + \alpha_2 = \alpha_1 + (\alpha_1 + \alpha_2)\beta_0 < 3\alpha_1 + \alpha_2\beta_0 < \alpha_0 = 3\alpha_1 + 2\alpha_2. \qquad \square$$

We now prove variants of several results in [Guralnick et al. 2012].

Lemma 4.2. *Suppose that $\mathcal{G}$ is a connected, simply connected, semisimple algebraic group over $\overline{\mathbb{F}}_p$ and $\Theta : \mathcal{G} \to \mathrm{GL}(V)$ a semisimple finite-dimensional representation. Let $\mathcal{G} > \mathcal{B} > \mathcal{T}$ denote a Borel subgroup and a maximal torus, and suppose that*

for any irreducible component V' of V and for any two distinct weights μ_1, μ_2 of $\mathcal{T}$ on V', we have $\mu_1 - \mu_2 \notin pX(\mathcal{T})$. (4-1)

Then there exist connected, simply connected, semisimple algebraic subgroups $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{I} \times \mathcal{J}$, $\Theta(\mathcal{J}) = 1$, and Θ induces a central isogeny of $\mathcal{I}$ onto its image, which is a semisimple algebraic group. Moreover, assumption (4-1) holds if for all irreducible constituents V' of V the highest weight of V' is p-restricted and either

(i) $\dim V' < p$, *or*

(ii) $\dim V' \leq p$ *and either $p \neq 2$ or $\mathcal{G}$ has no SL_2-factor.*

Proof. Write $V = \bigoplus V_i$ with V_i irreducible and $\mathscr{G} = \prod_{s\in S} \mathscr{G}_s$ with each $\mathscr{G}_s$ almost simple. The last sentence of the proof of Lemma 4 in [Guralnick et al. 2012] together with (4-1) show that the conclusion of that lemma holds for $\Theta_i\colon \mathscr{G} \to \mathrm{GL}(V_i)$ for all i. Hence there exists $S_i \subset S$ such that $\ker\Theta_i = \prod_{s\in S_i} \mathscr{G}_s \times Z_i$, where Z_i is a central subgroup of $\prod_{s\notin S_i} \mathscr{G}_s$ (maybe nonreduced). Then $\ker\Theta = \bigcap \ker\Theta_i = \prod_{s\in\bigcap S_i} \mathscr{G}_s \times Z$, where Z is a central subgroup of $\prod_{s\notin\bigcap S_i} \mathscr{G}_s$. So we can take $\mathscr{I} = \prod_{s\notin\bigcap S_i} \mathscr{G}_s$ and $\mathscr{J} = \prod_{s\in\bigcap S_i} \mathscr{G}_s$.

To prove the last part, we may suppose that V is irreducible. So $V \cong \bigotimes_{s\in S} V_s$, where V_s is an irreducible $\mathscr{G}_s$-representation. It is easy to see that if (4-1) fails, then it fails for $\mathscr{G}_s \to \mathrm{GL}(V_s)$ for some $s \in S$, so we may assume that $\mathscr{G}$ is almost simple.

(a) First suppose that $\mathscr{G} \cong \mathrm{SL}_2$. The highest weight of V is $\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right) \mapsto x^a$, for some $0 \le a \le p-1$, and $a \ne p-1$ if $p = 2$. Therefore the weights of $\operatorname{ad} V$ are $\left(\begin{smallmatrix} x & \\ & x^{-1}\end{smallmatrix}\right) \mapsto x^b$, where $b \in \{-2a, -2a+2, \dots, 2a-2, 2a\}$. It follows that (4-1) holds because $b \equiv 0 \pmod p$ implies that $b = 0$.

(b) Next suppose that $\mathscr{G} \not\cong \mathrm{SL}_2$. Let λ denote the highest weight of V; it is p-restricted by assumption. By Lemma 4.1(a) and Jantzen's inequality [1997, Lemma 1.2] we get

$$|\langle \mu, \beta^\vee\rangle| \le \langle \lambda, \alpha_0^\vee\rangle \le \tfrac{1}{2}\left\langle \lambda, \sum_{\alpha>0} \alpha^\vee \right\rangle < \tfrac{1}{2}\dim V \le \frac{p}{2}$$

for all weights μ of V and all roots β. Hence $|\langle \mu_1 - \mu_2, \beta^\vee\rangle| < p$ for all root β and all weights μ_i of V, so (4-1) holds. □

Lemma 4.3. *Suppose that $\mathscr{G} \le \prod \mathrm{GL}(W_i)$ is a connected reductive group over $\overline{\mathbb{F}}_p$, where for all i the representation W_i is irreducible with p-restricted highest weight and has dimension $\le p$. Let $\mathscr{T}$ be a maximal torus and $\mathscr{U}$ the unipotent radical of a Borel subgroup of $\mathscr{G}$ that contains $\mathscr{T}$. Let $V = \bigoplus W_i$.*

(i) *The maps* exp *and* log *induce inverse isomorphisms of varieties between* $\operatorname{Lie}\mathscr{U} \le \operatorname{End}(V)$ *and* $\mathscr{U} \le \mathrm{GL}(V)$.

(ii) *For any positive root α we have* $\exp(\operatorname{Lie}\mathscr{U}_\alpha) = \mathscr{U}_\alpha$.

(iii) *The map* $\exp : \operatorname{Lie}\mathscr{U} \to \mathscr{U}$ *depends only on $\mathscr{G}$ and $\mathscr{U}$, but not on V, W_i, or the representation* $\mathscr{G} \hookrightarrow \mathrm{GL}(V)$.

(iv) *If θ is an automorphism of $\mathscr{G}$ that preserves $\mathscr{T}$ and $\mathscr{U}$, then we have a commutative diagram*

$$\begin{array}{ccc} \operatorname{Lie}\mathscr{U} & \xrightarrow{d\theta} & \operatorname{Lie}\mathscr{U} \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ \mathscr{U} & \xrightarrow{\theta} & \mathscr{U} \end{array}$$

Proof. The proof is the same as that of [Guralnick et al. 2012, Lemma 5], where there was an extra assumption on the μ_i. The assumption on the weights μ_i is *only* used to prove that $X_{\alpha,n}$ acts trivially on $V = \bigoplus W_i$ for all $n \geq p$. Fix any i. It is enough to show that $X_{\alpha,n}$ acts trivially on W_i for all $n \geq p$. So it is enough to show that W_i cannot have two weights λ and $\lambda + n\alpha$ ($\alpha \in \Phi$, $n \geq p$). As $\dim W_i \leq p$, it follows from [Jantzen 1997] that the weights of W_i are the same as those of the irreducible characteristic-0 representation of the same highest weight. But in characteristic 0 it is known that if λ and $\lambda + n\alpha$ are weights of an irreducible representation, then so are λ, $\lambda + \alpha$, $\lambda + 2\alpha, \ldots, \lambda + n\alpha$, so $\dim W_i > n \geq p$, contradicting the assumption. □

Proposition 4.4. *Let $p > 3$ be prime. Suppose that V is a finite-dimensional vector space over $\bar{\mathbb{F}}_p$ and that $G \leq \mathrm{GL}(V)$ is a finite subgroup that acts semisimply on V. Let $G^+ \leq G$ be the subgroup generated by p-elements of G. Then V is a semisimple G^+-module. Let $d \geq 1$ be the maximal dimension of an irreducible G^+-submodule of V. Suppose that $p > d$ and that G^+ is a central product of quasisimple Chevalley groups in characteristic p. Then there exists an algebraic group $\mathcal{G}$ over $\mathbb{F}_p$ and a semisimple representation $\Theta : \mathcal{G}_{/\bar{\mathbb{F}}_p} \to \mathrm{GL}(V)$ with the following properties*:

(i) *The connected component $\mathcal{G}^0$ is semisimple simply connected.*

(ii) $\mathcal{G} \cong \mathcal{G}^0 \rtimes H$, *where H is a finite group of order prime to p.*

(iii) $\Theta(\mathcal{G}(\mathbb{F}_p)) = G$.

(iv) $\ker(\Theta) \cap \mathcal{G}^0(\mathbb{F}_p)$ *is central in* $\mathcal{G}^0(\mathbb{F}_p)$.

Moreover, any highest weight of $\mathcal{G}^0_{/\bar{\mathbb{F}}_p}$ on V is p-restricted. Also, G does not have any composition factor of order p.

Proof. The proof is essentially identical to the proof of [Guralnick et al. 2012, Proposition 7]. We do not get $\langle \lambda, \alpha^\vee \rangle < (p-1)/2$ in Step 2, but this was only used to apply Lemmas 4 and 5 in [Guralnick et al. 2012]. By Lemmas 4.2 and 4.3 above one can bypass this assumption, as we now explain. Both times Lemma 4 in [Guralnick et al. 2012] is applied, condition (ii) in Lemma 4.2 holds. In Step 4 we can apply Lemma 4.3 instead of Lemma 5 in [Guralnick et al. 2012] because $\bar{\mathcal{I}}$ acts irreducibly on W_i and its highest weight is p-restricted (as $\mathcal{I} \to \bar{\mathcal{I}}$ is a central isogeny). Similarly we can avoid Lemma 5 in [Guralnick et al. 2012] in Step 5, noting that the highest weights of V' are Galois-conjugate to the highest weights of V and recalling that $\psi_{/\bar{\mathbb{F}}_p}$ is a central isogeny onto its image. Finally, note that (iv) follows by construction. □

Theorem 4.5. *Suppose that $p > 3$, V is a finite-dimensional vector space over $\bar{\mathbb{F}}_p$, and $G \le \mathrm{GL}(V)$ is a finite subgroup that acts irreducibly on V. Let $G^+ \le G$ be the subgroup generated by p-elements of G. Let $d \ge 1$ be the maximal dimension of an irreducible G^+-submodule of V. Suppose that $p > d$ and that G^+ is a central product of quasisimple Chevalley groups in characteristic p. Then the set of p'-elements of G spans* $\operatorname{ad} V$ *as an $\bar{\mathbb{F}}_p$-vector space.*

Remark 4.6. Theorem 4.5 generalizes [Guralnick et al. 2012, Theorem 9]. We take the opportunity to point out a small gap in the last paragraph of the proof of that theorem. In the notation there, it is implicitly assumed that (i) $r(T(\mathbb{F}_l)) \subset r(H)$, so that the span of $r(H)$ equals the span of $r(T(\bar{\mathbb{F}}_l)H)$, and (ii) H normalizes the pair (B, T). Both assumptions are satisfied provided that when we apply [Guralnick et al. 2012, Proposition 7] in the proof of Theorem 9 there, we take r, $G = G^0 \rtimes H$, B, T, ... as constructed in the proof of that proposition.

Proof. Without loss of generality $d > 1$. Let $\Theta\colon \mathscr{G}_{/\bar{\mathbb{F}}_p} \to \mathrm{GL}(V)$ be as in Proposition 4.4. Then $V = \bigoplus W_i$, where W_i is an irreducible $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$-subrepresentation with p-restricted highest weight. Write $\mathscr{G}^0_{/\bar{\mathbb{F}}_p} \cong \mathscr{G}_1 \times \cdots \times \mathscr{G}_r$, where $\mathscr{G}_i$ is almost simple over $\bar{\mathbb{F}}_p$. Let $\mathscr{G}^0 > \mathscr{B} > \mathscr{T}$ denote a Borel subgroup and a maximal torus, and let Φ denote the roots of $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$ with respect to $\mathscr{T}_{/\bar{\mathbb{F}}_p}$.

(a) We consider the case where one of the W_i (equivalently any) is tensor-decomposable as a $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$-representation. Note that $W_i \cong X_{i1} \boxtimes \cdots \boxtimes X_{ir}$, where X_{ij} is an irreducible $\mathscr{G}_j$-representation with p-restricted highest weight. Since $\dim X_{ij} \le p-1$, its highest weight lies in the lowest alcove [Jantzen 1997; Serre 1994]; hence X_{ij} is tensor-indecomposable (as the highest weight is in the lowest alcove, we are reduced to the characteristic-0 case, where this is well known). Hence our assumption implies that $X_{ij} \not\cong \mathbb{1}$ for at least two values of j. Hence $\dim X_{ij} \le (p-1)/2$ for all i, j. Therefore $X^*_{ik} \otimes X_{jk}$ is a semisimple $\mathscr{G}_k$-representation by [Serre 1994], so $\operatorname{End} V$ is a semisimple $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$-representation. Moreover, all its highest weights are p-restricted: this follows exactly as in Step 2 of the proof of [Guralnick et al. 2012, Proposition 7] (use that $\dim X_{ij} \le (p-1)/2$). Hence any $\mathscr{G}^0(\mathbb{F}_p)$-submodule of $\operatorname{End} V$ is a $\mathscr{G}^0(\bar{\mathbb{F}}_p)$-submodule.

Furthermore, arguing as in Step 2 of the proof of [Guralnick et al. 2012, Proposition 7] for each $\mathscr{G}_k$, we deduce that for all weights μ of the maximal torus $\mathscr{T}_{/\bar{\mathbb{F}}_p}$ on V we have $|\langle \mu, \alpha^\vee \rangle| < (p-1)/2$ for all $\alpha \in \Phi$. We conclude as in the last paragraph of the proof of [Guralnick et al. 2012, Theorem 9].

(b) We consider the case when $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$ has no factors of type A_1, A_2, A_3, or B_2. We claim $|\langle \mu, \alpha^\vee \rangle| < (p-1)/4$ for all weights μ of $\mathscr{T}_{/\bar{\mathbb{F}}_p}$ on V and for all short coroots $\alpha^\vee \in \Phi^\vee$. It suffices to show that $\langle \lambda, \beta_0^\vee \rangle < (p-1)/4$ for all highest weights λ of $\mathscr{T}_{/\bar{\mathbb{F}}_p}$ on V and all highest short coroots $\beta_0^\vee$ (one for each component of $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$). So it is enough to show that if $\mathscr{G}'$ is an almost simple, simply connected group over

$\bar{\mathbb{F}}_p$, not of type A_1, A_2, A_3, or B_2, then $\langle\lambda, \beta_0^\vee\rangle < (p-1)/4$ for all p-restricted weights λ of $\mathscr{G}'$ such that $\dim L(\lambda) < p$, where $\beta_0^\vee$ is the highest short coroot of $\mathscr{G}'$. But this follows from Lemma 4.1(b) and Jantzen's inequality, and this proves the claim.

Since the short coroots span $X_*\big(\mathscr{T}_{/\bar{\mathbb{F}}_p}\big) \otimes \mathbb{Q}$ over $\mathbb{Q}$, Lemma 3 of [Guralnick et al. 2012] plus the claim show that distinct weights of $\mathscr{T}_{/\bar{\mathbb{F}}_p}$ on $\operatorname{End} V$ (and V) remain distinct on $\mathscr{T}(\mathbb{F}_p)$. Then [Guralnick 2012a, Lemma 1.1] shows that any $\mathscr{G}^0(\mathbb{F}_p)$-subrepresentation of $\operatorname{End} V$ is $\mathscr{G}^0(\bar{\mathbb{F}}_p)$-stable, so we can conclude as in the last paragraph of the proof of Theorem 9 in [Guralnick et al. 2012].

(c) If neither (a) nor (b) apply, then the W_i are tensor-indecomposable; in particular, the almost simple factors of $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$ are pairwise isomorphic. (Write $\mathscr{G}^0 \cong \prod \mathscr{H}_i$, where the subgroups $\mathscr{H}_i$ are almost simple over $\mathbb{F}_p$. Note that, for each i, $\mathscr{G}^0(\mathbb{F}_p)$ acts irreducibly on W_i with all but one $\mathscr{H}_j(\mathbb{F}_p)$ acting trivially. As $\mathscr{G}(\mathbb{F}_p)$ is irreducible on V and, by Proposition 4.4(iv), the subgroups $\mathscr{H}_i(\mathbb{F}_p)$ are pairwise isomorphic and, as $p > 3$, so are the $\mathscr{H}_i$.) Hence $\mathscr{G}^0_{/\bar{\mathbb{F}}_p} \cong \mathrm{SL}_2^n$, SL_3^n, SL_4^n, or Sp_4^n for some $n \geq 1$.

(d) We consider the case where $\mathscr{G}^0_{/\bar{\mathbb{F}}_p} \cong \mathrm{SL}_3^n$, SL_4^n, or Sp_4^n. We claim that for all weights μ of $\mathscr{T}_{/\bar{\mathbb{F}}_p}$ on V and for all $\alpha \in \Phi$,

$$|\langle\mu, \alpha^\vee\rangle| < \tfrac{1}{2}(p-1). \tag{4-2}$$

To see this, note that $|\langle\mu, \alpha^\vee\rangle| \leq \langle\lambda, \alpha_0^\vee\rangle$ for some highest weight λ of V and some highest coroot $\alpha_0^\vee$. Applying Lemma 4.1(a) to the component Φ_j of Φ such that $\alpha_0^\vee \in \Phi_j^\vee$ and using Jantzen's inequality, we get

$$\langle\lambda, \alpha_0^\vee\rangle \leq \tfrac{1}{2} \sum_{\Phi_{j,+}} \langle\lambda, \alpha^\vee\rangle < \tfrac{1}{2}(p-1).$$

By Lemma 3 in [Guralnick et al. 2012], (4-2) shows that distinct weights of $\mathscr{T}_{/\bar{\mathbb{F}}_p}$ on V remain distinct on $\mathscr{T}(\mathbb{F}_p)$. As usual, it thus suffices to show that $\operatorname{End} V$ is a semisimple $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$-module with p-restricted highest weights. We can argue independently for each factor of $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$, so it will suffice to show that if X, Y are nontrivial irreducible $\mathscr{G}'$-representations which are conjugate by $\mathrm{Aut}(\mathscr{G}')$ (with $\mathscr{G}' = \mathrm{SL}_3$, SL_4, or Sp_4) with p-restricted highest weights λ, λ' of dimension less than p, then $X \otimes Y$ is semisimple with p-restricted highest weights. By [Jantzen 1997; Serre 1994], λ and λ' lie in the lowest alcove, so $\operatorname{ch} L(\lambda)$ and $\operatorname{ch} L(\lambda')$ are given by Weyl's character formula.

In the following, note that $\langle\lambda, \beta_0^\vee\rangle = \langle\lambda', \beta_0^\vee\rangle$.

If $\mathscr{G}' \cong \mathrm{SL}_4$, write $\lambda = r\varpi_1 + s\varpi_2 + t\varpi_3$ $(r, s, t \geq 0)$, where ϖ_i is the i-th fundamental weight. Then

$$\begin{aligned} p-1 \geq \dim L(\lambda) &= \frac{[(r+1)(s+1)(t+1)][(r+s+2)(s+t+2)](r+s+t+3)}{2\cdot 2\cdot 3} \\ &\geq \frac{(r+s+t+1)(r+s+t+2)(r+s+t+3)}{2\cdot 3}. \end{aligned}$$

If $\langle \lambda, \beta_0^\vee \rangle = r+s+t \geq (p-1)/4$, then

$$p-1 \geq \frac{\frac{p+3}{4}\cdot\frac{p+7}{4}\cdot\frac{p+11}{4}}{6}.$$

Equivalently, $(p-5)[(p+13)^2-292] \leq 0$, i.e., $p=5$. In this case, equality holds throughout so $\lambda = \varpi_1$ or ϖ_3. The maximal weight of $X \otimes Y$, namely $2\varpi_1$ or $\varpi_1+\varpi_3$ or $2\varpi_3$, lies in the closure of the lowest alcove. Then $X \otimes Y$ is semisimple by the linkage principle (or just [Jantzen 2003, Proposition II.4.13]) and it has p-restricted highest weights. If $\langle \lambda, \beta_0^\vee \rangle < (p-1)/4$ the argument in (b) goes through instead.

If $\mathscr{G}' \cong \mathrm{Sp}_4$, write $\lambda = r\varpi_1 + s\varpi_2$ with $r, s \geq 0$ (type B_2). Then

$$\begin{aligned} p-1 \geq \dim L(\lambda) &= \frac{[(r+1)(s+1)](r+s+2)(2r+s+3)}{6} \\ &\geq \frac{(r+s+1)(r+s+2)(r+s+3)}{6}. \end{aligned}$$

If $\langle \lambda, \beta_0^\vee \rangle = r+s \geq (p-1)/4$, then $p=5$ as above and $\lambda = \varpi_2$. Again, $X \otimes Y$ has maximal weight $2\varpi_2$ lying in the closure of the lowest alcove; hence $X \otimes Y$ is semisimple with p-restricted highest weights. If $\langle \lambda, \beta_0^\vee \rangle < (p-1)/4$ we are done as in (b).

If $\mathscr{G}' \cong \mathrm{SL}_3$, write $\lambda = r\varpi_1 + s\varpi_2$ $(r, s \geq 0)$. If $r+s \geq (p-1)/2$, then

$$\begin{aligned} p-1 \geq \dim L(\lambda) &= \frac{[(r+1)(s+1)](r+s+2)}{2} \\ &\geq \frac{(r+s+1)(r+s+2)}{2} \geq \frac{\frac{p+1}{2}\cdot\frac{p+3}{2}}{2}. \end{aligned}$$

Equivalently $(p-2)^2+7 \leq 0$, which is impossible. Hence $r+s \leq (p-3)/2$, which implies that the maximal weight of $X \otimes Y$ lies in the lowest alcove. So $X \otimes Y$ is semisimple with p-restricted highest weights.

(e) We consider the case where $\mathscr{G}^0_{/\overline{\mathbb{F}}_p} \cong \mathrm{SL}_2^n$ and each W_i is tensor-indecomposable as a $\mathscr{G}^0_{/\overline{\mathbb{F}}_p}$-representation. Here, $\mathscr{G}^0(\mathbb{F}_p) \cong \mathrm{SL}_2(\mathbb{F}_q)^m$, where $[\mathbb{F}_q : \mathbb{F}_p]\cdot m = n$. Also, V is irreducible, each W_i is tensor-indecomposable, and SL_2 has no outer automorphism. It follows that $V \cong \left[\bigoplus_{i=1}^{\ell} V_i\right]^{\oplus k}$ as $\mathscr{G}^0_{/\overline{\mathbb{F}}_p}$-representations, where each V_i is of the form $\mathbb{1} \boxtimes \cdots \boxtimes V_0 \boxtimes \cdots \boxtimes \mathbb{1}$ (precisely one factor is V_0), the V_i are pairwise nonisomorphic, and V_0 is an irreducible SL_2-representation such that $1 < \dim V_0 < p$ with p-restricted highest weight.

(e1) We claim that the span of the p'-elements of $\mathscr{G}^0(\mathbb{F}_p)$ in $\operatorname{End} V$ contains the span of $\mathscr{T}(\bar{\mathbb{F}}_p)$ in $\operatorname{End} V$.

If $q > p$, note from the description of V_i above that distinct weights of $\mathscr{T}_{/\bar{\mathbb{F}}_p}$ on V remain distinct on $\mathscr{T}(\mathbb{F}_p)$. Hence the span of $\mathscr{T}(\mathbb{F}_p)$ in $\operatorname{End} V$ equals the span of $\mathscr{T}(\bar{\mathbb{F}}_p)$ in $\operatorname{End} V$.

If $q = p$, we will show that the p'-elements of $\mathscr{G}^0(\mathbb{F}_p)$ span the same subspace of $\operatorname{End} V$ as does all of $\mathscr{G}^0(\bar{\mathbb{F}}_p)$. First, from Proposition 4.4(iv), we deduce that $\ell = n$. As the V_i are distinct and irreducible $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$-representations, by the Artin–Wedderburn theorem we need to show that the p'-elements in $\mathscr{G}^0(\mathbb{F}_p)$ span $\prod_{i=1}^n \operatorname{End}(V_i)$, or equivalently its $\mathscr{G}^0(\mathbb{F}_p)$-head. (Note that the span of the p'-elements is $\mathscr{G}^0(\mathbb{F}_p)$-stable.) By Lemma 3.3, we see that the n representations $\operatorname{head}_{\mathscr{G}^0(\mathbb{F}_p)}(\operatorname{End}(V_i))$ have no $\mathscr{G}^0(\mathbb{F}_p)$-irreducible constituent in common except for the trivial direct summand of scalar matrices in $\operatorname{End}(V_i)$. By Proposition 3.1, the image of the p'-elements span $\operatorname{End}(V_i)$ for any i. Hence it suffices to show that the image of the p'-elements under the map

$$\begin{aligned} \mathscr{G}^0(\mathbb{F}_p) &\to \bar{\mathbb{F}}_p^n, \\ g &\mapsto (\operatorname{tr}(g|_{V_i}))_{i=1}^n \end{aligned}$$

spans $\mathbb{F}_p^n$. Note that as $1 < \dim V_0 < p$, the split torus $\begin{pmatrix} * & \\ & * \end{pmatrix} < \mathrm{SL}_2(\mathbb{F}_p)$ has a nontrivial eigenvalue χ on V_0 with multiplicity 1 or 2. Given $1 \le i \le n$, there exists an element in $\mathbb{F}_p[\mathscr{T}(\mathbb{F}_p)]$ that projects onto the $1 \otimes \cdots \otimes \chi \otimes \cdots \otimes 1$ eigenspace in any $\mathscr{T}(\mathbb{F}_p)$-representation, so as $p > 2$ it has nonzero trace on V_i but is zero on $\bigoplus_{j \ne i} V_j$. This proves the claim.

(e2) We claim that $\operatorname{head}_{\mathscr{G}^0_{/\bar{\mathbb{F}}_p}}(\operatorname{End} V) = \operatorname{head}_{\mathscr{G}^0(\mathbb{F}_p)}(\operatorname{End} V)$, and moreover that any highest weight of this representation is p-restricted.

If $d \le (p+1)/2$, then by [Serre 1994] $\operatorname{End} V$ is a semisimple $\mathscr{G}^0_{/\bar{\mathbb{F}}_p}$-module and clearly any highest weight of $\operatorname{End} V$ is p-restricted. The claim follows.

If $d \ge (p+3)/2$, note that head is compatible with direct sums, so we can consider each $V_i^* \otimes V_j$ separately. If $i \ne j$, then $V_i^* \otimes V_j$ is irreducible with p-restricted highest weight. If $i = j$, from Lemma 3.3 we get

$$\operatorname{head}_{\mathrm{SL}_2}(V_0^* \otimes V_0) \cong L(0) \oplus L(2) \oplus \cdots \oplus L(p-1).$$

In particular, any highest weight of $\operatorname{head}_{\mathscr{G}^0_{/\bar{\mathbb{F}}_p}}(V_i^* \otimes V_i)$ is p-restricted. By Lemma 3.3, showing

$$\operatorname{head}_{\mathscr{G}^0_{/\bar{\mathbb{F}}_p}}(V_i^* \otimes V_i) = \operatorname{head}_{\mathscr{G}^0(\mathbb{F}_p)}(V_i^* \otimes V_i)$$

is equivalent (after a Frobenius twist) to showing that

$$\mathrm{head}_{\mathrm{SL}_2}(T(2p-2-2j)) = \mathrm{head}_{\mathrm{SL}_2(\mathbb{F}_q)}(T(2p-2-2j))$$

for $0 \le j \le (p-3)/2$. If $q = p$ this follows from Lemma 3.3, as $d < p$. This in turn implies the statement for $q > p$, as any irreducible SL_2-constituent of $T(2p-2-2j)$ restricts irreducibly to $\mathrm{SL}_2(\mathbb{F}_q)$ if $q > p$ and semisimply to $\mathrm{SL}_2(\mathbb{F}_p)$. This proves the claim.

(e3) Now, let $\mathcal{M}$ denote the span of the images of the p'-elements of $\mathcal{G}(\mathbb{F}_p)$ in $\mathrm{head}_{\mathcal{G}^0(\mathbb{F}_p)}(\mathrm{End}(V))$. Note that $\mathcal{M}$ is a $\mathcal{G}^0(\mathbb{F}_p)$-subrepresentation. To prove weak adequacy, it suffices to show that $\mathcal{M} = \mathrm{head}_{\mathcal{G}^0(\mathbb{F}_p)}(\mathrm{End}(V))$. By (e2) we have that $\mathrm{head}_{\mathcal{G}^0_{/\bar{\mathbb{F}}_p}}(\mathrm{End}(V)) = \mathrm{head}_{\mathcal{G}^0(\mathbb{F}_p)}(\mathrm{End}(V))$ and that distinct irreducible $\mathcal{G}^0_{/\bar{\mathbb{F}}_p}$-subrepresentations of $\mathrm{head}_{\mathcal{G}^0_{/\bar{\mathbb{F}}_p}}(\mathrm{End}(V))$ restrict to distinct irreducible $\mathcal{G}^0(\mathbb{F}_p)$-representations. Hence, any $\mathcal{G}^0(\mathbb{F}_p)$-subrepresentation of $\mathrm{head}_{\mathcal{G}^0_{/\bar{\mathbb{F}}_p}}(\mathrm{End}(V))$ is $\mathcal{G}^0(\bar{\mathbb{F}}_p)$-stable. By (e1), we know that $\mathcal{M}$ contains the span of the image of $\mathcal{T}(\bar{\mathbb{F}}_p) \cdot H$. Therefore, by Lemma 8 in [Guralnick et al. 2012], $\mathcal{M}$ contains the span of the image of $\mathcal{G}(\bar{\mathbb{F}}_p)$. But the latter span equals $\mathrm{head}_{\mathcal{G}^0_{/\bar{\mathbb{F}}_p}}(\mathrm{End}(V))$ by the Artin–Wedderburn theorem. □

5. Weak adequacy in cross-characteristic

Recall that, given a finite-dimensional absolutely irreducible representation $\Phi : G \to \mathrm{GL}(V)$, the pair (G, V) is called *weakly adequate* if $\mathrm{End}(V)$ equals

$$\mathcal{M} := \langle \Phi(g) \in \Phi(G) : \Phi(g) \text{ semisimple} \rangle_k.$$

Assume $k = \bar{k}$ has characteristic p. First, we recall:

Lemma 5.1 [Guralnick 2012b, Lemma 2.3]. *If $G < \mathrm{GL}(V)$ is p-solvable and $p \nmid \dim V$, then (G, V) is weakly adequate.*

In general, a key tool to prove weak adequacy is provided by the following criterion:

Lemma 5.2. *Let V be a finite-dimensional vector space over k and $G \le \mathrm{GL}(V)$ a finite irreducible subgroup. Write $V|_{G^+} = e\sum_{i=1}^t W_i$, where the G^+-modules W_i are irreducible and pairwise nonisomorphic. Suppose there is a subgroup $Q \le G^+$ such that*

(i) $\{Q^g : g \in G\} = \{Q^x : x \in G^+\}$, *and*

(ii) *the Q-modules W_i are irreducible and pairwise nonisomorphic*,

then $\mathbf{N}_G(Q)$ is an irreducible subgroup of $\mathrm{GL}(V)$. If, furthermore,

(iii) *$\mathbf{N}_{G^+}(Q)$ is a p'-group,*

then (G, V) is weakly adequate.

Proof. The condition (i) is equivalent to the equality $G = NG^+$, where $N := \mathbf{N}_G(Q)$. Since G/G^+ is a p'-group, this implies that N is a p'-group if $\mathbf{N}_{G^+}(Q)$ is a p'-group. By the Artin–Wedderburn theorem, it therefore suffices to show that N is irreducible on V.

Set $V_i = eW_i$ so that $V = \bigoplus_{i=1}^m V_i$, $G_1 := I_G(W_1) = \mathrm{Stab}_G(V_1)$ the inertia group of the G^+-module W_1 in G, and $N_1 := N \cap G_1$. Then we have that $G_1 = N_1 G^+$ and $[N : N_1] = [G : G_1] = t$. Trivially, the condition (ii) implies that the N^+-modules W_i $(1 \le i \le t)$, are irreducible and pairwise nonisomorphic, where we set $N^+ := \mathbf{N}_{G^+}(Q)$. It now follows that $N_1 = I_N(W_1)$, the inertia group of the N^+-module W_1 in N; moreover, N acts transitively on $\{V_1, \dots, V_t\}$, and $V|_N = \mathrm{Ind}_{N_1}^N(V_1|_{N_1})$. By the Clifford correspondence, it suffices to show that the N_1-module V_1 is irreducible.

Let Φ denote the corresponding representation of G_1 on V_1 and let Ψ denote the corresponding representation of G^+ on W_1. By [Navarro 1998, Theorem 8.14], there is a projective representation Ψ_1 of G_1 such that

$$\Psi_1(n) = \Psi(n), \quad \Psi_1(xn) = \Psi_1(x)\Psi_1(n), \quad \Psi_1(nx) = \Psi_1(n)\Psi_1(x)$$

for all $n \in G^+$ and $x \in G_1$. Let α denote the factor set on G_1/G^+ induced by Ψ_1. By [Navarro 1998, Theorem 8.16], there is an e-dimensional projective representation Θ of G_1/G^+ with factor set α^{-1} such that $\Phi(g) = \Theta(g) \otimes \Psi_1(g)$ for all $g \in G_1$. (Here and in what follows, we will write $\Theta(g)$ instead of $\Theta(gG^+)$.) Since Φ is irreducible, Θ is irreducible.

Observe that N_1/N^+ is canonically isomorphic to G_1/G^+. Restricting to N_1, we then have that $\Phi(g) = \Theta(g) \otimes \Psi_1(g)$ for all $g \in N_1$, $\Psi_1(n) = \Psi(n)$ for all $n \in N^+$, $(\Psi_1)_{N_1}$ is a projective representation of N_1 with factor set α, and Θ_{N_1/N^+} is a projective representation of N_1/N^+ with factor set α^{-1}. Furthermore, Θ_{N_1/N^+} is irreducible. It follows by [Navarro 1998, Theorem 8.18] that Φ_{N_1} is irreducible, as stated. $\square$

In certain cases we will also need the following modification of Lemma 5.2:

Lemma 5.3. *Let V be a finite-dimensional vector space over k and $G \le \mathrm{GL}(V)$ a finite irreducible subgroup. Write $V|_{G^+} = e\sum_{i=1}^t W_i$, where the G^+-modules W_i are irreducible and pairwise nonisomorphic. Suppose there is a subgroup $Q \le G^+$ with the following properties*:

(i) $\{Q^g : g \in G\} = \{Q^x : x \in G^+\}$.

(ii) *$W_i \cong A_i \oplus B_i$ as Q-modules, where all the $2t$ Q-modules A_i and B_j are irreducible and pairwise nonisomorphic.*

If $\{A_1, \dots, A_t\}$ and $\{B_1, \dots, B_t\}$ are two disjoint N-orbits on $\mathrm{IBr}(Q)$ for $N := \mathbf{N}_G(Q)$, then we have that $V_N \cong A \oplus B$ as N-modules, where A and B are irreducible, $A_Q \cong e\big(\bigoplus_{i=1}^t A_i\big)$, and $B_Q \cong e\big(\bigoplus_{i=1}^t B_i\big)$. On the other hand, if $\{A_1, B_1, \dots, A_t, B_t\}$ forms a single N-orbit, then N is irreducible on V.

Proof. Again, the condition (i) implies that $G = NG^+$. Adopt the notation G_1, N_1, N^+, Φ, Ψ, Ψ_1, α of the proof of Lemma 5.2. As shown there, there is an irreducible e-dimensional projective representation Θ of G_1/G^+ with factor set α^{-1} such that $\Phi(g) = \Theta(g) \otimes \Psi_1(g)$ for all $g \in G_1$. Also, N_1/N^+ is canonically isomorphic to G_1/G^+. According to (ii), $(W_i)_Q \cong A_i \oplus B_i$, with $A_i \ncong B_i$. Hence we can decompose $(V_i)_Q = C_i \oplus D_i$, where $(C_i)_Q \cong eA_i$ and $(D_i)_Q \cong eB_i$, and define $A := \bigoplus_{i=1}^t C_i$, $B := \bigoplus_{i=1}^t D_i$.

(a) First we consider the case where $\{A_1, \dots, A_t\}$ and $\{B_1, \dots, B_t\}$ are two disjoint N-orbits. Then, for any $x \in N$, every composition factor of the Q-module xA is of the form A_j for some j, and every composition factor of B is of the form $B_{j'}$ for some j'. Hence we conclude that $xA = A$, and similarly $xB = B$. Thus A and B are N-modules. Certainly, N permutes $C_1, \dots, C_t$ transitively and N_1 fixes C_1. But $t = [N : N_1]$; hence $N_1 = \mathrm{Stab}_N(C_1)$ and $A = \mathrm{Ind}_{N_1}^N(C_1)$. Since $(C_i)_Q = eA_i$ and the Q-modules A_i are pairwise nonisomorphic, we also see that $N_1 = I_N(A_1)$. Similarly, $N_1 = I_N(B_1)$ and $B = \mathrm{Ind}_{N_1}^N(D_1)$. Therefore, by the Clifford correspondence, it suffices to prove that the N_1-modules C_1 and D_1 are irreducible.

Recall the decompositions $(W_1)_Q = A_1 \oplus B_1$ and $\Phi(g) = \Theta(g) \otimes \Psi_1(g)$ for all $g \in G_1$. Without loss, we may assume that the representation Ψ of G^+ on W_1 is written with respect to some basis $(v_1, \dots, v_{a+b})$ which is the union of a basis $(v_1, \dots, v_a)$ of A_1 and a basis $(v_{a+1}, \dots, v_{a+b})$ of B_1. Since $\Phi(g) = \Theta(g) \otimes \Psi_1(g)$ for all $g \in G_1$ acting on V_1, we can also choose a basis

$$\{u_i \otimes v_j : 1 \le i \le e,\ 1 \le j \le a + b\}$$

of V_1 such that $\Theta(g)$ is written with respect to $\{u_1, \dots, u_e\}$ and $\Psi_1(g)$ is written with respect to $\{v_1, \dots, v_{a+b}\}$. For any $x \in N_1$, writing $\Theta(x) = (\theta_{i'i})$ and $\Psi_1(x) = (\psi_{j'j})$, we then have that

$$\Phi(x)(u_i \otimes v_j) = \sum_{i',j'} \theta_{i'i}\psi_{j'j} u_{i'} \otimes v_{j'}.$$

Recall we are also assuming that the Q-modules A_1 and B_1 are not N-conjugate. Therefore, $\Phi(x)$ fixes each of

$$C_1 = \langle u_i \otimes v_j : 1 \le i \le e,\ 1 \le j \le a\rangle_k, \quad D_1 = \langle u_i \otimes v_j : 1 \le i \le e,\ a+1 \le j \le a+b\rangle_k.$$

In particular, $\theta_{i'i}\psi_{j'j} = 0$ whenever $j' > a$ and $j \le a$. Now if $\psi_{j'j} \neq 0$ for some $j \le a$ and some $j' > a$, we must have $\theta_{i'i} = 0$ for all i, i', i.e., $\Theta(x) = 0$, a contradiction. Similarly, $\psi_{j'j} = 0$ whenever $j > a$ and $j' \le a$. Therefore, we can write

$$\Psi_1(x) = \mathrm{diag}(\Psi_{1A}(x), \Psi_{1B}(x)) \tag{5-1}$$

in the chosen basis $\{v_1, \dots, v_{a+b}\}$. It also follows that $\Psi(y)$ fixes each of A_1 and B_1 for all $y \in N^+$, i.e., A_1 and B_1 are irreducible N^+-modules.

Now, $\Psi_1(x)\Psi_1(y) = \alpha(x, y)\Psi_1(xy)$ for any $x, y \in N_1$. Together with (5-1) this implies that

$$\Psi_{1A}(x)\Psi_{1A}(y) = \alpha(x, y)\Psi_{1A}(xy), \quad \Psi_{1B}(x)\Psi_{1B}(y) = \alpha(x, y)\Psi_{1B}(xy),$$

i.e., both Ψ_{1A} and Ψ_{1B} are projective representations of N_1 with factor set α. Since $\Psi_1(x) = \Psi(x)$ for all $x \in N^+$ and (5-1) certainly holds for $x \in N^+$, we also see that Ψ_{1A} extends the representation of N^+ on A_1, and similarly Ψ_{1B} extends the representation of N^+ on B_1. By [Navarro 1998, Theorem 8.18], the formulae

$$\Phi_A(g) := \Theta(g) \otimes \Psi_{1A}(g), \quad \Phi_B(g) := \Theta(g) \otimes \Psi_{1B}(g)$$

for $g \in N_1$ define irreducible (linear) representations of N_1 of dimension ea and eb (acting on C_1 and D_1, respectively), and so we are done.

(b) Next we consider the case N acts transitively on $\{A_1, \dots, B_t\}$. In this case, $N_1^\circ := I_N(A_1)$ has index $2t$ in N and is contained in N_1. Note that there is some $g \in N$ such that $B_1^g \cong A_1$ as Q-modules. Certainly, such g must belong to N_1, and also g interchanges C_1 and D_1. Applying the arguments of (a) to g, we see that $\Psi_1(g)$ interchanges A_1 and B_1. It follows that $(\Psi_1)_{N_1}$ is irreducible. In turn, this implies by [Navarro 1998, Theorem 8.18] that Φ_{N_1} is irreducible, i.e., N_1 is irreducible on V_1. But $[N_1 : N_1^\circ] = 2$ and $V_1 = C_1 \oplus D_1$ as N_1°-modules. Hence C_1 is an irreducible N_1°-module. Since $N_1^\circ = I_N(A_1)$ and C_1 is the A_1-isotypic component for Q on V, we conclude by Clifford's theorem that N is irreducible on V. □

Lemma 5.4. *Let V be a finite-dimensional vector space over k and $G \le \mathrm{GL}(V)$ a finite irreducible subgroup. Write $V|_{G^+} = e\sum_{i=1}^t W_i$, where the G^+-modules W_i are irreducible and pairwise nonisomorphic. Suppose there is a subgroup $Q \le G^+$ with the following properties*:

(a) $\{Q^g : g \in G\} = \{Q^x : x \in G^+\}$.

(b) $(W_i)_Q \cong A_i \oplus B_{i1} \oplus \cdots \oplus B_{is}$, *where $a := \dim A_i \ne \dim B_{il}$ for all $1 \le i \le t$ and all $1 \le l \le s$, the Q-modules A_i, B_{il} are irreducible, and the Q-modules A_i $(1 \le i \le t)$ are pairwise nonisomorphic.*

Then the following statements hold:

(i) *Denoting $N := \mathbf{N}_G(Q)$, we have that $V_N \cong A \oplus B$ as N-modules, where A is irreducible, $A_Q \cong e\big(\bigoplus_{i=1}^t A_i\big)$ and $B_Q \cong e\big(\bigoplus_{i,l} B_{il}\big)$.*

(ii) *Assume that N is a p'-subgroup, G^+ is perfect, and that, whenever $i \ne j$, no G^+-composition factor of $W_i^* \otimes W_j$ is trivial. If all G^+-composition factors of $\mathrm{End}(V)/\mathcal{M}$ (if there are any) are trivial, then in fact $\mathcal{M} = \mathrm{End}(V)$.*

Proof. (i) follows from same proof as Lemma 5.3. For (ii), note that since G^+ is perfect it must act trivially on $\mathscr{E}/\operatorname{End}(V)$, i.e., $\mathcal{M} \supseteq [\operatorname{End}(V), G^+]$. It follows that

$$\mathcal{M} \supseteq [\mathscr{E}_{1i}, G^+] \tag{5-2}$$

for $\mathscr{E}_{1i} := \operatorname{End}(V_i)$. On the other hand, $\operatorname{Hom}(V_i, V_j) = [\operatorname{Hom}(V_i, V_j), G^+]$, and so

$$\mathcal{M} \supseteq \bigoplus_{1 \leq i \neq j \leq t} \operatorname{Hom}(V_i, V_j).$$

It suffices to show that $\mathcal{M} \supseteq \mathscr{E}_{11}$ (and so by symmetry $\mathcal{M} \supseteq \mathscr{E}_{1i}$ for all i).

Applying the Artin–Wedderburn theorem to N, we see that

$$\mathcal{M} \supset \operatorname{End}(A) \supseteq \operatorname{End}(C_1), \tag{5-3}$$

where $(C_1)_Q \cong eA_1$. Also, as in the proof of Lemma 5.3, we can write

$$V_1 = U \otimes W_1, \quad C_1 = U \otimes A_1,$$

such that U affords a projective representation Θ of $G_1/G^+ \cong N_1/N^+$, W_1 affords a projective representation Ψ_1 of G_1 that extends the representation Ψ of G^+ on W_1, and $\Phi(g) = \Theta(g) \otimes \Psi_1(g)$ for the representation Φ of G_1 on V_1.

Note that the subspace $\operatorname{End}(W_1)^\circ$ consisting of all transformations with trace 0 is a G^+-submodule X of codimension 1 of $\operatorname{End}(W_1)$. Next, as a G^+-module,

$$\mathscr{E}_{11} = \operatorname{End}(V_1) \cong \operatorname{End}(U) \otimes \operatorname{End}(W_1) \cong e^2 \operatorname{End}(W_1).$$

So we see that $\mathscr{E}_{11}^+ := \operatorname{End}(U) \otimes \operatorname{End}(W_1)^\circ$ is a submodule of codimension e^2 in $\mathscr{E}_{11}$, and all G^+-composition factors of $\mathscr{E}_{11}/\mathscr{E}_{11}^+$ are trivial. Since G^+ is perfect, it follows that $\mathscr{E}_{11}^+ \supseteq [\mathscr{E}_{11}, G^+]$. But

$$\begin{aligned}\dim \operatorname{Hom}_{kG^+}(\mathscr{E}_{11}, k) &= e^2 \dim \operatorname{Hom}_{kG^+}(\operatorname{End}(W_1), k)\\ &= e^2 \dim \operatorname{Hom}_{kG^+}(W_1, W_1) = e^2.\end{aligned}$$

Hence, $\mathscr{E}_{11}^+ = [\mathscr{E}_{11}, G^+]$, and so by (5-2) we have that

$$\mathcal{M} \supset \mathscr{E}_{11}^+ = \operatorname{End}(U) \otimes \operatorname{End}(W_1)^\circ.$$

On the other hand, by (5-3) we also have that

$$\mathcal{M} \supset \operatorname{End}(C_1) = \operatorname{End}(U) \otimes \operatorname{End}(A_1).$$

Obviously, $\operatorname{End}(W_1)^\circ + \operatorname{End}(A_1) = \operatorname{End}(W_1)$ (as $\operatorname{End}(A_1)$ contains elements with nonzero trace). Hence we conclude that $\mathcal{M} \supseteq \mathscr{E}_{11}$, as stated. $\square$

We also record the following trivial observation:

Lemma 5.5. *Let E be a kG-module of finite length with submodules X and M. Suppose that $N \le G$ and that the N-modules X and E/X share no common composition factor (up to isomorphism). Suppose that the multiplicity of each composition factor C of X is at most its multiplicity as a composition factor of M (for instance, X is a subquotient of M). Then $M \supseteq X$.*

Proof. The hypothesis implies that the N-modules X and E/M have no common composition factor. On the other hand, $X/(M \cap X) \cong (X+M)/M \subseteq E/M$ as N-modules. It follows that $X = M \cap X$, as stated. □

Proposition 5.6. *Let (G, V) be as in the extraspecial case* (ii) *of Theorem 2.4. Then (G, V) is weakly adequate.*

Proof. Decompose $V_{G^+} = e\sum_{i=1}^{t} W_i$ as in Lemma 5.2. Recall by Theorem 2.4(ii) that $R := \boldsymbol{O}_{p'}(G^+) \lhd G$ acts irreducibly on each W_i. First we show that if $i \neq j$ then the R-modules W_i and W_j are nonisomorphic. Assume the contrary: $W_i \cong W_j$ as R-modules. Then the G^+-modules W_i and W_j are two extensions to $G^+ \rhd R$ of the R-module W_i. By [Navarro 1998, Corollary 8.20], $W_j \cong W_i \otimes U$ (as G^+-modules) for some one-dimensional G^+/R-module U. But G^+/R is perfect by Theorem 2.4(ii). It follows that U is the trivial module and $W_i \cong W_j$ as G^+-modules, a contradiction.

For future use, we also show that the G^+-module W_i has a unique complex lift. Indeed, the existence of a complex lift χ of W_i was established in [Blau and Zhang 1993, Theorem B]. Suppose that χ' is another complex lift. Then both χ and χ' are extensions of $\alpha := \chi_R$, and α is irreducible since R is irreducible on W_i. Then, again by [Navarro 1998, Corollary 8.20], $\chi' = \chi\lambda$ for some linear character λ of G^+/R, and so $\lambda = 1_{G^+/R}$ as G^+/R is perfect. Thus $\chi' = \chi$.

Now we write $G^+/R = S_1 \times \cdots \times S_n$ with $S_i \cong S$ as in Theorem 2.4(ii). We will define the subgroup $Q > R$ of G^+ with

$$Q/R = Q_1 \times \cdots \times Q_n$$

as follows. If $p = 17$ and $S = \mathrm{PSL}_2(17)$, then Q_i is a dihedral subgroup of order 16. If $S = \Omega^-_{2a}(2^b)'$ with $ab = n$ (and $a \ge 2$ as S is simple nonabelian), then Q_i is chosen to be the first parabolic subgroup (which is the normalizer of an isotropic 1-space in the natural module $\mathbb{F}_{2^b}^{2a}$, of index $(2^n+1)(2^{n-b}-1)/(2^b-1)$). If $S = \mathrm{Sp}_4(2)' \cong \mathsf{A}_6$, choose $Q_i \cong 3^2 : 4$, of order 36. If $S = \mathrm{Sp}_4(2^b)$ with $b \ge 2$, we fix a prime divisor r of b and choose $Q_i \cong \mathrm{Sp}_4(2^{b/r})$. For $S = \mathrm{Sp}_{2a}(2^b)$ with $a \ge 3$, we choose Q_i to be the first parabolic subgroup (which is the normalizer of a 1-space in the natural module $\mathbb{F}_{2^b}^{2a}$, of index $2^{2n}-1$). In all cases, our choice of Q_i ensures that the p'-subgroup Q_i is a maximal subgroup of S_i and, moreover, that the S_i-conjugacy class of Q_i is $\mathrm{Aut}(S_i)$-invariant. In particular, $\boldsymbol{N}_{G^+}(Q) = Q$. Also note that any

$g \in G$ normalizes R and permutes the simple factors S_i of G^+/R; in fact, its action on G^+/R belongs to $\mathrm{Aut}(S^n) = \mathrm{Aut}(S) \wr \mathsf{S}_n$. It follows that Q satisfies conditions (i) and (iii) of Lemma 5.2. Since $W_i \not\cong W_j$ as R-modules for $i \neq j$, $W_i \not\cong W_j$ as Q-modules as well. Hence we are done by Lemma 5.2. □

Theorem 5.7. *Suppose* (G, V) *is as in case* (i) *of Theorem 2.4. Then* (G, V) *is weakly adequate unless one of the following possibilities occurs for the group* $H < \mathrm{GL}(W)$ *induced by the action of* G^+ *on any irreducible* G^+*-submodule* W *of* V:

(i) $p = (q^n - 1)/(q - 1)$, *with* $n \geq 3$ *a prime, and* $H \cong \mathrm{PSL}_n(q)$.

(ii) $(p, H, \dim W) = (5, 2\mathsf{A}_7, 4), (7, 6_1{\cdot}\mathrm{PSL}_3(4), 6), (11, 2M_{12}, 10)$, *or* $(19, 3J_3, 18)$.

Proof. (a) Arguing as in part (b) of the proof of Theorem 2.4 (and using its notation), we see that for each i there is some k_i such that the kernel K_i of the action of G^+ on W_i contains $\prod_{j \neq k_i} L_j$, and so G^+ acts on W_i as $H_i = L_{k_i}/(L_{k_i} \cap K_i)$. We aim to define a subgroup $Q > \mathbf{Z}(G^+)$ of G^+ such that

$$Q = Q_1 * Q_2 * \cdots * Q_n,$$

where $Q_i/\mathbf{Z}(L_i) \leq L_i/\mathbf{Z}(L_i) =: S_i \cong S$ and Q satisfies the conditions of Lemma 5.2. In fact, we will find Q_i so that the p'-subgroup $Q_i/\mathbf{Z}(L_i)$ is a maximal subgroup of S_i and, moreover, the S_i-conjugacy class of $Q_i/\mathbf{Z}(L_i)$ is $\mathrm{Aut}(S_i)$-invariant. To this end, we first find Q_1; then for each $i > 1$, we can fix an element $g_i \in G$ conjugating S_1 to S_i and choose $Q_i = Q_1^{g_i}$. Since G fixes G^+ and $\mathbf{Z}(G^+)$ and induces a subgroup of $\mathrm{Aut}(S) \wr \mathsf{S}_n$ while acting on $G^+/\mathbf{Z}(G^+) \cong S^n$, it follows that Q satisfies conditions (i) and (iii) of Lemma 5.2. Moreover, in the cases where

$$G^+ = L_1 \times \cdots \times L_n \cong H^n, \tag{5-4}$$

then we can also write $Q = Q_1 \times \cdots \times Q_n$, which simplifies some parts of the arguments.

(b1) Suppose first that we are in the case (b1) of Theorem 2.1. Assume that $(H, p) = (\mathrm{Sp}_{2n}(q), (q^n + 1)/2)$. Here H is the full cover of S, so (5-4) holds. Then we choose Q_i to be the last parabolic subgroup of $\mathrm{Sp}_{2n}(q)$ (which is the stabilizer of a maximal totally isotropic subspace in the natural module $\mathbb{F}_q^{2n}$). Then $Q_i/\mathbf{Z}(L_i)$ is a maximal p'-subgroup of S_i and, moreover, the S_i-conjugacy class of $Q_i/\mathbf{Z}(L_i)$ is $\mathrm{Aut}(S_i)$-invariant. By [Guralnick et al. 2002, Theorem 2.1], the H-module W is one of the two Weil modules of dimension $(q^n - 1)/2$ of $H \cong \mathrm{Sp}_{2n}(q)$. Furthermore, by [Guralnick et al. 2002, Lemma 7.2], the restrictions of these two Weil modules of L_i to Q_i are irreducible and nonisomorphic. It follows that if $W_i \not\cong W_j$ as G^+-modules and $K_i = K_j$, then $W_i \not\cong W_j$ as Q-modules. On the other hand, if $K_i \neq K_j$, then $k_i \neq k_j$ (otherwise we would have $K_i = K_j = \prod_{a \neq k_i} L_a$ since L_{k_i} acts faithfully on V_i), whence $K_i \cap Q \neq K_j \cap Q$

and so $W_i \ncong W_j$ as Q-modules. Thus condition (ii) of Lemma 5.2 holds as well, and so we are done.

Consider the case $(H, p) = (2Ru, 29)$. Then H is the full cover of S and so (5-4) holds. Choose Q_i to be a unique (up to L_i-conjugacy) maximal subgroup of type $(2 \times \mathrm{PSU}_3(5)) : 2$ of L_i; see [Conway et al. 1985]. Note that L_i has a unique conjugacy class $3A$ of elements of order 3. By using [Jansen et al. 1995] and [Conway et al. 1985], and comparing the character values at this class $3A$, we see that L_i has two irreducible p-Brauer characters φ_1, φ_2, of degree 28, and their restrictions to Q_i yield the same irreducible character of Q_i. Now, if $K_i \neq K_j$, then $k_i \neq k_j$ (as W is a faithful kH-module), whence $K_i \cap Q \neq K_j \cap Q$ and so $W_i \ncong W_j$ as Q-modules. Suppose that $K_i = K_j$. By Clifford's theorem, there is some $g \in G$ such that $W_j = W_i^g$ as G^+-modules, and so as L_i-modules as well. In this case, g induces an automorphism of $L_i = 2Ru$. But all automorphisms of Ru are inner [Conway et al. 1985], so W_i and W_j afford the same Brauer L_i-character, whence $W_i \cong W_j$ as G^+-modules. Thus condition (ii) of Lemma 5.2 holds as well, and so we are done.

Next assume that $(H, p) = (\mathrm{SU}_n(q), (q^n+1)/(q+1))$; in particular $n \geq 3$ is odd. Since H is simple, (5-4) holds. Then we choose Q_i to be the last parabolic subgroup of $\mathrm{SU}_n(q)$ (which is the stabilizer of a maximal totally isotropic subspace in the natural module $\mathbb{F}_{q^2}^n$). Then the p'-subgroup Q_i is a maximal subgroup of S_i and the S_i-conjugacy class of Q_i is $\mathrm{Aut}(S_i)$-invariant. Next, if $n \geq 5$ then by [Guralnick et al. 2002, Theorem 2.7], $\mathrm{PSU}_n(q)$ has a unique irreducible module over k of dimension $p-1 = (q^n-q)/(q+1)$, which is again a Weil module. Furthermore, Lemmas 12.5 and 12.6 of [Guralnick et al. 2002] show that the restriction of this Weil module of L_i to Q_i is irreducible. The same conclusions hold in the case $n = 3$ by Theorem 4.2 and the proof of Remark 3.3 of [Geck 1990]. It follows that if $W_i \ncong W_j$ as G^+-modules, then $K_i \neq K_j$, $k_i \neq k_j$ (as W is a faithful kH-module), whence $K_i \cap Q \neq K_j \cap Q$ and so $W_i \ncong W_j$ as Q-modules. Thus condition (ii) of Lemma 5.2 holds, and so we are done again.

Note that we have listed the cases $(p, H) = (5, 2\mathsf{A}_7)$ and $(19, 3J_3)$ as possible exceptions in (ii).

(b2) Suppose now that we are in the case (b2) of Theorem 2.1; in particular, $p = 7$ and $\dim W = 6$. Assume first that $S = \mathsf{A}_7$. The arguments in the cases $L_i \cong 3\mathsf{A}_7$ and $6\mathsf{A}_7$ are the same, so we assume $L_i \cong 6\mathsf{A}_7$. Then we choose $Q_i/\mathbf{Z}(L_i)$ to be a unique (up to L_i-conjugacy) maximal subgroup of type A_6. Restricting the faithful reducible complex characters of degree 4 of $2\mathsf{A}_7$ and 6 of $3\mathsf{A}_7$ [Conway et al. 1985] to Q_i (and comparing character values at elements of order 3), we see that $Q_i \cong 6\mathsf{A}_6$. Now, using [Jansen et al. 1995], one can check that L_i has six irreducible p-Brauer characters of degree 6, and their restrictions to Q_i are irreducible and distinct. Now we can argue as in the case of $\mathrm{Sp}_{2n}(q)$.

Assume now that $H = 2J_2$, and so (5-4) holds. Choose $Q_i/\boldsymbol{Z}(L_i)$ to be a unique (up to L_i-conjugacy) maximal subgroup of type $3 \cdot \mathrm{PGL}_2(9)$ (see [Conway et al. 1985]). Also, using [Jansen et al. 1995], one can check that L_i has two irreducible p-Brauer characters of degree 6, and their restrictions to Q_i are irreducible and distinct. Now we can argue as in the case of $\mathrm{Sp}_{2n}(q)$.

Suppose that $H = 6_1 \cdot \mathrm{PSU}_4(3)$. We will prove weak adequacy of (G, V) in two steps. First, we choose $M_i/\boldsymbol{Z}(L_i)$ to be a unique (up to S_i-conjugacy) maximal subgroup of type $T \cong \mathrm{SU}_3(3)$ of S_i (see [Conway et al. 1985]). Since T has trivial Schur multiplier, we have that $M_i \cong Z_i \times T$, where $Z_i := \boldsymbol{Z}(L_i)$. According to [Jansen et al. 1995], L_i has two irreducible p-Brauer characters of degree 6, which have different central characters. It follows that their restrictions to M_i are irreducible and distinct. Setting

$$M := M_1 * \cdots * M_n,$$

we conclude by Lemma 5.2 that $N := \boldsymbol{N}_G(M)$ is irreducible on V; furthermore, $N/M \cong G/G^+$ is a p'-group. But note that M is *not* a p'-group. Now, at the second step, we note that $M \lhd N$ and $N^+ := \boldsymbol{O}^{p'}(N) = \boldsymbol{O}^{p'}(M) \cong T^n$, and, moreover, each irreducible N^+-submodule in V has dimension 6. Also, recall that $T = \mathrm{SU}_3(3)$ and $p = 7$. So we are done by applying the result of the case of $\mathrm{PSU}_n(q)$.

(b3) Consider the case (b3) of Theorem 2.1; in particular, $p = 11$ and $\dim W = 10$. Putting the possibility $H = 2M_{12}$ as a possible exception in (ii), we may assume that $H = M_{11}$ or $2M_{22}$. Then we choose $Q_i/\boldsymbol{Z}(L_i)$ to be a unique (up to S_i-conjugacy) maximal subgroup of type $M_{10} \cong \mathsf{A}_6 \cdot 2_3$ or $\mathrm{PSL}_3(4)$, respectively, of S_i (see [Conway et al. 1985]). In the former case, H is simple and so (5-4) holds. In the latter case, since $H_j \cong 2M_{22}$, we see that the cyclic group $\boldsymbol{Z}(L_i) \lhd G^+$ must act as a central subgroup of order 1 or 2 of H_j on each W_j. Hence the faithfulness of G on V implies that $L_i \cong 2M_{22}$. Since $\mathrm{PSL}_3(4)$ has no nontrivial representation of degree 10, we must have that $Q_i \cong 2 \cdot \mathrm{PSL}_3(4)$ is quasisimple in this case. Now, using [Jansen et al. 1995], one can check that L_i has two irreducible p-Brauer characters of degree 10, and their restrictions to Q_i are irreducible and distinct. Hence we can argue as in the case of $\mathrm{Sp}_{2n}(q)$.

(b4) Suppose we are in the case (b4) of Theorem 2.1; in particular, $p = 13$ and $\dim W = 12$. Since H is the full cover of S, (5-4) holds. Then we may choose $Q_i/\boldsymbol{Z}(L_i)$ to be a unique (up to S_i-conjugacy) maximal subgroup of type $J_2 : 2$ or $\mathrm{SL}_3(4) : 2_3$, respectively, of S_i (see [Conway et al. 1985]). Since J_2 has no nontrivial representation of degree 12, in the former case we must have that $Q_i \cong (C_3 \times 2J_2) \cdot C_2$, where $C_3 = \boldsymbol{O}_3(\boldsymbol{Z}(L_i))$ and the C_2 induces an outer automorphism of J_2. Also, according to [Breuer et al.], L_i has precisely two irreducible p-Brauer characters of degree 12, which differ at the central elements of order 3. Using [Jansen et al.

1995], we can now check that the restrictions of these two characters to Q_i are irreducible and distinct, and then finish as in the case of $\mathrm{Sp}_{2n}(q)$. In the latter case of $L_i = 2\mathrm{G}_2(4)$, since $\mathrm{SL}_3(4)$ has no nontrivial representation of degree 12 we must have that $Q_i \cong (6 \cdot \mathrm{PSL}_3(4)) \cdot 2_3$. Now, using [Jansen et al. 1995], one can check that L_i has a unique irreducible p-Brauer character of degree 12, and its restriction to Q_i is irreducible. Hence we can argue as in the case of $\mathrm{PSU}_n(q)$.

(c) Now we consider case (c) of Theorem 2.1; in particular, $\dim W = p-2$. Assume that $H = \mathsf{A}_p$ with $p \geq 5$. Since H is simple, (5-4) holds. Choosing $Q_i \cong \mathsf{A}_{p-1}$, we see that the p'-subgroup Q_i is a maximal subgroup of S_i and that the S_i-conjugacy class of Q_i is $\mathrm{Aut}(S_i)$-invariant. Also, using [Guralnick and Tiep 2005, Lemma 6.1] for $p \geq 17$ and [Jansen et al. 1995] for $p \leq 13$, we see that H has a unique irreducible kH-module of dimension $p-2$, and the restriction of this module to A_{p-1} is irreducible. Now we can argue as in the case of $\mathrm{PSU}_n(q)$.

Next suppose that $(H, p) = (\mathrm{SL}_2(q), q+1)$; in particular, p is a Fermat prime and H is simple so (5-4) holds. Choosing $Q_i < \mathrm{SL}_2(q)$ to be a Borel subgroup (of index p), we see that Q_i is a maximal p'-subgroup of S_i and that the S_i-conjugacy class of Q_i is $\mathrm{Aut}(S_i)$-invariant. Also, using [Burkhardt 1976], one can check that H has a unique irreducible kH-module of dimension $p-2$, and the restriction of this module to Q_i is irreducible. Now argue as above.

Suppose that $p = 5$ and $H = 3\mathsf{A}_6$ or $3\mathsf{A}_7$. First we note that $L_i \cong 3\mathsf{A}_s$ with $s = 6$ or $s = 7$ respectively. If not, then $L_i \cong 6\mathsf{A}_s$, but then, since $H_j \cong 3\mathsf{A}_s$, $\boldsymbol{O}_2(\boldsymbol{Z}(L_i))$ must act trivially on all W_i, contradicting the faithfulness of G on V. Now we choose Q_i to be the normalizer of a Sylow 3-subgroup in L_i, of order 108. It is straightforward to check that $\boldsymbol{N}_{S_i}(Q_i/\boldsymbol{Z}(L_i)) = Q_i/\boldsymbol{Z}(L_i)$ and that the S_i-conjugacy class of Q_i is $\mathrm{Aut}(S_i)$-invariant. Also, using [Jansen et al. 1995], one can check that H has two irreducible 5-Brauer characters of degree $p-2$, and the restrictions of them to Q_i are irreducible and distinct. Now we can argue as in the case of $\mathrm{Sp}_{2n}(q)$.

Suppose that $(p, H) = (11, M_{11})$ or $(23, M_{23})$. Again (5-4) holds as H is simple. Choosing Q_i to be $M_{10} \cong \mathsf{A}_6 \cdot 2_3$ (in the notation of [Conway et al. 1985]) or M_{22}, respectively, we have that Q_i is a unique maximal subgroup of L_i of the given p'-order up to L_i-conjugacy. Furthermore, L_i has a unique irreducible kH-module of dimension $p-2$, and the restriction of this module to Q_i is irreducible. Now argue as in the case of $\mathrm{PSU}_n(q)$.

(d) Finally, we consider case (d) of Theorem 2.1: $(p, H) = (11, J_1)$ or $(7, 2\mathsf{A}_7)$. Then we choose $Q_i/\boldsymbol{Z}(L_i)$ to be a unique (up to S_i-conjugacy) maximal subgroup of type $2^3 : 7 : 3$ or A_6, respectively (see [Conway et al. 1985]). In the former case, H is simple, and so (5-4) holds. In the latter case, note that L_i is $2\mathsf{A}_7$. If not, then $L_i \cong 6\mathsf{A}_7$, but then, since $H_j \cong 2\mathsf{A}_7$, $\boldsymbol{O}_3(\boldsymbol{Z}(L_i))$ must act trivially on all W_i, contradicting the faithfulness of G on V. It then follows that $Q_i \cong 2\mathsf{A}_6$ (as any

4-dimensional $k\mathsf{A}_6$-representation is trivial). Now, using [Jansen et al. 1995] one can check that H has a unique irreducible p-Brauer character of given degree, and its restriction to Q_i is irreducible. Now we can argue as in the case of $\mathrm{PSU}_n(q)$. □

Next we use Lemma 5.3 to handle three exceptions listed in Theorem 5.7:

Proposition 5.8. *In the case* $(p, H, \dim W) = (19, 3J_3, 18)$ *of* (ii) *of Theorem 5.7*, (G, V) *is weakly adequate.*

Proof. Since H is the full cover of S, we have $G^+ = L_1 \times \cdots \times L_n \cong H^n$. Since H acts faithfully on W, for each i there is some k_i such that the kernel K_i of the action of G^+ on W_i is precisely $\prod_{j \neq k_i} L_j$. We define a subgroup Q of G^+ such that

$$Q = Q_1 \times \cdots \times Q_n,$$

where $Q_i/\mathbf{Z}(L_i) \cong \mathrm{SL}_2(16) : 2$ is a maximal subgroup of $S_i = L_i/\mathbf{Z}(L_i) \cong J_3$. Since $\mathrm{SL}_2(16)$ has a trivial Schur multiplier and $\mathbf{Z}(L_i) \leq \mathbf{Z}(Q_i)$, we have that $Q_i \cong 3 \times (\mathrm{SL}_2(16) : 2)$. Furthermore, the S_i-conjugacy class of Q_i is $\mathrm{Aut}(S_i)$-invariant. Hence Q satisfies the condition (i) of Lemma 5.3.

Using [GAP 2004], one can check that L_i has exactly four irreducible 19-Brauer characters $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ of degree 18, and $(\varphi_j)_{Q_i} = \alpha_j + \beta_j$, with α_j of degree 1 with kernel $[Q_i, Q_i]$, β_j of degree 17, and the β_j are all distinct. Now we show that Q fulfills the condition (ii) of Lemma 5.3. Suppose that $W_i \not\cong W_j$ as G^+-modules. Then Q acts coprimely on W_i, with character $\tilde{\alpha}_i + \tilde{\beta}_i$, where $\tilde{\alpha}_i$ has degree 1 and $\tilde{\beta}_i$ has degree 17. If $k_i \neq k_j$, then $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ have different kernels and so are distinct, and likewise $\tilde{\beta}_i$ and $\tilde{\beta}_j$ are distinct. Suppose now that $k_i = k_j$. Then, because of the condition $W_i \not\cong W_j$, we may assume that W_i and W_j both have kernel $K := L_2 \times \cdots \times L_n$, and afford L_1-characters φ_k and φ_l with $1 \leq k \neq l \leq 4$. Since the G-module V is irreducible, we have $W_i \not\cong W_j \cong W_i^g$ for some $g \in G$ which stabilizes K and $G^+/K \cong L_1$ but does not induce an inner automorphism of L_1. The latter condition implies that g interchanges the two classes of elements of order 5 and inverts the central element of order 3 of L_1 [Conway et al. 1985]. The same is true for Q_1. It follows that $\alpha_k \neq \alpha_l$, $\beta_k \neq \beta_l$, and so

$$\tilde{\alpha}_i \neq \tilde{\alpha}_j, \quad \tilde{\beta}_i \neq \tilde{\beta}_j,$$

as claimed.

By Lemma 5.3, $V \cong A \oplus B$ as a module over the p'-group $N := \mathbf{N}_G(Q)$, where the N-modules A and B are irreducible of dimension e and $17e$, respectively. Hence, by the Artin–Wedderburn theorem applied to N,

$$\mathcal{M} := \langle \Phi(g) : g \in G, g \text{ semisimple } \rangle_k$$

contains $\mathscr{A} := \mathrm{End}(A) \oplus \mathrm{End}(B) = (A^* \otimes A) \oplus (B^* \otimes B)$ (if Φ denotes the representation of G on V). As in Lemma 5.3 and its proof, write $A = \bigoplus_{i=1}^t C_i = e\big(\bigoplus_{i=1}^t A_i\big)$ and $B = \bigoplus_{i=1}^t D_i = e\big(\bigoplus_{i=1}^t B_i\big)$ as Q-modules, where A_i affords $\tilde{\alpha}_i$ and B_i affords $\tilde{\beta}_i$. Hence, the complement to $\mathscr{A}$ in $\mathrm{End}(V)$ affords the Q-character

$$\Delta := e^2 \sum_{i,j=1}^{t} (\tilde{\alpha}_i \overline{\tilde{\beta}_j} + \tilde{\beta}_i \overline{\tilde{\alpha}_j}).$$

In particular, all irreducible constituents of $\Delta_{[Q,Q]}$ are of degree 17. The same must be true for the quotient $\mathrm{End}(V)/\mathscr{M}$.

As a G^+-module,

$$\mathrm{End}(V) = \bigoplus_{i,j=1}^{t} (V_i^* \otimes V_j) \cong e^2 \bigg(\bigoplus_{i,j=1}^{t} W_i^* \otimes W_j \bigg).$$

Observe that the G^+-module $W_i^* \otimes W_j$ is irreducible of dimension 324 if $k_i \neq k_j$. Assume that $k_i = k_j$, say $k_i = k_j = 1$. Using [GAP 2004] one can check that no irreducible constituent of $\varphi_k \overline{\varphi_l}$ for $1 \leq k, l \leq 4$ can consist of only irreducible characters of degree 17 while restricted to the subgroup $\mathrm{SL}_2(16)$ of $L_1 = 3J_3$. It follows that no irreducible constituent of the G^+-module $\mathrm{End}(V)$ can consist of only irreducible constituents of dimension 17 while restricted to $[Q, Q]$. Hence $\mathscr{M} = \mathrm{End}(V)$. □

Proposition 5.9. *In the case* $(p, H, \dim W) = (11, 2M_{12}, 10)$ *of* (ii) *of Theorem 5.7*, (G, V) *is weakly adequate.*

Proof. As H is the full cover of S, we have that $G^+ = L_1 \times \cdots \times L_n \cong H^n$. Since H acts faithfully on W, for each i there is some k_i such that the kernel K_i of the action of G^+ on W_i is precisely $\prod_{j \neq k_i} L_j$. We define a subgroup Q of G^+ such that

$$Q = Q_1 \times \cdots \times Q_n,$$

where $Q_i/\mathbf{Z}(L_i) \cong 2_+^{1+4} \cdot \mathsf{S}_3$ is a maximal subgroup of $S_i = L_i/\mathbf{Z}(L_i) \cong M_{12}$. Note that the S_i-conjugacy class of Q_i is $\mathrm{Aut}(S_i)$-invariant. Hence Q satisfies condition (i) of Lemma 5.3.

Using [GAP 2004], one can check that L_i has exactly two irreducible 11-Brauer characters φ_1, φ_2 of degree 10, and $(\varphi_j)_{Q_i} = \alpha + \beta_j$, with α of degree 4, β_j of degree 6, and $\beta_1 \neq \beta_2$. Furthermore, $Z_i := \mathbf{Z}(Q_i) \cong C_2^2$, and

$$\alpha_{Z_i} = 4\lambda, \quad (\beta_j)_{Z_i} = 6\mu, \tag{5-5}$$

where λ and μ are the two linear characters of Z_i that are faithful on $\mathbf{Z}(L_i) < Z_i$. In particular,

$$(\alpha\beta_j)_{Z_i} = 24\nu \tag{5-6}$$

with $\nu := \lambda\mu \neq 1_{Z_i}$.

Now we show that Q fulfills condition (ii) of Lemma 5.3. Suppose that $W_i \not\cong W_j$ as G^+-modules. Then Q acts on W_i, with character $\tilde{\alpha}_i + \tilde{\beta}_i$, where $\tilde{\alpha}_i(1) = 4$ and $\tilde{\beta}_i(1) = 6$. If $k_i \neq k_j$, then $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ have different kernels and so are distinct, and likewise $\tilde{\beta}_i$ and $\tilde{\beta}_j$ are distinct. In particular, in this case $W_i^* \otimes W_j$ is also irreducible. Suppose now that $k_i = k_j$. Then, we may assume that W_i and W_j both have kernel $K := L_2 \times \cdots \times L_n$, and afford L_1-characters φ_k and φ_l with $1 \leq k, l \leq 2$. Since the G-module V is irreducible, we have $W_j \cong W_i^g$ for some $g \in G$ which stabilizes K, and $G^+/K \cong L_1$. But φ_k is $\mathrm{Aut}(L_1)$-invariant [Jansen et al. 1995], whence $l = k$, i.e., $W_j \cong W_i$, a contradiction.

By Lemma 5.3, $V \cong A \oplus B$ as a module over the p'-group $N := \boldsymbol{N}_G(Q)$, where the N-modules A and B are irreducible of dimensions $4e$ and $6e$, respectively. Hence, by the Artin–Wedderburn theorem applied to N,

$$\mathcal{M} := \langle \Phi(g) : g \in G,\, g \text{ semisimple} \rangle_k$$

contains $\mathcal{A} := \mathrm{End}(A) \oplus \mathrm{End}(B) = (A^* \otimes A) \oplus (B^* \otimes B)$ (if Φ denotes the representation of G on V). As in Lemma 5.3 and its proof, write $A = \bigoplus_{i=1}^t C_i = e\big(\bigoplus_{i=1}^t A_i\big)$ and $B = \bigoplus_{i=1}^t D_i = e\big(\bigoplus_{i=1}^t B_i\big)$ as Q-modules, where A_i affords $\tilde{\alpha}_i$ and B_i affords $\tilde{\beta}_i$. Hence, the complement to $\mathcal{A}$ in $\mathrm{End}(V)$ affords the Q-character

$$\Delta := e^2 \sum_{i,j=1}^{t} (\tilde{\alpha}_i \overline{\tilde{\beta}_j} + \tilde{\beta}_i \overline{\tilde{\alpha}_j}).$$

Together with (5-5) and (5-6), this implies that the restriction of any irreducible constituents of Δ to $\boldsymbol{Z}(Q) = Z_1 \times \cdots \times Z_n$ does *not* contain $1_{\boldsymbol{Z}(Q)}$. Thus $\boldsymbol{Z}(Q)$ acts fixed-point-freely on the quotient $\mathrm{End}(V)/\mathcal{M}$. Furthermore, the Q-character of this quotient does not contain $\overline{\tilde{\beta}_i} \tilde{\beta}_j$ (as an irreducible constituent of degree 36) for any $i \neq j$.

As a G^+-module,

$$\mathrm{End}(V) = \bigoplus_{i,j=1}^{t} (V_i^* \otimes V_j) \cong e^2 \bigg(\bigoplus_{i,j=1}^{t} W_i^* \otimes W_j \bigg).$$

Now, if $i \neq j$ then the G^+-module $W_i^* \otimes W_j$ is irreducible and its Brauer character, while restricted to Q, contains $\overline{\tilde{\beta}_i} \tilde{\beta}_j$. On the other hand, the Brauer character of $W_i^* \otimes W_i$ is the direct sum of 1_{G^+} and another irreducible character of degree 99 (as one can check using [GAP 2004]), whose restriction to $\boldsymbol{Z}(Q)$ contains $1_{\boldsymbol{Z}(Q)}$ (which can be seen from (5-5)). Hence we conclude that $\mathcal{M} = \mathrm{End}(V)$. □

Lemma 5.10. *Let* $\mathrm{char}(k) = 5$ *and let* W *be a faithful irreducible* $k(2\mathsf{S}_7)$*-module of dimension* 8, *with corresponding representation* Θ. *Decompose* $W_L = W_1 \oplus W_2$ *as* L*-modules for* $L = 2\mathsf{A}_7$. *Then there is a* $5'$*-element* $z \in 2\mathsf{S}_7 \setminus L$ *and a set* $\mathcal{X} \subset L$ *such that*

(i) *x and xz are $5'$-elements for all $x \in \mathcal{X}$, and*

(ii) $\langle \Theta(x) : x \in \mathcal{X} \rangle_k = \mathrm{End}(W_1) \oplus \mathrm{End}(W_2)$.

Proof. Using [Wilson et al.] and [GAP 2004], K. Lux verified that one can find an element $h \in 2\mathsf{S}_7 \setminus L$ (of order 12) and a set $\mathscr{X} \subset L$ satisfying condition (i) such that $\langle \Theta(xz) : x \in \mathscr{X} \rangle_k$ has dimension 32. Since $\Theta(z) \in \mathrm{GL}(W)$, it follows that $\langle \Theta(x) : x \in \mathscr{X} \rangle_k$ is a subspace of dimension 32 in $\mathrm{End}(W_1) \oplus \mathrm{End}(W_2)$. Since the latter also has dimension 32, we are done. $\square$

Proposition 5.11. *In the case* $(p, H, \dim W) = (5, 2\mathsf{A}_7, 4)$ *of* (ii) *of Theorem 5.7*, (G, V) *is weakly adequate.*

Proof. (a) Recall that $G^+ = L_1 * \cdots * L_n$, and for each i there is some k_i such that the kernel K_i of G^+ contains $\prod_{j \neq k_i} L_j$. By relabeling the W_i, we may assume that $k_1 = 1$. Now, L_1 acts on each W_j either trivially or as the group $H_j \cong 2\mathsf{A}_7$. It follows that $\boldsymbol{O}_3(\boldsymbol{Z}(L_1))$ acts trivially on each W_j and so by faithfulness $\boldsymbol{O}_3(\boldsymbol{Z}(L_1)) = 1$, yielding $L_1 \cong 2\mathsf{A}_7$. On the other hand, $L_1/(K_1 \cap L_1) = H_1 \cong 2\mathsf{A}_7$, whence $K_1 \cap L_1 = 1$, $K_1 = \prod_{j \neq 1} L_j$. This is true for all i, so we have shown that

$$G^+ = L_1 \times L_2 \times \cdots \times L_n \cong H^n.$$

Certainly, G permutes the n components L_i, and this action is transitive by Theorem 2.4(i). Setting $J_1 := \boldsymbol{N}_G(L_1)$, one sees that $G_1 = I_G(W_1) = \mathrm{Stab}_G(V_1)$ is contained in J_1 (as it fixes $K_1 = \prod_{j>1} L_j$). Fix a decomposition $G = \bigcup_{i=1}^t g_i J_1$ with $g_1 = 1$ and $L_i = L_1^{g_i} = g_i L_1 g_i^{-1}$, and choose a subgroup $Q_1 < L_1$ such that $Q_1/\boldsymbol{Z}(L_1) \cong \mathrm{PSL}_2(7)$. Since involutions in A_7 lift to elements of order 4 in L_1, we see that $Q_1 \cong \mathrm{SL}_2(7)$. Now we define

$$Q = Q_1 \times Q_1^{g_2} \times \cdots \times Q_1^{g_n} < G^+.$$

Note that $\boldsymbol{N}_{G^+}(Q) = Q$ and so $N := \boldsymbol{N}_G(Q)$ is a p'-group. Also, L_1 has exactly two irreducible 5-Brauer characters φ_1, φ_2 of degree 4, restricting irreducibly and distinctly to Q_1.

(b) Consider the case where $k_i \neq k_j$ whenever $i \neq j$, i.e., $J_1 = G_1$ and $t = n$. We claim that Q satisfies the conditions of Lemma 5.2. Indeed, the condition $k_i \neq k_j$ implies that the Q-modules W_i and W_j are irreducible and nonisomorphic for $i \neq j$. Next, for any $x \in J_1$, since x fixes W_1 (up to isomorphism), x fixes the character φ of the L_1-module W_1 and so x cannot fuse the two classes $7A$ and $7B$ of elements of order 7 in L_1, whence x can induce only an inner automorphism of L_1. It follows that $Q_1^x = Q_1^t$ for some $t \in L_1$. Now we consider any $g \in G$. Then, for each i we can find j and $x_i \in J_1$ such that $gg_i = g_j x_i$. By the previous observation, there is some $t_i \in L_1$ such that $Q_1^{x_i} = Q_1^{t_i}$. Hence, setting $y_i = g_j t_i g_j^{-1} \in L_j$, we have that

$$Q_1^{gg_i} = Q_1^{g_j x_i} = g_j x_i Q_1 x_i^{-1} g_j^{-1} = g_j t_i Q_1 t_i^{-1} g_j^{-1} = y_i g_j Q_1 g_j^{-1} y_i^{-1} = (Q_1^{g_j})^{y_i}.$$

It follows that $Q^g = Q^y$ with $y = \prod_i y_i \in G^+$, i.e., Q fulfills condition (i) of Lemma 5.2. Now we can conclude by Lemma 5.2 that N is irreducible on V and so we are done.

(c) From now on we assume that, say, $k_1 = k_2$. Then W_1 and W_2 are nonisomorphic modules over $G^+/K_1 = L_1$. So we may assume that W_i affords the L_1-character φ_i for $i = 1, 2$. Note that any $x \in J_1$ sends W_1 to another irreducible G^+-module with the same kernel K_1, and so $\varphi_1^x \in \{\varphi_1, \varphi_2\}$. The irreducibility of G on V implies by Clifford's theorem that the induced action of J_1 on $\{\varphi_1, \varphi_2\}$ is transitive, with kernel G_1. We have shown that $[J_1 : G_1] = 2$ and $t = 2n$. We will label $g_i(W_1)$ as W_{2i-1} and $g_i(W_2)$ as W_{2i}. We also have that $W_2 \cong W_1^h$ for all $h \in J_1 \setminus G_1$. Comparing the kernels and the characters of Q on W_i, we see that the Q-modules W_i are all irreducible and pairwise nonisomorphic. Let

$$\mathscr{E}_1 := \bigoplus_{i=1}^{t} \mathrm{End}(V_i) = \bigoplus_{i=1}^{n} \mathscr{A}_i, \qquad \mathscr{A}_i := \mathrm{End}(V_{2i-1}) \oplus \mathrm{End}(V_{2i}),$$
$$\mathscr{E}_{21} := \bigoplus_{i=1}^{n} \mathscr{B}_i, \qquad \mathscr{B}_i := \mathrm{Hom}(V_{2i-1}, V_{2i}) \oplus \mathrm{Hom}(V_{2i}, V_{2i-1}),$$
$$\mathscr{E}_{22} := \bigoplus_{\substack{1 \le i \ne j \le 2n \\ \{i,j\} \ne \{2a-1, 2a\}}} \mathrm{Hom}(V_i, V_j)$$

so that $\mathrm{End}(V) = \mathscr{E}_1 \oplus \mathscr{E}_{21} \oplus \mathscr{E}_{22}$. Note that the G^+-composition factors of $\mathscr{E}_{21}$ are all of dimensions 6 and 10, whereas the G^+-composition factors of $\mathscr{E}_1$ are either trivial or of dimension 15, as one can check using [Jansen et al. 1995]. Furthermore, the G^+-composition factors of $\mathscr{E}_{22}$ are all of dimension 16. In particular, no G^+-composition factor of $\mathrm{Hom}(W_i, W_j)$ is trivial when $i \neq j$. Similarly, whenever $i \neq j$, the only common G^+-composition factor shared by $\mathscr{A}_i$ and $\mathscr{A}_j$ is k, and $\mathscr{B}_i$ and $\mathscr{B}_j$ share no common G^+-composition factor.

(d) Here we show that $\mathscr{A}_i \oplus \mathscr{B}_i$ is a subquotient of $\mathscr{M}$. To this end, note that J_1 acts irreducibly on $V_1 \oplus V_2$. There is no loss in replacing G by the image of J_1 in $\mathrm{End}(V_1 \oplus V_2)$ and V by $V_1 \oplus V_2$. In doing so, we also get that $n = 1$, $G^+ = L_1$, $[G : G_1] = 2$, $K_1 = 1$, and $G_1 = C * L_1$, where $C := \boldsymbol{C}_G(L_1)$ is a $5'$-group. So for $i = 1, 2$ we can write $V_i = U_i \otimes W_i$ as G_1-modules, where U_i is an irreducible kC-module with corresponding representation Λ_i. Hence for the representation Φ_i of G_1 on V_i, we have $\Phi_i = \Lambda_i \otimes \Theta_i$, where Θ_i is the representation of L_1 on W_i. Finally, for the representation Φ of G on $V = V_1 \oplus V_2$, we have $\Phi(g) = \mathrm{diag}(\Phi_1(g), \Phi_2(g))$ whenever $g \in G_1$.

Recall the element $z \in 2\mathsf{S}_7$ and the set $\mathscr{X} \subset L_1$ constructed in Lemma 5.10. Now we fix a $5'$-element $h \in G \setminus G_1$ such that h induces the same action on $L_1/\boldsymbol{Z}(L_1) \cong \mathsf{A}_7$

as the action of z on A_7. It follows that for all elements $x \in \mathscr{X}$ and for all $u \in C$, ux and uxh are $5'$-elements, whence $\mathcal{M}$ contains the subspaces

$$\mathscr{C} := \langle \Phi(ux) : u \in C, x \in \mathscr{X} \rangle_k, \quad \mathscr{C}\Phi(h) := \{v\Phi(h) : v \in \mathscr{C}\}.$$

We also have that $\Theta_2 \cong \Theta_1^h = \Theta_1^z$. Setting $\Theta(x) = \operatorname{diag}(\Theta_1(x), \Theta_2(x))$ for $x \in \mathscr{X}$, we have by the construction of $\mathscr{X}$ that

$$\langle \Theta(x) : x \in \mathscr{X} \rangle_k = \operatorname{End}(W_1) \oplus \operatorname{End}(W_2).$$

Thus, for $X \in \operatorname{End}(W_1)$, we can write the element $\operatorname{diag}(X, 0)$ of $\operatorname{End}(W_1) \oplus \operatorname{End}(W_2)$ as $\operatorname{diag}(X, 0) = \sum_{x \in \mathscr{X}} a_x \Theta(x)$ for some $a_x \in k$; i.e.,

$$\sum_{x \in \mathscr{X}} a_x \Theta_1(x) = X, \quad \sum_{x \in \mathscr{X}} a_x \Theta_2(x) = 0.$$

On the other hand, applying the Artin–Wedderburn theorem to the representation Λ_i of the $5'$-group C on U_i, we have that

$$\langle \Lambda_i(u) : u \in C \rangle_k = \operatorname{End}(U_i).$$

In particular, any $Y \in \operatorname{End}(U_1)$ can be written as $Y = \sum_{u \in C} b_u \Lambda_1(u)$ for some $b_u \in k$. It follows that the element $\operatorname{diag}(Y \otimes X, 0)$ of

$$\operatorname{End}(U_1) \otimes \operatorname{End}(W_1) \cong \operatorname{End}(U_1 \otimes W_1) = \operatorname{End}(V_1) \hookrightarrow \operatorname{End}(V)$$

can be written as

$$\begin{aligned} \operatorname{diag}\Bigg(\sum_{u \in C,\ x \in \mathscr{X}} b_u a_x \Lambda_1(u) \otimes \Theta_1(x), \sum_{u \in C,\ x \in \mathscr{X}} b_u a_x \Lambda_2(u) \otimes \Theta_2(x) \Bigg) \\ = \sum_{u \in C,\ x \in \mathscr{X}} a_x b_u \cdot \operatorname{diag}(\Phi_1(ux), \Phi_2(ux)) = \sum_{u \in C,\ x \in \mathscr{X}} a_x b_u \Phi(ux), \end{aligned}$$

and so it belongs to $\mathscr{C}$. Thus $\mathscr{C} \supseteq \operatorname{End}(V_1)$, and similarly $\mathscr{C} \supseteq \operatorname{End}(V_2)$. Since G_1 stabilizes each of V_1 and V_2, we then have that

$$\mathscr{C} = \operatorname{End}(V_1) \oplus \operatorname{End}(V_2) = \mathscr{A}_1.$$

But $\Phi(h)$ interchanges V_1 and V_2. It follows that $\mathcal{M}$ also contains

$$\mathscr{C}\Phi(h) = \operatorname{Hom}(V_1, V_2) \oplus \operatorname{Hom}(V_2, V_1) = \mathscr{B}_1,$$

as stated.

(e) Next we show that $\mathscr{E}_{22}$ is a subquotient of $\mathcal{M}$. Choose $R_i \cong 2 \times (7:3) < L_i$, the normalizer of some Sylow 7-subgroup of L_i. Note that $\mathbf{N}_{L_i}(R_i) = R_i$ and

$$(\varphi_j)_{R_1} = \alpha_j + \beta, \tag{5-7}$$

where $\alpha_j, \beta \in \mathrm{Irr}(R_1)$ are of degree 3 and 1, respectively, and $\alpha_1 \neq \alpha_2$. Defining

$$R = R_1 \times R_2 \times \cdots \times R_n < G^+,$$

we see that R satisfies the conditions of Lemma 5.4. Hence the subspace $A = e\big(\bigoplus_{i=1}^{t} A_i\big)$ defined in Lemma 5.4 (with A_1 affording the R_1-character α_1) is irreducible over the p'-group $N_G(R)$. By the Artin–Wedderburn theorem applied to $N_G(R)$ acting on $V = A \oplus B$, $\mathcal{M}$ contains

$$\mathrm{End}(A) \supset \mathcal{D} := \bigoplus_{\substack{1 \le i \neq j \le 2n \\ \{i,j\} \neq \{2a-1,2a\}}} \mathrm{Hom}(eA_i, eA_j).$$

As noted previously, each summand $\mathrm{Hom}(V_i, V_j)$ in $\mathcal{E}_{22}$ is acted on trivially by $\prod_{s \neq k_i, k_j} L_s$, and affords the $L_{k_i} \times L_{k_j}$-character $\varphi \otimes \varphi'$, where $\varphi, \varphi' \in \{\varphi_1, \varphi_2\}$. Working modulo $\mathcal{E}_1 \oplus \mathcal{E}_{21}$ and using this observation and (5-7), we then see that all irreducible constituents of the R-character of the complement to $\mathcal{D}$ in $\mathcal{E}_{22}$ are of the form $\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_n$, where $\gamma_i \in \mathrm{Irr}(R_i)$ and *all but at most one* of them have degree 1 (and the remaining, if any, is some α_j of degree 3). The same is true for the complement to $\mathcal{M}$ in $\mathcal{E}_{22}$ (again modulo $\mathcal{E}_1 \oplus \mathcal{E}_{21}$). On the other hand, (5-7) and the aforementioned observation imply that the R-character of the G^+-composition factor $\mathrm{Hom}(W_i, W_j)$ contains an irreducible R-character of degree 9 (namely, an $R_{k_i} \times R_{k_j}$-character of the form $\alpha \otimes \alpha'$, with $\alpha, \alpha' \in \{\alpha_1, \alpha_2\}$). It follows that $\mathcal{E}_{22}$ is a subquotient of $\mathcal{M}$.

(f) The results of (d) and (e), together with the remarks made at the end of (c), imply that all G^+-composition factors of $\mathrm{End}(V)/\mathcal{M}$ (if any) are trivial. Hence by Lemma 5.4 we conclude that $\mathcal{M} = \mathrm{End}(V)$. □

6. Weak adequacy for special linear groups

The exception (i) in Theorem 5.7 requires much more effort to resolve. We begin by setting up some notation. Let $n \ge 3$ and let q be a prime power such that $p = (q^n - 1)/(q - 1)$. In particular, n is a prime, $q = q_0^f$ for some prime q_0 and some odd f, $\gcd(n, q-1) = 1$ and so $\mathrm{PSL}_n(q) = \mathrm{SL}_n(q) =: S$ and $G_n := \mathrm{GL}_n(q) = S \times \mathbf{Z}(G_n)$. Consider the natural module

$$\mathcal{N} = \mathbb{F}_q^n = \langle e_1, \ldots, e_n \rangle_{\mathbb{F}_q}$$

for G_n, and let

$$Q = RL = \mathrm{Stab}_S(\langle e_2, \ldots, e_n \rangle_{\mathbb{F}_q}),$$

where R is elementary abelian of order q^{n-1} and $L \cong \mathrm{GL}_{n-1}(q)$. Note that Q is a p'-group. It is well known (see [Guralnick and Tiep 1999, Theorem 1.1]) that $G_n/\mathbf{Z}(G_n)$ has a unique irreducible p-Brauer character δ of degree $p-2$, where

$\delta(x) = \rho(x) - 2$ for all p'-elements $x \in G_n$, if we denote by ρ the permutation character of G_n acting on the set Ω of 1-spaces of $\mathcal{N}$. Let $\mathfrak{D}$ denote a kG_n-module affording δ.

Lemma 6.1. *In the above notation, $\delta_Q = \alpha + \beta$, where $\alpha \in \mathrm{Irr}(Q)$ has degree $q^{n-1} - 1$, $\beta \in \mathrm{Irr}(Q)$ has degree $(q^{n-1} - q)/(q - 1)$, and*

$$\alpha_R = \sum_{1_R \neq \lambda \in \mathrm{Irr}(R)} \lambda, \quad \beta_R = \beta(1)1_R.$$

Proof. Note that all nontrivial elements in R are L-conjugate to a fixed transvection $t \in R$, and $\delta(t) = \rho(t) - 2 = (q^{n-1} - q)/(q - 1) - 1$. It follows that

$$\delta_R = \sum_{1_R \neq \lambda \in \mathrm{Irr}(R)} \lambda + \frac{q^{n-1} - q}{q - 1} \cdot 1_R.$$

Next, Q acts doubly transitively on the 1-spaces of $\langle e_2, \dots, e_n \rangle_{\mathbb{F}_q}$, with kernel containing R and with character $\beta + 1_Q$, where $\beta \in \mathrm{Irr}(Q)$ of degree $(q^{n-1} - q)/(q - 1)$. Hence β is an irreducible constituent of δ, and the statement follows. □

In the subsequent treatment of $\mathrm{SL}_n(q)$, it is convenient to adopt the labeling of irreducible $\mathbb{C}G_n$-modules as given in [James 1986], which uses Harish-Chandra induction, denoted $\circ$. Each such module is labeled as $S(s_1, \lambda_1) \circ \cdots \circ S(s_m, \lambda_m)$, where $s_i \in \bar{\mathbb{F}}_q^\times$ has degree d_i (over $\mathbb{F}_q$), λ_i is a partition of k_i, and $\sum_{i=1}^m k_i d_i = n$ [James 1986; Kleshchev and Tiep 2009]. Similarly, irreducible kG_n-modules are labeled as $D(s_1, \lambda_1) \circ \cdots \circ D(s_m, \lambda_m)$, with some extra conditions including s_i being a p'-element. For $\lambda \vdash n$, let $\chi^\lambda = S(1, \lambda)$ denote the unipotent character of $\mathrm{GL}_n(q)$ labeled by λ. We set the convention that $\chi^{(n-2,2)} = 0$ for $n = 3$. Also, note that $1_{G_n} = \chi^{(n)}$ and $\rho = 1_{G_n} + \chi^{(n-1,1)}$ (see, e.g., [Guralnick and Tiep 1999, Lemma 5.1]). We next establish the following result, which holds for arbitrary $\mathrm{GL}_n(q)$ with $n \geq 3$ and which is interesting in its own right:

Lemma 6.2. *In the above notation, we have the following decomposition of ρ^2 into irreducible constituents over $G_n = \mathrm{GL}_n(q)$:*

$$\rho^2 = 2\chi^{(n)} + 4\chi^{(n-1,1)} + \chi^{(n-2,2)} + 2\chi^{(n-2,1^2)} + \sum_{\substack{a \in \mathbb{F}_q^\times \\ a^2 = 1 \neq a}} S(a, (1^2)) \circ S(1, (n-2))$$

$$+ \sum_{\substack{a \in \bar{\mathbb{F}}_q^\times \\ a^{q-1} = 1 \neq a^2}} S(a, (1)) \circ S(a^{-1}, (1)) \circ S(1, (n-2))$$

$$+ \sum_{\substack{a \in \bar{\mathbb{F}}_q^\times \\ b^{q+1} = 1 \neq b^2}} S(b, (1)) \circ S(1, (n-2)).$$

Proof. Recall that ρ is the permutation character of G_n acting on Ω and also on the diagonal $\{(x,x) : x \in \Omega\}$ of $\Omega \times \Omega$, whereas ρ^2 is the permutation character of G_n acting on $\Omega \times \Omega$. Letting $H_n := \mathrm{Stab}_{G_n}(\langle e_1\rangle_{\mathbb{F}_q}, \langle e_2\rangle_{\mathbb{F}_q})$, we then see that

$$\rho^2 = \rho + \mathrm{Ind}_{H_n}^{G_n}(1_{H_n}).$$

Notice that $\mathrm{Ind}_{H_n}^{G_n}(1_{H_n})$ is just the Harish-Chandra induction of the character $\mathrm{Ind}_{H_2}^{G_2}(1_{H_2}) \otimes 1_{G_{n-2}}$ of the Levi subgroup $G_2 \times G_{n-2}$ of the parabolic subgroup

$$P := \mathrm{Stab}_{G_n}(\langle e_1, e_2\rangle_{\mathbb{F}_q})$$

of G_n, i.e.,

$$\mathrm{Ind}_{H_n}^{G_n}(1_{H_n}) = \mathrm{Ind}_{H_2}^{G_2}(1_{H_2}) \circ 1_{G_{n-2}}. \tag{6-1}$$

Consider the case of odd q. Then, according to the proof of [Navarro and Tiep 2010, Proposition 5.5],

$$\begin{aligned}\mathrm{Ind}_{H_2}^{G_2}(1_{H_2}) = S(1,(2)) + 2S(1,(1^2)) + S(-1,(1^2))\\ +a \sum_{\substack{a\in\bar{\mathbb{F}}_q^\times\\ a^{q-1}=1\neq a^2}} S(a,(1)) \circ S(a^{-1},(1)) + \sum_{\substack{a\in\bar{\mathbb{F}}_q^\times\\ b^{q+1}=1\neq b^2}} S(b,(1)).\end{aligned} \tag{6-2}$$

Next, by [Guralnick and Tiep 1999, Lemma 5.1] we have

$$S(1,(2))\circ S(1,(n-2)) = \mathrm{Ind}_P^{G_n}(1_P) = \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)}, \tag{6-3}$$

$$S(1,(1))\circ S(1,(1))\circ S(1,(n-2)) = \chi^{(n)} + 2\chi^{(n-1,1)} + \chi^{(n-2,2)} + \chi^{(n-2,1^2)}. \tag{6-4}$$

Since $S(1,(1))\circ S(1,(1)) = S(1,(2)) + S(1,(1^2))$, the statement follows from (6-1)–(6-4) and properties of the Harish-Chandra induction in G_n (see [James 1986]).

The case q is even can be proved similarly, using

$$\begin{aligned}\mathrm{Ind}_{H_2}^{G_2}(1_{H_2}) = S(1,(2)) + 2S(1,(1^2)) + \sum_{\substack{a\in\bar{\mathbb{F}}_q^\times\\ a^{q-1}=1\neq a^2}} S(a,(1)) \circ S(a^{-1},(1))\\ + \sum_{\substack{a\in\bar{\mathbb{F}}_q^\times\\ b^{q+1}=1\neq b^2}} S(b,(1))\end{aligned}$$

instead of (6-2). □

Lemma 6.3. *In the above notation, if $p = (q^n - 1)/(q-1)$, we have the following decomposition of δ^2 into irreducible constituents over $S = \mathrm{SL}_n(q)$:*

$$\delta^2 = 2D(1, (n)) + 2D(1, (n-1, 1)) + D(1, (n-2, 2)) + 2D(1, (n-2, 1^2))$$
$$+ \sum_{\substack{a\in\mathbb{F}_q^\times \\ a^2=1\neq a}} D(a, (1^2)) \circ D(1, (n-2))$$
$$+ \sum_{\substack{a\in\overline{\mathbb{F}}_q^\times \\ a^{q-1}=1\neq a^2}} D(a, (1)) \circ D(a^{-1}, (1)) \circ D(1, (n-2))$$
$$+ \sum_{\substack{b\in\overline{\mathbb{F}}_q^\times \\ b^{q+1}=1\neq b^2}} D(b, (1)) \circ D(1, (n-2)).$$

In particular, if there is a composition factor U of the kS-module $\mathfrak{D} \otimes \mathfrak{D}$ with $U^R = 0$, then $n = 3$ and U affords the Brauer character $D(1, (1^3))$. Furthermore, the only composition factors of $\mathfrak{D} \otimes \mathfrak{D}$ that are not of p-defect zero are the ones with Brauer character $1_S = D(1, (n))$, $\delta = D(1, (n-1, 1))$, and $D(1, (n-2, 1^2))$.

Proof. Let us denote by χ° the restriction of any character χ of G_n to the set of p'-elements of G_n. Then

$$\delta^2 = (\rho^\circ - 2\cdot 1_{G_n})^2 = (\rho^\circ)^2 - 4(\chi^{(n-1,1)})^\circ,$$

and we can apply Lemma 6.2. Since $p = (q^n - 1)/(q-1)$ (or more generally, if p is a primitive prime divisor of $q^n - 1$), all complex characters in the decomposition for ρ^2 in Lemma 6.2 are of p-defect 0, except for $\chi^{(n)}$, $\chi^{(n-1,1)}$, and $\chi^{(n-2,1^2)}$. Furthermore, $(\chi^{(n-2,1^2)})^\circ = D(1, (n-1, 1)) + D(1, (n-2, 1^2))$ [Guralnick and Tiep 1999, Proposition 3.1 and §4]; in particular,

$$D(1, (n-2, 1^2))(1) = \frac{(q^n - q)(q^n - 2q^2 + 1)}{(q-1)(q^2-1)} + 1.$$

Since $G_n = S \times \mathbf{Z}(G_n)$, we arrive at the desired decomposition of δ^2. Also, the degree of any irreducible constituent ψ of δ^2 listed above is not divisible by $|R| - 1 = q^{n-1} - 1$, unless $n = 3$ and $\psi = D(1, (1^3))$, whence ψ_R must contain 1_R since L acts transitively on $\mathrm{Irr}(R) \setminus \{1_R\}$. In the exceptional case, ψ_R does not contain 1_R, as one can see by direct computation (or by using [Kleshchev and Tiep 2010, Theorem 5.4]). □

Corollary 6.4. *Assume that $p = (q^n - 1)/(q-1)$ and $n \geq 5$. Then $S = \mathrm{SL}_n(q)$ is weakly adequate on $\mathfrak{D}$.*

Proof. By Lemma 6.1 and the Artin–Wedderburn theorem applied to Q, $\mathcal{M}$ contains the subspace $\mathcal{A} := (A \otimes A) \oplus (B \otimes B)$ of $\mathfrak{D} \otimes \mathfrak{D} = \mathrm{End}(\mathfrak{D})$, with A affording α and B affording β. Thus, the complement to $\mathcal{A}$ in $\mathrm{End}(V)$ affords the Q-character $\Delta := 2\alpha\beta$. It follows by Lemma 6.1 that Δ_R does *not* contain 1_R, whence R does not have any nonzero fixed point while acting on this complement. The same must be true for the quotient $\mathrm{End}(V)/\mathcal{M}$, which is a semisimple Q-module. Since $n > 3$, by Lemma 6.3 this can happen only when $\mathcal{M} = \mathrm{End}(V)$. □

Next we will extend the result of Corollary 6.4 to the case $n = 3$.

Proposition 6.5. *Assume that* $p = (q^3 - 1)/(q - 1)$. *Then* $S = \mathrm{SL}_3(q)$ *is weakly adequate on* $\mathcal{D}$.

Proof. Note that δ is invariant under the graph automorphism τ of S, which interchanges the two conjugacy classes of the maximal parabolic subgroup

$$Q = RL = \mathrm{Stab}_S(\mathcal{U}) = \mathrm{Stab}_S(\langle e_1, e_2\rangle_{\mathbb{F}_q})$$

and its opposite

$$Q^\sharp = R^\sharp L^\sharp = \mathrm{Stab}_S(\langle e_1\rangle_{\mathbb{F}_q}).$$

Hence Lemma 6.1 also applies to $Q^\sharp$. To simplify the notation, we will drop the subscript $\mathbb{F}_q$ in various spans $\langle \cdot \rangle_{\mathbb{F}_q}$ in this proof.

First we will construct the Q-submodules $\mathcal{A}$, $\mathcal{B}$ affording the character α and β in $\mathcal{D}$. Clearly, R has $q + 1$ fixed points in $\mathbb{P}\mathcal{U}$ and one orbit of length q^2,

$$\mathcal{O} := \{\langle e_3 + y\rangle : y \in \mathcal{U}\},$$

on $\Omega = \mathbb{P}\mathcal{N}$. Denoting $\mathcal{I} := \left\langle \sum_{\omega\in\mathbb{P}\mathcal{N}} \omega\right\rangle_k$, we can now decompose $\mathcal{D} = \mathcal{A} \oplus \mathcal{B}$ as Q-modules, where

$$\mathcal{A} := [\mathcal{D}, R] = \left(\left\{\sum_{y\in\mathcal{U}} a_y\langle e_3 + y\rangle \;:\; a_y \in k, \sum_{y\in\mathcal{U}} a_y = 0\right\} \oplus \mathcal{I}\right) / \mathcal{I},$$

$$\mathcal{B} := \boldsymbol{C}_{\mathcal{D}}(R) = \left(\left\{\sum_{\omega\in\mathbb{P}\mathcal{U}} b_\omega\omega \;:\; b_\omega \in k, \sum_{\omega\in\mathbb{P}\mathcal{U}} b_\omega = 0\right\} \oplus \mathcal{I}\right) / \mathcal{I}.$$

Next, $R^\sharp$ has 1 fixed point $\langle e_1\rangle$ and $q + 1$ orbits of length q,

$$\mathcal{O}_\infty := \mathbb{P}\mathcal{U} \setminus \{\langle e_1\rangle\}, \quad \mathcal{O}_c := \{\langle e_3 + ce_2 + de_1\rangle : d \in \mathbb{F}_q\},\ c \in \mathbb{F}_q,$$

on $\mathbb{P}\mathcal{N}$. Then we can again decompose $\mathcal{D} = \mathcal{A}^\sharp \oplus \mathcal{B}^\sharp$ as $Q^\sharp$-modules, where $\mathcal{A}^\sharp = [\mathcal{D}, R^\sharp]$ and $\mathcal{B}^\sharp = \boldsymbol{C}_{\mathcal{D}}(R^\sharp)$. Note that $\mathcal{O} = \mathbb{P}\mathcal{N} \setminus \mathbb{P}\mathcal{U} = \bigcup_{c\in\mathbb{F}_q} \mathcal{O}_c$. Hence, the $q(q - 1)$ vectors

$$v_{c,d} = \langle e_3 + ce_2 + de_1\rangle - \langle e_3 + ce_2\rangle, \quad c \in \mathbb{F}_q,\ d \in \mathbb{F}_q^\times$$

belong to $\mathcal{A} \cap \mathcal{A}^\sharp$, and similarly the $q - 1$ vectors

$$u_a = \langle e_2 + ae_1\rangle - \langle e_2\rangle, \quad a \in \mathbb{F}_q^\times$$

belong to $\mathscr{B}\cap\mathscr{A}^\sharp$, and they are linearly independent. Thus

$$u_a\otimes v_{c,d}\in(\mathscr{A}^\sharp\otimes\mathscr{A}^\sharp)\cap(\mathscr{B}\otimes\mathscr{A})\quad\text{and}\quad v_{c,d}\otimes u_a\in(\mathscr{A}^\sharp\otimes\mathscr{A}^\sharp)\cap(\mathscr{A}\otimes\mathscr{B}),$$

and so both $(\mathscr{A}^\sharp\otimes\mathscr{A}^\sharp)\cap(\mathscr{B}\otimes\mathscr{A})$ and $(\mathscr{A}^\sharp\otimes\mathscr{A}^\sharp)\cap(\mathscr{A}\otimes\mathscr{B})$ have dimension at least $q(q-1)^2$. As a consequence,

$$\dim((\mathscr{A}^\sharp\otimes\mathscr{A}^\sharp)\cap(\mathscr{A}\otimes\mathscr{B}\oplus\mathscr{B}\otimes\mathscr{A}))\geq 2q(q-1)^2. \tag{6-5}$$

Since $\mathscr{D}$ is self-dual, it supports a nondegenerate S-invariant symmetric bilinear form $(\cdot,\cdot)$, with respect to which $\mathscr{A}$ and $\mathscr{B}$ are orthogonal, as are $\mathscr{A}^\sharp$ and $\mathscr{B}^\sharp$. As usual, we can now identify $\mathscr{D}\otimes\mathscr{D}$ with $\operatorname{End}(\mathscr{D})$ by sending $u\otimes v\in\mathscr{D}\otimes\mathscr{D}$ to

$$f_{u,v}:x\mapsto(x,u)v$$

for all $x\in\mathscr{D}$. Furthermore, in the proof of Corollary 6.4, we have already mentioned that $\mathscr{M}$ contains the subspaces $\operatorname{End}(\mathscr{A})\oplus\operatorname{End}(\mathscr{B})$ (arguing with Q) and $\operatorname{End}(\mathscr{A}^\sharp)$ (arguing with $Q^\sharp$). It now follows from (6-5) that

$$\dim(\operatorname{End}(\mathscr{A}^\sharp)\cap(\operatorname{Hom}(\mathscr{A},\mathscr{B})\oplus\operatorname{Hom}(\mathscr{B},\mathscr{A})))\geq 2q(q-1)^2.$$

Hence for $q\geq 5$ we have that

$$\begin{aligned}\dim\operatorname{End}(\mathscr{D})-\dim\mathscr{M}&\leq(q^2+q-1)^2-(q^2-1)^2-q^2-2q(q-1)^2\\&=4q(q-1)<(q-1)(q^2-1)=\dim D(1,(1^3)).\end{aligned}$$

On the other hand, Lemma 6.3 and the proof of Corollary 6.4 show that the only S-composition factor of $\operatorname{End}(\mathscr{D})/\mathscr{M}$ (if any) is $D(1,(1^3))$. Hence, we conclude that $\mathscr{M}=\operatorname{End}(V)$ if $q\geq 5$. Since $p=(q^3-1)/(q-1)$, in the remaining cases we have $q=2,3$. The case $q=2$ is already handled before as $S\cong\mathrm{PSL}_2(7)$, and the case $q=3$ has been checked with a computer by F. Lübeck. □

Now we can prove the weak adequacy of G on V in the case the G^+-module is homogeneous.

Proposition 6.6. *Assume that $t=1$, i.e., the G^+-module V is homogeneous in the case $(p,H,\dim W)=((q^n-1)/(q-1),\mathrm{SL}_n(q),p-2)$ of Theorem 5.7. Then (G,V) is weakly adequate.*

Proof. Since $V|_{G^+}=eW$, by Theorem 2.4 we have that $G^+=S=\mathrm{SL}_n(q)$. Recall that $\gcd(n,q-1)=1$ and $q=q_0^f$, where q_0 is a prime and f is odd; in particular, $\operatorname{Out}S\cong C_{2f}$ is cyclic. It follows that $L:=C\times S\lhd G=\langle L,\tau\rangle$ for some $\tau\in G$, and $C:=\boldsymbol{C}_G(S)$ is a p'-group. Let Ψ denote the corresponding representation of S on W and Φ denote the corresponding representation of G on V. Then, by Corollary 6.4 and Proposition 6.5, we have that

$$\langle\Psi(y):y\in S,\ y\text{ semisimple}\rangle_k=\operatorname{End}(W).$$

First we consider the case where V_L is irreducible. Then $V \cong U \otimes W$, where U is an irreducible kC-module and C acts trivially on W. Let Θ denote the corresponding representation of C on U. By the Artin–Wedderburn theorem, $\langle \Theta(x) : x \in C\rangle_k = \mathrm{End}(U)$. Since $\Phi(xy) = \Theta(x) \otimes \Psi(y)$ for $x \in C$, $y \in S$, and since C is a p'-group, we conclude that $\mathcal{M}$ contains $X \otimes Y$ for all $X \in \mathrm{End}(U)$ and $Y \in \mathrm{End}(W)$, i.e., $\mathcal{M} = \mathrm{End}(V)$.

Assume now that V_L is reducible. Note that V_L is semisimple and multiplicity-free, as G/L is cyclic. Since W is τ-invariant, it follows that

$$V_L = V_1 \oplus V_2 \oplus \cdots \oplus V_s \cong (U_1 \oplus U_2 \oplus \cdots \oplus U_s) \otimes W,$$

where $V_i = U_i \otimes W$ for some pairwise nonisomorphic irreducible kC-modules $U_1, \ldots, U_s$, $\langle\tau\rangle$ acts transitively on the set of isomorphism classes of $U_1, \ldots, U_s$, C acts trivially on W as before, and $\Phi(\tau)$ permutes the summands $V_1, \ldots, V_s$ transitively. Let Θ_i denote the corresponding representation of C on U_i, and let Θ denote the corresponding representation of C on $U := U_1 \oplus \cdots \oplus U_s$. Since $U_i \not\cong U_j$ for $i \neq j$, by the Artin–Wedderburn theorem, $\langle \Theta(x) : x \in C\rangle_k = \mathrm{End}(U_1) \oplus \cdots \oplus \mathrm{End}(U_s)$. It follows as above that $\mathcal{M}$ contains $X \otimes Y$ for all $Y \in \mathrm{End}(W)$ and all $X \in \mathrm{End}(U_i)$ (viewing X as an element of $\mathrm{End}(U)$ by letting it act as zero on U_j for all $j \neq i$). In other words, $\mathcal{M}$ contains the subspace $\mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_s)$ of $\mathrm{End}(V)$.

It remains to show that $\mathcal{M}$ contains $\mathrm{Hom}(V_i, V_j)$ for any $i \neq j$. Since $\Phi(\tau)$ permutes the summands $V_1, \ldots, V_s$ transitively, we can find $\sigma \in \langle\tau\rangle \setminus CS$ such that $\Phi(\sigma)$ sends V_i to V_j and such that σ induces a nontrivial outer automorphism of S. Observe that the condition $p = (q^n - 1)/(q - 1)$ implies that all elements in the coset $S\sigma$ are p'-elements. (Indeed, assume that $x\sigma$ has order divisible by p for some $x \in S$. Then some p'-power g of $x\sigma$ is a p-element in S. It follows that σ preserves the conjugacy class g^S, which is impossible by inspecting the eigenvalues of g.) So all elements in $L\sigma$ are p'-elements. Hence $\mathcal{M}$ also contains the subspace

$$\mathcal{A} := \langle \Phi(h\sigma) : h \in L\rangle_k = \langle \Phi(h) : h \in L\rangle_k \cdot \Phi(\sigma).$$

Again, by the Artin–Wedderburn theorem,

$$\langle \Phi(h) : h \in L\rangle_k = \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_s).$$

Since $\Phi(\sigma)$ sends V_i (isomorphically) to V_j, we conclude that

$$\mathcal{A} \supset \mathrm{End}(V_j, V_j)\Phi(\sigma) = \mathrm{Hom}(V_i, V_j),$$

and so $\mathcal{M} = \mathrm{End}(V)$. $\square$

Next we consider the subgroup

$$Q' = R'L' = \mathrm{Stab}_S(\langle e_n\rangle_{\mathbb{F}_q}, \langle e_2, \ldots, e_n\rangle_{\mathbb{F}_q}),$$

where R' is a q_0-group of special type of order q^{2n-3} and $L' \cong \mathrm{GL}_{n-2}(q) \times \mathrm{GL}_1(q)$. Note that the graph automorphism $x \mapsto {}^t x^{-1}$ of S sends Q' to $(Q')^g$, where $g \in S$ sends e_1 to e_n, e_n to $-e_1$, and fixes all other e_i. Since the S-conjugacy class of the p'-group Q' is fixed by all field automorphisms, it is $\mathrm{Aut}(S)$-invariant. Also, Q' is just the normalizer in S of the root subgroup $Z' := \mathbf{Z}(R') = [R', R']$ (of order q), whence $N_S(Q') = Q'$.

Lemma 6.7. *In the above notation, $\delta_{Q'} = \alpha' + \beta'_1 + \beta'_2 + \gamma' + 1_{Q'}$, where $\alpha' \in \mathrm{Irr}(Q')$ has degree $q^{n-2}(q-1)$, $\beta'_1, \beta'_2 \in \mathrm{Irr}(Q')$ have degree $q^{n-2}-1$, $\gamma' \in \mathrm{Irr}(Q')$ has degree $(q^{n-2}-q)/(q-1)$ if $n > 3$ and is zero if $n = 3$, and*

$$\alpha'_{Z'} = q^{n-2} \sum_{1_{Z'} \neq \lambda \in \mathrm{Irr}(Z')} \lambda, \qquad Z' \le \mathrm{Ker}(\beta'_1) \cap \mathrm{Ker}(\beta'_2) \cap \mathrm{Ker}(\gamma').$$

Proof. Note that all nontrivial elements in Z' are L'-conjugate to a fixed transvection $t \in Z'$, and $\delta(t) = \rho(t) - 2 = (q^{n-1}-q)/(q-1) - 1$. It follows that

$$\delta_{Z'} = q^{n-2} \sum_{1_{R'} \neq \lambda \in \mathrm{Irr}(Z')} \lambda + \left(2(q^{n-2}-1) + \frac{q^{n-2}-q}{q-1} + 1\right) \cdot 1_{Z'}.$$

Since R' is of special type, it also follows that $[\mathcal{D}, Z']$ gives rise to an irreducible Q'-module of dimension $q^{n-2}(q-1)$, with character α'. Now we can write $R'/Z' = (R'_1/Z') \times (R'_2/Z')$ as a direct product of two L'-invariant subgroups. Next, Q' acts on the subset Ω' of Ω consisting of all 1-spaces of $\langle e_2, \dots, e_n \rangle_{\mathbb{F}_q}$ (with kernel containing R'_1), with two orbits. Arguing as in the proof of Lemma 6.1, we see that this permutation action affords the Q'-character $\beta'_2 + \gamma' + 2 \cdot 1_{Q'}$, where the irreducible characters β'_2 and γ' (if $n > 3$; $\gamma' = 0$ if $n = 3$) have the indicated degrees. In general, Q' has three orbits on Ω, whence $1_{Q'}$ enters $\delta_{Q'}$. Also, note that t has an S-conjugate $t' \in R'_1 \setminus Z'$ and $\alpha'(t') = 0$. So if we set

$$\beta'_1(1) := \delta_{Q'} - (\alpha' + \beta'_2 + \gamma' + 1_{Q'}),$$

then we see that $\beta'_1 = \beta'_1(t) = q^{n-2} - 1$ and $\beta'_1(t') = -1$. Since L' acts transitively on the nontrivial elements of R'_1/Z', we conclude by Clifford's theorem that $\beta'_1 \in \mathrm{Irr}(Q')$. $\square$

As mentioned above, $S = \mathrm{SL}_n(q)$ has a unique irreducible kS-module $\mathcal{D}$ of dimension $p-2$. It follows by Theorem 2.4 that in the situation (i) of Theorem 5.7,

$$G^+ = S_1 \times \cdots \times S_t,$$

with $S_i \cong S$, and G^+ acts on W_i with kernel $K_i := \prod_{j \neq i} S_j$. Now, as G^+-modules, we have that

$$\mathcal{E} := \mathrm{End}(V) \cong \bigoplus_{1 \le i,j \le t} V_i^* \otimes V_j \cong e^2 \bigoplus_{1 \le i,j \le t} W_i^* \otimes W_j,$$

where $V_i^* \otimes V_i \cong \mathrm{End}(V_i)$ is acted on trivially by K_i, whereas $W_i^* \otimes W_j$ is an *irreducible* kG^+-module with kernel $K_i \cap K_j$ for $i \neq j$. It follows that the two G^+-submodules

$$\mathscr{E}_1 := \bigoplus_{1 \le i \le t} V_i^* \otimes V_i, \quad \mathscr{E}_2 := \bigoplus_{1 \le i \neq j \le t} V_i^* \otimes V_j$$

of $\mathrm{End}(V)$ share no common composition factor.

Now we can prove the main result of this section:

Theorem 6.8. *Suppose we are in the case* (i) *of Theorem 5.7, i.e.*, $(p, H, \dim W) = ((q^n-1)/(q-1), \mathrm{SL}_n(q), p-2)$. *Then* (G, V) *is weakly adequate.*

Proof. (a) Consider the subgroup

$$Q'^t = Q' \times \cdots \times Q' = Q'_1 \times \cdots \times Q'_t < S_1 \times \cdots \times S_t$$

of G^+. By Lemma 6.7 and the discussion preceding it, Q'^t satisfies the hypotheses of Lemma 5.4, with A_i affording the Q'-character α', and $N_G(Q'^t)$ is a p'-group. Note that $A_i \not\cong A_j$ for $i \neq j$ since $K_i \cap Q'^t \neq K_j \cap Q'^t$. Also, the summands A and B of the Q'^t-module V constructed in Lemma 6.7 have no common composition factor and A is irreducible. Hence,

$$\mathscr{M} \supseteq \mathrm{End}(A) \supset e^2 \bigoplus_{1 \le i \neq j \le t} A_i^* \otimes A_j =: \mathscr{A}$$

by the Artin–Wedderburn theorem. Note that $\mathscr{A} \subset \mathscr{E}_2$. Furthermore, if Δ is the Q'^t-character of the complement of $\mathscr{A}$ in $\mathscr{E}_2$, then, by Lemma 6.7, each irreducible constituent of Δ, when restricted to

$$Z'^t = Z' \times \cdots \times Z' = Z'_1 \times \cdots \times Z'_t,$$

is trivial on (at least) *all but one* Z'_i. The same is true for the G^+-module $\mathscr{E}/(\mathscr{E}_1 + \mathscr{M})$. On the other hand, as mentioned above, all G^+-composition factors of $\mathscr{E}/\mathscr{E}_1 \cong \mathscr{E}_2$ are of the form $W_i^* \otimes W_j$ with $i \neq j$. The Brauer character of any such $W_i^* \otimes W_j$, being restricted to $S_i \times S_j$, is $\delta \otimes \delta$, and so it contains the $Q'_i \times Q'_j$-irreducible constituent $\alpha' \otimes \alpha'$ which is nontrivial at both Z'_i and Z'_j by Lemma 6.7. It follows that $\mathscr{E}_1 + \mathscr{M} = \mathscr{E}$, i.e., $\mathscr{M}$ surjects onto $\mathscr{E}_2$. Applying Lemma 5.5 to the subgroup $G^+ \le G$, we conclude that $\mathscr{M} \supseteq \mathscr{E}_2$.

(b) We already mentioned that the G^+-modules $\mathscr{E}_1 = \bigoplus_{i=1}^t \mathscr{E}_{1i}$ and $\mathscr{E}_2$ share no common composition factor; in particular, k is not a composition factor of $\mathscr{E}_2$. Furthermore, since $\prod_{j \neq i} S_j$ acts trivially on V_i, we see that for distinct $i \neq j$ the only common G^+-composition factor that $\mathscr{E}_{1i}$ and $\mathscr{E}_{1j}$ can share is the principal character 1_{G^+}. Recall that $\mathscr{E}_{1i} \cong \mathfrak{D} \otimes \mathfrak{D}$ as S_i-modules. The irreducibility of G on V implies that $G_i := \mathrm{Stab}_G(V_i)$ acts irreducibly on V_i, and certainly $G^+ \lhd G_i$ acts homogeneously on V_i. By Proposition 6.6 applied to G_i, $\mathscr{E}_{1i}$ is a subquotient of $\mathscr{M}$. We have therefore shown that all *nontrivial* G^+-composition factors of $\mathscr{E} = \mathrm{End}(V)$ also occur in $\mathscr{M}$ with the same multiplicity, and so all the composition factors of the G^+-module $\mathscr{E}/\mathscr{M}$ (if any) are trivial. Applying Lemma 5.4 to the subgroup $Q'^t < G^+$, we conclude that $\mathscr{M} = \mathscr{E}$. □

Finally we can prove:

Theorem 6.9. *Suppose* (G, V) *is as in the case* (i) *of Theorem 2.4. Then* (G, V) *is weakly adequate.*

Proof. In view of Theorems 5.7, 6.8, and Propositions 5.8, 5.9, 5.11, we need to handle the case $(p, H, \dim W) = (7, 6{\cdot}\mathrm{PSL}_3(4), 6)$. In this case, L_i acts on each W_j either trivially or as $H_j \cong 6\cdot\mathrm{PSL}_3(4)$. It follows by the faithfulness of G on V that $\boldsymbol{Z}(L_i)$ has exponent 6, and so L_i is (isomorphic to) either $X := (2\times 2)\cdot 3\cdot\mathrm{PSL}_3(4)$ or a quotient $6\cdot\mathrm{PSL}_3(4)$ of X. We can also find k_i such that the kernel K_i of $G^+ = L_1 * \cdots * L_n$ acting on W_i contains $\prod_{j\neq k_i} L_j$. Without loss we may assume $k_1 = 1$.

(a) We claim that L_1 contains a subgroup $Q_1 = Z_1 \times \mathsf{A}_5$, whose conjugacy class is $\mathrm{Aut}(L_1)$-invariant (with $Z_1 := \boldsymbol{Z}(L_1)$). For this purpose, without loss of generality we may assume that $L_1 \cong X$. We consider a Levi subgroup $C_3 \times \mathrm{SL}_2(4) \cong C_3 \times \mathsf{A}_5$ of $\mathrm{SL}_3(4)$ which acts semisimply on the natural module $\mathbb{F}_4^3$. Then its conjugacy class in $\mathrm{SL}_3(4)$ is fixed by all the outer automorphisms of $\mathrm{SL}_3(4)$. Consider a faithful representation $\Lambda : X \to \mathrm{GL}_{18}(\mathbb{C})$ which is the sum of three irreducible representations, on which X acts with different kernels $\cong C_2$, and let Y be the full inverse image of A_5 in X. Note that involutions in $\mathrm{PSL}_3(4)$ lift to involutions in $6\cdot\mathrm{PSL}_3(4)$, whereas involutions in A_5 lift to elements of order 4 in $2\cdot\mathsf{A}_5$ [Conway et al. 1985]. It follows that $\Lambda(x)$ has order 2 for the inverse image $x \in X$ of any involution in A_5, and so $|x| = 2$. Hence $Y \cong (2\times 2)\times\mathsf{A}_5$, and the claim follows.

Defining $Q_i < L_i$ similarly, we see that

$$Q = Q_1 * Q_2 * \cdots * Q_n$$

satisfies condition (i) of Lemma 5.3. Since Q_1 is self-normalizing in L_1, we see that $\boldsymbol{N}_{G^+}(Q) = Q$ and that $N := \boldsymbol{N}_G(Q)$ is a p'-group.

We will now inflate Brauer characters of L_1 acting on W_1 to X and then replace L_1 by X. According to [Jansen et al. 1995], L_1 has exactly six irreducible 7-Brauer characters φ_s of degree 6, $1 \le s \le 6$, lying above the six distinct characters λ_s of Z_1 (with kernels the three distinct central subgroups of order 2), and $(\varphi_s)_{Q_1} = \lambda_s \otimes (\alpha + \beta)$, where $\alpha \neq \beta \in \mathrm{Irr}(\mathsf{A}_5)$, and either

$$\{\alpha, \beta\} = \{1a, 5a\} \tag{6-6}$$

or

$$\{\alpha, \beta\} = \{3a, 3b\}, \tag{6-7}$$

depending on whether φ_s takes value 2 or -2 on involutions in A_5. (Here we adopt the notation that $\mathrm{Irr}(\mathsf{A}_5) = \{1a, 3a, 3b, 4a, 5a\}$.) In either case, we have that $(W_1)_Q = A_1 \oplus B_1$, where the Q-modules A_1 and B_1 are irreducible and nonisomorphic. As shown in the proof of Lemma 5.2, $NG^+ = G$ and $N_1G^+ = G_1 := \mathrm{Stab}_G(V_1)$ for $N_1 := \boldsymbol{N}_{G_1}(Q)$. So we fix a decomposition $G = \bigcup_{i=1}^t g_i G_1$ with $g_i \in N$, $g_1 = 1$, and define $A_i := g_i(A_1) \subset W_i$ and $B_i := g_i(B_1) \subset W_i$. In particular, either (6-6) holds for all $(W_i)_Q$, or (6-7) holds for all $(W_i)_Q$.

We claim that Q also satisfies condition (ii) of Lemma 5.3. Indeed, assume that $W_i \not\cong W_j$. Now if $k_i \neq k_j$, then $L_{k_i} > Q_{k_i}$ acts trivially on W_j, but $\boldsymbol{Z}(Q_{k_i}) = Z_{k_i}$ acts nontrivially by scalars on W_i. In the case $k_i = k_j$, we may assume that $K_i \geq \prod_{s>1} L_s$, and so W_i and W_j afford the L_1-characters $\varphi, \varphi' \in \{\varphi_1, \dots, \varphi_6\}$, lying above *different* characters λ, λ' of Z_1. Now $\boldsymbol{Z}(Q_1) = Z_1$ acts on W_i and W_j by scalars but via different characters λ, λ', so we are done.

(b) Suppose we are in the case of (6-7) and, moreover, $G_1 = \mathrm{Stab}_G(V_1)$ interchanges the two classes $5A = x^{L_1}$ and $5B = (x^2)^{L_1}$ of elements of order 5 of $L_1 = 6_1 \cdot \mathrm{PSL}_3(4)$. Certainly, we can choose $x \in \mathsf{A}_5 < Q_1$. Since $N_1G^+ = G_1$, we can find some element $g \in N_1$ that interchanges the classes $5A$ and $5B$. In this case g also interchanges the characters $\alpha = 3a$ and $\beta = 3b$ of A_5, but fixes W_1 and the central character $\lambda \in \{\lambda_1, \dots, \lambda_6\}$ of Z_1. It follows that $\{A_1, \dots, B_t\}$ forms a single N-orbit, and so by Lemma 5.3 the p'-group N acts irreducibly on V, and we are done.

(c) From now on we may assume that we are *not* in the case considered in (b). We claim that $\{A_1, \dots, A_t\}$ and $\{B_1, \dots, B_t\}$ are two distinct N-orbits. Assume the contrary. Then by the construction of A_i and B_j there must be some $h \in N$ such that $B_1 \cong A_1^h$. This is clearly impossible in the case of (6-6). In the case of (6-7), $h \in G_1$ and furthermore h fuses the two classes of elements of order 5 in A_5. Hence $h \in G_1$ fuses the classes $5A$ and $5B$ of L_1, contrary to our assumption.

Now we can apply Lemma 5.3 to see that $V_N = A \oplus B$ and so

$$\mathcal{M} \supseteq \mathrm{End}(A) \oplus \mathrm{End}(B) \tag{6-8}$$

by the Artin–Wedderburn theorem. We also decompose $\mathrm{End}(V) = \mathscr{E}_1 \oplus \mathscr{E}_2$ as G^+-modules, and note that the Q-modules

$$\mathscr{E}_1 := \bigoplus_{i=1}^{t} \mathrm{End}(V_i) \cong e^2 \bigoplus_{i=1}^{t} W_i^* \otimes W_i,$$

$$\mathscr{E}_2 := \bigoplus_{1 \leq i \neq j \leq t} \mathrm{Hom}(V_i, V_j) \cong e^2 \bigoplus_{1 \leq i \neq j \leq t} W_i^* \otimes W_j$$

share no common composition factor. Indeed, the p'-group $\boldsymbol{Z}(G^+) = Z_1 * \cdots * Z_n \leq \boldsymbol{Z}(Q)$ acts trivially on $\mathscr{E}_1$ and nontrivially by scalars on each $W_i^* \otimes W_j$ when $i \neq j$.

Moreover, if $k_i \neq k_j$, say $K_i \geq \prod_{s\neq 1} L_s$ and $K_j \geq \prod_{s\neq 2} L_s$, then $W_i^* \otimes W_j$ and $W_j^* \otimes W_i$ are irreducible over $L_1 \times L_2$ (and are acted on trivially by $\prod_{s>2} L_s$), with nontrivial central characters $\nu_1^{-1} \otimes \nu_2$ and $\nu_1 \otimes \nu_2^{-1}$ over $Z_1 * Z_2$, where $\nu_1, \nu_2 \in \{\lambda_1, \dots, \lambda_6\}$ have order 6. If $W_i \not\cong W_j$ but $k_i = k_j$, say $k_i = k_j = 1$, then W_i and W_j afford the L_1-characters $\varphi \neq \varphi'$ lying above different characters $\lambda \neq \lambda'$ of Z_1. We distinguish different scenarios for λ and λ':

(c1) λ and λ' coincide at $\boldsymbol{O}_2(Z_1)$ (then they must be different at $\boldsymbol{O}_3(Z_1)$, and in fact $\lambda' = \lambda^{-1}$). Here, $W_i^* \otimes W_j$ and $W_j^* \otimes W_i$ are reducible over L_1 (and are acted on trivially by $\prod_{s>1} L_s$), with distinct nontrivial central characters λ^{-2} and λ^2 over Z_1. Furthermore, the L_1-character of $W_i^* \otimes W_j$ is $\gamma_3 + \delta_3$, where $\gamma_3 \in \mathrm{IBr}(L_1)$ has degree 15, $\delta_3 \in \mathrm{IBr}(L_1)$ has degree 21, and

$$(\gamma_3)_{\mathsf{A}_5} = 3a + 3b + 4a + 5a, \quad (\delta_3)_{\mathsf{A}_5} = 2 \cdot 1a + 4a + 3 \cdot 5a. \tag{6-9}$$

(c2) λ and λ' coincide at $\boldsymbol{O}_3(Z_1)$ (then they must be different at $\boldsymbol{O}_2(Z_1)$). Here, $W_i^* \otimes W_j$ and $W_j^* \otimes W_i$ again are reducible over L_1 (and are acted on trivially by $\prod_{s>1} L_s$), with the same nontrivial central character $\lambda^{-1}\lambda'$ over Z_1. Furthermore, the L_1-character of $W_i^* \otimes W_j$ is $\gamma_2 + \delta_2$, where $\gamma_2 \in \mathrm{IBr}(L_1)$ has degree 10, $\delta_2 \in \mathrm{IBr}(L_1)$ has degree 26, and

$$(\gamma_2)_{\mathsf{A}_5} = 1a + 4a + 5a, \quad (\delta_2)_{\mathsf{A}_5} = 1a + 3a + 3b + 4a + 3 \cdot 5a. \tag{6-10}$$

Here we have used the fact that the character of $W_i^* \otimes W_j$ takes value $(\pm 2)^2 = 4$ at involutions in A_5.

(c3) λ and λ' differ at both $\boldsymbol{O}_2(Z_1)$ and $\boldsymbol{O}_3(Z_1)$. Here, $W_i^* \otimes W_j$ and $W_j^* \otimes W_i$ are irreducible over L_1 (and are acted on trivially by $\prod_{s>1} L_s$), with distinct nontrivial central characters $\lambda^{-1}\lambda'$ and $\lambda(\lambda')^{-1}$ over Z_1. Furthermore, the L_1-character of $W_i^* \otimes W_j$ is γ_6, where $\gamma_6 \in \mathrm{IBr}(L_1)$ has degree 36 and

$$(\gamma_6)_{\mathsf{A}_5} = 2 \cdot 1a + 3a + 3b + 2 \cdot 4a + 4 \cdot 5a. \tag{6-11}$$

(d) According to (6-8), $\mathcal{M}$ contains the subspace $\mathcal{A} := \mathrm{End}(C_1) \oplus \mathrm{End}(D_1)$ of $\mathrm{End}(V_1)$, which affords the character $e^2(\alpha^2 + \beta^2)$ of $\mathsf{A}_5 < Q_1$ (and is acted on trivially by Z_1). Note that the L_1-character of $\mathrm{End}(W_1)$ is $\varphi_i \overline{\varphi}_i = 1_{L_1} + \psi$, where $\psi \in \mathrm{IBr}(L_1)$ of degree 35 and

$$\psi_{\mathsf{A}_5} = 1a + 3a + 3b + 2 \cdot 4a + 4 \cdot 5a.$$

On the other hand, the A_5-character of the complement to $\mathcal{A}$ in $\mathrm{End}(V_1)$ is

$$e^2(\alpha + \beta)^2 - e^2(\alpha^2 + \beta^2) = 2e^2\alpha\beta,$$

which is $2e^2 \cdot 5a$ in the case of (6-6) and $2e^2(4a+5a)$ in the case of (6-7); in particular, it does *not* contain $1a$. It follows by the observation right after (6-8) and Lemma 5.5 that $\mathcal{M} \supseteq \mathrm{End}(V_1)$ and so $\mathcal{M} \supseteq \mathscr{E}_1$.

(e) By (6-8), $\mathcal{M}$ contains the subspace $\mathscr{B}_{ij} := \mathrm{Hom}(C_i, C_j) \oplus \mathrm{Hom}(D_i, D_j)$ of $\mathscr{E}_{ij} := \mathrm{Hom}(V_i, V_j)$ whenever $i \neq j$ (recall that $(C_i)_Q \cong eA_i$ and $(D_i)_Q \cong eB_i$). We distinguish two cases according to whether k_i and k_j are equal or not.

First suppose that $k_i \neq k_j$, say $k_i = 1$ and $k_j = 2$. Then $\mathscr{E}_{ij}$ affords the $L_1 \times L_2$-character $e^2\overline{\theta}_1 \otimes \theta_2$ (where $\theta_i \in \mathrm{IBr}(L_i)$ has degree 6) and is acted on trivially by $\prod_{s>2} L_s$. Now the $Q_1 \times Q_2$-character of the complement to $\mathscr{B}_{ij}$ in $\mathrm{Hom}(V_i, V_j)$ when restricted to the subgroup $\mathsf{A}_5 \times \mathsf{A}_5$ is

$$e^2(\alpha_1+\beta_1)\otimes(\alpha_2+\beta_2) - e^2(\alpha_1\otimes\alpha_2+\beta_1\otimes\beta_2) = e^2(\alpha_1\otimes\beta_2+\beta_1\otimes\alpha_2)$$

(where α_1, β_1 play the role of α and β for the first factor A_5 and similarly for α_2, β_2). Also, the restriction of $\overline{\theta}_1 \otimes \theta_2$ to $\mathsf{A}_5 \times \mathsf{A}_5$ always contains an irreducible constituent distinct from $\alpha_1 \otimes \beta_2$ and $\beta_1 \otimes \alpha_2$, namely $\beta_1 \otimes \beta_2$.

Assume now that $k_i = k_j = 1$. Then the A_5-character of the complement to $\mathscr{B}_{ij}$ in $\mathscr{E}_{ij}$ is

$$e^2(\alpha+\beta)^2 - e^2(\alpha^2+\beta^2) = 2e^2\alpha\beta,$$

which is $2e^2 \cdot 5a$ in the case of (6-6) and $2e^2(4a+5a)$ in the case of (6-7). On the other hand, according to (6-9)–(6-11), the restriction to A_5 of each of the irreducible constituents γ and δ of $W_i^* \otimes W_j$ always contains either $1a$ or $3a$.

Now assume that $\mathcal{M} \neq \mathrm{End}(V)$. Working modulo $\mathscr{E}_1 \subset \mathcal{M}$, we see that $\mathcal{M} \supseteq \mathscr{B} := \bigoplus_{i\neq j} \mathscr{B}_{ij}$ has a nonzero complement in $\mathscr{E}_2 = \bigoplus_{i\neq j} \mathscr{E}_{ij}$. But the above analysis shows that *any* G^+-composition factor of $\mathscr{E}_2$ contains a Q-irreducible constituent which is *not* a Q-constituent of the complement to $\mathscr{B}$ in $\mathscr{E}_2$, a contradiction. □

Proof of Theorem 1.2. (a) First we consider the case where k is algebraically closed. Assume that G^+ is p-solvable. Then G is also p-solvable. Furthermore, $\dim V/\dim W$ divides $|G/G^+|$ by [Navarro 1998, Theorem 8.30], and so $p \nmid \dim V$. So we are done by Lemma 5.1. So we may now assume that G^+ is not p-solvable, $p > \dim W > 1$, and apply Theorem 2.4 to G. Then the statement follows from Theorem 4.5 in the case that G^+ is a central product of quasisimple groups of Lie type in characteristic p (if in addition $p > 3$), and from the results of Sections 5 and 6 in the remaining cases.

Suppose that $p = 3$ and $G^+ = L_1 * \cdots * L_n$ is a central product of quasisimple groups of Lie type in characteristic p (with $\mathbf{Z}(L_i)$ a p'-group for each i; see Theorem 2.4(iii)). Write $V_{G^+} = e\bigoplus_{i=1}^t W_i$ as usual. It is well known that the only quasisimple groups of Lie type in characteristic p that have a faithful representation of degree 2 over k are $\mathrm{SL}_2(p^a)$. Since $\dim W = 2$, we must have that $L_j \cong \mathrm{SL}_2(q)$ for a power $q > 3$ of 3 for all j (as the G^+-modules W_i are G-conjugate); moreover, for each i, there is a unique k_i such that L_j acts nontrivially on W_i precisely when $j = k_i$. Note that L_i contains a unique conjugacy class of cyclic subgroups T_i of order C_{q-1}. It is straightforward to check that the restrictions of all Brauer characters $\varphi \in \mathrm{IBr}_p(L_i)$ of degree 2 to $Q_i := N_{L_i}(T_i)$ are all irreducible and pairwise distinct. Letting $Q := Q_1 * \cdots * Q_n$ and arguing as in case (b1) of the proof of Theorem 5.7, we see that Q satisfies all the hypotheses of Lemma 5.2, whence we are done.

(b) Now we consider the general case. We will view G as a subgroup of $\mathrm{GL}(V)$ and let $\mathcal{M} := \langle g : g \in G \text{ semisimple}\rangle_k$ as usual. Since the kG-module V is absolutely irreducible, the $\bar{k}G$-module $\overline{V} := V \otimes_k \bar{k}$ is irreducible, and the condition $d < p$ implies that the dimension of any irreducible G^+-submodule in $\overline{V}$ is also less than p. By the previous case, $\mathcal{M} \otimes_k \bar{k} = \mathrm{End}(\overline{V})$. It follows that $\dim_k \mathcal{M} = (\dim V)^2$ and so $\mathcal{M} = \mathrm{End}(V)$.

□

7. Extensions and self-extensions, I: Generalities

First we record a convenient criterion about self-extensions in blocks of cyclic defect:

Lemma 7.1. *Suppose that G is a finite group and that V is an irreducible $\overline{\mathbb{F}}_pG$-representation that belongs to a block of cyclic defect. Then $\mathrm{Ext}^1_G(V, V) \neq 0$ if and only if V admits at least two nonisomorphic lifts to characteristic 0. In this case, $\dim \mathrm{Ext}^1_G(V, V) = 1$.*

Proof. Let B denote the block of V. If B has defect 0, V is projective and lifts uniquely to characteristic 0. Otherwise, B is a Brauer tree algebra. Note that $\mathrm{Ext}^1_G(V, V) \neq 0$ if and only if V embeds as subrepresentation of $\mathcal{P}(V)/V$. The Brauer tree shows that this happens if and only if either (i) B has an exceptional vertex and V is the unique edge incident with it, or (ii) B does not have an exceptional vertex and V is the unique edge of the tree. In (i), each exceptional representation in B lifts V, in (ii) both ordinary representations in B lift V, and it is clear that V has at most one lift in all other cases. To verify the final claim, note that $\mathrm{Hom}(V, \mathcal{P}(V)/V) \cong \mathrm{Ext}^1_G(V, V)$, and that in a Brauer tree algebra V occurs at most once in $\mathrm{soc}(\mathcal{P}(V)/V)$. □

In fact, as pointed out to us by V. Paskunas, one direction of Lemma 7.1 holds for any finite group G: if $\mathrm{Ext}^1_G(V, V) = 0$ then V has at most one characteristic-0 lift. Indeed, if V has no self-extension, we may first realize all characteristic-0 lifts over some finite extension $\mathbb{E}$ of $\mathbb{Q}_p$, as well as V over the residue field of $\mathbb{E}$. Then the universal deformation ring R of V over the ring $\mathbb{O}_{\mathbb{E}}$ is a quotient of $\mathbb{O}_{\mathbb{E}}$. But then $|\mathrm{Hom}_{\mathbb{O}_{\mathbb{E}}\text{-alg}}(R, \mathbb{O}_{\mathbb{E}})| \leq 1$, i.e., V has at most one characteristic-0 lift.

We will frequently use the following simple observations:

Lemma 7.2. *Let V be a finite-dimensional vector space over k and $G \le \mathrm{GL}(V)$ a finite absolutely irreducible subgroup. Write $V|_{G^+} = e\bigoplus_{i=1}^t W_i$, where the G^+-modules W_i are absolutely irreducible and pairwise nonisomorphic. Suppose that $\mathrm{Ext}^1_{G^+}(W_i, W_j) = 0$ for all i, j. Then $\mathrm{Ext}^1_G(V, V) = 0$.*

Proof. Since G^+ contains a Sylow p-subgroup of G, $\mathrm{Ext}^1_G(V, V)$ embeds in

$$\mathrm{Ext}^1_{G^+}(V_{G^+}, V_{G^+}) = \mathrm{Ext}^1_{G^+}\left(e\bigoplus_{i=1}^t W_i, e\bigoplus_{i=1}^t W_i\right) \cong e^2 \bigoplus_{i,j} \mathrm{Ext}^1_{G^+}(W_i, W_j) = 0. \quad \square$$

Lemma 7.3. *Let N be a normal subgroup of a finite group X and let A and B be finite-dimensional $k(X/N)$-modules. Consider $\mathrm{Ext}^1_X(A, B)$, where we inflate A and B to kX-modules.*

(i) *If $\mathrm{Ext}^1_X(A, B) = 0$, then $\mathrm{Ext}^1_{X/N}(A, B) = 0$.*

(ii) *If $\mathrm{Ext}^1_{X/N}(A, B) = 0$ and $\boldsymbol{O}^p(N) = N$, then $\mathrm{Ext}^1_X(A, B) = 0$.*

Proof. (i) is trivial. For (ii), let V be any extension of the kX-module A by the kX-module B, and let $\Phi : X \to \mathrm{GL}(V)$ denote the corresponding representation. Since N acts trivially on A and B, we see that $\Phi(N)$ is a p-group. But $\boldsymbol{O}^p(N) = N$; hence $\Phi(N) = 1$, i.e., N acts trivially on V. Now, $V \cong A \oplus B$ as $\mathrm{Ext}^1_{X/N}(A, B) = 0$. $\square$

Next we recall Holt's inequality in cohomology [1980]:

Lemma 7.4. *Let G be a finite group, $N \lhd G$, and let V be a finite-dimensional kG-module. Then for any integer $m \ge 0$ we have*

$$\dim H^m(G, V) \le \sum_{j=0}^m \dim H^j(G/N, H^{m-j}(N, V)).$$

From now on we again assume that k is algebraically closed.

Corollary 7.5. *Let $G = G_1 \times G_2$ be a direct product of finite groups and let V_i be a nontrivial irreducible kG_i-module for $i = 1, 2$.*

(i) *If we view $V_1 \otimes V_2$ as a kG-module, then $H^1(G, V_1 \otimes V_2) = 0$.*

(ii) *If we inflate V_1 and V_2 to kG-modules, then $\mathrm{Ext}^1_G(V_1, V_2) = 0$.*

Proof. For (i), applying Lemma 7.4 to $N := G_1$ we get

$$\dim H^1(G, V) \le \dim H^0(G_2, H^1(G_1, V)) + \dim H^1(G_2, H^0(G_1, V)).$$

Now the G_1-module V is a direct sum of $\dim V_2$ copies of V_1 and V_1 is nontrivial irreducible, whence $H^0(G_1, V) = 0$. Next, $H^1(G_1, V) \cong H^1(G_1, V_1) \otimes V_2$ as G_2-modules, with G_2 acting trivially on the first tensor factor. It follows that

$$H^0(G_2, H^1(G_1, V)) \cong H^1(G_1, V_1) \otimes H^0(G_2, V_2) = 0$$

as V_2 is nontrivial irreducible, and so we are done.

Part (ii) follows from (i) since $\mathrm{Ext}^1_G(V_1, V_2) \cong H^1(G, V_1^* \otimes V_2)$ and V_1^* is a nontrivial absolutely irreducible kG_1-module. □

Corollary 7.6. *Let the finite group H be a central product of quasisimple subgroups $H = H_1 * \cdots * H_n$, where $\mathbf{Z}(H_i)$ is a p'-group for all i. For $i = 1, 2$, let W_i be a nontrivial irreducible kH-module such that the action of H on W_i induces a quasisimple subgroup of* $\mathrm{GL}(W_i)$. *Suppose that the kernels of the actions of H on W_1 and on W_2 are different. Then* $\mathrm{Ext}^1_H(W_1, W_2) = 0$.

Proof. View H as a quotient of $L := H_1 \times \cdots \times H_n$ by a central p'-subgroup and inflate W_i to a kL-module. Next, write $W_i = W_1^i \otimes \cdots \otimes W_n^i$ for some absolutely irreducible kH_j-module W_j^i, $1 \le i \le 2$, $1 \le j \le n$. Since H_j is quasisimple, if $\dim W_j^i = 1$ then H_j acts trivially on W_i. On the other hand, if $\dim W_j^i > 1$, then H_j induces a quasisimple subgroup of $\mathrm{GL}(W_j^i)$. Hence, the condition that the action of H on W_i induces a quasisimple subgroup of $\mathrm{GL}(W_i)$ implies that $\dim W_j^i > 1$ for exactly one index $j = k_i$, whence the kernel of L on W_i is

$$H_1 \times \cdots \times H_{k_i-1} \times \mathbf{C}_{H_{k_i}}(W_{k_i}^i) \times H_{k_i+1} \times \cdots \times H_n.$$

Note that the hypothesis on H_i imply that $\prod_{j \ne k_1,\, k_2} H_j$ has no nontrivial p-quotient. Hence, by Lemma 7.3 there is no loss in taking the quotient of L by $\prod_{j \ne k_1,\, k_2} H_j$. If $k_1 \ne k_2$, then we are reduced to the case where $L = H_{k_1} \times H_{k_2}$, W_1 is a nontrivial H_{k_1}-module inflated to L and W_2 is a nontrivial H_{k_2}-module inflated to L, whence we are done by Corollary 7.5(ii). Suppose now that $k_1 = k_2$, say $k_1 = k_2 = 1$ for brevity. Then we are reduced to the case where $L = H_1$ and $K_1 \ne K_2$, with $K_i = \mathbf{C}_{H_1}(W_1^i) \le \mathbf{Z}(H_1)$. By Schur's lemma, $\mathbf{Z}(H_1)$ acts on W_i by scalars and semisimply, via a linear character λ_i. Since $K_1 \ne K_2$, we see that $\lambda_1 \ne \lambda_2$. It follows (by considering $\mathbf{Z}(H_1)$-blocks, or by considering λ_i-eigenspaces for $\mathbf{Z}(H_1)$ in any extension of W_1 by W_2) that $\mathrm{Ext}^1_L(W_1, W_2) = 0$. □

More generally, we record the following consequence of the Künneth formula:

Lemma 7.7 [Benson 1998, 3.5.6]. *Let H be a finite group. Assume that H is a central product of subgroups H_i for $1 \le i \le t$ and that $\mathbf{Z}(H)$ is a p'-group. Let X and Y be irreducible kH-modules. Write $X = X_1 \otimes \cdots \otimes X_t$ and $Y = Y_1 \otimes \cdots \otimes Y_t$, where X_i and Y_i are irreducible kH_i-modules.*

(i) *If $X_i \not\cong Y_i$ for at least two i, then* $\mathrm{Ext}^1_H(X, Y) = 0$.

(ii) *If $X_1 \not\cong Y_1$ but $X_i \cong Y_i$ for $i > 1$, then* $\mathrm{Ext}^1_H(X, Y) \cong \mathrm{Ext}^1_{H_1}(X_1, Y_1)$.

(iii) *If $X_i \cong Y_i$ for all i, then* $\mathrm{Ext}^1_H(X, Y) \cong \bigoplus_i \mathrm{Ext}^1_{H_i}(X_i, Y_i)$.

We continue with several general remarks:

Lemma 7.8. *Let V be a kG-module of finite length.*

(i) *Suppose that X is a composition factor of V such that V has no indecomposable subquotient of length 2 with X as a composition factor. Then $V \cong X \oplus M$ for some submodule $M \subset X$.*

(ii) *Suppose that $\mathrm{Ext}^1_G(X, Y) = 0$ for any two composition factors X, Y of V. Then V is semisimple.*

Proof. (i) We will assume that $V \not\cong X$. Let U be a submodule of V of smallest length that has X as a composition factor. First we show that $U \cong X$. If not, then U has a composition series $U = U_0 > U_1 > \cdots > U_m = 0$ for some $m \geq 2$. Note that $U/U_1 \cong X$, as otherwise X would be a composition factor of $U_1 \subset U$, contradicting the choice of U. Now U/U_2 is a subquotient of length 2 of V with X as a quotient. By the hypothesis, $U/U_2 = U'/U_2 \oplus U''/U_2$ with $U'/U_2 \cong X$ and $U'' \supset U_2$, again contradicting the choice of U.

Now let M be a submodule of V of largest length such that $M \cap U = 0$. In particular, $V/M \supseteq (M+U)/M \cong X$. Assume furthermore that $V \neq M+U$. Then we can find a submodule $V' \subseteq V$ such that $V'/(M+U)$ is simple. Again, V'/M is a subquotient of length 2 of V with X as a submodule. So by the hypothesis, $V'/M = (M+U)/M \oplus N/M$ for some submodule $N \subseteq V$ containing M properly. But then

$$N \cap U = (N \cap (M+U)) \cap U = M \cap U = 0,$$

contrary to the choice of M. Thus $V = M \oplus U$ is decomposable.

(ii) Induction on the length of V. If V is not simple, then by (i) we have $V \cong V' \oplus V''$ for some nonzero submodules V' and V''. Now apply the induction hypothesis to V' and V''. □

Lemma 7.9. *Let V be a kG-module. Suppose that U is a composition factor of V of multiplicity 1 and that U occurs both in* $\mathrm{soc}\, V$ *and* $\mathrm{head}\, V$. *Then $V \cong U \oplus M$ for some submodule $M \subset V$.*

Proof. Let $U_1 \cong U$ be a submodule of V. Since U occurs in head V, there is $M \subset V$ such that $V/M \cong U$. Now if $M \supseteq U_1$, then U would have multiplicity ≥ 2 in V. Hence $V = U_1 \oplus M$. □

Lemma 7.10. *Let V be a kG-module of finite length. Suppose the set of isomorphism classes of composition factors of V is a disjoint union $\mathscr{X} \cup \mathscr{Y}$ of nonempty subsets such that, for any $U \in \mathscr{X}$ and $W \in \mathscr{Y}$, there is no indecomposable subquotient of length 2 of V with composition factors U and W. Then V is decomposable.*

Proof. Let X and Y denote the largest submodules of V with all composition factors belonging to $\mathscr{X}$ and $\mathscr{Y}$, respectively. By definition, $X \cap Y = 0$. We claim that $V = X \oplus Y$. If not, we can find a submodule $Z \supset X \oplus Y$ of V such that

$U := Z/(X \oplus Y)$ is a simple G-module. Suppose for instance that $U \in \mathscr{X}$. Applying Lemma 7.8(i) to the G-module Z/X and its composition factor U, we see that $Z/X \cong U \oplus Y$. This implies that Z contains a submodule T with $T/X \cong U$, contradicting the choice of X.

Now $X, Y \neq 0$ as $\mathscr{X}, \mathscr{Y} \neq \varnothing$. It follows that V is decomposable. □

Lemma 7.11. *Let V be an indecomposable kG-module.*

(i) *If the G^+-module V_{G^+} admits a composition factor L of dimension* 1, *then all composition factors of V_{G^+} belong to $B_0(G^+)$.*

(ii) *Suppose a normal p'-subgroup N of G acts by scalars on a composition factor L of the G-module V. Then N acts by scalars on V. If in addition V is faithful then $N \leq \mathbf{Z}(G)$.*

Proof. (i) Since $G^+ = \boldsymbol{O}^{p'}(G^+)$, it must act trivially on L. Let X (resp. Y) denote the largest submodule of the G^+-module V with all composition factors belonging (resp. not belonging) to $B_0(G^+)$. By their definition and the definition of G^+-blocks, $V = X \oplus Y$. Note that both X and Y are G-stable as $G^+ \lhd G$. Since V is indecomposable, we see that $Y = 0$ and $V = X$.

(ii) Note that N acts completely reducibly on V and G permutes the N-homogeneous components of V. Since V is indecomposable, it follows that this action is transitive, whence all composition factors of the N-module V are G-conjugate. But, among them, the (unique) linear composition factor of L_N is certainly G-invariant. Hence this is the unique composition factor of V_N, and so N acts by scalars on V. □

8. Indecomposable representations of $\mathrm{SL}_2(q)$

We first prove a lemma:

Lemma 8.1. *Suppose that S, T are irreducible $\mathrm{SL}_2(\mathbb{F}_q)$-representations over $\overline{\mathbb{F}}_p$ with $q = p^n$, $n \geq 2$, and E is a nonsplit extension of T by S. Then $\dim E \geq p$ and $S \not\cong T$. Moreover, if $\dim S = \dim T$ then $\dim E \geq (p^2 - 1)/2$.*

Proof. This is immediate from Corollary 4.5(a) in [Andersen et al. 1983]. □

Proposition 8.2. *Suppose that V is a reducible, self-dual, indecomposable representation of $\mathrm{SL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$, where $q = p^n$. If $\dim V < 2p - 2$, then $q = p$ and one of the following holds*:

(i) $\dim V = p$ *and* $V \cong \mathscr{P}(\mathbb{1})$.

(ii) $\dim V = p + 1$ *and V is the unique nonsplit self-extension of* $L((p-1)/2)$.

(iii) $\dim V = p - 1$ *and V is the unique nonsplit self-extension of* $L((p-3)/2)$.

Conversely, all the listed cases give rise to examples.

Proof. Note that $p > 2$.

(a) Suppose first that $q = p$. If V is projective, then since $\dim V < 2p$, we must have $V \cong \mathcal{P}(\mathbb{1})$, which is uniserial of shape $(L(0) \mid L(p-3) \mid L(0))$ and of dimension p. (See for example [Alperin 1986].) If V is nonprojective, then, as $\mathrm{SL}_2(p)$ has a cyclic Sylow p-subgroup, V is one of the "standard modules" described in [Janusz 1969, §5]. As V is self-dual, the standard modules are described by paths in the Brauer tree as in [Janusz 1969, (5.2)(b)] with $P_0 = Q = P_{k+1}$. By inspecting the Brauer trees of $\mathrm{SL}_2(p)$ (see, e.g., [Alperin 1986]) and using that $\dim V < 2p-2$, we deduce moreover that $k = 1$ above, obtaining the modules in (ii), (iii).

In case (i), it is obvious that the module is self-dual since it is $\mathcal{P}(\mathbb{1})$. In cases (ii) and (iii) the uniqueness of the isomorphism class of the extension implies that it is self-dual.

(b) Now suppose that $q > p$. We need to show that no such V exists. (In fact we will show this holds even under the weaker bound $\dim V < 2p$.) Pick an irreducible subrepresentation $L(\lambda)$ of V, where $\lambda = \sum_{i=0}^{n-1} p^i \lambda_i$, $0 \le \lambda_i \le p-1$. Then V has a subquotient isomorphic to a nonsplit extension $0 \to L(\lambda) \to E \to L(\mu) \to 0$, where $\mu = \sum_{i=0}^{n-1} p^i \mu_i$, $0 \le \mu_i \le p-1$. By Lemma 8.1 we know that $\lambda \ne \mu$; hence $2 \dim L(\lambda) + \dim L(\mu) < 2p$. By Corollary 4.5(a) in [Andersen et al. 1983] we deduce that, up to a cyclic relabeling of the indices, $\lambda = \lambda_0 + p$, $\mu = p-2-\lambda_0$, and $\mu > (2p-3)/3 \ge 1$. In particular, μ uniquely determines λ. Hence, if $\operatorname{soc} V$ contains two nonisomorphic irreducible representations, then V admits indecomposable subrepresentations of length two that intersect in zero, so $\dim V \ge 2p$ by Lemma 8.1. Therefore, $\operatorname{soc} V \cong L(\lambda)^{\oplus r}$ for some $r \ge 1$.

Suppose first that $r \ge 2$. We claim that $\operatorname{soc}_2 V / \operatorname{soc} V \cong L(\mu)^{\oplus s}$ for some $0 \le \mu < p^n$ and some $s \ge 1$. (Here $\operatorname{soc}_i M$ is the increasing filtration determined by $\operatorname{soc}_0 M = 0$ and $\operatorname{soc}_i M / \operatorname{soc}_{i-1} M = \operatorname{soc}(M / \operatorname{soc}_{i-1} M)$. Note that the socle filtration is compatible with subobjects.) Note that any constituent of $\operatorname{soc}_2 V / \operatorname{soc} V$ extends $L(\lambda)$, and hence by above it is uniquely determined, unless $n = 2$ and $\lambda_0 = 1$. In the latter case, the constituents can be $L(\mu')$, $L(\mu'')$, where $\mu' = p-3$, $\mu'' = p(p-3)$. But only one of them can occur since $\dim L(\lambda) + \dim L(\mu') + \dim L(\mu'') = 2p$, and this proves the claim. Note that $L(\mu)$ can occur only once in V by Lemma 8.1; in particular, $s = 1$. We claim that $\dim \operatorname{Ext}^1(L(\mu), L(\lambda)) \ge r \ge 2$. Otherwise, $\operatorname{soc}_2 V$ is decomposable, so we obtain a splitting $\pi : \operatorname{soc}_2 V \to L(\lambda) \subset \operatorname{soc} V$. But $\operatorname{Ext}^1(V / \operatorname{soc}_2 V, L(\lambda)) = 0$, so we can extend π to a splitting of V, a contradiction. Hence $\dim \operatorname{Ext}^1(L(\mu), L(\lambda)) \ge 2$ and by Corollary 4.5(b) in [Andersen et al. 1983] we deduce that $n = 2$ and $\lambda_i, \mu_i \in \{(p-3)/2, (p-1)/2\}$ for all i. (Note that we can get all four combinations with $\lambda_i + \mu_i = p-2$, unlike what is claimed in that corollary.) This contradicts that $|\{\lambda_i, \mu_i : 0 \le i \le n-1\}| \ge 3$ (by above).

Suppose that $r = 1$, so $\operatorname{soc} V$ is irreducible. Note that $\operatorname{soc}_3 V = V$ by Lemma 8.1, as each constituent in a socle layer extends at least one constituent of the previous socle layer. As $\operatorname{soc} V$ is irreducible, V embeds in the projective indecomposable module $U_n(\lambda)$ whose socle is $L(\lambda)$. We have $V \subset \operatorname{soc}_3 U_n(\lambda)$. Note that $\lambda_i < p-1$ for all i, as $\dim V < 2p$. By Lemma 8.1, $L(\lambda)$ does not occur in $\operatorname{soc}_2 U_n(\lambda)/\operatorname{soc} U_n(\lambda)$. Also, $L(\lambda)$ occurs precisely n times in $\operatorname{soc}_3 U_n(\lambda)/\operatorname{soc}_2 U_n(\lambda)$. (Theorems 4.3 and 3.7 in [Andersen et al. 1983] imply that this is the case, unless $n = 2$ and $\lambda_i \in \{(p-3)/2, (p-1)/2\}$ for all i. But by above $\lambda_i < (p-3)/3 \le (p-3)/2$ for some i.) Let $M_i = L(\lambda_0) \otimes L(\lambda_1)^{(p)} \otimes \cdots \otimes Q_1(\lambda_i)^{(p^i)} \otimes \cdots \otimes L(\lambda_{n-1})^{(p^{n-1})}$ and $M := M_0 + \cdots + M_{n-1} \subset U_n(\lambda)$ in the notation of [Andersen et al. 1983, §3]. Note by Theorems 4.3 and 3.7 in [Andersen et al. 1983] that $\operatorname{soc}_2 U_n(\lambda) \subset M \subset \operatorname{soc}_3 U_n(\lambda)$ and that $M/\operatorname{soc}_2 U_n(\lambda) \cong L(\lambda)^{\oplus n}$. Therefore $V \subset M$, so

$$\frac{V}{L(\lambda)} \subset \frac{M}{L(\lambda)} = \frac{M_0}{L(\lambda)} \oplus \cdots \oplus \frac{M_{n-1}}{L(\lambda)}.$$

As $\operatorname{head}(M_i/L(\lambda)) \cong L(\lambda)$, there exists i such that $V/L(\lambda)$ surjects onto $M_i/L(\lambda)$. Thus $\dim V \ge \dim M_i \ge 2p$.

□

9. Finite groups with indecomposable modules of small dimension

Throughout this section, we assume that $k = \bar{k}$ is a field of characteristic $p > 3$. We want to describe the structure of finite groups G that admit reducible indecomposable modules of dimension $\le 2p-2$. The next results essentially reduce us to the case of quasisimple groups.

Lemma 9.1. *Let G be a finite group, $p > 3$, and V be a faithful kG-module of dimension $< 2p$. Suppose that $\boldsymbol{O}_p(G) = 1$ and $\boldsymbol{O}_{p'}(G) \le \boldsymbol{Z}(G)$. Then $F(G) = \boldsymbol{O}_{p'}(G) = \boldsymbol{Z}(G)$, $F^*(G) = E(G)\boldsymbol{Z}(G)$, and $G^+ = E(G)$ is either trivial or a central product of quasisimple groups of order divisible by p. In particular, G has no composition factor isomorphic to C_p, and so $H^1(G, k) = 0$.*

Proof. (a) Since $\boldsymbol{O}_p(G) = 1$, $Z := \boldsymbol{Z}(G) \le F(G) \le \boldsymbol{O}_{p'}(G)$. It follows that $F(G) = Z = \boldsymbol{O}_{p'}(G)$, and $F^*(G) = E(G)Z$. If moreover $E(G) = 1$, then

$$Z = F(G) = F^*(G) \ge \boldsymbol{C}_G(F^*(G)) = G,$$

whence G is an abelian p'-group, and $G^+ = 1 = E(G)$.

(b) Assume now that $E(G) > 1$ and write $E(G) = L_1 * \cdots * L_t$, a central product of $t \ge 1$ quasisimple subgroups. Since $\boldsymbol{O}_{p'}(E(G)) \le \boldsymbol{O}_{p'}(G) = Z$, $p \mid |L_i|$ for all i.

Next we show that $\boldsymbol{N}_G(L_i)/\boldsymbol{C}_G(L_i)L_i$ is a p'-group for all i. Indeed, note that the L_i-module V admits a nontrivial composition factor U of dimension $< 2p$. Otherwise it has a composition series with all composition factors being trivial, whence L_i acts on V as a p-group. Since V is faithful and L_i is quasisimple, this is a contradiction. So we can apply Theorem 2.1 and [Guralnick et al. 2014, Theorem 2.1] to the image of L_i in $\mathrm{GL}(U)$. In particular, denoting $S_i := L_i/\boldsymbol{Z}(L_i)$, one can check that $\mathrm{Out}\, S_i$ is a p'-group, unless it is a simple group of Lie type in characteristic p. In the former case we are done since $\boldsymbol{N}_G(L_i)/\boldsymbol{C}_G(L_i)L_i \hookrightarrow \mathrm{Out}\, S_i$. Consider the latter case. Observe that $\boldsymbol{Z}(L_i) \le \boldsymbol{Z}(E(G)) \le F(G)$ is a p'-group. So we may replace L_i by its simply connected isogenous version $\mathcal{G}^F$, where $F : \mathcal{G} \to \mathcal{G}$ is a Steinberg endomorphism on a simple simply connected algebraic group $\mathcal{G}$ in characteristic p. If moreover p divides $|\boldsymbol{N}_G(L_i)/\boldsymbol{C}_G(L_i)L_i|$, then $\boldsymbol{N}_G(L_i)$ induces an outer automorphism σ of L_i of order p. As $p > 3$, this can happen only when σ is a field automorphism. More precisely, L_i is defined over a field $\mathbb{F}_{p^{bp}}$ (for some $b \ge 1$), where $\mathbb{F}_{p^{bp}}$ is the smallest splitting field for L_i [Kleidman and Liebeck 1990, Proposition 5.4.4] and σ is induced by the field automorphism $x \mapsto x^{p^b}$. Since $\dim U \ge 2 > (\dim V)/p$, U must be σ-invariant. In turn, this implies by [Kleidman and Liebeck 1990, Proposition 5.4.2] that U and its (p^b)-th Frobenius twist are isomorphic. In this case, the proofs of Proposition 5.4.6 and Remark 5.4.7 of [Kleidman and Liebeck 1990] show that $\dim U \ge 2^p > 2p$, a contradiction.

(c) Recall that $\boldsymbol{C}_G(E(G)) = \boldsymbol{C}_G(F^*(G)) \le F^*(G) = E(G)Z$, whence $\boldsymbol{C}_G(E(G)) = Z$. Also, G acts via conjugation on the set $\{L_1, \dots, L_t\}$, with kernel (say) N. We claim that $p \nmid |G/N|$. If not, then we may assume that some p-element $g \in G$ permutes $L_1, \dots, L_p$ cyclically. Arguing as in (b), we see that L_1 acts nontrivially on some composition factor U of the $E(G)$-module V, and we can write $U = U_1 \otimes \cdots \otimes U_t$, where $U_i \in \mathrm{IBr}_p(L_i)$. If U is not g-invariant, then $\dim V \ge p(\dim U) \ge 2p$, a contradiction. Hence U is g-invariant. It follows that $2 \le \dim U_1 = \cdots = \dim U_p$ and so $\dim U \ge 2^p > 2p$, again a contradiction.

Now $N/E(G)Z$ embeds in $\prod_{i=1}^t \mathrm{Out}\, L_i$. Furthermore, the projection of N into $\mathrm{Out}\, L_i$ induces a subgroup of $\boldsymbol{N}_G(L_i)/\boldsymbol{C}_G(L_i)L_i$, which is a p'-group by (b). It follows that $N/E(G)Z$ is a p'-group, and so $G^+ = E(G)$. The last statement also follows. □

The next result on H^1 follows from standard results on H^1 — see [Guralnick et al. 2007, Lemma 5.2] and the main result of [Guralnick 1999].

Lemma 9.2. *Let G be a finite group and let V be a faithful irreducible kG-module. Assume that $H^1(G, V) \ne 0$. Then $\boldsymbol{O}_{p'}(G) = \boldsymbol{O}_p(G) = 1$, $E(G) = L_1 \times \cdots \times L_t$ and $V_{E(G)} = W_1 \oplus \cdots \oplus W_t$, where the L_i are isomorphic nonabelian simple groups of order divisible by p, W_i is an irreducible kL_i-module, and L_j, $j \ne i$ acts trivially on W_i. Moreover,* $\dim H^1(G, V) \le \dim H^1(L_1, W_1)$, $\dim W_i \ge p - 2$ *and* $\dim V \ge t(p-2)$. *In particular, if G is not almost simple, then either* $\dim V = 2p-4, 2p-2$ *or* $\dim V \ge 2p$, *or* $(p, \dim V) = (5, 9)$.

Lemma 9.3. *Let V be a faithful indecomposable kG-module with two composition factors V_1, V_2. Assume that $\boldsymbol{O}_p(G) = 1$ and $\dim V \le 2p-2$. If $J := \boldsymbol{O}_{p'}(G^+) \not\le \boldsymbol{Z}(G^+)$, then*:

(i) $p = 2^a + 1$ *is a Fermat prime.*

(ii) $\dim V_1 = \dim V_2 = p-1$.

(iii) $J/\boldsymbol{Z}(J)$ *is elementary abelian of order* 2^{2a}.

(iv) $H^1(G^+, k) \neq 0$.

Proof. Since $\mathrm{Ext}^1_G(V_1, V_2) \hookrightarrow \mathrm{Ext}^1_{G^+}(V_1, V_2)$, there are irreducible G^+-submodules W_i of V_i for $i = 1, 2$ such that $\mathrm{Ext}^1_{G^+}(W_1, W_2) \neq 0$. Assume that J acts by scalars on at least one of the W_i. Then, by Lemma 7.11(ii), J acts by scalars on both W_1 and W_2. If W_1' is any G^+-composition factor of V_1, then W_1' is G-conjugate to W_1. But $J \lhd G$, so we see that J acts by scalars on W_1'. Thus J acts by scalars on all G^+-composition factors of V_1, and similarly for V_2. Consider a basis of V consistent with a G^+-composition series of V, and any $x \in J$ and $y \in G^+$. Then $[x, y]$ acts as the identity transformation on each G^+-composition factor in this series, and so it is represented by an upper unitriangular matrix in the chosen basis. The same is true for any element in $[J, G^+] \lhd G$. Since V is faithful, we see that $[J, G^+] \le \boldsymbol{O}_p(G) = 1$ and so $J \le \boldsymbol{Z}(G^+)$, a contradiction.

Thus J cannot act by scalars on any W_i. Let Φ_i denote the representation of G^+ on W_i. Then $H := \Phi_i(G^+) < \mathrm{GL}(W_i)$ has no nontrivial p'-quotient, and contains a nonscalar normal p'-subgroup $\Phi_i(J)$. Applying Theorem 2.1 and also [Blau and Zhang 1993, Theorem A] to H, we conclude that $p = 2^a + 1$ is a Fermat prime, $\dim W_i = p-1$, and $Q := \boldsymbol{O}_{p'}(H)$ acts irreducibly on W_i. Furthermore, $\boldsymbol{Z}(Q) = \boldsymbol{Z}(H)$, and H/Q acts irreducibly on $Q/\boldsymbol{Z}(Q)$, an elementary abelian 2-group of order 2^{2a}. Now $\Phi_i(J)$ is a normal p'-subgroup of H that is *not* contained in $\boldsymbol{Z}(Q)$. It follows that $\Phi_i(J)\boldsymbol{Z}(Q) = Q$, $\boldsymbol{Z}(\Phi_i(J)) = \Phi_i(J) \cap \boldsymbol{Z}(Q)$, J is irreducible on W_i, and $\Phi_i(J)/\boldsymbol{Z}(\Phi_i(J)) \cong Q/\boldsymbol{Z}(Q)$ is elementary abelian of order 2^{2a}. Since $\dim V \le 2p-2$, it also follows that $W_i = V_i$.

Letting $A := V_1^* \otimes V_2$, we then see that $A = [J, A] \oplus \boldsymbol{C}_A(J)$ as J-modules. Next,

$$0 \neq \mathrm{Ext}^1_G(V_1, V_2) \cong H^1(G, A) \cong H^1(G, \boldsymbol{C}_A(J)),$$

since $H^1(G, [J, A]) = 0$ by the inflation restriction sequence. It follows that $\boldsymbol{C}_A(J) \neq 0$. But J is irreducible on both V_1 and V_2, so we must have that $\dim \boldsymbol{C}_A(J) = 1$ and $V_1 \cong V_2$ as J-modules. Since G^+ acts trivially on any 1-dimensional module, it follows that $H^1(G^+, k) \neq 0$. Since $W_1 \cong W_2$ as J-modules and V is a faithful semisimple J-module, we also see that $\mathrm{Ker}(\Phi_1) \cap J = \mathrm{Ker}(\Phi_2) \cap J = 1$. Thus Φ_i is faithful on J, and so $J/\boldsymbol{Z}(J)$ is elementary abelian of order 2^{2a}. $\square$

Lemma 9.4. *Let V be a faithful indecomposable kG-module with two composition factors V_1, V_2 of dimension > 1, $p > 3$, and $\boldsymbol{O}_p(G) = 1$.*

(i) *Assume that $\boldsymbol{O}_{p'}(G^+) \le \boldsymbol{Z}(G^+)$, and either $\dim V < 2p-2$ or $\dim V_1 = \dim V_2 = p-1$. If G^+ is not quasisimple, then $G^+ = L_1 * L_2$ is a central product of two quasisimple groups, $\dim V_1 = \dim V_2 = p-1$ and, up to relabeling the L_i, one of the following holds*:

  (a) *$V_i = A_i \otimes B$ as G^+-modules, where $A_i \in \mathrm{IBr}_p(L_1)$ is of dimension $(p-1)/2$ and $B \in \mathrm{IBr}_p(L_2)$ is of dimension 2; furthermore, $\mathrm{Ext}^1_{L_1}(A_1, A_2) \neq 0$.*

  (b) *$V_i = (A_i \otimes k) \oplus (k \otimes B_i)$ as G^+-modules, where $A_i \in \mathrm{IBr}_p(L_1)$ has dimension $(p-1)/2$, and some $g \in G$ interchanges L_1 with L_2 and A_i with B_i. Furthermore, $\mathrm{Ext}^1_{L_1}(A_1, A_2) \neq 0$.*

(ii) *If $\dim V < 2p-2$, then G^+ is quasisimple.*

Proof. (i) By Lemma 9.1 applied to G^+, $G^+ = (G^+)^+ = E(G^+) = L_1 * L_2 * \cdots * L_t$, a central product of t quasisimple groups. Suppose $t > 1$. Since $\mathrm{Ext}^1_G(V_1, V_2) \hookrightarrow \mathrm{Ext}^1_{G^+}(V_1, V_2)$, there are irreducible G^+-submodules W_i of V_i for $i = 1, 2$ such that $\mathrm{Ext}^1_{G^+}(W_1, W_2) \neq 0$. Write $W_i = W_{i1} \otimes \cdots \otimes W_{it}$, where W_{ij} is an irreducible L_j-module. By Lemma 7.7, we may assume that $W_{1j} \cong W_{2j}$ for $j = 2, \dots, t$, and either $\mathrm{Ext}^1_{L_1}(W_{11}, W_{21}) \neq 0$, or $W_{11} \cong W_{21}$ and $\mathrm{Ext}^1_{L_j}(W_{1j}, W_{2j}) \neq 0$ for some j. Interchanging L_1 and L_j in the latter case, we can always assume that $\mathrm{Ext}^1_{L_1}(W_{11}, W_{21}) \neq 0$. By [Guralnick 1999, Theorem A], we then have

$$\dim W_{11} + \dim W_{21} \ge p - 1 > 2. \tag{9-1}$$

Now if W_{1j} is nontrivial for some $j \ge 2$, say $W_{12} \not\cong k$, then

$$\dim V \ge \dim W_1 + \dim W_2 \ge 2(\dim W_{11} + \dim W_{21}) = 2p - 2.$$

It follows that $V_i = W_i = W_{i1} \otimes W_{i2} \otimes k \otimes \cdots \otimes k$, $\dim W_{i1} = (p-1)/2$, and $\dim W_{i2} = 2$. Furthermore, $t = 2$ as V is faithful, and we arrive at (a).

We may now assume that $W_{1j} \cong W_{2j} \cong k$ for all $j > 1$. Suppose that G normalizes L_1. Since every G^+-composition factor of V_1 is G-conjugate to W_1, it follows that L_2 acts trivially on all composition factors of V_1. The same is true for V_2. As L_2 is quasisimple, we see that L_2 acts trivially on V, contrary to the faithfulness of V. Thus there must be some $g \in G$ conjugating L_1 to L_j for some $j > 1$, say $L_1^g = L_2$. By (9-1) we may assume that $W_{11} \not\cong k$. Then $g(W_1) \not\cong W_1$, as L_2 acts trivially on W_1 but not on $g(W_1)$. Thus $(V_1)_{G^+}$ has at least two distinct simple summands W_1 and $g(W_1)$. If furthermore $W_{21} \not\cong k$, then $(V_2)_{G^+}$ also has at least two distinct simple summands W_2 and $g(W_2)$, and so

$$\dim V \ge 2(\dim W_1 + \dim W_2) = 2(\dim W_{11} + \dim W_{21}) \ge 2p - 2.$$

In this case, we must have that $V_i = W_i \oplus g(W_i)$, $\dim W_i = (p-1)/2$, and $t = 2$ as V is faithful, and we arrive at (b).

Consider the case $W_{21} \cong k$. Now (9-1) implies that $\dim W_1 = \dim W_{11} \geq p-2$, whence $\dim V_1 \geq 2p-4$. On the other hand, $\dim V_2 \geq 2$. It follows that $2p-4 = 2$, again a contradiction.

(ii) By Lemma 9.3, $\boldsymbol{O}_{p'}(G^+) \leq \boldsymbol{Z}(G^+)$. Hence we are done by (i). □

Lemma 9.5. *Let H be a quasisimple finite group of Lie type in characteristic $p > 3$. Assume that $V_1, V_2 \in \mathrm{IBr}_p(H)$ satisfy $\dim V_1 + \dim V_2 < 2p$.*

(i) *If $H \ncong \mathrm{SL}_2(q)$, $\mathrm{PSL}_2(q)$, then $\mathrm{Ext}^1_H(V_1, V_2) = 0$. In particular, there is no reducible indecomposable kG-module with $G^+ \cong H$ and $\dim V < 2p$.*

(ii) *Suppose $H \cong \mathrm{SL}_2(q)$ or $\mathrm{PSL}_2(q)$, $\mathrm{Ext}^1_H(V_1, V_2) \neq 0$, and $\dim V_1 = \dim V_2$. Then $q = p$ and $V_1 = L((p-3)/2)$ or $L((p-1)/2)$.*

Proof. (i) Note that $\boldsymbol{Z}(H)$ is a p'-group as $p > 3$. Hence, we can replace H by the fixed-point subgroup $\mathscr{G}^F$ for some Steinberg endomorphism $F : \mathscr{G} \to \mathscr{G}$ on some simple simply connected algebraic group $\mathscr{G}$ defined over a field of characteristic p (see Lemma 7.3). Hence, if $H \ncong \mathrm{Sp}_{2n}(5)$, the result follows by [McNinch 1999, Theorem 1.1]. In the exceptional case $H = \mathrm{Sp}_{2n}(5)$, we have $p = 5$ and so we are only considering modules of dimension at most 9. If $n \geq 3$, then $\dim V_1 + \dim V_2 > 10$ unless at least one of the V_i is trivial and the other is either trivial or the natural module of dimension $2n$, and in both cases $\mathrm{Ext}^1_H(V_1, V_2) = 0$. If $n = 2$, one just computes that all the relevant $\mathrm{Ext}^1_H(V_1, V_2)$ are trivial (done by Lux).

Suppose now that V is a reducible indecomposable kG-module with $G^+ \cong H$ and $\dim V < 2p$. By Lemma 7.8(ii), there are composition factors V_1, V_2 of V such that $\mathrm{Ext}^1_G(V_1, V_2) \neq 0$. It then follows that $\mathrm{Ext}^1_H(W_1, W_2) \neq 0$ for some simple H-summands W_i of V_i for $i = 1, 2$ and $\dim W_1 + \dim W_2 < 2p$, a contradiction.

(ii) Again we can replace H by $\mathrm{SL}_2(q)$. The statement then follows from Lemma 8.1 when $q > p$, and from [Andersen et al. 1983] if $q = p$. □

There are a considerable number of examples of nonsplit extensions $(V_1|V_2)$ with G^+ nonquasisimple and $\dim V_1 + \dim V_2 = 2p-2$. For example, suppose that $G = \mathrm{SL}_2(p) \times \mathrm{SL}_2(p)$ and $V_1 = L(1) \otimes L(a)$ and $V_2 = L(1) \otimes L(p-a-3)$. Then by [Andersen et al. 1983] and Lemma 7.7, $\mathrm{Ext}^1_G(V_1, V_2) \neq 0$. For our adequacy results, we do need to consider the case where $\dim V_1 = \dim V_2 = p-1$ in more detail:

Lemma 9.6. *Let V be a faithful indecomposable kG-module with two composition factors V_1, V_2, both of dimension $p-1$. Assume that $p > 3$ and $\boldsymbol{O}_p(G) = 1$. Then one of the following holds:*

(i) *$\boldsymbol{O}_{p'}(G^+) \nleq \boldsymbol{Z}(G^+)$ and Lemma 9.3 applies.*

(ii) *G^+ is quasisimple.*

(iii) $G^+ = \mathrm{SL}_2(p) \times \mathrm{SL}_2(p^a)$ (*modulo some central subgroup*) *and one of the following holds*:

(a) $V_1 \cong V_2 \cong L((p-3)/2) \otimes L(1)^{(p^b)}$ *as* G^+*-modules* (*for some* $0 \le b < a$).

(b) $a = 1$ *and* $V_1 \cong V_2 \cong X \oplus Y$, *where* G^+ *acts as a quasisimple group on* X, Y *and* $\dim X = \dim Y = (p-1)/2$ (*so* $X, Y \cong L((p-3)/2)$ *for the copy of* $\mathrm{SL}_2(p)$ *acting nontrivially on* X *or* Y).

Proof. Assume that neither (i) nor (ii) holds. Then by Lemma 9.4(i), $E(G^+) = G^+ = L_1 * L_2$ is a central product of two quasisimple groups, and either (a) or (b) of Lemma 9.4(i) occurs. In either case, we see that L_1 admits an indecomposable module W of length 2 with composition factors A_1 and A_2, both of dimension $(p-1)/2$. By [Blau and Zhang 1993, Theorem A] applied to W, L_1 is of Lie type in characteristic p. Also, $\boldsymbol{Z}(L_1) \le \boldsymbol{Z}(G^+) \le \boldsymbol{O}_{p'}(G)$ is a p'-group. Hence $L_1 \cong \mathrm{SL}_2(p)$ (modulo a central subgroup) by Lemma 9.5 and $A_1 \cong A_2 \cong L((p-3)/2)$. In particular, $L_2 \cong \mathrm{SL}_2(p)$ in case (b), and (iii)(b) holds. In the case of (a), $B \in \mathrm{IBr}_p(L_2)$ has dimension 2. Since $p > 3$, by Theorem 2.1 we conclude that L_2 is of Lie type in characteristic p, and in fact that $L_2 \cong \mathrm{SL}_2(p^a)$ (modulo a central subgroup) and $B \cong L(1)^{(p^b)}$ for some $a \ge 1$ and $0 \le b < a$. Thus (iii)(a) holds. ☐

Proposition 9.7. *Let* $p > 3$ *and let* G *be a finite group with a faithful*, *reducible*, *indecomposable* kG*-module* V *of dimension* $\le 2p-3$. *Suppose in addition that* $\boldsymbol{O}_p(G) = 1$. *Then* $G^+ = E(G^+)$, G *has no composition factor isomorphic to* C_p, *and one of the following holds*:

(i) G^+ *is quasisimple.*

(ii) G^+ *is a central product of two quasisimple groups and* $\dim V = 2p-3$. *Furthermore*, V *has one composition factor of dimension* 1, *and either one of dimension* $2p-4$ *or two of dimension* $p-2$. *In either case*, $V \ncong V^*$.

Proof. (a) Note that $\boldsymbol{O}_p(G^+) \le \boldsymbol{O}_p(G) = 1$. Next we show that $J := \boldsymbol{O}_{p'}(G^+) \le \boldsymbol{Z}(G^+)$. As in the proof of Lemma 9.3, it suffices to show that J acts by scalars on every G^+-composition factor of V. So assume that there is a G^+-composition factor X of V on which J does not act by scalars. Again as in the proof of Lemma 9.3, we see by Theorem 2.1 that $\dim X \ge p-1$. Since $\dim V \le 2p-3$, it follows that X is a G^+-composition factor of multiplicity 1, and, moreover, J acts by scalars on any other G^+-composition factor Y of V. Also, X extends to a G-composition factor (of multiplicity 1) of V. Now, by Lemma 7.8(i), there is an indecomposable subquotient of length 2 of V with G-composition factors X and $T \ncong X$. In particular, by symmetry we may assume that $0 \ne \mathrm{Ext}^1_G(X, T) \hookrightarrow \mathrm{Ext}^1_{G^+}(X, T)$, and so $\mathrm{Ext}^1_{G^+}(X, Y) \ne 0$ for some simple G^+-summand Y of T. But this is impossible by Lemma 7.11(ii) (as J acts by scalars on Y but not on X).

Applying Lemma 9.1 to G^+, we see that

$$G^+ = (G^+)^+ = E(G^+) = L_1 * \cdots * L_t,$$

a central product of t quasisimple subgroups. Note that $t \geq 1$ as otherwise G is a p'-group and so V does not exist. Furthermore, G has no composition factors isomorphic to C_p.

(b) Assume now that $t \geq 2$. Suppose in addition that , for every composition factor V_i of V, at most one of the components L_j of G^+ acts nontrivially on V_i. For $1 \leq j \leq t$, let $\mathscr{X}_j$ denote the set of isomorphism classes of composition factors V_i of V on which L_j acts nontrivially. Also let $\mathscr{X}_0$ denote the set of isomorphism classes of composition factors V_i of V on which G^+ acts trivially. By the faithfulness of V, $\mathscr{X}_j \neq \varnothing$ for $j > 0$. Consider for instance $X \in \mathscr{X}_1$. By Lemma 7.8(i), there is some $X' \in \mathscr{X}_j$ (for some j) and some indecomposable subquotient W of length 2 of V with composition factors X, X'. Note that the p-radical of the group induced by the action of G on W is trivial, as C_p is not a composition factor of G. Applying Lemma 9.4(ii) to W, we see that $j = 0$ or 1. Moreover, if for *all* $X \in \mathscr{X}_1$ there is no such W with $X' \in \mathscr{X}_0$, then Lemma 7.10 applied to $\left(\mathscr{X} := \mathscr{X}_1, \mathscr{Y} := \bigcup_{i \neq 1} \mathscr{X}_i\right)$ implies that V is decomposable, a contradiction. Thus for some $X \in \mathscr{X}_1$, such a W exists with $X' \in \mathscr{X}_0$. Note that in this case $\dim X \geq p - 2$. Indeed, G^+ acts trivially on X', and by symmetry we may assume that

$$0 < \dim \operatorname{Ext}^1_G(X', X) \leq \dim \operatorname{Ext}^1_{G^+}(X', X).$$

Therefore, for some simple summand X_1 of the G^+-module X we have that $0 \neq \operatorname{Ext}^1_{G^+}(k, X_1) \cong H^1(G^+, X_1)$. Note that C_p is not a composition factor of G^+, so by Lemma 7.3 we may assume here that G^+ acts faithfully on X_1. Applying Lemma 9.2 to G^+, we get $\dim X \geq \dim X_1 \geq p - 2$.

Similarly, for some $Y \in \mathscr{X}_2$, we get an indecomposable subquotient T of length 2 of V with composition factors Y and $Y' \in \mathscr{X}_0$, and moreover $\dim Y \geq p - 2$. Since $\dim V \leq 2p - 3$ and $\mathscr{X}_0 \ni X', Y'$, we conclude that $\dim V = 2p - 3$, $\dim X = \dim Y = p - 2$, $t = 2$, and $X' \cong Y'$ has dimension 1. Suppose in addition that $V \cong V^*$. Observe that $X^* \ncong Y, X'$, so $X \cong X^*$. Similarly, Y and X' are self-dual. Thus all three composition factors of V have multiplicity 1 each and are self-dual. At least one of them occurs in $\operatorname{soc} V$, and then also in $\operatorname{head} V$ by duality. It follows by Lemma 7.9 that V is decomposable, a contradiction. Thus we arrive at (ii).

(c) Finally, we consider the case where at least two of the L_i act nontrivially on some composition factor V_i of V. By Lemma 7.8(i), there is some indecomposable subquotient W of length 2 of V with composition factors V_i and V_j. By Lemma 9.4(ii) applied to W, $\dim V_j = 1$. In turn this implies by Lemma 9.2 that $\dim V_i \geq 2p - 4$. Since $\dim V \leq 2p - 3$, we must have that $\dim V_i = 2p - 4$, $V = W$, $t = 2$ and

$\dim V = 2p-3$. Applying Lemma 7.9 and using the indecomposability of V as above, we see that $V \not\cong V^*$, and again arrive at (ii). $\square$

10. Extensions and self-extensions, II

Let q be any odd prime power. It is well known (see, e.g., [Tiep and Zalesskii 1997] and [Guralnick et al. 2002]) that the finite symplectic group $\mathrm{Sp}_{2n}(q)$ has two complex irreducible *Weil* characters ξ_1, ξ_2 of degree $(q^n+1)/2$, and two such characters η_1, η_2 of degree $(q^n-1)/2$, whose reductions modulo any *odd* prime $p \nmid q$ are absolutely irreducible and distinct and are called (p-*modular*) *Weil characters* of $\mathrm{Sp}_{2n}(q)$.

Lemma 10.1. *Let q be an odd prime power and p an odd prime divisor of q^n+1 which does not divide $\prod_{i=1}^{2n-1}(q^i-1)$. Let $S := \mathrm{Sp}_{2n}(q)$ and let W_1 and W_2 denote the irreducible kS-modules affording the two irreducible p-modular Weil characters of S of degree $(q^n-1)/2$. Then for $1 \le i, j \le 2$ we have that* $\mathrm{Ext}^1_S(W_i, W_j) = 0$, *unless $i \neq j$ and $n = 1$, in which case* $\dim(\mathrm{Ext}^1_S(W_i, W_j)) = 1$.

Proof. The conditions on (n, q) imply that $(n, q) \neq (1, 3)$. In this case, [Tiep and Zalesskii 1996, Theorem 1.1] implies that each W_i has a unique complex lift (a complex module affording some η_i). Also, the Sylow p-subgroups of S are cyclic of order $(q^n+1)_p$. Hence $\mathrm{Ext}^1_S(W_i, W_i) = 0$ by Lemma 7.1.

Note that an involutory diagonal automorphism σ of S fuses η_1 with η_2 and W_1 with W_2. Consider the semidirect product $H := S \rtimes \langle\sigma\rangle$ and the irreducible kH-module $V := \mathrm{Ind}_S^H(W_1)$ of dimension q^n-1. Certainly, $\mathrm{Ind}_S^H(\eta_1)$ is a complex lift of V.

Assume that $n > 1$. Now if $(n, q) \neq (2, 3)$, then by [Tiep and Zalesskii 1996, Theorem 5.2], S has exactly five irreducible complex characters of degree $\le (q^n-1)$: $1_S, \eta_1, \eta_2, \xi_1$, and ξ_2. When $(n, q) = (2, 3)$, there is one extra complex character of degree 6 [Conway et al. 1985]. It follows that if χ is any complex lift of V, then $\chi_S = \eta_1 + \eta_2$. Since σ fuses η_1 and η_2, we see that $\chi = \mathrm{Ind}_S^H(\eta_1)$. Thus V has a unique complex lift, and so by Lemma 7.1 and Frobenius reciprocity we have

$$\begin{aligned} 0 = \mathrm{Ext}^1_H(V, V) &= \mathrm{Ext}^1_H(\mathrm{Ind}_S^H(W_1), V) \cong \mathrm{Ext}^1_S(W_1, V_S) \\ &\cong \mathrm{Ext}^1_S(W_1, W_1) \oplus \mathrm{Ext}^1_S(W_1, W_2). \end{aligned}$$

In particular, $\mathrm{Ext}^1_S(W_1, W_2) = 0$.

Next suppose that $n = 1$. Inspecting the character table of $\mathrm{SL}_2(q)$ as given in [Digne and Michel 1991, Table 2], we see that S has a σ-invariant complex irreducible character χ of degree $q-1$ such that the restriction of χ to p'-elements of S is the Brauer character of V_S. Since H/S is cyclic and generated by σ, it follows that χ extends to a complex irreducible character $\tilde{\chi}$ of H. Now $\tilde{\chi} \neq \mathrm{Ind}_S^H(\eta_1)$ (since the latter is reducible over S), but both of them are complex lifts of V (by Clifford's theorem). Applying Lemma 7.1 and Frobenius reciprocity as above, we see that $\dim \mathrm{Ext}^1_H(V, V) = \dim \mathrm{Ext}^1_S(W_1, W_2) = 1$. □

Lemma 10.2. *Let H be a quasisimple group with $\mathbf{Z}(H)$ a p'-group. Let W and W' be absolutely irreducible kH-modules in characteristic p of dimension d, where (H, p, d) is one of the following triples*:

$$(2\mathsf{A}_7, 5, 4),\quad (3J_3, 19, 18),\quad (2Ru, 29, 28),\quad (6_1\cdot\mathrm{PSL}_3(4), 7, 6),$$
$$(6_1\cdot\mathrm{PSU}_4(3), 7, 6),\quad (2J_2, 7, 6),\quad (3\mathsf{A}_7, 7, 6),\quad (6\mathsf{A}_7, 7, 6),\quad (M_{11}, 11, 10),$$
$$(2M_{12}, 11, 10),\quad (2M_{22}, 11, 10),\quad (6Suz, 13, 12),\quad (2\mathrm{G}_2(4), 13, 12),\quad (3\mathsf{A}_6, 5, 3),$$
$$(3\mathsf{A}_7, 5, 3),\quad (M_{11}, 11, 9),\quad (M_{23}, 23, 21),\quad (2\mathsf{A}_7, 7, 4),\quad (J_1, 11, 7).$$

If $\mathbf{Z}(H)$ acts the same way on W and W', assume in addition that there is an automorphism of H which sends W to W'. Then $\mathrm{Ext}^1_H(W, W') = 0$, with the following two exceptions: $(H, p, d) = (3\mathsf{A}_6, 5, 3)$ *and* $(2\mathsf{A}_7, 7, 4)$, *where* $\dim \mathrm{Ext}^1_H(W, W) = 1$.

Proof. Note that the Sylow p-subgroups of H have order p. Hence, in the case $W \cong W'$ we can apply Lemma 7.1; in particular, we arrive at the two exceptions listed above. This argument settles the cases of $(M_{11}, 11, 9)$, $(M_{23}, 23, 21)$, $(J_1, 11, 7)$, and $(2\mathrm{G}_2(4), 13, 12)$.

If $W \not\cong W'$ and $\mathbf{Z}(H)$ acts differently on W and W', then we also get that $\mathrm{Ext}^1_H(W, W') = 0$ since $\mathbf{Z}(H)$ is a central p'-group. So it remains to consider the case where $W \not\cong W'$ and $\mathbf{Z}(H)$ acts the same way on both of them. Suppose in addition that there is an involutory automorphism σ of H that swaps W and W' and that the module $\mathrm{Ind}^J_H(W)$ of $J := H \rtimes \langle\sigma\rangle$ has at most one complex lift. Then we can apply Lemma 7.1 to J as in the proof of Lemma 10.1 to conclude that $\mathrm{Ext}^1(W, W') = 0$. These arguments are used to handle the cases of $(2\mathsf{A}_7, 5, 4)$, $(3\mathsf{A}_7, 5, 3)$, $(3\mathsf{A}_7, 7, 6)$, $(2J_2, 7, 6)$, $(6Suz, 13, 12)$, $(6_1\cdot\mathrm{PSL}_3(4), 7, 6)$, and $(6_1\cdot\mathrm{PSU}_4(3), 7, 6)$.

In the six remaining cases of $(6\mathsf{A}_7, 7, 6)$, $(3J_3, 19, 18)$, $(2Ru, 29, 28)$, $(M_{11}, 11, 10)$, $(2M_{12}, 11, 10)$, and $(2M_{22}, 11, 10)$, we note (using [Jansen et al. 1995] or [GAP 2004]) that the nonisomorphic H-modules W and W' with the same action of $\mathbf{Z}(H)$ are *not* $\mathrm{Aut}(H)$-conjugate. □

Corollary 10.3. *Suppose that $q > 3$ is an odd prime power such that $p = (q+1)/2$ prime. Then there is a finite absolutely irreducible linear group $G < \mathrm{GL}(V) = \mathrm{GL}_{q-1}(k)$ of degree $q-1$ over k such that $G^+ \cong \mathrm{SL}_2(q)$, all irreducible G^+-submodules in V are Weil modules of dimension $(q-1)/2$, and* $\dim \mathrm{Ext}^1_G(V, V) = 1$. *In particular, (G, V) is not adequate.*

Proof. Our conditions on p, q imply that $q \equiv 1 \pmod 4$. Now we can just appeal to the proof of Lemma 10.1, taking $H = \mathrm{GU}_2(q)/C$, where C is the unique subgroup of order $(q+1)/2$ in $\mathbf{Z}(\mathrm{GU}_2(q))$. □

Proposition 10.4. *Suppose (G, V) is as in the extraspecial case* (ii) *of Theorem 2.4. Then* $\mathrm{Ext}^1_G(V, V) = 0$.

Proof. Write $V|_{G^+} = e\sum_{i=1}^t W_i$ as usual and let K_i be the kernel of the action of G^+ on W_i. By Lemma 7.2, it suffices to show that $\mathrm{Ext}^1_{G^+}(W_i, W_j) = 0$ for all i, j. Recall that $R := \boldsymbol{O}_{p'}(G^+)$ acts irreducibly on W_i. By Theorem 2.4, K_i has no composition factor $\cong C_p$, whence $\mathrm{Ext}^1_{G^+}(W_i, W_i) = \mathrm{Ext}^1_{G^+/K_i}(W_i, W_i)$ by Lemma 7.3(ii). Next, G^+/K_i has cyclic Sylow p-subgroups (of order p) by Theorem 2.1(e), and we have shown in the proof of Proposition 5.6 that the G^+/K_i-module W_i has a unique complex lift. Hence $\mathrm{Ext}^1_{G^+/K_i}(W_i, W_i) = 0$ by Lemma 7.1.

Suppose now that $i \neq j$ and let M be any extension of the G^+-module W_i by the G^+-module W_j. Recall that the R-modules W_i and W_j are irreducible and nonisomorphic, as shown in the proof of Proposition 5.6. But R is a p'-group, so by Maschke's theorem $M = M_1 \oplus M_2$ with $M_i \cong W_i$ as R-modules. Now for any $g \in G^+$, $g(M_i) \cong (W_i)^g \cong W_i$ as R-modules, and so $g(M_i) = M_i$. Thus $M = M_1 \oplus M_2$ as a G^+-module. We have shown that $\mathrm{Ext}^1_{G^+}(W_i, W_j) = 0$. □

Proposition 10.5. *Suppose that (G, V) is as in case* (i) *of Theorem 2.4. Then* $\mathrm{Ext}^1_G(V, V) = 0$, *unless one of the following possibilities occurs for the group $H < \mathrm{GL}(W)$ induced by the action of G^+ on any irreducible G^+-submodule W of V*:

(i) $p = (q + 1)/2$, $\dim W = p - 1$, *and* $H \cong \mathrm{SL}_2(q)$.

(ii) $p = 2^f + 1$ *is a Fermat prime*, $\dim W = p - 2$, *and* $H \cong \mathrm{SL}_2(2^f)$.

(iii) $(H, p, d) = (3\mathsf{A}_6, 5, 3)$ *and* $(2\mathsf{A}_7, 7, 4)$.

Proof. Write $V|_{G^+} = e\sum_{i=1}^t W_i$ as usual and let K_i be the kernel of the action of G^+ on W_i. By Lemma 7.2, it suffices to show that $\mathrm{Ext}^1_{G^+}(W_i, W_j) = 0$ for all i, j. Note that neither G^+ nor K_i can have C_p as a composition factor, according to Theorem 2.4. Furthermore, if $K_i \neq K_j$ then we are done by Corollary 7.6. So we may assume that $K_i = K_j$ and then by Lemma 7.3 replace G^+ by $H = G^+/K_i = G^+/K_j$. Now we will go over the possibilities for (H, W_i) listed in Theorem 2.1(b)–(d).

Suppose we are in the case (b1) of Theorem 2.1. Assume first that $(p, H) = ((q^n + 1)/2, \mathrm{Sp}_{2n}(q))$. It is well known (see [Guralnick et al. 2002, Theorem 2.1]) that H has exactly two irreducible modules of dimension $(q^n - 1)/2$, namely the two Weil modules of that dimension. Hence we can apply Lemma 10.1 and arrive at the exception (i).

Next, assume that $(p, H) = ((q^n + 1)/(q + 1), \mathrm{PSU}_n(q))$; in particular, $n \geq 3$ is odd. Applying [Guralnick et al. 2002, Theorem 2.7 and Proposition 11.3], we see that there is a unique irreducible kH-module of dimension $p - 1 = (q^n - q)/(q + 1)$, and, furthermore, that this module has a unique complex lift. Hence we are done by Lemma 7.1.

Suppose now that we are in the case (c) of Theorem 2.1. If $H = \mathsf{A}_p$, then using [Guralnick and Tiep 2005, Lemma 6.1] for $p \geq 17$ and [Conway et al. 1985] for $p \leq 13$, we see that H has a unique irreducible kH-module of dimension $p-2$, and, furthermore, that this module has no complex lift unless $p = 5$, whence we are done by Lemma 7.1. Note that the exception $p = 5$ is recorded in (ii) (with $f = 2$).

Next, assume that $(p, H) = ((q^n - 1)/(q-1), \mathrm{PSL}_n(q))$. If $n = 2$, then $p = q + 1 = 2^f + 1$ is a Fermat prime, in which case $H = \mathrm{SL}_2(2^f)$ has a unique irreducible kH-module W of dimension $p-2$, with 2^{f-1} complex lifts, whence $\dim \mathrm{Ext}^1_H(W, W) = 1$ by Lemma 7.1. This exception is recorded in (ii). If $n \geq 3$, then by [Guralnick and Tiep 1999, Theorem 1.1], H has a unique irreducible kH-module W of dimension $p-2$ with no complex lifts, whence $\dim \mathrm{Ext}^1_H(W, W) = 0$ by Lemma 7.1.

It remains to consider the 19 cases listed in Lemma 10.2. Furthermore, by Corollary 7.6, we need only consider the case where G^+ acts on W_i and W_j with the same kernel. Since G^+ has no composition factor isomorphic to C_p, by Lemma 7.3(ii) we may view W_i and W_j as modules over the same quasisimple group H, with the same kernel. The irreducibility of G on V further implies that $W_j \cong W_i^g$ for some $g \in G$, whence the H-modules W_i and W_j are $\mathrm{Aut}(H)$-conjugate. Now we are done by applying Lemma 10.2. □

Corollary 10.6. *Suppose that $p = 2^f + 1$ is a Fermat prime. Then there is a finite absolutely irreducible linear group $G < \mathrm{GL}(V) = \mathrm{GL}_{p-2}(k)$ of degree $p-2$ over k such that $G = G^+ \cong \mathrm{SL}_2(2^f)$ and $\dim \mathrm{Ext}^1_G(V, V) = 1$. In particular, (G, V) is not adequate.*

Proof. See the proof of Proposition 10.5 and the exception (ii) listed therein. □

Proof of Theorem 1.3. (a) Assume first that G is not p-solvable. Then G^+ has no composition factor isomorphic to C_p, and $H^1(G, k) = 0$ by Theorem 2.4. By Lemma 7.2, we need to verify that $\mathrm{Ext}^1_{G^+}(W_i, W_j) = 0$ for any two simple G^+-submodules W_i and W_j of V, of dimension $1 < d < p$. Suppose for instance that $\mathrm{Ext}^1_{G^+}(W_1, W_2) \neq 0$.

Suppose in addition that $p > 3$. Then the perfect group G^+ admits a reducible indecomposable module U with two composition factors W_1 and W_2, of dimension $2d$, say with kernel K. Since G^+ has no composition factor isomorphic to C_p, $\boldsymbol{O}_p(X) = 1$ for the group $X := G^+/K$ induced by the action of G^+ on U. Suppose that X is not quasisimple. By Proposition 9.7, we have $d = p - 1$. Then by Lemma 9.6, either we arrive at the exception (b)(ii) listed in Theorem 1.3, or else Lemma 9.3 applies. In the latter case, we see that $H^1(X, k) \neq 0$, whence X and G^+ admit C_p as a composition factor, a contradiction. Thus X is quasisimple and $\boldsymbol{Z}(X)$ is a p'-group. If X is of Lie type in characteristic $p > 3$, then we must have $d = (p \pm 1)/2$ and arrive (using Lemma 9.5) at the exception (b)(i). Otherwise we are in the case (i) of Theorem 2.4, and so by Proposition 10.5 we arrive at the exceptions (b)(iii)–(v).

(b) Now we consider the case where $p = 3$ and G is not p-solvable. Then the perfect group G^+ acts nontrivially on W_1 and W_2, which are of dimension 2. Applying Theorem 2.4, we see that $G^+ = L_1 * \cdots * L_n$ is a central product of quasisimple groups; moreover, for all j we have that $L_j = \mathrm{SL}_2(q)$ with $q = 3^a > 3$ or $q = 5$. Also, for each i, there is a unique k_i such that L_j acts nontrivially on W_i precisely when $j = k_i$. Since $\mathrm{Ext}^1_X(k, k) = 0$ for any perfect group X, by Lemma 7.7 we may assume that $k_1 = k_2 = 1$ and $\mathrm{Ext}^1_{L_1}(W_1, W_2) \neq 0$. If $q = 5$, then the case (b)(iii) holds. Otherwise we arrive at (b)(vi) — indeed, $\mathrm{Ext}^1_{L_1}(L(3^{a-2}), L(3^{a-1})) \neq 0$ by [Andersen et al. 1983, Corollary 4.5].

(c) We may now assume that G^+ is p-solvable (and so is G). In particular, the subgroup $H < \mathrm{GL}(W_i)$ induced by the action of G^+ on W_i is p-solvable, whence p is a Fermat prime, and $H = \boldsymbol{O}_{p'}(H)P$ with $P \cong C_p$. Since G^+ projects onto H, G^+ also has C_p as a composition factor, and so $H^1(G^+, k) \neq 0$; in particular, $\mathrm{Ext}^1_{G^+}(V, V) \neq 0$. We arrive at the exception (a) of Theorem 1.3. □

Proof of Corollary 1.4. Suppose that (G, V) is *not* adequate, and let $\overline{V} := V \otimes_k \bar{k}$. By the assumptions, $\dim W < p$. Since $\dim \overline{V} / \dim W$ divides $|G/G^+|$ by [Navarro 1998, Theorem 8.30], $p \nmid \dim_{\bar{k}} \overline{V} = \dim_k V$. Next, (G, V) is weakly adequate by Theorem 1.2. It follows that $\mathrm{Ext}^1_G(V, V) \neq 0$ and so $\mathrm{Ext}^1_G(\overline{V}, \overline{V}) \neq 0$. Now we can apply Theorem 1.3. □

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