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The main goal of this paper is to construct noncommutative Hilbert modular symbols. However, we also construct commutative Hilbert modular symbols. Both the commutative and the noncommutative Hilbert modular symbols are generalizations of Manin's classical and noncommutative modular symbols. We prove that many cases of (non)commutative Hilbert modular symbols are periods in the Kontsevich–Zagier sense. Hecke operators act naturally on them.

Manin defined the noncommutative modular symbol in terms of iterated path integrals. In order to define noncommutative Hilbert modular symbols, we use a generalization of iterated path integrals to higher dimensions, which we call iterated integrals on membranes. Manin examined similarities between noncommutative modular symbol and multiple zeta values in terms of both infinite series and of iterated path integrals. Here we examine similarities in the formulas for noncommutative Hilbert modular symbol and multiple Dedekind zeta values, recently defined by the current author, in terms of both infinite series and iterated integrals on membranes.

1. Introduction

Classical elliptic modular symbols were introduced by Birch [1971] and Manin [1972] in connection with the Birch–Swinnerton-Dyer conjecture for certain congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We recall that a modular symbol $\{p, q\}$ is associated to a pair of cusp points $p, q \in \mathbb{P}^1(\mathbb{Q})$ on the completed upper half-plane $\mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q})$. One can think of the modular symbol $\{p, q\}$ as a homology class of the geodesic

connecting p and q, in $H_1(X_\Gamma, \{\text{cusps}\})$, where X_Γ is the modular curve associated to a congruence subgroup of $SL_2(\mathbb{Z})$. One can pair $\{p, q\}$ with a cusp form f by

$$\{p, q\} \times f \mapsto \int_p^q f \, dz$$

If f is a cusp form of weight 2 then $f \, dz$ can be viewed as a cohomology class in $H^1(X_\Gamma)$. This gives a pairing between homology (Betti) and cohomology (de Rham) that leads to periods. Modular symbols are a useful tool for studying L-functions and in the computation of cohomology groups. For a review of such topics, one can consult [Manin 2009].

Their theory was developed in [Manin 1972; Drinfeld 1973; Shokurov 1980; Mazur 1973]. Later the theory was extended to higher ranks in [Ash and Rudolph 1979; Ash and Borel 1990; Gunnells 2000b].

Elliptic modular symbols are important tool in the study of modular forms. They are particularly useful in computations with modular forms. J. Cremona [1997] designed algorithms for computations with elliptic curves, based on modular symbols ("modular symbol algorithms"). Some of the applications include the computation of homology and cohomology. Also, the study of special values of L-functions became a vast area of applications of classical modular symbols; see [Mazur and Swinnerton-Dyer 1974; Kazhdan et al. 2000].

Later, W. Stein also contributed to the difficult area of computations with modular forms. See his excellent book [Stein 2007], which contains both theory and computational methods. For higher-rank groups, one can consult the appendix of this book, by P. Gunnells.

Manin's noncommutative modular symbol [2006] is a generalization of both the classical modular symbol and of multiple zeta values in terms of Chen's iterated integral theory in the holomorphic setting. Manin showed that the noncommutative modular symbol is a noncommutative 1-cocycle. He also showed that each of the iterated integrals on Hecke eigenforms that enter in the noncommutative modular symbol are periods.

The main goal of this paper is to construct noncommutative Hilbert modular symbols. However, we also construct an analog of the classical modular symbol for Hilbert modular varieties. Both symbols are generalizations of the corresponding constructions by Manin.

We compute explicit integrals in terms of the noncommutative Hilbert modular symbol of type $\boldsymbol{b}$, and present similar formulas for the recently defined multiple Dedekind zeta values (see [Horozov 2014b]). We prove that the iterated integrals on membranes that enter in the noncommutative modular symbol of type $\boldsymbol{c}$ are periods. We also give some explicit and some categorical arguments in support of a

conjecture that a certain type of noncommutative Hilbert modular symbol satisfies a noncommutative 2-cocycle condition.

Before describing our results, let us recall the noncommutative modular symbol of Manin [2006]. Let $\nabla = d - \sum_{i=1}^{m} X_i f_i \, dz$ be a connection on the upper half-plane, where $f_1, \ldots, f_m$ are cusp forms and $X_1, \ldots, X_m$ are formal variables. One can think of $X_1, \ldots, X_m$ as constant square matrices of the same size.

Let J_a^b be the parallel transport to the point b of the identity matrix 1 at the point a. Alternatively, J_b^a can be written as a generating series of iterated path integrals of the forms $f_1 \, dz, \ldots, f_m \, dz$, (see [Chen 1977] and [Manin 2006]), namely,

$$J_a^b = 1 + \sum_{i=1}^{m} X_i \int_a^b f_i \, dz + \sum_{i,j=1}^{m} X_i X_j \int_a^b f_i \, dz \cdot f_j \, dz + \cdots$$

Then $J_a^b J_b^c = J_a^c$. This property leads to the 1-cocycle $c_a^1(\gamma) = J_{\gamma a}^a$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, which is the noncommutative modular symbol (see [Manin 2006] and Section 2 of this paper). If $f_1, \ldots, f_m$ are normalized cusp Hecke eigenforms, then each iterated integral appearing in the generating series J_a^b is a period. In this paper we introduce both commutative and noncommutative modular symbols for Hilbert modular surfaces. As it turned out we need some new tools, in comparison to the classical modular symbols. In particular, for the noncommutative Hilbert modular symbol we need iterated integrals in dimension higher than one. We introduce them and study their properties in the special case of Hilbert modular surfaces for the Hilbert modular group $\mathrm{SL}_2(\mathbb{O}_K)$. For the Hilbert modular group, one may consult [Bruinier et al. 2008] and [Freitag 1990]. In the case of Hilbert modular surface, it is not possible to repeat Manin's constructions for the noncommutative modular symbols, since the integration domain is two-dimensional over the complex numbers. Instead, we develop a new approach (Section 3), which we call iterated integrals on membranes. This is a higher-dimensional analogue of iterated path integrals. In Section 4G, we explore similar relations between noncommutative Hilbert modular symbols and multiple Dedekind zeta values (see [Horozov 2014b]).

In Section 4, we associate modular symbols for $\mathrm{SL}_2(\mathbb{O}_K)$ to geodesic triangles and geodesic diangles (2-cells whose boundaries have two vertices and two edges, which are geodesics). We are going to explain how the geodesic triangles and the geodesic diangles are constructed. Consider four cusp points in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. We can map any three of them to 0, 1 and ∞ with a linear fractional transformation $\gamma \in \mathrm{GL}_2(K)$. There is a diagonal map $\mathbb{H}^1 \to \mathbb{H}^2$, whose image Δ contains 0, 1 and ∞. We can take a pullback of Δ with respect to the map γ in order to obtain a holomorphic (or antiholomorphic) curve that passes through the given three points. If $\det \gamma$ is totally positive or totally negative then $\gamma^* \Delta$ is a holomorphic curve in $\mathbb{H}^2$. If $\det \gamma$ is not totally positive or totally negative (that is, in one of the real embeddings

it is positive and in the other it is negative) then $\gamma^*\Delta$ is antiholomorphic. This means that it is holomorphic in $\mathbb{H}^2$ if we conjugate the complex structure in one of the copies of $\mathbb{H}^1$. The same type of change of the complex structure is considered in [Freitag 1990].

On each holomorphic (or antiholomorphic) curve $\gamma^*\Delta$, there is a unique geodesic triangle connecting the three given points. However, if we take two of the points, we see that they belong to two geodesic triangles. Thus they belong to two holomorphic, (antiholomorphic) curves. Therefore, there are two geodesic connecting the two points — each lying on different holomorphic (antiholomorphic) curves, as faces of the corresponding geodesic triangles defining the curves. There are two pairings that we consider: the first is an integral of a cusp form over a geodesic triangle and the second is an integral of a cusp form over a geodesic diangle. If we integrate a holomorphic 2-form coming from a cusp form over a geodesic triangle, we obtain 0, if the triangle lies on an holomorphic curve. Thus the only nonzero pairings come from integration of a cusp form over a diangle or over a triangle lying on an antiholomorphic curve.

Now let us look again at the four cusp points together with the geodesics that we have just described. We obtain four geodesic triangles, corresponding to each triple of points among the four points, and six diangles, corresponding to the six "edges" of a tetrahedron with vertices the four given points. Thus, we obtain a "tetrahedron" with thickened edges. We will use tetrahedrons with thickened edges as an intuition for a noncommutative 2-cocycle relation (see Conjecture 4.15) for the noncommutative Hilbert modular symbol, which is an analogue of Manin's noncommutative 1-cocycle relation for the noncommutative modular symbol.

Usually, the four vertices are treated as a tetrahedron and a 2-cocycle is functional on the faces, considered as 2-chains. The boundary is defined as a sum of the 2-cocycles on each of the faces (which are triangles). The boundary of the tetrahedron gives a boundary relation of a 2-cocycle.

In our case the analogue of a 2-cocycle is a functional on diangles and on triangles. And the boundary map is a sum over the faces of the thickened tetrahedron. Thus, the faces of the thickened tetrahedron are four triangles and six diangles, corresponding to the six edges of a tetrahedron.

We show that the geodesics on the boundary of a diangle or of a geodesic triangle lie on a holomorphic curves $\gamma^*\Delta$ for various elements γ with totally positive or totally negative determinant. This implies that when we take the quotient by a Hilbert modular group the holomorphic curve $\gamma^*\Delta$ becomes a Hirzebruch–Zagier divisor [Hirzebruch and Zagier 1976]. Then we prove that the commutative Hilbert modular symbols paired with a cusp forms of weight $(2, 2)$ gives periods in the sense of [Kontsevich and Zagier 2001].

In order to construct a noncommutative Hilbert modular symbol, first we define a suitable generalization of iterated path integrals, which we call iterated integrals on membranes (see Section 3). We choose the word "membrane" since such integrals are invariant under suitable variation of the domain of integration.

There is a topological reason for considering a noncommutative Hilbert modular symbol as opposed to only a commutative one. Let us first make such a comparison for the case of $SL_2(\mathbb{Z})$. The commutative modular symbol captures $H_1(X_\Gamma)$, while the noncommutative symbol captures the rational homotopy type of the modular curve X_Γ. Now, let $\widetilde{X}$ be a smooth Hilbert modular surface, by which we mean the minimal desingularization of the Borel–Baily compactification due to Hirzebruch. Then the rational fundamental group of a Hilbert modular surface vanishes: $\pi_1(\widetilde{X})_\mathbb{Q} = 0$ (see [Bruinier et al. 2008]). The noncommutative Hilbert modular symbol is an attempt to capture more from the rational homotopy type than $H_2(\widetilde{X})$ captures.

For the convenience of the reader, we first define type a iterated integrals on membranes (Definition 3.3). They are simpler to define and more intuitive. However, they do not have enough properties. (For example, they do not have an integral shuffle relation.) Then we define type b iterated integrals on membranes (Definition 3.4), which involves two permutations. Type b has integral shuffle relation (Theorem 3.21(i)), and type a is a particular case of type b.

We are mostly interested in iterated integrals of type b. If there is no index specifying the type of iterated integral over membranes, we assume that it is of type b.

Similarly to Manin's approach, we define a generating series of iterated integrals over membrane of type b over U, which we denote by $J(U)$. We also define a shuffle product of generating series of iterated integrals over membranes of type b (see Theorem 3.21(iii)),

$$\phi(J(U_1) \times_{\text{Sh}} J(U_2)) = J(U_1 \cup U_2),$$

for U_1, U_2 disjoint 2-dimensional manifolds with corners contained in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ (see [Borel and Serre 1973]). This shuffle product generalizes the composition of generating series of iterated path integrals, namely, $J_a^b J_b^c = J_a^c$, to dimension 2. Note that a similar definition is also possible in higher dimensions. Also, $J(U)$ is invariant under homotopy. This allows us to consider cocycles and coboundaries, where the relations use homotopy invariance and values at different cells can be composed via the shuffle product.

We define noncommutative Hilbert modular symbols, which we call c^1 and c^2; c^1 is the functional J on certain geodesic diangles and c^2 is the functional J on geodesic triangles. Conjecturally, c^1 is a 1-cocycle such that if we change the base point of c^1 then c^1 is modified by a coboundary. Also, conjecturally, c^2 is a 2-cocycle up to finitely many multiples of different values of c^1. Also, if we

change the base point of c^2 then c^2 is modified by a coboundary up to a finitely many multiples of different values of c^1. In Section 4E we give explicit formulas in support of the interpretation of the noncommutative symbols as cocycles.

In Section 4F, we give a categorical construction, which might help to prove that the noncommutative symbols are cocycles.

In Section 4G, we define multiple L-values associated to cusp forms, and we compare them to multiple Dedekind zeta values (see [Horozov 2014b]).

We also briefly recall the Riemann zeta values and multiple zeta values (MZVs). The Riemann zeta values are defined as

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k},$$

where n is an integer. MZVs are defined as

$$\zeta(k_1, \ldots, k_m) = \sum_{0 < n_1 < \cdots < n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}},$$

where $n_1, \ldots, n_m$ are integers. The above MZV is of depth m. Riemann zeta values $\zeta(k)$ and MZV $\zeta(k_1, \ldots, k_m)$ were defined by Euler [1748] for $m = 1, 2$.

The common feature of MZVs and the noncommutative modular symbol is that they both can be written as iterated path integrals (see [Goncharov 2001a; 2001b]). Moreover, Manin's noncommutative modular symbol resembles the generating series of MZV, which is the Drinfeld associator. Let us recall that the Drinfeld associator is a generating series of iterated integrals of the type J_a^b associated to the connection

$$\nabla = d - A \frac{dx}{x} - B \frac{dx}{1-x}$$

on $Y_{\Gamma(2)} = \mathbb{P}^1 - \{0, 1, \infty\}$. One can think of $Y_{\Gamma(2)}$ as the modular curve associated to the congruence subgroup $\Gamma(2)$ of $SL_2(\mathbb{Z})$. Then the differential forms dx/x and $dx/(1-x)$ are Eisenstein series of weight 2 on the modular curve $Y_{\Gamma(2)}$.

Relations between MZV and modular forms have been examined by many authors. For example, Goncharov [2001b; 2001c] considered a mysterious relation between MZV (multiple zeta values) of given weight and depth 3 and the cohomology of $GL_3(\mathbb{Z})$, which is closely related to the cohomology of $SL_3(\mathbb{Z})$. In the pursuit of such a relation in depth 4, Goncharov suggested and the current author computed the group cohomology of $GL_4(\mathbb{Z})$ with coefficients in a family of representations [Horozov 2014a]. Another relation between modular forms and MZV is presented in [Gangl et al. 2006].

Similarly to Manin's approach, we explore relations between the noncommutative Hilbert modular symbols and multiple Dedekind zeta values (see [Horozov 2014b]). Let us recall multiple Dedekind zeta values. Let each of $C_1, \ldots, C_m$ be a suitable

subset of the ring of integers $\mathbb{O}_K$ of a number field K. We call each of $C_1, \ldots, C_m$ a cone. Then *multiple Dedekind zeta values* are defined as

$$\zeta_{K;C_1,\ldots,C_m}(k_1, \ldots, k_m) = \sum_{\alpha_i \in C_i} \frac{1}{N(\alpha_1)^{k_1} N(\alpha_1 + \alpha_2)^{k_2} \cdots N(\alpha_1 + \cdots + \alpha_m)^{k_m}}.$$

The connection between noncommutative Hilbert modular symbols and multiple Dedekind zeta values lies both in the similarities in the infinite sum formulas and in the definition in terms of iterated integrals on membranes (see [Horozov 2014b]).

We consider a noncommutative Hilbert modular symbol of type $\boldsymbol{b}$ over one diangle and compare it with (multiple) Dedekind zeta values with summation over one discrete cone [Horozov 2014b]. However, in this case the two series look very different. We obtain that the multiple L-values are noncommutative modular symbols defined as J evaluated at an infinite union of diangles. We obtain that such L-values are very similar to the sum of multiple Dedekind zeta values, in the same way that the integrals in Manin's noncommutative modular symbol are similar to the multiple zeta values (MZV). Then the sum of the multiple Dedekind zeta values is over an infinite union of cones. The idea of considering cones originated in [Zagier 1976] and more generally in [Shintani 1976].

Classical or commutative modular symbols for $SL_3(\mathbb{Z})$ and $SL_4(\mathbb{Z})$ were constructed in [Ash and Borel 1990] and [Gunnells 2000a]. For $GL_2(\mathbb{O}_K)$, where K is a real quadratic field, Gunnells and Yasaki [2008] defined a modular symbol based on Voronoi decomposition of a fundamental domain, in order to compute the third cohomology group of $GL_2(\mathbb{O}_K)$. (For the Hilbert modular group $SL_2(\mathbb{O}_K)$ one may consult [Bruinier et al. 2008; Freitag 1990].) In contrast, here we use a geodesic triangulation of $\mathbb{H}^2/SL_2(\mathbb{O}_K)$. We are interested mostly in 2-cells, whose boundaries are geodesics. One of the (commutative) symbols that we define here resembles combinatorially the symplectic modular symbol of [Gunnells 2000b]. However, the meanings of the two types of symbols and their approaches are different.

There are several different directions for further work on Hilbert modular symbols. First of all, the commutative Hilbert modular symbols behave well with respect to Hecke operators. It will be interesting to extend the Hecke operators to cases of higher equal weights (k, k). To apply Hecke operators to Hilbert modular groups one either assumes a trivial narrow class group or one has to work with adeles. Another possible continuation of the current work is to extend commutative Hilbert modular symbols to the adelic setting. Then, one may try to extend these properties — higher equal weight cusp forms and Hecke operators in the adelic setting — to the noncommutative Hilbert modular symbols. Hopefully, the abelian Hilbert modular symbol would lead to computational tools for cohomology of some Hilbert modular groups with coefficients in various representations.

For the noncommutative Hilbert modular symbols we expect that some of the continuations would be establishing the 2-categorical framework that define nonabelian 2-cohomology sets. This work would also have applications to noncommutative reciprocity laws on algebraic surfaces. In dimension 1, we have a noncommutative reciprocity law as a reciprocity law for a generating series of iterated path integrals on a complex curve [Horozov 2011]. In dimension 2 we have proven both the Parshin reciprocity and a new reciprocity for a 4-function local symbols [Horozov 2014c] defined by the author, which are particular cases in the generating series. A 2-categorical second cohomology set would capture algebraically the generating series of iterated integrals on membranes needed for the general reciprocity on algebraic surfaces.

Finally, we expect that the (non)commutative Hilbert modular symbols would be useful for studying L-functions and multiple L-functions together with their special values.

2. Manin's noncommutative modular symbol

In this section we recall the definition and main properties of Manin's [2006] noncommutative modular symbol. In this paper, Manin uses iterated path integrals on a modular curve and on its universal cover — the upper half-plane. Our main constructions are parallel to some extent to Manin's approach, and for that reason we recall it below. However, instead of iterated path integrals we introduce a new tool — iterated integrals on membranes (see Section 3). Only this notion is adequate for studying noncommutative Hilbert modular symbols, by generalizing the iteration process to higher dimensions.

2A. *Iterated path integrals.* Here we recall iterated path integrals (see also [Parshin 1966; Chen 1977; Goncharov 2001a; Manin 2006]). In Section 3, we generalize them to iterated integrals over membranes.

Definition 2.1. Let $\omega_1, \ldots, \omega_m$ be m holomorphic 1-forms on $\mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q})$, the upper half-plane together with the cusps. Let

$$g : [0, 1] \to \mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q}),$$

be a piecewise smooth path. We define an iterated integral

$$\int_g \omega_1 \cdots \omega_m = \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_m < 1} g^*\omega_1(t_1) \wedge \cdots \wedge g^*\omega_m(t_m).$$

Let $X_1, \ldots, X_m$ be formal variables. Consider the differential equation

$$dF(\Omega) = F(\Omega)(X_1\omega_1 + \cdots + X_n\omega_m) \tag{1}$$

with values in the associative but noncommutative ring of formal power series in the noncommuting variables $X_1, \ldots, X_m$ over the ring of holomorphic functions on the upper half-plane. There is a unique solution with initial condition $F(\Omega)(g(0)) = 1$; that is, equal to 1 at the starting point $g(0)$. Then at the end of the path, that is, at the point $g(1)$, $F(\Omega)$ has the value

$$F_g(\Omega) = 1 + \sum_{i=1}^{m} X_i \int_g \omega_i + \sum_{i,j=1}^{m} X_i X_j \int_g \omega_i \omega_j + \sum_{i,j,k=1}^{m} X_i X_j X_k \int_g \omega_i \omega_j \omega_k + \cdots. \quad (2)$$

Using the solution (2) to (1), we prove the following theorem:

Theorem 2.2. *Let g_1 and g_2 be two paths such that the end of g_1 (i.e., $g_1(1)$) is equal to the beginning of g_2 (i.e., $g_2(0)$). Let $g_1 g_2$ denote the concatenation of g_1 and g_2. Then*

$$F_{g_1 g_2}(\Omega) = F_{g_1}(\Omega) F_{g_2}(\Omega).$$

Proof. The left-hand side is the value of the solution of the linear first-order ordinary differential equation at the point $g_2(1)$. From the uniqueness of the solution, we have that the solution along g_2 gives the same result, when the initial condition at $g_2(0)$ is $F_{g_1}(\Omega)$. That result is $F_{g_1}(\Omega) F_{g_2}(\Omega)$. □

The same result can be proven via product formula for iterated integrals. We need this alternative proof in order to generalize to higher dimensions.

Lemma 2.3 (product formula). *Let $\omega_1, \ldots, \omega_m$ be holomorphic 1-forms on $\mathbb{C}$ and g_1, g_2 two paths such that the end of g_1 is the beginning of g_2, that is, $g_1(1) = g_2(0)$. As before we denote by $g_1 g_2$ the concatenation of the paths g_1 and g_2. Then*

$$\int_{g_1 g_2} \omega_1 \cdots \omega_m = \sum_{i=0}^{m} \int_{g_1} \omega_1 \cdots \omega_i \int_{g_2} \omega_{i+1} \cdots \omega_m.$$

Proof. Let $g_1 : [0, 1] \to \mathbb{C}$ and let $g_2 : [1, 2] \to \mathbb{C}$. We consider the concatenation $g_1 g_2$ to be a map $g_1 g_2 : [0, 2] \to \mathbb{C}$ whose restriction to the interval $[0, 1]$ gives the path g_1 and whose restriction to the interval $[1, 2]$ gives g_2. From Definition 2.1, we have that

$$\int_{g_1 g_2} \omega_1 \cdots \omega_m = \int \cdots \int_{0 < t_1 < \cdots < t_m < 2} (g_1 g_2)^* \omega_1(t_1) \wedge \cdots \wedge (g_1 g_2)^* \omega_m(t_m).$$

In the domain of integration $0 < t_1 < \cdots < t_m < 2$ insert the number 1. Geometrically, we cut the simplex $0 < t_1 < \cdots < t_m < 2$ into a disjoint union of products of pairs of simplices such that $t_k \in [0, 1]$ for $k \leq i$ and $t_k \in [1, 2]$ for $k > i$. Thus, the union is over distinct values of i for $i = 0, \ldots, m$. And for each fixed i the two simplices

are $0 < t_1 < \cdots < t_i < 1$ and $1 < t_{i+1} < \cdots < t_m < 2$. Then we have

$$
\int_{g_1 g_2} \omega_1 \cdots \omega_m = \sum_{i=0}^{n} \int \cdots \int_{\substack{0 < t_1 < \cdots < t_i < 1 \\ 1 < t_{i+1} < \cdots < t_m < 2}} (g_1 g_2)^* \omega_1(t_1) \wedge \cdots \wedge (g_1 g_2)^* \omega_m(t_m)
$$

$$
= \sum_{i=0}^{n} \left(\int \cdots \int_{0 < t_1 < \cdots < t_i < 1} g_1^* \omega_1(t_1) \wedge \cdots \wedge g_1^* \omega_i(t_i) \right)
$$

$$
\times \left(\int \cdots \int_{1 < t_{i+1} < \cdots < t_m < 2} g_2^* \omega_{i+1}(t_{i+1}) \wedge \cdots \wedge g_2^* \omega_m(t_m) \right)
$$

$$
= \sum_{i=0}^{m} \int_{g_1} \omega_1 \cdots \omega_i \int_{g_2} \omega_{i+1} \cdots \omega_m. \qquad \square
$$

Definition 2.4. The set of all shuffles $\mathrm{sh}(i, j)$ is a subset of all permutations σ of the set $\{1, 2, \ldots, i + j\}$ such that

$$
\sigma(1) < \cdots < \sigma(i)
$$

and

$$
\sigma(i + 1) < \cdots < \sigma(i + j).
$$

Such a permutation σ is called a shuffle.

Lemma 2.5 (shuffle relation). *Let $\omega_1, \ldots, \omega_m$ be holomorphic 1-forms on $\mathbb{C}$ and let g be a path. Then*

$$
\int_g \omega_1 \cdots \omega_i \int_g \omega_{i+1} \cdots \omega_m = \sum_{\sigma \in \mathrm{sh}(i, m-i)} \int_g \omega_{\rho(1)} \cdots \omega_{\rho(m)},
$$

where $\mathrm{sh}(i, j)$ is the set of shuffles from Definition 2.4.

2B. *Manin's noncommutative modular symbol.* Now let g be a geodesic connecting two cusps a and b in the completed upper half-plane $\mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q})$. Let $\Omega = \{f_1 \, dz, \ldots, f_m \, dz\}$ be a finite set of holomorphic forms with respect to a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ such that $f_1, \ldots, f_m$ are cusp forms of weight 2. Let

$$
J_a^b = F_g(\Omega).
$$

As a reformulation of Theorem 2.2, we obtain:

Lemma 2.6. $J_a^b J_b^c = J_a^c.$

The following is a direct consequence:

Corollary 2.7. $J_b^a = (J_a^b)^{-1}.$

Now we are ready to define Manin's noncommutative modular symbol. Note that there is a natural action of Γ on J_a^b. If $\gamma \in \Gamma$ then γJ_a^b is defined as $J_{\gamma a}^{\gamma b}$. If $f_1, \ldots, f_m$ are cusp forms of weight 2, then $\omega_1 = f_1 \, dz, \ldots, \omega_m = f_m \, dz$ are forms of weight 0, that is, they are invariant forms with respect to the group Γ. Then

$$\gamma J_a^b = F_{\gamma g}(\omega_1, \ldots, \omega_m) = F_g(g^* \omega_1, \ldots, g^* \omega_m) = F_g(\omega_1, \ldots, \omega_m) = J_a^b.$$

Let Π be the subgroup of invertible elements of $\mathbb{C}\langle\!\langle X_1, \ldots, X_m \rangle\!\rangle$ with constant term 1. We extend the action of Γ on J_a^b to a trivial action of Γ on Π.

Following Manin, we present the key theorem and definition for the noncommutative modular symbol:

Theorem 2.8. *Let*

$$c_a^1(\gamma) = J_{\gamma a}^a.$$

Then c_a^1 represent a cohomology class in $H^1(\Gamma, \Pi)$, independently of the base point a.

Proof. First, c_a^1 is a cocycle:

$$dc_a^1(\beta, \gamma) = J_{\beta a}^a (\beta \cdot J_{\gamma a}^a)(J_{\beta \gamma a}^a)^{-1} = J_{\beta a}^a J_{\beta \gamma a}^{\beta a} J_a^{\beta \gamma a} = 1.$$

Second, c_a^1 and c_b^1 are homologous:

$$c_a^1(\gamma) = J_{\gamma a}^a = J_b^a J_{\gamma b}^b J_{\gamma a}^{\gamma b} = J_b^a c_b^1(\gamma)(\gamma \cdot J_b^a)^{-1}. \qquad \square$$

Definition 2.9. A noncommutative modular symbol is a nonabelian cohomology class in $H^1(\Gamma, \Pi)$, with representative

$$c_a^1(\gamma) = J_{\gamma a}^a,$$

3. Iterated integrals on membranes

Iterated integrals on membranes are a higher-dimensional analogue of iterated path integrals. This technical tool was used in [Horozov 2014b] for constructing multiple Dedekind zeta values and in [Horozov 2014c] for proving new and classical reciprocity laws on algebraic surfaces. It appeared first in the preprint [Horozov 2006] for the purpose of noncommutative Hilbert modular symbols.

3A. *Definition and properties.* Let $\mathbb{H}^1$ be the upper half-plane. Let $\mathbb{H}^2$ be a product of two upper half-planes. We are interested in the action of $\mathrm{GL}_2(K)$, where K is a real quadratic field. This group acts on $\mathbb{H}^2$ by linear fractional transformations. It is convenient to introduce cusp points $\mathbb{P}^1(K)$ as boundary points of $\mathbb{H}^2$.

Let $\omega_1, \ldots, \omega_m$ be holomorphic 2-forms on $\mathbb{H}^2$ which are continuous at the cusps $\mathbb{P}^1(K)$. Let

$$g : [0, 1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$$

be a continuous map which is smooth almost everywhere. Denote by F^1 and F^2 the following coordinatewise foliations: for any $a \in [0, 1]$, define the leaves

$$F_a^1 = \{(t_1, t_2) \in [0, 1]^2 \mid t_1 = a\} \quad \text{and} \quad F_a^2 = \{(t_1, t_2) \in [0, 1]^2 \mid t_2 = a\}.$$

Definition 3.1. We call the above map $g : [0, 1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$ a membrane on $\mathbb{H}^2$ if it is a continuous and piecewise differentiable map such that $g(F_a^1)$ and $g(F_a^2)$ belong to a finite union of holomorphic curves in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ for all constants a.

Similarly, we define a membrane of a Hilbert modular variety. Let $\omega_1, \ldots, \omega_m$ be holomorphic 2-forms on $Y_\Gamma = \mathbb{H}^2/\Gamma$ which are continuous at the cusps $\mathbb{P}^1(K)/\Gamma$. Let

$$g : [0, 1]^2 \to X_\Gamma$$

be a continuous map which is smooth almost everywhere, where $X_\Gamma = \mathbb{H}^2 \cup \mathbb{P}^1(K)$. Let $f_i : X_\Gamma \to \mathbb{P}^1(\mathbb{C})$ for $i = 1, 2$ be two algebraically independent rational functions on the Hilbert modular surface X_Γ. Denote by F^1 and F^2 the following coordinatewise foliations: for any $a \in [0, 1]$, define the leaves

$$F_a^1 = \{(t_1, t_2) \in [0, 1]^2 \mid t_1 = a\} \quad \text{and} \quad F_a^2 = \{(t_1, t_2) \in [0, 1]^2 \mid t_2 = a\}.$$

Let also

$$P_x^1 = \{P \in X_\Gamma \mid f_1(P) = x\} \quad \text{and} \quad P_x^2 = \{P \in X_\Gamma \mid f_2(P) = x\}.$$

Definition 3.2. We call the above map $g : [0, 1]^2 \to X_\Gamma$ a membrane on X_Γ if it is a continuous and piecewise differentiable map such that for each a there are x_1 and x_2 such that $g(F_a^1) \subset P_{x_1}^1$ and $g(F_a^2) \subset P_{x_2}^2$.

We define three types of iterated integrals over membranes — type **a**, type **b** and type **c**. Type **a** consists of linear iterations, while type **b** is more general and involves permutations. Type **a** is less general, but more intuitive. The advantage of type **b** is that it satisfies integral shuffle relation (Theorem 3.21). In other words a product of two integrals of type **b** can be expresses as a finite sum of iterated integrals over membranes of type **b**. However, one might not be able to express a product of two integrals of type **a** as a sum of finitely many integrals of type **a**. Both type **a** and type **b** are defined on a product of two upper half-planes. Type **c**

is defined on a Hilbert modular surface; that is, on a quotient of a product of upper half-planes by an arithmetic group which is commensurable to $\mathrm{SL}_2(\mathbb{O}_K)$. Type *c* also satisfies a shuffle product, that is, a product of two integrals of this type can be expresses a finite sum of such integrals.

Definition 3.3 (type *a*, ordered iteration over membranes). Let

$$g : [0, 1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$$

be a membrane on $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. Then define

$$\int_g \omega_1 \cdots \omega_m = \int_D \bigwedge_{j=1}^m g^* \omega_i(t_{1,j}, t_{2,j}),$$

where

$$D = \left\{ (t_{1,1}, \ldots, t_{2,m}) \in [0, 1]^{2m} \mid 0 \le t_{1,1} \le \cdots \le t_{1,m} \le 1, \ 0 \le t_{2,1} \le \cdots \le t_{2,m} \le 1 \right\}.$$

Definition 3.4 (type *b*, two permutations). Let

$$g : [0, 1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$$

be a membrane on $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, and let ρ_1, ρ_2 be two permutations of the set $\{1, 2, \ldots, m\}$. Then define

$$\int_g^{\rho_1, \rho_2} \omega_1 \cdots \omega_m = \int_D \bigwedge_{j=1}^m g^* \omega_j(t_{1,\rho_1(j)}, t_{2,\rho_2(j)}),$$

where

$$D = \left\{ (t_{1,1}, \ldots, t_{2,m}) \in [0, 1]^{2m} \mid \ \le t_{1,1} \le \cdots \le t_{1,m} \le 1, \ 0 \le t_{2,1} \le \cdots \le t_{2,m} \le 1 \right\}.$$

Definition 3.5 (type *c*, two permutations). Let

$$g : [0, 1]^2 \to X_\Gamma$$

be a membrane on the Hilbert modular surface $X_\Gamma = (\mathbb{H}^2 \cup \mathbb{P}^1(K))/\Gamma$, and let ρ_1, ρ_2 be two permutations of the set $\{1, 2, \ldots, m\}$. Then define

$$\int_g^{\rho_1, \rho_2} \omega_1 \cdots \omega_m = \int_D \bigwedge_{j=1}^m g^* \omega_j(t_{1,\rho_1(j)}, t_{2,\rho_2(j)}),$$

where

$$D = \left\{ (t_{1,1}, \ldots, t_{2,m}) \in [0, 1]^{2m} \mid 0 \le t_{1,1} \le \cdots \le t_{1,m} \le 1, \ 0 \le t_{2,1} \le \cdots \le t_{2,m} \le 1 \right\}.$$

Examples (iterated integrals of type b). Let $\alpha_i(t_1, t_2) = g^*\omega_i(t_1, t_2)$. Denote by (1) the trivial permutation and by (1 2) the permutation exchanging 1 and 2.

(1) The four diagrams

$t_{2,2}$		$\alpha_2(t_{1,2}, t_{2,2})$
$t_{2,1}$	$\alpha_1(t_{1,1}, t_{2,1})$	
	$t_{1,1}$	$t_{1,2}$

$t_{2,2}$	$\alpha_2(t_{1,1}, t_{2,2})$	
$t_{2,1}$		$\alpha_1(t_{1,2}, t_{2,1})$
	$t_{1,1}$	$t_{1,2}$

$t_{2,2}$		$\alpha_1(t_{1,2}, t_{2,2})$
$t_{2,1}$	$\alpha_2(t_{1,1}, t_{2,1})$	
	$t_{1,1}$	$t_{1,2}$

$t_{2,2}$	$\alpha_1(t_{1,1}, t_{2,2})$	
$t_{2,1}$		$\alpha_2(t_{1,2}, t_{2,1})$
	$t_{1,1}$	$t_{1,2}$
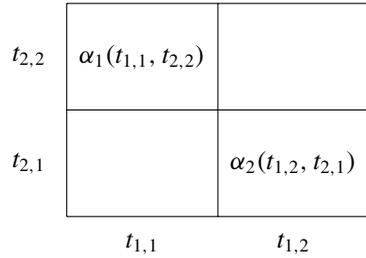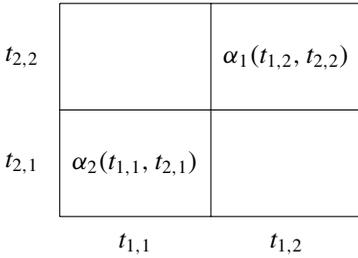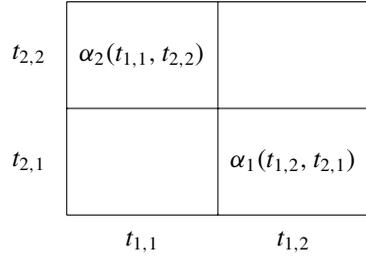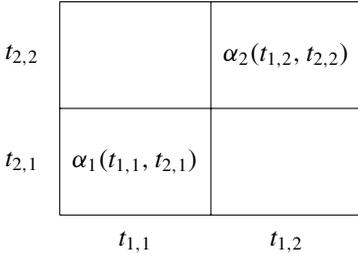
correspond, respectively, to the integrals

$$\int_g^{(2),(1)} \omega_1 \cdot \omega_2, \qquad \int_g^{(12),(1)} \omega_1 \cdot \omega_2,$$

$$\int_g^{(12),(12)} \omega_1 \cdot \omega_2, \qquad \int_g^{(1),(12)} \omega_1 \cdot \omega_2.$$

(2) The diagram

$t_{2,3}$			$\alpha_2(t_{1,3}, t_{2,3})$
$t_{2,2}$	$\alpha_2(t_{1,1}, t_{2,2})$		
$t_{2,1}$		$\alpha_1(t_{1,2}, t_{2,1})$	
	$t_{1,1}$	$t_{1,2}$	$t_{1,3}$
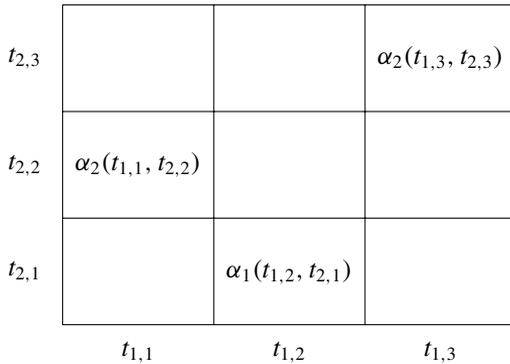
corresponds to the integral

$$\int_g^{(12),(1)} \omega_1 \cdot \omega_2 \cdot \omega_3.$$

Remark 3.6. Let us give more intuition for Definition 3.4. Each of the differential forms $g^*\omega_1, \ldots, g^*\omega_m$ has two arguments. Consider the set of first arguments for each of the differential forms $g^*\omega_1, \ldots, g^*\omega_m$. They are ordered as

$$0 < t_{1,1} < t_{1,2} < \cdots < t_{1,m} < 1 \tag{3}$$

(they are the coordinates of the domain D). Since $g^*\omega_j$ depends on $t_{1,\rho_1(j)}$, we have that $t_{1,k}$ is an argument of $g^*\omega_{\rho_1^{-1}(k)}$, where $k = \rho_1(j)$. Then we can order the differential forms $g^*\omega_1, \ldots, g^*\omega_m$ according to the order of their first arguments given by the inequalities (3), which is

$$g^*\omega_{\rho_1^{-1}(1)}, g^*\omega_{\rho_1^{-1}(2)}, \ldots, g^*\omega_{\rho_1^{-1}(m)}.$$

Similarly, we can order the differential forms $g^*\omega_1, \ldots, g^*\omega_m$ with respect to the order of their second arguments:

$$g^*\omega_{\rho_2^{-1}(1)}, g^*\omega_{\rho_2^{-1}(2)}, \ldots, g^*\omega_{\rho_2^{-1}(m)}.$$

We call the first ordering *horizontal* and the second ordering *vertical*.

Now we are going to examine the homotopy of a domain of integration and what that reflects about the integral. Let $g_s : [0, 1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$ be a family of membranes such that $g_s(0, 0) = \infty$ and $g_s(1, 1) = 0$. Assume that the parameter s is in the interval $[0, 1]$.

Let $h(s, t_1, t_2) = g_s(t_1, t_2)$ be a homotopy between g_0 and g_1. Let

$$G_s : [0, 1]^{2m} \to (\mathbb{H}^2 \cup \mathbb{P}^1(K))^m$$

be the map

$$G_s(t_{1,1}, \ldots, t_{2,m}) = \big(g_s(t_{1,\sigma_1(1)}, t_{2,\sigma_2(1)}), g_s(t_{1,\sigma_1(2)}, t_{2,\sigma_2(2)}), \ldots, g_s(t_{1,\sigma_1(m)}, t_{2,\sigma(m)})\big).$$

Let H be the induced homotopy between G_0 and G_1, defined by

$$H(s, t_{1,1}, \ldots, t_{2,m}) = G_s(t_{1,1}, \ldots, t_{2,m}).$$

We define diagonals in the domain $D \subset (0, 1)^{2m}$, where

$$D = \big\{(t_{1,1}, t_{2,1}, \ldots, t_{1,m}, t_{2,m}) \in (0, 1)^{2m} \mid 0 \leq t_{1,1} \leq t_{1,2} \leq \cdots \leq t_{1,m} \leq 1$$
$$\text{and } 0 \leq t_{2,1} \leq t_{2,2} \leq \cdots \leq t_{2,m} \leq 1\big\}.$$

We define $D_{1,k}$ for $k = 0, \ldots, m$ by $D_{1,0} = D|_{t_{1,1}=0}$, $D_{1,k} = D|_{t_{1,k}=t_{1,k+1}}$ for $k = 1, \ldots, m-1$ and $D_{1,m} = D|_{t_{1,m}=1}$. Similarly, we define $D_{2,k}$ for $k = 0, \ldots, m$ by $D_{2,0} = D|_{t_{2,1}=0}$, $D_{2,k} = D|_{t_{2,k}=t_{2,k+1}}$ for $k = 1, \ldots, m-1$ and $D_{2,m} = D|_{t_{2,m}=1}$.

For iterated integrals of types $\boldsymbol{a}$ and $\boldsymbol{b}$, we define diagonals in $V = (\mathbb{H}^2 \cup \mathbb{P}^1(K))^m$. We denote a generic coordinate of $V = (\mathbb{H}^2 \cup \mathbb{P}^1(K))^m$ by $(z_{1,1}, z_{2,1}, \ldots, z_{1,m}, z_{2,m})$ For $k = 1, \ldots, m-1$, let $V_{1,k} = V|_{z_{1,k}=z_{1,k+1}}$. Let also $V_{1,0} = V|_{z_{1,1}=0}$ and $V_{1,m} = V|_{z_{1,m}=1}$. Similarly, for $k = 1, \ldots, m-1$, let $V_{2,k} = V|_{z_{2,k}=z_{2,k+1}}$. Let also $V_{2,0} = V|_{z_{2,1}=0}$ and $V_{2,m} = V|_{z_{2,m}=1}$.

For iterated integrals of type $\boldsymbol{c}$, we define "diagonals" as fibers product of schemes corresponding to certain varieties (for fiber products of schemes one may look at the book [Hartshorne 1977]). Occasionally, it will be more natural to realize multiple fiber products as finite limits in the category of schemes of finite type over $\mathbb{C}$. Let $X_{i,j} = X_\Gamma$ for $i, j = 1, \ldots, n$. Let V be the universal scheme (finite limit) that maps to X_{ij} for each i and j as a part of a commutative diagram. The commutative diagram is defined as follows: $X_{i,j}$ and $X_{i+1,j}$ both map to $\mathbb{P}^1(\mathbb{C})$ via the morphism f_1 for $1 \le i \le n-1$ and all j, and $X_{i,j}$ and $X_{i,j+1}$ both map to $\mathbb{P}^1(\mathbb{C})$ via the morphism f_2 for $1 \le j \le n-1$ and all i. Define the following:

- Let $V_{1,0}$ be the subscheme of V defined by putting $\mathbb{P}^1(\mathbb{C})$ in the place of $X_{1,j}$, so that $f_1 : X_{1,j} \to \mathbb{P}^1(\mathbb{C})$ is replaced by the identity map and the corresponding $f_2 : X_{1,j} \to \mathbb{P}^1(\mathbb{C})$ is deleted.

- Let $V_{2,0}$ be the subscheme of V defined by putting $\mathbb{P}^1(\mathbb{C})$ in the place of $X_{i,1}$, so that $f_2 : X_{1,j} \to \mathbb{P}^1(\mathbb{C})$ is replaced by the identity map and the corresponding $f_1 : X_{1,j} \to \mathbb{P}^1(\mathbb{C})$ is deleted.

- Let $V_{1,n}$ be the subscheme of V defined by putting $\mathbb{P}^1(\mathbb{C})$ in the place of $X_{n,j}$, so that $f_1 : X_{n,j} \to \mathbb{P}^1(\mathbb{C})$ is replaced by the identity map and the corresponding $f_2 : X_{n,j} \to \mathbb{P}^1(\mathbb{C})$ is deleted.

- Let $V_{2,n}$ be the subscheme of V defined by putting $\mathbb{P}^1(\mathbb{C})$ in the place of $X_{i,n}$, so that $f_2 : X_{n,j} \to \mathbb{P}^1(\mathbb{C})$ is replaced by the identity map and the corresponding $f_1 : X_{n,j} \to \mathbb{P}^1(\mathbb{C})$ is deleted.

- Additionally, let $V_{1,i}$ be the subscheme of V obtained by replacing each factor $X_{i,j} \times_{\mathbb{P}^1(\mathbb{C})} X_{i+1,j}$ by the corresponding diagonal for fixed i and for all j.

- Finally, let $V_{2,j}$ be the subscheme of V obtained by replacing each factor $X_{i,j} \times_{\mathbb{P}^1(\mathbb{C})} X_{i,j+1}$ by the corresponding diagonal for fixed j and all i.

Theorem 3.7 (homotopy invariance theorem I). *The iterated integrals on membranes from* Definition 3.4 (*of type* $\boldsymbol{b}$) *are homotopy-invariant with respect to homotopies that preserve the boundary of the membrane.*

Proof. Let

$$\Omega = \bigwedge_{j=1}^m \omega_j(z_{1,\sigma_1(j)}, z_{2,\sigma_2(j)}).$$

Note that Ω is a closed form, since ω_i is a form of top dimension. By Stokes' theorem, we have

$$0 = \int_{s=0}^{s=1} \int_D H^* d\Omega \tag{4}$$

$$= \int_D G_1^* \Omega - \int_D G_0^* \Omega \tag{5}$$

$$\pm \int_{s=0}^{s=1} \sum_{k=1}^{m-1} \left(\int_{D_{1,k}} \pm \int_{D_{2,k}} \right) H^* \Omega \tag{6}$$

$$\pm \int_{s=0}^{s=1} \left(\int_{D_{1,0}} \pm \int_{D_{2,0}} \right) H^* \Omega \tag{7}$$

$$\pm \int_{s=0}^{s=1} \left(\int_{D_{1,m}} \pm \int_{D_{2,m}} \right) H^* \Omega. \tag{8}$$

We want to show that the difference in the terms in (5) is zero. It is enough to show that each of the terms (6), (7) and (8) are zero. If $z_{1,k} = z_{1,k+1}$ for types a and b (or on $V_{1,k}$ for type c), then the wedge of the corresponding differential forms will vanish. Thus the terms in (6) are zero. If $z_1 = 0$ then $dt_1 = 0$, defined via the pullback H^*. Then the terms (7) are equal to zero. Similarly, we obtain that the last integral (8) vanishes. $\qquad \square$

Let A be a 2-dimensional manifold with corners in $[0, 1]^2$. We recall the domain of integration

$$D = \left\{ (t_{1,1}, \ldots, t_{2,m}) \in [0, 1]^{2m} \mid 0 \le t_{1,1} \le \cdots \le t_{1,m} \le 1, \ 0 \le t_{2,1} \le \cdots \le t_{2,m} \le 1 \right\}.$$

Let us define

$$A^D = \{ (t_{1,1}, \ldots, t_{2,m}) \in D \mid (t_{1,i}, t_{2,j}) \in A \text{ for } i, j = 1, \ldots, m \}$$

Let ρ_1 and ρ_2 be two permutations of m elements. We define a function on A^D

$$G(t_{1,1}, \ldots, t_{2,m}) = \left(g(t_{1,\rho_1(1)}, t_{2,\rho_2(1)}), g(t_{1,\rho_1(2)}, t_{2,\rho_2(2)}), \ldots, g(t_{1,\rho_1(m)}, t_{2,\rho(m)}) \right).$$

Recall that

$$\Omega = \bigwedge_{j=1}^m \omega_j(z_{1,\rho_1(j)}, z_{2,\rho_2(j)}).$$

Definition 3.8. With the above notation, we define an iterated integral over a membrane of type b restricted to a domain $U = g(A)$ by

$$\,^b\!\!\int_{g,U}^{\rho_1,\rho_2} \omega_1 \cdots \omega_m = \int_{A^D} G^* \Omega.$$

Now we are going to define iterated integrals of type c:

Definition 3.9. Let $\Omega_0 = \bigwedge_{i,j=1}^m \Omega_{i,j}$, where $\Omega_{i,j} = \omega_i \delta_{i,j}$ on $X_{i,i} \equiv X_\Gamma$ and where $\Omega_{i,j} = 1$ for $i \neq j$. Let $\text{in} : X \to \prod_{i,j=1}^n X_{i,j}$ be the inclusion of the finite limit into the product of the schemes $X_{i,j}$. Let $\Omega = \text{in}^* \Omega_0$.

With this definition of Ω, we define iterated integrals of type c restricted to a domain $U = g(A)$ by

$$^c\!\!\int_{g,U}^{\rho_1,\rho_2} \omega_1 \cdots \omega_m = \int_{A^D} G^*\Omega.$$

Let A_1 and A_2 be two manifolds with corners which are subsets of $[0, 1]^2$, with a common component of the boundary. Let $A = A_1 \cup A_2$. Let s be a map of sets with values 1 or 2:

$$s : \{1, \ldots, m\} \to \{1, 2\}.$$

We define a certain subset A_s^D of A^D as follows: consider the image of the map G. It has m coordinates. The first coordinate, $g(t_{1,\rho_1(1)}, t_{2,\rho_2(1)})$, will be restricted to the set $A_{s(1)}$. The second coordinate, $g(t_{1,\rho_1(2)}, t_{2,\rho_2(2)})$, will be restricted to $A_{s(2)}$, and so on, and the last m coordinate $g(t_{1,\rho_1(m)}, t_{2,\rho_2(m)})$ will be restricted to $A_{s(m)}$. Formally, this can be written as

$$A_s^D = \{(t_{1,1}, \ldots, t_{2,m}) \in A^D \mid (t_{1,\rho_1(i)}, t_{2,\rho_2(i)}) \in A_{s(i)} \text{ for } i = 1, \ldots, m\}.$$

Note that the image of the map s is 1 or 2.

Definition 3.10. With the above notation, we define an iterated integral of type b or c over two domains U_1 and U_2, where $U_i = g(A_i)$ and $U = U_1 \cup U_2$, by

$$\int_{g,U,s}^{\rho_1\rho_2} \omega_1 \cdots \omega_m = \int_{A_s^D} G^*\Omega. \tag{9}$$

For type b we have that U is in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ and for type c we have that U is in $X_\Gamma = (\mathbb{H}^2 \cup \mathbb{P}^1(K))/\Gamma$.

Again we examine the homotopy of iterated integrals on membranes. Now we restrict the domain of integration to a manifold with corners A that is a subset of $[0, 1]^2$. Assume that for the boundary of a domain A, denoted by ∂A, we have that $g(\partial A)$ belongs to a finite union of complex analytic curves in $\mathbb{H}^2$ for type b and in X_Γ for type c. We call the minimal union of complex analytic (holomorphic) curves such that $g(\partial A)$ belongs to a finite union of complex analytic curves in $\mathbb{H}^2$ for type b and in X_Γ for type c the *complex boundary* of $g(\partial A)$.

Theorem 3.11 (homotopy invariance theorem II). *Iterated integrals over membranes are homotopy invariant with respect to homotopies that change the boundary ∂U of the domain of integration U so that the boundary varies on a finite union of complex analytic curves.*

Proof. Assume that $g_0(\partial A)$ and $g_1(\partial A)$ have the same complex boundary. Let h be a homotopy between g_0 and g_1, such that for each value of s we have that $h(s, \partial A)$ has the same complex boundary as $h(0, \partial A) = g_0(\partial A)$. Let $A \subset B$ be a strict inclusion of disks. Identify $B - A^\circ$ with $A \times [0, 1]$. Let $i : B - A^\circ \to [0, 1] \times \partial A$. Here A° is the interior of A and ∂A is the boundary of A. Let $\tilde{g}_0$ be a map from B to $\mathbb{H}^2$ such that $\tilde{g}_0(a) = g_0(a)$ for $a \in A$ and $\tilde{g}_0(b) \in h(i(b))$. Since the restriction of the pullback $(\tilde{g}_0^*\omega_i)|_{B-A} = 0$ is mapped to a finite union of complex curves, it vanishes. Therefore

$$\int_A g_0^*\Omega = \int_B \tilde{g}_0^*\Omega. \tag{10}$$

Let $\tilde{g}_1$ be a membrane from B defined by $\tilde{g}_1(a) = g_1(a)$ for $a \in A$ and $\tilde{g}_1(b) = \tilde{g}_1(a)$ for $i(b) = (s, a)$. (Note that $i(b) \in [0, 1] \times \partial A$.) Again,

$$\int_A g_1^*\Omega = \int_B \tilde{g}_1^*\Omega. \tag{11}$$

However, the boundary of B is mapped to the same set (pointwise) by both $\tilde{g}_0$ and $\tilde{g}_1$. Moreover, the homotopy between g_0 and g_1 extends to a homotopy between $\tilde{g}_0$ and $\tilde{g}_1$ that respects the inclusion into the complex boundary. Thus by Theorem 3.7, we have that

$$\int_B \tilde{g}_0^*\Omega = \int_B \tilde{g}_1^*\Omega.$$

Using (10) and (11), we complete the proof of this theorem. $\square$

3B. *Generating series.* We are going to define two types of generating series — type *a* and type *b*, corresponding to the iterated integrals on membranes of type *a* and type *b*.

Definition 3.12 (type *a*). Let A be a domain in $\mathbb{R}^2$. Let g be a membrane. Let $U = g(A) \subset \mathbb{H}^2$. And let $\omega_1, \ldots, \omega_m$ be holomorphic 2-forms on $\mathbb{H}^2$. We define a generating series of type *a* by

$$J^a(U) = 1 + \sum_{k=1}^{\infty} \sum_{c:\{1,\ldots,k\}\to\{1,\ldots,m\}} X_{c(1)} \otimes \cdots \otimes X_{c(k)} \int_{g,U} \omega_{c(1)} \cdots \omega_{c(k)},$$

where $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ is a map of sets.

Consider a map of sets $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ and two permutations ρ_1, ρ_2 of $\{1, 2, \ldots, k\}$. We call two triples (c', ρ_1', ρ_2') and $(c'', \rho_1'', \rho_2'')$ equivalent if they are in the same orbit of the permutation group S_k. That is, $(c'', \rho_1'', \rho_2'') \sim (c', \rho_1', \rho_2')$ if for some $\tau \in S_k$ we have $c'' = c'\tau^{-1}$, $\rho_1'' = \rho_1'\tau^{-1}$ and $\rho_2'' = \rho_2'\tau^{-1}$. Then for the equivalence class of a triple (c, ρ_1, ρ_2), we can associate a unique pair $(c \circ \rho_1, c \circ \rho_2)$ (which are precisely the indices of the X- and Y-variables in (12)

and (13), respectively.) The reason for using such an equivalence is that the integral in (13) is invariant by the above action of $\tau \in S_k$ on the triple (c, ρ_1, ρ_2).

Definition 3.13 (ring R, values of the generating series). The values of the generation series of iterated integrals on membranes will be in a ring R, which we define as follows. Let

$$R_0 = \mathbb{C}\langle\!\langle X_1, Y_1, \ldots, X_m, Y_m \rangle\!\rangle / I$$

be the quotient of the ring of formal power series modulo the two-sided ideal I generated by $X_i Y_j - Y_j X_i$ for $i, j = 1, \ldots, m$. Let $R \subset R_0$ be the subring of formal power series whose monomials have the following property: in every monomial of R, X_i occurs as many times as Y_i.

Definition 3.14 (type $\boldsymbol{b}$). We define the generating series of type $\boldsymbol{b}$ on U by

$$J^b(U) = 1 + \sum_{k=1}^{\infty} \sum_{(c, \rho_1, \rho_2)/\sim} X_{c(\rho_1^{-1}(1))} \otimes \cdots \otimes X_{c(\rho_1^{-1}(k))} \tag{12}$$

$$\otimes Y_{c(\rho_2^{-1}(1))} \otimes \cdots \otimes Y_{c(\rho_2^{-1}(k))} \int_{g,U}^{\rho_1, \rho_2} \omega_{c(1)} \cdots \omega_{c(k)}, \tag{13}$$

where the second summation is over all maps of sets $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ and all permutations ρ_1, ρ_2 of k elements, up to the above equivalence.

Let Y_Γ be a Hilbert modular surface. Let α and β be two rational functions on Y_Γ. We denote by D the union of the divisors $(\alpha)_\infty$ and $(\beta)_\infty$ at infinity. Let $F : Y_\Gamma - D \to \mathbb{C}^2$ be defined as $F(y) = (\alpha(y), \beta(y))$. Let $g : (0, 1)^2 \to Y_\Gamma - D$ be a membrane, so that the composition $F \circ g$ respects the coordinatewise foliations. Consider the differential forms ω_i from the definition of type $\boldsymbol{b}$. They are invariant under the action of the arithmetic group Γ. Thus, we can treat them as differential forms on the Hilbert modular variety Y_Γ.

Definition 3.15 (type $\boldsymbol{c}$). With the new definition of a membrane g and a domain $U \subset Y_\Gamma$, we define the generating series of type $\boldsymbol{c}$ by

$$J^c(U) = 1 + \sum_{k=1}^{\infty} \sum_{(c, \rho_1, \rho_2)/\sim} X_{c(\rho_1^{-1}(1))} \otimes \cdots \otimes X_{c(\rho_1^{-1}(k))} \tag{14}$$

$$\otimes Y_{c(\rho_2^{-1}(1))} \otimes \cdots \otimes Y_{c(\rho_2^{-1}(k))} \int_{g,U}^{\rho_1, \rho_2} \omega_{c(1)} \cdots \omega_{c(k)}, \tag{15}$$

where the second summation is over all maps of sets $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ and all permutations ρ_1, ρ_2 of k elements, up to the above equivalence.

Definition 3.16 (ring R', generating series $J(U_1, U_2)$). We define a generating series of iterated integrals on two disjoint domain U_1 and U_2 (see Definition 3.10). Let $U_i = g(A_i)$. Define

$$J(U_1, U_2)$$
$$= 1 + \sum_{k=1}^{\infty} \sum_{s:\{1,\ldots,k\}\to\{1,2\}} \sum_{(c,\rho_1,\rho_2)/\sim} X_{c(\rho_1^{-1}(1)),s(1)} \otimes \cdots \otimes X_{c(\rho_1^{-1}(k)),s(k)} \tag{16}$$

$$\otimes Y_{c(\rho_2^{-1}(1)),s(1)} \otimes \cdots \otimes Y_{c(\rho_2^{-1}(k)),s(k)} \int_{g,U,s}^{\rho_1,\rho_2} \omega_{c(1)} \cdots \omega_{c(k)}, \tag{17}$$

The generating series takes values in a ring R' defined as follows. Let

$$R'_0 = \mathbb{C}\langle\!\langle X_{1,1}, X_{1,2}, Y_{1,1}, Y_{1,2}, \ldots, X_{m,1}, X_{m,2}, Y_{m,1}, Y_{m,2}\rangle\!\rangle / I'$$

be a quotient of the ring of formal power series, where I' is the two-sided ideal generated by the Lie commutators of all the $X_{i,j}$ and $Y_{k,l}$. Let R' be the subring of R'_0 with the property that in every monomial of R', $X_{i,j}$ occurs as many times as $Y_{i,j}$.

Lemma 3.17. *Let $\phi : R' \to R$ be the homomorphism of rings defined by $\phi(X_{i,1}) = \phi(X_{i,2}) = X_i$ and $\phi(Y_{i,1}) = \phi(Y_{i,2}) = Y_i$. If $U = U_1 \cup U_2$ is in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, then*

$$\phi(J(U_1, U_2)) = J^b(U).$$

If $U = U_1 \cup U_2$ is in X_Γ, then

$$\phi(J(U_1, U_2)) = J^c(U).$$

Proof. After applying the homomorphism ϕ the formal variables on the left-hand side become independent of the map s. Therefore, we have to examine what happens when we sum over all possible maps s. The value $s(i)$ is 1 or 2. This has the following significance: if $s(i) = 1$, then we restrict the form $g^*\omega_{c(i)}$ to A_1 (instead of to A). Similarly, if $s(i) = 2$, we restrict $g^*\omega_{c(i)}$ to A_2. If we add both choices (restriction to A_1 and restriction to A_2) then we obtain the restriction of $g^*\omega_{c(i)}$ to $A = A_1 \cup A_2$. Thus, we obtain the formula

$$\sum_{s:\{1,\ldots,k\}\to\{1,2\}} \int_{g,U,s}^{\rho_1,\rho_2} \omega_{c(1)} \cdots \omega_{c(k)} = \int_{g,U}^{\rho_1,\rho_2} \omega_{c(1)} \cdots \omega_{c(k)}.$$

We do the same for every monomial in R. That proves the above lemma for the generating series. $\square$

3C. *Shuffle product of generating series.* The regions of integration that we are mostly interested in will be ideal diangles, that is, 2-cells whose boundaries have two vertices and two edges, and ideal triangles. All other regions that we will deal with are going to be a finite union of ideal diangles and ideal triangles. The first

type of decomposition is based on a union of two diangles with a common vertex. The second type of decomposition will be based on two of the cells (diangles or triangles) with a common edge.

Let g_1 and g_2 be two membranes. Let $P = (0, 0)$ and $Q = (1, 1)$ be the vertices of a diangle $A \subset \mathbb{R}^2$ and $Q = (1, 1)$ and $R = (2, 2)$ be the two points of a diangle $B \subset \mathbb{R}^2$. Assume that A lies within the rectangle with vertices $(0, 0), (0, 1), (1, 1), (1, 0)$. Similarly, assume that B lies within the rectangle $(1, 1), (1, 2), (2, 2), (2, 1)$. Let $U = g(A)$ and $V = g(B)$.

Theorem 3.18. (i) $\displaystyle\int_{g,U \cup V} \omega_1 \cdots \omega_m = \sum_{j=0}^{m} \int_{g,U} \omega_1 \cdots \omega_j \int_{g,V} \omega_{j+1} \cdots \omega_m.$

(ii) *The generating series of type **a** from* Definition 3.12 *satisfies*

$$J^a(g; A \cup B; \Omega) = J^a(g; A; \Omega) J^a(g; B; \Omega).$$

The proof of the first statement is essentially the same as the combinatorial proof for composition of paths, when one considers iterated path integrals (see Lemma 2.3). The proof of the second statement combines all compositions into generating series (see Definition 3.12), resembling Manin's approach for the noncommutative modular symbol.

For generating series of type **b**, we have a similar statement:

Definition 3.19. Let ρ' and ρ'' be two permutations of the sets $\{1, \ldots, i\}$ and $\{i+1, \ldots, i+j\}$, respectively. We define the permutation $\rho'^{-1} \cup \rho''^{-1}$ of $\{1, \ldots, i+j\}$, which acts on $\{1, \ldots, i\}$ as ρ'^{-1} and on $\{i+1, \ldots, i+j\}$ as ρ''^{-1}. We define the set of shuffles of two given permutations, denoted by $\mathrm{sh}(\rho', \rho'')$, as the set of all permutations ρ of the set $\{1, 2, \ldots, i+j\}$ such that ρ^{-1} is the composition of a shuffle of sets $\tau \in \mathrm{sh}(i, j)$ (see Definition 2.4) with $\rho'^{-1} \cup \rho''^{-1}$. That is,

$$\rho^{-1} = \tau \circ (\rho'^{-1} \cup \rho''^{-1}).$$

Definition 3.20. We define a shuffle of two monomials

$$M' = X_{c'(\rho_1'^{-1}(1))} \otimes \cdots \otimes X_{c'(\rho_1'^{-1}(i))} \otimes Y_{c'(\rho_2'^{-1}(1))} \otimes \cdots \otimes Y_{c'(\rho_2'^{-1}(i))} \int_{g,U'}^{\rho_1', \rho_2'} \omega_{c'(1)} \cdots \omega_{c'(i)}$$

and

$$M'' = X_{c''(\rho_1''^{-1}(1))} \otimes \cdots \otimes X_{c''(\rho_1''^{-1}(j))} \otimes Y_{c''(\rho_2''^{-1}(1))} \otimes \cdots \otimes Y_{c''(\rho_2''^{-1}(j))}$$
$$\times \int_{g,U''}^{\rho_1'', \rho_2''} \omega_{c''(i+1)} \cdots \omega_{c''(i+j)},$$

where ρ_1' and ρ_2' are permutations of $\{1, \ldots, i\}$, c' is a map of sets $\{1, \ldots, i\} \to \{1, \ldots, m\}$, ρ_1'' and ρ_2'' are permutations of $\{i+1, \ldots, i+j\}$, and c'' is a map of

sets $\{i+1, \ldots, i+j\} \to \{1, \ldots, m\}$. By a shuffle product of the monomials M' and M'', we mean the sum

$$M' \times_{\mathrm{Sh}} M'' = \sum_{\substack{\rho_1 \in \mathrm{sh}(\rho_1', \rho_1'') \\ \rho_2 \in \mathrm{sh}(\rho_2', \rho_2'')}} X_{c(\rho_1^{-1}(1)), s(1)} \otimes \cdots \otimes X_{c(\rho_1^{-1}(i+j)), s(i+j)}$$

$$\otimes\, Y_{c(\rho_2^{-1}(1)), s(1)} \otimes \cdots \otimes Y_{c(\rho_2^{-1}(i+j)), s(i+j)} \int_{g, U' \cup U'', s}^{\rho_1, \rho_2} \omega_{c(1)} \cdots \omega_{c(i+j)},$$

where $c : \{1, \ldots, i+j\} \to \{1, \ldots, m\}$ is such that its restriction to the first i elements is c' and its restriction to the last j elements is c''. Here the maps s takes the value 1 on the set $c^{-1}\{1, \ldots, i\} = c'^{-1}\{1, \ldots, i\}$ and the value 2 on the set $c^{-1}\{i+1, \ldots, i+j\} = c''^{-1}\{i+1, \ldots, i+j\}$.

Theorem 3.21 (shuffle product). *For iterated integrals of type b and the corresponding generating series, we have the following shuffle relations*:

(i) $$\int_{g, U}^{\rho_1', \rho_2'} \omega_1 \cdots \omega_j \int_{g, U}^{\rho_1'', \rho_2''} \omega_{j+1} \cdots \omega_m = \sum_{\substack{\rho_1 \in \mathrm{sh}(\rho_1', \rho_1'') \\ \rho_2 \in \mathrm{sh}(\rho_2', \rho_2'')}} \int_{g, U}^{\rho_1, \rho_2} \omega_1 \cdots \omega_m. \quad (18)$$

(ii) $$\int_{g, U'}^{\rho_1', \rho_2'} \omega_1 \cdots \omega_j \int_{g, U''}^{\rho_1'', \rho_2''} \omega_{j+1} \cdots \omega_m = \sum_{\substack{\rho_1 \in \mathrm{sh}(\rho_1', \rho_1'') \\ \rho_2 \in \mathrm{sh}(\rho_2', \rho_2'')}} \int_{g, U, s}^{\rho_1, \rho_2} \omega_1 \cdots \omega_m, \quad (19)$$

where s is a map from $\{1, \ldots, m\}$ to $\{1, 2\}$ such that $\{1, \ldots, j\}$ are mapped to 1 and the remaining elements are mapped to 2.

(iii) $$\phi(J^b(U') \times_{\mathrm{Sh}} J^b(U'')) = J^b(U' \cup U''). \quad (20)$$

(iv) $$\phi(J^c(U') \times_{\mathrm{Sh}} J^c(U'')) = J^c(U' \cup U''). \quad (21)$$

Proof. For part (i), it is useful to consider the two orderings of differential forms, given in Remark 3.6. Note that we need to order the forms both horizontally and vertically in the terminology of that remark. Let us consider first the horizontal order. That is the order with respect to the first variables of the differential forms $g^* \omega_{\rho_1'^{-1}(1)}, \ldots, g^* \omega_{\rho_1'^{-1}(j)}$ and $g^* \omega_{\rho_1''^{-1}(j+1)}, \ldots, g^* \omega_{\rho_1''^{-1}(m)}$, corresponding to the two integrals on the left-hand side of (18). In order to arrange both of the above orderings in one sequence of increasing first arguments, we need to shuffle them (similarly to shuffling a deck of cards). That leads to $\rho_1 \in \mathrm{sh}(\rho_1', \rho_1'')$ (see Definition 3.19). We proceed similarly with the second arguments and the permutations ρ_2', ρ_2'' and ρ_2.

For (ii), apply the equality from part (i) with the differential forms $g^* \omega_1, \ldots, g^* \omega_j$ multiplied by the function $\mathbf{1}_{A'}$, defined by

$$\mathbf{1}_{A'}(x) = \begin{cases} 1 & \text{for } x \in A', \\ 0 & \text{for } x \notin A', \end{cases}$$

and the differential forms $g^*\omega_{j+1}, \ldots, g^*\omega_m$ multiplied by $\mathbf{1}_{A''}$.

For part (iii), we are going to establish similar relation among generating series as elements of R'. Applying the homomorphism $\phi : R' \to R$ from Lemma 3.17, we obtain the desired equality. Every monomial from $J(U_1)$ is of the form

$$M' = X_{c'(\rho_1'^{-1}(1))} \otimes \cdots \otimes X_{c'(\rho_1'^{-1}(i))} \otimes Y_{c'(\rho_2'^{-1}(1))} \otimes \cdots \otimes Y_{c'(\rho_2'^{-1}(i))} \int_{g,U'}^{\rho_1',\rho_2'} \omega_{c'(1)} \cdots \omega_{c'(i)}$$

and similarly every monomial from $J(U_2)$ is of the form

$$M'' = X_{c''(\rho_1''^{-1}(1))} \otimes \cdots \otimes X_{c''(\rho_1''^{-1}(j))} \otimes Y_{c''(\rho_2''^{-1}(1))} \otimes \cdots \otimes Y_{c''(\rho_2''^{-1}(j))}$$
$$\times \int_{g,U''}^{\rho_1'',\rho_2''} \omega_{c''(i+1)} \cdots \omega_{c''(i+j)},$$

where ρ_1' and ρ_2' are permutations of $\{1, \ldots, i\}$, c' is a map of sets $\{1, \ldots, i\} \to \{1, \ldots, m\}$, ρ_1'' and ρ_2'' are permutations of $\{i+1, \ldots, i+j\}$, and c'' is a map of sets $\{i+1, \ldots, i+j\} \to \{1, \ldots, m\}$. We take the shuffle product of the monomials M' and M'' (see Definition 3.20):

$$M' \times_{\mathrm{Sh}} M'' = \sum_{\substack{\rho_1 \in \mathrm{sh}(\rho_1',\rho_1'') \\ \rho_2 \in \mathrm{sh}(\rho_2',\rho_2'')}} X_{c(\rho_1^{-1}(1)),s(1)} \otimes \cdots \otimes X_{c(\rho_1^{-1}(i+j)),s(i+j)}$$
$$\otimes Y_{c(\rho_2^{-1}(1)),s(1)} \otimes \cdots \otimes Y_{c(\rho_2^{-1}(i+j)),s(i+j)} \int_{g,U,s}^{\rho_1,\rho_2} \omega_{c(1)} \cdots \omega_{c(i+j)},$$

where the map s takes the value 1 on the set $c^{-1}\{1, \ldots, i\}$ and the value 2 on the set $c^{-1}\{i+1, \ldots, i+j\}$. This determines the map s uniquely.

In order to complete the proof, we have to show that every monomial in $J(U_1, U_2)$ can be obtained in exactly one way as a result (on the right-hand side) of a shuffle product of a pair of monomials (M_1, M_2) from $J(U_1)$ and $J(U_2)$. Every monomial from $J(U_1, U_2)$ is characterized by two permutation ρ_1, ρ_2 and two maps of sets $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ and $s : \{1, \ldots, k\} \to \{1, 2\}$. Let i be the number of elements in $s^{-1}(1)$ and j the number of elements in $s^{-1}(2)$. Then $i + j = k$. Then i is the number of differential forms among $g^*\omega_{c(1)}, \cdots, g^*\omega_{c(k)}$ which are restricted to the set A_1. The remaining j differential forms are restricted to A_2. Also, every permutation ρ_1 can be written in an unique way as a composition of a shuffle $\tau_1 \in \mathrm{sh}(i, j)$ and two disjoint permutations ρ_1' and ρ_1'' of i and of j elements, respectively (see Definition 3.19). Similarly, ρ_2 can be written in a unique way as a product of a shuffle $\tau_2 \in \mathrm{sh}(i, j)$ and two disjoint permutation ρ_2' and ρ_2''. The map of sets c_1 is defined as a restriction of the map c to the image of ρ_1'. Similarly, the map c_2 is defined as a restriction of the map c to the image of ρ_1''. Now we can define the monomials M' and M'' in $J(U_1)$ and $J(U_2)$ based on the triples ρ_1', ρ_2', c'

and ρ_1'', ρ_2'', c'', respectively. Such monomials are unique. One can show that the shuffle product of M' and M'' contains the monomial in $J(U_1, U_2)$ that we started with exactly once. The proof of part (iii) is complete after applying Lemma 3.17. $\square$

4. Hilbert modular symbols

In this section, we recall the Hilbert modular group and its action on the product of two upper half-planes. Then we define commutative Hilbert modular symbols (Section 4A) and its pairing with the cohomology of the Hilbert modular surface (Section 4B). In Sections 4C and 4D, we define noncommutative Hilbert modular symbols (Definition 4.13) as generating series of iterated integrals over membranes of type $\boldsymbol{b}$. We also examine relations among the noncommutative Hilbert modular symbols (Theorem 4.12), which we interpret as cocycle conditions or as a difference by a coboundary (Conjecture 4.14). In Section 4E, we consider a two-category C with a sheaf J on C. Then the noncommutative Hilbert modular symbol is a sheaf on a two-category. This is done in order to give a plausible approach to defining a suitable noncommutative cohomology set. In Section 4F, we make explicit computations and compare them to computations for multiple Dedekind zeta values.

4A. *Commutative Hilbert modular symbols.* In this subsection, we define commutative Hilbert modular symbols, using geodesics, geodesic triangles and geodesic diangles. Then, we prove certain relations among the commutative Hilbert modular symbols, which are generalized to relations among noncommutative Hilbert modular symbols (Section 4D).

Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic extension of $\mathbb{Q}$. Then the ring of integers in K is

$$\mathbb{O}_K = \begin{cases} \mathbb{Z}[(1 + \sqrt{d})/2] & \text{for } d = 1 \bmod 4, \\ \mathbb{Z}[\sqrt{d}] & \text{for } d = 2, 3 \bmod 4. \end{cases}$$

Then $\Gamma = \mathrm{SL}_2(\mathbb{O}_K)$ is called a Hilbert modular group. Let $\gamma \in \Gamma$. We recall the action of γ on a product of two upper half-planes $\mathbb{H}^2$. Let

$$\gamma = \gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

Let a_2, b_2, c_2, d_2 be the Galois conjugates of a_1, b_1, c_1, d_1, respectively. Let us define γ_2 by

$$\gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Let $z = (z_1, z_2)$ be any point of the product of two upper half-planes $\mathbb{H}^2$.

For an element $\gamma \in GL_2(K)$, we define the following action: If $\det \gamma$ is totally positive, that is $\det \gamma_1 > 0$ and $\det \gamma_2 > 0$, then the action of γ on $z = (z_1, z_2) \in \mathbb{H}^2$ is essentially the same as for $\gamma \in SL_2(K)$, namely,

$$\gamma z = (\gamma_1 z_1, \gamma_2 z_2),$$

where

$$\gamma_1 z_1 = \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1} \quad \text{and} \quad \gamma_2 z_2 = \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}$$

are linear fractional transforms. If $\det \gamma$ is totally negative, that is, $\det \gamma_1 < 0$ and $\det \gamma_2 < 0$, then we define

$$\gamma z = \left(-\frac{a_1 \bar{z}_1 + b_1}{c_1 \bar{z}_1 + d_1}, -\frac{a_2 \bar{z}_2 + b_2}{c_2 \bar{z}_2 + d_2} \right).$$

Similarly if one of the embeddings of $\det \gamma$ is positive and the other is negative, for example, $\det \gamma_1 > 0$ and $\det \gamma_2 < 0$, e.g., for $\det \gamma = \sqrt{d}$, then

$$\gamma z = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, -\frac{a_2 \bar{z}_2 + b_2}{c_2 \bar{z}_2 + d_2} \right).$$

We add cusp points $\mathbb{P}^1(K)$ to $\mathbb{H}^2$. Then the quotient $SL_2(\mathbb{O}_K) \backslash (\mathbb{P}^1(K) \cup \mathbb{H}^2)$ is compact.

We are going to carefully examine geodesics joining the cusps 0, 1 and ∞.

Let z_0, z_1, z_∞ be three distinct cusp points. There is a unique $\gamma \in PGL_2(K)$ that sends z_0, z_1 and z_∞ to 0, 1 and ∞, respectively.

Let

$$i : \mathbb{H} \to \mathbb{H}^2, \quad i(x) = (x, x)$$

be the diagonal map and Δ be its image. Consider the Hirzebruch–Zagier divisor $X = \gamma^* \Delta$. It is an analytic curve that passes through the points z_0, z_1 and z_∞. Then X is a holomorphic curve in $\mathbb{H}^2$ if $\det \gamma$ is totally positive or totally negative. If $\det \gamma$ is not totally positive or totally negative, then X is a holomorphic curve in $\mathbb{H}^1 \times \overline{\mathbb{H}}^1 \cup \mathbb{P}^1(K)$; in other words, it is an antiholomorphic curve in $\mathbb{H}^2$, such as $z_1 = -\bar{z}_2$. Let $\Delta_X = \gamma^* \Delta$ be the pullback of the geodesic triangle Δ formed by the points 0, 1, ∞ in the analytic curve X.

Given four points on the boundary in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, we are tempted to consider them as vertices of a geodesic tetrahedron in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, whose faces are triangles of the type Δ_X. However, there is one problem that we encounter: Two distinct cusps could be connected by two different geodesics in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. In particular, two triangles from the faces of the "tetrahedron" might not have a common edge, but only two common vertices. Thus, we are led to consider a *thickened tetrahedron* with two types of faces on the boundary: the first type is an ideal triangle that we have just defined and the other type is an ideal diangle — a union of geodesics

connecting two fixed points, which has the homotopy type of a disc with two vertices and two edges. The two edges of an ideal diangle in the boundary of a thickened tetrahedron correspond to the two geodesics connecting the same two cusps, where two geodesics belong to the geodesic triangles that have the two cusps in common.

Let us describe a diangle $D_{0,\infty;1,\alpha}$ whose two vertices are 0 and ∞ and whose two sides are geodesics that belong to each of the ideal triangles $0, 1, \infty$ and $0, \alpha, \infty$. The geodesic l_0 between the points 0 and ∞ that lie on the geodesic triangle $0, 1, \infty$ can be parametrized in the following way: $\{(it, it) \mid t \in \mathbb{R}, t \geq 0\} \subset \mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H})$. Here by $\mathrm{Im}(\mathbb{H})$ we mean the imaginary part of the upper half-plane. The element $\gamma \in \Gamma$ that sends $0, \alpha, \infty$ to $0, 1, \infty$ is $\gamma = \left(\begin{smallmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{smallmatrix} \right)$. Then $(\gamma^{-1})^*(it, it) = (|\alpha_1|it, |\alpha_2|it)$. Therefore, the geodesic l_α between the points 0 and ∞ that lie on the geodesic triangle $0, \alpha, \infty$ can be parametrized as $\{(|\alpha_1|it, |\alpha_2|it) \mid t \in \mathbb{R}, \ t \geq 0\} \subset \mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H})$. Then, we define the diangle $D_{0,\infty;1,\alpha}$ as the two-dimensional region in $\mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H})$ between the lines l_0 and l_α. We also consider the diangle with orientation. If $|\alpha_1| > |\alpha_2|$ then it is positively oriented. If the inequality is reversed then the diangle is negatively oriented; if $|\alpha_1| = |\alpha_2|$ then it is a degenerate diangle, which consists of a single geodesic. All other diangles that we will consider are translates of $D_{0,\infty;1,\alpha}$ via the action of any element $\gamma \in \mathrm{PGL}_2(K)$.

Lemma 4.1. (i) *Each geodesic triangle Δ_X lies either on a holomorphic curve or on an antiholomorphic curve.*

(ii) *Each geodesic in a geodesic triangle Δ_X belongs both to a holomorphic curve and to an antiholomorphic curve.*

Proof. Part (i) follows from the construction of a geodesic triangle before the lemma. For part (ii), consider the following: Let $\Delta(0, 1, \infty)$ be the geodesic triangle in the diagonal of $\mathbb{H}^2$ connecting the points $0, 1$ and ∞. It is a holomorphic curve. Thus, a geodesic $\{(it, it) \in \mathbb{H}^2 \mid t > 0\}$, connecting the points 0 and ∞ as a face of the geodesic triangle $\Delta(0, 1, \infty)$, lies on a holomorphic curve. Now consider the geodesic triangle $D(0, \sqrt{d}, \infty)$. It lies on an antiholomorphic curve in $\mathbb{H}^2$, by which we mean a complex curve in $\mathbb{H}^2$ (where we have taken the complex conjugate complex structure in one of the upper half-planes), since the linear fractional transformation that sends $D(0, \sqrt{d}, \infty)$ to $D(0, 1, \infty)$ does not have totally positive (or totally negative) determinant. Explicitly, the linear fractional transformation that sends $(0, \sqrt{d}, \infty)$ to $(0, 1, \infty)$ is

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{pmatrix}.$$

Then

$$(\gamma_1, \gamma_2) = \left(\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{d} \end{pmatrix} \right).$$

We have $\gamma_1(it) = (1/\sqrt{d})it$ and $\gamma_2(it) = -(1/\sqrt{d})\overline{i}t = \overline{\gamma_1(it)}$. Then the same geodesic (it, it) belongs to the antiholomorphic curve given by the pullback of the diagonal with respect to the linear fractional map γ. Thus, we obtain that the geodesic (it, it) connecting 0 and ∞ belongs to both a holomorphic curve and an antiholomorphic curve. Similarly, any translate of the geodesic (it, it) via a linear fractional map from $\mathrm{GL}_2(K)$ would belong to both a holomorphic curve and an antiholomorphic curve. That proves part (ii). $\qquad\square$

Definition 4.2. Let p_1, p_2, p_3, p_4 be cusp points in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. To each triple of points p_1, p_2, p_3, we associate the geodesic triangle $\{p_1, p_2, p_3\}$ with coefficient 1 as an element of the singular chain complex in $C_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$. Also, to each quadruple of points p_1, p_2, p_3, p_4, we associate the geodesic diangle between the two geodesic connecting p_1 and p_2 so that the first geodesic is a face of the geodesic triangle $\{p_1, p_2, p_3\}$ and the second geodesic is a face of the geodesic triangle $\{p_1, p_2, p_4\}$. We denote such a diangle by $\{p_1, p_2; p_3, p_4\}$. We call the geodesic triangle $\{p_1, p_2, p_3\}$ and the geodesic diangle $\{p_1, p_2; p_3, p_4\}$, considered as elements of $C_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$, *commutative Hilbert modular symbols*.

Theorem 4.3. *The commutative Hilbert modular symbols, modulo the boundary of singular 3-chains $\partial C_3(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$, satisfy the following properties*:

(1) *If σ is a permutation of the set $\{1, 2, 3\}$ then*

$$\{p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}\} = \mathrm{sign}(\sigma)\{p_1, p_2, p_3\}.$$

(2) *If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or antiholomorphic) curve of the type $\gamma^*\Delta$, then*

$$\{p_1, p_2, p_3\} + \{p_2, p_3, p_4\} = \{p_1, p_2, p_4\} + \{p_1, p_3, p_4\}.$$

For every four points p_1, p_2, p_3, p_4, we associate a diangle with vertices p_1 and p_2. Let $\{p_1, p_2; p_3, p_4\}$ be the corresponding symbol.

(3) *If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or antiholomorphic) curve of type $\gamma^*\Delta$, then*

$$0 = \{p_1, p_2; p_3, p_4\}.$$

(4) *For every distinct four points p_1, p_2, p_3, p_4, we have the following relations, based on the orientation of the domain:*

$$\{p_2, p_1; p_3, p_4\} = \{p_1, p_2; p_4, p_3\} = -\{p_2, p_1; p_4, p_3\} = -\{p_1, p_2; p_3, p_4\}.$$

(5) *For every five points p_1, p_2, p_3, p_4, p_5, we have*

$$\{p_1, p_2; p_3, p_4\} + \{p_1, p_2; p_4, p_5\} = \{p_1, p_2; p_3, p_5\}.$$

(6) *We also have a relation between the two types of commutative Hilbert modular symbols. For every four distinct points p_1, p_2, p_3, p_4, we have*

$$0 = \{p_1, p_2, p_3\} + \{p_2, p_3, p_4\} - \{p_1, p_2, p_4\} - \{p_1, p_3, p_4\}$$
$$+ \{p_1, p_2; p_3, p_4\} + \{p_2, p_3; p_1, p_4\} + \{p_3, p_1; p_2, p_4\}$$
$$+ \{p_3, p_4; p_1, p_2\} + \{p_1, p_4; p_2, p_3\} + \{p_2, p_4; p_3, p_1\}.$$

Proof. Part (1) follows from the orientation of the simplex in singular homology. Part (2) is an equality induced by two different triangulations on a holomorphic (or antiholomorphic) curve with four vertices. In that setting the diangles are trivial, which proves part (3). Part (4) follows from the orientation of the diangle. Part (5) corresponds to a union of two geodesic diangles with a common face, given by a third geodesic diangle. Part (5) will be used for a noncommutative 1-cocycle relation for the noncommutative Hilbert modular symbol (see Conjecture 4.14). Part (6) is a boundary relation for the boundary of a thickened tetrahedron. By a thickened tetrahedron, we mean a union of four geodesic triangles corresponding to each triple of points among the four points p_1, p_2, p_3, p_4 together with six geodesic diangles that correspond to the area between the faces of the geodesic triangles. They correspond exactly to the thickening of the six edges of a tetrahedron. $\square$

Part 6 will be used to derive explicit formulas for the noncommutative Hilbert modular symbol of type c' resembling a noncommutative 2-cocycle relation (see Conjecture 4.15).

4B. *Pairing of the modular symbols with cohomology.* In this subsection, we consider pairings between commutative Hilbert modular symbols and cusp forms. In some cases, we prove that such pairings give periods in the sense of [Kontsevich and Zagier 2001].

We are interested in holomorphic cusp forms with respect to Γ. Equivalently, we can consider the holomorphic 2-forms on $\widetilde{X}$, the minimal smooth algebraic compactification of X [Hirzebruch 1973]. At this point we should distinguish between geodesic triangles that lie on a holomorphic curve and those that lie on an antiholomorphic curve. The reason for this distinction is that a holomorphic 2-form restricted to a holomorphic curve vanishes. The way to distinguish the two types of geodesic triangles is the following: Let γ be a linear fractional transform that sends the points p_1, p_2, p_3 to 0, 1, ∞. If $\det \gamma$ is totally positive or totally negative, then the geodesic triangle p_1, p_2, p_3 lies on a holomorphic curve. If $\det \gamma$ is neither totally positive nor totally negative, then the geodesic triangle p_1, p_2, p_3 lies on an antiholomorphic curve.

Definition 4.4. Define $M_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$ to be the span of the Hilbert modular symbols $\{p_1, p_2, p_3\}$ and $\{p_1, p_2; p_3, p_4\}$ as a subspace of the singular chain $C_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$. We define the pairing

$$\langle \ , \ \rangle : M_2(\mathbb{H}^2 \cup \mathbb{P}^1(K)) \times S_{2,2}(\Gamma) \to \mathbb{C}$$

by setting

$$\langle \{p_1, p_2, p_3\}, f \, dz_1 \wedge dz_2 \rangle = \int_{\{p_1, p_2, p_3\}} f \, dz_1 \wedge dz_2$$

for geodesic triangles and

$$\langle \{p_1, p_2; p_3, p_4\}, f \, dz_1 \wedge dz_2 \rangle = \int_{\{p_1, p_2; p_3, p_4\}} f \, dz_1 \wedge dz_2$$

for geodesic diangles.

We are going to use that a Hilbert modular surface $X(\mathbb{C})$ can be realized as the complex points of an arithmetic surface defined over a number field F.

Theorem 4.5. *The image of the above pairing is a period over a number field F when we integrate a normalized cusp Hecke eigenform f of weight $(2, 2)$. (For Hecke eigenforms, see* [Shimura 1978; Berger et al. 2013].)

Proof. From Lemma 3.1(ii), the boundary of the geodesic triangles of the diangles are geodesics that lie on holomorphic curves in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. Therefore, in the quotient by the congruence group Γ, the geodesic lies in a Hirzebruch–Zagier divisor on the Hilbert modular surface. Thus, we integrate a closed algebraic differential 2-form (that is, a global differential 2-form with algebraic coefficients) on the Hilbert modular surface, with boundaries Hirzebruch–Zagier divisors. □

Conjecture 4.6. Let $f \in S_{k,k}(\Gamma)$ be a normalized cusp Hecke eigenform of weight (k, k). Then

$$\int_{\{p_1, p_2, p_3\}} f \, dz_1 \wedge dz_2$$

for geodesic triangles and

$$\int_{\{p_1, p_2; p_3, p_4\}} f \, dz_1 \wedge dz_2$$

for geodesic diangles are periods.

Theorem 4.5 is a proof of Conjecture 4.6 for the case of cusp form of weight $(2, 2)$.

4C. *Iteration — revisited.* We defined iterated integrals on diangles in Definitions 3.14 and 3.15. However, these definitions have to be extended to other domains of integration in order to consider iterated integrals on geodesic triangles.

A consequence of the results from this subsection is the following:

Theorem 4.7. *Iterated integrals of type **c** on a geodesic diangle and on a geodesic triangle of algebraic differential 2-forms on a Hilbert modular surface are periods in the sense of Kontsevich–Zagier.*

Before giving the proof, we need definitions of several objects, as well as their properties. In the process, we will be able to extend the definition of iterated integrals on membranes when the domain of integration is a geodesic triangle.

For type **b**, in Definition 3.14, we have a map $g : U \to \mathbb{H}^2$ that sends the two $\mathbb{R}$-foliations on U into two coordinatewise $\mathbb{C}$-foliations of $\mathbb{H}^2$. The same definition does not work when the domain U is a geodesic triangle. The reason is that a geodesic triangle is either a holomorphic curve or an antiholomorphic curve. In both cases, a pullback of one leaf to the geodesic triangle is a point not a line (which is the case for the diangles).

In order to extend Definitions 3.14 and 3.15 to the case when the domain U is a geodesic triangle, we are going to construct a new space using the fiber products multiple times.

Now, we are going to define a space Y_n associated to an iterated integral on n 2-forms on $\mathbb{H}^2$. We are going to use fiber products (see [Hartshorne 1977]). Let p_1 and p_2 be the projections of $\mathbb{H}^2$ on the first and the second component, respectively. Define $X_{ij} = \mathbb{H}^2$ for $1 \le i \le n$ and $1 \le j \le n$. (One should think of the component X_{ij} as the complexification of the real coordinate (s_i, t_j).) Let $C_i = \mathbb{H}$ for $1 \le i \le n$ and $C'_j = \mathbb{H}$ for $1 \le j \le n$. Let

$$X_j = X_{1j} \times_{C'_j} X_{2j} \times_{C'_j} \cdots \times_{C'_j} X_{nj}.$$

(X_j corresponds to the variable t_j.) Then

$$X_j \subset X_{1j} \times X_{2j} \times \cdots \times X_{nj}.$$

Let also

$$P_j = (p_1, \ldots, p_1) : X_{1j} \times X_{2j} \times \cdots \times X_{nj} \to C_1 \times \cdots \times C_n.$$

Let $P_j^\circ = P_j|_{X_j}$ be the restriction of P_j to the subset X_j. We define Y_n as the fiber product of $X_1, \ldots, X_n$ with respect to the morphisms $P_1^\circ, \ldots, P_n^\circ$ over the base $C_1 \times \cdots \times C_n$, namely

$$Y_n = X_1 \times_C \cdots \times_C X_n, \tag{22}$$

where $C = C_1 \times \cdots \times C_n$. Note that X_j is isomorphic to X_{j+1}. Let Z_j be the subspace of Y_n defined by setting the j- and the $(j+1)$-components of $Y_n = X_1 \times_C \cdots \times_C X_n$

to be equal. (The space Z_j corresponds to a boundary components obtained by letting $t_j = t_{j+1}$.) Similarly, we could have defined Y_n by defining first

$$X_i' = X_{i1} \times_{C_i} X_{i2} \times_{C_i} \cdots \times_{C_i} X_{in},$$

(X_i' corresponds to s_i) so that

$$X_i' \subset X_{i1} \times X_{i2} \times \cdots \times X_{in}.$$

Let

$$P_i' = (p_2, \ldots, p_2) : X_{i1} \times X_{i2} \times \cdots \times X_{in} \to C_1' \times \cdots \times C_n'.$$

Define $P_i'^{\circ} = P_i'|_{X_i'}$ to be the restriction of P_i' to X_i'. We define Y_n as the fiber product of $X_1', \ldots, X_n'$ with respect to the morphisms $P_1'^{\circ}, \ldots, P_n'^{\circ}$ over the base $C_1' \times \cdots \times C_n'$, namely,

$$Y_n = X_1' \times_{C'} \cdots \times_{C'} X_n', \tag{23}$$

where $C' = C_1' \times \cdots \times C_n'$. Similarly we define Z_i' to be the subspace of Y_n defined by setting the i- and the $(i+1)$-components of $Y_n = X_1' \times_{C'} \cdots \times_{C'} X_n'$ to be equal. (The space Z_i' corresponds to a boundary components obtained by letting $s_i = s_{i+1}$.)

We have given two definitions (22) and (23) of the space Y_n. In the two definitions we have only exchanged the role of p_1 and p_2. We will prove that both definitions lead to the same object in the case $n = 2$. The general case is left to the reader.

Lemma 4.8. *For $n = 2$, the two definitions* (22) *and* (23) *define isomorphic objects* Y_2.

Proof. The space Y_2 can be defined as a finite limit (in a categorical sense) of a diagram in the following way. Consider the commutative diagram

For any space W such that

$$
\begin{array}{ccccc}
& & C_1' & & \\
& \nearrow & & \nwarrow & \\
X_{11} & & & & X_{12} \\
& \searrow & & \nearrow & \\
C_1 & & W & & C_2 \\
& \nwarrow & & \searrow & \\
X_{21} & & & & X_{22} \\
& \searrow & & \swarrow & \\
& & C_2' & &
\end{array}
\tag{24}
$$
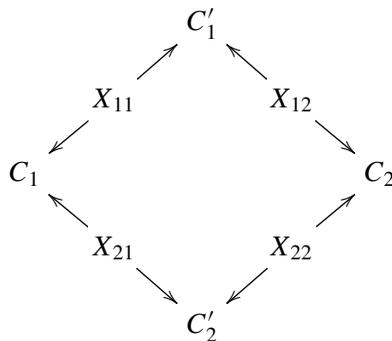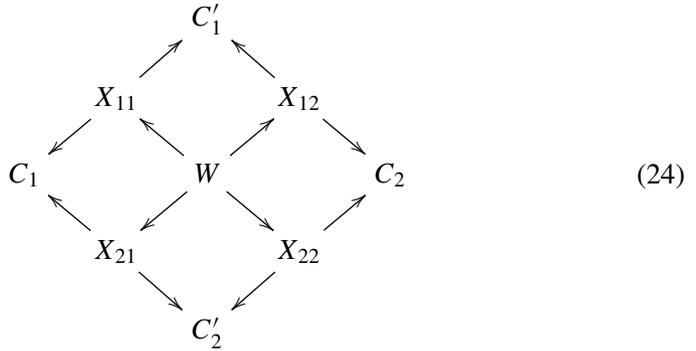
commutes, we have that the maps $f_{ij} : W \to X_{ij}$ factor through $g_{ij} : Y_2 \to X_{ij}$, so that $f_{ij} = g_{ij} \circ h$ for some $h : W \to Y_2$, and Y_2 is part of the commutative diagram

$$
\begin{array}{ccccc}
& & C_1' & & \\
& \nearrow & & \nwarrow & \\
X_{11} & & & & X_{12} \\
& \searrow & & \nearrow & \\
C_1 & & Y_2 & & C_2 \\
& \nwarrow & & \searrow & \\
X_{21} & & & & X_{22} \\
& \searrow & & \swarrow & \\
& & C_2' & &
\end{array}
$$
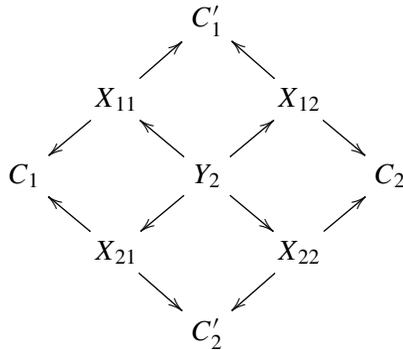
In order to prove this universal property of Y_2 we follow the first definition of Y_2. This leads to the commutative diagram

$$
\begin{array}{ccccccc}
& & & C_1' & & & \\
& & \nearrow & & \nwarrow & & \\
& X_{11} & \longleftarrow X_{11} \times_{C_1'} X_{12} \longrightarrow & X_{12} & & & \\
\nearrow & & \uparrow & & & \searrow & \\
C_1 & & W & & & & C_2 \\
\searrow & & \downarrow & & & \nearrow & \\
& X_{21} & \longleftarrow X_{21} \times_{C_2'} X_{22} \longrightarrow & X_{22} & & & \\
& & \searrow & & \swarrow & & \\
& & & C_2' & & &
\end{array}
$$
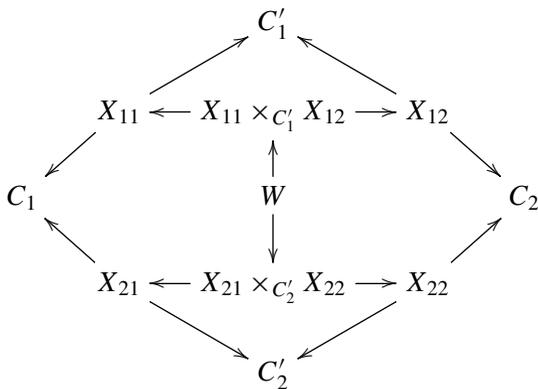
Then we have that $X_1 = X_{11} \times_{C_1'} X_{12}$ maps to $C = C_1 \times C_2$ and also $X_2 = X_{21} \times_{C_2'} X_{22}$ maps to $C = C_1 \times C_2$. Thus the maps from W to any element of the diagram

factors through $Y_2 = X_1 \times_C X_2$. Similarly, W factors through $X_1' \times_{C'} X_2'$, where $X_1' = X_{11} \times_{C_1} X_{21}$, $X_2' = X_{12} \times_{C_2} X_{22}$ and $C' = C_1' \times C_2'$. Since both $X_1 \times_C X_2$ and $X_1' \times_{C'} X_2'$ are universal objects with respect to the diagram (24), we have that they are isomorphic. □

Now, we return to the initial question of this subsection, namely, how to iterate over a geodesic triangle so that it is consistent with the current definition of iteration over a diangle.

For an n-fold iteration of 2-forms of types $\boldsymbol{b}$ or $\boldsymbol{c}$, we have to specify a domain $U \subset \mathbb{H}^2$, $\dim_{\mathbb{R}} U = 2$, and a pair of permutations ρ_1 and ρ_2 of n elements. We make an essential assumption that the boundary of $U \subset \mathbb{H}^2$, denoted by ∂U, projected onto the Hilbert modular surface Y_Γ lies on a finite union of Hirzebruch–Zagier divisors. We will denote the finite union of such Hirzebruch–Zagier divisors by HZ.

Let

$$P_{\rho_1,\rho_2} : X_{11} \times \cdots \times X_{nn} \to X_{\rho_1(1)\rho_2(1)} \times \cdots \times X_{\rho_1(n)\rho_2(n)}$$

be a projection to n of the factors. Let $U_{ij} \cong U$ for $1 \le i \le n$ and $1 \le j \le n$. Let

$$I : U_{\rho_1(1)\rho_2(1)} \times \cdots \times U_{\rho_1(n)\rho_2(n)} \to X_{\rho_1(1)\rho_2(1)} \times \cdots \times X_{\rho_1(n)\rho_2(n)}$$

be induced from the product of inclusion of the domains $U \to X$. We will use the notation

$$\underline{U}^\rho = U_{\rho_1(1)\rho_2(1)} \times \cdots \times U_{\rho_1(n)\rho_2(n)}$$

and

$$\underline{X}^\rho = X_{\rho_1(1)\rho_2(1)} \times \cdots \times X_{\rho_1(n)\rho_2(n)}.$$

Then the map I becomes

$$I : \underline{U}^\rho \to \underline{X}^\rho.$$

Let

$$J : Y_n \to \underline{X}^\rho$$

be the composition of the natural inclusion $Y_n \to X_{11} \times \cdots \times X_{nn}$ and the projection P_{ρ_1,ρ_2}. Then we define the domain of integration to be

$$\underline{U}^\rho_{Y_n} = \underline{U}^\rho \times_{\underline{X}^\rho} Y_n,$$

which is the fiber product of the maps I and J. Since $I : \underline{U}^\rho \to \underline{X}^\rho$ is an inclusion, we have that the induced map

$$\underline{U}^\rho_{Y_n} \to Y_n$$

is an inclusion.

In the above constructions, we have used a parallel between type $\boldsymbol{b}$ and type $\boldsymbol{c}$ iterated integrals on membranes. The following definition allows us to extend in some sense the two types when the domain of integration is an ideal triangle:

Definition 4.9 (iterated integrals on membranes of types b' or c'). For any manifold with corners of dimension 2 on a Hilbert modular variety, we define an iterated integral

$$\int_U^{\Sigma_n(\rho_1,\rho_2)} (f_1\, dz_1 \wedge dz_2) \cdots (f_n\, dz_1 \wedge dz_2)$$

$$= \int_{\underline{U}_{Y_n}^{\rho}} J^*(f_1\, dz_1 \wedge dz_2, \ldots, f_n\, dz_1 \wedge dz_2), \quad (25)$$

where $f_k\, dz_1 \wedge dz_2$ is a form defined on $X_{\rho_1(k)\rho_2(k)}$ for $1 \le k \le n$. If Y_n and $\underline{U}_{Y_n}^{\rho}$ are constructed in the setting of type b iterated integrals on membranes, then the above definition is of iterated integrals on membranes of type b'. Similarly, if Y_n and $\underline{U}_{Y_n}^{\rho}$ are constructed in the setting of type c iterated integrals on membranes, then the above definition is of iterated integrals on membranes of type c'.

If U is a diangle, then the relation of the above integral to the ones defined by iterated integrals over membranes is the following: The integral

$$\int_U^{\Sigma_n(\rho_1,\rho_2)} (f_1\, dz_1 \wedge dz_2) \cdots (f_n\, dz_1 \wedge dz_2)$$

is the sum of the integrals from Definitions 3.14 or 3.15, namely, the sum

$$\sum_{\rho \in \Sigma_n} \int_U^{(\rho\rho_1,\rho\rho_2)} (f_1\, dz_1 \wedge dz_2) \cdots (f_n\, dz_1 \wedge dz_2)$$

over the orbit of the diagonal action of the permutation group Σ_n on any chosen pair of permutations (ρ_1, ρ_2).

Proposition 4.10 (properties of the iterated integral (25)). (1) *The iterated integral (25) is well-defined when U is an ideal triangle both for types b and c.*

(2) *The iterated integral (25) for type c is a period if U is an ideal triangle or an ideal diangle, when $f_1, \ldots, f_n$ are normalized Hecke eigenforms of weight $(2, 2)$.*

(3) *The iterated integral (25), both for types b and c, is homotopy invariant with respect to homotopies that vary within the divisors*

$$J^{-1}\big(X_{\rho_1(1)\rho_2(1)} \times \cdots \times p_1^{-1}(q_i) \times \cdots \times X_{\rho_1(n)\rho_2(n)}\big),$$

where q_i is a point of $X_{\rho_1(i)\rho_2(i)}$ for fixed i and $p_1 : X_{\rho_1(i)\rho_2(i)} \to C$; or homotopies that vary within the divisors

$$J^{-1}\big(X_{\rho_1(1)\rho_2(1)} \times \cdots \times p_2^{-1}(q_i) \times \cdots \times X_{\rho_1(n)\rho_2(n)}\big),$$

where q_i is a point of $X_{\rho_1(i)\rho_2(i)}$ for fixed i and $p_2 : X_{\rho_1(i)\rho_2(i)} \to C'$.

Proof. (a) The integral (25) is well-defined for any two-dimensional submanifold with corners of the Hilbert modular variety [Borel and Serre 1973].

(b) The iterated integral (25) is a period since:

(1) A Hilbert modular variety can be defined over a number field.

(2) The normalized Hecke eigenforms $f_1, \ldots, f_n$ of weight $(2, 2)$ can be realized as algebraic differential forms on the Hilbert modular variety.

(3) The boundary of the region of integration $\overline{U}_{Y_n}^\rho$ is a divisor on Y_n, namely,

$$\bigcup_{i=1}^{n} \mathrm{HZ}_i,$$

where

$$\mathrm{HZ}_i = J^{-1}(X_{\rho_1(1)\rho_2(1)} \times \cdots \times \mathrm{HZ} \times \cdots \times X_{\rho_1(n)\rho_2(n)})$$

is a divisor of Y_n obtained as a pullback of a divisor whose i-th component is a Hirzebruch–Zagier divisor HZ, and the rest of the factors are $X_{\rho_1(k)\rho_2(k)}$ for $k \neq i$.

(c) The proof is essentially the same as that of Theorem 3.7.   □

The domain $\underline{U}_{Y_n}$ might be cut into disconnected components by the Z_i and Z'_j. In order to choose a connected component we need to define another region of integration. Recall that for the case of iterated integrals on membranes of type **b**, $p_1 : \mathbb{H}^2 \to C$ and $p_2 : \mathbb{H}^2 \to C'$ are projections onto the first and the second component, with $C \cong \mathbb{H}$ and $C' \cong \mathbb{H}$.

For the case of iterated integrals on membranes of type **c**, $p_1 = \alpha_1 \circ \pi$ and $p_2 = \alpha_2 \circ \pi$ are compositions of

$$\pi : \mathbb{H}^2 \to X_\Gamma,$$

the map from the universal cover to the Hilbert modular surface, with

$$\alpha_1, \alpha_2 : X_\Gamma \to \mathbb{P}^1,$$

two algebraically independent rational functions on the Hilbert modular surface, and $C_i \cong \mathbb{P}^1$ and $C'_j \cong \mathbb{P}^1$ for $1 \leq i \leq n$ and $1 \leq j \leq n$.

Let $q_0, q_1, r_0, r_1 \in \mathbb{P}^1$ be points. Let Q_0, Q_1, R_0 and R_1 be connected components of $p_1^{-1}(q_0)$, $p_1^{-1}(q_1)$, $p_2^{-1}(r_0)$, and $p_2^{-1}(r_1)$, respectively.

Let $V \to \mathbb{H}^2$ be a domain in $\mathbb{H}^2$ with boundary on the union

$$Q_0 \cup Q_1 \cup R_0 \cup R_1,$$

but with interior disjoint from this union. We define the divisors Z_0, Z_n, Z'_0, Z'_n of Y_n as follows: Z_0 will be the beginning of the integration of the t_1 variable ($t_1 = 0$),

Z_n will be the end of the integration of the t_n variable ($t_n = 1$), Z_0' will be the beginning of the integration of the s_1 variable ($s_1 = 0$), and Z_n' will be the end of the integration of the s_n variable ($s_n = 1$). We define them as the fiber product

$$Z_0 = Q_0 \times_C X_2 \times_C \cdots \times_C X_n, \qquad Z_n = X_1 \times_C \cdots \times_C X_{n-1} \times_C Q_1,$$
$$Z_0' = R_0 \times_{C'} X_2' \times_{C'} \cdots \times_{C'} X_n', \qquad Z_n' = X_1' \times_{C'} \cdots \times_{C'} X_{n-1}' \times_{C'} R_1.$$

We will use the notation

$$\underline{V}^\rho = V_{\rho_1(1)\rho_2(1)} \times \cdots \times V_{\rho_1(n)\rho_2(n)} \quad \text{and} \quad \underline{X}^\rho = X_{\rho_1(1)\rho_2(1)} \times \cdots \times X_{\rho_1(n)\rho_2(n)}.$$

Then the map I' becomes

$$I' : \underline{V}^\rho \to \underline{X}^\rho.$$

Let

$$J' : Y_n \to \underline{X}^\rho$$

be the composition of the natural inclusion $Y_n \to X_{11} \times \cdots \times X_{nn}$ and the projection P_{ρ_1,ρ_2}. Then we define the domain of integration to be

$$\underline{V}^\rho_{Y_n} = \underline{V}^\rho \times_{\underline{X}^\rho} Y_n,$$

which is the fiber product of the maps I' and J'. Since $I' : \underline{U}^\rho \to \underline{X}^\rho$ is an inclusion, we have that the induced map

$$\underline{V}^\rho_{Y_n} \to Y_n$$

is an inclusion.

Then the divisors $Z_0, Z_1, \ldots, Z_{n-1}, Z_n$ and $Z_0', Z_1', \ldots, Z_{n-1}', Z_n'$ cut out from $\underline{V}_{Y_n}$ a product of two n-simplices, which corresponds to the region where the product $\{0 \le s_1 \le \cdots \le s_n \le 1\} \times \{0 \le t_1 \le \cdots \le t_n \le 1\}$ is embedded. Denote by $\overline{V}^\rho_{Y_n}$ the connected components of $\overline{V}^\rho_{Y_n}$ that contains the image of the product $\{0 \le s_1 \le \cdots \le s_n \le 1\} \times \{0 \le t_1 \le \cdots \le t_n \le 1\}$ under the map g from Definition 3.5.

Proof of Theorem 4.7. We consider the type of iterated integrals defined in Definition 3.15. Using the above notation, the domain of integration is $U \subset V$. We define

$$\overline{U}^\rho_{Y_n} = \underline{U}^\rho_{Y_n} \cap \overline{V}^\rho_{Y_n}.$$

Then the boundary of $\overline{U}^\rho_{Y_n}$ lies on the union of divisors

$$\partial \overline{U}^\rho_{Y_n} \subset \left(\bigcup_{i=1}^n Z_i \right) \cup \left(\bigcup_{j=1}^n Z_j' \right).$$

The normalized Hecke eigenforms of weight $(2, 2)$ can be realized as algebraic differential forms on the Hilbert modular variety. Then the iterated integrals on a membrane of type c over the domain U are periods, since:

(1) A Hilbert modular variety can be defined over a number field.

(2) The normalized Hecke eigenforms $f_1, \ldots, f_n$ of weight $(2, 2)$ can be realized as algebraic differential forms on the Hilbert modular variety.

(3) The boundary of the region of integration $\overline{U}_{Y_n}^\rho$ is a divisor on Y_n, namely,

$$\left(\bigcup_{i=1}^n Z_i \right) \cup \left(\bigcup_{j=1}^n Z'_j \right). \qquad \qquad \square$$

4D. *Generating series and relations.* In this subsection, we examine the generating series of iterated integrals on membranes (of types $\boldsymbol{b}'$ or $\boldsymbol{c}'$), evaluated at geodesic triangles and geodesic diangles. We prove relations among them. Most importantly, the generating series J will be used in Section 4E to define noncommutative Hilbert modular symbols. Moreover, the relations that we prove in this subsection will be interpreted as cocycles or as coboundaries of the noncommutative Hilbert modular symbols satisfy in Section 4E.

Definition 4.11. Let $f_1, \ldots, f_m$ be m cusp forms with respect to a Hilbert modular group Γ. Let $f_1 \, dz_1 \wedge dz_2, \ldots, f_m \, dz_1 \wedge dz_2$ be the corresponding differential forms defining the generating series. Let $J(p_1, p_2, p_3)$ be the generating series J evaluated at the geodesic triangle with vertices p_1, p_2, p_3. Let $J(p_1, p_2; p_3, p_4)$ be the generating series J evaluated at the geodesic diangle $\{p_1, p_2; p_3, p_4\}$.

Both $J(p_1, p_2, p_3)$ and $J(p_1, p_2; p_3, p_4)$ will be called noncommutative Hilbert modular symbols after the action of the arithmetic group is included (see Definition 4.13).

Theorem 4.12. *The generating series J is one of the types $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{b}'$ or $\boldsymbol{c}'$. Note that $J(p_1, p_2; p_3, p_4)$ is defined for all types, while $J(p_1, p_2, p_3)$ is defined only for types $\boldsymbol{b}'$ or $\boldsymbol{c}'$. Then the generating series $J(p_1, p_2, p_3)$ and $J(p_1, p_2; p_3, p_4)$ satisfy the following relations:*

(1) *If σ is a permutation of the set $\{1, 2, 3\}$, then*

$$J(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = J^{\mathrm{sign}(\sigma)}(p_1, p_2, p_3).$$

(2) *If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or antiholomorphic) curve of type $\gamma^* \Delta$, then*

$$1 = J(p_1, p_2, p_3) J(p_2, p_3, p_4)$$
$$\times J(p_2, p_1, p_4) J(p_1, p_4, p_3).$$

(3) *If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or antiholomorphic) curve of type $\gamma^* \Delta$, then*

$$1 = J(p_1, p_2; p_3, p_4).$$

(4) *For every four points p_1, p_2, p_3, p_4, we have the following relation based on the orientation of the domain:*

$$J(p_2, p_1; p_3, p_4) = J(p_1, p_2; p_4, p_3)$$
$$= J^{-1}(p_2, p_1; p_4, p_3)$$
$$= J^{-1}(p_1, p_2; p_3, p_4).$$

(5) *For every five points p_1, p_2, p_3, p_4, p_5, we have*

$$J(p_1, p_2; p_3, p_4) J(p_1, p_2; p_4, p_5) = J(p_1, p_2; p_3, p_5).$$

(6) *For every four points p_1, p_2, p_3, p_4, we have the following relation, based on the boundary of a thickened tetrahedron:*

$$1 = J(p_1, p_2, p_3) J(p_2, p_3, p_4)$$
$$\times J(p_2, p_1, p_4) J(p_1, p_4, p_3)$$
$$\times J(p_1, p_2; p_3, p_4) J(p_2, p_3; p_1, p_4) J(p_3, p_1; p_2, p_4)$$
$$\times J(p_3, p_4; p_1, p_2) J(p_1, p_4; p_2, p_3) J(p_2, p_4; p_3, p_1).$$

Proof. For part (1), let σ be an odd permutation. Let U be the union of two triangles along one of their edges. Let the first triangle have vertices p_1, p_2, p_3 and the second triangle have vertices p_3, p_2, p_1 with the opposite orientation. We can glue the two triangles along the edge $p_1 p_2$. (Gluing along any other edge would lead to the same result for the corresponding generating series.) From the shuffle product formula (Theorem 3.21(iii)), it follows that $J(U) = J(p_1, p_2, p_3) J(p_3, p_2, p_1)$. (Note that the product is not the product in the ring R. It is induced by a shuffle product of iterated integrals on membranes.) From the second homotopy invariance theorem (Theorem 3.11) it follows that the generating series $J(U)$ depends on U up to homotopy, which keeps the boundary components $p_2 p_3$, $p_3 p_2$, $p_1 p_3$ and $p_3 p_1$ on fixed unions of holomorphic curves. We can contract U to its boundaries ∂U so that the contracting homotopy keeps the boundary components on a fixed union of holomorphic curves. Therefore, $J(U) = J(\partial U) = 1$.

Parts (2), (4) and (5) can be proven similarly.

For part (3), if p_1, p_2, p_3, p_4 belong to the same holomorphic (or antiholomorphic) curve, then the corresponding diangle has no interior, since the two edges will coincide. Recall that the edges of the diangle are defined via unique geodesic triangles lying on a holomorphic (or antiholomorphic) curve.

The proof of part (6) is essentially the same as the one for part (1); however, we will prove it independently, since it is a key property of the noncommutative Hilbert modular symbol. Consider a thickened tetrahedron with vertices p_1, p_2, p_3, p_4.

The faces of the thickened tetrahedron are precisely the ones listed in the product of part (6). The whole product is equal to $J(V)$, where V is the union of all faces of the thickened tetrahedron. From the second homotopy invariance theorem it follows that the generating series $J(V)$ depends on V up to homotopy, which keeps the boundary components on a fixed union holomorphic curves. Since V bounds a contractible 3-dimensional region (a thickened tetrahedron), from Theorem 3.11, it follows that $J(V) = J(\text{point}) = 1$.   □

4E. *Definition of noncommutative Hilbert modular symbols.* In this subsection, we define noncommutative Hilbert modular symbols. They are analogues of Manin's [2006] noncommutative modular symbol, applicable to the Hilbert modular group. Instead of the iterated path integrals that Manin uses, we use a higher-dimensional analogue, defined in Section 3.

Usually, a modular symbol represents a cohomology class. Manin's noncommutative modular symbol represents a noncommutative first cohomology class. We would like to say that the noncommutative Hilbert modular symbols represent noncommutative cohomology classes; this is formulated in Conjectures 4.14 and 4.15.

After defining the noncommutative Hilbert modular symbols, we prove some of their properties. These properties will be interpreted intuitively as cocycle or coboundary conditions. The approach in this subsection is more geometric. The purpose of presenting them here is to give many examples of relations and to help establish a suitable cohomology theory that truly captures these relations in a more structured way.

The cocycle interpretation is only for intuition; it is not precise. The formula holds for geometric reasons. Note that the composition is not the multiplication in the ring R; it is given by the shuffle product (see Theorem 3.21), which works for the generating series on iterated integrals on membranes. The multiplication is written linearly as we would multiply several elements in a group or in a ring; however, the multiplication is two-dimensional among regions with common boundaries.

In the next subsection will give some intuition about higher categories, for the purpose of giving more structure to the noncommutative Hilbert modular symbols and for a possible approach to defining a first and second noncommutative cohomology class.

For definitions of iterated integrals on membranes, see Definitions 3.4 and 3.5 for types b and c and Definition 4.9 for types b' and c'.

Definition 4.13. We define noncommutative Hilbert modular symbols as generating series of iterated integrals on membranes of types b, c, b', or c' over a geodesic diangle by

$$c^1_{p_1, p_2; p_3}(\gamma) = J(p_1, p_2; p_3, \gamma p_3).$$

We also define noncommutative Hilbert modular symbols as generating series of iterated integrals on membranes of types b' or c' over a geodesic triangle by

$$c_p^2(\gamma, \delta) = J(p, \gamma p, \gamma \delta p),$$

where p, p_1, p_2, p_3 are cusp points in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ and $\beta, \gamma, \delta \in \mathrm{SL}_2(\mathbb{O}_K)$.

We are going to define an action of $\mathrm{Mat}_2(\mathbb{O}_K)^+$ on the generating series J^c, where $\mathrm{Mat}_2(\mathbb{O}_K)^+$ is the semigroup of 2×2 matrices with totally positive determinant.

In order to interpret $c^1(\gamma)$ and $c^2(\gamma, \delta)$ as cocycles, we are going to define an action of the semigroup $\mathrm{Mat}_2(\mathbb{O}_K)^+$ on the whole ring R where the generating series take values. Such an action can be given via Hecke operators.

For simplicity, we shall assume that $\mathbb{O}_K$ has narrow class number 1. We consider all Hecke eigenforms of weight $(2, 2)$ with respect to $\mathrm{Mat}_2(\mathbb{O}_K)^+$. Now, let u be a unit such that $u_1 > 0$ and $u_2 < 0$, where u_1 and u_2 are the images of u under the two real embeddings of K into $\mathbb{R}$. It exists, since the narrow class group is trivial. (For example, $K = \mathbb{Q}(\sqrt{2})$ is such a field.) We define an action of $\gamma \in \mathrm{Mat}_2(\mathbb{O}_K)$ on the ring R (Definition 3.13) where the generating series takes values. We define

$$\gamma \bullet f \mapsto T_\gamma(f)$$

for $\gamma \in \mathrm{Mat}_2(\mathbb{O}_K)^+$. Let $f_1, \ldots, f_m$ be a basis of Hecke eigenforms of the space of cusp form of weight $(2, 2)$. Let $X_1, Y_1, \ldots, X_m, Y_m$ be generators of R, and to each f_i associate X_i and Y_i. Then the action of $\gamma \in \mathrm{Mat}_2(\mathbb{O}_K)^+$ is given by $T_\gamma(X_i) = c(\gamma, f_i) X_i$ and $T_\gamma(Y_i) = Y_i$, where $c(\gamma, f_i)$ is the eigenvalue of the Hecke operator.

In this setting the group action, namely, the action of the Hilbert modular group, is trivial. This trivial action extend to the action of $T_1 = \mathrm{id}$ on the whole ring R. In fact, for an element $\beta \in \mathrm{SL}_2(\mathbb{O}_K)$, the trivial action on $c^1_{p_1, p_2; p_3}$ and c_p^2 can be realized as

$$(\beta c^1_{p_1, p_2; p_3})(\gamma) = c^1_{\beta p_1, \beta p_2; \beta p_3}(\beta \gamma)$$

and

$$(\beta c_p^2)(\gamma, \delta) = c^2_{\beta p}(\beta p, \beta \gamma p, \beta \gamma \delta p).$$

The last two relations hold because for a cusp form of weight $(2, 2)$ the differential form $f \, dz_1 \wedge dz_2$ is invariant under the action of the Hilbert modular group Γ. Algebraically, for any geodesic diangle, we have

$$\beta J(p_1, p_2; p_3, p_4) = J(p_1, p_2; p_3, p_4) = J(\beta p_1, \beta p_2; \beta p_3, \beta p_4).$$

Similarly, for a geodesic triangle,

$$\beta J(p_1, p_2, p_3) = J(p_1, p_2, p_3) = J(\beta p_1, \beta p_2, \beta p_3).$$

The relations among the symbols are based on two properties: composition via shuffle product Theorem 3.21(iii) and homotopy invariance (Theorems 3.7 and 3.11).

Conjecture 4.14. The noncommutative Hilbert modular symbol $c^1_{p_1, p_2; p_3}$ is a 1-cocycle. Moreover, if we change the point p_3 to q_3, then the cocycle changes by a coboundary.

Property (5) of Theorem 4.12 can be interpreted as a 1-cocycle relation. Consider the analogy with a noncommutative 1-cocycle of a group acting on a noncommutative ring; we define the boundary of $c^1_{p_1, p_2; p_3}$ by

$$dc^1_{p_1, p_2; p_3}(\beta, \gamma) = c^1_{p_1, p_2; p_3}(\beta)(\beta c^1_{p_1, p_2; p_3})(\gamma)(c^1_{p_1, p_2; p_3}(\beta\gamma))^{-1}.$$

The action of β on the cocycle is given in Definition 4.13. In contrast to a first noncommutative cocycle (see for example [Brown 1982]), here we have two-dimensional composition of symbols, that is, one can compose the symbols as two-morphisms in a two-category.

Then

$$
\begin{aligned}
dc^1_{p_1, p_2; p_3}(\beta, \gamma) &= J(p_1, p_2; p_3, \beta p_3)(\beta J(p_1, p_2; p_3, \gamma p_3)) J^{-1}(p_1, p_2; p_3, \beta\gamma p_3) \\
&= J(p_1, p_2; p_3, \beta p_3) J(p_1, p_2; \beta p_3, \beta\gamma p_3) J^{-1}(p_1, p_2; p_3, \beta\gamma p_3) \\
&= 1.
\end{aligned}
\tag{26}
$$

If we change p_3 to q_3 then the cocycle changes by a coboundary. Let $b^0 = J(p_1, p_2; p_3, q_3)$ be a 0-cochain. Then

$$
\begin{aligned}
c^1_{p_1, p_2; q_3}(\gamma) &= J(p_1, p_2; p_3, \gamma p_3) \\
&= J(p_1, p_2; p_3, q_3) J(p_1, p_2; q_3, \gamma q_3) J(p_1, p_2; \gamma q_3, \gamma p_3) \\
&= J(p_1, p_2; p_3, q_3) J(p_1, p_2; q_3, \gamma q_3)(\gamma J(p_1, p_2; p_3, q_3))^{-1} \\
&= b^0 c^1_{p_1, p_2; q_3}(\gamma)(\gamma b^0)^{-1}
\end{aligned}
\tag{27}
$$

Conjecture 4.15. The noncommutative Hilbert modular symbol $c^2_p(\beta, \gamma)$ satisfies a 2-cocycle relation. Moreover, if we change the point p to q, then the cocycle changes by a coboundary up to terms involving c^1.

Recall

$$c^2_p(\beta, \gamma) = J(p, \beta p, \beta\gamma p).$$

Then c_p^2 satisfies a 2-cocycle condition up to a multiple of the 1-cocycle $c_{q_1,q_2;q_3}^1$ for various points q_1, q_2, q_3. For the 2-cocycle relation, we compute $dc_p^2(\beta, \gamma, \delta)$:

$$dc_p^2(\beta, \gamma, \delta) = c_p^2(\beta, \gamma)c^2(\beta, \gamma\delta)(c^2(\beta\gamma, \delta))^{-1}(\beta \cdot c^2(\gamma, \delta))^{-1}$$
$$= J(p, \beta p, \beta\gamma p)J(p, \beta p, \beta\gamma\delta p)$$
$$\times J(p, \beta\gamma p, \beta\gamma\delta p)^{-1}J(\beta p, \beta\gamma p, \beta\gamma\delta p)^{-1}. \qquad (28)$$

In order to have $dc_p^2(\beta, \gamma, \delta) = 1$, we must multiply by suitable values of c^1, corresponding to edges of a certain thickened tetrahedron. Then

$$dc_p^2(\beta, \gamma, \delta) \times \left[c_{p,\beta p;\beta\gamma p}^1((\beta\gamma)\delta(\beta\gamma)^{-1})c_{\beta p,\beta\gamma p;p}^1(\beta\gamma\delta)c_{\beta\gamma p,p;\beta p}^1((\beta)\gamma\delta\beta^{-1}) \right.$$
$$\left. \times c_{\beta\gamma p,\beta\gamma\delta p;p}^1(\beta)c_{p,\beta\gamma\delta p;\beta p}^1(\beta\gamma\beta^{-1})c_{\beta p,\beta\gamma\delta p;\beta\gamma p}^1((\beta\gamma)^{-1}) \right]$$
$$= \left[c_p^2(\beta, \gamma)c^2(\beta, \gamma\delta)(c^2(\beta\gamma, \delta))^{-1}(\beta \cdot c^2(\gamma, \delta))^{-1} \right]$$
$$\times \left[c_{p,\beta p;\beta\gamma p}^1((\beta\gamma)\delta(\beta\gamma)^{-1})c_{\beta p,\beta\gamma p;p}^1(\beta\gamma\delta)c_{\beta\gamma p,p;\beta p}^1((\beta)\gamma\delta\beta^{-1}) \right.$$
$$\left. \times c_{\beta\gamma p,\beta\gamma\delta p;p}^1(\beta)c_{p,\beta\gamma\delta p;\beta p}^1(\beta\gamma\beta^{-1})c_{\beta p,\beta\gamma\delta p;\beta\gamma p}^1((\beta\gamma)^{-1}) \right]$$
$$= \left[J(p, \beta p, \beta\gamma p)J(p, \beta p, \beta\gamma\delta p)J(p, \beta\gamma p, \beta\gamma\delta p)^{-1}J(\beta p, \beta\gamma p, \beta\gamma\delta p)^{-1} \right]$$
$$\times \left[J(p, \beta p; \beta\gamma p, \beta\gamma\delta p)J(\beta p, \beta\gamma p; p, \beta\gamma\delta p)J(\beta\gamma p, p; \beta p, \beta\gamma\delta p) \right.$$
$$\left. \times J(\beta\gamma p, \beta\gamma\delta p; p, \beta p)J(p, \beta\gamma\delta p; \beta p, \beta\gamma p)J(\beta p, \beta\gamma\delta p; \beta\gamma p, p) \right]$$
$$= 1.$$

The first equality follows from (28). The second equality follows from the definition of the symbols. And the last equality follows from property (6) of Theorem 4.12 with $(p_1, p_2, p_3, p_4) = (p, \beta p, \beta\gamma p, \beta, \gamma\delta p)$. Therefore, we obtain that $dc_p^2(\beta, \gamma, \delta)$ is 1 up to values of the 1-cocycle c^1.

Conjecture 4.16. The conjectural cocycles c_p^2 and c_q^2 are homologous:

$$c_p^2(\beta, \gamma) = c_q^2(\beta, \gamma)[db_{pq}^1(\beta, \gamma)] \prod_i J(D_i),$$

up to a product of $J(D_i)$, where the D_i are geodesic diangles.

Before we proceed, we would like to make an analogy between 1-dimensional and 2-dimensional cocycles. For the 1-dimensional cocycle, the property that it is a cocycle uses the geometry of a triangle, where the faces of the triangle are essentially the 1-cocycle. We want commutativity of the triangular diagram. We think of the commutativity of the diagram as follows: consider the interior of the triangle as a homotopy of paths and think of the 1-cocycle as a homotopy-invariant function. The 2-cocycle relation is represented by the faces of a tetrahedron. By "commutativity" of the diagram, we mean a homotopy invariant 2-cocycle and a homotopy from one of the faces to the union of the other three faces.

The comparison that $c^1_{p_1, p_2; p_3}$ and $c^1_{p_1, p_2; q_3}$ are homologous is given by a square-shaped diagram. The analogy with dimension 2 is that the cocycles c^2_p and c^2_q are two faces of an octahedron. The vertices associated to $c^2_p(\beta, \gamma)$ are $(p, \beta p, \beta \gamma p)$ and the vertices associated to c^2_q are $(q, \beta q, \beta \gamma q)$. The two faces will be opposite to each other on the octahedron Oct so that the three pairs of opposite vertices are $(p, \beta \gamma q)$, $(\beta p, q)$ and $(\beta \gamma p, \beta q)$. The remaining six faces are combined into two triples. Each of them corresponds to a coboundary of a 1-chain.

Let

$$b^1_{p,q}(\beta) = [J(p, q, \beta p) J(q, \beta q, \beta p)][J(q, \beta p; p, \beta q)].$$

Consider the action of $\gamma \in \Gamma$ on b^1 by acting on each point in the argument of J, denoted as before by $\gamma \cdot b^1$. Then, we define

$$db^1_{p,q}(\beta, \gamma) = b^1_{p,q}(\beta)[\beta \cdot b^1_{p,q}(\gamma)][b^1_{p,q}(\beta \gamma)]^{-1},$$

where $\beta \cdot b^1_{p,q}(\gamma) = [J(\beta p, \beta q, \beta \gamma p) J(\beta q, \beta \gamma q, \beta \gamma p)][J(\beta q, \beta \gamma p; p; \beta \gamma q)]$.

Consider the above octahedron Oct. Remove from it the tetrahedron T with vertices $(p, q, \beta \gamma q, \beta \gamma p)$. Then the triangles of the remaining geometric figure are precisely the triangles in the definitions of $c^2_p(\beta, \gamma)$, $c^2_q(\beta, \gamma)$ and $db^1_{p,q}(\beta, \gamma)$. Now, consider thickenings of the edges, which are common for two triangles. It can be done in the following way. Instead of any triangle, we can take a geodesic triangle. The two triangles that had a common edge might have only two common vertices. Then the region between the two geodesic, one for each of the geodesic triangles, forms the induced diangle. Take J of the induces diangles from the octahedron Oct and J^{-1} of the induced diangles from the tetrahedron T. Their product gives $\prod_i J(D_i)$. The equality holds because we apply J to the union of the faces of the thickened Oct $-T$, which gives 1.

4F. *A two-category.* Why do we need a two-category? Is there an example of a sheaf on this category/topology? How does the noncommutative Hilbert modular symbols represents a sheaf?

The ideas presented in this subsection will be developed in a follow-up paper. Here we present the basic constructions that give justification for the conjectures that the noncommutative Hilbert modular symbols c^1 and c^2 are cocycles in some categorical and sheaf-theoretic setting. For sheaves on 2-categories one may consult [Street 1982]. Since our 2-morphisms are invertible one may also use Lurie's constructions [2009] of sheaves on higher categories.

We are going to construct a 2-category C and a sheaf J on C. We define p to be an object of the 2-category C if p is a cusp point, that is $p \in \mathbb{P}^1(K)$. We define 1-morphisms in the following way. Let σ be the geodesic connecting 0 and ∞ that lies on the diagonal $\Delta = i(\mathbb{H}) \subset \mathbb{H} \times \mathbb{H}$. There is unique such geodesic. All geodesics $\gamma^* \sigma$ together with a choice of orientation are defined to be 1-morphisms,

where $\gamma \in \mathrm{PGL}_2(K)$. We define the 1-morphisms of C to be finite concatenations of geodesics of type $\gamma^*\sigma$ or the trivial path whose image coincides with a cusp point. Consider ideal triangles and ideal diangles as cells from which we build manifolds with corners. A 2-morphism is a finite union of manifolds with corners, made from finitely many ideal diangles and ideal triangles, which is path-connected and has orientation.

The boundary of a 1-morphism is a union of two objects — the starting point and the ending point of the directed path. The boundary of a 2-morphism (a 2-manifold with corners) is a finite union of 1-morphisms (oriented loops), where the orientation of the loops on the boundary is induced by the orientation of the 2-manifold with corners.

Now we are going to define a 2-sheaf J, whose values on a 2-morphism will be in a subset of the ring R and whose values on an object and on a 1-morphism will be a subset of a countable product of the ring R with itself.

As always, $S_{2,2}(\Gamma)$ denotes the space of cusp forms of weight $(2, 2)$ with respect to the group Γ. Here we will consider this space as the space of holomorphic 2-forms on $\mathbb{H} \times \mathbb{H}$ which vanish on the cusps and which can descend to the Hilbert modular surface $X_\Gamma = \Gamma \backslash (\mathbb{H}^2 \cup \mathbb{P}^1(K))$. Every n-tuple of such holomorphic forms $\Omega \in (S_{2,2}(\Gamma))^n$ defines a value of a 2-morphism f in C. Let this value be the generating series $J_f(\Omega)$. Let J_f be the collection of all values $J_f(\Omega)$ for $\Omega \in (S_{2,2}(\Gamma))^n$. Let e be a 1-morphism. We say that e is in the boundary of a 2-morphism f, denoted $e \subset \partial f$, if the image of the loop e is in the boundary of the image of the membrane f together with the induced orientation on e from f. We say that an object p is in the boundary of a 1-morphism e, denoted by $p \in \partial e$, if p is a source or a target of e. We define the values of J on a 1-morphism e to be the product

$$\prod_{e \subset \partial f} J_f \subset \prod_{e \subset \partial f} R.$$

We define the values of J on objects p to be

$$\prod_{\substack{p \in \partial e \\ e \subset \partial f}} J_f \subset \prod_{\substack{p \in \partial e \\ e \subset \partial f}} R.$$

The sheaf conditions for 1- and 2-morphisms resemble the classical conditions for a presheaf to be a sheaf.

Let $f_i : A_i \to [0, 1]^2$ be a finite collection of disjoint 2-morphisms, whose union is a morphism $f : A \to [0, 1]^2$. We define a finite collection f_{ij}^k of 1-morphisms and 0-morphisms (objects) such that the union $\bigcup_k \mathrm{im}(f_{ij}^k) = \mathrm{im}(f_i) \cap \mathrm{im}(f_j)$ is a disjoint union of the intersection.

Then the equalizer

$$J_f \to \prod_i J_{f_i} \rightrightarrows \prod_{ijk} J_{f_{ij}^k}$$

is exact (for a definition of equalizer one may consult [Borceux 1994]).

Similarly, let e be a 1-morphism and let $\{e_i\}$ be a finite set of disjoint 1-morphisms such that the union $\bigcup_i \mathrm{im}(e_i)$ is equal to $\mathrm{im}(e)$. We can write the intersection $\mathrm{im}(e_i) \cap \mathrm{im}(e_j)$ as a finite union of 0-morphisms $\bigcup_k \mathrm{im}(e_{ij}^k)$, for some 1-morphisms e_{ij}^k.

Then the equalizer

$$J_e \to \prod_i J_{e_i} \rightrightarrows \prod_{ijk} J_{e_{ij}^k}$$

is exact.

The cochain is defined as

$$\prod_{p:\,0\text{-morph}} J_p \to \prod_{e:\,1\text{-morph}} J_e \to \prod_{f:\,2\text{-morph}} J_f \to \prod_{\substack{g:\,2\text{-morph} \\ \partial g = \varnothing}} J_g$$

The maps $J_e \to J_p$ and $J_p \to J_f$ are surjective when they are defined, resembling flabby sheaves. Thus, we should have trivial zeroth and first cohomology set. The only nontrivial cohomology will be the second cohomology set. The cocycle conditions for both noncommutative Hilbert modular symbols c^1 and c^2 can be interpreted as a particular case of maps

$$\prod_{f:\,2\text{-morph}} J_f \to \prod_{\substack{g:\,2\text{-morph} \\ \partial g = \varnothing}} J_g.$$

For c^1, the boundary condition is that a union of two diangles with a common edge is a third diangle. One can think of the these three diangles as the boundary of a degenerate 3-dimensional region. One can realize this cocycle condition as a sheaf-theoretic one by modifying the above definition so that the 2-morphisms consists of a finite union of ideal diangles (without using the ideal triangles). Then the sheaf-theoretic second cocycle condition is the one for noncommutative Hilbert modular symbol c^1.

If we are able to quotient the 2-category described in the beginning of this subsection by the 2-morphisms generated by diangles, then we have only two morphisms generated by ideal triangles. The noncommutative Hilbert modular symbol c^2 is exactly the one that considers ideal triangles. Note that its cocycle relation for c^2 is satisfied up to 2-morphisms generated by diangles.

4G. *Explicit computations. Multiple Dedekind zeta values.* In this subsection, we make explicit computations of some ingredients in the noncommutative Hilbert modular symbol. Manin [2006] compared explicit formulas of integrals in the noncommutative modular symbol to multiple zeta values. The similarities are in terms of both infinite series formulas and formulas via iterated path integrals. Here we compare certain integrals in the noncommutative Hilbert modular symbol to multiple Dedekind zeta values (for multiple Dedekind zeta values, see [Horozov 2014b]). Again the similarities are in terms of both infinite series formulas and formulas via iterated integrals over membranes.

We are going to consider the Fourier expansion of two Hilbert cusp forms f and g. Let $\omega_f = f\, dz_1 \wedge dz_2$, $\omega_g = g\, dz_1 \wedge dz_2$ and $\omega_0 = dz_1 \wedge dz_2$. We are going to associate L-values to iterated integrals of the forms ω_f and ω_g. The L-values will be iterated integrals over an union of diangles. One can think of a diangle connecting 0 and ∞ as a segment or as a real cone. The union will be a disjoint union of all such real cones connecting 0 and ∞ or simply $\mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H})$. We also recall the definition of a multiple Dedekind zeta values via (discrete) cones. Finally, we prove analogous formulas for iterated L-values associated to Hilbert cusp forms and for multiple Dedekind zeta values.

We will be mostly interested in the modular symbol associated to a diangle. Let us recall what we mean by a diangle.

Let p_1, p_2, p_3, p_4 be four cusp points. Let $\gamma_1 \in \mathrm{GL}_2(K)$ be a linear fractional transformation that sends $\gamma_1(p_1)$ to 0, $\gamma_1(p_2)$ to ∞, and $\gamma_1(p_3)$ to 1. Let Δ be the image of the diagonal embedding of $\mathbb{H}^1$ into $\mathbb{H}^2$. Then 0, 1 and ∞ are boundary points of Δ. Let $\lambda(0, \infty)$ be the unique geodesic in Δ that connects 0 and ∞. And let

$$\lambda_1(p_1, p_2) = \gamma_1^{-1}\lambda(0, \infty)$$

be the pullback of the geodesic λ to a geodesic connecting p_1 and p_2.

Now consider the triple p_1, p_2 and p_4. Let $\gamma_2 \in \mathrm{GL}_2(K)$ be a linear fractional transformation that sends $\gamma_2(p_1)$ to 0, $\gamma_2(p_2)$ to ∞ and $\gamma_2(p_4)$ to 1. Let Δ be the image of the diagonal embedding of $\mathbb{H}^1$ into $\mathbb{H}^2$. Then 0, 1 and ∞ are boundary points of Δ. Let $\lambda(0, \infty)$ be the unique geodesic in Δ that connects 0 and ∞. And let

$$\lambda_2(p_1, p_2) = \gamma_2^{-1}\lambda(0, \infty)$$

be the pullback of the geodesic λ to a geodesic connecting p_1 and p_2.

By a diangle, we mean a region in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ with the homotopy type of a disc, bounded by the geodesics $\lambda_1(0, \infty)$ and $\lambda_2(0, \infty)$.

We are going to present a computation for the diangle D_u defined by the points $(0, \infty, u^1, u^{-1})$, where u is a generator for the group of units modulo ± 1 in K. Let (1) be the trivial permutation.

Lemma 4.17. *Let u be a totally positive unit. Then*

$$\iint_{D_u}^{(1)(1)} e^{2\pi i(\alpha_1 z_1 + \alpha_2 z_2)}\, dz_1 \wedge dz_2 = \frac{1}{(2\pi i)^2} \frac{u_2^2 - u_1^2}{(\alpha_1 u_1 + \alpha_2 u_2)(\alpha_1 u_2 + \alpha_2 u_1)}.$$

Proof. Let u_1 and u_2 be the two embeddings of u into $\mathbb{R}$. Then $(0, \infty, u)$ can be mapped to $(0, \infty, 1)$ by $\gamma_1 = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. The geodesic $\lambda(0, \infty)$ can be parametrized by (it, it) for $t \in \mathbb{R}$. Then the geodesic $\lambda_1(0, \infty)$ on the geodesic triangle $(0, \infty, u)$ can be parametrized by $\{(iu_1 t, iu_2 t) \mid t > 0\}$. Similarly, the geodesic $\lambda_2(0, \infty)$ on the geodesic triangle $(0, \infty, u^{-1})$ can be parametrized by $\{(iu_2 t, iu_1 t) \mid t > 0\}$. Then the diangle D_u can be parametrized by

$$\left\{ (z_1, z_2) \in \mathbb{H}^2 \mid \mathrm{Re}(z_1) = \mathrm{Re}(z_2) = 0,\ \mathrm{Im}(z_1) \in \left(\frac{u_1}{u_2} t, \frac{u_2}{u_1} t \right),\ \mathrm{Im}(z_2) = t \in (0, \infty) \right\}.$$

Then we have

$$
\begin{aligned}
\iint_{D_u}^{(1)(1)} e^{2\pi i(\alpha_1 z_1 + \alpha_2 z_2)}\, dz_1 \wedge dz_2 &= \int_{\infty}^{0} \left(\int_{\frac{u_2}{u_1}t}^{\frac{u_1}{u_2}t} e^{2\pi i(\alpha_1 z_1 + \alpha_2 t)}\, dz_1 \right) dt \\
&= \frac{1}{2\pi i \alpha_1} \int_{\infty}^{0} \left(e^{\alpha_1 \frac{u_1}{u_2} t + \alpha_2 t} - e^{\alpha_1 \frac{u_2}{u_1} t + \alpha_2 t} \right) dt \\
&= \frac{1}{(2\pi i)^2} \frac{1}{\alpha_1} \left(\frac{1}{\alpha_1 \frac{u_1}{u_2} + \alpha_2} - \frac{1}{\alpha_1 \frac{u_2}{u_1} + \alpha_2} \right) \\
&= \frac{1}{(2\pi i)^2} \frac{u_2^2 - u_1^2}{(\alpha_1 u_1 + \alpha_2 u_2)(\alpha_1 u_2 + \alpha_2 u_1)} \qquad \square
\end{aligned}
$$

Therefore, one term of the Fourier expansion of a Hilbert cusp form paired with a symbol given by one diangle does not resemble a norm of an algebraic integer. However, if we integrate over an infinite union of diangles, then a similarity with Dedekind zeta and multiple Dedekind zeta values occurs.

Consider the limit of D_{u^n} when $n \to \infty$. It is the product of the two imaginary axes of the two upper half-planes. Set

$$\mathrm{Im}(\mathbb{H}^2) = \mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H}).$$

One can think of this region as an infinite union of diangles.

Denote by αz the sum of products $\alpha_1 z_1 + \alpha_2 z_2$. Using the methods of [Horozov 2014b, Section 1], we obtain

$$\frac{(2\pi i)^{-2}}{N(\alpha)N(\alpha + \beta)} = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} e^{2\pi i \alpha z}\, dz_1 \wedge dz_2 \cdot e^{2\pi i \beta z}\, dz_1 \wedge dz_2$$

and

$$\frac{1}{(2\pi i)^2} \frac{1}{N(\alpha)^3 N(\alpha+\beta)^2}$$
$$= \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} e^{2\pi i\alpha z}\, dz_1 \wedge dz_2 \cdot (dz_1 \wedge dz_2) \cdot (dz_1 \wedge dz_2) \cdot e^{2\pi i\beta z}\, dz_1 \wedge dz_2 \cdot (dz_1 \wedge dz_2).$$

Let f and g be two cusp forms of weights $(2k, 2k)$ and $(2l, 2l)$, respectively. Consider the Fourier expansion of both of the cusp forms. Let

$$f = \sum_{\alpha \gg 0} a_\alpha e^{2\pi i\alpha z} \quad \text{and} \quad g = \sum_{\beta \gg 0} b_\beta e^{2\pi i\beta z}.$$

Since f is of weight $(2k, 2k)$, we have that $a_{u\alpha} = a_\alpha$, where u is a unit. For such a modular form the modular factor with respect to the transformation $z \to uz$ is 1. The L-values of f are

$$L_f(n) = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} \sum_{\alpha \in \mathbb{O}_K^+/U^+} a_\alpha e^{2\pi i\alpha z}\, dz_1 \wedge dz_2 \cdot (dz_1 \wedge dz_2)^{\cdot(n-1)}$$
$$= \frac{1}{(2\pi i)^{2n}} \sum_{\alpha \in \mathbb{O}_K^+/U^+} \frac{a_\alpha}{N(\alpha)^n}.$$

Here $\mathbb{O}_K^+$ denotes the totally positive algebraic integers in K and U^+ denotes the totally positive units.

We recall some of the definitions from [Horozov 2014b]. We fix a positive cone C in $\mathbb{O}_K$, by which we mean

$$C = \mathbb{N} \cup \{\alpha \in \mathbb{O}_K \mid a + b\epsilon,\ a, b \in \mathbb{N}\},$$

where ϵ is a generator of the group of totally positive units. By $\epsilon^k C$, we mean the collection of products $\epsilon^k \alpha$, where α varies in the cone C.

The following infinite sum is an example of a multiple Dedekind zeta value:

$$\zeta_{K;C,\epsilon^k C}(m, n) = \sum_{\alpha \in C} \sum_{\beta \in \epsilon^k C} \frac{1}{N(\alpha)^m N(\alpha + \beta)^n}.$$

Let $Z(m, n) = \sum_{k \in \mathbb{Z}} \zeta_{K;C,\epsilon^k C}(m, n)$, where C is any set representing the totally positive algebraic integers $\mathbb{O}_K^+$ modulo totally positive units U^+.

Lemma 4.18. *The values $Z(m, n)$ are finite for $m > n > 1$.*

Proof. Let ϵ be a generators of the group of totally positive units U^+ in K. For the two real embeddings ϵ_1 and ϵ_2 of ϵ, we can assume that $\epsilon_1 > 1 > \epsilon_2$. Otherwise we can take its reciprocal.

$$Z(m,n) = \sum_{k \in Z} \sum_{\alpha,\beta \in C} \frac{1}{N(\alpha)^m N(\alpha + \epsilon^k \beta)^n} \tag{29}$$

$$< \sum_{\alpha,\beta \in C} \frac{1}{N(\alpha)^m} \left(\frac{1}{N(\alpha+\beta)^n} + \sum_{k=1}^{\infty} \frac{2^n}{\epsilon_1^k} \left(\frac{1}{\alpha_1^n \beta_2^n} + \frac{1}{\alpha_2^n \beta_1^n} \right) \right) \tag{30}$$

$$< \sum_{\alpha,\beta \in C} \frac{1}{N(\alpha)^m} \left(\frac{1}{N(\alpha+\beta)^n} + \sum_{k=1}^{\infty} \frac{2}{\epsilon_1^k} \left(\frac{N(\alpha+\beta)^n - N(\alpha)^n}{N(\alpha+\beta)^n} \right) \right) \tag{31}$$

$$= \sum_{\alpha,\beta \in C} \frac{1}{N(\alpha)^m} \left(\frac{1}{N(\alpha+\beta)^n} + \frac{2}{\epsilon_1 - 1} \left(\frac{N(\alpha+\beta)^n - N(\alpha)^n}{N(\alpha+\beta)^n} \right) \right) \tag{32}$$

$$= \sum_{\alpha,\beta \in C} \frac{1}{N(\alpha)^m N(\alpha+\beta)^n} + \frac{2}{\epsilon_1 - 1} \frac{1}{N(\alpha)^n} - \frac{2}{N(\alpha)^{m-n} N(\alpha+\beta)^n} \tag{33}$$

$$= \zeta_K(C; m, n) - \frac{2}{\epsilon_1 - 1}(\zeta_K(C; n) + \zeta_K(C; m-n, n)). \tag{34}$$

Equation (29) is the definition. Inequality (30) is based on the following: $\epsilon_2 < 1$ is replaced with 1 when $k > 0$. For $k < 0$ we use $\epsilon_2^k = \epsilon_1^{-k}$. We put 1 for ϵ_1^k for $k < 0$. The case $k = 0$ is treated separately. Finally we group the terms with equal powers of ϵ_1. In inequality (31) we estimate the mixed terms in the brackets. In (32) we take the sum of the geometric series in ϵ_1^{-1}. Then in (33) we open the brackets. And finally, in (34), we express the sums as a finite linear combinations of a Dedekind zeta value and multiple Dedekind zeta values.   □

The following definition of an iterated L-value is a coefficient of one monomial from the noncommutative Hilbert modular symbol of type $\boldsymbol{b}$:

Definition 4.19. For a pair of Hilbert cusp forms f and g with Fourier expansions

$$f = \sum_{\alpha \gg 0} a_\alpha e^{2\pi i \alpha z} \quad \text{and} \quad g = \sum_{\beta \gg 0} b_\beta e^{2\pi i \beta z},$$

we define iterated L-values

$$L_{f,g}(m,n) = \int_{\text{Im}(\mathbb{H}^2)}^{(1)(1)} \sum_{(\alpha,\beta) \in (\mathbb{O}_K^+, \mathbb{O}_K^+)/U} (a_\alpha e^{2\pi i \alpha z} \, dz_1 \wedge dz_2) \cdot (dz_1 \wedge dz_2)^{\cdot(m-1)}$$
$$\cdot (b_\beta e^{2\pi i \beta z} \, dz_1 \wedge dz_2) \cdot (dz_1 \wedge dz_2)^{\cdot(n-1)}.$$

Theorem 4.20. *Using the above definition, we have*

$$L_{f,g}(m,n) = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in C, \beta \in \epsilon^k C} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\beta)^n}.$$

Proof.

$$L_{f,g}(m,n) = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} \sum_{(\alpha,\beta) \in (\mathbb{O}_K^+, \mathbb{O}_K^+)/U} (a_\alpha e^{2\pi i \alpha z} \, dz_1 \wedge dz_2) \cdot (dz_1 \wedge dz_2)^{\cdot(m-1)}$$
$$\cdot (b_\beta e^{2\pi i \beta z} \, dz_1 \wedge dz_2) \cdot (dz_1 \wedge dz_2)^{\cdot(n-1)}$$

$$= \sum_{(\alpha,\beta) \in (\mathbb{O}_K^+, \mathbb{O}_K^+)/U} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\beta)^n}$$

$$= \sum_{k \in \mathbb{Z}; \alpha, \beta \in C} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\epsilon^k \beta)^n}$$

$$= \sum_{k \in \mathbb{Z}} \sum_{\alpha \in C} \sum_{\beta \in \epsilon^k C} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\beta)^n}. \qquad \square$$

We would like to bring to the attention of the reader Definition 4.19, the definition of the multiple L- values. More specifically, we would like to point out that the region of integration is an infinite union of diangles (or equivalently an infinite union of real cones; see the beginning of this section). Note also that in Theorem 4.20 the values of the multiple L-functions are expressed as an infinite sums over different discrete cones, namely, over $\epsilon^k C$ for $k \in \mathbb{Z}$. However, a single real cone D_u, as in Lemma 4.18, does not correspond to a single discrete cone. Only a good union of real cones $\mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H})$ corresponds to a good union of discrete cones $\bigcup_{k \in \mathbb{Z}} (C, \epsilon^k C)$ as a fundamental domain of $(\mathbb{O}_K^+, \mathbb{O}_K^+)/U^+$.

Acknowledgements

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