Algebra & Number Theory

Volume 9
2015
No. 2

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ALGEBRA AND NUMBER THEORY 9:2 (2015)
dx.doi.org/10.2140/ant.2015.9.493

Finite-dimensional quotients of Hecke algebras

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Let W be a complex reflection group. We prove that there is a maximal finite-dimensional quotient of the Hecke algebra $\mathcal{H}_q(W)$ of W, and that the dimension of this quotient coincides with $|W|$. This is a weak version of a 1998 Broué–Malle–Rouquier conjecture. The proof is based on the categories $\mathcal{O}$ for rational Cherednik algebras.

1. Introduction

Let W be a complex reflection group. Recall that such groups are fully classified; see [Shephard and Todd 1954]. In this context, one can also define the braid group B_W. Namely, let $\mathfrak{h}$ denote the reflection representation of W. Inside $\mathfrak{h}$, one considers the open subset of regular points $\mathfrak{h}^{\mathrm{reg}} := \{x \in \mathfrak{h} \mid W_x = \{1\}\}$, so that W is the Galois group of the cover $\mathfrak{h}^{\mathrm{reg}} \twoheadrightarrow \mathfrak{h}^{\mathrm{reg}}/W$. By definition, the braid group B_W is the fundamental group $\pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$.

If W is a Coxeter group, one considers a flat deformation of $\mathbb{C}W$, called a *Hecke algebra*. These algebras are of importance in representation theory (e.g., that of finite groups of Lie type) and beyond (e.g., in knot theory). One can define Hecke algebras for complex reflection groups as well. In the general case, this was done in [Broué et al. 1998, Section 4C]. To recall the definition, we need some more notation. Namely, let $\mathfrak{H}$ denote the set of reflection hyperplanes for W. For $\Gamma \in \mathfrak{H}$, let W_Γ denote the pointwise stabilizer of Γ; this is a cyclic group. Set $\ell_\Gamma := |W_\Gamma|$. The group B_W is generated by elements T_Γ, $\Gamma \in \mathfrak{H}$, where, roughly speaking, T_Γ is the rotation around Γ by $2\pi/\ell_\Gamma$; see [loc. cit., Section 2]. Now, pick independent variables $u_{\Gamma,i}$, $i = 0, 1, \dots, \ell_\Gamma - 1$ with $u_{\Gamma,i} = u_{\Gamma',i}$ for W-conjugate Γ, Γ'. Set $\boldsymbol{u} := (u_{\Gamma,i})$. By definition ([loc. cit., Definition 4.21]), the Hecke algebra $\mathcal{H}_{\boldsymbol{u}}(W)$ is the quotient of $\mathbb{Z}[\boldsymbol{u}^{\pm 1}]B_W$ by the relations

$$\prod_{i=0}^{\ell_\Gamma - 1} (T_\Gamma - u_{\Gamma,i}) = 0.$$

MSC2010: primary 20C08; secondary 20F55, 16G99.
Keywords: Hecke algebras, rational Cherednik algebras, categories $\mathcal{O}$, KZ functor.

Broué, Malle and Rouquier [Broué et al. 1998, Section 4C] conjectured that $\mathcal{H}_{\boldsymbol{u}}(W)$ is a free $\mathbb{Z}[\boldsymbol{u}^{\pm 1}]$-module generated by $|W|$ elements. Currently, the proof is missing in the case of several exceptional complex reflection groups. In this paper, we are going to prove a weaker version of this conjecture.

First of all, we are dealing with specializations to $\mathbb{C}$. For a collection of nonzero complex numbers $(q_{\Gamma,i})$, where $\Gamma \in \mathfrak{H}/W$ and $i \in \{0, 1, \dots, \ell_\Gamma - 1\}$, consider the $\mathbb{C}$-algebra $\mathcal{H}_q(W)$, the specialization of $\mathcal{H}_{\boldsymbol{u}}(W)$ with $u_{\Gamma,i} \mapsto q_{\Gamma,i}$. Note that replacing the collection $q_{\Gamma,i}$ with $(\alpha_\Gamma q_{\Gamma,i})$ for $\alpha_\Gamma \in \mathbb{C}^\times$, we get isomorphic algebras; see, e.g., [Rouquier 2008, Section 3.3.3]. So the number of parameters actually equals $|S/W|$, where S denotes the set of complex reflections in W. Note that if $q_{\Gamma,j} = \exp(2\pi\sqrt{-1}j/\ell_\Gamma)$ for all Γ and j, we just have $\mathcal{H}_q(W) = \mathbb{C}W$. In general, however, it is even unclear whether the algebra $\mathcal{H}_q(W)$ is finite-dimensional or not. In a way, the infinite dimension is the only obstruction to the equality $\dim \mathcal{H}_q(W) = |W|$. More precisely, we have the following theorem, which is the main result of this paper:

Theorem 1.1. *There is a minimal two-sided ideal $I \subset \mathcal{H}_q(W)$ such that $\mathcal{H}_q(W)/I$ is finite-dimensional. Moreover, we have* $\dim \mathcal{H}_q(W)/I = |W|$.

Other results towards the Broué–Malle–Rouquier conjecture were known before; see [Marin 2014] for a review. One advantage of our approach is that it is fully conceptual and does not involve any case-by-case arguments.

The key idea of the proof is to use categories $\mathbb{O}$ for rational Cherednik algebras $H_c(W)$, introduced in [Ginzburg et al. 2003]. By definition, the algebra $H_c(W)$ is the subalgebra in the skew-group algebra $D(\mathfrak{h}^{\mathrm{reg}})\#W$ generated by $\mathbb{C}[\mathfrak{h}], \mathbb{C}W$ and the so-called Dunkl operators $D_a, a \in \mathfrak{h}$. These are differential operators with first-order poles along the reflection hyperplanes Γ. We have a triangular decomposition $H_c(W) = \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes S(\mathfrak{h})$ that allows to define the category $\mathbb{O}$. This is the category of all $H_c(W)$-modules that are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and have a locally nilpotent action of $\mathfrak{h} \subset S(\mathfrak{h})$.

Let us pick $M \in \mathbb{O}$. Its restriction to $\mathfrak{h}^{\mathrm{reg}}$ is a W-equivariant local system on $\mathfrak{h}^{\mathrm{reg}}$. So the fiber M_x carries a monodromy representation of B_W. It was shown in [Ginzburg et al. 2003] that the $\mathbb{C}B_W$-action on M_x factors though a certain quotient of $\mathcal{H}_q(W)$ that has dimension $|W|$. We will show that every finite-dimensional $\mathcal{H}_q(W)$-module can be represented in the form M_x for some $M \in \mathbb{O}$.

The assumption $\dim \mathcal{H}_q(W) = |W|$ is actually important for the representation theory of $H_c(W)$. Theorem 1.1 should make it possible to remove this assumption, but we are not going to elaborate on that.

The paper is organized as follows: In Section 2 we gather various generalities on the rational Cherednik algebras and their categories $\mathbb{O}$. Then in Section 3 we prove the main theorem.

2. Generalities

2A. ***Rational Cherednik algebras.*** Rational Cherednik algebras were introduced by Etingof and Ginzburg [2002]. In this subsection we recall their definition.

Let W be a complex reflection group and $\mathfrak{h}$ be its reflection representation. We denote the subset of W consisting of complex reflections by S. For $s \in S$, pick an eigenvector $\alpha_s \in \mathfrak{h}^*$ for s with eigenvalue $\lambda_s \neq 1$. We fix a W-invariant function $c : S \to \mathbb{C}$. Using this function, for $a \in \mathfrak{h}$, we can define the Dunkl operator $D_a \in D(\mathfrak{h}^{\mathrm{reg}})\#W$ by

$$D_a = \partial_a + \sum_{s \in S} \frac{2c(s)}{1-\lambda_s} \frac{\langle \alpha_s, a \rangle}{\alpha_s}(s-1).$$

The rational Cherednik algebra $H_c(W)$ is the subalgebra in $D(\mathfrak{h}^{\mathrm{reg}})\#W$ generated by $\mathbb{C}[\mathfrak{h}], \mathbb{C}W$ and the Dunkl operators $D_a, a \in \mathfrak{h}$. Alternatively, one can present $H_c(W)$ by generators and relations: $H_c(W)$ is the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$ by the relations

$$[x, x'] = [y, y'] = 0, [y, x] = \langle y, x \rangle - \sum_{s \in S} c(s) \langle \alpha_s, y \rangle \langle \alpha_s^\vee, x \rangle s, \quad x, x' \in \mathfrak{h}^*, \, y, y' \in \mathfrak{h}.$$

Here, we write $\alpha_s^\vee$ for the eigenvector of s in $\mathfrak{h}$ with eigenvalue λ_s^{-1} and $\langle \alpha_s^\vee, \alpha_s \rangle = 2$. To get from the second definition to the first one, we use the homomorphism $H_c(W) \to D(\mathfrak{h}^{\mathrm{reg}})\#W$ given by $x \mapsto x, w \mapsto w, y \mapsto D_y$. Set $\delta := \prod_{s \in S} \alpha_s^{\ell_s}$, where ℓ_s stands for the order of s (note that this is slightly different from the usual definition). This is a W-invariant element, and the operator $\operatorname{ad}\delta : H_c(W) \to H_c(W)$ is locally nilpotent, so the localization $H_c(W)[\delta^{-1}]$ is defined. The homomorphism $H_c(W) \to D(\mathfrak{h}^{\mathrm{reg}})\#W$ extends to an isomorphism $H_c(W)[\delta^{-1}] \xrightarrow{\sim} D(\mathfrak{h}^{\mathrm{reg}})\#W$.

The algebra $H_c(W)$ admits a triangular decomposition: a natural map $S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes S(\mathfrak{h}) \to H_c(W)$ is an isomorphism. Also $H_c(W)$ is graded with $\deg x = 1$, $\deg w = 0$, $\deg y = -1$, $x \in \mathfrak{h}^*$, $w \in W$, and $y \in \mathfrak{h}$. We call this grading the Euler grading. It is inner: it is given by the eigenvalues of $\operatorname{ad} h$, where

$$h = \sum_{i=1}^{n} x_i y_i + \frac{n}{2} - \sum_{s \in S} \frac{2c(s)}{1-\lambda_s} s. \tag{2-1}$$

Now let us discuss base change for $H_c(W)$. Let U be an affine algebraic variety equipped with an étale map $U \to \mathfrak{h}/W$. Then $\mathbb{C}[U] \otimes_{\mathbb{C}[\mathfrak{h}]^W} H_c(W)$ has a natural algebra structure; it is a subalgebra in $D(U \times_{\mathfrak{h}/W} \mathfrak{h}^{\mathrm{reg}})\#W$ generated by $\mathbb{C}[U] \otimes_{\mathbb{C}[\mathfrak{h}/W]} \mathbb{C}[\mathfrak{h}], \mathbb{C}W$ and the Dunkl operators. Similarly, if U is a Stein complex analytic manifold (again equipped with an étale map $U \to \mathfrak{h}/W$), then $\mathbb{C}_{\mathrm{an}}[U] \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c(W)$ is an algebra. Here and below $\mathbb{C}_{\mathrm{an}}[U]$ denotes the algebra of analytic functions on U.

2B. ***Category*** $\mathbb{O}$ ***and*** **KZ** ***functor.*** The category $\mathbb{O}$ for $H_c(W)$ was defined in [Ginzburg et al. 2003]. By definition, it consists of all $H_c(W)$-modules M that are finitely generated over $S(\mathfrak{h}^*) = \mathbb{C}[\mathfrak{h}]$ and where the action of $\mathfrak{h}$ is locally nilpotent. Equivalently, $\mathbb{O}$ consists of all $H_c(W)$-modules M that are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and that can be graded. This category $\mathbb{O}$ will be denoted by $\mathbb{O}_c(W)$.

Let us proceed to the KZ functor introduced in [loc. cit., Section 5]. Pick $M \in \mathbb{O}_c(W)$. Then $M[\delta^{-1}]$ is a W-equivariant local system on $D(\mathfrak{h}^{\mathrm{reg}})$ with regular singularities. The category of such local systems is equivalent to the category B_W-$\mathrm{mod}_{\mathrm{fin}}$ of finite-dimensional B_W-modules; to a local system M' one assigns its fiber (or, more precisely, the fiber of its descent to $\mathfrak{h}^{\mathrm{reg}}/W$) equipped with the monodromy representation. It turns out that the monodromy representation associated to $M[\delta^{-1}]$ factors through $\mathscr{H}_q(W)$ [loc. cit., Section 5.3], where the parameter q is computed as follows: For a reflection hyperplane Γ, set

$$h_{\Gamma,i} = \frac{1}{\ell_\Gamma} \sum_{s \in W_\Gamma \setminus \{1\}} \frac{2c(s)}{\lambda_s - 1} \lambda_s^{-i}, \tag{2-2}$$

$$q_{\Gamma,i} = \exp(2\pi\sqrt{-1}(h_{\Gamma,j} + j/\ell_H)). \tag{2-3}$$

So we get an exact functor $\mathrm{KZ} : \mathbb{O}_c(W) \to \mathscr{H}_q(W)$-$\mathrm{mod}_{\mathrm{fin}}$. This functor is given by $\mathrm{Hom}_{\mathbb{O}_c(W)}(P_{\mathrm{KZ}}, \bullet)$, where P_{KZ} is a projective object such that $\dim \mathrm{End}_{\mathbb{O}_c(W)}(P_{\mathrm{KZ}})$ is equal to $|W|$, equipped with a homomorphism $\mathscr{H}_q(W) \to \mathrm{End}_{\mathbb{O}_c(W)}(P_{\mathrm{KZ}})^{\mathrm{opp}}$. The proof of [loc. cit., Theorem 5.15] shows that this homomorphism is surjective.

Theorem 1.1 will follow if we show that the functor KZ is essentially surjective.

2C. ***Isomorphisms of étale lifts.*** Here we are going to recall some results of [Bezrukavnikov and Etingof 2009] regarding isomorphisms of completions.

Let $W' \subset W$ be a parabolic subgroup, i.e., the stabilizer of a point in $\mathfrak{h}$. Set $\mathfrak{h}^{\mathrm{reg}-W'} := \{b \in \mathfrak{h} \mid W_b \subset W'\}$. The complement of $\mathfrak{h}^{\mathrm{reg}-W'}$ in $\mathfrak{h}$ is the union of the hyperplanes $\ker \alpha_s$ for $s \notin W'$. So $\mathfrak{h}^{\mathrm{reg}-W'}$ is a principal open subset of $\mathfrak{h}$. Note that the natural morphism $\mathfrak{h}^{\mathrm{reg}-W'}/W' \to \mathfrak{h}/W$ is étale (and $\mathfrak{h}^{\mathrm{reg}-W'}/W'$ is precisely the unramified locus of $\mathfrak{h}/W' \to \mathfrak{h}/W$).

Consider the space $H_c(W)_{\mathrm{reg}-W'} := \mathbb{C}[\mathfrak{h}^{\mathrm{reg}-W'}]^{W'} \otimes_{\mathbb{C}[\mathfrak{h}]^W} H_c(W)$. As was mentioned in the end of Section 2A, $H_c(W)_{\mathrm{reg}-W'}$ is actually an algebra. Bezrukavnikov and Etingof [2009, Section 3.3] essentially found an alternative description of this algebra. Namely, consider the Cherednik algebra $H_c(W', \mathfrak{h})$ defined for the pair $W', \mathfrak{h}$; it decomposes into the tensor product $H_c(W', \mathfrak{h}) = D(\mathfrak{h}^{W'}) \otimes H_c(W')$ (here we abuse notation and write c for the restriction of c to $S \cap W'$; $D(\mathfrak{h}^{W'})$ stands for the algebra of differential operators on $\mathfrak{h}^{W'}$). Then consider its localization $H_c(W', \mathfrak{h})_{\mathrm{reg}-W'} := \mathbb{C}[\mathfrak{h}^{\mathrm{reg}-W'}]^{W'} \otimes_{\mathbb{C}[\mathfrak{h}]^{W'}} H_c(W', \mathfrak{h})$. Then, following [loc. cit., Section 3.2], we can form the centralizer algebra $Z(W, W', H_c(W', \mathfrak{h})_{\mathrm{reg}-W'})$. Recall that, by definition, for an algebra A equipped with a homomorphism $\mathbb{C}W' \to A$,

one defines the centralizer algebra $Z(W, W', A)$ as the endomorphism algebra of the right A-module $\mathrm{Map}_{W'}(W, A) = \{f : W \to A \mid f(w'w) = w' f(w)\}$. Choosing representatives of the left W'-cosets in W, we get an identification $Z(W, W', A) \cong \mathrm{Mat}_{|W/W'|}(A)$. The algebra A can be recovered from $Z(W, W', A)$ as follows: Consider the element $e(W') \in Z(W, W', A)$ given by $e(W') f(u) = f(u)$ if $u \in W'$ and 0 otherwise. Then $e(W')Z(W, W', A)e(W')$ is naturally identified with A.

The following is essentially [loc. cit., Theorem 3.2] (there, the authors considered completions instead of étale lifts, but the proof works in our situation as well).

Lemma 2.1. *There is a unique isomorphism*

$$\theta : H_c(W)_{\mathrm{reg}-W'} \xrightarrow{\sim} Z(W, W', H_c(W', \mathfrak{h})_{\mathrm{reg}-W'}) \tag{2-4}$$

such that the following hold for any $f \in \mathrm{Map}_{W'}(W, A)$ *and any* $u \in W$:

$$\begin{aligned}
&[\theta(g) f](u) = g f(u), \quad g \in \mathbb{C}[\mathfrak{h}^{\mathrm{reg}-W'}]^{W'},\\
&[\theta(\alpha) f](u) = (u\alpha) f(u), \quad \alpha \in \mathfrak{h}^*,\\
&[\theta(w) f](u) = f(uw), \quad w \in W,\\
&[\theta(a) f](u) = (ua) f(u) + \sum_{s \in S \setminus W'} \frac{2c(s)}{1 - \lambda_s} \frac{\langle \alpha_s, ua \rangle}{\alpha_s} (f(su) - f(u)), \quad a \in \mathfrak{h}.
\end{aligned}$$

Note that the algebras in (2-4) come equipped with $\mathbb{C}^\times$-actions by algebra automorphisms. For example, the action of $H_c(W)_{\mathrm{reg}-W'}$ comes from the action on $H_c(W)$ induced from the Euler grading and the action on $\mathfrak{h}^{\mathrm{reg}-W'}/W'$ induced from the scaling action ($t.x = t^{-1}x$) on $\mathfrak{h}$. The isomorphism θ is $\mathbb{C}^\times$-equivariant.

We can further restrict θ to some analytic submanifolds or formal subschemes of $\mathfrak{h}^{\mathrm{reg}-W'}/W'$. Choose a little disk $Y \subset \mathfrak{h}^{\mathrm{reg}-W'} \cap \mathfrak{h}^{W'}$ and also a little disk D around 0 in $\mathfrak{h}_{W'}/W'$, where $\mathfrak{h}_{W'}$ denotes the unique W'-stable complement to $\mathfrak{h}^{W'}$ in $\mathfrak{h}$. We set $\widehat{Y} := Y \times D$; this is an open submanifold in $\mathfrak{h}^{\mathrm{reg}-W'}/W'$, in $(\mathfrak{h}^{\mathrm{reg}-W'} \cap \mathfrak{h}^{W'}) \times \mathfrak{h}_{W'}/W'$ or in $\mathfrak{h}/W$ (under the natural morphism $\mathfrak{h}^{\mathrm{reg}-W'}/W' \to \mathfrak{h}/W$).

So we get an isomorphism

$$\theta_Y : \mathbb{C}_{\mathrm{an}}[\widehat{Y}] \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c(W) \xrightarrow{\sim} Z(W, W', \mathbb{C}_{\mathrm{an}}[\widehat{Y}] \otimes_{\mathbb{C}[\mathfrak{h}/W']} H_c(W', \mathfrak{h})). \tag{2-5}$$

Note that this isomorphism is compatible with the Euler derivations.

We can restrict even further. Pick a point $b \in Y$, and consider the completion $\mathbb{C}[\mathfrak{h}/W]^{\wedge_b}$ of $\mathbb{C}[\mathfrak{h}/W]$ with respect to the maximal ideal defined by b. Then θ_Y induces

$$\begin{aligned}
\theta_b : \mathbb{C}[\mathfrak{h}/W]^{\wedge_b} \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c(W)&\\
\xrightarrow{\sim} Z(W, W', \mathbb{C}[\mathfrak{h}/W']^{\wedge_b} \otimes_{\mathbb{C}[\mathfrak{h}/W']} H_c(W', \mathfrak{h})).&
\end{aligned} \tag{2-6}$$

This isomorphism was originally constructed in [Bezrukavnikov and Etingof 2009].

3. Proof of the main theorem

3A. ***Scheme of the proof.*** Let V be a finite-dimensional $\mathscr{H}_q(W)$-module and let N denote the corresponding W-equivariant D-module on $\mathfrak{h}^{\mathrm{reg}}$. Our goal is to show that there is an $M \in \mathbb{O}_c(W)$ such that $M[\delta^{-1}] \cong N$. This consists of two steps:

(I) Set $\mathfrak{h}^{\mathrm{sr}} := \{b \in W \mid \dim \mathfrak{h}^{W_b} \geqslant \dim \mathfrak{h} - 1\}$. This is an open subset that coincides with $\bigcup_{\Gamma \in \mathfrak{H}} \mathfrak{h}^{\mathrm{reg}-W_\Gamma}$; the codimension of its complement is bigger than 1. We will see that there is an $\mathbb{O}_{\mathfrak{h}^{\mathrm{sr}}/W}$-coherent $\mathbb{O}_{\mathfrak{h}^{\mathrm{sr}}/W} \otimes_{\mathbb{C}[\mathfrak{h}]^W} H_c(W)$-module $\widetilde{N}$ whose restriction to $\mathfrak{h}^{\mathrm{reg}}/W$ is isomorphic to N and that carries a locally finite derivation compatible with the Euler derivation of $H_c(W)$.

(II) We will see that $\widetilde{N}$ is a vector bundle. From here we will deduce that the global sections of $\widetilde{N}$ are finitely generated and hence lie in $\mathbb{O}_c(W)$. Then we take $M := \Gamma(\widetilde{N})$.

Let us elaborate on how we are going to achieve (I). First, in Section 3B we will check that the Euler vector field acts on N locally finitely. This will eventually prove that $\widetilde{N}$ comes equipped with a locally finite derivation that is compatible with the Euler one on $H_c(W)$.

Now let us explain how we produce $\widetilde{N}$; this is done in Section 3C. Take $W' = W_\Gamma$ and let $\widehat{Y}$ have the same meaning as in Section 2C. Set $\widehat{Y}^\times := \widehat{Y} \setminus Y$. Consider $N_\Gamma := e(W')(\mathbb{C}_{\mathrm{an}}[\widehat{Y}^\times] \otimes_{\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}/W]} N)$. This is a vector bundle on $\widehat{Y}^\times$ with a meromorphic connection that has pole of order 1 on Y (since N_Γ is obtained by restricting an algebraic vector bundle $e(W')\eta_\Gamma^* N$, where η_Γ is a natural morphism $\mathfrak{h}/W_\Gamma \to \mathfrak{h}/W$, it makes sense to speak about sections of N_Γ with poles on Y; here and below η_Γ denotes the projection $\mathfrak{h}/W_\Gamma \to \mathfrak{h}/W$). Our first step will be to see that N_Γ is obtained by restricting a $\mathbb{C}_{\mathrm{an}}[\widehat{Y}] \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c(W)$-module M_Γ. Then we will see that $[e(W')\eta_\Gamma^* N] \cap M_\Gamma$ (the intersection of subspaces in N_Γ) is finitely generated over $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}-W'}/W']$. We will get $\widetilde{N}$, roughly speaking, by taking the intersection of N and $\eta_\Gamma^* N \cap M_\Gamma$ over all possible Γ.[1]

3B. ***Locally finite derivation.*** Our goal here is to show that the Euler vector field acts on N locally finitely. Recall that N is a local system on $\mathfrak{h}^{\mathrm{reg}}/W$ with regular singularities. Our claim is a consequence of the following general result.

Lemma 3.1. *Let X be the complement to a $\mathbb{C}^\times$-stable divisor in $\mathbb{C}^d$, and let N be a local system with regular singularities on X. Then the Euler vector field* eu *acts on N locally finitely, meaning that every $n \in N$ is included into a finite-dimensional* eu*-stable subspace.*

[1] After this paper was written, I learned from Etingof that most of the proof is already contained in some form in [Wilcox 2011]. Lemmas 5.7 and 5.8 there are similar to what is done in Section 3C, while the main result of Section 3D has a somewhat easier proof in [Wilcox 2011, Lemma 3.6].

This claim should be standard, but we provide the proof for the sake of completeness. For a different proof, see [Wilcox 2011, Lemma 3.2].

Proof. If $N' \subset N$ is a $D(X)$-submodule and eu acts locally finitely on N/N' and N', then the same is true for N. So it is enough to assume that N (and hence V) is irreducible. Consider the element $\eta \in \pi_1(X)$ given by the loop $\exp(2\pi\sqrt{-1}t)x_0$, $t \in [0,1]$, where x_0 denotes the base point. The element η is central and hence has to act on V by a scalar. Under the Riemann–Hilbert correspondence, this translates to the claim that N is twisted equivariant with respect to the $\mathbb{C}^\times$-action. This implies our claim. □

3C. ***Extension to codimension*** **1.** We start by constructing M_Γ.

Set $\widehat{Y}^\times := Y \times D^\times = \widehat{Y} \setminus Y$. Consider the category $\mathrm{Loc}_{\mathrm{rs}}(\widehat{Y}, Y)$ of meromorphic local system on $\widehat{Y}^\times$ with regular singularities on Y (so an object in $\mathrm{Loc}_{\mathrm{rs}}(\widehat{Y}, Y)$ comes equipped with a lattice over the ring of meromorphic differential operators on $\widehat{Y}^\times$, and a morphism in the category is supposed to preserve such lattices). The category $\mathrm{Loc}_{\mathrm{rs}}(\widehat{Y}, Y)$ is equivalent to the category $\mathbb{C}[T^{\pm 1}]$-mod of finite-dimensional $\mathbb{C}[T^{\pm 1}]$-modules via taking the monodromy representation, because $\pi_1(\widehat{Y}^\times) = \pi_1(D^\times) = \mathbb{Z}$ (here we use the regular singularities condition). Under the equivalence $\mathbb{C}[T^{\pm 1}]$-mod $\cong \mathrm{Loc}_{\mathrm{rs}}(\widehat{Y}, Y)$, the KZ functor becomes

$$\mathcal{O}_c(W') \to \mathrm{Loc}_{\mathrm{rs}}(\widehat{Y}, Y), \quad M \mapsto \mathbb{C}_{\mathrm{an}}[\widehat{Y}^\times] \otimes_{\mathbb{C}[\mathfrak{h}_{W'}/W']} M$$

(with meromorphic lattice $\mathbb{C}_{\mathrm{an}}[\widehat{Y}][\nu^{-1}] \otimes_{\mathbb{C}[\mathfrak{h}_{W'}/W']} M$, where ν denotes a coordinate on $\mathfrak{h}_{W'}/W'$). The right adjoint KZ^* sends $N' \in \mathrm{Loc}_{\mathrm{rs}}(\widehat{Y}, Y)$ to the subspace of N' of all meromorphic elements annihilated by the vector fields on Y and lying in the generalized eigenspace for $\mathfrak{h}_{W'} \subset H_c(W')$ with eigenvalue 0.

Now we can produce a $\mathbb{C}[\widehat{Y}] \otimes_{\mathbb{C}[\mathfrak{h}/W']} H_c(W', \mathfrak{h})$-module $M_\Gamma \in \mathcal{O}_c(W', \widehat{Y})$. Set $N_\Gamma = e(W')(\mathbb{C}_{\mathrm{an}}[\widehat{Y}^\times] \otimes_{\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}/W]} N)$. This is an object in $\mathrm{Loc}_{\mathrm{rs}}(\widehat{Y}, Y)$. Note that, under the equivalence $\mathrm{Loc}_{\mathrm{rs}}(\widehat{Y}, Y) \cong \mathbb{C}[T^{\pm 1}]$-mod, we have $N_\Gamma \in \mathcal{H}_q(W')$-mod. Now set

$$M_\Gamma := \mathbb{C}_{\mathrm{an}}[\widehat{Y}] \otimes_{\mathbb{C}[\mathfrak{h}_{W'}/W']} \mathrm{KZ}^*(N_\Gamma). \tag{3-1}$$

Note that the description of $\mathrm{KZ}^*(N_\Gamma) \subset N_\Gamma$ implies that it is stable under the Euler vector field on N_Γ. So $M_\Gamma \subset N_\Gamma$ is also stable under the Euler vector field.

Let $\widetilde{N}_\Gamma := M_\Gamma \cap e(W')\eta_\Gamma^* N$ (the intersection is taken inside N_Γ). This is a submodule in the $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}-W'}]^{W'} \otimes_{\mathbb{C}[\mathfrak{h}/W']} H_c(W')$-module $e(W')\eta_\Gamma^* N$ which is stable under the Euler vector field.

Lemma 3.2. *The module $\widetilde{N}_\Gamma$ is finitely generated over $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}-W'}]^{W'}$ and satisfies $\widetilde{N}_\Gamma[\nu^{-1}] = e(W')\eta_\Gamma^* N$, where ν is a coordinate on $\mathfrak{h}_{W'}/W' \cong \mathbb{C}$.*

Proof. Note that the epimorphism $\mathcal{H}_q(W') \to \mathrm{End}_{\mathcal{O}_c(W')}(P_{\mathrm{KZ}})^{\mathrm{opp}}$ is an isomorphism. Let $N_\Gamma^{\mathrm{mer}} \subset N_\Gamma$ denote the meromorphic lattice. Then M_Γ is contained in N_Γ^{mer} and is a $\mathbb{C}_{\mathrm{an}}[\widehat{Y}]$-lattice there. Indeed, it is enough to show this fiberwise (i.e., at any point of Y), where this is clear (to prove that M_Γ is a lattice we use the observation that N_Γ is the image of M_Γ under the KZ functor). For any other lattice M', we have $\nu^d M' \subset M_\Gamma \subset \nu^{-d} M'$ for some $d > 0$. So it is enough to show that $e(W')\eta_\Gamma^* N \cap M'$ is finitely generated for some lattice M'. Let us construct such an M'.

As a $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}/W]$-module, N is projective and hence is a direct summand in a free module, say $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}/W]^{\oplus r}$. So N_Γ^{mer} is a direct summand in $\mathbb{C}_{\mathrm{an}}[\widehat{Y}][\nu^{-1}]^{\oplus r}$. The intersection $M' := N_\Gamma^{\mathrm{mer}} \cap \mathbb{C}_{\mathrm{an}}[\widehat{Y}]^{\oplus r}$ (inside $\mathbb{C}_{\mathrm{an}}[\widehat{Y}][\nu^{-1}]^{\oplus r}$) is clearly a lattice in N_Γ^{mer}. Further, the intersection $M' \cap e(W')\eta_\Gamma^* N$ coincides with

$$e(W')\eta_\Gamma^* N \cap \mathbb{C}[\mathfrak{h}^{\mathrm{reg}-W'}/W']^{\oplus r},$$

and hence is finitely generated (and clearly is a lattice in $e(W')\eta_\Gamma^* N$). □

Now we are ready to define a module $\widetilde{N}$ over $\mathcal{O}_{\mathfrak{h}^{\mathrm{sr}}/W} \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c(W)$. Abusing notation, we will write $\widetilde{N}_\Gamma$ for the corresponding (under the equivalence) $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}-W_\Gamma}/W_\Gamma] \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c(W)$-module. The restriction of $\widetilde{N}_\Gamma$ to $\eta_\Gamma^{-1}(\mathfrak{h}^{\mathrm{reg}}/W)$ coincides with $\eta_\Gamma^* N$ by construction. Let $\iota_\Gamma : \mathfrak{h}^{\mathrm{reg}-W_\Gamma} \hookrightarrow \mathfrak{h}$ be the inclusion and $\pi_\Gamma : \mathfrak{h}^{\mathrm{reg}-W_\Gamma} \twoheadrightarrow \mathfrak{h}^{\mathrm{reg}-W_\Gamma}/W_\Gamma$ be the quotient morphism. Also let $\pi : \mathfrak{h} \to \mathfrak{h}/W$ denote the quotient morphism and $\iota : \mathfrak{h}^{\mathrm{reg}} \hookrightarrow \mathfrak{h}^{\mathrm{sr}}$ the inclusion. Note that, by the construction, $\iota_{\Gamma *}\pi_\Gamma^* \widetilde{N}_\Gamma \subset \iota_* \pi^* N$ (recall that we view N as a coherent sheaf on $\mathfrak{h}^{\mathrm{reg}}/W$). The intersection $\widehat{N} := \bigcap_\Gamma \iota_{\Gamma *}\pi_\Gamma^* \widetilde{N}_\Gamma$ is a coherent sheaf on $\mathfrak{h}^{\mathrm{sr}}$ because of Lemma 3.2 and the equality $\mathfrak{h}^{\mathrm{sr}} = \bigcup_\Gamma \mathfrak{h}^{\mathrm{reg}-W_\Gamma}$. The intersection is stable under the Euler vector field because all the $\widetilde{N}_\Gamma$ are. Also $\widehat{N}$ is W-stable; this is because $w\pi_\Gamma^* \widetilde{N}_\Gamma = \pi_{w\Gamma}^* \widetilde{N}_{w\Gamma}$. Now set $\widetilde{N} := \pi_*(\widehat{N})^W = \pi_* \widehat{N} \cap \iota'_* N$ (where $\iota' : \mathfrak{h}^{\mathrm{sr}}/W \hookrightarrow \mathfrak{h}/W$ denotes the inclusion). This is a coherent sheaf on $\mathfrak{h}^{\mathrm{sr}}/W$, stable under the Euler vector field on $\iota'_* N$. It remains to show that $\widetilde{N} \subset \iota'_* N$ is stable under $H_c(W)$. But this follows from the equality

$$\widetilde{N} = \iota'_* N \cap \bigcap_\Gamma \eta_{\Gamma *} \widetilde{N}_\Gamma, \tag{3-2}$$

where now we view η_Γ as a morphism $\mathfrak{h}^{\mathrm{reg}-W'}/W' \to \mathfrak{h}^{\mathrm{sr}}/W$. Equation (3-2) follows from the observation that $\widetilde{N}_\Gamma = \pi_{\Gamma *}(\pi_\Gamma^* \widetilde{N}_\Gamma)^{W_\Gamma}$ and $\pi = \eta_\Gamma \circ \pi_\Gamma$. Since all sheaves in the right-hand side of (3-2) are stable under $H_c(W)$, we see that $\widetilde{N}$ is stable as well. It follows from the construction that $\widetilde{N}|_{\mathfrak{h}^{\mathrm{reg}}/W} \cong N$.

3D. *Global sections.*

Lemma 3.3. *The sheaf $\widetilde{N}$ is a vector bundle on $\mathfrak{h}^{\mathrm{sr}}/W$.*

Proof. The proof is inspired by [Etingof et al. 2013, Section 3.2]. We need to show that $\widetilde{N}$ is maximal Cohen–Macaulay when viewed as a coherent sheaf on $\mathfrak{h}^{\mathrm{sr}}/W$. Let Z denote the non-CM locus of $\widetilde{N}$ in $\mathfrak{h}^{\mathrm{sr}}/W$ and let d be the codimension of Z in $\mathfrak{h}^{\mathrm{sr}}/W$. Pick an open $\mathbb{C}^\times$-stable affine subvariety U of $\mathfrak{h}^{\mathrm{sr}}/W$ that intersects Z (or, more precisely, an irreducible component of maximal dimension in Z). Consider $H^i_{U\cap Z}(U,\widetilde{N})$ for $i<d$. As in [Etingof et al. 2013, Section 3.2], all these groups are $\mathbb{C}[U]\otimes_{\mathbb{C}[\mathfrak{h}/W]}H_c(W)$-modules, finitely generated over $\mathbb{C}[U]$ (this follows from [SGA 2 1968, Exposé VIII, Corollary 2.3] using equivalence of (ii) and (iii) there; note that the depth used in (ii) coincides with the codimension thanks to the choice of Z). Moreover, by the choice of Z, one of these modules is nonzero, as in [Etingof et al. 2013, Section 3.2]. The support of $R:=H^i_{U\cap Z}(U,\widetilde{N})$ is contained in $Z\cap U$.

Pick $b\in\mathfrak{h}$ lying over the support of R. Recall the isomorphism $\theta_b: H_c(W)^{\wedge_b}\cong Z(W,W',H_c(W',\mathfrak{h})^{\wedge_b})$, where we take W' to be W_b. So we get a nonzero $H_c(W',\mathfrak{h})^{\wedge_b}$-module $e(W')\theta_{b*}(R^{\wedge_b})$. This module is finitely generated over $\mathbb{C}[\mathfrak{h}]^{\wedge_b}$. So it is of the form $\mathbb{C}[\mathfrak{h}_{W'}]^{\wedge_b}\otimes R_0^{\wedge_0}$ for $R_0\in\mathcal{O}_c(W')$. It follows that $d=1$ and that $R=\Gamma_{Z\cap U}(U,\widetilde{N})$. But, by construction, $\Gamma(U,\widetilde{N})$ is embedded into $\Gamma(U\cap\mathfrak{h}^{\mathrm{reg}}/W,N)$ and so $\Gamma(U,\widetilde{N})$ has no torsion $\mathbb{C}[U]$-submodules. We get a contradiction, showing that $\widetilde{N}$ is Cohen–Macaulay. Since $\widetilde{N}$ is torsion-free, we see that it is maximal Cohen–Macaulay, and hence is a vector bundle. □

Now we can use [SGA 2 1968, Exposé VIII, Corollary 2.3(iv)] (applied to an extension of $\widetilde{N}$ to a coherent sheaf on $\mathfrak{h}/W$) to see that $M:=\Gamma(\mathfrak{h}^{\mathrm{sr}}/W,\widetilde{N})$ is finitely generated over $\mathbb{C}[\mathfrak{h}]^W$. Let us show that the $H_c(W)$-module M lies in $\mathcal{O}_c(W)$. By construction, M carries a locally finite derivation compatible with the derivation $\operatorname{ad}h$ of $H_c(W)$. It follows that M is gradable and hence lies in $\mathcal{O}$. Also, by construction, $M[\delta^{-1}]=N$. This completes the proof.

Acknowledgements

This project was sparked by a discussion with Raphael Rouquier in April 2014. I would like to thank him and Roman Bezrukavnikov, Pavel Etingof and Dennis Gaitsgory for stimulating discussions. Furthermore, I am grateful to Pavel Etingof for his comments on a preliminary version of this paper and for pointing out the reference [Wilcox 2011], and Daniel Thompson for his careful reading of the paper. My work was supported by the NSF under grant DMS-1161584.

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Communicated by J. Toby Stafford
Received 2014-08-13 Accepted 2015-02-18

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

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