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It is possible to talk about the étale homotopy equivalence of rational points on algebraic varieties by using a relative version of the étale homotopy type. We show that over $p$-adic fields rational points are homotopy equivalent in this sense if and only if they are étale-Brauer equivalent. We also show that over the real field rational points on projective varieties are étale homotopy equivalent if and only if they are in the same connected component. We also study this equivalence relation over number fields and prove that in this case it is finer than the other two equivalence relations for certain generalised Châtelet surfaces.

## 1. Introduction

For every field $K$, let $\overline{K}$ denote its separable closure. For every variety $X$ defined over a $K$ as above and every field extension $L|K$, let $X_L$ denote the base change

of $X$ to $L$. Let $K$ be a field, and let $X$ be a variety defined over $K$. Harpaz and Schlank [2013] defined a relative version $\mathrm{Et}_{/K}(X)$ of the étale homotopy type of $X$ by looking at the action of the absolute Galois group $\mathrm{Gal}(\overline{K}|K)$ of $K$ on the étale hypercoverings of $X_{\overline{K}}$. With the aid of the action of $\mathrm{Gal}(\overline{K}|K)$ on $\mathrm{Et}_{/K}(X)$, they define a pro-object $X(hK)$ in the category of sets, which they call the homotopy fixed point set of $X$, that serves as a certain homotopical approximation of the set $X(K)$ of rational points. By slight abuse of notation we will use the same symbol to denote the projective limit of $X(hK)$, which we will consider as a topological space equipped with its pro-discrete topology. It is possible to define a natural map

$$\iota_{X/K} : X(K) \to X(hK),$$

which can be thought of as a homotopy-theoretic version of the section map in Grothendieck's anabelian geometry, which it also happens to refine. We say that $x, y \in X(K)$ are $H$-equivalent if $\iota_{X/K}(x) = \iota_{X/K}(y)$.

   The aim of this paper is to describe the $H$-equivalence relation on $X(K)$ as explicitly as possible for many $K$ and $X$. Let us first turn to the case when $K$ is a finite extension of $\mathbb{Q}_p$. In this case, the map $\iota_{X/K}$ is not surjective in general; every abelian variety of positive dimension is a counterexample (see Proposition 9.7). However it is possible to describe the equivalence relation it induces on $X(K)$ in rather concrete terms. For any smooth variety $X$ over any field $K$ of characteristic zero, let $\mathrm{Br}(X) = H^2(X, \mathbb{G}_m)$ denote the cohomological Brauer group of $X$. We say that $x, y \in X(K)$ are Brauer equivalent if $x^*(b) = y^*(b)$ for all $b \in \mathrm{Br}(X)$. We say that $x, y \in X(K)$ are étale-Brauer equivalent if, for every finite, étale morphism $Y \to X$ of varieties over $K$ and each $\tilde{x} \in Y(K)$ mapping to $x$, there is a $\tilde{y} \in Y(K)$ which maps to $y$ and which is Brauer equivalent to $\tilde{x}$. Then we have the following:

**Theorem 1.1.** *Let $K$ be a finite extension of $\mathbb{Q}_p$, and let $X$ be a smooth quasi-projective variety over $K$. Then étale-Brauer equivalence and $H$-equivalence coincide on $X(K)$.*

   It is important to note that this claim is not true for more general fields; Châtelet surfaces over number fields provide counterexamples (see the remark following Theorem 14.8 below). The main ingredients of the theorem above, besides obstruction theory, are duality for the Galois cohomology of $K$ and Gabber's theorem on the existence of Azumaya algebras. We also provide examples to show that the theorem cannot be strengthened by substituting Brauer equivalence for étale-Brauer equivalence; see Theorem 9.5 below. We can also characterise $H$-equivalence for the field of real numbers:

**Theorem 1.2.** *Let $K$ be the field $\mathbb{R}$ of real numbers, and let $X$ be a smooth affine or projective variety over $K$. Then two $K$-rational points of $X$ are $H$-equivalent if and only if they are in the same connected component of the topological space $X(K)$.*

The main tools of the proof of this result are a celebrated theorem of Mahé (see [Houdebine and Mahé 1982; Mahé 1982]), the theory of Stiefel–Whitney classes for quadratic bundles (see [Esnault et al. 1993; Milnor 1970]), and an equivariant version of a basic comparison result of Artin and Mazur [1969]. The reader should note that Quick [2011] developed a general theory of homotopy fixed point spaces for simplicial pro-sets equipped with a continuous action of a profinite group, which can be applied to Friedlander's étale topological type functor [1982]. His construction offers an alternative route for the foundations of our investigations.

***Contents.*** In the next section, we review the relative étale homotopy type and homotopy fixed points of varieties as defined by Harpaz and Schlank and their relation to the Artin–Mazur construction. In the third section, we introduce a pointed version of the relative étale homotopy type and compare it with the previously defined constructions. In the fourth section, we study the relationship between the étale homotopy groups of finite étale coverings. In the following section, we show that the étale homotopy types of abelian varieties and smooth curves which are not projective of genus zero are Eilenberg–MacLane spaces over algebraically closed fields of characteristic zero. The fact presented in these two sections might be well-known to the experts, but we could not find a convenient reference. In the sixth section, we prove two useful lemmas about lifting a pair of rational points on certain principal bundles. Then we prove the fundamental theorem of obstruction theory for $H$-equivalence in the seventh section. We study the analogue of the Manin pairing for homotopy fixed points in the eighth section. In the ninth section, we prove that étale-Brauer equivalence is strictly finer than Brauer equivalence on $X(K)$ when $K$ is a $p$-adic field and $X$ is a bielliptic surface over $K$, using a rather standard set of tools. Theorem 1.1 is proved in the tenth section, while in the eleventh we prove Theorem 1.2. In the twelfth section, we introduce a natural homotopy version of Grothendieck's section and the Shafarevich–Tate conjectures over number fields by substituting the arithmetic fundamental group with the relative version of the étale homotopy type, which we call the homotopy section principle (HSP), and prove that it is equivalent to its well-established analogues in the special case of curves and abelian varieties. We provide further examples of varieties which satisfy HSP (see Theorems 13.3, 13.7 and 14.8) in the final two sections, including generalised Châtelet surfaces.

## 2. Basic definitions

**Definition 2.1.** Let $\Gamma$ be a profinite group. By a $\Gamma$-set we mean a set with a $\Gamma$-action such that each element has an open stabiliser. Let $\Gamma$-Sets denote the category whose objects are $\Gamma$-sets and whose morphisms are $\Gamma$-equivariant maps between them. By a simplicial $\Gamma$-set we mean a simplicial object in $\Gamma$-Sets. These form a category

$\Gamma$-SSets in the usual way. Note that, for every simplicial $\Gamma$-set $S$ and every $n \in \mathbb{N}$, the $n$-skeleton $\mathrm{sk}_n(S)$, the $n$-coskeleton $\mathrm{cosk}_n(S)$ and the Kan replacement $\mathrm{Ex}^\infty(S)$ are all naturally equipped with a $\Gamma$-action. Since with respect to this action the stabiliser of each simplex of $\mathrm{sk}_n(S)$, $\mathrm{cosk}_n(S)$ and $\mathrm{Ex}^\infty(S)$ is open, these constructions furnish three functors: the $n$-skeleton $\mathrm{sk}_n : \Gamma\text{-SSets} \to \Gamma\text{-SSets}$, the $n$-coskeleton $\mathrm{cosk}_n : \Gamma\text{-SSets} \to \Gamma\text{-SSets}$, and the Kan replacement $\mathrm{Ex}^\infty : \Gamma\text{-SSets} \to \Gamma\text{-SSets}$ functors. Moreover, let $P_n : \Gamma\text{-SSets} \to \Gamma\text{-SSets}$ denote the corresponding analogue of the simplicial version of the $n$-th Postnikov piece given by the rule

$$P_n(S) = \mathrm{cosk}_{n+1}(\mathrm{sk}_{n+1}(\mathrm{Ex}^\infty S))$$

for every simplicial $\Gamma$-set $S$.

**Notation 2.2.** For every category $\mathscr{C}$ let $\text{Pro-}\mathscr{C}$ be the category of pro-objects of $\mathscr{C}$. For every pair of categories $\mathscr{C}, \mathscr{D}$ let $\mathscr{C} \times \mathscr{D}$ denote their direct product, and for every category $\mathscr{C}$ let $\mathscr{C}^{\mathrm{op}}$ denote its opposite category. Clearly there is a natural equivalence between $(\mathscr{C} \times \mathscr{D})^{\mathrm{op}}$ and $\mathscr{C}^{\mathrm{op}} \times \mathscr{D}^{\mathrm{op}}$; for the sake of simplicity we will not distinguish between these categories. We will consider every directed set, and in particular every ordered set, to be a category in the usual way.

**Definition 2.3.** Goerss [1995] constructs a model category structure on $\Gamma\text{-SSets}$, called the strict model structure. The corresponding homotopy category will be denoted by $\mathrm{Ho}(\Gamma\text{-SSets})$ and will be called the homotopy category of simplicial $\Gamma$-sets. Similarly to the construction in Chapter 4 of [Artin and Mazur 1969], we may define a Postnikov tower functor

$$(\cdot)^{\natural} : \text{Pro-}\Gamma\text{-SSets} \to \text{Pro-}\Gamma\text{-SSets}$$

as follows: if $I$ is a small filtering index category and $\Pi : I^{\mathrm{op}} \to \Gamma\text{-SSets}$ is a pro-object of $\Gamma\text{-SSets}$, then the functor $\Pi^{\natural} : I^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}} \to \Gamma\text{-SSets}$ is given by

$$\Pi^{\natural}(\alpha, n) = P_n(\Pi(\alpha)) \quad \text{for all } \alpha \in \mathrm{ob}(I) \text{ and } n \in \mathbb{N}.$$

We will denote by the same symbol the variant of the Postnikov tower functor in the category $\text{Pro-Ho}(\Gamma\text{-SSets})$, by the usual abuse of notation.

**Definition 2.4.** Next we recall the definition of the relative étale homotopy type, following [Harpaz and Schlank 2013]. Let $K$ be a field, let $\Gamma_K = \mathrm{Gal}(\bar{K}|K)$ denote the absolute Galois group of $K$ and let $\mathrm{Sch}_{/K}$ denote the category of locally Noetherian schemes over $\mathrm{Spec}(K)$. Let

$$\pi_{0/K} : \mathrm{Sch}_{/K} \to \Gamma_K\text{-Sets}$$

denote the functor which takes the $K$-scheme $X$ to the $\Gamma_K$-set of connected components of $X_{\bar{K}}$. By applying this functor level-wise and composing it with the localisation functor $\Gamma_K\text{-SSets} \to \mathrm{Ho}(\Gamma_K\text{-SSets})$, we get a functor from the category

of étale hypercoverings of the $K$-scheme $X$ to the homotopy category of simplicial $\Gamma_K$-sets. This construction furnishes, similarly to what is done in Chapter 9 of [Artin and Mazur 1969], another functor

$$\mathrm{Et}_{/K} : \mathrm{Sch}_{/K} \to \mathrm{Pro\text{-}Ho}(\Gamma_K\text{-}\mathrm{SSets}),$$

which we will call the relative étale homotopy type of $X$ over $K$. Note that by functoriality we get a natural map

$$\iota_{X/K} : X(K) \to [\mathrm{Et}_{/K}(\mathrm{Spec}(K)), \mathrm{Et}_{/K}(X)] \to [\mathrm{Et}_{/K}(\mathrm{Spec}(K))^{\natural}, \mathrm{Et}_{/K}(X)^{\natural}],$$

where the second map is furnished by applying the Postnikov tower functor. We will call the pro-set $[\mathrm{Et}_{/K}(\mathrm{Spec}(K))^{\natural}, \mathrm{Et}_{/K}(X)^{\natural}]$ the homotopy fixed points of $X$ and we will denote it by the symbol $X(hK)$.

Our next aim is to describe the target of this map more explicitly (and to justify the terminology which we have just introduced).

**Definition 2.5.** Let Sets and SSets denote the category of sets and the category of simplicial sets, respectively. Let $\Gamma$ be as above. The category $\Gamma$-SSets is equipped with a natural concept of homotopy fixed points (see [Goerss 1995]); we will denote this functor $\Gamma$-SSets $\to$ SSets by $\Pi \mapsto \Pi^{h\Gamma}$. For every small filtering index category $I$ and pro-object $\Pi : I^{\mathrm{op}} \to \mathrm{Ho}(\Gamma\text{-}\mathrm{SSets})$, we define the $\Gamma$-homotopy fixed point set of $\Pi$, denoted by $\Pi(E\Gamma)$, to be

$$\Pi(E\Gamma) = \lim_{\alpha \in \mathrm{ob}(I)} \pi_0(\Pi(\alpha)^{h\Gamma}).$$

We will frequently consider the limit $\Pi(E\Gamma)$ as a topological space via its natural pro-discrete topology. This structure is enough to reconstruct the underlying pro-set. By a formula of Harpaz and Schlank there is a natural identification

$$[(E\Gamma)^{\natural}, \Pi^{\natural}] = \Pi(E\Gamma),$$

where $E\Gamma$ is an analogue of the total space of the universal $\Gamma$-bundle in this setting (see [Harpaz and Schlank 2013, Definition 2.3]). They also show that when $\Gamma = \Gamma_K$ is the absolute Galois group of a field $K$ then $\mathrm{Et}_{/K}(\mathrm{Spec}(K)) = E\Gamma_K$, hence we have $X(hK) = \mathrm{Et}_{/K}(X)(E\Gamma_K)$ for every $K$-scheme $X$, justifying our terminology.

**Notation 2.6.** Let $\mathrm{Ho}(\mathrm{SSets})$ denote the homotopy category of simplicial sets, let Sch denote the category of locally Noetherian schemes, and let

$$\mathrm{Et} : \mathrm{Sch} \to \mathrm{Pro\text{-}Ho}(\mathrm{SSets})$$

denote the Artin–Mazur étale homotopy type functor. For every $X$ as above and every $n \in \mathbb{N}$ let $\mathrm{Et}^n(X)$ denote the $n$-th Postnikov piece $P_n(\mathrm{Et}(X))$ and let $\mathrm{Et}(X)^{\natural}$ denote the Postnikov tower of $\mathrm{Et}(X)$. For every field $K$ and every scheme $X$ over $K$ let $\overline{X}$ denote the base change $X_{\overline{K}}$. Moreover, for every such $K$, every $X \in \mathrm{ob}(\mathrm{Sch}_{/K})$ and every $n \in \mathbb{N}$, let $\mathrm{Et}^n_{/K}(X)$ denote the $n$-th Postnikov piece by $P_n(\mathrm{Et}_{/K}(X))$.

**Lemma 2.7.** *Let $K$ be a field and $X$ be a variety over $K$. Then there are natural isomorphisms*

$$f^n(X) : \mathrm{Et}^n(\overline{X}) \to \mathrm{Et}^n_{/K}(X) \quad and \quad f^{\natural}(X) : \mathrm{Et}(\overline{X})^{\natural} \to \mathrm{Et}_{/K}(X)^{\natural}$$

*in the category* Pro-Ho(SSets).

*Proof.* The first half of the claim is Proposition 2.14 of [Harpaz and Schlank 2013]. The second half is an immediate consequence of the first half and the compatibility of the maps $f^n(X)$. □

**Notation 2.8.** For every $X \in \mathrm{ob}(\mathrm{Ho}(\mathrm{SSets}))$ let $X^{\wedge} \in \mathrm{ob}(\mathrm{Pro}\text{-}\mathrm{Ho}(\mathrm{SSets}))$ denote its profinite completion. The basic result about the homotopy type of complex algebraic varieties is the following classical theorem of Artin and Mazur:

**Theorem 2.9.** *Let $X$ be a geometrically unibranch algebraic variety defined over $\mathbb{C}$. Then there is a canonical weak homotopy equivalence*

$$\eta_X : \mathrm{Et}(X)^{\natural} \to (X(\mathbb{C})^{\wedge})^{\natural}$$

*in* Pro-Ho(SSets).

*Proof.* This is [Artin and Mazur 1969, Corollary 12.10, p. 143]. □

**Proposition 2.10.** *Assume that $K$ is algebraically closed. Then $\iota_{X/K}$ is surjective. Two points $x, y \in X(K)$ are $H$-equivalent if and only if they lie in the same connected component of $X$ with respect to the Zariski topology.*

*Proof.* When $K$ is algebraically closed its absolute Galois group is trivial. Therefore $\mathrm{Ho}(\Gamma_K\text{-}\mathrm{SSets})$ is the homotopy category of simplicial sets and $\mathrm{Et}_{/K}(X)$ is just the usual Artin–Mazur étale homotopy type of $X$. Because $K$ is algebraically closed, $\mathrm{Et}_{/K}(\mathrm{Spec}(K))$ is contractible. Therefore there is a natural bijection

$$X(hK) \cong [*, \mathrm{Et}_{/K}(X)^{\natural}] \cong \pi_0(\mathrm{Et}_{/K}(X)^{\natural}) \cong \pi_0(X),$$

where the third identification is the consequence of a fundamental comparison theorem of Artin and Mazur [1969, Corollary 10.8, p. 122]. The claim now follows from the naturality of $\iota_{X/K}$. □

## 3. The pointed relative étale homotopy type

**Definition 3.1.** Let $\Gamma$ be as above. By a pointed simplicial $\Gamma$-set we mean a simplicial $\Gamma$-set $S_*$ with a point $p \in S_0$ fixed by $\Gamma$. These form a category, $\Gamma\text{-SSets}_*$, in the usual way. Note that, for every pointed simplicial $\Gamma$-set $S$ and every $n \in \mathbb{N}$, the $n$-skeleton $\text{sk}_n(S)$, the $n$-coskeleton $\text{cosk}_n(S)$, the Kan replacement $\text{Ex}^\infty(S)$ and $n$-th Postnikov piece $P_n(S)$ are all naturally equipped with a point fixed by $\Gamma$, and we will denote the corresponding four functors, the $n$-skeleton $\text{sk}_n : \Gamma\text{-SSets}_* \to \Gamma\text{-SSets}_*$, the $n$-coskeleton $\text{cosk}_n : \Gamma\text{-SSets}_* \to \Gamma\text{-SSets}_*$, the Kan replacement $\text{Ex}^\infty : \Gamma\text{-SSets}_* \to \Gamma\text{-SSets}_*$ and the $n$-th Postnikov piece $P_n : \Gamma\text{-SSets}_* \to \Gamma\text{-SSets}_*$ by the same symbols by a slight abuse of notation.

**Definition 3.2.** The homotopy category of $\Gamma\text{-SSets}_*$ with respect to the pointed version of weak equivalences of Goerss' strict model structure, called the homotopy category of pointed simplicial $\Gamma$-sets, will be denoted by $\text{Ho}(\Gamma\text{-SSets}_*)$. Similarly to the construction recalled in Definition 2.3, we may define a Postnikov tower functor

$$(\cdot)^\natural : \text{Pro-}\Gamma\text{-SSets}_* \to \text{Pro-}\Gamma\text{-SSets}_*,$$

and we will denote by the same symbol the corresponding Postnikov tower functor in the category $\text{Pro-Ho}(\Gamma\text{-SSets}_*)$ by the usual abuse of notation. This is of course justified as the formations of these invariants commute with the forgetful functor $\text{Pro-Ho}(\Gamma\text{-SSets}_*) \to \text{Pro-Ho}(\Gamma\text{-SSets})$.

**Definition 3.3.** Since $\Gamma\text{-SSets}$ is a model category, it has all colimits, in particular pushouts and equalisers. If $Y_*$ is a simplicial $\Gamma$-set and $X_* \subseteq Y_*$ is a subsimplicial $\Gamma$-set, then let $Y_*/X_*$ denote the simplicial $\Gamma$-set which is the pushout of the inclusion map $X_* \to Y_*$. We call $Y_*/X_*$ the contraction of $Y_*$ by $X_*$. If $X_*, Y_*$ are simplicial $\Gamma$-sets and $f_* : X_* \to Y_*$, $g_* : X_* \to Y_*$ are maps of simplicial $\Gamma$-sets, then let $Y_*(f_* = g_*)$ denote the simplicial $\Gamma$-set which is the coequaliser of $f_*$ and $g_*$.

**Definition 3.4.** Note that, for every pair of simplicial $\Gamma$-sets $X_*$ and $Y_*$, the product $X_* \times Y_*$ equipped with the natural (diagonal) $\Gamma$-action is also a simplicial $\Gamma$-set. Similarly the coproduct (disjoint union) $X_* \sqcup Y_*$ of simplicial $\Gamma$-sets $X_*$ and $Y_*$ is also a simplicial $\Gamma$-set equipped with the tautological $\Gamma$-action. Let $I$ denote the 1-simplex $\Delta^1 = [0, 1]$ with the trivial $\Gamma$-action; this choice makes it into a simplicial $\Gamma$-set. For every morphism $f : X_* \to Y_*$ of simplicial $\Gamma$-sets the mapping cylinder $\text{Cyl}(f)$ is the coequaliser of the two maps

$$f' : X \to Y_* \sqcup X_* \times I \quad \text{and} \quad p : X \to Y_* \sqcup X_* \times I,$$

where $f'$ is the composition of $f$ and the tautological inclusion $Y_* \subseteq Y_* \sqcup X_* \times I$ and $p$ is the composition of the map identifying $X_*$ with $X_* \times \{1\} \subseteq X_* \times I$ and the tautological inclusion $X_* \times I \subset Y_* \sqcup X_* \times I$. We define the mapping cone Cone($f$) of an $f : X_* \to Y_*$ as above as the contraction of Cyl($f$) by the image of the map

$$q : X_* \to \mathrm{Cyl}(f),$$

where $q$ is the composition of the map identifying $X_*$ with $X_* \times \{0\} \subseteq X_* \times I$ with the tautological inclusion $X_* \times I \subset Y_* \sqcup X_* \times I$ composed with the natural surjection $Y_* \sqcup X_* \times I \mapsto \mathrm{Cyl}(f)$. Note that Cone($f$) is canonically a pointed simplicial $\Gamma$-set, where the base point is the image of $q(X_0) \subset \mathrm{Cyl}(f)_0$ under the contraction map $\mathrm{Cyl}(f) \to \mathrm{Cone}(f)$.

**Definition 3.5.** By a pointed $K$-scheme $(X, x)$ we will mean a locally Noetherian scheme $X$ over $K$ with a $K$-valued point $x : \mathrm{Spec}(K) \to X$ on $X$. These form the objects of a category $\mathrm{Sch}_{/K*}$, where a morphism $f$ from an object $(X, x)$ to another object $(Y, y)$ is a map $f : X \to Y$ of schemes over $K$ such that $f(x) = y$. Now let $(X, x)$ be a pointed $K$-scheme and let $H_*$ be an étale hypercovering of $X$. Then the pullback $x^*(H_*)$ is an étale hypercovering of $\mathrm{Spec}(K)$, and the map $x$ induces a morphism $x_*(H_*) : \pi_{0/K}(x^*(H_*)) \to \pi_{0/K}(H_*)$ of simplicial $\Gamma$-sets. Let $\pi_{0/K}(H_*, x)$ denote the mapping cone of the composition of this map $x_*(H_*)$ and the canonical inclusion $\pi_{0/K}(H_*) \subseteq \mathrm{Ex}^\infty(\pi_{0/K}(H_*))$; it is a pointed simplicial $\Gamma$-set. A map $f : H_* \to J_*$ between étale hypercoverings of $X$ induces a map $\pi_{0/K}(f, x) : \pi_{0/K}(H_*, x) \to \pi_{0/K}(J_*, x)$ between pointed simplicial $\Gamma$-sets, and a homotopy between two maps $f : H_* \to J_*, g : H_* \to J_*$ induces a pointed $\Gamma$-equivariant homotopy between $\pi_{0/K}(f, x)$ and $\pi_{0/K}(g, x)$. Therefore we may apply Corollary 8.13(i) of [Artin and Mazur 1969, p. 105] to conclude that the functor

$$H_* \mapsto \pi_{0/K}(H_*, x)$$

above induces an object $\mathrm{Et}_{/K}(X, x)$ of Pro-Ho($\Gamma$-SSets$_*$). We will call the latter the pointed relative étale homotopy type of $(X, x)$.

**Notation 3.6.** Let $(X, x)$ be a pointed $K$-scheme. For every étale hypercovering $H_*$ of $X$ let

$$i(H_*, x) : \pi_{0/K}(H_*) \to \pi_{0/K}(H_*, x)$$

be the composition of the natural inclusion map

$$\pi_{0/K}(H_*) \to \mathrm{Cone}(x_*(H_*))$$

and the map $\mathrm{Cone}(x_*(H_*)) \to \pi_{0/K}(H_*, x)$ induced by the functoriality of mapping cones. This is a natural transformation between two functors from the category

of étale hypercoverings over $X$ into $\Gamma$-SSets, and hence it induces a map

$$i(X, x) : \mathrm{Et}_{/K}(X, x) \to \mathrm{Et}_{/K}(X)$$

of pro-objects of the homotopy category $\mathrm{Ho}(\Gamma\text{-SSets})$, where by slight abuse of notation we let $\mathrm{Et}_{/K}(X, x)$ also denote the image of the pointed relative étale homotopy type of $(X, x)$ with respect to the forgetful functor

$$\mathrm{Pro\text{-}Ho}(\Gamma\text{-SSets}_*) \to \mathrm{Pro\text{-}Ho}(\Gamma\text{-SSets}).$$

The map $i(X, x)$ is obviously natural.

**Proposition 3.7.** *The map* $i(X, x) : \mathrm{Et}_{/K}(X, x) \to \mathrm{Et}_{/K}(X)$ *induces a bijection on homotopy fixed points.*

*Proof.* Note that the functor $\pi_{0/K}$ induces an equivalence between the category of étale coverings over $\mathrm{Spec}(K)$ and $\Gamma$-Sets, so $\pi_{0/K}(x^*(H_*))$ is a contractible simplicial $\Gamma$-set. Therefore $i(H_*, x)$ is a weak equivalence for every étale hypercovering $H_*$ of $X$ with respect to Goerss' *weak* model structure (see Theorem A on p. 189 and Definition 1.11 on p. 194 in [Goerss 1995]), and hence induces a bijection $\pi_0(\pi_{0/K}(H_*)^{h\Gamma}) \to \pi_0(\mathrm{Cone}(x_*(H_*))^{h\Gamma})$. So the same holds for $i(X, x)$, too. $\square$

**Definition 3.8.** Let $\mathrm{Sch}_*$ be the category of pointed locally Noetherian schemes, and, as usual, denote the objects of $\mathrm{Sch}_*$ by pairs $(X, x)$, where $X$ is a locally Noetherian scheme and $x$ is a geometric point of $X$. By slight abuse of notation let

$$\mathrm{Et} : \mathrm{Sch}_* \to \mathrm{Pro\text{-}Ho}(\mathrm{SSets}_*)$$

denote the pointed version of the Artin–Mazur étale homotopy type functor. For every object $(X, x)$ of $\mathrm{Sch}_*$ and every $n \geq 1$, let $\pi_n(X, x)$ denote the $n$-th homotopy group of $\mathrm{Et}(X, x)$ when $X$ is connected.

**Notation 3.9.** For every pointed scheme $(X, x)$ and every $n \in \mathbb{N}$, let $\mathrm{Et}^n(X, x)$ denote the $n$-th Postnikov piece $P_n(\mathrm{Et}(X, x))$ and let $\mathrm{Et}(X, x)^\natural$ denote the Postnikov tower of $\mathrm{Et}(X, x)$. Similarly, for every field $K$, every pointed $K$-scheme $(X, x)$ and every $n \in \mathbb{N}$, let $\mathrm{Et}^n_{/K}(X, x)$ denote the $n$-th Postnikov piece $P_n(\mathrm{Et}_{/K}(X, x))$ and let $\mathrm{Et}_{/K}(X, x)^\natural$ denote the Postnikov tower of $\mathrm{Et}_{/K}(X, x)$. Since we fixed a separable closure $\bar{K}$ of $K$, we may associate to every $K$-valued point $x : \mathrm{Spec}(K) \to X$ of a $K$-scheme $X$ a $\bar{K}$-valued point $\bar{x} : \mathrm{Spec}(K) \to X$ which is the composition of the map $\mathrm{Spec}(\bar{K}) \to \mathrm{Spec}(K)$ induced by the inclusion $K \subseteq \bar{K}$ and $x$.

Similarly to above, by slight abuse of notation we let $\mathrm{Et}_{/K}(X, x)$ and $\mathrm{Et}^n_{/K}(X, x)$ also denote respectively the image of the pointed relative étale homotopy type of $(X, x)$ and of the $n$-th truncation of the latter with respect to the forgetful functor

$$\mathrm{Pro\text{-}Ho}(\Gamma\text{-SSets}_*) \to \mathrm{Pro\text{-}Ho}(\mathrm{SSets}_*).$$

**Proposition 3.10.** *Let $K$ be a field and $(X, x)$ a pointed $K$-scheme such that $X$ is a variety over $K$. Then there are natural isomorphisms*

$$f^n(X, x) : \mathrm{Et}^n(\overline{X}, \overline{x}) \to \mathrm{Et}^n_{/K}(X, x) \quad and \quad f^\natural(X, x) : \mathrm{Et}(\overline{X}, \overline{x})^\natural \to \mathrm{Et}_{/K}(X, x)^\natural$$

*in the category* Pro-Ho(SSets$_*$).

*Proof.* Since the second half is an immediate consequence of the first half and the compatibility of the maps $f^n(X, x)$, it will be enough to prove the former. Let $\Delta^0$ denote the 0-simplex, as usual. Let $(H_*, h)$ be a pointed étale hypercovering of $(X, \overline{x})$. By definition $h$ is a map $\Delta^0 \to \pi_{0/K}(x^*(H_*))$ of simplicial sets. For every $(H_*, h)$ as above let $\pi_{0/K}(H_*, h, x)$ be the contraction of $\pi_{0/K}(H_*, x)$ by the image of the map

$$c_h : \Delta^1 \cong \Delta^0 \times \Delta^1 \to \pi_{0/K}(H_*, x)$$

of simplicial sets, where $c_h$ is the composition $\pi \circ \iota \circ (h \times \mathrm{id}_I)$, where $\iota$ is the inclusion

$$\pi_{0/K}(x^*(H_*)) \times \Delta^1 \subset \mathrm{Ex}^\infty(\pi_{0/K}(H_*)) \sqcup \pi_{0/K}(x^*(H_*)) \times \Delta^1$$

and $\pi$ is the canonical surjection

$$\mathrm{Ex}^\infty(\pi_{0/K}(H_*)) \sqcup \pi_{0/K}(x^*(H_*)) \times \Delta^1 \to \pi_{0/K}(H_*, x).$$

Let

$$a(H_*, h) : \pi_{0/K}(H_*, x) \to \pi_{0/K}(H_*, h, x)$$

be the contraction map. We will consider $\pi_{0/K}(H_*, h, x)$ a pointed simplicial set, where its distinguished point $b(h) \in \pi_{0/K}(H_*, h, x)_0$ is the image of the base point of the pointed simplicial $\Gamma$-set under $a(H_*, h)$. Note that this map is a weak equivalence in Pro $-$ Ho(SSets$_*$), since we contracted a contractible subsimplicial set. Therefore by [Artin and Mazur 1969, Corollary 8.13(i), p. 105] the functor

$$(H_*, h) \mapsto (\pi_{0/K}(H_*, h, x), b(h))$$

induces an object $\mathrm{Et}'(\overline{X}, \overline{x})$ of Pro-Ho(SSets$_*$) which is isomorphic to $\mathrm{Et}_{/K}(X, x)$ in this category. For every $(H_*, h)$ as above let $\pi_{0/K}(h) \in \pi_{0/K}(H_0)$ be the point corresponding to $h$, and let

$$b(H_*, h) : \pi_{0/K}(H_*) \to \pi_{0/K}(H_*, x)$$

be the map of pointed simplicial sets which is the composition of the natural inclusion map

$$\pi_{0/K}(H_*) \subset \pi_{0/K}(H_*) \sqcup \pi_{0/K}(x^*(H_*)) \times \Delta^1$$

with the natural surjection

$$\pi_{0/K}(H_*) \sqcup \pi_{0/K}(x^*(H_*)) \times \Delta^1 \to \mathrm{Cone}(x_*)$$

composed with the map $\text{Cone}(x_*) \to \pi_{0/K}(H_*, x)$ induced by the functoriality of mapping cones. Let $c(H_*, h)$ be the composition of $b(H_*, h)$ with $a(H_*, h)$; then this map is a morphism

$$c(H_*, h) : (\pi_{0/K}(H_*), \pi_{0/K}(h)) \to (\pi_{0/K}(H_*, h, x), b(h))$$

of pointed simplicial sets. Since $\pi_{0/K}(x^*(H_*))$ is a contractible simplicial $\Gamma$-set, the map $c(H_*, h)$ is a weak equivalence for every pointed étale hypercovering $(H_*, h)$ of $X$. Therefore by [Artin and Mazur 1969, Corollary 8.13(i), p. 105] the functor

$$(H_*, h) \mapsto (\pi_{0/K}(H_*), \pi_{0/K}(h))$$

induces an object $\text{Et}''(\overline{X}, \overline{x})$ of Pro-Ho(SSets$_*$) which is isomorphic to $\text{Et}'(\overline{X}, \overline{x})$, and hence to $\text{Et}_{/K}(X, x)$ in this category. Therefore it will be sufficient to prove that there are natural isomorphisms

$$g^n(X, x) : \text{Et}_n(\overline{X}, \overline{x}) \to P_n(\text{Et}''(\overline{X}, \overline{x}))$$

in the category Pro-Ho(SSets$_*$) which are compatible with each other and with truncation.

Note that the inclusion of the indexing category of $\text{Et}''(\overline{X}, \overline{x})$ in the indexing category of $\text{Et}(\overline{X}, \overline{x})$ furnishes a natural map

$$g(X, x) : \text{Et}(\overline{X}, \overline{x}) \to \text{Et}''(\overline{X}, \overline{x}).$$

In order to prove that $g(X, x)$ is an isomorphism after taking $n$-th truncations, we can argue the same way as in the proof of [Harpaz and Schlank 2013, Proposition 2.14].     □

## 4. Homotopy groups of finite étale covers

Recall that for every object $X$ of Sch with a geometric point $x$ and every $n \in \mathbb{N}$ the symbol $\pi_n(X, x)$ denotes the homotopy group $\pi_n(\text{Et}(X), x)$. In this section assume that $K$ is an algebraically closed field.

**Proposition 4.1.** *Let $f : (X, x) \to (Y, y)$ be a finite étale map of pointed smooth connected quasiprojective varieties over $K$. Then the induced map*

$$\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y)$$

*is an isomorphism for every $n \geq 2$.*

*Proof.* Because $\pi_1(Y, y)$ is topologically finitely generated, its open normal subgroups are cofinal, and hence there is a finite étale map $g : (Z, z) \to (X, x)$ of pointed smooth connected quasiprojective varieties over $K$ such that the image of $\pi_1(f \circ g) : \pi_1(Z, z) \to \pi_1(Y, y)$ is an open normal subgroup. In this case the

image of $\pi_1(f) : \pi_1(Z, z) \to \pi_1(X, x)$ is an open normal subgroup, too. It will be enough to show that the maps $\pi_n(f \circ g)$ and $\pi_n(f)$ are isomorphisms for every $n \geq 2$. Since the composition $f \circ g$ is also a finite étale map, we've reduced the claim to the special case when the image of $\pi_1(f)$ is an open normal subgroup.

In this case $f$ is a finite Galois covering; let $G$ denote the covering group and let $\alpha : G \to \mathrm{Aut}(X)$ be the action corresponding to the deck transformations. Let $(H_*, h)$ be a pointed hypercovering of $Y$ with respect to the étale site of $Y$ pointed with $y$. Let $C_*$ denote the étale Čech hypercovering

$$X \underset{\longleftarrow}{\overset{\longrightarrow}{\Longleftarrow}} X \times_Y X \overset{\Longrightarrow}{\Lleftarrow} X \times_Y X \times_Y X \overset{\Lleftarrow}{\Lleftarrow} \quad \cdots$$

generated by the cover $X \to Y$, and equip it with a point $c$ with respect to the same pointed site. Let $(I_*, i)$ be the fibre product of $(H_*, h)$ and $(C_*, c)$ over $Y$; this pointed simplicial object is also a hypercovering. Let $(J_*, j)$ be the pullback $f^*(I_*, i)$ of the pointed hypercovering $(I_*, i)$ onto the étale site of $X$ pointed with $x$ with respect to $f$ and let $f_* : (\pi_0(J_*), \pi_0(j)) \to (\pi_0(I_*), \pi_0(i))$ be the map induced by $f$ between the pointed simplicial sets of connected components of $J_*$ and $I_*$.

The action $\alpha$ induces an action of $G$ on $J_*$, and hence an action of $G$ on $\pi_0(J_*)$. If we equip $\pi_0(I_*)$ with the trivial action then $f_*$ is $G$-equivariant. Because $f$ is a finite étale cover the map $f_*$ is surjective. Moreover, for every $n$ the étale cover $I_n \to Y$ factors through $f : X \to Y$, and hence the action of $G$ on the connected components of the base change $J_n \to X$ of this map to $X$ is free. We get that $f_* : (\pi_0(J_*), \pi_0(j)) \to (\pi_0(I_*), \pi_0(i))$ is a $G$-cover and hence the induced maps

$$\pi_n(f_*) : \pi_n(\pi_0(J_*), \pi_0(j)) \to \pi_n(\pi_0(I_*), \pi_0(i))$$

are isomorphisms for all $n \geq 2$.

Let $(L_*, l)$ be a pointed hypercovering of $X$ with respect to the étale site of $X$ pointed with $x$. By composing the structure maps with $f$ we get a pointed hypercovering of $Y$ with respect to the étale site of $Y$ pointed with $y$, which we will denote by $(H_*, h)$ by slight abuse of notation. By applying the same construction to $(H_*, h)$ as above we get a pointed hypercovering $(J_*, j)$ of the étale site of $X$ pointed with $x$ which dominates $(L_*, l)$. Therefore pointed hypercovers of $X$ of the form as $(J_*, j)$ above are cofinal, so the injectivity of the maps $\pi_n(f)$ for all $n \geq 2$ follows.

Let $\gamma$ be an element of $\pi_n(Y, y)$ (where $n \geq 2$). For every pointed hypercovering $(L_*, l)$ of $X$ with respect to the étale site of $X$ pointed with $x$ we will construct an element $\gamma'_{(L_*, l)} \in \pi_n(\pi_0(L_*), \pi_0(l))$ as follows. Let $(H_*, h)$, $(I_*, i)$ and $(J_*, j)$ be the same as in the paragraph above. Let $\gamma_{(I_*, i)} \in \pi_n(\pi_0(I_*), \pi_0(i))$ be the image of $\gamma$ under the tautological map $\pi_n(Y, y) \to \pi_n(\pi_0(I_*), \pi_0(i))$, let $\pi_n(f_*)^{-1}(\gamma_{(I_*, i)})$

be the unique preimage of $\gamma_{(I_*,i)}$ under the isomorphism

$$\pi_n(f_*) : \pi_n(\pi_0(J_*), \pi_0(j)) \to \pi_n(\pi_0(I_*), \pi_0(i)),$$

and let $\gamma'_{(L_*,l)}$ be the image of $\pi_n(f_*)^{-1}(\gamma_{(I_*,i)})$ under the natural map

$$\pi_n(\pi_0(J_*), \pi_0(j)) \to \pi_n(\pi_0(L_*), \pi_0(l)).$$

It is easy to check that the elements $\gamma'_{(L_*,l)}$ glue together to an element $\gamma'$ of $\pi_n(X, x)$ whose image under $\pi_n(f)$ is $\gamma$. The surjectivity of the maps $\pi_n(f)$ for all $n \geq 2$ follows. □

**Notation 4.2.** For every group $\Gamma$ let $\widehat{\Gamma}$ be its profinite completion. For every object $X$ of Sch, for every $n \in \mathbb{N}$ and for every pro-abelian group $A$ let $H_n(X, A)$ denote the homology group $H_n(\text{Et}(X), A)$. For every object $(X, x)$ of Sch$_*$ and $n$ as above let

$$h_n(X, x) : \pi_n(X, x) \to H_n(X, \widehat{\mathbb{Z}})$$

denote the Hurewicz map. Let $X$ be a smooth, geometrically irreducible, quasiprojective variety over $K$. Let $x$ be a $K$-valued point of $X$ and let Fet$(X, x)$ denote the category of finite étale pointed connected covers $(Y, y)$ of $(X, x)$ such that the image of the induced map $\pi_1(f) : \pi_1(Y, y) \to \pi_1(Y, y)$ is an open characteristic subgroup for every object $f : (Y, y) \to (X, x)$. Since for every $f : (Y, y) \to (X, x)$ as above the induced map $\pi_2(f) : \pi_2(Y, y) \to \pi_2(X, x)$ is an isomorphism by Proposition 4.1, the projective limit of the inverses of these maps is an isomorphism

$$a_X : \pi_2(X, x) \to \lim_{(Y,y)\in\text{Fet}(X,x)} \pi_2(Y, y).$$

Moreover, we may take the projective limit of the Hurewicz maps

$$b_X : \lim_{(Y,y)\in\text{Fet}(X,x)} \pi_2(Y, y) \to \lim_{(Y,y)\in\text{Fet}(X,x)} H_2(Y, \widehat{\mathbb{Z}}).$$

**Theorem 4.3.** *The map*

$$b_X \circ a_X : \pi_2(X, x) \to \lim_{(Y,y)\in\text{Fet}(X,x)} H_2(Y, \widehat{\mathbb{Z}})$$

*is an isomorphism.*

*Proof.* First we are going to prove that $b_X$ is injective. Let $\gamma$ be a nonzero element of $\pi_2(X, x)$. Then there is a pointed hypercovering $(H_*, h)$ of $X$ with respect to the étale site of $X$ pointed with $x$ such that the image of $\gamma$ under the natural map $\pi_2(X, x) \to \pi_2(\pi_0(H_*), \pi_0(h))$ is nonzero. Let $N$ be the kernel of the natural map $\pi_1(X, x) \to \pi_1(\pi_0(H_*), \pi_0(h))$; it is an open normal subgroup. Because $\pi_1(X, x)$ is topologically finitely generated, its open characteristic subgroups are cofinal, and hence there is an object $f : (Y, y) \to (X, x)$ of Fet$(X, x)$ such that the image of $\pi_1(f) : \pi_1(Y, y) \to \pi_1(X, x)$ lies in $N$.

Note that $f$ is a finite Galois covering; let $G$ denote the covering group and let $\alpha : G \to \mathrm{Aut}(X)$ be the action corresponding to the deck transformations. Let $C_*$ denote the étale Čech hypercovering generated by the cover $Y \to X$, and equip it with a point $c$ with respect to the same pointed site. Let $(I_*, i)$ be the pointed simplicial object which is the fibre product of $(H_*, h)$ and $(C_*, c)$ and let $(J_*, j)$ be the pullback $f^*(I_*, i)$ of the pointed hypercovering $(I_*, i)$ onto the étale site of $X$ pointed with $x$ with respect to $f$. Let $f_* : (\pi_0(J_*), \pi_0(j)) \to (\pi_0(I_*), \pi_0(i))$ be the map induced by $f$ between the pointed simplicial sets of connected components of $J_*$ and $I_*$.

We may argue as above to conclude that $f_* : (\pi_0(J_*), \pi_0(j)) \to (\pi_0(I_*), \pi_0(i))$ is a $G$-cover with respect to the action induced by $\alpha$ on $\pi_0(J_*)$ and the trivial action on $\pi_0(I_*)$. Therefore the induced map

$$\pi_1(f_*) : \pi_1(\pi_0(J_*), \pi_0(j)) \to \pi_1(\pi_0(I_*), \pi_0(i))$$

is injective. There is a commutative diagram

$$
\begin{array}{ccc}
\pi_1(Y, y) & \longrightarrow & \pi_1(\pi_0(J_*), \pi_0(j)) \\
{\scriptstyle \pi_1(f)} \downarrow & & {\scriptstyle \pi_1(f_*)} \downarrow \\
\pi_1(X, x) & \longrightarrow & \pi_1(\pi_0(I_*), \pi_0(i)).
\end{array}
$$

By assumption the composition of $\pi_1(f)$ and the lower horizontal map has trivial image. Since the upper horizontal map is surjective by [Artin and Mazur 1969, Corollary 10.6, pp. 121–122], we get that $\pi_1(f_*)$ has trivial image, too, and hence $\pi_1(\pi_0(J_*), \pi_0(j))$ is the trivial group. There is a similar commutative diagram

$$
\begin{array}{ccc}
\pi_2(Y, y) & \longrightarrow & \pi_2(\pi_0(J_*), \pi_0(j)) \\
{\scriptstyle \pi_2(f)} \downarrow & & {\scriptstyle \pi_2(f_*)} \downarrow \\
\pi_2(X, x) & \longrightarrow & \pi_2(\pi_0(I_*), \pi_0(i))
\end{array}
$$

for $\pi_2$. The image of $\gamma$, considered as an element of $\pi_2(Y, y)$, is nonzero under the composition of $\pi_2(f)$ and the lower horizontal map by assumption. Therefore its image $\gamma'$ under the upper horizontal map is also nonzero. Since $\pi_1(\pi_0(J_*), \pi_0(j))$ is trivial, we get that the image of $\gamma'$ under the Hurewicz map $\pi_2(\pi_0(J_*), \pi_0(j)) \to H_2(\pi_0(J_*), \widehat{\mathbb{Z}})$ is nonzero. By naturality this implies that the image of $\gamma$ under the Hurewicz map $h_2(Y, y) : \pi_2(Y, y) \to H_2(Y, \widehat{\mathbb{Z}})$ is nonzero, too.

Next we are going to prove that $b_X$ is surjective. For every $(Z, z) \in \mathrm{Fet}(X, x)$ let $\mathscr{C}(Z, z)$ be the category of morphisms $(Y, y) \to (Z, z)$ in $\mathrm{Fet}(X, x)$, and let

$$c_{(Z,z)} : \varprojlim_{(Y,y) \to (Z,z) \in \mathscr{C}(Z,z)} H_2(Y, \widehat{\mathbb{Z}}) \to H_2(Z, \widehat{\mathbb{Z}})$$

be the tautological map. Because the preimage $(b_X \circ a_X)^{-1}(\mathrm{Im}(c_{(Z,z)})) \subseteq \pi_2(X, x)$ is closed and $\pi_2(X, x)$ is profinite, by compactness it will be enough to show that for every object $(Z, z)$ as above $(b_X \circ a_X)^{-1}(\mathrm{Im}(c_{(Z,z)}))$ is nonempty. Now fix an element $\gamma \in \mathrm{Im}(c_{(Z,z)})$ and choose a

$$\gamma' \in \lim_{(Y,y) \to (Z,z) \in \mathscr{C}(Z,z)} H_2(Y, \hat{\mathbb{Z}})$$

such that $c_{(Z,z)}(\gamma') = \gamma$. For every pointed hypercovering $(H_*, h)$ of $Z$ with respect to the étale site of $Z$ pointed with $z$, we are going to construct an element $\gamma_{(H_*,h)} \in \pi_2(\pi_0(H_*), \pi_0(h))$ as follows.

Let $N$ be the kernel of the natural map $\pi_1(Z, z) \to \pi_1(\pi_0(H_*), \pi_0(h))$. Using that $\pi_1(Z, z)$ is topologically finitely generated as above, we get that there is a morphism $f : (Y, y) \to (Z, z)$ of Fet$(X, x)$ such that the image of $\pi_1(f) : \pi_1(Y, y) \to \pi_1(Z, z)$ lies in $N$. Let $C_*$ denote the étale Čech hypercovering generated by the cover $Y \to Z$, and equip it with a point $c$ with respect to the same pointed site. Let $(I_*, i)$ be the pointed hypercovering which is the fibre product of $(H_*, h)$ and $(C_*, c)$, and let $(J_*, j)$ be the pullback $f^*(I_*, i)$ of $(I_*, i)$ onto the étale site of $Y$ pointed with $y$ with respect to $f$. Let $f_* : (\pi_0(J_*), \pi_0(j)) \to (\pi_0(I_*), \pi_0(i))$ be the map induced by $f$ between the pointed simplicial sets of connected components of $J_*$ and $I_*$.

As we saw in the proof of injectivity, the group $\pi_1(\pi_0(J_*), \pi_0(j))$ is trivial and hence the Hurewicz map $\pi_2(\pi_0(J_*), \pi_0(j)) \to H_2(\pi_0(J_*), \hat{\mathbb{Z}})$ is an isomorphism. Therefore there is a unique $\gamma_f \in \pi_2(\pi_0(J_*), \pi_0(j))$ whose image under this Hurewicz map is the image of $\gamma'$ under the composition of $c_{(Y,y)}$ and the natural map $H_2(Y, \hat{\mathbb{Z}}) \to H_2(\pi_0(J_*), \hat{\mathbb{Z}})$. Let $\gamma_{(H_*,h)} \in \pi_2(\pi_0(H_*), \pi_0(h))$ be the image of $\gamma_f$ under the composition of the functorial maps $\pi_2(f_*) : \pi_2(\pi_0(J_*), \pi_0(j)) \to \pi_2(\pi_0(I_*), \pi_0(i))$ and $\pi_2(\pi_0(I_*), \pi_0(i)) \to \pi_2(\pi_0(H_*), \pi_0(h))$.

First we are going to show that $\gamma_{(H_*,h)}$ is independent of the choice of the morphism $f$. Let $f' : (Y', y') \to (Z, z)$ be another morphism of Fet$(X, x)$ such that the image of $\pi_1(f') : \pi_1(Y', y') \to \pi_1(Z, z)$ lies in $N$. Then the fibre product of $f$ and $f'$ over $(Z, z)$ is another morphism of Fet$(X, x)$ with this property. Moreover, it can be factorised into a composition of a morphism $g$ of Fet$(X, x)$ and $f$, and into a composition of a morphism $g'$ of Fet$(X, x)$ and $f'$ too. Therefore it will be enough to show that this construction applied to the morphism $f \circ g$ of Fet$(X, x)$ will give the same element in $\pi_2(\pi_0(H_*), \pi_0(h))$ as $f$, where $g : (V, v) \to (Y, y)$ is a morphism of Fet$(X, x)$.

Let $(C_*, c)$ be the same Cech hypercovering as above, and let $C'_*$ denote the étale Čech hypercovering generated by the cover $V \to Z$, and equip it with a point $c'$ with respect to the same pointed site. Note that $g$ furnishes a map $\delta : (C'_*, c') \to (C_*, c)$ of pointed hypercoverings. Let $(C_*, c)$, $(I_*, i)$ and $(J_*, j)$ be as above,

let $(I'_*, i')$ be the fibre product of $(H_*, h)$ and $(C'_*, c')$, and let $(J'_*, j')$ be the pullback $(f \circ g)^*(I'_*, i')$. Let $f_* : (\pi_0(J_*), \pi_0(j)) \to (\pi_0(I_*), \pi_0(i))$ and $(f \circ g)_* : (\pi_0(J'_*), \pi_0(j')) \to (\pi_0(I'_*), \pi_0(i'))$ be the maps induced by $f$ and by $f \circ g$, respectively. Note that $g$ induces a map $g_* : g^*(\pi_0(J_*), \pi_0(j)) \to (\pi_0(J_*), \pi_0(j))$. Then there is a commutative diagram

$$
\begin{array}{ccccc}
\pi_2(\pi_0(J'_*), \pi_0(j')) & \xrightarrow{\pi_2((f \circ g)_*)} & \pi_2(\pi_0(I'_*), \pi_0(i')) & \longrightarrow & \pi_2(\pi_0(H_*), \pi_0(h)) \\
\downarrow{\scriptstyle \pi_2(g_* \circ (f \circ g)^*(\delta_H))} & & \downarrow{\scriptstyle \pi_2(\delta_H)} & & \| \\
\pi_2(\pi_0(J_*), \pi_0(j)) & \xrightarrow{\pi_2(f_*)} & \pi_2(\pi_0(I_*), \pi_0(i)) & \longrightarrow & \pi_2(\pi_0(H_*), \pi_0(h)),
\end{array}
$$

where $\delta_H : (I'_*, i') \to (I_*, i)$ is the fibre product of $\delta$ with $(H, h)$, the morphism $(f \circ g)^*(\delta_H) : (\pi_0(J'_*), \pi_0(j')) \to g^*(\pi_0(J_*), \pi_0(j)) = (f \circ g)^*(\pi_0(I_*), \pi_0(i))$ is the base change of $\delta_H$ with respect to $f \circ g$; while the middle and left vertical maps are induced by $\delta_H$ and $g_* \circ (f \circ g)^*(\delta_H)$, respectively. Since the image of $\gamma_{f \circ g}$ under the left vertical map is $\gamma_f$, the claim above follows.

In order to conclude the proof of the theorem itself, we only need to show that, for every morphism $\delta : (H_*, h) \to (H'_*, h')$ of pointed hypercoverings of $Z$ with respect to the étale site of $Z$ pointed with $z$, the induced map $\pi_2(\delta) : \pi_2(\pi_0(H_*), \pi_0(h)) \to \pi_2(\pi_0(H'_*), \pi_0(h'))$ takes $\gamma_{(H_*, h)}$ to $\gamma_{(H'_*, h')}$. Indeed, in this case these $\gamma_{(H_*, h)}$ glue together to an element of $\pi_2(Z, z)$ whose image is $\gamma$ under the Hurewicz map, by construction. Now let $f : (Y, y) \to (Z, z)$ be a morphism of $\mathrm{Fet}(X, x)$ such that the image of $\pi_1(f) : \pi_1(Y, y) \to \pi_1(Z, z)$ lies in the kernel of the natural map $\pi_1(Z, z) \to \pi_1(\pi_0(H_*), \pi_0(h))$. Let $(C_*, c), (I_*, i)$ and $(J_*, j)$ be as above. Note that the image of $\pi_1(f)$ lies in the kernel of the natural map $\pi_1(Z, z) \to \pi_1(\pi_0(H'_*), \pi_0(h'))$, too. Let $(I'_*, i')$ be the fibre product of $(H'_*, h')$ and $(C_*, c)$ and let $(J'_*, j')$ be the pullback $f^*(I'_*, i')$ onto the étale site of $Y$ pointed with $y$ with respect to $f$. By slight abuse of notation let $f_*$ denote both maps $(\pi_0(J_*), \pi_0(j)) \to (\pi_0(I_*), \pi_0(i))$ and $(\pi_0(J'_*), \pi_0(j')) \to (\pi_0(I'_*), \pi_0(i'))$ induced by $f$. Then there is a commutative diagram

$$
\begin{array}{ccccc}
\pi_2(\pi_0(J_*), \pi_0(j)) & \xrightarrow{\pi_2(f_*)} & \pi_2(\pi_0(I_*), \pi_0(i)) & \longrightarrow & \pi_2(\pi_0(H_*), \pi_0(h)) \\
\downarrow{\scriptstyle \pi_2(f^*(\delta_C))} & & \downarrow{\scriptstyle \pi_2(\delta_C)} & & \downarrow{\scriptstyle \pi_2(\delta)} \\
\pi_2(\pi_0(J'_*), \pi_0(j')) & \xrightarrow{\pi_2(f_*)} & \pi_2(\pi_0(I'_*), \pi_0(i')) & \longrightarrow & \pi_2(\pi_0(H'_*), \pi_0(h')),
\end{array}
$$

where the middle and left vertical maps are induced by the fibre product $\delta_C$ of $\delta$ with $(C, c)$ and the pullback $f^*(\delta_C)$ of $\delta_C$ with respect to $f$, respectively. By construction, the image of $\gamma_f \in \pi_2(\pi_0(J_*), \pi_0(j))$ constructed for $(H, h)$ under $\pi_2(f^*(\delta_C))$ is the element denoted by the same symbol and constructed for $(H', h')$. The claim is now clear. $\qquad \square$

## 5. The homotopy type of curves and abelian varieties

**Definition 5.1.** Following [Serre 1997, p. 16], we will say that a group $\Gamma$ is good if the homomorphism of cohomology groups $H^n(\widehat{\Gamma}, M) \to H^n(\Gamma, M)$ induced by the natural homomorphism $\Gamma \to \widehat{\Gamma}$ is an isomorphism for every finite $\Gamma$-module $M$. For every smooth, connected, quasiprojective variety $X$ over any field $K$ and every $n \geq 1$, let $\pi_n(X)$ denote the isomorphism class of the $n$-th homotopy group of $\pi_n(X, x)$ for some geometric point $x$. As the notation indicates, these isomorphism classes do not depend on the choice of the base point.

**Proposition 5.2.** *Let $X$ be a smooth variety over $\mathbb{C}$ such that $X(\mathbb{C})$ has the homotopy type of the Eilenberg–MacLane space $B\pi_1(X(\mathbb{C}))$ and the group $\pi_1(X(\mathbb{C}))$ is good. Then $\mathrm{Et}(X)$ is weakly homotopy equivalent to $B\pi_1(X)$.*

*Proof.* By [Artin and Mazur 1969, Corollary 6.6, p. 72] the profinite completion $(B\Gamma)^\wedge$ of $B\Gamma$ is weakly homotopy equivalent to $B\widehat{\Gamma}$ if and only if $\Gamma$ is a good group. Because we assumed that $X$ is smooth, $\mathrm{Et}(X)$ is weakly homotopy equivalent to $B\widehat{\pi_1(X(\mathbb{C}))}$ by Theorem 2.9. On the other hand, the profinite completion of $\pi_1(X(\mathbb{C}))$ is isomorphic to $\pi_1(X)$ by the Grauert–Remmert theorem. The claim is now clear. □

**Remark 5.3.** It is important to note that the condition requiring the fundamental group to be good is not only sufficient, but also necessary. In particular, there are algebraic varieties $X$ over $\mathbb{C}$ such that $X(\mathbb{C})$ has the homotopy type of the Eilenberg–MacLane space $B\pi_1(X(\mathbb{C}))$, but the group $\pi_1(X(\mathbb{C}))$ is not good, therefore $\mathrm{Et}(X)$ is not an Eilenberg–MacLane space. For an important class of examples see [Mochizuki 2003, Lemma 3.16, p. 146].

**Proposition 5.4.** *Let $X$ be a smooth geometrically irreducible quasiprojective variety over an algebraically closed field $K$ of characteristic zero, and let $F$ be another algebraically closed field containing $K$. Then $\mathrm{Et}(X)$ is weakly homotopy equivalent to $\mathrm{Et}(X_F)$.*

*Proof.* This claim is a special case of [Artin and Mazur 1969, Corollary 12.12, p. 144] when $X$ is proper, using also [Artin and Mazur 1969, Theorem 11.1, p. 124]. We only need to add a little bit more when $X$ is not proper. By Hironaka's resolution of singularities, there is a projective variety $Y$ over $K$ which contains $X$ as an open subvariety such that the complement $C \subset Y$ is a normal crossings divisor. By the tame invariance theorem, the tame fundamental groups $\pi_1^C(Y)$ and $\pi_1^{C_F}(Y_F)$ are isomorphic. But since the base fields have characteristic zero we have

$$\pi_1^C(Y) \cong \pi_1(X) \quad \text{and} \quad \pi_1^{C_F}(Y_F) \cong \pi_1(X_F).$$

Therefore $\pi_1(X) \cong \pi_1(X_F)$, so the argument presented in [Artin and Mazur 1969] can be applied in this case, too. □

**Corollary 5.5.** (a) *Let $X$ be a smooth, geometrically connected curve over an algebraically closed field $K$ of characteristic zero which is not a projective curve of genus zero. Then $\mathrm{Et}(X)$ is weakly homotopy equivalent to $B\pi_1(X)$.*

(b) *Let $X$ be an abelian variety over an algebraically closed field $K$ of characteristic zero. Then $\mathrm{Et}(X)$ is weakly homotopy equivalent to $B\pi_1(X)$.*

*Proof.* Recall that a smooth, geometrically connected curve $Y$ defined over a field has type $(g, d)$ if $g$ is the genus of the smooth projective completion $Y^c$ of $Y$ and $d$ is the number of geometric points in the complement of $Y$ in $Y^c$. Let $X$ be a smooth, geometrically connected curve of type $(g, d)$ such that $(g, d) \neq (0, 0)$ over an algebraically closed field $K$ of characteristic zero. There is a subfield $F \subset K$ which is finitely generated over $\mathbb{Q}$ with $X$ already defined over $F$, that is, there is a smooth, geometrically connected curve $Y$ of type $(g, d)$ over $F$ whose base change to $K$ is $X$.

Choose an embedding $i : \overline{F} \to \mathbb{C}$ of fields. Then the base change $Y_{\mathbb{C}}$ of the curve $Y_{\overline{F}}$ to $\mathbb{C}$ with respect to this embedding is also a smooth, geometrically connected curve of type $(g, d)$. The topological space $Y_{\mathbb{C}}(\mathbb{C})$ has the homotopy type of the Eilenberg–MacLane space $B\pi_1(Y_{\mathbb{C}}(\mathbb{C}))$. The topological fundamental group of a smooth, connected complex curve is good (this fact follows at once from [Serre 1997, Problem 1(a), p. 15]) so we get from Proposition 5.2 that $\mathrm{Et}(Y_{\mathbb{C}})$ is weakly homotopy equivalent to $B\pi_1(Y_{\mathbb{C}})$. By a repeated application of Proposition 5.4 we get that $\mathrm{Et}(X) = \mathrm{Et}(Y_K)$ is weakly homotopy equivalent to $\mathrm{Et}(Y_{\mathbb{C}})$, and hence $\pi_1(X) \cong \pi_1(Y_{\mathbb{C}})$, so $\mathrm{Et}(X)$ is weakly homotopy equivalent to $B\pi_1(X)$.

The proof of claim (b) is essentially the same as the proof of claim (a); we only need to add that finitely generated free abelian groups are good (see [Serre 1997, Problem 2(d), p. 16]), so $\mathrm{Et}(A)$ is weakly homotopy equivalent to $B\pi_1(A)$ for every abelian variety $A$ defined over $\mathbb{C}$ by Propositions 5.2 and 5.4.                                          □

## 6. Grothendieck's short exact sequence

**Notation 6.1.** Let $X$ be a geometrically connected variety defined over $K$. Let $\eta$ be a $\overline{K}$-valued point of $X$. Then Grothendieck's short exact sequence of étale fundamental groups for $X$ is

$$1 \longrightarrow \pi_1(\overline{X}, \eta) \longrightarrow \pi_1(X, \eta) \longrightarrow \Gamma_K \longrightarrow 1, \qquad (6.1.1)$$

which is an exact sequence of profinite groups in the category of topological groups. Every $K$-rational point $x \in X(K)$ induces a section $\Gamma_K \to \pi_1(X, \eta)$ of the sequence (6.1.1), well-defined up to conjugation. Let $\mathrm{Sec}(X/K)$ denote the set of conjugacy classes of sections of (6.1.1) (in the category of profinite groups where morphisms

are continuous homomorphisms). Then we have a map

$$s_{X/K} : X(K) \to \mathrm{Sec}(X/K)$$

which sends every point $x \in X(K)$ to the corresponding conjugacy class of sections.

**Definition 6.2.** For every characteristic open subgroup $N$ of $\pi_1(\overline{X}, \eta)$, consider the short exact sequence

$$1 \longrightarrow \pi_1(\overline{X}, \eta)/N \longrightarrow \pi_1(X, \eta)/N \longrightarrow \Gamma_K \longrightarrow 1, \qquad (6.2.1)$$

obtained by dividing out (6.1.1) by $N$. Let $\mathrm{Sec}(X/K, N)$ denote the set of conjugacy classes of sections of (6.2.1). Let

$$s_{X/K,N} : X(K) \to \mathrm{Sec}(X/K, N)$$

denote the composition of $s_{X/K,N}$ and the natural forgetful map

$$\phi_{X/K,N} : \mathrm{Sec}(X/K) \to \mathrm{Sec}(X/K, N).$$

Note that for every pair of characteristic open subgroups $N' \subseteq N$ of $\pi_1(\overline{X}, \eta)$ the composition of $\phi_{X/K,N'}$ and the forgetful map $\mathrm{Sec}(X/K, N') \to \mathrm{Sec}(X/K, N)$ is $\phi_{X/K,N}$. Therefore we may take the projective limit of the maps $\phi_{X/K,N}$ to get a map

$$\phi_{X/K} = \lim_{N} \phi_{X/K,N} : \mathrm{Sec}(X/K) \to \lim_{N} \mathrm{Sec}(X/K, N).$$

where the limit is over the set of characteristic open subgroups of $\pi_1(\overline{X}, \eta)$ directed with respect to reverse inclusion.

**Proposition 6.3.** *The map $\phi_{X/K}$ is a bijection.*

*Proof.* Let $r, s$ be two sections $\Gamma_K \to \pi_1(X, \eta)$ such that for every characteristic open subgroup $N$ of $\pi_1(\overline{X}, \eta)$ the compositions of $r$ and $s$ with the quotient map $\pi_1(X, \eta) \to \pi_1(X, \eta)/N$ are conjugates. Then for every such $N$ the set

$$C_N = \{g \in \pi_1(\overline{X}, \eta) \mid g^{-1}r(h)gs(h)^{-1} \in N \text{ for all } h \in \Gamma_K\}$$

is nonempty. Since the sets $C_N$ are closed in the compact topological space $\pi_1(\overline{X}, \eta)$, their intersection

$$\bigcap_{N} C_N = \left\{ g \in \pi_1(\overline{X}, \eta) \mid g^{-1}r(h)gs(h)^{-1} \in \bigcap_{N} N \text{ for all } h \in \Gamma_K \right\}$$

is also nonempty. Because in a topologically finitely generated profinite group, such as $\pi_1(\overline{X}, \eta)$, the intersection of all characteristic open subgroups is the identity element, we get that $r$ and $s$ are conjugates. Therefore $\phi_{X/K}$ is injective.

Now let $r$ be an element of $\lim_N \mathrm{Sec}(X/K, N)$, and for every $N$ as above let $r_N \in \mathrm{Sec}(X/K, N)$ be the image of $r$ under the projection $\lim_M \mathrm{Sec}(X/K, M) \to \mathrm{Sec}(X/K, N)$. For every positive integer $m$ let $N(m)$ be the intersection of all open subgroups of $\pi_1(\overline{X}, \eta)$ of index at most $m$. We are going to construct a section $s_m : \Gamma_K \to \pi_1(X, \eta)/N(m)$ whose conjugacy class is $r_{N(m)}$ for every $m$ by induction, as follows. When $m = 1$ this section is just the identity. Assume now that $s_{m-1}$ is already constructed. Let $s'_m$ be a section $\Gamma_K \to \pi_1(X, \eta)/N(m)$ whose conjugacy class is $r_{N(m)}$. Because $r_{N(m)}$ maps to $r_{N(m-1)}$ under the forgetful map $\mathrm{Sec}(X/K, N(m)) \to \mathrm{Sec}(X/K, N(m-1))$, we get that there is a $g \in \pi_1(\overline{X}, \eta)/N(m)$ such that the composition of $g^{-1} s'_m g$ and the quotient map $\pi_1(X, \eta)/N(m) \to \pi_1(X, \eta)/N(m-1)$ is $s_{m-1}$. Let $s_m$ be $g^{-1} s'_m g$. These sections are compatible and their limit is a section

$$s : \Gamma_K \to \lim_{m=1}^{\infty} \pi_1(X, \eta)/N(m) = \pi_1(X, \eta)$$

whose image is $r$ under $\phi_{X/K}$. So the latter is surjective, too. $\qquad\square$

**Definition 6.4.** We say that $X$ is well-equipped with $K$-rational points if the map $s_{X/K,N}$ is surjective for every characteristic open subgroup $N$ of $\pi_1(\overline{X}, \eta)$. Note that for a different choice of a base point $\eta'$ there is an isomorphism between $\pi_1(X, \eta)$ and $\pi_1(X, \eta')$ which maps $\pi_1(\overline{X}, \eta)$ onto $\pi_1(\overline{X}, \eta')$, canonical up to conjugacy. Therefore the sets $\mathrm{Sec}(X/K)$ and $\mathrm{Sec}(X/K, N)$ are independent of the choice of the base point $\eta$, as the notation indicates.

**Proposition 6.5.** *The algebraic groups* $\mathrm{GL}_n$ *and* $\mathrm{PGL}_n$ *are well-equipped with $K$-rational points over any characteristic-zero field $K$ and positive integer $n$.*

*Proof.* Let $1$ denote the unit of $\mathrm{GL}_n(\overline{K})$, and $\mathrm{PGL}_n(\overline{K})$ as well. The quotient map $p : \mathrm{GL}_n \to \mathrm{PGL}_n$ by the centre of $\mathrm{GL}_n$ induces a surjection

$$\pi_1(p) : \pi_1(\overline{\mathrm{GL}_n}, 1) \to \pi_1(\overline{\mathrm{PGL}_n}, 1),$$

and hence it will be enough to prove the claim for $\mathrm{GL}_n$ only. Let $i : \mathrm{GL}_1 \to \mathrm{GL}_n$ be the map which embeds $\mathrm{GL}_1$ into $\mathrm{GL}_n$ as diagonal matrices with $1$ on the diagonal except at the upper-left corner. This map induces an isomorphism

$$\pi_1(i) : \pi_1(\overline{\mathrm{GL}_1}, 1) \to \pi_1(\overline{\mathrm{GL}_n}, 1).$$

Therefore it will be enough to prove the claim for $\mathrm{GL}_1$ only. There is a natural isomorphism

$$\pi_1(\overline{\mathrm{GL}_1}, 1) \cong \widehat{\mathbb{Z}},$$

and for every $k \in \mathbb{N}$ there is a natural bijection

$$\mathrm{Sec}(\mathrm{GL}_1 / K, k\widehat{\mathbb{Z}}) \cong H^1(K, \mu_k),$$

where $\mu_k \subseteq \overline{K}^*$ is the module of $k$-th roots of unity. Moreover, under this identification $s_{\mathrm{GL}_1/K,k\widehat{\mathbb{Z}}}$ corresponds to the coboundary map furnished by Kummer theory. Since, by Hilbert's theorem 90, $H^1(K, \overline{K}^*)$ is zero, the claim now follows.    □

**Proposition 6.6.** *Assume that $K$ is an algebraically closed field of characteristic zero. Let $G$ be a geometrically connected algebraic group over $K$, and let $f : X \to Y$ be a principal $G$-bundle over a geometrically connected smooth variety $Y$ over $K$. Let $x \in X$ and set $y = f(x)$. Then the sequence*

$$\pi_1(f^{-1}(y), x) \to \pi_1(X, x) \xrightarrow{\;\pi_1(f)\;} \pi_1(Y, y) \to 1 \qquad (6.6.1)$$

*is exact, where the first map is induced by the inclusion $f^{-1}(y) \subseteq X$.*

*Proof.* There is a subfield $F \subset K$ which is finitely generated over $\mathbb{Q}$ and $G, X, Y$ and $f$ are already defined over $F$. Therefore, by the invariance theorem for the (tame) étale fundamental group, it will be sufficient to prove the claim for $\overline{F}$. By the axiom of choice there is an embedding $\overline{F} \to \mathbb{C}$ of fields, and hence we may assume that $K$ is $\mathbb{C}$ without loss of generality, again by the invariance theorem. Because the map $X(\mathbb{C}) \to Y(\mathbb{C})$ induced by $f$ is a Serre fibration, there is a short exact sequence

$$\pi_1(f^{-1}(y)(\mathbb{C}), x) \longrightarrow \pi_1(X(\mathbb{C}), x) \longrightarrow \pi_1(Y(\mathbb{C}), y) \longrightarrow 1 \qquad (6.6.2)$$

of topological fundamental groups of complex analytic spaces. The profinite completion functor is right-exact, so the completion of (6.6.2) is also exact. By the Grauert–Remmert theorem the latter is the sequence (6.6.1).    □

**Proposition 6.7.** *Let $G$ be a geometrically connected algebraic group well-equipped with $K$-rational points over a field $K$ of characteristic zero. Let $f : X \to Y$ be a principal $G$-bundle over a smooth variety $Y$ over $K$ and let $x, y \in Y(K)$ be such that:*

(i) $s_{Y/K}(x) = s_{Y/K}(y)$.

(ii) *The sets $f^{-1}(x)(K)$ and $f^{-1}(y)(K)$ are nonempty.*

*Then, for every characteristic open subgroup $N$ of $\pi_1(\overline{X}, \eta)$ (where $\eta \in X(\overline{K})$ is arbitrary), there are two points $x_N \in f^{-1}(x)(K)$ and $y_N \in f^{-1}(y)(K)$ such that $s_{X/K,N}(x_N) = s_{X/K,N}(y_N)$.*

*Proof.* Pick two points $x' \in f^{-1}(x)(K)$, $y' \in f^{-1}(y)(K)$ and let $\eta \in f^{-1}(x)(\overline{K})$. Let $r \in s_{X/K}(x'), s \in s_{X/K}(y')$ be two sections of the short exact sequence (6.1.1). Because both $x'$ and $\eta$ lie in $f^{-1}(x)(K)$, we may assume that the image of $r$ is in $\pi_1(\overline{f^{-1}(x)}, \eta)$ without loss of generality. Now let $\theta = f(\eta)$, and let $r_0, s_0$ be the composition of $r, s$ and $\pi_1(f) : \pi_1(X, \eta) \to \pi_1(Y, \theta)$, respectively. As $r_0 \in s_{Y/K}(x)$ and $s_0 \in s_{Y/K}(y)$, we get that these sections are conjugate. By Proposition 6.6 the map $\pi_1(f)$ is surjective, and therefore we may assume that

$r_0$ and $s_0$ are the same, by conjugating $s$ if necessary. This implies that $s$ lies in $\pi_1(f^{-1}(x), \eta)$, just as $r$ does, by Proposition 6.6. Let $N$ be now a characteristic open subgroup of $\pi_1(\overline{X}, \eta)$, and let $M'$ be the preimage of $N$ with respect to the map $\pi_1(\overline{f^{-1}(x)}, \eta) \to \pi_1(\overline{X}, \eta)$. As $M'$ is open and $\pi_1(\overline{f^{-1}(x)}, \eta)$ is topologically finitely generated, there is a characteristic open subgroup $M$ of $\pi_1(\overline{f^{-1}(x)}, \eta)$ lying in $M'$. Because $f^{-1}(x)$ has a $K$-rational point, it is isomorphic to $G$. Therefore it is well-equipped with $K$-rational points, so there is an $x_N \in f^{-1}(x)(K)$ such that the composition of $s$ and the quotient map $\pi_1(f^{-1}(x), \eta) \to \pi_1(f^{-1}(x), \eta)/M$ lies in $s_{f^{-1}(x)/K, M}(x_N)$. If we set $y_N = y'$ then it is clear that the pair $x_N, y_N$ satisfies the required properties. $\qquad\square$

**Proposition 6.8.** *Let $G$ be a geometrically connected algebraic group well-equipped with $K$-rational points over a field $K$ of characteristic zero such that $\pi_1(\overline{G})$ is finite. Let $f : X \to Y$ be a principal $G$-bundle over a smooth variety $Y$ over $K$ and let $x, y \in Y(K)$ be such that*:

(i) $s_{Y/K}(x) = s_{Y/K}(y)$.

(ii) *The sets $f^{-1}(x)(K)$ and $f^{-1}(y)(K)$ are nonempty.*

*Then there are two points $x' \in f^{-1}(x)(K)$ and $y' \in f^{-1}(y)(K)$ such that $s_{X/K}(x') = s_{X/K}(y')$.*

*Proof.* The proof is the same as above, except that we look at sections of the full Grothendieck short exact sequence (6.1.1) for $X$. We leave the details to the reader. $\qquad\square$

**Remark 6.9.** By Proposition 6.5 we may apply Proposition 6.7 to principal $GL_n$-bundles. Note that $SL_n \to PSL_n$ is a finite étale cover and $SL_n$ is simply connected. As $PSL_n$ and $PGL_n$ are isomorphic, we get that $\pi_1(\overline{PGL_n})$ is finite. Therefore by Proposition 6.5 we may apply Proposition 6.8 to principal $PGL_n$-bundles.

## 7. Basic consequences of obstruction theory

**Definition 7.1.** For every $n \in \mathbb{N}$, by functoriality we get a natural map

$$\iota_{X/K}^n : X(K) \to [\mathrm{Et}_{/K}(\mathrm{Spec}(K)), \mathrm{Et}_{/K}(X)] \to [\mathrm{Et}_{/K}(\mathrm{Spec}(K))^\natural, \mathrm{Et}_{/K}^n(X)],$$

where the second map is furnished by applying the Postnikov tower functor and composing with the $n$-th truncation map $\mathrm{Et}_{/K}(X)^\natural \to \mathrm{Et}_{/K}^n(X)$. We will denote $[\mathrm{Et}_{/K}(\mathrm{Spec}(K))^\natural, \mathrm{Et}_{/K}^n(X)]$ by the symbol $X^n(hK)$. The $n$-th truncation map $S^\natural \to P_n(S)$ is natural, and hence induces a natural map
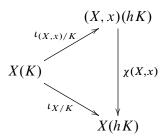
$$h_{X/K}^n : X(hK) \to X^n(hK)$$

such that $\iota^n_{X/K} = h^n_{X/K} \circ \iota_{X/K}$. For every positive integer $n$, let $\sim_n$ denote the following equivalence relation on $X(hK)$: for every pair $x, y \in X(hK)$ we have $x \sim_n y$ if and only if $h^n_{X/K}(x) = h^n_{X/K}(y)$. It is clear that the equivalence relation $\sim_{n+1}$ is finer than the equivalence relation $\sim_n$.

**Definition 7.2.** For every pointed $K$-scheme $(X, x)$, let

$$(X, x)(hK) = \mathrm{Et}_{/K}(X, x)(E\Gamma_K).$$

Let $\chi(X, x) : (X, x)(hK) \to X(hK)$ denote the bijection induced by the map $i(X, x)$ (see [Proposition 3.7](#)). Then we have a unique natural map $\iota_{(X,x)/K} : X(K) \to (X, x)(hK)$ such that the diagram



is commutative.

**Definition 7.3.** We are going to need a variant of the equivalence relations $\sim_n$ for pointed relative étale homotopy types, too. Let $(X, x)$ be a pointed $K$-scheme as above. We will denote $\mathrm{Et}^n_{/K}(X, x)(E\Gamma_K)$ by the symbol $(X, x)^n(hK)$. The $n$-th truncation map furnishes a natural map

$$h^n_{(X,x)/K} : (X, x)(hK) \to (X, x)^n(hK).$$

By slight abuse of notation, for every positive integer $n$ let $\sim_n$ denote the following equivalence relation on $(X, x)(hK)$: for every pair $x, y \in (X, x)(hK)$ we have $x \sim_n y$ if and only if $h^n_{(X,x)/K}(x) = h^n_{(X,x)/K}(y)$.

This notation is justified because of the following:

**Lemma 7.4.** *Under the map $\chi(X, x)$ the equivalence relation $\sim_n$ on $(X, x)(hK)$ corresponds to the equivalence relation $\sim_n$ on $X(hK)$.*

*Proof.* For every étale hypercovering $H_*$ of $X$ there is a natural commutative diagram

where we use Notation 3.6, and where the vertical maps are induced by truncation. Since the upper horizontal map is a *weak* equivalence, so is the lower horizontal map, so it induces a bijection on homotopy fixed points. The claim now follows by taking the limit, similarly to the proof of Proposition 3.7. □

**Notation 7.5.** For every profinite group $\Gamma$ and every pro-discrete $\Gamma$-module $M$, let $H^k(\Gamma, M)$ denote the projective limit of the (continuous) cohomology groups $H^k(\Gamma, N)$, where $N$ runs through the directed system of discrete quotients of $M$. For the sake of simple notation, for the rest of the paper for every field $K$ and every pro-discrete $\Gamma_K$-module $M$ let $H^k(K, M)$ denote the group $H^k(\Gamma_K, M)$ introduced above. Note that these groups commute with projective limits.

**Theorem 7.6.** *Assume that $X$ is a smooth geometrically connected variety over $K$, and assume that $X(K)$ is nonempty. For every $x, y \in X(hK)$:*

(i) *$x = y$ if and only if $x \sim_n y$ for every $n \in \mathbb{N}$.*

(ii) *There is a natural bijection*

$$j_{X/K} : X^1(hK) \to \mathrm{Sec}(X/K)$$

*such that for every $p \in X(K)$ we have $j_{X/K}(\iota^1_{X/K}(p)) = s_{X/K}(p)$.*

(iii) *For every positive integer $n$, if $x \sim_n y$ then there exists a natural obstruction class $\delta^X_n(x, y) \in H^{n+1}(K, \pi_{n+1}(\overline{X}))$ such that $x \sim_{n+1} y$ if and only if $\delta^X_n(x, y) = 0$.*

*Proof.* The first claim is an immediate consequence of the definition of $\mathrm{Et}_{/K}(X)^\natural$. Next we are going to prove (ii). Let $f : Y \to X$ be a torsor under a finite étale group $\mathcal{G}$ over $K$. Then the $\overline{K}$-valued points $G = \mathcal{G}(\overline{K})$ of $\mathcal{G}$ form a finite group equipped with an action of $\Gamma_K$. Let $C_*$ denote the étale Čech hypercovering

$$Y \mathrel{\substack{\longrightarrow \\ \longleftarrow \\ \longleftarrow}} Y \times_X Y \mathrel{\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}} Y \times_X Y \times_X Y \mathrel{\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}} \cdots$$

generated by the cover $Y \to X$. It is explained at the beginning of Section 9 of [Harpaz and Schlank 2013] that there is a natural map

$$c_Y : \pi_{0/K}(C_*)(E\Gamma_K) \to H^1(\Gamma_K, G).$$

Moreover, by [Harpaz and Schlank 2013, Lemma 9.1], for every $p \in X(K)$ the image of the corresponding homotopy fixed point in $\pi_{0/K}(C_*)(E\Gamma_K)$ with respect to $c_Y$ is the element which classifies the $\mathcal{G}$-torsor $Y_p = f^{-1}(p)$. When $Y$ is geometrically connected there is a *weak* equivalence $\pi_\Gamma(C_*) \to BG$, where we equip the latter with the tautological $\Gamma_K$-action, and hence the map $c_Y$ is a bijection (see the discussion after Lemma 9.7 of [Harpaz and Schlank 2013]). Moreover, the composition of the natural map $X(hK) \to \pi_{0/K}(C_*)(E\Gamma_K)$ and $c_Y$ factors through $h^1_{X/K}$.

Let $\eta \in X(K)$ be a $K$-rational point and let $\bar{\eta}$ denote the $\bar{K}$-valued point associated to $\eta$ (see Notation 3.9). Fix an element $s$ of $s_{X/K}(\eta)$. For every characteristic open subgroup $N$ of $\pi_1(X, \bar{x})$ let $N' \subseteq \pi_1(X, \bar{x})$ be the subgroup generated by $N$ and the image of $s$. Since $N'$ is an open subgroup, there is a connected finite étale cover $f_N : Y_N \to X$ such that the image of $\pi_1(Y_N)$ with respect to $\pi_1(f_N)$ is $N'$. Let $\mathcal{G}_N$ denote the unique finite étale group over $K$ such that $\mathcal{G}_N(\bar{K})$ is

$$\pi_1(\bar{X}, \bar{\eta})/N \cong \pi_1(X, \bar{\eta})/N',$$

equipped with its natural $\Gamma_K$-action induced by conjugation. Then $f_N : Y_N \to X$ is a torsor under $\mathcal{G}_N$, and hence by applying the construction above to the étale Čech hypercovering generated by $f_N$ we get a map

$$j_{X/K,N} : X^1(hK) \to \mathrm{Sec}(X/K, N)$$

such that for every $p \in X(K)$ we have $j_{X/K,N}(\iota^1_{X/K}(p)) = s_{X/K,N}(p)$. According to Proposition 6.3, by taking the limit over every characteristic open subgroup $N$ of $\pi_1(X, \bar{\eta})$ we get a continuous map

$$j_{X/K} : X^1(hK) \to \mathrm{Sec}(X/K)$$

between compact Hausdorff topological spaces such that for every $p \in X(K)$ we have $j_{X/K}(\iota^1_{X/K}(p)) = s_{X/K}(p)$. By [Harpaz and Schlank 2013, Lemma 9.11] this map is a bijection.

For every connected pointed $\Gamma_K$-space $S$ which has only finitely many nontrivial homotopy groups, there is a natural spectral sequence for homotopy groups of homotopy fixed points

$$E_2^{s,t} = H^s(K, \pi_t(S)) \Rightarrow \pi_{t-s}(S^{h\Gamma_K})$$

constructed by Goerss [1995, Theorem B, p. 189]. Therefore for objects of this category there is a natural obstruction class of the type described in the last claim. So claim (iii) follows from Lemma 7.4 and Proposition 3.10 by applying Goerss' results to $\mathrm{Et}_{/K}(X, \eta)$. □

**Lemma 7.7.** *Assume that $X$ is a smooth geometrically connected variety over $K$ and $\mathrm{Et}(\bar{X})$ is weakly homotopy equivalent to $B\pi_1(\bar{X})$. Also suppose that $X(K) \neq \varnothing$. Then for every $x, y \in X(hK)$ we have $x = y$ if and only if $j_{X/K}(x) = j_{X/K}(y)$.*

*Proof.* All the higher homotopy groups of $\bar{X}$ vanish, so the claim is immediate from the theorem above. □

We say that two points $x, y \in X(K)$ are directly $A$-equivalent if there is a map $f : \mathbb{A}^1_K \to X$ of $K$-varieties such that $f(0) = x$ and $f(1) = y$. The $A$-equivalence on $X(K)$ is the equivalence relation generated by direct $A$-equivalence.

**Proposition 7.8.** *Assume that $K$ has characteristic zero. Then for every $X$ over $K$ the map*

$$\iota_{X/K} : X(K) \to X(hK)$$

*factors through $A$-equivalence.*

*Proof.* It will be sufficient to show that for every two points $x, y \in X(K)$ which are directly $A$-equivalent we have $\iota_{X/K}(x) = \iota_{X/K}(y)$. Let $f : \mathbb{A}_K^1 \to X$ be a morphism of $K$-varieties such that $f(0) = x$ and $f(1) = y$. Both $\iota_{X/K}(x)$ and $\iota_{X/K}(y)$ lie in the image of the map $f_* : \mathbb{A}_K^1(hK) \to X(hK)$. By Corollary 5.5 and Lemma 7.7 the set $\mathbb{A}_K^1(hK)$ consists of one element, since $\mathbb{A}_{\overline{K}}^1$ has trivial étale fundamental group. The claim is now clear. $\qquad\square$

**Remark 7.9.** The validity of a such a claim was already suggested by Toën [2004], but his original claim is not true as stated in positive characteristic. In the special case when $K$ is a finite field it was observed in [Tamagawa 1997, Proposition 2.8, pp. 151-152] that the map $s_{\mathbb{A}_K^1/K}$, and hence the map $\iota_{\mathbb{A}_K^1/K}$, is injective. Thus it is not true in general that for two points $x, y \in X(K)$ which are directly $A$-equivalent we have $\iota_{X/K}(x) = \iota_{X/K}(y)$ when $K$ has positive characteristic.

## 8. The Manin pairing

**Notation 8.1.** For every object $X : I \to \mathrm{Ho}(\Gamma\text{-}\mathrm{SSets})$ of Pro-Ho($\Gamma$-SSets) such that $I$ is countable, let $X_{h\Gamma} \in \mathrm{ob}(\mathrm{Pro\text{-}Ho}(\mathrm{SSets}))$ denote the pro-homotopy quotient defined at the beginning of Section 6.2 of [Harpaz and Schlank 2013]. Note that this construction can be applied to the objects $\mathrm{Et}_{/K}(X), \mathrm{Et}_{/K}^n(X)$ and $\mathrm{Et}_{/K}(X)^{\natural}$ when $\Gamma = \Gamma_K$ is the absolute Galois group of the field $K$.

**Proposition 8.2.** *Let $K$ be a field and $X$ a variety over $K$. Then there are natural isomorphisms*

$$\mathrm{Et}_{/K}(X)_{h\Gamma_K} \cong \mathrm{Et}(X), \quad \mathrm{Et}_{/K}^n(X)_{h\Gamma_K} \cong \mathrm{Et}^n(X), \quad \mathrm{Et}_{/K}(X)_{h\Gamma_K}^{\natural} \cong \mathrm{Et}(X)^{\natural}$$

*in the category* Pro-Ho(SSets).

*Proof.* The second isomorphism is the content of Proposition 6.14 in [Harpaz and Schlank 2013]. The third isomorphism follows from the naturality of this isomorphism, and the first isomorphism is shown in the proof of the proposition mentioned above. $\qquad\square$

**Definition 8.3.** Let $X$ be again a variety over $K$. Note that by functoriality we get a natural map

$$\lambda_{X/K} : X(K) \to [\mathrm{Et}(\mathrm{Spec}(K)), \mathrm{Et}(X)] \to [\mathrm{Et}(\mathrm{Spec}(K))^{\natural}, \mathrm{Et}(X)^{\natural}],$$

where the second map is furnished by applying the Postnikov tower functor. By applying the pro-homotopy quotient functor and the proposition above, we get that there is a natural map

$$\kappa_{X/K} : X(hK) \to [\mathrm{Et}(\mathrm{Spec}(K))^{\natural}, \mathrm{Et}(X)^{\natural}]$$

such that $\lambda_{X/K} = \kappa_{X/K} \circ \iota_{X/K}$.

**Definition 8.4.** By [Artin and Mazur 1969, Corollary 10.8, pp. 122-123] there is a natural equivalence between the category of locally constant étale sheaves of finite abelian groups on $X$ and local coefficient systems of finite abelian groups on $\mathrm{Et}(X)^{\natural}$, and under this equivalence the étale cohomology of $X$ with coefficients in a locally constant étale sheaf $\mathcal{F}$ of finite abelian groups is the same as the cohomology of $\mathrm{Et}(X)^{\natural}$ with coefficients in the local coefficient system corresponding to $\mathcal{F}$. We will not distinguish between these two categories in what follows. In particular for a finite, étale group scheme $G$ over $K$ we will identify $H^*(\mathrm{Et}(X)^{\natural}, G)$ and $H^*(X, G)$.

A basic, but important, corollary of these observations is the following. Let $G$ and $X$ be as above, and let $c \in H^i(X, G)$ be a cohomology class for some $i \in \mathbb{N}$.

**Lemma 8.5.** *Assume that $x, y \in X(K)$ are $H$-equivalent. Then the cohomology classes $x^*(c), y^*(c) \in H^i(K, G)$ are equal.*

*Proof.* The lemma follows from the commutativity of the diagram

$$
\begin{array}{ccccc}
X(K) & \times & H^i(X, G) & \longrightarrow & H^i(K, G) \\
\lambda_{X/K} \downarrow & & \| & & \| \\
[\mathrm{Et}(\mathrm{Spec}(K))^{\natural}, \mathrm{Et}(X)^{\natural}] \times H^i(\mathrm{Et}(X)^{\natural}, G) & \longrightarrow & H^i(\mathrm{Et}(\mathrm{Spec}(K))^{\natural}, G)
\end{array}
$$

where the horizontal maps are the pairings furnished by pullback.    $\square$

For every $n \in \mathbb{N}$ not divisible by the characteristic of $K$, let $(\cdot, \cdot)_n$ denote the pairing

$$(\cdot, \cdot)_n : X(hK) \times H^2(X, \mu_n) \to H^2(K, \mu_n)$$

given by the rule $(x, c)_n = \kappa_{X/K}(x)^*(c)$.

**Lemma 8.6.** *The diagram*

$$
\begin{array}{ccccc}
X(K) \times & \mathrm{Br}(X) & \xrightarrow{(\cdot, \cdot)} & \mathrm{Br}(K) & \\
\iota_{X/K} \downarrow & \uparrow & & \uparrow \alpha_n & \quad (8.6.1) \\
X(hK) \times H^2(X, \mu_n) & \xrightarrow{(\cdot, \cdot)_n} & H^2(K, \mu_n) &
\end{array}
$$

*commutes, where the middle and right vertical arrows are induced by the inclusion $\mu_n \subset \mathbb{G}_m$ of sheaves.*

*Proof.* This is immediate from the functoriality of the constructions involved. □

**Lemma 8.7.** *Let $X$ be a geometrically connected variety over $K$ and $x, y \in X(K)$.*
*Then $\iota_{X/K}(x) \sim_1 \iota_{X/K}(y)$ if and only if for every finite, étale map $f : Y \to X$ of*
*geometrically connected varieties over $K$ such that there is an $\tilde{x} \in Y(K)$ with the*
*property $f(\tilde{x}) = x$ there is a $\tilde{y} \in Y(K)$ such that $f(\tilde{y}) = y$.*

*Proof.* By part (ii) of Theorem 7.6 we have $\iota_{X/K}(x) \sim_1 \iota_{X/K}(y)$ if and only if
$s_{X/K}(x) = s_{X/K}(y)$. It is well-known that the latter condition is equivalent to the
second condition of the claim (see for example [Tamagawa 1997]).          □

**Lemma 8.8.** *Let $f : Y \to X$ be a finite, étale map of varieties over $K$. Assume that*
*$x, y \in X(K)$ are $H$-equivalent and that there is an $\tilde{x} \in Y(K)$ such that $f(\tilde{x}) = x$.*
*Then there is a $\tilde{y} \in Y(K)$ such that $\tilde{x}, \tilde{y}$ are $H$-equivalent and $f(\tilde{y}) = y$.*

*Proof.* The connected component $X'$ of $X$ on which $x$ lies is geometrically con-
nected. By Proposition 2.10 the point $y$ must lie on the same component. The
connected component $Y'$ of $Y$ on which $\tilde{x}$ lies is also geometrically connected, and
the restriction $f|_{Y'} : Y' \to X'$ is a finite, étale map. Therefore we may assume
without loss of generality that $X$ and $Y$ are geometrically connected. Hence by
Lemma 8.7 there is a $\tilde{y} \in Y(K)$ such that $f(\tilde{y}) = y$, and we may even assume that
$\iota_{X/K}(\tilde{x}) \sim_1 \iota_{X/K}(\tilde{y})$.

It will be enough to show that $\tilde{x}$ and $\tilde{y}$ are $H$-equivalent. We will prove that
$\iota_{X/K}(\tilde{x}) \sim_n \iota_{X/K}(\tilde{y})$ for every $n \geq 2$ by induction. Since the map $\pi_n(f) : \pi_n(\overline{Y}) \to$
$\pi_n(\overline{X})$ is an isomorphism for every $n \geq 2$ by Proposition 4.1, we get that the
induced map

$$H^n(\pi_n(f)) : H^n(K, \pi_n(\overline{Y})) \to H^n(K, \pi_n(\overline{X}))$$

is also an isomorphism. By naturality of the obstruction classes we have

$$H^n(\pi_n(f))(\delta_n^Y(\tilde{x}, \tilde{y})) = \delta_n^X(x, y).$$

Since the right-hand side is zero we get that $\delta_n^Y(\tilde{x}, \tilde{y})$ is also zero.          □

**Proposition 8.9.** *Let $X$ be a regular variety over $K$, and assume that $x, y \in X(K)$*
*are $H$-equivalent. Then $x$ and $y$ are étale-Brauer equivalent.*

*Proof.* Let $Y \to X$ be a finite, étale morphism of varieties over $K$ such that there
is an $\tilde{x} \in Y(K)$ mapping to $x$. By Lemma 8.8 there is a $\tilde{y} \in Y(K)$ such that
$\tilde{x}, \tilde{y}$ are $H$-equivalent and $f(\tilde{y}) = y$. It will be enough to show that $\tilde{x}, \tilde{y}$ are
Brauer equivalent. Because $Y$ is the finite étale cover of a regular variety, it is also
regular, so it will be enough show that, given a regular variety $X$ over $K$, every
pair of points $x, y \in X(K)$ which are $H$-equivalent are also Brauer equivalent.
Because $X$ is regular, the group $H^2(X, \mathbb{G}_m)$ is torsion (see [Grothendieck 1971,
Proposition 1.4, p. 291]). Therefore for every $b \in H^2(X, \mathbb{G}_m)$ there is a natural

number $n \in \mathbb{N}$ and a $c \in H^2(X, \mu_n)$ such that $b$ is the image of $c$ under the natural map $H^2(X, \mu_n) \to H^2(X, \mathbb{G}_m)$. The claim now follows from the commutativity of the diagram (8.6.1). $\qquad\square$

## 9. Brauer equivalence versus étale-Brauer equivalence

**Definition 9.1.** Let $E$, $E'$ be two elliptic curves defined over a field $K$ whose characteristic is not two and let $t \in E'(K)$ be a point of order two. Let $\rho : E \to E$ be the multiplication by $-1$ map and let $\sigma : E' \to E'$ be the translation by $t$. Let $X$ denote the quotient of $E \times E'$ by the fixed-point-free involution $(\rho, \sigma)$. Then $X$ is a smooth projective geometrically irreducible surface over $K$. We call such surfaces bielliptic.

**Proposition 9.2.** *Let $K$ be a finite extension of $\mathbb{Q}_p$, and let $X$ be a bielliptic surface over $K$. Then the map $s_{X/K}$ is injective.*

*Proof.* Let $x, y \in X(K)$ be two different points and let $\eta$ be $\overline{K}$-valued point of $X$. Let $\mathcal{K} \subseteq \pi_1(\overline{X}, \eta)$ be the characteristic subgroup such that the quotient $\pi_1(\overline{X}, \eta)/\mathcal{K}$ is the maximal 2-torsion abelian quotient of $\pi_1(\overline{X}, \eta)$. Fix an element $s$ of $s_{X/K}(x)$ and let $\mathcal{K}' \subseteq \pi_1(X, \eta)$ be the subgroup generated by $\mathcal{K}$ and the image of $s$; this is an open subgroup. Let $f : Y \to X$ be the connected finite étale cover such that the image of $\pi_1(Y)$ with respect to $\pi_1(f)$ is $\mathcal{K}'$. Since $\overline{Y}$ is a finite étale cover of the abelian variety $\overline{E \times E'}$, where we use the notation of the definition above, it is also an abelian variety. Therefore $Y$ is a principal homogeneous space over an abelian variety defined over $K$.

By construction there is an $x' \in Y(K)$ such that $f(x') = x$. So $Y$ has a $K$-rational point, and hence it is also an abelian variety over $K$. If $y$ does not have a lift to a $K$-valued point of $Y$ then $s_{X/K}(x) \neq s_{X/K}(y)$. So we may assume that there is a $y' \in Y(K)$ such that $f(y') = y$. Let $G = \pi_1(X, \eta)/\pi_1(Y, \eta)$. Since we may take a finite extension of $K$ during the proof of injectivity of $s_{X/K}$, we may assume that the action of $\Gamma_K$ on $G$ is trivial without loss of generality. In this case this finite group is the Galois group of the connected finite étale cover $f : Y \to X$. It also acts on $\mathrm{Sec}(Y/K)$, and two elements of $\mathrm{Sec}(Y/K)$ are in the same $G$-orbit if and only if they have the same image under the map $\mathrm{Sec}(Y/K) \to \mathrm{Sec}(X/K)$ induced by $f$. Moreover, the section map $s_{Y/K} : Y(K) \to \mathrm{Sec}(Y/K)$ is $G$-equivariant. Therefore, if $s_{X/K}(x) = s_{X/K}(y)$ then $s_{Y/K}(x') = s_{Y/K}(g(y'))$ for some $g \in G$. For abelian varieties over $K$ the section map is injective, so $x' = g(y')$ in this case, which implies that $x = y$. This is a contradiction, therefore $s_{X/K}(x)$ is different from $s_{X/K}(y)$. $\qquad\square$

We continue to use the notation which we introduced above. Let $D$ denote the quotient of $E'$ by the fixed-point-free involution $\sigma$, and let $g : X \to D$ be the quotient map.

**Proposition 9.3.** *Assume that $K$ is a finite extension of $\mathbb{Q}_p$. Then the map $g^*:$ $H^2(D, \mathbb{G}_m) \to H^2(X, \mathbb{G}_m)$ induced by $g$ has finite cokernel.*

*Proof.* For every variety $Y$ over $K$, the Hochschild–Serre spectral sequence

$$E_{p,q}^2 = H^p(K, H^q(\overline{Y}, \mathbb{G}_m)) \Rightarrow H^{p+q}(Y, \mathbb{G}_m)$$

furnishes on $H^2(Y, \mathbb{G}_m)$ a natural filtration

$$0 = E_3^2 \subseteq E_2^2 \subseteq E_1^2 \subseteq E_0^2 = H^2(Y, \mathbb{G}_m) \tag{9.3.1}$$

such that

$$E_{p,2-p}^\infty \cong E_p^2/E_{p+1}^2, \quad p = 0, 1, 2.$$

The members $E_2^2$ and $E_1^2$ are usually denoted by $\mathrm{Br}_0(Y)$ and $\mathrm{Br}_1(Y)$, respectively. Because $H^3(K, \mathbb{G}_m) = 0$ (see [Serre 1997, Proposition 15, p. 93]), the coboundary map

$$d_{1,1}^2 : E_{1,1}^2 = H^1(K, H^1(\overline{Y}, \mathbb{G}_m)) \to H^3(K, \mathbb{G}_m)$$

is zero and therefore

$$E_{1,1}^\infty = E_{1,1}^3 = \mathrm{Ker}(d_{1,1}^2) = H^1(K, H^1(\overline{Y}, \mathbb{G}_m)).$$

In short, we have a natural exact sequence

$$0 \longrightarrow \mathrm{Br}_0(Y) \longrightarrow \mathrm{Br}_1(Y) \longrightarrow H^1(K, H^1(\overline{Y}, \mathbb{G}_m)) \longrightarrow 0.$$

The group $\mathrm{Br}_0(Y)$ is the image of the natural map $\mathrm{Br}(K) \to \mathrm{Br}(Y)$, therefore the map $g^* : \mathrm{Br}_0(D) \to \mathrm{Br}_0(X)$ is surjective. The Hochschild–Serre spectral sequence furnishes a natural injection

$$\mathrm{Br}(Y)/\mathrm{Br}_1(Y) \to \mathrm{Br}(\overline{Y}).$$

Since $\mathrm{Br}(\overline{X})$ is dual to the torsion subgroup of the Néron–Severi group $\mathrm{NS}(\overline{X})$ in our case (see [Skorobogatov 1999, p. 403]), which is finite, it will be enough to show that the map

$$g_* : H^1(K, H^1(\overline{D}, \mathbb{G}_m)) \to H^1(K, H^1(\overline{X}, \mathbb{G}_m))$$

induced by $g$ has finite cokernel. Since $g$ is the Albanese map for $X$ (see [loc. cit.]), the map $\mathrm{Pic}^0(\overline{D}) \to \mathrm{Pic}^0(\overline{X})$ induced by $g$ is an isomorphism. Therefore, by looking at the cohomological long exact sequence associated to the short exact sequence of $\Gamma_K$-modules

$$0 \longrightarrow \mathrm{Pic}^0(\overline{X})(\overline{K}) \longrightarrow H^1(\overline{X}, \mathbb{G}_m) \longrightarrow \mathrm{NS}(\overline{X}) \longrightarrow 0,$$

we are reduced to show that $H^1(K, \mathrm{NS}(\overline{X}))$ is finite. The abelian group $\mathrm{NS}(\overline{X})$ is finitely generated, so there is a finite Galois extension $L|K$ such that the action

of $\mathrm{Gal}(\overline{L}|L)$ on $\mathrm{NS}(\overline{X})$ is trivial. The abelianisation of $\mathrm{Gal}(\overline{L}|L)$ is isomorphic to the profinite completion of $L^*$, so it is topologically finitely generated. Therefore

$$H^1(L, \mathrm{NS}(\overline{X})) \cong \mathrm{Hom}(\mathrm{Gal}(\overline{L}|L), \mathrm{NS}(\overline{X}))$$

is finite. Therefore the inflation map

$$H^1(\mathrm{Gal}(L|K), \mathrm{NS}(\overline{X})) \to H^1(K, \mathrm{NS}(\overline{X}))$$

has finite cokernel. Since $\mathrm{Gal}(L|K)$ is finite we get that $H^1(K, \mathrm{NS}(\overline{X}))$ is finite. $\square$

Let $X$ be a smooth variety over a field $K$, and let $b \in H^2(X, \mathbb{G}_m)$. We say that $x, y \in X(K)$ are $b$-equivalent if $x^*(b) = y^*(b)$. This defines an equivalence relation of $X(K)$, which we will call $b$-equivalence.

**Proposition 9.4.** *Assume that $K$ is a finite extension of $\mathbb{Q}_p$. Then $b$-equivalence classes are open in the $p$-adic topology.*

*Proof.* Let $x \in X(K)$. It will be enough to show that $x$ has a $p$-adically open neighbourhood $U$ in $X(K)$ such that $x$ and $y$ is $b$-equivalent for every $y \in U$. We may assume that $X$ is affine by taking a Zariski-open neighbourhood of $x$. By a theorem of Gabber (see [de Jong 2013, Theorem 1.1]), there is an Azumaya algebra $A$ of some rank $n$ on $X$ which represents $b$. Let $\pi : Y \to X$ be the $\mathrm{PGL}_n$-torsor corresponding to $A$, that is, the torsor whose class in $H^1(X, \mathrm{PGL}_n)$ is the same as the class of $A$. Let $\sigma \in H^1(K, \mathrm{PGL}_n)$ be the class of the fibre of $Y$ over $x$ and let $\pi^\sigma : Y^\sigma \to X$ be the twist of $\pi : Y \to X$ by $\sigma$. Then the fibre of $Y^\sigma$ over $x$ is a trivial $\mathrm{PGL}_n$-torsor. Therefore there is an $x' \in Y^\sigma(K)$ such that $\pi^\sigma(x') = x$. Because $\pi^\sigma$ is a submersion, there is an $p$-adically open neighbourhood $U$ of $x$ in $X(K)$ and a $p$-adically analytical section $s : U \to Y^\sigma(K)$ of $\pi^\sigma$ with $s(x) = x'$. Therefore for every $y \in U$ the fibre of $Y^\sigma$ over $y$ has a $K$-rational point, so it is a trivial $\mathrm{PGL}_n$-torsor. The claim is now clear. $\square$

**Theorem 9.5.** *Let $K$ be as above, and let $X$ be a bielliptic surface over $K$. Then étale-Brauer equivalence is strictly finer than Brauer equivalence on $X(K)$.*

Of course our choice of example is motivated by the classical paper [Skorobogatov 1999], and the result above can be considered its natural local counterpart. (Also compare with [Harari 2000], which uses similar ideas.)

*Proof.* Let $X$ be a bielliptic surface over $K$, and let $g : X \to D$ be the map introduced above. By Proposition 9.2 the étale-Brauer equivalence-classes of $X(K)$ consists of points, so it will be enough to show that the Brauer equivalence-classes of $X(K)$ are infinite. Let $r \in D(K)$. Note that every $x, y \in g^{-1}(r)(K)$ are $b$-equivalent for every $b \in H^2(X, \mathbb{G}_m)$ in the image of the map $g^* : H^2(D, \mathbb{G}_m) \to H^2(X, \mathbb{G}_m)$. Therefore finitely many Brauer equivalence classes intersect $g^{-1}(r)(K)$ by Propositions 9.3

and 9.4. By the inverse function theorem both $D$ and $g^{-1}(r)$ have infinitely many $K$-valued points, so the claim holds. $\qquad\square$

In the rest of this section we study the somewhat independent question of the surjectivity of $\iota_{X/K}$ over $p$-adic fields.

**Notation 9.6.** Let Groups denote the category of groups. Let $\mathbb{N}_d$ denote the category whose objects are positive integers and whose morphisms are the following: for every pair of objects $m, n \in \mathrm{ob}(\mathbb{N}_d)$ the set of morphisms from $m$ to $n$ consists of the ordered pair $\phi_{m,n} = (m, n)$ if $n \mid m$, and otherwise is empty. For every abelian group $A$ and natural number $n$ let $A[n]$ denote the subgroup of $n$-torsion elements of $A$ and let $A^{\mathrm{tor}}$ denote the pro-group $A^{\mathrm{tor}} : \mathbb{N}_d \to$ Groups given by the rule

$$A^{\mathrm{tor}}(n) = A[n]$$

such that, for every pair of positive integers $m, n$ such that $n \mid m$, the homomorphism

$$A^{\mathrm{tor}}(\phi_{m,n}) : A[m] \to A[n]$$

is the multiplication by $m/n$ map.

**Proposition 9.7.** *Let $K$ be a finite extension of $\mathbb{Q}_p$. The following holds*:

(a) *For every smooth, geometrically connected projective curve $X$ of genus at least two over $K$ the map $\iota_{X/K}$ is injective, and it is surjective if the local version of Grothendieck's section conjecture holds for $X$.*

(b) *For every abelian variety $X$ over $K$ the map $\iota_{X/K}$ is injective, and it is surjective if and only if $X$ is zero-dimensional.*

*Proof.* First assume that $X$ is a smooth, geometrically connected projective curve of genus at least two over $K$. Recall that the local version of Grothendieck's section conjecture claims that the map $s_{X/K}$ is a bijection. We also know that in this case $s_{X/K}$ is injective. Therefore claim (a) follows at once from Corollary 5.5 and Lemma 7.7. Assume now that $X$ is an abelian variety over $K$. By Corollary 5.5 and Lemma 7.7 the map $j_{X/K}$ is a bijection. Moreover, there is a natural bijection

$$\mathrm{Sec}(X/K) \cong H^1\left(K, \prod_{l \text{ is prime}} T_l(X)\right), \qquad (9.7.1)$$

where $T_l(X)$ denotes the $l$-th Tate module of $X$, and under this identification $s_{X/K}$ corresponds to the coboundary map furnished by Kummer theory. In particular $\iota_{X/K}$ is injective, and the cokernel of $s_{X/K}$ is $H^1(K, X)^{\mathrm{tor}}$. By [Milne 1986, Corollary 3.4, p. 53] the groups $H^1(K, X)$ and $X^\vee(K)^\wedge$ are isomorphic, where the latter is the Pontryagin dual of the compact group $X^\vee(K)$ of $K$-valued points of the dual $X^\vee$ of $X$. Let $\mathbb{O}_K$ denote the valuation ring of $K$. The profinite group $X^\vee(K)$ is the direct sum of a finite group and $\dim(X^\vee) = \dim(X)$ copies of $\mathbb{O}_K$

by the inverse function theorem. Therefore the group $H^1(K, X)^{\text{tor}}$ is zero if and only if $X$ is zero-dimensional. Hence claim (b) is true.     $\square$

## 10. Étale-Brauer equivalence versus $H$-equivalence

**Definition 10.1.** Assume now that $K$ is a $p$-adic field, and for every $n \in \mathbb{N}$ let $c_{K,n} : H^2(K, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$ be the isomorphism furnished by local Tate duality. For every geometrically irreducible variety $X$ defined over $K$ let

$$\{\cdot, \cdot\}_n : H^2(K, H_2(\overline{X}, \mathbb{Z}/n\mathbb{Z})) \times H^0(K, H^2(\overline{X}, \mu_n)) \to \mathbb{Z}/n\mathbb{Z}$$

be the bilinear pairing given by the rule

$$\{x, y\}_n = c_{K,n}(x \cup y)$$

(for every $x \in H^2(K, H_2(\overline{X}, \mathbb{Z}/n\mathbb{Z}))$, $y \in H^0(K, H^2(\overline{X}, \mu_n)))$, where the cup product

$$\cup : H^2(K, H_2(\overline{X}, \mathbb{Z}/n\mathbb{Z})) \times H^0(K, H^2(\overline{X}, \mu_n)) \to H^2(K, \mu_n)$$

is induced by the evaluation pairing

$$H_2(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \times H^2(\overline{X}, \mu_n) \to \mu_n.$$

The following lemma is immediate from the functoriality of the constructions involved:

**Lemma 10.2.** *Let $f : X \to Y$ be a morphism of geometrically irreducible varieties over $K$. Then the diagram*

$$
\begin{array}{ccc}
H^2(K, H_2(\overline{X}, \mathbb{Z}/n\mathbb{Z})) \times H^0(K, H^2(\overline{X}, \mu_n)) & \xrightarrow{\{\cdot, \cdot\}_n} & \mathbb{Z}/n\mathbb{Z} \\
H_2(f)_* \downarrow \qquad\qquad (f^*)_* \uparrow & & \| \\
H^2(K, H_2(\overline{Y}, \mathbb{Z}/n\mathbb{Z})) \times H^0(K, H^2(\overline{Y}, \mu_n)) & \xrightarrow{\{\cdot, \cdot\}_n} & \mathbb{Z}/n\mathbb{Z}
\end{array}
\qquad (10.2.1)
$$

*commutes.*     $\square$

**Notation 10.3.** Let $X$ be a geometrically irreducible variety over $K$ with $X(K) \neq \varnothing$. For every positive integer $n$ and every $x, y \in X(hK)$ such that $x \sim_1 y$, let $\delta_1^X(x, y)_n$ denote the image of the obstruction class $\delta_1^X(x, y)$ under the composition of the natural map

$$H^2(K, \pi_2(\overline{X})) \to H^2(K, \pi_2(\overline{X})/n\pi_2(\overline{X}))$$

and the homomorphism

$$H_{*,n} : H^2(K, \pi_2(\overline{X})/n\pi_2(\overline{X})) \to H^2(K, H_2(\overline{X}, \mathbb{Z}/n\mathbb{Z}))$$

induced by the Hurewicz map

$$H_n : \pi_2(\overline{X})/n\pi_2(\overline{X}) \to H_2(\overline{X}, \mathbb{Z}/n\mathbb{Z}).$$

Let

$$\alpha_n : H^2(X, \mu_n) \to H^2(X, \mathbb{G}_m)$$

denote the map induced by the inclusion $\mu_n \to \mathbb{G}_m$, and finally let

$$\phi_n : H^2(X, \mu_n) \to H^0(K, H^2(\overline{X}, \mu_n))$$

be the map induced by base change.

**Lemma 10.4.** *Let $X$ be a geometrically irreducible smooth quasiprojective variety over $K$. For every $n \in \mathbb{N}$, every $c \in H^2(X, \mu_n)$ and every $x, y \in X(K)$ such that $\alpha_n(c) = 0$ and $\iota_{X/K}(x) \sim_1 \iota_{X/K}(y)$, we have*

$$\{\delta_1^X(\iota_{X/K}(x), \iota_{X/K}(y))_n, \phi_n(c)\}_n = 0.$$

*Proof.* We will need to introduce analogues of the concepts in [Definition 10.1](#) and [Notation 10.3](#) for hypercoverings. For every geometrically irreducible variety $Z$ defined over $K$ and every étale hypercovering $H_*$ of $Z$ let

$$\{\cdot, \cdot\}_n : H^2(K, H_2(\pi_{0/K}(H_*), \mathbb{Z}/n\mathbb{Z})) \times H^0(K, H^2(\pi_{0/K}(H_*), \mu_n)) \to \mathbb{Z}/n\mathbb{Z}$$

also denote the bilinear pairing given by the rule

$$\{a, b\}_n = c_{K,n}(a \cup b)$$

(for every $a \in H^2(K, H_2(\pi_{0/K}(H_*), \mathbb{Z}/n\mathbb{Z})), b \in H^0(K, H^2(\pi_{0/K}(H_*), \mu_n)))$, where the cup product

$$\cup : H^2(K, H_2(\pi_{0/K}(H_*), \mathbb{Z}/n\mathbb{Z})) \times H^0(K, H^2(\pi_{0/K}(H_*), \mu_n)) \to H^2(K, \mu_n)$$

is induced by the evaluation pairing

$$H_2(\pi_{0/K}(H_*), \mathbb{Z}/n\mathbb{Z}) \times H^2(\pi_{0/K}(H_*), \mu_n) \to \mu_n.$$

Assume now that $Z(K) \neq \varnothing$ and pick a point $z \in Z(K)$. In [Definition 3.5](#) we introduced a pointed simplicial $\Gamma_K$-set $\pi_{0/K}(H_*, z)$ such that there is a natural map $\pi_{0/K}(H_*) \to \pi_{0/K}(H_*, z)$ which is a weak equivalence. Let $a, b \in \pi_0(\pi_{0/K}(H_*)^{\Gamma_K})$ be such that their image is the same under the map

$$\pi_0(\pi_{0/K}(H_*)^{\Gamma_K}) \to \pi_0(P_1(\pi_{0/K}(H_*))^{\Gamma_K})$$

induced by the truncation morphism $\pi_{0/K}(H_*) \to P_1(\pi_{0/K}(H_*))$. By Theorem B on p. 189 of [Goerss 1995] there is an obstruction class $\delta_1^{H_*}(a, b) \in H^2\big(K, \pi_2(\pi_{0/K}(H_*, z))\big)$. For every positive integer $n$ and $a, b$ as above let $\delta_1^{H_*}(a, b)_n$ denote the image of $\delta_1^{H_*}(a, b)$ under the composition of the natural map

$$H^2\big(K, \pi_2(\pi_{0/K}(H_*, z))\big) \to H^2\big(K, \pi_2(\pi_{0/K}(H_*, z))/n\pi_2(\pi_{0/K}(H_*, z))\big),$$

the homomorphism

$$H^2\big(K, \pi_2(\pi_{0/K}(H_*, z))/n\pi_2(\pi_{0/K}(H_*, z))\big) \to H^2(K, H_2(\pi_{0/K}(H_*, z), \mathbb{Z}/n\mathbb{Z}))$$

induced by the Hurewicz map

$$\pi_{0/K}(H_*, z))/n\pi_2(\pi_{0/K}(H_*, z)) \to H_2(\pi_{0/K}(H_*, z), \mathbb{Z}/n\mathbb{Z}),$$

and the inverse of the isomorphism

$$H^2(K, H_2(\pi_{0/K}(H_*), \mathbb{Z}/n\mathbb{Z})) \to H^2(K, H_2(\pi_{0/K}(H_*, z), \mathbb{Z}/n\mathbb{Z}))$$

induced by the weak equivalence $\pi_{0/K}(H_*) \to \pi_{0/K}(H_*, z)$. Finally, for every $a \in X(hK)$ let $a^{H_*} \in \pi_0(\pi_{0/K}(H_*)^{\Gamma_K})$ denote its image under the canonical map $X(hK) \to \pi_0(\pi_{0/K}(H_*)^{\Gamma_K})$.

Now let us start the proof in earnest. Because $\alpha_n(c) = 0$ there is a line bundle $\mathcal{L}$ on $X$ such that the image of its isomorphism class under the coboundary map $\mathrm{Pic}(X) = H^1(X, \mathbb{G}_m) \to H^2(X, \mu_n)$ is $c$. Let $\pi : Y \to X$ denote the total space of $\mathcal{L}$ with the zero section removed; then $Y$ is a $\mathbb{G}_m$-torsor over $X$ whose class in $H^1(X, \mathbb{G}_m)$ is the same as the class of $\mathcal{L}$. It will be enough to show that

$$\big\{\delta_1^{H_*}\big(\iota_{X/K}(x)^{H_*}, \iota_{X/K}(y)^{H_*}\big)_n, d\big\}_n$$

is zero for every étale hypercovering $H_*$ of $X$ and every

$$d \in H^0(K, H^2(\pi_{0/K}(H_*), \mu_n))$$

whose image is $\phi_n(c)$ with respect to the homomorphism

$$H^0(K, H^2(\pi_{0/K}(H_*), \mu_n)) \to H^0(K, H^2(\overline{X}, \mu_n))$$

induced by the pullback map $H^2(\pi_{0/K}(H_*), \mu_n) \to H^2(\overline{X}, \mu_n)$.

Fix such a hypercovering $H_*$, and let $\pi^*(H_*)$ denote the pullback of $H_*$ onto $Y$ with respect to $\pi$. Note that $\phi_n(\pi^*(c)) = 0$; in fact even $\pi^*(c) = 0$. Indeed, the latter follows as the pullback of the torsor $\pi : Y \to X$ onto $Y$ with respect to $\pi$ is trivial: the diagonal $Y \to Y \times_X Y$ is a section. Let

$$d' \in H^0(K, H^2(\pi_{0/K}(\pi^*(H_*)), \mu_n))$$

be the image of $d$ with respect to the homomorphism

$$H^0(K, H^2(\pi_{0/K}(H_*), \mu_n)) \to H^0(K, H^2(\pi_{0/K}(\pi^*(H_*)), \mu_n))$$

induced by $\pi$. As the image of $d'$ is $\phi_n(\pi^*(c))$ under the pullback

$$H^0(K, H^2(\pi_{0/K}(\pi^*(H_*)), \mu_n)) \to H^0(K, H^2(\overline{Y}, \mu_n))$$

by naturality, we get that there is a morphism $f : I_* \to \pi^*(H_*)$ of étale hypercoverings of $Y$ such that $f^*(d') \in H^0(K, H^2(\pi_{0/K}(I_*), \mu_n))$ is zero.

Let $z \in Y(K)$ be arbitrary. (There are such points; for example, the fibre above $x$ contains $K$-rational points.) Let $\overline{z} \in Y(\overline{K})$ denote the geometric point lying above $z$, corresponding to the choice of algebraic closure $\overline{K} \supset K$. Note that the image of the canonical map from $\pi_1(\overline{Y}, \overline{z}) \to \pi_{0/K}(I_*, z)$, which is well-defined by [Proposition 3.10](), is finite. Therefore by [Remark 6.9]() there are two points $x'$ and $y'$ in $Y(K)$ whose images under $\pi$ are $x$ and $y$, respectively, such that $\iota_{X/K}(x')^{I_*}, \iota_{X/K}(y')^{I_*} \in \pi_0(\pi_{0/K}(I_*)^{\Gamma_K})$ have the same image under the map

$$\pi_0(\pi_{0/K}(I_*)^{\Gamma_K}) \to \pi_0(P_1(\pi_{0/K}(I_*))^{\Gamma_K})$$

induced by truncation. So by the above the obstruction class

$$\delta_1^{I_*}(\iota_{Y/K}(x')^{I_*}, \iota_{Y/K}(y')^{I_*}) \in H^2\big(K, \pi_2(\pi_{0/K}(I_*, z))\big)$$

is well-defined. By naturality of obstruction classes it will be enough to show that

$$\{\delta_1^{I_*}(\iota_{Y/K}(x')^{I_*}, \iota_{Y/K}(y')^{I_*})_n, f^*(d')\}_n = 0.$$

But this is clear since $f^*(d')$ is zero.                                                                    □

**Lemma 10.5.** *Let $X$ be a geometrically irreducible smooth quasiprojective variety over $K$. For every $n \in \mathbb{N}$, every $c \in H^2(X, \mu_n)$ and every $x, y \in X(K)$ such that $\iota_{X/K}(x) \sim_1 \iota_{X/K}(y)$, we have*

$$(\iota_{X/K}(x), c)_n = (\iota_{X/K}(y), c)_n \implies \{\delta_1^X(\iota_{X/K}(x), \iota_{X/K}(y))_n, \phi_n(c)\}_n = 0.$$

*Proof.* By a theorem of Gabber (see Theorem 1.1 of [[de Jong 2013]]()) there is an Azumaya algebra $A$ on $X$ which represents $\alpha_n(c) \in H^2(X, \mathbb{G}_m)$. Without loss of generality we may assume that $A$ has rank $n$ by enlarging $n$ if it is necessary, since for every pair of positive integers $n \mid m$ the map $H^2(K, \mu_n) \to H^2(K, \mu_m)$ induced by the inclusion map $\mu_n \subseteq \mu_m$ is injective. Let $\pi : Y \to X$ be the $\mathrm{PGL}_n$-torsor corresponding to $A$, that is, the torsor whose class in $H^1(X, \mathrm{PGL}_n)$ is the same as the class of $A$. Let $\sigma \in H^1(K, \mathrm{PGL}_n)$ be the class of the fibre of $Y$ over $x$ and let $\pi^\sigma : Y^\sigma \to X$ be the twist of $\pi : Y \to X$ by $\sigma$. Then the fibre of $Y^\sigma$ over $x$ is a trivial $\mathrm{PGL}_n$-torsor. Because $(x, c)_n = (y, c)_n$ and the natural map $H^1(K, \mathrm{PGL}_n) \to H^2(K, \mu_n)$ is injective, we get that the fibre of $Y^\sigma$ over $y$ is

also a trivial $\mathrm{PGL}_n$-torsor. So there are points $x'$ and $y'$ in $Y^\sigma(K)$ whose images under $\pi^\sigma$ are $x$ and $y$, respectively. By Remark 6.9 we may even assume that $\iota_{Y^\sigma/K}(x') \sim_1 \iota_{Y^\sigma/K}(y')$. So by Lemma 10.2 and the naturality of obstruction classes it will be enough to show that

$$\{\delta_1^{Y^\sigma}(\iota_{Y^\sigma/K}(x'), \iota_{Y^\sigma/K}(y'))_n, \phi_n((\pi^\sigma)^*(c))\}_n = 0.$$

In order to do so, it will be enough to show that

$$\alpha_n((\pi^\sigma)^*(c)) = (\pi^\sigma)^*(\alpha_n(c)) = 0$$

by Lemma 10.4. But this is clear since the pullback of the torsor $\pi : Y^\sigma \to X$ onto $Y^\sigma$ with respect to $\pi^\sigma$ is trivial: the diagonal $Y^\sigma \to Y^\sigma \times_X Y^\sigma$ is a section.     □

**Notation 10.6.** For every quasiprojective variety $Y$ over $K$ consider the Hochschild–Serre spectral sequence

$$E_{p,q}^2 = H^p(K, H^q(\overline{Y}, \mu_n)) \Rightarrow H^{p+q}(Y, \mu_n).$$

Because $H^3(K, \mu_n) = 0$ (for example by local Tate duality), the coboundary map

$$d_{0,2}^3 : E_{0,2}^3 \to E_{3,0}^3 \subseteq H^3(K, \mu_n)$$

is zero and therefore

$$E_{0,2}^\infty = E_{0,2}^3 = \mathrm{Ker}(d_{0,2}^2).$$

Therefore the spectral sequence furnishes an exact sequence

$$H^2(Y, \mu_n) \xrightarrow{\phi_n} H^0(K, H^2(\overline{Y}, \mu_n)) \xrightarrow{d_n} H^2(K, H^1(\overline{Y}, \mu_n)),$$

which is functorial. Here $d_n$ is the coboundary map $d_{0,2}^2 : E_{0,2}^2 \to E_{2,1}^2$ and $\phi_n$ is the quotient map $H^2(Y, \mu_n) \to E_{0,2}^\infty \cong \mathrm{Ker}(d_{0,2}^2)$ by the highest step in the filtration on $H^2(Y, \mu_n)$ induced by the spectral sequence.

**Lemma 10.7.** *Let $X$ be a geometrically irreducible smooth quasiprojective variety over $K$ and let $x, y \in X(K)$ be étale-Brauer equivalent. Then there is a connected finite étale cover $f : Y \to X$ such that:*

(i) *There are $\tilde{x}, \tilde{y} \in Y(K)$ such that $f(\tilde{x}) = x$, $f(\tilde{y}) = y$, and $\tilde{x}, \tilde{y}$ are Brauer-equivalent.*

(ii) *For every $c \in H^0(K, H^2(\overline{X}, \mu_n))$ we have $d_n(f^*(c)) = 0$.*

*Proof.* Let $\eta$ be a $\overline{K}$-valued point of $X$. Let $\mathcal{H}$ be the intersection of the kernels of all continuous homomorphisms $\pi_1(\overline{X}, \eta) \to \mathbb{Z}/n\mathbb{Z}$. Because $\pi_1(\overline{X}, \eta)$ is topologically finitely generated, the subgroup $\mathcal{H}$ is open and characteristic. Let $\mathcal{H}' \subseteq \pi_1(X, \eta)$ be the subgroup generated by $\mathcal{H}$ and by the image of an element of $s_{X/K}(x)$; this is an open subgroup. Let $f : Y \to X$ be the connected finite étale cover such that the image of $\pi_1(Y)$ with respect to $\pi_1(f)$ is $\mathcal{H}'$. By construction there is an $\tilde{x} \in Y(K)$

such that $f(\tilde{x}) = x$. Because $x$ and $y \in X(K)$ are étale-Brauer equivalent, there is a $\tilde{y} \in Y(K)$ such that $f(\tilde{y}) = y$, and $\tilde{x}, \tilde{y}$ are Brauer-equivalent. Recall that by the universal coefficient theorem

$$0 \longrightarrow \mathrm{Ext}^1(H_n(V, \widehat{\mathbb{Z}}), \mathbb{Z}/n\mathbb{Z}) \longrightarrow H^n(V, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{e_n} \mathrm{Hom}(H_n(V, \widehat{\mathbb{Z}}), \mathbb{Z}/n\mathbb{Z}) \longrightarrow 0$$

for every variety $V$ over $\overline{K}$, where the map $e_n$ is induced by the evaluation pairing and the Ext groups are for the category of pro-groups. In particular there is a natural isomorphism $H^1(V, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Hom}(H_1(V, \widehat{\mathbb{Z}}), \mathbb{Z}/n\mathbb{Z})$. Therefore the pullback map $f^* : H^1(\overline{X}, \mathbb{Z}/n\mathbb{Z}) \to H^1(\overline{Y}, \mathbb{Z}/n\mathbb{Z})$ is zero. Because the map $d_n$ is functorial, the claim now follows.                                                                                                 $\square$

Assume that $X(K) \neq \varnothing$. For every $x, y \in X(hK)$ such that $x \sim_1 y$, let $\epsilon_1^X(x, y)$ denote the image of the obstruction class $\delta_1^X(x, y)$ under the homomorphism

$$H_* : H^2(K, \pi_2(\overline{X})) \to H^2(K, H_2(\overline{X}, \widehat{\mathbb{Z}}))$$

induced by the Hurewicz map

$$H : \pi_2(\overline{X}) \to H_2(\overline{X}, \widehat{\mathbb{Z}}).$$

**Proposition 10.8.** *Let $X$ be a geometrically irreducible smooth quasiprojective variety over $K$, and let $x, y \in X(K)$ be étale-Brauer equivalent. Then we have*

$$\epsilon_1^X(\iota_{X/K}(x), \iota_{X/K}(y)) = 0.$$

Note that the claim is meaningful because $\iota_{X/K}(x) \sim_1 \iota_{X/K}(y)$ by Lemma 8.7.

*Proof.* It will be enough to show that

$$\delta_1^X(\iota_{X/K}(x), \iota_{X/K}(y))_n = 0$$

for every $n \in \mathbb{N}$. By the universal coefficient theorem quoted above and by local Tate duality, every element of $H^2(K, H_2(\overline{X}, \mathbb{Z}/n\mathbb{Z}))$ annihilated by the pairing $\{\cdot, \cdot\}_n$ must be zero. So it will be enough to show that

$$\{\delta_1^X(\iota_{X/K}(x), \iota_{X/K}(y))_n, c\}_n = 0$$

for every $c \in H^0(K, H^2(\overline{X}, \mu_n))$. Let $f : Y \to X$ and $\tilde{x}, \tilde{y} \in Y(K)$ be as in Lemma 10.7. By Lemma 10.2 and the naturality of obstruction classes it will be enough to show that

$$\{\delta_1^Y(\iota_{Y/K}(x'), \iota_{Y/K}(y'))_n, f^*(c)\}_n = 0$$

for every $c \in H^0(K, H^2(\overline{X}, \mu_n))$. Because $d_n(f^*(c)) = 0$, this claim follows from Lemmas 8.6 and 10.5.                                                                                 $\square$

The following result is Theorem 1.1 of the introduction:

**Theorem 10.9.** *Let $K$ be a finite extension of $\mathbb{Q}_p$, and let $X$ be a smooth quasiprojective variety over $K$. Then étale-Brauer equivalence and $H$-equivalence coincide on $X(K)$.*

*Proof.* Let $x, y \in X(K)$ be étale-Brauer equivalent. We need to show that they are $H$-equivalent. We may assume without loss of generality that $X$ is geometrically irreducible. We already noted that $\iota_{X/K}(x) \sim_1 \iota_{X/K}(y)$. Also note that by Theorem 7.6 it will be enough to show that $\delta_1^X(\iota_{X/K}(x), \iota_{X/K}(y)) = 0$ since the cohomological dimension of $K$ is 2, so in this case the obstruction classes $\delta_n^X(\iota_{X/K}(x), \iota_{X/K}(y))$ will be zero for every $n \geq 2$, too.

Fix an element $s$ of $s_{X/K}(x)$. For every open characteristic subgroup $\mathcal{H}$ of $\pi_1(\overline{X}, \eta)$ let $\mathcal{H}' \subseteq \pi_1(X, \eta)$ be the subgroup generated by $\mathcal{H}$ and the image of $s$; this is an open subgroup. Moreover, for every such $\mathcal{H}$ let $f_{\mathcal{H}} : Y_{\mathcal{H}} \to X$ be the connected finite étale cover such that the image of $\pi_1(Y)$ with respect to $\pi_1(f_{\mathcal{H}})$ is $\mathcal{H}'$. By construction there is an $x_{\mathcal{H}} \in Y_{\mathcal{H}}(K)$ such that $f_{\mathcal{H}}(x_{\mathcal{H}}) = x$. Because $x$ and $y$ are étale-Brauer equivalent, there is a $y_{\mathcal{H}} \in Y_{\mathcal{H}}(K)$ such that $f_{\mathcal{H}}(y_{\mathcal{H}}) = y$, and $x_{\mathcal{H}}, y_{\mathcal{H}}$ are étale-Brauer equivalent. By the naturality of the obstruction classes, the cohomology classes

$$\epsilon_1^{Y_{\mathcal{H}}}(x_{\mathcal{H}}, y_{\mathcal{H}}) \in H^2(K, H_2(\overline{Y}_{\mathcal{H}}, \widehat{\mathbb{Z}}))$$

furnish an element of $\lim_{(Y,y') \in \mathrm{Fet}(\overline{X}, x')} H^2(K, H_2(Y, \widehat{\mathbb{Z}}))$, where $x'$ is a $\overline{K}$-valued point of $X$, which is the image of $\delta_1^X(x, y)$ with respect to the map

$$H^2(K, \pi_2(\overline{X})) \cong H^2\left(K, \lim_{(Y,y') \in \mathrm{Fet}(\overline{X}, x')} H_2(Y, \widehat{\mathbb{Z}})\right)$$
$$\longrightarrow \lim_{(Y,y') \in \mathrm{Fet}(\overline{X}, x')} H^2(K, H_2(Y, \widehat{\mathbb{Z}}))$$

furnished by the map $b_X \circ a_X$, which is an isomorphism by Theorem 4.3. By Proposition 10.8 the classes $\epsilon_1^{Y_{\mathcal{H}}}(x_{\mathcal{H}}, y_{\mathcal{H}})$ are zero, and hence the theorem holds. $\square$

## 11. $H$-equivalence over the field of real numbers

**Definition 11.1.** Let $X$ be any scheme. Recall that a quadratic space over $X$ is a vector bundle $\mathcal{E}$ over $X$, that is, a locally free $\mathcal{O}_X$-module of finite rank, and an isomorphism $h : \mathcal{E} \to \mathcal{E}^*$, where $\mathcal{E}^*$ denotes the dual of $\mathcal{E}$, which is symmetric, that is, the composition

$$\mathcal{E} \longrightarrow \mathcal{E}^{**} \xrightarrow{h^*} \mathcal{E}^*$$

is equal to $h$, where the first map is the natural isomorphism of $\mathcal{E}$ with its bidual, and $h^*$ is the dual of $h$. In the special case when $X = \mathrm{Spec}(K)$, for $K$ a field, this concept is the same as the concept of a nondegenerate quadratic form over $K$.

**Definition 11.2.** Consider the case when $K = \mathbb{R}$. By Sylvester's theorem every nondegenerate quadratic form $q$ over $\mathbb{R}$ is isomorphic to a diagonal form

$$\langle \underbrace{1, 1, \ldots, 1}_{m}, \underbrace{-1, \ldots, -1}_{n} \rangle,$$

and the ordered pair $(m, n)$ only depends on the isomorphism class of $q$. Let $\rho(q) = m + n$ denote the rank of $q$ and let $\sigma(q) = m - n$ denote the signature of $q$, respectively. By the above, nondegenerate forms over $\mathbb{R}$ are classified by their rank and signature. Let $X$ be a smooth variety over $\mathbb{R}$, let $U \subset X(\mathbb{R})$ be a connected component, and let $q = (\mathscr{E}, h)$ be a quadratic space over $X$. Then for every $x \in U$ the pullback $x^*(q)$ has the same signature, which we will call the signature of $q$ on $U$.

**Theorem 11.3.** *Let $X$ be a smooth variety over $\mathbb{R}$ which is either affine or projective. Let $U, V \subset X(\mathbb{R})$ be two different connected components. Then there is a quadratic space $q = (\mathscr{E}, h)$ over $X$ such that the signature of $q$ on $U$ is zero and the signature of $q$ on $V$ is nonzero.*

*Proof.* See Theorem 1.1.1 in [Mahé 1982] for the affine case, and the main result of [Houdebine and Mahé 1982] for the projective case. □

**Definition 11.4.** Let $K$ be a field whose characteristic is not 2. Every nondegenerate quadratic form $q$ is isomorphic to a diagonal form

$$\langle a_1, a_2, \ldots, a_n \rangle.$$

The Stiefel–Whitney classes of the form $q$ above are defined as the cup product (see [Milnor 1970])

$$w(q) = 1 + w_1(q) + \cdots + w_n(q) = (1 + \delta(a_1))(1 + \delta(a_2)) \cdots (1 + \delta(a_n)),$$

where $w_i(q) \in H^i(K, \mathbb{Z}/2\mathbb{Z})$ and

$$\delta : K^* \to H^1(K, \mathbb{Z}/2\mathbb{Z})$$

is the boundary map of the Kummer exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{G}_m \xrightarrow{x \mapsto x^2} \mathbb{G}_m \longrightarrow 0.$$

The Stiefel–Whitney classes are independent of the diagonalisation of $q$.

**Remark 11.5.** Assume again that $K$ is the field of real numbers. Then as a graded algebra

$$H^*(\mathbb{R}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{F}_2[t],$$

where $t$ is the generator of the group $H^1(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ of order two. Let $q$ be a nondegenerate quadratic form $q$ over $\mathbb{R}$ isomorphic to a diagonal form

$$\langle \underbrace{1, 1, \ldots, 1}_{m}, \underbrace{-1, \ldots, -1}_{n} \rangle;$$

then by the above

$$w(q) = (1+t)^n = 1 + nt + \cdots + t^n.$$

Esnault, Kahn and Viehweg [Esnault et al. 1993] constructed Stiefel–Whitney classes for any quadratic space $q = (\mathscr{E}, h)$ over a $\mathbb{Z}[1/2]$-scheme $X$, which lives in mod-2 étale cohomology

$$w_i(q) \in H^i(X, \mathbb{Z}/2\mathbb{Z}),$$

is functorial over the category of $\mathbb{Z}[1/2]$-schemes, and specialises to the construction above when $X$ is the spectrum of a field. We will use these classes to separate connected components of real points of varieties defined over $\mathbb{R}$.

**Proposition 11.6.** *Let $X$ be a smooth variety over $\mathbb{R}$ which is either affine or projective. Let $U, V \subset X(\mathbb{R})$ be two different connected components. Then there is a natural number $i$ and a cohomology class $c \in H^i(X, \mathbb{Z}/2\mathbb{Z})$ over $X$ such that for every $x \in U$ the pullback $x^*(c) \in H^i(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ is zero and for every $x \in V$ the pullback $x^*(c) \in H^i(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ is nonzero.*

*Proof.* By Theorem 2.9 and Proposition 2.10, there is a $c \in H^0(X, \mathbb{Z}/2\mathbb{Z})$ such that $x^*(c) \in H^0(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ is zero for every $x \in U$ and $y^*(c) \in H^0(\mathbb{R}, \mathbb{Z}/2\mathbb{Z})$ is nonzero for every $y \in V$ if $U$ and $V$ lie on two different connected components of $X(\mathbb{C})$. Therefore we may assume that $X$ is geometrically connected without loss of generality. Let $q = (\mathscr{E}, h)$ be a quadratic space over $X$ such that the signature of $q$ on $U$ is zero and the signature of $q$ on $V$ is nonzero. We may assume that the signature of $q$ on $V$ is negative by taking $(\mathscr{E}, -h)$ instead, if necessary. Because $X$ is connected, the rank of the vector bundle $\mathscr{E}$ is constant on $X$. This rank is even, say $2m$, because the signature of $q$ on $U$ is zero. Then the signature of $q$ on $V$ is $2m-2n$, where $n$ is a positive integer bigger than $m$. By Remark 11.5 above, we have $x^*(w_n(q)) = w_n(x^*(q)) = 0$ for every $x \in U$ and $x^*(w_n(q)) = w_n(x^*(q)) = t^n$ for every $x \in V$. The claim follows. $\qquad\square$

**Definition 11.7.** For every morphism of sites $m : \mathscr{C} \to \mathscr{C}'$, let $m^* : \mathscr{C}' \to \mathscr{C}$ denote the functor underlying $m$. Let $\mathscr{C}$ be a Grothendieck site. A left action $\alpha$ of a group $\Gamma$ on $\mathscr{C}$ is a morphism $\alpha(g) : \mathscr{C} \to \mathscr{C}$ of sites for each $g \in \Gamma$ such that $\alpha(1)$ is the identity map of $\mathscr{C}$ and $\alpha(gh) = \alpha(g) \circ \alpha(h)$ for every $g, h \in \Gamma$. When $\Gamma$ is profinite, we say that the action $\alpha$ is continuous if for every morphism $h : U \to V$ in $\mathscr{C}$ there is an open subgroup $\Delta$ of $\Gamma$ such that $\alpha(g)^*(U) = U, \alpha(g)^*(V) = V$

and $\alpha(g)^*(h) = h$ for every $g \in \Delta$. Assume now that $\Gamma$ is a profinite group. By a $\Gamma$-site $(\mathscr{C}, \alpha)$ we mean a Grothendieck site $\mathscr{C}$ with a continuous left action $\alpha$ of $\Gamma$ on $\mathscr{C}$. As usual we will drop $\alpha$ from the notation whenever this is convenient.

**Example 11.8.** A basic example of a $\Gamma$-site is the Grothendieck site $\Gamma$-Sets, where the coverings are surjective maps, equipped with the left action $\alpha$ such that $\alpha(g)^*(U) = U$ for every object $U$ of $\Gamma$-Sets and every $g \in \Gamma$, and for every morphism $h : U \to V$ and $g \in \Gamma$ the map $\alpha(g)^*(h) : U \to V$ is given by the rule $x \mapsto gh(x)$. By a slight abuse of notation we will let $\Gamma$-Sets denote this $\Gamma$-site, too.

**Definition 11.9.** Let $(\mathscr{C}, \alpha)$ be a $\Gamma$-site. A $\Gamma$-invariant object of $(\mathscr{C}, \alpha)$ is an object $U$ of $\mathscr{C}$ such that $\alpha(g)^*(U) = U$ for every $g \in \Gamma$. A $\Gamma$-equivariant morphism of $(\mathscr{C}, \alpha)$ is a morphism $h : U \to V$ of $\mathscr{C}$ such that $U$ and $V$ are $\Gamma$-invariant objects and $h \circ \alpha(g)^*(\mathrm{id}_U) = \alpha(g)^*(\mathrm{id}_V) \circ h$ for every $g \in \Gamma$. Let $\mathscr{C}^\Gamma$ denote the category whose objects are $\Gamma$-invariant objects of $\mathscr{C}$ and whose morphisms are $\Gamma$-equivariant maps between these. Since the composition of $\Gamma$-equivariant morphisms are $\Gamma$-equivariant, with these morphisms $\mathscr{C}^\Gamma$ is indeed a category. Let $T$ denote the Grothendieck topology of the site $\mathscr{C}$; that is, for every object $U$ of $\mathscr{C}$ let $T(U)$ denote the collection of covering sieves of $U$. We say that a sieve $S$ on $U \in \mathrm{ob}(\mathscr{C}^\Gamma)$ is $\Gamma$-invariant if for every $V \in \mathrm{ob}(\mathscr{C})$ and every $h \in S(V)$ there is a $W \in \mathrm{ob}(\mathscr{C}^\Gamma)$, a morphism $h' \in S(W)$ which is $\Gamma$-equivariant, and a morphism $h'' : V \to W$ such that $h = h' \circ h''$. For every $\Gamma$-invariant $S$ as above, let $S^\Gamma$ denote the sieve on $U$ in the category $\mathscr{C}^\Gamma$ given by the rule

$$S^\Gamma(V) = S(V) \cap \mathrm{Hom}_{\mathscr{C}^\Gamma}(V, U).$$

For every $U \in \mathrm{ob}(\mathscr{C}^\Gamma)$ let $T^\Gamma(U)$ denote the following collection of sieves $S$ in the category $\mathscr{C}^\Gamma$:

$$T^\Gamma(U) = \{S^\Gamma \mid S \text{ is in } T(U) \text{ and is } \Gamma\text{-invariant}\}.$$

**Example 11.10.** Let $K$ be a field and let $\Gamma_K = \mathrm{Gal}(\overline{K}|K)$ denote the absolute Galois group of $K$ as above. Let $X$ be a locally Noetherian scheme over $K$ and let $\mathscr{C}$ denote the small étale site of the base change of $X$ to $\overline{K}$. Then $\mathscr{C}$ is naturally equipped with the structure of a $\Gamma_K$-site, induced by the action of $\Gamma_K$ on $\overline{K}$. By étale descent the category $\mathscr{C}^{\Gamma_K}$ is equivalent to the small étale category of $X$ and $T^{\Gamma_K}$ is the étale topology of $X$ on it. In particular $T^{\Gamma_K}$ is a Grothendieck topology.

**Definition 11.11.** Assume now that $\mathscr{C}$ satisfies the conditions in Chapters 8 and 9 of [Artin and Mazur 1969] and $T^\Gamma$ is a Grothendieck topology on $\mathscr{C}^\Gamma$. In particular we suppose that $\mathscr{C}$ is closed under finite coproducts and that it is locally connected in the sense of [Artin and Mazur 1969, Sections 9.1–9.2, pp. 111–112]. Let $U$ be a $\Gamma$-invariant object of $(\mathscr{C}, \alpha)$. Then $\Gamma$ acts on the set $\pi_0(U)$ of connected components of $U$ and this action makes $\pi_0(U)$ into a $\Gamma$-set. Let $\pi_\Gamma(U)$ denote

this $\Gamma$-set. We say that a simplicial object of $\mathscr{C}$ is $\Gamma$-invariant if it is a simplicial object of $\mathscr{C}^\Gamma$. If $X_*$ is a $\Gamma$-invariant simplicial object of $(\mathscr{C}, \alpha)$, then the face and degeneracy maps of $X_*$ induce $\Gamma$-equivariant maps between the $\Gamma$-set $\pi_\Gamma(X_n)$ which makes the collection $\{\pi_\Gamma(X_n)\}_{n=1}^\infty$ into an object of $\Gamma$-SSets, which we will denote by $\pi_\Gamma(X_*)$. Since the site $(\mathscr{C}^\Gamma, T^\Gamma)$ inherits the good properties of the site $\mathscr{C}$, we may apply [Artin and Mazur 1969, Corollary 8.13(i), p. 105] to conclude that the functor

$$X_* \mapsto \pi_\Gamma(X_*)$$

above induces an object of Pro-Ho($\Gamma$-SSets). We will call the later the $\Gamma$-equivariant homotopy type of $\mathscr{C}$ and denote it by $\Pi(\mathscr{C})$.

**Remark 11.12.** Let $K, \Gamma_K$ and $X$ be as in Example 11.10, and let $\mathscr{C}$ denote the small étale site of the base change of $X$ to $\bar{K}$, as above. Then the $\Gamma_K$-equivariant homotopy type of $\mathscr{C}$ is just the relative étale homotopy type $\mathrm{Et}_{/K}(X)$ of $X$ as defined by Harpaz and Schlank.

**Example 11.13.** Assume now that $\Gamma$ is a finite group and let $X$ be a locally connected, Hausdorff, paracompact topological space equipped with a continuous left $\Gamma$-action. Let $\mathscr{C}$ be the ordinary site on the coproducts of open subsets of $X$. Then $\mathscr{C}$ is naturally equipped with the structure of a $\Gamma$-site, induced by the action of $\Gamma$ on $X$. Moreover, the quotient $\Gamma \backslash X$ of $X$ by the action of $\Gamma$ is also a locally connected, Hausdorff, paracompact topological space, and the category $\mathscr{C}^\Gamma$ is the category of coproducts of open subsets of $\Gamma \backslash X$ and $T^\Gamma$ is the ordinary site of $\Gamma \backslash X$ on it. In particular $T^\Gamma$ is a Grothendieck topology.

**Definition 11.14.** Let $\Gamma$ and $X$ be as above. We say that $X$ is $\Gamma$-contractible if there is a subgroup $\Delta \leq \Gamma$ such that the quotient $\Gamma/\Delta$, equipped with the discrete topology and the natural left $\Gamma$-action, is $\Gamma$-equivariantly homotopy-equivalent to $X$. Assume now that every open subset of $X$ is paracompact, and that $X$ is locally $\Gamma$-equivariantly contractible; that is, for every (finite) orbit $O \in X$ and every $\Gamma$-invariant open $U \subseteq X$ containing $O$ there an open $\Gamma$-invariant and $\Gamma$-contractible $V \subseteq X$ such that $V \subseteq U$ and $O \subseteq V$.

Note that when $\Gamma$ is finite $\Gamma$-SSets is just the usual category of simplicial sets with a $\Gamma$-action, and moreover Goerss' notion of homotopy fixed point spaces coincides with the usual one. Also note that the singular complex $S_* X$ of $X$ is equipped with an action of $\Gamma$ which makes it an object of $\Gamma$-SSets.
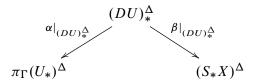
**Theorem 11.15.** *Let $U_*$ be a $\Gamma$-invariant hypercovering of $\mathscr{C}$ such that every $\Gamma$-orbit of connected components of every $U_n$ is $\Gamma$-contractible. Then the simplicial $\Gamma$-set $\pi_\Gamma(U_*)$ is isomorphic to the simplicial $\Gamma$-set $S_* X$ in* Ho($\Gamma$-SSets).

*Proof.* Let $S_* U_n$ denote the singular complex of $U_n$. Then $S_* U_*$ is a bisimplicial object in $\Gamma$-Sets. We denote by $(DU)_*$ its diagonal simplicial $\Gamma$-set $(DU)_n = S_n U_n$. Then we have obvious maps of simplicial $\Gamma$-sets
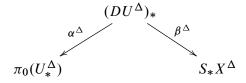
$$
\begin{array}{ccc}
 & (DU)_* & \\
{\scriptstyle \alpha} \swarrow & & \searrow {\scriptstyle \beta} \\
\pi_\Gamma(U_*) & & S_* X
\end{array}
$$

and we claim that these two maps are homotopy equivalences in $\mathrm{Ho}(\Gamma\text{-SSets})$, which will prove the theorem.

For every subgroup $\Delta \leq \Gamma$ and every $\Gamma$-set $Y$, let $Y^\Delta$ denote the subset of $Y$ fixed by $\Delta$. Similarly, for every simplicial $\Gamma$-set $Y_*$ let $Y_*^\Delta$ denote the simplicial set $\{Y_n^\Delta\}_{n=1}^\infty$ such that the face and degeneracy maps are the restrictions of such maps of the simplicial set $Y_*$. By the definition of the strict model structure we need to show that the maps

$$
\begin{array}{ccc}
 & (DU)_*^\Delta & \\
{\scriptstyle \alpha|_{(DU)_*^\Delta}} \swarrow & & \searrow {\scriptstyle \beta|_{(DU)_*^\Delta}} \\
\pi_\Gamma(U_*)^\Delta & & (S_* X)^\Delta
\end{array}
$$

of simplicial sets are homotopy equivalences in $\mathrm{Ho(SSets)}$ for every $\Delta$ as above.

Note that for every $\Gamma$-invariant hypercovering $V_*$ of $X$ the simplicial object $V_*^\Gamma$ in the category of disjoint unions of open sets of the closed subspace $X^\Delta$ of $X$ is a hypercovering, too, since it is the pullback of the hypercovering $V_*$ onto $X^\Delta$ with respect to the inclusion map. Because the $U_n$ are $\Gamma$-contractible we have $\pi_\Gamma(U_*)^\Delta = \pi_0(U_*^\Delta)$. Moreover, every $\Delta$-invariant singular simplex of $X$ must lie in $X^\Delta$, so $(S_* X)^\Delta$ is the singular complex $S_* X^\Delta$ of $X^\Delta$. Similarly let $S_* U_n^\Delta$ denote the singular complex of $U_n^\Delta$. Then $S_* U_*^\Delta$ is a bisimplicial set. We denote by $(DU^\Delta)_*$ its diagonal simplicial set $(DU^\Delta)_n = S_n U_n^\Delta$. Then we have $(DU)_*^\Delta = (DU^\Delta)_*$, so we only need to show that the analogues

$$
\begin{array}{ccc}
 & (DU^\Delta)_* & \\
{\scriptstyle \alpha^\Delta} \swarrow & & \searrow {\scriptstyle \beta^\Delta} \\
\pi_0(U_*^\Delta) & & S_* X^\Delta
\end{array}
$$

of the maps $\alpha$ and $\beta$ for the topological space $X^\Delta$ are homotopy equivalences in $\mathrm{Ho(SSets)}$ for every $\Delta$ as above. Since every $\Gamma$-orbit of connected components of every $U_n$ is $\Gamma$-contractible, the connected components of $U_n^\Delta$ are contractible. Similarly, $X^\Delta$ is locally contractible, since $X$ is locally $\Gamma$-contractible. Because

$X^\Delta$ is a closed subspace of a paracompact topological space, it is also paracompact. The claim now follows from [Artin and Mazur 1969, Theorem 12.1, p. 129]. □

**Corollary 11.16.** *Let $\mathscr{C}$ be the ordinary site on the coproducts of open subsets of a Hausdorff topological space $X$ equipped with a continuous left $\Gamma$-action. Assume that every open subset of $X$ is paracompact and that $X$ is locally $\Gamma$-equivariantly contractible. Then the pro-object $\Pi(\mathscr{C})$ is canonically isomorphic to the element $S_* X$ in* Pro-Ho($\Gamma$-SSets).

*Proof.* Because those $\Gamma$-invariant hypercoverings $U_*$ of $\mathscr{C}$ such that every connected component of every $U_n$ is $\Gamma$-contractible are cofinal by assumption, the claim follows immediately. □

**Definition 11.17.** Let again $\Gamma$ be an arbitrary profinite group. A morphism $m : (\mathscr{C}, \alpha) \to (\mathscr{C}', \alpha')$ of $\Gamma$-sites is a morphism of sites $m : \mathscr{C} \to \mathscr{C}'$ such that for every $g \in \Gamma$ and every morphism $h : U \to V$ of $\mathscr{C}'$ we have

$$\alpha(g)^*(m^*(U)) = m(\alpha'(g)^*(U)),$$
$$\alpha(g)^*(m^*(V)) = m^*(\alpha'(g)^*(V)),$$
$$\alpha(g)^*(m^*(h)) = m^*(\alpha'(g)^*(h)).$$

For every such $m$, the underlying functor $m^*$ carries $\Gamma$-invariant hypercoverings to $\Gamma$-invariant hypercoverings, and so it furnishes a map

$$\Pi(m) : \Pi(\mathscr{C}) \to \Pi(\mathscr{C}')$$

in Pro-Ho($\Gamma$-SSets) (when these are defined). The map $\Pi(m)$ in turn induces a map

$$\Pi(m)(E\Gamma) : \Pi(\mathscr{C})(E\Gamma) \to \Pi(\mathscr{C}')(E\Gamma)$$

of homotopy fixed points.

**Definition 11.18.** A $\Gamma$-invariant point (or more conveniently $\Gamma$-point) of a $\Gamma$-site $(\mathscr{C}, \alpha)$ is a morphism $p : \Gamma\text{-Sets} \to \mathscr{C}^\Gamma$ of $\Gamma$-sites. Note that the composition of $p^*$ and the functor $\Gamma\text{-Sets} \to$ Sets forgetting the $\Gamma$-action is a point of the site $\mathscr{C}$ in the sense of [Artin and Mazur 1969, Sections 8.3-8.4, pp. 95–96], which perhaps justifies our terminology. We will let $\mathscr{C}(\Gamma)$ denote the set of $\Gamma$-points of $(\mathscr{C}, \alpha)$. Note that the homotopy type $\Pi(\Gamma\text{-Sets})$ is contractible (this is clear from Corollary 11.16, too), so the set $\Pi(\Gamma\text{-Sets})(E\Gamma)$ has one element. For every $p \in \mathscr{C}(\Gamma)$ let $\iota_\mathscr{C}(p) \in \Pi(\mathscr{C})(E\Gamma)$ denote the image of $\Pi(\Gamma\text{-Sets})(E\Gamma)$ with respect to $\Pi(p)$. Clearly

$$\iota_\mathscr{C} : \mathscr{C}(\Gamma) \to \Pi(\Gamma\text{-Sets})(E\Gamma)$$

is a natural transformation.

**Example 11.19.** Let $K$, $\Gamma_K = \mathrm{Gal}(\overline{K}|K)$, $X$ and $\mathscr{C}$ be as in Example 11.10. Since the small étale site of $\mathrm{Spec}(\overline{K})$ is isomorphic to $\Gamma_K$-Sets as a $\Gamma_K$-site, every $K$-valued point of $X$ supplies a $\Gamma_K$-point of the site $\mathscr{C}$. Therefore the map $\iota_{\mathscr{C}}$ introduced above is a generalisation of the map $\iota_{X/K}$. Similarly, if $\Gamma$ is a finite group, $X$ is a locally connected, Hausdorff, paracompact topological space equipped with a continuous left $\Gamma$-action and $\mathscr{C}$ is as in Example 11.13, then every point of $X$ fixed by $\Gamma$ furnishes a $\Gamma$-point of the site $\mathscr{C}$, and hence the restriction of $\iota_{\mathscr{C}}$ onto $X^{\Gamma}$ is a map $X^{\Gamma} \to \Pi(\mathscr{C})(E\Gamma)$.

**Proposition 11.20.** *Let $K$ be the field $\mathbb{R}$ of real numbers, and let $X$ be a variety over $K$. Then two $K$-rational points of $X$ are $H$-equivalent if they are in the same connected component of the topological space $X(K)$.*

*Proof.* Let $\Gamma = \Gamma_K = \mathrm{Gal}(\overline{K}|K)$ be the group of two elements, and let $\mathscr{C}'$ denote the small étale site of the base change of $X$ to $\overline{K}$. Moreover, let $\mathscr{C}$ be the ordinary site on the coproducts of open subsets of $X(\overline{K})$ with respect to its usual topology. In addition to these $\Gamma$-sites we also introduce the $\Gamma$-site $\mathscr{C}''$ whose objects are topological spaces $X'$ lying over the topological space of $X(\overline{K})$ such that the map $X' \to X$ is a local isomorphism, i.e., such that every point $x \in X'$ has a neighbourhood which is isomorphic onto its image. Since any étale map of schemes $X' \to X$ over $\overline{K}$ is a local isomorphism on the underlying topological spaces, and since an open set is in $\mathscr{C}''$, we have morphisms of $\Gamma$-sites

$$\begin{array}{ccc} & \mathscr{C}'' & \\ {}^{m}\swarrow & & \searrow{}^{m'} \\ \mathscr{C} & & \mathscr{C}' \end{array}$$

Now it is clear from the definition of local isomorphisms that every $\Gamma$-invariant hypercovering of $\mathscr{C}''$ is dominated by a $\Gamma$-invariant hypercovering of $\mathscr{C}$. Thus the map $\Pi(m): \Pi(\mathscr{C}'') \to \Pi(\mathscr{C})$ is a homotopy equivalence in Pro-Ho($\Gamma$-SSets), and so

$$\Pi(m)(E\Gamma): \Pi(\mathscr{C}'')(E\Gamma) \to \Pi(\mathscr{C})(E\Gamma)$$

is a bijection. Therefore by the naturality of the maps $\iota_{\mathscr{C}}$, $\iota_{\mathscr{C}'}$ and $\iota_{\mathscr{C}''}$ it will be enough to show that for every pair $x$, $y$ of $K$-rational points of $X$ lying in the same connected component of the topological space $X(K)$ we have $\iota_{\mathscr{C}}(x) = \iota_{\mathscr{C}}(y)$. Let $f : [0,1] \to X(K)$ be a continuous path connecting $x$ with $y$; that is, we have $f(0) = x$ and $f(1) = y$. By naturality again it will be enough to show that $\iota_{\mathscr{D}}(0) = \iota_{\mathscr{D}}(1)$, where $\mathscr{D}$ is the ordinary site on the coproducts of open subsets of $[0,1]$ with respect to its usual topology, equipped with the trivial $\Gamma$-action. But the interval $[0,1]$ is contractible to a point $\Gamma$-equivariantly, so $\Pi(\mathscr{D})(E\Gamma)$ is a one-element set by Corollary 11.16. $\quad\square$

**Remark 11.21.** It is not difficult to push the arguments of this proof a little bit further to prove an equivariant version of [Artin and Mazur 1969, Theorem 12.9, p. 142], using an equivariant analogue of the profinite completion functor, but we will not pursue this further since it would take us too far away from our original project. However in a forthcoming publication we will in fact prove such a claim in a much more general context.

The following result is Theorem 1.2 of the introduction:

**Theorem 11.22.** *Let $K$ be the field $\mathbb{R}$ of real numbers, and let $X$ be a smooth affine or projective variety over $K$. Then two $K$-rational points of $X$ are $H$-equivalent if and only if they are in the same connected component of the topological space $X(K)$.*

*Proof.* By Lemma 8.5 and Proposition 11.6, two $H$-equivalent real points of $X$ must be in the same component. On the other hand, by Proposition 11.20 two real points of $X$ in the same connected component must be $H$-equivalent. □

## 12. The homotopy section principle

**Notation 12.1.** Recall that for every field of characteristic zero the topological $\mathrm{Gal}(\overline{K}|K)$-module $\widehat{\mathbb{Z}}(1)$ is defined as the projective limit $\varprojlim_{n\in\mathbb{N}} \mu_n$, where the directed set structure on $\mathbb{N}$ is furnished by divisibility and for every $m, n \in \mathbb{N}$ such that $m \mid n$ the transition map $\mu_n \to \mu_m$ is multiplication by $n/m$. For every number field $K$ let $|K|$ denote the set of places of $K$, and for every $v \in |K|$ let $K_v$ denote the completion of $K$ with respect to $v$. For every $v \in |K|$ fix an embedding $j_v : \overline{K} \to \overline{K}_v$ of $K$-extensions. For every $k \in \mathbb{N}$ and every discrete $\mathrm{Gal}(\overline{K}|K)$-module $M$ let $\mathrm{III}^k(K, M)$ denote the subgroup

$$\mathrm{III}^k(K, M) = \mathrm{Ker}\left( \prod_{v\in|K|} j_{v*} : H^k(K, M) \to \prod_{v\in|K|} H^k(K_v, M) \right)$$

of $H^k(K, M)$, where $j_{v*}$ denotes the restriction map induced by $j_v$ for every $v \in |K|$. For every topological $\mathrm{Gal}(\overline{K}|K)$-module $M$ which is a projective limit $\varprojlim_{i\in I} M_i$ of discrete $\mathrm{Gal}(\overline{K}|K)$-modules let $\mathrm{III}^k(K, M)$ denote $\varprojlim_{i\in I} \mathrm{III}^k(K, M_i)$.

**Lemma 12.2.** *We have $\mathrm{III}^2(K, \widehat{\mathbb{Z}}(1)) = 0$ for every number field $K$.*

*Proof.* By [Milne 1986, Theorem 4.10(a), p. 70] the group $\mathrm{III}^2(K, \widehat{\mathbb{Z}}(1))$ is zero if and only if $\mathrm{III}^1(K, \mathbb{Q}/\mathbb{Z})$ is, where we equip $\mathbb{Q}/\mathbb{Z}$ with the discrete topology and the trivial $\mathrm{Gal}(\overline{K}|K)$-action. We may identify $\mathrm{III}^1(K, \mathbb{Q}/\mathbb{Z})$ with the kernel of the map

$$\mathrm{Hom}(\Gamma_K, \mathbb{Q}/\mathbb{Z}) \to \prod_{v\in|K|} \mathrm{Hom}(\Gamma_{K_v}, \mathbb{Q}/\mathbb{Z})$$

furnished by restriction onto the family of subgroups $\Gamma_{K_v}$ of $\Gamma_K$ for all $v \in |K|$. The claim now follows from the Chebotarev density theorem. □

**Proposition 12.3.** *Assume that $K$ is a number field and $X(K)$ is nonempty. For every $x, y \in X(hK)$ such that $x \sim_2 y$ we have*

$$x = y \iff r_{v*}(x) = r_{v*}(y) \quad (v \in |K|, v \text{ is real}).$$

*Proof.* The product of the restriction maps

$$\prod_{\substack{v \in |K| \\ v \text{ is real}}} j_{v*} : H^n(K, M) \to \prod_{\substack{v \in |K| \\ v \text{ is real}}} H^n(K_v, M)$$

is injective for every integer $n \geq 3$ and every discrete Galois module $M$ over $K$ by [Milne 1986, Theorem 4.10(c), p. 70]. Hence for every $n \geq 2$ and every pair of sections $x, y \in X(hK)$ such that $x \sim_n y$ we get that

$$\delta_n^X(x, y) = 0 \iff \delta_n^{X_v}(r_{v*}(x), r_{v*}(y)) = 0 \quad (v \in |K|, v \text{ is real})$$

from naturality of the obstruction classes. The claim now follows from Theorem 7.6.
                                                                                  □

**Definition 12.4.** Let $K$ be for a moment any field of characteristic zero. Recall that two points $x, y \in X(K)$ are called directly $R$-equivalent if there is a rational map $f : \mathbb{P}_K^1 \dashrightarrow X$ of $K$-varieties such that $f(0) = x$ and $f(\infty) = y$. The $R$-equivalence on $X(K)$ is the equivalence relation generated by direct $R$-equivalence. Let $X(K)/R$ denote the equivalence classes of this relation. Note that $A$-equivalence coincides with $R$-equivalence when $X$ is projective by the valuative criterion of properness. In this case let

$$\iota_{X/K,R} : X(K)/R \to X(hK)$$

be the map furnished by Proposition 7.8.

**Notation 12.5.** It is particularly interesting to study $X(K)/R$ through the map $\iota_{X/K,R}$ when $K$ is a number field. For every variety $X$ defined over $K$ and every $v \in |K|$ let $X_v$ denote the base change of $X$ to $\mathrm{Spec}(K_v)$. For every $v \in |K|$ the embedding $j_v$ furnishes a map

$$r_{v*} : X(hK) \to X_v(hK_v)$$

by functoriality. Let $\mathbb{A}_K = \prod'_{v \in |K|} K_v$ denote the ring of adèles of $K$. Let $X(h\mathbb{A}_K)$ denote the image of $X(\mathbb{A}_K)$ with respect to the map

$$\prod_{v \in |K|} \iota_{X_v/K_v} : X_v(K_v) \to \prod_{v \in |K|} X_v(hK_v).$$

We define the set $\mathrm{Sel}(X/K)$ of Selmer homotopy fixed points of $X$ to be

$$\mathrm{Sel}(X/K) = \left( \prod_{v \in |K|} r_{v*} \right)^{-1} (X(h\mathbb{A}_k)) \subseteq X(hK).$$

We are interested in the following natural generalisation of the Shafarevich–Tate conjecture:

**Homotopy section principle** (HSP). *Assume that $X$ is smooth and projective. Then the map*

$$X(K)/R \xrightarrow{\iota_{X/K,R}} \mathrm{Sel}(X/K) \qquad (12.5.1)$$

*is injective and its image is dense with respect to the pro-discrete topology of $\mathrm{Sel}(X/K)$.*

The claim above is obviously true if $X$ does not have local points everywhere, and hence HSP should be considered as a new form of the local-global principle. The next proposition shows that HSP is indeed a generalisation of standard conjectures of this sort:

**Proposition 12.6.** *Let $K$ be a number field.*

(a) *HSP holds for Brauer–Severi varieties and for nonsingular quadratic hyper-surfaces $H \subset \mathbb{P}^n_K$ of positive dimension.*

(b) *Let $X$ be a smooth, geometrically connected projective curve $X$ of genus at least two over $K$. Then HSP holds for $X$ if and only if a weak (local-global) form of Grothendieck's section conjecture holds for $X$.*

(c) *Let $X$ be an abelian variety over $K$. Then HSP holds for $X$ if and only if the Shafarevich–Tate conjecture holds for $X$.*

*Proof.* First assume that $X$ is either a Brauer–Severi variety or a nonsingular quadratic hypersurface of positive dimension. When $\mathrm{Sel}(X/K)$ is empty there is nothing to prove. Assume now that $\mathrm{Sel}(X/K)$ is nonempty: then $X(\mathbb{A}_K)$ is nonempty, too. Because the local-global principle holds for $X$ we get that $X(K)$ is also nonempty. In this case $X(K)/R$ consists of one element, and hence it will be enough to show that $\mathrm{Sel}(X/K)$ also has one element. Let $x, y \in \mathrm{Sel}(X/K)$. Because for every $v \in |K|$ the set $X_v(K_v)/R$ has one element, we get that $r_{v*}(x) = r_{v*}(y)$ for every such $v$.

By [Artin and Mazur 1969, Corollary 12.13, p. 144] we know that $X_{\overline{K}}$ is weakly homotopy equivalent to $X_{\mathbb{C}}$. Because $\pi_1(X(\mathbb{C})) = \{1\}$ (either because $X_{\mathbb{C}}$ is isomorphic to $\mathbb{P}^n_{\mathbb{C}}$ for $n = \dim(X)$ or by the Lefschetz hyperplane section theorem), we get $\pi_1(X_{\overline{K}}) = \{1\}$ by Theorem 2.9. Therefore $x \sim_1 y$. Moreover,

$$\pi_2(X_{\overline{K}}) = H_2(X_{\overline{K}}, \widehat{\mathbb{Z}}) = \mathrm{Hom}(H^2(X_{\overline{K}}, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$$

by [Artin and Mazur 1969, Corollary 6.2, p. 70]. When $X$ is a Brauer–Severi variety, since it has a rational point it is isomorphic to $\mathbb{P}^n_K$. Therefore $H^2(X_{\overline{K}}, \hat{\mathbb{Z}}) \cong \hat{\mathbb{Z}}(-1)$, and hence $\pi_2(X_{\overline{K}}) \cong \hat{\mathbb{Z}}(1)$. Because $r_{v*}(x) = r_{v*}(y)$ for every $v \in |K|$, we have $\delta_2^X(x, y) \in \text{III}^2(K, \pi_2(X_{\overline{K}}))$, so this obstruction class vanishes by Lemma 12.2. So we get that $x \sim_2 y$.

When $X$ is a quadratic hypersurface of dimension at least 3, its embedding $X \to \mathbb{P}^{n+1}_K$ as such a hypersurface induces an isomorphism

$$H^2(X_{\overline{K}}, \hat{\mathbb{Z}}) \cong H^2(\mathbb{P}^{n+1}_{\overline{K}}, \hat{\mathbb{Z}}) \cong \hat{\mathbb{Z}}(-1)$$

by the Lefschetz hyperplane section theorem, and hence we may conclude as above that $x \sim_2 y$. The only remaining case is of a quadratic surface. In this case either

$$H^2(X_{\overline{K}}, \hat{\mathbb{Z}}) \cong \hat{\mathbb{Z}}(-1) \oplus \hat{\mathbb{Z}}(-1),$$

when both pencils of lines on $X$ are defined over $K$, or it is the induction of $\hat{\mathbb{Z}}(-1)$ from a quadratic extension of $K$. Clearly in the first case the group $\text{III}^2(K, \pi_2(X_{\overline{K}}))$ still vanishes by Lemma 12.2, while in the second case this claim follows from Shapiro's lemma and Lemma 12.2. Again we get that $x \sim_2 y$. Claim (a) now follows from Proposition 12.3.

Assume now that $X$ is a smooth, geometrically connected projective curve of genus at least two over $K$. Then there is a commutative diagram

$$
\begin{array}{ccc}
X(K) & \xrightarrow{\;s_{X/K}\;} & \text{Sec}(X/K) \\
\downarrow & & \downarrow \\
\prod_{v \in |K|} X_v(K_v) & \xrightarrow{\prod_{v \in |K|} s_{X_v/K_v}} & \prod_{v \in |K|} \text{Sec}(X_v/K_v)
\end{array}
$$

where the vertical maps are the obvious maps. The weak local-global form of Grothendieck's section conjecture asserts that the diagram above is cartesian. We also know that in this case $s_{X/K}$ is injective and by Faltings' theorem $X(K)$ is finite. In particular, in this case the map in (12.5.1) has dense image if and only if it is surjective.

Let's assume first that the weak local-global form of Grothendieck's section conjecture holds for $X$, and show that HSP holds for $X$. This is trivial when $\text{Sel}(X/K)$ is empty, so we may assume that $\text{Sel}(X/K)$ is nonempty. Then $\text{Sec}(X/K)$ has an element whose image in $\prod_{v \in |K|} \text{Sec}(X_v/K_v)$ lies in the image of $\prod_{v \in |K|} s_{X_v/K_v}$, and hence $X(K)$ is nonempty, by our assumption. The claim now follows from Corollary 5.5 and Lemma 7.7. Let us prove the converse. We may assume that $\text{Sec}(X/K)$ has an element whose image in $\prod_{v \in |K|} \text{Sec}(X_v/K_v)$ lies in the image of $\prod_{v \in |K|} s_{X_v/K_v}$ without loss of generality. By the main theorem of [Harpaz and

Schlank 2013] for smooth projective varieties, the set $\mathrm{Sel}(X/K)$ is nonempty if and only if the étale-Brauer set of $X$ is. In our case the latter is nonempty (see [Stix 2011]), so we get that $\mathrm{Sel}(X/K)$ is nonempty, too. So by our hypothesis $X(K)$ is nonempty, so the claim follows from Corollary 5.5 and Lemma 7.7. Claim (b) is settled.

Finally, consider the case when $X$ is an abelian variety over $K$. By Corollary 5.5 and Lemma 7.7 the map $j_{X/K}$ is a bijection. Moreover, there is a natural bijection

$$\mathrm{Sec}(X/K) \cong H^1\left(K, \prod_{l \text{ is prime}} T_l(X)\right), \tag{12.6.1}$$

where $T_l(X)$ denotes the $l$-th Tate module of $X$, and under this identification $s_{X/K}$ corresponds to the coboundary map furnished by Kummer theory. In particular $\iota_{X/K}$ is injective.

The image of $\mathrm{Sel}(X/K) \subseteq X(hK)$ with respect to the composition of $j_{X/K}$ and the isomorphism of (12.6.1) is $\mathrm{Sel}(K, X)^{\mathrm{tor}}$, where $\mathrm{Sel}(K, X)$ is the Selmer group of $X$ over $K$. The quotient of $\mathrm{Sel}(K, X)^{\mathrm{tor}}$ by the closure of the image of $X(K)$ under the coboundary map is $\mathrm{III}(K, X)^{\mathrm{tor}}$, where $\mathrm{III}(K, X)$ is the Tate–Shafarevich group of $X$ over $K$. Since the group $\mathrm{III}(K, X)[n]$ is finite for every positive integer $n$, the group $\mathrm{III}(K, X)^{\mathrm{tor}}$ is trivial if and only if $\mathrm{III}(K, X)$ is finite. So claim (c) holds. $\square$

**Remark 12.7.** It is tempting to believe that HSP should hold for every smooth projective variety because of its very general form, but this is not true. The fundamental reason is the Harpaz–Schlank theorem quoted above, which implies that if HSP holds for $X$ then the Brauer–Manin obstruction applied to étale covers is the only obstruction for the Hasse principle. Since now there are many counterexamples to this claim (see [Poonen 2010; Harpaz and Skorobogatov 2014; Colliot-Thélène et al. 2013]) we get that there are two- and three-dimensional counterexamples to HSP. However we can offer some positive results; see Theorems 13.3, 13.7 and 14.8 in the next two sections.

## 13. Geometrically rational and birational surfaces

**Lemma 13.1.** *Let $X$ be a smooth, projective, geometrically rational surface defined over a number field $K$. Then the group $\mathrm{III}^2(K, \pi_2(\overline{X}))$ is finite.*

*Proof.* There is a finite Galois extension $L|K$ such that the action of $\mathrm{Gal}(\overline{L}|L)$ on $\mathrm{Pic}(\overline{X})$ is trivial. Hence $\mathrm{III}^1(L, \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z}) = 0$ by the Chebotarev density theorem (see the proof of Lemma 12.2), where we equip $\mathbb{Q}/\mathbb{Z}$ with the discrete topology and the trivial $\mathrm{Gal}(\overline{K}|K)$-action. As the image of $\mathrm{III}^1(K, \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z})$ lies in $\mathrm{III}^1(L, \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z})$ with respect to the restriction map

$$H^1(K, \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z}) \to H^1(L, \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z}),$$

the group $\mathrm{III}^1(K, \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z})$ lies in the image of the inflation map

$$H^1(\mathrm{Gal}(L|K), \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z}) \to H^1(K, \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z}).$$

Since $H^1(\mathrm{Gal}(L|K), \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z})$ is finite, we get that $\mathrm{III}^1(K, \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z})$ is finite too. We have $\pi_1(\overline{X}) = \pi_1(\mathbb{P}^2_{\overline{K}}) = \{1\}$ because the étale fundamental group is a birational invariant. Hence

$$\pi_2(\overline{X}) = H_2(\overline{X}, \hat{\mathbb{Z}}) = \mathrm{Hom}(H^2(\overline{X}, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$
$$= \mathrm{Hom}(\mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z} \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}(1))$$

by [Artin and Mazur 1969, Corollary 6.2, p. 70] and the fact that for a geometrically rational surface $X$ the étale Chern class map

$$c_1 : \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z} \to H^2(\overline{X}, \mathbb{Q}/\mathbb{Z} \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Z}}(1))$$

is an isomorphism. Hence by [Milne 1986, Theorem 4.10(a), p. 70] there is a perfect duality between $\mathrm{III}^2(K, \pi_2(\overline{X}))$ and $\mathrm{III}^1(K, \mathrm{Pic}(\overline{X}) \otimes \mathbb{Q}/\mathbb{Z})$. The claim is now clear.                                                                                   $\square$

For every $s \in \mathrm{Sel}(X/K)$, let $\mathrm{Sel}_0(s) \subseteq \mathrm{Sel}(X/K)$ denote the preimage of $\prod_{v \in |K|} r_{v*}(s)$ with respect to the map $\prod_{v \in |K|} r_{v*}$.

**Lemma 13.2.** *Let $X$ be a smooth, projective, geometrically rational surface defined over a number field $K$ such that $X(K) \neq \varnothing$, and let $s \in \mathrm{Sel}(X/K)$. Then the cardinality of $\mathrm{Sel}_0(s)$ is at most the order of $\mathrm{III}^2(K, \pi_2(\overline{X}))$.*

*Proof.* By Proposition 12.3 it will be enough to show that the number of equivalence classes of the relation $\sim_2$ in $\mathrm{Sel}_0(s)$ is at most the order of $\mathrm{III}^2(K, \pi_2(\overline{X}))$. For every $x, y \in \mathrm{Sel}_0(s)$ at least we have $x \sim_1 y$ by Theorem 7.6, since $\pi_1(\overline{X}) = \{1\}$, and so $|\mathrm{Sec}(X/K)| = 1$. Moreover, $\delta_1^X(x, y) \in \mathrm{III}^2(K, \pi_2(\overline{X}))$ by the naturality of obstruction classes. But

$$\delta_1^X(x, y) = \delta_1^X(x, s) - \delta_1^X(y, s),$$

so the claim follows from the pigeonhole principle.                               $\square$

**Theorem 13.3.** *Let $X$ be a smooth, projective, geometrically rational surface defined over a number field $K$ such that $X(K) \neq \varnothing$. Then $\mathrm{Sel}(X/K)$ is finite.*

Because we expect that $X(K)/R$ is finite for such an $X$ and $K$, this result should be also expected, assuming that HSP holds for $X$.

*Proof.* Because for every archimedean $v \in |K|$ the topological space $X_v(K_v)$ has only finitely many connected components, the set $X_v(K_v)/H$ is finite for such $v$ by Theorem 1.2. Since $\pi_1(\overline{X}) = \{1\}$ we get that Brauer equivalence and $H$-equivalence coincide on $X_v(K_v)$ for every nonarchimedean $v \in |K|$ by Theorem 1.1. Hence by [Bloch 1981, Corollary A.2, p. 55] the set $X_v(K_v)/H$ is finite for every $v \in |K|$.

Moreover, $X_v(K_v)/H$ has at most one element when $v$ is nonarchimedean and $X$ has good reduction at $v$ by [Bloch 1981, Corollary A.3, p. 55]. Hence we may conclude that the set $\prod_{v \in |K|} X_v(K_v)/H$ is finite. So we only need to show that the map

$$\prod_{v \in |K|} r_{v*} : \mathrm{Sel}(X/K) \to \prod_{v \in |K|} X_v(K_v)/H$$

is finite-to-one. This follows from Lemmas 13.1 and 13.2.     □

**Lemma 13.4.** *Let $X$ and $Y$ be smooth projective surfaces over a field $K$, and let $\pi : X \to Y$ be a composition of monoidal transformations over $K$. Then the map $\pi_* : X(K)/R \to Y(K)/R$ induced by $\pi$ is a bijection.*

*Proof.* We may immediately reduce to the case when $\pi$ is the blow-up of an irreducible subvariety $S \subset Y$ of dimension zero by induction on the number of blow-ups in some sequence of contractions $X \to X_1 \to \cdots \to X_n = Y$ whose composition is $\pi$. If $S$ has no $K$-valued points then $\pi^{-1}(S)$ has no $K$-valued points either and the map $\pi_*$ is obviously a bijection. Otherwise $S$ consists of one $K$-valued point. In this case $\pi^{-1}(S)$ is isomorphic to $\mathbb{P}^1_K$ and the claim is clear.     □

**Lemma 13.5.** *Let $X$ and $Y$ be smooth projective surfaces over the field $\mathbb{R}$, and let $\pi : X \to Y$ be a composition of monoidal transformations over $\mathbb{R}$. Then the map $\pi_* : \pi_0(X(\mathbb{R})) \to \pi_0(Y(\mathbb{R}))$ induced by $\pi$ is a bijection.*

*Proof.* The argument is the same as above.     □

**Proposition 13.6.** *Let $X$ and $Y$ be smooth geometrically irreducible projective surfaces over a number field $K$ and let $\pi : X \to Y$ be a composition of monoidal transformations over $K$. Assume that $Y$ is simply connected and $Y(K) \neq \varnothing$. Then the map $\pi_* : \mathrm{Sel}(X/K) \to \mathrm{Sel}(Y/K)$ induced by $\pi$ is injective.*

*Proof.* Note that $X(K) \neq \varnothing$, since $X$ is birational to $Y$. Again we may assume without loss of generality that $\pi$ is the blow-up of an irreducible subvariety of dimension zero. Let $x, y \in \mathrm{Sel}(X/K)$ be such that $\pi_*(x) = \pi_*(y)$. Because the map induced by $\pi$ between the fundamental groups is an isomorphism, we get that $x \sim_1 y$. Also note that by Theorem 7.6 it will be enough to show that $\delta_1^X(x, y) = 0$, since by Lemma 13.5 and Theorem 1.2 we have $r_{v*}(x) = r_{v*}(y)$ for every real place $v$ of $K$, so the higher obstruction classes $\delta_n^X(x, y)$ will vanish for every $n \geq 2$ by Proposition 12.3.

By the Hurewicz theorem

$$\pi_2(\overline{X}) \cong H_2(\overline{X}, \widehat{\mathbb{Z}}) \cong H_2(\overline{Y}, \widehat{\mathbb{Z}}) \oplus \mathrm{Ker}(H_2(\pi)) \cong \pi_2(\overline{Y}) \oplus \mathrm{Ker}(H_2(\pi)),$$

where

$$H_2(\pi) : H_2(\overline{X}, \widehat{\mathbb{Z}}) \to H_2(\overline{Y}, \widehat{\mathbb{Z}})$$

is induced by $\pi$. Because $\pi$ is the blow-up of a single closed point, the module $\mathrm{Ker}(H_2(\pi))$ is the induction of $\hat{\mathbb{Z}}(1)$ from a finite extension of $K$. Therefore $\mathrm{III}^2(K, \mathrm{Ker}(H_2(\pi)))$ vanishes by Lemma 12.2 and Shapiro's lemma. Therefore $\delta_1^X(x, y)$ is zero by the naturality of obstruction classes. $\square$

**Theorem 13.7.** *Let $\pi : X \to Y$ be a composition of monoidal transformations between geometrically irreducible smooth projective surfaces over $K$. Assume that $\mathrm{Sel}(Y/K)$ is finite, $Y$ is simply connected, the set $Y(K)$ is nonempty, and HSP holds for $Y$ over $K$. Then $\mathrm{Sel}(X/K)$ is finite and HSP holds for $X$ over $K$, too.*

This result can be used to supply many examples of surfaces satisfying HSP, for example blow-ups of Châtelet surfaces; see Theorem 14.8 below.

*Proof.* By Proposition 13.6 the set $\mathrm{Sel}(X/K)$ injects into $\mathrm{Sel}(Y/K)$, so it is finite. Let $x, y \in X(K)$ be $H$-equivalent. Then $\pi(x)$ and $\pi(y)$ are $H$-equivalent elements of $Y(K)$, so they are $R$-equivalent, since HSP holds for $Y$. Therefore $x$ and $y$ are also $R$-equivalent by Lemma 13.4. We get that the map

$$\iota_{X/K} : X(K)/R \to \mathrm{Sel}(X/K) \tag{13.7.1}$$

is injective. Let $s$ be an element of $\mathrm{Sel}(X/K)$. Because $\mathrm{Sel}(Y/K)$ is finite, the topology on it is discrete. Therefore there is a $y \in Y(K)$ such that $\iota_{Y/K}(y) = \pi_*(s)$. Let $x \in X(K)$ be such that $\pi(x) = y$. Then $\pi_*(\iota_{X/K}(x)) = \iota_{Y/K}(y) = \pi_*(s)$, so by Proposition 13.6 we get that $\iota_{X/K}(x) = s$, so the map in (13.7.1) is also surjective. $\square$

## 14. Generalised Châtelet surfaces

**Notation 14.1.** For every torus $S$ defined over a field $K$ of characteristic zero let $C(S)$ denote the $\mathrm{Gal}(\bar{K}|K)$-module of its cocharacters, and for every scheme $X$ over $K$ let

$$\delta : H^1(X, S) \to H^2(X, C(S) \otimes \hat{\mathbb{Z}}(1))$$

be the projective limit of the coboundary maps

$$\delta_n : H^1(X, S) \to H^2(X, S[n])$$

furnished by the corresponding Kummer exact sequences, where $S[n]$ denotes the $n$-torsion subgroup scheme of $S$. Note that, when $K$ is a number field, $\delta$ maps $\mathrm{III}^1(K, S)$ into $\mathrm{III}^2(K, C(S) \otimes \hat{\mathbb{Z}}(1))$. Let

$$\delta_0 : \mathrm{III}^1(K, S) \to \mathrm{III}^2(K, C(S) \otimes \hat{\mathbb{Z}}(1))$$

be the restriction of $\delta$ onto $\mathrm{III}^1(K, S)$.

**Lemma 14.2.** *Let $S$ be a torus defined over a field $K$ of characteristic zero.*

(i) *The map $\delta : H^1(K, S) \to H^2(K, C(S) \otimes \hat{\mathbb{Z}}(1))$ is injective.*

(ii) *The map $\delta_0 : \text{III}^1(K, S) \to \text{III}^2(K, C(S) \otimes \hat{\mathbb{Z}}(1))$ is bijective when $K$ is a number field.*

*Proof.* First note that for every field $K$ of characteristic zero and for every torus $S$ defined over $K$ the cohomology group $H^1(K, S)$ has finite exponent. In fact there is a finite Galois extension $L|K$ such that the action of $\text{Gal}(\bar{L}|L)$ on $C(S)$ is trivial. Hence $H^1(L, S) = 0$ by Hilbert's theorem 90. Therefore $H^1(K, S)$ is the image of $H^1(\text{Gal}(L|K), S)$ under the inflation map. But the group $H^1(\text{Gal}(L|K), S)$ is annihilated by the order of $\text{Gal}(L|K)$. The first claim also follows.

Now we prove that $\delta_0$ is also surjective when $K$ is a number field. Let $L|K$ be a finite Galois extension of the type above and assume that the degree of this extension is $m$. By the Albert–Brauer–Hasse–Noether theorem the group $\text{III}^2(L, S)$ is trivial, and hence $\text{III}^2(K, S)$ is annihilated by multiplication by $m$, since it is a subquotient of $H^2(\text{Gal}(L|K), S)$. The cokernel of the restriction

$$\delta_n|_{\text{III}^1(X,S)} : \text{III}^1(X, S) \to \text{III}^2(X, S[n]) \qquad (14.2.1)$$

of $\delta_n$ onto $\text{III}^1(X, S)$ is a subgroup of $\text{III}^2(X, S)$. Therefore the map in (14.2.1) surjects onto $m\text{III}^2(X, S[n])$ by the above. Claim (ii) now follows by taking the limit. $\qquad\square$

**Definition 14.3.** We say that a smooth geometrically irreducible projective surface $X$ defined over a field $K$ of characteristic zero is a generalised Châtelet surface if there is an $a \in K^*$ and a separable polynomial $P \in K[x]$ of degree 4 such that $X$ is a smooth compactification of the affine surface given by the equation

$$y^2 - az^2 = P(x)$$

over $K$. For the sake of brevity we will frequently drop the adjective 'generalised' when we talk about generalised Châtelet surfaces.

**Proposition 14.4.** *Let $X$ be a Châtelet surface defined over a number field $K$. Then for every $s \in \text{Sel}(X/K)$ there is an $x \in X(K)$ such that $\iota_{X_v/K_v}(i_v(x)) = r_{v*}(s)$ for every $v \in |K|$.*

*Proof.* Let $S$ be a finite subset of $|K|$ which contains every archimedean place of $K$ and every nonarchimedean place of $K$ where $X$ does not have good reduction. For every $v \in |K|$ choose an $x_v \in X_v(K_v)$ such that $\iota_{X_v/K_v}(x_v) = r_{v*}(s)$. By [Colliot-Thélène et al. 1987b, Theorem 8.6(b), p. 87] for every $v \in |K|$ there is an open neighbourhood $U_v \subseteq X_v(K_v)$ of $x_v$ contained by the $R$-equivalence class of $x_v$ in $X_v(K_v)$. Because $\prod_{v \in |K|} x_v \in X(\mathbb{A}_K)^{\text{Br}}$ by the easy direction of the Harpaz–Schlank theorem, there is an $x \in X(K)$ such that $i_v(x) \in U_v$ for every

$v \in S$ by [ibid., Theorem 8.11(c), p. 92]. Clearly $\iota_{X_v/K_v}(i_v(x)) = r_{v*}(s)$ for every $v \in S$. Because for every $v \in |K| - S$ the set $X(K)/R$ consists of one element by [ibid., Theorem 8.6(c), p. 87], we get that $\iota_{X_v/K_v}(i_v(x)) = r_{v*}(s)$ for every $v \in |K| - S$ as well.                                                                          $\square$

**Definition 14.5.** For every smooth projective geometrically irreducible variety $X$ defined over a field $K$ let $\mathrm{CH}_0(X)$ denote the Chow group of zero-dimensional cycles on $X$, and let $A_0(X)$ denote the kernel of the degree map $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$. Fix a point $x \in X(K)$. Then there is a map $\Psi_x : X(K)/R \to A_0(X)$ which for every $y \in X(K)$ maps the $R$-equivalence class of $y$ to $[y] - [x]$. When $K$ is a number field let $\mathrm{III}^1 A_0(X)$ denote the subgroup of those elements $c$ of $A_0(X)$ such that the base change of $c$ to $K_v$ is the zero element of $A_0(X_v)$ for every $v \in |K|$. Now let $X$ be a Châtelet surface, let $S$ be a torus over $K$ whose group of characters is isomorphic to $\mathrm{Pic}(\overline{X})$ as a $\mathrm{Gal}(\overline{K}|K)$-module and let $\mathcal{T}$ be a universal torsor over $X$. Then we have a map

$$\rho_{\mathcal{T}} : X(K) \to H^1(K, S)$$

which associates to every $P \in X(K)$ the class of the fibre of $\mathcal{T}$ at $P$. Moreover, there is a unique homomorphism $\Phi_{\mathcal{T}} : A_0(X) \to H^1(K, S)$ such that for every $y \in X(K)$ we have $\Phi_{\mathcal{T}}([y] - [x]) = \rho_{\mathcal{T}}(y) - \rho_{\mathcal{T}}(x)$ (see [ibid., p. 88]). In particular the map $\rho_{\mathcal{T}}$ factors through $R$-equivalence; let $\rho_{\mathcal{T},R} : X(K)/R \to H^1(K, S)$ be the map which sends the $R$-equivalence class of every $y \in X(K)$ to $\rho_{\mathcal{T}}(y)$. When $K$ is a number field, $\Phi_{\mathcal{T}}$ maps $\mathrm{III}^1 A_0(X)$ into $\mathrm{III}^1(K, S)$. Let $\Phi_0 : \mathrm{III}^1 A_0(X) \to \mathrm{III}^1(K, S)$ be the restriction of $\Phi_{\mathcal{T}}$ onto $\mathrm{III}^1 A_0(X)$.

**Theorem 14.6** (Colliot-Thélène, Sansuc, Swinnerton-Dyer). *Let $X$ be a Châtelet surface, let $\mathcal{T}$ be a universal torsor over $X$ and let $x \in X(K)$.*

  (i) *The map $\rho_{\mathcal{T},R}$ is an injection.*

 (ii) *The map $\Psi_x$ is a bijection.*

(iii) *When $K$ is a number field the map $\Phi_0$ is a bijection.*

*Proof.* The map $\rho_{\mathcal{T},R}$ is injective by [Colliot-Thélène et al. 1987b, Theorem 8.5(a), p. 86]. Claim (ii) is true by [ibid., Theorem 8.8, p. 89] while claim (iii) holds by [ibid., Theorem 8.10, p. 91].                                                                          $\square$

Let $X$ be a Châtelet surface over a number field $K$. Then for every $x \in X(K)$ let $R_0(x) \subseteq X(K)/R$ denote the set of those $R$-equivalences classes $s$ such that for every $y \in s$ the points $x, y \in X_v(K_v)$ are $R$-equivalent for every $v \in |K|$.

**Corollary 14.7.** *For every $X$, $K$ and $x$ as above, the set $R_0(x)$ has the same cardinality as $\mathrm{III}^1(K, S)$.*

*Proof.* Note that $\Psi_x$ maps $R_0(x)$ into $\text{III}^1 A_0(X)$. Therefore it will be enough to show that the induced map

$$\Psi_x|_{R_0(x)} : R_0(x) \to \text{III}^1 A_0(X) \tag{14.7.1}$$

is a bijection by part (iii) of Theorem 14.6. It is injective by part (ii) of Theorem 14.6. Let $s \in \text{III}^1 A_0(X)$ be arbitrary; by part (ii) of Theorem 14.6 there is a $y \in X(K)$ such that $[y] - [x]$ is $s$. Because $\Psi_x$ is a bijection over $K_v$ for every $v \in |K|$ by part (ii) of Theorem 14.6, we get that the points $x, y \in X_v(K_v)$ are $R$-equivalent for every $v \in |K|$. Therefore the map in (14.7.1) is surjective, too. $\square$

**Theorem 14.8.** *Let $K$ be a number field and let $X$ be a generalised Châtelet surface over $K$. Then HSP holds for $X$.*

**Remark 14.9.** According to [Colliot-Thélène et al. 1987a, Remark 8.10.2, p. 91] there is a Châtelet surface $X$ defined over a number field $K$ on which $R$-equivalence is strictly finer than Brauer equivalence. Hence by Theorem 14.8 we get that $H$-equivalence is finer than étale-Brauer equivalence over number fields. Moreover, there are two rational points $x, y \in X(K)$ such that $x, y \in X_v(K_v)$ are $H$-equivalent for every $v \in |K|$, but $x$ and $y$ are not $H$-equivalent over $K$. The theorem above is also interesting because it covers a whole class of varieties $X$ for which HSP holds, but which are not homogeneous spaces; moreover, every $R$-equivalence class of $X(K)$ is Zariski-dense (see [Colliot-Thélène et al. 1987b, Theorem 8.5(b), p. 86]) and, by [Colliot-Thélène et al. 1987b, Theorem 8.13, p. 95], the set $X(K)/R$ could be arbitrarily large for $X$ defined over $\mathbb{Q}$.

*Proof of Theorem 14.8.* Because Châtelet surfaces are geometrically rational it will be both necessary and sufficient to show that the map

$$\iota_{X/K,R} : X(K)/R \to \text{Sel}(X/K)$$

is a bijection by Theorem 13.3. Let $c \in H^1(X, S)$ be the cohomology class corresponding to the universal torsor $\pi : \mathcal{T} \to X$. Let $x, y \in X(K)$ be in two different $R$-equivalence classes. By part (i) of Theorem 14.6 the pullback classes $x^*(c), y^*(c) \in H^1(K, S)$ are different. Therefore the classes

$$\delta(x^*(c)) = x^*(\delta(c)), \ \delta(y^*(c)) = y^*(\delta(c)) \in H^2(K, C(S) \otimes \hat{\mathbb{Z}}(1))$$

are also different by part (i) of Lemma 14.2. Therefore $x$ and $y$ are not $H$-equivalent by Lemma 8.5. The injectivity of $\iota_{X/K,R}$ follows.

Now we prove that it is surjective, too. Let $s \in \text{Sel}(X/K)$. By Proposition 14.4 there is an $x \in X(K)$ such that $\iota_{X_v/K_v}(i_v(x)) = r_{v*}(s)$ for every $v \in |K|$. Note that $\iota_{X/K,R}$ maps $R_0(x)$ into $\text{Sel}_0(s)$. Therefore it will be enough to show that the induced map

$$\iota_{X/K,R}|_{R_0(x)} : R_0(x) \to \text{Sel}_0(s) \tag{14.9.1}$$

is a bijection. By the above this map is injective, so it will be enough to show that the cardinality of $\mathrm{Sel}_0(s)$ is at most $\mathrm{III}^2(K, C(S) \otimes \hat{\mathbb{Z}}(1))$ by part (ii) of Lemma 14.2 and Corollary 14.7. By the definition of $S$ we have that

$$C(S) \otimes \hat{\mathbb{Z}}(1) = \mathrm{Hom}(\mathrm{Pic}(\overline{X}), \mathbb{Z}) \otimes \hat{\mathbb{Z}}(1) = \mathrm{Hom}(\mathrm{Pic}(\overline{X}) \otimes \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)).$$

We already noted in the proof of Lemma 13.1 that

$$\pi_2(\overline{X}) = \mathrm{Hom}(\mathrm{Pic}(\overline{X}) \otimes \hat{\mathbb{Z}}, \hat{\mathbb{Z}}(1)), \quad \text{so } \pi_2(\overline{X}) = C(S) \otimes \hat{\mathbb{Z}}(1).$$

Therefore it will be enough to show that the cardinality of $\mathrm{Sel}_0(s)$ is at most $\mathrm{III}^2(K, \pi_2(\overline{X}))$. But this is the content of Lemma 13.2.                                   □

## Acknowledgements

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