Towards local-global compatibility
for Hilbert modular forms of low weight

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We prove some new cases of local-global compatibility for the Galois representations associated to Hilbert modular forms of low weight. If $F/\mathbb{Q}$ is a totally real extension of degree d, we are interested in Hilbert modular forms for F of weight $(k_1, \ldots, k_d, w)$, with the k_i and w odd integers and some but not all k_i equal to 1 (the partial weight-one case). Recall that a Hecke eigenform f with such a weight has an associated compatible system $\rho_{f,p}$ of two-dimensional p-adic representations of $\mathrm{Gal}(\overline{F}/F)$, first constructed by Jarvis using congruences to forms of cohomological weight ($k_i \geq 2$ for all i).

One expects that the restriction of the representation $\rho_{f,p}$ to a decomposition group D_v at a finite place $v \nmid p$ of F should correspond (under the local Langlands correspondence) to the local factor at v, $\pi_{f,v}$, of the automorphic representation π_f generated by f. This expectation is what we refer to as local-global compatibility. For forms of cohomological weight, the compatibility was in most cases verified by Carayol using geometric methods. Combining this result with Jarvis's construction of Galois representations establishes many cases of local-global compatibility in the partial weight-one situation. However, when $\pi_{f,v}$ is a twist of the Steinberg representation, this method establishes a statement weaker that local-global compatibility. The difficulty in this case is to show that the Weil–Deligne representation associated to $\rho_{f,p}|_{D_v}$ has a nonzero monodromy operator. In this paper, we verify local-global compatibility in many of these 'missing' cases, using methods from the p-adic Langlands programme (including analytic continuation of overconvergent Hilbert modular forms, maps between eigenvarieties encoding Jacquet–Langlands functoriality and Emerton's completed cohomology).

1. Introduction

In this paper, we study the problem of local-global compatibility for the Galois representations attached to Hilbert modular forms of low weight. These Galois representations were constructed by Jarvis [1997]. We begin by recalling the main

theorem of that paper. Let F be a totally real finite extension of $\mathbb{Q}$. For a prime p, let ι be an isomorphism from a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ to $\mathbb{C}$. If v is a finite place of F and π_v is an admissible smooth irreducible representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{C}$, we denote the Frobenius-semisimple Weil–Deligne representation associated to π_v under the (Hecke normalised) local Langlands correspondence by $\sigma(\pi_v)$. We denote by $\sigma^\iota(\pi_v)$ the Weil–Deligne representation with coefficients in $\overline{\mathbb{Q}}_p$ obtained by composition with ι^{-1}. For a continuous $\overline{\mathbb{Q}}_p$-representation ρ of $\mathrm{Gal}(\overline{F}_v/F_v)$, with $v \nmid p$ a finite place of F, we denote by $\mathrm{WD}(\rho)$ the Weil–Deligne representation associated to ρ by Grothendieck's p-adic monodromy theorem, with Frobenius semisimplification $\mathrm{WD}(\rho)^{F\text{-ss}}$ and semisimplification $\mathrm{WD}(\rho)^{\mathrm{ss}}$ (i.e., we set the monodromy operator to zero to obtain $\mathrm{WD}(\rho)^{\mathrm{ss}}$).

Results of Carayol [1986b], Blasius and Rogawski [1993], Rogawski and Tunnell [1983], Taylor [1989] and Jarvis [1997] have shown the following:

Theorem 1.1. *Let π be a cuspidal algebraic[1] automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, such that for each infinite place τ of F the local factor π_τ is either discrete series or holomorphic limit of discrete series. Then there exists an irreducible representation*

$$r_{p,\iota}(\pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

such that if v is a finite place of F, with $v \nmid p$, and one of the following holds:

- π_τ *is discrete series for all infinite places τ,*
- π_v *is not special (i.e., π_v is not a twist of the Steinberg representation),*

then

$$\mathrm{WD}(r_{p,\iota}(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{F\text{-ss}} \cong \sigma^\iota(\pi_v).$$

Remark 1.2. The irreducibility of the Galois representation appearing in this theorem is proved using an argument of Ribet [1977] (see [Taylor 1995, Proposition 3.1]).

Remark 1.3. In the excluded case, where π_τ is holomorphic limit of discrete series for some τ and π_v is special, then Jarvis [1997, Remark 7.3] proved that

$$\mathrm{WD}(r_{p,\iota}(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{\mathrm{ss}} \cong \sigma^\iota(\pi_v)^{\mathrm{ss}}.$$

The main result of this paper addresses some of the excluded cases in Theorem 1.1. We prove:

Theorem 1.4. *Let π be as in Theorem 1.1, suppose that π_τ is holomorphic limit of discrete series for some infinite place τ and let $v \nmid p$ be a finite place of F such that π_v is special.*

Suppose the following technical hypotheses hold:

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[1]By algebraic, we just mean that the usual parity condition on the weights holds.

(1) *The prime p is absolutely unramified in F.*

(2) *For every place w of F with $w \mid p$, π_w is an unramified principal series representation.*

(3) *Moreover, for each $w \mid p$ we have $\pi_w \cong \mathrm{Ind}(\mu_1 \otimes \mu_2)$ (normalised parabolic induction from a Borel subgroup) with μ_1, μ_2 distinct unramified characters of $F_w^\times$.*

(4) *The residual representation*

$$\bar{r}_{p,\iota}(\pi) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is irreducible.

Then

$$\mathrm{WD}(r_{p,\iota}(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)})^{F\text{-ss}} \cong \sigma^\iota(\pi_v).$$

Remark 1.5. We will not discuss the question of local-global compatibility at places dividing p in this paper. However, under the 'p-distinguished' hypothesis (3) made above, an argument using analytic continuation of crystalline periods (as done in [Jorza 2010] for the case of low-weight Siegel modular forms) shows that the representations $r_{p,\iota}(\pi)$ are crystalline at places dividing p, with the expected associated Weil–Deligne representation.

The main novelty in this theorem is that we are able to establish nontriviality of the monodromy operator for a Galois representation which is only indirectly related (by congruences) to the cohomology of Shimura varieties.

1.6. *The technical hypotheses.* We will say a few words about the hypotheses in Theorem 1.4. Hypotheses (1) and (2) are satisfied for all but finitely many p. Hypothesis (1) appears because we use results of Kassaei [2013] on analytic continuation and gluing of overconvergent Hilbert modular forms. At least for forms of parallel weight, results have been announced by Pilloni and Stroh [2013] and Sasaki [2014] which apply without restriction on the ramification of p, so it seems reasonable to hope that this hypothesis could be removed in future. Once hypothesis (1) is removed, hypothesis (2) can be substantially relaxed by first making a base change to an extension F'/F in which p ramifies.

Hypothesis (3) is again present in [Kassaei 2013] and has been removed by Pilloni and Stroh [2013] and Sasaki [2014] in their setting, but we also make some independent use of this condition. In particular, we use (3) to keep track of the distinct p-stabilisations when we apply p-adic functoriality to move between different kinds of p-adic automorphic form. By contrast, in [Pilloni and Stroh 2013; Sasaki 2014], the situation is that the various p-stabilisations form a single generalised eigenspace for the Hecke operators. So extending our techniques to this setting would require more care when applying p-adic functoriality.

We assume hypothesis (4) in order to apply the results of [Newton 2013] (it simplifies the analysis of completed cohomology of Shimura curves). It should be possible to avoid this assumption by modifying the arguments of that paper to work with overconvergent cohomology [Ash and Stevens 2008].

To summarise, we believe that the techniques described in this paper could in principle prove a version of the above theorem with just one technical hypothesis: for each place $w \mid p$ of F, π_w is not a twist of the Steinberg representation. To handle the remaining cases seems to require a new idea or a different method.

Remark 1.7. Recent work of Kassaei, Sasaki and Tian [Kassaei et al. 2014] permits us to relax hypotheses (2) and (3) slightly (allowing tame ramification). We have kept to the 'unramified' setting for expository reasons, but here is the precise condition which can replace hypotheses (2) and (3) in Theorem 1.4:

(2′) For every place w of F with $w \mid p$, $\pi_w \cong \mathrm{Ind}(\mu_1 \otimes \mu_2)$ with μ_1, μ_2 distinct characters of $F_w^\times$ whose ratio is tamely ramified.

1.8. *Sketch of proof.* We now sketch the proof of Theorem 1.4. By twisting and base change we may assume that $[F : \mathbb{Q}]$ is even, that π_v is an unramified twist of the Steinberg representation, and that we have an auxiliary finite place w such that π_w is also an unramified twist of Steinberg. In light of Remark 1.3 we just need to show that the Weil–Deligne representation associated to the local representation

$$\rho_v = r_{p,\iota}(\pi)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$$

has a nonzero monodromy operator. Equivalently, we must show that ρ_v is not an unramified representation. We will assume that ρ_v is unramified and obtain a contradiction.

The auxiliary Steinberg place w allows us to find systems of Hecke eigenvalues attached to π in the p-adically completed cohomology of Shimura curves associated to indefinite quaternion algebras. In this context, a p-adic analogue of Mazur's principle [Newton 2013, Theorem 4.33] allows us to show that, since the representation ρ_v is unramified, we can strip v from the level of π. More precisely we can produce overconvergent Hilbert modular forms, with level prime to v, which share the same system of Hecke eigenvalues (outside v) as π. Finally, a generalisation, due to Kassaei, of Buzzard and Taylor's gluing and analytic continuation of overconvergent eigenforms allows us to produce a *classical* Hilbert modular form, with level prime to v, contributing to π. But we assumed that π_v was (a twist of) Steinberg, so π contains no nonzero $\mathrm{GL}_2(\mathbb{O}_{F_v})$-invariant vectors. Therefore we obtain the desired contradiction.

Readers familiar with the theory of p-adic and overconvergent automorphic forms may find it amusing that we make use of three different avatars of overconvergent automorphic forms in this paper — firstly geometrically defined overconvergent Hilbert

modular forms (sections of automorphic line bundles on strict neighbourhoods of the ordinary locus in Hilbert modular varieties as in, for example, [Kisin and Lai 2005]), secondly the overconvergent automorphic forms on definite quaternion algebras defined by Buzzard [2007] and finally the spaces obtained by applying Emerton's locally analytic Jacquet functor to the p-adically completed cohomology of Shimura curves [Emerton 2006b; Newton 2013].

The organisation of our paper is as follows: In Section 2 we reduce the statement of Theorem 1.4 to a special case. Then, in Section 3, we describe the various eigenvarieties we will make use of, together with their construction. We next explain how to find systems of Hecke eigenvalues arising from the automorphic representations π of interest in the completed cohomology of Shimura curves, where we apply the level optimisation results of [Newton 2013] (Section 4). Finally we return to overconvergent modular forms on Hilbert modular varieties and apply a crucial result of Kassaei [2013] (Section 5).

1.9. *Other approaches.*   Luu [2015, §3.2] has also recently described an approach to proving some cases of local-global compatibility for Hilbert modular forms of low weight when π satisfies an ordinarity hypothesis at places dividing p. We will use the same notation as in the previous subsection. Using the assumption that ρ_v is unramified, Luu applied a modularity lifting theorem to produce an ordinary p-adic Hilbert modular form g with level prime to v and the same system of Hecke eigenvalues (outside v) as π, and then imposed a hypothesis that amounts to ruling out the existence of g. It may also be possible to show that g is classical (hence obtaining a contradiction without an additional hypothesis) using a variant of the methods applied in the parallel weight-one case (as in, for example, [Kassaei 2013]), but it is not obvious to the author that these methods can be easily applied. One obstacle is that, unlike the parallel weight-one case, not all the p-stabilisations of the newform generating π are ordinary. The advantage of our method is that these p-stabilisations still have finite slope, so we can apply the level optimisation results of [Newton 2013] to obtain several overconvergent forms and then apply [Kassaei 2013].

1.10. *Examples.*   Until recently, not many examples of Hilbert modular forms of partial weight one were known — in particular, as far as the author knows, the only known examples were CM forms (and therefore not twists of Steinberg at any place). However, Moy and Specter [2015] have recently explicitly computed examples which are non-CM and Steinberg at a finite place.

2. Notation and reductions

We let $F/\mathbb{Q}$ be a totally real, finite extension of $\mathbb{Q}$, and fix a prime p such that p is absolutely unramified in F. We denote by Σ the set of Archimedean places of F.

We write $\mathbb{A}_F$ for the adeles of F and $\mathbb{A}_F^\infty$ for the finite adeles. For a finite place v of F we write $\mathbb{O}_v$ for the ring of integers in the local field F_v. For convenience, we fix once and for all a choice of uniformiser $\varpi_v \in \mathbb{O}_v$ for each finite place v.

We denote by Σ_p the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$, and for a place $\mathfrak{p} \mid p$ denote by $\Sigma_\mathfrak{p}$ those embeddings which extend to $F_\mathfrak{p}$. Our fixed isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ induces a bijection between Σ and Σ_p.

We will need some notation for (limits of) discrete series representations of $\mathrm{GL}_2(\mathbb{R})$. For $k \geq 2$ and w integers of the same parity, we let $D_{k,w}$ denote the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with central character $t \mapsto t^{-w}$ defined in [Carayol 1986b, Section 0.2]. For $k = 1$ and w an odd integer, we define $D_{1,w}$ to be the limit of discrete series $\mathrm{Ind}(\mu, \nu)$, where the induction is a normalised parabolic induction from a Borel subgroup and μ, ν are characters of $\mathbb{R}^\times$ defined by $\mu(t) = |t|^{-w/2} \mathrm{sgn}(t)$ and $\nu(t) = |t|^{-w/2}$.

We assume π is an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ as in the statement of Theorem 1.4, and that v is a finite place of F with π_v special. First we note that by twisting and making a quadratic base change (to an extension of F in which v splits) we can assume without loss of generality that $[F : \mathbb{Q}]$ is even, that π_v is an unramified twist of the Steinberg representation and that there is another finite place v' with $\pi_{v'}$ also an unramified twist of Steinberg.

With this in mind, for the rest of the paper we fix a cuspidal automorphic representation π_0 of $\mathrm{GL}_2(\mathbb{A}_F)$ as in the statement of Theorem 1.4, and moreover assume that:

- $[F : \mathbb{Q}]$ is even.
- $\pi_{0,v}$ is an unramified twist of Steinberg.
- There is a finite place v' coprime to p and v such that $\pi_{0,v'}$ is an unramified twist of Steinberg.
- $\pi_{0,\tau} \cong D_{k_\tau,w}$ for each $\tau \in \Sigma$ with k_τ an odd positive integer and w an odd integer (independent of τ).

2.1. *Hilbert modular forms and varieties.* We now proceed to reduce Theorem 1.4 to a statement about Hilbert modular forms. Let $L \subset \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ which contains the image of every embedding from F into $\overline{\mathbb{Q}}_p$. Let $\mathfrak{n}$ and $\mathfrak{m}$ be two coprime ideals of $\mathbb{O}_F$, both coprime to p, and suppose that $\mathfrak{n}$ is divisible by a rational integer which is ≥ 4.

Definition 2.2. Let $\mathfrak{c}$ be a fractional ideal of F such that its absolute norm $N\mathfrak{c}$ is coprime to $\mathfrak{mn}$. We denote by $\underline{\mathscr{M}}^{\mathfrak{c}}(\mathfrak{n}, \mathfrak{m})$ the functor from schemes over $\mathbb{O}_L$ to sets taking S to isomorphism classes of tuples $(A, \lambda, [\eta])$, where:

- A is a Hilbert–Blumenthal abelian scheme over S (in particular A is equipped with an action of $\mathbb{O}_F$ [Kisin and Lai 2005, 1.1]).

- λ is a $\mathfrak{c}$-polarisation of A [Kisin and Lai 2005, 1.2].

- η is an $\mathbb{O}_F$-equivariant closed immersion of S-group schemes

$$\eta : (\mathbb{O}_F/\mathfrak{mn})^\vee \hookrightarrow A[\mathfrak{mn}],$$

and $[\eta]$ is its equivalence class under the natural action of $(\mathbb{O}_F/\mathfrak{m})^\times$. Here $(\mathbb{O}_F/\mathfrak{mn})^\vee$ denotes the Cartier dual of the constant S-group scheme (with $\mathbb{O}_F$ action) $\mathbb{O}_F/\mathfrak{mn}$.

Under our assumptions, the functor $\underline{\mathscr{M}}^{\mathfrak{c}}(\mathfrak{n}, \mathfrak{m})$ is representable by a smooth $\mathbb{O}_L$-scheme $\mathscr{M}^{\mathfrak{c}}(\mathfrak{n}, \mathfrak{m})$, and there is a good theory of toroidal and minimal compactifications (see [loc. cit., 1.6, 1.8] — although we use a slightly modified level structure, everything in this reference goes through).

Having fixed a choice of representatives $\mathfrak{c}_1, \ldots, \mathfrak{c}_h$ for the narrow class group of F (which we may assume all satisfy $N\mathfrak{c}_i$ coprime to $p\mathfrak{nm}$), we define $\mathscr{M}(\mathfrak{n}, \mathfrak{m})$ to be the disjoint union of the $\mathscr{M}^{\mathfrak{c}_i}(\mathfrak{n}, \mathfrak{m})$.

For κ a homomorphism

$$\kappa : \mathrm{Res}_{\mathbb{Z}}^{\mathbb{O}_F} \mathbb{G}_{m/\mathbb{O}_L} \to \mathbb{G}_{m/\mathbb{O}_L}$$

we have line bundles $\underline{\omega}^\kappa$ as in [loc. cit., 1.4.2] on $\mathscr{M}(\mathfrak{n}, \mathfrak{m})$, which extend to suitable toroidal compactifications. We assume that w is an integer such that the character $N^w \cdot \kappa^{-1}$ admits a square root. Then, following [loc. cit., 1.11], we define Hecke operators $T_\mathfrak{a}$ for each prime ideal $\mathfrak{a}$ of $\mathbb{O}_F$ coprime to $\mathfrak{mn}$ and $U_\mathfrak{a}$ for each prime ideal $\mathfrak{a}$ dividing $\mathfrak{mn}$.

Denote by $\mathbb{T}(\mathfrak{n}, \mathfrak{m})$ the polynomial algebra over $\mathbb{Z}$ generated by symbols $T_\mathfrak{a}$ and $U_\mathfrak{a}$ as above. This algebra naturally acts on $H^0(\mathscr{M}(\mathfrak{n}, \mathfrak{m})_L, \underline{\omega}^\kappa)$, and we say that a nonzero element f of this L-vector space is a *Hecke eigenform* if the action of $\mathbb{T}(\mathfrak{n}, \mathfrak{m})$ preserves the one-dimensional space $L \cdot f$. If f is a cuspidal Hecke eigenform, we denote by θ_f the character $\mathbb{T}(\mathfrak{n}, \mathfrak{m}) \to L$ giving the action of the Hecke algebra on f. After projecting to a classical Hilbert modular form (as described in [loc. cit., 1.11.8]) f generates (a model over L for) the finite part π_f^∞ of a cuspidal automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_F)$. For $\tau \in \Sigma$ we have $\pi_\tau = D_{k_\tau, w}$, where the k_τ can be read off from κ (using the bijection between Σ and Σ_p). The central character of π_f is an algebraic Hecke character of the form $\chi_f \omega^{-w}$, where ω is the norm character and χ_f is a finite-order character. The Galois representation $r_{\pi_f, \iota}$ is the unique semisimple representation, unramified outside $p\mathfrak{mn}$, such that for $\mathfrak{q} \nmid p\mathfrak{mn}$ a geometric Frobenius element at $\mathfrak{q}$ has characteristic polynomial $X^2 - \theta_f(T_\mathfrak{q})X + (N\mathfrak{q})^{1-w}\iota^{-1}\chi_f(\varpi_\mathfrak{q})$

In the subsequent sections of the paper, we will prove the following 'level optimisation' result:

Proposition 2.3. *Suppose* $\mathfrak{m} = \mathfrak{q}\mathfrak{l}$ *is a product of two prime ideals. Let* $f \in H^0(\mathscr{M}(\mathfrak{n}, \mathfrak{m})_L, \underline{\omega}^\kappa)$ *be a Hecke eigenform. Suppose that:*

- *The local factor* $\pi_{f,\mathfrak{q}}$ *is an unramified twist of the Steinberg representation.*

- *The residual representation*

$$\bar{r}_{p,\iota}(\pi_f) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

  is irreducible.

- *The polynomials* $X^2 - \theta_f(T_\mathfrak{p})X + (N\mathfrak{p})^{1-w}\iota^{-1}\chi_f(\varpi_\mathfrak{p})$ *have distinct roots for all prime ideals* $\mathfrak{p} \mid p$.

- *The local representation*

$$r_{p,\iota}(\pi_f)|_{\mathrm{Gal}(\overline{F}_\mathfrak{l}/F_\mathfrak{l})}$$

  is unramified.

Then there exists a Hecke eigenform $g \in H^0(\mathscr{M}(\mathfrak{n}, \mathfrak{q})_L, \underline{\omega}^\kappa)$ *such that the Hecke eigenvalues of* f *and* g *coincide outside* $\mathfrak{l}$ *and* $\theta_f(U_\mathfrak{l})$ *is one of the roots of* $X^2 - \theta_g(T_\mathfrak{l})X + (N\mathfrak{l})^{1-w}\iota^{-1}\chi_f(\varpi_\mathfrak{l})$.

We now explain why this proposition is sufficient to deduce Theorem 1.4.

Lemma 2.4. *Proposition 2.3 implies Theorem 1.4.*

Proof. Recall that we have reduced the statement of Theorem 1.4 to the case of π_0 as described immediately before Section 2.1. We denote the finite part of π_0 by π_0^∞. For $\mathfrak{m}$ any nonzero ideal of $\mathbb{O}_F$ we define congruence subgroups $U_0(\mathfrak{m})$ and $U_1(\mathfrak{m})$ to be the subgroups of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ given by

$$U_0(\mathfrak{m}) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \mathfrak{m} \right\}$$

$$U_1(\mathfrak{m}) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : g = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{m} \right\}$$

Now we denote by $\mathfrak{l}$ and $\mathfrak{q}$ the prime ideals of $\mathbb{O}_F$ corresponding to the places v and v' respectively, and suppose that $\mathfrak{n}$ is an ideal of $\mathbb{O}_F$, coprime to $pq\mathfrak{l}$ and divisible by a rational integer ≥ 4, such that $(\pi_0^\infty)^{U_1(\mathfrak{n}) \cap U_0(\mathfrak{q}\mathfrak{l})} \neq 0$. The isomorphism $\iota : \overline{\mathbb{Q}}_p \to \mathbb{C}$ induces a bijection between Σ and the embeddings Σ_p from $F \hookrightarrow L$, so using this we can associate a character

$$\kappa : \mathrm{Res}_{\mathbb{Z}}^{\mathbb{O}_F} \mathbb{G}_{m/\mathbb{O}_L} \to \mathbb{G}_{m/\mathbb{O}_L}$$

to the Σ-tuple of integers $\{k_\tau\}_{\tau \in \Sigma}$ describing the Archimedean part of π_0. Now (possibly enlarging L) there is a Hecke eigenform $f \in H^0(\mathscr{M}(\mathfrak{n}, \mathfrak{q}\mathfrak{l})_L, \underline{\omega}^\kappa)$ with $\pi_f \cong \pi_0$, and so f satisfies the hypotheses of Proposition 2.3. This proposition gives us a Hecke eigenform $g \in H^0(\mathscr{M}(\mathfrak{n}, \mathfrak{q})_L, \underline{\omega}^\kappa)$ with $\pi_g \cong \pi_0$, and so the space

$(\pi_0^\infty)^{U_1(\mathfrak{n}) \cap U_0(\mathfrak{q})}$ is nonzero, which contradicts the assumption that $\pi_{0,\mathfrak{l}}$ is a twist of Steinberg. □

We end this section by discussing Hilbert modular varieties with Iwahori level at p and a definition of overconvergent Hilbert modular forms.

Definition 2.5. Let $\mathfrak{c}$ be a fractional ideal of F such that its absolute norm $N\mathfrak{c}$ is coprime to $p\mathfrak{m}\mathfrak{n}$. We denote by $\underline{\mathscr{M}}^{\mathfrak{c},\mathrm{Iw}}(\mathfrak{n}, \mathfrak{m})$ the functor from schemes over $\mathbb{O}_L$ to sets taking S to isomorphism classes of tuples $(A, \lambda, [\eta], H)$ up to isomorphism, where $(A, \lambda, [\eta])$ are as in Definition 2.2 and H is a finite flat subgroup scheme of $A[p]$, stable under the action of $\mathbb{O}_F$, of rank $p^{[F:\mathbb{Q}]}$ and isotropic with respect to the λ-Weil pairing.

The functor $\underline{\mathscr{M}}^{\mathfrak{c},\mathrm{Iw}}(\mathfrak{n}, \mathfrak{m})$ is represented by an $\mathbb{O}_L$-scheme $\mathscr{M}^{\mathfrak{c},\mathrm{Iw}}(\mathfrak{n}, \mathfrak{m})$. We denote by $\mathscr{M}^{\mathrm{Iw}}(\mathfrak{n}, \mathfrak{m})$ the disjoint union of these over suitable representatives $\mathfrak{c}_i$ of the narrow class group, as before.

Denoting the rigid generic fibre of $\mathscr{M}^{\mathrm{Iw}}(\mathfrak{n}, \mathfrak{m})$ by $M^{\mathrm{Iw}}(\mathfrak{n}, \mathfrak{m})_L$, we can consider sections of the line bundles $\underline{\omega}^\kappa$ over strict neighbourhoods of the locus in $M^{\mathrm{Iw}}(\mathfrak{n}, \mathfrak{m})_L$, where H is Cartier dual to $\mathbb{O}_F/p$ (i.e., the multiplicative ordinary locus), to obtain a space $M_\kappa^\dagger(\mathfrak{n}, \mathfrak{m})$ of overconvergent modular forms of weight κ (see [Kassaei 2013, §4]).

3. Eigenvarieties

We will need to make use of eigenvarieties constructed in different contexts. To clarify the relationship between these eigenvarieties, we are going to follow the abstract approach of [Bellaïche and Chenevier 2009, 7.2]. First we need to discuss the weight spaces over which our eigenvarieties will live.

3.1. *Weight spaces.* We set $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)(\mathbb{Q}_p) = \prod_{\mathfrak{p} \mid p} G_\mathfrak{p}$, where $G_\mathfrak{p} = \mathrm{GL}_2(F_\mathfrak{p})$. We denote by $B = \prod_{\mathfrak{p} \mid p} B_\mathfrak{p} \subset G$ the Borel subgroup comprising upper-triangular matrices and by $T = \prod_{\mathfrak{p} \mid p} T_\mathfrak{p}$ the maximal torus comprising diagonal matrices. We denote by $N_\mathfrak{p}$ the subgroup of $B_\mathfrak{p}$ whose elements have 1 on the diagonal. Finally, $T_0 = \prod_{\mathfrak{p} \mid p} T_{0,\mathfrak{p}} \subset T$ is the compact subgroup given by elements with integral entries.

Fix a finite extension $L \subset \overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, which we assume contains a normal closure of F. The functor taking an L-affinoid $\mathrm{Sp}\, A$ to the set of continuous $A^\times$-valued characters of

$$T_0 = (\mathbb{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p)^\times \times (\mathbb{O}_F \otimes_\mathbb{Z} \mathbb{Z}_p)^\times$$

is representable by a rigid analytic space $\widehat{T}_0$ over L [Buzzard 2004, §2]. Likewise, we have a rigid space $\widehat{T}$ representing continuous characters of T.

Definition 3.2. Denote by $\mathscr{W}$ the subspace of $\widehat{T}_0$ whose points correspond to continuous characters of T_0 which are trivial on a finite-index subgroup of $\mathbb{O}_F^\times$ (embedded diagonally in T_0).

Suppose we have an algebraic character $\kappa : \mathrm{Res}_{\mathbb{Z}}^{\mathbb{O}_F} \mathbb{G}_m / \mathbb{O}_L \to \mathbb{G}_m / \mathbb{O}_L$ such that $N^w \cdot \kappa^{-1}$ admits a square root. Recall that Σ_p denotes the set of embeddings from F to L, so κ corresponds to a Σ-tuple of integers $\{k_\tau\}_{\tau \in \Sigma}$, with the same parity as w.

The character

$$(t_1, t_2) \mapsto \prod_{\tau \in \Sigma_p} \tau(t_1)^{\frac{k_\tau - 2 - w}{2}} \tau(t_2)^{\frac{-w - k_\tau + 2}{2}}$$

is a point in $\mathscr{W}(L)$, which we also denote by κ.

Definition 3.3. Denote by $\mathscr{W}_\kappa$ the subspace of $\mathscr{W}$ such that maps from an L-affinoid Sp A to $\mathscr{W}_\kappa$ correspond to characters $(\chi_1, \chi_2) : (\mathbb{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \times (\mathbb{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to A^\times$ such that $\chi_1 \chi_2 = \mathrm{Nm}^{-w}$ and $\chi_1 \chi_2^{-1} = \kappa \, \mathrm{Nm}^{-2} \cdot (\epsilon \circ \mathrm{Nm})$, where ϵ is a continuous character $\epsilon : \mathbb{Z}_p^\times \to A^\times$ with

$$v_p(1 - \epsilon(a)) > \frac{1}{p-1}$$

for all $a \in \mathbb{Z}_p^\times$. Here Nm is the natural extension of the norm map to a continuous map $(\mathbb{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to \mathbb{Z}_p^\times$.

Taking ϵ to be the trivial character, we see that κ is a point of $\mathscr{W}_\kappa$. In fact, $\mathscr{W}_\kappa$ is a (one-dimensional) disc in $\mathscr{W}$, with centre κ.

Remark 3.4. The weight space $\mathscr{W}$ is isomorphic to the weight space (also denoted by $\mathscr{W}$) defined in [Buzzard 2007, §8]. Our character (χ_1, χ_2) of $(\mathbb{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \times (\mathbb{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ corresponds to the character $(n, v) = (\chi_1 \chi_2^{-1}, \chi_2)$ in Buzzard's weight space.

The weight space $\mathscr{W}_\kappa$ is isomorphic to the weight space (also denoted by $\mathscr{W}_\kappa$) defined in [Kisin and Lai 2005, §4.5].

We are going to build eigenvarieties interpolating classical Hilbert modular forms over the weight space $\mathscr{W}_\kappa$. These eigenvarieties will be constructed using three different notions of 'overconvergent automorphic form'.

3.5. *Refinements.* For $\mathfrak{p} \mid p$, denote by $I_\mathfrak{p}$ the Iwahori subgroup of $G_\mathfrak{p}$, comprising matrices which reduce to upper-triangular matrices mod $\varpi_\mathfrak{p}$. We write I for the product $\prod_{\mathfrak{p} \mid p} I_\mathfrak{p} \subset G$.

Definition 3.6. Let $\pi_\mathfrak{p}$ be an irreducible smooth representation of $G_\mathfrak{p}$ on a complex vector space which is either an irreducible principal series or a twist of the Steinberg representation. An *accessible refinement* of $\pi_\mathfrak{p}$ is a character χ of $T_\mathfrak{p}$ such that there is a $G_\mathfrak{p}$-equivariant embedding

$$\pi_\mathfrak{p} \hookrightarrow \mathrm{Ind}_{B_\mathfrak{p}}^{G_\mathfrak{p}} \chi.$$

Remark 3.7. We have the following explicit description of accessible refinements: we have $\pi_\mathfrak{p} \cong \mathrm{Ind}_{B_\mathfrak{p}}^{G_\mathfrak{p}} \mu_1 \otimes \mu_2$ or $\pi_\mathfrak{p} \cong \mathrm{St} \otimes \mu$, where μ, μ_i are characters of $F_\mathfrak{p}^\times$.

- Suppose $\pi_\mathfrak{p} \cong \mathrm{Ind}_{B_\mathfrak{p}}^{G_\mathfrak{p}} \mu_1 \otimes \mu_2$. Then the accessible refinements of $\pi_\mathfrak{p}$ are the characters $\mu_1 \otimes \mu_2$ and $\mu_2 \otimes \mu_1$. Note that if $\mu_1 = \mu_2$ these refinements are the same.

- Suppose $\pi_\mathfrak{p} \cong \mathrm{St} \otimes \mu$. Then $\pi_\mathfrak{p}$ is isomorphic to the unique irreducible sub-representation of the normalised induction $\mathrm{Ind}_{B_\mathfrak{p}}^{G_\mathfrak{p}} \mu \cdot |\cdot|_\mathfrak{p}^{1/2} \otimes \mu \cdot |\cdot|_\mathfrak{p}^{-1/2}$ and the unique accessible refinement of $\pi_\mathfrak{p}$ is $\mu \cdot |\cdot|_\mathfrak{p}^{1/2} \otimes \mu \cdot |\cdot|_\mathfrak{p}^{-1/2}$.

For an ideal $\mathfrak{n}$ of $\mathbb{O}_F$, coprime to p, we denote by $\mathscr{H}(\mathfrak{n})$ the free commutative polynomial ring over $\mathbb{Z}$ on generators labelled T_v and S_v for places v of F not dividing $p\mathfrak{n}$ and U_v for places $v \mid \mathfrak{n}$.

For $\mathfrak{m}, \mathfrak{n}$ ideals of $\mathbb{O}_F$ (coprime to each other and to p), suppose π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with $(\pi^{\infty p})^{U_1(\mathfrak{n}) \cap U_0(\mathfrak{m})} \neq 0$ and $\pi_\tau = D_{k_\tau, w}$ for each $\tau \in \Sigma$, with $k_\tau \geq 2$ and $k_\tau = w \bmod 2$ for each τ. We say that such a π is a classical automorphic representation of tame level $U_1(\mathfrak{n}) \cap U_0(\mathfrak{m})$ and weight (k, w). We moreover say that π has finite slope if the smooth Jacquet modules $J_{B_\mathfrak{p}}(\pi_\mathfrak{p})$ are nonzero for all $\mathfrak{p} \mid p$. Equivalently, the $\pi_\mathfrak{p}$ are all irreducible principal series or twists of Steinberg (since local factors of a cuspidal automorphic representation are generic).

Definition 3.8. Suppose π is a finite-slope representation. An *accessible refinement* of π is a character

$$\chi = \bigotimes_{\mathfrak{p} \mid p} \chi_\mathfrak{p} : T \to \mathbb{C}^\times$$

such that each $\chi_\mathfrak{p}$ is an accessible refinement of $\pi_\mathfrak{p}$.

Remark 3.9. The possible accessible refinements of finite-slope automorphic representations are completely classified by Remark 3.7.

Definition 3.10. We say that a finite-slope representation is *unramified* if $\pi_\mathfrak{p}$ is either an unramified principal series or an unramified twist of Steinberg for all $\mathfrak{p} \mid p$.

Remark 3.11. The following two conditions are easily seen to be equivalent to π being unramified:

- For every $\mathfrak{p} \mid p$, $\pi_\mathfrak{p}$ has nonzero invariants for the Iwahori subgroup $I_\mathfrak{p}$.

- Every accessible refinement of π factors through T/T_0.

Definition 3.12. Suppose π is a finite-slope representation of weight (k, w) and χ is an accessible refinement of π. We define a continuous character

$$\nu(\pi, \chi) : T \to \overline{\mathbb{Q}}_p^\times$$

by

$$\nu(\pi, \chi) = \bigotimes_{\mathfrak{p} \mid p} \iota^{-1} \chi_{\mathfrak{p}} |\cdot|_{\mathfrak{p}}^{1/2} \otimes |\cdot|_{\mathfrak{p}}^{-1/2} \prod_{\tau \in \Sigma_{\mathfrak{p}}} \tau^{(k_\tau - 2 - w)/2} \otimes \tau^{(-w - k_\tau + 2)/2}.$$

Remark 3.13. Our discussion of accessible refinements and the definition of the character $\nu(\pi, \chi)$ is parallel to that of [Chenevier 2009, §1.4]. We will see in Section 4.2 that these definitions are completely natural when constructing eigenvarieties using completed cohomology and Emerton's locally analytic Jacquet functor.

3.14. *Abstract eigenvarieties.*

Definition 3.15. Let $\mathscr{W}'$ be any subspace of $\mathscr{W}$ which is an admissible increasing union of affinoids (in practice it will be $\mathscr{W}_\kappa$). Let $\mathscr{H}$ be a commutative $\mathbb{Z}$-algebra and let $\mathscr{Z}$ be a subset of $\mathrm{Hom}(\mathscr{H}, \overline{\mathbb{Q}}_p) \times \widehat{T}(\overline{\mathbb{Q}}_p)$ whose image in $\widehat{T}_0(\overline{\mathbb{Q}}_p)$ is an accumulation[2] and Zariski-dense subset of $\mathscr{W}'$. Denote by Y the fibre product of $\widehat{T}$ and $\mathscr{W}'$ over $\widehat{T}_0$. Then an *eigenvariety* for the triple $(\mathscr{H}, \mathscr{Z}, \mathscr{W}')$ is a reduced rigid space X over L equipped with:

- a ring homomorphism $\psi : \mathscr{H} \to \mathcal{O}(X)$,
- a finite morphism $\nu : X \to Y$,
- an accumulation and Zariski-dense subset $Z \subset X(\overline{\mathbb{Q}}_p)$ (which we refer to as the 'classical subset'),

such that the following are satisfied:

(1) For all open affinoids $V \subset Y$ the natural map

$$\psi \otimes \nu^* : \mathscr{H} \otimes \mathcal{O}(V) \to \mathcal{O}(\nu^{-1}(V))$$

   is surjective.

(2) The natural evaluation map

$$X(\overline{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathscr{H}, \overline{\mathbb{Q}}_p),$$
$$x \mapsto \psi_x := (h \mapsto \psi(h)(x))$$

   induces a bijection $z \mapsto (\psi_z, \nu(z))$ from Z to $\mathscr{Z}$.

---

[2]A set is accumulation if each point $z \in Z$ has a basis of affinoid neighbourhoods U such that $Z \cap U$ is Zariski-dense in U.

The key property of the above definition is that an eigenvariety is unique up to unique isomorphism, by [Bellaïche and Chenevier 2009, Proposition 7.2.8].

Remark 3.16. An alternative way to abstractly characterise eigenvarieties in our context is as Zariski closures of sets of classical points in the rigid space given by the product of the rigid generic fibre of a Galois (pseudo)deformation ring and some affine spaces or copies of $\mathbb{G}_m$ (to keep track of additional Hecke eigenvalues).

For $\mathfrak{m}, \mathfrak{n}$ ideals of $\mathbb{O}_F$ (coprime to each other and to p), suppose π is an unramified representation of tame level $U_1(\mathfrak{n}) \cap U_0(\mathfrak{m})$.

There is a natural action of $\mathscr{H}(\mathfrak{m}\mathfrak{n})$ on $(\pi^\infty)^{U_1(\mathfrak{n}) \cap U_0(p\mathfrak{m})}$, where we let T_v and U_v act by double coset operators

$$\left[U \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} U \right]$$

and let S_v act by the double coset operator

$$\left[U \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U \right],$$

where $U = U_1(\mathfrak{n}) \cap U_0(p\mathfrak{m})$.

Given such a π, we obtain a subset $\mathscr{L}(\mathfrak{n}, \mathfrak{m})_\pi$ of $\mathrm{Hom}(\mathscr{H}(\mathfrak{m}\mathfrak{n}), \overline{\mathbb{Q}}_p) \times \hat{T}(\overline{\mathbb{Q}}_p)$ by taking pairs

$$(\psi, \nu(\pi, \chi)),$$

where χ is an accessible refinement of π and ψ is a character corresponding (via ι) to a Hecke eigenform in $(\pi^\infty)^{U_1(\mathfrak{n}) \cap U_0(p\mathfrak{m})}$. Note that any element of the set $\mathscr{L}(\mathfrak{n}, \mathfrak{m})_\pi$ determines π, by strong multiplicity one. The choice of accessible refinement χ corresponds to a choice of $U_\mathfrak{p}$-eigenvalue in the space $(\pi_\mathfrak{p})^{I_\mathfrak{p}}$ for each $\mathfrak{p} \mid p$. The character $\kappa_\pi := \nu(\pi, \chi)|_{T_0}$ is independent of the refinement χ.

Definition 3.17. Let $\mathfrak{m}$ and $\mathfrak{n}$ be a pair of coprime ideals in $\mathbb{O}_F$, both coprime to p.

(1) Denote by $\mathscr{L}(\mathfrak{n}, \mathfrak{m})$ the union of the $\mathscr{L}(\mathfrak{n}, \mathfrak{m})_\pi$ obtained from unramified π with tame level $U_1(\mathfrak{n}) \cap U_0(\mathfrak{m})$ such that $\kappa_\pi \in \mathscr{W}_\kappa$.

(2) For $\mathfrak{q}$ a prime divisor of $\mathfrak{m}$, we write $\mathscr{L}(\mathfrak{n}, \mathfrak{m})^{\mathfrak{q}\text{-sp}}$ for the subset of $\mathscr{L}(\mathfrak{n}, \mathfrak{m})$ arising from π with $\pi^{U_1(\mathfrak{n}) \cap U_0(p\mathfrak{m}/\mathfrak{q})} = 0$ (equivalently, the local factor $\pi_\mathfrak{q}$ is an unramified twist of the Steinberg representation).

(3) Similarly, we write $\mathscr{L}(\mathfrak{n}, \mathfrak{m})^{\mathfrak{q}\text{-ps}}$ for the subset of $\mathscr{L}(\mathfrak{n}, \mathfrak{m})$ arising from π with $\pi^{U_1(\mathfrak{n}) \cap U_0(p\mathfrak{m}/\mathfrak{q})} \neq 0$ (equivalently, the local factor $\pi_\mathfrak{q}$ is an unramified principal series representation).

(4) Now we consider the irreducible mod p Galois representation $\bar{\rho} = \bar{r}_{p,\iota}(\pi_f)$. We can define $\mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}$, $\mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-sp}}$ and $\mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-ps}}$ to be the subsets of $\mathscr{Z}(\mathfrak{n}, \mathfrak{m})$, etc. arising from those π with residual Galois representation $\bar{r}_{p,\iota}(\pi)$ isomorphic to $\bar{\rho}$.

3.18. *Buzzard's eigenvarieties.*

Theorem 3.19. *There exist eigenvarieties* $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}$, $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-sp}}$ *and* $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-ps}}$ *for the triples*

$$(\mathscr{H}(\mathfrak{mn}), \mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}, \mathscr{W}_{\kappa}),$$
$$(\mathscr{H}(\mathfrak{mn}), \mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-sp}}, \mathscr{W}_{\kappa}),$$
$$(\mathscr{H}(\mathfrak{mn}), \mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-ps}}, \mathscr{W}_{\kappa}).$$

We denote the classical subsets of these eigenvarieties by $Z(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}$, $Z(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-sp}}$ *and* $Z(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-ps}}$. *The following properties are satisfied by these eigenvarieties*:

- *There are closed immersions*

$$\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-sp}} \hookrightarrow \mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}} \quad and \quad \mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-ps}} \hookrightarrow \mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}$$

  commuting with the maps to weight space and respecting the homomorphisms ψ *from* $\mathscr{H}(\mathfrak{mn})$, *with images* X, Y, *respectively, given by unions of irreducible components* (in the sense of [Conrad 1999]).

- *Each irreducible component of* $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}$ *is contained in precisely one of* X *and* Y.

- *We have* $Z(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}} \cap X = Z(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-sp}}$ *and* $Z(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}} \cap Y = Z(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-ps}}$.

- *There is a map*

$$\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-ps}} \to \mathscr{E}(\mathfrak{n}, \mathfrak{m}/\mathfrak{q})_{\bar{\rho}},$$

  surjective on closed points, for which the preimage of a closed point $x \in \mathscr{E}(\mathfrak{n}, \mathfrak{m}/\mathfrak{q})_{\bar{\rho}}$ *is indexed by the roots of the Hecke polynomial* $X^2 - \psi(T_{\mathfrak{q}})(x)X + N\mathfrak{q}\psi(S_{\mathfrak{q}})(x)$.

Proof. We fix a definite quaternion algebra D/F, ramified precisely at the infinite places of F, and an isomorphism $(D \otimes_F \mathbb{A}_F^\infty)^\times \cong \mathrm{GL}_2(\mathbb{A}_F^\infty)$. Buzzard's definition [2007, Part III] of overconvergent automorphic forms on D, with tame level $U_1(\mathfrak{n}) \cap U_0(\mathfrak{m})$, allows us to construct an eigenvariety $\mathscr{E}(\mathfrak{n}, \mathfrak{m})$ for the triple $(\mathscr{H}(\mathfrak{mn}), \mathscr{Z}(\mathfrak{n}, \mathfrak{m}), \mathscr{W}_{\kappa})$. The Zariski density and accumulation properties for the classical points follow from a special case of the classicality criterion of [Loeffler 2011, Theorem 3.9.6]. To obtain the eigenvariety for $(\mathscr{H}(\mathfrak{mn}), \mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}, \mathscr{W}_{\kappa})$ we

just take the union of connected components in $\mathscr{E}(\mathfrak{n}, \mathfrak{m})$ whose closed points have associated residual Galois representation isomorphic to $\bar\rho$.

We then define $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar\rho}^{\text{q-sp}}$ to be the Zariski closure in $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar\rho}$ of the subset $Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}^{\text{q-sp}} \subset Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}$ corresponding to systems of Hecke eigenvalues in $\mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar\rho}^{\text{q-sp}}$. Similarly, we define $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar\rho}^{\text{q-ps}}$ to be the Zariski closure in $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar\rho}$ of the subset $Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}^{\text{q-ps}}$. We now need to check that $Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}^{\text{q-sp}}$ and $Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}^{\text{q-ps}}$ are accumulation subsets in their Zariski closures, together with the rest of the assertions of the theorem.

We can deduce everything we need by applying the results of [Bellaïche and Chenevier 2009, 7.8] on the family of Weil–Deligne representations carried by an eigenvariety (see also [Paulin 2011]). We proceed as follows: Denote by $X \subset \mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar\rho}$ the reduced closed subspace given by the union of irreducible components where the monodromy operator in the associated family of Weil–Deligne representations is generically nonzero — we call such a component 'generically special'. More precisely, in the notation of [Bellaïche and Chenevier 2009, 7.8], a generically special component W is one in which for all closed points $x \in W$ we have $N_{s(x)}^{\text{gen}}$ nonzero for $s(x)$ any germ of an irreducible component at x which is contained in W. Then we claim that

$$X \cap Z(\mathfrak{n}\mathfrak{m})_{\bar\rho} = Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}^{\text{q-sp}}.$$

Indeed, if $x \in Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}^{\text{q-sp}}$, then, by Proposition 7.8.19(iii) of [Bellaïche and Chenevier 2009] and local-global compatibility at $\mathfrak{q}$ for the automorphic representation π_x [Carayol 1986b], every irreducible component passing through x is generically special.

Conversely, if $x \in X \cap Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}$, then the Weil–Deligne representation at $\mathfrak{q}$ associated to ρ_x is forced to have the form $(W \oplus W(1), N)$, where W is a one-dimensional $\overline{k(x)}$-vector space with an unramified action of the Weil group $W_{\mathfrak{q}}$ and $W(1)$ denotes the twist of W by the p-adic cyclotomic character (N could be zero or nonzero). This means that either $x \in Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}^{\text{q-sp}}$ or the local factor $\pi_{\mathfrak{q}}$ of the automorphic representation π associated to x is one-dimensional. The latter situation cannot occur, since π is a cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ and therefore its local factors are infinite-dimensional (this follows from the existence of a global Whittaker model; for example, see the proof of Theorem 11.1 of [Jacquet and Langlands 1970]).

Now it is easy to deduce the accumulation property for $Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}^{\text{q-sp}}$ from the accumulation property for $Z(\mathfrak{n}\mathfrak{m})_{\bar\rho}$.

Similar arguments apply if we take $Y \subset \mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar\rho}$ to be the reduced closed subspace given by the union of irreducible components where the monodromy operator in the associated family of Weil–Deligne representations is generically zero — we call such components 'generically principal series'.

Finally we need to construct the map

$$\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\mathrm{q\text{-}ps}} \to \mathscr{E}(\mathfrak{n}, \mathfrak{m}/\mathfrak{q})_{\bar{\rho}}.$$

This can be done by giving an alternative construction of $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\mathrm{q\text{-}ps}}$ — indeed, by the uniqueness of abstract eigenvarieties, $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\mathrm{q\text{-}ps}}$ coincides with the nilreduction of the covering of $\mathscr{E}(\mathfrak{n}, \mathfrak{m}/\mathfrak{q})_{\bar{\rho}}$ given by the roots of $X^2 - \psi(T_{\mathfrak{q}})X + N\mathfrak{q}\psi(S_{\mathfrak{q}})(x)$ (this has a natural interpretation as a relative spectrum over $\mathscr{E}(\mathfrak{n}, \mathfrak{m}/\mathfrak{q})_{\bar{\rho}}$). $\qquad\square$

3.20. *Kisin and Lai's eigenvarieties.* Now we let f be a Hecke eigenform as in the statement of Proposition 2.3. For each prime ideal $\mathfrak{p} \mid p$, we denote the two distinct roots of $X^2 - \theta_f(T_{\mathfrak{p}})X + (N\mathfrak{p})^{1-w}\iota^{-1}\chi_f(\varpi_{\mathfrak{p}})$ by $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$. Then for each subset $S \subset \{\mathfrak{p} \mid p\}$ there is a unique Hecke eigenform

$$f_S \in H^0(\mathscr{M}^{\mathrm{Iw}}(\mathfrak{n}, \mathfrak{ql})_L, \underline{\omega}^{\kappa}),$$

whose Hecke eigenvalues away from p are the same as f, and for $\mathfrak{p} \mid p$ we have $\theta_{f_S}(U_{\mathfrak{p}}) = \alpha_{\mathfrak{p}}$ if $\mathfrak{p} \in S$ and $\theta_{f_S}(U_{\mathfrak{p}}) = \beta_{\mathfrak{p}}$ if $\mathfrak{p} \notin S$.

We moreover define a point $\nu_S \in \widehat{T}(\overline{\mathbb{Q}}_p)$ to be given by the character

$$\bigotimes_{\mathfrak{p} \mid p} \chi_{\mathfrak{p},1} \otimes \chi_{\mathfrak{p},2} \prod_{\tau \in \Sigma_{\mathfrak{p}}} \tau^{(k_\tau - 2 - w)/2} \otimes \tau^{(-w-k_\tau+2)/2},$$

where the $\chi_{\mathfrak{p},i}$ are characters of $F_{\mathfrak{p}}^{\times}/\mathcal{O}_{\mathfrak{p}}^{\times}$ defined by:

- $\chi_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}) = \alpha_{\mathfrak{p}}(N\mathfrak{p})^{-1}$ if $\mathfrak{p} \in T$ and $\chi_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}) = \beta_{\mathfrak{p}}(N\mathfrak{p})^{-1}$ otherwise.
- $\chi_{\mathfrak{p},2}(\varpi_{\mathfrak{p}}) = \beta_{\mathfrak{p}}$ if $\mathfrak{p} \in T$ and $\chi_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}) = \alpha_{\mathfrak{p}}$ otherwise.

Proposition 3.21. *For each $S \subset \{\mathfrak{p} \mid p\}$ there is a point x_S of $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}$ such that*

$$\nu(x_S) = \nu_S,$$

and the character

$$\psi_{x_S} : \mathscr{H}(\mathfrak{nql}) \to \overline{\mathbb{Q}}_p$$

induced by ψ is compatible with θ_{f_S}.

Proof. For this result, we need to use an alternative construction of the eigenvariety $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}$. This is given by the space $\mathscr{C}_{\kappa}(\mathfrak{m})$ of [Kisin and Lai 2005, Theorem 4.5.4] (with modified level structures). Here the $\mathfrak{m}$ corresponds to our choice of residual Galois representation $\bar{\rho}$. To show that $\mathscr{C}_{\kappa}(\mathfrak{m})$ coincides with $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}$, we need to verify that the subset of $\mathscr{C}_{\kappa}(\mathfrak{m})$ corresponding to the 'classical points' $\mathscr{Z}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}$ is Zariski-dense and accumulation. This follows from a classicality criterion for overconvergent Hilbert modular forms, which has recently been proved in two different ways — by Pilloni and Stroh [2011, Théorème 1.2] and by Tian and Xiao [2013, Proposition 6.3]. Note that Theorem 6.5 of the latter is the statement that the classical points are Zariski-dense in the Kisin–Lai eigenvarieties, but the

accumulation property also follows immediately from the proof. The existence of the points x_S is now immediate from the construction of the Kisin–Lai eigenvariety.

Alternatively, one can avoid the appeal to difficult classicality theorems and instead show that $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}$ is isomorphic (with its additional structures) to the Zariski closure of the classical points in $\mathscr{C}_\kappa(\mathfrak{m})$ by working with the set-up described in Remark 3.16. Applying [Kisin and Lai 2005, Theorem 4.5.6] then concludes the proof.   □

Since the local factor $\pi_{f,\mathfrak{q}}$ is an unramified twist of Steinberg, one naturally expects that the points x_S lie in the eigenvariety $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}^{\text{q-sp}}$. Proving that this is the case is a little delicate: the first thing to observe is that since we do not yet know local-global compatibility at $\mathfrak{q}$ for π_f, it is also possible that x_S could lie in $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}^{\text{q-ps}}$.

One can build $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}^{\text{q-sp}}$ using modules given by taking a kernel of trace maps to prime-to-$\mathfrak{q}$ level from the appropriate Banach modules of overconvergent forms. However, forming this kernel (for modules over the ring of functions on an affinoid open U in weight space) does not necessarily commute with base change from U to a point in weight space (the cokernel of the trace maps may not be flat). So the fact that $\pi_{f,\mathfrak{q}}$ is an unramified twist of Steinberg does not immediate imply that x_S lies in $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}^{\text{q-sp}}$. One needs to show that x_S lies in a *family* of forms which are 'q-new'. This problem is solved in [Newton 2014]:

Proposition 3.22. *The points x_S of Proposition 3.21 all lie in $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}^{\text{q-sp}}$.*

Proof. Take a point x_S and suppose that it does not lie in $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}^{\text{q-sp}}$. Then x_S is a point of $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}^{\text{q-ps}}$, so we may consider the image of x_S in $\mathscr{E}(\mathfrak{n}, \mathfrak{l})_{\bar{\rho}}$. This point satisfies the assumptions of [Newton 2014, Theorem 4.3], but the conclusion of this theorem tells us that x_S is indeed in $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}^{\text{q-sp}}$.   □

4. Completed cohomology and level optimisation

We give another construction of the eigenvariety $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-sp}}$, using the completed cohomology of Shimura curves. We will then apply [Newton 2013, Theorem 4.33] to deduce the following result:

Theorem 4.1. *Let $S \subset \{\mathfrak{p} \mid p\}$ and let x_S be the point of $\mathscr{E}(\mathfrak{n}, \mathfrak{ql})_{\bar{\rho}}^{\text{q-sp}}$ obtained from Propositions 3.21 and 3.22. Then there is a point y_S of $\mathscr{E}(\mathfrak{n}, \mathfrak{q})_{\bar{\rho}}^{\text{q-sp}}$ such that*:

- *The Hecke eigenvalues outside $\mathfrak{l}$ coincide with those of x_S.*

- $\psi_{x_S}(U_{\mathfrak{l}})$ *is one of the roots of* $X^2 - \psi_{y_S}(T_{\mathfrak{l}})X + (N\mathfrak{l})^{1-w}\iota^{-1}\chi_f(\varpi_{\mathfrak{l}})$.

The proof of this theorem occupies the rest of the section.

4.2. _Completed cohomology of Shimura curves._ For this section, we fix a quaternion algebra B/F such that B is nonsplit at precisely one infinite place, denoted τ_1, and one finite place, denoted $\mathfrak{q}$ (recall that $[F : \mathbb{Q}]$ is assumed to be even, so such quaternion algebras exist). Denote by G_B the reductive algebraic group over $\mathbb{Q}$ arising from the unit group of B. Note that G_B is an inner form of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$. For U a compact open subgroup of $G_B(\mathbb{A}_f)$, we have a complex (disconnected) Shimura curve

$$M(U)(\mathbb{C}) = G_B(\mathbb{Q})\backslash G_B(\mathbb{A}_f) \times (\mathbb{C} - \mathbb{R})/U,$$

where $G_B(\mathbb{Q})$ acts on $\mathbb{C} - \mathbb{R}$ via the τ_1 factor of $G_B(\mathbb{R})$.

These curves have canonical models over F, which we denote by $M(U)$. We follow the conventions of [Carayol 1986a] to define this canonical model.

Definition 4.3. We define $\tilde{H}^1(U^p, L)$ to be

$$\left(\varprojlim_n \varinjlim_{U_p} H^1_{\text{ét}}(M(U_p U^p)_{\overline{F}}, \mathbb{O}_L/\mathfrak{m}_L^n) \right) \otimes_{\mathbb{O}_L} L,$$

where U^p is any compact open subgroup of $G_B(\mathbb{A}^{\infty,p})$ and U_p runs over the compact open subgroups of $G_B(\mathbb{Q}_p)$.

The L-vector space $\tilde{H}^1(U^p, L)$ is naturally an L-Banach space with an admissible continuous action of $G_B(\mathbb{Q}_p) \cong \prod_{\mathfrak{p} \mid p} \mathrm{GL}_2(F_{\mathfrak{p}})$. Moreover, there is a direct summand $\tilde{H}^1(U^p, L)_{\overline{\rho}}$ such that all the systems of Hecke eigenvalues arising from $\tilde{H}^1(U^p, L)_{\overline{\rho}}$ correspond to Galois representations whose residual representations are isomorphic to $\overline{\rho}$.

We now explain how the systems of Hecke eigenvalues parametrised by the set $\mathscr{L}(\mathfrak{n}, \mathfrak{m})_{\overline{\rho}}^{\text{q-sp}}$ can be seen in the space $\tilde{H}^1(U^p, L)_{\overline{\rho}}$.

Suppose we have a Σ-tuple of integers $k = (k_\tau)_{\tau \in \Sigma}$ with each $k_\tau \geq 2$ and an integer w with $k_\tau = w \mod 2$ for all τ.

We denote by $W_{k,w}$ the L-representation of $\mathrm{GL}_2(F_p)$ defined by

$$\otimes_{\tau \in \Sigma_p} (\tau \circ \det)^{(w-k_\tau+2)/2} \mathrm{Sym}^{k_\tau - 2} V_\tau,$$

where V_τ is the representation of $\mathrm{GL}_2(F_p)$ acting via τ and the standard representation of $\mathrm{GL}_2(L)$. These representations then give rise to lisse étale L-sheaves $\mathscr{F}_{k,w}$ on the curves $M(U)$ (see, for example, [Newton 2013, §3.2]).

We set U^p to be the prime-to-p part of the compact open subgroup of $G_B(\mathbb{A}^\infty)$ given by $U_1(\mathfrak{n}) \cap U_0(\mathfrak{m}/\mathfrak{q})$. Now the Hecke algebra $\mathscr{H}(\mathfrak{mn})$ acts on $\tilde{H}^1(U^p, L)$ as follows: For places v prime to $\mathfrak{q}$, we have a standard action by double coset operators associated to our fixed uniformisers ϖ_v. For the place $\mathfrak{q}$, we choose a uniformiser $\varpi_{D_\mathfrak{q}}$ of the order $\mathbb{O}_{D_\mathfrak{q}}$ whose reduced norm is equal to the fixed

uniformiser $\varpi_{\mathfrak{q}}$ of $\mathbb{O}_{\mathfrak{q}}$, and let $U_{\mathfrak{q}}$ act on $\widetilde{H}^1(U^p, L)$ via the action of $\varpi_{D_{\mathfrak{q}}}$. This definition is explained by the following:

Lemma 4.4. *Let $\pi_{\mathfrak{q}} = \mathrm{St} \otimes \mu$ be an unramified twist of the Steinberg representation of $\mathrm{GL}_2(F_{\mathfrak{q}})$. The local Jacquet–Langlands correspondent $\mathrm{JL}(\pi_{\mathfrak{q}})$ of $\pi_{\mathfrak{q}}$ is the one-dimensional representation of $D_{\mathfrak{q}}^\times$ given by $\mu \circ \mathrm{Nrd}$, where Nrd denotes the reduced norm. The $U_{\mathfrak{q}}$-eigenvalue of the space of Iwahori-invariants in $\pi_{\mathfrak{q}}$ is equal to $\mu(\varpi_{\mathfrak{q}})$, and is therefore equal to the eigenvalue for the action of $\varpi_{D_{\mathfrak{q}}}$ on $\mathrm{JL}(\pi_{\mathfrak{q}})$.*

Proof. This follows from the standard computation of the $U_{\mathfrak{q}}$-eigenvalue of the space of Iwahori-invariants in $\pi_{\mathfrak{q}}$. $\qquad\square$

We have the following proposition, which is proved exactly the same way as [Newton 2013, Theorem 5.2]:

Proposition 4.5. *There is a $G_B(\mathbb{Q}_p)$-, $\mathrm{Gal}(\overline{F}/F)$- and Hecke-equivariant isomorphism*

$$\bigoplus_{(k,w)} \varinjlim_{U_p} H^1_{\text{ét}}(M(U_p U^p)_{\overline{F}}, \mathscr{F}_{k,w})_{\overline{\rho}} \otimes_L W_{k,w}^\vee \cong \widetilde{H}^1(U^p, L)_{\overline{\rho}}^{\mathrm{alg}},$$

where the right-hand side is the space of locally algebraic vectors (in the sense of [Emerton 2011, 4.2.6]) *in the L-Banach space representation $\widetilde{H}^1(U^p, L)_{\overline{\rho}}$.*

The above proposition allows us to determine the contribution of classical automorphic representations to the Jacquet module (in the sense of [Emerton 2006a]) $J_B(\widetilde{H}^1(U^p, L)_{\overline{\rho}})$ of the ($\mathbb{Q}_p$-)locally analytic vectors in $\widetilde{H}^1(U^p, L)_{\overline{\rho}}$:

Lemma 4.6. *There is a T-, $\mathrm{Gal}(\overline{F}/F)$- and Hecke-equivariant embedding*

$$\bigoplus_{(k,w)} \left(\varinjlim_{U_p} H^1_{\text{ét}}(M(U_p U^p)_{\overline{F}}, \mathscr{F}_{k,w})_{\overline{\rho}} \right)_N \otimes_L \chi_{k,w} \hookrightarrow J_B(\widetilde{H}^1(U^p, L)_{\overline{\rho}}),$$

where the subscript N denotes coinvariants (i.e., the classical Jacquet module) and $\chi_{k,w}$ is the character of T given by

$$\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \mapsto \prod_{\tau \in \Sigma_p} \tau(s_1)^{(k_\tau - 2 - w)/2} \tau(s_2)^{(-w - k_\tau + 2)/2}$$

Proof. This follows from left-exactness of the Jacquet module functor and [Emerton 2006a, Proposition 4.3.6], since the highest weight space $(W_{k,w}^\vee)^N$ has T-action given by $\chi_{k,w}$. $\qquad\square$

The following lemma is a standard result in the smooth representation theory of the groups $\mathrm{GL}_2(F_{\mathfrak{p}})$:

Lemma 4.7. *Let μ, μ_1, μ_2 be smooth complex characters of $F_{\mathfrak{p}}^{\times}$.*

(1) *The Jacquet module $\pi(\mu_1, \mu_2)_{N(\mathfrak{p})}$ is isomorphic as a $T_{\mathfrak{p}}$-representation to*

$$\mu_1 |\cdot|_{\mathfrak{p}}^{1/2} \otimes \mu_2 |\cdot|_{\mathfrak{p}}^{-1/2} \oplus \mu_2 |\cdot|_{\mathfrak{p}}^{1/2} \otimes \mu_1 |\cdot|_{\mathfrak{p}}^{-1/2}.$$

(2) *The Jacquet module $(\mathrm{St} \otimes \mu)_{N(\mathfrak{p})}$ is isomorphic as a $T_{\mathfrak{p}}$-representation to*

$$\mu |\cdot|_{\mathfrak{p}} \otimes \mu |\cdot|_{\mathfrak{p}}^{-1}.$$

Proof. See for example [Goldfeld and Hundley 2011, Theorem 8.12.15]. $\square$

As a consequence of Lemma 4.7, together with the Jacquet–Langlands correspondence and the contribution of automorphic representations of $G_B(\mathbb{A})$ to the cohomology of the curves $M(U)$, we obtain the following proposition:

Lemma 4.8. *Suppose $(\psi, \nu) \in \mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\mathrm{q}\text{-}\mathrm{sp}}$. Then there is a nonzero element*

$$v \in J_B(\widetilde{H}^1(U^p, L)_{\bar{\rho}}) \otimes_L \overline{\mathbb{Q}}_p$$

on which the Hecke operators away from p act via the character ψ and on which the torus T acts via the character ν.

The above lemma tells us that the 'classical set' $\mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\mathrm{q}\text{-}\mathrm{sp}}$ can be seen in the locally analytic T-representations $J_B(\widetilde{H}^1(U^p, L)_{\bar{\rho}})$. We now summarise Emerton's construction of an eigenvariety from this representation, and show that it is an eigenvariety for the triple $(\mathscr{H}(\mathfrak{m}\mathfrak{n}), \mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\mathrm{q}\text{-}\mathrm{sp}}, \mathscr{W}_\kappa)$.

The T-representation $J_B(\widetilde{H}^1(U^p, L)_{\bar{\rho}})$ is naturally dual to a coherent sheaf $\mathscr{M}$ on $\widehat{T}$ (see [Emerton 2006b, Proposition 2.3.2]). Denote by Y the fibre product $\widehat{T} \times_{\widehat{T}_0} \mathscr{W}_\kappa$, and let $\mathscr{M}_Y$ denote the pullback of $\mathscr{M}$ to a coherent sheaf on Y.

Taking the relative spectrum of the commutative subalgebra of endomorphisms of this sheaf generated by the Hecke algebra $\mathscr{H}(\mathfrak{m}\mathfrak{n})$ gives a rigid space with a finite map to Y. Passing to the nilreduction gives a reduced rigid space which we denote by $\mathscr{E}_\kappa$. By the above lemma, we have a subset $Z \subset \mathscr{E}_\kappa$ of classical points corresponding to the elements of $\mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\mathrm{q}\text{-}\mathrm{sp}}$.

Lemma 4.9. *The space $\mathscr{E}_\kappa$, together with the classical subset Z, is an eigenvariety for the triple*

$$(\mathscr{H}(\mathfrak{m}\mathfrak{n}), \mathscr{Z}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\mathrm{q}\text{-}\mathrm{sp}}, \mathscr{W}_\kappa).$$

Proof. The only condition we have to check is that Z is an accumulation and Zariski-dense subset of $\mathscr{E}_\kappa$ — everything else follows from the construction of $\mathscr{E}_\kappa$. To prove this, we have to interpret $\mathscr{E}_\kappa$ as part of (the nilreduction of) an eigenvariety constructed as in [Buzzard 2007], following [Emerton 2006a, Proposition 4.2.36; Newton 2013, Lemma 5.13]. The proof is slightly involved, the main reason being

that we can show that eigenvarieties constructed with completed cohomology have nice properties only (a priori) after composing the map to weight space $\widetilde{T}_0$ with a map corresponding to restriction to a finite-index subgroup of T_0. This comes about because [Emerton 2006a, Proposition 4.2.36] applies to 'cofree' modules over an Iwasawa algebra, not 'coprojective' modules.

Choose $U_p \subset G_B(\mathbb{Q}_p)$ a sufficiently small compact open subgroup such that $\widetilde{H}^1(U^p, L)_{\overline{\rho}}$ is a cofree representation of $U_p / \overline{F^\times \cap U_p U^p}$, in the sense of [Newton 2013, Definition 5.7]. Such a U_p exists by [Newton 2013, Corollary 5.8] — in fact, it suffices to take U_p pro-p, since our assumptions on the tame level already ensure that $U_p U^p$ is neat.

Denoting the closed subgroup $\overline{F^\times \cap U_p U^p}$ of U_p by X, we define a compact commutative p-adic analytic group by $S := T_0 \cap U_p / X$, and denote the rigid space parametrising its continuous characters by $\widehat{S}$. The characters corresponding to points of $\mathscr{W}_\kappa$ are trivial on the units of F with norm 1, so (possibly shrinking U_p if $p = 2$) these characters are trivial on X. Therefore restriction to $T_0 \cap U_p$ gives a map $\mathscr{W}_\kappa \to \widehat{S}$. In fact, the definition of $\mathscr{W}_\kappa$ implies that this map is an isomorphism onto its image, which we denote by $\mathscr{W}_S$.

We denote by $\widetilde{\mathscr{W}}_\kappa$ the preimage of $\mathscr{W}_S$ in $\mathscr{W}$. The space $\widetilde{\mathscr{W}}_\kappa$ is a finite disjoint union of open discs, whose components are indexed by characters of the finite group $T_0 / T_0 \cap U_p$. We denote by $\widetilde{Y}$ the fibre product $\widehat{T} \times_{\widehat{T}_0} \widetilde{\mathscr{W}}_\kappa$, and let $\mathscr{M}_{\widetilde{Y}}$ denote the pullback of $\mathscr{M}$ to a coherent sheaf on $\widetilde{Y}$. Mimicking the construction of $\mathscr{E}_\kappa$, we obtain a rigid space $\widetilde{\mathscr{E}}_\kappa$ with a finite map to $\widetilde{Y}$, such that $\mathscr{E}_\kappa$ is the open and closed subspace of $\widetilde{\mathscr{E}}_\kappa$ lying over $\mathscr{W}_\kappa$.

Consider an increasing sequence $X_n = \mathrm{Sp}(A_n)$ of admissible affinoid opens covering $\widehat{S}$, and write M for the space of global sections of $\mathscr{M}$ and M_n for the base change $M \widehat{\otimes}_{\mathscr{C}^{\mathrm{an}}(\widehat{S},L)} A_n$. It follows from [Emerton 2006a, Proposition 4.2.36] that M_n is the finite-slope part of an orthonormalisable Banach A_n-module with the action of a compact operator (coming from the action of an element $z \in T$). We may choose the X_n such that their inverse images in $\widetilde{\mathscr{W}}_\kappa$ are admissible affinoid opens $\widetilde{Y}_n = \mathrm{Sp}(B_n)$ (e.g., closed discs). It follows from [Buzzard 2007, Corollary 2.10] that the modules $M \widehat{\otimes}_{\mathscr{C}^{\mathrm{an}}(\widehat{S},L)} B_n$ are likewise finite-slope parts of orthonormalisable Banach B_n-modules with the action of a compact operator.

It now follows, as in [Chenevier 2004, Corollaire 6.4.4], that the image of each irreducible component of $\widetilde{\mathscr{E}}_\kappa$ in $\mathscr{W}_S$ is the image of a Fredholm hypersurface, and is therefore Zariski-open in $\mathscr{W}_S$. For each irreducible component of $\widetilde{\mathscr{E}}_\kappa$, the map to $\mathscr{W}_S$ factors through one of the connected components of $\widetilde{\mathscr{W}}_\kappa$, so each irreducible component of $\widetilde{\mathscr{E}}_\kappa$ has Zariski-open image in this connected component. In particular, each irreducible component of $\mathscr{E}_\kappa$ has Zariski-open image in $\mathscr{W}_\kappa$.

The classicality criterion of [Emerton 2006a, Theorem 4.4.5] now shows that Z is Zariski-dense and accumulation in $\mathscr{E}_\kappa$.   □

Now that we know that $\mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-sp}}$ and $\mathscr{E}_\kappa$ are eigenvarieties for the same triple, we have the following:

Corollary 4.10. *The closed points $x \in \mathscr{E}(\mathfrak{n}, \mathfrak{m})_{\bar{\rho}}^{\text{q-sp}}$ with $\nu(x) = \lambda \in \hat{T}(\overline{\mathbb{Q}}_p)$ correspond bijectively with systems of Hecke eigenvalues appearing in the* (*finite-dimensional*) $\overline{\mathbb{Q}}_p$*-vector space*

$$J_B(\tilde{H}^1(U^p, L)_{\bar{\rho}}) \otimes_L \overline{\mathbb{Q}}_p[\lambda],$$

defined to be the subspace where T acts via the character λ.

Proof. This follows from the construction of $\mathscr{E}_\kappa$ and Lemma 4.9.   □

Now Theorem 4.1 follows from combining Corollary 4.10 and [Newton 2013, Theorem 4.33].

5. Proof of Theorem 1.4

We can also construct $\mathscr{E}(\mathfrak{n}, \mathfrak{q})_{\bar{\rho}}$ using the overconvergent Hilbert modular forms of [Kisin and Lai 2005]. Therefore we conclude from Theorem 4.1 that there are overconvergent Hilbert modular eigenforms $g_S \in M_\kappa^\dagger(\mathfrak{n}, \mathfrak{q})$ whose systems of Hecke eigenvalues correspond to those of the points y_S.

Applying [Kassaei 2013, Theorem 7.10], we glue the g_S into a classical Hilbert modular eigenform

$$g \in H^0(\mathscr{M}(\mathfrak{n}, \mathfrak{q})_L, \underline{\omega}^\kappa)$$

as in the statement of Proposition 2.3 (we are using more general tame levels than Kassaei, but this presents no problem). This completes the proof of Proposition 2.3, and hence of Theorem 1.4.

Remark 5.1. Note that the situation in the above proof is that we have an overconvergent form g_S with level prime to $\mathfrak{l}$ such that one of its $\mathfrak{l}$-stabilisations is equal to the classical form f_S. One might guess that g_S can be obtained from f_S by applying a trace map, in which case it is immediate that g_S is classical. However, in the case we are considering all the trace maps vanish on f_S.

A simple example illustrating that some argument is required here is given by the classical weight-two Eisenstein series of level $\Gamma_0(l)$ (note that our level-lowering result [Newton 2013, Theorem 4.33] does not apply to this form). It is in the kernel of the trace maps, but it is also the l-stabilisation of a (nonoverconvergent) p-adic modular form.

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