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p-adic Hodge-theoretic properties of étale cohomology with mod p coefficients, and the cohomology of Shimura varieties

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We prove vanishing results for the cohomology of unitary Shimura varieties with integral coefficients at arbitrary level, and deduce applications to the weight part of Serre's conjecture. In order to do this, we show that the mod p cohomology of a smooth projective variety with semistable reduction over K, a finite extension of $\mathbb{Q}_p$, embeds into the reduction modulo p of a semistable Galois representation with Hodge–Tate weights in the expected range (at least after semisimplifying, in the case of the cohomological degree greater than 1).

The aim of this paper is to establish vanishing results for the cohomology of certain unitary similitude groups. For example, we prove the following result:

Theorem A. *Let X be a projective $U(2,1)$-Shimura variety of some sufficiently small level, and let $\mathcal{F}$ be a canonical local system of $\overline{\mathbb{F}}_p$-vector spaces on X. Let $\mathfrak{m}$ be a maximal ideal of the Hecke algebra acting on the cohomology $H^\bullet(X,\mathcal{F})$, and suppose that there is a Galois representation $\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ associated to $\mathfrak{m}$. If we suppose further that we have $\mathrm{SL}_3(k) \subset \rho_{\mathfrak{m}}(G_F) \subset \overline{\mathbb{F}}_p^\times \mathrm{SL}_3(k)$ for some finite extension $k/\mathbb{F}_p$, and that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}$ is 1-regular and irreducible, then the localisations $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathcal{F})_{\mathfrak{m}}$ vanish in degrees $i \neq 2$.*

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(See Corollary 3.5.1 and Lemma 4.1.9 below, and see Sections 2 and 3 for the precise definitions that we are using; for simplicity we work with $U(2,1)$-Shimura varieties[1] over a quadratic imaginary field F. Note that "sufficiently small level" means that the compact open subgroup defining the level is sufficiently small. We say that a Galois representation is associated to a maximal ideal of a Hecke algebra if there is the usual relation between Hecke polynomials and characteristic polynomials of Frobenius elements at unramified places; see Section 3.4 for a precise definition.)

In fact, we prove a version of this result for $U(n-1,1)$-Shimura varieties under weaker assumptions on $\rho_{\mathfrak{m}}$; however, in general we can only prove vanishing in degrees outside of the range $[n/2,(3n-4)/2]$.

We also prove the following result, which makes no explicit reference to a maximal ideal in the Hecke algebra:

Theorem B. *Let X and $\mathscr{F}$ be as in the statement of Theorem A. If ρ is a 3-dimensional irreducible sub-G_F-representation of the étale cohomology group $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathscr{F})$, then either every irreducible subquotient of $\rho_{|G_{\mathbb{Q}_p}}$ is 1-dimensional, or else $\rho_{|G_{\mathbb{Q}_p}}$ is not 1-regular, or else $\rho(G_F)$ is not generated by its subset of regular elements.*

Note that in neither theorem do we make any assumption on the level of the Shimura variety at p.

A Galois representation $\rho_{\mathfrak{m}}$ as in the statement of Theorem A is known to exist if $\mathfrak{m}$ corresponds to a system of Hecke eigenvalues arising from the reduction mod p of the Hecke eigenvalues attached to some automorphic Hecke eigenform. Furthermore, recent work of Scholze [2013] (which appeared after the first version of this paper was written) implies that such a representation exists for *any* maximal ideal $\mathfrak{m}$.

It seems reasonable to believe that any irreducible sub-G_F-representation of any of the étale cohomology groups $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathscr{F})$ for any of the Shimura varieties under consideration should in fact be a constituent of $\rho_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$ of the Hecke algebra. However, this doesn't seem to be known, and relating the "abstract" G_F-representations $\rho_{\mathfrak{m}}$ to the "physical" G_F-representations appearing on étale cohomology is one of the problems we have to deal with in proving our results.

Application to Serre-type conjectures. We are able to combine our results with those of [Emerton et al. 2013] so as to establish cases of the weight part of the Serre-type conjecture of [Herzig 2009] for $U(2,1)$. More precisely, we have the following result (where the assertion that $\bar{\rho}$ is modular means that the corresponding system of Hecke eigenvalues occurs in the mod p cohomology of some $U(2,1)$-Shimura variety; see Theorem 3.5.6 and Lemma 4.1.9.)

[1] These Shimura varieties might more properly be called GU(2, 1)-Shimura varieties; see Section 3 for their definition.

Theorem C. *Suppose $\rho : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ satisfies $\mathrm{SL}_3(k) \subset \rho(G_F) \subset \overline{\mathbb{F}}_p^\times \mathrm{SL}_3(k)$ for some finite extension $k/\mathbb{F}_p$, that $\rho|_{G_{\mathbb{Q}_p}}$ is irreducible and 1-regular, and that ρ is modular of some strongly generic weight. Then the set of generic weights for which ρ is modular is exactly the set predicted by the recipe of [Herzig 2009].*

Relationship with a mod p analogue of Arthur's conjecture. Arthur [1989, §9] made a quite precise conjecture regarding the systems of Hecke eigenvalues that appear in the L^2-automorphic spectrum of any reductive group over a number field, which has consequences for the nature of the Hecke eigenvalues appearing in the cohomology of Shimura varieties. For our purposes it suffices to describe a qualitative version of these consequences: namely, Arthur's conjecture implies that if λ is a system of Hecke eigenvalues appearing in the degree-i cohomology, where i is less than the middle dimension, then λ is attached (in the sense of, e.g., [Buzzard and Gee 2014; Johansson 2013]) to a reducible Galois representation (i.e., one which factors through a parabolic subgroup of the L-group).

The fragmentary evidence available suggests that a similar statement will be true for the mod p cohomology of Shimura varieties. Our Theorems A and B give further evidence in this direction.

p-adic Hodge theory. In order to prove these theorems, we establish some new results about the p-adic Hodge-theoretic properties of the étale cohomology of varieties over a number field or p-adic field with coefficients in a field of characteristic p. In the first section we establish results about the mod p étale cohomology of varieties over number fields or p-adic fields which, although weaker in their conclusions, are substantially broader in the scope of their application than previously known mod p comparison theorems. For example, we prove the following result (see Theorem 1.4.1 below):

Theorem D. *Let K be a finite extension of $\mathbb{Q}_p$, and write G_K for the absolute Galois group of K. If X is a smooth projective variety over K which has semistable reduction, and if ρ is an irreducible subquotient of the G_K-representation $H^i_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{F}}_p)$, then ρ also embeds as a subquotient of a G_K-representation over $\overline{\mathbb{F}}_p$ which is the reduction modulo the maximal ideal of a G_K-invariant $\overline{\mathbb{Z}}_p$-lattice in a G_K-representation over $\overline{\mathbb{Q}}_p$ which is semistable with Hodge–Tate weights contained in the interval* $[-i, 0]$.

Both the hypotheses and the conclusions of our theorems are rather precisely tailored to maximise (as far as we are able) their utility in applications to the analysis of Galois representations occurring in the cohomology of Shimura varieties, which we give in the third section.

The remaining two sections of the paper are devoted respectively to using integral p-adic Hodge theory (Breuil modules with descent data) to establish a result related to the reductions of tamely potentially semistable p-adic representations of $G_{\mathbb{Q}_p}$ (Section 2) and to proving some technical results about group representations (Section 4). The result of Section 2 is an essential ingredient in the arguments of Section 3, while the results of Section 4 provide sufficient conditions for the various representation-theoretic hypotheses appearing in the results of Section 3 to be satisfied.

Remark on related papers. Very general vanishing theorems for the mod p cohomology of Shimura varieties have been proved by Lan and Suh [2013]; however, their results apply only in situations of good reduction and for coefficients corresponding to small Serre weights, which makes them unsuitable for the kinds of applications we have in mind, e.g., to the weight part of Serre-type conjectures. In the ordinary case there is the work of Mokrane and Tilouine [2002, §9] in the Siegel case and Dimitrov [2005, §6.4] in the case of Hilbert modular varieties. Finally, in a recent preprint, Shin [2013] proved a general vanishing result for cohomology outside of middle degree for the part of the mod p cohomology which is supercuspidal at some prime $l \neq p$, by completely different methods from those of this paper. It seems plausible that, via the mod p local Langlands correspondence for $\mathrm{GL}_n(\mathbb{Q}_l)$, Shin's hypothesis could be interpreted as a condition on the restriction to a decomposition group at l of the relevant mod p Galois representations, whereas our conditions involve the restriction to a decomposition group at p, so our results appear to be complementary.

Conventions. For any field K we let G_K denote a choice of absolute Galois group of K.

If K is a finite field, then by a Frobenius element in G_K we will always mean a geometric Frobenius element. We extend this convention in an evident way to Frobenius elements at primes in Galois groups of number fields, and to Frobenius elements in Galois groups of local fields.

If K is a local field, then we denote by $\mathcal{O}_K$ the ring of integers of K, by I_K the inertia subgroup of G_K, by W_K the Weil group of K (the subgroup of G_K consisting of elements whose reduction modulo I_K is an integral power of Frobenius), and by WD_K the Weil–Deligne group of K.

If K is a number field and v is a finite place of K, then we will write K_v for the completion of K at v and $\mathcal{O}_{K_v}$ for its ring of integers. We will write $\mathcal{O}_{K,(v)}$ for the localisation of $\mathcal{O}_K$ at the prime ideal v.

We will write $\overline{\mathbb{Z}}_p$ for the ring of integers in $\overline{\mathbb{Q}}_p$ (a fixed algebraic closure of $\mathbb{Q}_p$), and $\mathfrak{m}\overline{\mathbb{Z}}_p$ for the maximal ideal of $\overline{\mathbb{Z}}_p$.

We let ω denote the mod p cyclotomic character. We will denote a Teichmüller lift with a tilde, so that for example $\tilde{\omega}$ is the Teichmüller lift of ω.

We use the traditional normalisation of Hodge–Tate weights, with respect to which the cyclotomic character has Hodge–Tate weight 1.

By a *closed geometric point* $\overline{x}$ of a scheme X, we mean a morphism of schemes $\overline{x} : \operatorname{Spec} \Omega \to X$ for a separably closed field Ω, whose image is a closed point x of X, and such that the induced embedding $\kappa(x) \hookrightarrow \Omega$ (where $\kappa(x)$ denotes the residue field of x) identifies Ω with a separable closure of $\kappa(x)$. If $\overline{x}$ is a closed geometric point of a Noetherian scheme X, then we let $\mathbb{O}_{X,\overline{x}}$ denote the local ring of X at $\overline{x}$, i.e., the stalk, in the étale topology on X, of the structure sheaf of X at $\overline{x}$; we let $(\mathbb{O}_{X,\overline{x}})^\wedge$ denote the completion of $\mathbb{O}_{X,\overline{x}}$, and we write $(X_{\overline{x}})^\wedge := \operatorname{Spf}((\mathbb{O}_{X,\overline{x}})^\wedge)$, and refer to $(X_{\overline{x}})^\wedge$ as the formal completion of X along the closed geometric point $\overline{x}$.

The symbol G will always denote a group; in Section 3 it will be a certain algebraic group, and in Section 4 it will be a finite group.

1. p-adic Hodge theoretic properties of mod p cohomology

1.1. ***Introduction.*** We now describe in more detail our results on the integral p-adic Hodge theory of the étale cohomology of projective varieties, which are perhaps the most novel part of this paper.

It is well-known that integral p-adic Hodge theory is less robust than the corresponding theory with rational coefficients; for example, the comparison theorems for integral and mod p étale cohomology due to Fontaine and Messing [1987] and Faltings [1989] involve restrictions both on the degrees of cohomology and the dimensions of the varieties considered, and they also require that the field K be absolutely unramified and that the variety under consideration be of good reduction. More recently, Caruso [2008] has proved an integral comparison theorem in the case of semistable reduction for possibly ramified fields K, but there are still restrictions: his result requires that $ei < p-1$, where e is the absolute ramification index of K, and i is the degree of cohomology under consideration.

These restrictions are unfortunate, since mod p and integral p-adic Hodge theory are among the most powerful local tools available for the analysis of Galois representations occurring in the mod p étale cohomology of varieties. The premise that underlies the present work is that frequently in such applications one does not need a precise comparison theorem relating the mod p étale cohomology to an analogous structure involving mod p de Rham or crystalline cohomology. Rather, one often uses the comparison theorem merely to draw much less specific conclusions, such as that the Galois representations occurring in certain mod p étale cohomology spaces are in the essential image of the Fontaine–Laffaille functor, applied to Fontaine–Laffaille modules whose Fontaine–Laffaille numbers lie in some prescribed range. Our aim is to establish results of the latter type in more general contexts than they have previously been proved.

The precise direction of our work is informed to a significant extent by the fairly recent development of a rich *internal* integral p-adic Hodge theory, by Breuil

[2000], Kisin [2006], Liu [2008] and others. What we mean here by the word "internal" is that these developments have been directed not so much at applications to comparison theorems, but rather at the purely Galois-theoretic problem of giving a p-adic Hodge-theoretic description of Galois-invariant lattices in crystalline or semistable Galois representations, and of the mod p Galois representations that appear in the reductions of such lattices. These tools, especially the theory of *Breuil modules* [2000], which provides the desired description of the mod p representations arising as reductions of such lattices, have proved very useful in arithmetic applications. Because of the availability of these tools, it has become both possible and worthwhile to move beyond the Fontaine–Laffaille context in integral p-adic Hodge theory. While Caruso's work mentioned above is a significant step in this direction, an important aspect of the present work will be the consideration of situations in which the bound $ei < p-1$, required for the validity of the comparison theorem of [Caruso 2008], does not hold.

Our goal, then, is to establish in various situations that a Galois representation appearing in the mod p étale cohomology of a variety can be embedded in the reduction of a Galois-invariant lattice contained in a crystalline or semistable Galois representation, with Hodge–Tate weights lying in some specified range (namely, the range that one would expect given the degree of the cohomology space under consideration). Since, in arithmetic situations, one frequently has to make a ramified base change in order to obtain good or semistable reduction, and since the resulting descent data on the associated Breuil module typically then play an important role in whatever analysis has to be undertaken, we also prove results in certain cases of potentially semistable reduction which allow us to gain some control over these descent data.

The idea underlying our approach is very simple. Suppose that X is a variety over a p-adic field K. If i is some degree of cohomology, then we have a short exact sequence

$$0 \longrightarrow H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)/pH^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \longrightarrow H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p) \longrightarrow H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)[p] \longrightarrow 0,$$

as well as an isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \xrightarrow{\sim} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p).$$

Thus, if both $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ and $H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ are torsion-free, then we see that $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)$ is the reduction mod p of $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$, which *is* a Galois-invariant lattice in $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Furthermore, the usual comparison theorems of *rational* p-adic Hodge theory [Faltings 1989; Tsuji 1999] can be applied to conclude that this latter representation is, e.g., crystalline (if X is proper with good reduction) or semistable (if X is proper with semistable reduction).

The obstruction to implementing this idea is that we have no reason to believe in general that $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ and $H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ will be torsion-free. To get around this difficulty, we engage in various dévissages using the weak Lefschetz theorem. To explain these, first consider the case when X is a projective curve and $i = 1$. In this case all the cohomology with $\mathbb{Z}_p$-coefficients is certainly torsion-free, and so $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)$ *is* the reduction of a Galois-invariant lattice in $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Now a simple induction using the weak Lefschetz theorem shows that for *any* smooth projective variety X over K there is an embedding

$$H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p) \hookrightarrow H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{F}_p),$$

where C is a smooth projective curve. Furthermore, if X has good (respectively semistable) reduction, we can ensure that the same is true of C. This gives the desired result in the case of H^1 (ignoring for a moment the problem of obtaining a refinement dealing with descent data in the potentially semistable case).

For higher degrees of cohomology, a more elaborate dévissage is required. The key point, again established via the weak Lefschetz theorem, is that if X is smooth and projective of dimension d, and if Y and Z are sufficiently generic hyperplane sections of X, then the cohomology of the pair $((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}})$, with either $\mathbb{Z}_p$ or $\mathbb{F}_p$ coefficients, vanishes in degrees other than d (see Section A.3 of the appendix; note that $X \setminus Y$ is affine), so that $H^d_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, \mathbb{Z}_p)$ is torsion-free and is thus a Galois-invariant lattice in $H^d_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, \mathbb{Q}_p)$, which is potentially semistable by [Yamashita 2011], and whose reduction is equal to $H^d_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, \mathbb{F}_p)$. Such relative cohomology spaces are the essential ingredient of the *basic lemma* of [Beilinson 1987], and we learned the idea of using them as building blocks for the cohomology of varieties from Nori [2002], who has used the basic lemma as the foundation of his approach to the construction of motives. Indeed, our present approach to integral p-adic Hodge theory was inspired by Beilinson's and Nori's work.

1.2. ***Bertini-type theorems.*** We begin by giving a straightforward generalisation of some of the results of [Jannsen and Saito 2012], which build on the results of [Poonen 2004] to prove Bertini-type theorems for varieties with semistable reduction over a discrete valuation ring. It will be convenient to allow K to denote either a number field, or a field of characteristic zero that is complete with respect to a discrete valuation with perfect residue field k_K of characteristic p. We abbreviate these two situations as "the global case" and "the local case" respectively, and in the former case we will let v denote a place of K dividing p.

We recall the following definition:

Definition 1.2.1. Suppose first that we are in the local case. We then say that a projective $\mathscr{O}_K$-scheme $\mathscr{X}$ is *semistable* if it is regular and flat over $\operatorname{Spec}\mathscr{O}_K$, and if the special fibre $\mathscr{X}_s$ is reduced and is a normal crossings divisor; equivalently, a finite-type $\mathscr{O}_K$-scheme $\mathscr{X}$ is semistable if, at each closed geometric point $\bar{x}$ of $\mathscr{X}_s$, there is an isomorphism of complete local rings

$$(\mathscr{O}_{\bar{x},\mathscr{X}})^\wedge \cong ((\mathscr{O}_K^{\mathrm{sh}})^\wedge)[\![x_1,\ldots,x_n]\!]/(x_1\cdots x_m - \varpi_K),$$

where $(\mathscr{O}_K^{\mathrm{sh}})^\wedge$ is the completion of the strict Henselisation of $\mathscr{O}_K$ (equivalently, the completion of the ring of integers in the maximal unramified algebraic extension of K), the element ϖ_K is a uniformiser of $(\mathscr{O}_K^{\mathrm{sh}})^\wedge$, and $1 \le m \le n$. We say that $\mathscr{X}$ is strictly semistable if it is semistable and if the special fibre $\mathscr{X}_s$ is a strict normal crossings divisor.

Again in the local case, we say that a smooth projective K-scheme has *good reduction* if it admits a smooth projective model over $\mathscr{O}_K$, and that it has (*strictly*) *semistable reduction* if it admits an extension to a projective $\mathscr{O}_K$-scheme which is (strictly) semistable in the sense of the preceding definition.

In the global case, we say that a smooth projective K-scheme has good reduction at v if it admits a smooth projective model over $\mathscr{O}_{K,(v)}$, and that it has (strictly) semistable reduction at v if it admits a (strictly) semistable projective model over $\mathscr{O}_{K,(v)}$, i.e., a projective model over $\mathscr{O}_{K,(v)}$ whose base change over $\mathscr{O}_{K_v}$ is (strictly) semistable in the sense of the preceding definition.

Remark 1.2.2. Note that our definition of a semistable $\mathscr{O}_K$-scheme (putting ourselves in the local case) includes the requirement that the scheme be regular. This is the definition that is frequently adopted in the theory of semistable reduction, and it is well-suited to our intended applications. Recall that, with this definition, semistability is *not* preserved under the base change to $\mathscr{O}_L$, if L is a finite extension of K, unless L/K is unramified or the original scheme is in fact smooth over $\mathscr{O}_K$; see also Remark 1.5.2 below.

Proposition 1.2.3. *Let X be a smooth projective variety over K with strictly semistable* (*respectively good*) *reduction* (*at v, in the global case*). *Then there are smooth hypersurface sections Y and Z of X* (*with respect to an appropriately chosen embedding of X into some projective space*) *such that Y and Z intersect transversely, and all of Y, Z, and $Y \cap Z$ have strictly semistable* (*respectively good*) *reduction* (*at v, in the global case*).

Proof. We first handle the local case. Choose an extension $\mathscr{X}$ of X to an $\mathscr{O}_K$-scheme that is projective and smooth (in the good reduction case) or strictly semistable (in the strictly semistable reduction case), and fix an embedding of $\mathscr{X}$ into some projective space over $\mathscr{O}_K$. By Corollaries 0 and 1 of [Jannsen and Saito 2012] (or, perhaps more precisely, by their proofs) we can find a hypersurface section $\mathcal{Y}$ of $\mathcal{X}$ such that $\mathcal{Y}$ is again either smooth or strictly semistable over $\mathcal{O}_K$. We take Y to be the generic fibre of $\mathcal{Y}$. By Remark 0(ii), together with Lemma 1 and the remark immediately before Corollary 1, of [Jannsen and Saito 2012], we see that in order to find Z it is enough to check that, given a finite collection $X_1, \dots, X_n$ of smooth projective schemes in $\mathbb{P}^N_{/k_K}$, there is a common hypersurface section meeting each of them transversely. This is an immediate consequence of Theorem 1.3 of [Poonen 2004], taking the set U_P there to be the subset of the completion $\widehat{\mathcal{O}}_P$ consisting of the f such that $f = 0$ is transverse to each X_i at P. (Since this set contains all the f which do not vanish at P, and in particular contains all the f congruent, modulo the maximal ideal, to a particular choice of f, it has positive Haar measure.)

We now pass to the global case. Let $\mathcal{X}$ be a smooth (in the good reduction case) or strictly semistable (in the strictly semistable reduction case) projective model of X over $\mathcal{O}_{K,(v)}$. Let P_d^* denote the projective space (over $\mathcal{O}_{K,(v)}$) of degree-d hypersurfaces in the ambient projective space containing $\mathcal{X}$. Applying the argument in the local case to the base change $\mathcal{X}_{/\mathcal{O}_{K_v}}$, we see that, for some $d \geq 1$, there is a K_v-valued point of P_d^* corresponding to a hypersurface section of $\mathcal{X}_{/\mathcal{O}_{K_v}}$ having either smooth or semistable intersection (depending on the case we are in) with $\mathcal{X}_{/\mathcal{O}_{K_v}}$. Furthermore, this point lies in an affinoid open subset of P^*_{d/K_v} (the preimage of an open set in the special fibre of P_d^*), all of whose points correspond to hyperplane sections of $\mathcal{X}_{/\mathcal{O}_{K_v}}$ with either smooth or strictly semistable intersection. (See Remark 0(i) and the proofs of Theorems 0 and 1 of [Jannsen and Saito 2012].) The set of K_v-points of this affinoid open set is a nonempty open subset of $P_d^*(K_v)$. Since K is dense in K_v, we see that this intersection also contains a K-point of P_d^*, which gives the required hypersurface section Y. We find the hypersurface section Z by applying the same argument. $\square$

1.3. *Cohomology in degree* 1. Our arguments in degree 1 are rather simpler than in general degree, so we warm up with this case. (In fact, our result in this case is slightly stronger than our result in general degree, as we do not need to semisimplify the representation, so this result is not completely subsumed by our later results in general degree.) Fix a prime p. Let K denote a field of characteristic zero, complete with respect to a discrete valuation, with ring of integers $\mathcal{O}_K$ and residue field k, assumed to be perfect of characteristic p. Let $\overline{K}$ denote an algebraic closure of K, and set $G_K := \mathrm{Gal}(\overline{K}/K)$.

Theorem 1.3.1. *If X is a smooth projective variety over K which has good (resp. strictly semistable) reduction, then $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)$ embeds G_K-equivariantly into the reduction modulo p of a G_K-invariant lattice in a crystalline (resp. semistable) p-adic representation of G_K whose Hodge–Tate weights are contained in* $[-1, 0]$.

Proof. We proceed by induction on the dimension d of X. If $d \leq 1$ then $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)$ is isomorphic to the reduction modulo p of $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$, and the latter space (being torsion-free, by virtue of our assumption on d) is in turn a lattice in $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, which is crystalline or semistable, respectively, with Hodge–Tate weights lying in $[-1, 0]$, by the main result of [Tsuji 1999].

Suppose now that $d > 1$. It follows from Corollary 0 (resp. Corollary 1) of [Jannsen and Saito 2012] that if X has good reduction (resp. strictly semistable reduction) then we may choose a smooth hypersurface section Y of X defined over K which has good (resp. strictly semistable) reduction. Our induction hypothesis applies to show that $H^1_{\text{ét}}(Y_{\overline{K}}, \mathbb{F}_p)$ embeds as a subobject of a G_K-representation over $\mathbb{F}_p$ which is the reduction modulo p of a G_K-invariant lattice in a crystalline (resp. semistable) p-adic representation of G_K whose Hodge–Tate weights are contained in $[-1, 0]$. On the other hand, the weak Lefschetz theorem with $\mathbb{F}_p$-coefficients [SGA 4_3 1973, Exposé XIV, Corollaire 3.3] implies that the natural (restriction) map $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p) \to H^1_{\text{ét}}(Y_{\overline{K}}, \mathbb{F}_p)$ is an embedding (because $1 \leq d - 1$ by assumption). This completes the proof. □

1.4. ***Cohomology of arbitrary degree.*** As always we fix a prime p. The necessary dévissages in this subsection will be more elaborate than in the previous one, and so, to maximise the utility of our results for later applications, it will be convenient to again allow K to denote either a number field or a field of characteristic zero that is complete with respect to a discrete valuation with perfect residue field of characteristic p. In applications it will also be useful to have flexibility in the choice of coefficients in the various cohomology spaces that we consider, and to this end we fix an algebraic extension E of $\mathbb{Q}_p$, with ring of integers $\mathbb{O}_E$ and residue field k_E. (In applications, E will typically either be a finite extension of $\mathbb{Q}_p$, or else will be $\overline{\mathbb{Q}}_p$.)

We now recall some consequences of the weak Lefschetz theorem. Among other notions, we will use the étale cohomology of a pair consisting of a variety and a closed subvariety; a precise definition of this cohomology, and a verification of its basic properties (such as those recalled in the next paragraph), is included in the Appendix.

Let X be a smooth projective variety of dimension d over K, and suppose that Y and Z are two smooth hypersurface sections of X, chosen so that $Y \cap Z$ is also smooth. Let A denote either E, $\mathbb{O}_E$, or k_E. In either the first or last case, the spaces $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, A)$ and $H^{2d-i}_{\text{ét}}((X \setminus Z)_{\overline{K}}, (Y \setminus Z)_{\overline{K}}, A)(d)$ are naturally dual to one another for each integer i. The weak Lefschetz theorem implies that the former space vanishes when $i > d$ and the latter space vanishes when $2d - i > d$, i.e., when $i < d$. Thus, in fact, both spaces vanish unless $i = d$. It then follows that both spaces vanish unless $i = d$ in the case when A is taken to be $\mathbb{O}_E$ as well, and hence that when $i = d$ both spaces are torsion-free.

Let $\overline{K}$ denote an algebraic closure of K, and set $G_K := \mathrm{Gal}(\overline{K}/K)$. Now let $\rho : G_K \to \mathrm{GL}_n(k_E)$ be irreducible and continuous. In the global case, we fix a place v of K lying over p, and a decomposition group $D_v \subset G_K$ for v.

Theorem 1.4.1. *If X is a smooth projective variety over K which has strictly semistable* (*resp. good*) *reduction* (*at v, if we are in the global case*), *and if ρ embeds as a subquotient of $H^i_{\text{ét}}(X_{\overline{K}}, k_E)$, then ρ also embeds as a subquotient of a G_K-representation over k_E which is the reduction modulo the uniformiser of a G_K-invariant $\mathbb{O}_E$-lattice in a G_K-representation which is semistable* (*resp. crystalline*) (*at v, in the global case*) *with Hodge–Tate weights contained in the interval* $[-i, 0]$.

Proof. We proceed by induction on the dimension of X. Suppose initially that we are in the strictly semistable reduction case. By Proposition 1.2.3, we can and do choose smooth hypersurface sections Y and Z, having smooth intersection, and such that Y, Z, and $Y \cap Z$ all have strictly semistable reduction.

We then consider the long exact sequences

$$\cdots \longrightarrow H^i_{Y_{\overline{K}},\text{ét}}(X_{\overline{K}}, A) \longrightarrow H^i_{\text{ét}}(X_{\overline{K}}, A) \longrightarrow H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, A) \longrightarrow \cdots,$$
$$\cdots \longrightarrow H^i_{(Y\cap Z)_{\overline{K}},\text{ét}}(Z_{\overline{K}}, A) \longrightarrow H^i_{\text{ét}}(Z_{\overline{K}}, A) \longrightarrow H^i_{\text{ét}}((Z \setminus Y)_{\overline{K}}, A) \longrightarrow \cdots,$$
$$\cdots \to H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, (Z\setminus Y)_{\overline{K}}, A) \to H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, A) \to H^i_{\text{ét}}((Z\setminus Y)_{\overline{K}}, A) \to \cdots,$$

with A taken to be either E or k_E (see [Milne 1980, Chapter III, Proposition 1.25] for the first two, which are local cohomology long exact sequences, and the Appendix for the third, which is the long exact sequence of the pair $(X \setminus Y, Z \setminus Y)$). We also recall (see [SGA 4_3 1973, Exposé XIV, §3]) that there are canonical isomorphisms

$$H^{i-2}_{\text{ét}}(Y_{\overline{K}}, A)(-1) \xrightarrow{\sim} H^i_{Y_{\overline{K}},\text{ét}}(X_{\overline{K}}, A),$$
$$H^{i-2}_{\text{ét}}((Y \cap Z)_{\overline{K}}, A)(-1) \xrightarrow{\sim} H^i_{(Y\cap Z)_{\overline{K}},\text{ét}}(Z_{\overline{K}}, A).$$

When $A = E$, all the cohomology spaces that appear are potentially semistable [Yamashita 2011]. Since $H^i_{\text{ét}}(X_{\overline{K}}, E)$, $H^i_{\text{ét}}(Y_{\overline{K}}, E)$, and $H^i_{\text{ét}}(Z_{\overline{K}}, E)$ are semistable with Hodge–Tate weights lying in $[-i, 0]$, we see that $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, E)$, $H^i_{\text{ét}}((Z \setminus Y)_{\overline{K}}, E)$, and $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, E)$ are semistable, with Hodge–Tate weights lying in $[-i, 0]$.

Now taking $A = k_E$, we see that, since ρ is irreducible, it appears as a subquotient of $H^{i-2}_{\text{ét}}(Y_{\overline{K}}, k_E)(-1)$, of $H^i_{\text{ét}}(Z_{\overline{K}}, k_E)$, of $H^{i-1}(Y \cap Z)(-1)$, or of $H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, (Z\setminus Y)_{\overline{K}}, k_E)$. In the first three cases, the theorem follows by induction on the dimension. In the final case, the conclusion follows from the vanishing theorem noted above; namely, $H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, (Z\setminus Y)_{\overline{K}}, E)$ is the desired semistable representation, with invariant lattice $H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, (Z\setminus Y)_{\overline{K}}, \mathbb{O}_E)$, whose reduction $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, k_E)$ contains ρ.

Finally, suppose we are in the good reduction case. Again, by Proposition 1.2.3, we can and do choose smooth hypersurface sections Y and Z, having smooth intersection, such that Y, Z, and $Y \cap Z$ all have good reduction. Applying the same argument as in the previous paragraph, we see by induction on the dimension of X that it is enough to check that $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, E)$ is crystalline, but this follows immediately from Theorem 1.2 of [Yamashita 2011]. (Note that if in the notation of that work we take $D^1 = Z$ and $D^2 = Y$, then by (A.1.2) below we see that $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, E)$ appears on the left side of the Hyodo–Kato isomorphism in the statement of Theorem 1.2 of [Yamashita 2011], and since we have already shown that $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, E)$ is semistable, it is enough to show that the monodromy operator N vanishes on the right side of the Hyodo–Kato isomorphism. This follows easily from the definition of this operator as a boundary map, as all objects concerned arise from base change from objects with trivial log structures.) □

1.5. ***Equivariant versions.*** In practice, we will need equivariant analogues of the preceding results. As in the preceding section, we let K denote either a number field ("the global case") or a field of characteristic zero that is complete with respect to a discrete valuation with perfect residue field of characteristic p ("the local case"). We let $\overline{K}$ denote an algebraic closure of K, and set $G_K := \mathrm{Gal}(\overline{K}/K)$. In the global case, we fix a place v of K lying over p, and a decomposition group $D_v \subset G_K$ for v.

We now put ourselves in the following (somewhat elaborate) situation, which we call a *tamely ramified semistable context*, or a *tame semistable context* for short.

We suppose that X_0 and X_1 are smooth projective varieties over K, that G is a finite group which acts on X_1, and that $\pi : X_1 \to X_0$ is a finite étale morphism which intertwines the given G-action on X_1 with the trivial G-action on X_0, making X_1 an étale G-torsor over X_0.

We suppose further that X_0 admits a semistable projective model $\mathscr{X}_0$ over $\mathcal{O}_K$ (in the local case) or over $\mathcal{O}_{K,(v)}$ (in the global case). We also suppose that there is a finite Galois extension L of K, and (in the global case) a prime w of L lying over v, such that $(X_1)_{/L}$ admits a semistable projective model $\mathscr{X}_1$ over $\mathcal{O}_L$ (in the local case) or over $\mathcal{O}_{L,(w)}$ (in the global case) to which the G-action extends, such that π extends to a morphism $\mathscr{X}_1 \to (\mathscr{X}_0)_{/\mathcal{O}_L}$ which intertwines the G-action on its source with the trivial G-action on its target, and such that the action of the (opposite group of) the inertia group $I(L/K)^{\mathrm{op}}$ (or $I(L_w/K_v)^{\mathrm{op}}$ in the global case) on $(X_1)_{/L}$ extends to an action on $\mathscr{X}_1$.[2]

[2]Note that the tameness condition that we are going to require below ensures that L/K is in fact tamely ramified, and hence that $I(L/K)$ is abelian. Thus passing to the opposite group is not actually necessary here when passing from the action on rings to the action on their Specs, but we will keep the superscript op in the notation for the sake of conceptual clarity.

Finally (and most importantly), we assume that the composite morphism

$$\mathscr{X}_1 \to (\mathscr{X}_0)_{/\mathbb{O}_L} \to \mathscr{X}_0 \tag{1.5.1}$$

(the first being the extension of π, and the second being the natural map) is tamely ramified along the special fibre $(\mathscr{X}_0)_s$, in the sense of [Grothendieck and Murre 1971, Definition 2.2.2].

In fact, in our applications we will consider the case that $\mathscr{X}_0$ is furthermore strictly semistable, in which case we will say that we are in a *tame strictly semistable context*.

Remark 1.5.2. The notion of a tame semistable context is somewhat rigid, as we will see in the following lemma, and would perhaps not be of much interest if it did not occur naturally in the Shimura variety context (as we will see Section 3.1). As one example of this rigidity, note that if $G = 1$, i.e., if X_0 and X_1 coincide, then the only way to achieve a tame semistable context is if L/K is unramified, or if $\mathscr{X}_0$ is smooth over $\mathbb{O}_K$. (Indeed, since a tamely ramified morphism is finite, and since the base change $\mathscr{X}_{0/\mathbb{O}_L}$ over the semistable $\mathbb{O}_K$-scheme $\mathscr{X}_0$ is normal, we see that if X_0 and X_1 coincide then the morphism $\mathscr{X}_1 \to \mathscr{X}_{0/\mathbb{O}_L}$ is necessarily an isomorphism. This implies that the semistable $\mathbb{O}_K$-scheme $\mathscr{X}_0$ has a semistable base change over $\mathbb{O}_L$, which, as we noted in Remark 1.2.2, is possible only if L/K is unramified or $\mathscr{X}_0$ is smooth over $\mathbb{O}_K$. Another point of view on this case is as follows: if $\mathscr{X}_0$ is semistable but not smooth, then in order to construct a semistable model $\mathscr{X}_1$ of $\mathscr{X}_{0/\mathbb{O}_L}$, we must perform some nontrivial blow-ups, and the resulting morphism $\mathscr{X}_1 \to \mathscr{X}_0$ is not finite, and in particular not tamely ramified.)

The following lemma gives a more concrete interpretation of the stipulation that (1.5.1) be tamely ramified along $(\mathscr{X}_0)_s$.

Lemma 1.5.3. *In the above setting, the morphism* (1.5.1) *is tamely ramified along* $(\mathscr{X}_0)_s$ *if and only if the following conditions hold*:

(1) L (*resp.* L_w *in the global case*) *is tamely ramified over* K (*resp.* K_v *in the global case*), *of ramification degree* e, *say.*

(2) *For each closed geometric point* $\bar{x}_1$ *of the special fibre* $(\mathscr{X}_1)_s$, *with image* $\bar{x}_0$ *in* $(\mathscr{X}_0)_s$, *and for some choice of isomorphism*

$$(\mathbb{O}_{\bar{x}_0,\mathscr{X}_0})^\wedge \cong (\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K), \tag{1.5.4}$$

where ϖ_K *is a uniformiser of* $(\mathbb{O}_K^{\mathrm{sh}})^\wedge$ *and* $1 \le m \le n$, *there is a corresponding isomorphism*

$$(\mathbb{O}_{\bar{x}_1,\mathscr{X}_1})^\wedge \cong (\mathbb{O}_L^{\mathrm{sh}})^\wedge[\![y_1,\dots,y_n]\!]/(y_1\cdots y_m - \varpi_L),$$

where ϖ_L *is a uniformiser of* $(\mathbb{O}_L^{\mathrm{sh}})^\wedge$, *such that the induced morphism*

$$((\mathscr{X}_1)_{\bar{x}_1})^\wedge \to ((\mathscr{X}_0)_{\bar{x}_0})^\wedge$$

is defined by the formula $x_j = y_j^e$ for $1 \leq j \leq m$ and $x_j = y_j$ for $m < j \leq n$.

Furthermore, if these equivalent conditions hold, then condition (2) *holds for* every *choice of isomorphism* (1.5.4).

Proof. We first note that if we are in the global case, then, relabelling K_v as K and L_w as L, we may reduce ourselves to proving the lemma in the local case. Thus we assume that we in the local case from now on.

If conditions (1) and (2) (for some choice of isomorphism (1.5.4)) hold, then the morphism (1.5.1) is certainly tamely ramified along $(\mathscr{X}_0)_s$. (This amounts to the claim that we can verify tame ramification by passing to formal completions of closed geometric points, which is indeed the case, as follows from [Grothendieck and Murre 1971, Corollary 4.1.5].)

Suppose conversely that (1.5.1) is tamely ramified along $(\mathscr{X}_0)_s$. Since this morphism factors through the natural morphism $(\mathscr{X}_0)_{/\mathbb{O}_L} \to \mathscr{X}_0$, it follows from [Grothendieck and Murre 1971, Lemma 2.2.5] that this latter morphism is tamely ramified, and hence (e.g., by Proposition 2.2.9 of that reference, although our particular situation is much simpler than the general case of faithfully flat descent for tamely ramified covers considered in that proposition) that $\operatorname{Spec} \mathbb{O}_L \to \operatorname{Spec} \mathbb{O}_K$ is tamely ramified, i.e., that L is tamely ramified over K, of some ramification degree e. Thus condition (1) holds.

Now choose a closed geometric point $\bar{x}_1$ of $(\mathscr{X}_1)_s$ lying over the closed geometric point $\bar{x}_0$ of $(\mathscr{X}_0)_s$, and fix an isomorphism of the form (1.5.4). Since $\mathscr{X}_1 \to \mathscr{X}_0$ is tamely ramified along the divisor $\varpi_K = 0$ of $\mathscr{X}_0$, Abhyankar's lemma [SGA 1 1971, Exposé XIII, Corollaire 5.6] (see also [Grothendieck and Murre 1971, Theorem 2.3.2] for a concise statement) implies that we may find regular elements $\{a_j\}_{j=1,\dots,k}$ of $(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K)$ such that $a_1 \cdots a_k$ generates the ideal (ϖ_K) of $(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K)$, exponents $e_1,\dots,e_k$ all coprime to p, and a subgroup $H \subset \mu_{e_1} \times \cdots \times \mu_{e_k}$, such that the $(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K)$-algebra $(\mathbb{O}_{\bar{x}_1,\mathscr{X}_1})^\wedge$ is isomorphic to

$$\big((\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!][T_1,\dots,T_k]/(x_1\cdots x_m - \varpi_K, T_1^{e_1} - a_1,\dots,T_k^{e_k} - a_k)\big)^H.$$

(Here $\mu_{e_1} \times \cdots \times \mu_{e_k}$, and hence H, acts on

$$(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!][T_1,\dots,T_k]/(x_1\cdots x_m - \varpi_K, T_1^{e_1} - a_1,\dots,T_k^{e_k} - a_k)$$

in the obvious manner: namely, an element $(\zeta_1,\dots,\zeta_k)$ acts on T_j via multiplication by ζ_j.)

Since each e_j is prime to p and $(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K)$ is strictly Henselian, any unit in this ring has an e_j-th root, and thus we are free to multiply any of the a_j by a unit. Consequently, we may assume that in fact $a_1 \cdots a_k = \varpi_K = x_1 \cdots x_m$, and hence (again taking advantage of our freedom to modify the a_j by units) that $\{1, \ldots, m\}$ is partitioned into k sets $J_1, \ldots, J_k$, such that $a_j = \prod_{i \in J_j} x_i$. Now if we extract the e_j-th roots of each x_i for $i \in J_j$, the resulting extension of $(\mathcal{O}_K^{\mathrm{sh}})^\wedge[\![x_1, \ldots, x_n]\!]/(x_1 \cdots x_m - \varpi_K)$ contains

$$(\mathcal{O}_K^{\mathrm{sh}})^\wedge[\![x_1, \ldots, x_n]\!][T_1, \ldots, T_k]/(x_1 \cdots x_m - \varpi_K, T_1^{e_1} - a_1, \ldots, T_k^{e_k} - a_k);$$

thus it is no loss of generality to assume that $k = m$ and that $a_j = x_j$, and so we conclude that $(\mathcal{O}_{\bar{x}_1, \mathcal{X}_1})^\wedge$ is isomorphic, as an $(\mathcal{O}_K^{\mathrm{sh}})^\wedge[\![x_1, \ldots, x_n]\!]/(x_1 \cdots x_m - \varpi_K)$-algebra, to

$$\left((\mathcal{O}_K^{\mathrm{sh}})^\wedge[\![x_1, \ldots, x_n]\!][T_1, \ldots, T_m]/(x_1 \cdots x_m - \varpi_K, T_1^{e_1} - x_1, \ldots, T_m^{e_m} - x_m)\right)^H,$$

for some subgroup $H \subset \mu_{e_1} \times \cdots \times \mu_{e_m}$.

Let I_j denote the subgroup $1 \times \cdots \times \mu_{e_j} \times \cdots \times 1$ of $\mu_{e_1} \times \cdots \times \mu_{e_m}$; this is the inertia group of the divisor (x_j) with respect to the cover

$$\mathrm{Spec}((\mathcal{O}_K^{\mathrm{sh}})^\wedge)[\![x_1, \ldots, x_n]\!][T_1, \ldots, T_m]/(x_1 \cdots x_m - \varpi_K, T_1^{e_1} - x_1, \ldots, T_m^{e_m} - x_m)$$
$$\to \mathrm{Spec}((\mathcal{O}_K^{\mathrm{sh}})^\wedge)[\![x_1, \ldots, x_n]\!]/(x_1 \cdots x_m - \varpi_K).$$

If we write $H_j = H \cap I_j$, then $H' := H_1 \times \cdots \times H_m$ is a subgroup of H, and the cover

$$\mathrm{Spec}\left((\mathcal{O}_K^{\mathrm{sh}})^\wedge[\![x_1, \ldots, x_n]\!][T_1, \ldots, T_m]/(x_1 \ldots x_m - \varpi_K, T_1^{e_1} - x_1, \ldots, T_m^{e_m} - x_m)\right)^{H'}$$
$$\to \mathrm{Spec}\left((\mathcal{O}_K^{\mathrm{sh}})^\wedge[\![x_1, \ldots, x_n]\!][T_1, \ldots, T_m]/(x_1 \ldots x_m - \varpi_K, T_1^{e_1} - x_1, \ldots, T_m^{e_m} - x_m)\right)^{H}$$

is unramified in codimension 1. Since $\mathcal{X}_1$ is regular, being semistable over $\mathcal{O}_L$, so is the target of this map (since we recall that this target is isomorphic to $((\mathcal{X}_1)_{\bar{x}_1})^\wedge$). The purity of the branch locus then implies that this cover is étale, and hence is an isomorphism (since its target is strictly Henselian). Consequently $H = H'$.

If we write

$$H_j = 1 \times \cdots \times \mu_{e'_j} \times \cdots \times 1 \subset 1 \times \cdots \times \mu_{e_j} \times \cdots \times 1 = I_j,$$

and set $d_j = e_j / e'_j$ and $S_j = T_j^{e'_j}$, then we conclude that

$$\begin{aligned}
&(\mathcal{O}_{\bar{x}_1, \mathcal{X}_1})^\wedge \\
&\quad \cong \left((\mathcal{O}_K^{\mathrm{sh}})^\wedge[\![x_1, \ldots, x_n]\!][T_1, \ldots, T_m]/(x_1 \cdots x_m - \varpi_K, T_1^{e_1} - x_1, \ldots, T_m^{e_m} - x_m)\right)^{H'} \\
&\quad \cong (\mathcal{O}_K^{\mathrm{sh}})^\wedge[\![x_1, \ldots, x_n]\!][S_1, \ldots, S_m]/(x_1 \cdots x_m - \varpi_K, S_1^{d_1} - x_1, \ldots, S_m^{d_m} - x_m).
\end{aligned}$$

Now $\mathcal{X}_1$ is an $\mathcal{O}_L$-scheme with reduced special fibre (again because it is semistable over $\mathcal{O}_L$). Since $(\mathcal{O}_{\bar{x}_1, \mathcal{X}_1})^\wedge$ is strictly Henselian, it contains $(\mathcal{O}_L^{\mathrm{sh}})^\wedge$, and we may choose a uniformiser ϖ_L of this ring such that $\varpi_L^e = \varpi_K$. Looking at the above description of $(\mathcal{O}_{\bar{x}_1, \mathcal{X}_1})^\wedge$, and taking into account that its reduction modulo ϖ_L

must be reduced, we see that this special fibre must be the zero locus of the element $S_1 \cdots S_m$, hence that $S_1 \cdots S_m = u\varpi_L$ for some unit u, and thus that $(S_1 \cdots S_m)^e = u^e \varpi_K$. We conclude that $d_1 = \cdots = d_m = e$ and that $u^e = 1$, and hence, replacing ϖ_L by $u\varpi_L$, we find that $S_1 \cdots S_m = \varpi_L$. This shows that (2) holds (for our given choice of isomorphism (1.5.4)). □

Remark 1.5.5. Note that we could avoid the appeal to the general theory of tame ramification (in particular, to Abhyankar's lemma) by just directly stipulating in our context that conditions (1) and (2) of Lemma 1.5.3 hold; indeed, in the proof of Theorem 1.5.15 below, we will work directly with these conditions, and in our applications to Shimura varieties we will also see directly that these conditions hold. Nevertheless, we have included Lemma 1.5.3 as an assurance to ourselves (and perhaps to the reader) that these conditions are somewhat natural.

Lemma 1.5.6. *In a tame strictly semistable context as above, the $\mathscr{O}_L$-scheme $\mathscr{X}_1$ is also strictly semistable.*

Proof. Since $\mathscr{X}_1$ is semistable by assumption, it is enough to show that the components of the special fibre $(\mathscr{X}_1)_s$ are regular. Suppose that D is a nonregular component of $(\mathscr{X}_1)_s$, and let $\bar{x}_1$ be a closed geometric point of D whose local ring on D is not regular. If we let $\bar{x}_0$ denote the image of $\bar{x}_1$ in $(\mathscr{X}_0)_s$, then Lemma 1.5.3(2) shows that we may find isomorphisms $(\mathscr{O}_{\bar{x}_1,(\mathscr{X}_1)_s})^\wedge \cong \overline{\mathscr{O}_K/\varpi_K}[\![y_1, \dots, y_n]\!]/(y_1 \cdots y_m)$ and $(\mathscr{O}_{\bar{x}_0,(\mathscr{X}_0)_s})^\wedge \cong \overline{\mathscr{O}_K/\varpi_K}[\![x_1, \dots, x_n]\!]/(x_1 \cdots x_m)$, with $1 \le m \le n$, such that the morphism $(\mathscr{O}_{\bar{x}_1,(\mathscr{X}_1)_s})^\wedge \to (\mathscr{O}_{\bar{x}_0,(\mathscr{X}_0)_s})^\wedge$ is given by $x_j = y_j^e$ for $1 \le j \le m$ and $x_j = y_j$ for $m < j \le n$. Since by assumption $\bar{x}_1$ is not a regular point of D, we find that necessarily $m \ge 2$, and that (possibly after permuting indices) there is an isomorphism $(\mathscr{O}_{\bar{x}_1,D})^\wedge \cong \overline{\mathscr{O}_K/\varpi_K}[\![y_1, \dots, y_n]\!]/(y_1 \cdots y_{m'})$, where $2 \le m' \le m$. If we let D' denote the image of D in $(\mathscr{X}_0)_s$, we conclude that there is an isomorphism $(\mathscr{O}_{\bar{x}_0,D'})^\wedge \cong \overline{\mathscr{O}_K/\varpi_K}[\![x_1, \dots, x_n]\!]/(x_1 \cdots x_{m'})$, and thus that D' is not regular. Hence $\mathscr{X}_0$ is not strictly semistable, a contradiction. □

We now suppose that we are in a tame semistable context, as described above, and suppose for the moment that we are in the local case. Then we have an action of $I(L/K)^{\mathrm{op}} \times G$ on the special fibre $(\mathscr{X}_1)_s$. Let D be an irreducible component of the special fibre of $(\mathscr{X}_0)_s$, and let $\widetilde{D}$ denote its preimage in $(\mathscr{X}_1)_s$, so that $\widetilde{D}$ is an $I(L/K)^{\mathrm{op}} \times G$-invariant union of irreducible components of $(\mathscr{X}_1)_s$.

Lemma 1.5.7. *If G is abelian, then there is a homomorphism $\psi : I(L/K) \to G$ such that the action of $I(L/K)^{\mathrm{op}}$ on $\widetilde{D}$ is given by composing the action of G with ψ; i.e., if $i \in I(L/K)$, then the action of i on $\widetilde{D}$ coincides with the action of $\psi(i)$.*

We first prove a general lemma:

Lemma 1.5.8. *Let S be a connected Noetherian scheme, let G and I be finite groups, and let $f : T \to S$ be a finite étale morphism with the property that $I^{\mathrm{op}} \times G$ acts on T over S in such a way that T becomes a G-torsor over S. If G is abelian, then there exists a morphism $\psi : I \to G$ such that the action of I on T is given by composing the action of G on T with the morphism ψ.*

Proof. For clarity, we will not impose the assumption that G is abelian until required.

If we fix a geometric point $\bar{s}$ of S, then the theory of the étale fundamental group [SGA 1 1971, Exposé V, Théorème 4.1] shows that passing to the fibre over $\bar{s}$ gives an equivalence of categories between the category of finite étale covers of S and the category of (discrete) finite sets with a continuous action of $\pi_1(S, \bar{s})$. In this way, T is classified by an object P of this latter category equipped with an action of $I^{\mathrm{op}} \times G$, with respect to which the G-action makes P a principal homogeneous G-set.

If we fix a base point $p \in P$, then we may identify P with G, thought of as a principal homogeneous G-set via left multiplication. As the automorphisms of G as a principal homogeneous G-set are naturally identified with G^{op} acting by right multiplication, we obtain a homomorphism $\psi_p : I^{\mathrm{op}} \to G^{\mathrm{op}}$, or equivalently a homomorphism $\psi_p : I \to G$, describing the action of I^{op} on P. If we replace p by $g \cdot p$ (for some $g \in G$), then one finds that $\psi_{gp} = g \psi_p g^{-1}$. Thus, if we now assume furthermore that G is abelian, then $\psi_p = \psi_{gp}$, and so it is reasonable in this case to write simply ψ for this homomorphism, which is well-defined independently of the choice of base point for P. Furthermore, when G is abelian, left and right multiplication by an element $g \in G$ coincide, and so the action of I^{op} on P is given by the formula $i \cdot p = \psi(i) \cdot p$ for all $p \in P$. Since the automorphisms of T over S induced by i and $\psi(i)$ coincide on P, they in fact coincide on all of T. $\square$

Proof of Lemma 1.5.7. The morphism $\mathscr{X}_1 \to (\mathscr{X}_0)_{/\mathbb{O}_L}$ is étale on generic fibres, and the explicit local formulas for this morphism provided by Lemma 1.5.3 show that it is in fact étale over an open subset $\mathscr{U}_0$ of $(\mathscr{X}_0)_{/\mathbb{O}_L}$ whose intersection with the special fibre $((\mathscr{X}_0)_{/\mathbb{O}_L})_s$ is Zariski dense. Replacing $\mathscr{U}_0$ with the intersection of all of its $I(L/K)^{\mathrm{op}}$-translates, we may furthermore assume that $\mathscr{U}_0$ is invariant under the action of $I(L/K)^{\mathrm{op}}$ on $(\mathscr{X}_0)_{/\mathbb{O}_L}$.

If we let $\mathscr{U}_1$ denote the preimage of $\mathscr{U}_0$ in $\mathscr{X}_1$, then $\mathscr{U}_1$ is invariant under the $I(L/K) \times G$-action on $\mathscr{X}_1$, and the morphism $\mathscr{U}_1 \to \mathscr{U}_0$ is a finite étale cover, for which the corresponding map $U_1 \to U_0$ on generic fibres realises U_1 as a G-torsor over U_0. It follows that the G-action on $\mathscr{U}_1$ realises $\mathscr{U}_1$ as an étale G-torsor over $\mathscr{U}_0$, and hence, passing to special fibres, that $(\mathscr{U}_1)_s$ is an étale G-torsor over $(\mathscr{U}_0)_s$.

Now the induced $I(L/K)^{\mathrm{op}}$-action on $(\mathscr{U}_0)_s$ is trivial, and so $I(L/K)^{\mathrm{op}}$ acts on $(\mathscr{U}_1)_s$ as a group of automorphisms of the G-torsor $(\mathscr{U}_1)_s$ over $(\mathscr{U}_0)_s$. If $D' := D \cap (\mathscr{U}_0)_s$, then D' is an irreducible component of $(\mathscr{U}_0)_s$, and $\widetilde{D}' := \widetilde{D} \cap (\mathscr{U}_1)_s$

is the restriction of $(\mathcal{U}_1)_s$ to D'. Thus $\widetilde{D}' \to \widetilde{D}$ is again an étale G-torsor, with an action of $I(L/K)^{\mathrm{op}}$ via automorphisms. Lemma 1.5.8 then shows that there is a homomorphism $\psi : I(L/K)^{\mathrm{op}} \to G^{\mathrm{op}}$, or equivalently a homomorphism $\psi : I(L/K) \to G$, such that the action of $I(L/K)^{\mathrm{op}}$ on the points of $\widetilde{D}'$ is given by composing the action of G with the homomorphism ψ. Since $\widetilde{D}$ is equal to the Zariski closure of $\widetilde{D}'$ in $(\mathcal{X}_1)_s$, the claim of the lemma follows. □

Lemma 1.5.9. *Suppose that we are in a tame strictly semistable context. If $g \in I(L/K) \times G$ and D is a component of $(\mathcal{X}_1)_s$, then D and gD either coincide or are disjoint.*

Proof. The images of D and gD in $(\mathcal{X}_0)_s$ coincide, and it follows from Lemma 1.5.3 that two distinct components of $(\mathcal{X}_1)_s$ that have nonempty intersection must have distinct images in $(\mathcal{X}_0)_s$. □

If we now suppose that we are in the global case, then the discussion applies with L/K everywhere replaced by L_w/K_v, and in particular for each component D we may define a character $\psi : I(L_w/K_v) \to G$ describing the action of $I(L_w/K_v)$ on D.

Our next result describes how our tame semistable context behaves upon passage to a semistable hypersurface section of $\mathcal{X}_0$. In its statement we assume for simplicity that we are in the local case.

Proposition 1.5.10. *Suppose that we are in the tamely ramified semistable context described above, and let $\mathcal{Y}_0$ be a regular hypersurface section of $\mathcal{X}_0$ such that the union of $\mathcal{Y}_0$ and $(\mathcal{X}_0)_s$ forms a divisor with normal crossings on $\mathcal{X}_0$. Let Y_0 denote the generic fibre of $\mathcal{Y}_0$, let Y_1 denote the preimage of Y_0 under the morphism $\pi : X_1 \to X_0$, and let $\mathcal{Y}_1$ be the preimage of $\mathcal{Y}_0$ under* (1.5.1) (*so that $(Y_1)_{/L}$ is the generic fibre of the $\mathcal{O}_L$-scheme $\mathcal{Y}_1$*). *Then*:

(1) *The complement of Y_1 in X_1 is affine.*

(2) *The generic fibre Y_0 of $\mathcal{Y}_0$ is smooth over K, the morphism $Y_1 \to Y_0$ is an étale G-torsor* (*so in particular Y_1 is also smooth over K*), *$\mathcal{Y}_0$ is a semistable model of Y_0 over $\mathcal{O}_K$, $\mathcal{Y}_1$ is a semistable model for $(Y_1)_{/L}$ over $\mathcal{O}_L$, and the morphism $\mathcal{Y}_1 \to \mathcal{Y}_0$ is tamely ramified*; *consequently $\mathcal{Y}_1 \to \mathcal{Y}_0$ is again a tamely ramified semistable context.*

(3) *Suppose that G is abelian. If D' is an irreducible component of $(\mathcal{Y}_1)_s$, contained in an irreducible component D of $(\mathcal{X}_1)_s$, then the homomorphism $\psi : I(L/K) \to G$, which describes the action of $I(L/K)$ on D, also describes the action of $I(L/K)$ on D'.*

Proof. Since $\mathcal{Y}_0$ is a hypersurface section of $\mathcal{X}_0$, its generic fibre Y_0 is a hypersurface section of X_0. Thus its complement is affine. Since π is a finite morphism by assumption, the complement of Y_1 in X_1 is again affine. Since Y_0 is a regular projective K-scheme (being the generic fibre of $\mathcal{Y}_0$, which is regular by assumption), it is in fact smooth over K. By definition, Y_1 is the preimage of Y_0 under the morphism $X_1 \to X_0$, which is an étale G-torsor by assumption. Thus $Y_1 \to Y_0$ is indeed an étale G-torsor (and so Y_1 is also smooth over K).

Let $\bar{x}_0$ denote a closed geometric point of the special fibre $(\mathcal{Y}_0)_s$. Since $\big(((\mathcal{X}_0)_{\bar{x}_0})^\wedge\big)_s \cup ((\mathcal{Y}_0)_{\bar{x}_0})^\wedge$ forms a divisor with normal crossings, since each component of $\big(((\mathcal{X}_0)_{\bar{x}_0})^\wedge\big)_s$ is regular, and since $\mathcal{Y}_0$ is regular by assumption, it follows from [Grothendieck and Murre 1971, Lemma 1.8.4] that $\big(((\mathcal{X}_0)_{\bar{x}_0})^\wedge\big)_s \cup ((\mathcal{Y}_0)_{\bar{x}_0})^\wedge$ is in fact a divisor with strictly normal crossings in $((\mathcal{X}_0)_{\bar{x}_0})^\wedge$, and hence the local equation ℓ of $((\mathcal{Y}_0)_{\bar{x}_0})^\wedge$, together with the elements $x_1, \ldots, x_m$ that cut out the irreducible components of $\big(((\mathcal{X}_0)_{\bar{x}_0})^\wedge\big)_s$, form part of a regular system of parameters for $(\mathcal{O}_{\bar{x}_0,\mathcal{X}_0})^\wedge$. Thus we may choose a model of the form (1.5.4) for $((\mathcal{X}_0)_{\bar{x}_0})^\wedge$ for which $m < n$ and in which ℓ is equal to the element x_n, i.e., in which $((\mathcal{Y}_0)_{\bar{x}_0})^\wedge$ is the zero locus of the element x_n.

We now choose a closed geometric point $\bar{x}_1$ of $(\mathcal{X}_1)_s$ lying over $\bar{x}_0$, as well as a model for the tamely ramified morphism $((\mathcal{X}_1)_{\bar{x}_1})^\wedge \to ((\mathcal{X}_0)_{\bar{x}_0})^\wedge$ as in part (2) of Lemma 1.5.3. Thus this morphism has the form

$$\begin{aligned}&\operatorname{Spec}((\mathcal{O}_L^{\mathrm{sh}})^\wedge)[\![y_1, \ldots, y_n]\!]/(y_1 \cdots y_m - \varpi_L)\\ &\qquad\qquad\to \operatorname{Spec}((\mathcal{O}_K^{\mathrm{sh}})^\wedge)[\![x_1, \ldots, x_n]\!]/(x_1 \cdots x_m - \varpi_K),\end{aligned}$$

with $x_j = y_j^e$ for $1 \le j \le m$ and $x_j = y_j$ for $m < j \le n$. In particular, we see that $x_n = y_n$, and thus we see that the induced morphism

$$((\mathcal{Y}_1)_{\bar{x}_1})^\wedge \to ((\mathcal{Y}_0)_{\bar{x}_0})^\wedge \tag{1.5.11}$$

can be written as

$$\begin{aligned}&\operatorname{Spec}((\mathcal{O}_L^{\mathrm{sh}})^\wedge)[\![y_1, \ldots, y_{n-1}]\!]/(y_1 \ldots y_m - \varpi_L)\\ &\qquad\to \operatorname{Spec}((\mathcal{O}_L^{\mathrm{sh}})^\wedge)[\![x_1, \ldots, x_{n-1}]\!]/(x_1 \ldots x_m - \varpi_K).\end{aligned} \tag{1.5.12}$$

Thus we see that $\mathcal{Y}_0$ and $\mathcal{Y}_1$ are indeed semistable models of their generic fibres (over $\mathcal{O}_K$ and $\mathcal{O}_L$ respectively), and that the morphism $\mathcal{Y}_1 \to \mathcal{Y}_0$ is tamely ramified. This completes the verification of (2). The claim of (3) follows from the fact that the action of $I(L/K)^{\mathrm{op}} \times G$ on $(\mathcal{Y}_1)_s$ is the restriction of the corresponding action on $(\mathcal{X}_1)_s$, together with the fact that any component of $(\mathcal{Y}_1)_s$ is contained in a component of $(\mathcal{X}_1)_s$. □

We now suppose that E is an algebraic extension of $\mathbb{Q}_p$ containing K_0. Recall that if $\rho : G_K \to \mathrm{GL}_n(E)$ is a potentially semistable representation, then we may attach a Weil–Deligne representation $\mathrm{WD}(\rho)$ to ρ by first passing to the potentially semistable Dieudonné module $D_{\mathrm{pst}}(\rho)$ of ρ, which is a module over $E \otimes_{\mathbb{Q}_p} K_0$, then fixing an embedding $K_0 \hookrightarrow E$, and hence a projection $\mathrm{pr} : E \otimes_{\mathbb{Q}_p} K_0 \to E$, and, finally, forming $\mathrm{WD}(\rho) := E \otimes_{E \otimes_{\mathbb{Q}_p} K_0, \mathrm{pr}} D_{\mathrm{pst}}$. Although $\mathrm{WD}(\rho)$ depends on the choice of the embedding $K_0 \hookrightarrow E$, up to isomorphism it is independent of this choice, as the Frobenius ϕ on $D_{\mathrm{pst}}(\rho)$ provides isomorphisms between the different choices. (See for example [Conrad et al. 1999, Appendix B] and [Taylor 2004, p. 78–79] for discussions of this construction and its properties.)

In the tame strictly semistable case, the following result will allow us to describe the inertial part of the Weil–Deligne representation associated to the p-adic étale cohomology of X_1, or of a pair (X_1, Y_1) that arises in the context of the preceding proposition. Before stating the result we introduce some additional notation, and an additional assumption.

Assume that G is abelian, and let J denote the set of $I(L/K) \times G$-orbits on the set of irreducible components of $(\mathscr{X}_1)_s$, and let D_j (for $j \in J$) denote the union of the components lying in the orbit labelled by j. Let $\psi_j : I(L/K) \to G$ be the homomorphism provided by Lemma 1.5.7, describing the action of $I(L/K)$ on the points of D_j.

Proposition 1.5.13. *Suppose that we are in a tame strictly semistable context as above. Either let W denote the Weil–Deligne representation associated to the potentially semistable G_K-representation $H^i_{\text{ét}}((X_1)_{/\overline{K}}, E)$, or else suppose that we are in the context of Proposition 1.5.10, and let W denote the Weil–Deligne representation associated to the potentially semistable G_K-representation $H^i_{\text{ét}}((X_1)_{/\overline{K}}, (Y_1)_{/\overline{K}}), E)$ (here i is some given degree of cohomology); in either case, W is a representation of the product* $\mathrm{WD}_K \times G$. *Assume furthermore that G is abelian.*

Then, if, as in the above discussion, $\widetilde{J}$ denotes the set of $I(L/K) \times G$-orbits of irreducible components of $(\mathscr{X}_1)_s$, we may decompose W as a direct sum $W = \bigoplus_{\tilde{j} \in \widetilde{J}} W_{\tilde{j}}$, such that on $W_{\tilde{j}}$ the action of the inertia group in W_K *is obtained by composing the G-action on $W_{\tilde{j}}$ with the homomorphism $I_K \to I(L/K) \xrightarrow[\psi_{\tilde{j}}]{} G$.*

Proof. Since the action of the inertia subgroup of W_K on W factors through a finite group, and representations of a finite group over a field of characteristic zero are semisimple, the claimed property of W is stable under the formation of subobjects, quotients, and extensions (in the category of $W_K \times G$-representations). A consideration of the long exact sequence of cohomology associated to the pair (X_1, Y_1) (see the Appendix) then reduces the claim for the cohomology of the pair to the claim for the cohomology of X_1 and Y_1 individually. Since Proposition 1.5.10 shows that the strictly semistable model $\mathscr{Y}_1$ of $(Y_1)_{/L}$ behaves in an identical manner to the strictly semistable model of $\mathscr{X}_1$ of $(X_1)_{/L}$, it in fact suffices to consider the case of X_1.

Thus we now restrict our attention to the W_K-representation W underlying the potentially semistable Dieudonné module associated to $H^i_{\text{ét}}((X_1)_{/\overline{K}}, E)$. By [Tsuji 1999], this Dieudonné module is naturally identified with the log-crystalline cohomology $H^i((\mathscr{X}_1)^\times_s/W(k)^\times)\otimes_{W(k)} E$ of the special fibre $(\mathscr{X}_1)_s$ with its natural log-structure. Lemma 1.5.9 shows that if an intersection of distinct components of the special fibre $(\mathscr{X}_1)_s$ is nonempty, then the various components appearing must lie in mutually distinct orbits of $I(L/K)\times G$ acting on the set of components. Recalling that J denotes the indexing set for the collection $\{D_j\}_{j\in J}$ of $I(L/K)\times G$-orbits of components of $(\mathscr{X}_1)_s$, this log-crystalline cohomology may be computed by the following spectral sequence of [Mokrane 1993]:

$$E_1^{-m,i+m} = \bigoplus_{\substack{l\geq\max\{0,-m\}\\ \{j_1,\dots,j_{2l+m+1}\}\subset J}} H^{i-2l-m}_{\text{cris}}(D_{j_1}\cap\cdots\cap D_{j_{2l+m+1}}/W(k))\otimes_{W(k)} E(-l-m) \Longrightarrow H^i((\mathscr{X}_1)^\times_s/W(k)^\times)\otimes_{W(k)} E.$$

The constructions of [Tsuji 1999; Mokrane 1993] are both functorial, so that everything here is compatible with the $I(L/K)\times G$-actions. Each of the summands in the E_1-term is naturally an $I(L/K)\times G$-representation, and furthermore the action of $I(L/K)$ is given by the composite of the action of G with one of the characters $\psi_{\tilde{j}}$. Thus the E_1-terms of this spectral sequence satisfy the claimed property of W. Thus W also satisfies this property, since it is obtained as a successive extension of subquotients of these E_1-terms. □

We are now ready to prove our equivariant versions of Theorems 1.3.1 and 1.4.1. For the first result, we place ourselves in the local case (since the global case immediately reduces to the local case by passing from K to K_v):

Theorem 1.5.14. *Suppose that we are in the tame strictly semistable context described above. Then the $G_K\times G$-representation $H^1_{\text{ét}}((X_1)_{\overline{K}}, \mathbb{F}_p)$ embeds $G_K\times G$-equivariantly into the reduction modulo the uniformiser of a $G_K\times G$-invariant $\mathbb{O}_E$-lattice in a representation V of $G_K\times G$ over E, having the following properties*:

(1) *The restriction of V to G_L is semistable, with Hodge–Tate weights contained in the interval* $[-1,0]$.

(2) *The Weil–Deligne representation associated to V, which is naturally a representation of* $\mathrm{WD}_K\times G$, *when restricted to a representation of $I_K\times G$ can be written as a direct sum $\bigoplus_{\tilde{j}\in\tilde{J}} W_{\tilde{j}}$ of $I_K\times G$-representations, where $\tilde{j}$ runs over the same index set that labels the set of $I(L/K)\times G$-orbits of irreducible*

components of $(\mathscr{X}_1)_s$, *such that on* $W_{\tilde{j}}$ *the action of the inertia group in* W_K *is obtained by composing the* G*-action on* $W_{\tilde{j}}$ *with the homomorphism* $I_K \to I(L/K) \xrightarrow[\psi_{\tilde{j}}]{} G$.

Proof. We follow the proof of Theorem 1.3.1, proceeding by descending induction on the dimension of X_0 and X_1, and passing to appropriately chosen hypersurface sections $\mathscr{Y}_0$ of $\mathscr{X}_0$ and their corresponding preimages Y_1 in X_1 and $\mathscr{Y}_1$ in $(\mathscr{X}_1)_{/L}$. Taking into account Proposition 1.5.10, we thus reduce to the case when X_0 and X_1 are curves, so that $H^1_{\text{ét}}((X_1)_{/\overline{K}}, \mathbb{F}_p)$ is the reduction mod p of $H^1_{\text{ét}}((X_1)_{/\overline{K}}, \mathbb{Z}_p)$, which is in turn a lattice in $H^1_{\text{ét}}((X_1)_{/\overline{K}}, \mathbb{Q}_p)$. This latter representation is potentially semistable with Hodge–Tate weights in $[-1, 0]$, by [Tsuji 1999]; the claim regarding Weil–Deligne representations follows from Proposition 1.5.13. □

For our second result, we allow ourselves to be in either the local or global context.

Theorem 1.5.15. *Suppose that we are in the tame strictly semistable context described above, and let* $\rho : G_K \times G \to \mathrm{GL}_n(k_E)$ *be an irreducible and continuous representation that embeds as a subquotient of* $H^i_{\text{ét}}((X_1)_{\overline{K}}, k_E)$. *Then* ρ *also embeds as a subquotient of a* $G_K \times G$*-representation over* k_E *which is the reduction modulo the uniformiser of a* $G_K \times G$*-invariant* $\mathbb{O}_E$*-lattice in a representation* V *of* $G_K \times G$ *over* E, *having the following properties*:

(1) *The representation* V *becomes semistable when restricted to* G_L (*resp. the decomposition group* $D_w \subset G_L$ *in the global case*), *with Hodge–Tate weights contained in the interval* $[-i, 0]$.

(2) *The Weil–Deligne representation associated to* V, *which is naturally a representation of* $\mathrm{WD}_K \times G$ (*resp.* $\mathrm{WD}_{K_v} \times G$ *in the global case*), *when restricted to a representation of* $I_K \times G$ (*resp.* $I_{K_v} \times G$ *in the global case*) *can be written as a direct sum* $\bigoplus_{\tilde{j}} W_{\tilde{j}}$ *of* $I_K \times G$*-representations* (*resp. of* $I_{K_v} \times G$*-representations*), *where* $\tilde{j}$ *runs over the same index set that labels the set of* $I(L/K) \times G$*-orbits* (*resp. of* $I_{K_v} \times G$*-orbits*) *of irreducible components of* $(\mathscr{X}_1)_s$, *such that on* $W_{\tilde{j}}$, *the action of the inertia group is obtained by composing the* G*-action on* $W_{\tilde{j}}$ *with the homomorphism* $I_K \to I(L/K) \xrightarrow[\psi_{\tilde{j}}]{} G$ (*resp. the homomorphism* $I_{K_v} \to I(L_w/K_v) \xrightarrow[\psi_{\tilde{j}}]{} G$).

Proof. We can be proved in exactly the same way as Theorem 1.4.1, taking into account Propositions 1.5.10 and 1.5.13. □

2. Breuil modules with descent data

In this section we establish a result (Theorem 2.2.4) which imposes some constraints on the reductions of certain tamely potentially semistable p-adic representations of $G_{\mathbb{Q}_p}$.

2.1. ***Preliminaries.*** We begin by recalling some results from Section 3 of [Emerton et al. 2013]. To this end, let p be an odd prime, let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$, and let E and K be finite extensions of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. Assume that E contains the images of all embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$. Let K_0 be the maximal absolutely unramified subfield of K, so that $K_0 = W(k)[1/p]$, where k is the residue field of K. Let K/K' be a Galois extension, with K' a field lying between $\mathbb{Q}_p$ and K. Assume further that K/K' is tamely ramified with ramification index e, and fix a uniformiser $\pi \in K$ with $\pi^e \in K'$. Let $E(u) \in W(k)[u]$ be the minimal polynomial of π over K_0.

Let k_E be the residue field of E, and let $0 \le r \le p-2$ be an integer. Recall that the category $k_E\text{-BrMod}^r_{\mathrm{dd}}$ of *Breuil modules of weight r with descent data* from K to K' and coefficients k_E consists of quintuples $(\mathcal{M}, \mathcal{M}_r, \varphi_r, \hat{g}, N)$, where:

- $\mathcal{M}$ is a finitely generated $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-module, free over $k[u]/u^{ep}$.
- $\mathcal{M}_r$ is a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-submodule of $\mathcal{M}$ containing $u^{er}\mathcal{M}$.
- $\varphi_r : \mathcal{M}_r \to \mathcal{M}$ is k_E-linear and φ-semilinear (where $\varphi : k[u]/u^{ep} \to k[u]/u^{ep}$ is the p-th power map) with image generating $\mathcal{M}$ as a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-module.
- $N : \mathcal{M} \to \mathcal{M}$ is $k \otimes_{\mathbb{F}_p} k_E$-linear and satisfies $N(ux) = uN(x) - ux$ for all $x \in \mathcal{M}$, $u^e N(\mathcal{M}_r) \subset \mathcal{M}_r$, and $\varphi_r(u^e N(x)) = cN(\varphi_r(x))$ for all $x \in \mathcal{M}_r$. Here, $c = \overline{F}(u)^p \in (k[u]/u^{ep})^\times$, where $E(u) = u^e + pF(u)$ in $W(k)[u]$.
- $\hat{g} : \mathcal{M} \to \mathcal{M}$ are additive bijections for each $g \in \mathrm{Gal}(K/K')$, preserving $\mathcal{M}_r$, commuting with the φ_r- and N-actions, and satisfying $\hat{g}_1 \circ \hat{g}_2 = (g_1 \circ g_2)^\wedge$ for all $g_1, g_2 \in \mathrm{Gal}(K/K')$. Furthermore, if $a \in k \otimes_{\mathbb{F}_p} k_E$ and $m \in \mathcal{M}$ then $\hat{g}(au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i \hat{g}(m)$.

There is a covariant functor $\mathrm{T}^{*,r}_{\mathrm{st}}$ from $k_E\text{-BrMod}^r_{\mathrm{dd}}$ to the category of k_E-representations of $G_{K'}$.

Lemma 2.1.1. *Suppose that $\mathcal{M} \in k_E\text{-BrMod}^r_{\mathrm{dd}}$ and T' is a $G_{K'}$-subrepresentation of $\mathrm{T}^{*,r}_{\mathrm{st}}(\mathcal{M})$ (so that in particular T' has the structure of a k_E-vector space). Then there is a unique subobject $\mathcal{M}'$ of $\mathcal{M}$ such that, if $f : \mathcal{M}' \to \mathcal{M}$ is the inclusion map, then $\mathrm{T}^{*,r}_{\mathrm{st}}(f)$ is identified with the inclusion $T' \hookrightarrow \mathrm{T}^{*,r}_{\mathrm{st}}(\mathcal{M})$. (Here $\mathcal{M}'$ is a subobject of $\mathcal{M}$ in the naive sense that it is a sub-$(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-module of $\mathcal{M}$, which inherits the structure of an object of $k_E\text{-BrMod}^r_{\mathrm{dd}}$ from $\mathcal{M}$ in the obvious way.)*

Proof. This is Corollary 3.2.9 of [Emerton et al. 2013]. □

We now specialise to the particular situation of interest to us in this paper; namely, we let K_0 be the unique unramified extension of $\mathbb{Q}_p$ of degree d, we take $K = K_0((-p)^{1/(p^d-1)})$, and we set $K' = K_0$, so that $e = p^d - 1$. Fix $\pi = (-p)^{1/(p^d-1)}$. We write $\widetilde{\omega}_d : \mathrm{Gal}(K/K_0) \to K_0^\times$ for the character $g \mapsto g(\pi)/\pi$, and we let ω_d be the reduction of $\widetilde{\omega}_d$ modulo π. (By inflation we can also think of $\widetilde{\omega}_d$ and ω_d as characters of $I_{K_0} = I_{\mathbb{Q}_p}$. Note that ω_d is a tame fundamental character of niveau d and that $\widetilde{\omega}_d$ is the Teichmüller lift of ω_d.) Note that when $d = 1$, we have $\omega_1 = \omega$, the mod p cyclotomic character.

Let φ be the arithmetic Frobenius on k, and let $\sigma_0 : k \hookrightarrow k_E$ be a fixed embedding. Inductively define $\sigma_1, \dots, \sigma_{d-1}$ by $\sigma_{i+1} = \sigma_i \circ \varphi^{-1}$; we will often consider the numbering to be cyclic, so that $\sigma_d = \sigma_0$. There are idempotents $e_i \in k \otimes_{\mathbb{F}_p} k_E$ such that if M is any $k \otimes_{\mathbb{F}_p} k_E$-module, then $M = \bigoplus_i e_i M$, and $e_i M$ is the subset of M consisting of elements m for which $(x \otimes 1)m = (1 \otimes \sigma_i(x))m$ for all $x \in k$. Note that $(\varphi \otimes 1)(e_i) = e_{i+1}$ for all i.

If $\rho : G_{K_0} \to \mathrm{GL}_n(E)$ is a potentially semistable representation which becomes semistable over K, then the associated inertial type (that is, the restriction to I_{K_0} of the Weil–Deligne representation associated to ρ) is a representation of I_{K_0} which becomes trivial when restricted to I_K, so we can and do think of it as a representation of $\mathrm{Gal}(K/K_0) \cong I_{K_0}/I_K$.

Proposition 2.1.2. *Maintaining our current assumptions on K, suppose that $\rho : G_{K_0} \to \mathrm{GL}_n(E)$ is a continuous representation whose restriction to G_K is semistable with Hodge–Tate weights contained in $[0, r]$, where $r \le p-2$, and let the inertial type of ρ be $\chi_1 \oplus \cdots \oplus \chi_n$, where each χ_i is a character of I_{K_0}/I_K. If $\bar{\rho}$ denotes the reduction modulo $\mathfrak{m}_E$ of a G_{K_0}-stable $\mathcal{O}_E$-lattice in ρ, then there is an element $\mathcal{M}$ of k_E-$\mathrm{BrMod}^r_{\mathrm{dd}}$, admitting a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-basis $v_1, \dots, v_n$ such that $\hat{g}(v_i) = (1 \otimes \overline{\chi}_i(g))v_i$ for all $g \in \mathrm{Gal}(K/K_0)$, and for which $\mathrm{T}^{*,r}_{\mathrm{st}}(\mathcal{M}) \cong \bar{\rho}$.*

Proof. This is Proposition 3.3.1 of [Emerton et al. 2013]. (Note that the conventions on the sign of the Hodge–Tate weights in that work are the opposite of the conventions in this paper.) ☐

Lemma 2.1.3. *Maintain our current assumptions on K, so that in particular we have $e = p^d - 1$. Then every rank-1 object of k_E-$\mathrm{BrMod}^r_{\mathrm{dd}}$ may be written in the form*

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}) \cdot m$,
- $e_i \mathcal{M}_r = u^{r_i} e_i \mathcal{M}$,
- $\varphi_r\big(\sum_{i=0}^{d-1} u^{r_i} e_i m\big) = \lambda m$ *for some* $\lambda \in (k \otimes_{\mathbb{F}_p} k_E)^\times$,
- $\hat{g}(m) = \big(\sum_{i=0}^{d-1} (\omega_d(g)^{k_i} \otimes 1)e_i\big)m$ *for all* $g \in \mathrm{Gal}(K/K_0)$, *and*
- $N(m) = 0$.

Here the integers $0 \le r_i \le (p^d - 1)r$ and k_i satisfy $k_i \equiv p(k_{i-1} + r_{i-1}) \bmod (p^d - 1)$ for all i. Conversely, any module $\mathcal{M}$ of this form is a rank-1 object of k_E-$\mathrm{BrMod}^r_{\mathrm{dd}}$. Furthermore,

$$\mathrm{T}^{*,r}_{\mathrm{st}}(\mathcal{M})|_{I_{K_0}} \cong \sigma_0 \circ \omega_d^{\kappa_0},$$

where $\kappa_0 \equiv k_0 + p(r_0 p^{d-1} + r_1 p^{d-2} + \cdots + r_{d-1})/(p^d - 1) \bmod (p^d - 1)$.

Proof. This is Lemma 3.3.2 of [Emerton et al. 2013]. □

Remark 2.1.4. In the sequel, we will only be interested in the case that for each i we have $k_i = (1 + p + \cdots + p^{d-1})x_i$ for some $0 \le x_i < p - 1$. In this case, the condition that $pr_i \equiv k_{i+1} - pk_i \bmod (p^d - 1)$ implies that $r_i \equiv (1 + p + \cdots + p^{d-1})(x_{i+1} - x_i) \bmod (p^d - 1)$, so the condition that $0 \le r_i \le (p^d - 1)r$ means that we can write $r_i = (1+p+\cdots+p^{d-1})(x_{i+1}-x_i)+(p^d-1)y_i$, with $0 \le y_i \le r$. An elementary calculation shows that we then have

$$\kappa_0 \equiv x_0 + y_0 + p^{d-1}(x_1 + y_1) + \cdots + p(x_{d-1} + y_{d-1}) \bmod (p^d - 1).$$

2.2. *Regularity.* Let $\mathbb{Q}_{p^n}$ denote the unique unramified extension of $\mathbb{Q}_p$ of degree n, with residue field $\mathbb{F}_{p^n}$. Regarding $\mathbb{F}_{p^n}$ as a subfield of $\overline{\mathbb{F}}_p$, we may then regard ω_n as a character $I_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times$.

Definition 2.2.1. Let $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be an irreducible representation, so that $\bar{\rho} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^n}}}^{G_{\mathbb{Q}_p}} \chi$ for some character $\chi : G_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times$. If we write

$$\chi|_{I_{\mathbb{Q}_p}} = \omega_n^{(a_0+pa_1+\cdots+p^{n-1}a_{n-1})},$$

where each $a_i \in [0, p-1]$ and not all the a_i are $p-1$, then the multiset of *exponents* of $\bar{\rho}$ is defined to be the multiset of residues of the a_i in $\mathbb{Z}/p\mathbb{Z}$.

Definition 2.2.2. Let $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a representation. Then the multiset of *exponents* of $\bar{\rho}$ is the union of the multisets of exponents of each of the Jordan–Hölder factors of $\bar{\rho}$.

Definition 2.2.3. Let $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a representation, and let r be a nonnegative integer. Then we say that $\bar{\rho}$ is *r-regular* if the exponents $a_1, \ldots, a_n$ of $\bar{\rho}$ are such that the residues $a_i + k \in \mathbb{Z}/p\mathbb{Z}$, $1 \le i \le n$, $0 \le k \le r + 1$, are pairwise distinct.

The following theorem is the main result we will need from explicit p-adic Hodge theory:

Theorem 2.2.4. *Let r be a nonnegative integer, and let $s : G_{\mathbb{Q}_p} \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ be a potentially semistable representation with Hodge–Tate weights contained in the interval $[0, r]$ and inertial type $\chi_1 \oplus \cdots \oplus \chi_m$. Suppose that there are (not necessarily distinct) integers $0 \le a_1, \ldots, a_n < p - 1$ such that each χ_i is equal to some $\widetilde{\omega}^{a_j}$.*

Suppose that $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is a subquotient of $\bar{s}$, the reduction mod $\mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_p}}$ of some $G_{\mathbb{Q}_p}$-stable $\overline{\mathbb{Z}}_p$-lattice in s. Suppose also that

- $\det \bar{\rho}|_{I_{\mathbb{Q}_p}} = \omega^{a_1+\cdots+a_n+n(n-1)/2}$,
- $r \le (n-1)/2$, *and*
- $p > n(n-1)/2 + 1$.

If $r = (n-1)/2$ then assume further that some irreducible subquotient of $\bar{\rho}$ has dimension greater than 1. *Then $\bar{\rho}$ is not r-regular.*

Proof. Modifying the choice of $\overline{\mathbb{Z}}_p$-lattice if necessary, it suffices to treat the case that $\bar{\rho}$ and $\bar{s}$ are semisimple. Let $\bar{\rho}' \cong \operatorname{Ind}_{G_{\mathbb{Q}_{p^d}}}^{G_{\mathbb{Q}_p}} \chi$ be an irreducible subrepresentation of $\bar{\rho}$. Take $K' = K_0 = \mathbb{Q}_{p^d}$ in the above notation, so that $e = p^d - 1$. Taking E to be sufficiently large so that s is defined over E and $\bar{\rho}$ is defined over k_E, and applying Proposition 2.1.2 to $s|_{G_{\mathbb{Q}_{p^d}}}$, we see that there is an element $\mathcal{M}$ of k_E-$\mathrm{BrMod}_{\mathrm{dd}}^r$ with

$$\mathrm{T}_{\mathrm{st}}^{*,r}(\mathcal{M}) \cong \bar{s}|_{G_{\mathbb{Q}_{p^d}}},$$

such that $\mathcal{M}$ has a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-basis $v_1, \dots, v_m$ with $\hat{g}(v_i) = (1 \otimes \overline{\chi}_i(g))v_i$ for all $g \in \mathrm{Gal}(K/K_0)$. Since $\bar{s}|_{G_{\mathbb{Q}_{p^d}}}$ contains a subrepresentation isomorphic to χ, we see from Lemma 2.1.1 that there is a rank-1 subobject $\mathcal{N}$ of $\mathcal{M}$ for which $\mathrm{T}_{\mathrm{st}}^{*,r}(\mathcal{N}) \cong \chi$.

Since $\mathcal{N}/u\mathcal{N}$ embeds into $\mathcal{M}/u\mathcal{M}$ (as $\mathcal{N}$ is a free $k[u]/u^{ep}$-submodule of the free $k[u]/u^{ep}$-module $\mathcal{M}$), we see from Lemma 2.1.3 and our assumption on the characters $\overline{\chi}_i$ that we may write $\mathcal{N}$ in the form

- $\mathcal{N} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{(p^d-1)p}) \cdot w$,
- $e_i \mathcal{N}_r = u^{r_i} e_i \mathcal{N}$,
- $\varphi_r\left(\sum_{i=0}^{d-1} u^{r_i} e_i w\right) = \lambda w$ for some $\lambda \in (k \otimes_{\mathbb{F}_p} k_E)^\times$,
- $\hat{g}(w) = \left(\sum_{i=0}^{d-1} (\omega_d(g)^{k_i} \otimes 1) e_i\right) w$ for all $g \in \mathrm{Gal}(K/K_0)$, and
- $N(w) = 0$.

Here the integers $0 \le r_i \le (p^d-1)r$ and k_i satisfy $k_i \equiv p(k_{i-1}+r_{i-1}) \pmod{p^d-1}$ for all i, and each k_i is equal to some $(1 + p + \cdots + p^{d-1})a_j$. (The conditions on the r_i come from Lemma 2.1.3, and the fact that each k_i is equal to some $(1+p+\cdots+p^{d-1})a_j$ comes from the fact that $\mathcal{N}$ is a submodule of $\mathcal{M}$ which has a basis $v_1, \dots, v_n$ such that $\hat{g}(v_i) = (1 \otimes \overline{\chi}_i(g))v_i$ and the assumption that each $\overline{\chi}_i$ is equal to some ω^{a_j}.) Writing $k_i = (1+p+\cdots+p^{d-1})x_i$, $0 \le x_i < p-1$, we see as in Remark 2.1.4 that we can write $r_i = (1+p+\cdots+p^{d-1})(x_{i+1}-x_i)+(p^d-1)y_i$, with $0 \le y_i \le r$. By Lemma 2.1.3 and Remark 2.1.4, we have

$$\chi|_{I_{\mathbb{Q}_p}} = \sigma_0 \circ \omega_d^{x_0+y_0+p^{d-1}(x_1+y_1)+\cdots+p(x_{d-1}+y_{d-1})}.$$

Since $r \le p-2$, we have $0 \le y_i \le p-2$, and we conclude (after allowing for "carrying") that each exponent of $\bar{\rho}$ is of the form $a_j + k$ with $0 \le k \le r+1$.

Suppose that $\bar{\rho}$ is r-regular. It must then be the case that the x_i above are all distinct. Applying this analysis to each irreducible subrepresentation of $\bar{\rho}$, we conclude that the a_i are all distinct. Since we have $\det(\bar{\rho}')|_{I_{\mathbb{Q}_p}} = \chi^{1+p+\cdots+p^{d-1}}|_{I_{\mathbb{Q}_p}} = \omega^{(x_0+y_0)+\cdots+(x_{d-1}+y_{d-1})}$, we conclude that $\det\bar{\rho}|_{I_{\mathbb{Q}_p}} = \omega^{a_1+\cdots+a_n+y}$ for some

$0 \le y \le nr \le n(n-1)/2$. The assumption on $\det \bar{\rho}$ then implies that $y \equiv n(n-1)/2 \pmod{p-1}$. If in fact $r < (n-1)/2$, then we have $0 \le y < n(n-1)/2$, which contradicts the assumption that $p > n(n-1)/2 + 1$.

It remains to treat the case that $r = (n-1)/2$, where we may assume (by the additional hypothesis that we have assumed in this case) that the representation $\bar{\rho}'$ above has dimension $d > 1$. By the above analysis we must have $y = n(n-1)/2$, so that each $y_i = r$. Since we have $r_i \le (p^d - 1)r$, we must have $x_{i+1} - x_i \le 0$ for each i, so that in fact $x_0 = x_1 = \cdots = x_{d-1}$, a contradiction (as we already showed that the x_i are distinct). ☐

3. The cohomology of Shimura varieties

3.1. ***The semistable reduction of certain $U(n-1,1)$-Shimura varieties.*** Fix $n \ge 2$, and fix an odd prime p.[3] We now recall the definitions of the $U(n-1,1)$-Shimura varieties with which we will work, and some associated integral models. For simplicity we work over $\mathbb{Q}$ (or rather an imaginary quadratic extension of $\mathbb{Q}$) rather than over a general totally real field.

For the most part we will follow Section 3 of [Haines and Rapoport 2012] (which uses a similar approach to [Harris and Taylor 2002]), with the occasional reference to [Harris and Taylor 2001]. Fix an imaginary quadratic field F in which the prime p splits, say $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ for some choice of $\mathfrak{p}$, let $x \mapsto \bar{x}$ be the nontrivial automorphism of F, and regard F as a subfield of $\mathbb{C}$ via a fixed embedding $F \hookrightarrow \mathbb{C}$.

Let D be a division algebra over F of dimension n^2, and let $*$ be an involution of D of the second kind (that is, $*|_F$ is nontrivial). Assume that D splits at $\mathfrak{p}$ (and hence at $\bar{\mathfrak{p}}$), and fix isomorphisms $D_{\mathfrak{p}} \cong M_n(\mathbb{Q}_p)$ and $D_{\bar{\mathfrak{p}}} \cong M_n(\mathbb{Q}_p)$ with the property that, under the induced isomorphism

$$D \otimes \mathbb{Q}_p \cong M_n(\mathbb{Q}_p) \times M_n(\mathbb{Q}_p)^{\mathrm{op}},$$

the involution $*$ corresponds to $(X, Y) \mapsto (Y^t, X^t)$.

Let $G_{/\mathbb{Q}}$ be the algebraic group whose R-points are

$$G(R) = \{x \in (D \otimes_{\mathbb{Q}} R)^{\times} \mid x \cdot x^* \in R^{\times}\}$$

for any $\mathbb{Q}$-algebra R. Thus our fixed isomorphism $D_{\mathfrak{p}} \cong M_n(\mathbb{Q}_p)$ induces an isomorphism $G \times_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{GL}_n \times \mathbb{G}_m$.

Now let $h_0 : \mathbb{C} \to D_{\mathbb{R}}$ be an $\mathbb{R}$-algebra homomorphism with the properties that $h_0(z)^* = h_0(\bar{z})$ and the involution $x \mapsto h_0(i)^{-1} x^* h_0(i)$ is positive (that is, $\mathrm{tr}_{B/\mathbb{Q}}(x h_0(i)^{-1} x^* h_0(i)) > 0$ for all nonzero x). Let $B = D^{\mathrm{op}}$ and let $V = D$, which we consider as a free left B-module of rank 1 by multiplication on the right.

[3] The reason for assuming that the prime p is odd is that below we will want to apply the discussion and results of Section 2, in which this assumption was made.

Then $\mathrm{End}_B(V) = D$, and one can find an element $\xi \in D^\times$ with the properties that $\xi^* = -\xi$ and such that the involution ι of B defined by $x^\iota = \xi x^* \xi^{-1}$ is positive (see Section I.7 of [Harris and Taylor 2001] or Section 5.2 of [Haines 2005] for the existence of such a ξ).

We have an alternating pairing $\psi(\cdot,\cdot) : D \times D \to \mathbb{Q}$ defined by $\psi(x, y) = \mathrm{tr}_{D/\mathbb{Q}}(x\xi y^*)$, and one sees easily that $\psi(bx, y) = \psi(x, b^\iota y)$ and that $\psi(\cdot, h_0(i)\cdot)$ is either positive- or negative-definite. After possibly replacing ξ by $-\xi$, we can and do assume that it is positive-definite.

It is easy to see that one has

$$G(\mathbb{R}) \cong \mathrm{GU}(r, s)$$

for some r, s with $r + s = n$. We impose the additional assumption that in fact $\{r, s\} = \{n - 1, 1\}$. Note that by Lemma I.7.1 of [Harris and Taylor 2001] one can find division algebras D for which this holds. We say that a compact open subgroup $K \subset G(\mathbb{A}^\infty)$ (resp. $K^p \subset G(\mathbb{A}^{p,\infty})$) is *sufficiently small* if for some prime q (resp. some prime $q \neq p$) the projection of K (resp. K^p) to $G(\mathbb{Q}_q)$ contains no element of finite order other than 1. If K is sufficiently small, we will consider the Shimura variety $\mathrm{Sh}(G, h_0|_{\mathbb{C}^\times}^{-1}, K)$. It has a canonical model over F, which we denote by $X(K)$ (note that if $n > 2$ the reflex field is F, while if $n = 2$ the reflex field is $\mathbb{Q}$, and we let $X(K)$ denote the base change of the canonical model from $\mathbb{Q}$ to F).

We say that a compact open subgroup K of $G(\mathbb{A})$ is of *level dividing* N, for some integer $N \geq 1$, if for all primes $l \nmid N$ we can write $K = K_l K^l$, where K_l is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_l)$ and K^l is a compact open subgroup of $G(\mathbb{A}^{\infty,l})$. (Note then that in fact $K = K_N \times \prod_{l \nmid N} K_l$, for some compact open subgroup K_N of $\prod_{l|N} G(\mathbb{Q}_l)$.) If K is of level dividing N, then we similarly refer to $X(K)$ as a $U(n-1, 1)$-*Shimura variety of level dividing* N.

We will now define integral models of these Shimura varieties for two specific kinds of level structure. We begin by introducing notation related to the level structures in question.

We write I_0 for the Iwahori subgroup of $\mathrm{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$; namely, I_0 is the subgroup of $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ consisting of elements whose first factor lies in the usual Iwahori subgroup of matrices which are upper-triangular mod p. We write I_1 to denote the pro-p-Iwahori subgroup of $\mathrm{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$; namely, I_1 is the (unique) pro-p Sylow subgroup of I_0, and consists of those elements of $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ whose first factor is upper-triangular unipotent mod p, and whose second factor is congruent to 1 mod p. We write I_1^* to denote the subgroup of I_0 consisting of those matrices in $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ whose first factor is upper-triangular unipotent mod p.

There is a natural isomorphism $\mathbb{Z}_p^\times = \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$, and this induces a natural isomorphism $I_1^* = \mathbb{F}_p^\times \times I_1$. If we let T denote the diagonal torus in GL_n, then there is also a natural isomorphism $I_0 = T(\mathbb{F}_p) \times I_1^*$.

We will define integral models for $X(I_0K^p)$ and $X(I_1^*K^p)$ over the local rings $\mathcal{O}_{F,(\mathfrak{p})}$ and $\mathcal{O}_{F(\zeta_{p-1}),(v)}$ respectively, where ζ_{p-1} denotes a primitive $(p-1)$-st root of unity, and v is some fixed place in $F(\zeta_{p-1})$ above $\mathfrak{p}$; here K^p is a sufficiently small compact open subgroup of $G(\mathbb{A}^{\infty,p})$, and we consider I_0 and I_1^* as subgroups of $G(\mathbb{Q}_p)\cong \mathrm{GL}_n(\mathbb{Q}_p)\times\mathbb{Q}_p^\times$. We will typically not include K^p in the notation, and we will write $\mathscr{X}_0(p)$, $\mathscr{X}_1(p)$ for our integral models of $X_0(p):=X(I_0K^p)$ and $X_1(p):=X(I_1^*K^p)$ respectively.

Remark 3.1.1. Note that in [Haines and Rapoport 2012], [Harris and Taylor 2002], and [Harris and Taylor 2001], the authors work over $\mathbb{Z}_p$, but we follow [Kottwitz 1992] in working over $\mathcal{O}_{F,(\mathfrak{p})}$, so as to satisfy the hypothesis required to be in the global case of Section 1. The appearance of ζ_{p-1} in the ring of definition of $\mathscr{X}_1(p)$ is a consequence of our use of Oort–Tate theory in the definition of the integral model in this case.

In order to define these integral models, we first recall a certain category of abelian schemes (up to isogeny) with polarisations and endomorphisms. If $\mathscr{F}$ is a set-valued functor on the category of connected, locally noetherian $\mathcal{O}_{F,(p)}$-schemes, we will also consider it to be a functor on the category of all locally noetherian $\mathcal{O}_{F(p)}$-schemes by setting

$$\mathscr{F}\Big(\coprod S_i\Big):=\prod\mathscr{F}(S_i).$$

Let $\mathcal{O}_B$ be the unique maximal $\mathbb{Z}_{(p)}$-order in B which under our fixed identification $B\otimes_{\mathbb{Q}}\mathbb{Q}_p=M_n(\mathbb{Q}_p)\times M_n(\mathbb{Q}_p)^{\mathrm{op}}$ is identified with $M_n(\mathbb{Z}_{(p)}\times M_n(\mathbb{Z}_{(p)})^{\mathrm{op}}$. Let S be a connected, locally noetherian $\mathcal{O}_{F,(\mathfrak{p})}$-scheme, and let AV_S be the category whose objects are pairs (A,i), where A is an abelian scheme over S of dimension n^2 and $i:\mathcal{O}_B\to\mathrm{End}_S(A)\otimes\mathbb{Z}_{(p)}$ is a homomorphism. We define homomorphisms in AV_S by

$$\mathrm{Hom}((A_1,i_1),(A_2,i_2))=\mathrm{Hom}_{\mathcal{O}_B}((A_1,i_1),(A_2,i_2))\otimes\mathbb{Z}_{(p)}$$

(that is, the elements of $\mathrm{Hom}_S(A_1,A_2)\otimes\mathbb{Z}_{(p)}$ which commute with the action of $\mathcal{O}_B$). The dual of an object (A,i) of AV_S is $(\widehat{A},\hat{i})$, where $\widehat{A}$ is the dual abelian scheme of A and $\hat{i}(b)=(i(b^\iota))^\wedge$. A polarisation of (A,i) is a homomorphism $\lambda:(A,i)\to(\widehat{A},\hat{i})$ in AV_S with the property that, for some $n\geq 1$, $n\lambda$ is induced by an ample line bundle on A. A principal polarisation is a polarisation which is also an isomorphism in AV_S. A $\mathbb{Q}$-class of polarisations is an equivalence class of homomorphisms $(A,i)\to(\widehat{A},\hat{i})$ which contains a polarisation, under the equivalence relation of differing by a $\mathbb{Q}^\times$-scalar.

Fix K^p a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$. Let $\mathscr{A}_0$ be the set-valued functor on the category of locally noetherian schemes over $\mathcal{O}_{F,(\mathfrak{p})}$

which sends a connected, locally noetherian scheme S over $\mathcal{O}_{F,(\mathfrak{p})}$ to the set of isomorphism classes of the following data:

- A commutative diagram of morphisms in the category AV_S of the form

$$\begin{tikzcd} A_0 \arrow[r, "\alpha_0"] \arrow[d, "\lambda_0"] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "\lambda_1"] & \cdots \arrow[r, "\alpha_{n-2}"] & A_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "\lambda_{n-1}"] & A_0 \arrow[d, "\lambda_0"] \\ \widehat{A}_0 & \widehat{A}_1 \arrow[l, "\hat{\alpha}_0"'] & \cdots \arrow[l, "\hat{\alpha}_1"'] & \widehat{A}_{n-1} \arrow[l, "\hat{\alpha}_{n-2}"'] & \widehat{A}_0 \arrow[l, "\hat{\alpha}_{n-1}"'] \end{tikzcd}$$

  where each α_i is an isogeny of degree p^{2n} and their composite is just multiplication by p. In addition, λ_0 is a $\mathbb{Q}$-class of polarisations containing a principal polarisation. Furthermore, we require that each A_i satisfies a compatibility between the two actions of $\mathcal{O}_F$ on the Lie algebra of A_i (one action coming from the structure morphism $\mathcal{O}_{F,(\mathfrak{p})} \to \mathcal{O}_S$, and the other from the $\mathcal{O}_B$-action; see [Harris and Taylor 2001, §III.4] for a discussion of this condition).

- A geometric point s of S, and a $\pi_1(S,s)$-invariant K^p-orbit of isomorphisms

$$\overline{\eta} : V \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty} \xrightarrow{\sim} H_1((A_0)_s, \mathbb{A}^{p,\infty})$$

  which are $\mathcal{O}_B$-linear and up to a constant in $(\mathbb{A}^{p,\infty})^\times$ take the ψ-pairing on the left side to the λ_0-Weil pairing on the right side. (This data is canonically independent of the choice of s; see the discussion on pp. 390–391 of [Kottwitz 1992].)

An isomorphism of this data is one induced by isomorphisms in AV_S which preserve the λ_i up to an overall $\mathbb{Q}^\times$-scalar.

The functor $\mathcal{A}_0$ is represented by a projective scheme $\mathcal{X}_0(p)$ over $\mathcal{O}_{F,(\mathfrak{p})}$, which is an integral model for $X(I_0K^p)$. (See the proof of Lemma 3.2 of [Taylor and Yoshida 2007], which shows that $\mathcal{X}_0(p)$ is projective over the usual integral model at hyperspecial level. At hyperspecial level, quasiprojectivity is proved on p. 391 of [Kottwitz 1992], and projectivity can be checked via the valuative criterion for properness as on p. 392 of the same work. More properly, $\mathcal{X}_0(p)$ is an integral model for a disjoint union of a number of copies of $X(I_0K^p)$, due to the possible failure of the Hasse principle; see for example Section 7 and the discussion on p. 400 of [Kottwitz 1992]. Since the cohomology of a disjoint union of spaces is the direct sum of the cohomologies of the individual spaces, this does not affect our arguments, and we will not dwell on this point in the following.) The proof of Proposition 3.4(3) of [Taylor and Yoshida 2007] (which goes over unchanged in our setting) shows that the special fibre of $\mathcal{X}_0(p)$ is a strict normal crossings divisor.

Our next goal is to describe an integral model $\mathcal{X}_1(p)$, over $\mathcal{O}_{F(\zeta_{p-1}),(v)}$, for $X(I_1^*K^p)$ (or rather, as in the previous paragraph, an integral model of a disjoint union of a number of copies of $X(I_1^*K^p)$). Recalling that T denotes the diagonal torus in GL_n, we let $\iota_i : \mathbb{G}_m \to T$ denote the embedding of tori identifying $\mathbb{G}_m$ with the subgroup of T consisting of elements which are 1 away from the i-th diagonal entry. We use the same notation ι_i to denote the map $\mathbb{F}_p^\times \to T(\mathbb{F}_p)$ induced by the map of tori.

The quotient I_0/I_1^* is naturally identified with $T(\mathbb{F}_p)$, and so $T(\mathbb{F}_p)$ acts on $X_1(p)$, with quotient isomorphic to $X_0(p)$.

Given an S-valued point of $\mathcal{A}_0$, let $A_i(p^\infty)$ be the p-divisible group associated to A_i for $i = 0, \ldots, n-1$. Each $A_i(p^\infty)$ has an action of $\mathbb{O}_B = M_n(\mathbb{Z}_{(p)}) \times M_n(\mathbb{Z}_p)^{\mathrm{op}}$. Let $X_i = e_{11} A_i(p^\infty)$, where e_{11} is the usual idempotent in $M_n(\mathbb{Z}_{(p)})$ (and is zero on the second factor). Then each X_i is a p-divisible group of height n and dimension 1, and we obtain a chain of isogenies of degree p

$$\mathscr{C} : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n := X_0,$$

whose composite is equal to multiplication by p.

We let OT denote the Artin stack over $\mathbb{O}_{F(\zeta_{p-1}),(v)}$ given by

$$\mathrm{OT} := \big[\operatorname{Spec} \mathbb{O}_{F(\zeta_{p-1}),(v)}[X, Y]/(XY - w_p)/\mathbb{G}_m\big],$$

where $\mathbb{G}_m$ acts via $\lambda \cdot (X, Y) = (\lambda^{p-1} X, \lambda^{1-p} Y)$ and w_p is some explicit element of $\mathbb{O}_{F(\zeta_{p-1}),(v)}$ of valuation 1. Oort–Tate theory shows that OT classifies finite flat group schemes of order p over $\mathbb{O}_{F(\zeta_{p-1}),(v)}$-schemes. The universal group scheme over OT is the stack

$$\mathscr{G} := \big[\operatorname{Spec} \mathbb{O}_{F(\zeta_{p-1}),(v)}[X, Y, Z]/(XY - w_p, Z^p - XZ)/\mathbb{G}_m\big],$$

where $\mathbb{G}_m$ acts on X and Y as above, and on Z via $\lambda \cdot Z = \lambda Z$. The morphism $\mathscr{G} \to \mathrm{OT}$ is the evident one, the zero section of $\mathscr{G}$ is cut out by the equation $Z = 0$, and we let $\mathscr{G}^\times$ denote the closed subscheme of $\mathscr{G}$ cut out by the equation $Z^{p-1} - X = 0$; this is the so-called *scheme of generators* of $\mathscr{G}$. (See Theorem 6.5.1 of [Genestier and Tilouine 2005] for these facts, which are a restatement of Theorem 2 of [Tate and Oort 1970] in the language of stacks.)

We define $\mathscr{X}_1(p)$ via the Cartesian diagram

$$\begin{array}{ccc} \mathscr{X}_1(p) & \longrightarrow & \mathscr{G}^\times \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \mathscr{G}^\times \\ \downarrow & & \downarrow \\ \mathscr{X}_0(p)_{/\mathbb{O}_{F(\zeta_{p-1}),(v)}} & \longrightarrow & \mathrm{OT} \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \mathrm{OT} \end{array} \tag{3.1.2}$$

where the bottom horizontal arrow is given by

$$\mathscr{C} \mapsto (\ker(\alpha_0), \ldots, \ker(\alpha_{n-1})).$$

Note that the right-hand vertical arrow is finite and relatively representable by construction, and so the left-hand vertical arrow is a finite morphism of schemes. The action of $T(\mathbb{F}_p)$ on $X_1(p)$ extends to an action on $\mathscr{X}_1(p)$, namely, the action pulled back from the action of $T(\mathbb{F}_p)$ on $\mathscr{G}^\times \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \mathscr{G}^\times$ over $\mathrm{OT} \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \mathrm{OT}$.

Let $\pi := (-p)^{1/(p-1)}$, and let w be the unique finite place of $L := F(\zeta_{p-1}, \pi)$ lying over our fixed place v of $F(\zeta_{p-1})$. We let $\mathbb{O}$ denote the localisation of $\mathbb{O}_L$ at w, and we let $\mathscr{X}_1(p)_{\mathbb{O}}$ denote the normalisation of the base change $\mathscr{X}_1(p)_{/\mathbb{O}}$ of $\mathscr{X}_1(p)$ over $\mathbb{O}$. We write $I := \mathrm{Gal}(L/F(\zeta_{p-1}))$; this is also the inertia group at w in $\mathrm{Gal}(L/F)$. The group I acts naturally on $\mathscr{X}_1(p)_{\mathbb{O}}$. The mod p cyclotomic character ω induces an isomorphism (which we continue to denote by ω)

$$\omega : I \xrightarrow{\sim} \mathbb{F}_p^\times.$$

For each $i = 0, \ldots, n-1$, we let $\alpha_i : I \to T$ denote the composite $\iota_i \circ \omega^{-1}$.

Lemma 3.1.3. *The scheme $\mathscr{X}_1(p)_{\mathbb{O}}$ is a semistable projective model for $X_1(p)_{/L}$ over $\mathbb{O}$, the natural morphism*

$$\mathscr{X}_1(p)_{\mathbb{O}} \to \mathscr{X}_0(p) \tag{3.1.4}$$

is tamely ramified, and the action of $I \times T$ on $X_1(p)_{/L}$ extends to an action on $\mathscr{X}_1(p)_{\mathbb{O}}$. Furthermore, on each irreducible component of its special fibre, the inertia group I acts through the composite of the T-action with one of the characters α_i.

Proof. We will apply a form of Deligne's homogeneity principle, as described in the proof of [Taylor and Yoshida 2007, Proposition 3.4], to the morphism (3.1.4). The scheme here denoted $\mathscr{X}_0(p)$ is there denoted X_U (and the integral model there is considered over $\mathbb{Z}_p$ rather than $\mathbb{O}_{F,(\mathfrak{p})}$, but this is immaterial for our present purposes), while the scheme there denoted X_{U_0} is an integral model of the Shimura variety (in the notation of the present paper) $X(K_pK^p)$, where $K_p = \mathrm{GL}_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$. We let $\mathscr{X}_0(p)_s^{(h)}$ (for $0 \le h \le n-1$) denote the locally closed subset of the special fibre $\mathscr{X}_0(p)_s$, obtained by pulling back the locally closed subset $\overline{X}_{U_0}^{(h)}$ defined in Section 3 of [Taylor and Yoshida 2007] under the natural projection $\mathscr{X}_0(p)_s = \overline{X}_U \to \overline{X}_{U_0}$.

We will show that the morphism (3.1.4) is tamely ramified in the formal neighbourhood of any closed geometric point of the special fibre $\mathscr{X}_0(p)_s$, and hence (by Lemma 1.5.3) that it is tamely ramified. We first note that the morphism $\mathscr{X}_0(p)_{/\mathbb{O}_{F(\zeta_{p-1}),(v)}} \to \mathscr{X}_0(p)$ induces an isomorphism on special fibres, and so we are free to replace $\mathscr{X}_0(p)$ by $\mathscr{X}_0(p)_{/\mathbb{O}_{F(\zeta_{p-1}),(v)}}$ in our considerations. We next note that the completion of the bottom arrow of (3.1.2) at a closed geometric point $\bar{x}_0$ of $\mathscr{X}_0(p)_s$ depends up to isomorphism only on the value of h for which $\bar{x}_0 \in \mathscr{X}_0(p)_s^{(h)}$ (since the p-divisible group attached to the point $\bar{x}_0$ depends only on the value of h), and hence that the restriction of (3.1.4) to a formal neighbourhood of $\bar{x}_0$ depends only on the value of h. Then, by [Taylor and Yoshida 2007, Lemma 3.1], the closure of $\mathscr{X}_0(p)_s^{(h)}$ contains $\mathscr{X}_0(p)_s^{(0)}$. Since being tamely ramified is an open condition, we conclude from these two conditions that, in order to prove the lemma, it suffices to show that the restriction of (3.1.4) to a formal neighbourhood of $\bar{x}_0$ is tamely ramified at closed geometric points $\bar{x}_0$ of $\mathscr{X}_0(p)_s^{(0)}$. (As already indicated, this argument is a variation on Deligne's homogeneity principle.)

Thus, consider a closed geometric supersingular point $\bar{x}_0$ of $\mathscr{X}_0(p)_s^{(0)}$, so that $\bar{x}_0$ admits a formal neighbourhood of the form

$$\operatorname{Spec} W(\overline{\mathbb{F}}_p)[\![T_1, \dots, T_n]\!]/(T_1 \cdots T_n - w_p).$$

The proof of [Taylor and Yoshida 2007, Proposition 3.4] shows that the T_i may be taken to be the matrix of α_{i-1} on tangent spaces, so that the map $\mathscr{X}_0(p) \to \mathrm{OT} \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \mathrm{OT}$ may be defined in the formal neighbourhood of $\bar{x}_0$ by the map

$$(T_1, \dots, T_n) \mapsto ((T_1, U_1), \dots, (T_n, U_n)),$$

where $U_i = T_1 \cdots \widehat{T_i} \cdots T_n$ (and, as is usual in these situations, a hat on a variable denotes that that variable is omitted in the expression). Thus a formal neighbourhood of a closed geometric point lying over $\bar{x}_0$ in $(\mathscr{X}_1(p)_{/\mathbb{O}})_s$ is isomorphic to

$$\operatorname{Spec} W(\overline{\mathbb{F}}_p)[\pi][\![V_1, \dots, V_n]\!]/((V_1 \cdots V_n)^{p-1} - w_p).$$

If we write $u := V_1 \cdots V_n/\pi$, then we see that $u^{p-1} = -w_p/p$, and hence that u lies in the normalisation of this formal neighbourhood. Furthermore, on each component of this normalisation, u is equal to one of the $(p-1)$-st roots of $-w_p/p$ lying in $W(\overline{\mathbb{F}}_p)$. Thus the normalisation of this formal neighbourhood is a union of components, each isomorphic to

$$\operatorname{Spec} \mathbb{O}[\![V_1, \dots, V_n]\!]/(V_1 \cdots V_n - u\pi),$$

with the morphism (3.1.4) being given by $T_i = V_i^{p-1}$. Thus this morphism is indeed tamely ramified in the formal neighbourhood of $\bar{x}_0$.

It is clear that the $I \times T$-action on $X_1(p)_L$ extends to an action on $\mathscr{X}_1(p)_{/\mathbb{O}}$, and hence to its normalisation $\mathscr{X}_1(p)_{\mathbb{O}}$. As for the final statement, note that I acts on π via ω, and hence on u via ω^{-1}, while I fixes each V_i. Also T acts on V_i through multiplication by the i-th diagonal entry, and so acts on u via multiplication by the determinant. Combining these facts, we see that I acts on the components of the special fibre on which $V_i = 0$ via $\imath_i \circ \omega^{-1}$. $\square$

Remark 3.1.5. This lemma (and the fact that $\mathscr{X}_0(p)$ is strictly semistable) shows that the map $\mathscr{X}_1(p)_{\mathbb{O}} \to \mathscr{X}_0(p)$ provides a tame strictly semistable context, in the sense of Section 1.5. In particular, we can apply Theorem 1.5.15 in this setting (of course taking the group G to be the abelian group T), and we see that each character ψ_j as in the statement of that theorem is equal to one of the characters α_i.

3.2. ***Canonical local systems.*** If K is a sufficiently small compact open subgroup of $G(\mathbb{A}^\infty)$, and V is a continuous representation of K on a finite-dimensional $\overline{\mathbb{F}}_p$-vector space (this vector space being equipped with its discrete topology), then we may associate to V an étale local system $\mathscr{F}_V$ of $\overline{\mathbb{F}}_p$-vector spaces on $X(K)$ as follows: Choose an open normal subgroup $K' \subset K$ lying in the kernel of V, and regard V as a representation of the quotient K/K'. Since $X(K')$ is naturally an étale K/K'-torsor over $X(K)$, we may form the étale local system of $\overline{\mathbb{F}}_p$-vector spaces over $X(K)$ associated to the K/K'-representation V. This local system is independent of the choice of K', up to canonical isomorphism, and we define it to be $\mathscr{F}_V$.

Definition 3.2.1. We refer to the étale local systems $\mathscr{F}_V$ that arise by the preceding construction as the *canonical local systems* on $X(K)$. If we may choose K' in the kernel of V to be of level dividing N (so that in particular $X(K)$ is of level dividing N), then we say that $\mathscr{F}_V$ can be *trivialised at level* N.

3.3. ***The Eichler–Shimura relation.*** Let X be a $U(n-1,1)$-Shimura variety of level dividing N. Let w be a place of F such that $l := w|_{\mathbb{Q}}$ splits in F and does not divide N. There is a natural action via correspondences on X of Hecke operators $T_w^{(i)}$, $0 \le i \le n$, where $T_w^{(i)}$ is the double coset operator corresponding to

$$\begin{pmatrix} l1_i & 0 \\ 0 & 1_{n-i} \end{pmatrix} \times 1 \in \mathrm{GL}_n(\mathbb{Q}_l) \times \mathbb{Z}_l^\times,$$

where we use the assumption that $l \nmid N$ and identify a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_l)$ with $\mathrm{GL}_n(\mathbb{Z}_l) \times \mathbb{Z}_l^\times$ via an isomorphism $D_w \cong M_n(\mathbb{Q}_l)$. These correspondences then act on the cohomology $H^j_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)$.

More generally if $\mathscr{F}_V$ is a canonical local system on X that can be trivialised at level N, then we obtain an action of the double coset operators $T_w^{(i)}$ on $\mathscr{F}_V$, and hence on the cohomology $H^j_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)$.

The following theorem regarding this action is then an immediate consequence of the main result of [Wedhorn 2000] (which proves the Eichler–Shimura relation for PEL Shimura varieties at places of good reduction at which the group is split).

Theorem 3.3.1. *Let X be a $U(n-1,1)$-Shimura variety of level dividing N and $\mathscr{F}_V$ a canonical local system on X. Let w be a place of F such that $w|_{\mathbb{Q}}$ splits in F and does not divide Np. Then $\sum_{i=0}^n (-1)^i (\mathrm{Norm}\, w)^{i(i-1)/2} T_w^{(i)} \mathrm{Frob}_w^{n-i}$ acts as 0 on each $H^j_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)$.*

3.4. ***Vanishing and torsion-freeness of cohomology for certain $U(n-1,1)$-Shimura varieties.*** Let X denote a $U(n-1,1)$-Shimura variety as above, and let $\mathcal{F}_V$ denote a canonical local system on X. Choose N so that X has level dividing N, so that $\mathcal{F}_V$ can be trivialised at level N, and so that p divides N. Assume that the projection of the corresponding level K to $G(\mathbb{A}^{p,\infty})$ is sufficiently small.

Let $\mathbb{T} = \overline{\mathbb{Z}}_p[T_w^{(i)}]$ be the polynomial ring in the variables $T_w^{(i)}$, $1 \le i \le n$, where w runs over the places of F such that $w|_{\mathbb{Q}}$ splits in F and does not divide N. Let $\mathfrak{m}$ be a maximal ideal in $\mathbb{T}$ with residue field $\overline{\mathbb{F}}_p$, and suppose that there exists a continuous irreducible representation $\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ which is unramified at all finite places not dividing N, and which satisfies $\mathrm{char}(\rho_{\mathfrak{m}}(\mathrm{Frob}_w)) \equiv \sum_{i=0}^{n}(-1)^i (\mathrm{Norm}\, w)^{i(i-1)/2} T_w^{(i)} X^{n-i} \mod \mathfrak{m}$ for all $w \nmid N$ such that $w|_{\mathbb{Q}}$ splits in F. Continue to fix a choice of a place $\mathfrak{p}$ of F dividing p, and write $G_{\mathbb{Q}_p}$ for $G_{F_{\mathfrak{p}}}$ from now on. Recall that the choice of $\mathfrak{p}$ also gives us an isomorphism $G(\mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$ as in Section 3.1.

We consider the following further hypothesis on $\rho_{\mathfrak{m}}$ (this is Hypothesis 4.1.1):

Hypothesis 3.4.1. If $\theta : G_F \to \mathrm{GL}_m(\overline{\mathbb{F}}_p)$ is any continuous, irreducible representation with the property that the characteristic polynomial of $\rho_{\mathfrak{m}}(g)$ annihilates $\theta(g)$ for every $g \in G_F$, then θ is equivalent to $\rho_{\mathfrak{m}}$.

We will now prove our first main result, a vanishing theorem for the cohomology of X with $\mathcal{F}_V$-coefficients:

Theorem 3.4.2. *Suppose that $\rho_{\mathfrak{m}}$ satisfies Hypothesis 3.4.1, that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}$ is r-regular for some $r \le (n-1)/2$, that $p > n(n-1)/2+1$ and, if $r = (n-1)/2$, suppose in addition that $\rho_{\mathfrak{m}|G_{\mathbb{Q}_p}}$ contains an irreducible subquotient of dimension greater than* 1. *Then the localisations $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}$ vanish for $i \le r$ and $i \ge 2(n-1)-r$.*

Remark 3.4.3. In Section 4.1 we will show that Hypothesis 3.4.1 is satisfied if either $\rho_{\mathfrak{m}}$ is induced from a character of G_K for some degree-n cyclic Galois extension $K/\mathbb{Q}$, or if $p \ge n$ and $\mathrm{SL}_n(k) \subseteq \rho_{\mathfrak{m}}(G_F) \subseteq \overline{\mathbb{F}}_p^\times \mathrm{GL}_n(k)$ for some subfield $k \subset \overline{\mathbb{F}}_p$.

Remark 3.4.4. While we work here with étale local systems and étale cohomology, by virtue of Artin's comparison theorem [SGA 4_3 1973, Exposé XI, Théorème 4.3] our vanishing results are equivalent to vanishing results for the cohomology of the complex $U(n-1,1)$-Shimura varieties with coefficients in the corresponding canonical local systems for the complex topology.

Proof of Theorem 3.4.2. First, note that it suffices to prove vanishing in degree $i \le r$ for all V, as vanishing in degree $i \ge 2(n-1)-r$ then follows by Poincaré duality. (Note that the dual of the canonical local system $\mathcal{F}_V$ attached to a representation V is the canonical local system attached to the contragredient representation $V^\vee$.)

We prove the theorem for $i \le r$ by induction on i, the case when $i < 0$ being trivial. We begin by reducing to the case when $\mathscr{F}_V$ is trivial. To this end, write $X = X(K)$, let K' be an open normal subgroup of K of level dividing N such that V is representation of K/K', and choose an embedding of K/K'-representations $V \hookrightarrow \overline{\mathbb{F}}_p[K/K']^n$ for some n; denote the cokernel by W. Let π denote the projection $X(K') \to X(K)$. Passing to the associated canonical local systems, we obtain a short exact sequence $0 \to \mathscr{F}_V \to \pi_* \overline{\mathbb{F}}_p^n \to \mathscr{F}_W \to 0$, which gives rise to an exact sequence of cohomology

$$H^{i-1}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_W) \longrightarrow H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V) \longrightarrow H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \pi_*\overline{\mathbb{F}}_p^n) = H^i_{\text{ét}}(X(K')_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)^n.$$

Localising at $\mathfrak{m}$ and applying our inductive hypothesis (with $\mathscr{F}_W$ in place of $\mathscr{F}_V$), we reduce to the case of constant coefficients (with $X(K')$ in place of X). We therefore turn to establishing the claimed vanishing in this case.

Suppose now that K' is any open normal subgroup of K of level dividing N. Combining the Hochschild–Serre spectral sequence

$$E_2^{m,n} = H^m_{\text{ét}}(K/K', H^n(X(K')_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}) \Longrightarrow H^{m+n}_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$$

with our inductive hypothesis, we find that vanishing of $H^i_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ implies the vanishing of $H^i_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$. Thus, without loss of generality, we may and do assume that $K = K_p K^p$, where K_p is an open normal subgroup of I_1 and K^p is a sufficiently small compact open subgroup of $G(\mathbb{A}^{\infty,p})$.

We again consider a Hochschild–Serre spectral sequence, this time the one relating the cohomology of $X(K)$ and $X(I_1 K^p)$, which takes the form

$$E_2^{m,n} = H^m(I_1/K_p, H^n_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}) \Longrightarrow H^{m+n}_{\text{ét}}(X(I_1 K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}.$$

Once more taking into account our inductive hypothesis, we obtain an isomorphism $H^i_{\text{ét}}(X(I_1 K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \xrightarrow{\sim} H^i_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}^{I_1/K_p}$. Since I_1/K_p is a p-group, while $H^i_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ is a vector space over a field of characteristic p, we see that the latter space vanishes if and only if its space of I_1/K_p-invariants does. Thus we are reduced to establishing the theorem in the case when $K = I_1 K^p$.

Now recall that $I_1^* = \mathbb{F}_p^\times \times I_1$, and that the projection onto the first factor arises from the similitude projection $\mathrm{GU}(n-1,1) \to \mathbb{G}_m$. From this it follows that $X(I_1 K^p)$ is isomorphic to the product $X(I_1^* K^p) \times_F \operatorname{Spec} A$, where $A := F[x]/\Phi_p(x)$ (where $\Phi_p(x)$ denotes the p-th cyclotomic polynomial; the action of $\mathbb{F}_p^\times = I_1^*/I_1$ on $X(I_1 K^p)$ is induced by the action of $\mathbb{F}_p^\times$ on A given by $x \mapsto x^a$ for $a \in \mathbb{F}_p^\times$). Consequently there is an isomorphism of Galois representations

$$H^i(X(I_1 K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \xrightarrow{\sim} \bigoplus_{j=0}^{p-2} H^i(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}_j} \otimes \omega^j,$$

where $\mathfrak{m}_j$ is the maximal ideal in $\mathbb{T}$ which corresponds to the twisted Galois representation $\rho_{\mathfrak{m}_j} := \rho_{\mathfrak{m}} \otimes \omega^{-j}$. Since each of the Galois representations $\rho_{\mathfrak{m}_j}$ satisfies the hypotheses of the theorem (as these hypotheses are invariant under twisting), we are reduced to proving the theorem in the case of $X(I_1^* K^p)$.

Consider an irreducible $G_F \times I_0/I_1^*$-subrepresentation of $H^i_{\text{ét}}(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$, which we may write in the form $\theta \otimes \beta$, where θ is an absolutely irreducible G_F-representation and β is a character of the abelian group $T(\mathbb{F}_p) = I_0/I_1^*$. Theorem 3.3.1 and Hypothesis 3.4.1 taken together imply that θ is equivalent to $\rho_{\mathfrak{m}}$. We consider β as an n-tuple of characters $\beta_1, \ldots, \beta_n$ of $\mathbb{F}_p^\times$. Write $\widetilde{\beta}_j$ for the Teichmüller lift of β_j and χ_j for the character of $I_{\mathbb{Q}_p}$ given by β_j. By Lemma 3.1.3, Remark 3.1.5 and Theorem 1.5.15, we see that θ can be embedded as the reduction mod p of a potentially semistable representation with Hodge–Tate weights in the range $[-i, 0]$, whose inertial type is a direct sum of characters belonging to the collection $\{\widetilde{\beta}_j\}$. (To see this, note that as θ is in the β-part of the cohomology, it necessarily occurs in the reduction of the $\widetilde{\beta}$-part of the $G_F \times T$-representation provided by Theorem 1.5.15.) We claim that $\det \theta|_{I_{\mathbb{Q}_p}} = \det \rho_{\mathfrak{m}}|_{I_{\mathbb{Q}_p}} = \overline{\chi}_1 \cdots \overline{\chi}_n \omega^{-n(n-1)/2}$. Admitting this for the moment, we may apply Theorem 2.2.4 to the representation $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}^\vee$, and we deduce that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}^\vee$ is not r-regular. Equivalently, we see that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}$ is not r-regular, which contradicts our assumptions.

It remains to establish the equality $\det \theta|_{I_{\mathbb{Q}_p}} = \det \rho_{\mathfrak{m}}|_{I_{\mathbb{Q}_p}} = \overline{\chi}_1 \cdots \overline{\chi}_n \omega^{-n(n-1)/2}$. To see this, note that Theorem 3.3.1 implies that, for each place $w \nmid N$ of F such that $w|_{\mathbb{Q}}$ splits in F, we have $\det \rho_{\mathfrak{m}}(\mathrm{Frob}_w) = (\mathrm{Norm}\, w)^{n(n-1)/2} T_w^{(n)}$. Let $\psi_{\mathfrak{m}}$ be the character of $\mathbb{A}_F^\times / F^\times$ corresponding to the character $\omega^{n(n-1)/2} \det \rho_{\mathfrak{m}}$ by global class field theory; then we need to prove that $\psi_{\mathfrak{m}}|_{\mathbb{Z}_p^\times} = \beta_1 \cdots \beta_n$. The centre of $G(\mathbb{A}_{\mathbb{Q}})$ is $\mathbb{A}_F^\times$, so by the definition of the Shimura variety X there is a natural action of $\mathbb{A}_F^\times / F^\times$ on $H^i_{\text{ét}}(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$. By the definition of the Hecke operators, we see that if ϖ_w is a uniformiser at a place $w \nmid N$ of F such that $w|_{\mathbb{Q}}$ splits in F, then ϖ_w acts as $T_w^{(n)}$. By the Chebotarev density theorem, we deduce that the action of $\mathbb{A}_F^\times / F^\times$ on the underlying vector space of θ (which by definition is a subspace of $H^i_{\text{ét}}(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$) is via the character $\psi_{\mathfrak{m}}$. In order to compute $\psi_{\mathfrak{m}}|_{\mathbb{Z}_p^\times}$, it is thus sufficient to compute the action of $\mathbb{Z}_p^\times$ on $H^i_{\text{ét}}(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$, and in particular sufficient to compute the action of the Iwahori subgroup I_0. Now, since θ is assumed to be in the β-part of the cohomology, I_0 acts via the character β of I_0/I_1^*, so that $\psi_{\mathfrak{m}}|_{\mathbb{Z}_p^\times} = \beta_1 \cdots \beta_n$, as required. □

Corollary 3.4.5. *Suppose that $\rho_{\mathfrak{m}}$ satisfies Hypothesis 3.4.1, that $\rho_{\mathfrak{m}|G_{\mathbb{Q}_p}}$ is r-regular for some $r \le \min\{(n-1)/2, p-2\}$, and, if $r = (n-1)/2$, suppose in addition that $\rho_{\mathfrak{m}|G_{\mathbb{Q}_p}}$ contains an irreducible subquotient of dimension greater than 1. Then the localisation $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$ vanishes for $i \le r$, while $H^{r+1}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$ is torsion-free.*

Proof. This follows at once from Theorem 3.4.2 and the short exact sequence

$$0 \longrightarrow H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}/\mathfrak{m}_{\overline{\mathbb{Z}}_p} H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \longrightarrow H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \longrightarrow H^{i+1}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}[\mathfrak{m}_{\overline{\mathbb{Z}}_p}] \longrightarrow 0. \qquad \square$$

Remark 3.4.6. As already remarked in the introduction, we expect some kind of mod p analogue of Arthur's conjectures to hold, and so in particular we expect that stronger results than Theorem 3.4.2 and Corollary 3.4.5 should hold. In particular, if $\mathfrak{m}$ is any maximal ideal in the Hecke algebra attached to an irreducible continuous representation $\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, then we expect that the localisations $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}$ and $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$ should vanish in degrees $i < n-1$.

On the other hand, it need not be the case that (for example) $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)$ vanishes in all degrees in which $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)$ vanishes. For example, in the case $n = 3$, for the unitary Shimura varieties that we consider here, namely those that are associated to division algebras, it is known that $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p) = 0$ [Rogawski 1990, Theorem 15.3.1]. (Under additional restrictions on the division algebra allowed, an analogous result is known for all values of n [Clozel 1993, Theorem 3.4].) On the other hand, one can construct examples for which $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p) \neq 0$, and hence for which $H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)$ is not torsion-free, via congruence cohomology. (See e.g., the proof of [Suh 2008, Theorem 3.4].) The existence of such classes does not contradict our theorems or expectations, since congruence cohomology is necessarily Eisenstein (i.e., gives rise to Eisenstein systems of Hecke eigenvalues, in the sense that the associated Galois representation $\rho_{\mathfrak{m}}$ is completely reducible).

3.5. ***On the mod p cohomology of certain $U(2,1)$-Shimura varieties.*** Let $X := X(K)$ denote a $U(2,1)$-Shimura variety, with K of level dividing N for some natural number N divisible by p, and such that the projection of K to $\mathbb{G}(\mathbb{A}^{p,\infty})$ is sufficiently small. Let $\mathcal{F}_V$ be a canonical local system on X, which may be trivialised at level N. The results of Section 3.4 are particularly powerful in this case, as we now demonstrate.

Corollary 3.5.1. *Suppose that $\rho_{\mathfrak{m}}$ satisfies Hypothesis 3.4.1, that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}$ is* 1-*regular, and that $\rho_{\mathfrak{m}\,|G_{\mathbb{Q}_p}}$ contains an irreducible subquotient of dimension greater than* 1. *Then the localisations $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}$ vanish for $i \neq 2$.*

Proof. This follows immediately from Theorem 3.4.2, noting that the hypothesis that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}$ is 1-regular implies that $p > 4$ (indeed, that $p \geq 11$). $\square$

We now prove a result which does not require the existence of a Galois representation $\rho_{\mathfrak{m}}$. We begin with a lemma:

Lemma 3.5.2. *If $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}$ with residue field $\overline{\mathbb{F}}_p$, such that $H^0_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)_{\mathfrak{m}} \neq 0$, then there is an abelian representation $\bar{\rho}_0 : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ such that* $\mathrm{char}(\bar{\rho}_0(\mathrm{Frob}_w)) = \sum_{i=0}^{3} (-1)^i (\mathrm{Norm}\, w)^{i(i-1)/2} T_w^{(i)} X^{n-i} \pmod{\mathfrak{m}}$ *for all split places w of E for which $w \nmid Np$.*

Proof. This is standard and follows for example from [Deligne 1979, Section 2.1]. □

Theorem 3.5.3. *If ρ is a 3-dimensional irreducible sub-G_F-representation of the étale cohomology group $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)$, then every irreducible subquotient of $\rho_{|G_{\mathbb{Q}_p}}$ is 1-dimensional, or else $\rho_{|G_{\mathbb{Q}_p}}$ is not 1-regular, or else $\rho(G_F)$ is not generated by its subset of regular elements.*

Remark 3.5.4. Recall that a square matrix is said to be regular if its minimal and characteristic polynomials coincide. In Section 4.2 we will show that $\rho(G_F)$ is generated by its subset of regular elements if either $\rho_{\mathfrak{m}}$ is induced from a character of G_K for some cubic Galois extension $K/\mathbb{Q}$, or if $\rho(G_F)$ contains a regular unipotent element.

Remark 3.5.5. In the proof of the theorem we use some of the results of Section 4.

Proof of Theorem 3.5.3. The argument follows similar lines to the proof of Theorem 3.4.2, although it is slightly more involved, since we are not giving ourselves the existence of the Galois representation $\rho_{\mathfrak{m}}$.[4] The key point will be that, in the Hochschild–Serre spectral sequences that appear, the only other cohomology to contribute besides H^1 will be H^0, and, for maximal ideals of $\mathbb{T}$ in the support of H^0, we do have associated Galois representations, by Lemma 3.5.2.

We first show that if $H^0_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_W)_{\mathfrak{m}} \neq 0$ for some canonical local system $\mathscr{F}_W$ on X and some maximal ideal $\mathfrak{m}$ of $\mathbb{T}$, then $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)[\mathfrak{m}]) = 0$. To see this, note that if $H^0_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_W)_{\mathfrak{m}} \neq 0$ and $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)[\mathfrak{m}]) \neq 0$, then Lemma 3.5.2 and Theorem 3.3.1 together imply that there exists an abelian representation $\bar{\rho}_0 : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ such that, for all $g \in G_F$, the characteristic polynomial of $\bar{\rho}_0(g)$ annihilates $\rho(g)$. By Lemma 4.1.3 this implies that ρ is abelian, which is impossible as ρ is irreducible.

Now $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)$ is the direct sum of its localisations at the various maximal ideals $\mathfrak{m}$ of $\mathbb{T}$, and hence, since $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X, \mathscr{F}_V)) \neq 0$ by hypothesis, we see that $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X, \mathscr{F}_V)_{\mathfrak{m}}) \neq 0$ for some maximal ideal $\mathfrak{m}$ of $\mathbb{T}$. Since this is a finite-length $\mathbb{T}_{\mathfrak{m}}$-module, we see that its $\mathbb{T}_{\mathfrak{m}}$-socle $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X, \mathscr{F}_V)[\mathfrak{m}])$ must also be nonzero, and hence, by the preceding paragraph, we conclude that $H^0_{\text{ét}}(X, \mathscr{F}_W)_{\mathfrak{m}} = 0$ for any canonical local system $\mathscr{F}_W$ on X.

[4]In fact, as noted in the introduction, recent work of Scholze [2013] implies that $\rho_{\mathfrak{m}}$ exists for any maximal ideal $\mathfrak{m}$ in the Hecke algebra. We have left our argument as originally written.

As in the proof of Theorem 3.4.2, choose a short exact sequence of canonical local systems $0 \to \mathscr{F}_V \to \pi_* \overline{\mathbb{F}}_p^n \to \mathscr{F}_W \to 0$, for some $\pi : X' \to X$. Passing to the long exact sequence

$$H^0_{\text{ét}}(X, \mathscr{F}_W)_{\mathfrak{m}} \longrightarrow H^1_{\text{ét}}(X, \mathscr{F}_V)_{\mathfrak{m}} \longrightarrow H^1_{\text{ét}}(X, \pi_* \overline{\mathbb{F}}_p)^n = H^1_{\text{ét}}(X', \overline{\mathbb{F}}_p)^n,$$

and using the result of the preceding paragraph, namely that $H^0_{\text{ét}}(X, \mathscr{F}_W)_{\mathfrak{m}} = 0$, we conclude that ρ embeds into $H^1_{\text{ét}}(X', \overline{\mathbb{F}}_p)$. Thus, replacing X by X', we reduce to the case when $\mathscr{F}_V$ is constant, which we assume from now on.

We now suppose that ρ is a subrepresentation of $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$. We will prove that ρ is then necessarily a subrepresentation of $H^1_{\text{ét}}(X(K^p I_1), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ for some sufficiently small open subgroup K^p of $G(\mathbb{A}^{\infty,p})$. The result will then follow from Lemmas 3.1.3 and 4.2.2 and Theorems 1.5.15, 3.3.1, and 2.2.4.

As in the proof of Theorem 3.4.2, we write $X = X(K)$, and choose a normal open subgroup $K' := K^p K_p$ of K, with $K_p \subset I_1$. The Hochschild–Serre spectral sequence associated to the cover $X(K') \to X(K)$ gives rise to an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(K/K', H^0(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}) \longrightarrow H^1_{\text{ét}}(X(K), \overline{\mathbb{F}}_p)_{\mathfrak{m}} \\ \longrightarrow H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}^{K/K'} \longrightarrow H^2(K/K', H^0(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}). \end{aligned}$$

The same argument as above, using Lemma 3.5.2 (applied now with $X(K')$ in place of X), Theorem 3.3.1, and Lemma 4.1.3 below, shows that

$$H^1(K/K', H^0(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}) = H^2(K/K', H^0(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}) = 0.$$

Thus in fact we have an isomorphism $H^1_{\text{ét}}(X(K), \overline{\mathbb{F}}_p) \xrightarrow{\sim} H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}^{K/K'}$, and hence an isomorphism

$$\begin{aligned} \operatorname{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K), \overline{\mathbb{F}}_p)) \xrightarrow{\sim} \operatorname{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}^{K/K'}) \\ = \operatorname{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}})^{K/K'}. \end{aligned}$$

In particular, if ρ appears as a subrepresentation of $H^1_{\text{ét}}(X(K), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$, then it appears as a subrepresentation of $H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$.

Now, considering the Hochschild–Serre spectral sequence for the cover $X(K') \to X(K^p I_1)$, and using the fact that if $\operatorname{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}) \neq 0$, then also $\operatorname{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}})^{I_1/K_p} \neq 0$ (since I_1/K_p is a p-group), we conclude that if ρ appears as a subrepresentation of $H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$, then it appears as a subrepresentation of $H^1_{\text{ét}}(X(K^p I_1), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$.

Arguing exactly as in the proof of Theorem 3.4.2, we then deduce that some twist of ρ appears in $H^1_{\text{ét}}(X(K^p I_1^*), \overline{\mathbb{F}}_p)$, and so, replacing ρ by this twist, it suffices to prove that if ρ is an irreducible 3-dimensional representation ρ of $H^1_{\text{ét}}(X(K^p I_1^*), \overline{\mathbb{F}}_p)$ that is generated by its regular elements, then either every irreducible subquotient of $\rho_{|G_{\mathbb{Q}_p}}$ is 1-dimensional, or else $\rho_{|G_{\mathbb{Q}_p}}$ is not 1-regular.

This follows from Lemma 3.1.3 and Theorems 1.5.15, 3.3.1, and 2.2.4 exactly as in the proof of Theorem 3.4.2, replacing the appeal to Hypothesis 3.4.1 with one to Lemma 4.2.2 below. $\square$

Our other main theorem concerns the weight part of the Serre-type conjecture of [Herzig 2009] for $U(2,1)$. It is proved by combining our techniques with those of [Emerton et al. 2013], where a similar theorem is proved for $U(3)$ (which is simpler, because one has vanishing of cohomology outside of degree 0). We begin by recalling some terminology from that work. We will call an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_3(\mathbb{F}_p)$ a *Serre weight*. Fix an irreducible representation $\bar{\rho} : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$. Let $X := X(K)$ be a $U(2,1)$-Shimura variety such that K is of level dividing N and has sufficiently small projection to $\mathbb{G}(\mathbb{A}^{p,\infty})$, where now we assume that $(N, p) = 1$. Assume furthermore that $\bar{\rho}$ is unramified at all places not dividing Np, and define a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ with residue field $\overline{\mathbb{F}}_p$ by demanding that, for each place $w \nmid Np$ of F such that $w|_{\mathbb{Q}}$ splits in F, the characteristic polynomial of $\bar{\rho}(\mathrm{Frob}_w)$ is equal to the reduction modulo $\mathfrak{m}$ of $\sum_{i=0}^{3}(-1)^i (\mathrm{Norm}\, w)^{i(i-1)/2} T_w^{(i)} X^{n-i}$.

Let V be a Serre weight; since $(N, p)=1$, we may write $K = K_p K^p$, where $K_p \subset G(\mathbb{Q}_p) \cong \mathrm{GL}_3(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$ is conjugate to $\mathrm{GL}_3(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$, and we may regard V as a representation of K_p via the projection $\mathrm{GL}_3(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GL}_3(\mathbb{F}_p)$. As usual, write $\mathcal{F}_V$ for the canonical local system associated to V. We say that $\bar{\rho}$ is *modular of weight* V if for some N, X as above and for some $0 \le i \le 4$ we have

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}} \neq 0.$$

Assume now that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible. Definition 6.2.2 of [Emerton et al. 2013] defines what it means for a Serre weight to be (strongly) generic, and Section 5.1 there (using the recipe of [Herzig 2009]) defines a set $W^?(\bar{\rho})$ of Serre weights in which it is predicted that $\bar{\rho}$ is modular. Let $W_{\text{gen}}(\bar{\rho})$ be the set of generic weights for which $\bar{\rho}$ is modular.

Theorem 3.5.6. *Suppose that $\bar{\rho}$ satisfies Hypothesis 4.1.1 and $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible and* 1*-regular. Suppose that $\bar{\rho}$ is modular of some strongly generic weight. Then $W_{\text{gen}}(\bar{\rho}) = W^?(\bar{\rho})$. In fact, for each $V \in W_{\text{gen}}(\bar{\rho})$, we have*

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}} \neq 0$$

if and only if $i = 2$ and $V \in W^?(\bar{\rho})$.

Proof. By the definition of $\mathfrak{m}$, the representation $\bar{\rho}$ satisfies the defining properties of the representation $\rho_{\mathfrak{m}}$ considered in Section 3.4. Applying Corollary 3.5.1, we see that for any Serre weight V we have $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}} = 0$ if $i \neq 2$. We will now deduce the result from Theorem 6.2.3 of [Emerton et al. 2013] (taking $\bar{r}$ there to be our $\bar{\rho}$). By Theorem 4.3.3 of that reference, we see that it suffices to show that we can define S and $\widetilde{S}$ as in Section 4 there, so that Axioms Ã1–Ã3 of Section 4.3 of that work are satisfied. Following [Emerton et al. 2013], we define S and $\widetilde{S}$ using completed cohomology in the sense of [Emerton 2006] (in [Emerton et al. 2013] the use of completed cohomology was somewhat disguised, but the constructions with algebraic modular forms there are equivalent to the use of completed cohomology of $U(3)$ in degree 0). In fact, given our vanishing results the verification of the axioms of [Emerton et al. 2013] is very similar to that carried out in that work for $U(3)$, and we content ourselves with sketching the arguments.

From now on we regard the prime-to-p level structure K^p of X as fixed, and we will vary K_p in our arguments. We will write $K_p(0)$ for $\mathrm{GL}_3(\mathbb{Z}_p)\times\mathbb{Z}_p^\times \subset G(\mathbb{Q}_p)$. We fix a sufficiently large extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$, residue field k_E, and uniformiser ϖ_E, and we define

$$S := \varinjlim_{K_p} H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}},$$

$$\widetilde{S} := \Bigl(\Bigl(\varprojlim_{s}\varinjlim_{K_p} H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s)_{\mathfrak{m}}\Bigr)\otimes_{\mathcal{O}_E}\overline{\mathbb{Z}}_p\Bigr)^{\text{l.alg}}$$

(that is, the locally algebraic vectors in the localisation at $\mathfrak{m}$ of the completed cohomology of degree 2). Using the Hochschild–Serre spectral sequence and the vanishing of $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}} = 0$ if $i \neq 2$, it is straightforward to verify Axioms Ã1–Ã3 of Section 4.3 of [Emerton et al. 2013], as we now explain.

First, we need to check that our definition of "modular" is consistent with that of Definition 4.2.2 of [Emerton et al. 2013]. This amounts to showing that for any Serre weight V

$$(S\otimes\overline{\mathbb{F}}_pV)^{K_p(0)} = H^2_{\text{ét}}(X(K_p(0)K^p)_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}.$$

To see this, note that since any sufficiently small K_p acts trivially on V, we have

$$S\otimes\overline{\mathbb{F}}_pV = \varinjlim_{K_p} H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}\otimes V \xrightarrow{\sim} \varinjlim_{K_p} H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}},$$

so it is enough to check that for all compact open subgroups $K_p \subset K_p(0)$ we have

$$H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}^{K_p(0)} = H^2_{\text{ét}}(X(K^pK_p(0))_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}},$$

which is an easy consequence of the Hochschild–Serre spectral sequence and our vanishing result. We also need an embedding $S\hookrightarrow \widetilde{S}\otimes\overline{\mathbb{Z}}_p\overline{\mathbb{F}}_p$ which is compatible with the actions of $\mathrm{GL}_3(\mathbb{Q}_p)$ and the Hecke algebra. In fact, it is easy to see that we have $\widetilde{S}\otimes\overline{\mathbb{Z}}_p\overline{\mathbb{F}}_p = S$. For example, there is a natural isomorphism

$$H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E)_{\mathfrak{m}}/\varpi_E H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E)_{\mathfrak{m}} = H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, k_E)_{\mathfrak{m}},$$

and hence the vanishing of $H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, k_E)_{\mathfrak{m}}$ implies that of the finitely generated $\mathcal{O}_E$-module $H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E)_{\mathfrak{m}}$. One then sees that for all s we have a natural isomorphism

$$H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E)_{\mathfrak{m}}/\varpi_E^s H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E)_{\mathfrak{m}} = H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s)_{\mathfrak{m}},$$

from which the claim follows easily.

We now examine Axiom Ã1. We must show that if $\widetilde{V}$ is a finite free $\overline{\mathbb{Z}}_p$-module with a locally algebraic action of $K_p(0)$ (acting through $\mathrm{GL}_3(\mathbb{Z}_p)$), then $(\widetilde{S} \otimes \overline{\mathbb{Z}}_p \widetilde{V})^{K_p(0)}$ is a finite free $\overline{\mathbb{Z}}_p$-module, and for $A = \overline{\mathbb{Q}}_p, \overline{\mathbb{F}}_p$ we have

$$(\widetilde{S} \otimes \overline{\mathbb{Z}}_p \widetilde{V})^{K_p(0)} \otimes \overline{\mathbb{Z}}_p A = (\widetilde{S} \otimes \overline{\mathbb{Z}}_p \widetilde{V} \otimes \overline{\mathbb{Z}}_p A)^{K_p(0)}.$$

This is straightforward, the key point being that if $\mathcal{F}_{\widetilde{V}}$ denotes the lisse étale sheaf attached to $\widetilde{V}$, then a straightforward argument with Hochschild–Serre as above gives

$$(\widetilde{S} \otimes \overline{\mathbb{Z}}_p \widetilde{V})^{K_p(0)} = H^2_{\text{ét}}(X(K^p K_p(0))_{\overline{\mathbb{Q}}}, \mathcal{F}_{\widetilde{V}})_{\mathfrak{m}},$$

which is certainly a finite free $\overline{\mathbb{Z}}_p$-module (it is torsion-free by the proof of Corollary 3.4.5).

The verification of Axioms Ã2 and Ã3 is now exactly the same as in Proposition 7.4.4 of [Emerton et al. 2013], as the Galois representations occurring in the localised cohomology module $H^2_{\text{ét}}(X(K^p K_p(0))_{\overline{\mathbb{Q}}}, \mathcal{F}_{\widetilde{V}})_{\mathfrak{m}}$ are associated to automorphic forms exactly as in that work.[5] (In fact, at least for Axiom Ã2 this is a rather roundabout way of proceeding, as the Galois representations in question are constructed in [Harris and Taylor 2001] by using $H^2_{\text{ét}}(X(K^p K_p(0))_{\overline{\mathbb{Q}}}, \mathcal{F}_{\widetilde{V}})$, and one can read off the required properties directly from the comparison theorems of p-adic Hodge theory. For Axiom Ã3 we are not aware of any comparison theorems in sufficient generality, so it is necessary at present to take a lengthier route through the theory of automorphic forms.) □

4. Group theory lemmas

The theorems of Section 3 contain certain hypotheses on the Galois representations involved. Our goal in this section is to establish some group-theoretic lemmas which give sufficient criteria for these hypotheses to be satisfied. Throughout the section G is a finite group and k is an algebraically closed field of characteristic p. For any square matrix A with entries in k, we write $\mathrm{char}(A)$ to denote the characteristic polynomial of A.

[5] In the interests of full disclosure, we are not aware of a reference in the literature giving the precise base change result from $U(2,1)$ to GL_3 that we need, but it seems to be well-known to the experts, and will follow from the much more general work in progress of Mok and Kaletha, Minguez, Shin and White.

4.1. ***Characterising representations by their characteristic polynomials.*** Let $\rho : G \to \mathrm{GL}_n(k)$ be an irreducible representation. In this subsection we establish some criteria for ρ to satisfy the following hypothesis:

Hypothesis 4.1.1. If $\theta : G \to \mathrm{GL}_m(k)$ is irreducible, and if $\mathrm{char}(\rho(g))$ annihilates $\theta(g)$ for every $g \in G$, then θ is equivalent to ρ.

Remark 4.1.2. Any irreducible ρ of dimension 2 satisfies Hypothesis 4.1.1, as was proved by Mazur [1977, proof of Proposition 14.2]. However, it is not satisfied in general if the dimension n of ρ is greater than 2 (for instance, this already fails if ρ is the irreducible 3-dimensional representation of A_4; see Section 5 of [Boston et al. 1991] as well as Remark 4.1.7 below).

Lemma 4.1.3. *Let $\rho : G \to \mathrm{GL}_n(k)$ and $\theta : G \to GL_m(k)$ be two representations. If θ is irreducible, and if* $\mathrm{char}(\rho(g))$ *annihilates $\theta(g)$ for every $g \in G$, then the kernel of θ contains the kernel of ρ.*

Proof. If $\rho(g)$ is trivial, then the assumption implies that every eigenvalue of $\theta(g)$ is equal to 1, and hence that $\theta(g)$ is unipotent, and so of order a power of p. Thus the image of $\ker(\rho)$ under θ is a normal subgroup H of $\theta(G)$ of p-power order, and we see that the space of invariants $(k^m)^H$ is a nontrivial subspace of k^m. Since H is normal in $\theta(G)$, we see that $\theta(G)$ leaves $(k^m)^H$ invariant, and hence, since θ is assumed to be irreducible, we see that in fact $(k^m)^H = k^m$. Thus H is trivial, which is to say that $\ker(\rho) \subset \ker(\theta)$, as claimed. □

Lemma 4.1.4. *If $\rho : G \to \mathrm{GL}_n(k)$ is a direct sum of 1-dimensional characters of G, and if $\theta(g) : G \to \mathrm{GL}_m(k)$ is an irreducible representation of G such that* $\mathrm{char}(\rho(g))$ *annihilates $\theta(g)$ for every $g \in G$, then $m = 1$, so that θ is a character, and every element of G lies in the kernel of at least one of the summands of $\rho \otimes \theta^{-1}$.*

Proof. Since ρ is a direct sum of characters, it factors through G^{ab}. Lemma 4.1.3 then shows that θ also factors through G^{ab}. Since θ is also assumed to be irreducible, we find that θ must be a character. Twisting by ρ by θ^{-1}, we may in fact assume that θ is trivial, and, writing $\rho = \chi_1 \oplus \cdots \oplus \chi_n$, we find that for each $g \in G$, the value $\chi_i(g)$ is equal to 1 for at least one value of i (since $\mathrm{char}(\rho(g)) = (X - \chi_1(g)) \cdots (X - \chi_n(g))$ annihilates $\theta(g) = 1$). Thus G is equal to the union of its subgroups $\ker(\chi_i)$. □

Remark 4.1.5. In the context of the preceding proposition, we can't conclude in general that θ coincides with one the summands of ρ. For example, if G denotes the Klein four-group, if p is odd, and if ρ denotes the 3-dimensional representation obtained by taking the direct sum of the three nontrivial characters of G, then, taking θ to be the trivial representation, the hypotheses of the proposition are satisfied, but θ is certainly not one of the summands of ρ.

Lemma 4.1.6. *If $\rho : G \to \mathrm{GL}_n(k)$ is irreducible, and is isomorphic to an induction $\mathrm{Ind}_H^G \psi$, where H is a cyclic normal subgroup of G of index n and $\psi : H \to k^\times$ is a character, then ρ satisfies Hypothesis 4.1.1.*

Proof. The restriction $\rho|_H$ is isomorphic to the direct sum $\bigoplus_{g \in G/H} \psi^g$. If we let θ' be a Jordan–Hölder constituent of the restriction $\theta_{|H}$, then Lemma 4.1.4 (applied to the representations $\rho_{|H}$ and θ' of H) implies that θ' is a character of H and (because H is cyclic) that $\theta' = \psi^g$ for some $g \in G/H$. The H-equivariant inclusion $\psi^g = \theta' \to \theta_{|H}$ then induces a nonzero G-equivariant map $\rho = \mathrm{Ind}_H^G \psi = \mathrm{Ind}_H^G \psi^g \to \theta$, which must be an isomorphism, since both its source and target are irreducible by assumption. This proves the lemma. □

Remark 4.1.7. If we take $G = A_4$ and H to be the normal subgroup of G of order four (so that H is a Klein four-group), then the induction of any nontrivial character of H gives an irreducible representation $\rho : G \to \mathrm{SO}_3(k)$. For every $g \in G$, the characteristic polynomial of $\rho(g)$ thus has 1 as an eigenvalue, and so, if θ denotes the trivial character of G, the element $\theta(g)$ is annihilated by $\mathrm{char}(\rho(g))$ for every $g \in G$. Thus the analogue of Lemma 4.1.6 does not hold in general if H is not cyclic.

We thank Florian Herzig for providing the proof of the following lemma:

Lemma 4.1.8. *Suppose that G is a finite subgroup of $\mathrm{GL}_n(k)$ which contains $\mathrm{SL}_n(k')$ for some subfield k' of k and is contained in $k^\times \mathrm{GL}_n(k')$.*

(1) *Any irreducible representation of G over k remains irreducible upon restriction to $\mathrm{SL}_n(k')$.*

(2) *Given any two irreducible representations of G which become isomorphic upon restriction to $\mathrm{SL}_n(k')$, one can be obtained from the other via twisting by a character of G that is trivial on $\mathrm{SL}_n(k')$.*

Proof. Let G act via θ on the k-vector space V, and let (θ, W) be an irreducible subrepresentation of $\theta|_{\mathrm{SL}_n(k')}$. Then W is obtained by restriction from a representation of the algebraic group SL_n / k' (see Section 1 of [Jantzen 1987]), so the action of $\mathrm{SL}_n(k')$ on W may be extended to an action of $\mathrm{GL}_n(k)$ and thus of G. By Frobenius reciprocity we obtain a surjective map $(\mathrm{Ind}_{\mathrm{SL}_n(k')}^G 1) \otimes W \to V$ of G-representations. Since $G/\mathrm{SL}_n(k')$ is a finite abelian group of prime-to-p order, we see that $(\mathrm{Ind}_{\mathrm{SL}_n(k')}^G 1)$ is a direct sum of 1-dimensional representations, so that V is a twist of W by some character which is trivial on $\mathrm{SL}_n(k')$. Thus the restriction of θ to $\mathrm{SL}_n(k')$ is just W, which is irreducible, proving (1).

This same argument also serves to establish (2). □

Lemma 4.1.9. *Assume that $p \geq n$. If $\rho : G \to \mathrm{GL}_n(k)$ is irreducible, and if $\mathrm{SL}_n(k') \subseteq \rho(G) \subseteq k^\times \mathrm{GL}_n(k')$ for some subfield k' of k, then ρ satisfies Hypothesis 4.1.1.*

Proof. The case $n = 1$ follows from Lemma 4.1.4, and, as remarked above, the case $n = 2$ is proved in the course of the proof of Proposition 14.2 of [Mazur 1977], so we may assume that $n \geq 3$. By Lemma 4.1.3, we may assume that ρ is faithful, so that we can identify G with $\rho(G)$. In particular, $\mathrm{SL}_n(k')$ is a subgroup of G, and, by Lemma 4.1.8, the restriction of θ to $\mathrm{SL}_n(k')$ remains irreducible.

Since G is finite, our assumption that $\mathrm{SL}_n(k') \subset G$ implies that k' is finite; suppose that k' has cardinality q. We recall some basic facts about the representation theory of $\mathrm{SL}_n(k')$; see for example Section 1 of [Jantzen 1987]. The irreducible k-representations of $\mathrm{SL}_n(k')$ are obtained by restriction from the algebraic group SL_n/k', and are precisely those representations whose highest weights are q-restricted. (With the usual choice of maximal torus T of SL_n, if we identify the weight lattice with $\mathbb{Z}^n$ modulo the diagonally embedded copy of $\mathbb{Z}$, a weight $(a_1, \ldots, a_n)$ is q-restricted if $0 \leq a_i - a_{i+1} \leq q-1$ for all $1 \leq i \leq n-1$.) Suppose that θ has highest weight $(a_1, \ldots, a_n)$. Let $g \in \mathrm{SL}_n(k')$ be a semisimple element with eigenvalues $\alpha_1, \ldots, \alpha_n$. Then, since g is conjugate to an element of $T(k)$, by considering the formal character of the corresponding representation of SL_n/k', we see that among the eigenvalues of $\theta(g)$ are each of the quantities

$$\prod_{i=1}^{n} \alpha_i^{x_i},$$

where the x_i are a permutation of $a_1, \ldots, a_n$.

Our assumption on θ and ρ implies that, for each such permutation, $\prod_{i=1}^n \alpha_i^{x_i}$ must be one of $\alpha_1, \ldots, \alpha_n$. In particular, if we let α be a primitive $(q^n-1)/(q-1)$-st root of unity, we may consider a semisimple element g with eigenvalues

$$\alpha, \alpha^q, \ldots, \alpha^{q^{n-1}}.$$

Then, for any $x_1, \ldots, x_n$ as above, there must be an integer $0 \leq \beta \leq n-1$ such that

$$q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n \equiv q^{\beta} \pmod{(q^n-1)/(q-1)}.$$

Fix some $1 \leq i \leq n-1$, and consider two permutations $x_1, \ldots, x_n$ and $x_1', \ldots, x_n'$ as above which satisfy $x_i = x_i'$ for $1 \leq i \leq n-2$, $x_{n-1} = a_i$, $x_n = a_{i+1}$, $x_{n-1}' = a_{i+1}$ and $x_n' = a_i$. Taking the difference of the two expressions

$$q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n \quad \text{and} \quad q^{n-1}x_1' + q^{n-2}x_2' + \cdots + x_n',$$

we conclude that there are integers $0 \leq \beta, \gamma \leq n-1$ such that

$$(q-1)(a_i - a_{i+1}) \equiv q^{\beta} - q^{\gamma} \pmod{(q^n-1)/(q-1)}.$$

Since $n \geq 3$, we have $0 \leq (q-1)(a_i - a_{i+1}) \leq (q-1)^2 < (q^n-1)/(q-1)$, and we conclude that either $\beta \geq \gamma$ and $(q-1)(a_i - a_{i+1}) = q^\beta - q^\gamma$, or $\beta < \gamma$ and $(q-1)(a_i - a_{i+1}) = (q^n-1)/(q-1) + q^\beta - q^\gamma$. In the second case, we have

$$\begin{aligned}(q-1)^2 &\geq (q-1)(a_i - a_{i+1})\\ &= (q^n-1)/(q-1) + q^\beta - q^\gamma\\ &\geq (q^n-1)/(q-1) + 1 - q^{n-1}\\ &= q^{n-2} + \cdots + q + 2.\end{aligned}$$

This is a contradiction if $n \geq 4$. If $n = 3$, $(q^3-1)/(q-1) \equiv 3 \pmod{q-1}$, so that $(q-1) \mid 3$, which is a contradiction as $p > 2$.

Thus it must be the case that $\beta \geq \gamma$ and $(q-1)(a_i - a_{i+1}) = q^\beta - q^\gamma$, so that $a_i - a_{i+1}$ is congruent mod p to 0 or 1, and thus $a_i - a_{i+1} = 0$ or 1. In particular, for each i we have $0 \leq a_i - a_n \leq n-1 \leq p-1$.

We now repeat the above analysis. Fix some $1 \leq i \leq n-1$, and consider two permutations $x_1, \dots, x_n$ and $x'_1, \dots, x'_n$ as above which satisfy $x_i = x'_i$ for $1 \leq i \leq n-2$, $x_{n-1} = a_i$, $x_n = a_n$, $x'_{n-1} = a_n$ and $x'_n = a_i$. Taking the difference of the two expressions $q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n$ and $q^{n-1}x'_1 + q^{n-2}x'_2 + \cdots + x'_n$, and using that $0 \leq a_i - a_n \leq p-1 \leq q-1$, we conclude as before that each $a_i - a_n = 0$ or 1. Thus there must be an integer $1 \leq r \leq n$ with $a_1 = \cdots = a_r = a_n + 1$, $a_{r+1} = \cdots = a_n$.

Returning to the original congruences

$$q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n \equiv q^\beta \pmod{(q^n-1)/(q-1)},$$

we see that the left side is congruent to a sum of precisely r distinct values q^j, $1 \leq j \leq n-1$. Thus $r = 1$, and $\theta|_{\mathrm{SL}_n(k')}$ is the standard representation of $\mathrm{SL}_n(k')$, i.e., $\theta|_{\mathrm{SL}_n(k')} \cong \rho|_{\mathrm{SL}_n(k')}$. By part (2) of Lemma 4.1.8, we see that there is a character $\chi : G \to k^\times$ with $\chi|_{\mathrm{SL}_n(k')} = 1$ such that $\theta \cong \rho \otimes \chi$.

To complete the proof, we must show that χ is trivial. Take $g \in G$; we will show that $\chi(g) = 1$. If $g' \in G$ has $\det g' = \det g$, then $g(g')^{-1} \in \mathrm{SL}_n(k')$, so $\chi(g') = \chi(g)$. First, note that by assumption we can write $g = \lambda h$, with $\lambda \in k^\times$, $h \in \mathrm{GL}_n(k')$. Choose $h' \in \mathrm{GL}_n(k')$ to have eigenvalues $\{1, \dots, 1, \det(h)\}$. Then $h(h')^{-1} \in \mathrm{SL}_n(k') \subset G$, so $g' := \lambda h'$ is an element of G. Then the hypothesis on θ and ρ shows that the eigenvalues of $\chi(g')g' = \chi(g)g'$ are contained in the set of eigenvalues of g', so that if $\chi(g) \neq 1$ we must have $\chi(g) = \det(h) = -1$. Assume for the sake of contradiction that this is the case. If n is odd, then we now choose h' to have eigenvalues $\{-1, \dots, -1\}$, and we immediately obtain a contradiction from the same argument. If n is even then since $p \geq n$ we have $p \geq 5$ (recall that we are assuming $n \geq 3$), and we may choose $a \in (k')^\times$, $a \neq \pm 1$. Then choosing h' to have eigenvalues $\{1, \dots, 1, a, -1/a\}$ gives a contradiction. $\square$

4.2. ***Representations whose image is generated by regular elements.*** Recall that a square matrix with entries in k is said to be *regular* if its minimal polynomial and characteristic polynomial coincide.

Lemma 4.2.1. *If $\rho : G \to \mathrm{GL}_n(k)$ and $\theta : G \to \mathrm{GL}_n(k)$ are representations such that the image $\theta(G)$ is generated by its subset of regular elements, and for every $g \in G$ the characteristic polynomial of $\rho(g)$ annihilates $\theta(g)$, then* $\det \rho = \det \theta$.

Proof. Let $g \in G$ be an element such that $\theta(g)$ is regular. Then the characteristic polynomials of $\rho(g)$ and $\theta(g)$ must be equal (since the minimal and characteristic polynomials of $\theta(g)$ coincide and the characteristic polynomial of $\rho(g)$ annihilates $\theta(g)$), so $\det \rho(g) = \det \theta(g)$. Since $\theta(G)$ is generated by its subset of regular elements, the result follows. □

In fact, we actually need a slight generalisation of this result, where we simply have a collection of characteristic polynomials, rather than a representation ρ. Suppose that for each $g \in G$ we have a monic polynomial

$$\rho_g(X) = X^n - a_1(g)X^{n-1} + \cdots + (-1)^n a_n(g) \in k[X]$$

of degree n with the property that for all $g, h \in G$, we have $a_n(gh) = a_n(g)a_n(h)$.

Lemma 4.2.2. *Suppose that $\theta : G \to \mathrm{GL}_n(k)$ is a representation with the property that $\theta(G)$ is generated by its subset of regular elements, and that for each $g \in G$ we have $\rho_g(\theta(g)) = 0$. Then for each $g \in G$ we have* $\det \theta(g) = a_n(g)$.

Proof. This may be proved in exactly the same way as Lemma 4.2.1. □

Let $\rho : G \to \mathrm{GL}_3(k)$ be irreducible. Our goal is to give criteria for ρ to satisfy the following hypothesis, in order to apply the previous lemmas:

Hypothesis 4.2.3. The image $\rho(G)$ is generated by its subset of regular elements.

Lemma 4.2.4. *If $\rho : G \to \mathrm{GL}_3(k)$ is irreducible, and if either*

(1) *ρ is isomorphic to an induction $\mathrm{Ind}_H^G \psi$, where H is a normal subgroup of index 3 in G and $\psi : H \to k^\times$ is a character, or*

(2) *$\rho(G)$ contains a regular unipotent element,*

then ρ satisfies Hypothesis 4.2.3.

Proof. Suppose first that ρ is isomorphic to an induction $\mathrm{Ind}_H^G \psi$. Since H is a proper subgroup of G, the set of elements $G - H$ generates G. If $g \in G - H$ then the characteristic polynomial of $\rho(g)$ is of the form $X^3 - \alpha$. If $p \neq 3$ then this has distinct roots, so $\rho(g)$ is regular, and if $p = 3$ then it is easy to check that $\rho(g)$ is the product of a scalar matrix and a unipotent matrix, and is regular.

Suppose now that $\rho(G)$ contains a regular unipotent element. For ease of notation, we will refer to $\rho(G)$ as G from now on. Let H be the subgroup of G generated by the regular elements, and assume for the sake of contradiction that H is a proper subgroup of G. We claim that H contains every scalar matrix in G; this is true because the product of a scalar matrix and a regular matrix is again a regular matrix. Consider an element $g \in G - H$; since it is not regular, and not scalar, it acts as a scalar on some unique plane in k^3. We write ℓ_g for the corresponding line in $\mathbb{P}^2(k)$.

Let h be the given regular unipotent element in H. Then h stabilises a unique line in k^3, so a unique point $P \in \mathbb{P}^2(k)$. As $g \in G - H$, we also have $gh \in G - H$. Then $\ell_g \cap \ell_{gh}$ is nonempty, so there is a point $Q \in \mathbb{P}^2(k)$ which is fixed by g and gh. It is thus also fixed by h, so in fact $Q = P$. Since g was an arbitrary element of $G - H$, we see that every element of $G - H$ fixes P, and since G is generated by $G - H$, this implies that every element of G fixes P. This contradicts the assumption that ρ is irreducible. □

Appendix: Cohomology of pairs

A.1. *Étale cohomology of a pair.* Let X be a scheme, finite-type and separated over a field, let Z be a closed subscheme, and write $j : U \hookrightarrow X$ for the open immersion of the complement $U := X \setminus Z$ into X. As in Section 1, we let E be an algebraic extension of $\mathbb{Q}_p$, where p is invertible on X, let k_E denote the residue field of E, and we let A be one of E or k_E. We define the étale cohomology of the pair (X, Z) with coefficients in A to be the étale cohomology of the sheaf $j_!A$ on X, i.e., we write

$$H^\bullet_{\text{ét}}(X, Z, A) := H^\bullet_{\text{ét}}(X, j_!A).$$

If $i : Z \hookrightarrow X$ is the closed immersion of Z, then the short exact sequence

$$0 \longrightarrow j_!A \longrightarrow A \longrightarrow i_*A \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \longrightarrow H^m_{\text{ét}}(Z, A) \longrightarrow H^{m+1}_{\text{ét}}(X, Z, A) \longrightarrow H^{m+1}_{\text{ét}}(X, A) \longrightarrow H^{m+1}_{\text{ét}}(Z, A) \longrightarrow \cdots,$$

which is the long exact cohomology sequence of the pair (X, Z).

We are particularly interested in the case of a pair $(X \setminus Y, Z \setminus Y)$, where X is a smooth projective variety over a separably closed field, and Y and Z are smooth divisors on X which meet transversely. In this case we have a Cartesian diagram of open immersions

$$\begin{tikzcd} X \setminus (Y \cup Z) \arrow[r, "j"] \arrow[d, "k'"] & X \setminus Y \arrow[d, "k"] \\ X \setminus Z \arrow[r, "j'"] & X \end{tikzcd} \tag{A.1.1}$$

According to the above definition, the cohomology of the pair $(X \setminus Y, Z \setminus Y)$ is computed as the cohomology of the sheaf $j_! A$ on $X \setminus Y$, which is canonically isomorphic to the cohomology of the complex $Rk_* j_! A$ on X. An important point is that there is a canonical isomorphism

$$j'_! Rk'_* A \xrightarrow{\sim} Rk_* j_! A. \tag{A.1.2}$$

(See the discussion of §III (b) on p. 44 of [Faltings 1989].)

A.2. *Verdier duality.* Verdier duality [SGA 4_3 1973, Exposé XVIII; Verdier 1967] states that if $f : X \to S$ is a morphism of finite-type and separated schemes over a separably closed field k, then, for any constructible étale A-sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on S, there is a canonical isomorphism (in the derived category) of complexes of étale sheaves on S

$$RHom(Rf_! \mathcal{F}, \mathcal{G}) \cong Rf_* RHom(\mathcal{F}, f^! \mathcal{G}).$$

We recall some standard special cases of this isomorphism, in the context of the diagram (A.1.1).

Taking f to be k' (and recalling that k' is an open immersion), we obtain an isomorphism

$$RHom(k'_! A, A) \cong Rk'_* RHom(A, A) = Rk'_* A,$$

and hence, by double duality, an isomorphism

$$RHom(Rk'_* A, A) \cong k'_! A. \tag{A.2.1}$$

Next, taking f to be j', and taking into account (A.1.2) and (A.2.1), we obtain isomorphisms

$$RHom(Rk_* j_! A, A) \cong RHom(j'_! Rk'_* A, A) \cong Rj'_* RHom(Rk'_* A, A) \cong Rj'_* k'_! A.$$

Finally, taking f to the natural map $X \to \operatorname{Spec} k$, and recalling that in this case we have

$$f^! A = A[2d](d)$$

(where d is the dimension of X; see [SGA 4_3 1973, Exposé XVIII, Théorème 3.2.5]) and that $Rf_* = Rf_!$ (since f is proper, the variety X being projective by assumption), we find that

$$\begin{aligned} RHom(Rf_* Rk_* j_! A, A) &\cong Rf_* RHom(Rk_* j_! A, A[2d](d)) \\ &\cong Rf_* Rj'_* k'_! A[2d](d). \end{aligned}$$

Passing to cohomology, we find that $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, A)$ is in natural duality with $H^{2d-m}(X \setminus Z, Y \setminus Z, A)(d)$.

A.3. ***Vanishing outside of, and torsion-freeness in, the middle degree.*** We continue to assume that X is a smooth projective variety of dimension d over the separably closed field k, and that Y and Z are smooth divisors on X which meet transversely, but, in addition, we now assume that the complements $X \setminus Y$ and $X \setminus Z$ are affine (and hence also that $Z \setminus Y$ and $Y \setminus Z$ are affine). This latter assumption implies that $H^m_{\text{ét}}(X \setminus Y, A)$ vanishes if $m > d$ and that $H^m_{\text{ét}}(Z \setminus Y, A)$ vanishes if $m \geq d$ [SGA 4_3 1973, Exposé XIV, Corollaire 3.3]. By the long exact cohomology sequence of the pair $(X \setminus Y, Y \setminus Z)$, we see that $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, A)$ vanishes if $m > d$. Similarly, we see that $H^{2d-m}(X \setminus Z, Y \setminus Z, A)(d)$ vanishes if $m < d$. Hence, by the duality between $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, A)$ and $H^{2d-m}(X \setminus Z, Y \setminus Z, A)(d)$, we find that both vanish unless $m = d$.

Let $\mathbb{O}_E$ denote the ring of integers in E. Suppose momentarily that $E/\mathbb{Q}_p$ is finite, and let ϖ be a uniformiser of $\mathbb{O}_E$. From a consideration of the cohomology long exact sequence arising from the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{O}_E/\varpi^n \xrightarrow{\varpi\cdot} \mathbb{O}_E/\varpi^{n+1} \longrightarrow k_E \longrightarrow 0,$$

and arguing inductively on n, we find that $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, \mathbb{O}_E/\varpi^n)$ vanishes in degrees other than $m = d$ for all n. Passing to the projective limit over n, we see that the same is true of $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, \mathbb{O}_E)$. Finally, a consideration of the cohomology long exact sequence arising from the short exact sequence

$$0 \longrightarrow \mathbb{O}_E \xrightarrow{\varpi\cdot} \mathbb{O}_E \longrightarrow k_E \longrightarrow 0$$

shows that $H^d_{\text{ét}}(X \setminus Y, Z \setminus Y, \mathbb{O}_E)$ is torsion-free.

By passage to the direct limit over subfields of E which are finite over $\mathbb{Q}_p$, we see that these properties continue to hold for arbitrary algebraic extensions $E/\mathbb{Q}_p$.

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Homotopy exact sequences and orbifolds

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We generalize the homotopy exact sequences of étale fundamental groups for proper separable fibrations to the case where fibrations are not necessarily proper and separable. To treat the case where fibrations admit nonreduced geometric fibers, we introduce orbifolds within the framework of schemes and study their fundamental groups. As an application, we give a criterion for simple-connectedness of elliptic surfaces over an algebraically closed field by classifying simply connected orbifold curves.

1. Introduction

In algebraic geometry, the determination of fundamental groups of algebraic varieties is a classical problem. However, the problem is difficult, especially in the positive-characteristic case, where few results are known except for the one-dimensional case. In this paper, we develop a method to compute étale fundamental groups of fibered regular schemes, and apply the method to study elliptic surfaces, which provides insight into the computation of étale fundamental groups of fibered varieties.

We denote the étale fundamental group of a pointed connected locally Noetherian scheme $(X, \bar{x})$ by $\pi_1(X, \bar{x})$. In the introduction, we omit the geometric point $\bar{x}$ for simplicity. Let $f : X \to S$ be a proper separable morphism between connected

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locally Noetherian schemes with $\mathbb{O}_S = f_*\mathbb{O}_X$. The fibration $f : X \to S$ may be characterized by the following conditions:

(1) S is a connected locally Noetherian scheme.

(2) $f : X \to S$ is a faithfully flat proper morphism.

(3) The homomorphism $\mathbb{O}_S \to f_*\mathbb{O}_X$ associated to f is an isomorphism.

(4) Any geometric fiber of f is reduced.

Choose a geometric fiber $i : X_0 \to X$ of f. In this case, Grothendieck showed in [SGA 1 1971, X.1] that the morphisms i and f induce a homotopy exact sequence

$$\pi_1(X_0) \xrightarrow{i_*} \pi_1(X) \xrightarrow{f_*} \pi_1(S) \longrightarrow 1.$$

However, Condition (4) is too strong to compute fundamental groups of fibered varieties, e.g., elliptic surfaces, which may admit nonreduced geometric fibers.

Introducing orbifolds and their fundamental groups, we generalize the above homotopy exact sequence to the case where fibrations admit nonreduced geometric fibers. Instead of considering all general connected locally Noetherian schemes, we restrict ourselves to regular integral schemes. We consider a fibration $f : X \to S$ satisfying the following conditions (see Definition 4.17 for slightly weaker conditions):

(1′) X and S are regular integral schemes.

(2′) $f : X \to S$ is a faithfully flat morphism of finite type.

(3′) $\mathbb{O}_S$ is integrally closed in $f_*\mathbb{O}_X$.

(4′) The geometric generic fiber of f is reduced.

For example, all elliptic fibrations over curves satisfy these conditions (Section 6A). In order to give a similar exact sequence with the same homomorphism i_* in the general case (Section 4C), we have to replace $\pi_1(S)$ by its extension. To this end, we introduce orbifolds within the framework of schemes and study their fundamental groups (Section 3).

An orbifold (S, B) is defined as a locally Noetherian normal scheme S with data of ramifications B (Definition 3.6). Any orbifold curve over an algebraically closed field may be regarded as a DM stack (Theorem B.1). However, in the higher dimensional case, our orbifolds are different from DM stacks and more suitable than DM stacks for our studies (Remark B.2). We denote the fundamental group of a pointed orbifold $(S, B, \bar{s})$ by $\pi_1(S, B, \bar{s})$ (Definition 3.22). The local invariants associated to the nonreduced geometric fibers of f (Section 4B) endow S with an orbifold structure. Our main result is the following (Section 4C):

Theorem 1.1. *Let (X, S, f) be a triple satisfying Condition* (C^*) (Definition 4.17). *Take the orbifold (S, B) associated to f* (Definition 4.23). *Choose a connected reduced geometric fiber $i : X_0 \to X$ of f over a regular point* (*e.g.*, *the geometric generic fiber of f*). *Take a geometric point $\bar{x}_0$ on X_0. Put $\bar{x} := i(\bar{x}_0)$ and $\bar{s} := f(\bar{x})$. The morphisms i and f induce canonical homomorphisms $i_* : \pi_1(X_0, \bar{x}_0) \to \pi_1(X, \bar{x})$ and $f_*^{\mathrm{orb}} : \pi_1(X, \bar{x}) \to \pi_1(S, B, \bar{s})$, respectively* (Definition 4.25). *Then the sequence*

$$\pi_1(X_0, \bar{x}_0) \xrightarrow{i_*} \pi_1(X, \bar{x}) \xrightarrow{f_*^{\mathrm{orb}}} \pi_1(S, B, \bar{s}) \longrightarrow 1$$

is exact.

Next, we apply the above theorem to the case where $f : X \to S$ is the structure morphism of an elliptic surface over an algebraically closed field (Section 6). We determine the data of ramifications B of the orbifold (S, B) induced by f (Section 6B), and determine which orbifold is induced by an elliptic surface (Section 6C). As a result, we obtain a criterion for simple-connectedness of elliptic surfaces (Section 6D):

Theorem 1.2. *Let k be an algebraically closed field of characteristic $p \geq 0$. Let C be a connected proper smooth k-curve. Let (X, C, f) be a relatively minimal elliptic fibration* (Definition 6.1). *For each closed point s on C, we set*

$$m_s := \begin{cases} m & \text{if } f^{-1}(s) \text{ is of type } {}_m\mathrm{I}_n \ (n \geq 0) \ (\text{the Kodaira symbol}), \\ 1 & \text{otherwise}. \end{cases}$$

By n_s we denote the maximum integer satisfying $p \nmid n_s$ and $n_s \mid m_s$ (if $p = 0$, then $n_s = m_s$). Then X is simply connected if and only if all of the following conditions are satisfied:

(1) $\chi(\mathcal{O}_X) > 0$.

(2) $C \cong \mathbb{P}_k^1$.

(3) $\#\{s \in C(k) \mid n_s > 1\} \leq 2$.

(4) $\gcd(n_s, n_t) = 1$ *for* $s \neq t$.

(5) *If $p > 0$ and $f^{-1}(s)$ is of type ${}_m\mathrm{I}_n$ $(n > 0)$, then $p \nmid m$.*

(6) *If $p > 0$ and $(f^{-1}(s))_{\mathrm{red}}$ is isomorphic to an ordinary elliptic curve*, *then the $\mathcal{O}_{C,s}$-module $(R^1 f_* \mathcal{O}_X)_s$ is torsion-free.*

Furthermore, *each of Conditions* (1)–(6) *is necessary.*

We make a remark on Conditions (5) and (6) in the above theorem, which appear only in the positive-characteristic case. Under certain technical assumptions, Katsura and Ueno [1985, §§6–7] observed that an elliptic surface admits a nontrivial étale covering if one of Conditions (5) and (6) is not satisfied. Localizing elliptic surfaces with respect to the base curve and using Galois cohomology groups, we study this phenomenon in a systematic way (Section 6B).

We may give plenty of examples of elliptic fibrations with multiple fibers by means of the algebraic analog [Lang 1986; Cossec and Dolgachev 1989, V, §4] (Section 6C) and the rigid analytic analog [Mitsui 2013] of Kodaira's logarithmic transformation [1964, §4] for complex analytic elliptic fibrations. As for topological fundamental groups in the complex analytic case, Moishezon [1977, Theorem 11, II, §2, p. 191] gave a similar criterion by means of deformations of elliptic fibrations in the category of differentiable manifolds. Although the determination of topological fundamental groups is an old problem, no references can be found for étale fundamental groups. Our proof is purely algebraic and applies in any characteristic.

Let us briefly review the studies on the fundamental group of an elliptic surface $f : X \to C$ in the complex analytic case. Take a smooth fiber $i : X_0 \to X$ of f. The morphism i induces a homomorphism $i_* : \pi_1(X_0) \to \pi_1(X)$. In order to study $\pi_1(X)$, Iitaka [1971, §4] determined $\operatorname{Coker} i_*$. He reduced the problem to the case where X does not admit any multiple fiber by using Kodaira's logarithmic transformation and van Kampen's theorem. In a similar way, Xiao [1991] studied the case of more general compact complex analytic fibered surfaces. In another point of view, the group $\operatorname{Coker} i_*$ is relatively easy to deal with because it may be interpreted as the fundamental group of the orbifold curve induced by the elliptic fibration f [Ue 1986, §1; Friedman and Morgan 1994, 1.3.6]. If $\chi(\mathbb{O}_X) = 0$, then the map f between the underlying topological spaces may be regarded as a higher dimensional analog of a Seifert fibration [Seifert 1933; Thornton 1967]. Thurston [1980, §13.4] studied circle bundles over two-dimensional orbifolds in the context of the geometry of three-manifolds, which clarified the structure of Seifert fibrations: a Seifert fibration may be regarded as a circle bundle over a two-dimensional orbifold. After these studies, Ue [1986, §1] showed that $\pi_1(X)$ is isomorphic to the fundamental group of the orbifold curve induced by f whenever $\chi(\mathbb{O}_X) > 0$. Using the orbifold curve, Friedman and Morgan [1994, 2.2.1 and 2.7.2] discussed the general case in a systematic way. In the present paper, we develop this idea of using orbifolds within the framework of schemes, and give homotopy exact sequences as explained above.

As for $\operatorname{Im} i_*$, no difference appears between the characteristic-zero case and the positive-characteristic case (Theorem 6.23). However, some differences appear between the algebraic case of characteristic zero and the complex analytic case (Remark 6.24). As for $\operatorname{Coker} i_*$, no difference appears between the algebraic case of characteristic zero and the complex analytic case. As mentioned above, the group $\operatorname{Coker} i_*$ is isomorphic to the fundamental group of the orbifold curve induced by f. Thus, it follows from the fact that any compact complex analytic curve is algebraic.

However, several differences appear between the characteristic-zero case and the positive-characteristic case, as explained below.

In the algebraic case of characteristic zero and in the complex analytic case, the local structure of the orbifold curve is determined by the multiplicities of the multiple fibers. On the other hand, in the positive-characteristic case, the local structure is more complicated. This is because the completion of the local ring of the base curve at any point admits lots of finite coverings even if we fix the degree of the covering. For example, the completion admits infinitely many Artin–Schreier coverings. In particular, the resolution of multiple fibers in the positive characteristic case is much more difficult [Katsura and Ueno 1985, §§6–7; 1986, §2; Liu et al. 2004, §8.6] than that in the algebraic case of characteristic zero and in the complex analytic case [Kodaira 1963, §6]. In order to determine the local structure of the orbifold, we develop the above resolution of multiple fibers and study the minimal regular models of torsors of elliptic curves (Section 6B). In conclusion, multiple fibers of additive type [Katsura and Ueno 1986] do not affect the local structure of the orbifold, and the local uniformizations of the orbifold are given by certain finite cyclic coverings (Proposition 6.11).

In order to show Theorem 1.2, we classify simply connected orbifold curves that are locally uniformized by finite cyclic coverings (Theorem 1.3(1)). More precisely, we prove a generalized Fenchel conjecture (Theorem 1.3(2)). The original conjecture states that any finitely generated Fuchsian group admits a torsion-free subgroup of finite index, which was proved by a purely group-theoretic approach in [Fox 1952] and [Chau 1983]. The conjecture is equivalent to the following: any compact complex analytic orbifold curve minus finitely many points may be trivialized by a finite branched covering except for some trivial cases. In other words, except for some trivial cases, there exists a finite branched covering of a given compact complex analytic curve minus finitely many points with prescribed ramifications. Using the geometry of orbifold curves, we generalize this result in any characteristic (Section 5):

Theorem 1.3. *Let* (C, B) *be a connected cyclic orbifold* k*-curve* (Definition 5.1). *Take the tame part* (C, B^t) *and the wild part* (C, B^w) *of* (C, B) (Definition 3.8). *Put* $M := \#\operatorname{Supp} B$ *and* $N := \#\operatorname{Supp} B^t$ (Definition 3.6). *For each* $s \in \operatorname{Supp} B^t$, *we put* $n_s := [B^t_s : K_s]$. *Then*:

(1) *The orbifold* (C, B) *is simply connected* (Definition 3.19) *if and only if one of the following conditions is satisfied*:

  (a) $C \cong \mathbb{A}^1_k$, $M = 0$, *and* $p = 0$.

  (b) $C \cong \mathbb{P}^1_k$, $B^t = B$, $M \leq 2$, *and* $\gcd(n_s, n_t) = 1$ *for* $s \neq t$.

(2) *There exists an orbifold trivialization of* (C, B) (Definition 3.10) *if and only if neither of the following conditions are satisfied*:

(a) $C \cong \mathbb{P}^1_k$, $M = N = 1$, *and* $e(C, B^w) > 0$ (Definition 5.14).
(b) $C \cong \mathbb{P}^1_k$, $B^t = B$, $M = 2$, *and* $n_s \neq n_t$ *where* $\operatorname{Supp} B = \{s, t\}$.

Let us explain our proof. The problem may be reduced to showing the existence or nonexistence of the following four types of coverings:

(1) a covering of the projective line with at most two tame branch points;

(2) a covering of the projective line with three tame branch points;

(3) a covering of a curve with one tame branch point;

(4) a covering of a curve with one wild branch point.

Case (1) is easy. In the other cases, difficulties arise when we construct a covering of a curve with prescribed ramifications. In Case (2), we use the techniques of degeneration of a covering of a curve over a mixed characteristic ring [Raynaud 1994, §6]. In Cases (3) and (4), we produce rational functions on étale coverings with prescribed zeros and poles in order to apply Kummer theory and Artin–Schreier–Witt theory.

Finally, the classification of simply connected orbifolds and the above studies on the homotopy exact sequences give the desired criterion for simple-connectedness of elliptic surfaces.

2. Notation and conventions

We denote the cardinality of a set A by $\#A$ and the degree of a finite field extension L/K by $[L : K]$. We denote the field of fractions of an integral domain R by $\operatorname{Frac} R$ and the strict Henselization of a local ring R by R^{sh}. For a ring R, an *R-curve* is a faithfully flat separated R-scheme of finite type and of pure relative dimension one. We denote the geometric genus of a proper curve C over a field by $g(C)$. The *multiplicity* of a nonzero Weil divisor D on a locally Noetherian normal scheme X is the maximum positive integer m such that there exists a Weil divisor D' on X satisfying $D = mD'$. A scheme Y over a scheme X is called an *étale covering space* of X if the structure morphism $Y \to X$ is finite, étale, and surjective. A scheme X is said to be *simply connected* if X is connected and does not admit any nontrivial connected étale covering space of X.

Let X be a connected locally Noetherian scheme. Take a geometric point $\bar{x} : \operatorname{Spec} \Omega \to X$ on X, where Ω is a separably closed field. The pair $(X, \bar{x})$ is called a *pointed connected locally Noetherian scheme*. We denote the étale fundamental group of $(X, \bar{x})$ by $\pi_1(X, \bar{x})$. We sometimes omit $\bar{x}$ and denote $\pi_1(X, \bar{x})$ by $\pi_1(X)$ for simplicity.

3. Orbifolds

Definition 3.1. A morphism f between schemes is said to be *separable* if f is flat and the fiber of f over any point is geometrically reduced [SGA 1 1971, X.1.1]. A morphism $f : X \to Y$ between schemes is said to be *generically separable* if f maps any point x of codimension zero to a point y of codimension zero and induces a separable morphism $\operatorname{Spec} \mathbb{O}_{X,x} \to \operatorname{Spec} \mathbb{O}_{Y,y}$. A morphism $f : X \to Y$ between schemes is called a *quasiseparable-covering* (*qsc*) *morphism* if f is a locally quasifinite generically separable morphism. We say that a morphism f between locally Noetherian schemes *preserves codimensions* if f maps any point to a point of the same codimension.

Remark 3.2. A separable-covering morphism between integral schemes is conventionally defined as a finite generically separable morphism. The notion of a qsc morphism is a generalization of this notion.

We frequently use the following:

Lemma 3.3. (1) *Any locally Noetherian normal scheme is the disjoint union of locally Noetherian integral schemes* [Matsumura 1989, Exercise 9.11].

(2) *The normalization of any locally Noetherian normal integral scheme X in any finite separable field extension of the function field of X is finite over X* [Matsumura 1989, §33, Lemma 1].

(3) *Any separated qsc morphism $X \to Y$ between connected locally Noetherian normal schemes decomposes into an open immersion $X \to Z$ and a finite qsc morphism $Z \to Y$ where Z is a connected locally Noetherian normal scheme* [EGA IV_3 1966, 8.12.11].

Lemma 3.4 [Matsumura 1989, 9.4 and 15.1]. *Let $\phi : A \to B$ be a homomorphism between Noetherian rings, and Q a prime ideal of B. Put $P := \phi^{-1}(Q)$. Then*:

(1) $\operatorname{ht} Q \leq \operatorname{ht} P + \dim B_Q/PB_Q$.

(2) *If ϕ is flat, then the equality in* (1) *holds.*

(3) *Any qsc morphism between locally Noetherian normal schemes preserves codimensions.*

Lemma 3.5 (Zariski–Nagata purity [SGA 1 1971, X.3.1]). *Let $f : X \to Y$ be a qsc morphism between locally Noetherian schemes. Assume that X is normal and Y is regular. Then any irreducible component of the non-étale locus of f is of codimension one.*

Definition 3.6. Let S be a locally Noetherian normal scheme. By $P(S)$ we denote the set of all points on S of codimension one. For each $s \in P(S)$ put $K_s := \operatorname{Frac} \mathbb{O}^{\mathrm{sh}}_{S,s}$. Take a separable closure $\overline{K_s}$ of K_s. Let B be a map that associates $s \in P(S)$ with a finite Galois extension B_s/K_s in $\overline{K_s}$ and satisfies the following condition: let $\operatorname{Supp} B := \{s \in P(S) \mid B_s \neq K_s\}$; then $\operatorname{Supp} B$ is locally finite. The pair (S, B) is called an *orbifold*. For a locally Noetherian normal scheme S, we denote the orbifold obtained by equipping S with the trivial orbifold structure by (S). Let P be a property of schemes (e.g., connected, quasicompact, regular). We say that an orbifold (S, B) *is* P if S has the property P. Let P be a property of finite field extensions (e.g., trivial, tame, wild, cyclic). We say that an orbifold (S, B) *is* P if B_s/K_s has the property P for any $s \in P(S)$.

Remark 3.7. In the above definition, we restrict ourselves to the case where the data of ramifications are given only in codimension one for our purpose of application to homotopy exact sequences (see Remark 4.26 for a generalization).

Definition 3.8. Let (S, B) be an orbifold. By B^t we denote the map that associates $s \in P(S)$ with the maximal tame field extension B^t_s/K_s in B_s. Then (S, B^t) is a cyclic orbifold. The orbifold (S, B^t) is called the *tame part* of (S, B). Assume that (S, B) is cyclic. By B^w we denote the map that associates $s \in P(S)$ with the minimum field extension B^w_s/K_s in B_s such that the equality $B^t_s B^w_s = B_s$ holds. Then (S, B^w) is a cyclic orbifold. The orbifold (S, B^t) is called the *wild part* of (S, B).

Lemma 3.9. *Let $u : S' \to S$ be a qsc morphism between locally Noetherian normal schemes. Take $s' \in P(S')$. Put $s := u(s')$. Then $s \in P(S)$. Put $K_s := \operatorname{Frac} \mathbb{O}^{\mathrm{sh}}_{S,s}$ and $K'_{s'} := \operatorname{Frac} \mathbb{O}^{\mathrm{sh}}_{S',s'}$. Then u induces a finite field extension $K'_{s'}/K_s$.*

Proof. We may assume that S and S' are affine. The first statement follows from Lemma 3.4(3). The last statement follows from Lemma 3.3(3). ☐

Definition 3.10. Let (S, B) and (S', B') be two orbifolds. We use the notation introduced in Lemma 3.9. Composing the field extensions $K'_{s'}/K_s$ and $B'_{s'}/K'_{s'}$, we obtain a field extension $B'_{s'}/K_s$. Assume that there exists a K_s-algebra homomorphism $\phi_{s'} : B_s \to B'_{s'}$ for all $s' \in P(S')$. Then u is called an *orbifold morphism* $(S', B') \to (S, B)$. If $\phi_{s'}$ is an isomorphism for all $s' \in P(S')$, then u is called an *orbifold étale morphism* $(S', B') \to (S, B)$. If u is finite, orbifold étale, and surjective, then (S', B') is called an *orbifold étale covering space* of (S, B). If (S', B') is a trivial orbifold and an orbifold étale covering space of (S, B), then u is called an *orbifold trivialization* of (S, B).

Remark 3.11. The homomorphisms $\phi_{s'}$ are *not* part of the data of an orbifold morphism. The image of $\phi_{s'}$ does not depend on the choice of $\phi_{s'}$ since B_s/K_s is Galois. The composite of any two orbifold (étale) morphisms is an orbifold (étale) morphism.

Definition 3.12. Let (S, B) be an orbifold. We use the notation introduced in Lemma 3.9. Assume that there exists a K_s-algebra homomorphism $\psi_{s'} : K'_{s'} \to B_s$

for all $s' \in P(S')$ (e.g., u is étale). Since $B_s/K'_{s'}$ is Galois, we may define an orbifold (S', B') by putting $B'_{s'} := B_s$ for all $s' \in P(S')$. Then u is an orbifold étale morphism $(S', B') \to (S, B)$. We say that u *induces* an orbifold étale morphism $(S', B') \to (S, B)$.

Lemma 3.13. *Let $f : X \to S$ and $g : Y \to S$ be two qsc morphisms between locally Noetherian normal schemes. Take the normalization Z of $X \times_S Y$ and the canonical projections $f' : Z \to Y$ and $g' : Z \to X$. Then f' and g' are qsc morphisms between locally Noetherian normal schemes.*

Proof. We may assume that X, Y, and S are affine. By (1) and (3) of Lemma 3.3, we have only to show the case where f and g are finite qsc morphisms between integral schemes. In this case, the lemma follows from Lemma 3.3(2). □

Lemma 3.14. *Take a separable closure $\overline{L}$ of a field L. Let M and N be two finite field extensions of L in $\overline{L}$. Put $P := M \cap N$ and $Q := MN$. By $\widetilde{Q}$ we denote the Galois closure of Q/L. Then the L-algebra $M \otimes_L N$ is L-isomorphic to a finite product of Q-subfields of $\widetilde{Q}$. If M/L is Galois then $M \otimes_L N \cong Q^{[P:L]}$ over L, where the right-hand side is the product of $[P : L]$ copies of Q. If M/L and N/L are Galois, then $\widetilde{Q} = Q$.*

Proof. This follows from the L-algebra isomorphism $M \otimes_L N \cong P \otimes_L Q$. □

By $\mathscr{C}$ (resp. $\mathscr{C}_{\text{ét}}$) we denote the category consisting of locally Noetherian normal schemes and qsc morphisms (resp. étale morphisms). By $\mathscr{C}_{\text{orb}}$ (resp. $\mathscr{C}_{\text{orb,ét}}$) we denote the category consisting of orbifolds and orbifold morphisms (resp. orbifold étale morphisms). We define a faithful functor $F_{\text{orb}} : \mathscr{C} \to \mathscr{C}_{\text{orb}}$ by $S \mapsto (S)$. In the same way, we define a faithful functor $F_{\text{orb,ét}} : \mathscr{C}_{\text{ét}} \to \mathscr{C}_{\text{orb,ét}}$. By $G : \mathscr{C}_{\text{ét}} \to \mathscr{C}$ and $G_{\text{orb}} : \mathscr{C}_{\text{orb,ét}} \to \mathscr{C}_{\text{orb}}$ we denote the canonical faithful functors. Then $F_{\text{orb}} \circ G$ is naturally isomorphic to $G_{\text{orb}} \circ F_{\text{orb,ét}}$.

Proposition 3.15. *The categories $\mathscr{C}$, $\mathscr{C}_{\text{ét}}$, $\mathscr{C}_{\text{orb}}$, and $\mathscr{C}_{\text{orb,ét}}$ admit any finite fiber product. In any case, the (underlying) scheme of any finite fiber product is isomorphic to the normalization of the fiber product of the (underlying) schemes, and any base change of any (orbifold) étale morphism is an (orbifold) étale morphism. The functors F_{orb}, $F_{\text{orb,ét}}$, G, and G_{orb} preserve any finite fiber product.*

Proof. Let us show the first statement. Lemma 3.13 shows the cases of $\mathscr{C}$ and $\mathscr{C}_{\text{ét}}$. Let (S_1, B_1) and (S_2, B_2) be two orbifolds over an orbifold (S_0, B_0). We define an orbifold (S_3, B_3) in the following way. Take the normalization S_3 of $S_1 \times_{S_0} S_2$ and the canonical projection $p_i : S_3 \to S_i$ for $i = 0$, 1, and 2. Take $s \in P(S_3)$. Put $K_i := \operatorname{Frac} \mathscr{O}^{\text{sh}}_{S_i, p_i(s)}$ and $K_3 := \operatorname{Frac} \mathscr{O}^{\text{sh}}_{S_3, s}$. The morphism p_i induces a field extension K_3/K_i. Put $L_i := B_{i, p_i(s)}$. Take a separable closure $\overline{K_3}$ of K_3. By L'_i/K_3 we denote the unique Galois extension in $\overline{K_3}$ that is K_3-isomorphic to the lifting of the Galois extension L_i/K_i via K_3/K_i. We define $B_{3,s}/K_3$ as the Galois extension $L_1' L_2'/K_3$. We apply the same procedure to all $s \in P(S_3)$. Then we obtain an orbifold (S_3, B_3) and an orbifold morphism $p_i : (S_3, B_3) \to (S_i, B_i)$ for $i = 0, 1,$ and 2. By construction, the orbifold (S_3, B_3) is the fiber product of (S_1, B_1) and (S_2, B_2) over (S_0, B_0) in $\mathscr{C}_{\mathrm{orb}}$. Thus, the category $\mathscr{C}_{\mathrm{orb}}$ admits any finite fiber product. Assume that (S_1, B_1) is orbifold étale over (S_0, B_0). By definition, we may regard L_0, L_1, and L_2 as field extensions of K_0 satisfying $L_1 = L_0 \subset L_2$. Since $K_3 \subset K_1 \otimes_{K_0} K_2 \subset L_1 \otimes_{K_0} L_2 \cong L_2^{[L_1:K_0]}$ (Lemma 3.14), we may regard K_3 as a K_2-subfield of L_2. Then $L_2 \cong B_{3,s}$ over K_2. Thus, the morphism $p_2 : (S_3, B_3) \to (S_2, B_2)$ is orbifold étale. In particular, the category $\mathscr{C}_{\mathrm{orb,ét}}$ admits any finite fiber product. The other statements follow from the construction. $\square$

Definition 3.16. Let (S, B_1) and (S, B_2) be two orbifolds. For $i = 1$ and 2, the identity on S is an orbifold morphism $(S, B_i) \to (S)$. We define the *composite orbifold* $(S, B_1 B_2)$ of (S, B_1) and (S, B_2) as the fiber product of (S, B_1) and (S, B_2) over (S) in $\mathscr{C}_{\mathrm{orb}}$.

Remark 3.17. By the proof of Proposition 3.15, the field $(B_1 B_2)_s$ is equal to the composite field of $B_{1,s}$ and $B_{2,s}$ for any $s \in P(S)$, where we regard the extensions $B_{1,s}$ and $B_{2,s}$ of K_s as subfields of a fixed separable closure $\overline{K_s}$.

Proposition 3.18. *For $i = 1$ and 2, let $u_i : (S_i) \to (S, B_i)$ be an orbifold trivialization. Take the normalization S_3 of $S_1 \times_S S_2$ and the canonical projection $p_i : S_3 \to S_i$ for $i = 1$ and 2, which is an orbifold morphism $(S_3) \to (S_i)$. Then the orbifold morphism $(S_3) \to (S, B_1 B_2)$ induced by $u_1 \circ p_1$ and $u_2 \circ p_2$ is an orbifold trivialization.*

Proof. The proposition follows from Lemma 3.14 and Remark 3.17. $\square$

Definition 3.19. An orbifold (S, B) is said to be *simply connected* if (S, B) is connected and does not admit any nontrivial connected orbifold étale covering space. Let (S, B) be a connected orbifold. Take a geometric point $\bar{s} : \operatorname{Spec} \Omega \to S$ on S, where Ω is a separably closed field. Assume that the image of $\bar{s}$ is a regular point on S and not contained in the closure of $\operatorname{Supp} B$ (e.g., the image is the generic point of S). The triple $(S, B, \bar{s})$ is called a *pointed connected orbifold*. Let $\mathscr{S} = (S, B, \bar{s} : \operatorname{Spec} \Omega \to S)$ and $\mathscr{S}' = (S', B', \bar{s}' : \operatorname{Spec} \Omega' \to S')$ be two pointed connected orbifolds. A *pointed orbifold* (*étale*) *morphism* $\mathscr{S}' \to \mathscr{S}$ is a pair of an orbifold (étale) morphism $u : (S', B') \to (S, B)$ and a morphism $\Phi : \operatorname{Spec} \Omega' \to \operatorname{Spec} \Omega$ between schemes such that the diagram

$$\begin{array}{ccc} \operatorname{Spec} \Omega' & \xrightarrow{\bar{s}'} & S' \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle u} \\ \operatorname{Spec} \Omega & \xrightarrow{\bar{s}} & S \end{array}$$

is commutative.

Remark 3.20. By Zariski–Nagata purity (Lemma 3.5), any orbifold étale morphism to (S, B) is étale over the image of $\bar{s}$.

Let $(S, B, \bar{s})$ be a pointed connected orbifold. By $\mathscr{C}_{(S,B)}$ we denote the category of finite orbifold étale (S, B)-orbifolds and orbifold étale (S, B)-morphisms. We define a functor $F_{\bar{s}}$ from $\mathscr{C}_{(S,B)}$ to the category of finite sets by sending an object $(S', B') \to (S, B)$ to the underlying set of $S' \times_S \bar{s}$. We refer to [SGA 1 1971, V.5] for the definition of a Galois category and a fiber functor (a fundamental functor).

Theorem 3.21. *The category $\mathscr{C}_{(S,B)}$ is a Galois category with the fiber functor $F_{\bar{s}}$.*

Proof. We show the theorem in the same way as in the case of the category of finite étale schemes over a pointed connected locally Noetherian scheme (see [SGA 1 1971, V.7]). We have only to verify that the pair $(\mathscr{C}_{(S,B)}, F_{\bar{s}})$ satisfies Axioms (G1)–(G6) in [loc. cit., V.4]:

(G1) The orbifold (S, B) is the final object in $\mathscr{C}_{(S,B)}$. The category $\mathscr{C}_{(S,B)}$ admits any finite fiber product (Proposition 3.15).

(G2) The empty set equipped with the trivial orbifold structure is the initial object in $\mathscr{C}_{(S,B)}$. Take an object $u : (S', B') \to (S, B)$ in $\mathscr{C}_{(S,B)}$. Assume that a finite group G acts on an orbifold (S', B') over (S, B) in $\mathscr{C}_{(S,B)}$. By S'' we denote the spectrum of the $\mathbb{O}_S$-algebra of G-invariant sections of $\mathbb{O}_{S'}$. Then S'' is the quotient of S' by G in the category of S-schemes [loc. cit., V.1.8]. By $v : S' \to S''$ we denote the morphism induced by the canonical inclusion homomorphism $\mathbb{O}_{S''} \to \mathbb{O}_{S'}$. The morphism u factors as $u = w \circ v$. Since u is finite, the morphism v is finite. Since S is locally Noetherian and u is finite, the morphism w is finite. Take a point s' on S' of codimension one. By $I(s') \subset G$ we denote the inertia group of s'. We use the notation $B'_{s'}/K'_{s'}$ introduced in Definition 3.10. Put $s'' := v(s')$, $B''_{s''} := B'_{s'}$, and $K''_{s''} := \operatorname{Frac} \mathbb{O}^{\mathrm{sh}}_{S'',s''}$. Then the image under the homomorphism $K''_{s''} \to K'_{s'}$ induced by v is equal to $(K'_{s'})^{I(s')}$. Furthermore, the extension $B''_{s''}/K''_{s''}$ is finite and Galois. By construction, the pair (S'', B'') is an orbifold that is a quotient of (S', B') by G in $\mathscr{C}_{(S,B)}$.

(G3) Lemma 3.3(1) implies that any morphism $u : (S', B') \to (S, B)$ in $\mathscr{C}_{(S,B)}$ factors as $w \circ v : (S', B') \to (S'', B'') \to (S, B)$ where v is a strict epimorphism, w is a monomorphism, and (S'', B'') is a direct summand of (S, B).

(G4) By definition, the functor $F_{\bar{s}}$ is left-exact.

(G5) By definition, the functor $F_{\bar{s}}$ preserves any finite direct sum and preserves the quotient by any action of any finite group. Take a strict epimorphism u in $\mathscr{C}_{(S,B)}$. We have to show that $F_{\bar{s}}(u)$ is surjective. By s we denote the image of $\bar{s}$ on S. By base change, we may replace S by $\operatorname{Spec} \mathbb{O}_{S,s}$. Then u is étale (Remark 3.20). Thus, the surjectivity follows from [loc. cit., V.3.5].

(G6) Take a morphism u in $\mathscr{C}_{(S,B)}$. Assume that $F_{\bar{s}}(u)$ is an isomorphism. We have to show that u is an isomorphism. By s we denote the image of $\bar{s}$ on S. By base change, we may replace S by $\operatorname{Spec} \mathbb{O}_{S,s}$. Then u is étale (Remark 3.20). Thus, it follows from [loc. cit., V.3.7].

Therefore, the pair $(\mathscr{C}_{(S,B)}, F_{\bar{s}})$ satisfies Axioms (G1)–(G6) in [loc. cit., V.4], which proves the theorem. □

Definition 3.22. Let $(S, B, \bar{s})$ be a pointed connected orbifold. The functor $F_{\bar{s}}$ is pro-representable by a profinite group (see [loc. cit., V.5]). This group is called the *fundamental group* of $(S, B, \bar{s})$ and denoted by $\pi_1(S, B, \bar{s})$. We sometimes omit $\bar{s}$ and denote $\pi_1(S, B, \bar{s})$ by $\pi_1(S, B)$ for simplicity.

Let $\mathcal{S} = (S, B, \bar{s})$ and $\mathcal{S}' = (S', B', \bar{s}')$ be two pointed connected orbifolds. Any pointed orbifold (étale) morphism $u : \mathcal{S}' \to \mathcal{S}$ induces an (injective) homomorphism $u_* : \pi_1(\mathcal{S}') \to \pi_1(\mathcal{S})$ (see [loc. cit., V.6]). Since any connected finite étale S-scheme induces a connected finite orbifold étale (S, B)-orbifold, we obtain a canonical surjective homomorphism $\phi_{\mathcal{S}} : \pi_1(\mathcal{S}) \to \pi_1(S, \bar{s})$. If the regular locus of S is open (e.g., S is excellent), then the singular locus of S is a closed subset of codimension at least two. Thus, Zariski–Nagata purity (Lemma 3.5) shows the following:

Proposition 3.23. *Let $\mathcal{S} = (S, B, \bar{s})$ be a pointed connected orbifold. By S_0 we denote the regular locus of S. Assume that S_0 is an open subset of S. By $u : \mathcal{S}_0 \to \mathcal{S}$ we denote the pointed orbifold étale morphism induced by the inclusion morphism $S_0 \to S$. Then the homomorphism $u_* : \pi_1(\mathcal{S}_0) \to \pi_1(\mathcal{S})$ induced by u is an isomorphism. If $\mathcal{S}$ is trivial and regular, then $\phi_{\mathcal{S}}$ is an isomorphism and, in particular, $\pi_1(\mathcal{S}) \cong \pi_1(S)$.*

Example 3.24. The homomorphism $\phi_{(S)} : \pi_1((S)) \to \pi_1(S)$ is not injective in general, where (S) is the trivial orbifold associated to S and we omit the geometric points. Let k be an algebraically closed field of characteristic zero, n an integer greater than one, and ζ a primitive n-th root of unity. Put $S' := \mathbb{A}^2_k$. Take the coordinate functions (x, y) of S'. We define an automorphism σ on S' by $(x, y) \mapsto (\zeta x, \zeta y)$. Take the quotient $u : S' \to S$ of σ. The scheme S is normal but not regular. The morphism u ramifies only at the origin o of S'. Since $\pi_1(S' \setminus \{o\}) \cong \pi_1(S') \cong 1$ by Zariski–Nagata purity (Lemma 3.5), we obtain the isomorphisms $\pi_1((S)) \cong \mathbb{Z}/n\mathbb{Z}$ and $\pi_1(S) \cong 1$. Thus, the homomorphism $\phi_{(S)}$ is not injective.

4. Homotopy exact sequences

4A. *Coverings of fibrations.*

Lemma 4.1. *Let $f : X \to S$ be a quasicompact morphism between locally Noetherian schemes. Assume that X is reduced. Then the following are equivalent*:

(1) *S is reduced and f is dominant.*

(2) *The homomorphism $\mathcal{O}_S \to f_*\mathcal{O}_X$ associated to f is injective.*

Assume that (1) *and* (2) *hold and that X is normal and integral. Then S is integral and the following statements are equivalent*:

(3) *S is normal and the function field of S is algebraically closed in that of X.*

(4) *$\mathcal{O}_S$ is integrally closed in $f_*\mathcal{O}_X$.*

If f is generically separable, *then* (3) *and* (4) *are equivalent to the following*:

(5) *S is normal and the generic fiber of f is geometrically integral.*

Proof. We may assume that S is affine. Put $R := \Gamma(S, \mathcal{O}_S)$ and $W := \Gamma(X, \mathcal{O}_X)$. Since W is reduced and the kernel of the homomorphism $R \to W$ associated to f is the defining ideal of the closure of $f(X)$, the first equivalence holds. Let us show the other statements. Since X is irreducible and f is dominant, the scheme S is irreducible, which implies that S is integral. By K and L we denote the function fields of S and X, respectively. By K' and R' we denote the algebraic closure of K in L and the integral closure of R in K', respectively. Then $R' = R$ if and only if R is normal and $K' = K$. Since X is normal and integral, the ring W is normal and integral [Liu 2002, 4.1.5], which implies that R' is the integral closure of R in W. Thus, the second equivalence holds. The last statement follows from [EGA IV_2 1965, 4.6.3]. □

Lemma 4.2. *Let $u : X \to Y$ and $v : Y \to Z$ be morphisms between integral schemes. Assume that $v \circ u$ is dominant and generically separable and that v is integral. Then u is dominant and generically separable and v is surjective and generically separable.*

Proof. Since $v \circ u$ is dominant and v is closed, the morphism v is surjective. Since v is integral and dominant, the preimage of the generic point of Z under v consists of the generic point of Y [Matsumura 1989, 9.3 (ii)]. Furthermore, since $v \circ u$ is dominant and generically separable, the morphism u is dominant and the morphism v is generically separable. Note the following: for any field extensions L/K, M/L, and N/L, the ring $M \otimes_L N$ may be regarded as a subring of $M \otimes_K N$ since $M \otimes_K N \cong M \otimes_L N \otimes_L (L \otimes_K L)$ and $L \cong L \otimes_K K \subset L \otimes_K L$. Thus, the morphism u is generically separable. □

Definition 4.3. *Condition* (*D*) on morphisms $f : X \to S$, $u' : X' \to X$, $u : S' \to S$, and $f' : X' \to S'$ between locally Noetherian schemes consists of the following conditions:

(1) f is quasicompact, surjective, and generically separable.

(2) u' is finite, surjective, and generically separable.

(3) u is integral.

(4) The homomorphism $\mathcal{O}_S \to f_*\mathcal{O}_X$ associated to f is injective and $\mathcal{O}_S$ is integrally closed in $f_*\mathcal{O}_X$.

(5) The homomorphism $\mathcal{O}_{S'} \to f'_*\mathcal{O}_{X'}$ associated to f' is injective and $\mathcal{O}_{S'}$ is integrally closed in $f'_*\mathcal{O}_{X'}$.

(6) The diagram

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "u"] & S \end{tikzcd}$$

is commutative.

Remark 4.4. Conditions (1)–(3) imply that f' is quasicompact. Conditions (1)–(6) imply that u is given by the integral closure of $\mathcal{O}_S$ in $(f \circ u')_*\mathcal{O}_{X'}$.

Proposition 4.5. *Let $f : X \to S$, $f' : X' \to S'$, $u : S' \to S$, and $u' : X' \to X$ be morphisms between locally Noetherian schemes satisfying Condition* (*D*). *Suppose that X is normal and that X' is connected and normal. Then*:

(1) *X, X', S, and S' are normal and integral.*

(2) *u is finite*, *surjective*, *and generically separable.*

(3) *f' is quasicompact*, *surjective*, *and generically separable.*

Proof. Since X' is connected and u' is surjective, the scheme X is connected. Lemma 3.3(1) shows that X and X' are integral. Thus, Lemma 4.1 shows that S and S' are integral and normal. Therefore, Statement (1) holds. Since $f \circ u'$ is surjective and generically separable and u is integral, Lemma 4.2 shows that f' is dominant and generically separable and u is surjective and generically separable. Thus, Lemma 3.3(2) shows that u is finite. Therefore, Statement (2) holds. Let us show that f' is surjective. We may assume that the finite covering X'/X is Galois after replacing X' and S' by finite coverings. The Galois group G of X'/X faithfully acts on the finite covering S'/S such that f' is G-equivariant. Since $f'(X')$ is stable under any element of G and the equalities $u(f'(X')) = f(u'(X')) = S$ hold, the morphism f' is surjective. Thus, Statement (3) holds. ☐

Proposition 4.6. *We use the same notation and assumption as in* Proposition 4.5. *Let $v : T \to S$ be one of the following morphisms between schemes*:

(a) *a smooth morphism*;

(b) *the localization at a point*;

(c) *the strict Henselization* (*or the Henselization*) *of the spectrum of a local ring*;

(d) *the completion of the spectrum of an excellent local ring at the closed point.*

By $f_T : X_T \to T$, $f'_T : X'_T \to S'_T$, $u_T : S'_T \to T$, and $u'_T : X'_T \to X_T$ we denote the base changes of the S-morphisms f, f', u, and u' via v, respectively. Take connected components Z and Z' of T and S'_T, respectively. Then:

(1) *Z, $f_T^{-1}(Z)$, Z', and $(f'_T)^{-1}(Z')$ are locally Noetherian, normal, and integral.*

(2) *f_T, f'_T, u_T, and u'_T satisfy Condition (D).*

(3) *If v is surjective, then u' is étale if and only if u'_T is étale.*

Proof. By U we denote any of T, X_T, S'_T, and X'_T. Then U is locally Noetherian. Since any fiber of v is geometrically regular, the scheme U is normal [Matsumura 1989, 23.9]. The schemes Z and Z' are integral (Lemma 3.3(1)). We may assume that $Z = T$ and $f_T^{-1}(Z) = X_T$. Since v is a dominant flat morphism between integral schemes, Lemma 4.1 implies that X_T is integral. In the same way, we may show that $(f'_T)^{-1}(Z')$ is integral. Thus, Statement (1) holds. Let us show Statement (2). We have only to show that $\mathcal{O}_T$ and $\mathcal{O}_{S'_T}$ are integrally closed in $f_{T*}\mathcal{O}_{X_T}$ and $f'_{T*}\mathcal{O}_{X'_T}$, respectively. Thus, Statement (2) follows from Lemma 4.1. Statement (3) follows from faithfully flat descent for étale morphisms. □

Lemma 4.7. *Let $u : X \to Y$ and $v : Y \to Z$ be morphisms between locally Noetherian schemes. Put $w := v \circ u$. Assume that Y and Z are normal. Suppose that u is dominant, v is affine, and w is finite, étale, and dominant. Then u and v are finite, étale, and surjective.*

Proof. Since Z is normal and w is étale, the scheme X is normal. We may assume that X, Y, and Z are integral (Lemma 3.3(1)). Then u and v are finite surjective morphisms between locally Noetherian normal integral schemes. We have only to show that u and v are étale over any point z on Z. Since w is finite, étale, and surjective, there exists an étale morphism $t : U \to Z$ such that $z \in t(U)$ and the restriction of the base change $X_U \to U$ of w via t to any connected component of X_U is an isomorphism [Bosch et al. 1990, 2.3.8]. Thus, by faithfully flat descent for étale morphisms, we may assume that w is an isomorphism. Then u and v induce isomorphisms between the function fields of X, Y, and Z. Since X, Y, and Z are normal and integral and u and v are finite, the morphisms u and v are isomorphisms, which implies that u and v are étale. □

Proposition 4.8. *Let $f : X \to S$ and $u : S' \to S$ be morphisms between locally Noetherian normal integral schemes. Assume that f is quasicompact, surjective, and generically separable, u is finite, surjective, and generically separable, and $\mathcal{O}_S$ is integrally closed in $f_*\mathcal{O}_X$. Then there exist a locally Noetherian normal integral scheme X' and morphisms $f' : X' \to S'$ and $u' : X' \to X$ satisfying the following*:

(a) *f, f', u, and u' satisfy Condition (D).*

(b) *For any normal integral scheme Y and any dominant morphisms $h : Y \to S'$ and $\xi : Y \to X$ satisfying $f \circ \xi = u' \circ h$, there exists a unique morphism $\xi' : Y \to X'$ such that $u' \circ \xi' = \xi$ and $f' \circ \xi' = h$.*

Furthermore, the following statements hold:

(1) *If ξ in* (b) *is finite and étale, then u' and ξ' are finite, étale, and surjective.*

(2) *If u is étale, then u' is étale.*

(3) *If f is separable and of finite type, then the converse of* (2) *holds.*

Proof. By K, K', and L we denote the function fields of S, S', and X, respectively. Put $L' := L \otimes_K K'$. Since K is algebraically closed in L (Lemma 4.1) and the extension K'/K is finite and separable, the ring L' is a field. Furthermore, the field K' is algebraically closed in L' and the extension L'/L is finite and separable. Take the normalization $u' : X' \to X$ of X in L'. Since u' is finite (Lemma 3.3(2)), the scheme X' is locally Noetherian, normal, and integral. Take the unique morphism $f' : X' \to S'$ such that $f \circ u' = u \circ f'$. Then $\mathcal{O}_{S'}$ is integrally closed in $f'_*\mathcal{O}_{X'}$ (Lemma 4.1). Thus, Condition (a) is satisfied. By construction, Condition (b) is satisfied. Statement (1) follows from Lemma 4.7. By $v : Z \to X$ and $w : Z \to S'$ we denote the base change of u via f and the base change of f via u, respectively. Then v is finite, and Z is an integral scheme with function field L'. Let us show Statement (2). Assume that u is étale. Then v is étale and Z is normal. Thus, the scheme Z is X-isomorphic to X', which implies that u' is étale. Let us show Statement (3). Assume that f is separable and of finite type and that u' is étale. Replacing X by the smooth locus of f, we may assume that f is smooth ([Bosch et al. 1990, 2.2.16] and Lemma 4.1). Then Z is normal. Thus, the scheme Z is X-isomorphic to X', which implies that v is étale. Therefore, Statement (3) follows from faithfully flat descent for étale morphisms. □

4B. ***Base spaces of local étale coverings.***

Definition 4.9. Let R be a strictly Henselian Noetherian normal local ring with field of fractions K. Take a separable closure $\overline{K}$ of K. Put $S := \operatorname{Spec} R$. Let $f : X \to S$ be a surjective generically separable morphism between connected Noetherian normal schemes. Assume that $\mathcal{O}_S$ is integrally closed in $f_*\mathcal{O}_X$. We define the *maximal base field* $\widetilde{K}$ (*of étale coverings of the total space*) of f in the following way. Let $\xi : Y \to X$ be a connected étale covering space. Then Y is normal. Take the normalization

$$Y \xrightarrow{h} S' \xrightarrow{u} S$$

of S in the composite $f \circ \xi : Y \to X \to S$. By K_ξ we denote the function field of S'. Then u induces a finite separable field extension K_ξ/K (Proposition 4.5).

We define $\widetilde{K}$ as the composite field of all K-embeddings of the finite separable extensions K_ξ/K in $\overline{K}$ for all connected étale covering spaces $\xi : Y \to X$ of X.

Remark 4.10. By definition, the field extension $\widetilde{K}/K$ is algebraic and Galois.

We use the notation introduced in Definition 4.9.

Proposition 4.11. *Let $g : X' \to X$ be a proper birational morphism between regular integral schemes. Then the maximal base field of $f \circ g$ in $\overline{K}$ is equal to $\widetilde{K}$.*

Proof. By Zariski–Nagata purity (Lemma 3.5; see also [SGA 1 1971, X.3.3]), the base change of finite étale X-schemes via g induces an equivalence of categories between the category of finite étale X-schemes and the category of finite étale X'-schemes, which proves the proposition. □

Lemma 4.12. *Let L/K be a finite field extension in $\overline{K}$. Then $L \subset \widetilde{K}$ if and only if there exists a connected étale covering space of X that induces the extension L/K.*

Proof. The "if" part follows from the definition of $\widetilde{K}$. Since any finite fiber product of étale covering spaces of X over X is an étale covering space of X, the "only if" part follows from Proposition 4.8(1). □

By k we denote the residue field of R. By $\{Z_i\}_{i \in I}$ we denote the set of all irreducible components of the special fiber X_k of f with the reduced structures. Take the integral closure k_i of k in $\Gamma(Z_i, \mathcal{O}_{Z_i})$.

Lemma 4.13. *The ring k_i is a field. If f is of finite type, then the field extension k_i/k is finite and purely inseparable for any $i \in I$.*

Proof. Since k_i is an integral extension of the field k, the integral domain k_i is a field. Assume that f is of finite type. Since Z_i is finite type over k, the function field K_i of Z_i is finitely generated over k. Since $k_i \subset K_i$ and k is separably closed, the last statement holds. □

Suppose that R is a discrete valuation ring. Then f is flat. The closed subscheme X_k is a divisor on X and, for any $i \in I$, the closed subscheme Z_i is a prime divisor on X (Lemma 3.4(2)). We may write $X_k = \sum_{i \in I} m_i Z_i$.

Lemma 4.14. *Suppose that R is a discrete valuation ring. Assume that f is of finite type. Put $n_i := [k_i : k]$, which is finite by* Lemma 4.13. *Let $\xi : Y \to X$ be a connected étale covering space. Take the normalization*

$$Y \xrightarrow{h} S' \xrightarrow{u} S$$

of S in the composite $f \circ \xi : Y \to X \to S$. Then the degree of u divides $\gcd(m_i)_{i \in I} \cdot \gcd(n_i)_{i \in I}$.

Proof. By m and n we denote the ramification index of S'/S and the degree of the residue field extension of S'/S, respectively. Then $m \mid m_i$ and $n \mid n_i$ for any $i \in I$ since ξ is étale and k is separably closed. Since R is a Henselian discrete valuation ring, the degree of u is equal to mn, which concludes the proof. $\square$

Proposition 4.15. *We use the notation introduced in* Definition 4.9. *By k we denote the residue field of R. Suppose that R is a discrete valuation ring. Assume that f is of finite type. Then*:

(1) *The field extension $\widetilde{K}/K$ is finite and Galois.*

(2) *If f is separable, then $\widetilde{K} = K$.*

(3) *If k is perfect, then $[\widetilde{K} : K]$ divides the multiplicity of the special fiber of f* (Section 2).

(4) *If f is proper and a finite field extension K'/K in $\overline{K}$ satisfies $X(K') \neq \varnothing$, then $\widetilde{K} \subset K'$.*

Proof. Statement (1) (resp. (2) and (3)) follows from Lemma 4.14 since $\widetilde{K}/K$ is Galois (Remark 4.10) (resp. $m_i = 1$ and $k_i = k$ for any $i \in I$, and $k_i = k$ for any $i \in I$ (Lemma 4.13)). Let us show Statement (4). By Proposition 4.8(1), we may assume that $\widetilde{K} \cap K' = K$. By S' and $\widetilde{S}$ we denote the normalizations of S in K' and $\widetilde{K}$, respectively. Take the scheme X' (resp. $\widetilde{X}$) and the morphism $u' : X' \to X$ (resp. $\tilde{u} : \widetilde{X} \to X$ and $\tilde{f} : \widetilde{X} \to \widetilde{S}$) given by Proposition 4.8. Then $\tilde{u} : \widetilde{X} \to X$ is an étale covering space. Since the base change of $\tilde{u}$ via u' induces $\widetilde{K}K'/K'$, we have only to show that $\widetilde{K} = K$ whenever $X(K) \neq \varnothing$. Assume that $X(K) \neq \varnothing$. Then f admits a section by the valuative criterion for properness. Since the pullback of any section of f via $\tilde{u}$ induces a section of $\tilde{f}$ and S is strictly Henselian, the degree of $\tilde{u}$ is equal to 1, which concludes the proof. $\square$

Example 4.16. Let us give an example of a morphism $f : X \to S$ of finite type with $[\widetilde{K} : K] = \infty$ when $\dim S > 1$. Assume that k is algebraically closed, the characteristic of k is not equal to 3, and $R = k[\![x, y, z]\!]/(x^3 + y^3 + z^3)$. By s we denote the closed point of S. Put $S_0 := S \setminus \{s\}$. Then $\pi_1((S)) \cong \pi_1(S_0)$ (Proposition 3.23). Take the blowing-up $f : X \to S$ of S at s. Then X is regular. Put $E := f^{-1}(s)$. The reduction E_{red} of E is k-isomorphic to an elliptic curve over k and the multiplicity of E is equal to 3. The morphism f is not flat at any point on E (Lemma 3.4(2)). The inclusion morphisms $S_0 \to X$, $E_{\mathrm{red}} \to E$, and $E \to X$ induce a surjective homomorphism $\pi_1(S_0) \to \pi_1(X)$, an injective homomorphism $\pi_1(E_{\mathrm{red}}) \to \pi_1(E)$, and an isomorphism $\pi_1(E) \cong \pi_1(X)$, respectively [SGA $4\frac{1}{2}$ 1977, IV.2.2]. Since $\pi_1(E_{\mathrm{red}})$ is not finite, the extension $\widetilde{K}/K$ is infinite.

4C. *Homotopy exact sequences.* In this subsection, we give homotopy exact sequences for fibrations satisfying the following conditions:

Definition 4.17. *Condition* (C) on a triple (X, S, f) consists of the following conditions:

(1) X and S are locally Noetherian normal integral schemes.

(2) $f : X \to S$ is a surjective morphism of finite type.

(3) $\mathbb{O}_S$ is integrally closed in $f_*\mathbb{O}_X$ (Conditions (1) and (2) imply that the homomorphism $\mathbb{O}_S \to f_*\mathbb{O}_X$ associated to f is injective (Lemma 4.1)).

(4) The geometric generic fiber of f is reduced.

Condition (C^*) on a triple (X, S, f) is Condition (C) and the following conditions:

(5) X is regular.

(6) f is flat in codimension one.

Remark 4.18. In the case where f is proper, Conditions (2) and (3) are equivalent to the following conditions:

(2′) $f : X \to S$ is proper.

(3′) The homomorphism $\mathbb{O}_S \to f_*\mathbb{O}_X$ associated to f is an isomorphism.

Remark 4.19. In our studies on homotopy exact sequences, Condition (C*) is necessary. This condition is used to describe the effect of the nonreduced geometric fibers of f on étale covering spaces of X in terms of an orbifold (S, B). Conditions (1)–(3) are used to produce a finite covering space of S by taking the normalization of S in the composite of a finite covering map of X and f. In the case where f is proper, this normalization may be given by the Stein factorization of the composite. Condition (5) is used to apply Zariski–Nagata purity (Lemma 3.5) to a finite covering space of X. In particular, the condition that the finite covering map is étale may be checked in codimension one. Condition (4) enables Condition (6) to encode this condition as the data of ramifications B of an orbifold (S, B). See Theorem 4.22 and Remark 4.26 for Condition (C).

Example 4.20. Take k, n, $u : S' \to S$, and σ as in Example 3.24. Let E be an elliptic curve over k. Put $X' := E \times_k S'$. By $f' : X' \to S'$ we denote the second projection. Choose a primitive n-torsion point P on E. We define an action τ on X' as the product of the translation by the addition of P on E and the action of σ on S'. We take the quotient $u' : X' \to X$ of τ. Since f' is equivariant with respect to τ and σ, we obtain a morphism $f : X \to S$. The triple (X, S, f) satisfies Condition (C). The morphisms f, f', u, and u' satisfy Condition (D) (Definition 4.3). Since X is regular and S is not regular, the morphism f is not flat [Matsumura 1989, 23.7 (i)]. However, the morphism f is flat in codimension one since f is flat over the regular locus of S [Matsumura 1989, 23.1]. In particular, the triple (X, S, f) satisfies Condition (C*).

Example 4.21. Take k, $u : S' \to S$, and o as in Example 3.24. Take the blowing-ups $f : X \to S$ and $f' : X' \to S'$ of S and S' at $u(o)$ and o, respectively. Then X and X' are regular. The universal property of blowing-up shows that there exists a unique morphism $u' : X' \to X$ such that $f \circ u' = u \circ f'$. The morphism u' ramifies along the exceptional divisor of f'. The morphisms f, f', u, and u' satisfy Condition (D) (Definition 4.3). The triple (X, S, f) satisfies Condition (C). However, the morphism f is not flat in codimension one (Lemma 3.4(2)). In particular, the triple (X, S, f) does not satisfy Condition (C*).

We first generalize Grothendieck's homotopy exact sequence to the case where fibrations are not necessarily proper:

Theorem 4.22. *Let (X, S, f) be a triple satisfying Condition* (*C*) (Definition 4.17). *Assume that f is separable. Choose a connected geometric fiber $i : X_0 \to X$ of f* (*e.g., the geometric generic fiber of f*). *Take a geometric point $\bar{x}_0$ on X_0. Put $\bar{x} := i(\bar{x}_0)$ and $\bar{s} := f(\bar{x})$. The morphisms i and f induce canonical homomorphisms $i_* : \pi_1(X_0, \bar{x}_0) \to \pi_1(X, \bar{x})$ and $f_* : \pi_1(X, \bar{x}) \to \pi_1(S, \bar{s})$, respectively. Then the sequence*

$$\pi_1(X_0, \bar{x}_0) \xrightarrow{i_*} \pi_1(X, \bar{x}) \xrightarrow{f_*} \pi_1(S, \bar{s}) \longrightarrow 1$$

is exact.

Proof. We have only to show the exactness at $\pi_1(X)$. Since X_0 is a geometric fiber of f, the relation $\operatorname{Im} i_* \subset \operatorname{Ker} f_*$ holds. Let us show that $\operatorname{Ker} f_* \subset \operatorname{Im} i_*$. Take $\eta \in \operatorname{Ker} f_*$. We have only to show the following: for any connected Galois étale covering space $\xi : Y \to X$, the element η acts trivially on $\pi_0(\xi^{-1}(X_0))$, where we denote the base change of X_0 via ξ by $\xi^{-1}(X_0)$. Take the normalization

$$Y \xrightarrow{h} S' \xrightarrow{u} S$$

of S in the composite $f \circ \xi : Y \to X \to S$. Then $u : S' \to S$ is an étale covering space (Proposition 4.5(2) and Proposition 4.8(3)). The action of η on Y/X induces an action of η on S'/S. Since $\eta \in \operatorname{Ker} f_*$, the element η acts trivially on S'/S. Thus, the element η acts trivially on $\pi_0(\xi^{-1}(X_0))$, which implies that $\eta \in \operatorname{Im} i_*$. Therefore, the sequence is exact. $\square$

Definition 4.23. Let (X, S, f) be a triple satisfying Condition (C) (Definition 4.17). We define the *orbifold* (S, B) *associated to* f in the following way. By $P(S)$ we denote the set of all points on S of codimension one. Take $s \in P(S)$. Put $K_s := \operatorname{Frac} \mathbb{O}^{\mathrm{sh}}_{S,s}$. By f_s we denote the base change of f via the composite $\operatorname{Spec} \mathbb{O}^{\mathrm{sh}}_{S,s} \to \operatorname{Spec} \mathbb{O}_{S,s} \to S$ of the canonical morphisms. Take the maximal base field $\widetilde{K_s}$ of f_s (Definition 4.9). Proposition 4.15(1) shows that the field extension $\widetilde{K_s}/K_s$ is finite and Galois. We define a map B on $P(S)$ by $s \mapsto \widetilde{K_s}/K_s$ (Definition 3.6). Let us show that the pair (S, B) is an orbifold. By S_0 we denote the open subscheme of S

that is the complement of the closure of Supp B. Take a nonempty open subscheme S_1 of S over which f is separable. Then $S_1 \subset S_0$ (Proposition 4.15(2)), which implies that Supp B is locally finite. Thus, the pair (S, B) is an orbifold.

Using the above orbifold, we give an étaleness criterion for finite coverings of X:

Theorem 4.24. *Let (X, S, f) be a triple satisfying Condition* (C^*) (Definition 4.17). *Take the orbifold (S, B) associated to f* (Definition 4.23). *Let $u : S' \to S$ be a finite surjective generically separable morphism between locally Noetherian normal integral schemes. Take the scheme X' and the morphism $u' : X' \to X$ given by* Proposition 4.8. *Then u' is étale if and only if u induces an orbifold étale morphism $(S', B') \to (S, B)$* (Definition 3.12).

Proof. First, we assume that $S = \operatorname{Spec} Q$ and $S' = \operatorname{Spec} Q'$ where Q and Q' are discrete valuation rings. The morphism u induces a finite flat extension Q'/Q of discrete valuation rings. Put $J := \operatorname{Frac} Q$, $J' := \operatorname{Frac} Q'$, $K := \operatorname{Frac} Q^{\mathrm{sh}}$, and $K' := J' \otimes_J K$. The field extension J'/J induced by u is finite and separable. Take the maximal unramified extension I of J in J'. We may embed I in K over J. By (1) and (2) of Proposition 4.8, we may assume that $J = I$. Then K' is a field. By Proposition 4.6, we may assume that $J = K$ and $J' = K'$. Then the theorem follows from Lemma 4.12.

Next, let us show the general case. The "only if" part follows from the first case and Proposition 4.6. Let us show the "if" part. Since f maps any point of codimension one to a point of codimension at most one (Lemma 3.4(2)), the first case and Proposition 4.6 show that u' is étale in codimension one. Thus, Zariski–Nagata purity (Lemma 3.5) shows that u' is étale, which proves the "if" part. □

Definition 4.25. Let (X, S, f) be a triple satisfying Condition (C^*) (Definition 4.17). Take the orbifold (S, B) associated to f (Definition 4.23). Choose a geometric point $\bar{x}$ on X. Put $\bar{s} := f(\bar{x})$. Assume that the image of $\bar{s}$ on S is a regular point on S (e.g., the generic point of S). We define the *homomorphism* $f_*^{\mathrm{orb}} : \pi_1(X, \bar{x}) \to \pi_1(S, B, \bar{s})$ *induced by* f in the following way. Let $u : (S', B') \to (S, B)$ be a connected orbifold étale covering space. Take the scheme X' and the morphism $u' : X' \to X$ given by Proposition 4.8. Then $u' : X' \to X$ is a connected étale covering space (Theorem 4.24). Thus, we obtain a surjective homomorphism $f_*^{\mathrm{orb}} : \pi_1(X, \bar{x}) \to \pi_1(S, B, \bar{s})$.

Proof of Theorem 1.1. We may show the theorem in the same way as in the proof of Theorem 4.22. We have only to use Theorem 4.24 instead of Proposition 4.8(3). □

Remark 4.26. We generalize the definition of an orbifold (S, B) (Definition 3.6) by the following two modifications: replace $P(S)$ by all points on S; remove the finiteness assumption on B_s/K_s. We may define the fundamental group of (S, B) in the same way as in the case of an orbifold. The morphism f induces a generalized orbifold (S, B). Note that if f is separable, then (S, B) is trivial (Proposition 4.8(3)). Replacing an orbifold by a generalized orbifold, we may show Theorems 4.24 and 1.1 for any triple (X, S, f) satisfying Condition (C) (Definition 4.17) without the regularity assumption on X and the flatness assumption on f in codimension one (Examples 4.16 and 4.21).

5. Orbifold trivializations of orbifold curves

In this section, we fix an algebraically closed field k of characteristic $p \geq 0$. We study orbifold trivializations of orbifold k-curves and classify simply connected cyclic orbifold k-curves.

Definition 5.1. An orbifold (C, B) (Definition 3.6) is called an *orbifold k-curve* (resp. a *proper orbifold k-curve*) if C is a k-curve (resp. a proper k-curve). If $p > 0$ and $[B_s : K_s]$ is power of p for any $s \in P(C)$, we say that an orbifold (C, B) is a *p-orbifold k-curve*.

Since the underlying scheme of any orbifold k-curve is a smooth k-curve, we study ramified coverings of smooth k-curves. The Riemann–Hurwitz formula shows the following:

Lemma 5.2. *Let $u : C \to \mathbb{P}^1_k$ be a finite tamely ramified k-morphism of degree d between connected proper smooth k-curves. By N we denote the number of the branched points of u. Then:*

(1) $N \neq 1$.

(2) *If $N = 2$, then C is isomorphic to $\mathbb{P}^1_k$, u ramifies at exactly two points, and both of the two ramification indices are equal to d.*

Proposition 5.3. *Let $(\mathbb{P}^1_k, B)$ be a proper tame orbifold k-curve. If* $\#\operatorname{Supp} B \leq 1$, *then $(\mathbb{P}^1_k, B)$ is simply connected. Assume that* $\operatorname{Supp} B = \{0, \infty\}$. *For $s = 0$ and ∞ we put $n_s := [B_s : K_s]$. Then:*

(1) *The orbifold $(\mathbb{P}^1_k, B)$ is simply connected if and only if* $\gcd(n_0, n_\infty) = 1$.

(2) *There exists an orbifold trivialization of $(\mathbb{P}^1_k, B)$ if and only if $n_0 = n_\infty$.*

In Statement (2), *the restriction of the orbifold trivialization to any connected component ramifies at exactly two points.*

Proof. The first statement follows from Lemma 5.2(1). Let us show the other statements. Put $d := \gcd(n_0, n_\infty)$. Take a parameter t of $\mathbb{P}^1_k$ so that $t(0) = 0$ and $t(\infty) = \infty$. Then the k-morphism $\mathbb{P}^1_k \to \mathbb{P}^1_k$, $t \mapsto t^d$ induces an orbifold étale morphism $(\mathbb{P}^1_k, B') \to (\mathbb{P}^1_k, B)$, where the equalities

$$[B'_s : K_s] = \begin{cases} n_i/d & \text{if } s = 0 \text{ or } \infty, \\ 1 & \text{otherwise.} \end{cases}$$

hold. Thus, Lemma 5.2 proves the proposition. □

Proposition 5.4. *Let* $(\mathbb{P}^1_k, B)$ *be a proper tame orbifold* k*-curve with* Supp $B = \{0, 1, \infty\}$. *For* $s = 0$, 1, *and* ∞, *we put* $n_s := [B_s : K_s]$. *Assume that* n_0, n_1, *and* n_∞ *are pairwise coprime. Then there exists a Galois orbifold trivialization of* $(\mathbb{P}^1_k, B)$ *with noncommutative simple Galois group.*

Proof. First, we consider the case $p = 0$. By $F_2 = \langle x_0, y_0 \rangle$ we denote the free group of rank two. Take the elements x, y, and z of the triangle group $\Delta(n_0, n_1, n_\infty)$ in the definition of $\Delta(n_0, n_1, n_\infty)$ in Section A. We define a homomorphism $\phi : F_2 \to \Delta(n_0, n_1, n_\infty)$ by $x_0 \mapsto x$ and $y_0 \mapsto y$. Then the equality $\phi(y_0^{-1} x_0^{-1}) = z$ holds. Since the étale fundamental group of $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$ is isomorphic to the profinite completion of F_2 (see [Lieblich and Olsson 2010] for a purely algebraic proof), Theorem A.9 shows the proposition.

Next, we consider the case $p > 0$. Take a complete discrete valuation ring R of characteristic zero whose residue field is isomorphic to k (e.g., the ring of Witt vectors over k). Put $K :=$ Frac R. The case $p = 0$ shows the following: replacing R by a finite extension of R, there exist a finite noncommutative simple group G and a K-morphism $w_K : Y_K \to \mathbb{P}^1_K$ between connected proper smooth K-curves satisfying the following condition:

(0) w_K is the quotient morphism $Y_K \to \mathbb{P}^1_K = Y_K/G$ whose branch points are equal to 0, 1, and ∞, over which each ramification index is equal to n_0, n_1, and n_∞, respectively.

In the following, we take an appropriate R-model w of w_K in the same way as in [Raynaud 1994, §§6.1–6.3]. Remark that in [Raynaud 1994, §6] the group G is a quasi-p-group and the ramification indices of w_K are equal to powers of p. However, these conditions are used only from Lemma 6.3.6. Replacing R by a finite extension of R, we obtain an R-morphism $w : Y \to P$ between R-schemes satisfying Conditions (1)–(6):

(1) Y is a projective normal semistable R-curve that is an R-model of Y_K.

(2) P is a projective normal semistable R-curve that is an R-model of $\mathbb{P}^1_K$.

(3) w is the quotient morphism $Y \to P = Y/G$.

(4) The restriction of w to the generic fibers is equal to w_K.

(5) The closure of each branch point of w_K in P is contained in the smooth locus P_{sm} of the R-scheme P.

To state Condition (6) below, we introduce notation. Condition (2) shows the following two statements on the special fiber P_k of P: any irreducible component is isomorphic to $\mathbb{P}^1_k$, and the dual graph of the irreducible components is a tree Γ_P. By 0_R, 1_R, and ∞_R we denote the closures of 0, 1, and ∞ on $\mathbb{P}^1_K$ in P, respectively.

For $s = 0$, 1, and ∞, we denote the reduction of s_R by s_k. Condition (5) implies that s_k is contained in exactly one irreducible component C_s. We denote the vertex of Γ_P corresponding to C_s by e_s. Any two vertices e_s and e_t are connected by a unique line l_{st} on Γ_P. The intersection $l_{01} \cap l_{1\infty} \cap l_{\infty 0}$ is exactly one vertex $e_{01\infty}$. We denote the irreducible component corresponding to $e_{01\infty}$ by $C_{01\infty}$.

(6) P is a successive blowing-up of $\mathbb{P}^1_R$, and the strict transform of the special fiber of $\mathbb{P}^1_R$ is equal to $C_{01\infty}$.

Take a subgroup H of G. By Condition (1), we obtain an R-morphism $u : Y \to X$ between R-schemes satisfying Conditions (7) and (8) [Raynaud 1990, Corollaire of Proposition 5]:

(7) X is a proper normal semistable R-curve with connected smooth generic fiber.

(8) u is the quotient morphism $Y \to X = Y/H$.

Conditions (3) and (8) give an R-morphism $v : X \to P$ between R-schemes such that $w = v \circ u$. Furthermore, the following condition is satisfied:

(9) u, v, and w are finite and surjective.

Since v and w are finite (Condition (9)), X and Y are Cohen–Macaulay (Conditions (1) and (7) and [EGA IV$_2$ 1965, 5.8.6]), and P_{sm} is regular (Condition (2)), the following condition is satisfied [Matsumura 1989, 23.1]:

(10) v and w are flat over P_{sm}.

By Γ_Y and Γ_X we denote the dual graphs of the irreducible components of the special fibers Y_k and X_k of Y and X. The group G acts on Γ_Y. The quotient morphisms u and v induce the quotient maps of the actions of G and H from the vertices of Γ_Y to the vertices of Γ_P and Γ_X, respectively. If an element g of G fixes an edge e of Γ_Y, then g does not exchange the two vertices on the edge e [Raynaud 1994, 6.3.5]. Thus, the quotients of the actions of G and H on Γ_Y are canonically isomorphic to Γ_P and Γ_X, respectively. Therefore, the morphisms u and v induce maps $\Gamma_Y \to \Gamma_X \to \Gamma_P$, respectively, that preserve the vertices and the edges.

Take the generic point λ of an irreducible component of P_k. Choose the generic point η of an irreducible component of Y_k over λ. Conditions (9) and (10) show that $\mathcal{O}_{Y,\eta}/\mathcal{O}_{P,\lambda}$ is a finite flat extension of discrete valuation rings. Conditions (1) and (2) imply that the ramification index of $\mathcal{O}_{Y,\eta}/\mathcal{O}_{P,\lambda}$ is equal to 1. Thus, the inertia group I_η of $\mathcal{O}_{Y,\eta}/\mathcal{O}_{P,\lambda}$ is a p-group. Since G is a noncommutative simple group and any simple p-group is commutative, the inequality $G \neq I_\eta$ holds. We take the above subgroup H of G so that $H = I_\eta$. Put $\theta := u(\eta)$. Then the extension of the residue fields of $\mathcal{O}_{X,\theta}/\mathcal{O}_{P,\lambda}$ is separable. Conditions (2) and (7) imply that the ramification index of $\mathcal{O}_{X,\theta}/\mathcal{O}_{P,\lambda}$ is equal to 1. Thus, the extension $\mathcal{O}_{X,\theta}/\mathcal{O}_{P,\lambda}$ is étale.

Take $g \in G$. Replacing η and H by $g\eta$ and gHg^{-1}, respectively, we obtain X^g and θ^g corresponding to X and θ, respectively, in the same way as in the above argument. Furthermore, the extension $\mathcal{O}_{X^g,\theta^g}/\mathcal{O}_{P,\lambda}$ is étale. Since v is finite (Condition (9)) and R is excellent, the normalization Y' of the fiber product of X^g for all $g \in G$ over P is finite over P. By Y'_K we denote the generic fiber of Y'. Since G is simple and $G \neq H$, the intersection $\bigcap_{g\in G} gHg^{-1}$ is the trivial group. Thus, any connected component of Y'_K is P_K-isomorphic to Y_K (Lemma 3.14). Since Y is normal (Condition (1)) and w is finite (Condition (9)), any connected component of Y' is P-isomorphic to Y. Thus, the product of θ^g for all $g \in G$ over λ gives a point η_0 on Y over λ such that the extension $\mathcal{O}_{Y,\eta_0}/\mathcal{O}_{P,\lambda}$ is étale. Since w is a Galois covering, the extension $\mathcal{O}_{Y,g\eta_0}/\mathcal{O}_{P,\lambda}$ is étale for any $g \in G$. Since λ is arbitrary, the morphism w is étale at the generic point of any irreducible component of Y_k. Put $P' := P_{\mathrm{sm}} \setminus (0_R \cup 1_R \cup \infty_R)$. By w' we denote the restriction $w|_{w^{-1}(P')} : w^{-1}(P') \to P'$. Condition (0) implies that the restriction of w' to the generic fibers is étale. Thus, the morphism w' is étale in codimension one. Therefore, Zariski–Nagata purity (Lemma 3.5) shows that w' is étale. Thus, by Lemma 5.2 and the same method as above, we may assume that $\Gamma_P = l_{01} \cup l_{1\infty} \cup l_{\infty 0}$ after successive blowing-down of exceptional curves on Y and P (Condition (6)).

The normalizations of the preimages of C_s and $C_{01\infty}$ under w are proper smooth k-curves C'_s and $C'_{01\infty}$, respectively. The covering C'_s/C_s branches at s_k, over which each ramification index is equal to n_s ([Raynaud 1994, 6.3.2] and Conditions (0), (4), and (5)). Since w' is étale, Lemma 5.2 implies that the preimage $C'_{01\infty}$ is connected and the covering $C'_{01\infty}/C_{01\infty}$ branches at exactly three points, over which each ramification index is equal to n_0, n_1, and n_∞, respectively. Therefore, the covering $C'_{01\infty}/C_{01\infty}$ induces a desired orbifold trivialization. □

Lemma 5.5. *Let C be a connected proper smooth k-curve of positive genus. Take a closed point s on C and an integer n. Assume that $p \nmid n$. Then there exists a connected étale covering space $u : C' \to C$, a divisor D on C', and a rational function h on C' such that $u^*[s] - nD = (h)$, where $[s]$ and (h) are the divisors defined by s and h, respectively.*

Proof. We may assume that n is positive. Since the genus of C is positive and $p \nmid n$, we may take a connected étale covering space $u : C' \to C$ of degree n. We denote the genus of C' by g. By J we denote the Jacobian variety of C' over k. Take a closed point s_0 on C'. Since the morphism $(C')^g \to J$ defined by $(s_i)_{i=1}^g \mapsto \sum_{i=1}^g([s_i] - [s_0])$ is surjective and the multiplication of J by n is surjective, there exists $(s_i)_{i=1}^g \in (C'(k))^g$ such that $u^*[s] - n[s_0]$ is linearly equivalent to $n\sum_{i=1}^g([s_i] - [s_0])$, which proves the lemma. □

Proposition 5.6. *Let (C, B) be a connected proper tame orbifold k-curve with $\# \operatorname{Supp} B = 1$. Assume that the genus of C is positive. Then there exists an orbifold trivialization of (C, B) that is the composite $u \circ v : C'' \to C' \to C$ of two finite coverings where u is étale and v is totally ramified over each branch point.*

Proof. Take $s \in \operatorname{Supp} B$. Put $n := [B_s : K_s]$. Take a connected étale covering space $u : C' \to C$ and a rational function h on C' given by Lemma 5.5. By $v : C'' \to C'$ we denote the normal model of the equation $z^n = h$. Then $u \circ v$ induces a desired orbifold trivialization. □

Proposition 5.7. *Let (C, B) be a connected tame orbifold k-curve. Put $M := \# \operatorname{Supp} B$. For each $s \in \operatorname{Supp} B$, we put $n_s := [B_s, K_s]$. Then there exists an orbifold trivialization of (C, B) if and only if neither of the following conditions are satisfied*: (a) $C \cong \mathbb{P}^1_k$ *and* $M = 1$; (b) $C \cong \mathbb{P}^1_k$, $M = 2$, *and* $n_s \neq n_t$ *where* $\operatorname{Supp} B = \{s, t\}$.

Proof. In the proof of the "if" part, we may replace C by the smooth compactification of C. Thus, we may assume that C is proper over k. By Proposition 3.18, we have only to consider the following cases: (1) $g(C) = 0$ and $M \leq 2$; (2) $g(C) = 0$, $M = 3$, and $\gcd(n_s, n_t) = 1$ for $s \neq t$; (3) $g(C) > 0$ and $M = 1$. Cases (1), (2), and (3) follow from Propositions 5.3, 5.4, and 5.6, respectively. □

In the following, we provide steps in order to prove Proposition 5.12. Assume that $p > 0$. Let C be a connected proper smooth k-curve. We denote the sheaf of rational functions on C by $\mathcal{M}_C$. Put $\mathcal{P}_C := \mathcal{M}_C / \mathcal{O}_C$. The exact sequence of abelian sheaves $0 \to \mathcal{O}_C \to \mathcal{M}_C \to \mathcal{P}_C \to 0$ induces a long exact sequence

$$H^0(C, \mathcal{M}_C) \xrightarrow{\phi_C} H^0(C, \mathcal{P}_C) \xrightarrow{\psi_C} H^1(C, \mathcal{O}_C).$$

By F_C we denote the absolute Frobenius endomorphism of C and its actions on the cohomology groups $H^0(C, \mathcal{P}_C)$ and $H^1(C, \mathcal{O}_C)$. We define F_C^0 as the identity map and, for each positive integer d, we inductively define F_C^d by $F_C^d := F_C^{d-1} \circ F_C$.

Lemma 5.8. *For any $\xi \in H^0(C, \mathcal{P}_C)$, there exists a connected étale covering space $u : C' \to C$, a nonnegative integer d, and a rational function h on C' such that $u^* F_C^d \xi = \phi_{C'}(h)$, where u^* is the homomorphism $H^0(C, \mathcal{P}_C) \to H^0(C', \mathcal{P}_{C'})$ induced by u.*

Proof. Since the k-vector space $H^1(C, \mathcal{O}_C)$ is finite-dimensional, we may take a nonnegative integer d and a polynomial $G(X) = \sum_{i=0}^n c_i X^i \in k[X]$ so that $c_0 \neq 0$ and $G(F_C)\eta = 0$, where we put $\eta := F_C^d \psi_C \xi$. Take an affine covering $\mathcal{U} = \{U_i\}$ of C and a representative $\{a_{ij}\} \in C^1(\mathcal{U}, \mathcal{O}_C)$ of η. We define a polynomial $G^{(p)}(X) \in k[X]$ by $G^{(p)}(X) := \sum_{i=0}^n c_i X^{p^i}$. Since $G(F_C)\eta = 0$, there exists

$\{a_i\} \in C^0(\mathcal{U}, \mathcal{O}_C)$ such that $G^{(p)}(a_{ij}) = a_j - a_i$ for all i and all j. Thus, we may define an étale covering space $u : C' \to C$ by the equations

$$\begin{cases} G^{(p)}(z_i) = a_i & \text{on } U_i, \\ z_j - z_i = a_{ij} & \text{on } U_i \cap U_j. \end{cases}$$

By definition, the pullback of η via u splits, which proves the lemma. □

We denote the cokernel of an endomorphism ϕ of a module M by M_ϕ.

Lemma 5.9. *Let R be an excellent discrete valuation ring of positive characteristic with separably closed residue field. By $\widehat{R}$ we denote the completion of R with respect to the maximal ideal. Put $K := \operatorname{Frac} R$, $K^{\mathrm{sh}} := \operatorname{Frac} R^{\mathrm{sh}}$, and $\widehat{K} := \operatorname{Frac} \widehat{R}$. We denote the Frobenius endomorphisms on these fields by F. The canonical inclusion homomorphisms $K \to K^{\mathrm{sh}}$ and $K^{\mathrm{sh}} \to \widehat{K}$ induce the canonical homomorphisms $\alpha : K \to K^{\mathrm{sh}}_{F-1}$ and $\beta : K^{\mathrm{sh}}_{F-1} \to \widehat{K}_{F-1}$, respectively. Then*:

(1) *α is surjective and $R \subset \operatorname{Ker} \alpha$.*

(2) *β is an isomorphism.*

Proof. Since K^{sh} is algebraically closed in $\widehat{K}$ by the approximation property [Bosch et al. 1990, 3.6.9], Artin–Schreier theory shows that β is injective. Since $\widehat{R}$ is isomorphic to the formal power series ring over the separably closed residue field of R, the relation $\widehat{R} \subset (F-1)\widehat{K}$ holds, which implies that $\beta \circ \alpha$ is surjective. Thus, Statement (2) holds, which implies Statement (1). □

We recall the definition of the addition of the ring of Witt vectors $W(A)$ with coefficient ring A. Let n be a nonnegative integer. Put

$$W_n(X_0, \dots, X_n) := \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \in \mathbb{Z}[X_0, \dots, X_n].$$

We inductively define S_n as the unique polynomial in $\mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$ satisfying the equality $W_n(S_0, \dots, S_n) = W_n(X_0, \dots, X_n) + W_n(Y_0, \dots, Y_n)$. For $a = (a_0, \dots, a_n, \dots) \in W(A)$ and $b = (b_0, \dots, b_n, \dots) \in W(A)$, the addition of Witt vectors is defined by

$$a + b := (S_0(a_0, b_0), \dots, S_n(a_0, \dots, a_n, b_0, \dots, b_n), \dots).$$

Lemma 5.10. *We denote the ideal of $\mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$ generated by $\{X_i Y_j\}_{0 \le i, j \le n}$ by I. Then the equality $S_n \equiv X_n + Y_n \mod I$ holds. In particular, the equality*

$$(a_0, \dots, a_{n-1}, a_n, \dots) + (0, \dots, 0, b_n, \dots) = (a_0, \dots, a_{n-1}, a_n + b_n, \dots)$$

holds in $W(A)$.

Proof. Let us show the first equality by induction on n. The case $n = 0$ is clear. Assume that the case i is proved for any $i < n$. By the induction hypothesis, the equality $p^n S_n \equiv p^n X_n + p^n Y_n \mod I$ holds, which proves the case n. Thus, the first equality holds for any n. The first equality shows the last equality. $\square$

For a positive integer n, we denote the ring of length-n Witt vectors with coefficient ring A by $W_n(A)$. We denote the Frobenius endomorphism on $W_n(A)$ by F. Take a connected étale covering space $u : C' \to C$, a closed point s on C, and $s' \in u^{-1}(s)$. Put $K_s := \operatorname{Frac} \mathcal{O}^{\mathrm{sh}}_{C,s}$ and $K'_{s'} := \operatorname{Frac} \mathcal{O}^{\mathrm{sh}}_{C',s'}$. The extensions $K'_{s'}/K_s$ for all $s' \in u^{-1}(s)$ induce a homomorphism

$$\Delta_{u,s,n} : W_n(K_s)_{F-1} \to \bigoplus_{s' \in u^{-1}(s)} W_n(K'_{s'})_{F-1}.$$

Put $A_{u,s} := \mathcal{O}_{C'}(C' - u^{-1}(s))$. The canonical homomorphisms $A_{u,s} \to K'_{s'}$ for all $s' \in u^{-1}(s)$ induce a homomorphism

$$\phi_{u,s,n} : W_n(A_{u,s})_{F-1} \to \bigoplus_{s' \in u^{-1}(s)} W_n(K'_{s'})_{F-1}.$$

Put $\Delta_{C,s,n} := \varinjlim_u \Delta_{u,s,n}$ and $\phi_{C,s,n} := \varinjlim_u \phi_{u,s,n}$, where $u : C' \to C$ runs through all connected étale covering spaces of C. By construction, the homomorphisms $\Delta_{C,s,n}$ and $\phi_{C,s,n}$ are compatible with the reductions of the rings of Witt vectors.

Lemma 5.11. *The relation* $\operatorname{Im} \Delta_{C,s,n} \subset \operatorname{Im} \phi_{C,s,n}$ *holds.*

Proof. Take $\xi \in W_n(K_s)_{F-1}$. Put $\eta := \Delta_{C,s,n}(\xi)$. We have to show that $\eta \in \operatorname{Im} \phi_{C,s,n}$. By induction on n, we have only to consider the following cases: (1) $n = 1$; (2) $n > 1$ and ξ is contained in the kernel of the reduction homomorphism $W_n(K_s) \to W_{n-1}(K_s)$. Note that $W_1(A) = A$ for any ring A. Lemma 5.10 reduces Case (2) to Case (1). Thus, we may assume that $n = 1$. Lemma 5.9(1) shows that the canonical homomorphism $\mathcal{M}_{C,s} \to K_s$ induces a surjective homomorphism $P : \mathcal{P}_{C,s} \to (K_s)_{F-1}$. Thus, we have only to show that $\Delta_{C,s,1}(P(\theta)) \in \operatorname{Im} \phi_{C,s,1}$ for any $\theta \in \mathcal{P}_{C,s}$. Since the equality $X^d = 1 + \bigl(\sum_{i=0}^{d-1} X^i\bigr)(X - 1)$ holds in the polynomial ring $k[X]$ for any positive integer d, we may replace θ by $F^d \theta$ for any positive integer d. Thus, the lemma follows from Lemma 5.8. $\square$

Proposition 5.12. *Assume that* $p > 0$. *Let* (C, B) *be a connected proper cyclic* p*-orbifold* k*-curve with* $\# \operatorname{Supp} B = 1$. *Then there exists an orbifold trivialization of* (C, B) *that is the composite* $u \circ v : C'' \to C' \to C$ *of two finite coverings where* u *is étale and* v *is totally ramified over each branch point.*

Proof. Take $s \in \operatorname{Supp} B$. Take an integer n so that $[B_s : K_s] = p^n$. Since $H^1(K_s, W_n) = 0$, the exact sequence of $\operatorname{Gal}(\overline{K_s}/K_s)$-modules

$$0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow W_n(\overline{K_s}) \xrightarrow{F-1} W_n(\overline{K_s}) \longrightarrow 0$$

induces an isomorphism $H^1(K_s, \mathbb{Z}/p^n\mathbb{Z}) \cong W_n(K_s)_{F-1}$. In particular, the field extension B_s/K_s is induced by an element $\xi \in W_n(K_s)_{F-1}$. Lemma 5.11 gives a connected étale covering space $u : C' \to C$ and an element $\eta \in W_n(A_{u,s})_{F-1}$ such that the equality $\Delta_{u,s,n}(\xi) = \phi_{u,s,n}(\eta)$ holds. By K' we denote the function field of C'. The image of η under the canonical homomorphism $W_n(A_{u,s})_{F-1} \to W_n(K')_{F-1}$ induces a cyclic extension K''/K' of degree p^n. Take the normalization $v : C'' \to C'$ of C' in K''. By the choice of η, the morphism $u \circ v$ induces a desired orbifold trivialization. $\square$

Proposition 5.13. *Let (C, B) be a connected p-orbifold k-curve. Then there exists an orbifold trivialization of (C, B).*

Proof. We may replace C by the smooth compactification of C. Thus, we may assume that C is proper over k. By Proposition 3.18, we may assume that $\#\operatorname{Supp} B = 1$. Take $s \in \operatorname{Supp} B$. Put $m := [B_s : K_s]$. Let us show the proposition by induction on m. The case $m = 1$ is clear. Assume that $m > 1$. Since B_s/K_s is solvable, we may take a Galois extension B'_s/K_s of degree p in B_s. For each $t \in P(C) \setminus \{s\}$, we put $B'_t := K_t$. We define a map B' on $P(C)$ by $t \mapsto B'_t/K_t$ (Definition 3.6). Then the pair (C, B') is an orbifold. Applying Proposition 5.12 to (C, B'), we may reduce the case m to the case m/p. Thus, the case m holds by the induction hypothesis. $\square$

Definition 5.14. The *Euler characteristic* $e(C)$ of a proper smooth k-curve C is the ℓ-adic Euler characteristic of C, which does not depend on the choice of the prime number ℓ that is prime to p. Let (C, B) be a proper orbifold k-curve. Take $s \in \operatorname{Supp} B$. We use the notation B_s/K_s introduced in Definition 3.6. By B_s° and K_s° we denote the valuation rings of the discrete valuation fields B_s and K_s, respectively. We define the *orbifold Euler characteristic* $e(C, B)$ of (C, B) by

$$e(C, B) := e(C) - \sum_{s \in \operatorname{Supp} B} \frac{1}{[B_s : K_s]} \operatorname{length}_{K_s^\circ}(\Omega^1_{B_s^\circ/K_s^\circ}).$$

The Riemann–Hurwitz formula shows the following:

Proposition 5.15. *Let (C, B) and (C', B') be two proper orbifold k-curves. If there exists an orbifold étale morphism $(C', B') \to (C, B)$ of degree n, then the equality $e(C', B') = ne(C, B)$ holds. In particular, if $e(C, B) > 0$ and C is connected, then the underlying curve of any connected orbifold étale covering space of (C, B) is isomorphic to $\mathbb{P}^1_k$.*

Proof of Theorem 1.3. Let us show Statement (2). Assume that C is not proper over k. Take the smooth compactification $\overline{C}$ of C. We may choose an extension $\overline{B}$ of B to $P(\overline{C})$ so that the orbifold $(\overline{C}, \overline{B})$ satisfies neither Condition (a) nor Condition (b). Thus, we may assume that C is proper over k. First, we consider the “if” part. By Propositions 3.18, 5.15, and 5.13, we may assume that $B^t = B$. In that case, the “if” part follows from Proposition 5.7. Next, we consider the “only if” part. Take $s \in \operatorname{Supp} B$ and an orbifold trivialization $C_1 \to (C, B)$. By Proposition 5.7, we have only to show that Condition (a) is not satisfied. Assume that Condition (a) is satisfied. Since C is simply connected, Proposition 5.12 gives an orbifold trivialization $u : C_2 \to (C, B^w)$ that is totally ramified over the unique branch point s. Condition (a) and Proposition 5.15 imply that $C_2 \cong \mathbb{P}^1_k$. Take the normalization C_3 of $C_1 \times_C C_2$ and the canonical projection $v : C_3 \to C_2$. Lemma 3.14 shows that v is tamely ramified over the unique branch point $u^{-1}(s)$, which contradicts Lemma 5.2. Thus, Condition (a) is not satisfied.

Let us show Statement (1). The “if part” follows from Proposition 5.3. We consider the “only if” part. Assume that Condition (a) is not satisfied. Statement (2) implies that $C \cong \mathbb{P}^1_k$ and $M \leq 2$. Proposition 5.13 implies that $B^t = B$. Proposition 5.3 implies that $\gcd(n_s, n_t) = 1$ for $s \neq t$. Thus, Condition (b) is satisfied. Therefore, the “only if” part holds. □

6. Fundamental groups of elliptic fibrations

6A. *Elliptic fibrations.* We study elliptic surfaces by localizing the fibrations with respect to the base curves. To this end, we generalize the definition of elliptic surfaces. We refer to [Liu 2002, §§8–9] for fibered surfaces.

Definition 6.1. An *elliptic fibration* is a triple (X, C, f) satisfying the following conditions:

(1) C and X are excellent regular integral schemes of dimension one and two, respectively.

(2) $f : X \to C$ is a proper morphism.

(3) The homomorphism $\mathcal{O}_C \to f_*\mathcal{O}_X$ associated to f is an isomorphism.

(4) The generic fiber of f is a proper smooth curve of genus one.

Let (X, C, f) be an elliptic fibration. A prime divisor D on X is said to be a (-1)-*curve* if the following conditions are satisfied. Put $k := \Gamma(D, \mathcal{O}_D)$. Then D is k-isomorphic to $\mathbb{P}^1_k$ and $\deg \mathcal{O}_X(D)|_D = -1$. If any fiber of f does not contain a (-1)-curve, then (X, C, f) is said to be *relatively minimal*. The *multiplicity* of a closed fiber F of f is the multiplicity of the divisor F on X (Section 2). The minimal regular C-model of the Jacobian of the generic fiber of f is called the *Jacobian fibration* of f.

Remark 6.2. Conditions (2) and (3) show that f is surjective. Thus, Condition (1) shows that f is flat. The multiplicity of F does not depend on the choice of the proper regular C-model of the generic fiber of f [Liu 2002, 9.2.7].

Lemma 6.3. *Let (X, C, f) be an elliptic fibration and $\xi : Y \to X$ a connected étale covering space. Take the Stein factorization*

$$Y \xrightarrow{h} D \xrightarrow{v} C$$

of the composite $f \circ \xi : Y \to X \to C$. Then:

(1) *(Y, D, h) is an elliptic fibration.*

(2) *v is finite, flat, surjective, and generically separable.*

Choose an integral scheme C' and a finite flat morphism $u : C' \to C$ such that v factors through u. Take the normalization X' of $X \times_C C'$, the canonical projections $u' : X' \to X$ and $f' : X' \to C'$, and the unique morphism $\xi' : Y \to X'$ satisfying $\xi = u' \circ \xi'$ and $h = f' \circ \xi'$. Then:

(3) *(X', C', f') is an elliptic fibration.*

(4) *u' and ξ' are finite, étale, and surjective.*

Proof. Since X is regular and ξ is étale, the scheme Y is regular. Thus, Statements (1) and (2) follow from Proposition 4.5. Statement (4) follows from Lemma 4.7. Since X is regular and u' is étale, the scheme X' is regular. Thus, Statement (3) follows from Proposition 4.5. □

We frequently use the following:

Proposition 6.4 [Liu et al. 2004, 6.6]. *Let C be the spectrum of a complete discrete valuation ring with algebraically closed residue field and field of fractions K. Let (X, C, f) be a relatively minimal elliptic fibration with Jacobian fibration (E, C, g). By X_K and E_K we denote the generic fibers of f and g, respectively. Then the special fiber of f is of type ${}_mT$ (the Kodaira symbol) if and only if the special fiber of g is of type T and the order of the torsor $[X_K] \in H^1(K, E_K)$ is equal to m.*

We refer to [Liu 2002, 8.3.39, 8.3.44, 9.3.31, and 9.3.32] for desingularizations and the minimal desingularizations of fibered surfaces.

Lemma 6.5. *Let (X, C, f) be an elliptic fibration with generic fiber X_K and $\xi_K : Y_K \to X_K$ a finite morphism between geometrically connected K-curves of genus one. Take the normalization $\xi : Y \to X$ of X in ξ_K. Assume that f is smooth and that the residue field at any closed point on C is algebraically closed. Then the triple $(Y, C, f \circ \xi)$ is a relatively minimal elliptic fibration.*

Proof. We may assume that C is the spectrum of a discrete valuation ring. Take the minimal desingularization $\lambda : \widehat{Y} \to Y$ of Y. By X_k, Y_k, and $\widehat{Y}_k$ we denote the special fibers of f, $f \circ \xi$, and $f \circ \xi \circ \lambda$, respectively. Choose an irreducible component D of Y_k and an irreducible component $\widehat{D}$ of $\widehat{Y}_k$ dominating D. Since $1 = g(X_k) \leq g(D) \leq g(\widehat{D})$, the Néron–Kodaira classification of singular fibers implies that $\widehat{D}$ is the unique irreducible component of $\widehat{Y}_k$ whose geometric genus is equal to 1. Thus, the component D is the unique irreducible component of Y_k, and the component $\widehat{D}$ is the unique irreducible component of $\widehat{Y}_k$ that dominates D. Since λ is the minimal desingularization, the morphism λ is an isomorphism. Therefore, the triple $(Y, C, f \circ \xi)$ is a relatively minimal elliptic fibration. □

Lemma 6.6. *Let (X, C, f) be an elliptic fibration with Jacobian fibration (E, C, g). Assume that the reduction of any closed fiber of f is isomorphic to an elliptic curve and that the residue field at any closed point on C is algebraically closed. By X_K and E_K we denote the generic fibers of f and g, respectively. Take a positive multiple n of the order of the torsor $[X_K] \in H^1(K, E_K)$. Then there exists a C-morphism $X \to E$ whose restriction to the generic fibers induces the multiplication of their Jacobian E_K by n.*

Proof. Take a finite Galois extension K'/K so that $X(K') \neq \varnothing$. Put $G := \mathrm{Gal}(K'/K)$ and $X_{K'} := X_K \times_K K'$. Choose a cocycle $c \in Z^1(G, E(K'))$ representing $[X_K]$. The curve X_K/K may be obtained as the quotient of a G-equivariant action on $X_{K'}/K'$ induced by c (see Section 6B). Moreover, an endomorphism of $Z^1(G, E(K'))$ induces a G-equivariant endomorphism of $X_{K'}$, whose quotient is a K-morphism between torsors of E_K. Since the endomorphism on $H^1(K, E_K)$ induced by the multiplication of E_K by n maps the torsor $[X_K]$ to the trivial torsor $[E_K]$, the endomorphism induces a K-morphism $\xi_K : X_K \to E_K$. Take the normalization $Y \to E$ of E in ξ_K. Proposition 6.4 shows that g is smooth. By Lemma 6.5, the scheme Y is the minimal regular C-model of X_K. Since the minimal regular C-model of X_K is unique up to unique C-isomorphism, the C-scheme Y is C-isomorphic to X, which concludes the proof. □

Corollary 6.7. *Let C be the spectrum of a complete discrete valuation ring with algebraically closed residue field. Let (X, C, f) be a relatively minimal elliptic fibration with Jacobian fibration (E, C, g). Then the reduction of the special fiber of f is isomorphic to an ordinary elliptic curve if and only if the special fiber of g is an ordinary elliptic curve.*

6B. ***Étale coverings of local elliptic fibrations.*** Let R be a complete discrete valuation ring with algebraically closed residue field k of characteristic $p \geq 0$ and field of fractions K. Put $C := \operatorname{Spec} R$. Let (X, C, f) be an elliptic fibration with Jacobian fibration (E, C, g). By X_K and E_K we denote the generic fibers of f and g, respectively. By $[X_K]$ we denote the element of $H^1(K, E_K)$ corresponding to the torsor X_K of E_K. Take a separable closure $\overline{K}$ of K. Take the maximal base field $\widetilde{K}$ of f in $\overline{K}$ (Definition 4.9). In this subsection, we determine the extension $\widetilde{K}/K$.

Take a finite Galois extension K'/K in $\overline{K}$ so that $X(K') \neq \varnothing$. Put $G_{K'/K} := \mathrm{Gal}(K'/K)$. The group $H^1(G_{K'/K}, E(K'))$ may be regarded as a subgroup of

$H^1(K, E_K)$ by the inflation homomorphism. Then the torsor $[X_K] \in H^1(K, E_K)$ is contained in $H^1(G_{K'/K}, E(K'))$ since $[X_K]$ splits over K'. Choose a cocycle $c \in Z^1(G_{K'/K}, E(K'))$ representing $[X_K]$. The extension K'/K induces a finite covering C'/C. By (E', C', g') we denote the Jacobian fibration of $X_K \times_K K'$. By the uniqueness of the normalization C' of C in K' and the Jacobian fibration g', we obtain a homomorphism $\rho : G_{K'/K} \to \mathrm{Aut}(C'/C) \to \mathrm{Aut}(E'/C)$, where the first arrow is induced by the automorphisms on the generic point of C' and the second arrow is induced by the base change of the automorphisms via g'. Furthermore, we obtain a map $\tau : G_{K'/K} \to E(K') \to \mathrm{Aut}(E'/C')$, where the first arrow is given by c and the second arrow is induced by the translation by addition. Since c is a cocycle, the map $\tilde{\tau} : G_{K'/K} \to \mathrm{Aut}(E'/C)$ defined by $\sigma \mapsto \tau(\sigma) \circ \rho(\sigma)$ is a homomorphism. By $\chi : E' \to Z := E'/\operatorname{Im} \tilde{\tau}$ we denote the quotient morphism of the action $\tilde{\tau}$. The quotient Z is a normal scheme over C whose generic fiber is isomorphic to X_K.

Lemma 6.8. *If any element of* $\operatorname{Im} \tilde{\tau}$ *fixes any closed point on* E', *then* $\widetilde{K} = K$.

Proof. We may assume that X is the minimal desingularization of Z. Since Z is normal, we may take a regular closed point z on Z. By $x \in X$ we denote the preimage of z. The extension $\widetilde{K}/K$ induces a finite covering $\widetilde{C}/C$. Take the minimal desingularization X' of $X \times_C C'$ (resp. $\widetilde{X}$ of $X \times_C \widetilde{C}$) and the canonical projection $u' : X' \to X$ (resp. $\tilde{u} : \widetilde{X} \to X$). By the choice of x, the preimage $(u')^{-1}(x)$ consists of one closed point on X'. By Proposition 4.15(4) and the definitions of u' and $\tilde{u}$, the morphism u' factors through the finite étale surjective morphism $\tilde{u}$. Thus, the degree of $\tilde{u}$ is equal to one, which implies that $\widetilde{K} = K$. □

Assume that (X, C, f) is relatively minimal. Put $G_K := \mathrm{Gal}(\overline{K}/K)$. By ${}_mT$ we denote the type of the special fiber of f (the Kodaira symbol). The type T is divided into the following three cases, (A), (M), and (E):

Case (*A*). *Additive type*: $T \neq \mathrm{I}_n$ $(n \geq 0)$. Since the residue field k of R is algebraically closed and the special fiber of f is simply connected, Lemma 6.3 implies that X is simply connected. In particular, the equality $\widetilde{K} = K$ holds.

Case (*M*). *Multiplicative type*: $T = \mathrm{I}_n$ $(n > 0)$. Tate's uniformization gives an exact sequence of G_K-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{G}_{m,K}(\overline{K}) \xrightarrow{\pi} E_K(\overline{K}) \longrightarrow 0,$$

where π maps 1 to $q \in K$ satisfying $0 < |q| < 1$. The exact sequence induces a long exact sequence

$$H^1(K, \mathbb{G}_{m,K}) \longrightarrow H^1(K, E_K) \longrightarrow H^2(K, \mathbb{Z}) \longrightarrow H^2(K, \mathbb{G}_{m,K}).$$

Since $H^1(K, \mathbb{G}_{m,K}) = H^2(K, \mathbb{G}_{m,K}) = 0$ and $H^2(K, \mathbb{Z}) \cong \mathrm{Hom}(G_K, \mathbb{Q}/\mathbb{Z})$, we obtain an isomorphism $\phi_M : H^1(K, E_K) \cong \mathrm{Hom}(G_K, \mathbb{Q}/\mathbb{Z})$. Put $\gamma_M := \phi_M([X_K])$. The group $\mathrm{Im}\,\gamma_M$ is finite and cyclic. The Galois extension L/K corresponding to $\mathrm{Ker}\,\gamma_M$ is the minimum separable field extension that splits $[X_K]$. By D we denote the normalization of C in L. Put $d := [L : K]$. Then the normalization Y of $X \times_C D$ is a relatively minimal elliptic fibration over D with special fiber of type I_{dn}, and the induced morphism $Y \to X$ is étale (see the proof of [Liu et al. 2004, 8.3(b)]). In particular, the relation $\widetilde{K} \supset L$ holds. Proposition 4.15(4) gives the relation $\widetilde{K} \subset L$. Thus, the equality $\widetilde{K} = L$ holds.

Case (*E*). *Elliptic type*: $T = \mathrm{I}_0$. By $\widehat{E}$ we denote the formal group law associated to E. By $\overline{R}$ and $\overline{\mathfrak{m}}$ we denote the integral closure of R in $\overline{K}$ and the maximal ideal of $\overline{R}$, respectively. Then $\widehat{E}$ gives a group structure on $\overline{\mathfrak{m}}$. By $\widehat{E}(\overline{\mathfrak{m}})$ we denote this group. Since the canonical homomorphism $E(\overline{R}) \to E(\overline{K})$ is a G_K-module isomorphism by the valuative criterion for properness, we obtain an exact sequence of G_K-modules

$$0 \longrightarrow \widehat{E}(\overline{\mathfrak{m}}) \longrightarrow E(\overline{K}) \longrightarrow E(k) \longrightarrow 0.$$

The exact sequence induces a long exact sequence

$$0 \longrightarrow H^1(K, \widehat{E}(\overline{\mathfrak{m}})) \xrightarrow{\iota} H^1(K, E_K) \longrightarrow H^1(K, E(k)) \xrightarrow{\psi} H^2(K, \widehat{E}(\overline{\mathfrak{m}})).$$

Since G_K acts trivially on $E(k)$, we obtain an isomorphism $H^1(K, E(k)) \cong \mathrm{Hom}(G_K, E(k))$.

Lemma 6.9. *If $p = 0$, then the group $H^i(K, \widehat{E}(\overline{\mathfrak{m}}))$ is trivial for any positive integer i. Otherwise the group $H^i(K, \widehat{E}(\overline{\mathfrak{m}}))$ is p-primary for any positive integer i.*

Proof. Any i-th Galois cohomology group is torsion for any positive integer i. Take an integer n so that $p \nmid n$. Then the multiplication of $\widehat{E}(\overline{\mathfrak{m}})$ by n is an isomorphism [Silverman 2009, IV.2.3(b)]. These facts show the lemma. □

Lemma 6.10. *The homomorphism ψ is the zero map.*

Proof. By Lemma 6.9, we may assume that $p > 0$. Furthermore, we have only to show the image of any element of $H^1(K, E(k))$ of p-power order under ψ is equal to zero. If the special fiber of E is ordinary, then the statement follows from [Raynaud 1970, 9.4.1(iii)]. Otherwise, the group $E(k)$ is p-torsion free. Thus, the group $H^1(K, E(k))$ is p-torsion free, which implies that the statement holds. □

From Lemma 6.10 we get a surjective homomorphism $\phi_E : H^1(K, E_K) \to \mathrm{Hom}(G_K, E(k))$. Put $\gamma_E := \phi_E([X_K])$. The group $\mathrm{Im}\,\gamma_E$ is finite and cyclic. The Galois extension L/K corresponding to $\mathrm{Ker}\,\gamma_E$ is the minimum separable field extension that splits the image of $[X_K]$ in $H^1(K, E(k))$. By D we denote the normalization of C in L. Put $d := [L : K]$. Then the normalization Y of $X \times_C D$ is a relatively minimal elliptic fibration over D with special fiber of type ${}_{m/d}\mathrm{I}_0$, and the canonical projection $Y \to X$ is étale. In particular, the relation $\widetilde{K} \supset L$ holds. If $[X_K] \in \iota(H^1(G_K, \widehat{E}(\overline{\mathfrak{m}})))$, then any element of $\operatorname{Im} \tilde{\tau}$ fixes any closed point on E'. Thus, Lemma 6.8 gives the equality $\widetilde{K} = L$.

We summarize the above results:

Proposition 6.11. *Assume that (X, C, f) is relatively minimal. By ${}_mT$ we denote the type of the special fiber of f. Take the maximal base field $\widetilde{K}$ of f* (Definition 4.9). *Then $[\widetilde{K} : K]$ divides m* (Proposition 4.15(3)). *Moreover*:

(A) *If $T \neq \mathrm{I}_n$ $(n \geq 0)$, then X is simply connected and $\widetilde{K} = K$.*

(M) *If $T = \mathrm{I}_n$ $(n > 0)$, then $\widetilde{K}/K$ corresponds to* $\operatorname{Ker} \gamma_M$ *in Case* (*M*) *and* $[\widetilde{K} : K] = \# \operatorname{Im} \gamma_M = m$.

(E) *If $T = \mathrm{I}_0$, then $\widetilde{K}/K$ corresponds to* $\operatorname{Ker} \gamma_E$ *in Case* (*E*), $[\widetilde{K} : K] = \# \operatorname{Im} \gamma_E$, *and one of the following statements holds*: (1) $p = 0$ *and* $[\widetilde{K} : K] = m$; (2) $p > 0$ *and $m/[\widetilde{K} : K]$ is a power of p.*

In particular, the extension $\widetilde{K}/K$ is finite and cyclic.

Lemma 6.12. *Assume that (X, C, f) is relatively minimal. By m we denote the multiplicity of the special fiber X_k of f. We define a divisor F on X by $F := X_k/m$. By n we denote the order of the normal bundle of F in the Picard group* $\operatorname{Pic} F$. *Then*:

(1) *The $\mathbb{O}_C$-module $R^1 f_* \mathbb{O}_X$ is torsion-free if and only if the equality $m = n$ holds.*

(2) *If $p = 0$, then the equality $m = n$ holds. Otherwise, there exists a nonnegative integer e such that the equality $m = np^e$ holds.*

(3) *We use the notation introduced in* Lemma 6.3. *Take m' and n' for (X', C', f') in the same way. By d we denote the degree of u. Assume that F is isomorphic to an elliptic curve and $p \nmid d$. Then the equalities $m = dm'$ and $n = dn'$ hold.*

Proof. Statements (1) and (2) follow from Proposition 1 in [Mitsui 2013]. Let us show Statement (3). Since u' is étale, the equality $u \circ f' = f \circ u'$ gives the equality $m = dm'$. By $\lambda : F' \to F$ we denote the base change of u' via the inclusion morphism $F \to X$. Since u' is a finite étale surjective morphism of degree d, the base change λ is a finite étale surjective morphism of degree d. Since F is isomorphic to an elliptic curve, the morphism λ may be regarded as a morphism between elliptic curves over k. The morphism λ induces a homomorphism $\lambda^* : \operatorname{Pic} F \to \operatorname{Pic} F'$. Since u' is étale, the divisor F' is equal to the pullback of the divisor F via u', which implies that $N_{F'/X} = \lambda^* N_{F/X}$. Since $p \nmid d$, the relation $p \nmid (n/n')$ holds. Thus, the equality $m = dm'$ and Statement (2) give the equality $n = dn'$. □

Lemma 6.13 [EGA III_1 1961, 7.7.5(II), 7.8.4, and 7.9.4]. *The following conditions are equivalent*:

(1) *f is cohomologically flat in dimension zero* [EGA III$_2$ 1963, 7.8.1]; *i.e., the formation of the direct image $f_*\mathbb{O}_X$ commutes with any base change.*

(2) *$R^1 f_*\mathbb{O}_X$ is torsion-free.*

Lemma 6.14. *We use the notation introduced in* Lemma 6.3. *Suppose that the reduction of the special fiber of f is isomorphic to an elliptic curve and that $p \nmid \deg v$. Then $R^1 f_*\mathbb{O}_X$ is torsion-free if and only if $R^1 h_*\mathbb{O}_Y$ is torsion-free.*

Proof. By [Raynaud 1970, 9.4.2] and Lemma 6.13, we may assume that ξ' is an isomorphism. Then the lemma follows from (1) and (3) of Lemma 6.12. □

Proposition 6.15. *Suppose that $p > 0$. Assume that (X, C, f) is relatively minimal. By ${}_mT$ we denote the type of the special fiber of f. Take the maximal base field $\widetilde{K}$ of f* (Definition 4.9). *By $(X_k)_{\mathrm{red}}$ we denote the reduction of the special fiber of f. Then $p \nmid [\widetilde{K} : K]$ if and only if one of the following conditions is satisfied*:

(1) $T \neq \mathrm{I}_n$ $(n \geq 0)$.

(2) *$p \nmid m$.*

(3) *$(X_k)_{\mathrm{red}}$ is isomorphic to a supersingular elliptic curve.*

(4) *$(X_k)_{\mathrm{red}}$ is isomorphic to an ordinary elliptic curve and $R^1 f_*\mathbb{O}_X$ is torsion-free.*

Proof. By (A) and (M) of Proposition 6.11, we have only to consider the case $T = \mathrm{I}_0$. By Proposition 6.11(E), the extension $\widetilde{K}/K$ corresponds to $\operatorname{Ker}\gamma_E$ and the equality $[\widetilde{K} : K] = \#\operatorname{Im}\gamma_E$ holds. If $(X_k)_{\mathrm{red}}$ is isomorphic to a supersingular elliptic curve, then the group $\operatorname{Hom}(G_K, E(k))$ is p-torsion free (Corollary 6.7). Thus, we may assume that $(X_k)_{\mathrm{red}}$ is isomorphic to an ordinary elliptic curve. By Lemmas 6.3 and 6.14, we may assume that $[\widetilde{K} : K]$ is a power of p. Then the proposition follows from [Raynaud 1970, 9.4.1(iii)] and Lemma 6.13. □

Proposition 6.16. *Let L/K be a finite cyclic extension in $\overline{K}$ and (E', C, g') a relatively minimal elliptic fibration with section. By E'_k we denote the special fiber of g'. Then the following two conditions are equivalent*:

(1) *There exists a relatively minimal elliptic fibration (X', C, f') satisfying the following conditions*:
   (a) *The maximal base field of f' is equal to L* (Definition 4.9).
   (b) *The Jacobian fibration of f' is given by g'.*

(2) *The following conditions are satisfied*:
   (a) *If E'_k is not of type I_n $(n \geq 0)$, then $L = K$.*
   (b) *If $p > 0$ and E'_k is isomorphic to a supersingular elliptic curve, then $p \nmid [L : K]$.*

Proof. Propositions 6.11 and 6.15 show that Condition (1) implies Condition (2). Let us show the converse. We may assume that E'_k is of type I_n ($n \geq 0$). Put $d := [L : K]$ and $G_{L/K} := \mathrm{Gal}(L/K)$. First, we consider the case $n > 0$. By assumption, there exists an element $\gamma_M \in \mathrm{Hom}(G_{L/K}, \mathbb{Q}/\mathbb{Z}) \subset \mathrm{Hom}(G_K, \mathbb{Q}/\mathbb{Z})$ of order d. Since ϕ_M is surjective, the case $n > 0$ follows from Proposition 6.11(M). Next, we consider the case $n = 0$. By assumption, there exists an element $\gamma_E \in \mathrm{Hom}(G_{L/K}, E(k)) \subset \mathrm{Hom}(G_K, E(k))$ of order d. Since ϕ_E is surjective, the case $n = 0$ follows from Proposition 6.11(E). □

Proposition 6.17. *Let (Y, C, h) be a relatively minimal elliptic fibration with special fiber of type I_n ($n \geq 0$). Let $\xi : Y \to X$ be a finite étale surjective C-morphism of degree d. We regard the restriction $\xi_K : Y_K \to X_K$ of ξ to the generic fibers as a homomorphism between elliptic curves, which is determined by the choice of an element of $Y(K)$. By G we denote the subgroup of $Y(K)$ consisting of all d-torsion elements. Put $H := \xi_K(G)$. By $\overline{G}$ and $\overline{H}$ we denote the sets of the closures of all elements of G and H in Y and X, respectively. Assume that $p \nmid d$. Then:*

(1) *$d \mid n$, $\#G = d^2$, and $\#H = d$.*

(2) *All elements of $\overline{G}$ are disjoint.*

(3) *All elements of $\overline{H}$ are disjoint.*

(4) *There exists an irreducible component of the special fiber of f that intersects with all elements of $\overline{H}$, and any other irreducible component of the special fiber of f is disjoint from all elements of $\overline{H}$.*

Proof. Since ξ is étale, the relation $d \mid n$ holds and the special fiber of f is of type I_l, where we set $l := n/d$. We may regard the smooth loci $\widehat{X}$ and $\widehat{Y}$ of f and h as the Néron models of X_K and Y_K, respectively. By the Néron mapping property, the homomorphism ξ_K induces the unique C-homomorphism $\widehat{\xi} : \widehat{Y} \to \widehat{X}$, which is the restriction of ξ to the smooth loci $\widehat{X}$ and $\widehat{Y}$. The restriction $\widehat{\xi}_k$ of $\widehat{\xi}$ to the special fibers of $\widehat{X}$ and $\widehat{Y}$ is a finite étale surjective k-homomorphism of degree d. If $n = 0$, then $\widehat{\xi}_k$ is a homomorphism between elliptic curves. If $n > 0$, then $\widehat{\xi}_k$ is the homomorphism

$$\mathbb{G}_{m,k} \times (\mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathbb{G}_{m,k} \times (\mathbb{Z}/l\mathbb{Z}), \quad (z, e \bmod n) \longmapsto (z, e \bmod l).$$

Thus, the closure of G in Y is finite and étale over C [Bosch et al. 1990, 7.3.2], which concludes the proof. □

6C. ***Elliptic surfaces with prescribed orbifolds.*** In this subsection, we use the following notation. Let k be an algebraically closed field of characteristic $p \geq 0$ and C a connected proper smooth k-curve with function field K. An elliptic fibration (X, C, f) is said to be *trivial* if there exists an elliptic curve F over k such that X is C-isomorphic to the C-scheme $F \times_k C$. Recall the following result on the global-to-local map:

Proposition 6.18 [Cossec and Dolgachev 1989, 5.4.6]. *Let E_K be an elliptic curve over K. Take the minimal regular C-model (E, C, g) of E_K. Assume that the elliptic fibration (E, C, g) is nontrivial. For each closed point s on C, we put $K_s := \operatorname{Frac} \mathbb{O}^{\mathrm{sh}}_{C,s}$ and $E_{K_s} := E_K \times_K K_s$. Then the global-to-local map $H^1(K, E_K) \to \bigoplus_{s \in C} H^1(K_s, E_{K_s})$ is surjective.*

Theorem 6.19. *Let (C, B) be a connected cyclic orbifold k-curve and (E, C, g) a nontrivial relatively minimal elliptic fibration with section. We use the notation B_s/K_s introduced in* Definition 3.6. *Then the following two conditions are equivalent*:

(1) *There exists a relatively minimal elliptic fibration (X, C, f) satisfying the following conditions*:
   (a) *The orbifold associated to f is isomorphic to* (C, B) (Definition 4.23).
   (b) *The Jacobian fibration of f is given by g.*
(2) *The following conditions are satisfied for any closed point s on C*:
   (a) *If $g^{-1}(s)$ is not of type* I_n $(n \geq 0)$, *then $B_s = K_s$.*
   (b) *If $p > 0$ and $g^{-1}(s)$ is isomorphic to a supersingular elliptic curve*, *then $p \nmid [B_s : K_s]$.*

Proof. The theorem follows from Propositions 6.16 and 6.18. □

Proposition 6.20. *Let (X, C, f) be a relatively minimal elliptic fibration with Jacobian fibration (E, C, g). Then the following conditions are equivalent*:

(1) $\chi(\mathbb{O}_X) \leq 0$.
(2) $\chi(\mathbb{O}_X) = 0$.
(3) *The reduction of any closed fiber of f is isomorphic to an elliptic curve.*
(4) *g is smooth.*

Proof. The equivalence of (1) and (2) follows from Proposition 2 in [Mitsui 2014]. The equivalence of (2) and (3) follows from Proposition 2 in [Mitsui 2013]. The equivalence of (3) and (4) follows from Proposition 6.4. □

Corollary 6.21. *Let (C, B) be a connected cyclic orbifold k-curve. Then there exists a relatively minimal elliptic fibration (X, C, f) with* $\chi(\mathbb{O}_X) > 0$ *such that the orbifold associated to f is isomorphic to* (C, B) (Definition 4.23).

Proof. Take a relatively minimal elliptic fibration (E, C, g) satisfying the following conditions: (1) g is not smooth; (2) g admits a section; (3) for any $s \in \operatorname{Supp} B$, the closed fiber $g^{-1}(s)$ is isomorphic to an ordinary elliptic curve. Since the elliptic fibration (E, C, g) is nontrivial, Theorem 6.19 gives a relatively minimal elliptic fibration (X, C, f) such that the orbifold associated to f is isomorphic to (C, B). Proposition 6.20 shows that the inequality $\chi(\mathbb{O}_X) > 0$ holds. □

6D. ***Fundamental groups of elliptic surfaces.*** In this subsection, we use the following notation. Let k be an algebraically closed field of characteristic $p \geq 0$, C a connected proper smooth k-curve with function field K, and (X, C, f) an elliptic fibration. We denote the intersection number of divisors D_1 and D_2 on X by $D_1 \cdot D_2$.

Lemma 6.22. *Assume that two sections D_1 and D_2 of f satisfy the following*:

(1) $\mathcal{O}_X(D_1 - D_2)|_{X_K}$ *is torsion in* $\mathrm{Pic}(X_K)$, *where X_K is the generic fiber of f.*

(2) *For any closed point x on C, there exists an irreducible component of the fiber $f^{-1}(x)$ that intersects with both of D_1 and D_2, and any other irreducible component of the fiber $f^{-1}(x)$ is disjoint from both of D_1 and D_2.*

Then the equality $D_1 \cdot D_2 = -\chi(\mathcal{O}_X)$ holds.

Proof. First, we assume that $D_1 = D_2$. Put $D := D_1 = D_2$ and $\mathcal{F} := \mathcal{O}_X(D)/\mathcal{O}_X$. Since the genus of X_K is equal to one and the effective divisor $D|_{X_K}$ on X_K is of degree one, the long exact sequence induced by the functor f_* and the exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{F} \longrightarrow 0$$

gives an isomorphism $f_*\mathcal{F} \to R^1 f_*\mathcal{O}_X$. In particular, the equalities

$$D \cdot D = \deg \mathcal{O}_X(D)|_D = \deg f_*\mathcal{F} = \deg R^1 f_*\mathcal{O}_X$$

hold. The Riemann–Roch theorem for the line bundle $R^1 f_*\mathcal{O}_X$ on C and the Leray spectral sequence for f give the equalities

$$\deg R^1 f_*\mathcal{O}_X = \chi(R^1 f_*\mathcal{O}_X) - \chi(\mathcal{O}_C) = -\chi(\mathcal{O}_X).$$

Thus, the equality $D \cdot D = -\chi(\mathcal{O}_X)$ holds.

Next, we consider the general case. By n we denote the order of $\mathcal{O}_X(D_1 - D_2)|_{X_K}$ in $\mathrm{Pic}(X_K)$ (Condition (1)). Then $n(D_1 - D_2)$ is linearly equivalent to a vertical divisor V. Condition (2) gives the equality $D_0 \cdot (D_1 - D_2) = 0$ for any vertical prime divisor D_0. Thus, the equality $V \cdot V = 0$ holds, which gives the equality $(D_1 - D_2) \cdot (D_1 - D_2) = 0$. Therefore, the first case shows the general case. □

Theorem 6.23. *Choose a smooth closed fiber $i : X_0 \to X$ of f. Take a geometric point $\bar{x}_0$ on X_0. Put $\bar{x} := i(\bar{x}_0)$ and $\bar{s} := f(\bar{x})$. By (C, B) we denote the connected proper orbifold k-curve associated to f* (Definition 4.23). *The morphisms i and f induce canonical homomorphisms $i_* : \pi_1(X_0, \bar{x}_0) \to \pi_1(X, \bar{x})$ and $f_*^{\mathrm{orb}} : \pi_1(X, \bar{x}) \to \pi_1(C, B, \bar{s})$, respectively* (Theorem 1.1). *Then*:

(1) *If $\chi(\mathcal{O}_X) > 0$, then i_* is trivial and f_*^{orb} is an isomorphism.*

(2) *If $\chi(\mathcal{O}_X) = 0$, then i_* is injective.*

Remark 6.24. In the complex analytic case, Statement (2) does not hold in general for topological fundamental groups although Statement (1) holds and i_* is nontrivial whenever $\chi(\mathbb{O}_X) = 0$ [Friedman and Morgan 1994, 2.2.1 and 2.7.2]. For example, if X is a Hopf surface, then $\chi(\mathbb{O}_X) = 0$, $\operatorname{Ker} i_* \cong \mathbb{Z}$, $\operatorname{Coker} i_* = 0$, and $\pi_1(X) \cong \operatorname{Im} i_* \cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ for some positive integer n.

Proof. By (E, C, g) we denote the Jacobian fibration of f. First, let us show Statement (1). We have only to show the following: for any connected étale covering space $\xi : Y \to X$, any connected component of $\xi^{-1}(X_0)$ is X_0-isomorphic to X_0. Assume that the above statement does not hold. Choose ξ that does not satisfy the above statement. Take the Stein factorization

$$Y \xrightarrow{h} C' \xrightarrow{u} C$$

of $f \circ \xi : Y \to X \to C$. Take the elliptic fibration (X', C', f') and the étale morphisms $\xi' : Y \to X'$ and $u' : X' \to X$ given by Lemma 6.3. By assumption, the morphism ξ' is not an isomorphism. Replacing ξ by ξ', we may assume that $(Y, C, f \circ \xi)$ is an elliptic fibration.

Since ξ is étale, any closed fiber is of type ${}_m\mathrm{I}_n$ $(n \geq 0)$. Since $\chi(\mathbb{O}_X) > 0$, Proposition 6.20 shows that f admits a closed fiber of type ${}_m\mathrm{I}_n$ $(n > 0)$. In particular, the j-invariant of g is nonconstant (Proposition 6.4). By d we denote the degree of ξ. If $p > 0$, then $p \nmid d$ since f admits a closed fiber that is isomorphic to a supersingular elliptic curve (Corollary 6.7) and ξ is étale. Choose a connected proper smooth k-curve C' and a finite morphism $u : C' \to C$ satisfying the following condition: the morphism u induces an extension of the function fields K'/K; by J_K we denote the Jacobian of the generic fiber of Y/C; then $Y(K') \neq \varnothing$ and $J_K(K')$ contains d^2 d-torsion elements. Take a desingularization X' of $X \times_C C'$ and the canonical projection $u' : X' \to X$. Since ξ' is étale, the base change $\xi' : Y' \to X'$ of ξ via u' is étale. Thus, we obtain an étale C'-morphism $\xi'' : Y'' \to X''$ between the minimal regular C'-models of Y' and X' after successive blowing-downs of (-1)-curves on Y' and X'. Replacing ξ by ξ'', we may assume that ξ is a morphism between Jacobian fibrations and $Y(K)$ contains d^2 d-torsion elements.

By H we denote the image of the d-torsion elements of $Y(K)$ under ξ. By $\overline{H}$ we denote the set of the closures of all elements of H in X. Proposition 6.17 shows the following: (a) $\#H = d > 1$; (b) all elements of $\overline{H}$ are disjoint; (c) for any closed point x on C, there exists an irreducible component of the fiber $f^{-1}(x)$ that intersects with all elements of $\overline{H}$, and any other irreducible component of the fiber $f^{-1}(x)$ is disjoint from all elements of $\overline{H}$. This contradicts Lemma 6.22 since $\chi(\mathbb{O}_X) > 0$. Therefore the homomorphism i_* is trivial.

Next, let us show Statement (2). We have only to show that, for any connected étale covering space $\tau : X_0' \to X_0$, there exists an étale covering $\xi : Y \to X$ such that any connected component of the preimage $\xi^{-1}(X_0)$ is X_0-isomorphic to X_0'. Since $\chi(\mathbb{O}_X) = 0$, Proposition 6.20 shows that g is smooth. Take an integer $n \geq 3$ so that $p \nmid n$. Since the n-torsion C-subgroup scheme of E/C is finite and étale, we may take a connected étale covering space $u : C' \to C$ satisfying the following condition: let $u' : E' \to E$ denote the base change of u via g; then $E'(C')$ contains n^2 n-torsion elements. Since g is smooth, the j-invariant of E is contained in k. Thus, the elliptic curve E' over C' induces a constant morphism from C' to the moduli scheme of elliptic curves with level n. Therefore, we obtain a C'-isomorphism $E' \cong X_0 \times_k C'$. Take the base change $\tau' : E'' \to E'$ of τ via the structure morphism $C' \to \operatorname{Spec} k$. Take the C-morphism $h : X \to E$ given by Lemma 6.6. Then the base change of $u' \circ \tau'$ via h is the desired morphism ξ. □

Lemma 6.25. *Let R be a strictly Henselian excellent discrete valuation ring of equicharacteristic. By $\widehat{R}$ we denote the completion of R with respect to the maximal ideal. Put $K := \operatorname{Frac} R$ and $\widehat{K} := \operatorname{Frac} \widehat{R}$. Let $\widehat{L}/\widehat{K}$ be a finite Galois extension. Then there exists a unique extension L/K in $\widehat{L}$ such that $\widehat{L} = \widehat{K}L$. Furthermore, the extension L/K is Galois, of degree $[\widehat{L} : \widehat{K}]$, and linearly disjoint from $\widehat{K}/K$.*

Proof. Since R is algebraically closed in $\widehat{R}$ by the approximation property [Bosch et al. 1990, 3.6.9], we have only to show the existence of L. We denote the characteristic of R by l. Put $d := [\widehat{L} : \widehat{K}]$. By assumption, the extension $\widehat{L}/\widehat{K}$ is solvable. Thus, by induction on d, we may assume that $l \nmid d$ or $d = l > 0$. The case $l \nmid d$ follows from Kummer theory since R contains a primitive d-th root of unity. The case $d = l > 0$ follows from Lemma 5.9(2) and Artin–Schreier theory. □

Finally, we give a proof of the criterion for simple-connectedness of elliptic surfaces:

Proof of Theorem 1.2. We use the notation introduced in Theorem 6.23. Theorems 1.1 and 6.23 show that $\pi_1(X)$ is trivial if and only if $\pi_1(C, B)$ is trivial and $\chi(\mathbb{O}_X) > 0$. Proposition 6.11 shows that the orbifold (C, B) is cyclic. Thus, Theorem 1.3 shows that $\pi_1(C, B)$ is trivial if and only if $C \cong \mathbb{P}^1_k$, $B^t = B$, $\# \operatorname{Supp} B \leq 2$, and $\gcd(n_s, n_t) = 1$ for $s \neq t$, where we put $n_s := [B_s : K_s]$ for each $s \in \operatorname{Supp} B$. Lemma 6.25 and Propositions 4.6, 6.11, and 6.15 imply that the above conditions on (C, B) are equivalent to Conditions (2)–(6).

Let us show that each of Conditions (1)–(6) is necessary. We remark that $\chi(\mathbb{O}_X) > 0$ if and only if the Jacobian fibration of f is not smooth (Proposition 6.20). The necessity of Conditions (1) and (2) is clear. The necessity of Conditions (3) and (4) follows from Corollary 6.21. The necessity of Conditions (5) and (6) follows from Proposition 6.15 and Theorem 6.19. □

Appendix A: Triangle groups and projective special linear groups

The result of this section is used in the proof of Proposition 5.4. Let a, b, and c be integers greater than 1. We define the *triangle group* $\Delta(a,b,c)$ by

$$\Delta(a,b,c) := \langle x, y, z \mid x^a = y^b = z^c = xyz = \mathrm{id}\rangle.$$

Let p be a prime number and q a power of p. In this section, we study homomorphisms $\Delta(a,b,c) \to \mathrm{SL}(2,\mathbb{F}_q)$ and $\Delta(a,b,c) \to \mathrm{PSL}(2,\mathbb{F}_q)$ that preserve the orders of x, y, and z. Take an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$. For each positive integer n prime to p, we take a primitive n-th root of unity ζ_n in $\overline{\mathbb{F}_q}$. Put $\mu_n := \zeta_n + \zeta_n^{-1}$. The proofs of Lemmas A.1–A.4 are straightforward.

Lemma A.1. *$\mu_n \in \mathbb{F}_q$ if and only if $n \mid (q^2-1)$.*

Lemma A.2. *We have equalities $\#\mathrm{SL}(2,\mathbb{F}_q) = q(q^2-1)$ and $\#\mathrm{PSL}(2,\mathbb{F}_q) = q(q^2-1)/\gcd(2,p-1)$. The projection $\mathrm{SL}(2,\mathbb{F}_q) \to \mathrm{PSL}(2,\mathbb{F}_q)$ maps any element of order n to an element of order $n/\gcd(2,p-1,n)$.*

Lemma A.3. *Take $X \in \mathrm{SL}(2,\mathbb{F}_q)$. Then the image of X in $\mathrm{SL}(2,\overline{\mathbb{F}_q})$ is conjugate to*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},$$

where $\alpha \in \overline{\mathbb{F}_q}^{\times}$. By n we denote the order of α in $\overline{\mathbb{F}_q}^{\times}$. If $p \neq 2$, then the order of X is equal to p, $2p$, or n, respectively. Otherwise, the order of X is equal to 2, 2, or n, respectively. The order of the image of X in $\mathrm{PSL}(2,\mathbb{F}_q)$ is equal to p, p, or $n/\gcd(2,p-1,n)$, respectively.

Lemma A.4. *Take $X \in \mathrm{SL}(2,\mathbb{F}_q)$. Let n be an integer prime to p. Assume that $n > 2$ and $\mathrm{tr}\, X = \mu_n$. Then the order of X is equal to n.*

Lemma A.5. *Assume that a, b, and c are greater than 2 and divide q^2-1. Then there exist X, Y, and Z in $\mathrm{SL}(2,\mathbb{F}_q)$, of orders a, b, and c, respectively, such that $XYZ = I$, where I is the identity matrix of $\mathrm{SL}(2,\mathbb{F}_q)$.*

Proof. Lemma A.1 shows that μ_a, μ_b, and μ_c are contained in $\mathbb{F}_q$. Put

$$X := \begin{pmatrix} 0 & -1 \\ 1 & \mu_a \end{pmatrix} \in \mathrm{SL}(2,\mathbb{F}_q).$$

We have only to construct Y and Z in $\mathrm{SL}(2,\mathbb{F}_q)$ so that $\mathrm{tr}\, Y = \mu_b$, $\mathrm{tr}\, Z = \mu_c$, and $XYZ = 1$ (Lemma A.4). We write

$$Y = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then we have only to choose α, β, γ, and δ in $\mathbb{F}_q$ so that $\alpha\delta-\beta\gamma=1$, $\alpha+\delta=\mu_b$, and $\beta-\gamma+\mu_a\delta=\mu_c$. Thus, we have only to show that there exists a solution $(\alpha,\beta)\in\mathbb{F}_q\times\mathbb{F}_q$ of the equation $F(\alpha,\beta)=0$, where we put

$$F(u,v):=u^2-\mu_a uv+v^2-\mu_b u+(\mu_a\mu_b-\mu_c)v+1.$$

If elements α_0, α_1, β_0, and β_1 in $\mathbb{F}_q$ satisfy $\alpha=\alpha_0+\alpha_1$ and $\beta=\beta_0+\beta_1$, then the equality

$$F(\alpha,\beta)=F(\alpha_0,\beta_0)+\frac{\partial F}{\partial u}(\alpha_0,\beta_0)\alpha_1+\frac{\partial F}{\partial v}(\alpha_0,\beta_0)\beta_1+\alpha_1^2-\mu_a\alpha_1\beta_1+\beta_1^2$$

holds. Note that the equalities

$$\begin{cases}(\partial F/\partial u)(u,v)=2u-\mu_a v-\mu_b,\\(\partial F/\partial v)(u,v)=-\mu_a u+2v+\mu_a\mu_b-\mu_c\end{cases}$$

hold. Since $a>2$ and $\gcd(p,a)=1$, the inequality $\mu_a^2\neq 4$ holds. Thus, we may take $(\alpha_0,\beta_0)\in\mathbb{F}_q\times\mathbb{F}_q$ so that $(\partial F/\partial u)(\alpha_0,\beta_0)=(\partial F/\partial v)(\alpha_0,\beta_0)=0$. Therefore, we have only to show that there exists a solution $(\alpha_1,\beta_1)\in\mathbb{F}_q\times\mathbb{F}_q$ of the equation $G(\alpha_1,\beta_1)=-F(\alpha_0,\beta_0)$, where we put $G(u,v):=u^2-\mu_a uv+v^2$. If $p\neq 2$ (resp. $p=2$), then the quadratic form $G(u,v)$ is nondegenerate (resp. nondefective), which concludes the proof. □

Lemma A.6. *Assume that the following conditions are satisfied*:

(1) *If $p\neq 2$, then $a=p$ or $2p$.*

(2) *If $p=2$, then $a=2$.*

(3) *b and c are greater than 2 and divide q^2-1.*

Then there exist X, Y, and Z in $\mathrm{SL}(2,\mathbb{F}_q)$ of orders a, b, and c, respectively, such that $XYZ=I$, where I is the identity matrix of $\mathrm{SL}(2,\mathbb{F}_q)$.

Proof. Lemma A.1 shows that μ_b and μ_c are contained in $\mathbb{F}_q$. First, we consider the case $a=p$. We define X, Y, and Z in $\mathrm{SL}(2,\mathbb{F}_q)$ by

$$X:=\begin{pmatrix}1&1\\0&1\end{pmatrix},\quad Y:=\begin{pmatrix}\zeta_b&0\\\mu_c-\mu_b&\zeta_b^{-1}\end{pmatrix},\quad\text{and}\quad Z:=\begin{pmatrix}\zeta_b^{-1}&-\zeta_b^{-1}\\\mu_b-\mu_c&\mu_c-\zeta_b^{-1}\end{pmatrix}.$$

Then the order of X is equal to p. Moreover, the equalities $\operatorname{tr}Y=\mu_b$, $\operatorname{tr}Z=\mu_c$, and $XYZ=I$ hold. Thus, the elements X, Y, and Z satisfy the desired conditions (Lemma A.4). Next, we consider the case $p\neq 2$ and $a=2p$. In that case, we have only to replace X, Y, and Z by

$$X:=\begin{pmatrix}-1&1\\0&-1\end{pmatrix},\quad Y:=\begin{pmatrix}\zeta_b&0\\\mu_b+\mu_c&\zeta_b^{-1}\end{pmatrix},\quad\text{and}\quad Z:=\begin{pmatrix}-\zeta_b^{-1}&-\zeta_b^{-1}\\\mu_b+\mu_c&\zeta_b^{-1}+\mu_c\end{pmatrix}.\ \square$$

Proposition A.7. *Assume that a, b, and c are pairwise coprime. Take the elements x, y, and z of $\Delta(a,b,c)$ in the definition of $\Delta(a,b,c)$. Put $G := \mathrm{SL}(2,\mathbb{F}_q)$ or $\mathrm{PSL}(2,\mathbb{F}_q)$. Then there exists a homomorphism $\phi : \Delta(a,b,c) \to G$ such that the orders of $\phi(x)$, $\phi(y)$, and $\phi(z)$ are equal to a, b, and c, respectively, if and only if the following three conditions are satisfied*:

(1) $abc \mid \#G$.

(2) *If $G = \mathrm{SL}(2,\mathbb{F}_q)$ and one of a, b, and c is equal to* 2, *then* $p = 2$.

(3) *If an integer u is equal to a, b, or c and is divisible by p, then u satisfies one of the following conditions*: (a) $u = p$: (b) $G = \mathrm{SL}(2,\mathbb{F}_q)$, $p \neq 2$, *and* $u = 2p$.

Proof. First, let us show the "only if" part. Since a, b, and c are pairwise coprime, the condition on ϕ implies Condition (1). Lemma A.3 and the condition on ϕ imply Condition (3). Assume that $G = \mathrm{SL}(2,\mathbb{F}_q)$, $a = 2$, and $p \neq 2$. Then $\phi(x) = -I$, where I is the identity matrix of $\mathrm{SL}(2,\mathbb{F}_q)$. Thus, the equality $\phi(y) = -\phi(z)$ holds, which contradicts the assumption that b is prime to c. Therefore, Condition (2) holds. The "if" part follows from Lemmas A.2, A.5, and A.6. □

Lemma A.8. *Assume that a, b, and c are pairwise coprime. Let G be a nontrivial finite group. If there exists a surjective homomorphism $\Delta(a,b,c) \to G$, then G is nonsolvable.*

Proof. Assume that G is solvable. Then there exists a nontrivial cyclic group H and a surjective homomorphism $\phi : \Delta(a,b,c) \to H$. We may write $H = \mathbb{Z}/n\mathbb{Z}$, where n is an integer greater than 1. Take the elements x, y, and z of $\Delta(a,b,c)$ in the definition of $\Delta(a,b,c)$. The orders of x, y, and z are equal to a, b, and c, respectively. By $\bar{a}$, $\bar{b}$, and $\bar{c}$ we denote the orders of $\phi(x)$, $\phi(y)$, and $\phi(z)$, respectively. Then $\bar{a} \mid a$, $\bar{b} \mid b$, $\bar{c} \mid c$ and $\overline{abc} \neq 1$. In particular, the integers $\bar{a}$, $\bar{b}$, and $\bar{c}$ are pairwise coprime. Since $xyz = \mathrm{id}$, the equality $\phi(x) + \phi(y) + \phi(z) = 0$ holds, which contradicts the facts that $\bar{a}$, $\bar{b}$, and $\bar{c}$ are pairwise coprime and $\overline{abc} \neq 1$. Therefore, the group G is nonsolvable. □

Theorem A.9. *Assume that a, b, and c are pairwise coprime. If $\{a,b,c\} = \{2,3,5\}$, then we suppose that $p = 5$. Otherwise, we suppose that $2abc \mid (p^2-1)$. Take the elements x, y, and z of $\Delta(a,b,c)$ in the definition of $\Delta(a,b,c)$. Then there exists a surjective homomorphism $\phi : \Delta(a,b,c) \to \mathrm{PSL}(2,\mathbb{F}_p)$ such that the orders of $\phi(x)$, $\phi(y)$, and $\phi(z)$ are equal to a, b, and c, respectively. Furthermore, there exists a prime number p such that $2abc \mid (p^2-1)$.*

Remark A.10. If $q > 3$, then the group $\mathrm{PSL}(2,\mathbb{F}_q)$ is noncommutative and simple.

Proof. Let us show the first statement. By Lemma A.2, we may take ϕ in Proposition A.7. We have only to show that ϕ is surjective. Lemma A.8 shows that the image of ϕ is nonsolvable. Thus, the case $\{a, b, c\} = \{2, 3, 5\}$ follows from the fact that any proper subgroup of $\mathrm{PSL}(2, \mathbb{F}_5)(\cong A_5)$ is solvable. Assume that $\{a, b, c\} \neq \{2, 3, 5\}$. By the classification of the subgroups of $\mathrm{PSL}(2, \mathbb{F}_p)$ [Dickson 1958, p. 285, XII, 260], any nonsolvable subgroup of $\mathrm{PSL}(2, \mathbb{F}_p)$ is isomorphic to $\mathrm{PSL}(2, \mathbb{F}_5)$ or $\mathrm{PSL}(2, \mathbb{F}_p)$. Suppose that ϕ is not surjective. Then the image of ϕ is isomorphic to $\mathrm{PSL}(2, \mathbb{F}_5)$. Since the order of any nontrivial element in $\mathrm{PSL}(2, \mathbb{F}_5)(\cong A_5)$ is equal to 2, 3 or 5, the equality $\{a, b, c\} = \{2, 3, 5\}$ holds, which contradicts the assumption. Thus, the homomorphism ϕ is surjective. The last statement follows from Dirichlet's theorem on arithmetic progressions. $\square$

Appendix B: Comparison between orbifolds and stacks

Let k be an algebraically closed field of characteristic $p \geq 0$. For a nonnegative integer n, a DM stack $\mathcal{S}$ is said to be *of dimension n* if an atlas of $\mathcal{S}$ is of dimension n. A *stack orbifold* $\mathcal{S}$ is a locally Noetherian normal DM stack that admits an open dense substack that is isomorphic to a scheme. A *stack orbifold k-curve* $\mathcal{S}$ is a separated smooth k-stack of dimension one that is a stack orbifold. In the following, we see that the notion of an orbifold k-curve coincides with the notion of a stack orbifold k-curve.

We construct a stack orbifold k-curve from an orbifold k-curve. Let (S, B) be a connected orbifold k-curve. Put $S_0 := S \setminus \mathrm{Supp}\, B$. Take $s \in S$. Choose an affine open subset U containing s. By $(U, B|_U)$ we denote the orbifold obtained by restricting (S, B) to U. We may take a Galois orbifold trivialization $U' \to (U, B|_U)$ of $(U, B|_U)$ (Theorem 1.3). We may regard the open subscheme $U \cap S_0$ of U as an open substack of the quotient stack $[U'/G]$. Pasting $[U'/G]$ for all $s \in S$, we obtain a stack orbifold k-curve with coarse moduli space S.

We construct an orbifold k-curve from a stack orbifold k-curve. Let $\mathcal{S}$ be a connected stack orbifold k-curve. Take an open dense substack $\mathcal{S}_0$ of $\mathcal{S}$ that is isomorphic to a scheme S_0. Take the coarse moduli space $\lambda : \mathcal{S} \to S$ of $\mathcal{S}$ [Rydh 2013]. We may regard S_0 as an open subscheme of S. Thus, the equality $\deg \lambda = 1$ holds [Vistoli 1989, 1.15]. Therefore, the scheme S is of finite type over k. Since $\mathcal{S}$ is connected and normal, the scheme S is connected and normal. Since k is perfect, the scheme S is a connected smooth k-curve.

By $P(S)$ we denote the set of all closed points on S. Take $s \in P(S)$. By $\mathrm{Aut}(s)$ we denote the automorphism group of s. If $s \in S_0$, then $\mathrm{Aut}(s)$ is trivial. By S_s and $\mathcal{S}_s$ we denote the schemes obtained by the strict Henselizations of S and $\mathcal{S}$ at s, respectively [Laumon and Moret-Bailly 2000, 6.2.1]. Take the quotient stack $\phi_s : \mathcal{S}_s \to T_s := [\mathcal{S}_s / \mathrm{Aut}(s)]$ and the canonical morphism $\mu_s : T_s \to \mathcal{S}$ [Laumon and Moret-Bailly 2000, 6.2.1]. By B_s, L_s, and K_s, we denote the fields of rational functions of $\mathcal{S}_s$, T_s, and S_s, respectively [Vistoli 1989, 1.14]. Since the composite

$\lambda \circ \mu_s : T_s \to S$ induces an isomorphism $K_s \to L_s$, the composite $\lambda \circ \mu_s \circ \phi_s : \mathcal{S}_s \to S$ induces a finite Galois extension B_s/K_s with Galois group $\mathrm{Aut}(s)$.

Take a separable closure $\overline{K_s}$ of K_s. We embed B_s in $\overline{K_s}$ over K_s. Since B_s/K_s is Galois, the image does not depend on the choice of the embedding. We define a map B on $P(S)$ by $s \mapsto B_s/K_s$ (Definition 3.6). Then Supp B is locally finite since $\mathcal{S}_0$ is open dense in $\mathcal{S}$. Thus, the pair (S, B) is a connected orbifold k-curve.

Theorem B.1. *The above two correspondences give an equivalence between the category of orbifold k-curves and orbifold (étale) k-morphisms and the category of stack orbifold k-curves and (étale) k-morphisms that induce qsc morphisms between coarse moduli spaces. In particular, the fundamental group of any orbifold k-curve coincides with the fundamental group of the corresponding stack orbifold k-curve in* [Noohi 2004, §4].

Remark B.2. The theorem does not hold for general orbifolds. There exists an orbifold étale covering space of a trivial orbifold (S, B) that is not an étale covering space of the scheme S (Examples 3.24 and 4.16).

Theorem B.1 follows from the local structure theorem on DM stacks [Laumon and Moret-Bailly 2000, 6.2]. The detail of the proof of the theorem is left to the readers.

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Factorially closed subrings of commutative rings

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We prove some new results about factorially closed subrings of commutative rings. We generalize this notion to quasifactorially closed subrings of commutative rings and prove some results about them from algebraic and geometric viewpoints. We show that quasifactorially closed subrings of polynomial and power series rings of dimension at most three are again polynomial (resp. power series) rings in a smaller number of variables. As an application of our results, we give a short proof of a result of Lê Dũng Tráng in connection with the Jacobian problem.

Introduction

We assume throughout the article that the base field k is an algebraically closed field of characteristic 0. Whenever we use topological arguments, k is tacitly assumed to be the field of complex numbers $\mathbb{C}$. By assuming naturally that k is embedded into $\mathbb{C}$, we can see that the results proved over $\mathbb{C}$ can be proved over k. For an integral domain S, the field of fractions of S is denoted by $Q(S)$, and the multiplicative group of units by S^*.

The present article grew out of the discussions we had during the workshop *Automorphisms of affine varieties*, held at the Kerala School of Mathematics, India (February 17–22, 2014). In particular, a part of our discussion was inspired by a talk given by Neena Gupta [2014] and a question asked by A. Kanel-Belov.

Let $A \subseteq B$ be integral domains. Then A is said to be *factorially closed*, or *fc*, in B if for any two nonzero elements $b_1, b_2 \in B$, $b_1 b_2 \in A$ implies that $b_1, b_2 \in A$. In some papers an fc subring is also called an *inert* subring. Factorially closed subrings appear naturally as the rings of invariants of the action of the additive group G_a, or a connected semisimple group on a polynomial ring.

The notion of fc subring is not well-behaved in the case of local rings due to the existence of too many units. Hence we have introduced a weaker notion: *quasifactorially closed* subrings. For any integral domains $A \subseteq B$, A is said to be quasifactorially closed, or qfc, in B if, for any nonzero $b \in B$, if there exists

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some nonzero $b' \in B$ such that $bb' \in A$, then there exists a unit $u \in B$ such that $bu \in A$. It turns out that quasifactorial closedness is more geometric and has several interesting applications. For example, we have proved that a qfc subring of a power series ring in at most three variables is again isomorphic to a power series ring in a smaller number of variables.

The fc property is also related to the property of the existence of nonconstant invertible regular functions on general fibers of the corresponding morphism of schemes.

We now mention the main results proved in this paper (with some hypothesis):

(1) An inclusion of graded domains $A \subseteq B$ is fc if and only if it is graded fc. Further, A is fc in B if and only if the localization of A at its irrelevant maximal ideal is fc in the corresponding localization of B (Theorems 2 and 3).

(2) For an inclusion of affine normal domains $A \subseteq B$ the fc locus is always open (Corollary 4.1).

(3) For an inclusion of affine UFDs $A \subseteq B$ the qfc locus is open if at most finitely many prime elements of A split in B (Theorem 5). (An example in Section 3 shows that the reverse implication is false.)

(4) If an inclusion of complete local normal domains $A \subseteq B$ over k is qfc then A is algebraically closed in B. Further, any irreducible element of A is irreducible in B (Theorem 6 and its corollaries).

(5) An fc subring of a polynomial ring in at most three variables is again a polynomial ring (Theorem 1). Similarly, a complete qfc subring of a power series ring in at most three variables is again a power series ring (Theorem 8).

(6) If an inclusion of affine normal domains $A \subseteq B$ (with a suitable hypothesis) is fc, then a general fiber of the corresponding morphism of affine varieties does not have any nonconstant invertible regular functions (Theorem 11).

Using this we give a new short proof of a result proved by many authors (M. Razar, R. Heitmann, S. Friedland, L. D. Tráng, C. Weber, W. Neumann, P. Norbury) in connection with the Jacobian problem [Neumann and Norbury 1998; Tráng 2008].

In Section 3 we give some examples of ring extensions which shed more light on the fc (and qfc) property.

In Section 4 we have listed some open problems about fc and qfc extensions.

1. Factorially closed subrings

We start with some basic properties of factorial closedness. Some easy proofs have been omitted.

Lemma 1 (local properties of factorial closedness). *The following statements are equivalent for an inclusion of integral domains $A \subseteq B$*:

(1) *The ring A is fc in B.*

(2) *For any multiplicatively closed set S in A, $S^{-1}A \subseteq S^{-1}B$ is fc.*

(3) *For any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is fc.*

(4) *For any maximal ideal $\mathfrak{m} \in \operatorname{Max} A$, $A_{\mathfrak{m}} \subseteq B_{\mathfrak{m}}$ is fc.*

(5) *There exist finitely many nonzero elements $a_1, a_2, \ldots, a_n \in A$, generating the unit ideal, such that A_{a_i} is fc in B_{a_i} for each $i = 1, 2, \ldots, n$.*

Moreover, if A is normal, the above statements are equivalent to the following one:

(7) *For each prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ of height 1, $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is fc.*

Proof. Omitted. □

Lemma 2 (transitive and sandwich properties of factorial closedness). *Let $A \subseteq B \subseteq C$ be integral domains.*

(1) *If the ring A is fc in B and B is fc in C, then A is fc in C.*

(2) *If A is fc in C then it is fc in B. However, in this case B need not be fc in C.*

The example $k \subseteq k[t^2] \subseteq k[t]$ shows that B need not be fc in C.

Lemma 3. *Let A be an fc subring of B.*

(1) *The ring A is algebraically closed in B.*

(2) *If $Q(A)$ is the field of fractions of A, then $Q(A) \cap B = A$. This is the same thing as saying that each principal ideal of A is a contracted ideal.*

(3) *If B is integrally closed (or a UFD), then so is A. In fact, in the case of Krull domains, the natural homomorphism of divisor class groups $\operatorname{Cl}(A) \to \operatorname{Cl}(B)$ is an injection whenever it is defined.*

(4) *Any unit of B is in A.*

Proof. The first assertion follows from the slightly more general fact that, for a pair of integral domains $A \subseteq B$, if $B \setminus A$ is closed under multiplication then A is algebraically closed in B.

The other three statements follow from the first one and the next observation. □

Remark. For an inclusion of Krull domains $A \subset B$, there is a natural homomorphism $\operatorname{Cl}(A) \to \operatorname{Cl}(B)$ if and only if no height 1 prime ideal of B contracts to a prime ideal of height > 1 in A [Samuel 1964].

If I is an ideal of A such that the ideal IB is principal, then, since A is fc in B, I itself must be principal. This observation will be implicitly used later.

Lemma 4. *Let A be an fc subring of B. Then the Jacobson radical of B, $\operatorname{Jac} B$, is contained in A. Moreover, if $\operatorname{Jac} B \neq 0$ then $A = B$. In particular, if B is semilocal then $A = B$.*

Proof. If $b \in \operatorname{Jac} B$, $1 + b \in B^*$. So, by Lemma 3(4), $1 + b \in A$ implies that $b \in A$. Now, if b is a nonzero element in $\operatorname{Jac} B$, for any $x \in B$, xb is also in $\operatorname{Jac} B$ and consequently in A. So $x \in A$. If B is semilocal, and not a field, then $\operatorname{Jac} B \neq 0$ and the rest of the assertion follows. If B is a field then every nonzero element in B is a unit, and since A is fc in B we again get $A = B$. $\square$

Lemma 5. *If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ and $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ are two sequences of integral domains, such that $A_i \subseteq B_i$ is an fc subring for each i, then $\bigcup_i A_i \subseteq \bigcup_i B_i$ is also factorially closed.*

Lemma 6. *If $A \subseteq B \subseteq C$ are integral domains with A an fc subring of B, then, for any subring D of C, $D \cap A \subseteq D \cap B$ is also factorially closed.*

Before looking into the ring-theoretic properties of factorial closedness, we would like to describe the structure of a factorially closed subalgebra of the polynomial ring $R = k[x_1, \ldots, x_n]$. This question can be answered if $n \leq 3$, and the answer is simply a polynomial subalgebra. We consider only the case where $n = 3$. The case $n = 2$ has a similar answer and is easier.

Theorem 1. *Let A be a factorially closed subring of $R = k[x, y, z]$.*

(1) *If $\dim A = 3$, then $A = R$.*

(2) *If $\dim A = 2$, then A is a polynomial ring in two variables.*

(3) *If $\dim A = 1$, then $A = k[f]$, where $f - c$ is irreducible in $k[x, y, z]$ for every $c \in k$.*

Proof. (1) In this case, the transcendence degree of A over k is 3. So A, being algebraically closed in $k[x, y, z]$ by Lemma 3(1), must be equal to $k[x, y, z]$.

In the other two cases, since A is a UFD (by Lemma 3(3)) of transcendence degree ≤ 2, by a result of Zariski [Nagata 1965, p. 52, Theorem 4] A is affine.

So the assertion (3) follows from the fact that, when A has dimension 1, A is an affine PID with trivial units.

(2) Note that A is a normal affine domain of dimension 2. We assume for simplicity that $k = \mathbb{C}$. Let $Y = \operatorname{Spec} R$ and $X = \operatorname{Spec} A$. The inclusion $A \hookrightarrow R$ defines a dominant morphism $p : Y \to X$. Then every fiber of p is either the empty set or is 1-dimensional. For, if there exists a fiber component D of dimension 2, let it be defined by $f = 0$ with $f \in R$. Since $p(D)$ is a closed point of X corresponding to a maximal ideal $\mathfrak{m}$ of A, we have $\mathfrak{m} \subseteq \mathfrak{m}R \subseteq fR$. This implies that any nonzero

element of $\mathfrak{m}$ is divisible by f, whence $f \in A$. This is a contradiction since A is 2-dimensional. Furthermore, a general fiber of p is irreducible since A is factorially closed in R. By [Miyanishi 1986, Theorem 3], X is isomorphic to either $\mathbb{A}^2$ or an affine hypersurface $x_1^2 + x_2^3 + x_3^5 = 0$ in $\mathbb{A}^3$. But, arguing as in the proof of [Miyanishi 1986, Theorem 4], we can show that the latter case cannot occur. $\square$

Let $B := \bigoplus_i B_i$ be a $\mathbb{Z}$-graded domain and $A := \bigoplus_i A_i$ a graded subring of B, i.e., $A_i \subseteq B_i$ for each i. We say that A is graded factorially closed or gfc, in short, in B if, given any two nonzero homogeneous elements $b_i, b_j \in B$, $b_i b_j \in A$ implies that $b_i, b_j \in A$. First, the following lemma shows that gfc is a local property:

Lemma 7. *Let $A \subseteq B$ be $\mathbb{Z}$-graded domains. Then the following statements are equivalent*:

(1) *The ring A is gfc in B.*

(2) *For any multiplicative set S in A, generated by homogeneous elements, $S^{-1}A \subseteq S^{-1}B$ is gfc.*

(3) *For any homogeneous prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, $A_{(\mathfrak{p})} \subseteq B_{(\mathfrak{p})}$, where $A_{(\mathfrak{p})}$ and $B_{(\mathfrak{p})}$ denote the localizations of A and B respectively at the multiplicative set consisting of all homogeneous elements of A not contained in $\mathfrak{p}$, is gfc.*

If, moreover, A happens to be positively graded, the above statements are equivalent to the following:

(4) *For any homogeneous maximal ideal $\mathfrak{m} \in \operatorname{Max} A$, $A_{(\mathfrak{m})} \subseteq B_{(\mathfrak{m})}$ is graded factorially closed.*

Proof. We only show (3) $\Longrightarrow$ (1). Let $x, y \in B$ be homogeneous elements such that $xy \in A$. So, $x, y \in A_{(\mathfrak{p})}$ for every homogeneous prime ideal $\mathfrak{p}$. But the set $(A : x) := \{a \in A \mid ax \in A\}$ is a homogeneous ideal in A. So, if $x \notin A$, then $(A : x)$ must be proper ideal and hence contained in a homogeneous prime ideal, leading to a contradiction. For positively graded rings, note that any homogeneous ideal is actually contained in a homogeneous maximal ideal. $\square$

Note that properties analogous to the fc property as expressed in Lemmas 1, 2 and 3 also hold for graded factorially closed subrings. The reader is invited to come up with the precise formulations and their proofs.

Next we take our first step in building a bridge between factorial closedness and graded factorial closedness.

Lemma 8. *Let $A \subseteq B$ be $\mathbb{Z}$-graded domains and $\mathfrak{p}$ a homogeneous prime ideal of A. If $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is fc then $A_{(\mathfrak{p})} \subseteq B_{(\mathfrak{p})}$ is gfc.*

Proof. Let $x, y \in B_{(\mathfrak{p})}$ be homogeneous elements such that $xy \in A_{(\mathfrak{p})}$. Let $x = x'/s$ and $y = y'/t$ with $x', y' \in B$ and $s, t \in \bigcup_i A_i - \mathfrak{p}$. Since $A_\mathfrak{p} \subseteq B_\mathfrak{p}$ is factorially closed, $x, y \in A_\mathfrak{p}$. So there exist nonzero elements in A, $a = \sum_j a_j$ and $\alpha = \sum_j \alpha_j$ with $\alpha \notin \mathfrak{p}$ such that $x'/s = a/\alpha$. Again, α being outside $\mathfrak{p}$ implies that $\alpha_{j^*} \notin \mathfrak{p}$ for some j^*. So $x'\alpha_{j^*} \in A$ and consequently $x \in A_{(\mathfrak{p})}$. Similarly $y \in A_{(\mathfrak{p})}$, and this finishes the proof. □

It is natural to ask whether gfc implies fc. Our next few results show that this is indeed true. We first treat the easy case of polynomial ring extensions and then show that the general case, under minor restrictions, reduces to this special case.

Lemma 9. *Let A be a factorially closed subring of an integral domain B. Then the polynomial ring $A[x]$ is also factorially closed in $B[x]$.*

Proof. Let $f(x), g(x) \in B[x] - \{0\}$ be such that $f(x)g(x) \in A[x]$. It is enough to show that $f(x) \in A[x]$. We consider the following two possible cases:

Case 1: *A is infinite.* Since f and g can have only finitely many roots, there exist infinitely many $a \in A$ such that $f(a), g(a)$ are nonzero elements of B and $f(a)g(a) \in A$, and consequently $f(a) \in A$, since A is factorially closed in B. In particular, if f has degree n, there exist $n+1$ distinct values in A, say $a_1, a_2, \ldots, a_{n+1}$, such that $f(a_i) \in A$ for each $i = 1, 2, \ldots, n+1$. So, treating the coefficients of f as variables, and plugging in the values a_i, we get $n+1$ linear equations in $n+1$ variables. The simultaneous linear equations have a solution in B. If we look at the corresponding Vandermonde matrix, it is obvious that the solution actually lies in $Q(A)$. Since $Q(A) \cap B = A$ by Lemma 3(2), $f(x) \in A[x]$.

Case 2: *A is a field.* Without any loss of generality we may assume that f and g are monic polynomials. Let L be a splitting field of fg over $Q(B)$. The roots of fg, and hence in particular the roots of f, are integral over $Q(A)$. Consequently, the coefficients of f, being symmetric functions of the roots, are integral over $Q(A)$ and hence algebraic over A. But since A is algebraically closed in B by Lemma 3(1), the coefficients are actually in A, and hence $f(x) \in A[x]$. □

For the general case, let A be a graded factorially closed subring of a $\mathbb{Z}$-graded domain B such that $A_i \neq 0$ and $A_{i+1} \neq 0$ for some integer i. We want to show that $A \subseteq B$ is factorially closed. Let S be the multiplicative set consisting of all nonzero homogeneous elements of A. Note that if $S^{-1}A \subseteq S^{-1}B$ is factorially closed then so is $A \subseteq B$. If $K := (S^{-1}A)_0$ and $\widetilde{B} := (S^{-1}B)_0$, then K is a field which is factorially closed in $\widetilde{B}$. Choose any nonzero elements $a_i \in A_i$, $a_{i+1} \in A_{i+1}$, and let $t := a_{i+1}/a_i$. Then $t \in (S^{-1}A)_1$ and $S^{-1}A = K[t, t^{-1}]$. To show that $K[t, t^{-1}]$ is factorially closed in $S^{-1}B$, let $b := b_{i_0} + b_{i_1} + \cdots + b_{i_r}$ and $c := c_{j_0} + c_{j_1} + \cdots + c_{j_s}$, with b_{i_0}, b_{i_r} and c_{j_0}, c_{j_s} nonzero, be elements of $S^{-1}B$ such that $bc \in S^{-1}A$. Writing $\overline{b_{i_\alpha}} := b_{i_\alpha} t^{-i_\alpha}$ for $\alpha = 0, 1, \ldots, r$ and $\overline{c_{j_\beta}} := c_{j_\beta} t^{-j_\beta}$ for $\beta = 0, 1, \ldots, s$, we get that

$b = \overline{b_{i_0}} t^{i_0} + \overline{b_{i_1}} t^{i_1} + \cdots + \overline{b_{i_r}} t^{i_r}$, $c = \overline{c_{j_0}} t^{j_0} + \overline{c_{j_1}} t^{j_1} + \cdots + \overline{c_{j_s}} t^{j_s} \in \widetilde{B}[t, t^{-1}]$. But since K is factorially closed in $\widetilde{B}$, so is $K[t] \subseteq \widetilde{B}[t]$, and consequently, by Lemmas 1 and 2, $K[t, t^{-1}]$ is factorially closed in $\widetilde{B}[t, t^{-1}]$. So $b, c \in S^{-1}A$, proving that $S^{-1}A \subseteq S^{-1}B$ is factorially closed, and hence $A \subseteq B$ is also factorially closed.

Therefore, we have proved the following result:

Theorem 2. *Let $A \subseteq B$ be $\mathbb{Z}$-graded domains with $A_i \neq 0$ and $A_{i+1} \neq 0$ for some integer i. Then A is factorially closed in B if it is graded factorially closed.*

Question. Is the hypothesis that A_i, A_{i+1} are nonzero for some i necessary?

We do not know the answer to the above question in general. But we sketch below a different proof of Theorem 2 without assuming the condition that A_i and A_{i+1} are nonzero for some i. However it works only when B is a UFD.

Let $A \subseteq B$ be $\mathbb{Z}$-graded domains with B a UFD. Now, assuming that A is graded factorially closed in B, we would like to show that A is factorially closed in B. After inverting all nonzero homogeneous elements of A, we may assume that A is of the form $k[t, t^{-1}]$, where t is a homogeneous prime element of positive degree in B. Further, since $k[t]$ is factorially closed in $B_0[t]$ by Theorem 9, it suffices to prove that $B_0[t]$ is factorially closed in $B^+ := \bigoplus_{i \geq 0} B_i$. So, if $f, g \in B^+$ are such that $fg \in B_0[t]$, we want to show that $f, g \in B_0[t]$. Let $f = f_0 + f_1 + \cdots + f_m$ and $g = g_0 + g_1 + \cdots + g_n$, with f_m and g_n nonzero. We can write f and g as

$$f = f_0' t^{\alpha_0} + f_1' t^{\alpha_1} + \cdots + f_m' t^{\alpha_m} \quad \text{and} \quad g = g_0' t^{\beta_0} + g_1' t^{\beta_1} + \cdots + g_m' t^{\beta_m},$$

where $f_i = f_i' t^{\alpha_i}$ and t does not divide f_i' unless it is zero, in which case we also take α_i to be zero, and similarly for g. If either $f_i' \in B_0$ for each i or $g_j' \in B_0$ for each j, we are done. Otherwise, we define α_* and β_* to be the minimums of the α_i for $f_i \neq 0$ and the β_j for $g_j \neq 0$, respectively. Let us also define $i^* := \max\{i \mid \alpha_i = \alpha_*\}$ and $j^* := \max\{j \mid \beta_j = \beta_*\}$. Note that $i^* < m$ and $j^* < n$. Looking at the homogeneous component of degree $i^* + j^*$ in fg, we get

$$(fg)_{i^*+j^*} = f_{i^*}' g_{j^*}' t^{\alpha_* + \beta_*} + \text{(elements divisible by } t^{\alpha_* + \beta_* + 1}).$$

But that means $f_{i^*}' g_{j^*}'$ must be divisible by t, which is a contradiction.

Finally, we put together the results connecting factorial closedness and graded factorial closedness in the form of the following theorem:

Theorem 3. *For positively graded domains $A \subseteq B$, if we assume that $A_1 \neq 0$, then the following statements are equivalent*:

(1) *$A \subseteq B$ is fc.*

(2) *For any homogeneous prime ideal $\mathfrak{p}$, $A_{(\mathfrak{p})} \subseteq B_{(\mathfrak{p})}$ is gfc.*

(3) *For any homogeneous prime ideal $\mathfrak{p}$, $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is fc.*

(4) *For any homogeneous maximal ideal $\mathfrak{m}$, $A_{(\mathfrak{m})} \subseteq B_{(\mathfrak{m})}$ is gfc.*

(5) *For any homogeneous maximal ideal* $\mathfrak{m}$, $A_{\mathfrak{m}} \subseteq B_{\mathfrak{m}}$ *is fc.*

Proof. Follows directly from Lemmas 7 and 8 and Theorem 2. □

In particular, if A_0 is a field, A is positively graded, $A_1 \neq (0)$ and $\mathfrak{m}$ is the irrelevant maximal ideal of A, then $A \subseteq B$ will be factorially closed if $A_{\mathfrak{m}} \subseteq B_{\mathfrak{m}}$ is factorially closed.

Given a subring A of an integral domain B, we define the factorially closed locus or fc locus of A in B to be $\mathfrak{FC}(A : B) := \{\mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is factorially closed$\}$. We intend to investigate the nature of this fc locus in the Zariski topology. We start with a few definitions: let $A/B := \{b \in B \mid bb' \in A$ for some $b' \in B - \{0\}\}$, which is an A-module. If $\bar{A}$ denotes the algebraic closure of A in B, it is easy to see that $A \subseteq Q(A) \cap B \subseteq \bar{A} \subseteq A/B \subseteq B$. Taking A to be the ring of integers $\mathbb{Z}$ and B to be $\mathbb{Z}[\sqrt{2}, x, y, 1/2y]$, one can see that the inclusions can be proper at each stage. By Lemma 1(2), factorial closedness is preserved under localization. So $\mathfrak{FC}(A : B)$ is closed under generalization. The following lemma gives a necessary and sufficient condition for the nonemptiness of the fc locus.

Lemma 10. *With notation as above, the following statements are equivalent*:

(1) *There exists a prime ideal* $\mathfrak{p} \in \operatorname{Spec} A$ *such that* $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ *is fc.*

(2) *The inclusion* $Q(A) \subseteq S^{-1}B$ *is fc, where* $S := A - \{0\}$.

(3) *There is an equality* $Q(A)^* = (S^{-1}B)^*$.

(4) *There is an equality* $A/B = Q(A) \cap B$.

Proof. We will only give the proof that (4) implies (2). The other implications are similar and easy.

Assume that $A/B = Q(A) \cap B$. We will show that $Q(A)$ is fc in $S^{-1}B$.

Let $(b_1/s_1) \cdot (b_2/s_2) \in Q(A)$, where the s_i are nonzero elements of A. Then there is a nonzero element $\alpha \in A$ such that $b_1 b_2 \alpha \in A$. This implies that $b_i \in A/B$, and hence $b_i \in Q(A)$. □

Note that $Q(A)^* = (S^{-1}B)^*$ implies that $B^* \subseteq Q(A)$. But the converse is false as the example $K[xz, yz] \subseteq K[x, y, z]$ with K a field shows.

To give conditions for the openness of the fc locus, we need a few auxiliary lemmas.

Lemma 11. *Let* $A \subseteq B$ *be integral domains. If* $\mathfrak{p} \in \operatorname{Spec} A$ *is a prime ideal of height* 1 *which is not in the image of* $\operatorname{Spec} B$, *then* $V(\mathfrak{p}) := \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{p} \subseteq \mathfrak{q}\}$ *does not meet* $\mathfrak{FC}(A : B)$.

Proof. Otherwise, if $\mathfrak{q} \in V(\mathfrak{p}) \cap \mathfrak{FC}(A : B)$, then $A_\mathfrak{p} \subseteq B_\mathfrak{p}$ is factorially closed by Lemma 1. But, since $\mathfrak{p}$ is not in the image of Spec B, each prime ideal of $B_\mathfrak{p}$ contracts to (0) in $A_\mathfrak{p}$. Consequently, every nonzero element of $A_\mathfrak{p}$ is a unit in $B_\mathfrak{p}$ which is a contradiction by Lemma 3. Hence we must have that $V(\mathfrak{p}) \cap \mathfrak{FC}(A : B) = \varnothing$. □

Lemma 12. *Let $A \subseteq B$ be integral domains with A noetherian and normal. If the image of* Spec B *contains all prime ideals* $\mathfrak{p} \in$ Spec A *of height* 1, *then either the fc locus $\mathfrak{FC}(A : B)$ is empty or A is factorially closed in B.*

Proof. If $A/B \neq Q(A) \cap B$, we know that the factorially closed locus will be empty. So we are interested in showing that, if $A/B = Q(A) \cap B$, then A is factorially closed in B. In order to prove factorial closedness, first note that it suffices to prove that any principal ideal of A is contracted from some ideal of B, or, equivalently, that $xB \cap A = xA$ for any $x \in A$. For, if it is true, let us consider $b_1, b_2 \in B - \{0\}$ such that $b_1 b_2 = a \in A$. Since $A/B = Q(A) \cap B$, $b_1 \in Q(A)$. Let $b_1 = \alpha/\beta$, with $\alpha, \beta \in A - \{0\}$. Now $\alpha \in \beta B \cap A = \beta A$, implying that $b_1 \in A$, and consequently A is factorially closed in B. So all we need to show is that any principal ideal of A is contracted from some ideal of B. But since A is a noetherian normal domain, any prime ideal associated to a principal ideal has height 1, and, as a result, using primary decomposition any principal ideal of A can be written as a finite intersection of primary ideals of height 1. So it is enough to prove that any height-1 primary ideal of A is a contracted ideal. Now, given any prime ideal $\mathfrak{p} \in$ Spec A of height 1, let us consider the commutative diagram

$$\begin{tikzcd} B \arrow[r, hook] & B_\mathfrak{p} \\ A \arrow[u, hook] \arrow[r, hook] & A_\mathfrak{p} \arrow[u, hook] \end{tikzcd}$$

The local ring $A_\mathfrak{p}$ is a DVR and $\mathfrak{p}A_\mathfrak{p}$ is a contracted ideal. So each $\mathfrak{p}A_\mathfrak{p}$-primary ideal is also contracted. Again, $\mathfrak{p}$-primary ideals of A are in a one-to-one correspondence with the $\mathfrak{p}A_\mathfrak{p}$-primary ideals of $A_\mathfrak{p}$. So we conclude that each $\mathfrak{p}$-primary ideal of A is contracted from some ideal in B, and this completes the proof. □

Note. Let $A \subseteq B$ be integral domains with A noetherian. We have proved that if $A_\mathfrak{p}$ is a DVR for some $\mathfrak{p} \in$ Spec A then $\mathfrak{p} \in \mathfrak{FC}(A : B)$ if and only if it is in the image of Spec B.

Now we are in a position to characterize the openness of the fc locus:

Theorem 4. *Let $A \subseteq B$ be integral domains with A noetherian and normal. Then $\mathfrak{FC}(A : B)$, if nonempty, is open in* Spec A *if and only if the image of* Spec B *misses only finitely many height-*1 *prime ideals.*

Proof. If $\mathfrak{FC}(A:B)$ is open, its complement contains at most finitely many prime ideals of height 1. So, by Lemma 12, the image of Spec B misses at most finitely many height-1 prime ideals. Conversely, assume that $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n$ are the only prime ideals of height 1 lying outside the image of Spec B. In view of Lemma 12, it is enough to show that any $\mathfrak{q} \in \operatorname{Spec} A$ not contained in $\bigcup_{i=1}^n V(\mathfrak{p}_i)$ is in $\mathfrak{FC}(A:B)$. So, let us choose any $\mathfrak{q} \in \operatorname{Spec} A - \bigcup_{i=1}^n V(\mathfrak{p}_i)$. We can find $x \in \bigcap_{i=1}^n \mathfrak{p}_i - \mathfrak{q}$. Considering the inclusion $A_x \subseteq B_x$, all height-1 prime ideals of A_x are in the image of Spec B_x. Since we are only interested in the case when $\mathfrak{FC}(A:B) \neq \varnothing$, we may assume by Lemma 10 that $A/B = Q(A) \cap B$. Consequently $A_x/B_x = Q(A_x) \cap B_x$. Therefore, Lemma 11 applies to show that A_x is factorially closed in B_x. Hence $\mathfrak{FC}(A:B) = \operatorname{Spec} A - \left(\bigcup_{i=1}^n V(\mathfrak{p}_i)\right)$ is open. □

Corollary 4.1. *With notation as in Theorem 4, if B is a finitely generated algebra over A then $\mathfrak{FC}(A:B)$ is always open.*

Proof. This follows from Theorem 4, since the corresponding dominant morphism of affine schemes Spec $B \to$ Spec A always contains a nonempty open set in its image, and consequently the image of Spec B can miss at most finitely many height-1 primes of Spec A. □

2. Quasifactorially closed subrings

If we attempt to generalize Lemma 9 to the case of formal power series rings $A[\![x]\!] \subsetneq B[\![x]\!]$ the attempt fails quite badly. For, suppose that $A \subsetneq B$ is factorially closed. If $b \in B - A$, then the element $1 + bx + x^2 + x^3 + x^4 + \cdots$ is a unit in $B[\![x]\!]$ which is not in $A[\![x]\!]$. So the extension $A[\![x]\!] \subseteq B[\![x]\!]$ is never factorially closed unless $A = B$. The presence of 'extra units' in the bigger ring turns out to be an obvious obstruction. To rectify this problem, we come up with a weaker notion of quasifactorially closedness.

Recall that, given an inclusion of integral domains $A \subseteq B$, A is said to be quasifactorially closed, or qfc for short, in B if, for any nonzero $b \in B$, if there exists some nonzero $b' \in B$ such that $bb' \in A$, then there exists a unit $u \in B$ such that $bu \in A$.

If A and B have the same units then the notions of factorial closedness and quasifactorial closedness coincide. Also note that $A \subseteq B$ is quasifactorially closed whenever either A or B is a field. But quasifactorial closedness, in general, is more of a geometric notion and does not behave well with algebraic operations. For example, although it is closed under localization, we are not yet sure if global information can be retrieved from local data as in Lemma 1. The *sandwich property*, as in Lemma 2, also fails, as any integral domain is always quasifactorially closed in any field containing it. The following example shows that the *transitive property* need not hold true either.

Example. Factorial closedness holds for $K[x] \subseteq K[x, y, z, w]/(xy - zw)$, and $A := K[x, y, z, w]/(xy - zw)$ is qfc in $A[1/y]$. But $K[x]$ is not qfc in $A[1/y]$, as there is no unit u in $A[1/y]$ such that $uz \in K[x]$.

Note that the above example also shows that in the definition of quasifactorial closedness it may not be possible to get a unit $u \in B$ such that $bu, b'u^{-1} \in A$. In fact, if it were true then one can check that, for integral domains $A \subseteq B \subseteq C$, if $A \subseteq B$ is fc and $B \subseteq C$ is qfc then $A \subseteq C$ would also be qfc, which is clearly not true, as the above example shows.

The following example shows that $A \subseteq B$ being qfc does not imply that $A[x] \subseteq B[x]$ is qfc:

Example. Take any nontrivial algebraic field extension L/K. Then $K \subseteq L$ is qfc but $K[x] \subseteq L[x]$ is not qfc. If $K = \mathbb{R}$ and $L = \mathbb{C}$, take $b = ix - 1$ and $b' = ix + 1$. Then $bb' \in \mathbb{R}[x]$, but there is no unit $u \in \mathbb{C}[x]$ such that $bu \in \mathbb{R}[x]$.

Let $A \subseteq B$ be fc. Then any irreducible element of A remains irreducible in B. If A is a UFD but B is not, then prime elements of A need not remain prime in B, as the example $k[x] \subseteq k[x, y, z]/(xy - z^2 - 1)$ shows, where the prime element x of $k[x]$ does not remain a prime in $k[x, y, z]/(xy - z^2 - 1)$. But if B is also a UFD then $A \subseteq B$ is fc if and only if each prime element of A remains a prime in B and $A^* = B^*$. For UFDs $A \subseteq B$, primes of A remaining primes in B is a sufficient condition for qfc. But it is not necessary, as the first example in Section 4 will show. However, it follows from Theorem 6 and its corollaries that the converse is also true in the case of complete local UFDs.

For integral domains $A \subseteq B$, we define the qfc locus of A in B by $\mathfrak{QFC}(A : B) := \{\mathfrak{p} \in \operatorname{Spec} A \mid A_\mathfrak{p} \subseteq B_\mathfrak{p} \text{ is qfc}\}$. Just like the fc locus, the qfc locus is also closed under generalization. Note that $\mathfrak{QFC}(A : B)$ is always nonempty since $(0) \in \mathfrak{QFC}(A : B)$. For, let $S = A - \{0\}$. If $(b_1/s_1).(b_2/c_2) \in Q(A)$ then b_i/s_i are units in $S^{-1}B$. Then $(b_i/s_i).(s_i/b_i) \in Q(A)$, implying that $Q(A)$ is qfc in $S^{-1}B$.

Next, we prove an openness criterion, analogous to Theorem 4, for the qfc locus, albeit for a somewhat restricted class of rings.

Theorem 5. *Let $A \subseteq B$ be affine UFDs. Assume that A and B have the same group of units and $Q(A)$ is algebraically closed in $Q(B)$. Then $\mathfrak{QFC}(A : B)$ is a nonempty open set if one, and hence all, of the following equivalent conditions hold*:

(1) *Given any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ of height ≥ 2, $\mathfrak{p}B$ has height ≥ 2.*

(2) *No prime element of B divides two distinct prime elements of A.*

(3) *Any two coprime elements of A continue to be coprime in B.*

(4) *There are only finitely many prime elements of A which either split in B or are units in B.*

Proof. Since A and B are UFDs, the proof of the equivalence of (1), (2) and (3) in the above statement is easy. This part does not need openness of $\mathfrak{QFC}(A : B)$.

Let V and W denote the irreducible affine varieties corresponding to A and B respectively, and let $f : W \to V$ be the induced morphism.

First we consider the case when $\dim A = 1$.

By assumption, $Q(A)$ is algebraically closed in $Q(B)$. Then it is well-known (by a suitable application of Bertini's theorem) that only finitely many scheme-theoretic fibers of the morphism $W \to V$ are either empty or not reduced and irreducible. This shows that (4) is also always true, so that conditions (1)–(4) are equivalent.

Now we will assume that $\dim A \geq 2$.

Assume now that the equivalent conditions (1), (2) and (3) hold. We will show that (4) holds.

Again, since $Q(A)$ is algebraically closed in $Q(B)$, there is a proper closed subvariety $S \subset V$ such that the inverse image of any point $p \notin S$ is scheme-theoretically reduced and irreducible. By (1), the inverse image of any closed subvariety of V of codim ≥ 2 does not contain any divisor in W. Now we can see that the only possible irreducible divisors $D \subset V$ which split in W are those contained in S.

The image $f(W)$ contains a nonempty Zariski-open subset since f is dominant. Hence $f(W)$ can miss at most finitely many divisors in V. This shows that (4) is true.

Next, we will show that (4) implies (1). Suppose that this is not true. Then there is a closed irreducible subvariety $S \subset V$ of codimension > 1 such that the inverse image of S in W contains an irreducible divisor Δ, defined by a prime element q. Now, if D is any irreducible divisor in V which contains S then the prime element defining D will split in B. Since $\dim B > 1$, there are infinitely many such prime elements in A.

This proves the equivalence of (1)–(4).

Now we will assume that the equivalent conditions (1)–(4) hold. We will show that $\mathfrak{QFC}(A : B)$ is a nonempty open set.

Let $p_1, \ldots, p_r$ be the prime elements in A such that p_i is a non-unit in B and not a prime element in B.

We will show that $\mathfrak{QFC}(A : B) = \operatorname{Spec} A \setminus \bigcup_{i=1}^{r} V(p_i A)$, and hence $\mathfrak{QFC}(A : B)$ is nonempty and open.

First we will show that if a prime element $p \in A$ is not a unit in B and does not remain a prime element in B, then $V(pA) \cap \mathfrak{QFC}(A : B) = \emptyset$.

So, let $p \in A$ be such a prime element and let $\mathfrak{q} \in V(pA)$. If $A_{\mathfrak{q}} \subseteq B_{\mathfrak{q}}$ is qfc then so is $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$, where $\mathfrak{p} := pA$. Since p is not a prime element in B, there exists $b_1, b_2 \in B - B^*$ such that $p = b_1 b_2$. But $A_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ being qfc implies that either b_1 or b_2 must be a unit in $B_{\mathfrak{p}}$. Without any loss of generality, let us assume that $b_1 \in B_{\mathfrak{p}}^*$. So b_1 divides some element $s \in A - \mathfrak{p}$. But s and p are coprime in A, and hence in B by (3). So b_1 must be a unit in B, which is a contradiction.

By assumption, there are only finitely many prime elements $p_i \in A$ such that p_i is a non-unit in B and not a prime element in B. We have already seen that $\bigcup_{i=1}^n V(p_i A) \subseteq \mathfrak{QFC}(A:B)^c$. So it suffices to show that any prime ideal of A which does not contain any of the p_i is in $\mathfrak{QFC}(A:B)$. Let $\mathfrak{q} \in \operatorname{Spec} A$ be such a prime ideal. Choose any $a \in \bigcap_{i=1}^n p_i A - \mathfrak{q}$. Then $A_a \subseteq B_a$ is qfc since each prime element in A_a continues to be a prime element in B_a. Consequently, $A_\mathfrak{q} \subseteq B_\mathfrak{q}$ is also qfc, and this completes the proof. $\square$

Remark. In Section 3, we will give an example to show that $\mathfrak{QFC}(A:B)$ can be open and nonempty even when there are infinitely many prime elements in A which are not units in B and are not prime elements in B.

The following theorem shows that the qfc property has some nice consequences in the case of complete local domains:

Theorem 6. *Let $(A, \mathfrak{m}_A) \subseteq (B, \mathfrak{m}_B)$ be local domains such that $\mathfrak{m}_A = \mathfrak{m}_B \cap A$ and $A/\mathfrak{m}_A = B/\mathfrak{m}_B$. Moreover, assume that A is complete in the $\mathfrak{m}_A$-adic topology and $\bigcap_{n=1}^\infty \mathfrak{m}_B^n = (0)$. If A is qfc in B then A is algebraically closed in B.*

Proof. Let $b \in B$ be algebraic over A. We will construct a sequence $(a_n) \in A^\mathbb{N}$ such that, for each n, $a_{n+1} = a_n + \alpha_1\alpha_2 \cdots \alpha_n\alpha_{n+1}$ and $b = a_n + \alpha_1\alpha_2 \cdots \alpha_n\beta_n$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1} \in \mathfrak{m}_A$ and $\beta_n \in \mathfrak{m}_B$. Any such sequence will be a Cauchy sequence in the $\mathfrak{m}_A$-adic topology of A which converges to b in the $\mathfrak{m}_B$-adic topology of B, implying that $b \in A$.

Since b is algebraic over A, there exist elements $c_0, c_1, \ldots, c_r \in A$, with c_0 and c_r nonzero, such that

$$c_0 + c_1 b + \cdots + c_r b^r = 0,$$

implying that $b(c_1 + c_2 b + \cdots + c_r b^{r-1}) \in A$. Since A is qfc in B, there exists a unit $u \in B^*$ such that $bu = a \in A$ or, equivalently, $b = au^{-1}$. We can write $u^{-1} = u_1' + b_1$ for some $u_1' \in A^*$ and $b_1 \in \mathfrak{m}_B$, so that $b = a(u_1' + b_1)$. Setting $a_1 := au_1'$, the induction hypothesis is satisfied for $n = 1$.

Next, suppose that we have already found elements $a_1, a_2, \ldots, a_n \in A$ satisfying the required conditions. To find a_{n+1}, note that $b = a_n + \alpha_1\alpha_2 \cdots \alpha_n\beta_n$, implying that $\alpha_1\alpha_2 \cdots \alpha_n\beta_n$ is also algebraic over A, and consequently there exists a unit $u_{n+1} \in B^*$ such that $\beta_n = \alpha_{n+1}u_{n+1}$ for some $\alpha_{n+1} \in \mathfrak{m}_A$. Writing u_{n+1} as $u_{n+1} = u_{n+1}' + \beta_{n+1}$, where $u_{n+1}' \in A^*$ and $\beta_{n+1} \in \mathfrak{m}_B$, we get

$$b = a_n + \alpha_1\alpha_2 \cdots \alpha_n\alpha_{n+1}(u_{n+1}' + \beta_{n+1}).$$

It is obvious that $a_{n+1} := a_n + \alpha_1\alpha_2 \cdots \alpha_n\alpha_{n+1}u_{n+1}'$ satisfies the required properties. This, together with induction, completes the proof. $\square$

With notation as in Theorem 6, we have the following easy corollaries:

Corollary 6.1. *There is an equality $Q(A) \cap B = A$.*

Corollary 6.2. *If B is normal, then so is A.*

Corollary 6.3. *If B satisfies the ascending chain condition for principal ideals, then so does A.*

Corollaries 6.1, 6.2 and 6.3 are immediate consequence of Theorem 6.

Corollary 6.4. *Any irreducible element of A remains irreducible in B.*

This can be proved in the same way as the proof of Theorem 6. We leave the details to the reader.

Corollary 6.5. *If $a \in A$ is a prime element of B, then it is already a prime element in A.*

Corollary 6.6. *Let $a, a' \in A$. Then $aA = a'A$ if and only if $aB = a'B$. In particular, if two elements of A are not associates in A, they cannot become associates in B.*

This follows easily from Corollary 6.1. For, if $a' \in aB$, then, writing $a' = ab$ with $b \in B$, by Corollary 6.1 we have $b \in Q(A) \cap B = A$.

Corollary 6.7. *If B is a UFD, then so is A.*

Proof. Corollary 6.3 shows that any element in A can be written as a product of irreducible elements in A. By Corollary 6.4, any irreducible element in A remains irreducible and hence a prime in B. Now Corollary 6.5 finishes the proof. □

Corollary 6.8. *If two elements of A have no common factor in A, they cannot have a common factor in B.*

Proof. If $b \in \mathfrak{m}_B$ is a common factor of $a, a' \in \mathfrak{m}_A$, then there is a unit $u \in B^*$ such that $bu \in \mathfrak{m}_A$. But then, by Corollary 6.1, bu is a common factor of a and a' in A, leading to a contradiction. □

Corollary 6.9. *If B is a UFD, then, for any prime ideal $\mathfrak{p} \in \operatorname{Spec} A$ of height ≥ 2, $\mathfrak{p}B$ has height ≥ 2.*

Now we prove an analogue of Theorem 1 for power series rings. First, we consider the 2-dimensional case.

Theorem 7. *Let $k \subsetneq A \subsetneq B := k[\![x, y]\!]$ be a noetherian complete (with respect to its maximal ideal) local qfc subring of B, the power series ring in two variables. Then A is isomorphic to a power series ring in one variable over k.*

Proof. By Corollary 6.7, A is a UFD. Let $p \in A$ be a prime element. By Corollary 6.4, p is a prime element in B, and since A is qfc in B we have $pB \cap A = pA$. This gives an inclusion of integral domains $A/pA \subseteq B/pB$. Now $\dim B/pB = 1$, and hence any two elements in B/pB are analytically dependent. Thus, $\dim A/pA \leq 1$. Now $\dim A \leq 2$. If $\dim A = 1$, then A is clearly isomorphic to a power series ring in one variable over k.

Now assume that $\dim A = 2$. We will show that $A = k[\![x, y]\!]$. For that, choose any two relatively prime elements u, v from the maximal ideal of A. By Corollaries 6.4 and 6.6, u and v are nonassociate prime elements of $k[\![x, y]\!]$. So, in particular, the extension of the maximal ideal of A to B is (x, y)-primary. Since A, B are complete, we infer that B is integral over A. But then by Theorem 6 A must be equal to $k[\![x, y]\!]$. □

Next we consider the case when $\dim B = 3$.

Theorem 8. *Let $k \subseteq A \subseteq B := k[\![X, Y, Z]\!]$, where A is a 2-dimensional noetherian complete* (*with respect to its maximal ideal*) *local qfc subring of B. Then A is isomorphic to a power series ring in two variables over k.*

Proof. The proof is similar to the proof of Theorem 7.

Note that A is a UFD by Corollary 6.7.

By Brieskorn's theorem [1968], either A is isomorphic to a power series in two variables over k, or $A \cong k[\![u, v, w]\!]/(u^2 + v^3 + w^5)$. We have to show that A cannot be isomorphic to $k[\![u, v, w]\!]/(u^2 + v^3 + w^5)$. By the argument in the proof of Theorem 7, the extended ideal $(\bar{u}, \bar{v}, \bar{w})B$ has height > 1. We know that A is the ring of invariants of the binary icosahedral group of order 120 acting on a power series ring $k[\![s, t]\!]$. The morphism $\operatorname{Spec} k[\![s, t]\!] \setminus \{(s, t)\} \to \operatorname{Spec} A \setminus \{(\bar{u}, \bar{v}, \bar{w})\}$ is finite unramified. Since $\operatorname{Spec} B \setminus V((\bar{u}, \bar{v}, \bar{w}))$ is simply connected, by covering space theory we have a factorization

$$\operatorname{Spec} B \setminus V((\bar{u}, \bar{v}, \bar{w})) \to \operatorname{Spec}\ k[\![s, t]\!] \setminus \{(s, t)\} \to \operatorname{Spec} A \setminus \{(\bar{u}, \bar{v}, \bar{w})\}.$$

By Hartog's theorem, we have $A \subset k[\![s, t]\!] \subseteq B$. But then A is not algebraically closed in B, contradicting Theorem 6. This shows that A is isomorphic to a power series ring in two variables over k. □

Question. In Theorem 8, is the assumption $\dim A = 2$ necessary, i.e., can a proper qfc subring of $k[\![x, y, z]\!]$ have dimension > 2?

It is well-known that, if $A \subseteq B$ are affine normal domains over an algebraically closed field of characteristic 0 such that $Q(A)$ is algebraically closed in $Q(B)$, then a general fiber of the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is irreducible. By Theorem 6, if $A \subseteq B$ are complete normal domains over an algebraically closed field of characteristic 0 such that A is qfc in B, then $Q(A)$ is algebraically closed in $Q(B)$. In view of the above observation we can ask the following question:

Question. Let (V, p), (W, q) be normal complex analytic germs and $f : (W, q) \to (V, p)$ a complex analytic morphism such that the analytic local ring of V is algebraically closed in that of W. Is a general fiber of f irreducible?

We have the following modest result as an affirmative answer to this question:

Theorem 9. *Let (W, q) be a normal complex analytic germ and $f : W \to \mathbb{C}$ a complex analytic morphism of germs. Assume that the ring $\mathbb{C}\{f\} \subset \mathcal{O}_{W,q}$ is qfc. Then a general fiber of f is connected.*

Proof. We will use a result of Tráng [1977] on the topology of singular points, which generalizes Milnor's results.

Tráng [1977] proved that there are positive numbers $0 < \delta \ll \epsilon \ll 1$ such that if D is a disc of radius δ in $\mathbb{C}$ then the morphism $B_\epsilon \cap W \cap f^{-1}(D - \{0\}) \to D - \{0\}$ is a topological fiber bundle, where B_ϵ is a ball of radius ϵ with center q in $\mathbb{C}^n$, such that $(W, q) \subseteq (\mathbb{C}^n, 0)$ is a closed embedding of germs. Since $\mathbb{C}\{f\} \subset \mathcal{O}_{W,q}$ is qfc, the fiber $\{f = 0\}$ is irreducible by Corollary 6.4. We have a long exact sequence of homotopy groups

$$\begin{aligned}\pi_1(F) \longrightarrow \pi_1(B_\epsilon \cap W \cap f^{-1}(D - \{0\})) \longrightarrow \pi_1(D - \{0\}) \longrightarrow \pi_0(F)\\ \longrightarrow \pi_0(B_\epsilon \cap W \cap f^{-1}(D - \{0\})) \longrightarrow \pi_0(D - \{0\}) \longrightarrow (1).\end{aligned}$$

Here F is a general fiber of $B_\epsilon \cap W \cap f^{-1}(D - \{0\}) \to D - \{0\}$. Both $B_\epsilon \cap W \cap f^{-1}(D - \{0\})$ and $D - \{0\}$ are connected. Since $\{f = 0\}$ is reduced and irreducible, a small transverse loop in $B_\epsilon \cap W \cap f^{-1}(D - \{0\})$ maps onto the generator of the fundamental group of $D - \{0\}$, hence the homomorphism

$$\pi_1(B_\epsilon \cap W \cap f^{-1}(D - \{0\})) \to \pi_1(D - \{0\})$$

is surjective. It follows that F is connected, and this proves the result. □

The next result is an interesting consequence of the property of being factorially closed. To state the result, we need a definition. Let $f : Y \to X$ be a dominant morphism of smooth algebraic varieties such that the general fibers are irreducible and reduced. Then there exist an open immersion $\iota : Y \hookrightarrow W$ and a projective morphism $\overline{f} : W \to X$ such that $f = \overline{f} \circ \iota$, where W is a smooth algebraic variety and $D := W \setminus Y$ is a divisor with simple normal crossings. Let $D = D_1 + \cdots + D_r$ be the irreducible decomposition. We further assume that D intersects transversally the fiber $F_P = \overline{f}^{-1}(P)$ for every closed point $P \in X$. We say that $\overline{f}$ is an SNC-completion of f. Suppose that for every $P \in X$ and every $1 \le i \le r$ the intersection $D_i \cdot F_P$ is irreducible and reduced. If there exists such an SNC-completion of f, we say that f is *fiberwise integral at infinity*. If there is an open set U of X such that $f : f^{-1}(U) \to U$ has a completion which is fiberwise integral at infinity, then we say that f is *generically fiberwise integral at infinity*. This condition is equivalent to saying that the generic fiber Y_η, with η the generic point of X, can be embedded into a projective smooth variety W_η defined over the field $k(\eta)$ in such a way that $D_\eta = W_\eta \setminus Y_\eta$ is a divisor consisting of geometrically integral smooth components with simple normal crossings.

Theorem 10. *Let A be an affine domain of dimension 1 over k. Assume that A is fc in a regular affine domain R over k. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} R$ and $f : Y \to X$ be the induced morphism. Assume that f has an SNC-completion which is generically fiberwise integral at infinity. Then there is a maximal ideal $\mathfrak{m}$ of A such that the affine domain $R/\mathfrak{m}R$ is regular and has no nontrivial units.*

Proof. Since A is fc in R and R is normal, it follows that A is normal, and hence regular as $\dim A = 1$. A general fiber of the morphism f is reduced and irreducible. In particular, by Bertini's theorem, $R/\mathfrak{m}R$ is a regular affine domain for all but finitely many maximal ideals in A. Removing from X the closed points corresponding to these maximal ideals, we may assume that f is a smooth morphism. Let $\overline{f} : W \to X$ be an SNC-completion which is fiberwise integral at infinity. Here we may have to replace X by a suitable open set. Let $D = W \setminus Y$ be the divisor at infinity and let $D = D_1 + D_2 + \cdots + D_r$ be the irreducible decomposition of D. If the result is not true, then we may assume that $R/\mathfrak{m}R$ has a nontrivial unit for every maximal ideal $\mathfrak{m}$ of A. Note that, by definition, each D_i meets each fiber F_P of $\overline{f}$ transversally and the intersection $D_i \cdot F_P$ is integral, i.e., irreducible and reduced.

Let P be a closed point of X. The fiber $f^{-1}(P)$ has a nonconstant unit u_P, and the divisor (u_P) in $F_P := \overline{f}^{-1}(P)$ has the form $\sum_i a(P)_i D_i|_{F_P}$ with $a(P)_i \in \mathbb{Z}$. Note that the subgroup $\sum_i \mathbb{Z} D_i$ of $\operatorname{Pic}(W)$, which is generated by the irreducible components of D, is a countable group. Choosing a nonconstant unit u_P for every $P \in X(k)$, we have a mapping $P \mapsto (u_P)$ from $X(k)$ to the group $\sum_i \mathbb{Z} D_i$, where $X(k)$ is the set of closed points of X and (u_P) is identified with $\sum_i a(P)_i D_i$. Since each $D_i \cap F_P$ is irreducible and reduced for each i, such an identification is possible. Then we can find a fixed divisor $D_0 = \sum_i a_i D_i$ and an infinite set Λ of $X(k)$ such that $D_0 \cdot F_P = (u_P)$ for each P of Λ. This means that the line bundle $\mathcal{O}(D_0)$ on W restricts to a trivial line bundle on F_P for each $P \in \Lambda$. By the upper-semicontinuity theorem [Hartshorne 1977, Chapter III, Theorem 12.8], the set of points in X such that the restriction of D_0 to F_P is trivial is a closed subvariety T of X containing the infinite set Λ. (Use the theorem for $\mathcal{L}$ and $\mathcal{L}^{-1}$ so that $\dim_k H^0(F_P, \mathcal{L}|_{F_P}) \geq 0$ and $\dim_k H^0(F_P, \mathcal{L}^{-1}|_{F_P}) \geq 0$.) Since $\dim X = 1$, $T = X$ and D_0 restricts to a trivial line bundle on every fiber of $\overline{f}$. By [Hartshorne 1977, Chapter III, Exercise 12.4], D_0 is linearly equivalent to the pullback by $\overline{f}$ of a divisor of the form $\sum_{j=1}^{s} b_j Q_j$ on X. Thus, the restriction of D_0 to $\overline{f}^{-1}(X \setminus \{Q_1, \ldots, Q_s\})$ is linearly equivalent to zero. Write D_0 as the divisor of a rational function (φ) on $\overline{f}^{-1}(X \setminus \{Q_1, \ldots, Q_s\})$. Then φ gives a nonconstant unit of $f^{-1}(X \setminus \{Q_1, \ldots, Q_s\})$. Since the units on $f^{-1}(X \setminus \{Q_1, \ldots, Q_s\})$ and $X \setminus \{Q_1, \ldots, Q_s\}$ are the same by the assumption of factorial closedness, φ is constant on each fiber of f. However, φ restricts onto the unit u_P up to a nonzero constant for every $P \in \Lambda$. This is a contradiction because u_P is not a constant. $\square$

Without the assumption that f has an SNC-completion which is fiberwise integral at infinity, Theorem 10 does not hold.

Example. Let W be the Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$ with vertical and horizontal $\mathbb{P}^1$-fibrations. Let $p_1 : W \to \mathbb{P}^1$ be the vertical one with a fiber L, and let the horizontal one p_2 be given by a linear system $|M|$. Let D_1 be an irreducible curve such that $D_1 \sim 2M + L$. Then the restriction $p_1|_{D_1} : D_1 \to \mathbb{P}^1$, being a double covering, has two branch points. Let L_1, L_2 be two fibers of p_1 over these branch points. Let $D = D_1 + L_1 + L_2$, and let $Y := W \setminus D$ and $X := \mathbb{P}^1 - \{\text{two branch points}\}$. Let $f = p_1|_Y$. Then every fiber of $f : Y \to X$ is irreducible; hence $k(X)$ is algebraically closed in $k(Y)$ and it is easy to see that A is factorially closed in R, where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} R$, because $A = k[t, t^{-1}]$ and every prime element of A is $t - c$ with some nonzero constant $c \in k$. Then the fiber over $t = c$ is irreducible. Hence $t - c$ is a prime element in R. Furthermore, the units of R are the same as the units of A because the only linear relation among the components of D is the one between L_1 and L_2. But every closed fiber of f has a nontrivial unit because it is isomorphic to $\mathbb{A}^1_*$. Note that R is not factorial since $\operatorname{Pic}(R) = \mathbb{Z}/2\mathbb{Z}$.

Remark. In this example, the fibration $f : Y \to X$ is a twisted $\mathbb{A}^1_*$-fibration. Let X' be the curve D_1 with two ramifying points for $p_1|_{D_1}$ removed, and let $f' : Y' \to X'$ be the base change of f by $X' \to X$. Then $Y' \cong A^1_* \times \mathbb{A}^1_*$. Write $Y = \operatorname{Spec} R$, $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} A'$. Then A is fc in R, but A' is not fc in $R' := R \otimes_A A'$. In fact, $R'^*/k^* \cong \mathbb{Z} \times \mathbb{Z}$ and $A'^*/k^* \cong \mathbb{Z}$. If A' were fc in R', then we must have $R'^* = A'^*$. Note that A'/A is a finite étale extension. Hence the factorial closedness is not preserved even by an étale base change.

Using Theorem 10 we can now give a very short proof of a result of [Neumann and Norbury 1998].

Theorem 11. *Let $f, g \in \mathbb{C}[X, Y]$ be a pair of polynomials in two variables with nonzero constant Jacobian determinant. Suppose that the following conditions are satisfied*:

(a) *For all $c \in \mathbb{C}$, the polynomial $f - c$ is irreducible and defines a rational curve.*

(b) *Let $\mathbb{C}^2 \subset Y$ be an open embedding in a smooth quasiprojective surface such that $f : \mathbb{C}^2 \to \mathbb{C}$ extends to a proper morphism $Y \to \mathbb{C}$ and $Y \setminus \mathbb{C}^2$ is a simple normal crossing divisor such that each irreducible component of $Y \setminus \mathbb{C}^2$ is a cross-section of the morphism $Y \to \mathbb{C}$.*

Then $\{f = 0\} \cong \mathbb{C}$, and hence the Jacobian Conjecture is true for the pair (f, g).

Remark. In [Neumann and Norbury 1998] f is called a simple rational polynomial.

Proof. By assumption, $f - c$ is an irreducible polynomial for all constants c. Also, $\mathbb{C}^2$ has no nonconstant invertible regular functions. Hence the morphism $\mathbb{C}^2 \to \mathbb{C}$ is an fc morphism. Condition (b) implies that $f : \mathbb{C}^2 \to \mathbb{C}$ is generically fiberwise integral at infinity. By Theorem 10 and condition (a), for all but finitely many $c \in \mathbb{C}$ the affine curve $\{f = c\}$ is smooth rational irreducible with no nonconstant invertible regular functions. Hence it is isomorphic to $\mathbb{C}$. By the Abhyankar–Moh–Suzuki theorem, after a suitable automorphism of $\mathbb{C}[X, Y]$ the polynomial f is mapped onto X. It is well-known that this implies that the Jacobian Conjecture is true for (f, g). $\square$

The next result is another interesting example of qfc subrings.

Theorem 12. *Let $A = \mathbb{C}\{z_1, z_2, \dots, z_n\}/P$ be an analytic local domain which is a UFD. Then A is qfc in $\hat{A}$.*

Proof. We use Artin's approximation theorem [1968].

It is known that $\hat{A}$ is also a UFD. To show that A is qfc in $\hat{A}$, it follows easily from the definition of a qfc subring that it is enough to show that any prime element of A remains a prime element in $\hat{A}$.

Suppose that $f \in A$ is a prime element. Assume that there are non-units g, h in $\hat{A}$ such that $f = gh$. Let P be generated by $f_1, f_2, \dots, f_r$. Let $w_1, w_2, \dots, w_r$, w_{r+1}, w_{r+2} be new indeterminates. Consider the system of equations in the variables $z_1, \dots, z_n, w_1, w_2, \dots, w_{r+2}$

$$f_1 - w_1 = 0 = f_2 - w_2 = \cdots = f_r - w_r = f - w_{r+1}w_{r+2}.$$

This system has solutions $w_1 = f_1, \dots, w_r = f_r$, $w_{r+1} = g$, $w_{r+2} = h$ in $\mathbb{C}[[z_1, \dots, z_n]]$. By Artin's theorem, we can find solutions g_0, h_0 in $\mathbb{C}\{z_1, \dots, z_n\}$ such that $f = g_0h_0$ modulo P and g_0, h_0 approximate g, h to any order. In particular, g_0 and h_0 cannot be units. Thus, every prime element in A remains a prime element in $\hat{A}$, and consequently A is qfc in $\hat{A}$. $\square$

Corollary 12.1. *Any element of $\mathbb{C}[[z_1, \dots, z_n]]$ which is algebraic over $\mathbb{C}\{z_1, \dots, z_n\}$ is itself convergent.* (Here, $z_1, z_2, \dots, z_n$ are indeterminates over $\mathbb{C}$.)

Question. Is Theorem 12 valid without assuming that A is a UFD?

3. Examples

We give some examples which shed more light on fc and qfc extensions.

(1) The following example shows that a qfc extension $A \subseteq B$ of local domains can be quite strange if A is not complete.

Let k be any field. Consider $A := k[x, e^x - 1]_{(x, e^x - 1)}$ and $B := k[[x]]$. Then $A \subseteq B$ is a local inclusion of local UFDs. Note that A is a regular local ring of dimension 2, whereas B has only one nonzero prime ideal, namely (x). From these observations we can easily deduce the following properties:

(a) Infinitely many primes of A split in B.

(b) Infinitely many distinct primes of A become associates in B.

(c) The dimension of A is bigger than the dimension of B.

(d) A prime element of B, namely x, divides infinitely many distinct prime elements of A.

(2) The properties of being a qfc extension and a flat extension are independent.

For, a ring of invariants A of a semisimple group acting on a polynomial ring B is fc in B, but the extension is not flat in general.

On the other hand, the extension $k[t^2] \subseteq k[t]$ is flat but not qfc.

(3) For an extension of normal affine domains $A \subseteq B$, the set of points $\mathfrak{m} \in \operatorname{Max} B$ such that $A_{\mathfrak{m}\cap A} \subseteq B_{\mathfrak{m}}$ is qfc is in general not Zariski-open in $\operatorname{Max} B$.

An example of this is the inclusion $A := k[x] \subseteq B := k[x, y, z]/(xy - z^2)$. If $\mathfrak{m}_0$ is the maximal ideal corresponding to the origin $(0, 0, 0)$ in B, then $A_{(x)}$ is qfc in $B_{\mathfrak{m}_0}$, but for maximal ideals corresponding to nearby points $(0, \lambda, 0)$ this is not true.

Remark. One may ask a similar question for fc extensions. But at least in the case of affine domains it is not very interesting, for then by Lemma 4 $A_{\mathfrak{m}\cap A} = B_{\mathfrak{m}}$. So A and B must be birational where the set of such points is clearly open.

(4) The ring extension $A := k[xy] \subseteq B := k[x, y]$ is such that A is algebraically closed in B and $A^* = B^*$, but the fc locus $\mathfrak{FC}(A : B)$ is empty.

(5) The ring extension $A := k[x] \subset B := k[x, y, z]/(x^2 + y^2 + z^2 - 1)$ has the property that any irreducible element of A remains irreducible in B, the extension is faithfully flat and both rings have same units, but A is not fc in B. Note that B is not factorial.

(6) Let $A := k[x, xy] \subset B := k[x, y]$, where x, y are indeterminates. Then $Q(A) \subset Q(B)$ is maximally algebraic. Any element of the form $x + axy$ is a prime element in A but not a prime element in B, where $a \in k^*$.

If $\mathfrak{q}$ is any prime ideal in A other than (x, xy), then either x or xy is a unit in $A_{\mathfrak{q}}$. Hence both x, y are units in $B_{\mathfrak{q}}$. It follows that any prime element in $A_{\mathfrak{q}}$ is either a prime element in $B_{\mathfrak{q}}$ or a unit in $B_{\mathfrak{q}}$. This shows that $\mathfrak{FC}(A : B) = \operatorname{Spec} A \setminus \{(x, xy)\}$ is nonempty and open, but infinitely many prime elements in A are non-units in B and are not prime elements in B.

If $\mathfrak{p}$ is any height-1 prime ideal in A, then at least one of x, xy does not lie in $\mathfrak{p}$. Hence, in $B_{\mathfrak{p}}$, x is always a prime element. From this we see that $A_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ is qfc.

Clearly A is not fc in B. Since A, B are UFDs and have the same units, A is not qfc in B.

This shows that the local analogue of Lemma 1(6) does not hold for the qfc property.

4. Open problems

(1) Let $A \subseteq B$ be normal complete local domains over k such that A is qfc in B. Is the power series ring in one variable $A[\![x]\!]$ qfc in $B[\![x]\!]$?

(2) Suppose that $A \subseteq B$ are normal affine domains such that for any maximal ideal $\mathfrak{m} \subset A$ the extension $A_{\mathfrak{m}} \subseteq B_{\mathfrak{m}}$ is qfc. Is A qfc in B?

(3) Let $A \subseteq B$ be an qfc inclusion of normal complete domains over k. Is $\dim A \leq \dim B$?

(4) Is any fc subring of a PID also a PID?

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Coherent analogues of matrix factorizations and relative singularity categories

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We define the triangulated category of relative singularities of a closed subscheme in a scheme. When the closed subscheme is a Cartier divisor, we consider matrix factorizations of the related section of a line bundle, and their analogues with locally free sheaves replaced by coherent ones. The appropriate exotic derived category of coherent matrix factorizations is then identified with the triangulated category of relative singularities, while the similar exotic derived category of locally free matrix factorizations is its full subcategory. The latter category is identified with the kernel of the direct image functor corresponding to the closed embedding of the zero locus and acting between the conventional (absolute) triangulated categories of singularities. Similar results are obtained for matrix factorizations of infinite rank; and two different "large" versions of the triangulated category of relative singularities, corresponding to the approaches of Orlov and Krause, are identified in the case of a Cartier divisor. A version of the Thomason–Trobaugh–Neeman localization theorem is proven for coherent matrix factorizations and disproven for locally free matrix factorizations of finite rank. Contravariant (coherent) and covariant (quasicoherent) versions of the Serre–Grothendieck duality theorems for matrix factorizations are established, and pull-backs and push-forwards of matrix factorizations are discussed at length. A number of general results about derived categories of the second kind for curved differential graded modules (CDG-modules) over quasicoherent CDG-algebras are proven on the way. Hochschild (co)homology of matrix factorization categories are discussed in an appendix.

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Introduction

A *matrix factorization* of an element w in a commutative ring R is a pair of square matrices (Φ, Ψ) of the same size, with entries from R, such that both the products $\Phi\Psi$ and $\Psi\Phi$ are equal to w times the identity matrix. In the coordinate-free language, a matrix factorization is a pair of finitely generated free R-modules M^0 and M^1 together with R-module homomorphisms $M^0 \to M^1$ and $M^1 \to M^0$ such that both the compositions $M^0 \to M^1 \to M^0$ and $M^1 \to M^0 \to M^1$ are equal to the multiplication with w. Matrix factorizations were introduced by Eisenbud [1980] and used by Buchweitz [1986] for the study of the maximal Cohen–Macaulay modules over hypersurface local rings.

Another name for this notion is "D-branes in the Landau–Ginzburg B model" (as suggested by Kontsevich) [Kapustin and Li 2003]; in this context, the element w is called the *potential*. One generalizes the above definition, replacing free modules with projective modules [Kapustin and Li 2003; Orlov 2004], with locally free sheaves [Orlov 2012], and finally with coherent sheaves [Lin and Pomerleano 2013]. The importance of the latter generalization is emphasized in the present paper.

Being particular cases of curved DG-modules over a curved DG-ring [Kapustin and Li 2003; Positselski 2011b], matrix factorizations form a DG-category. So one can consider the corresponding category of closed degree-zero morphisms up to chain homotopy, which is a triangulated category. Generally speaking, however, the homotopy category is "too big" for most purposes, and one would like to pass from it to an appropriately defined derived category. One can use the homotopy category in lieu of the derived one when dealing with projective modules [Kapustin and Li 2003; Orlov 2004]; for locally free matrix factorizations over a nonaffine scheme, there is an option of working with the quotient category of the homotopy category by the locally contractible objects [Polishchuk and Vaintrob 2011, Definition 3.13]. When dealing with coherent (analogues of) matrix factorizations, having some kind of derived category construction is apparently unavoidable.

The relevant concept of a derived category is that of the derived category of the second kind, as developed in [Positselski 2010; 2011b]. There are several versions of this notion; the appropriate one for quasicoherent sheaves is called the *coderived category* and for coherent sheaves it is the *absolute derived category*. The absolute derived category of locally free matrix factorizations was studied in [Orlov 2012]; for coherent matrix factorizations over a smooth variety, it was considered in [Lin and Pomerleano 2013]. These two absolute derived categories are equivalent for regular schemes, but can be different otherwise (as we show with an explicit counterexample).

The *triangulated category of singularities* of a Noetherian scheme was defined by D. Orlov [2004] as the quotient category of the bounded derived category of coherent sheaves by its full triangulated subcategory of perfect complexes, i.e., the objects locally presentable as finite complexes of locally free sheaves. This triangulated category vanishes if and only if the Noetherian scheme is regular. It was shown in [Orlov 2004, Theorem 3.9], under mild assumptions on an affine regular Noetherian scheme X and a potential (regular function) w on it, that the homotopy category of locally free matrix factorizations of w over X is equivalent to the triangulated category of singularities of the zero locus X_0 of w in X.

Orlov [2012] showed that the affineness assumption on X can be dropped in this result if one replaces the homotopy category of locally free matrix factorizations with their absolute derived category. He also considers the general case of a nonaffine singular scheme X, for which he obtains a fully faithful functor from the absolute derived category of locally free matrix factorizations over X to the triangulated category of singularities of X_0. The problem of studying the difference between these two triangulated categories was posed in the introduction to [Orlov 2012].

The first aim of the present paper is to provide an alternative proof of these results of Orlov for regular schemes, an alternative generalization of them to singular schemes, and a more precise version of Orlov's original generalization. We replace the triangulated category at the source of Orlov's fully faithful functor by a "larger" category (containing the original one) and the triangulated category at the target by a "smaller" category (a quotient of the original one), thereby transforming this functor into an equivalence of triangulated categories. We also describe the image of Orlov's fully faithful functor as the kernel of a certain other triangulated functor.

More precisely, we show that the absolute derived category of coherent matrix factorizations of w over X is equivalent to what we call the *triangulated category of singularities of X_0 relative to X*. The latter category is a certain quotient category of the triangulated category of singularities of X_0; it measures, roughly speaking, how much worse are the singularities of X_0 compared to those of X. As to the image of Orlov's fully faithful embedding, it consists precisely of those objects of the conventional (absolute) triangulated category of singularities of X_0 whose direct images vanish in the triangulated category of singularities of X.

The paper consists of three sections and two appendices. In Section 1, we prove three rather general technical assertions about derived categories of the second kind for curved differential graded modules (CDG-modules) over a quasicoherent CDG-algebra with a restriction on the homological dimension. One of them, claiming that certain embeddings of DG-categories of CDG-modules induce equivalences of the derived categories of the second kind, is a generalization of [Polishchuk and Positselski 2012, Theorem 3.2] based on a modification of the same argument, originally introduced for the proof of [Positselski 2010, Theorem 7.2.2].

The idea of the proof of the other assertion, according to which certain natural functors between derived categories of the second kind are fully faithful, is new.

The third technical assertion explains when the coderived category coincides with the absolute derived category of the same class of CDG-modules: e.g., for the locally projective CDG-modules this is true.

A version of (the former two of) these results is used in Section 2 to extend Orlov's cokernel functor from the absolute derived category of locally free matrix factorizations to the absolute derived category of coherent ones. This extension of the cokernel functor admits a simple construction of a functor in the opposite direction, suggested in [Lin and Pomerleano 2013]. We use these constructions to obtain a new proof of Orlov's theorem, and our own generalization of it to the singular case.

When X is regular, Orlov's and our results amount to the same assertion since the absolute derived categories of locally free and coherent matrix factorizations are equivalent by our Theorem 1.4. When X is singular, the natural functor between these two absolute derived categories is fully faithful by our Proposition 1.5, and Orlov's full-and-faithfulness theorem follows from ours by virtue of an appropriate semiorthogonality property.

We also compare a "large" version of the triangulated category of relative singularities with the coderived category of quasicoherent matrix factorizations, strengthening some results of Polishchuk and Vaintrob [2011]. A "large" version of the absolute triangulated category of singularities, defined by Orlov [2004], is identified with H. Krause's stable derived category [2005] in the case of a divisor in a regular scheme. A similar result is proven in the case of a Cartier divisor in a singular scheme, where we extend Krause's theory by defining the *relative stable derived category*. For any closed subscheme of finite flat dimension in a separated Noetherian scheme, the relative stable derived category is compactly generated by its full triangulated subcategory equivalent to the triangulated category of relative singularities.

The homotopy categories of unbounded complexes of projective modules over a ring and injective quasicoherent sheaves over a scheme were studied by Jørgensen [2005] and Krause [2005]; subsequently, Iyengar and Krause [2006] constructed an equivalence between these two categories for rings with dualizing complexes. These results were extended to quasicoherent sheaves over schemes by Neeman [2008] and Murfet [2007], who found a way to define a replacement of the homotopy category of (nonexistent) projective sheaves in terms of the flat ones. The equivalence between these two categories is a covariant version of Serre–Grothendieck duality [Hartshorne 1966]. It is also very similar to the derived comodule-contramodule correspondence theory, developed by the second author [Positselski 2010; 2011b].

Serre–Grothendieck duality for matrix factorizations in the situation of a smooth variety X (and an isolated singularity of X_0) was studied in [Murfet 2013]. In this paper we extend the duality to matrix factorizations over much more general schemes X, constructing an equivalence between two "large" exotic derived categories, namely, the coderived category of flat (or locally free) matrix factorizations of possibly infinite rank and the coderived category of quasicoherent matrix factorizations. Unless X is Gorenstein, this equivalence is *not* provided by the natural functor induced by the embedding of DG-categories, but rather differs from it in that the tensor product with the dualizing complex has to be taken along the way. A contravariant Serre duality in the form of an auto-antiequivalence of the absolute derived category of coherent matrix factorizations is also obtained.

There was some attention paid to pull-backs and push-forwards of matrix factorizations recently [Polishchuk and Vaintrob 2011; 2014; Dyckerhoff and Murfet 2013]. In Section 3, we approach this topic with our techniques, constructing the push-forwards of locally free matrix factorizations of infinite rank for any morphism of finite flat dimension between schemes of finite Krull dimension, and the push-forwards of locally free matrix factorizations of finite rank for any such morphism for which the induced morphism of the zero loci of w is proper. At the price of having to adjoin the images of idempotent endomorphisms, the preservation of finite rank under push-forwards is proven assuming only the support of the matrix factorization [Polishchuk and Vaintrob 2011] to be proper over the base.

Push-forwards of quasicoherent matrix factorizations are well-defined for any morphism of Noetherian schemes, and push-forwards of coherent matrix factorizations exist under properness assumptions similar to the above. A general study of category-theoretic and set-theoretic supports of quasicoherent and coherent CDG-modules is undertaken in this paper in order to obtain an independent proof of the preservation of coherence under the push-forwards not based on the passage to the triangulated categories of singularities.

The compatibility with pull-backs and push-forwards is an organic part of Serre–Grothendieck duality theory. The contravariant duality agrees with push-forwards of coherent sheaves (or matrix factorizations) with respect to proper morphisms [Hartshorne 1966], while the covariant duality transforms the conventional inverse image of flat matrix factorizations into the extraordinary inverse image of quasicoherent ones [Positselski 2012]. We use the latter result in order to construct the extraordinary inverse image functor of Hartshorne and Deligne, which is denoted by $f^!$ in [Hartshorne 1966] and which we denote by f^+, in the case of quasicoherent matrix factorizations.

Appendix A contains proofs of some basic facts about flat, locally projective, and injective quasicoherent graded modules which are occasionally used in the main body of the paper. Appendix B can be viewed as a complement to the paper [Polishchuk and Positselski 2012]. While Section B.1 contains some variations of and improvements on the results about Hochschild (co)homology of (C)DG-categories and (locally free) matrix factorizations in [loc. cit.], Section B.2 presents an alternative approach to the Hochschild (co)homology of coherent matrix factorizations based on the techniques developed in the main body of this paper.

1. Exotic derived categories of quasicoherent CDG-modules

1.1. ***CDG-rings and CDG-modules.*** A *CDG-ring* (curved differential graded ring) $B = (B, d, h)$ is defined as a graded ring $B = \bigoplus_{i\in\mathbb{Z}} B^i$ endowed with an odd derivation $d : B \to B$ of degree 1 and an element $h \in B^2$ such that $d^2(b) = [h, b]$ for all $b \in B$ and $d(h) = 0$. So one should have $d : B^i \to B^{i+1}$ and $d(ab) = d(a)b + (-1)^{|a|}ad(b)$; the brackets $[-,-]$ denote the supercommutator $[a, b] = ab - (-1)^{|a||b|}ba$. The element h is called the *curvature element*.

A morphism of CDG-rings $B \to A$ is a pair (f, a), with a morphism of graded rings $f : B \to A$ and an element $a \in A^1$ such that $f(d_B b) = d_A f(b) + [a, f(b)]$ for all $b \in B$ and $f(h_B) = h_A + d_A a + a^2$. The composition of morphisms of CDG-rings is defined by the obvious rule $(f, a) \circ (g, b) = (f \circ g,\ a + f(b))$. The element a is called the *change-of-connection element*. A discussion of the origins of these definitions can be found in the paper [Positselʹskiĭ 1993], where the above terminology first appeared (see also an earlier paper [Getzler and Jones 1990], where the motivation was entirely different).

A *left CDG-module* $M = (M, d_M)$ over a CDG-ring B is a graded B-module endowed with an odd derivation $d_M : M \to M$ compatible with the derivation d on B such that $d_M^2(m) = hm$ for all $m \in M$. Given a morphism of CDG-rings $(f, a) : B \to A$ and a CDG-module (M, d) over A, the CDG-module (M, d') over B is defined by the rule $d'(m) = d(m) + am$.

Given graded left B-modules M and N, homogeneous B-module morphisms $f : M \to N$ of degree n are defined as homogeneous maps supercommuting with the action of B; i.e., $f(bm) = (-1)^{n|b|}bf(m)$. When M and N are CDG-modules, the homogeneous B-module morphisms $M \to N$ form a complex of abelian groups with the differential $d(f)(m) = d(f(m)) - (-1)^{|f|} f(d(m))$. The curvature-related terms cancel out in the computation of the square of this differential, so one has $d^2(f) = 0$. Therefore, left CDG-modules over B form a DG-category.

Two aspects of the above definitions are worth pointing out. First, the CDG-rings or modules have *no* cohomology modules, as their differentials do not square to zero. Second, given a CDG-ring B, there is *no* natural way to define a CDG-module structure on the free graded B-module B (though B is naturally a CDG-bimodule over itself, in the appropriate sense).

We refer the reader to [Positselski 2011b, Section 3.1] or [Positselski 2010, Sections 0.4.3–0.4.5] for more detailed discussions of the above notions. We will not need to consider any gradings different from $\mathbb{Z}$-gradings in this paper, though all the general results will be equally applicable in the Γ-graded situation in the sense of [Polishchuk and Positselski 2012, Section 1.1].

1.2. ***Quasicoherent CDG-algebras.*** Throughout this paper, unless specified otherwise, X is a separated Noetherian scheme with enough vector bundles; in other words, it is assumed that every coherent sheaf on X is the quotient sheaf of a locally free sheaf of finite rank. Note that the class of all schemes satisfying these conditions is closed under the passages to open and closed subschemes [Orlov 2004, Section 1.2] and contains all regular separated Noetherian schemes [Hartshorne 1977, Exercise III.6.8].

Recall the definition of a *quasicoherent CDG-algebra* from [Positselski 2011b, Appendix B]. A quasicoherent CDG-algebra $\mathcal{B}$ over X is a graded quasicoherent $\mathcal{O}_X$-algebra such that for each affine open subscheme $U \subset X$, the graded ring $\mathcal{B}(U)$ is endowed with a structure of CDG-ring, i.e., a (not necessarily $\mathcal{O}_X$-linear) odd derivation $d : \mathcal{B}(U) \to \mathcal{B}(U)$ of degree 1 and an element $h \in \mathcal{B}^2(U)$. For each pair of embedded affine open subschemes $U \subset V \subset X$, an element $a_{UV} \in \mathcal{B}^1(U)$ is fixed such that the restriction morphism $\mathcal{B}(V) \to \mathcal{B}(U)$ together with the element a_{UV} form a morphism of CDG-rings. The obvious compatibility condition is imposed for triples of embedded affine open subschemes $U \subset V \subset W \subset X$.

A *quasicoherent left CDG-module* $\mathcal{M}$ over $\mathcal{B}$ is an $\mathcal{O}_X$-quasicoherent (or, equivalently, $\mathcal{B}$-quasicoherent) sheaf of graded left modules over $\mathcal{B}$ together with a family of differentials $d : \mathcal{M}(U) \to \mathcal{M}(U)$ defined for all affine open subschemes $U \subset X$ such that $\mathcal{M}(U)$ is a CDG-module over $\mathcal{B}(U)$ and the appropriate compatibility condition holds with respect to the restriction morphisms of CDG-rings $\mathcal{B}(V) \to \mathcal{B}(U)$. Specifically, for a quasicoherent left CDG-module $\mathcal{M}$, one should have

$$d(s)|_U = d(s|_U) + a_{UV} s|_U \quad \text{for any } s \in \mathcal{M}(V).$$

Quasicoherent left CDG-modules over a quasicoherent CDG-algebra $\mathcal{B}$ form a DG-category [Positselski 2011b]. The complex of morphisms between CDG-modules $\mathcal{N}$ and $\mathcal{M}$ is the graded abelian group of homogeneous $\mathcal{B}$-module morphisms $f : \mathcal{N} \to \mathcal{M}$ with the differential $d(f)$ defined locally as the supercommutator of f with the differentials in $\mathcal{N}(U)$ and $\mathcal{M}(U)$. We denote this DG-category by $\mathcal{B}$-qcoh.

We will call a quasicoherent graded algebra $\mathcal{B}$ over X *Noetherian* if the graded ring $\mathcal{B}(U)$ is left Noetherian for any affine open subscheme $U \subset X$. Equivalently, $\mathcal{B}$ is Noetherian if the abelian category of quasicoherent graded left $\mathcal{B}$-modules is a locally Noetherian Grothendieck category. In this case, the full DG-subcategory in $\mathcal{B}$-qcoh formed by CDG-modules whose underlying graded $\mathcal{B}$-modules are coherent (i.e., finitely generated over $\mathcal{B}$) is denoted by $\mathcal{B}$-coh.

Given a quasicoherent graded left $\mathcal{B}$-module $\mathcal{M}$ and a quasicoherent graded right $\mathcal{B}$-module $\mathcal{N}$, one can define their tensor product $\mathcal{N} \otimes_{\mathcal{B}} \mathcal{M}$, which is a quasicoherent graded $\mathcal{O}_X$-module. A quasicoherent graded left $\mathcal{B}$-module $\mathcal{M}$ is called *flat* if the functor $- \otimes_{\mathcal{B}} \mathcal{M}$ is exact on the abelian category of quasicoherent graded right $\mathcal{B}$-modules. Equivalently, $\mathcal{M}$ is flat if the graded left $\mathcal{B}(U)$-module $\mathcal{M}(U)$ is flat for any affine open subscheme $U \subset X$. The *flat dimension* of a quasicoherent graded module $\mathcal{M}$ is the minimal length of its flat left resolution.

The full DG-subcategory in $\mathcal{B}$-qcoh formed by CDG-modules whose underlying graded $\mathcal{B}$-modules are flat is denoted by $\mathcal{B}\text{-qcoh}_{\mathsf{fl}}$, and the full subcategory formed by CDG-modules whose underlying graded $\mathcal{B}$-modules have finite flat dimension is denoted by $\mathcal{B}\text{-qcoh}_{\mathsf{ffd}}$. The similarly defined DG-categories of coherent CDG-modules are denoted by $\mathcal{B}\text{-coh}_{\mathsf{fl}}$ and $\mathcal{B}\text{-coh}_{\mathsf{ffd}}$.

All the above DG-categories of quasicoherent CDG-modules (and the similar ones defined below in this paper) admit shifts and twists, and, in particular, cones. It follows that their homotopy categories $H^0(\mathcal{B}\text{-qcoh})$, $H^0(\mathcal{B}\text{-qcoh}_{\mathsf{fl}})$, $H^0(\mathcal{B}\text{-coh})$, etc. are triangulated. Besides, to any finite complex (of objects and closed morphisms) in one of these DG-categories, one can assign its total object, which is an object of (i.e., a CDG-module belonging to) the same DG-category [Positselski 2011b, Section 1.2].

The DG-categories $\mathcal{B}$-qcoh and $\mathcal{B}\text{-qcoh}_{\mathsf{fl}}$ also admit infinite direct sums. Hence in these two DG-categories one can totalize even an unbounded complex by taking infinite direct sums along the diagonals.

The DG-category $\mathcal{B}$-qcoh also admits infinite products (which one can obtain using the coherator construction from [Thomason and Trobaugh 1990, Section B.14]), but these are not well-behaved (neither exact nor local), so we will not use them.

1.3. ***Derived categories of the second kind.*** The nonexistence of the cohomology groups for curved structures stands in the way of the conventional definition of the derived category of CDG-modules, which therefore does not seem to make sense. The suitable class of constructions of derived categories for CDG-modules is that of the *derived categories of the second kind* [Positselski 2010; 2011b].

Let $\mathcal{B}$ be a quasicoherent CDG-algebra over X; assume that the quasicoherent graded algebra $\mathcal{B}$ is Noetherian. Then a coherent CDG-module over $\mathcal{B}$ is called *absolutely acyclic* if it belongs to the minimal thick subcategory of the homotopy category of coherent CDG-modules $H^0(\mathcal{B}\text{-coh})$ containing the total CDG-modules of all the short exact sequences of coherent CDG-modules over $\mathcal{B}$ (with closed morphisms between them). The quotient category of $H^0(\mathcal{B}\text{-coh})$ by the thick subcategory of absolutely acyclic CDG-modules is called the *absolute derived category* of coherent CDG-modules over $\mathcal{B}$ and denoted by $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-coh})$ [Positselski 2011b].

For any quasicoherent CDG-algebra $\mathcal{B}$ over X, a quasicoherent CDG-module over $\mathcal{B}$ is called *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category of quasicoherent CDG-modules $H^0(\mathcal{B}\text{-qcoh})$ containing the total CDG-modules of all the short exact sequences of quasicoherent CDG-modules over $\mathcal{B}$ and closed under infinite direct sums. The quotient category of $H^0(\mathcal{B}\text{-coh})$ by the thick subcategory of coacyclic CDG-modules is called the *coderived category* of quasicoherent CDG-modules over $\mathcal{B}$ and denoted by $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-qcoh})$ [Positselski 2010; 2011b].

Given an exact subcategory E in the abelian category of quasicoherent graded left $\mathcal{B}$-modules, one can define the *absolute derived category of left CDG-modules over $\mathcal{B}$ with the underlying graded $\mathcal{B}$-modules belonging to* E as the quotient category of the corresponding homotopy category by its minimal thick subcategory containing the total CDG-modules of all the exact triples of CDG-modules with the underlying graded $\mathcal{B}$-modules belonging to E. The objects of the latter subcategory are called *absolutely acyclic with respect to* E (or with respect to the DG-category of CDG-modules with the underlying graded modules belonging to E) [Polishchuk and Positselski 2012].

In particular, one defines the absolute derived categories $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{ffd}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}})$ as the quotient categories of the homotopy categories $H^0(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{ffd}})$ and $H^0(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}})$ by the thick subcategories of CDG-modules absolutely acyclic with respect to $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{ffd}}$ and $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}}$, respectively.

When the exact subcategory E is closed under infinite direct sums, the thick subcategory of CDG-modules *coacyclic with respect to* E is the minimal triangulated subcategory of the homotopy category CDG-modules with the underlying graded modules belonging to E, containing the total CDG-modules of all the exact triples of CDG-modules with the underlying graded modules belonging to E and closed under infinite direct sums. The quotient category by this thick subcategory is called the *coderived category of left CDG-modules over $\mathcal{B}$ with the underlying graded modules belonging to* E [Positselski 2010; Polishchuk and Positselski 2012].

Thus one defines the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ as the quotient categories of the homotopy category $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ by the thick subcategory of CDG-modules coacyclic with respect to $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}$. A little more care is needed for the definition of the coderived category $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ since the class of graded modules of finite flat dimension is not in general closed under infinite direct sums. An object $\mathcal{M} \in H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ is said to be *coacyclic with respect to $\mathcal{B}$*-$\mathsf{qcoh}_{\mathsf{ffd}}$ if there exists an integer $d \geq 0$ such that $\mathcal{M}$ is coacyclic with respect to the exact category of quasicoherent CDG-modules of flat dimension at most d. The coderived category of quasicoherent CDG-modules of finite flat dimension is, by the definition, the quotient category of $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ by the above-defined thick subcategory of coacyclic CDG-modules [Polishchuk and Positselski 2012, Section 3.2].

Remark 1.3. One may wonder whether coacyclicity (absolute acyclicity) of quasicoherent CDG-modules (of a certain class) is a local notion. One general approach to this kind of problem is to consider the Mayer–Vietoris/Čech exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \bigoplus_{\alpha} j_{U_\alpha *} j_{U_\alpha}^* \mathcal{M} \longrightarrow \bigoplus_{\alpha<\beta} j_{U_\alpha \cap U_\beta *} j_{U_\alpha \cap U_\beta}^* \mathcal{M} \longrightarrow \cdots \longrightarrow 0$$

for a finite affine open covering U_α of X. Since the inverse and direct images with respect to affine open embeddings are exact and compatible with direct sums, they preserve coacyclicity (absolute acyclicity). Hence if the restrictions of $\mathcal{M}$ to all U_α are coacyclic (absolutely acyclic), then so is $\mathcal{M}$ itself.

Alternatively, one can base this kind of argument on the implications of the Noetherianness assumption, rather than the separatedness assumption. For this purpose, one replaces a quasicoherent CDG-module $\mathcal{M}$ with its injective resolution (see Lemma 1.7(b)) before writing down its Čech resolution. In this approach, the covering need not be affine, as injective coacyclic objects are contractible, and direct images preserve contractibility; but it is important that the restrictions to open subschemes should preserve injectivity of quasicoherent graded $\mathcal{B}$-modules (see [Hartshorne 1966, Theorem II.7.18] and Theorem A.3; cf. [Thomason and Trobaugh 1990, Appendix B]).

When one is working with coherent CDG-modules, the Čech sequence argument is to be used in conjunction with Proposition 1.5 below. (Cf. Sections 1.10 and 3.2.)

1.4. *Finite flat dimension theorem.* The next theorem is our main technical result on which the proofs in Section 2 are based.

Though we generally prefer the coderived categories of (various classes of) infinitely generated CDG-modules over their absolute derived categories, technical considerations sometimes force us to deal with the latter (see Remark 1.5). Therefore, let $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$, $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$, and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh})$ denote the absolute derived categories of (flat, of finite flat dimension, or arbitrary) quasicoherent CDG-modules over a quasicoherent CDG-algebra $\mathcal{B}$.

Theorem 1.4. (a) *For any quasicoherent CDG-algebra $\mathcal{B}$ over X, the functor*

$$\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$$

induced by the embedding of DG-categories $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}$ is an equivalence of triangulated categories.

(b) *For any quasicoherent CDG-algebra $\mathcal{B}$ over X, the functor*

$$\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$$

induced by the embedding of DG-categories $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}$ is an equivalence of triangulated categories.

(c) *For any quasicoherent CDG-algebra $\mathcal{B}$ over X such that the underlying quasicoherent graded algebra $\mathcal{B}$ is Noetherian, the functor*

$$\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{ffd}})$$

induced by the embedding of DG-categories $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}} \to \mathcal{B}\text{-}\mathsf{coh}_{\mathsf{ffd}}$ is an equivalence of triangulated categories.

Proof. The proof follows that of [Polishchuk and Positselski 2012, Theorem 3.2] (see also [Positselski 2010, Theorem 7.2.2]) with some modifications. We will prove part (a); the proofs of parts (b) and (c) are similar. (Alternatively, parts (b) and (c) can be deduced from Proposition 1.5(a) and (b) below.)

Given an affine open subscheme $U \subset X$ and a graded module P over the graded ring $\mathcal{B}(U)$, one can construct the freely generated CDG-module $G^+(P)$ over the CDG-ring $\mathcal{B}(U)$ in the way explained in [Positselski 2011b, proof of Theorem 3.6]. The elements of $G^+(P)$ are formal expressions of the form $p+dq$, where $p, q \in P$. Given a quasicoherent graded module $\mathcal{P}$ over $\mathcal{B}$, the CDG-modules $G^+(\mathcal{P}(U))$ glue together to form a quasicoherent CDG-module $G^+(\mathcal{P})$ over $\mathcal{B}$. For any quasicoherent CDG-module $\mathcal{M}$ over $\mathcal{B}$, there is a bijective correspondence between morphisms of graded $\mathcal{B}$-modules $\mathcal{P} \to \mathcal{M}$ and closed morphisms of CDG-modules $G^+(\mathcal{P}) \to \mathcal{M}$ over $\mathcal{B}$. There is a natural short exact sequence of quasicoherent graded $\mathcal{B}$-modules $\mathcal{P} \to G^+(\mathcal{P}) \to \mathcal{P}[-1]$. The quasicoherent CDG-module $G^+(\mathcal{P})$ is naturally contractible with the contracting homotopy $t_{\mathcal{P}}$ given by the composition

$$G^+(\mathcal{P}) \longrightarrow \mathcal{P}[-1] \longrightarrow G^+(\mathcal{P})[-1].$$

Due to our assumption on X, for any quasicoherent $\mathcal{O}_X$-module $\mathcal{K}$ over X there exists a surjective morphism $\mathcal{E} \to \mathcal{K}$ onto $\mathcal{K}$ from a direct sum $\mathcal{E}$ of locally free sheaves of finite rank on X. Hence for any quasicoherent graded $\mathcal{B}$-module $\mathcal{M}$ there is a surjective morphism onto $\mathcal{M}$ from a flat quasicoherent graded $\mathcal{B}$-module $\mathcal{P} = \bigoplus_n \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}_n[n]$, and for any quasicoherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ there is a surjective closed morphism onto $\mathcal{M}$ from the CDG-module $G^+(\mathcal{P}) \in \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}$. (In fact, parts (a) and (b) of this theorem can be proven without the assumption of enough vector bundles on X since there are always enough flat sheaves; see Remark 2.6 and Lemma A.1.)

Now the construction from [loc. cit., proof of Theorem 3.6] provides for any object $\mathcal{M}$ of $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}$ a closed morphism onto $\mathcal{M}$ from an object of $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}$ with the cone absolutely acyclic with respect to $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}$. To obtain this morphism, one picks a finite left resolution of $\mathcal{M}$ consisting of objects from $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}$ with closed morphisms between them, and takes the total CDG-module of this resolution. By [loc. cit., Lemma 1.6], it follows that the triangulated category $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ is equivalent to the quotient category of $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ by its intersection in $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ with the thick subcategory of CDG-modules coacyclic with respect to $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}$. It only remains to show that any object of $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ that is coacyclic with respect to $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}$ is coacyclic with respect to $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}$.

Let us call a quasicoherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ *d-flat* if its underlying quasicoherent graded $\mathcal{B}$-module $\mathcal{M}$ has flat dimension not exceeding d. A d-flat quasicoherent CDG-module is said to be *d-coacyclic* if it is homotopy equivalent to a CDG-module obtained from the total CDG-modules of exact triples of d-flat

CDG-modules using the operations of cone and infinite direct sum. Our goal is to show that any 0-flat d-coacyclic CDG-module is 0-coacyclic. For this purpose, we will prove that any $(d-1)$-flat d-coacyclic CDG-module is $(d-1)$-coacyclic; the desired assertion will then follow by induction.

It suffices to construct for any d-coacyclic CDG-module $\mathcal{M}$ a $(d-1)$-coacyclic CDG-module $\mathcal{L}$ with a $(d-1)$-coacyclic CDG-submodule $\mathcal{K}$ such that the quotient CDG-module $\mathcal{L}/\mathcal{K}$ is isomorphic to $\mathcal{M}$. Then if $\mathcal{M}$ is $(d-1)$-flat, it would follow that both the cone of the morphism $\mathcal{K} \to \mathcal{L}$ and the total CDG-module of the exact triple $\mathcal{K} \to \mathcal{L} \to \mathcal{M}$ are $(d-1)$-coacyclic, so $\mathcal{M}$ also is. The construction is based on four lemmas similar to those in [Polishchuk and Positselski 2012, Section 3.2].

Lemma A. *Let $\mathcal{M}$ be the total CDG-module of an exact triple of d-flat quasicoherent CDG-modules $\mathcal{M}' \to \mathcal{M}'' \to \mathcal{M}'''$ over $\mathcal{B}$. Then there exists a surjective closed morphism onto $\mathcal{M}$ from a contractible* 0*-flat CDG-module $\mathcal{P}$ with a* $(d-1)$*-coacyclic kernel $\mathcal{K}$.*

Proof. Choose 0-flat quasicoherent CDG-modules $\mathcal{P}'$ and $\mathcal{P}'''$ such that there exist surjective closed morphisms $\mathcal{P}' \to \mathcal{M}'$ and $\mathcal{P}''' \to \mathcal{M}''$. Then there exists a surjective morphism from the exact triple of CDG-modules $\mathcal{P}' \to \mathcal{P}' \oplus \mathcal{P}''' \to \mathcal{P}'''$ onto the exact triple $\mathcal{M}' \to \mathcal{M}'' \to \mathcal{M}'''$. The rest of the proof is similar to that in [Polishchuk and Positselski 2012]. □

Lemma B. (a) *Let $\mathcal{K}' \subset \mathcal{L}'$ and $\mathcal{K}'' \subset \mathcal{L}''$ be* $(d-1)$*-coacyclic CDG-submodules in* $(d-1)$*-coacyclic CDG-modules, and let $\mathcal{L}'/\mathcal{K}' \to \mathcal{L}''/\mathcal{K}''$ be a closed morphism of CDG-modules. Then there exists a* $(d-1)$*-coacyclic CDG-module $\mathcal{L}$ with a* $(d-1)$*-coacyclic CDG-submodule $\mathcal{K}$ such that*

$$\mathcal{L}/\mathcal{K} \simeq \operatorname{cone}(\mathcal{L}'/\mathcal{K}' \to \mathcal{L}''/\mathcal{K}'').$$

(b) *In the situation of* (a), *assume that the morphism $\mathcal{L}'/\mathcal{K}' \to \mathcal{L}''/\mathcal{K}''$ is injective with a d-flat cokernel $\mathcal{M}_0$. Then there exists a* $(d-1)$*-coacyclic CDG-module $\mathcal{L}_0$ with a* $(d-1)$*-coacyclic CDG-submodule $\mathcal{K}_0$ such that $\mathcal{L}_0/\mathcal{K}_0 \simeq \mathcal{M}_0$.*

Proof. The proof is similar to that in [Polishchuk and Positselski 2012]. □

Lemma C. *For any contractible d-flat CDG-module $\mathcal{M}$ there exists a surjective closed morphism onto $\mathcal{M}$ from a contractible* 0*-flat CDG-module $\mathcal{L}$ with a* $(d-1)$*-coacyclic kernel $\mathcal{K}$.*

Proof. Let $p : \mathcal{P} \to \mathcal{M}$ be a surjective morphism onto the quasicoherent graded $\mathcal{B}$-module $\mathcal{M}$ from a flat quasicoherent graded $\mathcal{B}$-module $\mathcal{P}$, and $\tilde{p} : G^+(\mathcal{P}) \to \mathcal{M}$ be the induced surjective closed morphism of quasicoherent CDG-modules. Let $t : \mathcal{M} \to \mathcal{M}$ be a contracting homotopy for $\mathcal{M}$ and $t_{\mathcal{P}} : G^+(\mathcal{P}) \to G^+(\mathcal{P})$ be the natural contracting homotopy for $G^+(\mathcal{P})$. Then $\tilde{u} = \tilde{p}t_{\mathcal{P}} - t\tilde{p} : G^+(\mathcal{P}) \to \mathcal{M}$ is a closed morphism of quasicoherent CDG-modules of degree -1. Denote by u

the restriction of $\tilde{u}$ to $\mathcal{P} \subset G^+(\mathcal{P})$. There exists a surjective morphism from a flat quasicoherent graded $\mathcal{B}$-module $\mathcal{Q}$ onto the fibered product of the morphisms $p:\mathcal{P} \to \mathcal{M}$ and $u:\mathcal{P} \to \mathcal{M}$. Hence we obtain a surjective morphism of quasicoherent graded $\mathcal{B}$-modules $q : \mathcal{Q} \to \mathcal{P}$ and a morphism of quasicoherent graded $\mathcal{B}$-modules $v : \mathcal{Q} \to \mathcal{P}$ of degree -1 such that $uq = pv$.

The morphism q induces a surjective closed morphism of quasicoherent CDG-modules $\tilde{q} : G^+(\mathcal{Q}) \to G^+(\mathcal{P})$. The morphism $\tilde{q}$ is homotopic to zero with the natural contracting homotopy $\tilde{q}t_{\mathcal{Q}} = t_{\mathcal{P}}\tilde{q}$. The morphism v induces a closed morphism of CDG-modules $\tilde{v} : G^+(\mathcal{Q}) \to G^+(\mathcal{P})$ of degree -1. The morphism $t_{\mathcal{P}}\tilde{q} - \tilde{v}$ is another contracting homotopy for $\tilde{q}$. The latter homotopy forms a commutative square with the morphisms $\tilde{p}$, $\tilde{p}\tilde{q}$, and the contracting homotopy t for the CDG-module $\mathcal{M}$.

Let $\mathcal{N}$ be the kernel of the morphism $\tilde{p}\tilde{q} : G^+(\mathcal{Q}) \to \mathcal{M}$ and $\mathcal{K}$ be the kernel of the morphism $\tilde{p} : G^+(\mathcal{P}) \to \mathcal{M}$. Then the natural surjective closed morphism $r : \mathcal{N} \to \mathcal{K}$ is homotopic to zero; the restriction of the map $t_{\mathcal{P}}\tilde{q} - \tilde{v}$ provides the contracting homotopy that we need. In addition, the kernel $G^+(\ker q)$ of the morphism r is contractible. So the cone of the morphism r is isomorphic to $\mathcal{K} \oplus \mathcal{N}[1]$, and on the other hand, there is an exact triple $G^+(\ker q)[1] \to \operatorname{cone}(r) \to \operatorname{cone}(\mathrm{id}_{\mathcal{K}})$. Since $\mathcal{K}$ is $(d-1)$-flat and $\ker q$ is flat, this proves that $\mathcal{K}$ is $(d-1)$-coacyclic. It remains to take $\mathcal{L} = G^+(\mathcal{P})$. □

Lemma D. *Let $\mathcal{M} \to \mathcal{M}'$ be a homotopy equivalence of d-flat CDG-modules such that $\mathcal{M}'$ is the quotient CDG-module of a $(d-1)$-coacyclic CDG-module by a $(d-1)$-coacyclic CDG-submodule. Then $\mathcal{M}$ is also such a quotient.*

Proof. The proof is similar to that in [Polishchuk and Positselski 2012]. □

It is clear that the property of a CDG-module to be presentable as the cokernel of an injective closed morphism of $(d-1)$-coacyclic CDG-modules is stable under infinite direct sums. This finishes our construction and the proof of Theorem. □

Remark 1.4. The assertion of part (c) of Theorem 1.4 can be equivalently rephrased with flat modules replaced by locally projective ones. Indeed, a finitely presented module over a ring is flat if and only if it is projective.

In the infinitely generated situation of parts (a) and (b), flatness of quasicoherent sheaves is different from their local projectivity (which is a stronger condition), but the assertions remain true after one replaces the former with the latter. The same applies to Proposition 1.5(a) below. Indeed, by Theorem A.2, for any quasicoherent graded algebra $\mathcal{B}$ over an affine scheme U, projectivity of a graded module over the graded ring $\mathcal{B}(U)$ is a local notion. Taking this fact into account, our proof goes through for locally projective quasicoherent graded modules in place of flat ones and the locally projective dimension (defined as the minimal length of a locally projective resolution) in place of the flat dimension.

When $\mathcal{B} = \mathcal{O}_X$, local projectivity of quasicoherent modules is equivalent to local freeness [Bass 1963, Corollary 4.5]. Furthermore, in this case, assuming additionally that X has finite Krull dimension, the classes of quasicoherent sheaves of finite flat dimension and of finite locally projective dimension coincide [Raynaud and Gruson 1971, Corollaire II.3.3.2].

1.5. *Fully faithful embedding.* The next proposition is stronger than Theorem 1.4 in some respects, and is proven by an entirely different technique.

Proposition 1.5. (a) *For any quasicoherent CDG-algebra $\mathcal{B}$ over X, the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to \mathcal{B}\text{-}\mathsf{qcoh}$ *is fully faithful.*

Furthermore, let $\mathcal{B}$ be a quasicoherent CDG-algebra over X such that the underlying quasicoherent graded algebra $\mathcal{B}$ is Noetherian. Then

(b) *the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh})$ *induced by the embedding of DG-categories* $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}} \to \mathcal{B}\text{-}\mathsf{coh}$ *is fully faithful*;

(c) *the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $\mathcal{B}\text{-}\mathsf{coh} \to \mathcal{B}\text{-}\mathsf{qcoh}$ *is fully faithful*;

(d) *the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $\mathcal{B}\text{-}\mathsf{coh} \to \mathcal{B}\text{-}\mathsf{qcoh}$ *is fully faithful and its image forms a set of compact generators for* $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$.

Proof. The proof of part (d) in the case when X is affine can be found in [Positselski 2011b, Section 3.11] (the part concerning compact generation belongs to D. Arinkin). The proof in the general case is similar, and part (c) can be also proven in the way similar to [loc. cit., Theorem 3.11.1]. Part (b) in the affine case is easy and follows from the semiorthogonality property of CDG-modules with projective underlying graded modules and absolutely acyclic/contraacyclic CDG-modules [loc. cit., Theorem 3.5(b)] since finitely generated flat modules over a Noetherian ring are projective. A detailed proof of part (b) in the general case is given below; and the proof of part (a) (which does not automatically simplify in the affine case) is similar.

We will show that any morphism $\mathcal{E} \to \mathcal{L}$ from a CDG-module $\mathcal{E} \in H^0(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}})$ to a CDG-module $\mathcal{L} \in H^0(\mathcal{B}\text{-}\mathsf{coh})$ absolutely acyclic with respect to $\mathcal{B}\text{-}\mathsf{coh}$ can be annihilated by a morphism $\mathcal{P} \to \mathcal{E}$ from a CDG-module $\mathcal{P} \in H^0(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}})$ with a cone of the morphism $\mathcal{P} \to \mathcal{E}$ being absolutely acyclic with respect to $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}}$. By the definition, the CDG-module $\mathcal{L}$ is a direct summand of a CDG-module homotopy equivalent to a CDG-module obtained from the totalizations of exact triples of CDG-modules in $\mathcal{B}\text{-}\mathsf{coh}$ using the operation of passage to the cone of a closed morphism repeatedly. It suffices to consider the case when $\mathcal{L}$ itself is obtained from totalizations of exact triples using cones. We proceed by induction in the number of operations of passage to the cone in such a construction of $\mathcal{L}$.

So we assume that there is a distinguished triangle $\mathcal{K} \to \mathcal{L} \to \mathcal{M} \to \mathcal{K}[1]$ in $H^0(\mathcal{B}\text{-coh})$ such that $\mathcal{M}$ is the total CDG-module of an exact triple of CDG-modules in $\mathcal{B}$-coh, while the CDG-module $\mathcal{K}$ has the desired property with respect to morphisms into it from all CDG-modules $\mathcal{F} \in H^0(\mathcal{B}\text{-coh}_{\mathrm{fl}})$. If we knew that the object $\mathcal{M}$ also has the same property, it would follow that the composition $\mathcal{E} \to \mathcal{L} \to \mathcal{M}$ can be annihilated by a morphism $\mathcal{F} \to \mathcal{E}$ with $\mathcal{F} \in H^0(\mathcal{B}\text{-coh}_{\mathrm{fl}})$ and a cone absolutely acyclic with respect to $\mathcal{B}\text{-coh}_{\mathrm{fl}}$. The composition $\mathcal{F} \to \mathcal{E} \to \mathcal{L}$ then factorizes through $\mathcal{K}$, and the morphism $\mathcal{F} \to \mathcal{K}$ can be annihilated by a morphism $\mathcal{P} \to \mathcal{F}$ with $\mathcal{P} \in H^0(\mathcal{B}\text{-coh}_{\mathrm{fl}})$ and a cone absolutely acyclic with respect to $\mathcal{B}\text{-coh}_{\mathrm{fl}}$. The composition $\mathcal{P} \to \mathcal{F} \to \mathcal{E}$ provides the desired morphism $\mathcal{P} \to \mathcal{E}$.

Thus it remains to construct a morphism $\mathcal{F} \to \mathcal{E}$ with the required properties annihilating a morphism $\mathcal{E} \to \mathcal{M}$, where $\mathcal{M}$ is the total CDG-module of an exact triple of CDG-modules $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$. For any graded module $\mathcal{N}$ over $\mathcal{B}$, morphisms of graded $\mathcal{B}$-modules $\mathcal{N} \to \mathcal{M}$ of degree n are represented by triples (f, g, h), where $f : \mathcal{N} \to \mathcal{U}$ is a morphism of degree $n+1$, $g : \mathcal{N} \to \mathcal{V}$ is a morphism of degree n, and $h : \mathcal{N} \to \mathcal{W}$ is a morphism of degree $n-1$. Denote the closed morphisms in the exact triple $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$ by $j : \mathcal{U} \to \mathcal{V}$ and $k : \mathcal{V} \to \mathcal{W}$.

Lemma E. *Let $\mathcal{N}$ be a CDG-module over $\mathcal{B}$ and $\mathcal{M}$ be the total CDG-module of an exact triple of CDG-modules $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$ as above. Then*

(a) *the differential of a morphism of graded $\mathcal{B}$-modules $\mathcal{N} \to \mathcal{M}$ of degree n represented by a triple (f, g, h) is given by the rule*

$$d(f, g, h) = (-df,\ -jf + dg,\ kg - dh);$$

(b) *when (f, g, h) is a closed morphism of CDG-modules of degree n and the morphism of graded $\mathcal{B}$-modules $h : \mathcal{N} \to \mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $t : \mathcal{N} \to \mathcal{V}$ of degree $n-1$, the morphism (f, g, h) is homotopic to zero.*

Proof. We know that the complex of morphisms in the DG-category of CDG-modules $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{M})$ is the total complex of the bicomplex of abelian groups

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{U}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{V}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{W}).$$

The formula in (a) is the formula for the differential of a total complex.

Furthermore, the sequence $0 \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{U}) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{V}) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$ is exact. Let $\mathrm{Hom}'_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$ denote the cokernel of the morphism of complexes $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{U}) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{V})$; then $\mathrm{Hom}'_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$ is a subcomplex of $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$ and the total complex of the bicomplex

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{U}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathcal{N}, \mathcal{V}) \longrightarrow \mathrm{Hom}'_{\mathcal{B}}(\mathcal{N}, \mathcal{W})$$

is an acyclic subcomplex of $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N},\mathcal{M})$. Hence any cocycle in $\mathrm{Hom}_{\mathcal{B}}(\mathcal{N},\mathcal{M})$ that belongs to this subcomplex is a coboundary.

To present the same argument using our letter notation for morphisms, assume that $kt = h$. Then $k(dt-g) = dh-kg = 0$, so there exists a morphism of graded $\mathcal{B}$-modules $s:\mathcal{N}\to\mathcal{U}$ of degree n such that $dt-g = js$. Then $jds = -dg = -jf$; hence $ds = -f$ and $d(s,t,0) = (f,g,h)$. $\square$

Recall the notation $G^+(\mathcal{Q})$ for the CDG-module freely generated by a graded $\mathcal{B}$-module $\mathcal{Q}$ (see the beginning of the proof of Theorem 1.4).

Lemma F. *Let $\mathcal{M}$ be the total CDG-module of an exact triple of CDG-modules $\mathcal{U}\to\mathcal{V}\to\mathcal{W}$ as above, and let $\mathcal{Q}$ be a graded $\mathcal{B}$-module. Assume that a morphism of graded $\mathcal{B}$-modules $p:\mathcal{Q}\to\mathcal{M}$ of degree n with the components (f,g,h) is given such that the component $h:\mathcal{Q}\to\mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $t:\mathcal{Q}\to\mathcal{V}$ of degree $n-1$. Let $\tilde{p}:G^+(\mathcal{Q})\to\mathcal{M}$ be the induced closed morphism of CDG-modules of degree n and $(\tilde{f},\tilde{g},\tilde{h})$ be its three components. Then the morphism of graded $\mathcal{B}$-modules $\tilde{h}:G^+(\mathcal{Q})\to\mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $\tilde{t}:G^+(\mathcal{Q})\to\mathcal{V}$ of degree $n-1$.*

Proof. Notice that any closed morphism of CDG-modules $G^+(\mathcal{Q})\to\mathcal{M}$ is homotopic to zero since the CDG-module $G^+(\mathcal{Q})$ is contractible. The conclusion of the lemma is stronger, and we will need its full strength. The argument consists of a computation in the letter notation for morphisms.

For any CDG-module $\mathcal{N}$ over $\mathcal{B}$, morphisms of graded $\mathcal{B}$-modules

$$\tilde{r}:G^+(\mathcal{Q})\longrightarrow\mathcal{N}$$

of degree $n-1$ are uniquely determined by their restriction to $\mathcal{Q}$ and the restriction to $\mathcal{Q}$ of their differential $d\tilde{r}$, which can be arbitrary morphisms of graded $\mathcal{B}$-modules $\mathcal{Q}\to\mathcal{N}$ of degrees $n-1$ and n, respectively. Extend our morphism $t:\mathcal{Q}\to\mathcal{V}$ to a morphism of graded $\mathcal{B}$-modules $\tilde{t}:G^+(\mathcal{Q})\to\mathcal{V}$ of degree $n-1$ such that $(d\tilde{t})|_{\mathcal{Q}} = g$. Then $k\tilde{t}|_{\mathcal{Q}} = kt = h = \tilde{h}|_{\mathcal{Q}}$ and $(d(k\tilde{t}))|_{\mathcal{Q}} = k(d\tilde{t})|_{\mathcal{Q}} = kg = k\tilde{g}|_{\mathcal{Q}} = (d\tilde{h})|_{\mathcal{Q}}$ by Lemma E(a), and hence $k\tilde{t} = \tilde{h}$. $\square$

Now represent a closed morphism $\mathcal{E}\to\mathcal{M}$ by a triple (f,g,h) of morphisms of degrees 1, 0, and -1, respectively. Let $\mathcal{Q}$ be a flat coherent graded $\mathcal{B}$-module mapping surjectively onto the fibered product of the morphisms $k:\mathcal{V}\to\mathcal{W}$ and $h:\mathcal{E}\to\mathcal{W}$ (see the beginning of the proof of Theorem 1.4 again). Then there is a surjective morphism of graded $\mathcal{B}$-modules $q:\mathcal{Q}\to\mathcal{E}$ and its composition with the morphism $h:\mathcal{E}\to\mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $t:\mathcal{Q}\to\mathcal{V}$ of degree -1. Consider the induced morphism of CDG-modules $\tilde{q}:G^+(\mathcal{Q})\to\mathcal{E}$. By Lemma F, the composition $h\tilde{q}:G^+(\mathcal{Q})\to\mathcal{W}$ can be lifted to a morphism of graded $\mathcal{B}$-modules $\tilde{t}:G^+(\mathcal{Q})\to\mathcal{V}$ of degree -1.

Let $\mathcal{R}$ denote the kernel of the closed morphism $\tilde{q}$. Then the cone $\mathcal{F}$ of the embedding $\mathcal{R} \to G^+(\mathcal{Q})$ maps naturally onto $\mathcal{E}$ with the cone absolutely acyclic with respect to $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}}$. As a graded $\mathcal{B}$-module, the CDG-module $\mathcal{F}$ is isomorphic to $G^+(\mathcal{Q})\oplus\mathcal{R}[1]$; the composition $\mathcal{F}\to\mathcal{E}\to\mathcal{M}$ factorizes through the direct summand $G^+(\mathcal{Q})$, where it is defined by the triple $(f\tilde{q},\ g\tilde{q},\ h\tilde{q})$. Since the morphism $h\tilde{q}$ can be lifted to $\mathcal{V}$, so can the corresponding component $\mathcal{F} \to \mathcal{W}$ of the morphism $\mathcal{F} \to \mathcal{M}$. Thus the latter morphism is homotopic to zero by Lemma E(b). ☐

In some cases, the use of Lemma F in the above proof of part (b) can be avoided. Assume that X is a projective scheme over a Noetherian ring and the category of coherent graded $\mathcal{B}$-modules is equivalent to the category of coherent modules over some coherent (graded) $\mathcal{O}_X$-algebra $\mathcal{A}$. In this situation, one takes $\mathcal{Q}$ to be the graded $\mathcal{B}$-module corresponding to the (graded) $\mathcal{A}$-module induced from a large enough finite direct sum of (shifts of) copies of a sufficiently negative invertible $\mathcal{O}_X$-module; then there is a surjective morphism of graded $\mathcal{B}$-modules $\mathcal{Q} \to \mathcal{E}$ and any morphism of graded $\mathcal{B}$-modules $G^+(\mathcal{Q}) \to \mathcal{W}$ lifts to $\mathcal{V}$.

Remark 1.5. We do *not* know how to extend the proof of Proposition 1.5 (a) and (b) to the coderived categories of quasicoherent CDG-modules. Instead, this argument appears to be well-suited for use with the *contraderived* categories (see [Positselski 2011b, Section 3.3] for the definition). In particular, it allows to show that the contraderived category of left CDG-modules over a CDG-ring B with a right coherent underlying graded ring is equivalent to the contraderived category of CDG-modules whose underlying graded B-modules are flat (cf. [loc. cit., paragraph after the proof of Theorem 3.8]).

This is the main reason why we sometimes find it easier to deal with the absolute derived rather than the coderived categories of infinitely generated CDG-modules (cf. Remark 2.8). On the other hand, for the coderived category of quasicoherent CDG-modules we have the compact generation result (part (d) of Proposition 1.5), the results and arguments of Sections 1.7, 1.10, 2.5, 2.9, etc. The conditions under which these two versions of the construction of the derived category of the second kind for a given class of CDG-modules lead to the same triangulated category are discussed below in Section 1.6.

1.6. *Finite homological dimension theorem.* Let $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ denote the DG-category of quasicoherent CDG-modules over $\mathcal{B}$ whose underlying graded $\mathcal{B}$-modules are locally projective (see Remark 1.4 and Theorem A.2). Denote by $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ the corresponding coderived and absolute derived categories.

Theorem 1.6. *The triangulated categories* $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ *and* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ *coincide*; *i.e.*, *every CDG-module over* $\mathcal{B}$ *that is coacyclic with respect to* $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ *is also absolutely acyclic with respect to* $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$.

Proof. The reason for this assertion to be true is that the exact category of locally projective graded $\mathcal{B}$-modules has finite homological dimension [Orlov 2004, Lemma 1.12] and exact functors of infinite direct sums. If this exact category also had enough injectives, the simple argument from [Positselski 2011b, Theorem 3.6(a) and Remark 3.6] would suffice to establish the desired $\mathsf{D}^{\mathsf{co}} = \mathsf{D}^{\mathsf{abs}}$ isomorphism for it (see also [Positselski 2010, Remark 2.1]). The lengthy argument below is designed to provide a way around the injective objects issue in this kind of proof.

Our aim is to show that for any closed morphism $\mathcal{P} \to \mathcal{L}$ from a CDG-module $\mathcal{P} \in \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ to a CDG-module $\mathcal{L}$ absolutely acyclic with respect to $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$, there exists an exact sequence $0 \to \mathcal{Q}_d \to \mathcal{Q}_{d-1} \to \cdots \to \mathcal{Q}_0 \to \mathcal{P} \to 0$ of CDG-modules and closed morphisms in $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ such that the induced morphism from the total CDG-module of $\mathcal{Q}_d \to \cdots \to \mathcal{Q}_0$ to $\mathcal{L}$ is homotopic to zero. Here d is a fixed integer equal to the homological dimension of the exact category of locally projective graded $\mathcal{B}$-modules, which does not exceed the number of open subsets in an affine covering of X minus one.

Taking $\mathcal{P} = \mathcal{L}$ and the morphism $\mathcal{P} \to \mathcal{L}$ to be the identity, we will then conclude that $\mathcal{P}$ is isomorphic to a direct summand of the total CDG-module of $\mathcal{Q}_d \to \cdots \to \mathcal{Q}_0 \to \mathcal{P}$ in $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$. Hence an object of $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ is absolutely acyclic with respect to $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ if and only if it is isomorphic to a direct summand of the total CDG-module of a $(d+2)$-term exact sequence of CDG-modules from $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ with closed morphisms between them. It will immediately follow that the class of CDG-modules absolutely acyclic with respect to $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ is closed under infinite direct sums, so it coincides with the class of coacyclic CDG-modules.

We can suppose that there exists a sequence of distinguished triangles

$$\mathcal{K}_{i-1} \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{M}_i \longrightarrow \mathcal{K}_{i-1}[1]$$

in $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ such that $\mathcal{K}_0 = 0$, $\mathcal{K}_n = \mathcal{L}$, and $\mathcal{M}_i$ is the total CDG-module of an exact triple $\mathcal{U}_i \to \mathcal{V}_i \to \mathcal{W}_i$ of CDG-modules from $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ for all $1 \le i \le n$. We will start with constructing an exact sequence $0 \to \mathcal{Q}'_n \to \cdots \to \mathcal{Q}'_0 \to \mathcal{P} \to 0$ with the above properties, but of the length n rather than d. Then we will use the finite homological dimension property of locally projective graded $\mathcal{B}$-modules in order to obtain the desired resolution $\mathcal{Q}_\bullet$ of a fixed length d from a resolution $\mathcal{Q}'_\bullet$.

Lemma G. *Let $\mathcal{M}$ be the total CDG-module of an exact triple $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$ of CDG-modules from $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ and $\mathcal{K} \to \mathcal{L} \to \mathcal{M} \to \mathcal{K}[1]$ be a distinguished triangle in $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$. Then for any CDG-module $\mathcal{P} \in \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ and a morphism $\mathcal{P} \to \mathcal{L}$ in $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ there exists an exact triple $\mathcal{R} \to \mathcal{Q} \to \mathcal{P}$ of CDG-modules from $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ and a morphism $\mathcal{R}[1] \to \mathcal{K}$ in $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ such that the composition $\mathcal{F} \to \mathcal{P} \to \mathcal{L}$, where $\mathcal{F}$ is the cone of the closed morphism $\mathcal{R} \to \mathcal{Q}$, is equal to the composition $\mathcal{F} \to \mathcal{R}[1] \to \mathcal{K} \to \mathcal{L}$ in $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$.*

Proof. The argument is based on Lemmas E and F from Section 1.5. We can assume that $\mathcal{L}$ is the cone of a closed morphism $\mathcal{M}[-1] \to \mathcal{K}$ and fix a closed morphism $\mathcal{P} \to \mathcal{L}$ representing the given morphism in the homotopy category. Arguing as in the proof of Proposition 1.5, we can construct a surjective closed morphism $\mathcal{Q}' \to \mathcal{P}$ onto $\mathcal{P}$ from a CDG-module $\mathcal{Q}' \in \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ such that the composition $\mathcal{Q}' \to \mathcal{P} \to \mathcal{L} \to \mathcal{M} \to \mathcal{W}[-1]$ lifts to a morphism of graded $\mathcal{B}$-modules $\mathcal{Q}' \to \mathcal{V}[-1]$. Here it suffices to apply the functor G^+ to the fibered product of the morphisms of graded $\mathcal{B}$-modules $\mathcal{P} \to \mathcal{W}[-1]$ and $\mathcal{V}[-1] \to \mathcal{W}[-1]$ and use Lemma F.

Then the morphism $\mathcal{Q}' \to \mathcal{M}$ is homotopic to zero with a natural contracting homotopy (provided by the proof of Lemma E), so the morphism $\mathcal{Q}' \to \mathcal{L}$ factorizes, up to a homotopy, as the composition of a naturally defined closed morphism $\mathcal{Q}' \to \mathcal{K}$ and the closed morphism $\mathcal{K} \to \mathcal{L}$. Set $\mathcal{Q}$ to be the cocone of the closed morphism $\mathcal{Q}' \to \mathcal{K}$; then we have a surjective closed morphism $\mathcal{Q} \to \mathcal{Q}'$ such that the composition $\mathcal{Q} \to \mathcal{Q}' \to \mathcal{K}$ is homotopic to zero.

Let $\mathcal{R}$ be kernel of the morphism $\mathcal{Q} \to \mathcal{P}$ and $\mathcal{F}$ be the cone of the morphism $\mathcal{R} \to \mathcal{Q}$; then there is a natural closed morphism $\mathcal{F} \to \mathcal{P}$. Using Lemma E and arguing as in the end of the proof of Proposition 1.5 again, we can conclude that the composition $\mathcal{F} \to \mathcal{P} \to \mathcal{L} \to \mathcal{M}$ is homotopic to zero. Indeed, the composition $\mathcal{F} \to \mathcal{M} \to \mathcal{W}[-1]$ lifts to a graded $\mathcal{B}$-module morphism $\mathcal{F} \to \mathcal{V}[-1]$ since $\mathcal{F} \simeq \mathcal{Q} \oplus \mathcal{R}[-1]$ as a graded $\mathcal{B}$-module, the morphism $\mathcal{F} \to \mathcal{M}$ factorizes through the projection of $\mathcal{F}$ onto $\mathcal{Q}$, and the morphism $\mathcal{Q} \to \mathcal{Q}' \to \mathcal{W}[-1]$ lifts to a graded $\mathcal{B}$-module morphism $\mathcal{Q} \to \mathcal{Q}' \to \mathcal{V}[-1]$ by our construction.

Notice that the contracting homotopy that we have obtained for the closed morphism $\mathcal{F} \to \mathcal{M}$ forms a commutative diagram with the closed morphisms $\mathcal{Q} \to \mathcal{F}$, $\mathcal{Q} \to \mathcal{Q}'$, and the contracting homotopy that we have previously had for the closed morphism $\mathcal{Q}' \to \mathcal{M}$ (since so do the liftings $\mathcal{F} \to \mathcal{V}[-1]$ and $\mathcal{Q}' \to \mathcal{V}[-1]$). This allows to factorize, up to a homotopy, the closed morphism $\mathcal{F} \to \mathcal{L}$ as the composition of a closed morphism $\mathcal{F} \to \mathcal{K}$ and the closed morphism $\mathcal{K} \to \mathcal{L}$ in such a way that the morphism $\mathcal{F} \to \mathcal{K}$ forms a commutative diagram with the closed morphisms $\mathcal{Q} \to \mathcal{F}$, $\mathcal{Q} \to \mathcal{Q}'$, and the closed morphism $\mathcal{Q}' \to \mathcal{K}$ that we have previously constructed. The composition $\mathcal{Q} \to \mathcal{F} \to \mathcal{K}$, being equal to the composition $\mathcal{Q} \to \mathcal{Q}' \to \mathcal{K}$, is homotopic to zero; hence the morphism $\mathcal{F} \to \mathcal{K}$ factorizes through the closed morphism $\mathcal{F} \to \mathcal{R}[1]$ in $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$. ☐

Applying Lemma G to the morphism $\mathcal{P} \to \mathcal{L}$ and the distinguished triangle $\mathcal{K}_{n-1} \to \mathcal{L} \to \mathcal{M}_n \to \mathcal{K}_{n-1}$, we obtain an exact triple $\mathcal{R}'_0 \to \mathcal{Q}'_0 \to \mathcal{P}$ and a morphism $\mathcal{R}'_0[1] \to \mathcal{K}_{n-1}$ in $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$. Applying the same lemma again to the morphism $\mathcal{R}'_0[1] \to \mathcal{K}_{n-1}$ and the distinguished triangle $\mathcal{K}_{n-2} \to \mathcal{K}_{n-1} \to \mathcal{M}_{n-1} \to \mathcal{K}_{n-2}[1]$, we construct an exact triple $\mathcal{R}'_1 \to \mathcal{Q}'_1 \to \mathcal{R}'_0$ and a morphism $\mathcal{R}'_1[2] \to \mathcal{K}_{n-2}$, etc. Finally we obtain an exact triple $\mathcal{R}'_{n-1} \to \mathcal{Q}'_{n-1} \to \mathcal{R}'_{n-2}$ and a morphism $\mathcal{R}'_{n-1}[n] \to \mathcal{K}_0 = 0$.

Let us check that the natural morphism from the total CDG-module of the complex $0 \to \mathcal{R}'_{n-1} \to \mathcal{Q}'_{n-1} \to \cdots \to \mathcal{Q}'_0$ to the CDG-module $\mathcal{L}$ is homotopic to zero. Denote this morphism by f_n. It factorizes naturally through the cone $\mathcal{F}_0$ of the closed morphism $\mathcal{R}'_0 \to \mathcal{Q}'_0$, and the morphism $\mathcal{F}_0 \to \mathcal{L}$ is homotopic to the composition $\mathcal{F}_0 \to \mathcal{R}'_0[1] \to \mathcal{K}_{n-1} \to \mathcal{L}$. Hence, up to the homotopy, the morphism f_n factorizes through the morphism f_{n-1} from the total CDG-module of the complex $0 \to \mathcal{R}'_{n-1} \to \mathcal{Q}'_{n-1} \to \cdots \to \mathcal{Q}'_1$ to $\mathcal{K}_{n-1}$ induced by the morphism $\mathcal{R}'_0[1] \to \mathcal{K}_{n-1}$. Continuing to argue in this way, we conclude that the morphism f factorizes, up to a homotopy, through the morphism $f_0 : \mathcal{R}'_{n-1}[n] \to \mathcal{K}_0 = 0$.

It remains to "cut" our exact sequence of an unknown length n to a fixed size d. For this purpose, we will assume that $n > d$ and construct from our exact sequence of length n another exact sequence with the same properties, but of the length $n-1$. This part of the argument is based on the following lemma.

Lemma H. *For any CDG-module $\mathcal{M} \in \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$, locally projective graded $\mathcal{B}$-module $\mathcal{E}$, and a homogeneous surjective morphism of locally projective graded $\mathcal{B}$-modules $\mathcal{E} \to \mathcal{M}$, there exist a CDG-module $\mathcal{Q} \in \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$, a surjective closed morphism of CDG-modules $\mathcal{Q} \to \mathcal{M}$, and a homogeneous surjective morphism of locally projective graded $\mathcal{B}$-modules $\mathcal{Q} \to \mathcal{E}$ such that the triangle $\mathcal{Q} \to \mathcal{E} \to \mathcal{M}$ commutes.*

Proof. For any open subscheme $U \subset X$, one can simply define $\mathcal{Q}^i(U)$ as the abelian group of all pairs $(e' \in \mathcal{E}^{i+1}(U), e \in \mathcal{E}^i(U))$ such that $df(e) = f(e')$, where f denotes the morphism of graded $\mathcal{B}$-modules $\mathcal{E} \to \mathcal{M}$ and d is the differential in $\mathcal{M}$. The action of $\mathcal{B}$ in $\mathcal{Q}$ is defined by the formula $b(e', e) = ((-1)^{|b|}be' + d(b)e,\ be)$; the differential in $\mathcal{Q}$ is given by the obvious rule $d(e', e) = (he, e')$. The morphism $\mathcal{Q} \to \mathcal{E}$ is defined as $(e', e) \longmapsto e$; the morphism $\mathcal{Q} \to \mathcal{M}$, given by $(e', e) \longmapsto f(e)$, obviously commutes with the differentials.

It remains to check that the graded $\mathcal{B}$-module $\mathcal{Q}$ is locally projective. This can be done by comparing the above construction with the constructions of the freely (co)generated CDG-modules $G^+(\mathcal{E})$ and $G^-(\mathcal{E})$ from [Positselski 2011b, proof of Theorem 3.6] (see the beginning of the proof of Theorem 1.4). One can simply define $G^-(\mathcal{E})$ as being isomorphic to $G^+(\mathcal{E})[1]$. Since $\mathcal{M}$ is a CDG-module, there is a natural closed morphism of CDG-modules $\mathcal{M} \to G^-(\mathcal{M})$. The CDG-module $\mathcal{Q}$ is the fibered product of the surjective closed morphism of CDG-modules $G^-(\mathcal{E}) \to G^-(\mathcal{M})$ and the closed morphism $\mathcal{M} \to G^-(\mathcal{M})$; hence the graded $\mathcal{B}$-module $\mathcal{Q}$ is locally projective. The morphism $\mathcal{Q} \to \mathcal{E}$ is induced by the natural morphism of graded $\mathcal{B}$-modules $G^-(\mathcal{E}) \to \mathcal{E}$. It forms a commutative diagram with the morphism $\mathcal{E} \to \mathcal{M}$, since the composition $\mathcal{M} \to G^-(\mathcal{M}) \to \mathcal{M}$ is the identity morphism. $\square$

The exact sequence of CDG-modules

$$0 \longrightarrow \mathcal{R}'_{n-1} \longrightarrow \mathcal{Q}'_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}'_0 \longrightarrow \mathcal{P} \longrightarrow 0$$

represents a certain Yoneda Ext class of degree n between the locally projective graded $\mathcal{B}$-modules $\mathcal{P}$ and $\mathcal{R}'_{n-1}$. Since the homological dimension of the exact category of such $\mathcal{B}$-modules is equal to d and we assume that $n > d$, this Ext class has to vanish. This means that there exists an exact sequence of locally projective graded $\mathcal{B}$-modules $0 \to \mathcal{R}'_{n-1} \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{P} \to 0$ mapping to our original exact sequence, with the maps on the rightmost and leftmost terms being the identity maps, such that the embedding of $\mathcal{B}$-modules $\mathcal{R}'_{n-1} \to \mathcal{E}_{n-1}$ splits.

As explained in [Positselski 2011a, proof of Lemma 4.4], one can assume the morphisms $\mathcal{E}_i \to \mathcal{Q}'_i$ to be surjective. Applying Lemma H, we obtain a surjective closed morphism of CDG-modules $\mathcal{Q}_0 \to \mathcal{Q}'_0$ and a morphism of graded $\mathcal{B}$-modules $\mathcal{Q}_0 \to \mathcal{E}_0$ forming a commutative triangle with the morphism $\mathcal{E}_0 \to \mathcal{Q}'_0$. Applying Lemma H to the surjective morphism of fibered products $\mathcal{Q}_0 \times_{\mathcal{E}_0} \mathcal{E}_1 \to \mathcal{Q}_0 \times_{\mathcal{Q}'_0} \mathcal{Q}'_1$, we obtain a surjective closed morphism $\mathcal{Q}_1 \to \mathcal{Q}'_1$ and a closed morphism $\mathcal{Q}_1 \to \mathcal{Q}_0$ forming a commutative square with the closed morphisms $\mathcal{Q}_0 \to \mathcal{Q}'_0$ and $\mathcal{Q}'_1 \to \mathcal{Q}'_0$. Besides, the sequence $\mathcal{Q}_1 \to \mathcal{Q}_0 \to \mathcal{P}$ is exact at $\mathcal{Q}_0$. We also obtain a morphism of graded $\mathcal{B}$-modules $\mathcal{Q}_1 \to \mathcal{E}_1$ forming a commutative triangle with the morphisms to $\mathcal{Q}'_1$ and a commutative square with the morphisms to $\mathcal{E}_0$.

Proceeding in this way, we construct a sequence $\mathcal{Q}_{n-2} \to \cdots \to \mathcal{Q}_0 \to \mathcal{P} \to 0$, which is exact at all the middle terms, maps onto the sequence $\mathcal{Q}'_{n-2} \to \cdots \to \mathcal{Q}'_0 \to \mathcal{P}$ by closed morphisms, and maps into the sequence $\mathcal{E}_{n-2} \to \cdots \to \mathcal{E}_0 \to \mathcal{P}$ so that the triangle of the maps of sequences commutes. Finally, notice that $\mathcal{E}_{n-1} \simeq \mathcal{E}_{n-2} \times_{\mathcal{Q}'_{n-2}} \mathcal{Q}'_{n-1}$, and set $\mathcal{Q}_{n-1} = \mathcal{Q}_{n-2} \times_{\mathcal{Q}'_{n-2}} \mathcal{Q}'_{n-1}$. Then the exact sequence of CDG-modules $0 \to \mathcal{R}'_{n-1} \to \mathcal{Q}_{n-1} \to \cdots \to \mathcal{Q}_0 \to \mathcal{P} \to 0$ maps onto the exact sequence $0 \to \mathcal{R}'_{n-1} \to \mathcal{Q}'_{n-1} \to \cdots \to \mathcal{Q}'_0 \to \mathcal{P} \to 0$ by closed morphisms, and this map of exact sequences factorizes through the exact sequence of graded $\mathcal{B}$-modules $0 \to \mathcal{R}'_{n-1} \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{P} \to 0$. The composition of the morphism $\mathcal{Q}_{n-1} \to \mathcal{E}_{n-1}$ with the splitting $\mathcal{E}_{n-1} \to \mathcal{R}'_{n-1}$ of the embedding $\mathcal{R}'_{n-1} \to \mathcal{E}_{n-1}$ provides a graded $\mathcal{B}$-module splitting $\mathcal{Q}_{n-1} \to \mathcal{R}'_{n-1}$ of the embedding of CDG-modules $\mathcal{R}'_{n-1} \to \mathcal{Q}_{n-1}$.

Denote by $\mathcal{R}_{n-2}$ the image of the morphism of CDG-modules $\mathcal{Q}_{n-1} \to \mathcal{Q}_{n-2}$. The morphism from the total CDG-module of the complex $\mathcal{R}'_{n-1} \to \mathcal{Q}'_{n-1} \to \cdots \to \mathcal{Q}'_0$ to the CDG-module $\mathcal{L}$ is homotopic to zero; hence so is the morphism to $\mathcal{L}$ from the total CDG-module of the complex $\mathcal{R}'_{n-1} \to \mathcal{Q}_{n-1} \to \cdots \to \mathcal{Q}_0$. The latter morphism factorizes naturally through the total CDG-module of the complex $\mathcal{R}_{n-2} \to \mathcal{Q}_{n-2} \to \cdots \to \mathcal{Q}_0$. The cone of this closed morphism between two total CDG-modules is homotopy equivalent to the total CDG-module of the exact triple $\mathcal{R}'_{n-1} \to \mathcal{Q}_{n-1} \to \mathcal{R}_{n-2}$. Since this exact triple splits as an exact triple of graded $\mathcal{B}$-modules, its total CDG-module is contractible. Consequently, the morphism between the total CDG-modules of $\mathcal{R}'_{n-1} \to \mathcal{Q}_{n-1} \to \cdots \to \mathcal{Q}_0$ and $\mathcal{R}_{n-2} \to \mathcal{Q}_{n-2} \to \cdots \to \mathcal{Q}_0$ is a homotopy equivalence.

It follows that the natural morphism from the total CDG-module of the resolution $\mathcal{R}_{n-2} \to \mathcal{Q}_{n-2} \to \cdots \to \mathcal{Q}_0$ of the CDG-module $\mathcal{P}$ to the CDG-module $\mathcal{L}$ is homotopic to zero, and we are done. □

So far we have only considered flat coherent CDG-modules over quasicoherent CDG-algebras $\mathcal{B}$ whose underlying quasicoherent graded algebras are Noetherian. But the latter restriction is not necessary, as flat and locally finitely presented (or, which is equivalent, locally projective and finitely generated) quasicoherent graded $\mathcal{B}$-modules always form an exact subcategory of flat (or locally projective) graded $\mathcal{B}$-modules. The notation $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{lp}}$ (understood in the obvious sense as the DG-category of CDG-modules over $\mathcal{B}$ with coherent and locally projective underlying graded $\mathcal{B}$-modules) is synonymous with $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}}$ (see Remark 1.4).

Corollary 1.6. *The functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{lp}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ *induced by the embedding of DG-categories* $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{lp}} \to \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$ *is fully faithful.*

Proof. When $\mathcal{B}$ is Noetherian, one can show that the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{lp}}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}})$ is fully faithful by comparing parts (a)–(c) of Proposition 1.5 (with the flatness condition replaced by the local projectivity). In the general case, one proves this assertion directly, using an argument similar to the proof of Proposition 1.5(a) and (b). Then it remains to use Theorem 1.6. □

When every flat quasicoherent graded module over $\mathcal{B}$ has finite locally projective dimension (see Remark 1.4), one has

$$\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}),$$
$$\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}) \simeq \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$$

by appropriate versions of Theorem 1.4. Consequently, it follows from Theorem 1.6 that $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) = \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) = \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ in this case. Thus the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ is fully faithful; when $\mathcal{B}$ is Noetherian, so is the functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{ffd}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$.

1.7. ***Gorenstein case.*** Here we establish a sufficient condition for the functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ to be an equivalence of triangulated categories.

Let $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{inj}}$ denote the full DG-subcategory in $\mathcal{B}\text{-}\mathsf{qcoh}$ consisting of the CDG-modules whose underlying quasicoherent graded $\mathcal{B}$-modules are injective. Furthermore, let $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fid}}$ be the full DG-subcategory in $\mathcal{B}\text{-}\mathsf{qcoh}$ consisting of the CDG-modules whose underlying quasicoherent graded $\mathcal{B}$-modules have finite injective dimension (i.e., admit a finite right resolution by injective quasicoherent graded $\mathcal{B}$-modules). Let $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fid}})$ and $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fid}})$ denote the corresponding derived categories of the second kind. (The difficulty in the definition of the latter category, similar to the difficulty in the definition of $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ discussed in Section 1.3, does not actually arise, as it is clear from part (a) of the next lemma.)

Lemma 1.7. (a) *For any quasicoherent CDG-algebra $\mathcal{B}$ over X, the natural functors $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{inj}}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fid}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fid}})$ are equivalences of triangulated categories.*

(b) *Let $\mathcal{B}$ be a quasicoherent CDG-algebra over X whose underlying quasicoherent graded algebra $\mathcal{B}$ is Noetherian. Then the functor $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{inj}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ induced by the embedding $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{inj}} \to \mathcal{B}\text{-}\mathsf{qcoh}$ is an equivalence of triangulated categories.*

Proof. Part (a) is provided by [Positselski 2011b, Theorem and Remark in Section 3.6]. Part (b) is a particular case of [loc. cit., Theorem and Remark in Section 3.7] since the class of injective quasicoherent graded $\mathcal{B}$-modules is closed under infinite direct sums in its assumptions. (Cf. [Lin and Pomerleano 2013, Proposition 2.4].) □

Proposition 1.7. *Let $\mathcal{B}$ be a quasicoherent CDG-algebra over X such that the quasicoherent graded algebra $\mathcal{B}$ is Noetherian and the classes of quasicoherent graded $\mathcal{B}$-modules of finite flat dimension and of finite injective dimension coincide. Then the functors $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ induced by the embedding $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to \mathcal{B}\text{-}\mathsf{qcoh}$ are equivalences of triangulated categories.*

Proof. Since $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}} = \mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fid}}$, the isomorphism of categories $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) = \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ follows from part (a) of Lemma 1.7. Applying Theorem 1.4, we obtain the isomorphism of categories $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$. Similarly, it suffices to compare parts (a) and (b) of Lemma 1.7 in order to conclude that the functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fid}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ is an equivalence of categories; hence so are the functors $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$. (Cf. [Positselski 2011b, Section 3.9].) □

1.8. ***Pull-backs and push-forwards.*** Let $f : Y \to X$ be a morphism of separated Noetherian schemes, $\mathcal{B}_X$ be a quasicoherent CDG-algebra over X, and $\mathcal{B}_Y$ be a quasicoherent CDG-algebra over Y. A *morphism of quasicoherent CDG-algebras $\mathcal{B}_X \to \mathcal{B}_Y$ compatible with the morphism $Y \to X$* is the data of a CDG-ring morphism $\mathcal{B}_X(U) \to \mathcal{B}_Y(V)$ for each pair of affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$. This data should satisfy the obvious compatibility condition: for any affine open subschemes $U' \subset U$ and $V' \subset V$ such that $f(V') \subset U'$, the square diagram of CDG-ring morphisms between the CDG-rings $\mathcal{B}_X(U)$, $\mathcal{B}_X(U')$, $\mathcal{B}_Y(V)$, and $\mathcal{B}_Y(V')$ must be commutative.

Let $\mathcal{B}_X \to \mathcal{B}_Y$ be a morphism of quasicoherent CDG-algebras compatible with a morphism of schemes $Y \to X$. Then for any quasicoherent left CDG-module $\mathcal{M}$ over $\mathcal{B}_X$, the quasicoherent graded left module $f^*\mathcal{M} = \mathcal{B}_Y \otimes_{f^{-1}\mathcal{B}_X} f^{-1}\mathcal{M}$ over $\mathcal{B}_Y$ has a natural structure of quasicoherent CDG-module over $\mathcal{B}_Y$. Similarly, for any quasicoherent left CDG-module $\mathcal{N}$ over $\mathcal{B}_Y$ the quasicoherent graded left module

$f_*\mathcal{N}$ over $\mathcal{B}_X$ has a natural structure of quasicoherent CDG-module over $\mathcal{B}_X$. These CDG-module structures are defined in terms of the CDG-ring morphisms $\mathcal{B}_X(U) \to \mathcal{B}_Y(V)$. The above constructions provide the underived direct and inverse image functors, which can be viewed as triangulated functors $f^* : H^0(\mathcal{B}_X\text{-}\mathsf{qcoh}) \to H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh})$ and $f_* : H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh}) \to H^0(\mathcal{B}_X\text{-}\mathsf{qcoh})$. The functor f_* is right adjoint to the functor f^*.

The derived inverse image functor $\mathbb{L}f^*$ is in general only defined on CDG-modules satisfying certain finite flat dimension conditions. Restricting the functor f^* to flat CDG-modules, we obtain a triangulated functor

$$H^0(\mathcal{B}_X\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh}_{\mathsf{fl}}),$$

which takes objects coacyclic with respect to $\mathcal{B}_X\text{-}\mathsf{qcoh}_{\mathsf{fl}}$ to objects coacyclic with respect to $\mathcal{B}_Y\text{-}\mathsf{qcoh}_{\mathsf{fl}}$ since the inverse image preserves infinite direct sums and short exact sequences of flat quasicoherent graded modules. Hence there is the induced triangulated functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}_{\mathsf{fl}})$. Applying Theorem 1.4(a), we construct the derived inverse image functor

$$\mathbb{L}f^* : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}_{\mathsf{ffd}}).$$

Assuming that there are enough vector bundles on X and Y, and restricting the functor f^* to flat coherent CDG-modules, we obtain a triangulated functor $H^0(\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{fl}}) \to H^0(\mathcal{B}_Y\text{-}\mathsf{coh}_{\mathsf{fl}})$, which induces a triangulated functor

$$\mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_Y\text{-}\mathsf{coh}_{\mathsf{fl}}).$$

Assuming additionally that the quasicoherent graded algebras $\mathcal{B}_X$ and $\mathcal{B}_Y$ are Noetherian and applying Theorem 1.4(c), we construct the derived inverse image functor

$$\mathbb{L}f^* : \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_Y\text{-}\mathsf{coh}_{\mathsf{ffd}}).$$

When f is an affine morphism, the direct image of quasicoherent sheaves is an exact functor (preserving also infinite direct sums), so the functor $f_* : H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh}) \to H^0(\mathcal{B}_X\text{-}\mathsf{qcoh})$ induces a triangulated functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh})$. To construct the derived direct image functor between the coderived categories in the general case, we need to use injective resolutions.

From now on we assume that $\mathcal{B}_X$ and $\mathcal{B}_Y$ are Noetherian; so Lemma 1.7(b) is applicable to $\mathcal{B}_Y$. Restricting the functor f_* to the full subcategory $H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh}_{\mathsf{inj}}) \subset H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh})$ and composing it with the localization functor $H^0(\mathcal{B}_X\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh})$, we obtain the derived direct image functor

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}).$$

Proposition 1.8. *Assume that there are enough vector bundles on X and Y. Then the functors $\mathbb{L}f^* : \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{ffd}}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_Y\text{-}\mathsf{coh}_{\mathsf{ffd}})$ and $\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh})$ are partially adjoint to each other in the following sense: for any objects $\mathcal{M} \in \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{ffd}})$ and $\mathcal{N} \in \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh})$, there is a natural isomorphism of abelian groups*

$$\operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh})}(\iota_X\mathcal{M},\, \mathbb{R}f_*\mathcal{N}) \simeq \operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh})}(\iota_Y\mathbb{L}f^*\mathcal{M},\, \mathcal{N}),$$

where

$$\iota_X : \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}),$$
$$\iota_Y : \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh})$$

are the natural fully faithful triangulated functors.

Proof. The functors ι_X and ι_Y are fully faithful by Theorem 1.4(c) and Proposition 1.5(b) and (d). Using Theorem 1.4(c), let us assume that $\mathcal{M} \in \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{fl}})$. We can also assume that $\mathcal{N} \in H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh}_{\mathsf{inj}})$.

Then the left-hand side is the (filtered) inductive limit of

$$\operatorname{Hom}_{H^0(\mathcal{B}_X\text{-}\mathsf{qcoh})}(\mathcal{M}'',\, f_*\mathcal{N})$$

over all morphisms $\mathcal{M}'' \to \mathcal{M}$ in $H^0(\mathcal{B}_X\text{-}\mathsf{qcoh})$ with a cone coacyclic with respect to $\mathcal{B}_X\text{-}\mathsf{qcoh}$. According to the proofs of Proposition 1.5(b) and [Positselski 2011b, Theorem 3.11.1], any morphism from $\mathcal{M}$ to an object coacyclic with respect to $\mathcal{B}_X\text{-}\mathsf{qcoh}$ factorizes through an object absolutely acyclic with respect to $\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{fl}}$. Thus the above inductive limit coincides with the similar limit taken over all morphisms $\mathcal{M}' \to \mathcal{M}$ in $H^0(\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{fl}})$ with a cone absolutely acyclic with respect to $\mathcal{B}_X\text{-}\mathsf{coh}_{\mathsf{fl}}$.

By [loc. cit., Theorem 3.5(a), Remark 3.5, and Lemma 1.3], the right-hand side is isomorphic to $\operatorname{Hom}_{H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh})}(f^*\mathcal{M},\mathcal{N})$ and to $\operatorname{Hom}_{H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh})}(f^*\mathcal{M}',\mathcal{N})$ since the objects of $H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh}_{\mathsf{inj}})$ are right orthogonal to any coacyclic objects in $H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh})$. So the assertion follows from the adjointness of the functors f^* and f_* on the level of the homotopy categories of quasicoherent CDG-modules. □

Remark 1.8. It is not immediately obvious from the above construction that the derived functor $\mathbb{R}f_*$ is compatible with the compositions; i.e., for $g : Z \to Y$ and $f : Y \to X$, one has $\mathbb{R}(fg)_* \simeq \mathbb{R}f_* \circ \mathbb{R}g_*$. The problem is that the direct image functor f_* does not preserve injectivity of quasicoherent graded modules in general. When the derived direct image functors are adjoint to appropriately defined derived inverse images (see Section 1.9 below for some results of this kind), the problem reduces to checking that the derived inverse images are compatible with the compositions, which may be easier to see from our definitions.

One general approach to this problem is to replace injective quasicoherent graded $\mathcal{B}$-modules with quasicoherent graded $\mathcal{B}$-modules that are flabby as sheaves of graded abelian groups in our construction of the derived direct images. The class of flabby sheaves of abelian groups is closed under infinite direct sums since the underlying topological space of the scheme is Noetherian; it is also always closed under extensions and cokernels of injective morphisms. Whenever the quasicoherent graded algebra $\mathcal{B}$ is Noetherian, all injective quasicoherent graded $\mathcal{B}$-modules are flabby by Theorem A.3. Therefore, the coderived category of flabby quasicoherent CDG-modules over $\mathcal{B}$ is equivalent to the homotopy category $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{inj}})$ by a version of Lemma 1.7(b); hence it is also equivalent to the coderived category of all quasicoherent CDG-modules $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ (cf. the proof of Proposition 1.7).

The direct images preserve exact triples of flabby sheaves, so derived direct images can be defined using flabby resolutions. The direct images also take flabby sheaves to flabby sheaves; hence the desired compatibility of their derived functors with the compositions of scheme morphisms follows.

Moreover, assuming additionally that the scheme has finite Krull dimension, the absolute derived category of flabby quasicoherent CDG-modules is equivalent to $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh})$ by a dual version of Theorem 1.4(b), as the "flabby dimension" of any quasicoherent graded $\mathcal{B}$-module is finite. This allows us to define the derived direct images on the absolute derived categories of quasicoherent CDG-modules (another approach to this question is to use the construction from the proof of Proposition 1.9 below). Notice that all our constructions of derived inverse images are also applicable to the categories $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh})$.

Finally, let us point out that for any morphism of quasicoherent CDG-algebras $\mathcal{B}_X \to \mathcal{B}_Y$ with Noetherian underlying quasicoherent graded algebras $\mathcal{B}_X$ and $\mathcal{B}_Y$ compatible with a morphism of separated Noetherian schemes $f : Y \to X$, the functor $\mathbb{R}f_*$ has a right adjoint functor

$$f^! : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}).$$

Indeed, the triangulated category $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh})$ is compactly generated by Proposition 1.5(d), and the functor $\mathbb{R}f_*$ preserves infinite direct sums since the class of injective quasicoherent graded $\mathcal{B}_Y$-modules is closed under infinite direct sums, due to Noetherianness of $\mathcal{B}_Y$. So it remains to apply [Neeman 1996, Theorem 4.1].

There is a special situation when one can construct the above functor $f^!$ explicitly. Assume that $f : Y \to X$ is an affine morphism. Let us say that the quasicoherent graded algebra $\mathcal{B}_Y$ is *finite* over $\mathcal{B}_X$ if for any affine open subscheme $U \subset X$, the graded $\mathcal{B}_X(U)$-module $\mathcal{B}_Y(f^{-1}(U))$ is finitely generated, or in other words, if the quasicoherent graded $\mathcal{B}_X$-module $f_*\mathcal{B}_Y$ is coherent.

Let $\mathcal{B}_X \to \mathcal{B}_Y$ be a morphism of Noetherian quasicoherent CDG-algebras compatible with an affine morphism of separated Noetherian schemes $f : Y \to X$ such that the quasicoherent graded algebra $\mathcal{B}_Y$ is finite over $\mathcal{B}_X$. Given a quasicoherent graded left module $\mathcal{M}$ over $\mathcal{B}_X$, we set $(f^!\mathcal{M})(f^{-1}(U))$ to be the graded left module of homogeneous morphisms (of various degrees) $\mathrm{Hom}_{\mathcal{B}_X(U)}(\mathcal{B}_Y(f^{-1}(U)), \mathcal{M}(U))$ over the graded ring $\mathcal{B}_Y(f^{-1}(U))$ for any affine open subscheme $U \subset X$. Due to the finiteness condition on $\mathcal{B}_Y$ over $\mathcal{B}_X$, for any affine open subscheme $V \subset U$, there are natural isomorphisms $(f^!\mathcal{M})(f^{-1}(V)) \simeq \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} (f^!\mathcal{M})(f^{-1}(U)) \simeq \mathcal{O}_Y(f^{-1}(V)) \otimes_{\mathcal{O}_Y(f^{-1}(U))} (f^!\mathcal{M})(f^{-1}(U))$, which allow us to extend the assignment $f^{-1}(U) \longmapsto (f^!\mathcal{M})(f^{-1}(U))$ to a quasicoherent graded module $f^!(\mathcal{M})$ over the quasicoherent graded algebra $\mathcal{B}_Y$.

Given a quasicoherent CDG-module $\mathcal{M}$ over $\mathcal{B}_X$, the conventional rule

$$d(g)(m) = d(g(m)) - (-1)^{|g|} g(d(m))$$

(with the usual change-of-connection modifications) defines the structure of a quasicoherent CDG-module over $\mathcal{B}_Y$ on the quasicoherent graded module $f^!(\mathcal{M})$. This construction provides a triangulated functor $f^! : H^0(\mathcal{B}_X\text{-}\mathsf{qcoh}) \to H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh})$ right adjoint to the triangulated functor $f_* : H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh}) \to H^0(\mathcal{B}_X\text{-}\mathsf{qcoh})$. Restricting the functor $f^! : H^0(\mathcal{B}_X\text{-}\mathsf{qcoh}) \to H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh})$ to the full subcategory of injective quasicoherent CDG-modules in $H^0(\mathcal{B}_X\text{-}\mathsf{qcoh})$ and taking into account Lemma 1.7(b), we obtain the right derived functor

$$\mathbb{R}f^! : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}),$$

which is right adjoint to the (underived, as the morphism f is affine) direct image functor $f_* : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh})$. In other words, the functor $\mathbb{R}f^!$ coincides with the above adjoint functor $f^! : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh})$ in our special case.

1.9. ***Morphisms of finite flat dimension.*** Let $f : Y \to X$ be a morphism of separated Noetherian schemes, and let $\mathcal{B}_X \to \mathcal{B}_Y$ be a compatible morphism of quasicoherent CDG-algebras. We will say that the quasicoherent graded algebra $\mathcal{B}_Y$ *has finite flat dimension over* $\mathcal{B}_X$ if (the left derived functor of) the functor of inverse image f^* acting between the abelian categories of quasicoherent graded modules over $\mathcal{B}_X$ and $\mathcal{B}_Y$ has finite homological dimension. Equivalently, for any affine open subschemes $U \subset X$ and $V \subset Y$ such that $f(V) \subset U$, the graded right $\mathcal{B}_X(U)$-module $\mathcal{B}_Y(V)$ should have finite flat dimension.

A quasicoherent graded $\mathcal{B}_X$-module is said to be *adjusted to* f^* if its derived inverse image under f, as an object of the derived category of the abelian category of quasicoherent graded $\mathcal{B}_Y$-modules, coincides with the underived inverse image. Denote the DG-category of quasicoherent CDG-modules over $\mathcal{B}_X$ whose underlying graded $\mathcal{B}_X$-modules are adjusted to f^* by $\mathcal{B}_X\text{-}\mathsf{qcoh}_{f\text{-}\mathsf{adj}}$. When $\mathcal{B}_X$ is Noetherian, let $\mathcal{B}_X\text{-}\mathsf{coh}_{f\text{-}\mathsf{adj}}$ denote the similarly defined DG-category of coherent CDG-modules. We will use our usual notation for the absolute derived and coderived categories of these DG-categories of CDG-modules.

Lemma 1.9. *Assume that the quasicoherent graded algebra $\mathcal{B}_Y$ has finite flat dimension over $\mathcal{B}_X$. Then*

(a) *the functor* $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}_{f\text{-}\mathsf{adj}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $\mathcal{B}_X\text{-}\mathsf{qcoh}_{f\text{-}\mathsf{adj}} \to \mathcal{B}_X\text{-}\mathsf{qcoh}$ *is an equivalence of triangulated categories*;

(b) *the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{qcoh}_{f\text{-}\mathsf{adj}}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $\mathcal{B}_X\text{-}\mathsf{qcoh}_{f\text{-}\mathsf{adj}} \to \mathcal{B}_X\text{-}\mathsf{qcoh}$ *is an equivalence of triangulated categories*;

(c) *if there are enough vector bundles on X and $\mathcal{B}_X$ is Noetherian*, *the functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh}_{f\text{-}\mathsf{adj}}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh})$ *induced by the embedding of DG-categories* $\mathcal{B}_X\text{-}\mathsf{coh}_{f\text{-}\mathsf{adj}} \to \mathcal{B}_X\text{-}\mathsf{coh}$ *is an equivalence of triangulated categories.*

Proof. This is a version of Theorem 1.4, provable in the same way (cf. Corollary 2.6 below). The assertions hold, because any quasicoherent graded $\mathcal{B}_X$-module has a finite left resolution consisting of quasicoherent CDG-modules adjusted to f^*, and it is similar for coherent CDG-modules. ☐

The functor of inverse image $f^* : H^0(\mathcal{B}_X\text{-}\mathsf{qcoh}) \to H^0(\mathcal{B}_Y\text{-}\mathsf{qcoh})$ takes CDG-modules coacyclic with respect to $\mathcal{B}_X\text{-}\mathsf{qcoh}_{f\text{-}\mathsf{adj}}$ to CDG-modules coacyclic with respect to $\mathcal{B}_Y\text{-}\mathsf{qcoh}$, and hence induces a triangulated functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}_{f\text{-}\mathsf{adj}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh})$. Taking Lemma 1.9 into account, we construct the derived inverse image functor

$$\mathbb{L}f^* : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}).$$

One shows that this functor is left adjoint to the functor $\mathbb{R}f_*$ constructed in Section 1.8 in the way analogous to (but simpler than) the proof of Proposition 1.8.

When there are enough vector bundles on X, and $\mathcal{B}_X$ and $\mathcal{B}_Y$ are Noetherian, we construct the derived inverse image functor

$$\mathbb{L}f^* : \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_X\text{-}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathcal{B}_Y\text{-}\mathsf{coh})$$

in a similar way.

Let $\mathcal{B}_X^{\mathrm{op}}$ and $\mathcal{B}_Y^{\mathrm{op}}$ denote the quasicoherent graded algebras with the opposite multiplication to $\mathcal{B}_X$ and $\mathcal{B}_Y$.

Proposition 1.9. *When $\mathcal{B}_Y^{\mathrm{op}}$ has finite flat dimension over $\mathcal{B}_X^{\mathrm{op}}$, the derived inverse image functor* $\mathbb{L}f^* : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ *constructed in Section 1.8 has a right adjoint functor*

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh}_{\mathsf{ffd}}).$$

Proof. Let $\{U_\alpha\}$ be a finite affine covering of Y. To any object $\mathcal{N} \in \mathcal{B}_Y\text{-qcoh}_{\mathsf{ffd}}$, assign the total CDG-module $\mathbb{R}_{\{U_\alpha\}} f_*\mathcal{N}$ of the finite Čech complex

$$\bigoplus_\alpha f|_{U_\alpha *}(\mathcal{N}|_{U_\alpha}) \longrightarrow \bigoplus_{\alpha<\beta} f|_{U_\alpha\cap U_\beta *}(\mathcal{N}|_{U_\alpha\cap U_\beta}) \longrightarrow \cdots$$

of CDG-modules over $\mathcal{B}_X$.

The terms of this complex belong to $\mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}}$ since the morphism $f|_V : V \to X$ is affine for any intersection V of a nonempty subset of affine open subschemes $U_\alpha \subset Y$ and the quasicoherent graded algebra $\mathcal{B}_Y^{\mathrm{op}}$ has finite flat dimension over $\mathcal{B}_X^{\mathrm{op}}$. Hence one has $\mathbb{R}_{\{U_\alpha\}} f_*\mathcal{N} \in \mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}}$; it is clear that $\mathbb{R}_{\{U_\alpha\}} f_*$ is a DG-functor $\mathcal{B}_Y\text{-qcoh}_{\mathsf{ffd}} \to \mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}}$ taking coacyclic objects to coacyclic objects. So we have the induced functor $\mathbb{R} f_*$ between the coderived categories.

It remains to obtain the adjunction isomorphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}})}(\mathcal{M}, \mathbb{R} f_*\mathcal{N}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-qcoh}_{\mathsf{ffd}})}(\mathbb{L} f^*\mathcal{M}, \mathcal{N})$$

for $\mathcal{M} \in \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}})$. Denote by $\mathcal{N}_+$ the total CDG-module of the finite complex

$$C^\bullet_{\{U_\alpha\}}\mathcal{N} = \left(\bigoplus_\alpha j_{U_\alpha *} j^*_{U_\alpha}\mathcal{N} \longrightarrow \bigoplus_{\alpha<\beta} j_{U_\alpha\cap U_\beta *} j^*_{U_\alpha\cap U_\beta}\mathcal{N} \longrightarrow \cdots\right)$$

of CDG-modules over $\mathcal{B}_Y$ (where $j_V : V \to Y$ denotes the embedding of an affine open subscheme). Then we have $\mathbb{R}_{\{U_\alpha\}} f_*\mathcal{N} \simeq f_*\mathcal{N}_+$. There is a natural closed morphism $\mathcal{N} \to \mathcal{N}_+$ of CDG-modules over $\mathcal{B}_Y$ with the cone coacyclic (and even absolutely acyclic) with respect to $\mathcal{B}_Y\text{-qcoh}_{\mathsf{ffd}}$.

For any CDG-module $\mathcal{Q} \in \mathcal{B}_Y\text{-qcoh}_{\mathsf{ffd}}$ such that $f_*\mathcal{Q} \in \mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}}$, there is a natural map

$$\psi : \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}})}(\mathcal{M}, f_*\mathcal{Q}) \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-qcoh}_{\mathsf{ffd}})}(\mathbb{L} f^*\mathcal{M}, \mathcal{Q}).$$

Indeed, by the proof of Theorem 1.4(a), any morphism $\mathcal{M} \to f_*\mathcal{Q}$ in $\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}})$ can be represented as a fraction formed by a morphism $\mathcal{M}' \to \mathcal{M}$ in $H^0(\mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}})$, with $\mathcal{M}' \in \mathcal{B}_X\text{-qcoh}_{\mathsf{fl}}$ and a cone coacyclic with respect to $\mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}}$, and a morphism $\mathcal{M}' \to f_*\mathcal{Q}$ in $H^0(\mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}})$. To such a fraction, the map ψ assigns the related morphism $\mathbb{L} f^*\mathcal{M} = f^*\mathcal{M}' \to \mathcal{Q}$.

For a fixed $\mathcal{M}$, the map ψ is a morphism of cohomological functors of the argument $\mathcal{Q} \in H^0(\mathcal{B}_Y\text{-qcoh}_{\mathsf{ffd}})$ with $f_*\mathcal{Q} \in \mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}}$. Thus in order to show that it is an isomorphism for $\mathcal{Q} = \mathcal{N}_+$, it suffices to check that it is an isomorphism for $\mathcal{Q} = j_{V*}\mathcal{P}$ for every affine $V \subset Y$ and $\mathcal{P} \in \mathcal{B}_Y|_V\text{-qcoh}_{\mathsf{ffd}}$. This follows from the adjunction isomorphism

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-qcoh}_{\mathsf{ffd}})}(\mathcal{M}, f|_{V*}\mathcal{P}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y|_V\text{-qcoh}_{\mathsf{ffd}})}(\mathbb{L} f|_V^*\mathcal{M}, \mathcal{P})$$

and the similar isomorphism for the embedding j_V, which hold because the functors $f|_{V*}$ and j_{V*} are exact, the morphisms $f|_V$ and j_V being affine. $\square$

Remark 1.9. One can also use the above Čech complex approach in order to construct a version of the derived functor $\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}(\mathcal{B}_Y\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}_X\text{-}\mathsf{qcoh})$. One can check that this construction agrees with the injective resolution construction from Section 1.8, using the fact that the restrictions of injective quasicoherent graded $\mathcal{B}_Y$-modules to open subschemes are injective (Theorem A.3). Alternatively, in the assumption of finite flat dimension of $\mathcal{B}_Y$ over $\mathcal{B}_X$, one checks that both constructions provide functors right adjoint to $\mathbb{L}f^*$, hence they are isomorphic.

This allows us to conclude that the derived functors $\mathbb{R}f_*$ acting on arbitrary quasicoherent CDG-modules and quasicoherent CDG-modules of finite flat dimension form a commutative diagram with the natural functors from the coderived categories of the latter to the coderived categories of the former.

1.10. ***Supports of CDG-modules.*** Let X be a Noetherian scheme. The *set-theoretic support* of a quasicoherent sheaf $\mathcal{M}$ on X is the minimal closed subset $T \subset X$ such that the restriction of $\mathcal{M}$ to the open subscheme $X \setminus T$ vanishes. Given a Noetherian quasicoherent graded algebra $\mathcal{B}$ over X and a quasicoherent graded $\mathcal{B}$-module $\mathcal{M}$, the set-theoretic support $T = \operatorname{Supp}\mathcal{M}$ of $\mathcal{M}$ is defined similarly. It only depends on the underlying quasicoherent $\mathcal{O}_X$-module of $\mathcal{M}$.

Let $\mathcal{B}$ be a quasicoherent CDG-algebra over X whose underlying quasicoherent graded algebra $\mathcal{B}$ is Noetherian. Fix a closed subset $T \subset X$. Denote by $\mathcal{B}\text{-}\mathsf{qcoh}_T$ the full DG-subcategory in $\mathcal{B}\text{-}\mathsf{qcoh}$ consisting of all the quasicoherent CDG-modules whose underlying quasicoherent graded $\mathcal{B}$-modules have their set-theoretic supports contained in T. The DG-category $\mathcal{B}\text{-}\mathsf{coh}_T$ of coherent CDG-modules with set-theoretic support in T is defined similarly.

Let $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_T)$ and $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_T)$ denote the coderived and the absolute derived category of these DG-categories of CDG-modules. Finally, let $\mathcal{B}\text{-}\mathsf{qcoh}_{T,\mathsf{inj}}$ denote the DG-category of quasicoherent CDG-modules over $\mathcal{B}$ whose underlying quasicoherent graded modules are injective objects of the abelian category of quasicoherent graded $\mathcal{B}$-modules with set-theoretic support contained in T.

Proposition 1.10. (a) *The functor $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{T,\mathsf{inj}}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_T)$ induced by the embedding of DG-categories $\mathcal{B}\text{-}\mathsf{qcoh}_{T,\mathsf{inj}} \to \mathcal{B}\text{-}\mathsf{qcoh}_T$ is an equivalence of triangulated categories.*

(b) *The functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_T) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_T)$ induced by the embedding of DG-categories $\mathcal{B}\text{-}\mathsf{coh}_T \to \mathcal{B}\text{-}\mathsf{qcoh}_T$ is fully faithful and its image is a set of compact generators of the target category.*

(c) *The functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_T) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ induced by the embedding of DG-categories $\mathcal{B}\text{-}\mathsf{qcoh}_T \to \mathcal{B}\text{-}\mathsf{qcoh}$ is fully faithful.*

(d) *The functor* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_T) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh})$ *induced by the embedding of DG-categories* $\mathcal{B}\text{-}\mathsf{coh}_T \to \mathcal{B}\text{-}\mathsf{coh}$ *is fully faithful.*

Proof. Part (a) is essentially a particular case of [Positselski 2011b, Theorem and Remark in Section 3.7]. It is only important here that there are enough injective objects in the abelian category of quasicoherent graded $\mathcal{B}$-modules supported set-theoretically in T and the class of such injective objects is closed under infinite direct sums. This is so because the abelian category in question is a locally Noetherian Grothendieck category (since X and $\mathcal{B}$ are Noetherian). Part (b) can be proven in the same way as the results of [loc. cit., Section 3.11]. Part (d) follows from parts (b) and (c) and Proposition 1.5(d).

Finally, part (c) follows from part (a), Lemma 1.7(b), and the fact that any injective object $\mathcal{J}$ in the category of quasicoherent graded $\mathcal{B}$-modules supported set-theoretically in T is also an injective object in the category of arbitrary quasicoherent graded $\mathcal{B}$-modules. The latter is essentially a reformulation of the Artin–Rees lemma.

Indeed, it suffices to check that for any coherent graded $\mathcal{B}$-module $\mathcal{M}$ and its coherent graded $\mathcal{B}$-submodule $\mathcal{N}$, any morphism of quasicoherent graded $\mathcal{B}$-modules $\phi : \mathcal{N} \to \mathcal{J}$ can be extended to $\mathcal{M}$. Let Z be a closed subscheme structure on the closed subset $T \subset X$. Then there is an integer $n \geq 0$ such that the morphism ϕ annihilates $\mathcal{I}_Z^n\mathcal{N}$ (where $\mathcal{I}_Z$ is the sheaf of ideals of the closed subscheme Z). By Lemma A.3, there exists $m \geq 0$ such that $\mathcal{I}_Z^m\mathcal{M} \cap \mathcal{N} \subset \mathcal{I}_Z^n\mathcal{N}$. Then there exists a morphism $\mathcal{M}/\mathcal{I}_Z^m\mathcal{M} \to \mathcal{J}$ of quasicoherent graded $\mathcal{B}$-modules supported set-theoretically in T which extends the given morphism into $\mathcal{J}$ from the quasicoherent graded $\mathcal{B}$-submodule $\mathcal{N}/(\mathcal{I}_Z^m\mathcal{M} \cap \mathcal{N}) \subset \mathcal{M}/\mathcal{I}_Z^m\mathcal{M}$. □

Let $U \subset X$ denote the open subscheme $X \setminus T$.

Theorem 1.10. (a) *The functor of restriction to the open subscheme* $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}|_U\text{-}\mathsf{qcoh})$ *is the Verdier localization functor by the thick subcategory*

$$\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_T) \subset \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}).$$

In particular, the kernel of the restriction functor coincides with the subcategory $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_T)$.

(b) *The functor of restriction to the open subscheme* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}|_U\text{-}\mathsf{coh})$ *is the Verdier localization functor by the triangulated subcategory*

$$\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_T) \subset \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}).$$

In particular, the kernel of the restriction functor coincides with the thick envelope of (*i.e., the minimal thick subcategory containing*) $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_T)$ *in* $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh})$.

Proof. Let $j : U \to X$ denote the natural open embedding. To prove part (a), consider the functor $\mathbb{R}j_* : \mathsf{D}^{\mathrm{co}}(\mathcal{B}|_U\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathrm{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ as constructed in Section 1.8. Since the quasicoherent graded algebra $\mathcal{B}|_U$ is flat over $\mathcal{B}$, the functor $\mathbb{R}j_*$ is right adjoint to the restriction functor $j^* : \mathsf{D}^{\mathrm{co}}(\mathcal{B}\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathrm{co}}(\mathcal{B}|_U\text{-}\mathsf{qcoh})$. Obviously, the composition $j^*\mathbb{R}j_*$ is the identity functor. It follows that the functor j^* is a Verdier localization functor by its kernel, which is the full subcategory consisting of all the cones of the adjunction morphisms $\mathcal{M} \to \mathbb{R}j_*j^*\mathcal{M}$, where $\mathcal{M} \in \mathsf{D}^{\mathrm{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$.

Represent the object $\mathcal{M}$ by a CDG-module with an injective underlying quasicoherent graded $\mathcal{B}$-module. By Theorem A.3, the quasicoherent graded $\mathcal{B}|_U$-module $j^*\mathcal{M}$ is then also injective, so we have $\mathbb{R}j_*j^*\mathcal{M} = j_*j^*\mathcal{M}$. Obviously, both the kernel and the cokernel of the closed morphism of CDG-modules $\mathcal{M} \to j_*j^*\mathcal{M}$ belong to $\mathcal{B}\text{-}\mathsf{qcoh}_T$, and it follows, in view of Proposition 1.10(c), that the cone also belongs to $\mathsf{D}^{\mathrm{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_T)$.

To prove part (b), notice first that any coherent CDG-module over $\mathcal{B}|_U$ can be extended to a coherent CDG-module over $\mathcal{B}$ (because a coherent sheaf $\mathcal{K}$ on U can be extended to a coherent subsheaf of $j_*\mathcal{K}$), so the restriction functor is essentially surjective. Taking this observation into account, part (b) follows from part (a), Proposition 1.10(b), Proposition 1.5(d), and the standard results about localization of compactly generated triangulated categories [Neeman 1992, Lemma 2.5 to Theorem 2.1]. ☐

Define the *category-theoretic support* $\operatorname{supp}\mathcal{M}$ of a quasicoherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ as the minimal closed subset $T \subset X$ such that the restriction $\mathcal{M}|_U$ of $\mathcal{M}$ to the open subscheme $U = X \setminus T$ is a coacyclic CDG-module over $\mathcal{B}|_U$. In other words, $X \setminus \operatorname{supp}\mathcal{M}$ is the union of all open subschemes $V \subset X$ such that $\mathcal{M}|_V$ is a coacyclic CDG-module over $\mathcal{B}|_V$ (see Remark 1.3). Obviously, one has $\operatorname{supp}\mathcal{M} \subset \operatorname{Supp}\mathcal{M}$.

The category-theoretic support of a coherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ can be equivalently defined as the minimal closed subset $T \subset X$ such that the restriction $\mathcal{M}|_U$ of $\mathcal{M}$ to the open subscheme $U = X \setminus T$ is absolutely acyclic. Indeed, any CDG-module from $\mathcal{B}|_U\text{-}\mathsf{coh}$ that is coacyclic with respect to $\mathcal{B}|_U\text{-}\mathsf{qcoh}$ is also absolutely acyclic with respect to $\mathcal{B}|_U\text{-}\mathsf{coh}$ by Proposition (d).

Corollary 1.10. (a) *For any quasicoherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ with category-theoretic support* $\operatorname{supp}\mathcal{M}$ *contained in T, there exists a quasicoherent CDG-module $\mathcal{M}'$ over $\mathcal{B}$ such that $\mathcal{M}$ is isomorphic to $\mathcal{M}'$ in* $\mathsf{D}^{\mathrm{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ *and set-theoretic support* $\operatorname{Supp}\mathcal{M}'$ *is contained in T.*

(b) *For any coherent CDG-module $\mathcal{M}$ over $\mathcal{B}$ with category-theoretic support* $\operatorname{supp}\mathcal{M}$ *contained in T, there exists a coherent CDG-module $\mathcal{M}'$ over $\mathcal{B}$ such that $\mathcal{M}$ is isomorphic to a direct summand of $\mathcal{M}'$ in* $\mathsf{D}^{\mathrm{abs}}(\mathcal{B}\text{-}\mathsf{coh})$ *and set-theoretic support* $\operatorname{Supp}\mathcal{M}'$ *is contained in T.*

Proof. Follows immediately from Theorem 1.10. □

Remark 1.10. One can prove that the restriction functor in Theorem 1.10(a) is a Verdier localization functor without assuming the quasicoherent graded algebra $\mathcal{B}$ to be Noetherian. Indeed, one can construct a right adjoint functor $\mathbb{R}j_*$ to the restriction functor j^* in the way similar to that of Proposition 1.9; then it is easy to see that $j^*\mathbb{R}j_*$ is the identity functor.

When $\mathcal{B}$ is Noetherian, Theorem 1.10 can be generalized as follows. Let S and T be closed subsets in X; set $U = X\setminus T$. Then the restriction functor $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_S) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}|_U\text{-}\mathsf{qcoh}_{U\cap S})$ is the Verdier localization functor by the thick subcategory $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}_{T\cap S})$, and the restriction functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_S) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}|_U\text{-}\mathsf{coh}_{U\cap S})$ is the Verdier localization functor by the triangulated subcategory $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{T\cap S})$. The proof is similar to the above.

It is not difficult to deduce from the latter assertions, using the result of [Neeman 1996, Theorem 2.1(5)], that the property of an object of $\mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$ to belong to the thick envelope of $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh})$ is local in X. Using the Čech exact sequence as in Remark 1.3, one can easily see that the property of an object of $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh})$ to belong to $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ is also local.

We do not know whether the property of an object of $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh})$ or $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ to belong to $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}})$ is local in general. In the particular case of matrix factorizations, such results will be proven in Section 3.2 using the connection with singularity categories (cf. Remark 3.6).

Notice that the theory of localization for compactly generated triangulated categories, on which the proof of Theorem 1.10(b) is based, was originally applied in algebraic geometry for the purposes of the Thomason–Trobaugh localization theory of perfect complexes. In this section we apply it to coherent CDG-modules. In fact, we will see in Section 3.3 that the localization theory *fails* for locally free matrix factorizations of finite rank.

2. Triangulated categories of relative singularities

2.1. ***Relative singularity category.*** Recall that X denotes a separated Noetherian scheme with enough vector bundles. The *triangulated category of singularities* $\mathsf{D}^{\mathsf{b}}_{Sing}(X)$ of the scheme X is defined [Orlov 2004, Section 1.2] as the quotient category of the bounded derived category $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh})$ of coherent sheaves on X by its thick subcategory $\mathit{Perf}(X)$ of perfect complexes on X.

The perfect complexes, in our assumptions, can be simply defined as bounded complexes of locally free sheaves of finite rank, so $\mathit{Perf}(X) = \mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh}_{\mathsf{lf}})$ is the bounded derived category of the exact category $X\text{-}\mathsf{coh}_{\mathsf{lf}}$ of locally free sheaves of finite rank on X. Equivalently, the perfect complexes are the compact objects of the unbounded derived category of quasicoherent sheaves $\mathsf{D}(X\text{-qcoh})$ on the scheme X [Neeman 1996, Examples 1.10–1.11 and Corollary 2.3].

Let $Z \subset X$ be a closed subscheme such that $\mathcal{O}_Z$ has finite flat dimension as an $\mathcal{O}_X$-module. In this case the derived inverse image functor $\mathbb{L}i^*$ for the closed embedding $i : Z \to X$ acts on the bounded derived categories of coherent sheaves, $\mathsf{D}^{\mathsf{b}}(X\text{-coh}) \to \mathsf{D}^{\mathsf{b}}(Z\text{-coh})$. We call the quotient category of $\mathsf{D}^{\mathsf{b}}(Z\text{-coh})$ by the thick subcategory generated by the objects in the image of this functor the *triangulated category of singularities of* Z *relative to* X and denote it by $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$.

Note that the triangulated category of relative singularities $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ is a quotient category of the conventional (absolute) triangulated category of singularities $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$ of the scheme Z. Indeed, the thick subcategory $\mathit{Perf}(Z) \subset \mathsf{D}^{\mathsf{b}}(Z\text{-coh})$ is generated by any ample family of vector bundles on Z since any such family is a set of compact generators of the unbounded derived category of quasicoherent sheaves $\mathsf{D}(Z\text{-qcoh})$ on Z [Neeman 1996]; in particular, it is generated by the restrictions to Z of vector bundles from X (see also Lemma 2.8).

The functor $\mathbb{L}i^* : \mathsf{D}^{\mathsf{b}}(X\text{-coh}) \to \mathsf{D}^{\mathsf{b}}(Z\text{-coh})$ induces a triangulated functor

$$i^\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Z).$$

Furthermore, since the sheaf $i_*\mathcal{O}_Z$ belongs to $\mathit{Perf}(X)$, the functor $i_* : \mathsf{D}^{\mathsf{b}}(Z\text{-coh}) \to \mathsf{D}^{\mathsf{b}}(X\text{-coh})$ takes $\mathit{Perf}(Z)$ to $\mathit{Perf}(X)$ (cf. [Orlov 2004, paragraphs before Proposition 1.14]). Hence the functor i_* induces a triangulated functor $i_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(Z) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X)$ right adjoint to i°. The triangulated category $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ is the quotient category of $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$ by the thick subcategory generated by the image of the functor i°.

When X is regular, any coherent sheaf on X has a finite resolution by locally free sheaves of finite rank. So $\mathsf{D}^{\mathsf{b}}_{Sing}(X) = 0$, and hence the triangulated categories $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ coincide. The converse is also true: the structure sheaf of the reduced scheme structure on the closure of any singular point of X is not a perfect complex on X, so $\mathsf{D}^{\mathsf{b}}_{Sing}(X) \neq 0$ when X is not regular.

Remark 2.1. Roughly speaking, the triangulated category of relative singularities $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ measures how much worse the singularities of Z are compared to the singularities of X in a neighborhood of Z.

The basic formal properties of $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ are similar to those of $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$. When the $\mathcal{O}_X$-module $\mathcal{O}_Z$ has finite flat dimension, the derived category $\mathsf{D}^{\mathsf{b}}(X\text{-coh})$ is generated by coherent sheaves adjusted to i^*. Let $\mathsf{E}_{Z/X}$ denote the minimal full subcategory of the abelian category of coherent sheaves on Z containing the restrictions of such coherent sheaves from X and closed under extensions and the kernels of epimorphisms of sheaves. Then $\mathsf{E}_{Z/X}$ is naturally an exact category and its bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{Z/X})$ is equivalent to the thick subcategory of

$\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})$ generated by the derived restrictions of coherent sheaves from X, so $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X) = \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})/\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{Z/X})$. One can define the E-homological dimension of a coherent sheaf (or bounded complex) on Z as the minimal length of a left resolution consisting of objects from $\mathsf{E}_{Z/X}$. This dimension does not depend on the choice of a resolution (in the same sense that the conventional flat dimension doesn't). The thick subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{Z/X})$ consists of those objects of $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})$ that have finite E-homological dimensions.

Unlike in the case of perfect complexes, we do not know whether the property to belong to $\mathsf{E}_{Z/X}$ or $\mathsf{D}^{\mathsf{b}}(\mathsf{E}_{Z/X})$ is local, though. In the case when Z is a Cartier divisor, locality can be established using Theorem 2.7 below and Remark 1.3.

2.2. ***Matrix factorizations.*** Following [Polishchuk and Vaintrob 2011], we will consider matrix factorizations of a global section of a line bundle. So let $\mathcal{L}$ be a line bundle (invertible sheaf) on X and $w \in \mathcal{L}(X)$ be a fixed section, called the *potential.*

Let $\mathcal{B} = (X, \mathcal{L}, w)$ denote the following $\mathbb{Z}$-graded quasicoherent CDG-algebra over X. The component $\mathcal{B}^n$ is isomorphic to $\mathcal{L}^{\otimes n/2}$ for $n \in 2\mathbb{Z}$ and vanishes for $n \in 2\mathbb{Z}+1$, the multiplication in $\mathcal{B}$ being given by the natural isomorphisms $\mathcal{L}^{\otimes n/2} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m/2} \to \mathcal{L}^{\otimes (n+m)/2}$. For any affine open subscheme $U \subset X$, the differential on $\mathcal{B}(U)$ is zero, and the curvature element is $w|_U \in \mathcal{B}^2(U) = \mathcal{L}(U)$. The elements a_{UV} defining the restriction morphisms of CDG-rings $\mathcal{B}(V) \to \mathcal{B}(U)$ all vanish.

The category of quasicoherent $\mathbb{Z}$-graded $\mathcal{B}$-modules is equivalent to the category of quasicoherent $\mathbb{Z}/2$-graded $\mathcal{O}_X$-modules, the equivalence assigning to a graded $\mathcal{B}$-module $\mathcal{M}$ the pair of $\mathcal{O}_X$-modules which we denote symbolically by $\mathcal{U}^0 = \mathcal{M}^0$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} = \mathcal{M}^1$. Conversely, $\mathcal{M}^n \simeq \mathcal{U}^{n \bmod 2} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n/2}$ for all $n \in \mathbb{Z}$ (the meaning of the notation in the right-hand side being the obvious one). This equivalence of abelian categories preserves all the properties of coherence, flatness, flat dimension, local projectivity/local freeness, etc. that we were interested in in Section 1.

Following [Lin and Pomerleano 2013], we will consider CDG-modules over $\mathcal{B} = (X, \mathcal{L}, w)$ whose underlying graded $\mathcal{B}$-modules correspond to coherent or quasicoherent $\mathcal{O}_X$-modules, rather than just locally free sheaves (as in the conventional matrix factorizations). A quasicoherent CDG-module over $(X, \mathcal{L}, w)$ is the same thing as a pair of quasicoherent $\mathcal{O}_X$-modules $\mathcal{U}^0$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2}$ endowed with $\mathcal{O}_X$-linear morphisms $\mathcal{U}^0 \to \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2}$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L}$ such that both compositions

$$\mathcal{U}^0 \to \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L} \quad \text{and} \quad \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L} \to \mathcal{U}^1 \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes 3/2}$$

are equal to the multiplications with w.

2.3. ***Exotic derived categories of matrix factorizations.*** The following corollary is a restatement of the results of Section 1 in the application to the quasicoherent CDG-algebra $\mathcal{B} = (X, \mathcal{L}, w)$. We will use the notation $(X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}$ (instead of the previously introduced $\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{fl}}$) for the DG-category of locally free matrix factorizations of finite rank, and the notation $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}$ (instead of the previously introduced $\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}$) for the DG-category of locally free matrix factorizations of possibly infinite rank (see Remark 1.4). The rest of our notation system for various classes of quasicoherent CDG-modules over $\mathcal{B} = (X, \mathcal{L}, w)$ remains in use.

In addition, we also denote by $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}}$ the DG-category of quasicoherent CDG-modules of finite locally free/locally projective dimension over $(X, \mathcal{L}, w)$ (see Remark 1.4 again). Let $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}})$ and $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}})$ be the corresponding derived categories of the second kind.

Corollary 2.3. (a) *The functor* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

(b) *The functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

(c) *The functors*

$$\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}}),$$
$$\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}})$$

induced by the embedding of DG-categories $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}}$ *are equivalences of triangulated categories.*

(d) *The triangulated categories* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$ *and* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$ *coincide, as do the categories* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}})$ *and* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}})$. *The natural functors between these four categories form a commutative square of equivalences of triangulated categories.*

(e) *When the scheme X has finite Krull dimension, the functors*

$$\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}),$$
$$\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$$

induced by the embedding of DG-categories $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}$ *are equivalences of triangulated categories. The natural functors between these four categories form a commutative square of equivalences.*

(f) *When the scheme X has finite Krull dimension, the triangulated category* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ *coincides with* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ *and the triangulated* *category* $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ *coincides with* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$. *The natural functors between these four categories form a commutative square of equivalences.*

(g) *The functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{ffd}})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}} \to (X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

(h) *The triangulated functors*

$$\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$$

induced by the embeddings of DG-categories $(X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}} \to (X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to (X,\mathcal{L},w)\text{-}\mathsf{qcoh}$ *are fully faithful.*

(i) *The triangulated functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}} \to (X,\mathcal{L},w)\text{-}\mathsf{coh}$ *is fully faithful.*

(j) *The triangulated functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}} \to (X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}$ *is fully faithful.*

(k) *The triangulated functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{-}\mathsf{coh} \to (X,\mathcal{L},w)\text{-}\mathsf{qcoh}$ *is fully faithful.*

(l) *The triangulated functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{-}\mathsf{coh} \to (X,\mathcal{L},w)\text{-}\mathsf{qcoh}$ *is fully faithful and its image forms a set of compact generators for* $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$.

Proof. Parts (a), (b) and (g) are particular cases of Theorem 1.4, and the proof of part (c) is similar (see Remark 1.4). Part (g) also essentially follows from Proposition 1.5(b) (and part (b) can be proven similarly). Parts (h), (i), (k) and (l) are particular cases of Proposition 1.5 (except for "locally free half" of part (h), which is similar to the "flat half"). Part (d) is Theorem 1.6 together with part (c). Part (j) is Corollary 1.6. Part (e) follows from parts (a)–(c) and Remark 1.4 (cf. the discussion in the end of Section 1.6). Part (f) follows from parts (a), (b), (d) and (e); alternatively, it can be proven directly in the way similar to part (d), using the fact that the exact category of flat quasicoherent sheaves on X has finite homological dimension when the Krull dimension of X is finite. ☐

2.4. ***Regular and Gorenstein scheme cases.*** When the scheme X is regular or Gorenstein, the assertions of Corollary 2.3 simplify as follows.

Corollary 2.4. (a) *When the scheme X is Gorenstein of finite Krull dimension, the functors*

$$\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$$

induced by the embedding of DG-categories $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}$ *are equivalences of triangulated categories.*

(b) *When the scheme X is regular of finite Krull dimension, the natural functors between the categories* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$, $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$, $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$, *and* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ *form a commutative square of equivalences of triangulated categories.*

(c) *When the scheme X is regular, the natural functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$ *is an equivalence of triangulated categories.*

Proof. Part (a) is a particular case of Proposition 1.7. Part (c) follows from Corollary 2.3(g) since any coherent sheaf on a regular scheme has finite flat dimension. In the assumptions of part (b), the functor $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ is an isomorphism of triangulated categories by [Positselski 2011b, Theorem 3.6(a) and Remark 3.6] since the abelian category of quasicoherent sheaves on a regular scheme of finite Krull dimension has finite homological dimension and enough injectives (cf. Theorem 1.6). The remaining assertions of part (b) follow from Corollary 2.3(a) and (b), or alternatively from part (a). □

Assuming that X has finite Krull dimension, the assertions of Corollaries 2.3 and 2.4 may be summarized by the following commutative diagram of triangulated functors. Here, as above, $\mathcal{B}$ denotes the quasicoherent CDG-algebra $(X, \mathcal{L}, w)$:

$$
\begin{tikzcd}
\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{lf}}) \arrow[rr, equal] \arrow[ddd, tail] & & \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}_{\mathsf{ffd}}) \arrow[drrr, tail, "= \text{when } X \text{ regular}"] \arrow[ddd, tail] & & & \\
& & & & & \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{coh}) \arrow[dd, tail, "\text{comp. gener.}"] \arrow[dddl, tail] \\
& \mathsf{D}^{\mathsf{co=abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lfd}}) \arrow[dl, equal] \arrow[dr, equal] & & & & \\
\mathsf{D}^{\mathsf{co=abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \arrow[rr, equal] & & \mathsf{D}^{\mathsf{co=abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \arrow[rrr, "= \text{when } X \text{ Gorenstein}"] \arrow[drr, tail, "= \text{when } X \text{ regular}"'] & & & \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}) \\
& \mathsf{D}^{\mathsf{co=abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \arrow[ul, equal] \arrow[ur, equal] & & & \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}) \arrow[ur, two heads, "= \text{when } X \text{ regular}"'] &
\end{tikzcd}
$$

The four categories in the left lower area are coderived categories coinciding with absolute derived categories (of the same classes of quasicoherent CDG-modules). The five double lines between these four categories are equivalences, as is the upper left horizontal line. All the arrows going down are fully faithful functors. The image of the rightmost vertical arrow is a set of compact generators in the target category. The only arrow going up is a Verdier localization functor.

Nothing is claimed about the long horizontal arrow in the right lower area of the diagram in general; but when X is Gorenstein, this functor is an equivalence of categories. When X is regular, all the arrows going right are equivalences of categories (so the whole diagram reduces to one triangulated category with infinite direct sums, containing a full triangulated subcategory of compact generators).

Recall also that, by Lemma 1.7, for any X we have a commutative diagram of triangulated functors

$$\begin{tikzcd} H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{inj}}) \arrow[r, equal] & \mathsf{D}^{\mathsf{co}=\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fid}}) \arrow[rr, equal] \arrow[dr, tail] & & \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh}) \\ & & \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}) \arrow[ur, two heads] & \end{tikzcd}$$

with equivalences of categories in the upper line. The fully faithful embedding $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fid}}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh})$, which in the Gorenstein case (of finite Krull dimension) coincides with the embedding $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \to \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh})$, is always right adjoint to the localization functor $\mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(\mathcal{B}\text{-}\mathsf{qcoh})$.

Remark 2.4. When X is an affine Noetherian scheme of finite Krull dimension, the embeddings of DG-categories $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lp}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}$ induce equivalences $H^0(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{lp}}) \simeq \mathsf{D}^{\mathsf{abs}}(\mathcal{B}\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{ctr}}(\mathcal{B}\text{-}\mathsf{qcoh})$ between the homotopy category of (locally) projective matrix factorizations of infinite rank, the absolute derived category of flat matrix factorizations, and the contraderived category of arbitrary quasicoherent matrix factorizations (see [Positselski 2011b, Section 3.8]; cf. Remark 1.5).

2.5. ***Serre–Grothendieck duality.*** The aim of this section is to show that the somewhat mysterious long horizontal arrow in the above large diagram is actually a functor between two equivalent triangulated categories, for a rather wide class of schemes X. The functor $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ in the above diagram, which is induced by the embedding of DG-categories $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}$, is *not* the equivalence that we have in mind, however (unless the scheme is Gorenstein). Instead, the equivalence between the categories $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ is constructed using a dualizing complex on X [Hartshorne 1966, Section V.2].

Before recalling the definition of a dualizing complex, let us discuss the notion of the *quasicoherent internal Hom*. Given quasicoherent sheaves $\mathcal{M}$ and $\mathcal{N}$ over X, the quasicoherent sheaf $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{M}, \mathcal{N})$ is defined by the isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(- \otimes_{\mathcal{O}_X} \mathcal{M},\, \mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{M}, \mathcal{N}))$ of functors from the category of quasicoherent sheaves to the category of abelian groups. Equivalently, the quasicoherent sheaf $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{M}, \mathcal{N})$ can be obtained by applying the coherator functor [Thomason and Trobaugh 1990, Sections B.12–B.14] to the sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$. Whenever $\mathcal{M}$ is a coherent sheaf, the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ of $\mathcal{O}_X$-module internal Hom is quasicoherent, and $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{M}, \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$.

Notice that the construction of the sheaf $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{M}, \mathcal{N})$ is *not* local in general; i.e., it does not commute with the restrictions of quasicoherent sheaves to open subschemes; when the sheaf $\mathcal{M}$ is coherent, it does.

Lemma 2.5. (a) *For any injective quasicoherent sheaf $\mathcal{J}$ over a separated Noetherian scheme X, the functor $\mathcal{M} \longmapsto \mathcal{H}om_{X\text{-qc}}(\mathcal{M}, \mathcal{J})$ is exact.*

(b) *For any flat quasicoherent sheaf $\mathcal{F}$ and injective quasicoherent sheaf $\mathcal{J}$ over X, the quasicoherent sheaves $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{J}$ and $\mathcal{H}om_{X\text{-qc}}(\mathcal{F}, \mathcal{J})$ are injective.*

(c) *For any injective quasicoherent sheaves $\mathcal{J}'$ and $\mathcal{J}$ over X, the quasicoherent sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{J}', \mathcal{J})$ is flat.*

Proof. The second assertion of part (b) is obvious from the universal property defining $\mathcal{H}om_{X\text{-qc}}$. To prove the first one, notice that injectivity of quasicoherent sheaves over a Noetherian scheme is a local property ([Hartshorne 1966, Lemma II.7.16 and Theorem II.7.18] or Theorem A.3), a flat quasicoherent sheaf over an affine scheme is a filtered inductive limit of locally free sheaves of finite rank [Bourbaki 1980, Numéros 1.5–6], and injectivity of modules over a Noetherian ring is preserved by filtered inductive limits.

The proof of parts (a) and (c) follows the argument in [Murfet 2007, Lemma 8.7]. Choose a finite affine covering U_α of the scheme X and consider the morphism $\mathcal{J} \to \bigoplus_\alpha j_{U_\alpha *} j_{U_\alpha}^* \mathcal{J}$. Being an embedding of injective quasicoherent sheaves, it splits, so $\mathcal{J}$ is a direct summand of the direct sum of $j_{U_\alpha *} j_{U_\alpha}^* \mathcal{J}$. Hence it suffices to prove both assertions in the case when $\mathcal{J} = j_{V*}\mathcal{J}''$, where $\mathcal{J}''$ is an injective quasicoherent sheaf on an affine open subscheme $V \subset X$.

Now we have $\mathcal{H}om_{X\text{-qc}}(\mathcal{M}, j_{V*}\mathcal{J}'') \simeq j_{V*}\, \mathcal{H}om_{V\text{-qc}}(j_V^*\mathcal{M}, \mathcal{J}'')$. Since $V \to X$ is a flat affine morphism, the functor j_{V*} is exact and preserves the flatness of quasicoherent sheaves. This proves part (a) and reduces (c) to the case of an affine scheme $X = V$. It remains to apply [Cartan and Eilenberg 1956, Proposition VI.5.3]. □

For our purposes, a *dualizing complex* $\mathcal{D}_X^\bullet$ on X is a finite complex of injective quasicoherent sheaves such that the cohomology sheaves of $\mathcal{D}_X^\bullet$ are coherent and for any coherent sheaf $\mathcal{M}$ over X, the natural morphism of finite complexes of quasicoherent sheaves $\mathcal{M} \to \mathcal{H}om_{X\text{-qc}}(\mathcal{H}om_{X\text{-qc}}(\mathcal{M}, \mathcal{D}_X^\bullet), \mathcal{D}_X^\bullet)$ is a quasi-isomorphism. Note that it follows from the former two conditions on $\mathcal{D}_X^\bullet$ that the complex $\mathcal{H}om_{X\text{-qc}}(\mathcal{M}, \mathcal{D}_X^\bullet)$ has coherent cohomology sheaves. This makes the conditions imposed on $\mathcal{D}_X^\bullet$ actually local in X, so the restriction $\mathcal{D}_U^\bullet = \mathcal{D}_X^\bullet|_U$ of the complex of sheaves $\mathcal{D}_X^\bullet$ to an open subscheme $U \subset X$ is a dualizing complex on U.

Given a quasicoherent CDG-algebra $\mathcal{B}$ over X, a quasicoherent left CDG-module $\mathcal{M}$ over $\mathcal{B}$, and a complex of quasicoherent sheaves $\mathcal{F}^\bullet$ on X, consider the complexes of quasicoherent left CDG-modules $\mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}$ and $\mathcal{H}om_{X\text{-qc}}(\mathcal{F}^\bullet, \mathcal{M})$ over $\mathcal{B}$. Taking their totalizations (formed, if necessary, by taking infinite direct sums along the diagonals), construct two triangulated functors $H^0(\mathcal{B}\text{-qcoh}) \to H^0(\mathcal{B}\text{-qcoh})$ depending on a complex $\mathcal{F}^\bullet$. Given a right CDG-module $\mathcal{N}$ over $\mathcal{B}$ (see [Positselski 2011b, Sections 3.1 and B.1]), similarly construct a complex of quasicoherent left

CDG-modules $\mathcal{H}om_{X\text{-qc}}(\mathcal{N},\mathcal{F}^\bullet)$ over $\mathcal{B}$, obtaining a triangulated functor from the homotopy category of right CDG-modules $H^0(\mathsf{qcoh}\text{-}\mathcal{B})$ to $H^0(\mathcal{B}\text{-}\mathsf{qcoh})$.

In the particular case of matrix factorizations, we conclude that the covariant functors $\mathcal{F}^\bullet\otimes_{\mathcal{O}_X}-$ and $\mathcal{H}om_{X\text{-qc}}(\mathcal{F}^\bullet,-)$ take quasicoherent matrix factorizations of a potential $w\in\mathcal{L}(X)$ to (complexes of) quasicoherent matrix factorizations of w, while the contravariant functor $\mathcal{H}om_{X\text{-qc}}(-,\mathcal{F}^\bullet)$ transforms quasicoherent matrix factorizations of the opposite potential $-w\in\mathcal{L}(X)$ into (complexes of) quasicoherent matrix factorizations of w. Of course, the quasicoherent CDG-algebras $(X,\mathcal{L},w)$ and $(X,\mathcal{L},-w)$ over a scheme X are naturally isomorphic, but we prefer to keep the distinction between the two.

The next proposition provides the matrix factorization version of the conventional (contravariant) Serre–Grothendieck duality for bounded complexes of coherent sheaves. We assume that X is a separated Noetherian scheme with a dualizing complex $\mathcal{D}_X^\bullet$. Recall that any such scheme has finite Krull dimension [Hartshorne 1966, Corollary V.7.2]. We denote by D^{op} the opposite category to a category D.

Proposition 2.5. *The triangulated functor*

$$\mathcal{H}om_{X\text{-qc}}(-,\mathcal{D}_X^\bullet): H^0((X,\mathcal{L},-w)\text{-}\mathsf{qcoh})^{\mathsf{op}}\longrightarrow H^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$$

induces a well-defined triangulated functor between the absolute derived categories $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{qcoh})^{\mathsf{op}}$ *and* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$ *taking the full triangulated subcategory* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{coh})^{\mathsf{op}}\subset\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{qcoh})^{\mathsf{op}}$ *into the full subcategory* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})\subset\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$. *The composition of the duality functors* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})\to\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{coh})^{\mathsf{op}}\to\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})$ *is the identity functor.*

Proof. The functor $\mathcal{H}om_{X\text{-qc}}(-,\mathcal{D}_X^\bullet)$ preserves absolute acyclicity, because $\mathcal{D}_X^\bullet$ is a complex of injective quasicoherent sheaves, so Lemma 2.5(a) applies. Given a coherent matrix factorization $\mathcal{M}$, the finite complex of matrix factorizations $\mathcal{H}om_{X\text{-qc}}(-,\mathcal{D}_X^\bullet)$ has coherent cohomology matrix factorizations, so one can use its canonical truncations in order to prove by induction that its totalization belongs to the triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})$.

Finally, for any quasicoherent matrix factorization $\mathcal{M}$, consider the bicomplex of matrix factorizations $\mathcal{H}om_{X\text{-qc}}(\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{D}_X^\bullet),\mathcal{D}_X^\bullet)$ and take its totalization in the two directions where it is a complex, obtaining a complex of matrix factorizations. Then there is a natural morphism of finite complexes of matrix factorizations $\mathcal{M}\to\mathcal{H}om_{X\text{-qc}}(\mathcal{H}om_{X\text{-qc}}(\mathcal{M},\mathcal{D}_X^\bullet),\mathcal{D}_X^\bullet)$, which is a quasi-isomorphism of complexes of matrix factorizations when $\mathcal{M}$ is coherent. The induced closed morphism of the total matrix factorizations is an isomorphism in $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$ since the totalization of a finite acyclic complex of matrix factorizations is absolutely acyclic. It remains to use the fact that the functor $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$ is fully faithful (see Corollary 2.3(k)) again. □

The next result is our covariant Serre–Grothendieck duality theorem for matrix factorizations. It is the matrix factorization analogue of the similar results for complexes of projective and injective modules [Iyengar and Krause 2006, Theorem 4.2] and sheaves [Murfet 2007, Theorem 8.4]. It also strongly resembles the *derived comodule-contramodule correspondence* theory (see [Positselski 2010, Corollaries 5.4 and 6.3; 2011b, Theorem 5.2]; cf. Remark 2.4 above). Notice that our proof is more akin to the arguments in [Positselski 2010; 2011b] than those of [Iyengar and Krause 2006; Murfet 2007] in that we give a direct proof of the covariant duality independent of both the contravariant duality and any descriptions of the compact objects in the categories to be compared.

Theorem 2.5. *The functors*

$$\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} - : H^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow H^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{inj}}),$$
$$\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, -) : H^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{inj}}) \longrightarrow H^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$$

induce mutually inverse equivalences between the coderived categories

$$\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \quad \textit{and} \quad \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}).$$

Proof. Recall that $H^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{inj}}) \simeq \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$ by Lemma 1.7(b) and $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) = \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ by Corollary 2.3(f) (though we will reprove the latter fact rather than use it in the following argument; see also Remark 2.6 below and Lemma A.1). The functor

$$\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} - : H^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow H^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{inj}})$$

obviously takes matrix factorizations coacyclic with respect to $(X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}$ to matrix factorizations coacyclic with respect to $(X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{inj}}$, which are all contractible. It remains to check that the induced functors are mutually inverse.

Let $\mathcal{E}$ be a matrix factorization from $(X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}$. As in the previous proof, consider the bicomplex of matrix factorizations $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})$ and take its total complex of matrix factorizations. Then there is a natural morphism $\mathcal{E} \to \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})$ of finite complexes of matrix factorizations from $(X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}$. To prove that the induced morphism of the total matrix factorizations is an isomorphism in $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$, we once again use the fact that the totalization of a finite acyclic complex of matrix factorizations is absolutely acyclic. So it suffices to check that for any flat quasicoherent sheaf $\mathcal{F}$ over X, the natural morphism $\mathcal{F} \to \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$ is a quasi-isomorphism of complexes of flat quasicoherent sheaves. This will be done below.

Similarly, let $\mathcal{M}$ be a matrix factorization from $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{inj}}$. Consider the morphism of finite complexes of injective matrix factorizations given by $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{M}) \to \mathcal{M}$. To prove that the cone of the induced morphism of the total matrix factorizations is contractible, it suffices to check that for any injective quasicoherent sheaf $\mathcal{J}$ over X, the natural morphism of complexes of injective sheaves $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{J}) \to \mathcal{J}$ is a quasi-isomorphism.

Let $'\mathcal{D}_X^\bullet$ denote a finite complex of coherent sheaves over X endowed with a quasi-isomorphism $'\mathcal{D}_X^\bullet \to \mathcal{D}_X^\bullet$. Then the morphism $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}) \to \mathcal{H}om_{X\text{-}\mathsf{qc}}('\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$ is a quasi-isomorphism for any flat quasicoherent sheaf $\mathcal{F}$. The construction of the composition

$$\mathcal{F} \longrightarrow \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}) \longrightarrow \mathcal{H}om_{X\text{-}\mathsf{qc}}('\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{F})$$

is local in X, so it suffices to check that the composition is a quasi-isomorphism when X is affine. Then, using the passage to the filtered inductive limit, we may assume that $\mathcal{F}$ is locally free of finite rank, and further that $\mathcal{F} = \mathcal{O}_X$. It remains to recall that the morphism $\mathcal{O}_X \to \mathcal{H}om_{X\text{-}\mathsf{qc}}('\mathcal{D}_X^\bullet, \mathcal{D}_X^\bullet)$ is a quasi-isomorphism by the definition of $\mathcal{D}_X^\bullet$.

Let $''\mathcal{D}_X^\bullet$ be a bounded-above complex of flat quasicoherent sheaves mapping quasi-isomorphically to $'\mathcal{D}_X^\bullet$. Then for any injective quasicoherent sheaf $\mathcal{J}$ over X there are quasi-isomorphisms

$$''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{J}) \longrightarrow \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{J}),$$
$$''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{J}) \longrightarrow {}''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}('\mathcal{D}_X^\bullet, \mathcal{J})$$

forming a commutative diagram with the evaluation morphisms into $\mathcal{J}$. Hence it remains to check that the morphism $''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}('\mathcal{D}_X^\bullet, \mathcal{J}) \to \mathcal{J}$ is a quasi-isomorphism, which is a local question. Assume further that $''\mathcal{D}_X^\bullet$ is a bounded-above complex of locally free sheaves of finite rank. Then there is a natural isomorphism of complexes of sheaves

$$''\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}('\mathcal{D}_X^\bullet, \mathcal{J}) \simeq \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{H}om_{X\text{-}\mathsf{qc}}(''\mathcal{D}_X^\bullet, '\mathcal{D}_X^\bullet), \mathcal{J}).$$

The related morphism

$$\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{H}om_{X\text{-}\mathsf{qc}}(''\mathcal{D}_X^\bullet, '\mathcal{D}_X^\bullet), \mathcal{J}) \longrightarrow \mathcal{J}$$

is induced by the natural morphism of complexes $\mathcal{O}_X \to \mathcal{H}om_{X\text{-}\mathsf{qc}}(''\mathcal{D}_X^\bullet, '\mathcal{D}_X^\bullet)$. The latter is again a quasi-isomorphism essentially by the definition of $\mathcal{D}_X^\bullet$. □

From this point on we resume assuming that X has enough vector bundles.

Notice that the equivalence functor

$$\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} - : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$$

that we constructed takes the full triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$ into the full triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$. This is so because the dualizing complex $\mathcal{D}_X^{\bullet}$ has bounded coherent cohomology sheaves.

Now we will use Proposition 2.5 and Theorem 2.5 in order to construct compact generators of the triangulated category $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$ (cf. [Jørgensen 2005; Neeman 2008]).

Consider the abelian category $Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh})$ of coherent matrix factorizations of $-w$ and closed morphisms of degree 0 between them, and its exact subcategory of locally free matrix factorizations of finite rank $Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh}_{\mathsf{lf}})$. The natural functor between the bounded-above derived categories of our abelian category and its exact subcategory

$$\mathsf{D}^-(Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh}_{\mathsf{lf}})) \longrightarrow \mathsf{D}^-(Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh}))$$

is an equivalence of triangulated categories. The vector bundle duality functor $\mathcal{H}om_{X\text{-}\mathsf{qc}}(-,\mathcal{O}_X) : Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh}_{\mathsf{lf}})^{\mathsf{op}} \to Z^0((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ induces a triangulated functor $\mathsf{D}^-(Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh}_{\mathsf{lf}}))^{\mathsf{op}} \to \mathsf{D}^+(Z^0((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}}))$ taking bounded-above complexes to bounded-below ones.

Let $\mathsf{D}^+(Z^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}))$ denote the bounded-below derived category of the exact category of locally free matrix factorizations of possibly infinite rank. Since the bounded-below acyclic complexes over any exact category with infinite direct sums are coacyclic [Positselski 2010, Lemma 2.1], there is a well-defined, triangulated direct sum totalization functor $\mathsf{D}^+(Z^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})) \to \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$. Consider the composition

$$\begin{aligned} Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh})^{\mathsf{op}} &\longrightarrow \mathsf{D}^-(Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh}))^{\mathsf{op}} \\ &\simeq \mathsf{D}^-(Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh}_{\mathsf{lf}}))^{\mathsf{op}} \longrightarrow \mathsf{D}^+(Z^0((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}})) \\ &\longrightarrow \mathsf{D}^+(Z^0((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})) \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}), \end{aligned}$$

where two of the functors are the duality and the totalization discussed above, while the other two are the natural embedding and the functor induced by such.

One easily checks that this composition takes cones of closed morphisms in $Z^0((X,\mathcal{L},-w)\text{-}\mathsf{coh})$ to cocones in $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$; hence it induces a triangulated functor $H^0((X,\mathcal{L},-w)\text{-}\mathsf{coh})^{\mathsf{op}} \to \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$. Similarly, the above composition takes the totalizations of short exact sequences in $(X,\mathcal{L},-w)\text{-}\mathsf{coh}$ to objects corresponding to the totalizations of short exact sequences in $(X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}$; one checks this by considering a left locally free resolution of a short exact sequence of coherent matrix factorizations. Thus we obtain a triangulated functor

$$\Omega : \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}).$$

Corollary 2.5. *The functor* Ω *is fully faithful, and its image forms a set of compact generators in* $\mathsf{D^{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh_{lf}})$. *The following diagram of triangulated functors is commutative*:

$$
\begin{tikzcd}
\mathsf{D^{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{coh_{lf}})^{\mathsf{op}} \arrow[r, tail, "\upsilon^{\mathsf{op}}"] \arrow[d, equal, "\mathcal{H}om_{X\text{-}\mathsf{qc}}(-{,}\mathcal{O}_X)"'] & \mathsf{D^{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{coh})^{\mathsf{op}} \arrow[d, equal, "\mathcal{H}om_{X\text{-}\mathsf{qc}}(-{,}\mathcal{D}_X^\bullet)"] \arrow[ddl, tail, "\Omega"'] \\
\mathsf{D^{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh_{lf}}) \arrow[r, tail, "\mathcal{D}_X^\bullet\otimes_{\mathcal{O}_X}-"] \arrow[d, tail, "\kappa"'] & \mathsf{D^{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}) \arrow[d, tail, "\gamma"', "\text{comp. gener.}"] \\
\mathsf{D^{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh_{lf}}) \arrow[r, equal, "\mathcal{D}_X^\bullet\otimes_{\mathcal{O}_X}-", "\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet{,}-)"'] & \mathsf{D^{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})
\end{tikzcd}
$$

Here υ, κ, *and* γ *denote the fully faithful functors induced by the natural embeddings of DG-categories of CDG-modules. The two upper vertical lines are the natural contravariant dualities* (*antiequivalences*) *on the* (*absolute derived*) *categories of locally free matrix factorizations of finite rank and coherent matrix factorizations. The lower horizontal line is the equivalence of categories from Theorem 2.5*, *and the middle horizontal arrow is the fully faithful functor discussed after the proof of Theorem 2.5.*

The above diagram is to be compared with the following subdiagram of the large diagram in the end of Section 2.4:

$$
\begin{tikzcd}
\mathsf{D^{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh_{lf}}) \arrow[r, tail, "\upsilon"] \arrow[d, tail, "\kappa"'] & \mathsf{D^{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}) \arrow[d, tail, "\gamma"', "\text{comp. gener.}"] \\
\mathsf{D^{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh_{lf}}) \arrow[r, "\lambda"] & \mathsf{D^{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})
\end{tikzcd}
$$

Here λ denotes the triangulated functor induced by the embedding of DG-categories of CDG-modules $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh_{lf}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}$.

Notice that it is clear from these two diagrams that the functor λ is an equivalence of triangulated categories whenever the functor υ is. Indeed, if υ is an equivalence of categories, then the image of κ is a set of compact generators in the target category, and λ is an infinite direct sum-preserving triangulated functor identifying triangulated subcategories of compact generators, and hence λ is an equivalence. In this case, the functor $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -$ becomes an autoequivalence of the triangulated category $\mathsf{D^{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ and restricts to an autoequivalence of its full subcategory of compact generators $\mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$.

Proof of Corollary 2.5. The assertions in the first sentence follow from the second one, as we know γ to be fully faithful and its image to be a set of compact generators by Corollary 2.3(l). The commutativity of both squares and the upper left triangle is clear. To check commutativity of the lower right triangle, consider a coherent matrix factorization $\mathcal{M}$ of the potential $-w$; let $\mathcal{E}_\bullet$ be its left resolution in the abelian category $Z^0((X, \mathcal{L}, -w)\text{-}\mathsf{coh})$ whose terms $\mathcal{E}_n$ belong to $Z^0((X, \mathcal{L}, -w)\text{-}\mathsf{coh}_{\mathsf{lf}})$. Then the finite complex of matrix factorizations $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{M}, \mathcal{D}_X^\bullet)$ maps quasi-isomorphically to the bounded-below complex of injective matrix factorizations $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{E}_\bullet, \mathcal{D}_X^\bullet) \simeq \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{E}_\bullet, \mathcal{O}_X)$, so the cone of the corresponding morphism of the total matrix factorizations is coacyclic. □

2.6. *w-flat matrix factorizations.* From now on we will assume that for any affine open subscheme $U \subset X$ the element $w|_U$ is not a zero divisor in the $\mathcal{O}(U)$-module $\mathcal{L}(U)$; in other words, the morphism of sheaves $w : \mathcal{O}_X \to \mathcal{L}$ is injective.

The following results will be used in the proof of the main theorem and its analogues below. Let us call a quasicoherent $\mathcal{O}_X$-module $\mathcal{E}$ *w-flat* if the map $w : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$ is injective. Notice that any submodule of a w-flat module is w-flat, so the "w-flat dimension" of a quasicoherent sheaf over X never exceeds 1.

Denote by $(X, \mathcal{L}, w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}}$ the DG-category of coherent CDG-modules over $(X, \mathcal{L}, w)$ with w-flat underlying graded $\mathcal{O}_X$-modules and by $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}}$ the similar DG-category of quasicoherent CDG-modules. Let $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}})$, $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}}$, and $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}})$ denote the corresponding derived categories of the second kind.

Furthermore, denote by $(X, \mathcal{L}, w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}\cap\mathsf{ffd}}$ the DG-category of coherent CDG-modules over $(X, \mathcal{L}, w)$ whose underlying graded $\mathcal{O}_X$-modules are both w-flat and of finite flat dimension, and by $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}\cap\mathsf{lfd}}$ the DG-category of w-flat quasicoherent CDG-modules of finite locally free dimension. Let the corresponding exotic derived categories be denoted by $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}\cap\mathsf{ffd}})$, $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}\cap\mathsf{lfd}})$, and $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}\cap\mathsf{lfd}})$.

Corollary 2.6. (a) *The functor* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}$ *is an equivalence of triangulated categories.*

(b) *The functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}}) \to \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}$ *is an equivalence of triangulated categories.*

(c) *The functor* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}}) \to \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}} \to (X, \mathcal{L}, w)\text{-}\mathsf{coh}$ *is an equivalence of triangulated categories.*

(d) *The functor* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}\cap\mathsf{lfd}}) \to \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}})$ *induced by the embedding of DG-categories* $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}\cap\mathsf{lfd}} \to (X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}}$ *is an equivalence of triangulated categories.*

(e) *The functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}\cap\mathsf{lfd}}) \to \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}\cap\mathsf{lfd}} \to (X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lfd}}$ *is an equivalence of triangulated categories.*

(f) *The functor* $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}\cap\mathsf{ffd}}) \to \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{ffd}})$ *induced by the embedding of DG-categories* $(X,\mathcal{L},w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}\cap\mathsf{ffd}} \to (X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{ffd}}$ *is an equivalence of triangulated categories.*

Proof. The proofs are analogous to those of Corollary 2.3(a)–(c) and (g) (except that no induction in d is needed, as it suffices to consider the case $d=1$). Parts (d), (e), (f) are analogous to parts (a), (b), (c), respectively. Parts (b), (c), (e), and (f) can be also proven in the way similar to Corollary 2.3(h) and (i). □

Remark 2.6. The assertions of parts (a) and (b) hold under somewhat weaker assumptions than above: namely, one does not need to assume the existence of enough vector bundles on X. And one can make parts (d) and (e) hold without vector bundles by replacing the finite locally free dimension condition in their formulation with the finite flat dimension condition. The reason is that there are enough flat sheaves on any reasonable scheme (see Lemma A.1).

In fact, even part (c) does not depend on the existence of vector bundles since a surjective morphism onto a given coherent sheaf $\mathcal{M}$ from a w-flat coherent sheaf can be easily constructed, e.g., by starting from a surjective morphism onto $\mathcal{M}$ from a flat quasicoherent sheaf $\mathcal{F}$ and picking a large enough coherent subsheaf in $\mathcal{F}$. Accordingly, one does not need vector bundles to prove the equivalence of categories in the lower horizontal line in Theorem 2.7 below and the other two equivalences in Theorem 2.8. Replacing locally free sheaves with flat ones in the relevant definitions and assuming the Krull dimension to be finite, one can have the whole of Proposition 2.8 hold without vector bundles as well.

Another alternative is to use *very flat* quasicoherent sheaves, which there are always enough of and which always form a category of finite homological dimension on a quasicompact semiseparated scheme [Positselski 2012, Section 4.1]. Similarly, the existence of vector bundles is not needed for the validity of Theorem 1.4(a) and (b), Proposition 1.5(a), (c), and (d), all the assertions of Sections 1.7 and 1.10, Corollary 2.3(a), (b), (f), (k), and (l), Corollary 2.4(a) and (b), Proposition 2.5, Theorem 2.5, and some other results.

2.7. ***Main theorem.*** Let $X_0 \subset X$ be the closed subscheme defined locally by the equation $w=0$, and $i : X_0 \to X$ be the natural closed embedding. The next theorem is the main result of this paper.

Theorem 2.7. *There is a natural equivalence of triangulated categories*

$$\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X).$$

Together with the functor $\Sigma : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ *constructed in [Orlov 2012], this equivalence forms the following diagram of triangulated functors*:

$$\begin{tikzcd}
 & & & \mathsf{D}^{\mathsf{b}}_{Sing}(X) \arrow[dl, shift left, "i^\circ"] \\
0 \arrow[r] & \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \arrow[r, tail, "\Sigma"] \arrow[d, tail] & \mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \arrow[ur, shift left, "{i_\bullet, i_\circ}"] \arrow[d, two heads] & \\
 & \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \arrow[r, equal, "\mathbb{L}\Xi", "\Upsilon"'] & \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X) \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

where the upper horizontal arrow Σ *is fully faithful, the left vertical arrow is fully faithful, the right vertical arrow is a Verdier localization functor, and the lower horizontal line* $\mathbb{L}\Xi = \Upsilon^{-1}$ *is an equivalence of categories. The square is commutative; the three diagonal arrows* $i_\bullet$, i°, $i_\circ$ (*the middle one pointing down and the two other ones pointing up*) *are adjoint.*

Furthermore, the image of the functor Σ *is precisely the full subcategory of objects annihilated by the functor* $i_\circ$, *or equivalently, by the functor* $i_\bullet$. *In other words, the image of* Σ *is equal both to the left and to the right orthogonal complements to the thick subcategory generated by the image of the functor* i°; *that is, an object* $\mathcal{F} \in \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ *is isomorphic to* $\Sigma(\mathcal{M})$ *for some* $\mathcal{M} \in \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ *if and only if for every* $\mathcal{E} \in \mathsf{D}^{\mathsf{b}}_{Sing}(X)$, *one has*

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)}(i^\circ\mathcal{E}, \mathcal{F}) = 0,$$

or equivalently, for every $\mathcal{E} \in \mathsf{D}^{\mathsf{b}}_{Sing}(X)$, *one has* $\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)}(\mathcal{F}, i^\circ\mathcal{E}) = 0$.

The thick subcategory generated by the image of the functor i° is the kernel of the right vertical arrow. So the upper horizontal arrow and the right vertical arrow are included into "exact sequences" of triangulated categories (as marked by the zeros at the ends; there is no exactness at the uppermost rightmost end).

When X is a regular scheme, the functor

$$\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$$

is an equivalence of categories by Corollary 2.4(c), and so is the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ (as explained in Section 2.1). Hence it follows that the functor Σ is an equivalence of categories, too. Thus we recover the result of Orlov [2012, Theorem 3.5] claiming the equivalence of triangulated categories $\mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh_{lf}}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ for a regular X.

A counterexample in Section 3.3 will show that when X is not regular, the functor $\mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh_{lf}}) \to \mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$ does not have to be an equivalence, and indeed, the thick subcategory generated by $\mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh_{lf}})$ can be a proper strictly full subcategory in $\mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$.

Proof of the lower horizontal equivalence. To obtain the equivalence of triangulated categories in the lower horizontal line, we will construct triangulated functors in both directions and then check that they are mutually inverse. Given a bounded complex of coherent sheaves $\mathcal{F}^\bullet$ over X_0, consider the CDG-module $\Upsilon(\mathcal{F}^\bullet)$ over $(X, \mathcal{L}, w)$ with the underlying coherent graded module given by the rule

$$\Upsilon^n(\mathcal{F}^\bullet) = \bigoplus\nolimits_{m\in\mathbb{Z}} i_*\mathcal{F}^{n-2m} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}$$

and the differential induced by the differential on $\mathcal{F}^\bullet$. Since $d^2 = 0$ on $\mathcal{F}^\bullet$ and w acts by zero in $i_*\mathcal{F}^j$, this is a CDG-module. It is clear that Υ is a well-defined triangulated functor $\mathsf{D^b}(X_0\text{-}\mathsf{coh}) \to \mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$ since the derived category of bounded complexes over an abelian category coincides with their absolute derived category.

Let us check that Υ annihilates the image of the functor $\mathbb{L}i^*$. It suffices to consider a w-flat coherent sheaf $\mathcal{E}$ on X and check that $\Upsilon(\operatorname{coker} w) = 0$, where $w : \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1} \to \mathcal{E}$. Indeed, $\Upsilon(\operatorname{coker} w)$ is the cokernel of the injective morphism of contractible coherent CDG-modules $\mathcal{N} \to \mathcal{M}$, where $\mathcal{N}^{2n+1} = \mathcal{M}^{2n+1} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ and $\mathcal{N}^{2n} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n-1}$, while $\mathcal{M}^{2n} = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ for $n \in \mathbb{Z}$.

This provides the desired triangulated functor

$$\Upsilon : \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X) \longrightarrow \mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}).$$

The functor in the opposite direction is a version of Orlov's cokernel functor, but in our situation it has to be constructed as a derived functor since the functor of the cokernel of an arbitrary morphism is not exact. Recall the equivalence of triangulated categories $\mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}}) \to \mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$ from Corollary 2.6(c).

Define the functor $\Xi : Z^0((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}}) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ from the category of w-flat coherent CDG-modules over $(X, \mathcal{L}, w)$ and closed morphisms of degree 0 between them to the triangulated category of relative singularities by the rule

$$\Xi(\mathcal{M}) = \operatorname{coker}(d : \mathcal{M}^{-1} \to \mathcal{M}^0) = \operatorname{coker}(i^*d : i^*\mathcal{M}^{-1} \to i^*\mathcal{M}^0),$$

where the former cokernel, which is by definition a coherent sheaf on X annihilated by w, is considered as a coherent sheaf on X_0. One can immediately see that the functor Ξ transforms morphisms homotopic to zero into morphisms factorizable through the restrictions to X_0 of w-flat coherent sheaves on X. Hence the functor Ξ factorizes through the homotopy category $H^0((X,\mathcal{L},w)\text{-}\mathsf{coh}_{w\text{-fl}})$.

It is explained in [Polishchuk and Vaintrob 2011, Lemma 3.12] that the functor Ξ is triangulated (see also Lemma 3.6 below) and in [Orlov 2012, Proposition 3.2] that the functor Ξ factorizes through $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{w\text{-fl}})$. The latter assertion can be also deduced by considering the complex (1.3) from [Polishchuk and Vaintrob 2011]. Indeed, the complex $i^*\mathcal{M}$ corresponding to the total CDG-module $\mathcal{M}$ of an exact triple in $\mathcal{B}\text{-}\mathsf{coh}_{w\text{-fl}}$ is the total complex of an exact triple of complexes in the exact category $\mathsf{E}_{X_0/X}$ from Remark 2.1; hence the complex $i^*\mathcal{M}$ is exact with respect to $\mathsf{E}_{X_0/X}$ and the cokernels of its differentials belong to this exact subcategory in the abelian category of coherent sheaves over X_0. So we obtain the triangulated functor

$$\Xi : \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{w\text{-fl}}) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X),$$

and consequently, the left derived functor

$$\mathbb{L}\Xi : \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X).$$

Let us check that the two functors Υ and $\mathbb{L}\Xi$ are mutually inverse. For any w-flat coherent CDG-module $\mathcal{M}$ over $(X,\mathcal{L},w)$, there is a natural surjective closed morphism of CDG-modules $\phi : \mathcal{M} \to \Upsilon\Xi(\mathcal{M})$ with a contractible kernel. Clearly, $\phi : \mathrm{Id} \to \Upsilon\mathbb{L}\Xi$ is an (iso)morphism of functors.

Conversely, any object of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ can be represented by a coherent sheaf on X_0, and any morphism in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ is isomorphic to a morphism coming from the abelian category of such coherent sheaves. Indeed, the bounded-above derived category $\mathsf{D}^-(X_0\text{-}\mathsf{coh})$ of coherent sheaves over X_0 is equivalent to the bounded-above derived category $\mathsf{D}^-(X_0\text{-}\mathsf{coh}_{\mathsf{lf}})$ of locally free sheaves; using a truncation far enough to the left, one can represent any object or morphism in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ by a long enough shift of a coherent sheaf or a morphism of coherent sheaves. Now for any coherent sheaf $\mathcal{F}$ on X_0, there is a natural distinguished triangle

$$\mathcal{F}\otimes_{\mathcal{O}_{X_0}} i^*\mathcal{L}^{\otimes -1}[1] \longrightarrow \mathbb{L}i^*i_*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}\otimes_{\mathcal{O}_{X_0}} i^*\mathcal{L}^{\otimes -1}[2]$$

in $\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh})$, which provides a natural isomorphism $\mathcal{F} \simeq \mathcal{F}\otimes_{\mathcal{O}_{X_0}} i^*\mathcal{L}^{\otimes -1}[2]$ in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$.

Let $\mathcal{F}$ be a coherent sheaf on X_0; pick a vector bundle $\mathcal{E}$ on X together with a surjective morphism $\mathcal{E} \to i_*\mathcal{F}$ with the kernel $\mathcal{E}'$. Then the CDG-module $\mathcal{M}$ over $(X,\mathcal{L},w)$ with the components $\mathcal{M}^{2n} = \mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ and $\mathcal{M}^{2n-1} = \mathcal{E}'\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ maps surjectively onto $\Upsilon(\mathcal{F})$ with a contractible kernel, and $\mathbb{L}\Xi\Upsilon(\mathcal{F}) = \Xi(\mathcal{M}) = \mathcal{F}$ (cf. [Lin and Pomerleano 2013, Lemma 2.18]). Denote the isomorphism we have constructed by $\psi : \mathbb{L}\Xi\Upsilon(\mathcal{F}) \to \mathcal{F}$. The composition $\Upsilon\psi \circ \phi\Upsilon : \Upsilon(\mathcal{F}) \to \Upsilon\mathbb{L}\Xi\Upsilon(\mathcal{F}) \to \Upsilon(\mathcal{F})$ is clearly the identity morphism. It is obvious that ψ commutes with any morphism of coherent sheaves $\mathcal{F}$ on X_0, but checking that it commutes with all morphisms, or all isomorphisms, in $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0/X)$ is a little delicate (cf. Remark 2.7 below).

Notice that $\Upsilon\psi$ is an (iso)morphism of functors since $\phi\Upsilon$ is, and consequently $\mathbb{L}\Xi\Upsilon\psi$ is an (iso)morphism of functors. Thus it remains to check that the functor $\mathbb{L}\Xi\Upsilon$ is faithful, i.e., does not annihilate any morphisms. Indeed, any morphism in $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0/X)$ is isomorphic to a morphism coming from the abelian category of coherent sheaves on X_0, and the functor $\mathbb{L}\Xi\Upsilon$ transforms such morphisms into isomorphic ones. The construction of the equivalence of categories in the lower horizontal line is finished. One still has to check that the isomorphisms ϕ commute with the isomorphisms $\Upsilon\Xi(\mathcal{M}[1]) \simeq \Upsilon\Xi(\mathcal{M})[1]$, but this is straightforward.

Alternatively, one can use w-flat coherent sheaves on X or objects of the exact category $\mathsf{E}_{X_0/X}$ of coherent sheaves on X_0 (as applicable) instead of the locally free sheaves everywhere in the above argument. □

Proof of "exactness" in the upper line. We start with a discussion of the three adjoint functors in the right upper corner. The functor $i_\circ$ right adjoint to the functor $i^\circ : \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X) \to \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)$ was constructed in Section 2.1.

To construct the left adjoint functor to i°, notice that the right derived functor of the subsheaf with scheme-theoretic support in the closed subscheme

$$\mathbb{R}i^! : \mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh})$$

only differs from the functor $\mathbb{L}i^*$ by a shift and a twist; $\mathbb{R}i^!\mathcal{E}^\bullet \simeq \mathbb{L}i^*\mathcal{E}^\bullet \otimes_{\mathcal{O}_{X_0}} \mathcal{L}|_{X_0}[-1]$. One can check this first for w-flat coherent sheaves $\mathcal{E}$, when both objects to be identified are shifts of sheaves, so it suffices to compare their direct images under i, which are both computed by the same two-term complex $\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{L}$; then replace a complex $\mathcal{E}^\bullet$ with a finite complex of w-flat coherent sheaves (for a general result of this kind, see [Neeman 1996, Theorem 5.4]).

Hence the functor $\mathbb{R}i^!$ takes $\mathit{Perf}(X)$ to $\mathit{Perf}(X_0)$ and induces a triangulated functor $i^\bullet : \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X) \to \mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)$ right adjoint to $i_\circ$. It follows that the functor $i_\bullet(\mathcal{F}) = i_\circ(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{L}[-1]$ is left adjoint to the functor i°.

To prove the vanishing of the composition of functors in the upper line and the orthogonality assertions, notice that

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)}(i^\circ\mathcal{E}, \Sigma\mathcal{M}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X)}(\mathcal{E}, i_\circ\Sigma\mathcal{M})$$

and $i_*\Sigma(\mathcal{M}) = \mathrm{coker}(\mathcal{M}^{-1} \to \mathcal{M}^0) \in \mathit{Perf}(X)$ for any $\mathcal{M} \in \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ since the morphism $\mathcal{M}^{-1} \to \mathcal{M}^0$ of locally free sheaves on X is injective. Similarly,

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X_0)}(\Sigma\mathcal{M}, i^\circ\mathcal{E}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X)}(i_\bullet\Sigma\mathcal{M}, \mathcal{E})$$

and $i_\bullet\Sigma(\mathcal{M}) = i_\circ\Sigma(\mathcal{M}) \otimes_{\mathcal{O}_X} \mathcal{L}[-1] = 0$ in $\mathsf{D}^{\mathsf{b}}_{\mathit{Sing}}(X)$.

Obviously, our derived cokernel functor $\mathbb{L}\Xi$ makes a commutative diagram with the cokernel functor Σ from [Orlov 2012]. The left vertical arrow is fully faithful by Corollary 2.3(i). The assertion that the upper horizontal arrow is fully faithful is due to Orlov [2012, Theorem 3.4]. We have just obtained a new proof of it with our methods. Indeed, it follows from orthogonality that the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ induces isomorphisms on the groups of morphisms between any two objects, one of which comes from $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}})$. Conversely, Orlov's theorem together with the orthogonality argument and the equivalence of categories in the lower horizontal line imply that the left vertical arrow is fully faithful.

Now assume that $i_\circ\mathcal{F}=0$ for some $\mathcal{F}\in\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$. Clearly, there exists $m\geq 0$ and a coherent sheaf $\mathcal{K}$ on X_0 such that $\mathcal{F}\simeq\mathcal{K}[m]$ in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$. Then $i_*\mathcal{K}$ is a perfect complex, i.e., a coherent sheaf of finite flat dimension on X. Let us view it as an object of $(X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{ffd}}$; i.e., consider the CDG-module $\mathcal{N}$ over $(X,\mathcal{L},w)$ with the components $\mathcal{N}^{2n}=i_*\mathcal{K}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ and $\mathcal{N}^{2n+1}=0$.

The construction of the cokernel functor Σ can be straightforwardly extended to w-flat coherent matrix factorizations of finite flat dimension, providing a triangulated functor

$$\widetilde{\Sigma}:\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}\cap\mathsf{ffd}})\longrightarrow\mathsf{D}^{\mathsf{b}}_{Sing}(X_0).$$

The functor $\widetilde{\Sigma}$ is well-defined since one has $i^*\mathcal{M}\in Perf(X_0)$ for any w-flat coherent sheaf $\mathcal{M}$ of finite flat dimension on X. Using the equivalence of triangulated categories $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}\cap\mathsf{ffd}})\simeq\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{ffd}})$ from Corollary 2.6(f), one constructs the derived functor

$$\mathbb{L}\widetilde{\Sigma}:\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{ffd}})\longrightarrow\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$$

in the same way as it was done above for the derived functor $\mathbb{L}\Xi$. Since the functor $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}})\to\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{ffd}})$ is an equivalence of categories by Corollary 2.3(g), the (essential) images of the functors Σ and $\mathbb{L}\widetilde{\Sigma}$ coincide.

Let us check that $\mathbb{L}\widetilde{\Sigma}(\mathcal{N})\simeq\mathcal{K}$ as an object of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$. We argue as above, picking a vector bundle $\mathcal{E}$ on X together with a surjective morphism $\mathcal{E}\to i_*\mathcal{K}$ with the kernel $\mathcal{E}'$. Then the CDG-module $\mathcal{M}$ over $(X,\mathcal{L},w)$ with the components $\mathcal{M}^{2n}=\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ and $\mathcal{M}^{2n-1}=\mathcal{E}'\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes n}$ maps surjectively onto $\mathcal{N}$ with a contractible kernel. Hence the object $\mathcal{M}\in(X,\mathcal{L},w)\text{-}\mathsf{coh}_{w\text{-}\mathsf{fl}\cap\mathsf{ffd}}$ is isomorphic to $\mathcal{N}$ in $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{ffd}})$, and we have $\mathbb{L}\widetilde{\Sigma}(\mathcal{N})=\widetilde{\Sigma}(\mathcal{M})=\mathcal{K}$. Therefore, the object $\mathcal{K}\in\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ belongs to the (essential) image of the functor Σ, and it follows that so does the object $\mathcal{F}\simeq\mathcal{K}[m]$.

One can strengthen the above argument so as to obtain a construction of the (partial) inverse functor Δ to the functor Σ similar to the above construction of the functor Υ inverse to the functor $\mathbb{L}\Xi$. Consider the full subcategory $\mathsf{F}_{X_0/X}\subset X_0\text{-}\mathsf{coh}$ in the abelian category of coherent sheaves on X_0 consisting of all the sheaves $\mathcal{F}$

such that the sheaf $i_*\mathcal{F}$ has finite flat dimension (i.e., is a perfect complex) on X. The category $\mathsf{F}_{X_0/X}$ contains all the locally free sheaves on X_0 and is closed under the kernels of surjections, the cokernels of embeddings, and the extensions.

Hence $\mathsf{F}_{X_0/X}$ is an exact subcategory in X_0-coh. The natural functor

$$\mathsf{D}^{\mathsf{b}}(\mathsf{F}_{X_0/X}) \longrightarrow \mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh})$$

is fully faithful; its image coincides with the kernel of the composition of the direct image and Verdier localization functors $\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X)$. Accordingly, the quotient category $\mathsf{D}^{\mathsf{b}}(\mathsf{F}_{X_0/X})/\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh}_{\mathsf{lf}})$ is identified with the kernel of the direct image functor $i_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X)$.

Now the functor

$$\Delta : \mathsf{D}^{\mathsf{b}}(\mathsf{F}_{X_0/X})/\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{ffd}})$$

is constructed in the way similar to the construction of the functor Υ, by taking the direct image from X_0 to X and applying the periodicity summation. That is,

$$\Delta^n(\mathcal{F}^\bullet) = \textstyle\bigoplus_{m\in\mathbb{Z}} i_*\mathcal{F}^{n-2m} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}$$

for any $\mathcal{F}^\bullet \in \mathsf{D}^{\mathsf{b}}(\mathsf{F}_{X_0/X})$. One checks that the functor Δ is inverse to the functor $\mathbb{L}\widetilde{\Sigma}$, the latter being viewed as a functor taking values in the triangulated subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{F}_{X_0/X})/\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$, in the same way as it was done above for the functors Υ and $\mathbb{L}\Xi$. This provides yet another proof of the fact that the functor Σ is fully faithful, together with another proof of our description of its image. It is also obvious from the constructions that the functor Δ makes a commutative diagram with the functor Υ. □

Remark 2.7. The somewhat tricky technical argument in the first part of the above proof can be clarified and generalized using the approach developed by the first author in [Efimov 2013, Appendix A].

Let C be an abelian category, $L : \mathsf{C} \to \mathsf{C}$ be its covariant autoequivalence, and $w : \mathrm{Id} \to L$ be a natural transformation commuting with L (that is for any object $B \in \mathsf{C}$, one has $w_{L(B)} = L(w_B)$). Let $MF(\mathsf{C}, L, w)$ denote the abelian category of "matrix factorizations of w in C", i.e., pairs of objects U^0, $L^{1/2}(U^1) \in \mathsf{C}$ endowed with pairs of morphisms $U^0 \to L^{1/2}(U^1)$, $L^{1/2}(U^1) \to L(U^0)$ such that the compositions

$$U^0 \longrightarrow L^{1/2}(U^1) \longrightarrow L(U^0) \quad \text{and} \quad L^{1/2}(U^1) \longrightarrow L(U^0) \longrightarrow L^{3/2}(U^1)$$

are equal to w_{U^0} and $w_{L^{1/2}(U^1)}$, respectively. Given a matrix factorization $M = (U^0, U^1)$, one sets

$$M^n = L^{n/2}(U^{n \bmod 2}).$$

Passing to the quotient category by the ideal of morphisms homotopic to zero, one obtains the homotopy category of matrix factorizations of w in C, and their absolute derived category, denoted by $\mathsf{D}^{\mathsf{abs}}(\mathsf{C}, L, w)$, is produced by the Verdier localization procedure similar to the one discussed in Section 1.3. (Cf. [Positselski 2011a, Remark 4.3].)

Let $\mathsf{C}_0 \subset \mathsf{C}$ denote the full subcategory formed by all the objects $A \in \mathsf{C}$ for which $w_A = 0$; so C_0 is an abelian subcategory in C closed under subobjects and quotient objects. An object $B \in \mathsf{C}$ is said to have no w-torsion if the morphism w_B is injective; and one says that the potential (natural transformation) w does not divide zero in C if every object of C is the quotient object of an object without w-torsion. Let $i_* : \mathsf{C}_0 \to \mathsf{C}$ denote the exact identity embedding functor and $i^* : \mathsf{C} \to \mathsf{C}_0$ be the functor left adjoint to i_*, so $i^*(B) = \operatorname{coker}(w_{L^{-1}(B)} : L^{-1}(B) \to B)$. Assuming that w does not divide zero (as we do in the sequel), one can construct the left derived functor $\mathbb{L}i^* : \mathsf{D}^{\mathsf{b}}(\mathsf{C}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0)$ with $\mathbb{L}_s i^*(B) = 0$ for all $s \neq 0, 1$ and any object $B \in \mathsf{C}$. The functor $\mathbb{L}i^*$ is left adjoint to the triangulated functor $i_* : \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0) \to \mathsf{D}^{\mathsf{b}}(\mathsf{C})$ induced by the identity embedding $i_* : \mathsf{C}_0 \to \mathsf{C}$.

Similarly, let $\upsilon^n : \mathsf{C}_0 \to MF(\mathsf{C}, L, w)$ denote the exact functor assigning to an object $A \in \mathsf{C}_0$ the matrix factorization M with $M^n = A$ and $M^{n+1} = 0$, and let $\xi^n : MF(\mathsf{C}, L, w) \to \mathsf{C}_0$ be the functor left adjoint to υ^n, assigning the object $\operatorname{coker}(M^{n-1} \to M^n) \in \mathsf{C}_0$ to a matrix factorization M. Considering the bounded derived category $\mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w)$ of the abelian category $MF(\mathsf{C}, L, w)$, one can construct the left derived functor

$$\mathbb{L}\xi^n : \mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0);$$

once again, the functor $\mathbb{L}\xi^n$ is left adjoint to $\upsilon^n : \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0) \to \mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w)$ and one has $\mathbb{L}_s\xi^n(M) = 0$ for all $s \neq 0, 1$ and any matrix factorization $M \in MF(\mathsf{C}, L, w)$.

Then the composition of the functor $\upsilon^n : \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0) \to \mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w)$ with the totalization functor $\mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w) \to \mathsf{D}^{\mathsf{abs}}(\mathsf{C}, L, w)$ induces an equivalence of triangulated categories

$$\Upsilon^n : \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0)/\langle \mathbb{L}i^*\mathsf{D}^{\mathsf{b}}(\mathsf{C})\rangle \longrightarrow \mathsf{D}^{\mathsf{abs}}(\mathsf{C}, L, w)$$

between the quotient category of the derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{C}_0)$ by the thick subcategory generated by the image of the functor $\mathbb{L}i^*$ and the absolute derived category of matrix factorizations. The composition of the functor $\mathbb{L}\xi^n : \mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w) \to \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0)$ with the Verdier localization functor $\mathsf{D}^{\mathsf{b}}(\mathsf{C}_0) \to \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0)/\langle \mathbb{L}i^*\mathsf{D}^{\mathsf{b}}(\mathsf{C})\rangle$ factorizes through the totalization functor $\mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w) \to \mathsf{D}^{\mathsf{abs}}(\mathsf{C}, L, w)$, providing the triangulated functor

$$\mathbb{L}\Xi^n : \mathsf{D}^{\mathsf{abs}}(\mathsf{C}, L, w) \longrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0)/\langle \mathbb{L}i^*\mathsf{D}^{\mathsf{b}}(\mathsf{C})\rangle$$

inverse to Υ^n.

Indeed, let $F^n : MF(\mathsf{C}, L, w) \to \mathsf{C}$ denote the forgetful functor taking a matrix factorization M to the object $M^n \in \mathsf{C}$, and let $G^{n-} : \mathsf{C} \to MF(\mathsf{C}, L, w)$ denote the functor left adjoint to F^{n-1} (and right adjoint to F^n); so the functor G^{n-} takes an object $B \in \mathsf{C}$ to a contractible matrix factorization M with $M^{n-1} = M^n = B$ (cf. the constructions of the functors G^+ and G^- in the proofs in Sections 1.4 and 1.6). It is claimed that the induced triangulated functors $G^{n-} : \mathsf{D}^{\mathsf{b}}(\mathsf{C}) \to \mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w)$ and $\upsilon^n : \mathsf{D}^{\mathsf{b}}(\mathsf{C}_0) \to \mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w)$ are fully faithful and their images form a semiorthogonal decomposition of the derived category $\mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w)$.

To check the first assertion, it suffices to notice that the triangulated functor G^{n-} is left adjoint to the functor $F^{n-1} : \mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w) \to \mathsf{D}^{\mathsf{b}}(\mathsf{C})$, and their composition $F^{n-1} \circ G^{n-}$ is the identity endofunctor on $\mathsf{D}^{\mathsf{b}}(\mathsf{C})$. Similarly, the composition of triangulated functors $\mathbb{L}\xi^n \circ \upsilon^n$ is the identity endofunctor on $\mathsf{D}^{\mathsf{b}}(\mathsf{C}_0)$, so υ^n is fully faithful as a functor between the derived categories. Furthermore, one has $F^{n-1} \circ \upsilon^n = 0 = \mathbb{L}\xi^n \circ G^{n-}$, implying the semiorthogonality. Finally, for any matrix factorization M whose terms are objects without w-torsion, there is a short exact sequence

$$0 \longrightarrow G^{n-}F(M) \longrightarrow M \longrightarrow \upsilon^n \xi^n M \longrightarrow 0$$

in $MF(\mathsf{C}, L, w)$ and $\mathbb{L}\xi^n M = \xi^n M$, proving the decomposition claim.

Now we notice that for any object $B \in \mathsf{C}$ having no w-torsion, there is a short exact sequence

$$0 \longrightarrow G^{(n+2)-}(B) \longrightarrow G^{(n+1)-}(B) \longrightarrow \upsilon^n j^* B \longrightarrow 0$$

in $MF(\mathsf{C}, L, w)$. According to (the proof of) [Efimov 2013, Proposition A.3(1) and (2)], the totalization functor $\mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w) \to \mathsf{D}^{\mathsf{abs}}(\mathsf{C}, L, w)$ is the Verdier localization functor by the thick subcategory generated by the objects of the form $G^{n-}(B) = G^{(n+2)-}L(B)$ and $G^{(n+1)-}(B) \in MF(\mathsf{C}, L, w) \subset \mathsf{D}^{\mathsf{b}}MF(\mathsf{C}, L, w)$. The assertions about the existence of triangulated functors Υ^n and $\mathbb{L}\Xi^n$ and their being mutually inverse equivalences of categories follow from these observations.

Returning to a separated Noetherian scheme X with enough vector bundles and the Cartier divisor $X_0 \subset X$ of a global section w of a line bundle $\mathcal{L}$ on X, the above approach based on [loc. cit., Proposition A.3] provides an elegant construction of Orlov's triangulated cokernel functor $\Sigma : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ in addition to a proof of our equivalence of categories $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$.

2.8. ***Infinite matrix factorizations.*** Following [Orlov 2004, paragraphs after Remark 1.9], one can define a "large" version of the triangulated category of singularities $\mathsf{D}'_{Sing}(X)$ of a scheme X as the quotient category of the bounded derived category of quasicoherent sheaves $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh})$ by the thick subcategory $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh}_{\mathsf{lf}})$ of bounded complexes of locally free sheaves (of infinite rank). When X has finite Krull dimension, the latter subcategory coincides with the thick subcategory $\mathsf{D^b}(X\text{-}\mathsf{qcoh_{fl}})$ of bounded complexes of flat sheaves (see Remark 1.4).

Similarly, let $Z \subset X$ be a closed subscheme such that $\mathcal{O}_Z$ has finite flat dimension as an $\mathcal{O}_X$-module. Let us define a "large" triangulated category of relative singularities $\mathsf{D}'_{Sing}(Z/X)$ as the quotient category of $\mathsf{D^b}(Z\text{-}\mathsf{qcoh})$ by the minimal thick subcategory containing the image of the functor $\mathbb{L}i^* : \mathsf{D^b}(X\text{-}\mathsf{qcoh}) \to \mathsf{D^b}(Z\text{-}\mathsf{qcoh})$ and closed under those infinite direct sums that exist in $\mathsf{D^b}(Z\text{-}\mathsf{qcoh})$. The quotient category of $\mathsf{D^b}(Z\text{-}\mathsf{qcoh})$ by the minimal thick subcategory containing $\mathbb{L}i^*\mathsf{D^b}(X\text{-}\mathsf{qcoh})$ (without the direct sum closure) will be also of interest to us; let us denote it by $\mathsf{D}''_{Sing}(Z/X)$.

Lemma 2.8. *The triangulated categories $\mathsf{D}'_{Sing}(Z/X)$ and $\mathsf{D}''_{Sing}(Z/X)$ are quotient categories of $\mathsf{D}'_{Sing}(Z)$. When the scheme X is regular of finite Krull dimension, these three triangulated categories coincide.*

Proof. To prove the first assertion, let us show that any locally free sheaf on Z, considered as an object of $\mathsf{D^b}(Z\text{-}\mathsf{qcoh})$, is a direct summand of a bounded complex whose terms are direct sums of locally free sheaves of finite rank restricted from X. Indeed, pick a finite left resolution of a given locally free sheaf on Z with the middle terms as above, long enough compared to the number of open subsets in an affine covering of Z. Then the corresponding Ext class between the cohomology sheaves at the rightmost and leftmost terms has to vanish in view of the Mayer–Vietoris sequence for Ext groups between quasicoherent sheaves [Orlov 2004, Lemma 1.12]. Hence the rightmost term is a direct summand of the complex formed by the middle terms.

The second assertion holds for the categories $\mathsf{D}''_{Sing}(Z/X)$ and $\mathsf{D}'_{Sing}(Z)$ since any quasicoherent sheaf on a regular scheme of finite Krull dimension has a finite left resolution consisting of locally free sheaves. To identify these two categories with $\mathsf{D}'_{Sing}(Z/X)$, one needs to know that the subcategory of bounded complexes of locally free sheaves on Z is closed under those infinite direct sums that exist in $\mathsf{D^b}(Z\text{-}\mathsf{qcoh})$. The latter is true for any Noetherian scheme Z of finite Krull dimension with enough vector bundles since the finitistic projective dimension of a commutative ring of finite Krull dimension is finite [Raynaud and Gruson 1971, Théorème II.3.2.6]. □

Now let $\mathcal{L}$ be a line bundle on X, $w \in \mathcal{L}(X)$ be a global section corresponding to an injective morphism of sheaves $\mathcal{O}_X \to \mathcal{L}$, and $X_0 \subset X$ be the locus of $w = 0$.

Proposition 2.8. *There is a natural equivalence of triangulated categories*

$$\mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \simeq \mathsf{D}''_{Sing}(X_0/X).$$

Together with the infinite-rank version $\Sigma' : \mathsf{D^{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh_{lf}}) \to \mathsf{D}'_{Sing}(X_0)$ of Orlov's cokernel functor Σ from [Orlov 2012], this equivalence forms the following diagram of triangulated functors:

$$\begin{tikzcd}
 & & & \mathsf{D}'_{Sing}(X) \arrow[dl, shift left, "i^\circ"] \\
0 \arrow[r] & \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \arrow[r, tail, "\Sigma'"] \arrow[d, tail] & \mathsf{D}'_{Sing}(X_0) \arrow[ur, shift left, "i_\bullet, i_\circ"] \arrow[d, two heads] & \\
 & \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}) \arrow[r, equal] & \mathsf{D}''_{Sing}(X_0/X) \arrow[d] & \\
 & & 0 &
\end{tikzcd}$$

where the upper horizontal arrow Σ' is fully faithful, the left vertical arrow is fully faithful, the right vertical arrow is the Verdier localization functor by the thick subcategory generated by the image of the diagonal down arrow i°, and the lower horizontal line is an equivalence of categories. The square is commutative; the three diagonal arrows $i_\bullet$, i°, $i_\circ$ are adjoint.

Furthermore, the image of the functor Σ' is precisely the full subcategory of objects annihilated by the functor $i_\circ$, or equivalently, by the functor $i_\bullet$. In other words, the image of Σ' is equal both to the left and to the right orthogonal complements to (the thick subcategory generated by) the image of the functor i°.

Proof. The proof is completely similar to that of Theorem 2.7. It uses Corollaries 2.6(b), 2.3(h), 2.6(e), and 2.3(c). The first assertion can be also obtained as a particular case of the result of Remark 2.7.

Alternatively, one can prove that the functor Σ' is fully faithful in the same way as it was done for the functor Σ in [Orlov 2012, Theorem 3.4], and deduce the assertion that the left vertical arrow is fully faithful from the orthogonality.

Note that one can check in a straightforward way that the functor Σ' annihilates the objects coacyclic with respect to $(X,\mathcal{L},w)$-$\mathsf{qcoh}_{\mathsf{lf}}$. This provides another proof of Corollary 2.3(d), working in the assumption that w is a local nonzero-divisor. □

The functors Σ and Σ' together with the direct image functors $i_\circ$ form the commutative diagram of an embedding of "exact sequences" of triangulated functors:

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \arrow[r, tail, "\Sigma"] \arrow[d, tail] & \mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \arrow[r, "i_\circ"] \arrow[d, tail] & \mathsf{D}^{\mathsf{b}}_{Sing}(X) \arrow[d, tail] \\
0 \arrow[r] & \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \arrow[r, tail, "\Sigma'"] & \mathsf{D}'_{Sing}(X_0) \arrow[r, "i_\circ"] & \mathsf{D}'_{Sing}(X)
\end{tikzcd}$$

The leftmost vertical arrow is fully faithful by Corollary 2.3(j). The other two vertical arrows are fully faithful by Orlov's theorem [2004, Proposition 1.13] claiming that the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X) \to \mathsf{D}'_{Sing}(X)$ is fully faithful for any separated Noetherian scheme X with enough vector bundles. The leftmost nontrivial terms in both lines are the kernels of the rightmost arrows by Theorem 2.7 and Proposition 2.8.

Theorem 2.8. *There is a natural equivalence of triangulated categories*

$$\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \simeq \mathsf{D}'_{Sing}(X_0/X)$$

forming a commutative diagram of triangulated functors:

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \arrow[r, equal] \arrow[d, tail] \arrow[dd, tail, bend right=60, "\substack{\text{comp.}\\ \text{gener.}}"'] & \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X) \arrow[d, tail] \arrow[dd, tail, bend left=60, "\substack{\text{comp.}\\ \text{gener.}}"] \\
\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \arrow[r, equal] \arrow[d, two heads] & \mathsf{D}''_{Sing}(X_0/X) \arrow[d, two heads] \\
\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \arrow[r, equal] & \mathsf{D}'_{Sing}(X_0/X)
\end{tikzcd}$$

with the equivalences of categories from Theorem 2.7 *and Proposition 2.8. The upper vertical arrows are fully faithful, the lower ones are Verdier localization functors, and the vertical compositions are fully faithful. The categories in the lower line admit arbitrary direct sums, and the images of the vertical compositions are sets of compact generators in the target categories.*

Proof. The construction of the desired equivalence of categories is very similar to the construction of the equivalence of categories in Theorem 2.7 and Proposition 2.8. Using Corollary 2.6(a), one defines the infinite-rank version of the functor $\mathbb{L}\Xi$, then shows that the obvious infinite-rank version of the functor Υ is inverse to it. Notice that the functor $\Xi : Z^0((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}}) \to \mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{qcoh})$ preserves infinite direct sums and the functor $\Upsilon : \mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ preserves those infinite direct sums that exist in $\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{qcoh})$, so the functors

$$\Xi : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}}) \longrightarrow \mathsf{D}'_{Sing}(X_0/X),$$
$$\Upsilon : \mathsf{D}'_{Sing}(X_0/X) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$$

are well-defined.

The upper left vertical arrow is fully faithful by Corollary 2.3(k); it follows that the upper right vertical arrow is fully faithful, too. The assertions about the vertical compositions are proved similarly. The category $\mathsf{D}'_{Sing}(X_0/X)$ admits arbitrary direct sums, since the category $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ does. By Corollary 2.3(l), the left vertical composition is fully faithful and its image is a set of compact generators in the target, so the right vertical composition has the same properties. $\square$

The following square diagram of triangulated functors is commutative:

$$\begin{tikzcd} \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \arrow[r, tail, "\Sigma'"] \arrow[d] & \mathsf{D}'_{Sing}(X_0) \arrow[d, two heads] \\ \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}) \arrow[r, equal] & \mathsf{D}'_{Sing}(X_0/X) \end{tikzcd}$$

The upper horizontal arrow Σ' is fully faithful; the right vertical arrow is a Verdier localization functor. The lower line is an equivalence of triangulated categories. Nothing is claimed about the left vertical arrow in general.

When the scheme X is Gorenstein of finite Krull dimension, the left vertical arrow is an equivalence of categories by Corollary 2.4(a). When X is also regular, the right vertical arrow is an equivalence of categories by Lemma 2.8. So Σ' is an equivalence of categories $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \simeq \mathsf{D}'_{Sing}(X_0)$ and we have obtained a strengthened version of [Polishchuk and Vaintrob 2011, Theorem 4.2] (in the scheme case).

Remark 2.8. It is well-known that the Verdier localization functor of a triangulated category with infinite direct sums by a thick subcategory closed under infinite direct sums preserves infinite direct sums [Neeman 2001, Lemma 3.2.10]. This result is not applicable to the localization functors $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh}) \to \mathsf{D}'_{Sing}(X)$ and $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}'_{Sing}(Z/X)$, as the category $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh})$ does not admit arbitrary infinite direct sums.

Using the equivalence of categories from Theorem 2.8 and the observation that the functor Υ preserves infinite direct sums, one can show that the localization functor $\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{qcoh}) \to \mathsf{D}'_{Sing}(X_0/X)$ takes those infinite direct sums that exist in $\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{qcoh})$ into direct sums in the triangulated category of relative singularities $\mathsf{D}'_{Sing}(X_0/X)$ of the zero locus of w in X. However, there is *no* obvious reason why the localization functor $\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{qcoh}) \to \mathsf{D}'_{Sing}(X_0)$ should take those infinite direct sums that exist in $\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{qcoh})$ into direct sums in the absolute triangulated category of singularities $\mathsf{D}'_{Sing}(X_0)$.

That is the problem one encounters attempting to prove that the kernel of the localization functor $\mathsf{D}'_{Sing}(X_0) \to \mathsf{D}'_{Sing}(X_0/X)$ is semiorthogonal to the image of the functor Σ'.

2.9. *Stable derived category.* Following Krause [2005], we define the *stable derived category* of a Noetherian scheme X as the homotopy category of acyclic unbounded complexes of injective quasicoherent sheaves on X. As explained below, this is another (and in some respects better) "large" version of the triangulated category of singularities of X; for this reason, we denote it by $\mathsf{D}^{\mathsf{st}}_{Sing}(X)$.

In view of Lemma 1.7(b) (see also [Positselski 2010, Remark 5.4]), one can equivalently define $\mathsf{D}^{\mathsf{st}}_{Sing}(X)$ as the quotient category of the homotopy category of acyclic complexes of quasicoherent sheaves over X by the thick subcategory of coacyclic complexes, or as the full subcategory of acyclic complexes in the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$ of (complexes of) quasicoherent sheaves over X. It is the latter definition that will be used in the sequel.

Clearly, the category $\mathsf{D}^{\mathsf{st}}_{Sing}(X)$ has arbitrary infinite direct sums. Krause [2005, Corollary 5.4] constructs a fully faithful functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X) \to \mathsf{D}^{\mathsf{st}}_{Sing}(X)$ and proves that its image is a set of compact generators of the target category.

Theorem 2.9. *For any separated Noetherian scheme Z with enough vector bundles, there is a natural triangulated functor $\mathsf{D}'_{Sing}(Z) \to \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$ forming a commutative diagram with the natural functors from $\mathsf{D}^{\mathsf{b}}_{Sing}(Z)$ into both these categories. The composition*

$$\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}'_{Sing}(Z) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$$

preserves those infinite direct sums that exist in $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$. When $Z = X_0$ is a divisor in a regular separated Noetherian scheme of finite Krull dimension, the functor $\mathsf{D}'_{Sing}(X_0) \to \mathsf{D}^{\mathsf{st}}_{Sing}(X_0)$ is an equivalence of triangulated categories.

Proof. The construction of the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(Z) \to \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$ in [Krause 2005] is given in terms of the Verdier localization functor $Q : \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}(Z\text{-}\mathsf{qcoh})$ by the triangulated subcategory $\mathsf{D}^{\mathsf{st}}_{Sing}(Z) \subset \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ and its adjoint functors on both sides, which exist according to [loc. cit., Corollary 4.3]. The proof of our theorem is based on explicit constructions of the restrictions of these adjoint functors to some subcategories of bounded complexes in $\mathsf{D}(Z\text{-}\mathsf{qcoh})$.

It is well known that the Verdier localization functor $H^0(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}(Z\text{-}\mathsf{qcoh})$ from the homotopy category of (complexes of) quasicoherent sheaves on Z to their derived category has a right adjoint functor $\mathsf{D}(Z\text{-}\mathsf{qcoh}) \to H^0(Z\text{-}\mathsf{qcoh})$. The objects in the image of this functor are called *homotopy injective complexes* of quasicoherent sheaves on Z. The composition $\mathsf{D}(Z\text{-}\mathsf{qcoh}) \to H^0(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ provides the functor $Q_\rho : \mathsf{D}(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ right adjoint to Q. In particular, any bounded-below complex in $\mathsf{D}(Z\text{-}\mathsf{qcoh})$ has a bounded-below injective resolution and any bounded-below complex of injectives is homotopy injective. Furthermore, any bounded-below acyclic complex is coacyclic [Positselski 2010, Lemma 2.1]. It follows that any bounded-below complex from $\mathsf{D}^+(Z\text{-}\mathsf{qcoh})$, considered as an object of $\mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$, represents its own image under the functor Q_ρ.

On the other hand, any bounded-above complex from $\mathsf{D}(Z\text{-}\mathsf{qcoh})$ has a locally free left resolution defined uniquely up to a quasi-isomorphism of complexes in the exact category of locally free sheaves; i.e., there is an equivalence of bounded above derived categories $\mathsf{D}^-(Z\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \simeq \mathsf{D}^-(Z\text{-}\mathsf{qcoh})$. Since the exact category $Z\text{-}\mathsf{qcoh}_{\mathsf{lf}}$ has finite homological dimension, any acyclic complex in it is coacyclic (and even absolutely acyclic [loc. cit., Remark 2.1]), so there are natural functors $\mathsf{D}^-(Z\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \to \mathsf{D}(Z\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \simeq \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$.

Lemma 2.9. *The composition of the embedding* $\mathsf{D}^-(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}(Z\text{-}\mathsf{qcoh})$ *with the functor* $Q_\lambda : \mathsf{D}(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ *left adjoint to* Q *is isomorphic to the functor* $\mathsf{D}^-(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ *constructed above.*

Proof. We have to show that $\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})}(\mathcal{L}^\bullet, \mathcal{E}^\bullet) = 0$ for any bounded-above complex of locally free sheaves $\mathcal{L}^\bullet$ and any acyclic complex $\mathcal{E}^\bullet$ of quasicoherent sheaves on Z. Let us check that any morphism $\mathcal{L}^\bullet \to \mathcal{E}^\bullet$ in $H^0(Z\text{-}\mathsf{qcoh})$ factorizes through a coacyclic complex of quasicoherent sheaves. Clearly, we can assume that the complex $\mathcal{E}^\bullet$ is bounded above, too. Let $\mathcal{K}^\bullet$ be the cocone of a closed morphism of complexes $\mathcal{L}^\bullet \to \mathcal{E}^\bullet$; then $\mathcal{K}^\bullet$ is bounded above and the composition $\mathcal{K}^\bullet \to \mathcal{L}^\bullet \to \mathcal{E}^\bullet$ is homotopic to zero. Pick a bounded-above complex of locally free sheaves $\mathcal{F}^\bullet$ together with a quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{K}^\bullet$. Then the cone of the composition $\mathcal{F}^\bullet \to \mathcal{K}^\bullet \to \mathcal{L}^\bullet$, being a bounded-above acyclic complex of locally free sheaves, is coacyclic. Since the composition $\mathcal{F}^\bullet \to \mathcal{L}^\bullet \to \mathcal{E}^\bullet$ is homotopic to zero, the morphism $\mathcal{L}^\bullet \to \mathcal{E}^\bullet$ factorizes, up to homotopy, through this cone. □

Now we can describe the action of the functor $I_\lambda : \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{st}}_{Sing}(Z\text{-}\mathsf{qcoh})$ left adjoint to the embedding $\mathsf{D}^{\mathsf{st}}_{Sing}(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ on bounded-above complexes in $\mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$. If $\mathcal{K}^\bullet$ is a bounded-above complex of quasicoherent sheaves and $\mathcal{F}^\bullet$ is its locally free left resolution, then the cone of the closed morphism $\mathcal{F}^\bullet \to \mathcal{K}^\bullet$ represents the object $I_\lambda(\mathcal{K}^\bullet) \in \mathsf{D}^{\mathsf{st}}_{Sing}(Z\text{-}\mathsf{qcoh})$. In view of Lemma 2.9, this cone is functorial and does not depend on the choice of $\mathcal{F}^\bullet$ for the usual semiorthogonality reasons.

The embedding of compact generators $\mathsf{D}^{\mathsf{b}}_{Sing}(Z) \to \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$ is constructed in [Krause 2005] as the functor induced by the restriction of the composition $I_\lambda \circ Q_\rho : \mathsf{D}(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$ to the full subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}) \subset \mathsf{D}(Z\text{-}\mathsf{qcoh})$. Let us explain why this is so. By Proposition 1.5(d) (cf. [loc. cit., Proposition 2.3 and Remark 3.8]), the natural functor $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ is fully faithful and its image is a set of compact generators in the target. This is the image of $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}) \subset \mathsf{D}(Z\text{-}\mathsf{qcoh})$ under the functor Q_ρ, as constructed above. It is clear from the above construction of the functor Q_λ that it preserves compactness (and in fact coincides with the functor Q_ρ on perfect complexes in $\mathsf{D}(Z\text{-}\mathsf{qcoh})$ [loc. cit., Lemma 5.2]). Since the functors Q_λ and I_λ, being left adjoints, preserve infinite direct sums, and I_λ is a Verdier localization functor by the image of Q_λ, it follows that the image of any set of compact generators of $\mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ under I_λ is a set of compact generators of $\mathsf{D}^{\mathsf{st}}_{Sing}(Z)$ [Neeman 1996, Theorem 2.1(4)].

In order to define the desired functor

$$\mathsf{D}'_{Sing}(Z) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z),$$

restrict the same composition $I_\lambda \circ Q_\rho$ to the full subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}) \subset \mathsf{D}(Z\text{-}\mathsf{qcoh})$. According to the above, this restriction assigns to any bounded complex of quasicoherent sheaves $\mathcal{K}^\bullet$ on Z the cone of a morphism $\mathcal{F}^\bullet \to \mathcal{K}^\bullet$ into it from its locally free left resolution $\mathcal{F}^\bullet$. Clearly, the functor

$$\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$$

that we have constructed preserves those infinite direct sums that exist in $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$ and annihilates the triangulated subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$. So we have the induced functor $\mathsf{D}'_{Sing}(Z) \to \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$, and the first two assertions of the theorem are proven.

To prove the last assertion, we use the results of Section 2.8. Assume that $Z = X_0$ is the zero locus of a section $w \in \mathcal{L}(X)$ of a line bundle on X; as usual, $w : \mathcal{O}_X \to \mathcal{L}$ has to be an injective morphism of sheaves. Then by Theorem 2.8 and Lemma 2.8, the category $\mathsf{D}'_{Sing}(Z)$ admits infinite direct sums and the image of the fully faithful functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \to \mathsf{D}'_{Sing}(X_0)$ is a set of compact generators in the target. Furthermore, it follows from the proof of Theorem 2.8 that any object of $\mathsf{D}'_{Sing}(X_0)$ can be represented by a quasicoherent sheaf on X_0 and the direct sum of an infinite family of such objects is represented by the direct sums of such sheaves (see Remark 2.8). Thus the functor $\mathsf{D}'_{Sing}(Z) \to \mathsf{D}^{\mathsf{st}}_{Sing}(Z)$, being an infinite direct sum-preserving triangulated functor identifying triangulated subcategories of compact generators, is an equivalence of triangulated categories. ☐

We keep the assumptions of Theorem 2.9 and the notation of the last paragraph of its proof; i.e., X is a regular separated Noetherian scheme of finite Krull dimension with enough vector bundles and $X_0 \subset X$ is the divisor of zeros of a locally nonzero-dividing section $w \in \mathcal{L}(X)$. The closed embedding $X_0 \to X$ is denoted by i.

Corollary 2.9. *The functor*

$$\Lambda : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \simeq \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(X_0)$$

assigning to a locally free (*or just w-flat*) *quasicoherent matrix factorization $\mathcal{M}$ the acyclic complex of locally free* (*or quasicoherent*) *sheaves $i^*\mathcal{M}$ on X_0 is an equivalence of triangulated categories.*

Proof. Given a w-flat matrix factorization $\mathcal{M}$, the complex of sheaves $i^*\mathcal{M}$ on X_0 is acyclic by [Polishchuk and Vaintrob 2011, Lemma 1.5]. Clearly, the assignment $\mathcal{M} \longmapsto i^*\mathcal{M}$ defines a triangulated functor $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}}) \to \mathsf{D}^{\mathsf{st}}_{Sing}(X_0)$.

To prove that this functor is an equivalence of categories, it suffices to identify it, up to a shift, with the composition of the equivalences

$$\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}'_{Sing}(X_0) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(X_0).$$

Here one simply notices that for any $\mathcal{M} \in \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$ the complex $i^*\mathcal{M}$ is isomorphic in $\mathsf{D}^{\mathsf{st}}_{Sing}(X_0)$ to its canonical truncation $\tau_{\le 1} i^*\mathcal{M}$, and the latter complex is the cocone of the morphism into $\Sigma(\mathcal{M})$ from one of its left locally free resolutions. So the functor Λ is identified with $\Sigma[-1]$. □

2.10. ***Relative stable derived category.*** The goal of this section is to generalize the results of the previous one to the case of a singular Noetherian scheme X. The relative version of stable derived category, defined for a closed embedding of finite flat dimension $i : Z \to X$, is equivalent to the categories $\mathsf{D}'_{Sing}(X_0/X)$ and $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ in the case of the Cartier divisor $Z = X_0$ corresponding to a locally nonzero-dividing section w of a line bundle $\mathcal{L}$ on X.

Let X be a separated Noetherian scheme of finite Krull dimension and $i : Z \to X$ be a closed embedding of schemes such that $i_*\mathcal{O}_Z$ has finite flat dimension as an $\mathcal{O}_X$-module. According to Section 1.9, there is a left derived inverse image functor $\mathbb{L}i^* : \mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$. This functor forms a commutative diagram with the similar functor $\mathbb{L}i^* : \mathsf{D}(X\text{-}\mathsf{qcoh}) \to \mathsf{D}(Z\text{-}\mathsf{qcoh})$, and consequently, takes acyclic complexes in $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$ to acyclic complexes in $\mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$.

Proposition 2.10. *The following four triangulated categories are naturally equivalent*:

(a) *the full subcategory in* $\mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ *consisting of all the objects annihilated by the direct image functor* $i_* : \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$;

(b) *the quotient category of the homotopy category of complexes over* Z-qcoh *whose direct images are coacyclic complexes over* X-qcoh *by the thick subcategory of coacyclic complexes over* Z-qcoh;

(c) *the quotient category of* $\mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ *by its minimal triangulated subcategory, containing the objects in* $\mathbb{L}i^*\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$ *and closed under infinite direct sums*;

(d) *the quotient category of the full subcategory of acyclic complexes in* $\mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ *by its minimal triangulated subcategory, containing the left derived inverse images of acyclic complexes in* $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$ *and closed under infinite direct sums.*

Proof. The equivalence of (a) and (b) is obvious. To show that the natural functor from the category (d) to the category (c) is an equivalence, notice that the minimal triangulated subcategory containing flat quasicoherent sheaves and closed under infinite direct sums together with the triangulated subcategory of acyclic complexes form a semiorthogonal decomposition of $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$, and similarly for Z [Positselski 2012, Corollary A.4.7]. Since flat quasicoherent sheaves on Z belong to the thick subcategory in $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}) \subset \mathsf{D}^{\mathsf{co}}(Z\text{-}\mathsf{qcoh})$ generated by the inverse images of flat quasicoherent sheaves from X (see the proof of Lemma 2.8), the assertion follows.

Finally, the functor $\mathbb{L}i^*$ preserves infinite direct sums and compactness of objects since its right adjoint functor i_* preserves infinite direct sums. Hence the minimal triangulated subcategory in $\mathsf{D}^{\mathsf{co}}(Z\text{-qcoh})$ containing $\mathbb{L}i^*\mathsf{D}^{\mathsf{co}}(X\text{-qcoh})$ and closed under infinite direct sums is compactly generated by some objects which are compact in $\mathsf{D}^{\mathsf{co}}(Z\text{-qcoh})$. By Brown representability, the quotient category in (c) is equivalent to the right orthogonal complement to this triangulated subcategory, which is the kernel category in (a). ☐

We call any of the equivalent triangulated categories in Proposition 2.10 the *relative stable derived category of Z over X* and denote it by $\mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$ (cf. [Becker 2014, Section 2]). In particular, defining the relative stable derived category by the construction (c), we have natural triangulated functors

$$\mathsf{D}^{\mathsf{b}}(Z\text{-qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(Z\text{-qcoh}) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z/X).$$

Clearly, the composition $\mathsf{D}^{\mathsf{b}}(Z\text{-qcoh}) \to \mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$ factorizes through the relative singularity category $\mathsf{D}'_{Sing}(Z/X)$, providing a natural functor $\mathsf{D}'_{Sing}(Z/X) \to \mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$.

Lemma 2.10. *The composition of triangulated functors*

$$\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X) \longrightarrow \mathsf{D}'_{Sing}(Z/X) \longrightarrow \mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$$

is fully faithful and its image forms a set of compact generators for the triangulated category $\mathsf{D}^{\mathsf{st}}_{Sing}(Z/X)$.

Proof. By Proposition 1.5(d), the full triangulated subcategory $\mathsf{D}^{\mathsf{abs}}(Z\text{-coh})$ compactly generates the triangulated category $\mathsf{D}^{\mathsf{co}}(Z\text{-qcoh})$, and similarly this holds for X. In view of the construction (c) and the argument in the proof of Proposition 2.10, the assertion follows from [Neeman 1992, Theorem 2.1]. ☐

Now let $\mathcal{L}$ be a line bundle on X, let $w \in \mathcal{L}(X)$ be a locally nonzero-dividing section of $\mathcal{L}$, and let $i: X_0 \to X$ be closed embedding of the zero locus of w. Defining the category $\mathsf{D}^{\mathsf{st}}_{Sing}(X_0/X)$ by the construction (d), let $\mathbb{L}\Lambda : \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-qcoh}) \to \mathsf{D}^{\mathsf{st}}_{Sing}(X_0/X)$ be the triangulated functor assigning to a w-flat quasicoherent matrix factorization $\mathcal{M}$ the acyclic complex $i^*\mathcal{M}$ over X_0-qcoh.

Since any bounded-below acyclic complex over X_0-qcoh is coacyclic, and any bounded-above complex belongs to the minimal triangulated subcategory in $\mathsf{D}^{\mathsf{co}}(X_0\text{-qcoh})$ generated by its terms and closed under infinite direct sums, the following diagram of triangulated functors is commutative (cf. Corollary 2.9):

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-qcoh}) \arrow[rr, "\mathbb{L}\Xi[-1]"] \arrow[dr, "\mathbb{L}\Lambda"'] & & \mathsf{D}'_{Sing}(X_0/X) \arrow[dl] \\
& \mathsf{D}^{\mathsf{st}}_{Sing}(X_0/X) &
\end{tikzcd}$$

Theorem 2.10. *For any locally nonzero-dividing section w of a line bundle $\mathcal{L}$ on a separated Noetherian scheme X of finite Krull dimension, all the three functors on the above diagram are equivalences of triangulated categories.*

Proof. The functor $\mathbb{L}\Xi$ is an equivalence by Theorem 2.8. To show that the functor $\mathbb{L}\Lambda$ is an equivalence, let us check that it identifies compact generators. By Proposition 1.5(d), the category $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$ is compactly generated by its full triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})$, while according to Lemma 2.10 the category $\mathsf{D}^{\mathsf{st}}_{Sing}(X_0/X)$ is compactly generated by its full triangulated subcategory $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$. The restriction of the functor $\mathbb{L}\Lambda$ being an equivalence between these two subcategories (in view of commutativity of the diagram and) by Theorem 2.7, it follows that the functor $\mathbb{L}\Lambda$ itself is an equivalence, too. ☐

Remark 2.10. Another proof of Theorem 2.10 can be obtained using the approach based on [Efimov 2013, Appendix A]. In the notation and assumptions of Remark 2.7, suppose that C is an abelian category with exact functors of arbitrary infinite direct sums. Then so is the abelian category $MF(\mathsf{C},L,w)$; the full abelian subcategory $\mathsf{C}_0\subset\mathsf{C}$ is closed under infinite direct sums; and the triangulated functors i_*, $\mathbb{L}i^*$, υ^n, $\mathbb{L}\xi^n$, F^n, G^{n-} act between the coderived categories $\mathsf{D}^{\mathsf{co}}(\mathsf{C})$, $\mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)$, and $\mathsf{D}^{\mathsf{co}}MF(\mathsf{C},L,w)$.

As in Remark 2.7, one proves that the functors $G^{n-}:\mathsf{D}^{\mathsf{co}}(\mathsf{C})\to\mathsf{D}^{\mathsf{co}}MF(\mathsf{C},L,w)$ and $\upsilon^n:\mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)\to\mathsf{D}^{\mathsf{co}}MF(\mathsf{C},L,w)$ are fully faithful and their images form a semiorthogonal decomposition of the coderived category $\mathsf{D}^{\mathsf{co}}MF(\mathsf{C},L,w)$. By (the proof of) [loc. cit., Proposition A.3(3) and (4)], the totalization functor

$$\mathsf{D}^{\mathsf{co}}MF(\mathsf{C},L,w)\longrightarrow\mathsf{D}^{\mathsf{co}}(\mathsf{C},L,w)$$

acting between the coderived category of the abelian category $MF(\mathsf{C},L,w)$ and the coderived category of matrix factorizations $\mathsf{D}^{\mathsf{co}}(\mathsf{C},L,w)$ (defined as in Section 1.3) is the Verdier localization by the minimal triangulated subcategory containing the objects $G^{n-}(B)$ and $G^{(n+1)-}(B)$ for all $B\in\mathsf{C}$ and closed under infinite direct sums.

It follows that the composition of the functor $\upsilon^n:\mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)\to\mathsf{D}^{\mathsf{co}}MF(\mathsf{C},L,w)$ with the totalization functor $\mathsf{D}^{\mathsf{co}}MF(\mathsf{C},L,w)\to\mathsf{D}^{\mathsf{co}}(\mathsf{C},L,w)$ induces an equivalence of triangulated categories

$$\mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)/\langle\mathbb{L}i^*\mathsf{D}^{\mathsf{co}}(\mathsf{C})\rangle_\oplus\longrightarrow\mathsf{D}^{\mathsf{co}}(\mathsf{C},L,w)$$

between the quotient category of the coderived category $\mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)$ by its minimal triangulated subcategory containing the image of the functor $\mathbb{L}i^*:\mathsf{D}^{\mathsf{co}}(\mathsf{C})\to\mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)$ and closed under infinite direct sums, and the coderived category of matrix factorizations. The composition of the functor $\mathbb{L}\xi^n:\mathsf{D}^{\mathsf{co}}MF(\mathsf{C},L,w)\to\mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)$ with the Verdier localization functor $\mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)\to\mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)/\langle\mathbb{L}i^*\mathsf{D}^{\mathsf{co}}(\mathsf{C})\rangle_\oplus$ factorizes through the totalization functor, providing the inverse equivalence $\mathsf{D}^{\mathsf{co}}(\mathsf{C}, L, w) \to \mathsf{D}^{\mathsf{co}}(\mathsf{C}_0)/\langle \mathbb{L}i^*\mathsf{D}^{\mathsf{co}}(\mathsf{C})\rangle_{\oplus}$.

Returning to quasicoherent matrix factorizations of a global section $w \in \mathcal{L}(X)$ of a line bundle $\mathcal{L}$ on a separated Noetherian scheme X with the zero locus $X_0 \subset X$, we obtain direct constructions of two mutually inverse triangulated equivalences between the coderived category $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ and the relative stable derived category $\mathsf{D}^{\mathsf{st}}_{Sing}(X_0/X)$ as defined in part (c) of Proposition 2.10.

3. Supports, pull-backs, and push-forwards

3.1. ***Supports.*** This section paves the ground for the results about preservation of finite rank or coherence by the push-forwards of matrix factorizations with proper supports, which will be proven in Sections 3.5–3.6.

Let X be a separated Noetherian scheme and $T \subset X$ be a Zariski closed subset. Denote by $X\text{-}\mathsf{coh}_T$ the abelian category of coherent sheaves on X with set-theoretic support in T; and we will use similar notation for quasicoherent sheaves.

It is a well-known fact (essentially, a reformulation of the Artin–Rees lemma) that the embedding of abelian categories $X\text{-}\mathsf{qcoh}_T \to X\text{-}\mathsf{qcoh}$ takes injectives to injectives. It follows that the functor $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh}_T) \to \mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh})$ is fully faithful. Clearly, its image is a thick subcategory and the corresponding quotient category can be naturally identified with $\mathsf{D}^{\mathsf{b}}(U\text{-}\mathsf{coh})$, where $U = X \setminus T$ (cf. Section 1.10).

Assume additionally that X has enough vector bundles. Let $\mathit{Perf}_T(X) \subset \mathit{Perf}(X)$ denote the full subcategory of perfect complexes with the cohomology sheaves set-theoretically supported in T. By the above result, $\mathit{Perf}_T(X)$ can be considered as a thick subcategory in $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh}_T)$. According to [Orlov 2011, Lemma 2.6], the functor $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh}_T)/\mathit{Perf}_T(X) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X)$ induced by the embedding $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh}_T) \to \mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh})$ is fully faithful. We denote the source (or the image) category of this functor by $\mathsf{D}^{\mathsf{b}}_{Sing}(X, T)$.

By [Chen 2010, Theorem 1.3], the restriction functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X) \to \mathsf{D}^{\mathsf{b}}_{Sing}(U)$ is the Verdier localization functor by the triangulated subcategory $\mathsf{D}^{\mathsf{b}}_{Sing}(X, T)$. In particular, the kernel of the restriction functor coincides with the thick envelope of (i.e., the minimal thick subcategory containing) $\mathsf{D}^{\mathsf{b}}_{Sing}(X, T)$ in $\mathsf{D}^{\mathsf{b}}_{Sing}(X)$.

Now we are going to establish the similar results for the triangulated categories of relative singularities. Let $i : Z \to X$ be a closed subscheme such that $i_*\mathcal{O}_Z \in \mathit{Perf}(X)$, and let $\mathit{Perf}(Z/X) = \mathsf{D}^{\mathsf{b}}(\mathsf{E}_{Z/X})$ (see Remark 2.1) denote the thick subcategory in $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})$ generated by $\mathbb{L}i^*\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh})$. Let $T \subset Z$ be a Zariski closed subset; put $U = X \setminus T$ and $V = Z \setminus T$. We denote by $\mathit{Perf}_T(Z/X)$ the full subcategory of all objects of $\mathit{Perf}(Z/X)$ with the cohomology sheaves set-theoretically supported in T. Consider it as a thick subcategory in $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}_T)$, and denote by $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X, T)$ the quotient category $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}_T)/\mathit{Perf}_T(Z/X)$.

Lemma 3.1. (a) *The functor* $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X, T) \to \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ *induced by the embedding* $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}_T) \to \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})$ *is fully faithful.*

(b) *The restriction functor* $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X) \to \mathsf{D}^{\mathsf{b}}_{Sing}(V/U)$ *is the Verdier localization functor by the triangulated subcategory* $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X, T)$*. In particular, the kernel of the restriction functor coincides with the thick envelope of* $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X, T)$ *in* $\mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$.

Proof. The proof of (a) is similar to that of [Orlov 2011, Lemma 2.6]. One only needs to notice that the tensor product of an object of $Perf(Z/X)$ with an object of $Perf(Z)$ belongs to $Perf(Z/X)$. This follows from the fact that $Perf(Z)$ as a thick subcategory in $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})$ is generated by the restrictions of vector bundles from X (see Section 2.1). Part (b) is true since the thick subcategory $Perf(V/U) \subset \mathsf{D}^{\mathsf{b}}(V\text{-}\mathsf{coh})$ is generated by the image of the restriction functor $Perf(Z/X) \to Perf(V/U)$, which is true because any coherent sheaf on U can be extended to a coherent sheaf on X. $\square$

Let $\mathcal{L}$ be a line bundle over X and $w \in \mathcal{L}(X)$ be a section; set $X_0 = \{w = 0\} \subset X$. The definitions of the set-theoretic and category-theoretic supports $\operatorname{Supp} \mathcal{M}$ and $\operatorname{supp} \mathcal{M}$ of a coherent matrix factorization $\mathcal{M} \in (X, \mathcal{L}, w)\text{-}\mathsf{coh}$ were given (in a greater generality of coherent CDG-modules) in Section 1.10.

Given a locally free matrix factorization of finite rank $\mathcal{M} \in (X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}$, define the *(category-theoretic) support* $\operatorname{supp} \mathcal{M} \subset X$ as the minimal closed subset $T \subset X$ such that the restriction $\mathcal{M}|_U$ of $\mathcal{M}$ to the open subscheme $U = X \setminus T$ is absolutely acyclic with respect to $(U, \mathcal{L}|_U, w|_U)\text{-}\mathsf{coh}_{\mathsf{lf}}$. By Corollary 2.3(i), the definitions of category-theoretic supports of coherent matrix factorizations and of locally free matrix factorizations of finite rank agree when they are both applicable.

Equivalently, for a locally free matrix factorization $\mathcal{M}$ of finite rank over X, the open subscheme $X \setminus \operatorname{supp} \mathcal{M}$ is the union of all affine open subschemes $U \subset X$ such that the matrix factorization $\mathcal{M}|_U$ is contractible (see Remark 1.3). For any coherent matrix factorization $\mathcal{M}$, one has $\operatorname{supp} \mathcal{M} \subset X_0$ since any matrix factorization of an invertible potential is contractible (cf. [Polishchuk and Vaintrob 2011, Section 5]).

Let $T \subset X$ be a closed subset. Denote by $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ (respectively $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)\text{-}\mathsf{coh})$) the quotient category of the homotopy category of locally free matrix factorizations of finite rank (resp. coherent matrix factorizations) supported category-theoretically inside T by the thick subcategory of matrix factorizations absolutely acyclic with respect to $(X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}$ (resp. $(X, \mathcal{L}, w)\text{-}\mathsf{coh}$). Clearly, the functors $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ and $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$ are fully faithful [loc. cit.].

By the definition, the thick subcategories

$$\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}),$$
$$\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$$

only depend on the intersection $X_0 \cap T$ (rather than the whole of T). Equivalently, they can be defined as the full subcategories of objects annihilated by the restriction functors $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{abs}}((U, \mathcal{L}|_U, w|_U)\text{-}\mathsf{coh}_{\mathsf{lf}})$ and $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{abs}}((U, \mathcal{L}|_U, w|_U)\text{-}\mathsf{coh})$, where $U = X \setminus T$.

As in Section 1.10, we denote by $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_T)$ the absolute derived category of coherent matrix factorizations with set-theoretic support in T. The functor $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_T) \to \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$ is fully faithful by Proposition 1.10(d). By Corollary 1.10(b), the full subcategory

$$\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$$

is the thick envelope of the full subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_T)$.

Now assume that $w : \mathcal{O}_X \to \mathcal{L}$ is an injective morphism of sheaves.

Proposition 3.1. (a) *The equivalence of categories*

$$\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$$

identifies the triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_T)$ *with the triangulated subcategory* $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X,\ X_0 \cap T)$. *In particular, the former triangulated subcategory only depends on the intersection* $X_0 \cap T$.

(b) *The full preimage of the thick envelope of the triangulated subcategory*

$$\mathsf{D}^{\mathsf{b}}_{Sing}(X_0,\ X_0 \cap T) \subset \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$$

under the fully faithful functor $\Sigma : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ *coincides with the triangulated subcategory* $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$.

Proof. Part (b) follows from the fact that the thick envelope of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0,\ X_0 \cap T)$ is the kernel of the restriction functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X_0 \setminus T)$, the similar fact for $\mathsf{D}^{\mathsf{abs}}_T((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$, and the compatibility of the functors Σ with the restrictions to open subschemes, together with their full-and-faithfulness.

To prove part (a), notice first that the functor Υ obviously takes $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X,\ X_0 \cap T)$ into $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_T)$. Let us check that the functor $\mathbb{L}\Xi$ takes $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_T)$ into $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X,\ X_0 \cap T)$. Let $\mathcal{M}$ be a coherent matrix factorization supported set-theoretically in T. Present $\mathcal{M}$ as the cokernel of an injective morphism of w-flat coherent matrix factorizations $\mathcal{K} \to \mathcal{N}$. Since the functor $\mathbb{L}\Xi$ is triangulated, the object $\mathbb{L}\Xi(\mathcal{M}) \in \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ is isomorphic to the cone of the morphism $\Xi(\mathcal{K}) \to \Xi(\mathcal{N})$ (cf. Lemma 3.6). The morphism $\Xi(\mathcal{K}) \to \Xi(\mathcal{N})$ of coherent sheaves on X_0 is an isomorphism outside T, so its kernel and cokernel are supported in $X_0 \cap T$. Thus the cone is quasi-isomorphic to a two-term complex of coherent sheaves on X_0 with the terms supported set-theoretically in $X_0 \cap T$. $\square$

3.2. ***Locality of local freeness.*** The aim of this section is to show that the property of an object of $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ or $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})$ to be a direct summand of an object from $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ is local in a separated Noetherian scheme X with a dualizing complex and enough vector bundles, assuming that the potential $w \in \mathcal{L}(X)$ is not locally zero-dividing.

Let Z be a Noetherian scheme of finite Krull dimension with enough vector bundles. Recall that the natural functor $\mathsf{D}^{\mathsf{b}}_{Sing}(Z) \to \mathsf{D}'_{Sing}(Z)$ is fully faithful [Orlov 2004, Proposition 1.13] (cf. Section 2.8).

Proposition 3.2. *Let $Z = U \cup V$ be a covering by two open subschemes. Then any object of $\mathsf{D}'_{Sing}(Z)$ whose restrictions to U and V belong to the full subcategories $\mathsf{D}^{\mathsf{b}}_{Sing}(U) \subset \mathsf{D}'_{Sing}(U)$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(V) \subset \mathsf{D}'_{Sing}(V)$, respectively, is a direct summand of an object belonging to the full subcategory $\mathsf{D}^{\mathsf{b}}_{Sing}(Z) \subset \mathsf{D}'_{Sing}(Z)$.*

Proof. Consider the bounded derived category of quasicoherent sheaves $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$ on Z and two full triangulated subcategories $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})$ and $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ in it. Clearly, the intersection $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}) \cap \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ coincides with the full subcategory of perfect complexes $\mathit{Perf}(Z) = \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$.

Lemma 3.2. *Any morphism from an object of the full subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ into an object of the full subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$ factorizes through an object belonging to $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}_{\mathsf{lf}})$.*

Proof. See the proof of [Orlov 2004, Proposition 1.13]. □

It follows from Lemma 3.2 (by the way of the octahedron axiom) that any object $\mathcal{K}^\bullet$ of the full triangulated subcategory $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})_{\mathsf{fl}\text{-}\mathsf{c}}$ generated by $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})$ in $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$ can be included in a distinguished triangle

$$\mathcal{F}^\bullet \longrightarrow \mathcal{K}^\bullet \longrightarrow \mathcal{M}^\bullet \longrightarrow \mathcal{F}^\bullet[1],$$

with $\mathcal{F}^\bullet \in \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathcal{M}^\bullet \in \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})$. Besides, the natural functor $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}_{\mathsf{fl}})/\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})/\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{coh})$ is fully faithful.

To prove Proposition 3.2, one has to show that any object $\mathcal{K}^\bullet \in \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$ whose restrictions to U and V belong to the subcategories $\mathsf{D}^{\mathsf{b}}(U\text{-}\mathsf{qcoh})_{\mathsf{fl}\text{-}\mathsf{c}}$ and $\mathsf{D}^{\mathsf{b}}(V\text{-}\mathsf{qcoh}_{\mathsf{fl}\text{-}\mathsf{c}})$, respectively, is a direct summand of an object from $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})_{\mathsf{fl}\text{-}\mathsf{c}} \subset \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$. According to the above, there exist two objects $\mathcal{F}^\bullet_U \in \mathsf{D}^{\mathsf{b}}(U\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathcal{F}^\bullet_V \in \mathsf{D}^{\mathsf{b}}(V\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and two morphisms $\mathcal{F}^\bullet_U \to \mathcal{K}^\bullet|_U$ and $\mathcal{F}^\bullet_V \to \mathcal{K}^\bullet|_V$ whose cones belong to $\mathsf{D}^{\mathsf{b}}(U\text{-}\mathsf{coh})$ and $\mathsf{D}^{\mathsf{b}}(V\text{-}\mathsf{coh})$, respectively.

Set $W = U \cap V \subset Z$; then the restrictions of $\mathcal{F}^\bullet_U$ and $\mathcal{F}^\bullet_V$ to W are isomorphic in $\mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{qcoh})/\mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{coh})$, and consequently, in $\mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{qcoh}_{\mathsf{fl}})/\mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{coh}_{\mathsf{lf}})$, too. Notice that the category $\mathit{Perf}(W) = \mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{coh}_{\mathsf{lf}})$ is idempotent complete, and therefore, a thick subcategory in $\mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{qcoh}_{\mathsf{fl}})$. It follows that there exists a finite complex of flat quasicoherent sheaves $\mathcal{F}^\bullet_W$ on W together with two morphisms

$\mathcal{F}_U^\bullet|_W \to \mathcal{F}_W^\bullet$ and $\mathcal{F}_V^\bullet|_W \to \mathcal{F}_W^\bullet$ whose cones are perfect complexes. Denote the cocones of these morphisms by $\mathcal{G}_W^\bullet$ and $\mathcal{H}_W^\bullet$.

For any object A of a triangulated category D, let us denote by $'A$ the object $A \oplus A[1]$. For any triangulated subcategory $\mathsf{C} \subset \mathsf{D}$, whenever an object $A \in \mathsf{D}$ is a direct summand of an object from C, the object $'A$ belongs to C, as $A \oplus B \in \mathsf{C}$ implies $A \oplus A[1] \in \mathsf{C}$ in view of the distinguished triangle $A \oplus B \to A \oplus B \to A \oplus A[1] \to A[1] \oplus B[1]$ [Thomason 1997, Theorem 2.1].

By the Thomason–Trobaugh theorem [1990, Section 5], the objects $'\mathcal{G}_W^\bullet$ and $'\mathcal{H}_W^\bullet$ can be extended to perfect complexes on U and V, respectively. Moreover, these extensions $\mathcal{G}_U^\bullet \in \mathsf{D}^{\mathsf{b}}(U\text{-}\mathsf{coh}_{\mathsf{lf}})$ and $\mathcal{H}_V^\bullet \in \mathsf{D}^{\mathsf{b}}(V\text{-}\mathsf{coh}_{\mathsf{lf}})$ can be chosen in such a way that the morphisms $'\mathcal{G}_W^\bullet \to {}'\mathcal{F}_U^\bullet|_W$ and $'\mathcal{H}_W^\bullet \to {}'\mathcal{F}_V^\bullet|_W$ would be extendable to morphisms $\mathcal{G}_U^\bullet \to {}'\mathcal{F}_U^\bullet$ and $\mathcal{H}_V^\bullet \to {}'\mathcal{F}_V^\bullet$ [Neeman 1996, Theorem 2.1(4) and (5)].

Furthermore, the objects $'\mathcal{G}_U^\bullet$ and $'\mathcal{H}_V^\bullet$ can be extended to perfect complexes $\mathcal{G}^\bullet$ and $\mathcal{H}^\bullet$ on the whole scheme Z so that the compositions of morphisms

$$'\mathcal{G}_U^\bullet \longrightarrow {}''\mathcal{F}_U^\bullet \longrightarrow {}''\mathcal{K}^\bullet|_U \quad \text{and} \quad '\mathcal{H}_V^\bullet \longrightarrow {}''\mathcal{F}_V^\bullet \longrightarrow {}''\mathcal{K}^\bullet|_V$$

would be extendable to morphisms $\mathcal{G}^\bullet \to {}''\mathcal{K}^\bullet$ and $\mathcal{H}^\bullet \to {}''\mathcal{K}^\bullet$. Denote by $\mathcal{K}_{(1)}^\bullet$ a cone of the morphism $\mathcal{G}^\bullet \oplus \mathcal{H}^\bullet \to {}''\mathcal{K}^\bullet$, by $\mathcal{F}_{U,(1)}^\bullet$ a cone of the morphism $'\mathcal{G}_U^\bullet \to {}''\mathcal{F}_U^\bullet$, and by $\mathcal{F}_{V,(1)}^\bullet$ a cone of the morphism $'\mathcal{H}_V^\bullet \to {}''\mathcal{F}_V^\bullet$. We have come back to the original situation with an object $\mathcal{K}_{(1)}^\bullet \in \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$, two objects $\mathcal{F}_{U,(1)}^\bullet \in \mathsf{D}^{\mathsf{b}}(U\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathcal{F}_{V,(1)}^\bullet \in \mathsf{D}^{\mathsf{b}}(V\text{-}\mathsf{qcoh}_{\mathsf{fl}})$, and two morphisms

$$\mathcal{F}_{U,(1)}^\bullet \longrightarrow \mathcal{K}_{(1)}^\bullet|_U \quad \text{and} \quad \mathcal{F}_{V,(1)}^\bullet \longrightarrow \mathcal{K}_{(1)}^\bullet|_V$$

whose cones belong to $\mathsf{D}^{\mathsf{b}}(U\text{-}\mathsf{coh})$ and $\mathsf{D}^{\mathsf{b}}(V\text{-}\mathsf{coh})$, respectively. In addition, the objects $\mathcal{F}_{U,(1)}^\bullet|_W$ and $\mathcal{F}_{V,(1)}^\bullet|_W$ are now isomorphic in $\mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{qcoh}_{\mathsf{fl}})$.

The construction does not guarantee commutativity of the diagram formed by the isomorphism

$$\mathcal{F}_{U,(1)}^\bullet|_W = \mathcal{F}_{W,(1)}^\bullet \simeq \mathcal{F}_{V,(1)}^\bullet|_W$$

and the restrictions of the morphisms $\mathcal{F}_{U,(1)}^\bullet \to \mathcal{K}_{(1)}^\bullet$ and $\mathcal{F}_{V,(1)}^\bullet \to \mathcal{K}_{(1)}^\bullet$ to W. However, the original choice of the morphisms

$$\mathcal{F}_U^\bullet|_W \longrightarrow \mathcal{F}_W^\bullet \quad \text{and} \quad \mathcal{F}_V^\bullet|_W \longrightarrow \mathcal{F}_W^\bullet$$

makes this diagram commute in the quotient category $\mathsf{D}^b(W\text{-}\mathsf{qcoh})/\mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{coh})$. Hence the difference of two morphisms $\mathcal{F}_{W,(1)}^\bullet \rightrightarrows \mathcal{K}_{(1)}^\bullet|_W$ factorizes through a bounded complex of coherent sheaves on W, and consequently (according to Lemma 3.2) also through a perfect complex on W. Denote the latter by $\mathcal{E}^\bullet \in \mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{coh}_{\mathsf{lf}})$.

Now let $j : U \to Z$, $k : V \to Z$, and $h : W \to Z$ denote the natural open embeddings. Consider the square diagram formed by the morphisms

$$\begin{gathered}\mathbb{R}j_*\mathcal{F}^\bullet_{U,(1)} \oplus \mathbb{R}k_*\mathcal{F}^\bullet_{V,(1)} \longrightarrow \mathbb{R}h_*\mathcal{F}^\bullet_{U,(1)}|_W, \\ \mathbb{R}j_*\mathcal{K}^\bullet_{(1)}|_U \oplus \mathbb{R}k_*\mathcal{K}^\bullet_{(1)}|_V \longrightarrow \mathbb{R}h_*\mathcal{K}^\bullet_{(1)}|_W.\end{gathered}$$

According to the above, this diagram is not necessarily commutative; but it can be made commutative by adding the new direct summand $\mathbb{R}h_*\mathcal{E}^\bullet$ to the term $\mathbb{R}j_*\mathcal{K}^\bullet_{(1)}|_U \oplus \mathbb{R}k_*\mathcal{K}^\bullet_{(1)}|_V$ with the morphism $\mathbb{R}h_*\mathcal{E}^\bullet \to \mathbb{R}h_*\mathcal{K}^\bullet_{(1)}|_W$ induced by the morphism $\mathcal{E}^\bullet \to \mathcal{K}^\bullet_{(1)}|_W$ and the morphism $\mathbb{R}j_*\mathcal{F}^\bullet_{U,(1)} \oplus \mathbb{R}k_*\mathcal{F}^\bullet_{V,(1)}$ equal to zero on the first direct summand and induced by the morphism $\mathcal{F}^\bullet_{V,(1)}|_W \simeq \mathcal{F}^\bullet_{W,(1)} \to \mathcal{E}^\bullet$ on the second one.

Let $\mathcal{F}^\bullet$ denote a cocone of the morphism

$$\mathbb{R}j_*\mathcal{F}^\bullet_{U,(1)} \oplus \mathbb{R}k_*\mathcal{F}^\bullet_{V,(1)} \longrightarrow \mathbb{R}h_*\mathcal{F}^\bullet_{U,(1)}|_W$$

and $\mathcal{L}^\bullet$ denote a cocone of the morphism

$$\mathbb{R}j_*\mathcal{K}^\bullet_{(1)}|_U \oplus \mathbb{R}k_*\mathcal{K}^\bullet_{(1)}|_V \oplus \mathbb{R}h_*\mathcal{E}^\bullet \longrightarrow \mathbb{R}h_*\mathcal{K}^\bullet_{(1)}|_W.$$

Then the commutative square can be extended to a morphism of distinguished triangles, so we obtain a morphism $\mathcal{F}^\bullet \to \mathcal{L}^\bullet$. Since $\mathcal{K}^\bullet_{(1)}$ is a cocone of the morphism

$$\mathbb{R}j_*\mathcal{K}^\bullet_{(1)}|_U \oplus \mathbb{R}k_*\mathcal{K}^\bullet_{(1)}|_V \longrightarrow \mathbb{R}h_*\mathcal{K}^\bullet_{(1)}|_W,$$

there is also a distinguished triangle $\mathcal{K}^\bullet_{(1)} \to \mathcal{L}^\bullet \to \mathbb{R}h_*\mathcal{E} \to \mathcal{K}^\bullet_{(1)}[1]$.

Notice that the complexes $\mathcal{F}^\bullet$ and $\mathbb{R}h_*\mathcal{E}^\bullet$ belong to $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ (since the class of bounded complexes of flat quasicoherent sheaves is preserved by the derived direct images with respect to flat morphisms of Noetherian schemes; cf. Proposition 1.9). Furthermore, the complex $\mathbb{R}h_*\mathcal{E}^\bullet$ is perfect over W. Restricting to W our morphism of distinguished triangles, and recalling that cones of the morphisms

$$\mathcal{F}^\bullet_{U,(1)} \longrightarrow \mathcal{K}^\bullet_{(1)}|_U \quad \text{and} \quad \mathcal{F}^\bullet_{V,(1)} \longrightarrow \mathcal{K}^\bullet_{(1)}|_V$$

are coherent complexes over U and V, one easily concludes that a cone of the morphism $\mathcal{F}^\bullet \to \mathcal{L}^\bullet$ is a coherent complex over W.

Denote this cone temporarily by $\mathcal{K}^\bullet_{(2)}$. Clearly, in order to show that the original complex $\mathcal{K}^\bullet$ is a direct summand of an object from $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$ in $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$ (which is our goal), it suffices to check that the complex $\mathcal{K}^\bullet_{(2)}$ is as well. It also follows from the constructions that the restrictions of the complex $\mathcal{K}^\bullet_{(2)}$ to U and V belong to $\mathsf{D}^{\mathsf{b}}(U\text{-}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$ and $\mathsf{D}^{\mathsf{b}}(V\text{-}\mathsf{qcoh}_{\mathsf{fl\text{-}c}})$, respectively. Dropping the lower index and redenoting $\mathcal{K}^\bullet_{(2)}$ simply by $\mathcal{K}^\bullet$, we are coming back to the situation in the beginning of the proof with the new knowledge that $\mathcal{K}^\bullet$ may be assumed to be a coherent complex over W.

The next segment of our proof is based on the localization theory for coderived categories of quasicoherent sheaves on Noetherian schemes (similar to the Thomason–Trobaugh–Neeman theorem for the conventional derived categories, the difference being that arbitrary bounded complexes of coherent sheaves play the role of perfect complexes). What we need is a particular case of the theory developed in Section 1.10 (corresponding to the choice of the quasicoherent CDG-algebra $\mathcal{O}_Z$ over Z).

Specifically, it follows from Proposition 1.5(d) and Theorem 1.10 together with [Neeman 1996, Theorem 2.1(5)] that any morphism from an object of $\mathsf{D^b}(W\text{-}\mathsf{coh})$ into a restriction to W of an object $\mathcal{K}^\bullet$ from $\mathsf{D^b}(Z\text{-}\mathsf{qcoh})$ (or even from $\mathsf{D^{co}}(Z\text{-}\mathsf{qcoh})$) can be extended to a morphism to $\mathcal{K}^\bullet$ from an object of $\mathsf{D^b}(Z\text{-}\mathsf{coh})$. Applying this assertion to the identity morphism $\mathcal{K}^\bullet|_W \to \mathcal{K}^\bullet|_W$ in the above situation, we obtain a morphism $\mathcal{M}^\bullet \to \mathcal{K}^\bullet$ into $\mathcal{K}^\bullet$ from a coherent complex $\mathcal{M}^\bullet$ over Z that is a quasi-isomorphism over W. Passing to a cone of this morphism, we may assume $\mathcal{K}^\bullet$ to be acyclic over W.

By Corollary 1.10, such a complex $\mathcal{K}^\bullet$ is quasi-isomorphic to a (bounded) complex of quasicoherent sheaves on Z whose terms are concentrated set-theoretically in the complement $Z \setminus W$. The latter is a disjoint union of two nonintersecting closed subsets in Z, namely, the complements $S = Z \setminus U$ and $T = Z \setminus V$. Now the complex $\mathcal{K}^\bullet$ decomposes into a direct sum of two complexes with set-theoretic supports inside S and T, respectively.

One can consider the two direct summands separately. We have to show that any bounded complex of quasicoherent sheaves $\mathcal{K}^\bullet$ on Z, which is supported set-theoretically in T and whose restriction to U belongs to $\mathsf{D^b}(U\text{-}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$, itself belongs to $\mathsf{D^b}(Z\text{-}\mathsf{qcoh})_{\mathsf{fl\text{-}c}}$. Arguing as in the beginning of this proof, we have an object $\mathcal{G}^\bullet \in \mathsf{D^b}(U\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ together with a morphism $\mathcal{G}^\bullet \to \mathcal{K}^\bullet|_U$ whose cone belongs to $\mathsf{D^b}(U\text{-}\mathsf{coh})$. The restriction $\mathcal{G}^\bullet|_W$ then belongs to both $\mathsf{D^b}(W\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D^b}(W\text{-}\mathsf{coh})$, and is, therefore, a perfect complex on W.

Again by the Thomason–Trobaugh theorem, the object $'\mathcal{G}^\bullet|_W$ can be extended to a perfect complex $\mathcal{H}^\bullet$ on V. A cocone of the morphism

$$\mathbb{R}j_*{}'\mathcal{G}^\bullet \oplus \mathbb{R}k_*\mathcal{H}^\bullet \longrightarrow \mathbb{R}h_*{}'\mathcal{G}^\bullet|_W$$

provides an object $\mathcal{F}^\bullet \in \mathsf{D^b}(Z\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ isomorphic to $'\mathcal{G}^\bullet$ over U and to $\mathcal{H}^\bullet$ over V. Now the morphism $'\mathcal{G}^\bullet \to {}'\mathcal{K}^\bullet|_U$ over U extends uniquely to a morphism $\mathcal{F}^\bullet \to {}'\mathcal{K}^\bullet$ over Z since the set-theoretic support of $'\mathcal{K}^\bullet$ is contained in a closed subset lying inside U. A cone of the morphism $\mathcal{F}^\bullet \to {}'\mathcal{K}^\bullet$ is a coherent complex on Z since it is so in restrictions to U and V. Thus, the proposition is proven. □

Now let X be a separated Noetherian scheme of finite Krull dimension with enough vector bundles, $\mathcal{L}$ be a line bundle on X, and $w \in \mathcal{L}(X)$ be a locally nonzero-dividing potential. Let $X_0 \subset X$ be the zero locus of w.

Corollary 3.2. *Let $X = U \cap V$ be a covering by two open subschemes. Then any object of* $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ *whose restrictions to U and V belong to the full triangulated subcategories*

$$\mathsf{D}^{\mathsf{abs}}((U, \mathcal{L}|_U, w|_U)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((U, \mathcal{L}|_U, w|_U)\text{-}\mathsf{qcoh}_{\mathsf{fl}}),$$
$$\mathsf{D}^{\mathsf{abs}}((V, \mathcal{L}|_V, w|_V)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((V, \mathcal{L}|_V, w|_V)\text{-}\mathsf{qcoh}_{\mathsf{fl}}),$$

respectively, is a direct summand of an object from the full triangulated subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$.

Proof. By Proposition 2.8, the category $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ is a full triangulated subcategory of the triangulated category $\mathsf{D}'_{Sing}(X_0)$. The (essential) intersection of the full subcategories $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ in $\mathsf{D}'_{Sing}(X_0)$ is the triangulated category $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$.

Indeed, an object of $\mathcal{F} \in \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ belongs to $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ if and only if the object $i_\circ\mathcal{F}$ vanishes in $\mathsf{D}^{\mathsf{b}}_{Sing}(X)$ (Theorem 2.7); an object $\mathcal{F} \in \mathsf{D}'_{Sing}(X_0)$ belongs to $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ if and only if the object $i_\circ\mathcal{F}$ vanishes in $\mathsf{D}'_{Sing}(X)$ (Proposition 2.8); and the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X) \to \mathsf{D}'_{Sing}(X)$ is fully faithful.

Moreover, the (essential) intersection of $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ with the thick envelope of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ in $\mathsf{D}'_{Sing}(X_0)$ is the thick envelope of $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ in $\mathsf{D}'_{Sing}(X_0)$. Indeed, let $\mathcal{M}$ be an object of the intersection; then $\mathcal{M}\oplus\mathcal{M}[1]$ belongs to both $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$, hence also to $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$, and consequently $\mathcal{M}$ belongs to the thick envelope of $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$.

Now let $\mathcal{K}$ be our object of $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$; it can be also viewed as an object of $\mathsf{D}'_{Sing}(X_0)$. If its restrictions to U and V belong to $\mathsf{D}^{\mathsf{abs}}((U, \mathcal{L}|_U, w|_U)\text{-}\mathsf{coh}_{\mathsf{lf}})$ and $\mathsf{D}^{\mathsf{abs}}((V, \mathcal{L}|_V, w|_V)\text{-}\mathsf{coh}_{\mathsf{lf}})$, they also belong to $\mathsf{D}^{\mathsf{b}}_{Sing}(U_0) \subset \mathsf{D}'_{Sing}(U_0)$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(V_0) \subset \mathsf{D}'_{Sing}(V_0)$ (where we set $U_0 = U \cap X_0$ and $V_0 = V \cap X_0$).

Applying Proposition 3.2, we can conclude that $\mathcal{K}$ belongs to the thick envelope of $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ in $\mathsf{D}'_{Sing}(X_0)$. The assertion of Corollary 3.2 follows from the above. ☐

Assume additionally that the scheme X admits a dualizing complex $\mathcal{D}^\bullet_X$.

Theorem 3.2. *Let $X = U \cap V$ be a covering by two open subschemes. Then any object of* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$ *whose restrictions to U and V belong to the thick envelopes of the triangulated subcategories*

$$\mathsf{D}^{\mathsf{abs}}((U, \mathcal{L}|_U, w|_U)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{abs}}((U, \mathcal{L}|_U, w|_U)\text{-}\mathsf{coh}),$$
$$\mathsf{D}^{\mathsf{abs}}((V, \mathcal{L}|_V, w|_V)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{abs}}((V, \mathcal{L}_V, w|_V)\text{-}\mathsf{coh})$$

itself belongs to the thick envelope of the triangulated subcategory

$$\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}).$$

Proof. The argument is based on the Serre–Grothendieck duality theory for matrix factorizations as developed in Section 2.5, which allows us to reduce the question to the result of Corollary 3.2. Specifically, let $\mathcal{M}$ be our coherent matrix factorization over X. Replacing, if necessary, $\mathcal{M}$ with $\mathcal{M}\oplus\mathcal{M}[1]$, we may assume the restrictions of $\mathcal{M}$ to U and V to be isomorphic to locally free matrix factorizations of finite rank.

Let us apply the construction of functor

$$\Omega : \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},-w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$$

from Section 2.5 to the matrix factorization $\mathcal{M}$. That is, we pick a left resolution of $\mathcal{M}$ by locally free matrix factorizations of finite rank, dualize by applying $\mathrm{Hom}_{X\text{-}\mathsf{qc}}(-,\mathcal{O}_X)$, and totalize using infinite direct sums. By Corollary 2.5, the functor Ω is fully faithful; it also identifies $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}})^{\mathsf{op}}$ with $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{coh}_{\mathsf{lf}})$. Hence it suffices to check that the matrix factorization $\Omega(\mathcal{M})$ belongs to the thick envelope of $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ in $\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},-w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$. But we know as much from Corollary 3.2. □

3.3. ***Nonlocalization of local freeness.*** The lack of a workable notion of the conventional derived category (as opposed to the coderived category) for quasicoherent matrix factorizations stands in the way of a direct extension of the Thomason–Trobaugh–Neeman localization theorem for perfect complexes [Thomason and Trobaugh 1990; Neeman 1992; 1996] to locally free matrix factorizations of finite rank. We have seen in Section 1.10 how the localization theory can be developed for coherent matrix factorizations. In this section we demonstrate a counterexample showing that the localization theory, in its conventional form, actually does *not* hold for locally free matrix factorizations.

In other words, the restriction

$$\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((U,\mathcal{L}|_U,w|_U)\text{-}\mathsf{coh}_{\mathsf{lf}})$$

for an open subscheme $U\subset X$ is not always a Verdier quotient functor, even up to the direct summands. Moreover, the triangulated category $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ may fail to be generated by a single object, unlike in the case of the categories of perfect complexes on quasicompact quasiseparated schemes.

All the potentials in our example will be simply regular functions, i.e., sections of the trivial line bundle $\mathcal{O}_X$ or $\mathcal{O}_U$, etc.; so we drop the line bundle $\mathcal{L}$ from our notation in the rest of the section and write simply $\mathsf{D}^{\mathsf{abs}}((X,w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ or $\mathsf{D}^{\mathsf{abs}}((X,w)\text{-}\mathsf{coh})$, etc. For simplicity, we will work over the basic field of complex numbers $\mathbb{C}$.

Consider the 3-dimensional affine quadratic cone

$$X=\{xy=zw\}\subset\mathbb{A}^4=\operatorname{Spec}\mathbb{C}[x,y,z,w].$$

Further, let us take the open subset

$$U=\{z\neq 0\}\subset X.$$

Clearly, we have an isomorphism of pairs (algebraic variety, regular function on it)

$$(U, w) \xrightarrow{\sim} (\mathbb{A}^2_{t_1,t_2} \times \mathbb{G}_m,\ t_1t_2), \quad (x, y, z, w) \longmapsto \Big(\Big(x, \frac{y}{z}\Big), z\Big), \tag{1}$$

where we denote $\mathbb{A}^2_{t_1,t_2} = \operatorname{Spec} \mathbb{C}[t_1, t_2]$ and, as usual, $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$.

Lemma 3.3. (a) *We have a natural equivalence of triangulated categories*

$$\mathsf{D}^{\mathsf{abs}}((U, w)\text{-}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh}).$$

(b) *The restriction functor*

$$\mathsf{D}^{\mathsf{abs}}((X, w)\text{-}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((U, w)\text{-}\mathsf{coh})$$

is an equivalence.

Here the category of matrix factorizations of the zero potential $\mathsf{D}^{\mathsf{abs}}((Y, 0)\text{-}\mathsf{coh})$ is, of course, simply the derived category of 2-periodic complexes of coherent sheaves on a smooth variety Y.

Proof. Part (a): By (1), we have the equivalence

$$\mathsf{D}^{\mathsf{abs}}((U, w)\text{-}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{abs}}((\mathbb{A}^2_{t_1,t_2} \times \mathbb{G}_m, t_1t_2)\text{-}\mathsf{coh}).$$

By Knörrer periodicity (cf. [Orlov 2006, Theorem 3.1]), we have the equivalence

$$\mathsf{D}^{\mathsf{abs}}((\mathbb{A}^2_{t_1,t_2} \times \mathbb{G}_m, t_1t_2)\text{-}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh}).$$

Part (b): Let us put $D = X \setminus U$. By Theorem 1.10(b) (see also Section 3.1), we have the short exact sequence of triangulated categories

$$0 \longrightarrow \mathsf{D}^{\mathsf{abs}}_D((X, w)\text{-}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, w)\text{-}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((U, w)\text{-}\mathsf{coh}) \longrightarrow 0$$

Thus, we need to show that the category $\mathsf{D}^{\mathsf{abs}}_D((X, w))\text{-}\mathsf{coh})$ is zero. It suffices to check that the category $\mathsf{D}^{\mathsf{abs}}((D, w)\text{-}\mathsf{coh})$ is zero.

Let us put $S = \{xy = 0\} \subset \mathbb{A}^2$. Then we have the isomorphism

$$(D, w) \xrightarrow{\sim} (S \times \mathbb{A}^1_t,\ t), \quad (x, y, 0, w) \longmapsto ((x, y), w).$$

Since $\mathsf{D}^{\mathsf{abs}}((\mathbb{A}^1_t, t)\text{-}\mathsf{coh}) = 0$, it follows that

$$\mathsf{D}^{\mathsf{abs}}((D, w)\text{-}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{abs}}((S \times \mathbb{A}^1_t, t)\text{-}\mathsf{coh}) = 0. \qquad \square$$

Since U is smooth, we have the equivalence

$$\mathsf{D}^{\mathsf{abs}}((U, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \simeq \mathsf{D}^{\mathsf{abs}}((U, w)\text{-}\mathsf{coh}).$$

Now we turn to the category $\mathsf{D}^{\mathsf{abs}}((X, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$. As usual, we put

$$X_0 = \{w = 0\} \subset X.$$

According to Theorem 2.7, the triangulated category $\mathsf{D}^{\mathsf{abs}}((X, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ is equivalent to the kernel of the direct image functor $i_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X)$ acting between the triangulated categories of singularities of the schemes X_0 and X. This can be rephrased by saying that $\mathsf{D}^{\mathsf{abs}}((X, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ is equivalent to the quotient category of the category of bounded complexes of coherent sheaves on X_0 whose direct images are perfect complexes on X by the category of perfect complexes on X_0. Denoting the triangulated category of coherent complexes on X_0 whose direct images are perfect on X by $\mathit{Perf}(X_0, X) \subset \mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh})$, we have the equivalence of triangulated categories

$$\mathsf{D}^{\mathsf{abs}}((X, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \simeq \mathit{Perf}(X_0, X)/\mathit{Perf}(X_0). \tag{2}$$

Note that we have the natural isomorphism

$$X_0 \simeq S \times \mathbb{A}^1, \quad (x, y, z, 0) \longmapsto ((x, y), z).$$

It follows immediately that

$$\mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \simeq \mathsf{D}^{\mathsf{abs}}((\mathbb{A}^1, 0)\text{-}\mathsf{coh}). \tag{3}$$

Proposition 3.3. (a) *We have the natural equivalence of triangulated categories*

$$\mathit{Perf}(X_0, X)/\mathit{Perf}(X_0) \simeq \mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh})_{\text{0-dim}},$$

where $\mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh})_{\text{0-dim}} \subset \mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh})$ *is the subcategory of complexes with zero-dimensional support.*

Moreover, we have a commutative diagram of fully faithful triangulated functors:

$$\begin{tikzcd}
\mathit{Perf}(X_0, X)/\mathit{Perf}(X_0) \arrow[r, tail] \arrow[d, equal] & \mathsf{D}^{\mathsf{b}}_{Sing}(X_0) \arrow[d, equal] \\
\mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh})_{\text{0-dim}} \arrow[r, tail, "j_*"] & \mathsf{D}^{\mathsf{abs}}((\mathbb{A}^1, 0)\text{-}\mathsf{coh}),
\end{tikzcd}$$

where $j : \mathbb{G}_m \to \mathbb{A}^1$ *is the open embedding.*

(b) *We have a commutative diagram of fully faithful triangulated functors and equivalences*:

$$\begin{tikzcd}
\mathsf{D}^{\mathsf{abs}}((X, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \arrow[r, tail] \arrow[d, equal] & \mathsf{D}^{\mathsf{abs}}((U, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \arrow[d, equal] \\
\mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh})_{\text{0-dim}} \arrow[r, tail, "\iota"] & \mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh}),
\end{tikzcd}$$

where ι *is the tautological embedding.*

Proof. Part (a): Indeed, from the equivalence (3) we have the natural fully faithful triangulated functor

$$\mathit{Perf}(X_0, X)/\mathit{Perf}(X_0) \longrightarrow \mathsf{D}^{\mathsf{abs}}((\mathbb{A}^1, 0)\text{-}\mathsf{coh}).$$

Let us denote by $\mathsf{T} \subset \mathsf{D}^{\mathsf{abs}}((\mathbb{A}^1, 0)\text{-}\mathsf{coh})$ the essential image of this functor. For each $z_0 \in \mathbb{C} \setminus \{0\}$ we have a line $l_{z_0} := \{y = 0,\ z = z_0\} \subset X_0$. Since $l_{z_0} \subset U$ and U is smooth, the coherent sheaf $\mathcal{O}_{l_{z_0}}$ is contained in $\mathit{Perf}(X_0, X)$. Further, its image in $D^{\mathsf{b}}_{Sing}(X_0)$ corresponds to the skyscraper $\mathcal{O}_{z_0} \in \mathsf{D}^{\mathsf{abs}}((\mathbb{A}^1, 0)\text{-}\mathsf{coh})$ under the equivalence (3). It follows that the triangulated category T contains $j_*(\mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh})_{\mathsf{0\text{-}dim}})$ as a full subcategory.

Suppose that T is strictly bigger than $j_*(\mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh})_{\mathsf{0\text{-}dim}})$. Then it contains an object $\mathcal{F}_0 = \mathcal{O}_0 \oplus \mathcal{O}_0[1] \in \mathsf{D}^{\mathsf{abs}}((\mathbb{A}^1, 0)\text{-}\mathsf{coh})$, where $\mathcal{O}_0$ is the structure sheaf of the origin. Denote by $O \in X_0$ the origin $(0, 0, 0, 0)$. Then the image of the coherent sheaf $\mathcal{O}_O \in X_0\text{-}\mathsf{coh}$ in $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ corresponds to $\mathcal{F}_0$ under the equivalence (3). But the object $\mathcal{O}_O \in \mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh})$ is not relatively perfect under the inclusion $X_0 \to X$ (i.e., it does not belong to $\mathit{Perf}(X_0, X)$) since O is the singular point of X. We get a contradiction.

Thus, we have an equivalence $\mathsf{T} \simeq j_*(\mathsf{D}^{\mathsf{abs}}((\mathbb{G}_m, 0)\text{-}\mathsf{coh})_{\mathsf{0\text{-}dim}})$. This proves (a).

Part (b) follows immediately from part (a) and the equivalence (2). □

In particular, we see that the functor $\mathsf{D}^{\mathsf{abs}}((X, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{abs}}((U, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ is not essentially surjective, even up to the direct summands. Moreover, the triangulated category $\mathsf{D}^{\mathsf{abs}}((X, w)\text{-}\mathsf{coh}_{\mathsf{lf}})$ does not even have a countable set of generators.

3.4. ***Pull-backs and push-forwards in singularity categories.*** Let $f : Y \to X$ be a morphism of separated Noetherian schemes with enough vector bundles. The morphism f is said to have *finite flat dimension* if the derived inverse image functor $\mathbb{L}f^* : \mathsf{D}^-(X\text{-}\mathsf{qcoh}) \to \mathsf{D}^-(Y\text{-}\mathsf{qcoh})$ takes $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh})$ to $\mathsf{D}^{\mathsf{b}}(Y\text{-}\mathsf{qcoh})$.

In this case, the functor $\mathbb{L}f^*$ induces the inverse image functors on the triangulated categories of singularities

$$f^\circ : \mathsf{D}'_{Sing}(X) \longrightarrow \mathsf{D}'_{Sing}(Y)$$
$$f^\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Y).$$

Under the same assumption of finite flat dimension, the derived direct image functor $\mathbb{R}f_* : \mathsf{D}^{\mathsf{b}}(Y\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh})$ takes $\mathsf{D}^{\mathsf{b}}(Y\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ to $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh}_{\mathsf{fl}})$, as one can see by computing $\mathbb{R}f_*$ in terms of an affine covering of Y in the spirit of the proof of Proposition 1.9. When the scheme X has finite Krull dimension, one has $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh}_{\mathsf{fl}}) = \mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh}_{\mathsf{lf}})$, so the functor $\mathbb{R}f_*$ induces the direct image functor

$$f_\circ : \mathsf{D}'_{Sing}(Y) \longrightarrow \mathsf{D}'_{Sing}(X),$$

which is right adjoint to f°.

Whenever the morphism f is proper of finite type and has finite flat dimension, the functor $\mathbb{R}f_*$ takes $\mathsf{D}^{\mathsf{b}}(Y\text{-}\mathsf{coh})$ to $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh})$ [Grothendieck 1961, Théorème 3.2.1] and induces the direct image functor

$$f_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(Y) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X),$$

which is right adjoint to f° [Orlov 2004, paragraphs before Proposition 1.14]. More generally, for a morphism f of finite flat dimension and any closed subset $T \subset Y$ such that (a closed subscheme structure on) T is proper of finite type over X, the functor $\mathbb{R}f_*$ takes $\mathsf{D}^{\mathsf{b}}(Y\text{-}\mathsf{coh}_T)$ to $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh})$ and induces the direct image functor

$$f_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(Y, T) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X).$$

Indeed, the intersection of $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh})$ in $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{qcoh})$ is equal to $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh}_{\mathsf{lf}})$, as any complex of finite flat dimension with bounded coherent cohomology is easily seen to be perfect.

Let $Z \subset X$ and $W \subset Y$ be closed subschemes such that $\mathcal{O}_Z$ is a perfect $\mathcal{O}_X$-module, $\mathcal{O}_W$ is a perfect $\mathcal{O}_Y$-module, and $f(W) \subset Z$. Assume that both morphisms $f : Y \to X$ and $f|_W : W \to Z$ have finite flat dimensions. Then the derived inverse image functor $\mathbb{L}f|_W^* : \mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{b}}(W\text{-}\mathsf{qcoh})$ induces the inverse image functors on the triangulated categories of relative singularities

$$f^\circ : \mathsf{D}'_{Sing}(Z/X) \longrightarrow \mathsf{D}'_{Sing}(W/Y)$$
$$f^\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y).$$

Now let $Z \subset X$ be a closed subscheme; set $W = Z \times_X Y$. Denote the closed embeddings $Z \to X$ and $W \to Y$ by i and i', respectively; also let f' denote the morphism $f|_W : W \to Z$. Assume that W coincides with the derived product of Z and Y over X; i.e., $\mathbb{L}f^* i_*\mathcal{O}_Z = i'_*\mathcal{O}_W$. Assume further that $i_*\mathcal{O}_Z$ is a perfect $\mathcal{O}_X$-module; then also $i'_*\mathcal{O}_W$ is a perfect $\mathcal{O}_Y$-module.

For any $\mathcal{M} \in \mathsf{D}^{\mathsf{b}}(Y\text{-}\mathsf{qcoh})$, there is a natural morphism

$$\phi_{\mathcal{M}} : \mathbb{L}i^*\mathbb{R}f_*\mathcal{M} \longrightarrow \mathbb{R}f'_*\mathbb{L}i'^*\mathcal{M}$$

in $\mathsf{D}^{\mathsf{b}}(Z\text{-}\mathsf{qcoh})$. Using the projection formula for tensor products with perfect complexes, one easily checks that the morphism $i_*\phi_{\mathcal{M}}$ is an isomorphism. Hence, so is the morphism $\phi_{\mathcal{M}}$ since the functor i_* does not annihilate any objects of the derived category. Hence we obtain the induced functor of direct image

$$f_\circ : \mathsf{D}'_{Sing}(W/Y) \longrightarrow \mathsf{D}'_{Sing}(Z/X).$$

When the morphism f is proper of finite type, there is also the induced functor

$$f_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X).$$

Assume additionally that the morphism f has finite flat dimension; then so does the morphism f'. In this case, the functor $f_\circ : \mathsf{D}'_{Sing}(W/Y) \to \mathsf{D}'_{Sing}(Z/X)$ is right adjoint to the functor $f^\circ : \mathsf{D}'_{Sing}(Z/X) \to \mathsf{D}'_{Sing}(W/Y)$. When the morphism f is proper of finite type, the functor $f_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y) \to \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ is right adjoint to the functor $f^\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X) \to \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y)$.

Remark 3.4. In the case when Z is a Cartier divisor in X, we will construct the functor $f_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(W/Y) \to \mathsf{D}^{\mathsf{b}}_{Sing}(Z/X)$ under somewhat weaker assumptions below in Section 3.5. Namely, it will suffice that the morphism $f' : W \to Z$ be proper of finite type, while the morphism $f : Y \to Z$ need not be. A generalization to the case of proper support will also be obtained.

3.5. *Push-forwards of matrix factorizations.* Let $f : Y \to X$ be a morphism of separated Noetherian schemes with enough vector bundles, $\mathcal{L}$ be a line bundle on X, and $w \in \mathcal{L}(X)$ be a section.

Set $\mathcal{B}_X = (X, \mathcal{L}, w)$ and $\mathcal{B}_Y = (Y, f^*\mathcal{L}, f^*w)$; then there is a natural morphism of CDG-algebras $\mathcal{B}_X \to \mathcal{B}_Y$ compatible with the morphism of schemes $f : Y \to X$. Therefore, according to Section 1.8, there are the derived inverse image functors

$$\mathbb{L}f^* : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}}),$$
$$\mathbb{L}f^* : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_{\mathsf{ffd}})$$

and the derived direct image functor

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}).$$

The latter two functors are "partially adjoint" to each other.

Given a triangulated category D, we denote by $\overline{\mathsf{D}}$ its idempotent completion. By [Balmer and Schlichting 2001, Section 1], the category $\overline{\mathsf{D}}$ has the natural structure of a triangulated category.

Lemma 3.5. *For any closed subset $T \subset Y$ such that (for a closed subscheme structure on T) the morphism $f|_T : T \to X$ is proper of finite type, the functor $\mathbb{R}f_*$ takes the full subcategory $\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_T) \subset \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh})$ into the full subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$, thus defining a triangulated functor of direct image*

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_T) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}).$$

Consequently, there is the triangulated functor

$$\overline{\mathbb{R}f_*} : \overline{\mathsf{D}^{\mathsf{abs}}_T((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh})} \longrightarrow \overline{\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})}.$$

Proof. We will use the construction of the functor

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$$

similar to the one in the proof of Proposition 1.9 (see Remark 1.9). According to this construction, given a matrix factorization $\mathcal{M} \in (Y, f^*\mathcal{L}, f^*w)\text{-qcoh}$, the object $\mathbb{R}f_*\mathcal{M} \in \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-qcoh})$ is represented by the total matrix factorization $\mathbb{R}_{\{U_\alpha\}} f_*\mathcal{M}$ of the finite Čech complex $f_* C^\bullet_{\{U_\alpha\}}\mathcal{M}$ of matrix factorizations on X. The derived functor of direct image of complexes of quasicoherent sheaves $\mathbb{R}f_* : \mathsf{D}^{\mathsf{b}}(Y\text{-qcoh}) \to \mathsf{D}^{\mathsf{b}}(X\text{-qcoh})$ can be constructed in the same way.

By [Grothendieck 1961, Théorème 3.2.1], the latter functor takes $\mathsf{D}^{\mathsf{b}}(Y\text{-coh}_T)$ into $\mathsf{D}^{\mathsf{b}}(X\text{-coh})$. Hence the cohomology matrix factorizations of the finite complex of matrix factorizations $f_* C^\bullet_{\{U_\alpha\}}\mathcal{M}$ belong to $(X, \mathcal{L}, w)\text{-coh}$ when the matrix factorization $\mathcal{M}$ belongs to $(Y, f^*\mathcal{L}, f^*w)\text{-coh}_T$. It follows that the object $\mathbb{R}f_*\mathcal{M}$ belongs to $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-coh}) \subset \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-qcoh})$ in this case.

To prove the last assertion, it remains to apply Corollary 1.10(b). $\square$

Now assume that both morphisms of sheaves $w : \mathcal{O}_X \to \mathcal{L}$ and $f^*w : \mathcal{O}_Y \to f^*\mathcal{L}$ are injective. Let $X_0 \subset X$ and $Y_0 \subset Y$ denote the closed subschemes defined locally by the equations $w = 0$ and $f^*w = 0$, respectively. In this setting, we will compare the constructions of direct image functors for matrix factorizations and for the triangulated categories of relative singularities, and prove the assertions of Lemma 3.5 in a different way. Recall that in Section 3.4 we constructed the functor of direct image $f_\circ : \mathsf{D}'_{Sing}(Y_0/Y) \to \mathsf{D}'_{Sing}(X_0/X)$.

Proposition 3.5. (a) *Whenever the morphism $f_0 = f|_{Y_0} : Y_0 \to X_0$ is proper of finite type, the functor $\mathbb{R}f_*$ takes the full subcategory*

$$\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-coh}) \subset \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-qcoh})$$

into the full subcategory

$$\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-coh}) \subset \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-qcoh}),$$

thus defining a triangulated functor

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-coh}).$$

(b) *For any closed subset $T \subset Y_0$ such that (for a closed subscheme structure on T) the morphism $f_0|_T : T \to X_0$ is proper of finite type, the functor $f_\circ$ takes the full subcategory $\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y, T) \subset \mathsf{D}'_{Sing}(Y_0/Y)$ into the full subcategory $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X) \subset \mathsf{D}'_{Sing}(X_0/X)$, thus defining a triangulated functor*

$$f_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y, T) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X).$$

(c) *The equivalences of categories $\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-coh}_T) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X, T)$ from Proposition* 3.1(*a*) *and $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-coh}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ from Theorem* 2.7 *transform the direct image functor*

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-coh}_T) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-coh})$$

from Lemma 3.5 into the direct image functor $f_\circ$ from part (b).

Proof. Part (a) follows from Lemma 3.5 and Proposition 3.1(a), or alternatively, from part (b) and the proof of part (c) below. In part (b), the fact of key importance is that the functor $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X) \to \mathsf{D}'_{Sing}(X_0/X)$ is fully faithful (by Theorem 2.8). The functor $f_\circ$ takes $\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y, T)$ into $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ because the functor $\mathbb{R}f_{0*} : \mathsf{D}^{\mathsf{b}}(Y_0\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{qcoh})$ takes $\mathsf{D}^{\mathsf{b}}(Y_0\text{-}\mathsf{coh}_T)$ into $\mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{coh})$ [Grothendieck 1961]. To prove part (c), we will check that the equivalences of categories from Theorem 2.8 transform the functor $\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$ into the functor $f_\circ : \mathsf{D}'_{Sing}(Y_0/Y) \to \mathsf{D}'_{Sing}(X_0/X)$. (Together with part (b) and Proposition 3.1(a), this will also provide another proof of Lemma 3.5.)

For this purpose, extend the functor

$$\Upsilon_Y : \mathsf{D}^{\mathsf{b}}(Y_0\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh})$$

to the functor $\widetilde{\Upsilon}_Y : \mathsf{D}^+(Y_0\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh})$ in the obvious way (taking infinite direct sums of quasicoherent sheaves in the construction of the matrix factorization $\widetilde{\Upsilon}_Y(\mathcal{F}^\bullet)$). The functor $\widetilde{\Upsilon}_Y$ is well-defined since any bounded-below acyclic complex of quasicoherent sheaves is coacyclic [Positselski 2010, Lemma 2.1]. Furthermore, the functor $\widetilde{\Upsilon}_Y$ can be presented as the composition of the "periodicity summation" functor $\mathsf{D}^+(Y_0\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}((Y_0, i'^*f^*\mathcal{L}, 0)\text{-}\mathsf{qcoh})$ taking values in the coderived category of quasicoherent matrix factorizations of the zero potential on Y_0, and the functor of direct image $i'_* : \mathsf{D}^{\mathsf{co}}((Y_0, i'^*f^*\mathcal{L}, 0)\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh})$ with respect to the closed embedding i'.

The functors

$$\mathbb{R}f_{0*} : \mathsf{D}^+(Y_0\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^+(X_0\text{-}\mathsf{qcoh}),$$
$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$$

form a commutative diagram with the functors $\widetilde{\Upsilon}_X$ and $\widetilde{\Upsilon}_Y$. Indeed, the "periodicity summations" of bounded-below complexes of quasicoherent sheaves on Y_0 and X_0, taking injective resolutions to injective resolutions, obviously commute with the derived direct images with respect to f', as the direct image preserves infinite direct sums. Furthermore, the derived direct images of quasicoherent matrix factorizations are compatible with the compositions of morphisms of schemes (see Remark 1.8), and hence also commute with each other. It follows that the functors $\mathbb{R}f_*$ and $f_\circ$ agree as they should. (Alternatively, one can prove this in the way similar to the proof of Proposition 3.6 below.) $\square$

3.6. *Push-forwards for morphisms of finite flat dimension.* Let $f : Y \to X$ be a morphism of finite flat dimension between separated Noetherian schemes with enough vector bundles, $\mathcal{L}$ be a line bundle on X, and $w \in \mathcal{L}(X)$ be a section. As in Section 3.5, we have a natural morphism of CDG-algebras $\mathcal{B}_X = (X, \mathcal{L}, w) \to \mathcal{B}_Y = (Y, f^*\mathcal{L}, f^*w)$ compatible with the morphism of schemes $Y \to X$.

The quasicoherent graded algebra $\mathcal{B}_Y$ has finite flat dimension over $\mathcal{B}_X$. Therefore, according to Section 1.9, there are derived inverse image functors

$$\mathbb{L}f^* : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh})$$
$$\mathbb{L}f^* : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}),$$

the former of which is left adjoint to the functor

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$$

from Section 3.5.

Furthermore, according to Proposition 1.9, there is a derived direct image functor

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \simeq \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \simeq \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}),$$

which is right adjoint to the functor

$$\mathbb{L}f^* : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$$

from Section 3.5.

Now assume that X and Y have finite Krull dimensions. Recall that the natural triangulated functors $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \to \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ are fully faithful by Corollary 2.3(e) and (j).

As in the second half of Section 3.5, assume that both morphisms of sheaves $w : \mathcal{O}_X \to \mathcal{L}$ and $f^*w : \mathcal{O}_Y \to f^*\mathcal{L}$ are injective, and denote by $f_0 : Y_0 \to X_0$ the induced morphism between the zero loci schemes of f^*w and w. Since the morphism f has finite flat dimension, so does the morphism f_0.

Proposition 3.6. (a) *Whenever the morphism f_0 is proper of finite type, the functor*

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$$

takes the full subcategory $\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ into the full subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$. Besides, the functor

$$f_{0\circ} : \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0) \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$$

takes the full subcategory $\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0)$ into the full subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$. Both restrictions define the same triangulated functor

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}).$$

(b) *For any closed subset $T \subset Y_0$ such that* (*for a closed subscheme structure on T*) *the morphism $f_0|_T : T \to X_0$ is proper of finite type*, *the functor*

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$$

takes the full subcategory $\mathsf{D}^{\mathsf{abs}}_T((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ into the thick envelope of the full subcategory

$$\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}).$$

Besides, *the triangulated functor*

$$\overline{f_{0\circ}} : \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0, T)} \longrightarrow \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)}$$

takes the full subcategory $\mathsf{D}^{\mathsf{abs}}_T((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0, T)}$ into the thick envelope of the full subcategory

$$\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)}.$$

Both restrictions define the same triangulated functor

$$\overline{\mathbb{R}f_*} : \overline{\mathsf{D}^{\mathsf{abs}}_T((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_{\mathsf{lf}})} \longrightarrow \overline{\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}})}.$$

Proof. Both categories $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ are full triangulated subcategories of the triangulated category $\mathsf{D}'_{Sing}(X_0)$ (see Proposition 2.8 and [Orlov 2004, Proposition 1.13]). According to the proof of Corollary 3.2, the intersection of $\mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ with (the thick envelope of) $\mathsf{D}^{\mathsf{b}}_{Sing}(X_0)$ in $\mathsf{D}'_{Sing}(X_0)$ (is the thick envelope of) the subcategory $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_{\mathsf{lf}}) \subset \mathsf{D}'_{Sing}(X_0)$.

Thus it suffices to show that the direct image functor

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$$

agrees with the direct image functor $f_{0\circ} : \mathsf{D}'_{Sing}(Y_0) \to \mathsf{D}'_{Sing}(X_0)$. The latter assertion does not depend on any properness assumptions.

Recall that the derived functor $\mathbb{R}f_*$ was constructed in the proof of Proposition 1.9 in terms of the Čech complex whose terms are direct sums of the CDG-modules $f|_{V*}\mathcal{M}|_V$, where $\mathcal{M} \in \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{ffd}})$ and $V \subset Y$. The derived direct image $\mathbb{R}f_{0*} : \mathsf{D}^{\mathsf{b}}(Y_0\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{b}}(X_0\text{-}\mathsf{qcoh})$ can be constructed in the similar way; moreover, one can use for this purpose the restriction to Y_0 of an affine open covering U_α of the scheme Y.

We will make use of the flat dimension analogue of Corollary 2.6(d). Let $\widetilde{\Sigma}'_X$ and $\widetilde{\Sigma}'_Y$ denote the obvious extensions of the functors Σ' from $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}$ to the category of w-flat matrix factorizations of finite flat dimension $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-}\mathsf{fl}\cap\mathsf{ffd}}$ and from $(Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{lf}}$ to $(Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{f^*w\text{-}\mathsf{fl}\cap\mathsf{ffd}}$ (see the proofs of Proposition 2.8 and Theorem 2.7). Notice that the direct image functors $f|_{V*}$

take f^*w-flat sheaves to w-flat sheaves and $(V, f^*\mathcal{L}|_V, f^*w|_V)\text{-}\mathsf{qcoh}_{f^*w\text{-fl}\cap\mathsf{ffd}}$ to $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-fl}\cap\mathsf{ffd}}$.

Let $\mathcal{N}$ be a matrix factorization from $(Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{f^*w\text{-fl}\cap\mathsf{ffd}}$. Since the open subschemes V are presumed to be affine, there are natural isomorphisms

$$\widetilde{\Sigma}'_X(f|_{V*}\mathcal{N}|_V) \simeq f_0|_{V\cap Y_0\,*}\widetilde{\Sigma}'_Y(\mathcal{N})|_{V\cap Y_0}$$

of quasicoherent sheaves on X_0. Now it remains to use the next lemma. $\square$

Lemma 3.6. *Let $\mathcal{M}^{-n} \to \cdots \to \mathcal{M}^N$ be a finite complex of matrix factorizations from $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-fl}\cap\mathsf{ffd}}$ and $\mathcal{M}$ be its totalization. Then the complex $\widetilde{\Sigma}'(\mathcal{M}^{-n}) \to \cdots \to \widetilde{\Sigma}'(\mathcal{M}^N)$ and the quasicoherent sheaf $\widetilde{\Sigma}'(\mathcal{M})$ on X_0 represent naturally isomorphic objects in the triangulated category of singularities $\mathsf{D}'_{Sing}(X_0)$. The same applies to a finite complex of matrix factorizations from $(X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{w\text{-fl}}$, the functor Ξ, and the triangulated category of relative singularities $\mathsf{D}''_{Sing}(X_0/X)$.*

Proof. For each $-n \le p \le N$, the restriction of the matrix factorization $\mathcal{M}^p$ to the closed subscheme $X_0 \subset X$ is an unbounded complex of quasicoherent sheaves $i^*\mathcal{M}^{p,\bullet}$. By [Polishchuk and Vaintrob 2011, Lemma 1.5], this complex is acyclic.

The complex $\widetilde{\Sigma}'(\mathcal{M}^{-n}) \to \cdots \to \widetilde{\Sigma}'(\mathcal{M}^N)$ of quasicoherent sheaves on X_0 is quasi-isomorphic to the total complex of the bicomplex $\mathcal{K}^{\bullet,\bullet}$ with the terms $\mathcal{K}^{p,0} = i^*\mathcal{M}^{p,0}$, $\mathcal{K}^{p,-1} = i^*\mathcal{M}^{p,-1}$, $\mathcal{K}^{p,-2} = \ker(i^*\mathcal{M}^{p,-1} \to i^*\mathcal{M}^{p,0})$, and $\mathcal{K}^{p,q} = 0$ for $q \ne 0, -1, -2$. Similarly, the quasicoherent sheaf $\widetilde{\Sigma}'(\mathcal{M})$ on X_0 is quasi-isomorphic to the total complex of the bicomplex $\mathcal{E}^{\bullet,\bullet}$ with the terms $\mathcal{E}^{p,p} = i^*\mathcal{M}^{p,p}$, $\mathcal{E}^{p,p-1} = i^*\mathcal{M}^{p,p-1}$, $\mathcal{E}^{p,p-2} = \ker(i^*\mathcal{M}^{p,p-1} \to i^*\mathcal{M}^{p,p})$, and $\mathcal{E}^{p,q} = 0$ for $q - p \ne 0, -1, -2$.

We can assume that $N, n \ge 0$. Consider the bicomplex $\mathcal{F}^{\bullet,\bullet}$ with the terms $\mathcal{F}^{p,q} = i^*\mathcal{M}^{p,q}$ for $-n-1 \le q \le N$, $\mathcal{F}^{p,-n-2} = \ker(i^*\mathcal{M}^{p,-n-1} \to i^*\mathcal{M}^{p,-n})$, and $\mathcal{F}^{p,q} = 0$ for $q < -n-2$ or $q > N$. Then there are natural surjective morphisms of bicomplexes $\mathcal{F}^{\bullet,\bullet} \to \mathcal{K}^{\bullet,\bullet}$ and $\mathcal{F}^{\bullet,\bullet} \to \mathcal{E}^{\bullet,\bullet}$. The kernels of both morphisms are the direct sums of a finite bicomplex of quasicoherent sheaves of finite flat dimension on X_0 and a finite bicomplex of quasicoherent sheaves on X_0 with acyclic columns. Thus both morphisms become isomorphisms in $\mathsf{D}'_{Sing}(X_0)$. $\square$

Remark 3.6. One would like to have a theory of set-theoretic supports for locally free matrix factorizations of finite rank that would allow us to prove Proposition 3.6 in the way similar to the proof of Lemma 3.5. However, we do not know how to do this. In particular, we do *not* know whether every locally free matrix factorization of finite rank with category-theoretic support in T is isomorphic in the absolute derived category to a direct summand of an object represented by a coherent matrix factorization of finite flat dimension with set-theoretic support in T (cf. Corollary 1.10 and Section 3.3).

Another alternative approach to proving Proposition 3.6 would be to show that the intersection of the full subcategories $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})$ and $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh}_{\mathsf{lf}})$ in the absolute derived category $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})$ coincides with the full subcategory $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}_{\mathsf{lf}})$. We do *not* know whether this is true.

3.7. ***Duality and push-forwards.*** In the following two sections we discuss the compatibility properties of the derived direct and inverse image functors for matrix factorizations with the Serre–Grothendieck duality functors from Section 2.5.

Let X be a separated Noetherian scheme with a dualizing complex $\mathcal{D}_X^\bullet$, and let $f: Y \to X$ be a separated morphism of finite type. As usually, we set $\mathcal{D}_Y^\bullet = f^+\mathcal{D}_X^\bullet$, where f^+ is the functor denoted by $f^!$ in [Hartshorne 1966] (right adjoint to $\mathbb{R}f_*$ for proper morphisms f and left adjoint to $\mathbb{R}f_*$ for open embeddings f; see [Neeman 1996, Example 4.2] and [Hartshorne 1966, Remark before Proposition V.8.5 and Deligne's Appendix]). This formula defines the dualizing complex $\mathcal{D}_Y^\bullet$ up to a natural quasi-isomorphism only, and we presume this derived category object (as well as $\mathcal{D}_X^\bullet$) to be represented by a finite complex of injective quasicoherent sheaves.

Proposition 3.7. *Let $T \subset Y_0$ be a closed subset such that (for some closed subscheme structure on T) the morphism $f|_T : T \to X_0$ is proper. Then the derived direct image functor*

$$\overline{\mathbb{R}f_*} : \overline{\mathsf{D}_T^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh})} \longrightarrow \overline{\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh})}$$

and the similar functor for the potential $-w$ form a commutative diagram with the Serre duality functors

$$\mathcal{H}om_{X\text{-qc}}(-,\mathcal{D}_X^\bullet) : \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},-w)\text{-}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w)\text{-}\mathsf{coh}),$$
$$\mathcal{H}om_{Y\text{-qc}}(-,\mathcal{D}_Y^\bullet) : \mathsf{D}_T^{\mathsf{abs}}((Y, f^*\mathcal{L}, -f^*w)\text{-}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D}_T^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}).$$

Two proofs of Proposition 3.7 are given below. One of them is based on the theory of set-theoretic supports of coherent CDG-modules developed in Section 1.10 and the arguments similar to the proof of Lemma 3.5. It does not depend on the assumption about w and f^*w being local nonzero-divisors and does not mention the zero loci. The other proof is based on the passage to the triangulated categories of relative singularities and uses Proposition 3.5(c).

First proof. First of all, the duality functor

$$\mathcal{H}om_{Y\text{-qc}}(-,\mathcal{D}_Y^\bullet) : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, -f^*w)\text{-}\mathsf{qcoh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh})$$

obviously takes the full subcategory $\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, -f^*w)\text{-}\mathsf{coh}_T)^{\mathsf{op}}$ into

$$\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_T)$$

and vice versa. Furthermore, for any quasicoherent sheaf $\mathcal{K}$ on Y denote by $\Gamma_T\mathcal{K} \subset \mathcal{K}$ the maximal quasicoherent subsheaf with set-theoretic support in T.

Then for any matrix factorization $\mathcal{M} \in \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, -f^*w)\text{-}\mathsf{coh}_T)$, the natural morphism $\mathcal{H}om_{Y\text{-}\mathsf{qc}}(\mathcal{M},\, \Gamma_T\mathcal{D}_Y^\bullet) \to \mathcal{H}om_{Y\text{-}\mathsf{qc}}(\mathcal{M}, \mathcal{D}_Y^\bullet)$ is an isomorphism in $\mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh}_T)$.

As in the proof of Lemma 3.5, we will use the construction of the functor

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$$

similar to the one from the proof of Proposition 1.9 (see Remarks 1.8 and 1.9). Let $\{U_\alpha\}$ and $\{V_\beta\}$ be two affine open coverings of the scheme Y. For any matrix factorization $\mathcal{N} \in (Y, f^*\mathcal{L}, -f^*w)\text{-}\mathsf{qcoh}$, there is a natural morphism of bicomplexes of matrix factorizations

$$f_*C^\bullet_{\{U_\alpha\}}\,\mathcal{H}om_{Y\text{-}\mathsf{qc}}(\mathcal{N},\, \Gamma_T\mathcal{D}_Y^\bullet) \longrightarrow \mathcal{H}om_{X\text{-}\mathsf{qc}}(f_*\mathcal{N},\, f_*C^\bullet_{\{U_\alpha\}}\Gamma_T\mathcal{D}_Y^\bullet).$$

Passing to the total complexes and taking the composition with the adjunction morphism $f_*C_{\{U_\alpha\}}{}^\bullet\Gamma_T\mathcal{D}_Y^\bullet = \mathbb{R}f_*(\Gamma_T\mathcal{D}_Y^\bullet) \to \mathcal{D}_X^\bullet$, we obtain a natural morphism of complexes of matrix factorizations

$$f_*C^\bullet_{\{U_\alpha\}}\,\mathcal{H}om_{Y\text{-}\mathsf{qc}}(\mathcal{N},\, \Gamma_T\mathcal{D}_Y^\bullet) \longrightarrow \mathcal{H}om_{X\text{-}\mathsf{qc}}(f_*\mathcal{N},\, \mathcal{D}_X^\bullet)$$

(cf. [Neeman 1996, beginning of Section 6]).

Substituting $\mathcal{N} = C^\bullet_{\{V_\beta\}}\mathcal{M}$ for some $\mathcal{M} \in (Y, f^*\mathcal{L}, -f^*w)\text{-}\mathsf{qcoh}$, we get a natural morphism of bicomplexes of matrix factorizations

$$f_*C^\bullet_{\{U_\alpha\}}\,\mathcal{H}om_{Y\text{-}\mathsf{qc}}(C^\bullet_{\{V_\beta\}}\mathcal{M},\, \Gamma_T\mathcal{D}_Y^\bullet) \longrightarrow \mathcal{H}om_{X\text{-}\mathsf{qc}}(f_*C^\bullet_{\{V_\beta\}}\mathcal{M},\, \mathcal{D}_X^\bullet).$$

When $\mathcal{M}$ is a coherent matrix factorization supported set-theoretically in T, the induced morphism of the total complexes is a quasi-isomorphism of complexes of matrix factorizations by the conventional Serre–Grothendieck duality theorem for bounded derived categories of coherent sheaves and proper morphisms of schemes (see [Hartshorne 1966, Theorem VII.3.3] or [Neeman 1996, Section 6]). Hence the induced morphism of the total matrix factorizations is an isomorphism in $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$, and consequently also in $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$. □

Second proof. Assume that w and f^*w are locally nonzero-dividing sections of the respective line bundles. Let $i : X_0 \to X$ be the zero locus of w and $i' : Y_0 \to Y$ be the zero locus of f^*w. As above, we set $\mathcal{D}^\bullet_{X_0} = \mathbb{R}i^!\mathcal{D}^\bullet_X$ and $\mathcal{D}^\bullet_{Y_0} = \mathbb{R}i'^!\mathcal{D}^\bullet_Y$ [Hartshorne 1966, Proposition V.2.4], and presume all these dualizing complexes to be finite complexes of injective quasicoherent sheaves.

The duality functor

$$\mathcal{H}om_{Y\text{-}\mathsf{qc}}(-, \mathcal{D}_Y^\bullet) : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, -f^*w)\text{-}\mathsf{coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh})$$

is compatible with the restrictions to the open subscheme $Y \setminus T$ and thus identifies the full subcategories $\mathsf{D}^{\mathsf{abs}}_T((Y, f^*\mathcal{L}, -f^*w)\text{-}\mathsf{coh})^{\mathsf{op}}$ and $\mathsf{D}^{\mathsf{abs}}_T((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{coh})$. To prove the proposition, we will define the Serre duality functors on the triangulated categories of relative singularities $\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y)$ and $\mathsf{D}^{\mathsf{b}}_{Sing}(X/X_0)$, then check that the equivalences of triangulated categories $\mathbb{L}\Xi = \Upsilon^{-1}$ commute with the dualities, and finally reduce to the conventional Serre–Grothendieck duality theorem for bounded complexes of coherent sheaves.

The duality functor $\mathcal{H}om_{X_0\text{-qc}}(-, \mathcal{D}^\bullet_{X_0}) : \mathsf{D}^{\mathsf{b}}(X_0\text{-coh})^{\mathsf{op}} \to \mathsf{D}^{\mathsf{b}}(X_0\text{-coh})$ takes objects of the form $\mathbb{L}i^*\mathcal{K}^\bullet$, where $\mathcal{K}^\bullet \in \mathsf{D}^{\mathsf{b}}(X\text{-coh})$, to similar objects. Indeed, one has

$$\mathcal{H}om_{X_0\text{-qc}}(\mathbb{L}i^*\mathcal{K}^\bullet, \mathcal{D}^\bullet_{X_0}) \simeq \mathbb{R}i^!\,\mathcal{H}om_{X\text{-qc}}(\mathcal{K}^\bullet, \mathcal{D}^\bullet_X)$$

[loc. cit., Proposition V.8.5] and $\mathbb{R}i^! \simeq \mathcal{L}|_{X_0}[-1] \otimes_{\mathcal{O}_{X_0}} \mathbb{L}i^*$ (see the proof of Theorem 2.7). Therefore, we have the induced duality functor

$$\mathcal{H}om_{X_0\text{-qc}}(-, \mathcal{D}^\bullet_{X_0}) : \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X).$$

Similarly, the duality functor

$$\mathcal{H}om_{Y_0\text{-qc}}(-, \mathcal{D}^\bullet_{Y_0}) : \mathsf{D}^{\mathsf{b}}(Y_0\text{-coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{b}}(Y_0\text{-coh})$$

takes the full subcategory $\mathsf{D}^{\mathsf{b}}(Y_0\text{-coh}_T)^{\mathsf{op}}$ into $\mathsf{D}^{\mathsf{b}}(Y_0\text{-coh}_T)$ and $\mathit{Perf}_T(Y_0/Y)^{\mathsf{op}}$ into $\mathit{Perf}_T(Y_0/Y)$. Hence the induced duality functor

$$\mathcal{H}om_{Y_0\text{-qc}}(-, \mathcal{D}^\bullet_{Y_0}) : \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y, T)^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y, T).$$

Checking that the equivalence of categories $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-coh}) \simeq \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ commutes with the dualities is easily done using the functor Υ. It suffices to notice the functorial quasi-isomorphism $\mathcal{H}om_{X\text{-qc}}(i_*\mathcal{F}^\bullet, \mathcal{D}^\bullet_X) \simeq i_*\mathcal{H}om_{X_0\text{-qc}}(\mathcal{F}^\bullet, \mathcal{D}^\bullet_{X_0})$ for any complex $\mathcal{F}^\bullet \in \mathsf{D}^{\mathsf{b}}(X_0\text{-coh})$ [loc. cit., Theorem III.6.7]. The same applies to the equivalence of categories

$$\overline{\mathsf{D}^{\mathsf{abs}}_T((Y, f^*\mathcal{L}, f^*w)\text{-coh})} \simeq \overline{\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y, T)}.$$

Furthermore, by Proposition 3.5(c), the equivalences of categories $\mathbb{L}\Xi = \Upsilon^{-1}$ transform the derived direct image functor

$$\overline{\mathbb{R}f_*} : \overline{\mathsf{D}^{\mathsf{abs}}_T((Y, f^*\mathcal{L}, f^*w)\text{-coh})} \longrightarrow \overline{\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-coh})}$$

into (the idempotent closure of) the direct image functor $f_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y, T) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$.

Finally, the direct image functor $f_\circ : \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/Y, T) \to \mathsf{D}^{\mathsf{b}}_{Sing}(X_0/X)$ commutes with the Serre duality functors since so do the derived direct image functors $\mathbb{R}f|_{T*} : \mathsf{D}^{\mathsf{b}}(\widetilde{T}\text{-coh}) \to \mathsf{D}^{\mathsf{b}}(X_0\text{-coh})$ for all the closed subscheme structures $\widetilde{T} \subset Y_0$ on the closed subset T and the similar functors related to the closed embeddings $\widetilde{T}' \to \widetilde{T}''$ of various such subscheme structures into each other. This is the conventional Serre–Grothendieck duality theorem for proper morphisms of schemes. $\square$

3.8. ***Duality and pull-backs.*** Let X be a separated Noetherian scheme with a dualizing complex $\mathcal{D}_X^\bullet$ and $f : Y \to X$ be a separated morphism of finite type; set $\mathcal{D}_Y^\bullet = f^+\mathcal{D}_X^\bullet$. Let $\mathcal{L}$ be a line bundle on X and $w \in \mathcal{L}(X)$ be a section.

Let us first suppose that the morphism f is smooth of relative dimension n. Then the functor $f^+ : \mathsf{D}^+(X\text{-qcoh}) \to \mathsf{D}^+(Y\text{-qcoh})$ is naturally isomorphic to $\omega_{Y/X}[n] \otimes_{\mathcal{O}_Y} f^*$, where $\omega_{Y/X}$ is the line bundle of relative top forms.

In particular, $\mathcal{D}_Y^\bullet \simeq \omega_{Y/X}[n] \otimes_{\mathcal{O}_Y} f^*\mathcal{D}_X^\bullet$ (where $f^*\mathcal{D}_X^\bullet$ is also presumed to have been replaced by a complex of injectives). Then it is clear that the equivalences of categories

$$\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} - : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-qcoh}),$$
$$f^*\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_Y} - : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-qcoh})$$

from Section 2.5 transform the inverse image functor for flat matrix factorizations $f^* : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-qcoh}_{\mathsf{fl}})$ into the (underived, as the morphism f is flat) inverse image functor for quasicoherent matrix factorizations $f^* : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-qcoh}) \to \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-qcoh})$.

Furthermore, for any quasicoherent matrix factorization $\mathcal{M}$ on X there is a natural morphism of finite complexes of matrix factorizations $f^*\,\mathcal{H}om_{X\text{-qc}}(\mathcal{M}, \mathcal{D}_X^\bullet) \to \mathcal{H}om_{Y\text{-qc}}(f^*\mathcal{M}, f^*\mathcal{D}_X^\bullet)$ on Y. When $\mathcal{M}$ is a coherent matrix factorization, this is a quasi-isomorphism of complexes of matrix factorizations (since the similar assertion holds for coherent sheaves [Hartshorne 1966, Proposition II.5.8]), so the related morphism of total matrix factorizations has an absolutely acyclic cone. Thus the antiequivalences of categories

$$\mathcal{H}om_{X\text{-qc}}(-, \mathcal{D}_X^\bullet) : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, -w)\text{-coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-coh}),$$
$$\mathcal{H}om_{Y\text{-qc}}(-, f^*\mathcal{D}_X^\bullet) : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, -f^*w)\text{-coh})^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-coh})$$

form a commutative diagram with the inverse image functors f^* for coherent matrix factorizations.

Now suppose that f is a proper morphism of finite type. The following theorem describes the compatibility property of the covariant Serre–Grothendieck duality with the inverse images of matrix factorizations (cf. [Positselski 2012, Theorem 5.15.3], where a similar result is proven for complexes of quasicoherent sheaves).

Theorem 3.8. *The equivalences of categories*

$$\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} - : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-qcoh}),$$
$$\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} - : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-qcoh})$$

transform the inverse image functor

$$f^* : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-qcoh}_{\mathsf{fl}})$$

into the functor $f^! : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh})$ *right adjoint to the direct image functor*

$$\mathbb{R}f_* : \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh})$$

(*see the end of Section* 1.8).

Proof. For any quasicoherent matrix factorization $\mathcal{N}$ on Y and any flat quasicoherent matrix factorization $\mathcal{E}$ on X, we have to construct an isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w)\text{-}\mathsf{qcoh})}(\mathbb{R}f_*\mathcal{N},\ \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})& \\ \simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((Y,f^*\mathcal{L},f^*w)\text{-}\mathsf{qcoh})}&(\mathcal{N},\ \mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{E}).\end{aligned}$$

The composition

$$\begin{aligned}\mathrm{Hom}_Y(\mathcal{N},\ \mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{E}) &\longrightarrow \mathrm{Hom}_X(\mathbb{R}f_*\mathcal{N},\ \mathbb{R}f_*(\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{E})) \\ \simeq \mathrm{Hom}_X(\mathbb{R}f_*\mathcal{N},\ f_*\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_X} \mathcal{E}) &\longrightarrow \mathrm{Hom}_X(\mathbb{R}f_*\mathcal{N},\ \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{E})\end{aligned}$$

provides a morphism from the right-hand to the left-hand side. Here all the Hom functors are taken in the coderived categories of quasicoherent matrix factorizations on Y and X; the middle isomorphism holds since $\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} f^*\mathcal{E}$ is an injective matrix factorization on Y (so the derived direct image can be computed for it by applying the underived direct image functor f_* termwise) and by the projection formula; the last morphism is induced by the adjunction $f_*\mathcal{D}_Y^\bullet \to \mathcal{D}_X^\bullet$.

Furthermore, on both sides of the desired isomorphism we have injective matrix factorizations in the second arguments of the Hom functors; hence the Hom can be computed in the homotopy category of matrix factorizations instead of the coderived category in both cases. Finally, one can assume $\mathcal{N}$ to be an injective matrix factorization, too, and compute $\mathbb{R}f_*\mathcal{N} = f_*\mathcal{N}$ termwise (alternatively, one could use the Čech construction). Similarly, the tensor products in the second arguments are totalizations of termwise tensor products.

Now one can fix the components involved for both matrix factorizations $\mathcal{N}$ and $\mathcal{E}$, obtaining a morphism of finite complexes of abelian groups of the same kind as above, but related to (one-term) complexes of quasicoherent sheaves rather than matrix factorizations. The latter is an isomorphism by [Positselski 2012, Theorem 5.15.3]. It remains to notice that the totalization of an acyclic finite complex of (unbounded) complexes of abelian groups is acyclic. □

The next corollary is a matrix factorization version of the main result of Deligne's appendix to [Hartshorne 1966] (see also [Positselski 2012, Section 5.16]).

Corollary 3.8. *For any morphism of finite type between separated Noetherian schemes with dualizing complexes* $f : Y \to X$*, a line bundle* $\mathcal{L}$ *on* X*, and a section*

$w \in \mathcal{L}(X)$, *one can define a triangulated functor*

$$f^+ : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w))\text{-}\mathsf{qcoh})$$

in such a way that

(i) *for an open embedding f, one has $f^+ = f^*$, and more generally, for a smooth morphism f of relative dimension n one has $f^+ = \omega_{Y/X}[n] \otimes_{\mathcal{O}_Y} f^*$;*

(ii) *for a proper morphism f, the functor $f^+ = f^!$ is right adjoint to $\mathbb{R}f_*$;*

(iii) *the construction is compatible with the compositions of the morphisms f.*

Proof. It suffices to define $f^+ : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh})$ as the functor corresponding to the inverse image of flat quasicoherent matrix factorizations $f^* : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$ under the identifications of categories

$$\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} - : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w)\text{-}\mathsf{qcoh}),$$
$$\mathcal{D}_Y^\bullet \otimes_{\mathcal{O}_Y} - : \mathsf{D}^{\mathsf{abs}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{co}}((Y, f^*\mathcal{L}, f^*w)\text{-}\mathsf{qcoh}),$$

where $\mathcal{D}_X^\bullet$ is any dualizing complex on X and $\mathcal{D}_Y^\bullet = f^+\mathcal{D}_X^\bullet$. ☐

Appendix A. Quasicoherent graded modules

A.1. ***Flat quasicoherent sheaves.*** I am grateful to A. Neeman for suggesting that a result of the following kind can be proven without much difficulty.

Lemma A.1. *On any quasicompact semiseparated scheme, any quasicoherent sheaf is the quotient sheaf of a flat quasicoherent sheaf.*

Proof. Let X be our scheme. Assume that a quasicoherent sheaf $\mathcal{M}$ over X is flat over an open subscheme $V \subset X$; given an affine open subscheme $U \subset X$, we will construct a surjective morphism $\mathcal{N} \to \mathcal{M}$ onto $\mathcal{M}$ from a quasicoherent sheaf $\mathcal{N}$ over X that is flat over $U \cup V$. Let j denote the embedding $U \to X$. There exists a surjective morphism onto $j^*\mathcal{M}$ from a flat quasicoherent sheaf $\mathcal{F}$ over U; let $\mathcal{K}$ denote the kernel of this morphism of sheaves.

Since the morphism $j : U \to X$ is affine and flat, the functor j_* is exact and preserves flatness. Consider the pull-back of the exact triple $j_*\mathcal{K} \to j_*\mathcal{F} \to j_*j^*\mathcal{M}$ with respect to the morphism $\mathcal{M} \to j_*j^*\mathcal{M}$; denote the middle term of the resulting exact triple by $\mathcal{N}$. One has $\mathcal{N}|_U = \mathcal{F}|_U$, so $\mathcal{N}$ is flat over U. Furthermore, the sheaf $j^*\mathcal{M}$ is flat over $V \cap U$; hence, so is the sheaf $\mathcal{K}$. The embedding $U \cap V \to V$ is an affine flat morphism, so the sheaf $j_*\mathcal{K}$ is flat over V. From the exact triple $j_*\mathcal{K} \to \mathcal{N} \to \mathcal{M}$, we conclude that $\mathcal{N}$ is flat over V. ☐

It follows immediately that any quasicoherent graded module over a quasicoherent graded algebra $\mathcal{B}$ over X is a quotient module of a flat quasicoherent graded module.

A.2. ***Locally projective quasicoherent graded modules.*** The following result is essentially due to Raynaud and Gruson [1971] (for a discussion, see [Drinfeld 2006, Section 2]); here we just briefly explain how to deduce the formulation that interests us from their assertions.

Theorem A.2. *Let X be an affine scheme and $\{U_\alpha\}$ be its finite affine covering. Let $\mathcal{B}$ be a quasicoherent graded algebra over X and $\mathcal{P}$ be a quasicoherent graded module over $\mathcal{B}$. Then the graded $\mathcal{B}(X)$-module $\mathcal{P}(X)$ is projective if and only if the graded $\mathcal{B}(U_\alpha)$-module $\mathcal{P}(U_\alpha)$ is projective for every α.*

Proof. First of all, a graded module P over a graded ring B is projective if and only if it is projective as an ungraded module. Indeed, if P is graded projective, then it is a homogeneous direct summand of a free graded module; hence P is also ungraded projective. Conversely, pick a homogeneous (of degree 0) surjective homomorphism $F \to P$ onto a given graded module P from a free graded module F. If P is ungraded projective, this homomorphism has a (perhaps nonhomogeneous) section s, and the homogeneous component of s of degree 0 provides a homogeneous section. Hence it suffices to consider ungraded modules over an ungraded quasicoherent algebra $\mathcal{B}$.

It is clear that if $\mathcal{P}(X)$ is a projective $\mathcal{B}(X)$-module, then $\mathcal{P}(V)$ is a projective $\mathcal{B}(V)$-module for any affine open subscheme $V \subset X$. Conversely, assume that the $\mathcal{B}(U_\alpha)$-module $\mathcal{P}(U_\alpha)$ is projective for every α. Then by the result of [Kaplansky 1958], the $\mathcal{B}(U_\alpha)$-modules $\mathcal{P}(U_\alpha)$ are direct sums of countably generated modules, and it follows easily that so is the $\mathcal{B}(X)$-module $\mathcal{P}(X)$ (essentially, since a connected graph with an at most countable set of edges at each vertex has a countable number of vertices). Hence we can assume the $\mathcal{B}(X)$-module $\mathcal{P}(X)$ to be countably generated.

Besides, the $\mathcal{B}(U_\alpha)$-modules $\mathcal{P}(U_\alpha)$ are flat; hence so is the $\mathcal{B}(X)$-module $\mathcal{P}(X)$. By [Raynaud and Gruson 1971, Corollaire II.2.2.2], it remains to show that the $\mathcal{B}(X)$-module $\mathcal{P}(X)$ satisfies the Mittag-Leffler condition; this can be easily deduced from the similar property of the $\mathcal{B}(U_\alpha)$-modules $\mathcal{P}(U_\alpha)$ using the formulation of this condition given in Proposition II.2.1.4(iii) or Propositions II.2.1.4(ii) and II.2.1.1(i) of [Raynaud and Gruson 1971] (cf. Sections II.2.5 and II.3.1 of the same paper). □

A.3. ***Injective quasicoherent graded modules.*** The following result is a noncommutative generalization of a theorem of Hartshorne [1966, Theorem II.7.18] about injective quasicoherent sheaves on Noetherian schemes. Our proof method, based on the Artin–Rees lemma, is different from the one in [loc. cit.].

Theorem A.3. *Let $\mathcal{B}$ be a Noetherian quasicoherent graded algebra over a Noetherian scheme X. Then any injective object in the category of quasicoherent graded left modules over $\mathcal{B}$ is also an injective object of the category of arbitrary sheaves of graded $\mathcal{B}$-modules over X.*

Consequently, the restriction $\mathcal{J}|_U$ of an injective quasicoherent graded module $\mathcal{J}$ over $\mathcal{B}$ to an open subscheme $U \subset X$ is an injective quasicoherent graded module over $\mathcal{B}|_U$. Conversely, if U_α is an open covering of X and the quasicoherent graded $\mathcal{B}|_{U_\alpha}$-modules $\mathcal{J}|_{U_\alpha}$ are injective, then a quasicoherent graded $\mathcal{B}$-module $\mathcal{J}$ is injective. Besides, the underlying sheaf of graded abelian groups of any injective quasicoherent graded $\mathcal{B}$-module $\mathcal{J}$ is flabby.

Proof. First of all, notice that the abelian category $\mathcal{B}$-qcoh of quasicoherent graded modules over $\mathcal{B}$ is a locally Noetherian Grothendieck category with coherent graded modules forming the subcategory of Noetherian generators [Hartshorne 1977, Exercise II.5.15]; so, in particular, $\mathcal{B}$-qcoh has enough injectives and the assertions of Theorem A.3 are not vacuous. The category of sheaves of graded $\mathcal{B}$-modules $\mathcal{B}$-mod has similar properties, with the extensions by zero of the restrictions of $\mathcal{B}$ to (small) open subschemes of X forming a set of Noetherian generators [Hartshorne 1966, Theorem II.7.8].

Secondly, let us check that the main result in the first paragraph implies the assertions in the second one. Indeed, injective sheaves of graded $\mathcal{B}$-modules have all the properties we are interested in. They remain injective after being restricted to an open subscheme since the extension by zero from an open subscheme is an exact functor. They are flabby since given two open subschemes $U \subset V \subset X$ and j_U, j_V being their identity embeddings $U, V \to X$, the morphism of sheaves of graded $\mathcal{B}$-modules $j_{U!}\mathcal{B}|_U \to j_{V!}\mathcal{B}|_V$ is injective. And their property is local [loc. cit., Lemma II.7.16] because sheaves of graded $\mathcal{B}$-modules supported inside one of the subschemes U_α form a set of generators of the category $\mathcal{B}$-mod.

Now let $\mathcal{J}$ be an injective quasicoherent graded module over $\mathcal{B}$. To prove the main assertion, we have to show that for any open subscheme $U \subset X$ and a subsheaf of graded $\mathcal{B}$-modules $\mathcal{G} \subset j_{U!}\mathcal{B}|_U$, any homogeneous morphism of sheaves of graded $\mathcal{B}$-modules $\mathcal{G} \to \mathcal{J}$ can be extended to a similar morphism $j_{U!}\mathcal{B}|_U \to \mathcal{J}$. Indeed, $\mathcal{G}$ is a subsheaf of graded $\mathcal{B}$-modules in the coherent graded $\mathcal{B}$-module $\mathcal{B}$; hence according to the following proposition, there exists a quasicoherent graded $\mathcal{B}$-module $\mathcal{G} \subset \mathcal{F} \subset \mathcal{B}$ such that the morphism $\mathcal{G} \to \mathcal{J}$ can be extended to a homogeneous morphism of quasicoherent graded $\mathcal{B}$-modules $\mathcal{F} \to \mathcal{J}$.

Since $\mathcal{J}$ is injective in $\mathcal{B}$-qcoh, the latter morphism can in turn be extended to a similar morphism $\mathcal{B} \to \mathcal{J}$. Restricting to $j_{U!}\mathcal{B}|_U$, we obtain the desired morphism of sheaves of graded $\mathcal{B}$-modules $j_{U!}\mathcal{B}|_U \to \mathcal{J}$. □

Proposition A.3. *In the assumptions of Theorem A.3, let $\mathcal{E}$ be a coherent graded left $\mathcal{B}$-module, $\mathcal{G} \subset \mathcal{E}$ be a subsheaf of graded $\mathcal{B}$-modules, $\mathcal{M}$ be a quasicoherent graded $\mathcal{B}$-module, and $\phi : \mathcal{G} \to \mathcal{M}$ be a morphism of sheaves of graded $\mathcal{B}$-modules. Then there exists a coherent graded $\mathcal{B}$-module $\mathcal{G} \subset \mathcal{F} \subset \mathcal{E}$ such that the morphism ϕ can be extended to $\mathcal{F}$.*

Proof. Before proving Proposition A.3, let us reformulate its conclusion as follows. In the same setting, there exists a quasicoherent graded $\mathcal{B}$-module $\mathcal{K}$ together with an injective morphism $\mathcal{M} \to \mathcal{K}$ and a morphism $\mathcal{E} \to \mathcal{K}$ forming a commutative diagram with the embedding $\mathcal{G} \to \mathcal{E}$ and the morphism $\phi : \mathcal{G} \to \mathcal{M}$. Indeed, if a coherent $\mathcal{B}$-module $\mathcal{F}$ exists, one can take $\mathcal{K}$ to be the fibered coproduct of $\mathcal{E}$ and $\mathcal{M}$ over $\mathcal{F}$; conversely, if a quasicoherent $\mathcal{B}$-module $\mathcal{K}$ exists, one can take $\mathcal{F}$ to be the full preimage of $\mathcal{M} \subset \mathcal{K}$ under the morphism $\mathcal{E} \to \mathcal{K}$. Notice also that one can always replace $\mathcal{M}$ with its sufficiently big coherent graded $\mathcal{B}$-submodule.

Now let us state the version of Artin–Rees lemma that we will use.

Lemma A.3. *In the assumptions of Theorem A.3, let $\mathcal{M}$ be a coherent graded $\mathcal{B}$-module, $\mathcal{N} \subset \mathcal{M}$ a coherent graded $\mathcal{B}$-submodule, and $Z \subset X$ a closed subscheme with the sheaf of ideals $\mathcal{I}_Z \subset \mathcal{O}_X$. Then for any $n \geq 0$, there exists $m \geq 0$ such that the intersection $\mathcal{I}_Z^m \mathcal{M} \cap \mathcal{N}$ is contained in $\mathcal{I}_Z^n \mathcal{N}$.*

Proof. Clearly, the question is local, so it suffices to consider the case of an affine scheme X. Then (the graded version of) the Artin–Rees lemma for ideals generated by central elements in noncommutative Noetherian rings [Goodearl and Warfield 1989, Theorem 13.3] applies. $\square$

Being a Noetherian object, the sheaf of graded $\mathcal{B}$-modules $\mathcal{G}$ is generated by a finite number of homogeneous sections $s_n \in \mathcal{G}(U_n)$, where $U_n \subset X$ are some open subschemes. If all of these subschemes coincide with X, the sheaf $\mathcal{G}$, being a subsheaf of a coherent sheaf generated by global sections, is itself coherent, so there is nothing to prove. In the general case, we will argue by induction on the number of open subschemes U_n that are not equal to X.

Let $U = U_1 \subsetneq X$ be one such open subscheme, and $T = X \setminus U$ be its closed complement. We can assume that $\mathcal{M}$ is a coherent graded $\mathcal{B}$-module. Let $\mathcal{N}$ denote its maximal coherent graded $\mathcal{B}$-submodule supported set-theoretically in T. Applying Lemma A.3 to $\mathcal{N} \subset \mathcal{M}$, we conclude that there is a closed subscheme structure $i : Z \to X$ on T such that the morphism $\mathcal{N} \to i_*i^*\mathcal{M}$ is injective. Consequently, so is the morphism $\mathcal{M} \to i_*i^*\mathcal{M} \oplus j_*j^*\mathcal{M}$, where j denotes the open embedding $U \to X$.

Let us show that there is a thicker closed subscheme structure $i' : Z' \to X$ on T such that the kernel of the morphism of sheaves $i'_*i'^*\mathcal{G} \to i'_*i'^*\mathcal{E}$ is contained in the kernel of the morphism of sheaves $i'_*i'^*\mathcal{G} \to i_*i^*\mathcal{G}$. Indeed, there exists a finite collection of subsheaves of graded $\mathcal{B}$-modules in $\mathcal{G}$, each of them an extension by zero of a coherent graded $\mathcal{B}|_V$-module from some open subscheme $V \subset X$ such that the stalk of $\mathcal{G}$ at each point of X coincides with the stalk of one of these subsheaves. So the assertion reduces to the case when $\mathcal{G}$ is a coherent graded $\mathcal{B}$-submodule in $\mathcal{E}$ when it is an equivalent reformulation of Lemma A.3.

Let $\mathcal{H} \subset i'^*\mathcal{E}$ denote the image of the morphism of sheaves of graded $i'^*\mathcal{B}$-modules $i'^*\mathcal{G} \to i'^*\mathcal{E}$ over the scheme Z'. Let $\iota : Z \to Z'$ be the natural closed embedding. Then, according to the above, the morphism of sheaves of graded $i'^*\mathcal{B}$-modules $i'^*\mathcal{G} \to \iota_* i^*\mathcal{G}$ induces a morphism $\mathcal{H} \to \iota_* i^*\mathcal{G}$.

The sheaf of graded $i'^*\mathcal{B}$-modules $\mathcal{H}$ is generated by the images of the restrictions of the sections s_n, $n \geq 2$, to the closed subschemes $Z' \cap U_n \subset U_n$. Hence the induction assumption is applicable to $\mathcal{H}$, and we can conclude that there exists a quasicoherent graded $i'^*\mathcal{B}$-module $\mathcal{K}$ on the scheme Z' together with an injective morphism $\iota_* i^*\mathcal{M} \to \mathcal{K}$ and a morphism $i'^*\mathcal{E} \to \mathcal{K}$ forming a commutative diagram with the embedding $\mathcal{H} \to i'^*\mathcal{E}$ and the composition $\mathcal{H} \to \iota_* i^*\mathcal{G} \to \iota_* i^*\mathcal{M}$.

Similarly, the sheaf of graded $\mathcal{B}|_U$-modules $j^*\mathcal{G}$ is generated by the restrictions of the sections s_n to the open subschemes $U_1 \cap U_n \subset U_n$, among which the (restriction of) the section s_1 is a global section over $U = U_1$. Hence the induction assumption is applicable to $j^*\mathcal{G}$, and there exists a quasicoherent graded $\mathcal{B}|_U$-module $\mathcal{L}$ together with an injective morphism $j^*\mathcal{M} \to \mathcal{L}$ and a morphism $j^*\mathcal{E} \to \mathcal{L}$ forming a commutative diagram with the embedding $j^*\mathcal{G} \to j^*\mathcal{E}$ and the morphism $j^*\mathcal{G} \to j^*\mathcal{M}$.

Now the injective morphism $\mathcal{M} \to i'_*\mathcal{K} \oplus j_*\mathcal{L}$ (whose first component is the composition $\mathcal{M} \to i_* i^*\mathcal{M} \simeq i'_* \iota_* i^*\mathcal{M} \to i'_*\mathcal{K}$) and the morphism $\mathcal{E} \to i'_*\mathcal{K} \oplus j_*\mathcal{L}$ provide the desired commutative diagram of morphisms of sheaves of graded $\mathcal{B}$-modules over X. □

Appendix B. Hochschild (co)homology of matrix factorizations

This appendix complements the paper [Polishchuk and Positselski 2012] in two ways. Section B.1 contains some modifications and improvements of the main results of [loc. cit.] generally, and as applied to locally free matrix factorizations of finite rank in particular. The main thrust consists of replacing the finite homological dimension conditions in [loc. cit.] with the Noetherianness conditions to the (limited) extent possible.

Section B.2, on the other hand, presents an elementary approach to the computation of Hochschild (co)homology of coherent matrix factorizations, entirely unrelated to that in [loc. cit.] and not based on any notion of Hochschild (co)homology of the second kind, but rather on the Serre–Grothendieck duality theory.

B.1. ***Locally free matrix factorizations of finite rank.*** In Sections B.1.1–B.1.4, we start with a bit of categorical nonsense, following the lines of [Polishchuk and Positselski 2012, Sections 3.3–3.5], but with the additional coherence/Noetherianness conditions imposed from the very beginning. We use the notation from [loc. cit.] rather than that of the main body of this paper. Then in Section B.1.5, we turn to locally free matrix factorizations of finite rank over certain possibly singular, affine algebraic varieties. Finally, Section B.1.6 presents an improvement over the discussion of matrix factorizations over smooth affine varieties in [loc. cit., Section 4.8]. An example of an application of our techniques to nonaffine varieties can be found in the preprint [Efimov 2012].

B.1.1. *Coherent and Noetherian CDG-categories.* Let $(\Gamma, \sigma, \mathbf{1})$ be a grading group data [Polishchuk and Positselski 2012, Section 1.1] and $B^\#$ be a small Γ-graded preadditive category [Positselski 2011a, Section A.1]. Both left and right Γ-graded $B^\#$-modules form abelian categories.

A Γ-graded $B^\#$-module is said to be *finitely generated* (respectively, *finitely presented*) if it is a quotient module of a finitely generated free Γ-graded $B^\#$-module [Polishchuk and Positselski 2012, Section 1.5] (respectively, the cokernel of a morphism of finitely generated free Γ-graded $B^\#$-modules).

A Γ-graded preadditive category $B^\#$ is called *left Noetherian* if any submodule of a finitely generated Γ-graded left $B^\#$-module is finitely generated, or equivalently, if the abelian category of Γ-graded left $B^\#$-modules is locally Noetherian. A Γ-graded preadditive category $B^\#$ is called *left coherent* if any submodule of a finitely presented Γ-graded left $B^\#$-module is finitely presented.

Let B be a small (Γ-graded) CDG-category [loc. cit., Section 1.2] and $B^\#$ be its underlying Γ-graded preadditive category. Following [loc. cit.], we denote the DG-categories of left and right CDG-modules over B by $B\text{-mod}^{\mathrm{cdg}}$ and $\text{mod}^{\mathrm{cdg}}\text{-}B$. The DG-subcategories of left CDG-modules whose underlying Γ-graded $B^\#$-modules are flat or injective are denoted by $B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fl}}$ and $B\text{-mod}^{\mathrm{cdg}}_{\mathrm{inj}} \subset B\text{-mod}^{\mathrm{cdg}}$. Similarly, the DG-subcategories of left and right CDG-modules over B whose underlying Γ-graded $B^\#$-modules are projective and finitely generated are denoted by $B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}}$ and $\text{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B$.

Assuming that the Γ-graded category $B^\#$ is left Noetherian, the DG-subcategory of left CDG-modules whose underlying Γ-graded $B^\#$-modules are finitely generated is denoted by $B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}} \subset B\text{-mod}^{\mathrm{cdg}}$. Assuming that the Γ-graded category $B^\#$ is right coherent, the DG-subcategory of right CDG-modules whose underlying Γ-graded $B^\#$-modules are finitely presented is denoted by $\text{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B$.

The coderived and contraderived categories of left CDG-modules over B are denoted by $\mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})$ and $\mathrm{D}^{\mathrm{ctr}}(B\text{-mod}^{\mathrm{cdg}})$, respectively [loc. cit., Section 3.2]. Assuming that the Γ-graded category $B^\#$ is right coherent, the class of flat Γ-graded left B-modules [loc. cit., Section 2.2] is closed under infinite products, so the contraderived category $\mathrm{D}^{\mathrm{ctr}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fl}})$ is well-defined. The homotopy category of the DG-category $B\text{-mod}^{\mathrm{cdg}}_{\mathrm{inj}}$ is denoted, as usually, by $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{inj}})$.

In the respective assumptions of left Noetherianness or right coherence of the Γ-graded category $B^\#$, the absolute derived categories of CDG-modules with finitely generated or finitely presented underlying Γ-graded $B^\#$-modules are denoted by $\mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ and $\mathrm{D}^{\mathrm{abs}}(\text{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B)$, respectively.

B.1.2. *Derived functors of the second kind.* Let k be a commutative ring and B be a small k-linear CDG-category. Assume that the Γ-graded category $B^{\#}$ is left Noetherian. Let L and M be left CDG-modules over B; suppose that the Γ-graded left $B^{\#}$-module $L^{\#}$ underlying the CDG-module L over B is finitely generated.

As in [Polishchuk and Positselski 2012, §§ 2.1–2], we denote by $Z^0(B\text{-mod}^{\mathrm{cdg}})$ and $Z^0(\mathrm{mod}^{\mathrm{cdg}}\text{-}B)$ the abelian categories of left and right CDG-modules over B. Let $Z^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}}) \subset Z^0(B\text{-mod}^{\mathrm{cdg}})$ and $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}}) \subset H^0(B\text{-mod}^{\mathrm{cdg}})$ denote the abelian and homotopy categories of left CDG-modules over B with finitely generated underlying Γ-graded $B^{\#}$-modules, and $Z^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B) \subset Z^0(\mathrm{mod}^{\mathrm{cdg}}\text{-}B)$ and $H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B) \subset H^0(\mathrm{mod}^{\mathrm{cdg}}\text{-}B)$ be the similar categories of right CDG-modules with finitely presented underlying Γ-graded modules.

Let $J^{\bullet}$ be a right resolution of M in $Z^0(B\text{-mod}^{\mathrm{cdg}})$ such that the Γ-graded left $B^{\#}$-modules $J^{i\#}$ are injective, and let J be the total CDG-module of the complex of CDG-modules $J^{\bullet}$ constructed by taking infinite direct sums along the diagonals. Then the complex $\mathrm{Tot}^{\oplus}\,\mathrm{Hom}^B(L, J^{\bullet})$ computing $\mathrm{Ext}_B^{II}(L, M)$ [loc. cit., Section 2.2] is isomorphic to the complex $\mathrm{Hom}^B(L, J)$ [loc. cit., formula (6)], which computes the k-modules of morphisms from L into $M[*]$ in the coderived category $\mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})$ [Positselski 2011b, Theorems 3.5(a) and 3.7]. Thus,

$$H^*\,\mathrm{Ext}_B^{II}(L, M) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})}(L, M[*]).$$

Just as in [Polishchuk and Positselski 2012, Section 3.3], one can lift this isomorphism from the level of cohomology modules to that of the derived category $\mathrm{D}(k\text{-mod})$ in the following way. Consider the functor

$$\mathrm{Hom}^B : H^0(B\text{-mod}^{\mathrm{cdg}})^{\mathrm{op}} \times H^0(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-mod}),$$

and restrict it to the full subcategory $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{inj}})$ in the second argument. This restriction factorizes through the coderived category $\mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})$ in the first argument. Taking into account [Positselski 2011b, Theorem 3.7], we obtain a right derived functor

$$\mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})^{\mathrm{op}} \times \mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-mod}).$$

Restricting to the full subcategory $\mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})^{\mathrm{op}} \subset \mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})^{\mathrm{op}}$ [loc. cit., Theorem 3.11.1] in the first argument, we have the derived functor

$$\mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})^{\mathrm{op}} \times \mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-mod}). \tag{4}$$

The composition of this functor with the localization functors

$$Z^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}}) \longrightarrow \mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}}) \quad \text{and} \quad Z^0(B\text{-mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})$$

agrees with the derived functor Ext_B^{II} where the former is defined.

Now assume that the Γ-graded category $B^\#$ is right coherent. Consider the functor [Polishchuk and Positselski 2012, formula (5)]

$$\otimes_B : H^0(\mathrm{mod}^{\mathrm{cdg}}\text{-}B) \times H^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-}\mathrm{mod})$$

and restrict it to the Cartesian product of full subcategories

$$H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B) \times H^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fl}}) \subset H^0(\mathrm{mod}^{\mathrm{cdg}}\text{-}B) \times H^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}).$$

Since the tensor product with a finitely presented Γ-graded right $B^\#$-module commutes with infinite products of Γ-graded left $B^\#$-modules, this restriction factorizes through the contraderived category $\mathrm{D}^{\mathrm{ctr}}(B\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fl}})$ in the second argument. Clearly, it also factorizes through the absolute derived category $\mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B)$ in the first argument.

By Remark 1.5 of the main body of this paper (see also [Positselski 2012, Proposition A.3.1(b)]), the natural functor $\mathrm{D}^{\mathrm{ctr}}(B\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fl}}) \to \mathrm{D}^{\mathrm{ctr}}(B\text{-}\mathrm{mod}^{\mathrm{cdg}})$ is an equivalence of triangulated categories. Hence we obtain the left derived functor

$$\mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B) \times \mathrm{D}^{\mathrm{ctr}}(B\text{-}\mathrm{mod}^{\mathrm{cdg}}) \longrightarrow \mathrm{D}(k\text{-}\mathrm{mod}). \tag{5}$$

Up to composing with the localization functors $Z^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B) \to \mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B)$ and $Z^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}}) \to \mathrm{D}^{\mathrm{ctr}}(B\text{-}\mathrm{mod}^{\mathrm{cdg}})$, this functor agrees with the derived functor $\mathrm{Tor}^{B,II}$ from [Polishchuk and Positselski 2012, Section 2.2] where the former is defined.

Indeed, let N be an object of $Z^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B)$. Let $P_\bullet$ be a left resolution of an object $M \in Z^0(B\text{-}\mathrm{mod}^{\mathrm{cdg}})$ by left CDG-modules over B with flat underlying Γ-graded $B^\#$-modules, and let P be the total CDG-module of the complex $P_\bullet$ constructed by taking infinite products along the diagonals. Then the complex $\mathrm{Tot}^{\sqcap}(N \otimes_B P_\bullet)$ computing $\mathrm{Tor}^{B,II}(N, M)$ is isomorphic to the complex $N \otimes_B P$ computing the derived functor (5) on the objects N and M.

B.1.3. *Comparison of the two theories.* Let C be a small k-linear (Γ-graded) DG-category. The above constructions applicable to CDG-categories and CDG-modules over them can be also applied to DG-categories and DG-modules as a particular case. Following [Polishchuk and Positselski 2012], we denote the DG-categories of left and right DG-modules over C by $C\text{-}\mathrm{mod}^{\mathrm{dg}}$ and $\mathrm{mod}^{\mathrm{dg}}\text{-}C$, and generally use the upper index “dg” instead of “cdg” in the notation related to DG-modules.

As in [loc. cit., Sections 2.1, 3.1 and 3.4], we denote by $H^0(C\text{-}\mathrm{mod}^{\mathrm{dg}})_{\mathrm{inj}}$ and $H^0(C\text{-}\mathrm{mod}^{\mathrm{dg}})_{\mathrm{fl}}$ the homotopy categories of h-injective and h-flat left DG-modules over C. The notation $H^0(C\text{-}\mathrm{mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}}$ and $H^0(C\text{-}\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fl}})_{\mathrm{fl}}$ stands for the full triangulated subcategories in $H^0(C\text{-}\mathrm{mod}^{\mathrm{dg}})$ formed by h-injective DG-modules over C whose underlying Γ-graded $C^\#$-modules are injective, or h-flat DG-modules whose underlying Γ-graded $C^\#$-modules are flat, respectively. Finally, we let

$H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fgp}})_{\mathrm{prj}} \subset H^0(C\text{-mod}^{\mathrm{dg}})$ and $H^0(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fgp}}\text{-}C)_{\mathrm{fl}} \subset H^0(\mathrm{mod}^{\mathrm{dg}}\text{-}C)$ denote the full triangulated subcategories of h-projective left and h-flat right DG-modules whose underlying Γ-graded $C^\#$-modules are projective and finitely generated.

Assume that the Γ-graded category $C^\#$ is left Noetherian. Let L be an object of $Z^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}})$. Given a left DG-module M over C, pick its injective resolution $J^\bullet$ in the exact category $Z^0(C\text{-mod}^{\mathrm{dg}})$ [loc. cit., Section 2.1]. Let $\mathrm{Tot}^\oplus(J^\bullet) \to \mathrm{Tot}^\sqcap(J^\bullet)$ be the natural closed morphism between the total DG-modules of the complex $J^\bullet$ constructed by taking infinite direct sums and infinite products along the diagonals. Then the induced morphism of complexes of k-modules

$$\mathrm{Hom}^C(L,\ \mathrm{Tot}^\oplus(J^\bullet)) \longrightarrow \mathrm{Hom}^C(L,\ \mathrm{Tot}^\sqcap(J^\bullet))$$

represents the comparison morphism $\mathrm{Ext}^{II}_C(L, M) \to \mathrm{Ext}_C(L, M)$ [loc. cit., formula (10)] in $\mathsf{D}(k\text{-mod})$ between the two kinds of Ext objects for the DG-modules L and M.

Similarly, assume that the Γ-graded category $C^\#$ is right coherent. Let N be an object of $Z^0(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fp}}\text{-}C)$. Given a left DG-module M over C, pick its projective resolution $P_\bullet$ in the exact category $Z^0(C\text{-mod}^{\mathrm{dg}})$. Let $\mathrm{Tot}^\oplus(P_\bullet) \to \mathrm{Tot}^\sqcap(P_\bullet)$ be the natural closed morphism between the total DG-modules of the complex $P_\bullet$ constructed by taking infinite direct sums and infinite products along the diagonals. Then the induced morphism of complexes of k-modules

$$N \otimes_C \mathrm{Tot}^\oplus(P_\bullet) \longrightarrow N \otimes_C \mathrm{Tot}^\sqcap(P_\bullet)$$

represents the comparison morphism $\mathrm{Tor}^C(N, M) \to \mathrm{Tor}^{C,II}(N, M)$ [loc. cit., formula (9)] in $\mathsf{D}(k\text{-mod})$ between the two kinds of Tor objects for the DG-modules N and M.

Proposition A. *Assume that the Γ-graded category $C^\#$ is left Noetherian. Let L be a left DG-module over C whose underlying Γ-graded left $C^\#$-module is finitely generated, and let M be a left DG-module over C. Then the natural morphism $\mathrm{Ext}^{II}_C(L, M) \to \mathrm{Ext}_C(L, M)$ is an isomorphism provided that either*

(i) *the object $M \in \mathsf{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}})$ belongs to the image of the fully faithful functor $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}} \to \mathsf{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}})$; or*

(ii) *the object $L \in \mathsf{D}^{\mathrm{abs}}(C\text{-mod}^{\mathrm{dg}})$ belongs to the image of the fully faithful functor $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fgp}})_{\mathrm{prj}} \to \mathsf{D}^{\mathrm{abs}}(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}})$.*

Proof. Let $J^\bullet$ be an injective resolution of the DG-module M in the exact category $Z^0(C\text{-mod}^{\mathrm{dg}})$. Then the natural morphism $M \to \mathrm{Tot}^\oplus(J^\bullet)$ is always an isomorphism in $\mathsf{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}})$ [Positselski 2011b, proof of Theorem 3.7], while the morphism $M \to \mathrm{Tot}^\sqcap(J^\bullet)$ is an isomorphism in the conventional derived category $\mathsf{D}(C\text{-mod}^{\mathrm{dg}})$ [loc. cit., proofs of Theorems 1.4-5]. Furthermore, one has $\mathrm{Tot}^\oplus(J^\bullet) \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})$ and $\mathrm{Tot}^\sqcap(J^\bullet) \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}}$.

Part (i): The functor is fully faithful by [loc. cit., Theorem 3.5(a) and Lemma 1.3]. According to formula (4) from Section B.1.2 and [Polishchuk and Positselski 2012, Section 3.1], both kinds of Ext involved are well-defined as functors of the argument $M \in \mathsf{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}})$. Hence one can assume $M \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}}$. Then both morphisms $M \to \mathrm{Tot}^{\oplus}(J^\bullet)$ and $M \to \mathrm{Tot}^{\sqcap}(J^\bullet)$ are homotopy equivalences by semiorthogonality; hence so is the morphism $\mathrm{Tot}^{\oplus}(J^\bullet) \to \mathrm{Tot}^{\sqcap}(J^\bullet)$ and the assertion follows.

Part (ii): In view of the first paragraph of this proof, a cone K of the morphism $\mathrm{Tot}^{\oplus}(J^\bullet) \to \mathrm{Tot}^{\sqcap}(J^\bullet)$ in $H^0(C\text{-mod}^{\mathrm{dg}})$ is an acyclic DG-module over C whose underlying Γ-graded $C^\#$-module is injective. Hence the complex of morphisms $\mathrm{Hom}^C(-, K)$ is a well-defined functor $\mathsf{D}^{\mathrm{abs}}(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}})^{\mathrm{op}} \to \mathsf{D}(k\text{-mod})$ annihilating $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fgp}})_{\mathrm{prj}}$. □

Proposition B. *Assume that the Γ-graded category $C^\#$ is right coherent. Let N be a right DG-module over C whose underlying Γ-graded right $C^\#$-module is finitely presented, and let M be a left DG-module over C. Then the natural morphism $\mathrm{Tor}^C(N, M) \to \mathrm{Tor}^{C,II}(N, M)$ is an isomorphism provided that either*

(i) *there exists a closed morphism $P \to M$ into M from a DG-module $P \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}})_{\mathrm{fl}}$ with a cone contraacyclic with respect to C-mod$^{\mathrm{dg}}$ or completely acyclic with respect to C-mod$^{\mathrm{dg}}_{\mathrm{fl}}$* (see [Polishchuk and Positselski 2012, Sections 3.2 and 4.7]); *or*

(ii) *the object $N \in \mathsf{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{dg}}\text{-}C)$ belongs to the image of the fully faithful functor $H^0(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fgp}}\text{-}C)_{\mathrm{fl}} \to \mathsf{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fp}}\text{-}C)$.*

Proof. Let $P_\bullet$ be a projective resolution of the DG-module M in the exact category $Z^0(C\text{-mod}^{\mathrm{dg}})$. Then the natural morphism $\mathrm{Tot}^{\sqcap}(P_\bullet) \to M$ is always an isomorphism in $\mathsf{D}^{\mathrm{ctr}}(C\text{-mod}^{\mathrm{dg}})$ [Positselski 2011b, proof of Theorem 3.8], while the morphism $\mathrm{Tot}^{\oplus}(P_\bullet) \to M$ is an isomorphism in $\mathsf{D}(C\text{-mod}^{\mathrm{dg}})$ [loc. cit., proof of Theorem 1.4]. Furthermore, one has $\mathrm{Tot}^{\sqcap}(P_\bullet) \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}})$ and $\mathrm{Tot}^{\oplus}(P_\bullet) \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}})_{\mathrm{fl}}$.

Part (i): Acyclic DG-modules in the second argument are annihilated by the functor Tor^C by [Polishchuk and Positselski 2012, Section 3.1], while contraacyclic DG-modules in the second argument are annihilated by the functor $\mathrm{Tor}^{C,II}(N, -)$ according to the formula (5). The latter also applies to DG-modules completely acyclic with respect to C-mod$^{\mathrm{dg}}_{\mathrm{fl}}$ since the functor of a tensor product with a finitely presented DG-module preserves infinite direct sums and products. So one can replace M with P and assume that $M \in H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}})_{\mathrm{fl}}$.

Then a cone of the morphism $\mathrm{Tot}^{\sqcap}(P_\bullet) \to M$ is contraacyclic with respect to C-mod$^{\mathrm{dg}}$ with a flat underlying Γ-graded $C^\#$-module, and hence also contraacyclic with respect to $C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}}$. On the other hand, a cone of the morphism $\mathrm{Tot}^{\oplus}(P_\bullet) \to M$ is acyclic and h-flat. It follows that the functor $N \otimes_C -$ transforms both of these morphisms, and therefore also the morphism $\mathrm{Tot}^{\oplus}(P_\bullet) \to \mathrm{Tot}^{\sqcap}(P_\bullet)$, into quasi-isomorphisms of complexes of k-modules.

Part (ii): A cone K of the morphism $\mathrm{Tot}^{\oplus}(P_\bullet) \to \mathrm{Tot}^{\sqcap}(P_\bullet)$ in $H^0(C\text{-mod}^{\mathrm{dg}})$ is an acyclic DG-module over C whose underlying Γ-graded $C^\#$-module is flat. Hence the tensor product $- \otimes_C K$ is a well-defined functor $\mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fp}}\text{-}C) \to \mathrm{D}(k\text{-mod})$ annihilating $H^0(\mathrm{mod}^{\mathrm{dg}}_{\mathrm{fgp}}\text{-}C)_{\mathrm{fl}}$. □

In particular, assuming that the category $C^\#$ is left Noetherian, the natural morphism $\mathrm{Ext}^{II}_C(L, M) \to \mathrm{Ext}_C(L, M)$ is an isomorphism for all $L \in C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}}$ and $M \in C\text{-mod}^{\mathrm{dg}}$ provided that the Verdier localization functor $\mathrm{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}}) \to \mathrm{D}(C\text{-mod}^{\mathrm{dg}})$ is an equivalence of triangulated categories. Assuming that the category $C^\#$ is right coherent, the natural morphism $\mathrm{Tor}^C(N, M) \to \mathrm{Tor}^{C,II}(N, M)$ is an isomorphism for all $N \in \mathrm{mod}^{\mathrm{dg}}_{\mathrm{fp}}\text{-}C$ and $M \in C\text{-mod}^{\mathrm{dg}}$ provided that the Verdier localization functor $\mathrm{D}^{\mathrm{ctr}}(C\text{-mod}^{\mathrm{dg}}) \to \mathrm{D}(C\text{-mod}^{\mathrm{dg}})$ is an equivalence of categories, or alternatively, that any acyclic DG-module from $C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}}$ is completely acyclic with respect to $C\text{-mod}^{\mathrm{dg}}_{\mathrm{fl}}$.

B.1.4. *Comparison for the DG-category of CDG-modules.* Let B be a k-linear CDG-category and $C = \mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B$ be the DG-category of right CDG-modules over B whose underlying Γ-graded $B^\#$-modules are projective and finitely generated. The DG-categories of (left or right) CDG-modules over B and DG-modules over C are naturally equivalent [Polishchuk and Positselski 2012, Sections 1.5 and 2.6] (as are the categories of Γ-graded modules over $B^\#$ and $C^\#$).

Following [loc. cit., Section 3.5], we denote by M_C the DG-module over C corresponding to a CDG-module M over B.

Let $k^\vee$ be an injective cogenerator of the abelian category of k-modules. Introduce the notation $B\text{-mod}^{\mathrm{cdg}}_{\mathrm{prj}} \subset B\text{-mod}^{\mathrm{cdg}}$ for the DG-category of left CDG-modules over B with projective underlying Γ-graded $B^\#$-modules. The results below in this section are to be compared with those from [loc. cit., Sections 3.5 and 4.7].

Proposition A′. *Assume that the Γ-graded category $B^\#$ is left Noetherian. Let L be a left CDG-module over B whose underlying Γ-graded left $B^\#$-module $L^\#$ is finitely generated, and let M be a left CDG-module over B. Then the natural morphism $\mathrm{Ext}^{II}_C(L_C, M_C) \to \mathrm{Ext}_C(L_C, M_C)$ is an isomorphism provided that either*

(i) *the object M belongs to the minimal triangulated subcategory of* $\mathrm{D}^{\mathrm{co}}(B\text{-mod}^{\mathrm{cdg}})$ *containing the objects* $\mathrm{Hom}_k(F, k^\vee)$ *for all* $F \in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ *and closed under infinite products*; *or*

(ii) *the object L belongs to the minimal thick subcategory of* $\mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ *containing the image of* $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$.

Proof. Part (i): The equivalence of categories

$$H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}} \simeq \mathrm{D}(C\text{-mod}^{\mathrm{dg}})$$

makes the embedding functor $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}} \to \mathrm{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}})$ right adjoint to the localization functor $\mathrm{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}}) \to \mathrm{D}(C\text{-mod}^{\mathrm{dg}})$. It follows that the functor $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}} \to \mathrm{D}^{\mathrm{co}}(C\text{-mod}^{\mathrm{dg}})$ preserves infinite products (also, all infinite products exist in the coderived category since it is compactly generated [Positselski 2011b, Theorem 3.11.2]). Since the category $H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{inj}})_{\mathrm{inj}}$ is the minimal triangulated subcategory of $H^0(C\text{-mod}^{\mathrm{dg}})$ containing the objects $\mathrm{Hom}_k(F_C, k^\vee)$ and closed under infinite products [loc. cit., Theorem 1.5], the assertion follows from Proposition A(i).

Part (ii): The equivalence of absolute derived categories

$$\mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}}) \simeq \mathrm{D}^{\mathrm{abs}}(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}})$$

takes objects of the full subcategory $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}}) \subset \mathrm{D}^{\mathrm{abs}}(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ to representable (and, consequently, perfect and h-projective) DG-modules in

$$H^0(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fgp}}) \subset \mathrm{D}^{\mathrm{abs}}(C\text{-mod}^{\mathrm{dg}}_{\mathrm{fg}}),$$

so it remains to apply Proposition A(ii). □

Proposition B′. *Assume that the Γ-graded category $B^\#$ is right coherent. Let N be a right CDG-module over B whose underlying Γ-graded right $B^\#$-module $N^\#$ is finitely presented, and let M be a left CDG-module over B. Then the natural morphism* $\mathrm{Tor}^C(N_C, M_C) \to \mathrm{Tor}^{C,II}(N_C, M_C)$ *is an isomorphism provided that either*

(i) *the object M belongs to the minimal triangulated subcategory of*

$$H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{prj}}) \subset \mathrm{D}^{\mathrm{ctr}}(B\text{-mod}^{\mathrm{cdg}})$$

containing the image of $H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$ *and closed under infinite direct sums*; *or*

(ii) *the object N belongs to the minimal thick subcategory of* $\mathrm{D}^{\mathrm{abs}}(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fp}}\text{-}B)$ *containing the image of* $H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$.

Proof. Similar to that of Proposition A′ and based on Proposition B. □

Now assume that the commutative ring k has finite weak homological dimension and all the Γ-graded k-modules of morphisms in the category $B^\#$ are flat. Clearly, the DG-categories of left and right CDG-modules over the CDG-category $B \otimes_k B^{\mathrm{op}}$ are naturally equivalent, as are the DG-categories of left and right DG-modules over the DG-category $C \otimes_k C^{\mathrm{op}}$. The DG-category of CDG-modules over $B \otimes_k B^{\mathrm{op}}$ is also naturally equivalent to the DG-category of DG-modules over $C \otimes_k C^{\mathrm{op}}$

[Polishchuk and Positselski 2012, Section 2.6]. As above, we denote by M_C the DG-module over $C \otimes_k C^{\mathrm{op}}$ corresponding to a CDG-module M over $B \otimes_k B^{\mathrm{op}}$.

To any left CDG-module G and right CDG-module F over B, one can assign the left CDG-module $G \otimes_k F$ and the right CDG-module $F \otimes_k G$ (corresponding to each other under the above equivalence) over the CDG-category $B \otimes_k B^{\mathrm{op}}$. There are also the natural *diagonal* CDG-module B over $B \otimes_k B^{\mathrm{op}}$ and DG-module C over $C \otimes_k C^{\mathrm{op}}$ [loc. cit., Section 2.4]; these also correspond to each other with respect to the above equivalence of DG-categories.

For any DG-module M_C over $C \otimes_k C^{\mathrm{op}}$, we are interested in the comparison morphisms between the two kinds of Hochschild cohomology $HH^{II,*}(C, M_C) \to HH^*(C, M_C)$ and Hochschild homology $HH_*(C, M_C) \to HH_*^{II}(C, M_C)$ [loc. cit., formula (23)].

Proposition C. *Assume that the Γ-graded category $B^{\#} \otimes_k B^{\#\mathrm{op}}$ is Noetherian and the diagonal Γ-graded module $B^{\#}$ over it is finitely generated. Let M be a CDG-module over $B \otimes_k B^{\mathrm{op}}$. Then the natural morphism $HH^{II,*}(C, M_C) \to HH^*(C, M_C)$ is an isomorphism provided that either*

(i) *the object M belongs to the minimal triangulated subcategory of*

$$\mathrm{D}^{\mathrm{co}}(B \otimes_k B^{\mathrm{op}}\text{-mod}^{\mathrm{cdg}})$$

containing the CDG-modules $Hom_k(F \otimes_k G, k^{\vee})$ for all $F \in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ and $G \in H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$ and closed under infinite products; *or*

(ii) *the diagonal CDG-module B over $B \otimes_k B^{\mathrm{op}}$ belongs to the minimal thick subcategory of*

$$\mathrm{D}^{\mathrm{abs}}(B \otimes_k B^{\mathrm{op}}\text{-mod}^{\mathrm{cdg}}_{\mathrm{fg}})$$

containing the CDG-modules $G \otimes_k F$ for all $F \in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ and $G \in H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$.

Proposition D. *Assume that the Γ-graded category $B^{\#} \otimes_k B^{\#\mathrm{op}}$ is coherent and the diagonal Γ-graded module $B^{\#}$ over it is finitely presented. Let M be a CDG-module over $B \otimes_k B^{\mathrm{op}}$. Then the natural morphism $HH_*(C, M_C) \to HH_*^{II}(C, M_C)$ is an isomorphism provided that either*

(i) *the object M belongs to the minimal triangulated subcategory of*

$$H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{prj}}) \subset \mathrm{D}^{\mathrm{ctr}}(B \otimes_k B^{\mathrm{op}}\text{-mod}^{\mathrm{cdg}})$$

containing the CDG-modules $G \otimes_k F$ for all $F \in H^0(\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B)$ and $G \in H^0(B\text{-mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$ and closed under infinite direct sums; *or*

(ii) *the diagonal CDG-module B over $B \otimes_k B^{\mathrm{op}}$ belongs to the minimal thick subcategory of* $\mathrm{D}^{\mathrm{abs}}(B \otimes_k B^{\mathrm{op}}\text{-mod}_{\mathrm{fg}}^{\mathrm{cdg}})$ *containing the CDG-modules $G \otimes_k F$ for all $F \in H^0(\mathrm{mod}_{\mathrm{fgp}}^{\mathrm{cdg}}\text{-}B)$ and $G \in H^0(B\text{-mod}_{\mathrm{fgp}}^{\mathrm{cdg}})$.*

Proofs of Propositions C and D. Similar to the proofs of Propositions A′ and B′. □

In particular, assume that the Γ-graded category $B^{\#} \otimes_k B^{\#\mathrm{op}}$ is Noetherian and the diagonal Γ-graded module $B^{\#}$ over it is finitely generated. Suppose that the diagonal CDG-module B over $B \otimes_k B^{\mathrm{op}}$ belongs to the minimal thick subcategory of $\mathrm{D}^{\mathrm{abs}}(B \otimes_k B^{\mathrm{op}}\text{-mod}_{\mathrm{fg}}^{\mathrm{cdg}})$ containing the CDG-modules $G \otimes_k F$ for all $F \in H^0(\mathrm{mod}_{\mathrm{fgp}}^{\mathrm{cdg}}\text{-}B)$ and $G \in H^0(B\text{-mod}_{\mathrm{fgp}}^{\mathrm{cdg}})$. Then, according to [Polishchuk and Positselski 2012, formulas (44-45) in Section 2.6] and parts (ii) of Propositions C and D, there are natural isomorphisms

$$HH^*(C, M_C) \simeq HH^{II,*}(C, M_C) \simeq HH^{II,*}(B, M), \tag{6}$$

$$HH_*(C, M_C) \simeq HH_*^{II}(C, M_C) \simeq HH_*^{II}(B, M) \tag{7}$$

for any CDG-module M over $B \otimes_k B^{\mathrm{op}}$. Specializing to the case of the diagonal CDG-module $M = B$ and DG-module $M_C = C$, we obtain

$$\begin{gathered} HH^*(C) \simeq HH^{II,*}(C) \simeq HH^{II,*}(B), \\ HH_*(C) \simeq HH_*^{II}(C) \simeq HH_*^{II}(B). \end{gathered} \tag{8}$$

B.1.5. *Locally free matrix factorizations.* Let k be a regular commutative Noetherian ring of finite Krull dimension and X be an affine scheme of finite type over $\operatorname{Spec} k$. Let $w \in \mathcal{O}(X)$ be a global regular function on X. Consider the $\mathbb{Z}/2$-graded CDG-algebra B over k with $B^0 = \mathcal{O}(X)$, $B^1 = 0$, $d = 0$, and $h = -w \in B^0$. We will find it convenient to denote the CDG-algebra B simply by $(X, h) = (X, -w)$ (cf. Section 2.2 of the main body of this paper).

Then $C = \mathrm{mod}_{\mathrm{fgp}}^{\mathrm{cdg}}\text{-}B$ is the $\mathbb{Z}/2$-graded DG-category of locally free matrix factorizations of finite rank of the potential w on X. Furthermore, one has $B \otimes_k B^{\mathrm{op}} = (X \times_k X,\ w_2 - w_1)$, where $w_i = p_i^* w \in \mathcal{O}(X \times_k X)$, $i = 1, 2$, and $p_i : X \times_k X \to X$ denote the coordinate projections. Let $\Delta : X \to X \times_k X$ be the diagonal embedding and $\Delta_* \mathcal{O}_X$ be the corresponding coherent sheaf on $X \times_k X$.

Consider the coherent matrix factorization of the potential $w_2 - w_1$ on $X \times X$ whose even-degree component is the sheaf $\Delta_* \mathcal{O}_X$, while the odd-degree component vanishes. We will denote this "diagonal" matrix factorization simply by $\Delta_* \mathcal{O}_X \in H^0((X \times_k X,\ w_2 - w_1)\text{-mod}_{\mathrm{fg}}^{\mathrm{cdg}})$. Applying the machinery of the previous sections leads to the following result (cf. [Polishchuk and Positselski 2012, Sections 4.8–4.10]).

Corollary B.1.5. *Suppose that the diagonal matrix factorization $\Delta_*\mathcal{O}_X$ belongs to the minimal thick subcategory of* $\mathsf{D}^{\mathrm{abs}}((X\times_k X,\ w_2-w_1)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fg}})$ *containing the external tensor products of locally free matrix factorizations of finite rank $p_1^*G\otimes_k p_2^*F$ for all $G\in H^0((X,-w)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$ and $F\in H^0((X,w)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})$. Then the natural isomorphisms* (8) *hold for the CDG-algebra $B=(X,w)$ and the DG-category of locally free matrix factorizations $C=\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}}\text{-}B$.* □

Notice that the condition under which the conclusion of Corollary B.1.5 has been proven is a rather strong one, particularly when X is not assumed to be a regular scheme. Then it is not even clear when or why the diagonal matrix factorization $\Delta_*\mathcal{O}_X$ should belong to the thick envelope of the full triangulated subcategory of locally free matrix factorizations

$$H^0((X\times_k X,\ w_2-w_1)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fgp}})\subset \mathsf{D}^{\mathrm{abs}}((X\times_k X,\ w_2-w_1)\text{-}\mathrm{mod}^{\mathrm{cdg}}_{\mathrm{fg}})$$

on $X\times_k X$, let alone to the thick subcategory generated by external tensor products of locally free matrix factorizations from the two copies of X.

B.1.6. *Smooth stratifications.* A scheme X of finite type over a field k is said to admit a *smooth stratification* [Efimov 2013] if it can be presented as a disjoint union of its locally closed subsets $X=\coprod_\alpha S_\alpha$ so that each S_α, when endowed with the structure of a reduced locally closed subscheme in X, becomes a smooth scheme over k. In particular, every scheme of finite type over a perfect field k admits a smooth stratification, as any regular scheme of finite type over a perfect field is smooth over it [Grothendieck 1967, Corollaires 17.15.2 and 17.15.13]. Notice that a scheme of finite type over a field admits a smooth stratification if and only if its maximal reduced closed subscheme does.

The definition of a *regular stratification* of a Noetherian scheme is similar, except that the strata S_α are only required to be regular schemes in their reduced locally closed subscheme structures. Any scheme of finite type over a field admits a regular stratification [Grothendieck 1965, Scholie 7.8.3(iii)–(iv) and Proposition 7.8.6(i)].

Let X be a smooth affine scheme over a field k and $w\in\mathcal{O}(X)$ be a regular function on X. Set $X_0=\{w=0\}\subset X$ to be the zero locus of w. The following result is a slight generalization of [Polishchuk and Positselski 2012, Corollary 4.8.A] based on the above definitions.

Corollary B.1.6. *Assume that there exists a closed subscheme $Z\subset X$ such that $w: X\setminus Z\to\mathbb{A}^1_k$ is a smooth morphism, $w|_Z=0$, and the scheme Z admits a smooth stratification over k. Then the conditions of Corollary* B.1.5 *are satisfied, so its conclusions apply.*

Proof. According to the argument in [Polishchuk and Positselski 2012, Section 4.8], it suffices to show that the bounded derived category of coherent sheaves on $Z\times Z$

is generated by external tensor products of coherent sheaves on the two Cartesian factors. This is a particular case of the following lemma. ☐

Lemma B.1.6. *Let Z' and Z'' be schemes of finite type over a field k. Assume that the scheme Z' admits a smooth stratification. Then the bounded derived category of coherent sheaves $\mathsf{D}^{\mathsf{b}}((Z'\times Z'')\text{-}\mathsf{coh})$ on the Cartesian product $Z'\times_k Z''$ coincides with its minimal thick subcategory containing the external tensor products $\mathcal{K}'\otimes_k\mathcal{K}''$ of coherent sheaves on $\mathcal{K}'$ on Z' and $\mathcal{K}''$ on Z''.*

Proof. One proceeds by induction on the total number of strata in a smooth stratification of Z' and a regular stratification of Z''. Clearly, one can replace Z' and Z'' with their maximal reduced closed subschemes. Now if S_{α_0} is an open stratum in Z' and T_{β_0} is an open stratum in Z'', then S_{α_0} is smooth as an open subscheme in Z' and T_{β_0} is regular as an open subscheme in Z'', while the induction assumption applies to $(Z'\setminus S_{\alpha_0})\times_k Z''$ and $Z'\times_k(Z''\setminus T_{\beta_0})$. The scheme $S_{\alpha_0}\times_k T_{\beta_0}$ is regular since it is smooth over a regular scheme. The rest of the argument is based on [Orlov 2011, Proposition 2.7] and follows the lines of [Lin and Pomerleano 2013, proof of Theorem 3.7]. ☐

B.2. ***Coherent matrix factorizations.*** In this section, we return to the notation system typical for the main body of this paper. The notion of a critical value of a regular function on a singular variety is defined in Section B.2.1. In Section B.2.2 we show that the external tensor product of coherent matrix factorizations is a fully faithful functor between the absolute derived categories and provide a sufficient condition for the pretriangulated extension of its DG-category version to be a quasiequivalence. The Hochschild cohomology of the DG-category corresponding to the absolute derived category of coherent matrix factorizations of a potential having no critical values but zero is computed in Section B.2.4.

The notion of cotensor product of complexes of quasicoherent sheaves and quasicoherent matrix factorizations is discussed in Sections B.2.5–B.2.6 and used in order to compute the Hochschild homology of the (same) DG-category of coherent matrix factorizations in Section B.2.7. The direct sum formula for the Hochschild (co)homology of the DG-categories of coherent matrix factorizations of a potential with several critical values is established in Section B.2.8.

In some sense, the results of this section (as compared to those of Section B.1) suggest that the DG-category corresponding to the absolute derived category of coherent matrix factorizations on a singular variety may be better behaved than the similar category of locally free matrix factorizations of finite rank. Other (and in some way related) arguments in support of the same conclusion are provided by the results of the papers [Lunts 2010; Efimov 2013] showing that the DG-category corresponding to the absolute derived category of coherent matrix factorizations is smooth (and even homotopically finitely presented), under suitable conditions on the field k. (Cf. the counterexample in Section 3.3.)

B.2.1. *Noncritical functions.* Let k be a field and X be a scheme of finite type over $\operatorname{Spec} k$. Let $f \in \mathcal{O}(X)$ be a global regular function on X.

Let Y be a scheme of finite type over $\operatorname{Spec} k$ and $g \in \mathcal{O}(Y)$ be a global regular function. Let $p_1 : X \times_k Y \to X$ and $p_2 : X \times_k Y \to Y$ be the natural projections. Consider the regular function $f_1 + g_2 = p_1^* f + p_2^* g$ on $X \times_k Y$.

Suppose that $f : X \to \mathbb{A}^1_k$ is a flat morphism from X to the affine line (when k is algebraically closed, this means that the function $f - c$ is a local nonzero-divisor on X for every $c \in k$). Then the morphism $f_1 + g_2 : X \times_k Y \to \mathbb{A}^1_k$ is also flat as it is the composition of two flat morphisms

$$X \times_k Y \longrightarrow \mathbb{A}^1_k \times_k Y \longrightarrow \mathbb{A}^1_k$$

(the former morphism being flat since the morphism $f : X \to \mathbb{A}^1_k$ is and the latter one because the polynomial $x + g$ does not divide zero in $B[x]$ for any commutative ring B and element $g \in B$). In particular, it follows that the function $f_1 + g_2$ is a local nonzero-divisor on the Cartesian product $X \times_k Y$.

A function $f \in \mathcal{O}(X)$ is said to be *noncritical* (or to *have no critical values*) if for any regular function $g \in \mathcal{O}(Y)$ on a scheme Y of finite type over $\operatorname{Spec} k$ the absolute derived category of coherent matrix factorizations $\mathsf{D}^{\mathsf{abs}}((X \times_k Y, \mathcal{O}, f_1 + g_2)\text{-}\mathsf{coh})$ vanishes (i.e., is equivalent to the zero category). According to Remark 1.3 and Theorem 1.10(b), this condition is local in both X and Y.

Therefore, given a scheme X of finite type over $\operatorname{Spec} k$ and a regular function $f \in \mathcal{O}(X)$, there is a unique maximal open subscheme $X'_f \subset X$ where the function f is noncritical. We will see below that the open subscheme X'_f is always dense in X if the morphism $f : X \to \mathbb{A}^1_k$ is flat and the field k has zero characteristic.

Similarly, there is a unique maximal open subscheme $\mathbb{A}^1_{k,f} \subset \mathbb{A}^1_k$ such that the restriction of f to its full preimage in X is noncritical. The scheme $\mathbb{A}^1_{k,f}$ is always nonempty if the field k has zero characteristic. The points in the complement $\mathbb{A}^1_k \setminus \mathbb{A}^1_{k,f}$ are called the *critical values* of f. In particular, one says that f *has no critical values but zero* if the restriction of f to $f^{-1}(\mathbb{A}^1_k \setminus \{0\}) \subset X$ is noncritical.

Notice that when the schemes X and Y are separated and the morphism of schemes $f : X \to \mathbb{A}^1_k$ is flat, the category $\mathsf{D}^{\mathsf{abs}}((X \times_k Y, \mathcal{O}, f_1 + g_2)\text{-}\mathsf{coh})$ is equivalent to the triangulated category $\mathsf{D}^{\mathsf{b}}_{Sing}(\{f_1 + g_2 = 0\}/X \times_k Y)$ of relative singularities of the zero locus of the function $f_1 + g_2$ on $X \times_k Y$ (see Theorem 2.7).

Remark B.2.1. It would be interesting to have a geometric characterization of noncriticality of functions on singular schemes. For example, how does our definition of noncriticality relate to the condition that the differential of f at every closed point $x \in X$ be a nonzero element of the Zariski cotangent space $T^*_x X$? We do *not* know this; cf. the smooth stratification approach below.

Lemma B.2.1. *Let $X = \coprod_\alpha S_\alpha$ be a scheme of finite type over* $\operatorname{Spec} k$ *presented as a disjoint union of its locally closed subsets, endowed with their reduced locally closed subscheme structures. Let $\mathcal{L}$ be a line bundle on X and $w \in \mathcal{L}(X)$ be its global section. In this setting, if the absolute derived categories* $\mathsf{D}^{\mathsf{abs}}((S_\alpha, \mathcal{L}|_{S_\alpha}, w|_{S_\alpha})\text{-}\mathsf{coh})$ *vanish for all α, then so does the absolute derived category* $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh})$.

Proof. Proceeding by induction on the number of strata in the stratification S_α, it suffices to consider the case when there are only two of them, namely, a closed subset $S \subset X$ and its open complement $X \setminus S$. One can also replace X with its maximal reduced closed subscheme. Then the desired assertion follows from Theorem 1.10(b) since the triangulated category $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_S)$ is generated by the image of the natural functor $\mathsf{D}^{\mathsf{abs}}((S, \mathcal{L}|_S, w|_S)\text{-}\mathsf{coh}) \to \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w)\text{-}\mathsf{coh}_S)$. □

Proposition B.2.1. *Let X be a scheme of finite type over* $\operatorname{Spec} k$ *and $f \in \mathcal{O}(X)$ be a regular function on X. Let $X = \coprod_\alpha S_\alpha$ be a smooth stratification of the scheme X over k (see Section* B.1.6) *such that the morphisms of schemes $f|_{S_\alpha} : S_\alpha \to \mathbb{A}^1_k$ are smooth for all α. Then the function f is noncritical on X.*

Proof. Let Y be a scheme of finite type over $\operatorname{Spec} k$ and $g \in \mathcal{O}(Y)$ be a regular function. We have to show that the triangulated category $\mathsf{D}^{\mathsf{abs}}((X \times_k Y, \mathcal{O}, f_1 + g_2)\text{-}\mathsf{coh})$ vanishes. Choosing a stratification of Y by regular locally closed subschemes and applying Lemma B.2.1, one can assume that X is smooth over k and Y is regular.

Then the scheme $X \times_k Y$ is also regular, the derivative of the function $f_1 + g_2 \in \mathcal{O}(X \times_k Y)$, viewed as an element of the Zariski cotangent space, does not vanish at any points where the function itself does (and, in a sense, at any other closed points, too), and it follows that the zero locus of $f_1 + g_2$ in $X \times_k Y$ is also a regular scheme. It remains to use Theorem 2.7 (or [Orlov 2012, Theorem 3.5] and Corollary 2.4(c)). □

It follows from Proposition B.2.1 that for any scheme of finite type X with a smooth stratification $X = \coprod_\alpha S_\alpha$ over $\operatorname{Spec} k$ and any regular function $f \in \mathcal{O}(X)$, the set of critical values of the function f on X is contained in the union of the sets of critical values of the functions $f|_{S_\alpha}$. In particular, if the characteristic of k is zero, then all of these sets are finite.

B.2.2. *External tensor products.* Let X' and X'' be separated schemes of finite type over a field k, and let $w' \in \mathcal{O}(X')$ and $w'' \in \mathcal{O}(X'')$ be regular functions. Let $X' \times_k X''$ be the Cartesian product, p_1 and p_2 be its natural projections onto the factors X' and X'', and $w'_1 + w''_2 = p_1^* w' + p_2^* w''$ be the related regular function on $X' \times_k X''$. Then there is the external tensor product functor

$$\otimes_k : \mathsf{D}^{\mathsf{co}}((X', \mathcal{O}, w')\text{-qcoh}) \times \mathsf{D}^{\mathsf{co}}((X'', \mathcal{O}, w'')\text{-qcoh}) \\ \longrightarrow \mathsf{D}^{\mathsf{co}}((X' \times_k X'',\ \mathcal{O},\ w'_1 + w''_2)\text{-qcoh}), \quad (9)$$

which restricts to the similar functor

$$\otimes_k : \mathsf{D}^{\mathsf{abs}}((X', \mathcal{O}, w')\text{-coh}) \times \mathsf{D}^{\mathsf{abs}}((X'', \mathcal{O}, w'')\text{-coh}) \\ \longrightarrow \mathsf{D}^{\mathsf{abs}}((X' \times_k X'',\ \mathcal{O},\ w'_1 + w''_2)\text{-coh}) \quad (10)$$

on coherent matrix factorizations.

Proposition B.2.2. *Let $\mathcal{K}'$ and $\mathcal{M}'$ be coherent matrix factorizations of the potential w' on the scheme X', and let $\mathcal{K}''$ and $\mathcal{M}''$ be coherent matrix factorizations of the potential w'' on the scheme X''. Then the natural map of $\mathbb{Z}/2$-graded k-vector spaces of morphisms*

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{abs}}((X',\mathcal{O},w')\text{-coh})}(\mathcal{K}', \mathcal{M}'[*]) \otimes_k \mathrm{Hom}_{\mathsf{D}^{\mathsf{abs}}((X'',\mathcal{O},w'')\text{-coh})}(\mathcal{K}'', \mathcal{M}''[*]) \\ \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{abs}}((X'\times_k X'',\mathcal{O},w'_1+w''_2)\text{-coh})}(\mathcal{K}' \otimes_k \mathcal{K}'',\ \mathcal{M}' \otimes_k \mathcal{M}''[*]) \quad (11)$$

induced by the additive functor of two arguments (10) *is an isomorphism.*

Proof. By Proposition 1.5(d), it suffices to show that the natural map

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((X',\mathcal{O},w')\text{-qcoh})}(\mathcal{K}', \mathcal{M}'[*]) \otimes_k \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((X'',\mathcal{O},w'')\text{-qcoh})}(\mathcal{K}'', \mathcal{M}''[*]) \\ \longrightarrow \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((X'\times_k X'',\mathcal{O},w'_1+w''_2)\text{-qcoh})}(\mathcal{K}' \otimes_k \mathcal{K}'',\ \mathcal{M}' \otimes_k \mathcal{M}''[*]) \quad (12)$$

induced by the functor (9) is an isomorphism for any coherent matrix factorizations $\mathcal{K}'$, $\mathcal{K}''$ and quasicoherent matrix factorizations $\mathcal{M}'$, $\mathcal{M}''$ of the potentials w' and w''. One easily checks that the desired assertion holds for the Hom spaces in the homotopy categories of matrix factorizations (since it holds for morphisms between the external tensor products of coherent and quasicoherent sheaves).

Furthermore, one can assume the quasicoherent matrix factorizations $\mathcal{M}'$ and $\mathcal{M}''$ to be injective. Then the Hom spaces in the left-hand side of the map (12) coincide with the similar Hom spaces computed in the homotopy categories of matrix factorizations. Let $\mathcal{I}^\bullet$ be a right resolution of $\mathcal{M}' \otimes_k \mathcal{M}''$ in the abelian category of quasicoherent matrix factorizations (and closed morphisms between them) consisting of injective matrix factorizations, and let $\mathcal{J}$ be the total matrix factorization of the complex $\mathcal{I}^\bullet$ constructed by taking infinite direct sums along the diagonals. Then the k-vector spaces of morphisms from $\mathcal{K}' \otimes_k \mathcal{K}''$ into $\mathcal{J}$ in the homotopy category of matrix factorizations are isomorphic to the right-hand side of (12) [Positselski 2011b, Theorem 3.7].

It remains to show that the spaces of morphisms from $\mathcal{K}'\otimes_k\mathcal{K}''$ to $\mathcal{M}'\otimes_k\mathcal{M}''$ in the homotopy category of matrix factorizations are isomorphic to the similar spaces of morphisms from $\mathcal{K}'\otimes_k\mathcal{K}''$ to $\mathcal{J}$. Indeed, taking the termwise Hom from $\mathcal{K}'\otimes_k\mathcal{K}''$ preserves exactness of the sequence $0\to\mathcal{M}'\otimes_k\mathcal{M}''\to\mathcal{I}^\bullet$ since the higher Ext spaces from the components of $\mathcal{K}'\otimes_k\mathcal{K}''$ into those of $\mathcal{M}'\otimes_k\mathcal{M}''$ in the abelian category of quasicoherent sheaves on $X'\otimes_k X''$ vanish. The latter assertion can be checked for affine schemes X', X'' using projective resolutions and then globally for the cohomology of quasicoherent sheaves using, e.g., the Čech approach. $\square$

Theorem B.2.2. *Assume that the morphisms of schemes*

$$w':X'\to\mathbb{A}^1_k\quad\textit{and}\quad w'':X''\to\mathbb{A}^1_k$$

are flat. Suppose that there exist closed subschemes $Z'\subset X'$ and $Z''\subset X''$ such that $w'|_{Z'}=0=w''_{Z''}$, the functions w' and w'' are noncritical on $X'\setminus Z'$ and $X''\setminus Z''$, and the scheme Z' admits a smooth stratification over k. Then the absolute derived category $\mathsf{D}^{\mathsf{abs}}((X'\times_k X'',\,\mathcal{O},\,w'_1+w''_2)\text{-}\mathsf{coh})$ coincides with its minimal thick subcategory containing the image of the functor (10).

Proof. By the definition of noncriticality, one has

$$\mathsf{D}^{\mathsf{abs}}((((X'\setminus Z')\times_k X''),\,\mathcal{O},\,w'_1+w''_2)\text{-}\mathsf{coh})=0$$
$$=\mathsf{D}^{\mathsf{abs}}(((X'\times_k(X''\setminus Z'')),\,\mathcal{O},\,w'_1+w''_2)\text{-}\mathsf{coh}).$$

Therefore, any coherent matrix factorization of the potential $w'_1+w''_2$ on $X'\times_k X''$ has its category-theoretic support inside $Z'\times_k Z''$, and is consequently isomorphic in $\mathsf{D}^{\mathsf{abs}}((X'\times_k X'',\,\mathcal{O},\,w'_1+w''_2)\text{-}\mathsf{coh})$ to a direct summand of an object represented by a coherent matrix factorization supported set-theoretically inside $Z'\times_k Z''$ (see Corollary 1.10(b)). It follows that the triangulated category

$$\mathsf{D}^{\mathsf{abs}}((X'\times_k X'',\,O,\,w'_1+w''_2)\text{-}\mathsf{coh})$$

is generated by the direct images of coherent matrix factorizations of the zero potential from the closed embedding $Z'\times_k Z''\to X'\times_k X''$.

Furthermore, let X'_0, X''_0, and Y_0 denote the zero loci of the functions w', w'', and $w'_1+w''_2$ on X', X'', and $X'\times_k X''$, respectively. Denote the natural closed embeddings by $i':X'_0\to X'$, $i'':X''_0\to X''$, $\iota:X'_0\times X''_0\to Y_0$, and $h:Y_0\to X'\times X''$. The external tensor product functor (cf. [Polishchuk and Positselski 2012, Lemma 4.8.B])

$$\otimes_k:\mathsf{D}^{\mathsf{b}}_{Sing}(X'_0/X')\times\mathsf{D}^{\mathsf{b}}_{Sing}(X''_0/X'')\longrightarrow\mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/(X'\times_k X''))\qquad(13)$$

is well-defined since for any bounded complexes of coherent sheaves $\mathcal{F}^\bullet$ on X' and $\mathcal{K}^\bullet$ on X''_0 one has $\iota_*(\mathbb{L}i'^*\mathcal{F}^\bullet\otimes_k\mathcal{K}^\bullet)\simeq\mathbb{L}h^*((\mathrm{id}_{X'}\times i'')_*(\mathcal{F}^\bullet\otimes_k\mathcal{K}^\bullet))$. Indeed, the square diagram of closed embeddings

$$\begin{array}{ccc} X'_0 \times_k X''_0 & \longrightarrow & X' \times_k X''_0 \\ \downarrow & & \downarrow \\ Y_0 & \longrightarrow & X' \times X'' \end{array}$$

is Cartesian and the higher derived tensor products related to the construction of this relative Cartesian product of schemes all vanish.

The functor $\Upsilon : \mathsf{D}^{\mathsf{b}}_{Sing}(Y_0/(X' \times_k X'')) \to \mathsf{D}^{\mathsf{abs}}((X' \times_k X'',\, \mathcal{O},\, w'_1 + w''_2)\text{-}\mathsf{coh})$ (see Section 2.7) and the similar functors for the potentials w' and w'' on X' and X'' transform the external product functor (10) into the external tensor product functor (13). By the assumption, one has $Z' \subset X'_0$ and $Z'' \subset X''_0$. It remains to apply Lemma B.1.6 in order to finish the proof of the theorem. □

B.2.3. *Internal Hom of matrix factorizations.* Let X be a separated Noetherian scheme. Let $\mathcal{L}$ be a line bundle on X and $w', w'' \in \mathcal{L}(X)$ be its global sections. Then given a matrix factorization $\mathcal{U}^0 \to \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L}$ of the potential w' and a matrix factorization $\mathcal{V}^0 \to \mathcal{V}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{V}^0 \otimes_{\mathcal{O}_X} \mathcal{L}$ of the potential w'' on the scheme X (in the symbolic notation of Section 2.2), one can construct the matrix factorization

$$\begin{aligned} &\mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{V}^0 \oplus \mathcal{U}^1 \otimes_{\mathcal{O}_X} \mathcal{V}^1 \\ &\qquad \longrightarrow \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \otimes_{\mathcal{O}_X} \mathcal{V}^0 \oplus \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{V}^1 \otimes \mathcal{L}^{\otimes 1/2} \\ &\qquad\qquad \longrightarrow \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{V}^0 \otimes_{\mathcal{O}_X} \mathcal{L} \oplus \mathcal{U}^1 \otimes_{\mathcal{O}_X} \mathcal{V}^1 \otimes_{\mathcal{O}_X} \mathcal{L} \end{aligned}$$

of the potential $w' + w''$ on X. Here the tensor product $\mathcal{U}^1 \otimes_{\mathcal{O}_X} \mathcal{V}^1$ is defined as the sheaf $(\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2}) \otimes_{\mathcal{O}_X} (\mathcal{V}^1 \otimes \mathcal{L}^{\otimes 1/2}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes -1}$ on X, while the differential on the tensor product of matrix factorizations is given by the conventional rule $d(u \otimes v) = d(u) \otimes v + (-1)^{|u|} u \otimes d(v)$.

We denote the matrix factorization so obtained by $\mathcal{U} \otimes_{\mathcal{O}_X} \mathcal{V}$ and call it the *tensor product* of two matrix factorizations $\mathcal{U}$ and $\mathcal{V}$ of two sections w' and w'' of the same line bundle $\mathcal{L}$ on a scheme X. Restricting to the cases when one or both matrix factorizations are flat, and passing to the coderived categories, one obtains the induced tensor product functors

$$\begin{aligned} \otimes_{\mathcal{O}_X} : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w')\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \times \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w'')\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \\ \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\, \mathcal{L},\, w' + w'')\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \end{aligned} \tag{14}$$

and

$$\begin{aligned} \otimes_{\mathcal{O}_X} : \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w')\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \times \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w'')\text{-}\mathsf{qcoh}) \\ \longrightarrow \mathsf{D}^{\mathsf{co}}((X,\, \mathcal{L},\, w' + w'')\text{-}\mathsf{qcoh}). \end{aligned} \tag{15}$$

The functors (14) and (15) are well-defined since the tensor product with a flat (quasicoherent) matrix factorization takes a short exact sequence of flat matrix factorizations to a short exact sequence of flat matrix factorizations, the tensor product with a flat matrix factorization takes a short exact sequence of quasicoherent matrix factorizations to a short exact sequence of quasicoherent matrix factorizations, and the tensor product with a quasicoherent matrix factorization takes a short exact sequence of flat matrix factorizations to a short exact sequence of quasicoherent matrix factorizations. Also, the tensor product functor preserves infinite direct sums.

Given a quasicoherent matrix factorization $\mathcal{U}^0 \to \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{U}^0 \otimes_{\mathcal{O}_X} \mathcal{L}$ of a potential $w' \in \mathcal{L}(X)$ and a quasicoherent matrix factorization $\mathcal{V}^0 \to \mathcal{V}^1 \otimes \mathcal{L}^{\otimes 1/2} \to \mathcal{V}^1 \otimes_{\mathcal{O}_X} \mathcal{L}$ of a potential $w'' \in \mathcal{L}(X)$ on the scheme X, one can construct the quasicoherent matrix factorization

$$\begin{aligned}&\mathcal{H}om_{X\text{-qc}}(\mathcal{U}^0, \mathcal{V}^0) \oplus \mathcal{H}om_{X\text{-qc}}(\mathcal{U}^1, \mathcal{V}^1) \\ &\qquad \longrightarrow \mathcal{H}om_{X\text{-qc}}(\mathcal{U}^0,\, \mathcal{V}^1 \otimes \mathcal{L}^{\otimes 1/2}) \oplus \mathcal{H}om_{X\text{-qc}}(\mathcal{U}^1, \mathcal{V}^0) \otimes \mathcal{L}^{\otimes 1/2} \\ &\qquad\qquad \longrightarrow \mathcal{H}om_{X\text{-qc}}(\mathcal{U}^0, \mathcal{V}^0) \otimes_{\mathcal{O}_X} \mathcal{L} \oplus \mathcal{H}om_{X\text{-qc}}(\mathcal{U}^1, \mathcal{V}^1) \otimes_{\mathcal{O}_X} \mathcal{L}\end{aligned}$$

of the potential $w'' - w'$ on X. Here the sheaf $\mathcal{H}om_{X\text{-qc}}(\mathcal{U}^1, \mathcal{V}^0) \otimes \mathcal{L}^{\otimes 1/2}$ is defined as the tensor product $\mathcal{H}om_{X\text{-qc}}(\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2},\, \mathcal{V}^0) \otimes_{\mathcal{O}_X} \mathcal{L}$, while the differential on the internal Hom is given by the conventional rule $d(g)(u) = d(g(u)) - (-1)^{|g|} g(d(u))$.

We denote the matrix factorization so obtained by $\mathcal{H}om_{X\text{-qc}}(\mathcal{U}, \mathcal{V})$ and call it the matrix factorization of *quasicoherent internal Hom* between the quasicoherent matrix factorizations $\mathcal{U}$ and $\mathcal{V}$ of two sections w' and w'' of the same line bundle $\mathcal{L}$ on a scheme X. Restricting to the case when the matrix factorization in the second argument is injective, one obtains the induced internal Hom functor

$$\begin{aligned}\mathcal{H}om_{X\text{-qc}} : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w')\text{-}\mathsf{qcoh})^{\mathsf{op}} \times H^0((X, \mathcal{L}, w'')\text{-}\mathsf{qcoh}_{\mathsf{inj}}) \\ \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\, \mathcal{L},\, w'' - w')\text{-}\mathsf{qcoh}), \quad (16)\end{aligned}$$

which can be also viewed as the right derived internal Hom functor

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{X\text{-qc}} : \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w')\text{-}\mathsf{qcoh})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w'')\text{-}\mathsf{qcoh}) \\ \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\, \mathcal{L},\, w'' - w')\text{-}\mathsf{qcoh}). \quad (17)\end{aligned}$$

Remark B.2.3. Alternatively, one could restrict the quasicoherent internal Hom functor to pairs of quasicoherent matrix factorizations which are *both* injective, obtaining the triangulated functor

$$\begin{aligned}\mathcal{H}om_{X\text{-qc}} : H^0((X, \mathcal{L}, w')\text{-}\mathsf{qcoh}_{\mathsf{inj}})^{\mathsf{op}} \times H^0((X, \mathcal{L}, w'')\text{-}\mathsf{qcoh}_{\mathsf{inj}}) \\ \longrightarrow H^0((X,\, \mathcal{L},\, w'' - w')\text{-}\mathsf{qcoh}_{\mathsf{fl}}), \quad (18)\end{aligned}$$

which can be also viewed as a derived internal Hom functor

$$\mathbb{LR}\,\mathcal{H}om_{X\text{-qc}} : \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w')\text{-}\mathsf{qcoh})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w'')\text{-}\mathsf{qcoh}) \\ \longrightarrow \mathsf{D}^{\mathsf{abs}}((X,\,\mathcal{L},\,w''-w')\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \quad (19)$$

that is a left derived functor in its first argument and a right derived functor in the second one. Notice that the derived functor so obtained does *not* agree with the right derived functor defined above; i.e., the composition of the functor (19) with the natural fully faithful functor $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w''-w')\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \to \mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w''-w')\text{-}\mathsf{qcoh})$ and the Verdier localization functor $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{L},w')\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w')\text{-}\mathsf{qcoh})$ is *not* isomorphic to the functor (17).

In particular, when $w' = w''$, the functors (16) and (17) take values in the absolute derived category of quasicoherent matrix factorizations of the zero potential $0 \in \mathcal{L}(X)$. The objects of this category are simply complexes of quasicoherent sheaves $\mathcal{M}^\bullet$ on X endowed with a 2-periodicity isomorphism $\mathcal{M}^\bullet[2] \simeq \mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{L}$. So there is a natural forgetful functor

$$\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},0)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}) \quad (20)$$

and the similar functors acting on the homotopy, absolute derived, etc. categories of flat, coherent, locally free, etc. matrix factorizations.

Furthermore, there is the derived global sections functor

$$\mathbb{R}\Gamma(X,-) : \mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}(\mathbb{Z}\text{-}\mathsf{mod}) \quad (21)$$

taking values in the derived category of abelian groups and defined using either the injective resolutions or the Čech construction (see Sections 1.8–1.9). In fact, the functor (21) factorizes through the conventional derived category $\mathsf{D}(X\text{-}\mathsf{qcoh})$.

Composing the forgetful functor with the functor of underived global sections of complexes of quasicoherent sheaves, one obtains a triangulated functor

$$\Gamma(X,-) : H^0((X,\mathcal{L},0)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}(\mathbb{Z}\text{-}\mathsf{mod}). \quad (22)$$

Alternatively, the functor (22) can be defined as the functor $\mathrm{Hom}_{(X,\mathcal{L},0)\text{-}\mathsf{qcoh}}(\mathcal{O}_X,-)$, where the structure sheaf $\mathcal{O}_X$ is viewed as a matrix factorization $(\mathcal{U}^0,\mathcal{U}^1)$ of the potential $0 \in \mathcal{L}(X)$ with the components $\mathcal{U}^0 = \mathcal{O}_X$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} = 0$.

Similarly, composing the functors (20) and (21), one obtains a triangulated functor

$$\mathbb{R}\Gamma(X,-) : \mathsf{D}^{\mathsf{co}}((X,\mathcal{L},0)\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}(\mathbb{Z}\text{-}\mathsf{mod}), \quad (23)$$

which can be alternatively described as the functor $\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},0)\text{-}\mathsf{qcoh})}(\mathcal{O}_X,-[*])$. In the case when $\mathcal{L} = \mathcal{O}_X$, the functors (22) and (23) can be viewed as taking values in the derived category of $\mathbb{Z}/2$-graded (2-periodic) complexes of abelian groups.

For any quasicoherent matrix factorizations $\mathcal{K}$ and $\mathcal{M}$ of a potential $w \in \mathcal{L}(X)$ on the scheme X there is a natural isomorphism of complexes of abelian groups

$$\mathrm{Hom}_{(X,\mathcal{L},w)\text{-qcoh}}(\mathcal{K},\mathcal{M}) \simeq \Gamma(X, \mathcal{H}om_{X\text{-qc}}(\mathcal{K},\mathcal{M})), \tag{24}$$

and more generally, for any quasicoherent matrix factorizations $\mathcal{K}$ and $\mathcal{E}$ of potentials w' and $w'' \in \mathcal{L}(X)$ and a quasicoherent matrix factorization $\mathcal{M}$ of the potential $w' + w''$ on the scheme X there is a natural isomorphism of complexes

$$\mathrm{Hom}_{(X,\,\mathcal{L},\,w'+w'')\text{-qcoh}}(\mathcal{K} \otimes_{\mathcal{O}_X} \mathcal{E},\ \mathcal{M}) \simeq \mathrm{Hom}_{(X,\mathcal{L},w'')\text{-qcoh}}(\mathcal{E}, \mathcal{H}om_{X\text{-qc}}(\mathcal{K},\mathcal{M})). \tag{25}$$

Lemma B.2.3. *Let $\mathcal{K}$ be a quasicoherent matrix factorization and $\mathcal{M}$ be an injective quasicoherent matrix factorization of a potential $w \in \mathcal{L}(X)$. Let*

$$\mathcal{H}om_{X\text{-qc}}(\mathcal{K},\mathcal{M}) \longrightarrow \mathcal{J}$$

be a closed morphism with a coacyclic cone between quasicoherent matrix factorizations of the potential $0 \in \mathcal{L}(X)$ from the matrix factorization of quasicoherent internal Hom into an injective matrix factorization $\mathcal{J}$. Then the induced morphism

$$\Gamma(X, \mathcal{H}om_{X\text{-qc}}(\mathcal{K},\mathcal{M})) \longrightarrow \Gamma(X,\mathcal{J})$$

is a quasi-isomorphism of complexes of abelian groups.

Proof. Let $0 \to \mathcal{H}om_{X\text{-qc}}(\mathcal{K},\mathcal{M}) \to \mathcal{I}^\bullet$ be a right resolution of the matrix factorization $\mathcal{H}om_{X\text{-qc}}(\mathcal{K},\mathcal{M})$ by injective matrix factorization $\mathcal{I}^i$. Then one can take $\mathcal{J}$ to be the total matrix factorization of the complex $\mathcal{I}^\bullet$ constructed by passing to the infinite direct sums along the diagonals. Notice that the functor of global sections of quasicoherent sheaves on X commutes with the infinite direct sums. It remains to show that the functor $\Gamma(X,-) = \mathrm{Hom}_{(X,\mathcal{L},w'')\text{-qcoh}}(\mathcal{O}_X,-)$ preserves the exactness of the sequence $0 \to \mathcal{H}om_{X\text{-qc}}(\mathcal{K},\mathcal{M}) \to \mathcal{I}^\bullet$ (cf. the proof of Proposition B.2.2).

In fact, we claim that the Ext groups from flat quasicoherent sheaves to the components of $\mathcal{H}om_{X\text{-qc}}(\mathcal{K},\mathcal{M})$ vanish in the abelian category X-qcoh. This assertion follows from the results of [Positselski 2012, Lemma 2.5.3(c) and Corollary 4.1.9(b)] (the argument is based essentially on the above Lemma A.1). ☐

We recall the constructions of the (underived and derived) direct and inverse image functors for matrix factorizations from Sections 1.8–1.9 and 3.5–3.6. In addition to the conventional adjunction of the (underived) direct and inverse image functors f_* and f^* (as mentioned in Section 1.8), there is also the "internal Hom adjunction", formulated as follows.

Let $f : Z \to Y$ be a morphism of separated Noetherian schemes, $\mathcal{L}$ be a line bundle on Y, and w', $w'' \in \mathcal{L}(Y)$ be two global sections. Let $\mathcal{K} \in H^0((Y,\mathcal{L},w')\text{-qcoh})$

and $\mathcal{M} \in H^0((Z,\ f^*\mathcal{L},\ f^*w'')\text{-}\mathsf{qcoh})$ be quasicoherent matrix factorizations on Y and Z. Then there is a natural isomorphism

$$f_* \mathcal{H}om_{Z\text{-}\mathsf{qc}}(f^*\mathcal{K}, \mathcal{M}) \simeq \mathcal{H}om_{Y\text{-}\mathsf{qc}}(\mathcal{K}, f_*\mathcal{M}) \tag{26}$$

of quasicoherent matrix factorizations of the potential $w'' - w'$ on Y.

Now let X be a separated scheme of finite type over a field k, and let $w', w'' \in \mathcal{O}(X)$ be two global regular functions on X. Denote by p_1 and p_2 the natural projections $X \times_k X \rightrightarrows X$, and consider the regular function $w_1' + w_2'' = p_1^* w' + p_2^* w''$ on $X \times_k X$. Let $\Delta : X \to X \times_k X$ denote the diagonal map.

Let $\mathcal{N}$ and $\mathcal{K}$ be quasicoherent matrix factorizations of the potentials w' and w'' on X. Then there is a natural isomorphism $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{K} \simeq \Delta^*(\mathcal{N} \otimes_k \mathcal{K})$ of matrix factorizations of the potential $w' + w''$ on X. Therefore, given a quasicoherent matrix factorization $\mathcal{M}$ of the potential $w' + w'' \in \mathcal{O}(X)$, one has a natural isomorphism of $\mathbb{Z}/2$-graded complexes of abelian groups

$$\begin{aligned} &\operatorname{Hom}_{(X,\mathcal{O},w'')\text{-}\mathsf{qcoh}}(\mathcal{K}, \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{N}, \mathcal{M})) \\ &\qquad\qquad \simeq \operatorname{Hom}_{(X\times_k X,\, \mathcal{O},\, w_1' + w_2'')\text{-}\mathsf{qcoh}}(\mathcal{N} \otimes_k \mathcal{K},\ \Delta_*\mathcal{M}). \end{aligned} \tag{27}$$

Proposition B.2.3. (a) *Assume that the matrix factorization $\mathcal{N}$ is coherent and the matrix factorization $\mathcal{M}$ is injective. Let $\Delta_*\mathcal{M} \to \mathcal{J}$ be a closed morphism with a coacyclic cone between quasicoherent matrix factorizations of the potential $w_1' + w_2''$ on $X \times_k X$ from the direct image $\Delta_*\mathcal{M}$ into an injective matrix factorization $\mathcal{J}$. Then there is a natural closed morphism with a coacyclic cone $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{N}, \mathcal{M}) \to p_{2*} \mathcal{H}om_{X\times_k X\text{-}\mathsf{qc}}(p_1^*\mathcal{N}, \mathcal{J})$ of quasicoherent matrix factorizations of the potential w'' on X, and the matrix factorization $p_{2*} \mathcal{H}om_{X\times_k X\text{-}\mathsf{qc}}(p_1^*\mathcal{N}, \mathcal{J})$ is injective.*

(b) *There is a natural isomorphism of $\mathbb{Z}/2$-graded complexes of abelian groups*

$$\begin{aligned} &\operatorname{Hom}_{(X,\mathcal{O},w'')\text{-}\mathsf{qcoh}}(\mathcal{K},\ p_{2*} \mathcal{H}om_{X\times_k X\text{-}\mathsf{qc}}(p_1^*\mathcal{N}, \mathcal{J})) \\ &\qquad\qquad \simeq \operatorname{Hom}_{(X\times_k X,\, \mathcal{O},\, w_1' + w_2'')\text{-}\mathsf{qcoh}}(\mathcal{N} \otimes_k \mathcal{K},\ \mathcal{J}). \end{aligned}$$

Proof. Part (a): The desired closed morphism is provided by the composition

$$\begin{aligned} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{N}, \mathcal{M}) &\simeq p_{2*}\Delta_* \mathcal{H}om_{X\text{-}\mathsf{qc}}(\Delta^* p_1^*\mathcal{N}, \mathcal{M}) \\ &\simeq p_{2*} \mathcal{H}om_{X\times_k X\text{-}\mathsf{qc}}(p_1^*\mathcal{N}, \Delta_*\mathcal{M}) \longrightarrow p_{2*} \mathcal{H}om_{X\times_k X\text{-}\mathsf{qc}}(p_1^*\mathcal{N}, \mathcal{J}). \end{aligned}$$

To prove that this morphism has a coacyclic cone, pick a right resolution $\mathcal{I}^\bullet$ of the matrix factorization $\Delta_*\mathcal{M}$ on $X \times_k X$ by injective matrix factorizations, and take $\mathcal{J}$ to be the totalization of the complex of matrix factorizations $\mathcal{I}^\bullet$ constructed by passing to the infinite direct sums along the diagonals.

Then the complex of matrix factorizations $0 \to \mathcal{H}om_{X\times_k X\text{-}\mathsf{qc}}(p_1^*\mathcal{N}, \Delta_*\mathcal{M}) \to \mathcal{H}om_{X\times_k X\text{-}\mathsf{qc}}(p_1^*\mathcal{N}, \mathcal{I}^\bullet)$ is acyclic since for any affine open subscheme $U \subset X$ the higher Ext spaces between the components of the restrictions of $p_1^*\mathcal{N}$ and $\Delta_*\mathcal{M}$ to $U \times_k U$ vanish. The latter assertion follows from the adjunction of derived functors $\mathbb{L}\Delta^*$ and $\Delta_* = \mathbb{R}\Delta_*$ or $p_1^* = \mathbb{L}p_1^*$ and $\mathbb{R}p_{1*}$ together with the agreement of the derived direct/inverse images of (complexes of) quasicoherent sheaves with the compositions of morphisms of separated Noetherian schemes.

It remains to show that our complex will stay acyclic after applying the direct image functor p_{2*}. According to the argument in the proof of Lemma B.2.3, the components of the matrix factorizations $\mathcal{H}om_{X\times_k X\text{-qc}}(p_1^*\mathcal{N}, \mathcal{I}^i)$ are acyclic for the direct image; so are the components of the matrix factorization $\mathcal{H}om_{X\times_k X\text{-qc}}(p_1^*\mathcal{N}, \Delta_*\mathcal{M})$, in view of the above local argument and since for any affine open subscheme $U \subset X$, the higher Ext spaces between the components of the restrictions of $p_1^*\mathcal{N}$ and $\Delta_*\mathcal{M}$ to $X \times_k U$ vanish. The latter assertion is checked in the same way as above.

Finally, the claim that the matrix factorization in question is injective follows from the computation in part (b), which shows that the left-hand side is an exact functor of the argument $\mathcal{K}$, because the right-hand side is.

Part (b) is straightforward:

$$\begin{aligned}
&\operatorname{Hom}_{(X,\mathcal{O},w'')\text{-qcoh}}(\mathcal{K},\ p_{2*}\,\mathcal{H}om_{X\times_k X\text{-qc}}(p_1^*\mathcal{N}, \mathcal{J})) \\
&\qquad \simeq \operatorname{Hom}_{(X\times_k X,\,\mathcal{O},\,w_2'')\text{-qcoh}}(p_2^*\mathcal{K},\ \mathcal{H}om_{X\times_k X\text{-qc}}(p_1^*\mathcal{N}, \mathcal{J})) \\
&\qquad \simeq \operatorname{Hom}_{(X\times_k X,\,\mathcal{O},\,w_1'+w_2'')\text{-qcoh}}(p_1^*\mathcal{N} \otimes_{\mathcal{O}_{X\times_k X}} p_2^*\mathcal{K},\ \mathcal{J}) \\
&\qquad \simeq \operatorname{Hom}_{(X\times_k X,\,\mathcal{O},\,w_1'+w_2'')\text{-qcoh}}(\mathcal{N} \otimes_k \mathcal{K},\ \mathcal{J}). \qquad \square
\end{aligned}$$

B.2.4. *Hochschild cohomology.* Our goal in the rest of this appendix is to compute the Hochschild (co)homology of the DG-category $\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh})$ corresponding to the triangulated category $\mathsf{D}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh})$. The word "corresponding" here means, first of all, that there is a natural equivalence of (triangulated) categories $H^0\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh}) \simeq \mathsf{D}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh})$ (see [Positselski 2011b, Section 1.2]).

As the absolute derived category is constructed from the homotopy category of matrix factorizations using the Verdier localization procedure, so the DG-category $\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh})$ is obtained by applying a DG-version of localization to the DG-category of coherent matrix factorizations $(X,\mathcal{O},w)\text{-}\mathsf{coh}$ of the potential w on the scheme X (see Section 1.2). Several such localization procedures are known, leading to naturally quasiequivalent DG-categories. As the Hochschild (co)homology of DG-categories are preserved by quasiequivalences [Polishchuk and Positselski 2012, Sections 2.1 and 2.4], it is not very important which localization procedure to choose. To be specific, let us say that we prefer Drinfeld's localization [Drinfeld 2004]. Similarly one localizes the DG-category $(X,\mathcal{O},w)\text{-}\mathsf{qcoh}$ and obtains a DG-category $\mathsf{DG}^{\mathsf{co}}((X, \mathcal{O}, w)\text{-}\mathsf{qcoh})$ "corresponding" to the coderived category $\mathsf{D}^{\mathsf{co}}((X, \mathcal{O}, w)\text{-}\mathsf{qcoh})$.

Our method will naturally allow us to compute the Hochschild cohomology $HH^*(\mathsf{DG}^{\mathsf{abs}}((X, \mathcal{O}, w)\text{-}\mathsf{coh}))$ together with its structure of an associative (in fact, supercommutative, but we will neither prove nor use this fact) $\mathbb{Z}/2$-graded algebra over k. Similarly, the Hochschild homology $HH_*(\mathsf{DG}^{\mathsf{abs}}((X, \mathcal{O}, w)\text{-}\mathsf{coh}))$ will be computed together with its structure of a $\mathbb{Z}/2$-graded module over the $\mathbb{Z}/2$-graded associative algebra $HH^*(\mathsf{DG}^{\mathsf{abs}}((X, \mathcal{O}, w)\text{-}\mathsf{coh}))$.

Let X be a separated scheme of finite type over a field k and $w \in \mathcal{O}(X)$ be a global regular function. Assume that the morphism of schemes $w : X \to \mathbb{A}^1_k$ is flat. Consider the Cartesian square $X \times_k X$ and endow it with the potential (global function) $w_2 - w_1 = p_2^*(w) - p_1^*(w)$.

Any $\mathbb{Z}/2$-graded complex of quasicoherent sheaves $\mathcal{K}^\bullet$ on X can be viewed as a matrix factorization of the potential $0 \in \mathcal{O}(X)$. Furthermore, one can take its direct image $\Delta_*\mathcal{K}^\bullet$ with respect to the diagonal embedding $\Delta : X \to X \times_k X$ and consider it as a quasicoherent matrix factorization of the potential $w_2 - w_1$ on $X \times_k X$. Given a bounded $\mathbb{Z}$-graded complex of quasicoherent sheaves $\mathcal{K}^\bullet$, one can associate a $\mathbb{Z}/2$-complex with it (by taking direct sums of all terms with the same parity) and then apply the above constructions.

Theorem B.2.4. *Assume that there exists a closed subscheme $Z \subset X$ such that $w|_Z = 0$, the function w is noncritical on $X \setminus Z$, and the scheme Z admits a smooth stratification over k. In particular, if the field k is perfect, it suffices to require that the function w on X have no critical values but zero (and take $Z = \{w = 0\}$). Then there is a natural isomorphism between the Hochschild cohomology algebra $HH^*(\mathsf{DG}^{\mathsf{abs}}((X, \mathcal{O}, w)\text{-}\mathsf{coh}))$ and the Ext algebra*

$$\operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}((X \times_k X, \mathcal{O}, w_2 - w_1)\text{-}\mathsf{qcoh})}(\Delta_*\mathcal{D}_X^\bullet,\ \Delta_*\mathcal{D}_X^\bullet[*]),$$

where $\mathcal{D}_X^\bullet$ denotes a dualizing complex on X.

Proof. By the definition, the Hochschild cohomology algebra of a $\mathbb{Z}/2$-graded DG-category DG is the $\mathbb{Z}/2$-graded algebra $\operatorname{Hom}_{\mathsf{D}(\mathsf{DG} \otimes_k \mathsf{DG}^{\mathsf{op}}\text{-}\mathsf{mod})}(\mathsf{DG}, \mathsf{DG}[*])$, where the Hom is taken in the conventional derived category $\mathsf{D}(\mathsf{DG} \otimes_k \mathsf{DG}^{\mathsf{op}}\text{-}\mathsf{mod})$ of DG-bimodules over DG (or DG-modules over $\mathsf{DG} \otimes_k \mathsf{DG}^{\mathsf{op}}$) between two copies of the diagonal DG-bimodule DG over DG [Polishchuk and Positselski 2012, Sections 2.4 and 3.1].

Specializing to the case of the DG-category $\mathsf{DG}_w = \mathsf{DG}^{\mathsf{abs}}((X, \mathcal{O}, w)\text{-}\mathsf{coh})$, we notice, first of all, that the contravariant Serre–Grothendieck duality functor $\mathcal{M} \longmapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{D}_X^\bullet)$ (see Section 2.5) provides a quasiequivalence between the DG-categories $\mathsf{DG}^{\mathsf{abs}}((X, \mathcal{O}, w)\text{-}\mathsf{coh})^{\mathsf{op}}$ and $\mathsf{DG}^{\mathsf{abs}}((X, \mathcal{O}, -w)\text{-}\mathsf{coh})$. Furthermore, the external tensor product is a DG-functor

$$\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},-w)\text{-}\mathsf{coh})\otimes_k \mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh}) \\ \longrightarrow \mathsf{DG}^{\mathsf{abs}}((X\times_k X,\ \mathcal{O},\ w_2-w_1)\text{-}\mathsf{coh}), \quad (28)$$

which, according to Proposition B.2.2 and Theorem B.2.2, induces an equivalence between the derived categories of (left or right) DG-modules over the two DG-categories in the left-hand and right-hand sides. Composing the Serre–Grothendieck duality with the external tensor product, we obtain (perhaps, after replacing our DG-categories with naturally quasiequivalent ones) a DG-functor

$$\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh})^{\mathsf{op}}\otimes_k \mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh}) \\ \longrightarrow \mathsf{DG}^{\mathsf{abs}}((X\times_k X,\ \mathcal{O},\ w_2-w_1)\text{-}\mathsf{coh}) \quad (29)$$

having the same property with respect to the derived categories of DG-modules over the left-hand and right-hand sides as the DG-functor (28).

We are interested specifically in the diagonal right DG-module over $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w$, that is, the contravariant functor from $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w$ to the DG-category of $\mathbb{Z}/2$-graded complexes of k-vector spaces taking an object $(\mathcal{M}^{\mathrm{op}},\mathcal{K})$ to the complex $\mathrm{Hom}_{\mathsf{DG}}(\mathcal{K},\mathcal{M})$. It is claimed that the diagonal DG-module is naturally quasi-isomorphic to the DG-module obtained by composing the DG-functor (29) with the right DG-module over the right-hand side represented by the object

$$\Delta_*\mathcal{D}_X^\bullet\in\mathsf{DG}^{\mathsf{abs}}((X\times_k X,\ \mathcal{O},\ w_2-w_1)\text{-}\mathsf{coh})\subset\mathsf{DG}^{\mathsf{co}}((X\times_k X,\ \mathcal{O},\ w_2-w_1)\text{-}\mathsf{qcoh}).$$

Indeed, for any quasicoherent matrix factorizations $\mathcal{K}$ and $\mathcal{N}$ of the potentials w and $-w$ on X there is a natural isomorphism of $\mathbb{Z}/2$-graded complexes of abelian groups (see (27))

$$\mathrm{Hom}_{(X,\mathcal{O},w)\text{-}\mathsf{qcoh}}(\mathcal{K},\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N},\mathcal{D}_X^\bullet)) \\ \simeq \mathrm{Hom}_{(X\times_k X,\,\mathcal{O},\,w_2-w_1)\text{-}\mathsf{qcoh}}(\mathcal{N}\otimes_k\mathcal{K},\ \Delta_*\mathcal{D}_X^\bullet).$$

Proposition B.2.3 shows how one can pass from this isomorphism to a quasi-isomorphism of the similar complexes of morphisms in the DG-categories $\mathsf{DG}^{\mathsf{co}}((X,\mathcal{O},w)$ -$\mathsf{qcoh})$ and $\mathsf{DG}^{\mathsf{co}}((X\times_k X,\ \mathcal{O},\ w_2-w_1)\text{-}\mathsf{qcoh})$.

Now morphisms between representable DG-modules in the derived category of DG-modules over a DG-category DG are computed by the complex of morphisms in DG between the representing objects, so our proof is finished. ☐

Remark B.2.4. Given a separated scheme X of finite type over a field k, let $\mathcal{D}_X^\bullet$ be a dualizing complex on X and $\mathcal{D}_{X\times_k X}^\bullet$ be a dualizing complex on $X\times_k X$ such that $\mathcal{D}_X^\bullet \simeq \mathbb{R}\Delta^!(\mathcal{D}_{X\times_k X}^\bullet)$. Then the antiequivalence of absolute derived categories

$$\mathcal{H}om_{X\times_k X\text{-qc}}(-,\,\mathcal{D}^\bullet_{X\times_k X})\colon \mathsf{D}^{\mathsf{abs}}((X\times_k X,\,\mathcal{O},\,w_1-w_2)\text{-}\mathsf{coh})^{\mathsf{op}} \\ \simeq \mathsf{D}^{\mathsf{abs}}((X\times_k X,\,\mathcal{O},\,w_2-w_1)\text{-}\mathsf{coh})$$

from Proposition 2.5 transforms the object $\Delta_*\mathcal{O}_X\in\mathsf{D}^{\mathsf{abs}}((X\times_k X,\mathcal{O},w_1-w_2)\text{-}\mathsf{coh})$ into the object $\Delta_*\mathcal{D}_X^\bullet\in\mathsf{D}^{\mathsf{abs}}((X\times_k X,\,\mathcal{O},\,w_2-w_1)\text{-}\mathsf{coh})$ (see Proposition 3.7). Therefore, in the assumptions of Theorem B.2.4, the Hochschild cohomology algebra $HH^*(\mathsf{D}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh}))$ of the DG-category $\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh})$ can be also identified with the Ext algebra

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{abs}}((X\times_k X,\mathcal{O},w_1-w_2)\text{-}\mathsf{coh})}(\Delta_*\mathcal{O}_X,\,\Delta_*\mathcal{O}_X[*])^{\mathrm{op}}$$

(cf. Remark B.2.7 below).

B.2.5. *Cotensor product of complexes of quasicoherent sheaves.* Let X be a separated Noetherian scheme. Then there is a tensor product functor on the coderived category of ($\mathbb{Z}$-graded complexes of) flat quasicoherent sheaves on X (cf. [Murfet 2007, Chapter 6])

$$\otimes_{\mathcal{O}_X}\colon \mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}_{\mathsf{fl}})\times\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}_{\mathsf{fl}})\longrightarrow\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}_{\mathsf{fl}}),\tag{30}$$

and a similar functor of the tensor product on the coderived categories of flat and arbitrary quasicoherent sheaves (see [Positselski 2012, Section 4.12])

$$\otimes_{\mathcal{O}_X}\colon \mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}_{\mathsf{fl}})\times\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})\longrightarrow\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}).\tag{31}$$

Now let $\mathcal{D}_X^\bullet$ be a dualizing complex on X (viewed, as usual, as a finite complex of injective quasicoherent sheaves). Then the equivalence of triangulated categories $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}_{\mathsf{fl}})\simeq\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$ constructed using the dualizing complex $\mathcal{D}_X^\bullet$ (see [Murfet 2007, Chapter 8]) transforms the tensor product functor (30) into the tensor product functor (31). One can use the same equivalence of categories to define a tensor triangulated category structure with the unit object $\mathcal{D}_X^\bullet$ on the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$. We call this operation the *cotensor product* of complexes of quasicoherent sheaves on X and denote it by

$$\square_{\mathcal{D}_X^\bullet}\colon \mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})\times\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})\longrightarrow\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh}).\tag{32}$$

Explicitly, $\mathcal{N}^\bullet\,\square_{\mathcal{D}_X^\bullet}\,\mathcal{M}^\bullet=\mathcal{D}_X^\bullet\otimes_{\mathcal{O}_X}\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet,\mathcal{N}^\bullet)\otimes_{\mathcal{O}_X}\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet,\mathcal{M}^\bullet)$ for any complexes of injective quasicoherent sheaves $\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ on X (cf. Lemma 1.7(b)) and also $\mathcal{N}^\bullet\,\square_{\mathcal{D}_X^\bullet}\,\mathcal{M}^\bullet=\mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet,\mathcal{N}^\bullet)\otimes_{\mathcal{O}_X}\mathcal{M}^\bullet$ for any complex of injective quasicoherent sheaves $\mathcal{N}^\bullet$ and any complex of quasicoherent sheaves $\mathcal{M}^\bullet$ on X.

Recall that the full triangulated subcategory of bounded-below complexes in $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$ is equivalent to $\mathsf{D}^+(X\text{-}\mathsf{qcoh})$ (see [Positselski 2010, Lemma 2.1 and

Remark 4.1] or [Positselski 2012, Lemma A.1.2]). Denote by $\mathsf{D}^+_{\mathsf{coh}}(X\text{-}\mathsf{qcoh})$ the full triangulated subcategory in $\mathsf{D}^+(X\text{-}\mathsf{qcoh})$ consisting of complexes with coherent cohomology sheaves; then the category $\mathsf{D}^+_{\mathsf{coh}}(X\text{-}\mathsf{qcoh})$ can be also viewed as a full triangulated subcategory in $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$.

For any complexes of quasicoherent sheaves $\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ on X there is a natural morphism of complexes of quasicoherent sheaves

$$\begin{aligned}\mathcal{D}^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}^\bullet_X, \mathcal{N}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}^\bullet_X, \mathcal{M}^\bullet)&\\ \longrightarrow \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{N}^\bullet, \mathcal{D}^\bullet_X) \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{M}^\bullet, \mathcal{D}^\bullet_X),\, \mathcal{D}^\bullet_X)&\end{aligned} \tag{33}$$

on X defined in terms of the composition morphisms

$$\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}^\bullet_X, \mathcal{K}^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{K}^\bullet, \mathcal{D}^\bullet_X) \longrightarrow \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}^\bullet_X, \mathcal{D}^\bullet_X)$$

for complexes of quasicoherent sheaves $\mathcal{K}^\bullet$ on X and the natural quasi-isomorphism

$$\mathcal{D}^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}^\bullet_X, \mathcal{D}^\bullet_X) \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}^\bullet_X, \mathcal{D}^\bullet_X) \longrightarrow \mathcal{D}^\bullet_X.$$

Theorem B.2.5. *For any bounded-below complexes of injective quasicoherent sheaves $\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ with coherent cohomology sheaves on a separated Noetherian scheme X with a dualizing complex $\mathcal{D}^\bullet_X$, the natural morphism* (33) *is a homotopy equivalence of bounded-below complexes of injective quasicoherent sheaves on X with coherent cohomology sheaves.*

Proof. By Lemma 2.5(b) and (c), both sides of (33) are bounded-below complexes of injective quasicoherent sheaves. Since the functor

$$\mathcal{H}om_{X\text{-}\mathsf{qc}}(-\,, \mathcal{D}^\bullet_X) : \mathsf{D}(X\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}(X\text{-}\mathsf{qcoh})$$

takes $\mathsf{D}^+_{\mathsf{coh}}(X\text{-}\mathsf{qcoh}) \subset \mathsf{D}^+(X\text{-}\mathsf{qcoh})$ into $\mathsf{D}^-(X\text{-}\mathsf{coh}) \subset \mathsf{D}^-(X\text{-}\mathsf{qcoh})$ and vice versa, while the derived tensor product functor

$$\otimes^{\mathbb{L}}_X : \mathsf{D}^-(X\text{-}\mathsf{qcoh}) \times \mathsf{D}^-(X\text{-}\mathsf{qcoh}) \longrightarrow \mathsf{D}^-(X\text{-}\mathsf{qcoh})$$

takes $\mathsf{D}^-(X\text{-}\mathsf{coh}) \times \mathsf{D}^-(X\text{-}\mathsf{coh})$ into $\mathsf{D}^-(X\text{-}\mathsf{coh})$, the right-hand side has coherent cohomology sheaves.

It remains to prove the homotopy equivalence claim. Since the homotopy category of bounded-below complexes of injectives is equivalent to $\mathsf{D}^+(X\text{-}\mathsf{qcoh})$, one only has to check that the map is a quasi-isomorphism. Let us first show that it suffices to do so for complexes of sheaves on affine open subschemes $U \subset X$.

Indeed, for any quasicoherent sheaves $\mathcal{E}$ and $\mathcal{K}$ on X there is a natural morphism of quasicoherent sheaves $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{E}, \mathcal{K})|_U \to \mathcal{H}om_{U\text{-}\mathsf{qc}}(\mathcal{E}|_U, \mathcal{K}|_U)$ on U. The morphism of complexes of quasicoherent sheaves $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{E}^\bullet, \mathcal{K}^\bullet)|_U \to$

$\mathcal{H}om_{U\text{-qc}}(\mathcal{E}^\bullet|_U, \mathcal{K}^\bullet|_U)$ is a quasi-isomorphism whenever the complex $\mathcal{E}^\bullet$ has coherent cohomology, $\mathcal{K}^\bullet$ is a complex of injective quasicoherent sheaves, and one of the complexes $\mathcal{E}^\bullet$ and $\mathcal{K}^\bullet$ is finite.

Finally, the tensor product in the right-hand side preserves quasi-isomorphisms of bounded-above complexes of flat quasicoherent sheaves, while the one in the left-hand side is well-defined on the coderived category of (complexes of) flat quasicoherent sheaves. It remains to notice that the functor $\mathrm{Hom}_{X\text{-qc}}(\mathcal{D}_X^\bullet, -)$ in the equivalence of categories in Theorem 2.5 agrees with the restrictions to open subschemes since so does its inverse functor $\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} -$.

Now that we are on an affine scheme U, pick bounded-above complexes of vector bundles $'\mathcal{N}^\bullet$ and $'\mathcal{M}^\bullet$ isomorphic to $\mathcal{H}om_{U\text{-qc}}(\mathcal{N}^\bullet, \mathcal{D}_U^\bullet)$ and $\mathcal{H}om_{U\text{-qc}}(\mathcal{M}^\bullet, \mathcal{D}_U^\bullet)$, respectively, in $\mathsf{D}^-(U\text{-coh})$. Given the isomorphisms

$$\begin{aligned}\mathcal{H}om_{U\text{-qc}}(\mathcal{D}_U^\bullet, \mathcal{H}om_{U\text{-qc}}('\mathcal{N}^\bullet, \mathcal{D}_U^\bullet)) &\simeq \mathcal{H}om_{U\text{-qc}}('\mathcal{N}^\bullet, \mathcal{H}om_{U\text{-qc}}(\mathcal{D}_U^\bullet, \mathcal{D}_U^\bullet)) \\ &\simeq \mathcal{H}om_{U\text{-qc}}('\mathcal{N}^\bullet, \mathcal{O}_U),\end{aligned}$$

and similar isomorphisms for $'\mathcal{M}^\bullet$ in $\mathsf{D}^{\mathsf{co}}(U\text{-qcoh}_{\mathsf{fl}})$, the assertion reduces to the obvious isomorphism of complexes

$$\begin{aligned}\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{U\text{-qc}}('\mathcal{N}^\bullet, \mathcal{O}_U) \otimes_{\mathcal{O}_U} \mathcal{H}om_{U\text{-qc}}('\mathcal{M}^\bullet, \mathcal{O}_U) \\ \simeq \mathrm{Hom}_{U\text{-qc}}('\mathcal{N}^\bullet \otimes_{\mathcal{O}_X} {}'\mathcal{M}^\bullet,\ \mathcal{D}_U^\bullet). \quad \square\end{aligned}$$

For any complexes of quasicoherent sheaves $\mathcal{K}^\bullet$ and $\mathcal{M}^\bullet$ on X we denote by $\mathcal{H}om^{\oplus}_{X\text{-qc}}(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$ the complex of quasicoherent sheaves on X obtained by totalizing the bicomplex of quasicoherent internal Hom sheaves $\mathcal{H}om_{X\text{-qc}}(\mathcal{K}^i, \mathcal{M}^j)$ by taking infinite direct sums along the diagonals. Assuming that $\mathcal{M}^\bullet$ is a complex of injective quasicoherent sheaves, the complex $\mathcal{H}om^{\oplus}_{X\text{-qc}}(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$ is absolutely acyclic with respect to X-qcoh whenever the complex $\mathcal{K}^\bullet$ is (see Lemma 2.5(a)).

In the same assumption, the complex $\mathcal{H}om^{\oplus}_{X\text{-qc}}(\mathcal{K}^\bullet, \mathcal{M}^\bullet)$ is also coacyclic with respect to X-qcoh whenever the complex of quasicoherent sheaves $\mathcal{K}^\bullet$ is acyclic and bounded from above [Positselski 2010, Lemma 2.1]. Therefore, representing the second argument of $\mathcal{H}om^{\oplus}_{X\text{-qc}}$ by complexes of injectives, one can construct the right derived functors

$$\mathbb{R}\,\mathcal{H}om^{\oplus}_{X\text{-qc}} : \mathsf{D}^{\mathsf{abs}}(X\text{-qcoh})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(X\text{-qcoh}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(X\text{-qcoh}) \tag{34}$$

and

$$\mathbb{R}\,\mathcal{H}om^{\oplus}_{X\text{-qc}} : \mathsf{D}^{-}(X\text{-qcoh})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{co}}(X\text{-qcoh}) \longrightarrow \mathsf{D}^{\mathsf{co}}(X\text{-qcoh}) \tag{35}$$

of the functor $\mathcal{H}om^{\oplus}_{X\text{-qc}}$.

For any complexes of quasicoherent sheaves $\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ on X there is a natural morphism of complexes of quasicoherent sheaves

$$\mathcal{N}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{M}^\bullet) \longrightarrow \mathcal{H}om^{\oplus}_{X\text{-qc}}(\mathcal{H}om_{X\text{-qc}}(\mathcal{N}^\bullet, \mathcal{D}_X^\bullet),\ \mathcal{M}^\bullet) \tag{36}$$

on X defined in terms of the composition/evaluation morphism

$$\mathcal{N}^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{N}^\bullet, \mathcal{D}_X^\bullet) \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{M}^\bullet) \longrightarrow \mathcal{M}^\bullet.$$

Proposition B.2.5. *For any bounded-below complex of injective quasicoherent sheaves $\mathcal{N}^\bullet$ with coherent cohomology sheaves and any complex of injective quasicoherent sheaves $\mathcal{M}^\bullet$ on X, the natural morphism* (36) *is a homotopy equivalence of complexes of injective quasicoherent sheaves on X.*

Proof. It suffices to check that the morphism (36) is an isomorphism in $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$. Since both sides of the desired isomorphism are well-defined as functors of the argument $\mathcal{N}^\bullet \in \mathsf{D}^+(X\text{-}\mathsf{qcoh})$ taking values in $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$, one can freely replace $\mathcal{N}^\bullet$ with any quasi-isomorphic bounded-below complex of quasicoherent sheaves. The same applies to the bounded-above complex $\mathcal{H}om_{X\text{-qc}}(\mathcal{N}^\bullet, \mathcal{D}_X^\bullet)$ in the right-hand side of (36).

Since all the functors involved are local in X up to isomorphism in the relevant triangulated categories, it suffices to consider complexes of sheaves over affine open subschemes $U \subset X$ (see Remark 1.3). Representing the object $\mathcal{H}om_{U\text{-qc}}(\mathcal{N}^\bullet, \mathcal{D}_U^\bullet) \in \mathsf{D}^-(U\text{-}\mathsf{coh}) \subset \mathsf{D}^-(U\text{-}\mathsf{qcoh})$ by a bounded-above complex of vector bundles $'\mathcal{N}^\bullet$, it remains to notice the isomorphism of complexes

$$\mathcal{H}om_{U\text{-qc}}('\mathcal{N}^\bullet, \mathcal{D}_U^\bullet) \otimes_{\mathcal{O}_U} \mathcal{F}^\bullet \simeq \mathcal{H}om^{\oplus}_{U\text{-qc}}('\mathcal{N}^\bullet,\ \mathcal{D}_U^\bullet \otimes_{\mathcal{O}_U} \mathcal{F}^\bullet)$$

for any complex of quasicoherent sheaves $\mathcal{F}^\bullet$ on U and point out that the functor $\mathcal{H}om^{\oplus}_{U\text{-qc}}('\mathcal{N}^\bullet, -)$ takes a homotopy equivalence

$$\mathcal{D}_U^\bullet \otimes_{\mathcal{O}_U} \mathcal{H}om_{U\text{-qc}}(\mathcal{D}_U^\bullet, \mathcal{M}^\bullet) \longrightarrow \mathcal{M}^\bullet$$

to a homotopy equivalence. □

In the particular cases when either $\mathcal{N}^\bullet$ is a finite complex of quasicoherent sheaves with coherent cohomology sheaves, or $\mathcal{N}^\bullet$ is a bounded-below complex of quasicoherent sheaves with coherent cohomology sheaves and $\mathcal{M}^\bullet$ is a bounded-below complex of quasicoherent sheaves, the direct sum totalization of the bicomplex $\mathcal{H}om_{X\text{-qc}}$ in the right-hand side of the isomorphism (36) in the coderived category $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$ is no different from the conventional direct product totalization.

Finally, let X be a separated scheme of finite type over a field k and $\pi : X \to \operatorname{Spec} k$ be its structure morphism. Then $\mathcal{D}_X^\bullet \simeq \pi^+\mathcal{O}_{\operatorname{Spec} k}$ (see Section 3.7) is a natural choice of the dualizing complex on X. Let $\pi \times_k \pi : X \times_k X \to \operatorname{Spec} k$ be the structure morphism of the Cartesian square of X over k. Then the dualizing complex

$\mathcal{D}^\bullet_{X\times_k X} = (\pi\times_k\pi)^+\mathcal{O}_{\operatorname{Spec} k}$ on $X\times_k X$ is quasi-isomorphic to the external tensor product $\mathcal{D}^\bullet_X\otimes_k\mathcal{D}^\bullet_X$, and one has $\mathcal{D}^\bullet_X\simeq\Delta^+(\mathcal{D}^\bullet_{X\times_k X})\simeq\mathbb{R}\Delta^!(\mathcal{D}^\bullet_X\otimes_k\mathcal{D}^\bullet_X)$, where $\Delta : X\to X\times_k X$ denotes the diagonal map.

The equivalence of categories $\mathsf{D}^{\mathsf{co}}(X\times_k X\text{-}\mathsf{qcoh_{fl}})\simeq\mathsf{D}^{\mathsf{co}}(X\times_k X\text{-}\mathsf{qcoh})$ constructed using the dualizing complex $\mathcal{D}^\bullet_{X\times_k X}$ and the similar equivalence $\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh_{fl}})\simeq\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})$ constructed using the dualizing complex $\mathcal{D}^\bullet_X$ transform the external tensor product functor

$$\otimes_k : \mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh_{fl}})\times\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh_{fl}})\longrightarrow\mathsf{D}^{\mathsf{co}}(X\times_k X\text{-}\mathsf{qcoh_{fl}})$$

into the external tensor product functor

$$\otimes_k : \mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})\times\mathsf{D}^{\mathsf{co}}(X\text{-}\mathsf{qcoh})\longrightarrow\mathsf{D}^{\mathsf{co}}(X\times_k X\text{-}\mathsf{qcoh})$$

since so do the functors $\mathcal{D}^\bullet_X\otimes_{\mathcal{O}_X}-$ and $\mathcal{D}^\bullet_{X\times_k X}\otimes_{\mathcal{O}_{X\times_k X}}-$.

Let $\mathcal{N}^\bullet$ and $\mathcal{M}^\bullet$ be two complexes of injective quasicoherent sheaves on X, and let $\mathcal{J}^\bullet$ be a complex of injective quasicoherent sheaves on $X\times_k X$ isomorphic to $\mathcal{N}^\bullet\otimes_k\mathcal{M}^\bullet$ in $\mathsf{D}^{\mathsf{co}}(X\times_k X\text{-}\mathsf{qcoh})$. Then in the coderived categories of quasicoherent sheaves one has

$$\begin{aligned}\mathcal{N}^\bullet\,\square_{\mathcal{D}^\bullet_X}\,\mathcal{M}^\bullet = \mathcal{D}^\bullet_X\otimes_{\mathcal{O}_X}\Delta^*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}^\bullet_X,\mathcal{N}^\bullet)\otimes_k\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}^\bullet_X,\mathcal{M}^\bullet))\\ \simeq\mathcal{D}^\bullet_X\otimes_{\mathcal{O}_X}\Delta^*\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}^\bullet_{X\times_k X},\mathcal{J}^\bullet)\simeq\mathbb{R}\Delta^!(\mathcal{N}^\bullet\otimes_k\mathcal{M}^\bullet)\end{aligned}$$

by [Positselski 2012, Theorem 5.15.3] applied to the proper morphism (actually, closed embedding) Δ. We have obtained the formula

$$\mathcal{N}^\bullet\,\square_{\pi^+\mathcal{O}_{\operatorname{Spec} k}}\,\mathcal{M}^\bullet\simeq\mathbb{R}\Delta^!(\mathcal{N}^\bullet\otimes_k\mathcal{M}^\bullet) \tag{37}$$

for the cotensor product of complexes of quasicoherent sheaves on the scheme X (see the end of Section 1.8 for the notation $\mathbb{R}f^!$ as applied to objects of the coderived category of quasicoherent sheaves).

B.2.6. *Cotensor product of matrix factorizations.* The equivalences of triangulated categories

$$\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w'')\text{-}\mathsf{qcoh_{fl}})\simeq\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w'')\text{-}\mathsf{qcoh})$$
$$\mathsf{D}^{\mathsf{co}}((X,\,\mathcal{L},\,w'+w'')\text{-}\mathsf{qcoh_{fl}})\simeq\mathsf{D}^{\mathsf{co}}((X,\,\mathcal{L},\,w'+w'')\text{-}\mathsf{qcoh})$$

constructed using a dualizing complex $\mathcal{D}^\bullet_X$ (see Section 2.5) transform the tensor product functor (14) into the tensor product functor (15). So one can use the same equivalences of categories together with the similar equivalence

$$\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w')\text{-}\mathsf{qcoh_{fl}})\simeq\mathsf{D}^{\mathsf{co}}((X,\mathcal{L},w')\text{-}\mathsf{qcoh})$$

(constructed using the same dualizing complex $\mathcal{D}_X^\bullet$) in order to define a triangulated functor of two arguments

$$\Box_{\mathcal{D}_X^\bullet} : \mathsf{D^{co}}((X, \mathcal{L}, w')\text{-qcoh}) \times \mathsf{D^{co}}((X, \mathcal{L}, w'')\text{-qcoh}) \longrightarrow \mathsf{D^{co}}((X,\ \mathcal{L},\ w' + w'')\text{-qcoh}), \quad (38)$$

which we call the *cotensor product* of matrix factorizations.

As in the case of complexes of quasicoherent sheaves, one explicitly has

$$\mathcal{N} \Box_{\mathcal{D}_X^\bullet} \mathcal{M} = \mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{N}) \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{M})$$

for any injective quasicoherent matrix factorizations $\mathcal{N}$ and $\mathcal{M}$ on X, and also

$$\mathcal{N} \Box_{\mathcal{D}_X^\bullet} \mathcal{M} = \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{N}) \otimes_{\mathcal{O}_X} \mathcal{M}$$

for any injective quasicoherent matrix factorization $\mathcal{N}$ and any quasicoherent matrix factorization $\mathcal{M}$ on X. As in Section B.2.3, $\mathcal{N}$ and $\mathcal{M}$ must be matrix factorizations of two sections w' and w'' of the same line bundle $\mathcal{L}$ on a scheme X; then the cotensor product $\mathcal{N} \Box_{\mathcal{D}_X^\bullet} \mathcal{M}$ is a matrix factorization of the section $w' + w''$ of the line bundle $\mathcal{L}$.

Remark B.2.6. While a matrix factorization version of Proposition B.2.5 is presented below, Remark B.2.3 explains the reason why a matrix factorization version of Theorem B.2.5 cannot be formulated in the way similar to the version for complexes of quasicoherent sheaves above. Still, let $\mathcal{N}$ and $\mathcal{M}$ be coherent matrix factorizations of sections w' and w'' of the same line bundle $\mathcal{L}$ on a separated Noetherian scheme X with enough vector bundles. Let $\mathcal{P}$ and $\mathcal{Q}$ be coherent matrix factorizations of the potentials $-w'$ and $-w'' \in \mathcal{L}(X)$ isomorphic to $\mathcal{H}om_{X\text{-qc}}(\mathcal{N}, \mathcal{D}_X^\bullet)$ and $\mathcal{H}om_{X\text{-qc}}(\mathcal{M}, \mathcal{D}_X^\bullet)$ in the respected coderived categories.

Let $\mathcal{E}_\bullet$ and $\mathcal{F}_\bullet$ be left resolutions of the matrix factorizations $\mathcal{P}$ and $\mathcal{Q}$ by locally free matrix factorizations of finite rank (of the respected potentials). Then the totalizations of the bounded-below complexes of matrix factorizations $\mathcal{H}om_{X\text{-qc}}(\mathcal{E}_\bullet, \mathcal{D}_X^\bullet)$, $\mathcal{H}om_{X\text{-qc}}(\mathcal{F}_\bullet, \mathcal{D}_X^\bullet)$, and $\mathcal{H}om_{X\text{-qc}}(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{F}_\bullet,\ \mathcal{D}_X^\bullet)$ represent objects naturally isomorphic to $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{N} \Box_{\mathcal{D}_X^\bullet} \mathcal{M}$ in the coderived categories of matrix factorizations of the potentials w', w'', and $w' + w''$ (cf. Corollary 2.5).

For any quasicoherent matrix factorizations $\mathcal{M}$ and $\mathcal{N}$ of sections w' and w'' of the same line bundle $\mathcal{L}$ on the scheme X, there is a natural morphism of quasicoherent matrix factorizations of the section $w' + w''$ of the line bundle $\mathcal{L}$ on X

$$\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-qc}}(\mathcal{D}_X^\bullet, \mathcal{M}) \longrightarrow \mathcal{H}om_{X\text{-qc}}(\mathcal{H}om_{X\text{-qc}}(\mathcal{N}, \mathcal{D}_X^\bullet),\ \mathcal{M}) \qquad (39)$$

constructed in the same way as it was done for complexes of quasicoherent sheaves in Section B.2.5.

Proposition B.2.6. *For any coherent matrix factorization $\mathcal{N}$ and injective quasicoherent matrix factorization $\mathcal{M}$ of sections w' and w'' of the same line bundle $\mathcal{L}$ on a separated Noetherian scheme X, the natural morphism* (39) *is an isomorphism in the coderived category of quasicoherent matrix factorizations of the potential $w' + w'' \in \mathcal{L}(X)$.*

Proof. The argument follows the lines of the proof of Proposition B.2.5. The left-hand side of the desired isomorphism is well-defined as a functor of the argument $\mathcal{N} \in \mathsf{D}^{\mathsf{co}}((X, \mathcal{L}, w'')\text{-}\mathsf{qcoh})$ taking values in $\mathsf{D}^{\mathsf{co}}((X,\ \mathcal{L},\ w' + w'')\text{-}\mathsf{qcoh})$, while the right-hand side is well-defined as a functor of the argument $\mathcal{N} \in \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w'')\text{-}\mathsf{coh})$ taking values, say, in the same coderived category. Besides, the right-hand side, viewed as an object of the coderived category, only depends on the matrix factorization $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{N}, \mathcal{D}_X^\bullet)$ viewed as an object of the absolute derived category.

Furthermore, the contravariant Serre–Grothendieck duality $\mathcal{H}om_{X\text{-}\mathsf{qc}}(-, \mathcal{D}_X^\bullet)$ is well-defined as a functor $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w'')\text{-}\mathsf{qcoh}) \to \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, -w'')\text{-}\mathsf{qcoh})$ and takes $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w'')\text{-}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, w'')\text{-}\mathsf{qcoh})$ into $\mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, -w'')\text{-}\mathsf{coh}) \subset \mathsf{D}^{\mathsf{abs}}((X, \mathcal{L}, -w'')\text{-}\mathsf{coh})$, inducing an equivalence between these two subcategories (see Proposition 2.5). In particular, one can conclude that all the functors involved are local in X, and it suffices to prove the desired assertion for matrix factorizations over affine open subschemes $U \subset X$.

Now let $\mathcal{K}$ be a coherent matrix factorization of the potential $-w''$ isomorphic to $\mathcal{H}om_{U\text{-}\mathsf{qc}}(\mathcal{N}, \mathcal{D}_U^\bullet)$ in $\mathsf{D}^{\mathsf{abs}}((U, \mathcal{L}, -w'')\text{-}\mathsf{qcoh})$, and let $\mathcal{E}_\bullet$ be its left resolution by locally free matrix factorizations of the same potential $-w'' \in \mathcal{L}(U)$. Then the matrix factorization $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{K}, \mathcal{M})$ is isomorphic in $\mathsf{D}^{\mathsf{co}}((X,\ \mathcal{L},\ w' + w'')\text{-}\mathsf{qcoh})$ to the totalization of the complex of matrix factorizations $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{E}_\bullet, \mathcal{M})$ constructed by taking infinite direct sums along the diagonals; and the matrix factorization $\mathcal{N} \simeq \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{K}, \mathcal{D}_X^\bullet)$ can be described similarly (cf. the proof of Corollary 2.5).

It remains to notice that the functor of tensoring with $\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{E}_\bullet, \mathcal{O}_X)$ and totalizing by taking infinite direct sums along the diagonals takes the homotopy equivalence

$$\mathcal{D}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{D}_X^\bullet, \mathcal{M}) \longrightarrow \mathcal{M}$$

to a homotopy equivalence of matrix factorizations. $\square$

As in Section B.2.5, we finish by discussing the case of a separated scheme X of finite type over a field k. From now on we also assume that $\mathcal{L} = \mathcal{O}_X$. So let $w', w'' \in \mathcal{O}(X)$ be two global regular functions on X; as in Section B.2.2, we consider the regular function $w_1' + w_2'' = p_1^* w' + p_2^* w''$ on $X \times_k X$. We use the dualizing complexes $\mathcal{D}_X^\bullet = \pi^+ \mathcal{O}_{\operatorname{Spec} k}$ and $\mathcal{D}_{X \times_k X}^\bullet = (\pi \times_k \pi)^+ \mathcal{O}_{\operatorname{Spec} k}$.

The equivalence of categories

$$\mathsf{D}^{\mathsf{co}}((X \times_k X,\ \mathcal{O},\ w_1' + w_2'')\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \simeq \mathsf{D}^{\mathsf{co}}((X \times_k X,\ \mathcal{O},\ w_1' + w_2'')\text{-}\mathsf{qcoh})$$

constructed using the dualizing complex $\mathcal{D}^\bullet_{X\times_k X}$ and the similar equivalences of coderived categories of matrix factorizations of the potentials w' and w'' on X constructed using the dualizing complex $\mathcal{D}^\bullet_X$ transform the external tensor product functor (cf. (9))

$$\otimes_k : \mathsf{D}^{\mathsf{co}}((X,\mathcal{O},w')\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \times \mathsf{D}^{\mathsf{co}}((X,\mathcal{O},w'')\text{-}\mathsf{qcoh}_{\mathsf{fl}}) \\ \longrightarrow \mathsf{D}^{\mathsf{co}}((X\times_k X,\ \mathcal{O},\ w'_1+w''_2)\text{-}\mathsf{qcoh}_{\mathsf{fl}})$$

into the external tensor product functor

$$\otimes_k : \mathsf{D}^{\mathsf{co}}((X,\mathcal{O},w')\text{-}\mathsf{qcoh}) \times \mathsf{D}^{\mathsf{co}}((X,\mathcal{O},w'')\text{-}\mathsf{qcoh}) \\ \longrightarrow \mathsf{D}^{\mathsf{co}}((X\times_k X,\ \mathcal{O},\ w'_1+w''_2)\text{-}\mathsf{qcoh})$$

since so do the functors $\mathcal{D}^\bullet_X\otimes_{\mathcal{O}_X} -$ and $\mathcal{D}^\bullet_{X\times_k X}\otimes_{\mathcal{O}_{X\times_k X}} -$.

Let $\mathcal{N}$ and $\mathcal{M}$ be injective quasicoherent matrix factorizations of the potentials w' and w'' on X, and let $\mathcal{J}$ be an injective quasicoherent matrix factorization of the potential $w'_1+w''_2$ on $X\times_k X$ isomorphic to $\mathcal{N}\otimes_k\mathcal{M}$ in

$$\mathsf{D}^{\mathsf{co}}((X\times_k X,\ \mathcal{O},\ w'_1+w''_2)\text{-}\mathsf{qcoh}).$$

Then in the coderived categories of quasicoherent matrix factorizations one has

$$\mathcal{N}\,\square_{\mathcal{D}^\bullet_X}\,\mathcal{M} = \mathcal{D}^\bullet_X\otimes_{\mathcal{O}_X}\Delta^*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}^\bullet_X,\mathcal{N})\otimes_k\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}^\bullet_X,\mathcal{M})) \\ \simeq \mathcal{D}^\bullet_X\otimes_{\mathcal{O}_X}\Delta^*\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}^\bullet_{X\times_k X},\mathcal{J}) \simeq \mathbb{R}\Delta^!(\mathcal{N}\otimes_k\mathcal{M})$$

by the result of Theorem 3.8 applied to the proper morphism Δ. We have obtained the formula

$$\mathcal{N}\,\square_{\pi^+\mathcal{O}_{\operatorname{Spec} k}}\,\mathcal{M} \simeq \mathbb{R}\Delta^!(\mathcal{N}\otimes_k\mathcal{M}) \tag{40}$$

for the cotensor product of quasicoherent matrix factorizations on the scheme X.

B.2.7. *Hochschild homology.* Let X be a separated scheme of finite type over a field k and $\pi : X\to\operatorname{Spec} k$ be its structure morphism. Let $w\in\mathcal{O}(X)$ be a global regular function; as in Section B.2.4, we assume that the morphism of schemes $w : X\to\mathbb{A}^1_k$ is flat. Consider the scheme $X\times_k X$ and endow it with the potential $w_2-w_1 = p_2^*(w)-p_1^*(w)$. Let $\Delta : X\to X\times_k X$ denote the diagonal morphism.

Theorem B.2.7. *In the assumptions of Theorem* B.2.4, *there is a natural isomorphism between the Hochschild homology module* $HH_*(\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh}))$ *over the algebra* $HH^*(\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh}))$ *and the Ext module*

$$\operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}((X\times_k X,\mathcal{O},w_2-w_1)\text{-}\mathsf{qcoh})}(\Delta_*\mathcal{O}_X,\ \Delta_*\mathcal{D}^\bullet_X[*])$$

over the algebra

$$\mathrm{Hom}_{\mathsf{D^{co}}((X\times_k X,\,\mathcal{O},\,w_2-w_1)\text{-}\mathsf{qcoh})}(\Delta_*\mathcal{D}_X^\bullet,\ \Delta_*\mathcal{D}_X^\bullet[*]).$$

Here $\mathcal{D}_X^\bullet$ denotes the dualizing complex $\pi^+\mathcal{O}_{\operatorname{Spec} k}$ on X.

Proof. By the definition, the Hochschild homology of a $\mathbb{Z}/2$-graded DG-category DG is the $\mathbb{Z}/2$-graded vector space $\mathrm{Tor}_*^{\mathsf{DG}\otimes_k\mathsf{DG}^{\mathsf{op}}}(\mathsf{DG},\mathsf{DG})$ for the diagonal right and left DG-modules DG over the DG-category $\mathsf{DG}\otimes_k\mathsf{DG}^{\mathsf{op}}$ [Polishchuk and Positselski 2012, Sections 2.4 and 3.1]. This is the conventional derived tensor product ("of the first kind") of a left and a right DG-module over a small DG-category. The Hochschild cohomology algebra $\mathrm{Hom}_{\mathsf{D}(\mathsf{DG}\otimes_k\mathsf{DG}^{\mathsf{op}})}(\mathsf{DG},\mathsf{DG}[*])$ acts on the Hochschild homology space via its action on, say, the first argument of the Tor.

As in the proof of Theorem B.2.4, we set $\mathsf{DG}_w=\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh})$; accordingly, $\mathsf{DG}_{-w}=\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},-w)\text{-}\mathsf{coh})$ and

$$\mathsf{DG}_{w_2-w_1}=\mathsf{DG}^{\mathsf{abs}}((X\times_k X,\ \mathcal{O},\ w_2-w_1)\text{-}\mathsf{coh}).$$

The DG-functor $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w\to\mathsf{DG}_{w_2-w_1}$ (29) induces a fully faithful functor between the homotopy categories $H^0(\mathsf{DG}_w)^{\mathsf{op}}\otimes_k H^0(\mathsf{DG}_w)\to H^0(\mathsf{DG}_{w_2-w_1})$ such that every object in the target category can be obtained from objects in the image using the operations of a cone and the passage to a direct summand.

Let $\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)$ denote the DG-category version of the (conventional) derived category of right DG-modules over the DG-category $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}$ (i.e., contravariant DG-functors from $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}$ into the DG-category $\mathsf{DG}(k\text{-}\mathsf{vect})$ of $\mathbb{Z}/2$-graded complexes of k-vector spaces). Let $\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)^0\subset\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)$ denote the full DG-subcategory of DG-modules corresponding to compact objects of the derived category $\mathsf{D}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)$ of right DG-modules.

The derived tensor product with the left DG-module DG_w over $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w$ can be viewed as a covariant DG-functor $\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)\to\mathsf{DG}(k\text{-}\mathsf{vect})$. We are interested in the restriction of this DG-functor to the DG-subcategory $\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)^0$; let us denote it by

$$F:\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)^0\longrightarrow\mathsf{D}(k\text{-}\mathsf{vect}).$$

There is a natural DG-functor $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w\to\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)^0$ assigning to any object of $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w$ the contravariant DG-functor represented by it. Similarly one constructs a DG-functor $\mathsf{DG}_{w_2-w_1}\to\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)^0$ whose composition with the DG-functor $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w\to\mathsf{DG}_{w_2-w_1}$ is naturally quasi-isomorphic to the DG-functor $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w\to\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)^0$.

It is claimed that the composition of the DG-functor

$$\mathsf{DG}_{w_2-w_1}\longrightarrow\mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w)^0$$

with the DG-functor $F : \mathsf{DG}(\mathsf{mod}\text{-}\mathsf{DG}_w^{\mathsf{op}} \otimes_k \mathsf{DG}_w)^0 \to \mathsf{D}(k\text{-vect})$ is naturally quasi-isomorphic to the DG-functor $\mathrm{Hom}_{\mathsf{DG}_{w_2-w_1}}(\Delta_*\mathcal{O}_X, -)$. Since the derived categories of left DG-modules over $\mathsf{DG}_{w_1-w_2}$ and $\mathsf{DG}_w^{\mathsf{op}} \otimes_k \mathsf{DG}_w$ are equivalent, it suffices to construct a quasi-isomorphism between the compositions of the two DG-functors in question with the DG-functor $\mathsf{DG}_w^{\mathsf{op}} \otimes_k \mathsf{DG}_w \to \mathsf{DG}_{w_2-w_1}$.

Indeed, let $(\mathcal{K}^{\mathrm{op}}, \mathcal{M})$ be an object of $\mathsf{DG}_w^{\mathsf{op}} \otimes_k \mathsf{DG}_w$. Then the functor of the (derived or underived) tensor product with the diagonal left DG-module DG_w takes the right DG-module over $\mathsf{DG}_w^{\mathsf{op}} \otimes_k \mathsf{DG}_w$ represented by $(\mathcal{K}^{\mathrm{op}}, \mathcal{M})$ to the complex of k-vector spaces $\mathrm{Hom}_{\mathsf{DG}_w}(\mathcal{K}, \mathcal{M})$. Substituting $\mathcal{K} = \mathcal{H}om_{X\text{-qc}}(\mathcal{N}, \mathcal{D}_X^\bullet)$ with $\mathcal{N} \in \mathsf{DG}_{-w}$ and assuming $\mathcal{M}$ to be represented by an injective matrix factorization isomorphic to the given coherent one in $\mathsf{DG}^{\mathsf{co}}((X, \mathcal{O}, w)\text{-qcoh})$, we have to compute the complex of k-vector spaces $\mathrm{Hom}_{(X,\mathcal{O},w)\text{-qcoh}}(\mathcal{H}om_{X\text{-qc}}(\mathcal{N}, \mathcal{D}_X^\bullet), \mathcal{M})$.

Now the formula (24) together with Lemma B.2.3 allow us to interpret this complex as $\mathbb{R}\Gamma(X, \mathcal{H}om_{X\text{-qc}}(\mathcal{H}om_{X\text{-qc}}(\mathcal{N}, \mathcal{D}_X^\bullet), \mathcal{M}))$. According to Proposition B.2.6 together with the formula (40), this is the same as $\mathbb{R}\Gamma(X, \mathbb{R}\Delta^!(\mathcal{N} \otimes_k \mathcal{M}))$, or, in other notation, $\mathrm{Hom}_{\mathsf{DG}^{\mathsf{co}}((X,\mathcal{O},0)\text{-qcoh})}(\mathcal{O}_X, \mathbb{R}\Delta^!(\mathcal{N} \otimes_k \mathcal{M}))$. Finally, the adjunction of Δ_* and $\mathbb{R}\Delta^!$ allows us to rewrite the complex in question as

$$\mathrm{Hom}_{\mathsf{DG}^{\mathsf{co}}((X \times_k X, \mathcal{O}, w_2-w_1)\text{-qcoh})}(\Delta_*\mathcal{O}_X, \mathcal{N} \otimes_k \mathcal{M}).$$

The desired quasi-isomorphism of DG-functors is obtained.

It remains to recall that, according to the proof of Theorem B.2.4, the diagonal right DG-module DG_w over $\mathsf{DG}_w^{\mathsf{op}} \otimes_k \mathsf{DG}_w$ is represented by the object $\Delta_*\mathcal{D}_X^\bullet \in \mathsf{DG}_{w_2-w_1}$, in order to finish our proof here. $\square$

Remark B.2.7. The Hochschild homology module $HH_*((\mathsf{DG}^{\mathsf{abs}}(X, \mathcal{O}, w)\text{-coh}))$ over the Hochschild cohomology algebra $HH^*((\mathsf{DG}^{\mathsf{abs}}(X, \mathcal{O}, w)\text{-coh}))$ can be also computed as the Ext module

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((X \times_k X, \mathcal{O}, w_1-w_2)\text{-qcoh})}(\Delta_*\mathcal{O}_X, \ \Delta_*\mathcal{D}_X^\bullet[*])$$

over the Ext algebra

$$\mathrm{Hom}_{\mathsf{D}^{\mathsf{abs}}((X \times_k X, \mathcal{O}, w_1-w_2)\text{-coh})}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X[*])^{\mathrm{op}},$$

according to Remark B.2.4. Moreover, the contravariant Serre duality for matrix factorizations over $X \times_k X$ can be used in order to obtain an alternative proof of our Hochschild homology computation. Indeed, for any coherent matrix factorizations

$\mathcal{N}$ and $\mathcal{M}$ of the potentials $-w$ and w on X there are natural quasi-isomorphisms

$$\begin{aligned}
&\operatorname{Hom}_{\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh})}(\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{N},\mathcal{D}_X^\bullet),\mathcal{M})\\
&\ \simeq \operatorname{Hom}_{\mathsf{DG}^{\mathsf{abs}}((X,\mathcal{O},w)\text{-}\mathsf{coh})}(\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{N},\mathcal{D}_X^\bullet),\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{M},\mathcal{D}_X^\bullet),\mathcal{D}_X^\bullet))\\
&\ \simeq \operatorname{Hom}_{\mathsf{DG}^{\mathsf{abs}}((X\times_k X,\mathcal{O},w_1-w_2)\text{-}\mathsf{coh})}(\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{N},\mathcal{D}_X^\bullet)\otimes_k\mathcal{H}om_{X\text{-}\mathsf{qc}}(\mathcal{M},\mathcal{D}_X^\bullet),\Delta_*\mathcal{D}_X^\bullet)\\
&\ \simeq \operatorname{Hom}_{\mathsf{DG}^{\mathsf{abs}}((X\times_k X,\mathcal{O},w_1-w_2)\text{-}\mathsf{coh})}(\mathcal{H}om_{X\times_k X\text{-}\mathsf{qc}}(\mathcal{N}\otimes_k\mathcal{M},\mathcal{D}_{X\times_k X}^\bullet),\Delta_*\mathcal{D}_X^\bullet)\\
&\ \simeq \operatorname{Hom}_{\mathsf{DG}^{\mathsf{abs}}((X\times_k X,\mathcal{O},w_2-w_1)\text{-}\mathsf{coh})}(\Delta_*\mathcal{O}_X,\mathcal{N}\otimes_k\mathcal{M})
\end{aligned}$$

by Proposition 2.5 and the proof of Theorem B.2.4. In other words, while the right diagonal DG-module DG_w over $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w$ is represented by the object

$$\Delta_*\mathcal{D}_X^\bullet\in\mathsf{DG}^{\mathsf{abs}}((X\times_k X,\ \mathcal{O},\ w_2-w_1)\text{-}\mathsf{coh})$$

as a contravariant DG-functor on $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w\subset\mathsf{DG}_{w_2-w_1}$, the left diagonal DG-bimodule DG_w over $\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w$ is represented by the object

$$\Delta_*\mathcal{O}_X^\bullet\in\mathsf{DG}^{\mathsf{abs}}((X\times_k X,\ \mathcal{O},\ w_2-w_1)\text{-}\mathsf{coh})$$

as a covariant DG-functor on

$$\mathsf{DG}_w^{\mathsf{op}}\otimes_k\mathsf{DG}_w\subset\mathsf{DG}_{w_2-w_1}=\mathsf{DG}^{\mathsf{abs}}((X\times_k X,\ \mathcal{O},\ w_2-w_1)\text{-}\mathsf{coh}).$$

B.2.8. *Direct sum over the critical values.* Let X be a separated scheme of finite type over a field k and $\pi : X\to\operatorname{Spec}k$ be its structure morphism. As in Sections B.2.5–B.2.7 (see also Section 3.7), we choose the dualizing complex $\mathcal{D}_X^\bullet\simeq\pi^+\mathcal{O}_{\operatorname{Spec}k}$ on X. Let $w\in\mathcal{O}(X)$ be a global regular function on X such that the morphism of schemes $w : X\to\mathbb{A}_k^1$ is flat (cf. [Orlov 2004; 2012]).

Let $c_1,\dots,c_n\in k$ be a finite number of different elements of the ground field. Assume that there exist closed subschemes $Z_i\subset X$ such that the function w is noncritical on $X\setminus(Z_1\cup\cdots\cup Z_n)$, the restriction of w to Z_i is equal to the constant c_i, and the schemes Z_i admit smooth stratifications over k.

In particular, if the field k is perfect, it suffices to require that the function w has only a finite number of critical values $c_1,\dots,c_n\in\mathbb{A}_k^1$ (i.e., the open subscheme $\mathbb{A}_{k,f}^1\subset\mathbb{A}_k^1$ is nonempty; see Section B.2.1), and all of these values belong to the field k (rather than its algebraic closure). When the field k has zero characteristic, the former condition holds automatically. Then one simply takes Z_i to be the zero locus of the function w_i-c_i on X.

Consider the Cartesian square $X\times_k X$ with the global function $w_2-w_1=p_2^*(w)-p_1^*(w)$ on it. Let $\Delta : X\to X\times_k X$ denote the diagonal morphism. The following result is to be compared with [Polishchuk and Positselski 2012, Corollary 4.10].

Corollary B.2.8. *There are natural isomorphisms of $\mathbb{Z}/2$-graded k-algebras*

$$\bigoplus_{i=1}^{n} HH^*(\mathsf{DG}^{\mathsf{abs}}((X,\,\mathcal{O},\,w-c_i)\text{-}\mathsf{coh}))$$
$$\simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((X\times_k X,\mathcal{O},w_2-w_1)\text{-}\mathsf{qcoh})}(\Delta_*\mathcal{D}_X^\bullet,\ \Delta_*\mathcal{D}_X^\bullet[*])$$
$$\simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{abs}}((X\times_k X,\mathcal{O},w_1-w_2)\text{-}\mathsf{coh})}(\Delta_*\mathcal{O}_X,\Delta_*\mathcal{O}_X[*])^{\mathrm{op}}. \quad (41)$$

There are also natural isomorphisms of $\mathbb{Z}/2$-graded k-modules

$$\bigoplus_{i=1}^{n} HH_*(\mathsf{DG}^{\mathsf{abs}}((X,\,\mathcal{O},\,w-c_i)\text{-}\mathsf{coh}))$$
$$\simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((X\times_k X,\mathcal{O},w_2-w_1)\text{-}\mathsf{qcoh})}(\Delta_*\mathcal{O}_X,\ \Delta_*\mathcal{D}_X^\bullet[*])$$
$$\simeq \mathrm{Hom}_{\mathsf{D}^{\mathsf{co}}((X\times_k X,\mathcal{O},w_1-w_2)\text{-}\mathsf{qcoh})}(\Delta_*\mathcal{O}_X,\ \Delta_*\mathcal{D}_X^\bullet[*]) \quad (42)$$

over the $\mathbb{Z}/2$-graded k-algebra (41).

Proof. For each $i=1,\dots,n$, let Y_i denote the open subscheme $X\setminus\bigcup_{j\neq i}Z_i\subset X$. Let $w_i\in\mathcal{O}(Y_i)$ denote the restriction of the regular function $w-c_i$ to Y_i. The argument is based on the results of Sections B.2.4 and B.2.7 applied to the schemes Y_i (or their open subschemes) endowed with the potentials w_i.

The restriction of morphisms (in the coderived categories) of quasicoherent matrix factorizations to the open subschemes $Y_i\subset X$ defines a $\mathbb{Z}/2$-graded k-algebra morphism from the (middle or) right-hand side to the left-hand side of (41), and a $\mathbb{Z}/2$-graded k-module morphism from the (middle or) right-hand side to the left-hand side of (42). It remains to show that these morphisms are isomorphisms.

For this purpose, one can start with replacing $\Delta_*\mathcal{D}_X^\bullet$ or $\Delta_*\mathcal{O}_X$ in the second argument of the Hom spaces in the middle or right-hand sides of (41) and (42) with an injective matrix factorization $\mathcal{J}$ on $X\times_k X$ representing the same object in the coderived category. Then one notices that the restriction from $X\times_k X$ to its open subscheme $V=\bigcup_{i=1}^{n}Y_i\times_k Y_i$ does not change the Hom spaces in the right-hand sides, as the image of Δ is contained in V.

Finally, one writes down the Čech resolution of the matrix factorization $\mathcal{J}|_V$ corresponding to the covering of the scheme V by its open subschemes $Y_i\times_k Y_i$. This is a finite acyclic complex of injective matrix factorizations, so applying the functor $\mathrm{Hom}_{(V,\mathcal{O},(w_2-w_1)|_V)\text{-}\mathsf{qcoh}}(\mathcal{K},-)$ from any quasicoherent matrix factorization $\mathcal{K}$ preserves its acyclicity. Since the Hom spaces on any intersection of at least two different open subschemes in the covering are zero by Theorems B.2.4– B.2.7 (as w is noncritical on $Y_i\cap Y_j$ for any $i\neq j$), the desired isomorphisms follow. □

Remark B.2.8. The Hochschild cohomology algebra and the Hochschild homology module of the DG-category version $\mathsf{DG}^{\mathsf{b}}(X\text{-}\mathsf{coh})$ of the bounded derived category $\mathsf{D}^{\mathsf{b}}(X\text{-}\mathsf{coh})$ of (complexes of) coherent sheaves on a separated scheme X of finite type over a field k can be computed in the way similar to (but simpler than) the above. The answers are the same as in Theorems B.2.4 and B.2.7:

$$HH^*(\mathsf{DG^b}(X\text{-}\mathsf{coh})) \simeq \mathrm{Hom}_{\mathsf{D}(X\times_k X\text{-}\mathsf{qcoh})}(\Delta_*\mathcal{D}_X^\bullet,\ \Delta_*\mathcal{D}_X^\bullet[*]) \\ \simeq \mathrm{Hom}_{\mathsf{D^b}(X\times_k X\text{-}\mathsf{coh})}(\Delta_*\mathcal{O}_X,\Delta_*\mathcal{O}_X[*])^{\mathrm{op}} \quad (43)$$

and

$$HH_*(\mathsf{DG^b}(X\text{-}\mathsf{coh})) \simeq \mathrm{Hom}_{\mathsf{D}(X\times_k X\text{-}\mathsf{qcoh})}(\Delta_*\mathcal{O}_X,\ \Delta_*\mathcal{D}_X^\bullet[*]), \quad (44)$$

the only difference being that $\mathsf{DG^b}(X\text{-}\mathsf{coh})$ is a $\mathbb{Z}$-graded DG-category and the right-hand sides describe the Hochschild (co)homology as a $\mathbb{Z}$-graded algebra and module. The only assumption is that the scheme X should admit a smooth stratification over k (i.e., it suffices that the field k be perfect).

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