Algebra & Number Theory

Volume 9
2015
No. 5

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Matthew Emerton and Toby Gee

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ALGEBRA AND NUMBER THEORY 9:5 (2015)
dx.doi.org/10.2140/ant.2015.9.1035

p-adic Hodge-theoretic properties of étale cohomology with mod p coefficients, and the cohomology of Shimura varieties

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We prove vanishing results for the cohomology of unitary Shimura varieties with integral coefficients at arbitrary level, and deduce applications to the weight part of Serre's conjecture. In order to do this, we show that the mod p cohomology of a smooth projective variety with semistable reduction over K, a finite extension of $\mathbb{Q}_p$, embeds into the reduction modulo p of a semistable Galois representation with Hodge–Tate weights in the expected range (at least after semisimplifying, in the case of the cohomological degree greater than 1).

The aim of this paper is to establish vanishing results for the cohomology of certain unitary similitude groups. For example, we prove the following result:

Theorem A. *Let X be a projective $U(2,1)$-Shimura variety of some sufficiently small level, and let $\mathcal{F}$ be a canonical local system of $\overline{\mathbb{F}}_p$-vector spaces on X. Let $\mathfrak{m}$ be a maximal ideal of the Hecke algebra acting on the cohomology $H^\bullet(X,\mathcal{F})$, and suppose that there is a Galois representation $\rho_\mathfrak{m}: G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ associated to $\mathfrak{m}$. If we suppose further that we have $\mathrm{SL}_3(k) \subset \rho_\mathfrak{m}(G_F) \subset \overline{\mathbb{F}}_p^\times \mathrm{SL}_3(k)$ for some finite extension $k/\mathbb{F}_p$, and that $\rho_\mathfrak{m}|_{G_{\mathbb{Q}_p}}$ is 1-regular and irreducible, then the localisations $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathcal{F})_\mathfrak{m}$ vanish in degrees $i \neq 2$.*

The first author was supported in part by NSF grant DMS-1003339, and the second author was supported in part by NSF grant DMS-0841491 and EPSRC Mathematics Platform grant EP/I019111/1.
MSC2010: 11F33.
Keywords: p-adic Hodge theory, Shimura varieties.

(See Corollary 3.5.1 and Lemma 4.1.9 below, and see Sections 2 and 3 for the precise definitions that we are using; for simplicity we work with $U(2,1)$-Shimura varieties[1] over a quadratic imaginary field F. Note that "sufficiently small level" means that the compact open subgroup defining the level is sufficiently small. We say that a Galois representation is associated to a maximal ideal of a Hecke algebra if there is the usual relation between Hecke polynomials and characteristic polynomials of Frobenius elements at unramified places; see Section 3.4 for a precise definition.)

In fact, we prove a version of this result for $U(n-1,1)$-Shimura varieties under weaker assumptions on $\rho_{\mathfrak{m}}$; however, in general we can only prove vanishing in degrees outside of the range $[n/2, (3n-4)/2]$.

We also prove the following result, which makes no explicit reference to a maximal ideal in the Hecke algebra:

Theorem B. *Let X and $\mathcal{F}$ be as in the statement of Theorem A. If ρ is a 3-dimensional irreducible sub-G_F-representation of the étale cohomology group $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F})$, then either every irreducible subquotient of $\rho_{|G_{\mathbb{Q}_p}}$ is 1-dimensional, or else $\rho_{|G_{\mathbb{Q}_p}}$ is not 1-regular, or else $\rho(G_F)$ is not generated by its subset of regular elements.*

Note that in neither theorem do we make any assumption on the level of the Shimura variety at p.

A Galois representation $\rho_{\mathfrak{m}}$ as in the statement of Theorem A is known to exist if $\mathfrak{m}$ corresponds to a system of Hecke eigenvalues arising from the reduction mod p of the Hecke eigenvalues attached to some automorphic Hecke eigenform. Furthermore, recent work of Scholze [2013] (which appeared after the first version of this paper was written) implies that such a representation exists for *any* maximal ideal $\mathfrak{m}$.

It seems reasonable to believe that any irreducible sub-G_F-representation of any of the étale cohomology groups $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F})$ for any of the Shimura varieties under consideration should in fact be a constituent of $\rho_{\mathfrak{m}}$ for some maximal ideal $\mathfrak{m}$ of the Hecke algebra. However, this doesn't seem to be known, and relating the "abstract" G_F-representations $\rho_{\mathfrak{m}}$ to the "physical" G_F-representations appearing on étale cohomology is one of the problems we have to deal with in proving our results.

Application to Serre-type conjectures. We are able to combine our results with those of [Emerton et al. 2013] so as to establish cases of the weight part of the Serre-type conjecture of [Herzig 2009] for $U(2,1)$. More precisely, we have the following result (where the assertion that $\bar{\rho}$ is modular means that the corresponding system of Hecke eigenvalues occurs in the mod p cohomology of some $U(2,1)$-Shimura variety; see Theorem 3.5.6 and Lemma 4.1.9.)

[1] These Shimura varieties might more properly be called $\mathrm{GU}(2,1)$-Shimura varieties; see Section 3 for their definition.

Theorem C. *Suppose $\rho : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ satisfies $\mathrm{SL}_3(k) \subset \rho(G_F) \subset \overline{\mathbb{F}}_p^\times \mathrm{SL}_3(k)$ for some finite extension $k/\mathbb{F}_p$, that $\rho|_{G_{\mathbb{Q}_p}}$ is irreducible and* 1*-regular, and that ρ is modular of some strongly generic weight. Then the set of generic weights for which ρ is modular is exactly the set predicted by the recipe of [Herzig 2009].*

Relationship with a mod p analogue of Arthur's conjecture. Arthur [1989, §9] made a quite precise conjecture regarding the systems of Hecke eigenvalues that appear in the L^2-automorphic spectrum of any reductive group over a number field, which has consequences for the nature of the Hecke eigenvalues appearing in the cohomology of Shimura varieties. For our purposes it suffices to describe a qualitative version of these consequences: namely, Arthur's conjecture implies that if λ is a system of Hecke eigenvalues appearing in the degree-i cohomology, where i is less than the middle dimension, then λ is attached (in the sense of, e.g., [Buzzard and Gee 2014; Johansson 2013]) to a reducible Galois representation (i.e., one which factors through a parabolic subgroup of the L-group).

The fragmentary evidence available suggests that a similar statement will be true for the mod p cohomology of Shimura varieties. Our Theorems A and B give further evidence in this direction.

p-adic Hodge theory. In order to prove these theorems, we establish some new results about the p-adic Hodge-theoretic properties of the étale cohomology of varieties over a number field or p-adic field with coefficients in a field of characteristic p. In the first section we establish results about the mod p étale cohomology of varieties over number fields or p-adic fields which, although weaker in their conclusions, are substantially broader in the scope of their application than previously known mod p comparison theorems. For example, we prove the following result (see Theorem 1.4.1 below):

Theorem D. *Let K be a finite extension of $\mathbb{Q}_p$, and write G_K for the absolute Galois group of K. If X is a smooth projective variety over K which has semistable reduction, and if ρ is an irreducible subquotient of the G_K-representation $H^i_{\text{ét}}(X_{\overline{K}}, \overline{\mathbb{F}}_p)$, then ρ also embeds as a subquotient of a G_K-representation over $\overline{\mathbb{F}}_p$ which is the reduction modulo the maximal ideal of a G_K-invariant $\overline{\mathbb{Z}}_p$-lattice in a G_K-representation over $\overline{\mathbb{Q}}_p$ which is semistable with Hodge–Tate weights contained in the interval* $[-i, 0]$.

Both the hypotheses and the conclusions of our theorems are rather precisely tailored to maximise (as far as we are able) their utility in applications to the analysis of Galois representations occurring in the cohomology of Shimura varieties, which we give in the third section.

The remaining two sections of the paper are devoted respectively to using integral p-adic Hodge theory (Breuil modules with descent data) to establish a result related to the reductions of tamely potentially semistable p-adic representations of $G_{\mathbb{Q}_p}$ (Section 2) and to proving some technical results about group representations (Section 4). The result of Section 2 is an essential ingredient in the arguments of Section 3, while the results of Section 4 provide sufficient conditions for the various representation-theoretic hypotheses appearing in the results of Section 3 to be satisfied.

Remark on related papers. Very general vanishing theorems for the mod p cohomology of Shimura varieties have been proved by Lan and Suh [2013]; however, their results apply only in situations of good reduction and for coefficients corresponding to small Serre weights, which makes them unsuitable for the kinds of applications we have in mind, e.g., to the weight part of Serre-type conjectures. In the ordinary case there is the work of Mokrane and Tilouine [2002, §9] in the Siegel case and Dimitrov [2005, §6.4] in the case of Hilbert modular varieties. Finally, in a recent preprint, Shin [2013] proved a general vanishing result for cohomology outside of middle degree for the part of the mod p cohomology which is supercuspidal at some prime $l \neq p$, by completely different methods from those of this paper. It seems plausible that, via the mod p local Langlands correspondence for $\mathrm{GL}_n(\mathbb{Q}_l)$, Shin's hypothesis could be interpreted as a condition on the restriction to a decomposition group at l of the relevant mod p Galois representations, whereas our conditions involve the restriction to a decomposition group at p, so our results appear to be complementary.

Conventions. For any field K we let G_K denote a choice of absolute Galois group of K.

If K is a finite field, then by a Frobenius element in G_K we will always mean a geometric Frobenius element. We extend this convention in an evident way to Frobenius elements at primes in Galois groups of number fields, and to Frobenius elements in Galois groups of local fields.

If K is a local field, then we denote by $\mathcal{O}_K$ the ring of integers of K, by I_K the inertia subgroup of G_K, by W_K the Weil group of K (the subgroup of G_K consisting of elements whose reduction modulo I_K is an integral power of Frobenius), and by WD_K the Weil–Deligne group of K.

If K is a number field and v is a finite place of K, then we will write K_v for the completion of K at v and $\mathcal{O}_{K_v}$ for its ring of integers. We will write $\mathcal{O}_{K,(v)}$ for the localisation of $\mathcal{O}_K$ at the prime ideal v.

We will write $\overline{\mathbb{Z}}_p$ for the ring of integers in $\overline{\mathbb{Q}}_p$ (a fixed algebraic closure of $\mathbb{Q}_p$), and $\mathfrak{m}\overline{\mathbb{Z}}_p$ for the maximal ideal of $\overline{\mathbb{Z}}_p$.

We let ω denote the mod p cyclotomic character. We will denote a Teichmüller lift with a tilde, so that for example $\widetilde{\omega}$ is the Teichmüller lift of ω.

We use the traditional normalisation of Hodge–Tate weights, with respect to which the cyclotomic character has Hodge–Tate weight 1.

By a *closed geometric point* $\bar{x}$ of a scheme X, we mean a morphism of schemes $\bar{x} : \operatorname{Spec} \Omega \to X$ for a separably closed field Ω, whose image is a closed point x of X, and such that the induced embedding $\kappa(x) \hookrightarrow \Omega$ (where $\kappa(x)$ denotes the residue field of x) identifies Ω with a separable closure of $\kappa(x)$. If $\bar{x}$ is a closed geometric point of a Noetherian scheme X, then we let $\mathbb{O}_{X,\bar{x}}$ denote the local ring of X at $\bar{x}$, i.e., the stalk, in the étale topology on X, of the structure sheaf of X at $\bar{x}$; we let $(\mathbb{O}_{X,\bar{x}})^\wedge$ denote the completion of $\mathbb{O}_{X,\bar{x}}$, and we write $(X_{\bar{x}})^\wedge := \operatorname{Spf}((\mathbb{O}_{X,\bar{x}})^\wedge)$, and refer to $(X_{\bar{x}})^\wedge$ as the formal completion of X along the closed geometric point $\bar{x}$.

The symbol G will always denote a group; in Section 3 it will be a certain algebraic group, and in Section 4 it will be a finite group.

1. p-adic Hodge theoretic properties of mod p cohomology

1.1. ***Introduction.*** We now describe in more detail our results on the integral p-adic Hodge theory of the étale cohomology of projective varieties, which are perhaps the most novel part of this paper.

It is well-known that integral p-adic Hodge theory is less robust than the corresponding theory with rational coefficients; for example, the comparison theorems for integral and mod p étale cohomology due to Fontaine and Messing [1987] and Faltings [1989] involve restrictions both on the degrees of cohomology and the dimensions of the varieties considered, and they also require that the field K be absolutely unramified and that the variety under consideration be of good reduction. More recently, Caruso [2008] has proved an integral comparison theorem in the case of semistable reduction for possibly ramified fields K, but there are still restrictions: his result requires that $ei < p-1$, where e is the absolute ramification index of K, and i is the degree of cohomology under consideration.

These restrictions are unfortunate, since mod p and integral p-adic Hodge theory are among the most powerful local tools available for the analysis of Galois representations occurring in the mod p étale cohomology of varieties. The premise that underlies the present work is that frequently in such applications one does not need a precise comparison theorem relating the mod p étale cohomology to an analogous structure involving mod p de Rham or crystalline cohomology. Rather, one often uses the comparison theorem merely to draw much less specific conclusions, such as that the Galois representations occurring in certain mod p étale cohomology spaces are in the essential image of the Fontaine–Laffaille functor, applied to Fontaine–Laffaille modules whose Fontaine–Laffaille numbers lie in some prescribed range. Our aim is to establish results of the latter type in more general contexts than they have previously been proved.

The precise direction of our work is informed to a significant extent by the fairly recent development of a rich *internal* integral p-adic Hodge theory, by Breuil

[2000], Kisin [2006], Liu [2008] and others. What we mean here by the word "internal" is that these developments have been directed not so much at applications to comparison theorems, but rather at the purely Galois-theoretic problem of giving a p-adic Hodge-theoretic description of Galois-invariant lattices in crystalline or semistable Galois representations, and of the mod p Galois representations that appear in the reductions of such lattices. These tools, especially the theory of *Breuil modules* [2000], which provides the desired description of the mod p representations arising as reductions of such lattices, have proved very useful in arithmetic applications. Because of the availability of these tools, it has become both possible and worthwhile to move beyond the Fontaine–Laffaille context in integral p-adic Hodge theory. While Caruso's work mentioned above is a significant step in this direction, an important aspect of the present work will be the consideration of situations in which the bound $ei < p-1$, required for the validity of the comparison theorem of [Caruso 2008], does not hold.

Our goal, then, is to establish in various situations that a Galois representation appearing in the mod p étale cohomology of a variety can be embedded in the reduction of a Galois-invariant lattice contained in a crystalline or semistable Galois representation, with Hodge–Tate weights lying in some specified range (namely, the range that one would expect given the degree of the cohomology space under consideration). Since, in arithmetic situations, one frequently has to make a ramified base change in order to obtain good or semistable reduction, and since the resulting descent data on the associated Breuil module typically then play an important role in whatever analysis has to be undertaken, we also prove results in certain cases of potentially semistable reduction which allow us to gain some control over these descent data.

The idea underlying our approach is very simple. Suppose that X is a variety over a p-adic field K. If i is some degree of cohomology, then we have a short exact sequence

$$0 \longrightarrow H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)/pH^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \longrightarrow H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p) \longrightarrow H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)[p] \longrightarrow 0,$$

as well as an isomorphism

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p) \xrightarrow{\sim} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p).$$

Thus, if both $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ and $H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ are torsion-free, then we see that $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)$ is the reduction mod p of $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$, which *is* a Galois-invariant lattice in $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Furthermore, the usual comparison theorems of *rational* p-adic Hodge theory [Faltings 1989; Tsuji 1999] can be applied to conclude that this latter representation is, e.g., crystalline (if X is proper with good reduction) or semistable (if X is proper with semistable reduction).

The obstruction to implementing this idea is that we have no reason to believe in general that $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ and $H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ will be torsion-free. To get around this difficulty, we engage in various dévissages using the weak Lefschetz theorem. To explain these, first consider the case when X is a projective curve and $i = 1$. In this case all the cohomology with $\mathbb{Z}_p$-coefficients is certainly torsion-free, and so $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)$ *is* the reduction of a Galois-invariant lattice in $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Now a simple induction using the weak Lefschetz theorem shows that for *any* smooth projective variety X over K there is an embedding

$$H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p) \hookrightarrow H^1_{\text{ét}}(C_{\overline{K}}, \mathbb{F}_p),$$

where C is a smooth projective curve. Furthermore, if X has good (respectively semistable) reduction, we can ensure that the same is true of C. This gives the desired result in the case of H^1 (ignoring for a moment the problem of obtaining a refinement dealing with descent data in the potentially semistable case).

For higher degrees of cohomology, a more elaborate dévissage is required. The key point, again established via the weak Lefschetz theorem, is that if X is smooth and projective of dimension d, and if Y and Z are sufficiently generic hyperplane sections of X, then the cohomology of the pair $((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}})$, with either $\mathbb{Z}_p$ or $\mathbb{F}_p$ coefficients, vanishes in degrees other than d (see Section A.3 of the appendix; note that $X \setminus Y$ is affine), so that $H^d_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, \mathbb{Z}_p)$ is torsion-free and is thus a Galois-invariant lattice in $H^d_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, \mathbb{Q}_p)$, which is potentially semistable by [Yamashita 2011], and whose reduction is equal to $H^d_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, \mathbb{F}_p)$. Such relative cohomology spaces are the essential ingredient of the *basic lemma* of [Beilinson 1987], and we learned the idea of using them as building blocks for the cohomology of varieties from Nori [2002], who has used the basic lemma as the foundation of his approach to the construction of motives. Indeed, our present approach to integral p-adic Hodge theory was inspired by Beilinson's and Nori's work.

1.2. ***Bertini-type theorems.*** We begin by giving a straightforward generalisation of some of the results of [Jannsen and Saito 2012], which build on the results of [Poonen 2004] to prove Bertini-type theorems for varieties with semistable reduction over a discrete valuation ring. It will be convenient to allow K to denote either a number field, or a field of characteristic zero that is complete with respect to a discrete valuation with perfect residue field k_K of characteristic p. We abbreviate these two situations as "the global case" and "the local case" respectively, and in the former case we will let v denote a place of K dividing p.

We recall the following definition:

Definition 1.2.1. Suppose first that we are in the local case. We then say that a projective $\mathcal{O}_K$-scheme $\mathcal{X}$ is *semistable* if it is regular and flat over $\operatorname{Spec} \mathcal{O}_K$, and if the special fibre $\mathcal{X}_s$ is reduced and is a normal crossings divisor; equivalently, a finite-type $\mathcal{O}_K$-scheme $\mathcal{X}$ is semistable if, at each closed geometric point $\bar{x}$ of $\mathcal{X}_s$, there is an isomorphism of complete local rings

$$(\mathcal{O}_{\bar{x},\mathcal{X}})^\wedge \cong ((\mathcal{O}_K^{\mathrm{sh}})^\wedge)[\![x_1, \dots, x_n]\!]/(x_1 \cdots x_m - \varpi_K),$$

where $(\mathcal{O}_K^{\mathrm{sh}})^\wedge$ is the completion of the strict Henselisation of $\mathcal{O}_K$ (equivalently, the completion of the ring of integers in the maximal unramified algebraic extension of K), the element ϖ_K is a uniformiser of $(\mathcal{O}_K^{\mathrm{sh}})^\wedge$, and $1 \le m \le n$. We say that $\mathcal{X}$ is strictly semistable if it is semistable and if the special fibre $\mathcal{X}_s$ is a strict normal crossings divisor.

Again in the local case, we say that a smooth projective K-scheme has *good reduction* if it admits a smooth projective model over $\mathcal{O}_K$, and that it has (*strictly*) *semistable reduction* if it admits an extension to a projective $\mathcal{O}_K$-scheme which is (strictly) semistable in the sense of the preceding definition.

In the global case, we say that a smooth projective K-scheme has good reduction at v if it admits a smooth projective model over $\mathcal{O}_{K,(v)}$, and that it has (strictly) semistable reduction at v if it admits a (strictly) semistable projective model over $\mathcal{O}_{K,(v)}$, i.e., a projective model over $\mathcal{O}_{K,(v)}$ whose base change over $\mathcal{O}_{K_v}$ is (strictly) semistable in the sense of the preceding definition.

Remark 1.2.2. Note that our definition of a semistable $\mathcal{O}_K$-scheme (putting ourselves in the local case) includes the requirement that the scheme be regular. This is the definition that is frequently adopted in the theory of semistable reduction, and it is well-suited to our intended applications. Recall that, with this definition, semistability is *not* preserved under the base change to $\mathcal{O}_L$, if L is a finite extension of K, unless L/K is unramified or the original scheme is in fact smooth over $\mathcal{O}_K$; see also Remark 1.5.2 below.

Proposition 1.2.3. *Let X be a smooth projective variety over K with strictly semistable* (*respectively good*) *reduction* (*at v, in the global case*). *Then there are smooth hypersurface sections Y and Z of X* (*with respect to an appropriately chosen embedding of X into some projective space*) *such that Y and Z intersect transversely, and all of Y, Z, and $Y \cap Z$ have strictly semistable* (*respectively good*) *reduction* (*at v, in the global case*).

Proof. We first handle the local case. Choose an extension $\mathcal{X}$ of X to an $\mathcal{O}_K$-scheme that is projective and smooth (in the good reduction case) or strictly semistable (in the strictly semistable reduction case), and fix an embedding of $\mathcal{X}$ into some projective space over $\mathcal{O}_K$. By Corollaries 0 and 1 of [Jannsen and Saito 2012] (or, perhaps more precisely, by their proofs) we can find a hypersurface section $\mathcal{Y}$ of $\mathcal{X}$ such that $\mathcal{Y}$ is again either smooth or strictly semistable over $\mathcal{O}_K$. We take Y to be the generic fibre of $\mathcal{Y}$. By Remark 0(ii), together with Lemma 1 and the remark immediately before Corollary 1, of [Jannsen and Saito 2012], we see that in order to find Z it is enough to check that, given a finite collection $X_1, \dots, X_n$ of smooth projective schemes in $\mathbb{P}^N_{/k_K}$, there is a common hypersurface section meeting each of them transversely. This is an immediate consequence of Theorem 1.3 of [Poonen 2004], taking the set U_P there to be the subset of the completion $\widehat{\mathcal{O}}_P$ consisting of the f such that $f = 0$ is transverse to each X_i at P. (Since this set contains all the f which do not vanish at P, and in particular contains all the f congruent, modulo the maximal ideal, to a particular choice of f, it has positive Haar measure.)

We now pass to the global case. Let $\mathcal{X}$ be a smooth (in the good reduction case) or strictly semistable (in the strictly semistable reduction case) projective model of X over $\mathcal{O}_{K,(v)}$. Let P_d^* denote the projective space (over $\mathcal{O}_{K,(v)}$) of degree-d hypersurfaces in the ambient projective space containing $\mathcal{X}$. Applying the argument in the local case to the base change $\mathcal{X}_{/\mathcal{O}_{K_v}}$, we see that, for some $d \geq 1$, there is a K_v-valued point of P_d^* corresponding to a hypersurface section of $\mathcal{X}_{/\mathcal{O}_{K_v}}$ having either smooth or semistable intersection (depending on the case we are in) with $\mathcal{X}_{/\mathcal{O}_{K_v}}$. Furthermore, this point lies in an affinoid open subset of P^*_{d/K_v} (the preimage of an open set in the special fibre of P_d^*), all of whose points correspond to hyperplane sections of $\mathcal{X}_{/\mathcal{O}_{K_v}}$ with either smooth or strictly semistable intersection. (See Remark 0(i) and the proofs of Theorems 0 and 1 of [Jannsen and Saito 2012].) The set of K_v-points of this affinoid open set is a nonempty open subset of $P_d^*(K_v)$. Since K is dense in K_v, we see that this intersection also contains a K-point of P_d^*, which gives the required hypersurface section Y. We find the hypersurface section Z by applying the same argument. □

1.3. *Cohomology in degree* 1. Our arguments in degree 1 are rather simpler than in general degree, so we warm up with this case. (In fact, our result in this case is slightly stronger than our result in general degree, as we do not need to semisimplify the representation, so this result is not completely subsumed by our later results in general degree.) Fix a prime p. Let K denote a field of characteristic zero, complete with respect to a discrete valuation, with ring of integers $\mathcal{O}_K$ and residue field k, assumed to be perfect of characteristic p. Let $\overline{K}$ denote an algebraic closure of K, and set $G_K := \mathrm{Gal}(\overline{K}/K)$.

Theorem 1.3.1. *If X is a smooth projective variety over K which has good* (*resp. strictly semistable*) *reduction, then $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)$ embeds G_K-equivariantly into the reduction modulo p of a G_K-invariant lattice in a crystalline* (*resp. semistable*) *p-adic representation of G_K whose Hodge–Tate weights are contained in* $[-1, 0]$.

Proof. We proceed by induction on the dimension d of X. If $d \leq 1$ then $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)$ is isomorphic to the reduction modulo p of $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$, and the latter space (being torsion-free, by virtue of our assumption on d) is in turn a lattice in $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, which is crystalline or semistable, respectively, with Hodge–Tate weights lying in $[-1, 0]$, by the main result of [Tsuji 1999].

Suppose now that $d > 1$. It follows from Corollary 0 (resp. Corollary 1) of [Jannsen and Saito 2012] that if X has good reduction (resp. strictly semistable reduction) then we may choose a smooth hypersurface section Y of X defined over K which has good (resp. strictly semistable) reduction. Our induction hypothesis applies to show that $H^1_{\text{ét}}(Y_{\overline{K}}, \mathbb{F}_p)$ embeds as a subobject of a G_K-representation over $\mathbb{F}_p$ which is the reduction modulo p of a G_K-invariant lattice in a crystalline (resp. semistable) p-adic representation of G_K whose Hodge–Tate weights are contained in $[-1, 0]$. On the other hand, the weak Lefschetz theorem with $\mathbb{F}_p$-coefficients [SGA 4_3 1973, Exposé XIV, Corollaire 3.3] implies that the natural (restriction) map $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p) \to H^1_{\text{ét}}(Y_{\overline{K}}, \mathbb{F}_p)$ is an embedding (because $1 \leq d - 1$ by assumption). This completes the proof. □

1.4. ***Cohomology of arbitrary degree.*** As always we fix a prime p. The necessary dévissages in this subsection will be more elaborate than in the previous one, and so, to maximise the utility of our results for later applications, it will be convenient to again allow K to denote either a number field or a field of characteristic zero that is complete with respect to a discrete valuation with perfect residue field of characteristic p. In applications it will also be useful to have flexibility in the choice of coefficients in the various cohomology spaces that we consider, and to this end we fix an algebraic extension E of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_E$ and residue field k_E. (In applications, E will typically either be a finite extension of $\mathbb{Q}_p$, or else will be $\overline{\mathbb{Q}}_p$.)

We now recall some consequences of the weak Lefschetz theorem. Among other notions, we will use the étale cohomology of a pair consisting of a variety and a closed subvariety; a precise definition of this cohomology, and a verification of its basic properties (such as those recalled in the next paragraph), is included in the Appendix.

Let X be a smooth projective variety of dimension d over K, and suppose that Y and Z are two smooth hypersurface sections of X, chosen so that $Y \cap Z$ is also smooth. Let A denote either E, $\mathcal{O}_E$, or k_E. In either the first or last case, the spaces $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, A)$ and $H^{2d-i}_{\text{ét}}((X \setminus Z)_{\overline{K}}, (Y \setminus Z)_{\overline{K}}, A)(d)$ are naturally dual to one another for each integer i. The weak Lefschetz theorem implies that the former space vanishes when $i > d$ and the latter space vanishes when $2d - i > d$, i.e., when $i < d$. Thus, in fact, both spaces vanish unless $i = d$. It then follows that both spaces vanish unless $i = d$ in the case when A is taken to be $\mathcal{O}_E$ as well, and hence that when $i = d$ both spaces are torsion-free.

Let $\overline{K}$ denote an algebraic closure of K, and set $G_K := \mathrm{Gal}(\overline{K}/K)$. Now let $\rho : G_K \to \mathrm{GL}_n(k_E)$ be irreducible and continuous. In the global case, we fix a place v of K lying over p, and a decomposition group $D_v \subset G_K$ for v.

Theorem 1.4.1. *If X is a smooth projective variety over K which has strictly semistable* (*resp. good*) *reduction* (*at v, if we are in the global case*), *and if ρ embeds as a subquotient of $H^i_{\text{ét}}(X_{\overline{K}}, k_E)$, then ρ also embeds as a subquotient of a G_K-representation over k_E which is the reduction modulo the uniformiser of a G_K-invariant $\mathcal{O}_E$-lattice in a G_K-representation which is semistable* (*resp. crystalline*) (*at v, in the global case*) *with Hodge–Tate weights contained in the interval* $[-i, 0]$.

Proof. We proceed by induction on the dimension of X. Suppose initially that we are in the strictly semistable reduction case. By Proposition 1.2.3, we can and do choose smooth hypersurface sections Y and Z, having smooth intersection, and such that Y, Z, and $Y \cap Z$ all have strictly semistable reduction.

We then consider the long exact sequences

$$\cdots \longrightarrow H^i_{Y_{\overline{K}},\text{ét}}(X_{\overline{K}}, A) \longrightarrow H^i_{\text{ét}}(X_{\overline{K}}, A) \longrightarrow H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, A) \longrightarrow \cdots,$$

$$\cdots \longrightarrow H^i_{(Y\cap Z)_{\overline{K}},\text{ét}}(Z_{\overline{K}}, A) \longrightarrow H^i_{\text{ét}}(Z_{\overline{K}}, A) \longrightarrow H^i_{\text{ét}}((Z \setminus Y)_{\overline{K}}, A) \longrightarrow \cdots,$$

$$\cdots \to H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, (Z\setminus Y)_{\overline{K}}, A) \to H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, A) \to H^i_{\text{ét}}((Z\setminus Y)_{\overline{K}}, A) \to \cdots,$$

with A taken to be either E or k_E (see [Milne 1980, Chapter III, Proposition 1.25] for the first two, which are local cohomology long exact sequences, and the Appendix for the third, which is the long exact sequence of the pair $(X \setminus Y, Z \setminus Y)$). We also recall (see [SGA 4_3 1973, Exposé XIV, §3]) that there are canonical isomorphisms

$$H^{i-2}_{\text{ét}}(Y_{\overline{K}}, A)(-1) \xrightarrow{\sim} H^i_{Y_{\overline{K}},\text{ét}}(X_{\overline{K}}, A),$$
$$H^{i-2}_{\text{ét}}((Y \cap Z)_{\overline{K}}, A)(-1) \xrightarrow{\sim} H^i_{(Y\cap Z)_{\overline{K}},\text{ét}}(Z_{\overline{K}}, A).$$

When $A = E$, all the cohomology spaces that appear are potentially semistable [Yamashita 2011]. Since $H^i_{\text{ét}}(X_{\overline{K}}, E)$, $H^i_{\text{ét}}(Y_{\overline{K}}, E)$, and $H^i_{\text{ét}}(Z_{\overline{K}}, E)$ are semistable with Hodge–Tate weights lying in $[-i, 0]$, we see that $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, E)$, $H^i_{\text{ét}}((Z \setminus Y)_{\overline{K}}, E)$, and $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, E)$ are semistable, with Hodge–Tate weights lying in $[-i, 0]$.

Now taking $A = k_E$, we see that, since ρ is irreducible, it appears as a subquotient of $H^{i-2}_{\text{ét}}(Y_{\overline{K}}, k_E)(-1)$, of $H^i_{\text{ét}}(Z_{\overline{K}}, k_E)$, of $H^{i-1}(Y \cap Z)(-1)$, or of $H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, (Z\setminus Y)_{\overline{K}}, k_E)$. In the first three cases, the theorem follows by induction on the dimension. In the final case, the conclusion follows from the vanishing theorem noted above; namely, $H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, (Z\setminus Y)_{\overline{K}}, E)$ is the desired semistable representation, with invariant lattice $H^i_{\text{ét}}((X\setminus Y)_{\overline{K}}, (Z\setminus Y)_{\overline{K}}, \mathcal{O}_E)$, whose reduction $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, k_E)$ contains ρ.

Finally, suppose we are in the good reduction case. Again, by Proposition 1.2.3, we can and do choose smooth hypersurface sections Y and Z, having smooth intersection, such that Y, Z, and $Y \cap Z$ all have good reduction. Applying the same argument as in the previous paragraph, we see by induction on the dimension of X that it is enough to check that $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, E)$ is crystalline, but this follows immediately from Theorem 1.2 of [Yamashita 2011]. (Note that if in the notation of that work we take $D^1 = Z$ and $D^2 = Y$, then by (A.1.2) below we see that $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, E)$ appears on the left side of the Hyodo–Kato isomorphism in the statement of Theorem 1.2 of [Yamashita 2011], and since we have already shown that $H^i_{\text{ét}}((X \setminus Y)_{\overline{K}}, (Z \setminus Y)_{\overline{K}}, E)$ is semistable, it is enough to show that the monodromy operator N vanishes on the right side of the Hyodo–Kato isomorphism. This follows easily from the definition of this operator as a boundary map, as all objects concerned arise from base change from objects with trivial log structures.) □

1.5. ***Equivariant versions.*** In practice, we will need equivariant analogues of the preceding results. As in the preceding section, we let K denote either a number field ("the global case") or a field of characteristic zero that is complete with respect to a discrete valuation with perfect residue field of characteristic p ("the local case"). We let $\overline{K}$ denote an algebraic closure of K, and set $G_K := \text{Gal}(\overline{K}/K)$. In the global case, we fix a place v of K lying over p, and a decomposition group $D_v \subset G_K$ for v.

We now put ourselves in the following (somewhat elaborate) situation, which we call a *tamely ramified semistable context*, or a *tame semistable context* for short.

We suppose that X_0 and X_1 are smooth projective varieties over K, that G is a finite group which acts on X_1, and that $\pi : X_1 \to X_0$ is a finite étale morphism which intertwines the given G-action on X_1 with the trivial G-action on X_0, making X_1 an étale G-torsor over X_0.

We suppose further that X_0 admits a semistable projective model $\mathscr{X}_0$ over $\mathscr{O}_K$ (in the local case) or over $\mathscr{O}_{K,(v)}$ (in the global case). We also suppose that there is a finite Galois extension L of K, and (in the global case) a prime w of L lying over v, such that $(X_1)_{/L}$ admits a semistable projective model $\mathscr{X}_1$ over $\mathscr{O}_L$ (in the local case) or over $\mathscr{O}_{L,(w)}$ (in the global case) to which the G-action extends, such that π extends to a morphism $\mathscr{X}_1 \to (\mathscr{X}_0)_{/\mathscr{O}_L}$ which intertwines the G-action on its source with the trivial G-action on its target, and such that the action of the (opposite group of) the inertia group $I(L/K)^{\text{op}}$ (or $I(L_w/K_v)^{\text{op}}$ in the global case) on $(X_1)_{/L}$ extends to an action on $\mathscr{X}_1$.[2]

[2]Note that the tameness condition that we are going to require below ensures that L/K is in fact tamely ramified, and hence that $I(L/K)$ is abelian. Thus passing to the opposite group is not actually necessary here when passing from the action on rings to the action on their Specs, but we will keep the superscript op in the notation for the sake of conceptual clarity.

Finally (and most importantly), we assume that the composite morphism

$$\mathscr{X}_1 \to (\mathscr{X}_0)_{/\mathbb{O}_L} \to \mathscr{X}_0 \tag{1.5.1}$$

(the first being the extension of π, and the second being the natural map) is tamely ramified along the special fibre $(\mathscr{X}_0)_s$, in the sense of [Grothendieck and Murre 1971, Definition 2.2.2].

In fact, in our applications we will consider the case that $\mathscr{X}_0$ is furthermore strictly semistable, in which case we will say that we are in a *tame strictly semistable context*.

Remark 1.5.2. The notion of a tame semistable context is somewhat rigid, as we will see in the following lemma, and would perhaps not be of much interest if it did not occur naturally in the Shimura variety context (as we will see Section 3.1). As one example of this rigidity, note that if $G = 1$, i.e., if X_0 and X_1 coincide, then the only way to achieve a tame semistable context is if L/K is unramified, or if $\mathscr{X}_0$ is smooth over $\mathbb{O}_K$. (Indeed, since a tamely ramified morphism is finite, and since the base change $\mathscr{X}_{0/\mathbb{O}_L}$ over the semistable $\mathbb{O}_K$-scheme $\mathscr{X}_0$ is normal, we see that if X_0 and X_1 coincide then the morphism $\mathscr{X}_1 \to \mathscr{X}_{0/\mathbb{O}_L}$ is necessarily an isomorphism. This implies that the semistable $\mathbb{O}_K$-scheme $\mathscr{X}_0$ has a semistable base change over $\mathbb{O}_L$, which, as we noted in Remark 1.2.2, is possible only if L/K is unramified or $\mathscr{X}_0$ is smooth over $\mathbb{O}_K$. Another point of view on this case is as follows: if $\mathscr{X}_0$ is semistable but not smooth, then in order to construct a semistable model $\mathscr{X}_1$ of $\mathscr{X}_{0/\mathbb{O}_L}$, we must perform some nontrivial blow-ups, and the resulting morphism $\mathscr{X}_1 \to \mathscr{X}_0$ is not finite, and in particular not tamely ramified.)

The following lemma gives a more concrete interpretation of the stipulation that (1.5.1) be tamely ramified along $(\mathscr{X}_0)_s$.

Lemma 1.5.3. *In the above setting, the morphism* (1.5.1) *is tamely ramified along* $(\mathscr{X}_0)_s$ *if and only if the following conditions hold*:

(1) L (*resp.* L_w *in the global case*) *is tamely ramified over* K (*resp.* K_v *in the global case*), *of ramification degree* e, *say.*

(2) *For each closed geometric point* $\bar{x}_1$ *of the special fibre* $(\mathscr{X}_1)_s$, *with image* $\bar{x}_0$ *in* $(\mathscr{X}_0)_s$, *and for some choice of isomorphism*

$$(\mathbb{O}_{\bar{x}_0,\mathscr{X}_0})^\wedge \cong (\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K), \tag{1.5.4}$$

where ϖ_K *is a uniformiser of* $(\mathbb{O}_K^{\mathrm{sh}})^\wedge$ *and* $1 \le m \le n$, *there is a corresponding isomorphism*

$$(\mathbb{O}_{\bar{x}_1,\mathscr{X}_1})^\wedge \cong (\mathbb{O}_L^{\mathrm{sh}})^\wedge[\![y_1,\dots,y_n]\!]/(y_1\cdots y_m - \varpi_L),$$

where ϖ_L *is a uniformiser of* $(\mathbb{O}_L^{\mathrm{sh}})^\wedge$, *such that the induced morphism*

$$((\mathscr{X}_1)_{\bar{x}_1})^\wedge \to ((\mathscr{X}_0)_{\bar{x}_0})^\wedge$$

is defined by the formula $x_j = y_j^e$ *for* $1 \le j \le m$ *and* $x_j = y_j$ *for* $m < j \le n$.

Furthermore, *if these equivalent conditions hold*, *then condition* (2) *holds for* every *choice of isomorphism* (1.5.4).

Proof. We first note that if we are in the global case, then, relabelling K_v as K and L_w as L, we may reduce ourselves to proving the lemma in the local case. Thus we assume that we in the local case from now on.

If conditions (1) and (2) (for some choice of isomorphism (1.5.4)) hold, then the morphism (1.5.1) is certainly tamely ramified along $(\mathscr{X}_0)_s$. (This amounts to the claim that we can verify tame ramification by passing to formal completions of closed geometric points, which is indeed the case, as follows from [Grothendieck and Murre 1971, Corollary 4.1.5].)

Suppose conversely that (1.5.1) is tamely ramified along $(\mathscr{X}_0)_s$. Since this morphism factors through the natural morphism $(\mathscr{X}_0)_{/\mathbb{O}_L} \to \mathscr{X}_0$, it follows from [Grothendieck and Murre 1971, Lemma 2.2.5] that this latter morphism is tamely ramified, and hence (e.g., by Proposition 2.2.9 of that reference, although our particular situation is much simpler than the general case of faithfully flat descent for tamely ramified covers considered in that proposition) that $\operatorname{Spec} \mathbb{O}_L \to \operatorname{Spec} \mathbb{O}_K$ is tamely ramified, i.e., that L is tamely ramified over K, of some ramification degree e. Thus condition (1) holds.

Now choose a closed geometric point $\bar{x}_1$ of $(\mathscr{X}_1)_s$ lying over the closed geometric point $\bar{x}_0$ of $(\mathscr{X}_0)_s$, and fix an isomorphism of the form (1.5.4). Since $\mathscr{X}_1 \to \mathscr{X}_0$ is tamely ramified along the divisor $\varpi_K = 0$ of $\mathscr{X}_0$, Abhyankar's lemma [SGA 1 1971, Exposé XIII, Corollaire 5.6] (see also [Grothendieck and Murre 1971, Theorem 2.3.2] for a concise statement) implies that we may find regular elements $\{a_j\}_{j=1,\dots,k}$ of $(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K)$ such that $a_1 \cdots a_k$ generates the ideal (ϖ_K) of $(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K)$, exponents $e_1,\dots,e_k$ all coprime to p, and a subgroup $H \subset \mu_{e_1} \times \cdots \times \mu_{e_k}$, such that the $(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K)$-algebra $(\mathbb{O}_{\bar{x}_1,\mathscr{X}_1})^\wedge$ is isomorphic to

$$\big((\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!][T_1,\dots,T_k]/(x_1\cdots x_m - \varpi_K, T_1^{e_1} - a_1, \dots, T_k^{e_k} - a_k)\big)^H.$$

(Here $\mu_{e_1} \times \cdots \times \mu_{e_k}$, and hence H, acts on

$$(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!][T_1,\dots,T_k]/(x_1\cdots x_m - \varpi_K, T_1^{e_1} - a_1, \dots, T_k^{e_k} - a_k)$$

in the obvious manner: namely, an element $(\zeta_1,\dots,\zeta_k)$ acts on T_j via multiplication by ζ_j.)

Since each e_j is prime to p and $(\mathbb{O}_K^{\mathrm{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m - \varpi_K)$ is strictly Henselian, any unit in this ring has an e_j-th root, and thus we are free to multiply any of the a_j by a unit. Consequently, we may assume that in fact $a_1 \cdots a_k = \varpi_K = x_1 \cdots x_m$, and hence (again taking advantage of our freedom to modify the a_j by units) that $\{1,\dots,m\}$ is partitioned into k sets $J_1,\dots,J_k$, such that $a_j=\prod_{i\in J_j} x_i$. Now if we extract the e_j-th roots of each x_i for $i\in J_j$, the resulting extension of $(\mathcal{O}_K^{\text{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m-\varpi_K)$ contains

$$(\mathcal{O}_K^{\text{sh}})^\wedge[\![x_1,\dots,x_n]\!][T_1,\dots,T_k]/(x_1\cdots x_m-\varpi_K, T_1^{e_1}-a_1,\dots,T_k^{e_k}-a_k);$$

thus it is no loss of generality to assume that $k=m$ and that $a_j=x_j$, and so we conclude that $(\mathcal{O}_{\bar{x}_1,\mathscr{X}_1})^\wedge$ is isomorphic, as an $(\mathcal{O}_K^{\text{sh}})^\wedge[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m-\varpi_K)$-algebra, to

$$\big((\mathcal{O}_K^{\text{sh}})^\wedge[\![x_1,\dots,x_n]\!][T_1,\dots,T_m]/(x_1\cdots x_m-\varpi_K, T_1^{e_1}-x_1,\dots,T_m^{e_m}-x_m)\big)^H,$$

for some subgroup $H\subset\mu_{e_1}\times\cdots\times\mu_{e_m}$.

Let I_j denote the subgroup $1\times\cdots\times\mu_{e_j}\times\cdots\times 1$ of $\mu_{e_1}\times\cdots\times\mu_{e_m}$; this is the inertia group of the divisor (x_j) with respect to the cover

$$\operatorname{Spec}((\mathcal{O}_K^{\text{sh}})^\wedge)[\![x_1,\dots,x_n]\!][T_1,\dots,T_m]/(x_1\cdots x_m-\varpi_K, T_1^{e_1}-x_1,\dots,T_m^{e_m}-x_m)$$
$$\to \operatorname{Spec}((\mathcal{O}_K^{\text{sh}})^\wedge)[\![x_1,\dots,x_n]\!]/(x_1\cdots x_m-\varpi_K).$$

If we write $H_j=H\cap I_j$, then $H':=H_1\times\cdots\times H_m$ is a subgroup of H, and the cover

$$\operatorname{Spec}\big((\mathcal{O}_K^{\text{sh}})^\wedge[\![x_1,\dots,x_n]\!][T_1,\dots,T_m]/(x_1\dots x_m-\varpi_K, T_1^{e_1}-x_1,\dots,T_m^{e_m}-x_m)\big)^{H'}$$
$$\to\operatorname{Spec}\big((\mathcal{O}_K^{\text{sh}})^\wedge[\![x_1,\dots,x_n]\!][T_1,\dots,T_m]/(x_1\dots x_m-\varpi_K, T_1^{e_1}-x_1,\dots,T_m^{e_m}-x_m)\big)^{H}$$

is unramified in codimension 1. Since $\mathscr{X}_1$ is regular, being semistable over $\mathcal{O}_L$, so is the target of this map (since we recall that this target is isomorphic to $((\mathscr{X}_1)_{\bar{x}_1})^\wedge$). The purity of the branch locus then implies that this cover is étale, and hence is an isomorphism (since its target is strictly Henselian). Consequently $H=H'$.

If we write

$$H_j=1\times\cdots\times\mu_{e_j'}\times\cdots\times 1\subset 1\times\cdots\times\mu_{e_j}\times\cdots\times 1=I_j,$$

and set $d_j=e_j/e_j'$ and $S_j=T_j^{e_j'}$, then we conclude that

$$\begin{aligned}(\mathcal{O}_{\bar{x}_1,\mathscr{X}_1})^\wedge &\cong \big((\mathcal{O}_K^{\text{sh}})^\wedge[\![x_1,\dots,x_n]\!][T_1,\dots,T_m]/(x_1\cdots x_m-\varpi_K, T_1^{e_1}-x_1,\dots,T_m^{e_m}-x_m)\big)^{H'}\\ &\cong (\mathcal{O}_K^{\text{sh}})^\wedge[\![x_1,\dots,x_n]\!][S_1,\dots,S_m]/(x_1\cdots x_m-\varpi_K, S_1^{d_1}-x_1,\dots,S_m^{d_m}-x_m).\end{aligned}$$

Now $\mathscr{X}_1$ is an $\mathcal{O}_L$-scheme with reduced special fibre (again because it is semistable over $\mathcal{O}_L$). Since $(\mathcal{O}_{\bar{x}_1,\mathscr{X}_1})^\wedge$ is strictly Henselian, it contains $(\mathcal{O}_L^{\text{sh}})^\wedge$, and we may choose a uniformiser ϖ_L of this ring such that $\varpi_L^e=\varpi_K$. Looking at the above description of $(\mathcal{O}_{\bar{x}_1,\mathscr{X}_1})^\wedge$, and taking into account that its reduction modulo ϖ_L

must be reduced, we see that this special fibre must be the zero locus of the element $S_1 \cdots S_m$, hence that $S_1 \cdots S_m = u\varpi_L$ for some unit u, and thus that $(S_1 \cdots S_m)^e = u^e \varpi_K$. We conclude that $d_1 = \cdots = d_m = e$ and that $u^e = 1$, and hence, replacing ϖ_L by $u\varpi_L$, we find that $S_1 \cdots S_m = \varpi_L$. This shows that (2) holds (for our given choice of isomorphism (1.5.4)). □

Remark 1.5.5. Note that we could avoid the appeal to the general theory of tame ramification (in particular, to Abhyankar's lemma) by just directly stipulating in our context that conditions (1) and (2) of Lemma 1.5.3 hold; indeed, in the proof of Theorem 1.5.15 below, we will work directly with these conditions, and in our applications to Shimura varieties we will also see directly that these conditions hold. Nevertheless, we have included Lemma 1.5.3 as an assurance to ourselves (and perhaps to the reader) that these conditions are somewhat natural.

Lemma 1.5.6. *In a tame strictly semistable context as above, the $\mathcal{O}_L$-scheme $\mathscr{X}_1$ is also strictly semistable.*

Proof. Since $\mathscr{X}_1$ is semistable by assumption, it is enough to show that the components of the special fibre $(\mathscr{X}_1)_s$ are regular. Suppose that D is a nonregular component of $(\mathscr{X}_1)_s$, and let $\bar{x}_1$ be a closed geometric point of D whose local ring on D is not regular. If we let $\bar{x}_0$ denote the image of $\bar{x}_1$ in $(\mathscr{X}_0)_s$, then Lemma 1.5.3(2) shows that we may find isomorphisms $(\mathcal{O}_{\bar{x}_1,(\mathscr{X}_1)_s})^\wedge \cong \overline{\mathcal{O}_K/\varpi_K}[[y_1,\ldots,y_n]]/(y_1\cdots y_m)$ and $(\mathcal{O}_{\bar{x}_0,(\mathscr{X}_0)_s})^\wedge \cong \overline{\mathcal{O}_K/\varpi_K}[[x_1,\ldots,x_n]]/(x_1\cdots x_m)$, with $1 \le m \le n$, such that the morphism $(\mathcal{O}_{\bar{x}_1,(\mathscr{X}_1)_s})^\wedge \to (\mathcal{O}_{\bar{x}_0,(\mathscr{X}_0)_s})^\wedge$ is given by $x_j = y_j^e$ for $1 \le j \le m$ and $x_j = y_j$ for $m < j \le n$. Since by assumption $\bar{x}_1$ is not a regular point of D, we find that necessarily $m \ge 2$, and that (possibly after permuting indices) there is an isomorphism $(\mathcal{O}_{\bar{x}_1,D})^\wedge \cong \overline{\mathcal{O}_K/\varpi_K}[[y_1,\ldots,y_n]]/(y_1\cdots y_{m'})$, where $2 \le m' \le m$. If we let D' denote the image of D in $(\mathscr{X}_0)_s$, we conclude that there is an isomorphism $(\mathcal{O}_{\bar{x}_0,D'})^\wedge \cong \overline{\mathcal{O}_K/\varpi_K}[[x_1,\ldots,x_n]]/(x_1\cdots x_{m'})$, and thus that D' is not regular. Hence $\mathscr{X}_0$ is not strictly semistable, a contradiction. □

We now suppose that we are in a tame semistable context, as described above, and suppose for the moment that we are in the local case. Then we have an action of $I(L/K)^{\mathrm{op}} \times G$ on the special fibre $(\mathscr{X}_1)_s$. Let D be an irreducible component of the special fibre of $(\mathscr{X}_0)_s$, and let $\widetilde{D}$ denote its preimage in $(\mathscr{X}_1)_s$, so that $\widetilde{D}$ is an $I(L/K)^{\mathrm{op}} \times G$-invariant union of irreducible components of $(\mathscr{X}_1)_s$.

Lemma 1.5.7. *If G is abelian, then there is a homomorphism $\psi : I(L/K) \to G$ such that the action of $I(L/K)^{\mathrm{op}}$ on $\widetilde{D}$ is given by composing the action of G with ψ; i.e., if $i \in I(L/K)$, then the action of i on $\widetilde{D}$ coincides with the action of $\psi(i)$.*

We first prove a general lemma:

Lemma 1.5.8. *Let S be a connected Noetherian scheme, let G and I be finite groups, and let $f : T \to S$ be a finite étale morphism with the property that $I^{\mathrm{op}} \times G$ acts on T over S in such a way that T becomes a G-torsor over S. If G is abelian, then there exists a morphism $\psi : I \to G$ such that the action of I on T is given by composing the action of G on T with the morphism ψ.*

Proof. For clarity, we will not impose the assumption that G is abelian until required.

If we fix a geometric point $\bar{s}$ of S, then the theory of the étale fundamental group [SGA 1 1971, Exposé V, Théorème 4.1] shows that passing to the fibre over $\bar{s}$ gives an equivalence of categories between the category of finite étale covers of S and the category of (discrete) finite sets with a continuous action of $\pi_1(S, \bar{s})$. In this way, T is classified by an object P of this latter category equipped with an action of $I^{\mathrm{op}} \times G$, with respect to which the G-action makes P a principal homogeneous G-set.

If we fix a base point $p \in P$, then we may identify P with G, thought of as a principal homogeneous G-set via left multiplication. As the automorphisms of G as a principal homogeneous G-set are naturally identified with G^{op} acting by right multiplication, we obtain a homomorphism $\psi_p : I^{\mathrm{op}} \to G^{\mathrm{op}}$, or equivalently a homomorphism $\psi_p : I \to G$, describing the action of I^{op} on P. If we replace p by $g \cdot p$ (for some $g \in G$), then one finds that $\psi_{gp} = g \psi_p g^{-1}$. Thus, if we now assume furthermore that G is abelian, then $\psi_p = \psi_{gp}$, and so it is reasonable in this case to write simply ψ for this homomorphism, which is well-defined independently of the choice of base point for P. Furthermore, when G is abelian, left and right multiplication by an element $g \in G$ coincide, and so the action of I^{op} on P is given by the formula $i \cdot p = \psi(i) \cdot p$ for all $p \in P$. Since the automorphisms of T over S induced by i and $\psi(i)$ coincide on P, they in fact coincide on all of T. □

Proof of Lemma 1.5.7. The morphism $\mathscr{X}_1 \to (\mathscr{X}_0)_{/\mathbb{O}_L}$ is étale on generic fibres, and the explicit local formulas for this morphism provided by Lemma 1.5.3 show that it is in fact étale over an open subset $\mathscr{U}_0$ of $(\mathscr{X}_0)_{/\mathbb{O}_L}$ whose intersection with the special fibre $((\mathscr{X}_0)_{/\mathbb{O}_L})_s$ is Zariski dense. Replacing $\mathscr{U}_0$ with the intersection of all of its $I(L/K)^{\mathrm{op}}$-translates, we may furthermore assume that $\mathscr{U}_0$ is invariant under the action of $I(L/K)^{\mathrm{op}}$ on $(\mathscr{X}_0)_{/\mathbb{O}_L}$.

If we let $\mathscr{U}_1$ denote the preimage of $\mathscr{U}_0$ in $\mathscr{X}_1$, then $\mathscr{U}_1$ is invariant under the $I(L/K) \times G$-action on $\mathscr{X}_1$, and the morphism $\mathscr{U}_1 \to \mathscr{U}_0$ is a finite étale cover, for which the corresponding map $U_1 \to U_0$ on generic fibres realises U_1 as a G-torsor over U_0. It follows that the G-action on $\mathscr{U}_1$ realises $\mathscr{U}_1$ as an étale G-torsor over $\mathscr{U}_0$, and hence, passing to special fibres, that $(\mathscr{U}_1)_s$ is an étale G-torsor over $(\mathscr{U}_0)_s$.

Now the induced $I(L/K)^{\mathrm{op}}$-action on $(\mathscr{U}_0)_s$ is trivial, and so $I(L/K)^{\mathrm{op}}$ acts on $(\mathscr{U}_1)_s$ as a group of automorphisms of the G-torsor $(\mathscr{U}_1)_s$ over $(\mathscr{U}_0)_s$. If $D' := D \cap (\mathscr{U}_0)_s$, then D' is an irreducible component of $(\mathscr{U}_0)_s$, and $\widetilde{D}' := \widetilde{D} \cap (\mathscr{U}_1)_s$

is the restriction of $(\mathcal{U}_1)_s$ to D'. Thus $\widetilde{D}' \to \widetilde{D}$ is again an étale G-torsor, with an action of $I(L/K)^{\mathrm{op}}$ via automorphisms. Lemma 1.5.8 then shows that there is a homomorphism $\psi : I(L/K)^{\mathrm{op}} \to G^{\mathrm{op}}$, or equivalently a homomorphism $\psi : I(L/K) \to G$, such that the action of $I(L/K)^{\mathrm{op}}$ on the points of $\widetilde{D}'$ is given by composing the action of G with the homomorphism ψ. Since $\widetilde{D}$ is equal to the Zariski closure of $\widetilde{D}'$ in $(\mathscr{X}_1)_s$, the claim of the lemma follows. □

Lemma 1.5.9. *Suppose that we are in a tame strictly semistable context. If $g \in I(L/K) \times G$ and D is a component of $(\mathscr{X}_1)_s$, then D and gD either coincide or are disjoint.*

Proof. The images of D and gD in $(\mathscr{X}_0)_s$ coincide, and it follows from Lemma 1.5.3 that two distinct components of $(\mathscr{X}_1)_s$ that have nonempty intersection must have distinct images in $(\mathscr{X}_0)_s$. □

If we now suppose that we are in the global case, then the discussion applies with L/K everywhere replaced by L_w/K_v, and in particular for each component D we may define a character $\psi : I(L_w/K_v) \to G$ describing the action of $I(L_w/K_v)$ on D.

Our next result describes how our tame semistable context behaves upon passage to a semistable hypersurface section of $\mathscr{X}_0$. In its statement we assume for simplicity that we are in the local case.

Proposition 1.5.10. *Suppose that we are in the tamely ramified semistable context described above, and let $\mathscr{Y}_0$ be a regular hypersurface section of $\mathscr{X}_0$ such that the union of $\mathscr{Y}_0$ and $(\mathscr{X}_0)_s$ forms a divisor with normal crossings on $\mathscr{X}_0$. Let Y_0 denote the generic fibre of $\mathscr{Y}_0$, let Y_1 denote the preimage of Y_0 under the morphism $\pi : X_1 \to X_0$, and let $\mathscr{Y}_1$ be the preimage of $\mathscr{Y}_0$ under* (1.5.1) (*so that $(Y_1)_{/L}$ is the generic fibre of the $\mathscr{O}_L$-scheme $\mathscr{Y}_1$*). *Then*:

(1) *The complement of Y_1 in X_1 is affine.*

(2) *The generic fibre Y_0 of $\mathscr{Y}_0$ is smooth over K, the morphism $Y_1 \to Y_0$ is an étale G-torsor* (*so in particular Y_1 is also smooth over K*), *$\mathscr{Y}_0$ is a semistable model of Y_0 over $\mathscr{O}_K$, $\mathscr{Y}_1$ is a semistable model for $(Y_1)_{/L}$ over $\mathscr{O}_L$, and the morphism $\mathscr{Y}_1 \to \mathscr{Y}_0$ is tamely ramified*; *consequently $\mathscr{Y}_1 \to \mathscr{Y}_0$ is again a tamely ramified semistable context.*

(3) *Suppose that G is abelian. If D' is an irreducible component of $(\mathscr{Y}_1)_s$, contained in an irreducible component D of $(\mathscr{X}_1)_s$, then the homomorphism $\psi : I(L/K) \to G$, which describes the action of $I(L/K)$ on D, also describes the action of $I(L/K)$ on D'.*

Proof. Since $\mathscr{Y}_0$ is a hypersurface section of $\mathscr{X}_0$, its generic fibre Y_0 is a hypersurface section of X_0. Thus its complement is affine. Since π is a finite morphism by assumption, the complement of Y_1 in X_1 is again affine. Since Y_0 is a regular projective K-scheme (being the generic fibre of $\mathscr{Y}_0$, which is regular by assumption), it is in fact smooth over K. By definition, Y_1 is the preimage of Y_0 under the morphism $X_1 \to X_0$, which is an étale G-torsor by assumption. Thus $Y_1 \to Y_0$ is indeed an étale G-torsor (and so Y_1 is also smooth over K).

Let $\bar{x}_0$ denote a closed geometric point of the special fibre $(\mathscr{Y}_0)_s$. Since $\big(((\mathscr{X}_0)_{\bar{x}_0})^\wedge\big)_s \cup ((\mathscr{Y}_0)_{\bar{x}_0})^\wedge$ forms a divisor with normal crossings, since each component of $\big(((\mathscr{X}_0)_{\bar{x}_0})^\wedge\big)_s$ is regular, and since $\mathscr{Y}_0$ is regular by assumption, it follows from [Grothendieck and Murre 1971, Lemma 1.8.4] that $\big(((\mathscr{X}_0)_{\bar{x}_0})^\wedge\big)_s \cup ((\mathscr{Y}_0)_{\bar{x}_0})^\wedge$ is in fact a divisor with strictly normal crossings in $((\mathscr{X}_0)_{\bar{x}_0})^\wedge$, and hence the local equation ℓ of $((\mathscr{Y}_0)_{\bar{x}_0})^\wedge$, together with the elements $x_1, \dots, x_m$ that cut out the irreducible components of $\big(((\mathscr{X}_0)_{\bar{x}_0})^\wedge\big)_s$, form part of a regular system of parameters for $(\mathscr{O}_{\bar{x}_0,\mathscr{X}_0})^\wedge$. Thus we may choose a model of the form (1.5.4) for $((\mathscr{X}_0)_{\bar{x}_0})^\wedge$ for which $m < n$ and in which ℓ is equal to the element x_n, i.e., in which $((\mathscr{Y}_0)_{\bar{x}_0})^\wedge$ is the zero locus of the element x_n.

We now choose a closed geometric point $\bar{x}_1$ of $(\mathscr{X}_1)_s$ lying over $\bar{x}_0$, as well as a model for the tamely ramified morphism $((\mathscr{X}_1)_{\bar{x}_1})^\wedge \to ((\mathscr{X}_0)_{\bar{x}_0})^\wedge$ as in part (2) of Lemma 1.5.3. Thus this morphism has the form

$$\begin{aligned}\operatorname{Spec}((\mathscr{O}_L^{\mathrm{sh}})^\wedge)[\![y_1, \dots, y_n]\!]/(y_1 \cdots y_m - \varpi_L)&\\ \to \operatorname{Spec}((\mathscr{O}_K^{\mathrm{sh}})^\wedge)[\![x_1, \dots, x_n]\!]/(x_1 \cdots x_m - \varpi_K),&\end{aligned}$$

with $x_j = y_j^e$ for $1 \le j \le m$ and $x_j = y_j$ for $m < j \le n$. In particular, we see that $x_n = y_n$, and thus we see that the induced morphism

$$((\mathscr{Y}_1)_{\bar{x}_1})^\wedge \to ((\mathscr{Y}_0)_{\bar{x}_0})^\wedge \tag{1.5.11}$$

can be written as

$$\begin{aligned}\operatorname{Spec}((\mathscr{O}_L^{\mathrm{sh}})^\wedge)[\![y_1, \dots, y_{n-1}]\!]/(y_1 \dots y_m - \varpi_L)&\\ \to \operatorname{Spec}((\mathscr{O}_L^{\mathrm{sh}})^\wedge)[\![x_1, \dots, x_{n-1}]\!]/(x_1 \dots x_m - \varpi_K).&\end{aligned} \tag{1.5.12}$$

Thus we see that $\mathscr{Y}_0$ and $\mathscr{Y}_1$ are indeed semistable models of their generic fibres (over $\mathscr{O}_K$ and $\mathscr{O}_L$ respectively), and that the morphism $\mathscr{Y}_1 \to \mathscr{Y}_0$ is tamely ramified. This completes the verification of (2). The claim of (3) follows from the fact that the action of $I(L/K)^{\mathrm{op}} \times G$ on $(\mathscr{Y}_1)_s$ is the restriction of the corresponding action on $(\mathscr{X}_1)_s$, together with the fact that any component of $(\mathscr{Y}_1)_s$ is contained in a component of $(\mathscr{X}_1)_s$. □

We now suppose that E is an algebraic extension of $\mathbb{Q}_p$ containing K_0. Recall that if $\rho : G_K \to \mathrm{GL}_n(E)$ is a potentially semistable representation, then we may attach a Weil–Deligne representation $\mathrm{WD}(\rho)$ to ρ by first passing to the potentially semistable Dieudonné module $D_{\mathrm{pst}}(\rho)$ of ρ, which is a module over $E \otimes_{\mathbb{Q}_p} K_0$, then fixing an embedding $K_0 \hookrightarrow E$, and hence a projection $\mathrm{pr} : E \otimes_{\mathbb{Q}_p} K_0 \to E$, and, finally, forming $\mathrm{WD}(\rho) := E \otimes_{E \otimes_{\mathbb{Q}_p} K_0, \mathrm{pr}} D_{\mathrm{pst}}$. Although $\mathrm{WD}(\rho)$ depends on the choice of the embedding $K_0 \hookrightarrow E$, up to isomorphism it is independent of this choice, as the Frobenius ϕ on $D_{\mathrm{pst}}(\rho)$ provides isomorphisms between the different choices. (See for example [Conrad et al. 1999, Appendix B] and [Taylor 2004, p. 78–79] for discussions of this construction and its properties.)

In the tame strictly semistable case, the following result will allow us to describe the inertial part of the Weil–Deligne representation associated to the p-adic étale cohomology of X_1, or of a pair (X_1, Y_1) that arises in the context of the preceding proposition. Before stating the result we introduce some additional notation, and an additional assumption.

Assume that G is abelian, and let J denote the set of $I(L/K) \times G$-orbits on the set of irreducible components of $(\mathscr{X}_1)_s$, and let D_j (for $j \in J$) denote the union of the components lying in the orbit labelled by j. Let $\psi_j : I(L/K) \to G$ be the homomorphism provided by Lemma 1.5.7, describing the action of $I(L/K)$ on the points of D_j.

Proposition 1.5.13. *Suppose that we are in a tame strictly semistable context as above. Either let W denote the Weil–Deligne representation associated to the potentially semistable G_K-representation $H^i_{\text{ét}}((X_1)_{/\overline{K}}, E)$, or else suppose that we are in the context of Proposition 1.5.10, and let W denote the Weil–Deligne representation associated to the potentially semistable G_K-representation $H^i_{\text{ét}}((X_1)_{/\overline{K}}, (Y_1)_{/\overline{K}}), E)$ (here i is some given degree of cohomology); in either case, W is a representation of the product* $\mathrm{WD}_K \times G$. *Assume furthermore that G is abelian.*

Then, if, as in the above discussion, $\widetilde{J}$ denotes the set of $I(L/K) \times G$-orbits of irreducible components of $(\mathscr{X}_1)_s$, we may decompose W as a direct sum $W = \bigoplus_{\tilde{j} \in \widetilde{J}} W_{\tilde{j}}$, such that on $W_{\tilde{j}}$ the action of the inertia group in W_K *is obtained by composing the G-action on $W_{\tilde{j}}$ with the homomorphism $I_K \to I(L/K) \xrightarrow[\psi_{\tilde{j}}]{} G$.*

Proof. Since the action of the inertia subgroup of W_K on W factors through a finite group, and representations of a finite group over a field of characteristic zero are semisimple, the claimed property of W is stable under the formation of subobjects, quotients, and extensions (in the category of $W_K \times G$-representations). A consideration of the long exact sequence of cohomology associated to the pair (X_1, Y_1) (see the Appendix) then reduces the claim for the cohomology of the pair to the claim for the cohomology of X_1 and Y_1 individually. Since Proposition 1.5.10 shows that the strictly semistable model $\mathscr{Y}_1$ of $(Y_1)_{/L}$ behaves in an identical manner to the strictly semistable model of $\mathscr{X}_1$ of $(X_1)_{/L}$, it in fact suffices to consider the case of X_1.

Thus we now restrict our attention to the W_K-representation W underlying the potentially semistable Dieudonné module associated to $H^i_{\text{ét}}((X_1)_{/\overline{K}}, E)$. By [Tsuji 1999], this Dieudonné module is naturally identified with the log-crystalline cohomology $H^i((\mathscr{X}_1)^\times_s/W(k)^\times) \otimes_{W(k)} E$ of the special fibre $(\mathscr{X}_1)_s$ with its natural log-structure. Lemma 1.5.9 shows that if an intersection of distinct components of the special fibre $(\mathscr{X}_1)_s$ is nonempty, then the various components appearing must lie in mutually distinct orbits of $I(L/K)\times G$ acting on the set of components. Recalling that J denotes the indexing set for the collection $\{D_j\}_{j\in J}$ of $I(L/K)\times G$-orbits of components of $(\mathscr{X}_1)_s$, this log-crystalline cohomology may be computed by the following spectral sequence of [Mokrane 1993]:

$$E_1^{-m,i+m} = \bigoplus_{\substack{l\geq \max\{0,-m\} \\ \{j_1,\dots,j_{2l+m+1}\}\subset J}} H^{i-2l-m}_{\text{cris}}(D_{j_1}\cap\cdots\cap D_{j_{2l+m+1}}/W(k))\otimes_{W(k)} E(-l-m) \Longrightarrow H^i((\mathscr{X}_1)^\times_s/W(k)^\times)\otimes_{W(k)} E.$$

The constructions of [Tsuji 1999; Mokrane 1993] are both functorial, so that everything here is compatible with the $I(L/K)\times G$-actions. Each of the summands in the E_1-term is naturally an $I(L/K)\times G$-representation, and furthermore the action of $I(L/K)$ is given by the composite of the action of G with one of the characters $\psi_{\tilde{j}}$. Thus the E_1-terms of this spectral sequence satisfy the claimed property of W. Thus W also satisfies this property, since it is obtained as a successive extension of subquotients of these E_1-terms. □

We are now ready to prove our equivariant versions of Theorems 1.3.1 and 1.4.1. For the first result, we place ourselves in the local case (since the global case immediately reduces to the local case by passing from K to K_v):

Theorem 1.5.14. *Suppose that we are in the tame strictly semistable context described above. Then the $G_K\times G$-representation $H^1_{\text{ét}}((X_1)_{\overline{K}}, \mathbb{F}_p)$ embeds $G_K\times G$-equivariantly into the reduction modulo the uniformiser of a $G_K\times G$-invariant $\mathscr{O}_E$-lattice in a representation V of $G_K\times G$ over E, having the following properties*:

(1) *The restriction of V to G_L is semistable, with Hodge–Tate weights contained in the interval* $[-1, 0]$.

(2) *The Weil–Deligne representation associated to V, which is naturally a representation of* $\mathrm{WD}_K\times G$, *when restricted to a representation of $I_K\times G$ can be written as a direct sum $\bigoplus_{\tilde{j}\in\tilde{J}} W_{\tilde{j}}$ of $I_K\times G$-representations, where $\tilde{j}$ runs over the same index set that labels the set of $I(L/K)\times G$-orbits of irreducible*

components of $(\mathscr{X}_1)_s$, *such that on* $W_{\tilde{j}}$ *the action of the inertia group in* W_K *is obtained by composing the* G*-action on* $W_{\tilde{j}}$ *with the homomorphism* $I_K \to I(L/K) \xrightarrow[\psi_{\tilde{j}}]{} G$.

Proof. We follow the proof of Theorem 1.3.1, proceeding by descending induction on the dimension of X_0 and X_1, and passing to appropriately chosen hypersurface sections $\mathscr{Y}_0$ of $\mathscr{X}_0$ and their corresponding preimages Y_1 in X_1 and $\mathscr{Y}_1$ in $(\mathscr{X}_1)_{/L}$. Taking into account Proposition 1.5.10, we thus reduce to the case when X_0 and X_1 are curves, so that $H^1_{\text{ét}}((X_1)_{/\overline{K}}, \mathbb{F}_p)$ is the reduction mod p of $H^1_{\text{ét}}((X_1)_{/\overline{K}}, \mathbb{Z}_p)$, which is in turn a lattice in $H^1_{\text{ét}}((X_1)_{/\overline{K}}, \mathbb{Q}_p)$. This latter representation is potentially semistable with Hodge–Tate weights in $[-1, 0]$, by [Tsuji 1999]; the claim regarding Weil–Deligne representations follows from Proposition 1.5.13. □

For our second result, we allow ourselves to be in either the local or global context.

Theorem 1.5.15. *Suppose that we are in the tame strictly semistable context described above*, *and let* $\rho : G_K \times G \to \mathrm{GL}_n(k_E)$ *be an irreducible and continuous representation that embeds as a subquotient of* $H^i_{\text{ét}}((X_1)_{\overline{K}}, k_E)$. *Then* ρ *also embeds as a subquotient of a* $G_K \times G$*-representation over* k_E *which is the reduction modulo the uniformiser of a* $G_K \times G$*-invariant* $\mathscr{O}_E$*-lattice in a representation* V *of* $G_K \times G$ *over* E, *having the following properties*:

(1) *The representation* V *becomes semistable when restricted to* G_L (*resp. the decomposition group* $D_w \subset G_L$ *in the global case*), *with Hodge–Tate weights contained in the interval* $[-i, 0]$.

(2) *The Weil–Deligne representation associated to* V, *which is naturally a representation of* $\mathrm{WD}_K \times G$ (*resp.* $\mathrm{WD}_{K_v} \times G$ *in the global case*), *when restricted to a representation of* $I_K \times G$ (*resp.* $I_{K_v} \times G$ *in the global case*) *can be written as a direct sum* $\bigoplus_{\tilde{j}} W_{\tilde{j}}$ *of* $I_K \times G$*-representations* (*resp. of* $I_{K_v} \times G$*-representations*), *where* $\tilde{j}$ *runs over the same index set that labels the set of* $I(L/K) \times G$*-orbits* (*resp. of* $I_{K_v} \times G$*-orbits*) *of irreducible components of* $(\mathscr{X}_1)_s$, *such that on* $W_{\tilde{j}}$, *the action of the inertia group is obtained by composing the* G*-action on* $W_{\tilde{j}}$ *with the homomorphism* $I_K \to I(L/K) \xrightarrow[\psi_{\tilde{j}}]{} G$ (*resp. the homomorphism* $I_{K_v} \to I(L_w/K_v) \xrightarrow[\psi_{\tilde{j}}]{} G$).

Proof. We can be proved in exactly the same way as Theorem 1.4.1, taking into account Propositions 1.5.10 and 1.5.13. □

2. Breuil modules with descent data

In this section we establish a result (Theorem 2.2.4) which imposes some constraints on the reductions of certain tamely potentially semistable p-adic representations of $G_{\mathbb{Q}_p}$.

2.1. ***Preliminaries.*** We begin by recalling some results from Section 3 of [Emerton et al. 2013]. To this end, let p be an odd prime, let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$, and let E and K be finite extensions of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. Assume that E contains the images of all embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$. Let K_0 be the maximal absolutely unramified subfield of K, so that $K_0 = W(k)[1/p]$, where k is the residue field of K. Let K/K' be a Galois extension, with K' a field lying between $\mathbb{Q}_p$ and K. Assume further that K/K' is tamely ramified with ramification index e, and fix a uniformiser $\pi \in K$ with $\pi^e \in K'$. Let $E(u) \in W(k)[u]$ be the minimal polynomial of π over K_0.

Let k_E be the residue field of E, and let $0 \le r \le p-2$ be an integer. Recall that the category $k_E\text{-BrMod}^r_{\text{dd}}$ of *Breuil modules of weight r with descent data* from K to K' and coefficients k_E consists of quintuples $(\mathcal{M}, \mathcal{M}_r, \varphi_r, \hat{g}, N)$, where:

- $\mathcal{M}$ is a finitely generated $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-module, free over $k[u]/u^{ep}$.
- $\mathcal{M}_r$ is a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-submodule of $\mathcal{M}$ containing $u^{er}\mathcal{M}$.
- $\varphi_r : \mathcal{M}_r \to \mathcal{M}$ is k_E-linear and φ-semilinear (where $\varphi : k[u]/u^{ep} \to k[u]/u^{ep}$ is the p-th power map) with image generating $\mathcal{M}$ as a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-module.
- $N : \mathcal{M} \to \mathcal{M}$ is $k \otimes_{\mathbb{F}_p} k_E$-linear and satisfies $N(ux) = uN(x) - ux$ for all $x \in \mathcal{M}$, $u^e N(\mathcal{M}_r) \subset \mathcal{M}_r$, and $\varphi_r(u^e N(x)) = cN(\varphi_r(x))$ for all $x \in \mathcal{M}_r$. Here, $c = \overline{F}(u)^p \in (k[u]/u^{ep})^\times$, where $E(u) = u^e + pF(u)$ in $W(k)[u]$.
- $\hat{g} : \mathcal{M} \to \mathcal{M}$ are additive bijections for each $g \in \text{Gal}(K/K')$, preserving $\mathcal{M}_r$, commuting with the φ_r- and N-actions, and satisfying $\hat{g}_1 \circ \hat{g}_2 = (g_1 \circ g_2)^\wedge$ for all $g_1, g_2 \in \text{Gal}(K/K')$. Furthermore, if $a \in k \otimes_{\mathbb{F}_p} k_E$ and $m \in \mathcal{M}$ then $\hat{g}(au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i \hat{g}(m)$.

There is a covariant functor $\mathrm{T}^{*,r}_{\text{st}}$ from $k_E\text{-BrMod}^r_{\text{dd}}$ to the category of k_E-representations of $G_{K'}$.

Lemma 2.1.1. *Suppose that $\mathcal{M} \in k_E\text{-BrMod}^r_{\text{dd}}$ and T' is a $G_{K'}$-subrepresentation of $\mathrm{T}^{*,r}_{\text{st}}(\mathcal{M})$ (so that in particular T' has the structure of a k_E-vector space). Then there is a unique subobject $\mathcal{M}'$ of $\mathcal{M}$ such that, if $f : \mathcal{M}' \to \mathcal{M}$ is the inclusion map, then $\mathrm{T}^{*,r}_{\text{st}}(f)$ is identified with the inclusion $T' \hookrightarrow \mathrm{T}^{*,r}_{\text{st}}(\mathcal{M})$. (Here $\mathcal{M}'$ is a subobject of $\mathcal{M}$ in the naive sense that it is a sub-$(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-module of $\mathcal{M}$, which inherits the structure of an object of $k_E\text{-BrMod}^r_{\text{dd}}$ from $\mathcal{M}$ in the obvious way.)*

Proof. This is Corollary 3.2.9 of [Emerton et al. 2013]. □

We now specialise to the particular situation of interest to us in this paper; namely, we let K_0 be the unique unramified extension of $\mathbb{Q}_p$ of degree d, we take $K = K_0((-p)^{1/(p^d-1)})$, and we set $K' = K_0$, so that $e = p^d - 1$. Fix $\pi = (-p)^{1/(p^d-1)}$. We write $\widetilde{\omega}_d : \text{Gal}(K/K_0) \to K_0^\times$ for the character $g \mapsto g(\pi)/\pi$, and we let ω_d be the reduction of $\widetilde{\omega}_d$ modulo π. (By inflation we can also think of $\widetilde{\omega}_d$ and ω_d as characters of $I_{K_0} = I_{\mathbb{Q}_p}$. Note that ω_d is a tame fundamental character of niveau d and that $\widetilde{\omega}_d$ is the Teichmüller lift of ω_d.) Note that when $d = 1$, we have $\omega_1 = \omega$, the mod p cyclotomic character.

Let φ be the arithmetic Frobenius on k, and let $\sigma_0 : k \hookrightarrow k_E$ be a fixed embedding. Inductively define $\sigma_1, \ldots, \sigma_{d-1}$ by $\sigma_{i+1} = \sigma_i \circ \varphi^{-1}$; we will often consider the numbering to be cyclic, so that $\sigma_d = \sigma_0$. There are idempotents $e_i \in k \otimes_{\mathbb{F}_p} k_E$ such that if M is any $k \otimes_{\mathbb{F}_p} k_E$-module, then $M = \bigoplus_i e_i M$, and $e_i M$ is the subset of M consisting of elements m for which $(x \otimes 1)m = (1 \otimes \sigma_i(x))m$ for all $x \in k$. Note that $(\varphi \otimes 1)(e_i) = e_{i+1}$ for all i.

If $\rho : G_{K_0} \to \mathrm{GL}_n(E)$ is a potentially semistable representation which becomes semistable over K, then the associated inertial type (that is, the restriction to I_{K_0} of the Weil–Deligne representation associated to ρ) is a representation of I_{K_0} which becomes trivial when restricted to I_K, so we can and do think of it as a representation of $\mathrm{Gal}(K/K_0) \cong I_{K_0}/I_K$.

Proposition 2.1.2. *Maintaining our current assumptions on K, suppose that $\rho : G_{K_0} \to \mathrm{GL}_n(E)$ is a continuous representation whose restriction to G_K is semistable with Hodge–Tate weights contained in $[0, r]$, where $r \le p-2$, and let the inertial type of ρ be $\chi_1 \oplus \cdots \oplus \chi_n$, where each χ_i is a character of I_{K_0}/I_K. If $\bar{\rho}$ denotes the reduction modulo $\mathfrak{m}_E$ of a G_{K_0}-stable $\mathcal{O}_E$-lattice in ρ, then there is an element $\mathcal{M}$ of k_E-$\mathrm{BrMod}^r_{\mathrm{dd}}$, admitting a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-basis $v_1, \ldots, v_n$ such that $\hat{g}(v_i) = (1 \otimes \bar{\chi}_i(g))v_i$ for all $g \in \mathrm{Gal}(K/K_0)$, and for which $\mathrm{T}^{*,r}_{\mathrm{st}}(\mathcal{M}) \cong \bar{\rho}$.*

Proof. This is Proposition 3.3.1 of [Emerton et al. 2013]. (Note that the conventions on the sign of the Hodge–Tate weights in that work are the opposite of the conventions in this paper.) □

Lemma 2.1.3. *Maintain our current assumptions on K, so that in particular we have $e = p^d - 1$. Then every rank-1 object of k_E-$\mathrm{BrMod}^r_{\mathrm{dd}}$ may be written in the form*

- $\mathcal{M} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}) \cdot m$,
- $e_i \mathcal{M}_r = u^{r_i} e_i \mathcal{M}$,
- $\varphi_r\big(\sum_{i=0}^{d-1} u^{r_i} e_i m\big) = \lambda m$ *for some* $\lambda \in (k \otimes_{\mathbb{F}_p} k_E)^\times$,
- $\hat{g}(m) = \big(\sum_{i=0}^{d-1} (\omega_d(g)^{k_i} \otimes 1)e_i\big)m$ *for all* $g \in \mathrm{Gal}(K/K_0)$, *and*
- $N(m) = 0$.

Here the integers $0 \le r_i \le (p^d - 1)r$ and k_i satisfy $k_i \equiv p(k_{i-1} + r_{i-1}) \bmod (p^d - 1)$ for all i. Conversely, any module $\mathcal{M}$ of this form is a rank-1 object of k_E-$\mathrm{BrMod}^r_{\mathrm{dd}}$. Furthermore,

$$\mathrm{T}^{*,r}_{\mathrm{st}}(\mathcal{M})|_{I_{K_0}} \cong \sigma_0 \circ \omega_d^{\kappa_0},$$

where $\kappa_0 \equiv k_0 + p(r_0 p^{d-1} + r_1 p^{d-2} + \cdots + r_{d-1})/(p^d - 1) \bmod (p^d - 1)$.

Proof. This is Lemma 3.3.2 of [Emerton et al. 2013]. □

Remark 2.1.4. In the sequel, we will only be interested in the case that for each i we have $k_i = (1 + p + \cdots + p^{d-1})x_i$ for some $0 \le x_i < p-1$. In this case, the condition that $pr_i \equiv k_{i+1} - pk_i \bmod (p^d - 1)$ implies that $r_i \equiv (1 + p + \cdots + p^{d-1})(x_{i+1} - x_i) \bmod (p^d - 1)$, so the condition that $0 \le r_i \le (p^d-1)r$ means that we can write $r_i = (1+p+\cdots+p^{d-1})(x_{i+1}-x_i)+(p^d-1)y_i$, with $0 \le y_i \le r$. An elementary calculation shows that we then have

$$\kappa_0 \equiv x_0 + y_0 + p^{d-1}(x_1 + y_1) + \cdots + p(x_{d-1} + y_{d-1}) \bmod (p^d - 1).$$

2.2. *Regularity.* Let $\mathbb{Q}_{p^n}$ denote the unique unramified extension of $\mathbb{Q}_p$ of degree n, with residue field $\mathbb{F}_{p^n}$. Regarding $\mathbb{F}_{p^n}$ as a subfield of $\overline{\mathbb{F}}_p$, we may then regard ω_n as a character $I_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times$.

Definition 2.2.1. Let $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be an irreducible representation, so that $\bar{\rho} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^n}}}^{G_{\mathbb{Q}_p}} \chi$ for some character $\chi : G_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times$. If we write

$$\chi|_{I_{\mathbb{Q}_p}} = \omega_n^{(a_0+pa_1+\cdots+p^{n-1}a_{n-1})},$$

where each $a_i \in [0, p-1]$ and not all the a_i are $p-1$, then the multiset of *exponents* of $\bar{\rho}$ is defined to be the multiset of residues of the a_i in $\mathbb{Z}/p\mathbb{Z}$.

Definition 2.2.2. Let $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a representation. Then the multiset of *exponents* of $\bar{\rho}$ is the union of the multisets of exponents of each of the Jordan–Hölder factors of $\bar{\rho}$.

Definition 2.2.3. Let $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a representation, and let r be a nonnegative integer. Then we say that $\bar{\rho}$ is *r-regular* if the exponents $a_1, \dots, a_n$ of $\bar{\rho}$ are such that the residues $a_i + k \in \mathbb{Z}/p\mathbb{Z}$, $1 \le i \le n$, $0 \le k \le r+1$, are pairwise distinct.

The following theorem is the main result we will need from explicit p-adic Hodge theory:

Theorem 2.2.4. *Let r be a nonnegative integer, and let $s : G_{\mathbb{Q}_p} \to \mathrm{GL}_m(\overline{\mathbb{Q}}_p)$ be a potentially semistable representation with Hodge–Tate weights contained in the interval $[0, r]$ and inertial type $\chi_1 \oplus \cdots \oplus \chi_m$. Suppose that there are (not necessarily distinct) integers $0 \le a_1, \dots, a_n < p-1$ such that each χ_i is equal to some $\widetilde{\omega}^{a_j}$.*

Suppose that $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is a subquotient of $\bar{s}$, the reduction mod $\mathfrak{m}_{\mathcal{O}_{\overline{\mathbb{Q}}_p}}$ of some $G_{\mathbb{Q}_p}$-stable $\overline{\mathbb{Z}}_p$-lattice in s. Suppose also that

- $\det \bar{\rho}|_{I_{\mathbb{Q}_p}} = \omega^{a_1+\cdots+a_n+n(n-1)/2}$,
- $r \le (n-1)/2$, *and*
- $p > n(n-1)/2 + 1$.

If $r = (n-1)/2$ then assume further that some irreducible subquotient of $\bar{\rho}$ has dimension greater than 1. *Then $\bar{\rho}$ is not r-regular.*

Proof. Modifying the choice of $\overline{\mathbb{Z}}_p$-lattice if necessary, it suffices to treat the case that $\bar{\rho}$ and $\bar{s}$ are semisimple. Let $\bar{\rho}' \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^d}}}^{G_{\mathbb{Q}_p}} \chi$ be an irreducible subrepresentation of $\bar{\rho}$. Take $K' = K_0 = \mathbb{Q}_{p^d}$ in the above notation, so that $e = p^d - 1$. Taking E to be sufficiently large so that s is defined over E and $\bar{\rho}$ is defined over k_E, and applying Proposition 2.1.2 to $s|_{G_{\mathbb{Q}_{p^d}}}$, we see that there is an element $\mathcal{M}$ of k_E-$\mathrm{BrMod}_{\mathrm{dd}}^r$ with

$$\mathrm{T}_{\mathrm{st}}^{*,r}(\mathcal{M}) \cong \bar{s}|_{G_{\mathbb{Q}_{p^d}}},$$

such that $\mathcal{M}$ has a $(k \otimes_{\mathbb{F}_p} k_E)[u]/u^{ep}$-basis $v_1, \dots, v_m$ with $\hat{g}(v_i) = (1 \otimes \bar{\chi}_i(g))v_i$ for all $g \in \mathrm{Gal}(K/K_0)$. Since $\bar{s}|_{G_{\mathbb{Q}_{p^d}}}$ contains a subrepresentation isomorphic to χ, we see from Lemma 2.1.1 that there is a rank-1 subobject $\mathcal{N}$ of $\mathcal{M}$ for which $\mathrm{T}_{\mathrm{st}}^{*,r}(\mathcal{N}) \cong \chi$.

Since $\mathcal{N}/u\mathcal{N}$ embeds into $\mathcal{M}/u\mathcal{M}$ (as $\mathcal{N}$ is a free $k[u]/u^{ep}$-submodule of the free $k[u]/u^{ep}$-module $\mathcal{M}$), we see from Lemma 2.1.3 and our assumption on the characters $\bar{\chi}_i$ that we may write $\mathcal{N}$ in the form

- $\mathcal{N} = ((k \otimes_{\mathbb{F}_p} k_E)[u]/u^{(p^d-1)p}) \cdot w$,
- $e_i \mathcal{N}_r = u^{r_i} e_i \mathcal{N}$,
- $\varphi_r\big(\sum_{i=0}^{d-1} u^{r_i} e_i w\big) = \lambda w$ for some $\lambda \in (k \otimes_{\mathbb{F}_p} k_E)^\times$,
- $\hat{g}(w) = \big(\sum_{i=0}^{d-1} (\omega_d(g)^{k_i} \otimes 1) e_i\big) w$ for all $g \in \mathrm{Gal}(K/K_0)$, and
- $N(w) = 0$.

Here the integers $0 \le r_i \le (p^d-1)r$ and k_i satisfy $k_i \equiv p(k_{i-1}+r_{i-1}) \pmod{p^d-1}$ for all i, and each k_i is equal to some $(1 + p + \cdots + p^{d-1})a_j$. (The conditions on the r_i come from Lemma 2.1.3, and the fact that each k_i is equal to some $(1+p+\cdots+p^{d-1})a_j$ comes from the fact that $\mathcal{N}$ is a submodule of $\mathcal{M}$ which has a basis $v_1, \dots, v_n$ such that $\hat{g}(v_i) = (1 \otimes \bar{\chi}_i(g))v_i$ and the assumption that each $\bar{\chi}_i$ is equal to some ω^{a_j}.) Writing $k_i = (1+p+\cdots+p^{d-1})x_i$, $0 \le x_i < p-1$, we see as in Remark 2.1.4 that we can write $r_i = (1+p+\cdots+p^{d-1})(x_{i+1}-x_i) + (p^d-1)y_i$, with $0 \le y_i \le r$. By Lemma 2.1.3 and Remark 2.1.4, we have

$$\chi|_{I_{\mathbb{Q}_p}} = \sigma_0 \circ \omega_d^{x_0+y_0+p^{d-1}(x_1+y_1)+\cdots+p(x_{d-1}+y_{d-1})}.$$

Since $r \le p-2$, we have $0 \le y_i \le p-2$, and we conclude (after allowing for "carrying") that each exponent of $\bar{\rho}$ is of the form $a_j + k$ with $0 \le k \le r+1$.

Suppose that $\bar{\rho}$ is r-regular. It must then be the case that the x_i above are all distinct. Applying this analysis to each irreducible subrepresentation of $\bar{\rho}$, we conclude that the a_i are all distinct. Since we have $\det(\bar{\rho}')|_{I_{\mathbb{Q}_p}} = \chi^{1+p+\cdots+p^{d-1}}|_{I_{\mathbb{Q}_p}} = \omega^{(x_0+y_0)+\cdots+(x_{d-1}+y_{d-1})}$, we conclude that $\det\bar{\rho}|_{I_{\mathbb{Q}_p}} = \omega^{a_1+\cdots+a_n+y}$ for some

$0 \le y \le nr \le n(n-1)/2$. The assumption on $\det \bar{\rho}$ then implies that $y \equiv n(n-1)/2 \pmod{p-1}$. If in fact $r < (n-1)/2$, then we have $0 \le y < n(n-1)/2$, which contradicts the assumption that $p > n(n-1)/2 + 1$.

It remains to treat the case that $r = (n-1)/2$, where we may assume (by the additional hypothesis that we have assumed in this case) that the representation $\bar{\rho}'$ above has dimension $d > 1$. By the above analysis we must have $y = n(n-1)/2$, so that each $y_i = r$. Since we have $r_i \le (p^d - 1)r$, we must have $x_{i+1} - x_i \le 0$ for each i, so that in fact $x_0 = x_1 = \cdots = x_{d-1}$, a contradiction (as we already showed that the x_i are distinct). □

3. The cohomology of Shimura varieties

3.1. ***The semistable reduction of certain $U(n-1,1)$-Shimura varieties.*** Fix $n \ge 2$, and fix an odd prime p.[3] We now recall the definitions of the $U(n-1,1)$-Shimura varieties with which we will work, and some associated integral models. For simplicity we work over $\mathbb{Q}$ (or rather an imaginary quadratic extension of $\mathbb{Q}$) rather than over a general totally real field.

For the most part we will follow Section 3 of [Haines and Rapoport 2012] (which uses a similar approach to [Harris and Taylor 2002]), with the occasional reference to [Harris and Taylor 2001]. Fix an imaginary quadratic field F in which the prime p splits, say $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ for some choice of $\mathfrak{p}$, let $x \mapsto \bar{x}$ be the nontrivial automorphism of F, and regard F as a subfield of $\mathbb{C}$ via a fixed embedding $F \hookrightarrow \mathbb{C}$.

Let D be a division algebra over F of dimension n^2, and let $*$ be an involution of D of the second kind (that is, $*|_F$ is nontrivial). Assume that D splits at $\mathfrak{p}$ (and hence at $\bar{\mathfrak{p}}$), and fix isomorphisms $D_{\mathfrak{p}} \cong M_n(\mathbb{Q}_p)$ and $D_{\bar{\mathfrak{p}}} \cong M_n(\mathbb{Q}_p)$ with the property that, under the induced isomorphism

$$D \otimes \mathbb{Q}_p \cong M_n(\mathbb{Q}_p) \times M_n(\mathbb{Q}_p)^{\mathrm{op}},$$

the involution $*$ corresponds to $(X, Y) \mapsto (Y^t, X^t)$.

Let $G_{/\mathbb{Q}}$ be the algebraic group whose R-points are

$$G(R) = \{x \in (D \otimes_{\mathbb{Q}} R)^\times \mid x \cdot x^* \in R^\times\}$$

for any $\mathbb{Q}$-algebra R. Thus our fixed isomorphism $D_{\mathfrak{p}} \cong M_n(\mathbb{Q}_p)$ induces an isomorphism $G \times_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{GL}_n \times \mathbb{G}_m$.

Now let $h_0 : \mathbb{C} \to D_{\mathbb{R}}$ be an $\mathbb{R}$-algebra homomorphism with the properties that $h_0(z)^* = h_0(\bar{z})$ and the involution $x \mapsto h_0(i)^{-1} x^* h_0(i)$ is positive (that is, $\mathrm{tr}_{B/\mathbb{Q}}(x h_0(i)^{-1} x^* h_0(i)) > 0$ for all nonzero x). Let $B = D^{\mathrm{op}}$ and let $V = D$, which we consider as a free left B-module of rank 1 by multiplication on the right.

[3]The reason for assuming that the prime p is odd is that below we will want to apply the discussion and results of Section 2, in which this assumption was made.

Then $\mathrm{End}_B(V) = D$, and one can find an element $\xi \in D^\times$ with the properties that $\xi^* = -\xi$ and such that the involution ι of B defined by $x^\iota = \xi x^* \xi^{-1}$ is positive (see Section I.7 of [Harris and Taylor 2001] or Section 5.2 of [Haines 2005] for the existence of such a ξ).

We have an alternating pairing $\psi(\cdot,\cdot) : D \times D \to \mathbb{Q}$ defined by $\psi(x, y) = \mathrm{tr}_{D/\mathbb{Q}}(x\xi y^*)$, and one sees easily that $\psi(bx, y) = \psi(x, b^\iota y)$ and that $\psi(\cdot, h_0(i)\cdot)$ is either positive- or negative-definite. After possibly replacing ξ by $-\xi$, we can and do assume that it is positive-definite.

It is easy to see that one has

$$G(\mathbb{R}) \cong \mathrm{GU}(r, s)$$

for some r, s with $r + s = n$. We impose the additional assumption that in fact $\{r, s\} = \{n - 1, 1\}$. Note that by Lemma I.7.1 of [Harris and Taylor 2001] one can find division algebras D for which this holds. We say that a compact open subgroup $K \subset G(\mathbb{A}^\infty)$ (resp. $K^p \subset G(\mathbb{A}^{p,\infty})$) is *sufficiently small* if for some prime q (resp. some prime $q \neq p$) the projection of K (resp. K^p) to $G(\mathbb{Q}_q)$ contains no element of finite order other than 1. If K is sufficiently small, we will consider the Shimura variety $\mathrm{Sh}(G, h_0|_{\mathbb{C}^\times}^{-1}, K)$. It has a canonical model over F, which we denote by $X(K)$ (note that if $n > 2$ the reflex field is F, while if $n = 2$ the reflex field is $\mathbb{Q}$, and we let $X(K)$ denote the base change of the canonical model from $\mathbb{Q}$ to F).

We say that a compact open subgroup K of $G(\mathbb{A})$ is of *level dividing* N, for some integer $N \geq 1$, if for all primes $l \nmid N$ we can write $K = K_l K^l$, where K_l is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_l)$ and K^l is a compact open subgroup of $G(\mathbb{A}^{\infty,l})$. (Note then that in fact $K = K_N \times \prod_{l \nmid N} K_l$, for some compact open subgroup K_N of $\prod_{l|N} G(\mathbb{Q}_l)$.) If K is of level dividing N, then we similarly refer to $X(K)$ as a $U(n-1,1)$*-Shimura variety of level dividing* N.

We will now define integral models of these Shimura varieties for two specific kinds of level structure. We begin by introducing notation related to the level structures in question.

We write I_0 for the Iwahori subgroup of $\mathrm{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$; namely, I_0 is the subgroup of $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ consisting of elements whose first factor lies in the usual Iwahori subgroup of matrices which are upper-triangular mod p. We write I_1 to denote the pro-p-Iwahori subgroup of $\mathrm{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^\times$; namely, I_1 is the (unique) pro-p Sylow subgroup of I_0, and consists of those elements of $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ whose first factor is upper-triangular unipotent mod p, and whose second factor is congruent to 1 mod p. We write I_1^* to denote the subgroup of I_0 consisting of those matrices in $\mathrm{GL}_n(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ whose first factor is upper-triangular unipotent mod p.

There is a natural isomorphism $\mathbb{Z}_p^\times = \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$, and this induces a natural isomorphism $I_1^* = \mathbb{F}_p^\times \times I_1$. If we let T denote the diagonal torus in GL_n, then there is also a natural isomorphism $I_0 = T(\mathbb{F}_p) \times I_1^*$.

We will define integral models for $X(I_0K^p)$ and $X(I_1^*K^p)$ over the local rings $\mathcal{O}_{F,(\mathfrak{p})}$ and $\mathcal{O}_{F(\zeta_{p-1}),(v)}$ respectively, where ζ_{p-1} denotes a primitive $(p-1)$-st root of unity, and v is some fixed place in $F(\zeta_{p-1})$ above $\mathfrak{p}$; here K^p is a sufficiently small compact open subgroup of $G(\mathbb{A}^{\infty,p})$, and we consider I_0 and I_1^* as subgroups of $G(\mathbb{Q}_p)\cong \mathrm{GL}_n(\mathbb{Q}_p)\times\mathbb{Q}_p^\times$. We will typically not include K^p in the notation, and we will write $\mathscr{X}_0(p)$, $\mathscr{X}_1(p)$ for our integral models of $X_0(p):=X(I_0K^p)$ and $X_1(p):=X(I_1^*K^p)$ respectively.

Remark 3.1.1. Note that in [Haines and Rapoport 2012], [Harris and Taylor 2002], and [Harris and Taylor 2001], the authors work over $\mathbb{Z}_p$, but we follow [Kottwitz 1992] in working over $\mathcal{O}_{F,(\mathfrak{p})}$, so as to satisfy the hypothesis required to be in the global case of Section 1. The appearance of ζ_{p-1} in the ring of definition of $\mathscr{X}_1(p)$ is a consequence of our use of Oort–Tate theory in the definition of the integral model in this case.

In order to define these integral models, we first recall a certain category of abelian schemes (up to isogeny) with polarisations and endomorphisms. If $\mathscr{F}$ is a set-valued functor on the category of connected, locally noetherian $\mathcal{O}_{F,(p)}$-schemes, we will also consider it to be a functor on the category of all locally noetherian $\mathcal{O}_{F(p)}$-schemes by setting

$$\mathscr{F}\Big(\coprod S_i\Big):=\prod\mathscr{F}(S_i).$$

Let $\mathcal{O}_B$ be the unique maximal $\mathbb{Z}_{(p)}$-order in B which under our fixed identification $B\otimes_{\mathbb{Q}}\mathbb{Q}_p=M_n(\mathbb{Q}_p)\times M_n(\mathbb{Q}_p)^{\mathrm{op}}$ is identified with $M_n(\mathbb{Z}_{(p)}\times M_n(\mathbb{Z}_{(p)})^{\mathrm{op}}$. Let S be a connected, locally noetherian $\mathcal{O}_{F,(\mathfrak{p})}$-scheme, and let AV_S be the category whose objects are pairs (A,i), where A is an abelian scheme over S of dimension n^2 and $i:\mathcal{O}_B\to\mathrm{End}_S(A)\otimes\mathbb{Z}_{(p)}$ is a homomorphism. We define homomorphisms in AV_S by

$$\mathrm{Hom}((A_1,i_1),(A_2,i_2))=\mathrm{Hom}_{\mathcal{O}_B}((A_1,i_1),(A_2,i_2))\otimes\mathbb{Z}_{(p)}$$

(that is, the elements of $\mathrm{Hom}_S(A_1,A_2)\otimes\mathbb{Z}_{(p)}$ which commute with the action of $\mathcal{O}_B$). The dual of an object (A,i) of AV_S is $(\widehat{A},\hat{i})$, where $\widehat{A}$ is the dual abelian scheme of A and $\hat{i}(b)=(i(b^\iota))^\wedge$. A polarisation of (A,i) is a homomorphism $\lambda:(A,i)\to(\widehat{A},\hat{i})$ in AV_S with the property that, for some $n\geq 1$, $n\lambda$ is induced by an ample line bundle on A. A principal polarisation is a polarisation which is also an isomorphism in AV_S. A $\mathbb{Q}$-class of polarisations is an equivalence class of homomorphisms $(A,i)\to(\widehat{A},\hat{i})$ which contains a polarisation, under the equivalence relation of differing by a $\mathbb{Q}^\times$-scalar.

Fix K^p a sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$. Let $\mathscr{A}_0$ be the set-valued functor on the category of locally noetherian schemes over $\mathcal{O}_{F,(\mathfrak{p})}$

which sends a connected, locally noetherian scheme S over $\mathcal{O}_{F,(\mathfrak{p})}$ to the set of isomorphism classes of the following data:

- A commutative diagram of morphisms in the category AV_S of the form

$$\begin{array}{ccccccccc} A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_{n-2}} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & A_0 \\ \downarrow{\scriptstyle\lambda_0} & & \downarrow{\scriptstyle\lambda_1} & & & & \downarrow{\scriptstyle\lambda_{n-1}} & & \downarrow{\scriptstyle\lambda_0} \\ \widehat{A}_0 & \xleftarrow{\hat{\alpha}_0} & \widehat{A}_1 & \xleftarrow{\hat{\alpha}_1} & \cdots & \xleftarrow{\hat{\alpha}_{n-2}} & \widehat{A}_{n-1} & \xleftarrow{\hat{\alpha}_{n-1}} & \widehat{A}_0 \end{array}$$

  where each α_i is an isogeny of degree p^{2n} and their composite is just multiplication by p. In addition, λ_0 is a $\mathbb{Q}$-class of polarisations containing a principal polarisation. Furthermore, we require that each A_i satisfies a compatibility between the two actions of $\mathcal{O}_F$ on the Lie algebra of A_i (one action coming from the structure morphism $\mathcal{O}_{F,(\mathfrak{p})} \to \mathcal{O}_S$, and the other from the $\mathcal{O}_B$-action; see [Harris and Taylor 2001, §III.4] for a discussion of this condition).

- A geometric point s of S, and a $\pi_1(S,s)$-invariant K^p-orbit of isomorphisms

$$\bar{\eta} : V \otimes_{\mathbb{Q}} \mathbb{A}^{p,\infty} \xrightarrow{\sim} H_1((A_0)_s, \mathbb{A}^{p,\infty})$$

  which are $\mathcal{O}_B$-linear and up to a constant in $(\mathbb{A}^{p,\infty})^\times$ take the ψ-pairing on the left side to the λ_0-Weil pairing on the right side. (This data is canonically independent of the choice of s; see the discussion on pp. 390–391 of [Kottwitz 1992].)

An isomorphism of this data is one induced by isomorphisms in AV_S which preserve the λ_i up to an overall $\mathbb{Q}^\times$-scalar.

The functor $\mathcal{A}_0$ is represented by a projective scheme $\mathcal{X}_0(p)$ over $\mathcal{O}_{F,(\mathfrak{p})}$, which is an integral model for $X(I_0K^p)$. (See the proof of Lemma 3.2 of [Taylor and Yoshida 2007], which shows that $\mathcal{X}_0(p)$ is projective over the usual integral model at hyperspecial level. At hyperspecial level, quasiprojectivity is proved on p. 391 of [Kottwitz 1992], and projectivity can be checked via the valuative criterion for properness as on p. 392 of the same work. More properly, $\mathcal{X}_0(p)$ is an integral model for a disjoint union of a number of copies of $X(I_0K^p)$, due to the possible failure of the Hasse principle; see for example Section 7 and the discussion on p. 400 of [Kottwitz 1992]. Since the cohomology of a disjoint union of spaces is the direct sum of the cohomologies of the individual spaces, this does not affect our arguments, and we will not dwell on this point in the following.) The proof of Proposition 3.4(3) of [Taylor and Yoshida 2007] (which goes over unchanged in our setting) shows that the special fibre of $\mathcal{X}_0(p)$ is a strict normal crossings divisor.

Our next goal is to describe an integral model $\mathcal{X}_1(p)$, over $\mathcal{O}_{F(\zeta_{p-1}),(v)}$, for $X(I_1^*K^p)$ (or rather, as in the previous paragraph, an integral model of a disjoint union of a number of copies of $X(I_1^*K^p)$). Recalling that T denotes the diagonal torus in GL_n, we let $\iota_i : \mathbb{G}_m \to T$ denote the embedding of tori identifying $\mathbb{G}_m$ with the subgroup of T consisting of elements which are 1 away from the i-th diagonal entry. We use the same notation ι_i to denote the map $\mathbb{F}_p^\times \to T(\mathbb{F}_p)$ induced by the map of tori.

The quotient I_0/I_1^* is naturally identified with $T(\mathbb{F}_p)$, and so $T(\mathbb{F}_p)$ acts on $X_1(p)$, with quotient isomorphic to $X_0(p)$.

Given an S-valued point of $\mathcal{A}_0$, let $A_i(p^\infty)$ be the p-divisible group associated to A_i for $i = 0, \ldots, n-1$. Each $A_i(p^\infty)$ has an action of $\mathcal{O}_B = M_n(\mathbb{Z}_{(p)}) \times M_n(\mathbb{Z}_p)^{\mathrm{op}}$. Let $X_i = e_{11}A_i(p^\infty)$, where e_{11} is the usual idempotent in $M_n(\mathbb{Z}_{(p)})$ (and is zero on the second factor). Then each X_i is a p-divisible group of height n and dimension 1, and we obtain a chain of isogenies of degree p

$$\mathscr{C} : X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n := X_0,$$

whose composite is equal to multiplication by p.

We let OT denote the Artin stack over $\mathcal{O}_{F(\zeta_{p-1}),(v)}$ given by

$$\mathrm{OT} := \left[\operatorname{Spec} \mathcal{O}_{F(\zeta_{p-1}),(v)}[X, Y]/(XY - w_p)/\mathbb{G}_m\right],$$

where $\mathbb{G}_m$ acts via $\lambda \cdot (X, Y) = (\lambda^{p-1}X, \lambda^{1-p}Y)$ and w_p is some explicit element of $\mathcal{O}_{F(\zeta_{p-1}),(v)}$ of valuation 1. Oort–Tate theory shows that OT classifies finite flat group schemes of order p over $\mathcal{O}_{F(\zeta_{p-1}),(v)}$-schemes. The universal group scheme over OT is the stack

$$\mathcal{G} := \left[\operatorname{Spec} \mathcal{O}_{F(\zeta_{p-1}),(v)}[X, Y, Z]/(XY - w_p, Z^p - XZ)/\mathbb{G}_m\right],$$

where $\mathbb{G}_m$ acts on X and Y as above, and on Z via $\lambda \cdot Z = \lambda Z$. The morphism $\mathcal{G} \to \mathrm{OT}$ is the evident one, the zero section of $\mathcal{G}$ is cut out by the equation $Z = 0$, and we let $\mathcal{G}^\times$ denote the closed subscheme of $\mathcal{G}$ cut out by the equation $Z^{p-1} - X = 0$; this is the so-called *scheme of generators* of $\mathcal{G}$. (See Theorem 6.5.1 of [Genestier and Tilouine 2005] for these facts, which are a restatement of Theorem 2 of [Tate and Oort 1970] in the language of stacks.)

We define $\mathscr{X}_1(p)$ via the Cartesian diagram

$$\begin{array}{ccc} \mathscr{X}_1(p) & \longrightarrow & \mathcal{G}^\times \times_{\mathcal{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathcal{O}_{F(\zeta_{p-1}),(v)}} \mathcal{G}^\times \\ \downarrow & & \downarrow \\ \mathscr{X}_0(p)_{/\mathcal{O}_{F(\zeta_{p-1}),(v)}} & \longrightarrow & \mathrm{OT} \times_{\mathcal{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathcal{O}_{F(\zeta_{p-1}),(v)}} \mathrm{OT} \end{array} \tag{3.1.2}$$

where the bottom horizontal arrow is given by

$$\mathscr{C} \mapsto (\ker(\alpha_0), \ldots, \ker(\alpha_{n-1})).$$

Note that the right-hand vertical arrow is finite and relatively representable by construction, and so the left-hand vertical arrow is a finite morphism of schemes. The action of $T(\mathbb{F}_p)$ on $X_1(p)$ extends to an action on $\mathcal{X}_1(p)$, namely, the action pulled back from the action of $T(\mathbb{F}_p)$ on $\mathcal{G}^\times \times_{\mathcal{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathcal{O}_{F(\zeta_{p-1}),(v)}} \mathcal{G}^\times$ over $\mathrm{OT} \times_{\mathcal{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathcal{O}_{F(\zeta_{p-1}),(v)}} \mathrm{OT}$.

Let $\pi := (-p)^{1/(p-1)}$, and let w be the unique finite place of $L := F(\zeta_{p-1}, \pi)$ lying over our fixed place v of $F(\zeta_{p-1})$. We let $\mathcal{O}$ denote the localisation of $\mathcal{O}_L$ at w, and we let $\mathcal{X}_1(p)_{\mathcal{O}}$ denote the normalisation of the base change $\mathcal{X}_1(p)_{/\mathcal{O}}$ of $\mathcal{X}_1(p)$ over $\mathcal{O}$. We write $I := \mathrm{Gal}(L/F(\zeta_{p-1}))$; this is also the inertia group at w in $\mathrm{Gal}(L/F)$. The group I acts naturally on $\mathcal{X}_1(p)_{\mathcal{O}}$. The mod p cyclotomic character ω induces an isomorphism (which we continue to denote by ω)

$$\omega : I \xrightarrow{\sim} \mathbb{F}_p^\times.$$

For each $i = 0, \ldots, n-1$, we let $\alpha_i : I \to T$ denote the composite $\iota_i \circ \omega^{-1}$.

Lemma 3.1.3. *The scheme $\mathcal{X}_1(p)_{\mathcal{O}}$ is a semistable projective model for $X_1(p)_{/L}$ over $\mathcal{O}$, the natural morphism*

$$\mathcal{X}_1(p)_{\mathcal{O}} \to \mathcal{X}_0(p) \tag{3.1.4}$$

is tamely ramified, and the action of $I \times T$ on $X_1(p)_{/L}$ extends to an action on $\mathcal{X}_1(p)_{\mathcal{O}}$. Furthermore, on each irreducible component of its special fibre, the inertia group I acts through the composite of the T-action with one of the characters α_i.

Proof. We will apply a form of Deligne's homogeneity principle, as described in the proof of [Taylor and Yoshida 2007, Proposition 3.4], to the morphism (3.1.4). The scheme here denoted $\mathcal{X}_0(p)$ is there denoted X_U (and the integral model there is considered over $\mathbb{Z}_p$ rather than $\mathcal{O}_{F,(\mathfrak{p})}$, but this is immaterial for our present purposes), while the scheme there denoted X_{U_0} is an integral model of the Shimura variety (in the notation of the present paper) $X(K_pK^p)$, where $K_p = \mathrm{GL}_n(\mathbb{Z}_p)\times\mathbb{Z}_p^\times$. We let $\mathcal{X}_0(p)_s^{(h)}$ (for $0 \le h \le n-1$) denote the locally closed subset of the special fibre $\mathcal{X}_0(p)_s$, obtained by pulling back the locally closed subset $\overline{X}_{U_0}^{(h)}$ defined in Section 3 of [Taylor and Yoshida 2007] under the natural projection $\mathcal{X}_0(p)_s = \overline{X}_U \to \overline{X}_{U_0}$.

We will show that the morphism (3.1.4) is tamely ramified in the formal neighbourhood of any closed geometric point of the special fibre $\mathcal{X}_0(p)_s$, and hence (by Lemma 1.5.3) that it is tamely ramified. We first note that the morphism $\mathcal{X}_0(p)_{/\mathcal{O}_{F(\zeta_{p-1}),(v)}} \to \mathcal{X}_0(p)$ induces an isomorphism on special fibres, and so we are free to replace $\mathcal{X}_0(p)$ by $\mathcal{X}_0(p)_{/\mathcal{O}_{F(\zeta_{p-1}),(v)}}$ in our considerations. We next note that the completion of the bottom arrow of (3.1.2) at a closed geometric point $\bar{x}_0$ of $\mathcal{X}_0(p)_s$ depends up to isomorphism only on the value of h for which $\bar{x}_0 \in \mathcal{X}_0(p)_s^{(h)}$ (since the p-divisible group attached to the point $\bar{x}_0$ depends only on the value of h), and hence that the restriction of (3.1.4) to a formal neighbourhood of $\bar{x}_0$ depends only on the value of h. Then, by [Taylor and Yoshida 2007, Lemma 3.1], the closure of $\mathscr{X}_0(p)_s^{(h)}$ contains $\mathscr{X}_0(p)_s^{(0)}$. Since being tamely ramified is an open condition, we conclude from these two conditions that, in order to prove the lemma, it suffices to show that the restriction of (3.1.4) to a formal neighbourhood of $\bar{x}_0$ is tamely ramified at closed geometric points $\bar{x}_0$ of $\mathscr{X}_0(p)_s^{(0)}$. (As already indicated, this argument is a variation on Deligne's homogeneity principle.)

Thus, consider a closed geometric supersingular point $\bar{x}_0$ of $\mathscr{X}_0(p)_s^{(0)}$, so that $\bar{x}_0$ admits a formal neighbourhood of the form

$$\operatorname{Spec} W(\overline{\mathbb{F}}_p)[\![T_1, \dots, T_n]\!]/(T_1 \cdots T_n - w_p).$$

The proof of [Taylor and Yoshida 2007, Proposition 3.4] shows that the T_i may be taken to be the matrix of α_{i-1} on tangent spaces, so that the map $\mathscr{X}_0(p) \to \mathrm{OT} \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \cdots \times_{\mathbb{O}_{F(\zeta_{p-1}),(v)}} \mathrm{OT}$ may be defined in the formal neighbourhood of $\bar{x}_0$ by the map

$$(T_1, \dots, T_n) \mapsto ((T_1, U_1), \dots, (T_n, U_n)),$$

where $U_i = T_1 \cdots \widehat{T_i} \cdots T_n$ (and, as is usual in these situations, a hat on a variable denotes that that variable is omitted in the expression). Thus a formal neighbourhood of a closed geometric point lying over $\bar{x}_0$ in $(\mathscr{X}_1(p)_{/\mathbb{O}})_s$ is isomorphic to

$$\operatorname{Spec} W(\overline{\mathbb{F}}_p)[\pi][\![V_1, \dots, V_n]\!]/((V_1 \cdots V_n)^{p-1} - w_p).$$

If we write $u := V_1 \cdots V_n/\pi$, then we see that $u^{p-1} = -w_p/p$, and hence that u lies in the normalisation of this formal neighbourhood. Furthermore, on each component of this normalisation, u is equal to one of the $(p-1)$-st roots of $-w_p/p$ lying in $W(\overline{\mathbb{F}}_p)$. Thus the normalisation of this formal neighbourhood is a union of components, each isomorphic to

$$\operatorname{Spec} \mathbb{O}[\![V_1, \dots, V_n]\!]/(V_1 \cdots V_n - u\pi),$$

with the morphism (3.1.4) being given by $T_i = V_i^{p-1}$. Thus this morphism is indeed tamely ramified in the formal neighbourhood of $\bar{x}_0$.

It is clear that the $I \times T$-action on $X_1(p)_L$ extends to an action on $\mathscr{X}_1(p)_{/\mathbb{O}}$, and hence to its normalisation $\mathscr{X}_1(p)_{\mathbb{O}}$. As for the final statement, note that I acts on π via ω, and hence on u via ω^{-1}, while I fixes each V_i. Also T acts on V_i through multiplication by the i-th diagonal entry, and so acts on u via multiplication by the determinant. Combining these facts, we see that I acts on the components of the special fibre on which $V_i = 0$ via $\iota_i \circ \omega^{-1}$. □

Remark 3.1.5. This lemma (and the fact that $\mathscr{X}_0(p)$ is strictly semistable) shows that the map $\mathscr{X}_1(p)_{\mathbb{C}} \to \mathscr{X}_0(p)$ provides a tame strictly semistable context, in the sense of Section 1.5. In particular, we can apply Theorem 1.5.15 in this setting (of course taking the group G to be the abelian group T), and we see that each character ψ_j as in the statement of that theorem is equal to one of the characters α_i.

3.2. ***Canonical local systems.*** If K is a sufficiently small compact open subgroup of $G(\mathbb{A}^\infty)$, and V is a continuous representation of K on a finite-dimensional $\overline{\mathbb{F}}_p$-vector space (this vector space being equipped with its discrete topology), then we may associate to V an étale local system $\mathscr{F}_V$ of $\overline{\mathbb{F}}_p$-vector spaces on $X(K)$ as follows: Choose an open normal subgroup $K' \subset K$ lying in the kernel of V, and regard V as a representation of the quotient K/K'. Since $X(K')$ is naturally an étale K/K'-torsor over $X(K)$, we may form the étale local system of $\overline{\mathbb{F}}_p$-vector spaces over $X(K)$ associated to the K/K'-representation V. This local system is independent of the choice of K', up to canonical isomorphism, and we define it to be $\mathscr{F}_V$.

Definition 3.2.1. We refer to the étale local systems $\mathscr{F}_V$ that arise by the preceding construction as the *canonical local systems* on $X(K)$. If we may choose K' in the kernel of V to be of level dividing N (so that in particular $X(K)$ is of level dividing N), then we say that $\mathscr{F}_V$ can be *trivialised at level* N.

3.3. ***The Eichler–Shimura relation.*** Let X be a $U(n-1,1)$-Shimura variety of level dividing N. Let w be a place of F such that $l := w|_{\mathbb{Q}}$ splits in F and does not divide N. There is a natural action via correspondences on X of Hecke operators $T_w^{(i)}$, $0 \le i \le n$, where $T_w^{(i)}$ is the double coset operator corresponding to

$$\begin{pmatrix} l1_i & 0 \\ 0 & 1_{n-i} \end{pmatrix} \times 1 \in \mathrm{GL}_n(\mathbb{Q}_l) \times \mathbb{Z}_l^\times,$$

where we use the assumption that $l \nmid N$ and identify a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_l)$ with $\mathrm{GL}_n(\mathbb{Z}_l) \times \mathbb{Z}_l^\times$ via an isomorphism $D_w \cong M_n(\mathbb{Q}_l)$. These correspondences then act on the cohomology $H^j_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)$.

More generally if $\mathscr{F}_V$ is a canonical local system on X that can be trivialised at level N, then we obtain an action of the double coset operators $T_w^{(i)}$ on $\mathscr{F}_V$, and hence on the cohomology $H^j_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)$.

The following theorem regarding this action is then an immediate consequence of the main result of [Wedhorn 2000] (which proves the Eichler–Shimura relation for PEL Shimura varieties at places of good reduction at which the group is split).

Theorem 3.3.1. *Let X be a $U(n-1,1)$-Shimura variety of level dividing N and $\mathscr{F}_V$ a canonical local system on X. Let w be a place of F such that $w|_{\mathbb{Q}}$ splits in F and does not divide Np. Then $\sum_{i=0}^n (-1)^i (\mathrm{Norm}\, w)^{i(i-1)/2} T_w^{(i)} \mathrm{Frob}_w^{n-i}$ acts as 0 on each $H^j_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)$.*

3.4. ***Vanishing and torsion-freeness of cohomology for certain $U(n-1,1)$-Shimura varieties.*** Let X denote a $U(n-1,1)$-Shimura variety as above, and let $\mathcal{F}_V$ denote a canonical local system on X. Choose N so that X has level dividing N, so that $\mathcal{F}_V$ can be trivialised at level N, and so that p divides N. Assume that the projection of the corresponding level K to $G(\mathbb{A}^{p,\infty})$ is sufficiently small.

Let $\mathbb{T} = \overline{\mathbb{Z}}_p[T_w^{(i)}]$ be the polynomial ring in the variables $T_w^{(i)}$, $1 \le i \le n$, where w runs over the places of F such that $w|_{\mathbb{Q}}$ splits in F and does not divide N. Let $\mathfrak{m}$ be a maximal ideal in $\mathbb{T}$ with residue field $\overline{\mathbb{F}}_p$, and suppose that there exists a continuous irreducible representation $\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ which is unramified at all finite places not dividing N, and which satisfies $\mathrm{char}(\rho_{\mathfrak{m}}(\mathrm{Frob}_w)) \equiv \sum_{i=0}^{n} (-1)^i (\mathrm{Norm}\, w)^{i(i-1)/2} T_w^{(i)} X^{n-i} \mod \mathfrak{m}$ for all $w \nmid N$ such that $w|_{\mathbb{Q}}$ splits in F. Continue to fix a choice of a place $\mathfrak{p}$ of F dividing p, and write $G_{\mathbb{Q}_p}$ for $G_{F_{\mathfrak{p}}}$ from now on. Recall that the choice of $\mathfrak{p}$ also gives us an isomorphism $G(\mathbb{Q}_p) \cong \mathrm{GL}_n(\mathbb{Q}_p) \times \mathbb{Q}_p^{\times}$ as in Section 3.1.

We consider the following further hypothesis on $\rho_{\mathfrak{m}}$ (this is Hypothesis 4.1.1):

Hypothesis 3.4.1. If $\theta : G_F \to \mathrm{GL}_m(\overline{\mathbb{F}}_p)$ is any continuous, irreducible representation with the property that the characteristic polynomial of $\rho_{\mathfrak{m}}(g)$ annihilates $\theta(g)$ for every $g \in G_F$, then θ is equivalent to $\rho_{\mathfrak{m}}$.

We will now prove our first main result, a vanishing theorem for the cohomology of X with $\mathcal{F}_V$-coefficients:

Theorem 3.4.2. *Suppose that $\rho_{\mathfrak{m}}$ satisfies Hypothesis 3.4.1, that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}$ is r-regular for some $r \le (n-1)/2$, that $p > n(n-1)/2+1$ and, if $r = (n-1)/2$, suppose in addition that $\rho_{\mathfrak{m}|G_{\mathbb{Q}_p}}$ contains an irreducible subquotient of dimension greater than* 1. *Then the localisations $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}$ vanish for $i \le r$ and $i \ge 2(n-1)-r$.*

Remark 3.4.3. In Section 4.1 we will show that Hypothesis 3.4.1 is satisfied if either $\rho_{\mathfrak{m}}$ is induced from a character of G_K for some degree-n cyclic Galois extension $K/\mathbb{Q}$, or if $p \ge n$ and $\mathrm{SL}_n(k) \subseteq \rho_{\mathfrak{m}}(G_F) \subseteq \overline{\mathbb{F}}_p^{\times} \mathrm{GL}_n(k)$ for some subfield $k \subset \overline{\mathbb{F}}_p$.

Remark 3.4.4. While we work here with étale local systems and étale cohomology, by virtue of Artin's comparison theorem [SGA 4_3 1973, Exposé XI, Théorème 4.3] our vanishing results are equivalent to vanishing results for the cohomology of the complex $U(n-1,1)$-Shimura varieties with coefficients in the corresponding canonical local systems for the complex topology.

Proof of Theorem 3.4.2. First, note that it suffices to prove vanishing in degree $i \le r$ for all V, as vanishing in degree $i \ge 2(n-1)-r$ then follows by Poincaré duality. (Note that the dual of the canonical local system $\mathcal{F}_V$ attached to a representation V is the canonical local system attached to the contragredient representation $V^{\vee}$.)

We prove the theorem for $i \leq r$ by induction on i, the case when $i < 0$ being trivial. We begin by reducing to the case when $\mathscr{F}_V$ is trivial. To this end, write $X = X(K)$, let K' be an open normal subgroup of K of level dividing N such that V is representation of K/K', and choose an embedding of K/K'-representations $V \hookrightarrow \overline{\mathbb{F}}_p[K/K']^n$ for some n; denote the cokernel by W. Let π denote the projection $X(K') \to X(K)$. Passing to the associated canonical local systems, we obtain a short exact sequence $0 \to \mathscr{F}_V \to \pi_* \overline{\mathbb{F}}_p^n \to \mathscr{F}_W \to 0$, which gives rise to an exact sequence of cohomology

$$H^{i-1}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_W) \longrightarrow H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V) \longrightarrow H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \pi_* \overline{\mathbb{F}}_p^n) = H^i_{\text{ét}}(X(K')_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)^n.$$

Localising at $\mathfrak{m}$ and applying our inductive hypothesis (with $\mathscr{F}_W$ in place of $\mathscr{F}_V$), we reduce to the case of constant coefficients (with $X(K')$ in place of X). We therefore turn to establishing the claimed vanishing in this case.

Suppose now that K' is any open normal subgroup of K of level dividing N. Combining the Hochschild–Serre spectral sequence

$$E_2^{m,n} = H^m_{\text{ét}}(K/K', H^n(X(K')_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}) \Longrightarrow H^{m+n}_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$$

with our inductive hypothesis, we find that vanishing of $H^i_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ implies the vanishing of $H^i_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$. Thus, without loss of generality, we may and do assume that $K = K_p K^p$, where K_p is an open normal subgroup of I_1 and K^p is a sufficiently small compact open subgroup of $G(\mathbb{A}^{\infty,p})$.

We again consider a Hochschild–Serre spectral sequence, this time the one relating the cohomology of $X(K)$ and $X(I_1 K^p)$, which takes the form

$$E_2^{m,n} = H^m(I_1/K_p, H^n_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}) \Longrightarrow H^{m+n}_{\text{ét}}(X(I_1 K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}.$$

Once more taking into account our inductive hypothesis, we obtain an isomorphism $H^i_{\text{ét}}(X(I_1 K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \xrightarrow{\sim} H^i_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}^{I_1/K_p}$. Since I_1/K_p is a p-group, while $H^i_{\text{ét}}(X(K)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ is a vector space over a field of characteristic p, we see that the latter space vanishes if and only if its space of I_1/K_p-invariants does. Thus we are reduced to establishing the theorem in the case when $K = I_1 K^p$.

Now recall that $I_1^* = \mathbb{F}_p^\times \times I_1$, and that the projection onto the first factor arises from the similitude projection $\mathrm{GU}(n-1,1) \to \mathbb{G}_m$. From this it follows that $X(I_1 K^p)$ is isomorphic to the product $X(I_1^* K^p) \times_F \operatorname{Spec} A$, where $A := F[x]/\Phi_p(x)$ (where $\Phi_p(x)$ denotes the p-th cyclotomic polynomial; the action of $\mathbb{F}_p^\times = I_1^*/I_1$ on $X(I_1 K^p)$ is induced by the action of $\mathbb{F}_p^\times$ on A given by $x \mapsto x^a$ for $a \in \mathbb{F}_p^\times$). Consequently there is an isomorphism of Galois representations

$$H^i(X(I_1 K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \xrightarrow{\sim} \bigoplus_{j=0}^{p-2} H^i(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}_j} \otimes \omega^j,$$

where $\mathfrak{m}_j$ is the maximal ideal in $\mathbb{T}$ which corresponds to the twisted Galois representation $\rho_{\mathfrak{m}_j} := \rho_{\mathfrak{m}} \otimes \omega^{-j}$. Since each of the Galois representations $\rho_{\mathfrak{m}_j}$ satisfies the hypotheses of the theorem (as these hypotheses are invariant under twisting), we are reduced to proving the theorem in the case of $X(I_1^* K^p)$.

Consider an irreducible $G_F \times I_0/I_1^*$-subrepresentation of $H^i_{\text{ét}}(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$, which we may write in the form $\theta \otimes \beta$, where θ is an absolutely irreducible G_F-representation and β is a character of the abelian group $T(\mathbb{F}_p) = I_0/I_1^*$. Theorem 3.3.1 and Hypothesis 3.4.1 taken together imply that θ is equivalent to $\rho_{\mathfrak{m}}$. We consider β as an n-tuple of characters $\beta_1, \ldots, \beta_n$ of $\mathbb{F}_p^\times$. Write $\widetilde{\beta}_j$ for the Teichmüller lift of β_j and χ_j for the character of $I_{\mathbb{Q}_p}$ given by β_j. By Lemma 3.1.3, Remark 3.1.5 and Theorem 1.5.15, we see that θ can be embedded as the reduction mod p of a potentially semistable representation with Hodge–Tate weights in the range $[-i, 0]$, whose inertial type is a direct sum of characters belonging to the collection $\{\widetilde{\beta}_j\}$. (To see this, note that as θ is in the β-part of the cohomology, it necessarily occurs in the reduction of the $\widetilde{\beta}$-part of the $G_F \times T$-representation provided by Theorem 1.5.15.) We claim that $\det \theta|_{I_{\mathbb{Q}_p}} = \det \rho_{\mathfrak{m}}|_{I_{\mathbb{Q}_p}} = \overline{\chi}_1 \cdots \overline{\chi}_n \omega^{-n(n-1)/2}$. Admitting this for the moment, we may apply Theorem 2.2.4 to the representation $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}^\vee$, and we deduce that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}^\vee$ is not r-regular. Equivalently, we see that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}$ is not r-regular, which contradicts our assumptions.

It remains to establish the equality $\det \theta|_{I_{\mathbb{Q}_p}} = \det \rho_{\mathfrak{m}}|_{I_{\mathbb{Q}_p}} = \overline{\chi}_1 \cdots \overline{\chi}_n \omega^{-n(n-1)/2}$. To see this, note that Theorem 3.3.1 implies that, for each place $w \nmid N$ of F such that $w|_{\mathbb{Q}}$ splits in F, we have $\det \rho_{\mathfrak{m}}(\mathrm{Frob}_w) = (\mathrm{Norm}\, w)^{n(n-1)/2} T_w^{(n)}$. Let $\psi_{\mathfrak{m}}$ be the character of $\mathbb{A}_F^\times / F^\times$ corresponding to the character $\omega^{n(n-1)/2} \det \rho_{\mathfrak{m}}$ by global class field theory; then we need to prove that $\psi_{\mathfrak{m}}|_{\mathbb{Z}_p^\times} = \beta_1 \cdots \beta_n$. The centre of $G(\mathbb{A}_{\mathbb{Q}})$ is $\mathbb{A}_F^\times$, so by the definition of the Shimura variety X there is a natural action of $\mathbb{A}_F^\times / F^\times$ on $H^i_{\text{ét}}(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$. By the definition of the Hecke operators, we see that if ϖ_w is a uniformiser at a place $w \nmid N$ of F such that $w|_{\mathbb{Q}}$ splits in F, then ϖ_w acts as $T_w^{(n)}$. By the Chebotarev density theorem, we deduce that the action of $\mathbb{A}_F^\times / F^\times$ on the underlying vector space of θ (which by definition is a subspace of $H^i_{\text{ét}}(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$) is via the character $\psi_{\mathfrak{m}}$. In order to compute $\psi_{\mathfrak{m}}|_{\mathbb{Z}_p^\times}$, it is thus sufficient to compute the action of $\mathbb{Z}_p^\times$ on $H^i_{\text{ét}}(X(I_1^* K^p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$, and in particular sufficient to compute the action of the Iwahori subgroup I_0. Now, since θ is assumed to be in the β-part of the cohomology, I_0 acts via the character β of I_0/I_1^*, so that $\psi_{\mathfrak{m}}|_{\mathbb{Z}_p^\times} = \beta_1 \cdots \beta_n$, as required. □

Corollary 3.4.5. *Suppose that $\rho_{\mathfrak{m}}$ satisfies Hypothesis 3.4.1, that $\rho_{\mathfrak{m}|G_{\mathbb{Q}_p}}$ is r-regular for some $r \leq \min\{(n-1)/2, p-2\}$, and, if $r = (n-1)/2$, suppose in addition that $\rho_{\mathfrak{m}|G_{\mathbb{Q}_p}}$ contains an irreducible subquotient of dimension greater than* 1. *Then the localisation $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$ vanishes for $i \leq r$, while $H^{r+1}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$ is torsion-free.*

Proof. This follows at once from Theorem 3.4.2 and the short exact sequence

$$0 \longrightarrow H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}/\mathfrak{m}_{\overline{\mathbb{Z}}_p} H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \longrightarrow H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \longrightarrow H^{i+1}_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}[\mathfrak{m}_{\overline{\mathbb{Z}}_p}] \longrightarrow 0. \qquad \square$$

Remark 3.4.6. As already remarked in the introduction, we expect some kind of mod p analogue of Arthur's conjectures to hold, and so in particular we expect that stronger results than Theorem 3.4.2 and Corollary 3.4.5 should hold. In particular, if $\mathfrak{m}$ is any maximal ideal in the Hecke algebra attached to an irreducible continuous representation $\rho_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, then we expect that the localisations $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}$ and $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)_{\mathfrak{m}}$ should vanish in degrees $i < n-1$.

On the other hand, it need not be the case that (for example) $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)$ vanishes in all degrees in which $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p)$ vanishes. For example, in the case $n = 3$, for the unitary Shimura varieties that we consider here, namely those that are associated to division algebras, it is known that $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_p) = 0$ [Rogawski 1990, Theorem 15.3.1]. (Under additional restrictions on the division algebra allowed, an analogous result is known for all values of n [Clozel 1993, Theorem 3.4].) On the other hand, one can construct examples for which $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p) \neq 0$, and hence for which $H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Z}}_p)$ is not torsion-free, via congruence cohomology. (See e.g., the proof of [Suh 2008, Theorem 3.4].) The existence of such classes does not contradict our theorems or expectations, since congruence cohomology is necessarily Eisenstein (i.e., gives rise to Eisenstein systems of Hecke eigenvalues, in the sense that the associated Galois representation $\rho_{\mathfrak{m}}$ is completely reducible).

3.5. *On the mod p cohomology of certain $U(2,1)$-Shimura varieties.* Let $X := X(K)$ denote a $U(2,1)$-Shimura variety, with K of level dividing N for some natural number N divisible by p, and such that the projection of K to $\mathbb{G}(\mathbb{A}^{p,\infty})$ is sufficiently small. Let $\mathcal{F}_V$ be a canonical local system on X, which may be trivialised at level N. The results of Section 3.4 are particularly powerful in this case, as we now demonstrate.

Corollary 3.5.1. *Suppose that $\rho_{\mathfrak{m}}$ satisfies Hypothesis 3.4.1, that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}$ is* 1-*regular, and that $\rho_{\mathfrak{m}|G_{\mathbb{Q}_p}}$ contains an irreducible subquotient of dimension greater than* 1. *Then the localisations $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}$ vanish for $i \neq 2$.*

Proof. This follows immediately from Theorem 3.4.2, noting that the hypothesis that $\rho_{\mathfrak{m}}|_{G_{\mathbb{Q}_p}}$ is 1-regular implies that $p > 4$ (indeed, that $p \geq 11$). $\square$

We now prove a result which does not require the existence of a Galois representation $\rho_{\mathfrak{m}}$. We begin with a lemma:

Lemma 3.5.2. *If* $\mathfrak{m}$ *is a maximal ideal of* $\mathbb{T}$ *with residue field* $\overline{\mathbb{F}}_p$*, such that* $H^0_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}} \neq 0$*, then there is an abelian representation* $\bar{\rho}_0 : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ *such that* $\mathrm{char}(\bar{\rho}_0(\mathrm{Frob}_w)) = \sum_{i=0}^{3}(-1)^i (\mathrm{Norm}\, w)^{i(i-1)/2} T_w^{(i)} X^{n-i} \pmod{\mathfrak{m}}$ *for all split places* w *of* E *for which* $w \nmid Np$.

Proof. This is standard and follows for example from [Deligne 1979, Section 2.1]. □

Theorem 3.5.3. *If* ρ *is a* 3*-dimensional irreducible sub-*G_F*-representation of the étale cohomology group* $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)$*, then every irreducible subquotient of* $\rho_{|G_{\mathbb{Q}_p}}$ *is* 1*-dimensional, or else* $\rho_{|G_{\mathbb{Q}_p}}$ *is not* 1*-regular, or else* $\rho(G_F)$ *is not generated by its subset of regular elements.*

Remark 3.5.4. Recall that a square matrix is said to be regular if its minimal and characteristic polynomials coincide. In Section 4.2 we will show that $\rho(G_F)$ is generated by its subset of regular elements if either $\rho_{\mathfrak{m}}$ is induced from a character of G_K for some cubic Galois extension $K/\mathbb{Q}$, or if $\rho(G_F)$ contains a regular unipotent element.

Remark 3.5.5. In the proof of the theorem we use some of the results of Section 4.

Proof of Theorem 3.5.3. The argument follows similar lines to the proof of Theorem 3.4.2, although it is slightly more involved, since we are not giving ourselves the existence of the Galois representation $\rho_{\mathfrak{m}}$.[4] The key point will be that, in the Hochschild–Serre spectral sequences that appear, the only other cohomology to contribute besides H^1 will be H^0, and, for maximal ideals of $\mathbb{T}$ in the support of H^0, we do have associated Galois representations, by Lemma 3.5.2.

We first show that if $H^0_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_W)_{\mathfrak{m}} \neq 0$ for some canonical local system $\mathcal{F}_W$ on X and some maximal ideal $\mathfrak{m}$ of $\mathbb{T}$, then $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)[\mathfrak{m}]) = 0$. To see this, note that if $H^0_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_W)_{\mathfrak{m}} \neq 0$ and $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)[\mathfrak{m}]) \neq 0$, then Lemma 3.5.2 and Theorem 3.3.1 together imply that there exists an abelian representation $\bar{\rho}_0 : G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ such that, for all $g \in G_F$, the characteristic polynomial of $\bar{\rho}_0(g)$ annihilates $\rho(g)$. By Lemma 4.1.3 this implies that ρ is abelian, which is impossible as ρ is irreducible.

Now $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)$ is the direct sum of its localisations at the various maximal ideals $\mathfrak{m}$ of $\mathbb{T}$, and hence, since $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X, \mathcal{F}_V)) \neq 0$ by hypothesis, we see that $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X, \mathcal{F}_V)_{\mathfrak{m}}) \neq 0$ for some maximal ideal $\mathfrak{m}$ of $\mathbb{T}$. Since this is a finite-length $\mathbb{T}_{\mathfrak{m}}$-module, we see that its $\mathbb{T}_{\mathfrak{m}}$-socle $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X, \mathcal{F}_V)[\mathfrak{m}])$ must also be nonzero, and hence, by the preceding paragraph, we conclude that $H^0_{\text{ét}}(X, \mathcal{F}_W)_{\mathfrak{m}} = 0$ for any canonical local system $\mathcal{F}_W$ on X.

[4]In fact, as noted in the introduction, recent work of Scholze [2013] implies that $\rho_{\mathfrak{m}}$ exists for any maximal ideal $\mathfrak{m}$ in the Hecke algebra. We have left our argument as originally written.

As in the proof of Theorem 3.4.2, choose a short exact sequence of canonical local systems $0 \to \mathcal{F}_V \to \pi_* \overline{\mathbb{F}}_p^n \to \mathcal{F}_W \to 0$, for some $\pi : X' \to X$. Passing to the long exact sequence

$$H^0_{\text{ét}}(X, \mathcal{F}_W)_{\mathfrak{m}} \longrightarrow H^1_{\text{ét}}(X, \mathcal{F}_V)_{\mathfrak{m}} \longrightarrow H^1_{\text{ét}}(X, \pi_* \overline{\mathbb{F}}_p)^n = H^1_{\text{ét}}(X', \overline{\mathbb{F}}_p)^n,$$

and using the result of the preceding paragraph, namely that $H^0_{\text{ét}}(X, \mathcal{F}_W)_{\mathfrak{m}} = 0$, we conclude that ρ embeds into $H^1_{\text{ét}}(X', \overline{\mathbb{F}}_p)$. Thus, replacing X by X', we reduce to the case when $\mathcal{F}_V$ is constant, which we assume from now on.

We now suppose that ρ is a subrepresentation of $H^1_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$. We will prove that ρ is then necessarily a subrepresentation of $H^1_{\text{ét}}(X(K^p I_1), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ for some sufficiently small open subgroup K^p of $G(\mathbb{A}^{\infty,p})$. The result will then follow from Lemmas 3.1.3 and 4.2.2 and Theorems 1.5.15, 3.3.1, and 2.2.4.

As in the proof of Theorem 3.4.2, we write $X = X(K)$, and choose a normal open subgroup $K' := K^p K_p$ of K, with $K_p \subset I_1$. The Hochschild–Serre spectral sequence associated to the cover $X(K') \to X(K)$ gives rise to an exact sequence

$$\begin{aligned} 0 \longrightarrow H^1(K/K', H^0(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}) \longrightarrow H^1_{\text{ét}}(X(K), \overline{\mathbb{F}}_p)_{\mathfrak{m}} \\ \longrightarrow H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}^{K/K'} \longrightarrow H^2(K/K', H^0(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}). \end{aligned}$$

The same argument as above, using Lemma 3.5.2 (applied now with $X(K')$ in place of X), Theorem 3.3.1, and Lemma 4.1.3 below, shows that

$$H^1(K/K', H^0(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}) = H^2(K/K', H^0(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}) = 0.$$

Thus in fact we have an isomorphism $H^1_{\text{ét}}(X(K), \overline{\mathbb{F}}_p) \xrightarrow{\sim} H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}^{K/K'}$, and hence an isomorphism

$$\begin{aligned} \mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K), \overline{\mathbb{F}}_p)) \xrightarrow{\sim} \mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}^{K/K'}) \\ = \mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}})^{K/K'}. \end{aligned}$$

In particular, if ρ appears as a subrepresentation of $H^1_{\text{ét}}(X(K), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$, then it appears as a subrepresentation of $H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$.

Now, considering the Hochschild–Serre spectral sequence for the cover $X(K') \to X(K^p I_1)$, and using the fact that if $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}) \neq 0$, then also $\mathrm{Hom}_{G_F}(\rho, H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}})^{I_1/K_p} \neq 0$ (since I_1/K_p is a p-group), we conclude that if ρ appears as a subrepresentation of $H^1_{\text{ét}}(X(K'), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$, then it appears as a subrepresentation of $H^1_{\text{ét}}(X(K^p I_1), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$.

Arguing exactly as in the proof of Theorem 3.4.2, we then deduce that some twist of ρ appears in $H^1_{\text{ét}}(X(K^p I_1^*), \overline{\mathbb{F}}_p)$, and so, replacing ρ by this twist, it suffices to prove that if ρ is an irreducible 3-dimensional representation ρ of $H^1_{\text{ét}}(X(K^p I_1^*), \overline{\mathbb{F}}_p)$ that is generated by its regular elements, then either every irreducible subquotient of $\rho_{|G_{\mathbb{Q}_p}}$ is 1-dimensional, or else $\rho_{|G_{\mathbb{Q}_p}}$ is not 1-regular.

This follows from Lemma 3.1.3 and Theorems 1.5.15, 3.3.1, and 2.2.4 exactly as in the proof of Theorem 3.4.2, replacing the appeal to Hypothesis 3.4.1 with one to Lemma 4.2.2 below. □

Our other main theorem concerns the weight part of the Serre-type conjecture of [Herzig 2009] for $U(2,1)$. It is proved by combining our techniques with those of [Emerton et al. 2013], where a similar theorem is proved for $U(3)$ (which is simpler, because one has vanishing of cohomology outside of degree 0). We begin by recalling some terminology from that work. We will call an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_3(\mathbb{F}_p)$ a *Serre weight.* Fix an irreducible representation $\bar{\rho}: G_F \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$. Let $X := X(K)$ be a $U(2,1)$-Shimura variety such that K is of level dividing N and has sufficiently small projection to $\mathbb{G}(\mathbb{A}^{p,\infty})$, where now we assume that $(N, p) = 1$. Assume furthermore that $\bar{\rho}$ is unramified at all places not dividing Np, and define a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ with residue field $\overline{\mathbb{F}}_p$ by demanding that, for each place $w \nmid Np$ of F such that $w|_{\mathbb{Q}}$ splits in F, the characteristic polynomial of $\bar{\rho}(\mathrm{Frob}_w)$ is equal to the reduction modulo $\mathfrak{m}$ of $\sum_{i=0}^{3}(-1)^i(\mathrm{Norm}\, w)^{i(i-1)/2}T_w^{(i)}X^{n-i}$.

Let V be a Serre weight; since (N, p)=1, we may write $K = K_pK^p$, where $K_p \subset G(\mathbb{Q}_p) \cong \mathrm{GL}_3(\mathbb{Q}_p)\times\mathbb{Q}_p^\times$ is conjugate to $\mathrm{GL}_3(\mathbb{Z}_p)\times\mathbb{Z}_p^\times$, and we may regard V as a representation of K_p via the projection $\mathrm{GL}_3(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GL}_3(\mathbb{F}_p)$. As usual, write $\mathscr{F}_V$ for the canonical local system associated to V. We say that $\bar{\rho}$ is *modular of weight* V if for some N, X as above and for some $0 \le i \le 4$ we have

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)_{\mathfrak{m}} \neq 0.$$

Assume now that $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible. Definition 6.2.2 of [Emerton et al. 2013] defines what it means for a Serre weight to be (strongly) generic, and Section 5.1 there (using the recipe of [Herzig 2009]) defines a set $W^?(\bar{\rho})$ of Serre weights in which it is predicted that $\bar{\rho}$ is modular. Let $W_{\text{gen}}(\bar{\rho})$ be the set of generic weights for which $\bar{\rho}$ is modular.

Theorem 3.5.6. *Suppose that $\bar{\rho}$ satisfies Hypothesis 4.1.1 and $\bar{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible and* 1*-regular. Suppose that $\bar{\rho}$ is modular of some strongly generic weight. Then $W_{\text{gen}}(\bar{\rho}) = W^?(\bar{\rho})$. In fact, for each $V \in W_{\text{gen}}(\bar{\rho})$, we have*

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)_{\mathfrak{m}} \neq 0$$

if and only if $i = 2$ and $V \in W^?(\bar{\rho})$.

Proof. By the definition of $\mathfrak{m}$, the representation $\bar{\rho}$ satisfies the defining properties of the representation $\rho_{\mathfrak{m}}$ considered in Section 3.4. Applying Corollary 3.5.1, we see that for any Serre weight V we have $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathscr{F}_V)_{\mathfrak{m}} = 0$ if $i \neq 2$. We will now deduce the result from Theorem 6.2.3 of [Emerton et al. 2013] (taking $\bar{r}$ there to be our $\bar{\rho}$). By Theorem 4.3.3 of that reference, we see that it suffices to show that we can define S and $\widetilde{S}$ as in Section 4 there, so that Axioms Ã1–Ã3 of Section 4.3 of that work are satisfied. Following [Emerton et al. 2013], we define S and $\widetilde{S}$ using completed cohomology in the sense of [Emerton 2006] (in [Emerton et al. 2013] the use of completed cohomology was somewhat disguised, but the constructions with algebraic modular forms there are equivalent to the use of completed cohomology of $U(3)$ in degree 0). In fact, given our vanishing results the verification of the axioms of [Emerton et al. 2013] is very similar to that carried out in that work for $U(3)$, and we content ourselves with sketching the arguments.

From now on we regard the prime-to-p level structure K^p of X as fixed, and we will vary K_p in our arguments. We will write $K_p(0)$ for $\mathrm{GL}_3(\mathbb{Z}_p)\times\mathbb{Z}_p^\times \subset G(\mathbb{Q}_p)$. We fix a sufficiently large extension $E/\mathbb{Q}_p$ with ring of integers $\mathbb{O}_E$, residue field k_E, and uniformiser ϖ_E, and we define

$$S := \varinjlim_{K_p} H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}},$$

$$\widetilde{S} := \Big(\Big(\varprojlim_{s}\varinjlim_{K_p} H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \mathbb{O}_E/\varpi_E^s)_{\mathfrak{m}}\Big)\otimes_{\mathbb{O}_E}\overline{\mathbb{Z}}_p\Big)^{\text{l.alg}}$$

(that is, the locally algebraic vectors in the localisation at $\mathfrak{m}$ of the completed cohomology of degree 2). Using the Hochschild–Serre spectral sequence and the vanishing of $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}} = 0$ if $i \neq 2$, it is straightforward to verify Axioms Ã1–Ã3 of Section 4.3 of [Emerton et al. 2013], as we now explain.

First, we need to check that our definition of "modular" is consistent with that of Definition 4.2.2 of [Emerton et al. 2013]. This amounts to showing that for any Serre weight V

$$(S\otimes_{\overline{\mathbb{F}}_p} V)^{K_p(0)} = H^2_{\text{ét}}(X(K_p(0)K^p)_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}.$$

To see this, note that since any sufficiently small K_p acts trivially on V, we have

$$S\otimes_{\overline{\mathbb{F}}_p} V = \varinjlim_{K_p} H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \overline{\mathbb{F}}_p)_{\mathfrak{m}}\otimes V \xrightarrow{\sim} \varinjlim_{K_p} H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}},$$

so it is enough to check that for all compact open subgroups $K_p \subset K_p(0)$ we have

$$H^2_{\text{ét}}(X(K^pK_p)_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}}^{K_p(0)} = H^2_{\text{ét}}(X(K^pK_p(0))_{\overline{\mathbb{Q}}}, \mathcal{F}_V)_{\mathfrak{m}},$$

which is an easy consequence of the Hochschild–Serre spectral sequence and our vanishing result. We also need an embedding $S \hookrightarrow \widetilde{S}\otimes_{\overline{\mathbb{Z}}_p}\overline{\mathbb{F}}_p$ which is compatible with the actions of $\mathrm{GL}_3(\mathbb{Q}_p)$ and the Hecke algebra. In fact, it is easy to see that we have $\widetilde{S}\otimes_{\overline{\mathbb{Z}}_p}\overline{\mathbb{F}}_p = S$. For example, there is a natural isomorphism

$$H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{O}_E)_{\mathfrak{m}}/\varpi_E H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{O}_E)_{\mathfrak{m}} = H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, k_E)_{\mathfrak{m}},$$

and hence the vanishing of $H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, k_E)_{\mathfrak{m}}$ implies that of the finitely generated $\mathcal{O}_E$-module $H^3_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E)_{\mathfrak{m}}$. One then sees that for all s we have a natural isomorphism

$$H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E)_{\mathfrak{m}}/\varpi_E^s H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E)_{\mathfrak{m}} = H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s)_{\mathfrak{m}},$$

from which the claim follows easily.

We now examine Axiom Ã1. We must show that if $\tilde{V}$ is a finite free $\overline{\mathbb{Z}}_p$-module with a locally algebraic action of $K_p(0)$ (acting through $\mathrm{GL}_3(\mathbb{Z}_p)$), then $(\tilde{S} \otimes \overline{\mathbb{Z}}_p \tilde{V})^{K_p(0)}$ is a finite free $\overline{\mathbb{Z}}_p$-module, and for $A = \overline{\mathbb{Q}}_p, \overline{\mathbb{F}}_p$ we have

$$(\tilde{S} \otimes \overline{\mathbb{Z}}_p \tilde{V})^{K_p(0)} \otimes \overline{\mathbb{Z}}_p A = (\tilde{S} \otimes \overline{\mathbb{Z}}_p \tilde{V} \otimes \overline{\mathbb{Z}}_p A)^{K_p(0)}.$$

This is straightforward, the key point being that if $\mathcal{F}_{\tilde{V}}$ denotes the lisse étale sheaf attached to $\tilde{V}$, then a straightforward argument with Hochschild–Serre as above gives

$$(\tilde{S} \otimes \overline{\mathbb{Z}}_p \tilde{V})^{K_p(0)} = H^2_{\text{ét}}(X(K^p K_p(0))_{\overline{\mathbb{Q}}}, \mathcal{F}_{\tilde{V}})_{\mathfrak{m}},$$

which is certainly a finite free $\overline{\mathbb{Z}}_p$-module (it is torsion-free by the proof of Corollary 3.4.5).

The verification of Axioms Ã2 and Ã3 is now exactly the same as in Proposition 7.4.4 of [Emerton et al. 2013], as the Galois representations occurring in the localised cohomology module $H^2_{\text{ét}}(X(K^p K_p(0))_{\overline{\mathbb{Q}}}, \mathcal{F}_{\tilde{V}})_{\mathfrak{m}}$ are associated to automorphic forms exactly as in that work.[5] (In fact, at least for Axiom Ã2 this is a rather roundabout way of proceeding, as the Galois representations in question are constructed in [Harris and Taylor 2001] by using $H^2_{\text{ét}}(X(K^p K_p(0))_{\overline{\mathbb{Q}}}, \mathcal{F}_{\tilde{V}})$, and one can read off the required properties directly from the comparison theorems of p-adic Hodge theory. For Axiom Ã3 we are not aware of any comparison theorems in sufficient generality, so it is necessary at present to take a lengthier route through the theory of automorphic forms.) □

4. Group theory lemmas

The theorems of Section 3 contain certain hypotheses on the Galois representations involved. Our goal in this section is to establish some group-theoretic lemmas which give sufficient criteria for these hypotheses to be satisfied. Throughout the section G is a finite group and k is an algebraically closed field of characteristic p. For any square matrix A with entries in k, we write $\mathrm{char}(A)$ to denote the characteristic polynomial of A.

[5]In the interests of full disclosure, we are not aware of a reference in the literature giving the precise base change result from $U(2, 1)$ to GL_3 that we need, but it seems to be well-known to the experts, and will follow from the much more general work in progress of Mok and Kaletha, Minguez, Shin and White.

4.1. ***Characterising representations by their characteristic polynomials.*** Let $\rho : G \to \mathrm{GL}_n(k)$ be an irreducible representation. In this subsection we establish some criteria for ρ to satisfy the following hypothesis:

Hypothesis 4.1.1. If $\theta : G \to \mathrm{GL}_m(k)$ is irreducible, and if $\mathrm{char}(\rho(g))$ annihilates $\theta(g)$ for every $g \in G$, then θ is equivalent to ρ.

Remark 4.1.2. Any irreducible ρ of dimension 2 satisfies Hypothesis 4.1.1, as was proved by Mazur [1977, proof of Proposition 14.2]. However, it is not satisfied in general if the dimension n of ρ is greater than 2 (for instance, this already fails if ρ is the irreducible 3-dimensional representation of A_4; see Section 5 of [Boston et al. 1991] as well as Remark 4.1.7 below).

Lemma 4.1.3. *Let $\rho : G \to \mathrm{GL}_n(k)$ and $\theta : G \to GL_m(k)$ be two representations. If θ is irreducible, and if* $\mathrm{char}(\rho(g))$ *annihilates $\theta(g)$ for every $g \in G$, then the kernel of θ contains the kernel of ρ.*

Proof. If $\rho(g)$ is trivial, then the assumption implies that every eigenvalue of $\theta(g)$ is equal to 1, and hence that $\theta(g)$ is unipotent, and so of order a power of p. Thus the image of $\ker(\rho)$ under θ is a normal subgroup H of $\theta(G)$ of p-power order, and we see that the space of invariants $(k^m)^H$ is a nontrivial subspace of k^m. Since H is normal in $\theta(G)$, we see that $\theta(G)$ leaves $(k^m)^H$ invariant, and hence, since θ is assumed to be irreducible, we see that in fact $(k^m)^H = k^m$. Thus H is trivial, which is to say that $\ker(\rho) \subset \ker(\theta)$, as claimed. □

Lemma 4.1.4. *If $\rho : G \to \mathrm{GL}_n(k)$ is a direct sum of 1-dimensional characters of G, and if $\theta(g) : G \to \mathrm{GL}_m(k)$ is an irreducible representation of G such that* $\mathrm{char}(\rho(g))$ *annihilates $\theta(g)$ for every $g \in G$, then $m = 1$, so that θ is a character, and every element of G lies in the kernel of at least one of the summands of $\rho \otimes \theta^{-1}$.*

Proof. Since ρ is a direct sum of characters, it factors through G^{ab}. Lemma 4.1.3 then shows that θ also factors through G^{ab}. Since θ is also assumed to be irreducible, we find that θ must be a character. Twisting by ρ by θ^{-1}, we may in fact assume that θ is trivial, and, writing $\rho = \chi_1 \oplus \cdots \oplus \chi_n$, we find that for each $g \in G$, the value $\chi_i(g)$ is equal to 1 for at least one value of i (since $\mathrm{char}(\rho(g)) = (X - \chi_1(g)) \cdots (X - \chi_n(g))$ annihilates $\theta(g) = 1$). Thus G is equal to the union of its subgroups $\ker(\chi_i)$. □

Remark 4.1.5. In the context of the preceding proposition, we can't conclude in general that θ coincides with one the summands of ρ. For example, if G denotes the Klein four-group, if p is odd, and if ρ denotes the 3-dimensional representation obtained by taking the direct sum of the three nontrivial characters of G, then, taking θ to be the trivial representation, the hypotheses of the proposition are satisfied, but θ is certainly not one of the summands of ρ.

Lemma 4.1.6. *If $\rho : G \to \mathrm{GL}_n(k)$ is irreducible, and is isomorphic to an induction $\mathrm{Ind}_H^G \psi$, where H is a cyclic normal subgroup of G of index n and $\psi : H \to k^\times$ is a character, then ρ satisfies Hypothesis 4.1.1.*

Proof. The restriction $\rho|_H$ is isomorphic to the direct sum $\bigoplus_{g \in G/H} \psi^g$. If we let θ' be a Jordan–Hölder constituent of the restriction $\theta_{|H}$, then Lemma 4.1.4 (applied to the representations $\rho_{|H}$ and θ' of H) implies that θ' is a character of H and (because H is cyclic) that $\theta' = \psi^g$ for some $g \in G/H$. The H-equivariant inclusion $\psi^g = \theta' \to \theta_{|H}$ then induces a nonzero G-equivariant map $\rho = \mathrm{Ind}_H^G \psi = \mathrm{Ind}_H^G \psi^g \to \theta$, which must be an isomorphism, since both its source and target are irreducible by assumption. This proves the lemma. □

Remark 4.1.7. If we take $G = A_4$ and H to be the normal subgroup of G of order four (so that H is a Klein four-group), then the induction of any nontrivial character of H gives an irreducible representation $\rho : G \to \mathrm{SO}_3(k)$. For every $g \in G$, the characteristic polynomial of $\rho(g)$ thus has 1 as an eigenvalue, and so, if θ denotes the trivial character of G, the element $\theta(g)$ is annihilated by $\mathrm{char}(\rho(g))$ for every $g \in G$. Thus the analogue of Lemma 4.1.6 does not hold in general if H is not cyclic.

We thank Florian Herzig for providing the proof of the following lemma:

Lemma 4.1.8. *Suppose that G is a finite subgroup of $\mathrm{GL}_n(k)$ which contains $\mathrm{SL}_n(k')$ for some subfield k' of k and is contained in $k^\times \mathrm{GL}_n(k')$.*

(1) *Any irreducible representation of G over k remains irreducible upon restriction to $\mathrm{SL}_n(k')$.*

(2) *Given any two irreducible representations of G which become isomorphic upon restriction to $\mathrm{SL}_n(k')$, one can be obtained from the other via twisting by a character of G that is trivial on $\mathrm{SL}_n(k')$.*

Proof. Let G act via θ on the k-vector space V, and let (θ, W) be an irreducible subrepresentation of $\theta|_{\mathrm{SL}_n(k')}$. Then W is obtained by restriction from a representation of the algebraic group SL_n / k' (see Section 1 of [Jantzen 1987]), so the action of $\mathrm{SL}_n(k')$ on W may be extended to an action of $\mathrm{GL}_n(k)$ and thus of G. By Frobenius reciprocity we obtain a surjective map $(\mathrm{Ind}_{\mathrm{SL}_n(k')}^G 1) \otimes W \to V$ of G-representations. Since $G/\mathrm{SL}_n(k')$ is a finite abelian group of prime-to-p order, we see that $(\mathrm{Ind}_{\mathrm{SL}_n(k')}^G 1)$ is a direct sum of 1-dimensional representations, so that V is a twist of W by some character which is trivial on $\mathrm{SL}_n(k')$. Thus the restriction of θ to $\mathrm{SL}_n(k')$ is just W, which is irreducible, proving (1).

This same argument also serves to establish (2). □

Lemma 4.1.9. *Assume that $p \geq n$. If $\rho : G \to \mathrm{GL}_n(k)$ is irreducible, and if $\mathrm{SL}_n(k') \subseteq \rho(G) \subseteq k^\times \mathrm{GL}_n(k')$ for some subfield k' of k, then ρ satisfies Hypothesis 4.1.1.*

Proof. The case $n = 1$ follows from Lemma 4.1.4, and, as remarked above, the case $n = 2$ is proved in the course of the proof of Proposition 14.2 of [Mazur 1977], so we may assume that $n \geq 3$. By Lemma 4.1.3, we may assume that ρ is faithful, so that we can identify G with $\rho(G)$. In particular, $\mathrm{SL}_n(k')$ is a subgroup of G, and, by Lemma 4.1.8, the restriction of θ to $\mathrm{SL}_n(k')$ remains irreducible.

Since G is finite, our assumption that $\mathrm{SL}_n(k') \subset G$ implies that k' is finite; suppose that k' has cardinality q. We recall some basic facts about the representation theory of $\mathrm{SL}_n(k')$; see for example Section 1 of [Jantzen 1987]. The irreducible k-representations of $\mathrm{SL}_n(k')$ are obtained by restriction from the algebraic group SL_n /k', and are precisely those representations whose highest weights are q-restricted. (With the usual choice of maximal torus T of SL_n, if we identify the weight lattice with $\mathbb{Z}^n$ modulo the diagonally embedded copy of $\mathbb{Z}$, a weight $(a_1, \dots, a_n)$ is q-restricted if $0 \leq a_i - a_{i+1} \leq q - 1$ for all $1 \leq i \leq n-1$.) Suppose that θ has highest weight $(a_1, \dots, a_n)$. Let $g \in \mathrm{SL}_n(k')$ be a semisimple element with eigenvalues $\alpha_1, \dots, \alpha_n$. Then, since g is conjugate to an element of $T(k)$, by considering the formal character of the corresponding representation of SL_n /k', we see that among the eigenvalues of $\theta(g)$ are each of the quantities

$$\prod_{i=1}^{n} \alpha_i^{x_i},$$

where the x_i are a permutation of $a_1, \dots, a_n$.

Our assumption on θ and ρ implies that, for each such permutation, $\prod_{i=1}^{n} \alpha_i^{x_i}$ must be one of $\alpha_1, \dots, \alpha_n$. In particular, if we let α be a primitive $(q^n - 1)/(q-1)$-st root of unity, we may consider a semisimple element g with eigenvalues

$$\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}.$$

Then, for any $x_1, \dots, x_n$ as above, there must be an integer $0 \leq \beta \leq n-1$ such that

$$q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n \equiv q^\beta \pmod{(q^n - 1)/(q-1)}.$$

Fix some $1 \leq i \leq n-1$, and consider two permutations $x_1, \dots, x_n$ and $x'_1, \dots, x'_n$ as above which satisfy $x_i = x'_i$ for $1 \leq i \leq n-2$, $x_{n-1} = a_i$, $x_n = a_{i+1}$, $x'_{n-1} = a_{i+1}$ and $x'_n = a_i$. Taking the difference of the two expressions

$$q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n \quad \text{and} \quad q^{n-1}x'_1 + q^{n-2}x'_2 + \cdots + x'_n,$$

we conclude that there are integers $0 \leq \beta, \gamma \leq n-1$ such that

$$(q-1)(a_i - a_{i+1}) \equiv q^\beta - q^\gamma \pmod{(q^n - 1)/(q-1)}.$$

Since $n \geq 3$, we have $0 \leq (q-1)(a_i - a_{i+1}) \leq (q-1)^2 < (q^n-1)/(q-1)$, and we conclude that either $\beta \geq \gamma$ and $(q-1)(a_i - a_{i+1}) = q^\beta - q^\gamma$, or $\beta < \gamma$ and $(q-1)(a_i - a_{i+1}) = (q^n-1)/(q-1) + q^\beta - q^\gamma$. In the second case, we have

$$\begin{aligned}(q-1)^2 &\geq (q-1)(a_i - a_{i+1}) \\ &= (q^n-1)/(q-1) + q^\beta - q^\gamma \\ &\geq (q^n-1)/(q-1) + 1 - q^{n-1} \\ &= q^{n-2} + \cdots + q + 2.\end{aligned}$$

This is a contradiction if $n \geq 4$. If $n = 3$, $(q^3-1)/(q-1) \equiv 3 \pmod{q-1}$, so that $(q-1) \mid 3$, which is a contradiction as $p > 2$.

Thus it must be the case that $\beta \geq \gamma$ and $(q-1)(a_i - a_{i+1}) = q^\beta - q^\gamma$, so that $a_i - a_{i+1}$ is congruent mod p to 0 or 1, and thus $a_i - a_{i+1} = 0$ or 1. In particular, for each i we have $0 \leq a_i - a_n \leq n-1 \leq p-1$.

We now repeat the above analysis. Fix some $1 \leq i \leq n-1$, and consider two permutations $x_1, \dots, x_n$ and $x'_1, \dots, x'_n$ as above which satisfy $x_i = x'_i$ for $1 \leq i \leq n-2$, $x_{n-1} = a_i$, $x_n = a_n$, $x'_{n-1} = a_n$ and $x'_n = a_i$. Taking the difference of the two expressions $q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n$ and $q^{n-1}x'_1 + q^{n-2}x'_2 + \cdots + x'_n$, and using that $0 \leq a_i - a_n \leq p-1 \leq q-1$, we conclude as before that each $a_i - a_n = 0$ or 1. Thus there must be an integer $1 \leq r \leq n$ with $a_1 = \cdots = a_r = a_n + 1$, $a_{r+1} = \cdots = a_n$.

Returning to the original congruences

$$q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n \equiv q^\beta \pmod{(q^n-1)/(q-1)},$$

we see that the left side is congruent to a sum of precisely r distinct values q^j, $1 \leq j \leq n-1$. Thus $r = 1$, and $\theta|_{\mathrm{SL}_n(k')}$ is the standard representation of $\mathrm{SL}_n(k')$, i.e., $\theta|_{\mathrm{SL}_n(k')} \cong \rho|_{\mathrm{SL}_n(k')}$. By part (2) of Lemma 4.1.8, we see that there is a character $\chi : G \to k^\times$ with $\chi|_{\mathrm{SL}_n(k')} = 1$ such that $\theta \cong \rho \otimes \chi$.

To complete the proof, we must show that χ is trivial. Take $g \in G$; we will show that $\chi(g) = 1$. If $g' \in G$ has $\det g' = \det g$, then $g(g')^{-1} \in \mathrm{SL}_n(k')$, so $\chi(g') = \chi(g)$. First, note that by assumption we can write $g = \lambda h$, with $\lambda \in k^\times$, $h \in \mathrm{GL}_n(k')$. Choose $h' \in \mathrm{GL}_n(k')$ to have eigenvalues $\{1, \dots, 1, \det(h)\}$. Then $h(h')^{-1} \in \mathrm{SL}_n(k') \subset G$, so $g' := \lambda h'$ is an element of G. Then the hypothesis on θ and ρ shows that the eigenvalues of $\chi(g')g' = \chi(g)g'$ are contained in the set of eigenvalues of g', so that if $\chi(g) \neq 1$ we must have $\chi(g) = \det(h) = -1$. Assume for the sake of contradiction that this is the case. If n is odd, then we now choose h' to have eigenvalues $\{-1, \dots, -1\}$, and we immediately obtain a contradiction from the same argument. If n is even then since $p \geq n$ we have $p \geq 5$ (recall that we are assuming $n \geq 3$), and we may choose $a \in (k')^\times$, $a \neq \pm 1$. Then choosing h' to have eigenvalues $\{1, \dots, 1, a, -1/a\}$ gives a contradiction. $\square$

4.2. ***Representations whose image is generated by regular elements.*** Recall that a square matrix with entries in k is said to be *regular* if its minimal polynomial and characteristic polynomial coincide.

Lemma 4.2.1. *If $\rho : G \to \mathrm{GL}_n(k)$ and $\theta : G \to \mathrm{GL}_n(k)$ are representations such that the image $\theta(G)$ is generated by its subset of regular elements, and for every $g \in G$ the characteristic polynomial of $\rho(g)$ annihilates $\theta(g)$, then* $\det \rho = \det \theta$.

Proof. Let $g \in G$ be an element such that $\theta(g)$ is regular. Then the characteristic polynomials of $\rho(g)$ and $\theta(g)$ must be equal (since the minimal and characteristic polynomials of $\theta(g)$ coincide and the characteristic polynomial of $\rho(g)$ annihilates $\theta(g)$), so $\det \rho(g) = \det \theta(g)$. Since $\theta(G)$ is generated by its subset of regular elements, the result follows. □

In fact, we actually need a slight generalisation of this result, where we simply have a collection of characteristic polynomials, rather than a representation ρ. Suppose that for each $g \in G$ we have a monic polynomial

$$\rho_g(X) = X^n - a_1(g)X^{n-1} + \cdots + (-1)^n a_n(g) \in k[X]$$

of degree n with the property that for all $g, h \in G$, we have $a_n(gh) = a_n(g)a_n(h)$.

Lemma 4.2.2. *Suppose that $\theta : G \to \mathrm{GL}_n(k)$ is a representation with the property that $\theta(G)$ is generated by its subset of regular elements, and that for each $g \in G$ we have $\rho_g(\theta(g)) = 0$. Then for each $g \in G$ we have* $\det \theta(g) = a_n(g)$.

Proof. This may be proved in exactly the same way as Lemma 4.2.1. □

Let $\rho : G \to \mathrm{GL}_3(k)$ be irreducible. Our goal is to give criteria for ρ to satisfy the following hypothesis, in order to apply the previous lemmas:

Hypothesis 4.2.3. The image $\rho(G)$ is generated by its subset of regular elements.

Lemma 4.2.4. *If $\rho : G \to \mathrm{GL}_3(k)$ is irreducible, and if either*

(1) *ρ is isomorphic to an induction $\mathrm{Ind}_H^G \psi$, where H is a normal subgroup of index* 3 *in G and $\psi : H \to k^\times$ is a character, or*

(2) *$\rho(G)$ contains a regular unipotent element,*

then ρ satisfies Hypothesis 4.2.3.

Proof. Suppose first that ρ is isomorphic to an induction $\mathrm{Ind}_H^G \psi$. Since H is a proper subgroup of G, the set of elements $G - H$ generates G. If $g \in G - H$ then the characteristic polynomial of $\rho(g)$ is of the form $X^3 - \alpha$. If $p \neq 3$ then this has distinct roots, so $\rho(g)$ is regular, and if $p = 3$ then it is easy to check that $\rho(g)$ is the product of a scalar matrix and a unipotent matrix, and is regular.

Suppose now that $\rho(G)$ contains a regular unipotent element. For ease of notation, we will refer to $\rho(G)$ as G from now on. Let H be the subgroup of G generated by the regular elements, and assume for the sake of contradiction that H is a proper subgroup of G. We claim that H contains every scalar matrix in G; this is true because the product of a scalar matrix and a regular matrix is again a regular matrix. Consider an element $g \in G - H$; since it is not regular, and not scalar, it acts as a scalar on some unique plane in k^3. We write ℓ_g for the corresponding line in $\mathbb{P}^2(k)$.

Let h be the given regular unipotent element in H. Then h stabilises a unique line in k^3, so a unique point $P \in \mathbb{P}^2(k)$. As $g \in G - H$, we also have $gh \in G - H$. Then $\ell_g \cap \ell_{gh}$ is nonempty, so there is a point $Q \in \mathbb{P}^2(k)$ which is fixed by g and gh. It is thus also fixed by h, so in fact $Q = P$. Since g was an arbitrary element of $G - H$, we see that every element of $G - H$ fixes P, and since G is generated by $G - H$, this implies that every element of G fixes P. This contradicts the assumption that ρ is irreducible. ☐

Appendix: Cohomology of pairs

A.1. ***Étale cohomology of a pair.*** Let X be a scheme, finite-type and separated over a field, let Z be a closed subscheme, and write $j : U \hookrightarrow X$ for the open immersion of the complement $U := X \setminus Z$ into X. As in Section 1, we let E be an algebraic extension of $\mathbb{Q}_p$, where p is invertible on X, let k_E denote the residue field of E, and we let A be one of E or k_E. We define the étale cohomology of the pair (X, Z) with coefficients in A to be the étale cohomology of the sheaf $j_! A$ on X, i.e., we write

$$H^\bullet_{\text{ét}}(X, Z, A) := H^\bullet_{\text{ét}}(X, j_! A).$$

If $i : Z \hookrightarrow X$ is the closed immersion of Z, then the short exact sequence

$$0 \longrightarrow j_! A \longrightarrow A \longrightarrow i_* A \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \longrightarrow H^m_{\text{ét}}(Z, A) \longrightarrow H^{m+1}_{\text{ét}}(X, Z, A) \longrightarrow H^{m+1}_{\text{ét}}(X, A) \longrightarrow H^{m+1}_{\text{ét}}(Z, A) \longrightarrow \cdots,$$

which is the long exact cohomology sequence of the pair (X, Z).

We are particularly interested in the case of a pair $(X \setminus Y, Z \setminus Y)$, where X is a smooth projective variety over a separably closed field, and Y and Z are smooth divisors on X which meet transversely. In this case we have a Cartesian diagram of open immersions

$$\begin{array}{ccc} X \setminus (Y \cup Z) & \xrightarrow{j} & X \setminus Y \\ \downarrow{\scriptstyle k'} & & \downarrow{\scriptstyle k} \\ X \setminus Z & \xrightarrow{j'} & X \end{array} \tag{A.1.1}$$

According to the above definition, the cohomology of the pair $(X \setminus Y, Z \setminus Y)$ is computed as the cohomology of the sheaf $j_! A$ on $X \setminus Y$, which is canonically isomorphic to the cohomology of the complex $Rk_* j_! A$ on X. An important point is that there is a canonical isomorphism

$$j'_! Rk'_* A \xrightarrow{\sim} Rk_* j_! A. \tag{A.1.2}$$

(See the discussion of §III (b) on p. 44 of [Faltings 1989].)

A.2. ***Verdier duality.*** Verdier duality [SGA 4_3 1973, Exposé XVIII; Verdier 1967] states that if $f : X \to S$ is a morphism of finite-type and separated schemes over a separably closed field k, then, for any constructible étale A-sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on S, there is a canonical isomorphism (in the derived category) of complexes of étale sheaves on S

$$RHom(Rf_! \mathcal{F}, \mathcal{G}) \cong Rf_* RHom(\mathcal{F}, f^! \mathcal{G}).$$

We recall some standard special cases of this isomorphism, in the context of the diagram (A.1.1).

Taking f to be k' (and recalling that k' is an open immersion), we obtain an isomorphism

$$RHom(k'_! A, A) \cong Rk'_* RHom(A, A) = Rk'_* A,$$

and hence, by double duality, an isomorphism

$$RHom(Rk'_* A, A) \cong k'_! A. \tag{A.2.1}$$

Next, taking f to be j', and taking into account (A.1.2) and (A.2.1), we obtain isomorphisms

$$RHom(Rk_* j_! A, A) \cong RHom(j'_! Rk'_* A, A) \cong Rj'_* RHom(Rk'_* A, A) \cong Rj'_* k'_! A.$$

Finally, taking f to the natural map $X \to \operatorname{Spec} k$, and recalling that in this case we have

$$f^! A = A[2d](d)$$

(where d is the dimension of X; see [SGA 4_3 1973, Exposé XVIII, Théorème 3.2.5]) and that $Rf_* = Rf_!$ (since f is proper, the variety X being projective by assumption), we find that

$$\begin{aligned} RHom(Rf_* Rk_* j_! A, A) &\cong Rf_* RHom(Rk_* j_! A, A[2d](d)) \\ &\cong Rf_* Rj'_* k'_! A[2d](d). \end{aligned}$$

Passing to cohomology, we find that $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, A)$ is in natural duality with $H^{2d-m}(X \setminus Z, Y \setminus Z, A)(d)$.

A.3. ***Vanishing outside of, and torsion-freeness in, the middle degree.*** We continue to assume that X is a smooth projective variety of dimension d over the separably closed field k, and that Y and Z are smooth divisors on X which meet transversely, but, in addition, we now assume that the complements $X \setminus Y$ and $X \setminus Z$ are affine (and hence also that $Z \setminus Y$ and $Y \setminus Z$ are affine). This latter assumption implies that $H^m_{\text{ét}}(X \setminus Y, A)$ vanishes if $m > d$ and that $H^m_{\text{ét}}(Z \setminus Y, A)$ vanishes if $m \geq d$ [SGA 4_3 1973, Exposé XIV, Corollaire 3.3]. By the long exact cohomology sequence of the pair $(X \setminus Y, Y \setminus Z)$, we see that $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, A)$ vanishes if $m > d$. Similarly, we see that $H^{2d-m}(X \setminus Z, Y \setminus Z, A)(d)$ vanishes if $m < d$. Hence, by the duality between $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, A)$ and $H^{2d-m}(X \setminus Z, Y \setminus Z, A)(d)$, we find that both vanish unless $m = d$.

Let $\mathbb{O}_E$ denote the ring of integers in E. Suppose momentarily that $E/\mathbb{Q}_p$ is finite, and let ϖ be a uniformiser of $\mathbb{O}_E$. From a consideration of the cohomology long exact sequence arising from the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{O}_E/\varpi^n \xrightarrow{\varpi\cdot} \mathbb{O}_E/\varpi^{n+1} \longrightarrow k_E \longrightarrow 0,$$

and arguing inductively on n, we find that $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, \mathbb{O}_E/\varpi^n)$ vanishes in degrees other than $m = d$ for all n. Passing to the projective limit over n, we see that the same is true of $H^m_{\text{ét}}(X \setminus Y, Z \setminus Y, \mathbb{O}_E)$. Finally, a consideration of the cohomology long exact sequence arising from the short exact sequence

$$0 \longrightarrow \mathbb{O}_E \xrightarrow{\varpi\cdot} \mathbb{O}_E \longrightarrow k_E \longrightarrow 0$$

shows that $H^d_{\text{ét}}(X \setminus Y, Z \setminus Y, \mathbb{O}_E)$ is torsion-free.

By passage to the direct limit over subfields of E which are finite over $\mathbb{Q}_p$, we see that these properties continue to hold for arbitrary algebraic extensions $E/\mathbb{Q}_p$.

Acknowledgements

We would like to thank Brian Conrad, Florian Herzig, Mark Kisin, Kai-Wen Lan, Madhav Nori, Bjorn Poonen, Michael Rapoport, Sug Woo Shin, Junecue Suh, and Teruyoshi Yoshida for helpful correspondence and conversations on the subject of this article. We also thank an anonymous referee for their critical reading of the paper, which led to the filling of some gaps in the arguments of Section 1.5.

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Communicated by Brian Conrad
Received 2013-10-24 Revised 2015-03-13 Accepted 2015-04-10

emerton@math.uchicago.edu *Mathematics Department, University of Chicago, Chicago, IL 60637, United States*

toby.gee@imperial.ac.uk *Imperial College London, London SW7 2AZ, United Kingdom*

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

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Algebra & Number Theory

Volume 9 No. 5 2015

1937-0652(2015)9:5;1-4