Algebra & Number Theory

Volume 9

2015

No. 6

Noncommutative geometry and Painlevé equations

Andrei Okounkov and Eric Rains

msp

ALGEBRA AND NUMBER THEORY 9:6 (2015)
dx.doi.org/10.2140/ant.2015.9.1363

Noncommutative geometry and Painlevé equations

Andrei Okounkov and Eric Rains

We construct the elliptic Painlevé equation and its higher dimensional analogs as the action of line bundles on 1-dimensional sheaves on noncommutative surfaces.

1. Introduction

1.1. The classical Painlevé equations are very special 2-dimensional dynamical systems; they and their generalizations (including discretizations) appear in many applications. Their theory is very well developed, in fact, from many different angles; see for example [Conte 1999] for an introduction. Many of these approaches are very geometric, and some can be interpreted in terms of noncommutative geometry.[1] A full discussion of the relation between the two topics in the title is outside of the scope of the present paper.

Our goals here are very practical. The dynamical systems we discuss appear in a very simple, yet challenging, problem of probability theory and mathematical physics: planar dimer (or lattice fermion) with a changing boundary; see [Kenyon 2009] for an introduction and [Okounkov 2009; 2010a; 2010b; 2010c] for the developments that lead to the present paper. The link to Artin-style noncommutative geometry, which is the subject of this paper, turns out to be very useful for dynamical and probabilistic applications.

Our hope is to promote further interaction between the two fields, and with that goal in mind, we state most of our results in the minimal interesting generality, with only a hint of the bigger picture. We also emphasize explicit examples.

MSC2010: 14A22.
Keywords: noncommutative geometry, Painlevé equations.

[1] In particular, Arinkin and Borodin [2006] gave an algebrogeometric interpretation of a degenerate discrete Painlevé equation. Their dynamics takes place not on the moduli spaces of sheaves but rather on moduli of discrete analogs of connections. Their construction may, in fact, be interpreted in terms of ours, as will be shown in [Rains $\geq$ 2015].

1.2. In algebraic geometry, there is an abundance of group actions of the following kind. Let $S \subset \mathbb{P}^N$ be a projective algebraic variety (it will be a surface in what follows, whence the choice of notation), and let

$$\mathsf{A} = \mathbb{C}[x_0, \ldots, x_N]/(\text{equations of } S)$$

be its homogeneous coordinate ring. Coherent sheaves $\mathcal{M}$ on S may be described as

$$\operatorname{Coh} S = \frac{\text{finitely generated graded }\mathsf{A}\text{-modules }\mathsf{M}}{\text{those of finite dimension}}. \tag{1}$$

They depend on discrete as well as continuous parameters so that

$$X = \text{moduli space of } \mathcal{M}$$

is a countable union of algebraic varieties. While there is a very developed general theory of such moduli spaces (see, e.g., [Huybrechts and Lehn 2010]), one can get a very concrete sense of X by giving generators and relations for M, as we will do below.

The group $\operatorname{Pic} S$ of line bundles $\mathcal{L}$ on S acts on X by

$$\mathcal{M} \mapsto \mathcal{L} \otimes \mathcal{M}. \tag{2}$$

If $\mathcal{L}$ is topologically nontrivial, this permutes connected components of X. A great many *integrable* actions of abelian groups can be understood from this perspective, an obvious invariant of the dynamics being the cycle in S given by the support of $\mathcal{M}$.

1.3. Our point of departure in this paper is the observation that A need not be commutative for the constructions of Section 1.2. In fact, noncommutative projective geometry in the sense of M. Artin [Stafford and Van den Bergh 2001] is precisely the study of graded algebras with a good category (1).

The key new feature of the noncommutative situation is that for tensor products like (2) one needs a *right* A-module L and then

$$\mathsf{L} \otimes_{\mathsf{A}} \mathsf{M} \in \operatorname{Mod} \mathsf{A}', \quad \mathsf{A}' = \operatorname{End}_{\mathsf{A}} \mathsf{L}.$$

If L is a deformation of a line bundle, then A' is closely related to A, but in general,

$$\mathsf{A}' \ncong \mathsf{A},$$

as can be already seen in very simple examples; see Section 2.3. As a result, we have

$$X \xrightarrow{\mathsf{L}\otimes} X' \tag{3}$$

where X' is the corresponding moduli space for A'.

While this sounds very abstract, we will be talking about a very concrete special case in which S is a blowup of another surface S_0,

$$S = \text{blowup of } S_0 \text{ at } p \in S_0,$$

and L is the exceptional divisor. In the noncommutative case, tensoring with L will make the point p move in S_0 by an amount proportional to the strength of noncommutativity; see Section 2.3.

1.4. Noncommutativity deforms the dynamics in two ways. First, the action (3) happens on a larger space that parametrizes both the module M and the algebra A, with an invariant fibration given by forgetting M. Specifically, we will be talking about sheaves on blowups of $\mathbb{P}^2$, where the centers $p_1, \dots, p_n$ of the blowup are allowed to move on a fixed cubic curve $E \subset \mathbb{P}^2$. There will be a $\mathbb{Z}^n$-action on these that covers a $\mathbb{Z}^n$-action on E^n by translations.

Second, the notion of a support of a sheaf is lost in noncommutative geometry, so noncommutative deformation destroys whatever algebraic integrability that the action (2) may have. It is sometimes replaced by local analytic integrals (given, e.g., by monodromy of certain linear difference equations), but even then the orbits of the dynamics are typically dense; see also Section 4.8 below.

1.5. In noncommutative projective geometry, the 3-generator Sklyanin algebra, or the elliptic quantum $\mathbb{P}^2$, occupies a special place. In this paper, we focus on this key special case and discuss the corresponding dynamics from several points of view, including an explicit linear algebra description of it; see Section 5. This explicit description may be reformulated as addition on a moving Jacobian, generalizing the dynamics of [Kajiwara et al. 2006, §7].

In the first nontrivial case, we find the elliptic difference Painlevé equation of [Sakai 2001], the one that gives all other Painlevé equations by degenerations and continuous limits. A particularly detailed discussion of this example may be found in Section 6. In particular, we will see that, in this case, our system of isomorphisms between moduli spaces agrees (for sufficiently general parameters) with the corresponding system of isomorphisms between rational surfaces considered by Sakai.

In the semiclassical limit, the elliptic quantum $\mathbb{P}^2$ degenerates to a Poisson structure on a commutative $\mathbb{P}^2$, which induces a Poisson structure on suitable moduli spaces of sheaves [Tyurin 1988; Bottacin 1995; Hurtubise and Markman 2002], and the moduli spaces we consider in the commutative case are particularly simple instances of symplectic leaves in these Poisson spaces. In Section 7, we show that these Poisson structures on moduli spaces carry over to the noncommutative setting.

2. Blowups and Hecke modifications

2.1.1. In this paper, we work with 1-dimensional sheaves on noncommutative projective planes. They closely resemble their commutative ancestors, which we briefly review now.

A coherent sheaf $\mathcal{M}$ on $\mathbb{P}^2$ is an object in the category (1) for $\mathsf{A} = \mathbb{C}[x_0, x_1, x_2]$. A basic invariant of $\mathcal{M}$ is its Hilbert polynomial

$$h_{\mathcal{M}}(n) = \dim \mathsf{M}_n, \quad n \gg 0.$$

The dimension of $\mathcal{M}$ is the degree of this polynomial, so for 1-dimensional sheaves,

$$h_{\mathcal{M}}(n) = dn + \chi$$

where d is the degree of the scheme-theoretic support of $\mathcal{M}$ and χ is the Euler characteristic of $\mathcal{M}$. The ratio χ/d is called the slope of $\mathcal{M}$. Sheaves with

$$\operatorname{slope} \mathcal{M}' < \operatorname{slope} \mathcal{M}$$

for all proper subsheaves $\mathcal{M}'$ are called *stable*; the moduli spaces of stable sheaves are particularly nice.

2.1.2. We will be content with *birational* group actions; hence, it will be enough for us to consider open dense subsets of the moduli spaces formed by sheaves of the form

$$\mathcal{M} = \iota_* L$$

where $\iota : C \hookrightarrow S$ is an inclusion of a smooth curve of degree d and L is a line bundle on C. All such sheaves are stable with

$$\chi = \deg L + 1 - g.$$

Here $g = (d-1)(d-2)/2$ is the genus of C. Their moduli space is a fibration over the base

$$B = \mathbb{P}^{d(d+3)/2} \setminus \{\text{singular curves}\}$$

of nonsingular curves C with the fiber $\operatorname{Jac}_{\deg L} C$, the Jacobian of line bundles of degree $\deg L$. In particular, this moduli space has dimension

$$\dim X = d^2 + 1.$$

2.1.3. Curves C meeting a point $p \in \mathbb{P}^2$ form a hyperplane in B. Incidence to p may be rephrased in terms of the blowup

$$\mathrm{Bl} : S \to \mathbb{P}^2$$

with center p. Namely, C meets p if and only if

$$C = \mathrm{Bl}\,\widetilde{C}$$

where $\widetilde{C} \subset S$ is a curve of degree

$$[\widetilde{C}] = d \cdot [\text{line}] - [\mathscr{E}] \in H_2(S, \mathbb{Z}).$$

Here $\mathscr{E} = \mathrm{Bl}^{-1} p$ is the exceptional divisor of the blowup.

Line bundles $\widetilde{L}$ on $\widetilde{C}$ may be pushed forward to $\mathbb{P}^2$ to give sheaves that surject to the structure sheaf $\mathbb{O}_p$ of p. If $\widetilde{\mathcal{M}}$ is such a line bundle viewed as a sheaf on S, then the sheaves

$$\mathcal{M} = \mathrm{Bl}_* \widetilde{\mathcal{M}}, \qquad \mathcal{M}' = \mathrm{Bl}_* \widetilde{\mathcal{M}}(-\mathscr{E}) \tag{4}$$

fit into an exact sequence of the form

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathbb{O}_p \to 0. \tag{5}$$

When two sheaves $\mathcal{M}$ and $\mathcal{M}'$ differ by (5), one says that one is a *Hecke modification* of another. Thus, Hecke modifications at p correspond to twists by the exceptional divisor on the blowup with center p. For noncommutative algebras, the language of Hecke modifications will be more convenient.

2.2. A fundamental fact that goes back to Mukai and Tyurin is that a Poisson structure ω^{-1} on a surface induces a Poisson structure on moduli of sheaves; see for example Chapter 10 of [Huybrechts and Lehn 2010]. Let (ω^{-1}) denote the divisor of the Poisson structure. For $\mathbb{P}^2$, this is a curve E of degree 3. Fix

$$p_1, \ldots, p_{3d} \in E \tag{6}$$

that lie on a curve d; that is,

$$\sum_{i=1}^{3d} [p_i] = \mathbb{O}_{\mathbb{P}^2}(d)|_E \in \mathrm{Jac}_{3d}\, E.$$

Let

$$B' = \mathbb{P}^{(d-1)(d-2)/2} \setminus \{\text{singular curves}\} \subset B$$

parametrize curves C meeting (6) or, equivalently, curves $\widetilde{C}$ in

$$S = \mathrm{Bl}_{p_1,\ldots,p_{3d}}\, \mathbb{P}^2$$

of degree $d \cdot [\text{line}] - \sum [\mathscr{E}_i]$. It is by now a classical fact [Beauville 1991] that the fibration

$$\begin{tikzcd} \mathrm{Pic}\, C \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ {[C]} \arrow[r, hook] & B' \end{tikzcd} \tag{7}$$

is Lagrangian and that these are the symplectic leaves of the Poisson structure on X. Further, the group $\operatorname{Pic} S$ acts on (7) preserving the fibers and the symplectic form. Here $\operatorname{Pic} C$ is a countable union of algebraic varieties that parametrize line bundles on C of arbitrary degree.

The noncommutative deformation will perturb this discrete integrable system. In particular, the points p_i will have to move on the cubic curve E. The following model example illustrates this phenomenon.

2.3.1. The effect of noncommutativity may already be seen in the affine situation. Let R be a noncommutative deformation of $\mathbb{C}[x, y]$, and let us examine the effect of Hecke modifications (5) on R-modules.

The point modules for R are 0-dimensional modules. These are annihilated by the 2-sided ideal generated by commutators in R, so this ideal has to be nontrivial for point modules to exist. We consider

$$R = \mathbb{C}\langle x, y\rangle/(xy - yx = \hbar y). \tag{8}$$

Here $\hbar$ is a parameter that measures the strength of the noncommutativity. Setting $\hbar = 0$, one recovers the commutative ring $\mathbb{C}[x, y]$ with the Poisson bracket

$$\{x, y\} = \lim_{\hbar\to 0} \frac{xy - yx}{\hbar} = y. \tag{9}$$

The line $\{y = 0\}$ is formed by 0-dimensional leaves of this Poisson bracket; they correspond to point modules

$$\mathcal{O}_s = R/\mathfrak{m}_s, \quad \mathfrak{m}_s = (y, x - s),\ s \in \mathbb{C},$$

for R. All of them are annihilated by y, which generates the commutator ideal of R.

Let M be an R-module of the form

$$M = R/f, \quad f = f_0(x) + f_1(x, y)\cdot y \in R,\ f_0 \neq 0.$$

The maps $M \to \mathcal{O}_s$ factor through the map

$$M \to M/yM = \mathbb{C}[x]/f_0(x)$$

and hence correspond to the roots of $f_0(x)$. So far, this is entirely parallel to the commutative case, except there are a lot fewer points — those are confined to the divisor of the Poisson bracket (9).

2.3.2. The following simple lemma shows Hecke correspondences move the points of intersection with this divisor.

Lemma 1. *Let $s_1, s_2, \ldots$ be the roots of $f_0(x)$, and let $M' = \mathfrak{m}_s M$ be the kernel in the exact sequence*

$$0 \to M' \to M \to \mathcal{O}_{s_1} \to 0.$$

Then $M' = R/f'$ where the roots of $f_0'(x)$ are $s_1 - \hbar, s_2, s_3, \ldots$.

In particular, the iteration of Hecke correspondences gives a chain of submodules of the form

$$M \supset \mathfrak{m}_s M \supset \mathfrak{m}_{s-\hbar}\mathfrak{m}_s M \supset \mathfrak{m}_{s-2\hbar}\mathfrak{m}_{s-\hbar}\mathfrak{m}_s M \supset \cdots .$$

A more general statement will be shown in Proposition 1.

2.3.3. For a noncommutative analog of the correspondence between Hecke modifications (5) and twists on the blowup (4), we need to retrace geometric constructions in module-theoretic terms.

Let S be the blowup of an affine surface $S_0 = \operatorname{Spec} R$ with center in an ideal $I \subset R$. Sheaves on S correspond to the quotient category (1) for

$$\mathsf{A} = \bigoplus_{n=0}^{\infty} \mathsf{A}_n, \quad \mathsf{A}_0 = R, \tag{10}$$

and $\mathsf{A}_n = I^n \subset R$. This quotient category

$$\operatorname{Coh} S = \operatorname{Tails} \mathsf{A}$$

is informally known as the category of tails; the morphisms in it are

$$\operatorname{Hom}_{\text{tails}}(\mathsf{M}, \mathsf{M}') = \varinjlim \operatorname{Hom}_{\text{graded }\mathsf{A}\text{-modules}}(\mathsf{M}_{\geq k}, \mathsf{M}')$$

where $\mathsf{M}_{\geq k} \subset \mathsf{M}$ is the submodule of elements of degree k and higher. The pushforward Bl_* of sheaves is the functor

$$\mathsf{M} \mapsto \mathrm{Bl}_* \mathsf{M} = \operatorname{Hom}_{\text{tails}}(\mathsf{A}, \mathsf{M}) \in \operatorname{Mod} R. \tag{11}$$

In the opposite direction, we have the pullback $\mathrm{Bl}^* M = \mathsf{A} \otimes_R M$ of modules as well as their proper transform

$$\mathrm{Bl}^{-1} M = \bigoplus I^n M \in \operatorname{Coh} S.$$

2.3.4. Now for a noncommutative ring R as in (8), we look for a graded module M over a graded algebra A such that

$$\mathrm{Bl}_* \mathsf{M}(n) = \mathfrak{m}_{s-(n-1)\hbar} \cdots \mathfrak{m}_{s-\hbar}\mathfrak{m}_s M \tag{12}$$

where $\mathsf{M}(n)_k = \mathsf{M}_{n+k}$ is the shift of the grading and the pushforward is defined as in (11). Here $r \in R$ acts on $\phi \in \operatorname{Hom}_{\text{tails}}(\mathsf{A}, \mathsf{M})$ by

$$[r \cdot \phi](a) = \phi(ar).$$

The algebra A, known as Van den Bergh's noncommutative blowup [Artin 1997], is constructed as

$$\mathsf{A} = \bigoplus_{n \geq 0} (T\mathfrak{m}_s)^n$$

where T is a new generator subject to

$$T^{-1} r T = y r y^{-1} \quad \text{for all } r \in R,$$

which means that

$$xT = T(x - \hbar), \qquad yT = Ty,$$

and hence

$$(T\mathfrak{m}_s)^n = T^n \mathfrak{m}_{s-(n-1)\hbar} \cdots \mathfrak{m}_{s-\hbar}\mathfrak{m}_s .$$

It is easy to see that the A-module

$$\mathsf{M} = \mathrm{Bl}^{-1} M = \bigoplus T^n \mathfrak{m}_{s-(n-1)\hbar} \cdots \mathfrak{m}_{s-\hbar}\mathfrak{m}_s M$$

satisfies (12) provided M has no 0-dimensional submodules supported on s, $s - \hbar, \dots$.

2.3.5. To relate Hecke modifications to tensor products, we note that

$$\mathrm{Bl}^{-1}_{s-\hbar}(\mathfrak{m}_s M) = \mathsf{L} \otimes_{\mathsf{A}_s} (\mathrm{Bl}^{-1}_s M)$$

where

$$\mathsf{L} = T^{-1}\mathsf{A}_s(1) \in \mathrm{Bimod}(\mathsf{A}_{s-\hbar}, \mathsf{A}_s).$$

Here we indicated the centers of the blowup by subscripts s and $s - \hbar$, respectively.

The functor $\mathsf{L}\otimes_{\mathsf{A}_s}$ is the noncommutative version of $\mathcal{O}_S(-\mathcal{E})\otimes$, and we see that it moves the center of the blowup by minus (to match the minus in $\mathcal{O}_S(-\mathcal{E})$) the noncommutativity parameter $\hbar$.

3. Sheaves on quantum planes

3.1. One of the most interesting noncommutative surfaces is associated with the 3-dimensional *Sklyanin algebra* A, which is a graded algebra, generated over $\mathsf{A}_0 = \mathbb{C}$ by three generators $x_1, x_2, x_3 \in \mathsf{A}_1$ subject to three quadratic relations.

The relations in A may be written in the superpotential form

$$\frac{\partial}{\partial x_i} \mathcal{W} = 0$$

where

$$\mathcal{W} = a x_1 x_2 x_3 + b x_3 x_2 x_1 + \frac{c}{3} \sum x_i^3$$

and the derivative is applied cyclically, that is,

$$\frac{\partial}{\partial x_1} x_{i_0} \cdots x_{i_{p-1}} = \sum_{k=0}^{p-1} \delta_{1,i_k} x_{i_{k+1}} \cdots x_{i_{p-1+k}},$$

where the subsubscripts are taken modulo p. The parameters a, b, and c will be assumed generic in what follows.

3.2. The structure of A has been much studied; see for example [Stafford and Van den Bergh 2001]. In particular, it is a Noetherian domain and

$$\sum_n \dim \mathsf{A}_n t^n = (1-t)^{-3}.$$

By definition, the category Tails A is the category of coherent sheaves on a quantum $\mathbb{P}^2$. The Grothendieck group of this category is the same as the K-theory of $\mathbb{P}^2$; that is,

$$K(\text{Tails } \mathsf{A}) = \mathbb{Z}^3,$$

corresponding to the 3 coefficients in the Hilbert polynomial. In particular, for 1-dimensional sheaves, we have

$$\dim \mathsf{M}_n = n \deg \mathsf{M} + \chi(\mathsf{M}), \quad n \gg 0. \tag{13}$$

3.3. Modules M such that $\dim \mathsf{M}_n = 1$, $n \gg 1$, are called *point modules* and play a very special role. Choosing a nonzero $v_n \in \mathsf{M}_n$, we get a sequence of points

$$p_n = (p_{1,n} : p_{2,n} : p_{3,n}) \in \mathbb{P}^2 = \mathbb{P}(\mathsf{A}_1)^*$$

such that

$$x_i v_n = p_{i,n} v_{n+1}.$$

The relations in A then imply that the locus

$$\{(p_n, p_{n+1})\} \in \mathbb{P}^2 \times \mathbb{P}^2$$

is a graph of an automorphism $p_{n+1} = \tau(p_n)$ of a plane cubic curve $E \subset \mathbb{P}^2$ [Artin et al. 1990]. The assignment

$$\mathsf{M} \mapsto p_0 \in E \tag{14}$$

identifies E with the moduli space of point modules M and τ with the automorphism[2]

$$\tau : \mathsf{M} \mapsto \mathsf{M}(1)$$

of the shift of grading $\mathsf{M}(1)_n = \mathsf{M}_{n+1}$. The inverse of (14) is given by

$$p \mapsto \mathsf{A}/\mathsf{A}p^{\perp},$$

where $p^{\perp} \subset \mathsf{A}_1$ is the kernel of $p \in \mathbb{P}(\mathsf{A}_1)^*$.

[2] Note that, if τ' is any other automorphism of E such that $\tau'^3 = \tau^3$, then the Sklyanin algebra associated with the pair (E, τ') has an equivalent category of coherent sheaves. Indeed, one has a natural isomorphism

$$\bigoplus_n \mathsf{A}_{3n} \cong \bigoplus_n \mathsf{A}'_{3n},$$

though this does not extend to an isomorphism $\mathsf{A} \cong \mathsf{A}'$. (Here 3 is the degree of the anticanonical bundle on $\mathbb{P}^2$.) This is why all key formulas below depend only on τ^3.

3.4. The action of A on point modules factors through the surjection in

$$0 \to (\mathsf{E}) \to \mathsf{A} \to \mathsf{B} \to 0, \tag{15}$$

where $\mathsf{E} \in \mathsf{A}_3$ is a distinguished normal (in fact, central) element and B is the twisted homogeneous coordinate ring of E. By definition,

$$\mathsf{B} = B(E, \mathcal{O}(1), \tau) = \bigoplus_{n \geq 0} H^0(E, \mathcal{L}_0 \otimes \cdots \otimes \mathcal{L}_{n-1}),$$

where $\mathcal{L}_k = (\tau^{-k})^* \mathcal{O}(1)$. The multiplication

$$\mathrm{mult}_\tau : \mathsf{B}_n \otimes \mathsf{B}_m \to \mathsf{B}_{n+m}$$

is the usual multiplication precomposed with $\tau^{-m} \otimes 1$. See for example [Stafford and Van den Bergh 2001] for a general discussion of such algebras.

The map

$$\mathrm{Coh}\, E \ni \mathcal{F} \mapsto \Gamma(\mathcal{F}) = \bigoplus_n H^0(E, \mathcal{F} \otimes \mathcal{L}_0 \otimes \cdots \otimes \mathcal{L}_{n-1})$$

induces an equivalence between the category of coherent sheaves on E and finitely generated graded B-modules up to torsion; see Theorem 2.1.5 in [Stafford and Van den Bergh 2001]. Note in particular that

$$\mathsf{B}(k) \cong \begin{cases} \Gamma(\mathcal{L}_{-k} \otimes \cdots \otimes \mathcal{L}_{-1}), & k \geq 0, \\ \Gamma(\mathcal{L}_0^{-1} \otimes \cdots \otimes \mathcal{L}_{-k-1}^{-1}), & k < 0. \end{cases} \tag{16}$$

3.5. It is shown in [Artin et al. 1991, Theorem 7.3] that the algebra $\mathsf{A}[\mathsf{E}^{-1}]_0$ is simple. If M is a 0-dimensional A-module, then $\mathsf{M}[\mathsf{E}^{-1}]_0$ is a finite-dimensional $\mathsf{A}[\mathsf{E}^{-1}]_0$-module, hence zero. It follows that any 0-dimensional A-module has a filtration with point quotients.

3.6. Moduli spaces of stable $\mathsf{M} \in \mathrm{Tails}\,\mathsf{A}$ may be constructed using the standard tools of geometric invariant theory, as in, e.g., [Nevins and Stafford 2007], or using the existence of an exceptional collection

$$\mathsf{A}, \mathsf{A}(1), \mathsf{A}(2) \in \mathrm{Tails}\,\mathsf{A},$$

as in [de Naeghel and Van den Bergh 2004]. In any event, at least for generic parameters of A, the moduli space $\mathfrak{M}(d, \chi)$ of 1-dimensional sheaves of degree d and Euler characteristic χ is irreducible of dimension

$$\dim \mathfrak{M}(d, \chi) = d^2 + 1.$$

It is enough to see this in the commutative case, where a generic M has a presentation of the form

$$0 \to \mathsf{A}(-2)^{d-\chi} \xrightarrow{L} \mathsf{A}(-1)^{d-2\chi} \oplus \mathsf{A}^{\chi} \to \mathsf{M} \to 0, \tag{17}$$

assuming

$$0 \le \chi \le \tfrac{1}{2}d;$$

see in particular [Beauville 2000]. When $\frac{1}{2}d < \chi \le d$, the $\mathsf{A}(-1)$ term moves from the generators to syzygies. The values of χ outside $[0, d]$ are obtained by a shift of grading.

It follows that (17) also gives a presentation of a generic stable 1-dimensional M for Sklyanin algebras.

3.7. The letter L is chosen in (17) to connect with the so-called L-operators in theory of integrable systems. In (17), L is a just a matrix with linear and quadratic entries in the generators $x_1, x_2, x_3 \in \mathsf{A}_1$. The space of possible L's, therefore, is just a linear space that needs to be divided by the action of

$$\operatorname{Aut}\operatorname{Source} L \times \operatorname{Aut}\operatorname{Target} L \cong GL(d-\chi) \times GL(d-2\chi) \times GL(\chi) \ltimes \mathbb{C}^{3\chi(d-2\chi)}.$$

In particular, we have a birational map

$$\operatorname{Mat}(d \times d)^3 / \, GL(d) \times GL(d) \dashrightarrow \mathfrak{M}(d, 0), \tag{18}$$

which is literally unchanged from the commutative situation.

4. Weyl group action on parabolic sheaves

4.1. Our goal in this section is to examine the action of Hecke correspondences (5) on 1-dimensional sheaves M on quantum planes. For this, the language of parabolic sheaves will be convenient.

In what follows, we assume $\mathsf{M} \in \operatorname{Tails}\mathsf{A}$ is stable 1-dimensional without E-torsion. This means the sequence

$$0 \to \mathsf{M}(-3) \xrightarrow{\mathsf{E}} \mathsf{M} \to \mathsf{M}/\mathsf{EM} \to 0 \tag{19}$$

is exact, and comparing the Hilbert polynomials, we see M/EM has a filtration with $3\deg\mathsf{M}$ point quotients. A choice of such filtration

$$\mathsf{M} = \mathsf{M}_0 \supset \mathsf{M}_1 \supset \mathsf{M}_2 \supset \cdots \supset \mathsf{M}_{3\deg\mathsf{M}} = \mathsf{EM} \cong \mathsf{M}(-3)$$

is called a *parabolic* structure on M.

4.2. Moduli spaces $\mathfrak{P}(d, \chi)$ of parabolic sheaves may be constructed as in the commutative situation. The forgetful map

$$\mathfrak{P}(d, \chi) \dashrightarrow \mathfrak{M}(d, \chi)$$

is generically finite of degree $(3\deg\mathsf{M})!$ corresponding to the generic module M/EM being a direct sum of nonisomorphic point modules.

4.3. Given a parabolic module M, we denote by

$$\partial\mathsf{M} = (\mathsf{M}_0/\mathsf{M}_1, \ldots, \mathsf{M}_{3d-1}/\mathsf{M}_{3d}) \in E^{3d}$$

the isomorphism class of its point factors.

In the commutative case, the sum of $\partial\mathsf{M}$ in $\operatorname{Pic} E$ equals $\mathcal{O}(\deg M)$. The analogous noncommutative statement reads:

Proposition 1. *Let* M *be* 1*-dimensional and have no* E*-torsion. Then*

$$\sum_{p\in\partial\mathsf{M}} p = \mathcal{O}(\deg \mathsf{M}) + 3(\chi(\mathsf{M}) - \deg \mathsf{M})\tau \in \operatorname{Pic}_{3d} E. \tag{20}$$

Here we identify the automorphism τ with the element $\tau(p) - p \in \operatorname{Pic}_0 E$. This does not depend on the choice of $p \in E$.

Proof. Let

$$\cdots \to F_1 \to F_0 \to \mathsf{M} \to 0$$

be a graded free resolution of a module M. The cohomology groups of

$$\cdots \to \mathsf{B} \otimes F_1 \to \mathsf{B} \otimes F_0 \to 0$$

are, by definition, the groups $\operatorname{Tor}^i(\mathsf{B}, \mathsf{M})$. The class of the Euler characteristic

$$[\operatorname{Tor}(\mathsf{B}, \mathsf{M})] = \sum (-1)^i [\operatorname{Tor}^i(\mathsf{B}, \mathsf{M})] \in K(\mathsf{B}) \cong K(E)$$

may be computed using only the K-theory class of M. In fact,

$$c_1([\operatorname{Tor}(\mathsf{B}, \mathsf{M})]) = \mathcal{O}(\deg \mathsf{M}) + 3(\chi(\mathsf{M}) - \deg \mathsf{M} - \operatorname{rk} \mathsf{M})\tau.$$

It is enough to check this for $\mathsf{M} = \mathsf{A}(k)$, which follows from (16).

Alternatively, the groups $\operatorname{Tor}^i(\mathsf{B}, \mathsf{M})$ may be computed from a free resolution of B. From (19), we find

$$\operatorname{Tor}^0(\mathsf{B}, \mathsf{M}) = \mathsf{M}/\mathsf{E}\mathsf{M},$$

while all higher ones vanish. The proposition follows. □

4.4. Let

$$\mathsf{W} = S(3d) \ltimes \mathbb{Z}^{3d}$$

be the extended affine Weyl group of $GL(3d)$. Weyl group actions on moduli of parabolic objects is a classic of geometric representation theory. In our context, the lattice subgroup may be interpreted as

$$\mathbb{Z}^{3d} \cong \operatorname{Pic} \operatorname{Bl}_{p_1,\ldots,p_{3d}} \mathbb{P}^2 / \operatorname{Pic} \mathbb{P}^2,$$

while $S(3d)$ acts on it by monodromy as the centers of the blowup move around.

The group W is generated by reflections $s_0, \dots, s_{3d-1}$ in the hyperplanes

$$\{a_{3d} = a_1 - 1\}, \{a_1 = a_2\}, \dots, \{a_{3d-1} = a_{3d}\}$$

together with the transformation

$$g \cdot (a_1, \dots, a_{3d}) \mapsto (a_2, \dots, a_{3d}, a_1 - 1).$$

The involutions s_i satisfy the Coxeter relations

$$(s_i s_{i+1})^3 = 1$$

of the affine Weyl group of $GL(3d)$ while g acts on them as the Dynkin diagram automorphism

$$g s_i g^{-1} = s_{i-1}.$$

Here and above the indices are taken modulo $3d$.

4.5. On the open locus where

$$\mathsf{M}/\mathsf{EM} = \bigoplus_{i=1}^{3d} \mathbb{O}_{p_i}$$

and all p_i are distinct, the symmetric group $S(3d)$ acts on parabolic structures by permuting the factors.

This extends to a birational action of $S(3d)$ on $\mathfrak{P}(d, \chi)$. The closure of the graph of s_k may be described as the nondiagonal component of the correspondence

$$\{(\mathsf{M}, \mathsf{M}') \mid \mathsf{M}_i = \mathsf{M}'_i,\ i \neq k\} \subset \mathfrak{P} \times \mathfrak{P}.$$

4.6. We define

$$g \cdot \mathsf{M} = \mathsf{M}_1$$

with the parabolic structure

$$\mathsf{M}_1 \supset \cdots \supset \mathsf{M}_{3d} \supset \mathsf{EM}_1,$$

where, as before, we assume that M has no E-torsion. This gives a birational map

$$g : \mathfrak{P}(d, \chi) \to \mathfrak{P}(d, \chi - 1).$$

4.7. We make W act on E^{3d} by

$$\mathbb{Z}^{3d} \ni (a_1, a_2, \dots) \mapsto (\tau^{3a_1}, \tau^{3a_2}, \dots) \in \operatorname{Aut} E^{3d} \tag{21}$$

while $S(3d)$ permutes the factors. Then we have

$$\partial \mathsf{EM} = \partial \mathsf{M}(-3) = (-1, \dots, -1) \cdot \partial \mathsf{M}. \tag{22}$$

Theorem 1. *The transformations $s_1, \dots, s_{3d-1}$ and g generate an action of* W *by birational transformations of* $\bigsqcup_\chi \mathfrak{P}(d, \chi)$. *The map*

$$\partial : \mathfrak{P}(d, \chi) \to E^{3d}$$

is equivariant with respect to this action.

Proof. Clearly, $s_1, \dots, s_{3d-1}$ generate the symmetric group $S(3d)$, as do their conjugates under the action of g. Setting

$$s_0 = g s_1 g^{-1},$$

one sees that g permutes $s_0, \dots, s_{3d-1}$ cyclically, verifying all relations in Λ. Equivariance of ∂ follows from (22). □

4.8. Evidently, the $\mathsf{A}[\mathsf{E}^{-1}]$ module $\mathsf{M}[\mathsf{E}^{-1}]$ is not changed by the dynamics; that is to say, its isomorphism class is an invariant of the dynamics. From a dynamical viewpoint, however, this is not very useful information since no reasonable moduli space for $\mathsf{A}[\mathsf{E}^{-1}]$-modules exists, which is just another way of stating the fact that generic orbits of our dynamical system are dense in the analytic topology.

Local analytic integrals of the dynamics may be constructed in this setting if a representation of the noncommutative algebra by linear difference operators is given. (This will be done in [Rains $\geq$ 2015].) The monodromy of the difference equation corresponding to a module is the required invariant. An important virtue of such local invariants is their convergence to algebraic invariants as the noncommutative deformation is removed, which is very useful, for example, for the study of averaging of perturbations.

5. A concrete description of the action

5.1. The goal of this section to make the action in Theorem 1 as explicit as possible. Consider the exact sequence

$$1 \to \mathsf{W}_0 \to \mathsf{W} \xrightarrow{\chi} \mathbb{Z} \to 1$$

where χ is the sum of entries on $\mathbb{Z}^{3d}$ and 0 on $S(3d)$. We have

$$\chi(w \cdot \mathsf{M}) = \chi(\mathsf{M}) + \chi(w),$$

so the subgroup W_0 acts on $\mathfrak{P}(d, \chi)$ for any $\chi \in \mathbb{Z}$. Since all of them are birational, we can focus on one, for example

$$X = \mathfrak{P}(d, d+1).$$

5.2. We will see that there is a diagram of maps, with birational top row,

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dr, "\partial"'] & S^g\mathbb{P}^2 \times E^{3d-1} \arrow[d] \\ & E^{3d-1} \end{tikzcd} \tag{23}$$

where $g = \binom{d-1}{2}$ is the genus of a smooth curve of degree d and $S^g\mathbb{P}^2$ parametrizes unordered collections $D \subset \mathbb{P}^2$ of g points. We view the product E^{3d-1} as embedded in E^{3d} via

$$E^{3d-1} = \left\{ \sum_{i=1}^{3d} p_i = \mathbb{O}(d) + 3\tau \right\} \subset E^{3d}. \tag{24}$$

This subset is W_0-invariant.

5.3. The action of W_0 has a particularly nice description in terms of (23), and it agrees with the action on E^d already defined in Section 4.7.

The symmetric group $S(3d)$ permutes the points $p_i \in E$ and does nothing to $D \subset \mathbb{P}^2$. It remains to define the action of the lattice generators

$$\alpha_{ij} = \delta_i - \delta_j \in \mathbb{Z}^{3d},$$

where $\delta_i = (0, \dots, 0, 1, 0, \dots, 0)$ form the standard basis of $\mathbb{Z}^n$. We claim

$$\alpha_{ij}(D, P) = (D', P')$$

where $P' = \alpha_{ij} P$ as in Section 4.7, while the points D' are found from the following construction.

Let $C \subset \mathbb{P}^2$ be the degree-d curve that meets D and $(-\delta_j) \cdot P$. Because of the condition (24), such a curve exists and is generically unique. The divisor $D' \subset C$ is found from

$$D + p_i = D' + p'_j \in \operatorname{Pic} C.$$

Again, since D' is of degree $g(C)$, generically, there is a unique effective divisor satisfying this equation.

Theorem 2. *This defines a birational action of* W_0 *that is birationally isomorphic to the action from Theorem 1.*

Remark. The case $d = 3$ of the above dynamical system was considered in [Kajiwara et al. 2006, §7] as a description of the elliptic Painlevé equation in terms of the arithmetic on a moving elliptic curve. Since we only consider this in terms of birational maps, this only shows that our dynamics is birationally equivalent to elliptic Painlevé; we will see in Section 6 that (for generic parameters) the description in terms of sheaves agrees *holomorphically* with elliptic Painlevé.

We break up the proof into a sequence of propositions.

5.4. A general point $\mathsf{M} \in X$ is of the form $\mathsf{M} = \operatorname{Coker} L$ where

$$L : \mathsf{A}(-1)^{d-1} \to \mathsf{A}(1) \oplus \mathsf{A}^{d-2};$$

see Section 3.6. Consider the submatrix

$$\bar{L} : \mathsf{A}(-1)^{d-1} \to \mathsf{A}^{d-2},$$

and let $\operatorname{div} \mathsf{M} \subset \mathbb{P}^2$ be the subscheme cut out by the maximal minors of $\bar{L}$. Generically,

$$\operatorname{div} \mathsf{M} = \binom{d-1}{2} \text{ distinct points.}$$

Proposition 2. *The map*

$$X \ni \mathsf{M} \mapsto (\operatorname{div} \mathsf{M}, \partial \mathsf{M}) \in S^g \mathbb{P}^2 \times E^{3d-1},$$

where $E^{3d-1} \subset E^{3d}$ *as in* (24), *is birational.*

Proof. The statement about $\sum p$ follows from (20). Since the source and the target have the same dimension, it is enough to show that the map has degree 1. For this, we may assume that A is commutative, in which case the claim is classically known; see, e.g., [Beauville 2000]. □

5.5. In fact, in the commutative case, M is generically a line bundle $\mathscr{L}$ on a smooth curve $C = \operatorname{supp} \mathsf{M}$. From its resolution, we see that $\mathscr{L}(-1)$ has a unique section. The divisor of this section on C is $\operatorname{div} \mathsf{M}$.

5.6. Similarly, a general point $\mathsf{M} \subset \mathfrak{P}(d, d)$ is of the form $\mathsf{M} = \operatorname{Coker} L$, where

$$L : \mathsf{A}(-1)^d \to \mathsf{A}^d$$

is a $d \times d$ matrix L of linear forms. The matrix L is unique up to left and right multiplication by elements of $GL(d)$. The description of this $GL(d) \times GL(d)$ quotient is classically known [Beauville 2000] and given by

$$L \mapsto (C, \mathscr{L})$$

where

$$C = \{\det L = 0\} \subset \mathbb{P}^2$$

is a degree-d curve cut out by the usual, commutative determinant and $\mathscr{L}$ is the cokernel of the commutative morphism, viewed as a sheaf on C. Generically, C is smooth, $\mathscr{L}$ is a line bundle, and

$$g(C) = \binom{d-1}{2}, \qquad \deg \mathscr{L} = g(C) - 1 + d.$$

Proposition 3. *The point modules $\mathcal{O}_p$ in $\partial\mathsf{M}$ correspond to points $p \in C \cap E$.*

Proof. Let $f_1, \ldots, f_d$ be the images in M of the generators of A^d. We may assume that $f_2, \ldots, f_d$ are in the kernel of $\mathsf{M} \to \mathcal{O}_p$. The coefficient of f_1 in any relation among the f_i must belong to $\mathsf{A}p^\perp$, whence the claim. $\square$

5.7. Now suppose $\mathsf{M} \in \mathfrak{P}(d, d+1)$ and $p \in \partial\mathsf{M}$, and define

$$\mathsf{M}_p = \operatorname{Ker}(\mathsf{M} \to \mathcal{O}_p) = \mathfrak{m}_p\mathsf{M}.$$

Then $\mathsf{M}_p \in \mathfrak{P}(d, d)$, and we denote by $(C_p, \mathcal{L}_p)$ the curve and the line bundle that correspond to M_p.

Proposition 4. *The curve C_p meets* $\operatorname{div}\mathsf{M}$.

Proof. Let f and $g_1, \ldots, g_{d-2}$ be the images in M of the generators of $\mathsf{A}(1)$ and A^{d-2}, respectively. We may chose them so that all g_i are mapped to 0 in $\mathcal{O}_p$. Let

$$0 \to \mathsf{A}(-1) \xrightarrow{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} \mathsf{A}^{\oplus 2} \xrightarrow{\begin{bmatrix} l_1 & l_2 \end{bmatrix}} \mathsf{A}(1) \to \mathcal{O}_p \to 0$$

be a free resolution. All relations in M must be of the form

$$(r_1 l_1 + r_2 l_2) f + \sum s_i g_i = 0, \quad r_1, r_2, s_i \in \mathsf{A}_1.$$

There are $d-1$ linearly independent relations like this, and the coefficients s_i in them form the matrix $\bar{L}$.

We observe that $l_1 f, l_2 f, g_1, \ldots, g_{d-2}$ generate M_p, and for this presentation, the matrix L_p has the block form

$$L_p = \begin{bmatrix} u & r \\ 0 & \bar{L} \end{bmatrix}. \tag{25}$$

The proposition follows. $\square$

5.8. Denote

$$\partial\mathsf{M} = (p, p_2, \ldots, p_{3d}).$$

Since $\mathsf{M}_p = \mathfrak{m}_p\mathsf{M}$ still surjects to $\mathcal{O}_{p_i}$, $i \geq 2$, we have

$$\partial\mathsf{M}_p = (p', p_2, \ldots, p_{3d})$$

for some $p' \in E$ and from (20) we see that

$$p' = \tau^{-3}(p).$$

From the proof of Proposition 4, we note that

$$p' = \{u_1 = u_2 = 0\}.$$

5.9. The following proposition concludes the proof of Theorem 2.

Proposition 5. *We have* $\mathscr{L}_p = \operatorname{div} \mathsf{M} + \mathbb{O}(1) - p'$ *in* $\operatorname{Pic} C_p$.

Proof. This is a purely commutative statement, in fact, just a restatement of the remark in Section 5.5. $\square$

6. The elliptic Painlevé equation

Let us consider the case $d = 3$ in more detail. In this case, we can be fairly explicit about the moduli space, at least for sufficiently general parameters. We suppose that the $3d = 9$ points of $\partial \mathsf{M}$ are distinct (and ordered) so that specifying a point in $\mathfrak{P}(3, \chi)$ is equivalent to specifying the corresponding point in $\mathfrak{M}(3, \chi)$. We consider the case $\chi = 1$ as in that case we need to consider only one shape of presentation. (Of course, this also holds for $\chi = -1$ by duality; the case $\chi = 0$ is somewhat trickier, though the calculation below of the action of the Hecke modifications implies a similar description for that moduli space.)

We naturally restrict our attention to semistable sheaves and note that a sheaf in $\mathfrak{M}(3, 1)$ is semistable if and only if it is stable, which holds if and only if it has no proper subsheaf with positive Euler characteristic.

Lemma 2. *Suppose the sheaf* $\mathsf{M} \in \mathfrak{M}(3, 1)$ *has a free resolution of the form*

$$0 \to \mathsf{A}(-2)^2 \to \mathsf{A}(-1) \oplus \mathsf{A} \to \mathsf{M} \to 0$$

or in other words is generated by elements f *and* g *of degrees* 1 *and* 0 *satisfying relations*

$$v_1 f + w_1 g = v_2 f + w_2 g = 0,$$

with $v_1, v_2 \in A_1$ *and* $w_1, w_2 \in A_2$. *If* M *is stable, then* v_1 *and* v_2 *are linearly independent and there is no element* $x \in A_1$ *such that* $v_1 x = w_1$ *and* $v_2 x = w_2$. *Conversely, the cokernel of any morphism* $\mathsf{A}(-2)^2 \to \mathsf{A}(-1) \oplus \mathsf{A}$ *satisfying these conditions is a stable sheaf* $\mathsf{M} \in \mathfrak{M}(3, 1)$.

Proof. If v_1 and v_2 are not linearly independent, then without loss of generality we may assume $v_2 = 0$. But then $w_2 \neq 0$ (by injectivity) and thus the submodule generated by the image of A is the cokernel of the map $w_2 : \mathsf{A}(-2) \to \mathsf{A}$ and has Euler characteristic 1, violating stability.

Similarly, if the second condition is violated, then we may replace f by $f - xg$ and thus eliminate the dependence of the relations on g. But then A is a direct summand of M, violating the condition that M have rank 0.

For the converse, note first that, if the map is not injective, then there is in particular a nonzero morphism $\mathsf{A}(-d) \to \mathsf{A}(-2)^2$ exhibiting the failure of injectivity. In particular, the composition

$$0 \to \mathsf{A}(-d) \to \mathsf{A}(-2)^2 \to \mathsf{A}(-1)$$

must be 0. Since by assumption v_1 and v_2 are linearly independent, the existence of a kernel implies that v_1 and v_2 have a single point in common. The kernel of the map $\begin{bmatrix} v_1 & v_2 \end{bmatrix} : \mathsf{A}(-2)^2 \to \mathsf{A}(-1)$ is thus isomorphic to $\mathsf{A}(-3)$ (since point sheaves have resolutions of this form). As the map from $\mathsf{A}(-d)$ factors through this kernel, we may as well take $d = 3$. But then the dual argument shows that the map $\begin{bmatrix} w_1 & w_2 \end{bmatrix}$ must factor through $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$, contradicting our second condition.

Thus, in particular, the conditions ensure that the cokernel is in $\mathfrak{M}(3, 1)$, and it remains to show stability. Note that any destabilizing subsheaf has positive Euler characteristic and thus has a map from A. Moreover, since it has degree < 3, it must be globally generated so that, if M is unstable, the subsheaf generated by g is destabilizing. Furthermore, to have degree < 3, g must in particular satisfy a relation $wg = 0$, implying that v_1 and v_2 are linearly dependent, giving a contradiction. □

Remark. One can show that every stable sheaf in $\mathfrak{M}(3, 1)$ must have a presentation of the above form, but for present purposes, we simply restrict our attention to the corresponding open subset of the stable moduli space, which by the following proof is projective.

Theorem 3. *Suppose $p_1, \dots, p_9$ is a sequence of 9 distinct points of E such that $p_1 + \cdots + p_9 = \mathbb{O}(3) - 6\tau$. Then the moduli space of stable sheaves in $\mathfrak{M}(3, 1)$ with $\mathsf{M}|_E \supset (p_1, \dots, p_9)$ is canonically isomorphic to the blowup of $\mathbb{P}^2$ in the images of $p_1, \dots, p_9$ under the embedding $E \to \mathbb{P}^2$ coming from $\mathscr{L}_1 \sim \mathbb{O}(1) - 3\tau$.*

Proof. We may view the coefficients v_1, v_2, w_1, and w_2 as global sections of line bundles on E; to be precise,

$$v_1, v_2 \in \mathrm{Hom}(\mathsf{A}(-2), \mathsf{A}(-1)) \cong \mathrm{Hom}(\mathsf{B}(-2), \mathsf{B}(-1)) \cong H^0(\mathscr{L}_1)$$

and

$$w_1, w_2 \in \mathrm{Hom}(\mathsf{A}(-2), \mathsf{A}) \cong \mathrm{Hom}(\mathsf{B}(-2), \mathsf{B}) \cong H^0(\mathscr{L}_0 \otimes \mathscr{L}_1).$$

Now, $H^0(\mathscr{L}_1)$ is 3-dimensional, and v_1 and v_2 are linearly independent by stability, and we thus obtain a morphism from the moduli space of stable sheaves with the above presentation to $\mathbb{P}^2$. Note also that, in this identification, the constraint on $\partial\mathsf{M}$ reduces to a requirement that

$$w_1 v_2 - w_2 v_1 \in H^0((\mathscr{L}_0 \otimes \mathscr{L}_1) \otimes \mathscr{L}_1)$$

vanish at $p_1, \dots, p_9$.

We need to show that this morphism has 0-dimensional fibers except over the points $p_1, \dots, p_9$, where the fiber is $\mathbb{P}^1$; this together with smoothness will imply the identification with the blowup.

Note that a point $p \in E$ corresponds to the subspace

$$H^0(\mathscr{L}_1(-p)) \subset H^0(\mathscr{L}_1)$$

and thus the cases to consider are those in which v_1 and v_2 have no common zero, those in which they have a single common zero not of the form p_i, and those in which they have a common zero at p_i. The key fact is the following statement about global sections of line bundles on elliptic curves.

Lemma 3. *Let v_1 and v_2 be linearly independent global sections of $\mathscr{L}_1$. Then the map*

$$H^0(\mathscr{L}_0 \otimes \mathscr{L}_1)^2 \to H^0(\mathscr{L}_0 \otimes \mathscr{L}_1^2)$$

given by $(w_1, w_2) \mapsto v_2 w_1 - v_1 w_2$ is surjective if v_1 and v_2 have no common zero and otherwise has image of codimension 1, *consisting of those global sections vanishing at said common zero.*

Proof. Consider the complex

$$0 \to H^0(\mathscr{L}_0) \to H^0(\mathscr{L}_0 \otimes \mathscr{L}_1)^2 \to H^0(\mathscr{L}_0 \otimes \mathscr{L}_1^2) \to 0,$$

with the left map being $x \mapsto (v_1 x, v_2 x)$. This has Euler characteristic $3-6-6+9=0$ and is exact on the left, so it will suffice to understand the middle cohomology. Now, the kernel of the above determinant map consists of pairs (w_1, w_2) with $v_2 w_1 = v_1 w_2$. Assuming neither w_1 nor w_2 is 0 (which would clearly imply $w_1 = w_2 = 0$), we find that

$$\operatorname{div} v_2 + \operatorname{div} w_1 = \operatorname{div} v_1 + \operatorname{div} w_2.$$

If v_1 and v_2 have no common zero, we conclude that

$$\operatorname{div} w_1 - \operatorname{div} v_1 = \operatorname{div} w_2 - \operatorname{div} v_2$$

is an effective divisor, and thus,

$$w_1/v_1 = w_2/v_2 \in H^0(\mathscr{L}_0).$$

But this gives exactness in the middle.

Similarly, if v_1 and v_2 both vanish at p, then the same reasoning shows that

$$w_1/v_1 = w_2/v_2 \in H^0(\mathscr{L}_0(p)).$$

In particular, we find that the middle cohomology has dimension at most 1; since the right map is clearly not surjective in this case, its image must therefore have codimension 1 as required. □

In particular, if v_1 and v_2 have no common zero, the map from pairs (w_1, w_2) to the corresponding determinant is surjective, and thus, there is a unique pair (w_1, w_2) up to equivalence compatible with the constraints on $\partial\mathsf{M}$. If v_1 and v_2 have a common zero not of the form p_i, then the map fails to be surjective, and the only allowed determinant is 0. We thus obtain a 1-dimensional space of possible pairs (modulo multiples of (v_1, v_2)), giving rise to a single equivalence class of stable sheaves. Finally, if v_1 and v_2 have a common zero at p_i, then the 1-dimensional space of allowed determinants pulls back to a 2-dimensional space of pairs (w_1, w_2) modulo multiples of (v_1, v_2) and thus gives rise to a $\mathbb{P}^1$-worth of stable sheaves.

It remains to show smoothness. The tangent space to a sheaf with presentation $v_1 f + w_1 g = v_2 f + w_2 g = 0$ consists of the set of quadruples (v_1', v_2', w_1', w_2') such that

$$v_1' w_2 - v_2' w_1 + v_1 w_2' - v_2 w_1'$$

vanishes at $p_1, \ldots, p_9$. (More precisely, it is the quotient of this space by the space of trivial deformations, induced by infinitesimal automorphisms of $\mathsf{A}(-2)^2$ and $\mathsf{A}(-1) \oplus \mathsf{A}$; stability implies that the trivial subspace has dimension $4+5-1=8$, independent of M.) It will thus suffice to show that this surjects onto $H^0(\mathscr{L}_0 \otimes \mathscr{L}_1^2)$ since then the dimension will be independent of M (and equal to $18-8-(9-1)=2$). If v_1 and v_2 have no common zero, this follows directly from the lemma. If they have a common zero at p, but $v_1 w_2 - v_2 w_1 \neq 0$, then since p is equal to at most one p_i, we conclude that one of w_1 or w_2 must not vanish at p, so again the lemma gives surjectivity.

Finally, if v_1, v_2, w_1, and w_2 all vanish at p, then $v_1 w_2 - v_2 w_1 = 0$. But then we again find as in the lemma that

$$w_1/v_1 = w_2/v_2 \in H^0(\mathscr{L}_0),$$

and this violates stability. □

Remark. Presumably this result could be extended to remove the constraint that the base points are distinct, except that the blowup will no longer be smooth since the base locus of the blowup is then singular.

We also wish to understand how the action of the affine Weyl group $\Lambda_0 \cong \tilde{A}_8$ translates to this explicit description of the moduli space. The action of S_9 is essentially trivial as this simply permutes the points $p_1, \ldots, p_9$. It remains to consider the generator s_0. This can be performed in two steps: first shift down p_1 to obtain a sheaf with Euler characteristic 0 and $\partial \mathsf{M} = (p_2, \ldots, p_9, \tau^{-3}(p_1))$, and then shift up p_9 to obtain a sheaf with Euler characteristic 1 and $\partial \mathsf{M} = (\tau^3(p_9), p_2, \ldots, p_8, \tau^{-3}(p_1))$ as required.

Let $\mathbb{O}_{p_1}$ have the free resolution

$$0 \to \mathsf{A} \xrightarrow{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} \mathsf{A}(-1)^{\oplus 2} \xrightarrow{\begin{bmatrix} l_1 & l_2 \end{bmatrix}} \mathsf{A} \to \mathbb{O}_{p_1} \to 0.$$

Then there are two cases to consider. If $\langle v_1, v_2 \rangle \neq \langle l_1, l_2 \rangle$, then we may choose our generators f and g of M in such a way that g maps to 0 in $\mathbb{O}_{p_1}$. Then we have relations of the form

$$(r_{11}l_1 + r_{21}l_2) f + v_1 g = (r_{21}l_1 + r_{22}l_2) f + v_2 g = 0,$$

and the submodule generated by $l_1 f$, $l_2 f$, and g has a presentation of the form

$$0 \to \mathsf{A}(-2)^3 \xrightarrow{L} \mathsf{A}(-1)^3 \to \mathsf{M}_{p_1}$$

with

$$L = \begin{bmatrix} u_1 & r_{11} & r_{12} \\ u_2 & r_{21} & r_{22} \\ 0 & v_1 & v_2 \end{bmatrix}.$$

Generically, $\det L$ has divisor $\tau^{-3}(p_1) + p_2 + \cdots + p_9$ and thus has rank 2 at p_9. Then suitable row and column operations recover a new matrix of the above form except with u_1' and u_2' vanishing at p_9; thus, $\mathsf{M}_{p_1} \cong \mathsf{M}'_{\tau^3(p_9)}$ for suitable M' as required.

This can fail in only two ways: either the rank of $L(p_9)$ could be smaller than 2, or the corresponding column could have rank only 1. Suppose first that $\det L \neq 0$. As long as $\tau^{-3}(p_1) \neq p_9$, we find that p_9 is a simple zero of $\det L$, so the rank cannot drop below 2. If after the row and column operations we find that u_1' and u_2' are linearly dependent, then we find that $\det L$ must factor as a product uv with $u \in H^0(\mathscr{L}_1(-p_9))$ and $v \in H^0(\mathscr{L}_1^2)$. In particular, this can only happen if we have

$$\mathscr{L}_1(p_i + p_j + p_9) \cong \mathcal{O}_E$$

for some $2 \leq i < j \leq 8$ or

$$\mathscr{L}_1(\tau^{-3} p_1 + p_i + p_9) \cong \mathcal{O}_E$$

for some $2 \leq i \leq 8$.

If $\det L = 0$ on E, we may consider the cubic polynomial obtained by viewing L as a matrix over $\mathbb{P}^2$, and observe that this must be the equation of E. In particular, since E is smooth, we still cannot have rank < 2 at any point, and since E is irreducible, L cannot be made block-upper-triangular.

We thus conclude that, as long as the points

$$\tau^{-3}(p_1), p_1, p_2, \ldots, p_9, \tau^3(p_9)$$

are all distinct and no three add to a divisor representing $\mathscr{L}_1$, then s_0 induces a morphism between the complement of the fiber over p_1 in the original moduli space and the complement of the fiber over $\tau^3(p_9)$ in the new moduli space.

It remains to consider the fiber over p_1. In this case, we note that f maps to 0 in $\mathcal{O}_{p_1}$, and thus, we can no longer proceed as above. In a suitable basis, the relations of M now read

$$r_1 f + l_1 g = r_2 f + l_2 g = 0,$$

and thus, M_{p_1} is generated by f, with the single relation

$$(u_1r_1 + u_2r_2)f = 0.$$

In particular, we obtain a sheaf with presentation of the form

$$0 \to \mathsf{A}(-3) \to \mathsf{A} \to \mathsf{M}_{p_1} \to 0.$$

It remains only to show that every sheaf with such a presentation arises in this way and that we can recover the original sheaf from the presentation.

Lemma 4. *Let $p \in E$ be any point, cut out by linear equations $l_1 = l_2 = 0$. Then the central element $\mathsf{E} \in \mathsf{A}_3$ can be expressed in the form*

$$\mathsf{E} = l_1 f_1 + l_2 f_2$$

with $f_1, f_2 \in \mathsf{A}_2$, and this expression is unique up to adding a pair

$$(v_1 g, v_2 g)$$

to (f_1, f_2), where $v_1, v_2, g \in \mathsf{A}_1$ satisfy $l_1 v_2 + l_2 v_2 = 0$.

Proof. Consider the map

$$\mathsf{A}_2^2 \to \mathsf{A}_3$$

given by $(f_1, f_2) \mapsto l_1 f_1 + l_2 f_2$. Since (l_1, l_2) cuts out a point module, the image of this map must be codimension-1, and since E annihilates every point module, the image contains E. Uniqueness follows by dimension-counting since the specified kernel is 3-dimensional. $\square$

We conclude that, for any element of Λ_0, there is a finite collection of linear inequalities on the base points, which guarantee that both the domain and range are blowups of $\mathbb{P}^2$, and the element of Λ_0 acts as a morphism. In particular, we see that it suffices to have

$$\tau^{3k} p_i \neq p_j$$

for $i < j$, $k \in \mathbb{Z}$, and

$$\mathscr{L}_{3l+1} \not\cong \mathscr{O}_E(p_i + p_j + p_k)$$

for $i < j < k$, $l \in \mathbb{Z}$, in order for the entire group Λ_0 to act as isomorphisms between blowups of $\mathbb{P}^2$.

In particular, we find that the translation subgroup of Λ_0 acts in the same way as the translation subgroup of $\widetilde{E}_8$ in Sakai's description of the elliptic Painlevé equation. Note that both groups have the same rank, and by comparing determinants under the intersection form in the commutative limit, we conclude that the translation subgroup of Λ_0 has index 3 in the translation subgroup of $\widetilde{E}_8$. It is straightforward to see that we can generate the entire lattice by including the operation

$$\mathsf{M} \mapsto g^3 \mathsf{M}(1),$$

though it is more difficult to see how this acts in terms of presentations of sheaves.

We note in passing that the above calculation of M_{p_i} shows that the -1-curve corresponding to the fiber over $p_i \in \mathbb{P}^2$ can be described as the subscheme of moduli space where the sheaf M_{p_i} of Euler characteristic 0 has a global section; this should be compared to the cohomological description of τ-divisors in [Arinkin and Borodin 2009]. In fact, every -1-curve on the moduli space has a similar description: act by a suitable element of Λ_{E_8}, and then ask for the Hecke modification at p_1 to have a global section.

We also note that the results of [Rains 2013b, Theorem 7.1] suggest that one should consider the moduli space of stable sheaves M on A such that $h(\mathsf{M}) = 3rt + r$ and

$$\mathsf{M}|_E = (\mathcal{O}_{p_1} \oplus \cdots \oplus \mathcal{O}_{p_9})^r,$$

where now $p_1 + \cdots + p_9 - \mathcal{O}(3) + 6\tau$ is a torsion point of order r. This moduli space remains 2-dimensional and is expected to again be a 9-point blowup of $\mathbb{P}^2$. (A variant of this will be considered in [Rains $\geq$ 2015].)

7. Poisson structures

7.1. The Poisson structure on the moduli space of sheaves on a commutative Poisson surface has a purely categorical definition (originally constructed by [Tyurin 1988], shown to satisfy the Jacobi identity for vector bundles in [Bottacin 1995], and extended to general sheaves of homological dimension 1 in [Hurtubise and Markman 2002]). This definition can be carried over to the noncommutative case, and we will see that the analogous bivector again gives a Poisson structure, and the Hecke modifications again act as symplectomorphisms. The main qualitative difference in the noncommutative case is (as we have seen) that the Hecke modifications are no longer automorphisms of a given symplectic leaf but rather give maps between related symplectic leaves.

Tyurin's construction in the commutative setting relies on the observation that the tangent space at a sheaf M on a Poisson surface X, or equivalently the space of infinitesimal deformations, is given by the self-Ext group

$$\mathrm{Ext}^1(M, M).$$

By Serre duality, the cotangent space is given by

$$\mathrm{Ext}^1(M, M \otimes \omega_X).$$

This globalizes to general sheaves such that $\dim \operatorname{End} M = 1$; the cotangent sheaf on the moduli space is given by $\operatorname{Ext}^1(M, M \otimes \omega_X)$, where M is the universal sheaf on the moduli space. A nontrivial Poisson structure on X corresponds to a nonzero morphism $\wedge^2\Omega_X \to \mathcal{O}_X$ or equivalently to a nonzero morphism $\alpha : \omega_X \to \mathcal{O}_X$. We thus obtain a map

$$\operatorname{Ext}^1(M, M \otimes \omega_X) \xrightarrow{1\otimes\alpha} \operatorname{Ext}^1(M, M)$$

and a bilinear form on $\operatorname{Ext}^1(M, M \otimes \omega_X)$. One can then show [Bottacin 1995; Hurtubise and Markman 2002] that this bilinear form induces a Poisson structure. In addition, the resulting Poisson variety has a natural foliation by *algebraic* symplectic leaves: if C_α is the curve $\alpha = 0$, then for any sheaf M_α on C_α, the (Poisson) subspace of sheaves M with $M\otimes\mathcal{O}_{C_\alpha} \cong M_\alpha$ and $\operatorname{Tor}_1(M, \mathcal{O}_{C_\alpha}) = 0$ is a smooth symplectic leaf.

In our noncommutative setting, there is again an analogue of Serre duality; one finds that $H^2(\mathsf{A}(-3)) \cong \mathbb{C}$ (just as in the commutative case), and for any M and M', we have canonical pairings

$$\langle -, -\rangle : \operatorname{Ext}^i(\mathsf{M}', \mathsf{M}(-3)) \otimes \operatorname{Ext}^{2-i}(\mathsf{M}, \mathsf{M}') \to H^2(\mathsf{A}(-3)).$$

Moreover, these pairings are (super)symmetric in the following sense. If $\alpha \in \operatorname{Ext}^i(\mathsf{M}', \mathsf{M}(-3))$ and $\beta \in \operatorname{Ext}^{2-i}(\mathsf{M}, \mathsf{M}')$, then

$$\langle\alpha, \beta\rangle = (-1)^i \langle\beta(-3), \alpha\rangle = (-1)^i \langle\beta, \alpha(3)\rangle.$$

In addition, the pairing factors through the Yoneda product in that

$$\langle\alpha, \beta\rangle = \langle\alpha \cup \beta, 1\rangle =: \operatorname{tr}(\alpha \cup \beta).$$

As in the commutative case, infinitesimal deformations of M are classified by $\operatorname{Ext}^1(\mathsf{M}, \mathsf{M})$, and the map

$$\operatorname{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \xrightarrow{E\cup -} \operatorname{Ext}^1(\mathsf{M}, \mathsf{M})$$

induces a skew-symmetric pairing

$$\operatorname{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \otimes \operatorname{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \xrightarrow{\operatorname{tr}(-\cup E\cup -)} H^2(\mathsf{A}(-3)) \cong \mathbb{C},$$

and this should be our desired Poisson structure.

Note here that the Poisson structure depends on the choice of E and the choice of automorphism $H^2(\mathsf{A}(-3)) \cong \mathbb{C}$; both are unique up to a scalar, and only the product of the scalars matters. The cohomology long exact sequence associated with

$$0 \to \mathsf{A}(-3) \xrightarrow{E} \mathsf{A} \to \mathsf{B} \to 0$$

induces (since $H^1(\mathsf{A}) = H^2(\mathsf{A}) = 0$) an isomorphism

$$H^1(\mathsf{B}) \cong H^2(\mathsf{A}(-3)),$$

depending linearly on E, so that the composition

$$H^1(\mathsf{B}) \cong H^2(\mathsf{A}(-3)) \cong \mathbb{C}$$

scales in the same way as the Poisson structure. In other words, the scalar freedom in the Poisson structure corresponds to a choice of isomorphism $H^1(\mathsf{B}) \cong \mathbb{C}$; the canonical equivalence $\operatorname{Tails} \mathsf{B} \cong \operatorname{Coh} E$ turns this into an isomorphism $H^1(\mathbb{O}_E) \cong \mathbb{C}$ or equivalently a choice of nonzero holomorphic differential on E.

We will see that this construction remains Poisson in the noncommutative setting and that the description of the symplectic leaves carries over mutatis mutandis. Note that, in this section, we will refer to the "moduli space of simple sheaves on A", where a sheaf is *simple* if $\operatorname{End} \mathsf{M} = \mathbb{C}$ (in the commutative setting, this is a weakened form of the constraint that a sheaf is stable). One expects following [Altman and Kleiman 1980] that this should be a quasiseparated algebraic space $\mathfrak{M}_{\mathsf{A}}$. Per [Rains 2013a], a Poisson structure on such a space is just a compatible system of Poisson structures on the domains of étale morphisms to the space; in the moduli space setting, we must thus assign a Poisson structure to every formally universal family of simple sheaves on A. The above bivector is clearly compatible so will be Poisson if and only if it is Poisson on every formally universal family. Any statement below about $\mathfrak{M}_{\mathsf{A}}$ should be interpreted as a statement about formally universal families in this way.

We will sketch two proofs of the following result below.

Theorem 4. *The above construction defines a Poisson structure on* $\mathfrak{M}_{\mathsf{A}}$, *and on the open subspace of sheaves transverse to* E (*i.e.*, *such that* $\operatorname{Tor}_1(\mathsf{M}, \mathsf{B}) = 0$), *the fibers of the map* $\mathsf{M} \to \mathsf{M} \otimes \mathsf{B} \in \operatorname{Coh} E$ *are unions of* (*smooth*) *symplectic leaves of this Poisson structure.*

Remark. One expects that, as in [Rains 2013a], one should have a covering by algebraic symplectic leaves even without the transversality assumption; in general, the symplectic leaves should be the preimages of the *derived* restriction $\mathsf{M} \to \mathsf{M} \otimes^{\mathsf{L}} \mathsf{B}$, taking sheaves on A to the derived category of $\operatorname{Coh} E$.

We should note here that, in the case of M torsion-free (and stable), an alternate construction of a Poisson structure was given in [Nevins and Stafford 2007]; their Poisson structure is presumably a constant multiple of the Tyurin-style Poisson structure.

7.2. Although the above construction is somewhat difficult to deal with computationally (but see below), it has significant advantages in terms of functoriality.

In particular, it is quite straightforward to show that Hecke modifications give symplectomorphisms on the relevant symplectic leaves. Curiously, the argument ends up depending crucially on noncommutativity!

With an eye to future applications, we consider a generalization of Hecke modifications as follows. Let M be a simple 1-dimensional sheaf on A. We define the "downward pseudotwist" at $p \in E$ of M to be the kernel of the natural map $\mathsf{M} \to \mathsf{M} \otimes \mathcal{O}_p$; similarly, the "upward pseudotwist" is the universal extension of $\mathcal{O}_p \otimes \operatorname{Ext}^1(\mathcal{O}_p, \mathsf{M})$ by M. If the restriction $\mathsf{M}|_E$ of M to E (i.e., $\mathsf{M} \otimes \mathsf{B}$, viewed as a sheaf on E) is not equal to the sum over p of $\mathsf{M} \otimes \mathcal{O}_p$, then one could consider some other natural modifications along these lines; in the commutative case, these correspond to twists by line bundles on iterated blowups in which we have blown up the same point on E multiple times. These will always be limits of the above operations so will again be symplectic by the limiting argument considered below.

Proposition 6. *The two pseudotwists define (inverse) birational maps between symplectic leaves of the open subspace of $\mathfrak{M}_{\mathsf{A}}$ classifying 1-dimensional sheaves transverse to E. Where the maps are defined, they are symplectic.*

Remark. Note that we need to merely prove that the morphisms preserve the above bivector; this can be verified independently of whether the bivector satisfies the Jacobi identity. In addition, it suffices to prove that the pseudotwists are Poisson on suitable open subsets of the moduli space.

Proof. We first consider the downward pseudotwist M' of M, corresponding to the short exact sequence

$$0 \to \mathsf{M}' \to \mathsf{M} \to \mathsf{M} \otimes \mathcal{O}_p \to 0.$$

We impose the additional conditions that

$$\operatorname{Hom}(\mathsf{M}', \mathcal{O}_p) = \operatorname{Hom}(\mathsf{M}, \mathcal{O}_p(-3)) = 0.$$

Observe that this is really just a condition on the commutative sheaf $\mathsf{M}|_E$, stating that it is 0 near $\tau^{-3}(p)$ and near p is a sum of copies of $\mathcal{O}_p$. Indeed, the first condition is precisely that $\operatorname{Hom}(\mathsf{M}, \mathcal{O}_p(-3)) = 0$ and implies $\operatorname{Tor}_1(\mathsf{M}, \mathsf{B}) = 0$ while the second condition follows from the 4-term exact sequence

$$0 \to (\mathsf{M} \otimes \mathcal{O}_p)(-3) \to \mathsf{M}'|_E \to \mathsf{M}|_E \to \mathsf{M} \otimes \mathcal{O}_p \to 0.$$

Note that, if $\tau^{-3}(p) = p$, then the above conditions imply $\mathsf{M}' \cong \mathsf{M}$ and thus eliminate any interesting examples of pseudotwists. Of course, since E is smooth, $\tau^{-3}(p) = p$ if and only if $\tau^3 = 1$, and this is equivalent to the existence of an equivalence $\operatorname{Tails} \mathsf{A} \cong \operatorname{Coh} \mathbb{P}^2$. Away from the commutative case, the conditions are not particularly hard to satisfy; in particular, the generic sheaf in any component of the moduli space of 1-dimensional sheaves will satisfy this condition at every point of E.

By Serre duality, we have $\mathrm{Ext}^2(\mathsf{M}, \mathcal{O}_p(-3)) \cong \mathrm{Hom}(\mathcal{O}_p, \mathsf{M}) = 0$ and similarly $\mathrm{Ext}^2(\mathsf{M}', \mathcal{O}_p) = 0$. It then follows by an Euler characteristic calculation that $\mathrm{Ext}^1(\mathsf{M}, \mathcal{O}_p(-3)) = \mathrm{Ext}^1(\mathsf{M}', \mathcal{O}_p) = 0$ as well. Since $\mathsf{M} \otimes \mathcal{O}_p$ is a sum of copies of $\mathcal{O}_p$, we find that the natural maps

$$\mathrm{Ext}^i(\mathsf{M}', \mathsf{M}') \to \mathrm{Ext}^i(\mathsf{M}', \mathsf{M}),$$
$$\mathrm{Ext}^i(\mathsf{M}, \mathsf{M}'(-3)) \to \mathrm{Ext}^i(\mathsf{M}, \mathsf{M}(-3))$$

are isomorphisms. (In particular, M' is simple if and only if M is simple.) By Serre duality, the same applies to

$$\mathrm{Ext}^i(\mathsf{M}, \mathsf{M}) \to \mathrm{Ext}^i(\mathsf{M}', \mathsf{M}),$$
$$\mathrm{Ext}^i(\mathsf{M}, \mathsf{M}'(-3)) \to \mathrm{Ext}^i(\mathsf{M}', \mathsf{M}'(-3)).$$

By the functoriality of Ext, we find that the compositions

$$\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}'(-3)) \cong \mathrm{Ext}^1(\mathsf{M}', \mathsf{M}'(-3)) \xrightarrow{E} \mathrm{Ext}^1(\mathsf{M}', \mathsf{M}') \cong \mathrm{Ext}^1(\mathsf{M}', \mathsf{M})$$

and

$$\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}'(-3)) \cong \mathrm{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \xrightarrow{E} \mathrm{Ext}^1(\mathsf{M}, \mathsf{M}) \cong \mathrm{Ext}^1(\mathsf{M}', \mathsf{M})$$

agree, and thus, the induced isomorphism

$$\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}) \cong \mathrm{Ext}^1(\mathsf{M}', \mathsf{M}')$$

respects the Poisson structure.

It remains only to show that this isomorphism is the differential of the pseudotwist. Note that the pseudotwist is only a morphism on the strata of the moduli space with fixed $\dim \mathrm{Hom}(\mathsf{M}, \mathcal{O}_p)$. Thus, we only need to consider those classes in $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M})$ that preserve this dimension. In other words, we must consider extensions

$$0 \to \mathsf{M} \to \mathsf{N} \to \mathsf{M} \to 0$$

that remain exact when tensored with $\mathcal{O}_p$. Then the corresponding extension N' of M' by M' is the kernel of the natural map $\mathsf{N} \to \mathsf{N} \otimes \mathcal{O}_p$. That both extensions have the same image in $\mathrm{Ext}^1(\mathsf{M}', \mathsf{M})$ follows from exactness of the complex

$$0 \to \mathsf{M}' \to \mathsf{M} \oplus \mathsf{N}' \to \mathsf{N} \otimes \mathsf{M}' \to \mathsf{M} \to 0$$

(the two extensions are the cokernel of the map from M' and the kernel of the map to M), and this is the total complex of a double complex with exact rows.

Note that we also have $\mathrm{Ext}^*(\mathcal{O}_p, \mathsf{M}) = 0$, and thus, the connecting map

$$\mathrm{Hom}(\mathcal{O}_p, \mathsf{M} \otimes \mathcal{O}_p)) \to \mathrm{Ext}^1(\mathcal{O}_p, \mathsf{M}')$$

is an isomorphism, implying that M is the upward pseudotwist of M'. Since we can restate the conditions on M and M' in terms of $\mathsf{M}'|_C$, we find that the upward pseudotwist is also Poisson.

In fact, the hypotheses on M and M' are significantly stronger than necessary. The point is that, once we constrain $\dim \operatorname{Hom}(\mathsf{M}, \mathbb{O}_p)$, the further constraints in the above argument are dense open conditions. If we replace this by the weaker open condition that M' is simple, we still obtain a morphism between Poisson spaces. The failure of such a morphism to be Poisson is measured by a form on the cotangent sheaf, which by the above argument vanishes on a dense open subset and thus vanishes identically. □

Remark. This limiting argument also lets us deduce the commutative case from the noncommutative case, though in the commutative setting we can also use an interpretation involving twists on blowups [Rains 2013a]; this actually works for arbitrary sheaves of homological dimension 1, and presumably the same holds in the noncommutative setting. The above argument fails for torsion-free sheaves, however, as does the fact that the upward and downward pseudotwists are inverse to each other.

7.3. We now turn our attention to showing that the above actually defines a Poisson structure, i.e., that the corresponding biderivation on the structure sheaf satisfies the Jacobi identity. Unfortunately, the existing arguments in the commutative setting involve working with explicit Čech cocycles for extensions of vector bundles; while both Čech cocycles and vector bundles have noncommutative analogues, neither is particularly easy to compute with. It turns out, however, that in many cases we can reduce the computation of the pairing to a computation on the commutative curve E. (In fact, combined with the construction of [Hurtubise and Markman 2002], this is enough to verify the Jacobi identity in general.)

We assume here that M is a simple sheaf transverse to E; we also assume $\mathsf{M}/E\mathsf{M} \neq 0$. (In our case, we could equivalently just assume $\mathsf{M} \neq 0$, but this is the form in which the condition appears below; for commutative surfaces, the two conditions are not equivalent, and the conditions are likely to deviate from each other for more general noncommutative surfaces as well.) The map giving the Poisson structure then fits into a long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathsf{M}, \mathsf{M}) \to \operatorname{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \to \operatorname{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \\ \to \operatorname{Ext}^1(\mathsf{M}, \mathsf{M}) \to \operatorname{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \to \operatorname{Ext}^2(\mathsf{M}, \mathsf{M}(-3)) \to 0, \end{aligned}$$

where $\operatorname{Hom}(\mathsf{M}, \mathsf{M}(-3)) \subset \operatorname{Hom}(\mathsf{M}, \mathsf{M})$ is trivial since $\operatorname{Hom}(\mathsf{M}, \mathsf{M}) \cong \mathbb{C}$ injects in $\operatorname{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M})$, and $\operatorname{Ext}^2(\mathsf{M}, \mathsf{M}) = 0$ by duality. Now,

$$\operatorname{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \cong \operatorname{Hom}_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M})$$

and may thus be computed entirely inside $\mathrm{Coh}\, E$ so via commutative geometry. Since the sequence is essentially self-dual, we should also expect to have $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \cong \mathrm{Ext}^1_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M})$. We can make this explicit as follows: a class in $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M})$ is represented by an extension

$$0 \to \mathsf{M}/E\mathsf{M} \to \mathsf{N} \to \mathsf{M} \to 0,$$

and since M is B-flat, this induces an extension

$$0 \to \mathsf{M}/E\mathsf{M} \to \mathsf{N}/E\mathsf{N} \to \mathsf{M}/E\mathsf{M} \to 0,$$

and pulling back recovers the original extension. Conversely, any extension of $\mathsf{M}/E\mathsf{M}$ by $\mathsf{M}/E\mathsf{M}$ over B can be viewed as an extension of $\mathsf{M}/E\mathsf{M}$ by $\mathsf{M}/E\mathsf{M}$ in $\mathrm{Tails}\, \mathsf{A}$ and pulled back to an extension of M by $\mathsf{M}/E\mathsf{M}$ that restricts back to the original extension. In other words, "tensor with B" and "pull back" give inverse maps as required.

Since the map $R\,\mathrm{Hom}(\mathsf{M}, \mathsf{M}(-3)) \to R\,\mathrm{Hom}(\mathsf{M}, \mathsf{M})$ in the derived category is self-dual (subject to our choice of isomorphism $H^2(\mathsf{A}(-3)) \cong \mathbb{C}$), it follows that the corresponding exact triangle is self-dual and thus that the remaining maps

$$R\,\mathrm{Hom}(\mathsf{M}, \mathsf{M}) \to R\,\mathrm{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \cong R\,\mathrm{Hom}_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M})$$

and

$$R\,\mathrm{Hom}_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M}) \cong R\,\mathrm{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \to R\,\mathrm{Hom}(\mathsf{M}, \mathsf{M}(-3))[1]$$

in the exact triangle are dual. In particular, it follows that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Ext}^1_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M}) \arrow[r, "\mathrm{tr}"] & H^1(\mathbb{O}_E) \arrow[d, "\wr"] \\
\mathrm{Ext}^2(\mathsf{M}, \mathsf{M}(-3)) \arrow[r, "\mathrm{tr}"] & H^2(\mathsf{A}(-3)) \arrow[r, "\sim"] & \mathbb{C}
\end{tikzcd}$$

Since the map from $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M})$ to $\mathrm{Ext}^2(\mathsf{M}, \mathsf{M}(-3))$ is surjective, to compute the trace of any class in $\mathrm{Ext}^2(\mathsf{M}, \mathsf{M}(-3))$, we need to simply choose a preimage in $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M})$, interpret it as an extension of sheaves on the commutative curve E, and take the trace there.

Since we only need to consider classes in $\mathrm{Ext}^2(\mathsf{M}, \mathsf{M}(-3))$ that arise via the Yoneda product, it will be particularly convenient to use the Yoneda interpretation of such classes via 2-extensions. If N' is an extension of M by M and N is an extension of M by $\mathsf{M}(-3)$, then $\mathsf{N} \cup \mathsf{N}'$ is represented by the 2-extension

$$0 \to \mathsf{M}(-3) \to \mathsf{N} \to \mathsf{N}' \to \mathsf{M} \to 0, \tag{26}$$

where $\mathsf{N} \to \mathsf{N}'$ is the composition $\mathsf{N} \to \mathsf{M} \to \mathsf{N}'$. Recall that two 2-extensions are equivalent if and only if the complexes $\mathsf{N} \to \mathsf{N}'$ are quasi-isomorphic. The functoriality of $\mathrm{Ext}^2(-,-)$ is expressed via pullback and pushforward, as appropriate; the connecting maps are more complicated but are again amenable to explicit description [Mitchell 1965].

In our case, we have the following. The pushforward of (26) under the map $\mathsf{M}(-3) \xrightarrow{\mathsf{E}} \mathsf{M}$ has the form

$$0 \to \mathsf{M} \to \mathsf{N}'' \to \mathsf{N}' \to \mathsf{M} \to 0, \tag{27}$$

where $\mathsf{N}'' \cong (\mathsf{N} \oplus \mathsf{M})/\mathsf{M}(-3)$. Since $\mathrm{Ext}^2(\mathsf{M}, \mathsf{M}) = 0$, this 2-extension is trivial, and thus, there exists a sheaf Z and a filtration

$$0 \subset \mathsf{Z}_1 \subset \mathsf{Z}_2 \subset \mathsf{Z}$$

such that the sequence

$$0 \to \mathsf{Z}_1 \to \mathsf{Z}_2 \to \mathsf{Z}/\mathsf{Z}_1 \to \mathsf{Z}/\mathsf{Z}_2 \to 0$$

agrees with (27) or equivalently such that

$$0 \to \mathsf{M} \to \mathsf{N}' \to \mathsf{M} \to 0$$

is the pushforward under $\mathsf{N}'' \to \mathsf{M}$ of an extension

$$0 \to \mathsf{N}'' \to \mathsf{Z} \to \mathsf{M} \to 0.$$

It follows that the 2-extension (26) is equivalent to

$$0 \to \mathsf{M}(-3) \to \mathsf{Z}' \to \mathsf{Z} \to \mathsf{M} \to 0,$$

where Z' is the pullback of N under $\mathsf{N}'' \to \mathsf{M}$. Now, since N'' was itself obtained by pushing N forward, we have $\mathsf{N} \subset \mathsf{N}''$ in a natural way, giving $\mathsf{N} \subset \mathsf{Z}', \mathsf{Z}$ in compatible ways. Quotienting by this gives an equivalent 2-extension

$$0 \to \mathsf{M}(-3) \xrightarrow{\mathsf{E}} \mathsf{M} \to \mathsf{Z}/\mathsf{N} \to \mathsf{M} \to 0,$$

expressing (26) as the image under the connecting map of

$$0 \to \mathsf{M}/\mathsf{E}\mathsf{M} \to \mathsf{Z}/\mathsf{N} \to \mathsf{M} \to 0.$$

The corresponding class in $\mathrm{Ext}^1_{\mathsf{B}}(\mathsf{M}/\mathsf{E}\mathsf{M}, \mathsf{M}/\mathsf{E}\mathsf{M})$ is then obtained by tensoring with B:

$$0 \to \mathsf{M}/\mathsf{E}\mathsf{M} \to \mathsf{Z}/(\mathsf{E}\mathsf{Z} + \mathsf{N}) \to \mathsf{M}/\mathsf{E}\mathsf{M} \to 0.$$

It will be helpful to think of this last extension in a slightly different way. Since $\mathrm{Tor}_1(\mathsf{B}, \mathsf{M}) = 0$, the quotient $\mathsf{Z}/\mathsf{E}\mathsf{Z}$ inherits a filtration

$$0 \subset \mathsf{M}/\mathsf{E}\mathsf{M} \subset \mathsf{N}''/\mathsf{E}\mathsf{N}'' \subset \mathsf{Z}/\mathsf{E}\mathsf{Z} \to 0,$$

so to compute $\mathsf{Z}/(\mathsf{EZ}+\mathsf{N})$, we only need to understand the map $\mathsf{N} \to \mathsf{N}''/\mathsf{EN}''$. Since $\mathsf{M}(-3) \subset \mathsf{N}$ maps to $\mathsf{EM} \subset \mathsf{EN}''$, the map $\mathsf{N} \to \mathsf{N}''/\mathsf{EN}''$ factors through the natural map $\mathsf{N} \to \mathsf{M}$ and then through the quotient map $\mathsf{M} \to \mathsf{M}/\mathsf{EM}$. In other words, the map $\mathsf{N} \to \mathsf{N}''/\mathsf{EN}''$ precisely gives a splitting of the short exact sequence

$$0 \to \mathsf{M}/\mathsf{EM} \to \mathsf{N}''/\mathsf{EN}'' \to \mathsf{M}/\mathsf{EM} \to 0.$$

Note in particular that the pairing of N and N' depends only on the two extensions $\mathsf{N}'', \mathsf{N}' \in \operatorname{Ext}^1(\mathsf{M}, \mathsf{M})$ and a splitting of $\mathsf{N}''/\mathsf{EN}''$. We need to simply combine N'' and N' into a filtered sheaf, quotient by E, then mod out by the submodule M/EM coming from the splitting to obtain the desired extension. Finally, given this resulting extension, we simply compute the trace in the usual commutative algebraic geometry sense. If we were given a splitting of N'/EN', we could instead take the kernel of the resulting map $\mathsf{Z}/\mathsf{EZ} \to \mathsf{M}/\mathsf{EM}$; a splitting of both makes M/EM a direct summand.

7.4. At this point, we can understand the Poisson structure entirely in terms of extensions of M by M together with commutative data; to proceed further, we will need a more explicit description of self-extensions of M. Suppose that M is given by a presentation

$$0 \to V \xrightarrow{L} W \to \mathsf{M} \to 0,$$

and consider an extension $0 \to \mathsf{M} \to \mathsf{N} \to \mathsf{M} \to 0$.

We first note that, if $\operatorname{Ext}^2(W, V) = 0$, then there exists a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & V \arrow[r, "L"] \arrow[d] & W \arrow[r] \arrow[d] & \mathsf{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V' \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & \mathsf{N} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V \arrow[r, "L"] \arrow[d] & W \arrow[r] \arrow[d] & \mathsf{M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with short exact rows and columns. Indeed, we may pull N back to an extension of W by M, which is in the kernel of the connecting map $\operatorname{Ext}^1(W, \mathsf{M}) \to \operatorname{Ext}^2(W, V) = 0$ and thus is the pushforward of an extension W', giving a surjective map of short exact sequences, the kernel of which is as required.

If we further have $\mathrm{Ext}^1(V, V) = \mathrm{Ext}^1(W, W) = 0$, then both V' and W' are trivial extensions, and we find that N has a presentation

$$0 \to V \oplus V \xrightarrow{\begin{bmatrix} L & L' \\ 0 & L \end{bmatrix}} W \oplus W \to \mathsf{N} \to 0.$$

(This corresponds to the deformation $\mathrm{Coker}(L + \epsilon L')$ over $\mathbb{C}[\epsilon]/\epsilon^2$.)

With this in mind, we assume

$$\mathrm{Ext}^2(W, V) = \mathrm{Ext}^1(V, V) = \mathrm{Ext}^1(W, W) = 0$$

so that extensions of M by M are represented by maps $L' : V \to W$. (Of course, this representation is by no means unique!) Given two such extensions, it is trivial to construct the desired filtered sheaf: Z is simply the kernel of the morphism

$$\begin{bmatrix} L & L' & 0 \\ 0 & L & L'' \\ 0 & 0 & L \end{bmatrix} : V^3 \to W^3.$$

(We could equally well take the $(1, 3)$ entry to be an arbitrary map $L''' : V \to W$; this corresponds to the fact that the class in $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}/\mathsf{E}\mathsf{M})$ we obtain is only determined modulo the image of $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M})$.)

If we further assume that $\mathrm{Tor}_1(\mathsf{B}, W) = 0$, so $\mathrm{Tor}_1(\mathsf{B}, V) = 0$ (and recall we have already assumed $\mathrm{Tor}_1(\mathsf{B}, \mathsf{M}) = 0$), then we have an exact sequence

$$0 \to \mathrm{Hom}(V, W(-3)) \to \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W/\mathsf{E}W)$$

and $\mathrm{Hom}(V, W(-3)) \cong \mathrm{Ext}^2(W, V)^* = 0$, and thus, the extension L' only depends on its restriction to $\mathrm{Hom}(V, W/\mathsf{E}W) \cong \mathrm{Hom}_{\mathsf{B}}(V/\mathsf{E}V, W/\mathsf{E}W)$. (Note that, if we also assumed $\mathrm{Ext}^1(W, V) = 0$, every map in $\mathrm{Hom}_{\mathsf{B}}(V/\mathsf{E}V, W/\mathsf{E}W)$ would come from a deformation, but we will not need this assumption.)

We thus obtain the following, purely commutative construction. Given sheaves (which for our purposes will always be locally free) V_E and W_E on E and an injective morphism $L_E : V_E \to W_E$, say that $L'_E : V_E \to W_E$ is isotrivial if the corresponding deformation of the cokernel is trivial or in other words if the extension

$$\mathrm{Coker} \begin{bmatrix} L_E & L'_E \\ 0 & L_E \end{bmatrix}$$

of $\mathrm{Coker}\, L_E$ by $\mathrm{Coker}\, L_E$ splits. Then we may define a bilinear form on the space of isotrivial morphisms (or between the space of isotrivial morphisms and the space of all morphisms) by combining the two morphisms into a triangular matrix

$$\begin{bmatrix} L_E & L'_E & 0 \\ 0 & L_E & L''_E \\ 0 & 0 & L \end{bmatrix},$$

splitting off $\operatorname{Coker} L_E$ as a direct summand of the cokernel, and then taking the trace of the class of the corresponding extension.

It turns out this is already enough to let us prove Poissonness in several important cases. Suppose, for instance, that $V \cong \mathsf{A}^n$ and $W \cong \mathsf{A}[1]^m$; this implies the various vanishing statements we require. Then $V_E \cong \mathbb{O}_E^n$ is independent of τ while $W_E \cong \mathscr{L}^m$ for a degree-3 line bundle $\mathscr{L}$; the latter depends on τ, but any two such bundles are related under pulling back through a translation of E. Moreover, a given map $L_E : V_E \to W_E$ lifts to a unique morphism $L : V \to W$, and L is injective if and only if L_E is injective. (Even the condition that $\operatorname{Coker} L$ is simple turns out to be reducible to a question on L_E, but in any case, this is an open condition.) In particular, given any value of τ, we have an open subspace of the moduli space parametrizing sheaves with such a presentation, and for any other value τ', the corresponding open subspace is birational in a way preserving the Poisson structure. In particular, we may take $\tau' = 1_E$, at which point the corresponding moduli space is just a moduli space of sheaves on $\mathbb{P}^2$. Since the Jacobi identity is known to hold there, it holds on an open subspace for any τ and thus (since the failure of the Jacobi identity is measured by a morphism $\wedge^3\Omega \to \mathbb{O}$) on the closure of that open subspace so for any sheaf with a presentation of the given form.

In fact, with a bit more work, we can extend Poissonness to *any* simple sheaf (apart from point sheaves). The point is that, if $\mathsf{M}(d)$ is acyclic for $d \geq -3$, then M has a resolution

$$0 \to \mathsf{A}(-2)^{n_2} \to \mathsf{A}(-1)^{n_1} \to \mathsf{A}^{n_0} \to \mathsf{M} \to 0,$$

and as in [Hurtubise and Markman 2002], we can recover M from the cokernel of the map $\mathsf{A}(-2)^{n_2} \to \mathsf{A}(-1)^{n_1}$. The Poisson structure satisfies the Jacobi identity in the neighborhood of the latter sheaf (since this is just a twist of the kind of presentation we have already considered), and the calculation of [Hurtubise and Markman 2002] shows that the map from a neighborhood of M to this neighborhood simply negates the Poisson structure.

Note that it follows from this construction that we do not obtain any new symplectic varieties; every symplectic leaf in the noncommutative setting is mapped in this way to an open subset of a symplectic leaf in the moduli space of vector bundles on $\mathbb{P}^2$.

7.5. The above argument is somewhat unsatisfactory, as it depends on a somewhat delicate reduction to the commutative case, so is likely to be difficult to generalize to other noncommutative surfaces (e.g., deformations of del Pezzo surfaces). We thus continue our investigation of the pairing.

Since we are now in a completely commutative setting, we may use Čech cocycles to perform computations. In particular, a splitting of the extension N'_E corresponding to L'_E is a cocycle for $\operatorname{Hom}(N'_E, M_E)$ while the filtered sheaf Z_E is represented by a cocycle for $\operatorname{Ext}^1(M_E, N'_E)$. The desired trace is then simply the trace pairing of these two classes, which reduces to the trace pairing on matrices.

By the structure of Z_E, we find that the cocycle representing Z_E is simply the (global) morphism

$$\begin{bmatrix} L''_E & 0 \end{bmatrix} \in \operatorname{Hom}(V_E, W_E^2).$$

The splitting of N'_E is slightly more complicated. If we write $E = U_1 \cup U_2$ with U_1 and U_2 affine opens, then the relevant map $N'_E \to M_E$ is represented over U_i by

$$\begin{bmatrix} B'_i \\ 0 \end{bmatrix} \in \operatorname{Hom}_{U_i}(W_E^2, W_E), \qquad \begin{bmatrix} A'_i \\ 0 \end{bmatrix} \in \operatorname{Hom}_{U_i}(V_E^2, V_E)$$

such that

$$L'_E = B'_i L_E - L_E A'_i,$$

and there exists

$$\begin{bmatrix} \Phi'_{12} \\ 0 \end{bmatrix} \in \operatorname{Hom}_{U_1 \cap U_2}(W_E^2, V_E)$$

such that

$$B'_2 - B'_1 = L_E \Phi'_{12}, \qquad A'_2 - A'_1 = \Phi'_{12} L_E.$$

Note that, since L_E is assumed injective, Φ'_{12} is uniquely determined. Combining this, we find that the trace pairing is given by

$$-\operatorname{Tr}(L''_E \Phi'_{12}) \in \Gamma(U_1 \cap U_2, \mathbb{O}_E),$$

viewed as a cocycle for $H^1(\mathbb{O}_E)$.

Essentially the same formula (possibly up to sign) was given by Polishchuk [1998], who constructed a Poisson structure on the moduli space of *stable* morphisms between vector bundles on E. Although he allows the vector bundles to vary, it is easy to check that any deformation in the image of the cotangent space induces the trivial deformation of the two bundles. As a result, Polishchuk's proof of the Jacobi identity carries over to our case. (Note that he imposes a stability condition, which is typically stronger than the natural stability condition in Tails A. However, all he really uses is that $\operatorname{Hom}(W, V) = 0$ and that the complex has no nonscalar automorphisms, i.e., the natural analogue of "simple".) Note that the interpretation of Polishchuk's Poisson structure coming from our calculation makes it straightforward to identify the symplectic leaves: each symplectic leaf classifies the ways of representing a particular sheaf as the cokernel of a map $V \to W$ with V and W fixed.

In the 1-dimensional case, the bundles V_E and W_E have the same rank, and thus, L_E is generically invertible. If we choose U_1 such that L_E is invertible on U_1, then we can arrange that

$$A_1' = L_E^{-1} L_E', \qquad B_1' = 0,$$

at which point

$$\Phi_{12}' = L_E^{-1} B_2',$$

so the pairing is given by the cocycle

$$-\operatorname{Tr}(L_E'' L_E^{-1} B_2').$$

Given a holomorphic differential ω, the corresponding map to $\mathbb{C}$ is given by

$$\sum_{x \in U_2} \operatorname{Res}_x \operatorname{Tr}(L_E'' L_E^{-1} B_2')\omega.$$

The contributions come only from those points where L_E fails to be invertible, i.e., from the support of M_E. Moreover, we readily see that the local contribution at x will not change if we replace (A_2', B_2') by any other splitting holomorphic at x.

Acknowledgments

The principal results contained in this paper were obtained in September 2008 during our stay at the CRM in Montreal. It is a special pleasure to thank John Harnad and Jacques Hurtubise for making this possible and to acknowledge their fundamental contribution to the subject that is being deformed here in the noncommutative direction.

We received valuable feedback, in particular, from D. Kaledin and D. Kazhdan during Okounkov's 2009 Zabrodsky Lecture at Hebrew University as well as from many other people on other occasions.

Okounkov thanks the NSF for financial support under FRG DMS-1159416. Rains was supported by NSF grants DMS-0833464 and DMS-1001645.

References

[Altman and Kleiman 1980] A. B. Altman and S. L. Kleiman, "Compactifying the Picard scheme", *Adv. Math.* **35**:1 (1980), 50–112. MR 81f:14025a Zbl 0427.14015

[Arinkin and Borodin 2006] D. Arinkin and A. Borodin, "Moduli spaces of d-connections and difference Painlevé equations", *Duke Math. J.* **134**:3 (2006), 515–556. MR 2007h:14047 Zbl 1109.39019

[Arinkin and Borodin 2009] D. Arinkin and A. Borodin, "τ-function of discrete isomonodromy transformations and probability", *Compos. Math.* **145**:3 (2009), 747–772. MR 2010m:39028 Zbl 1210.14036

[Artin 1997] M. Artin, "Some problems on three-dimensional graded domains", pp. 1–19 in *Representation theory and algebraic geometry* (Waltham, MA, 1995), edited by A. Martsinkovsky and G. Todorov, London Math. Soc. Lecture Note Ser. **238**, Cambridge Univ., 1997. MR 99a:16037 Zbl 0888.16025

[Artin et al. 1990] M. Artin, J. Tate, and M. Van den Bergh, "Some algebras associated to automorphisms of elliptic curves", pp. 33–85 in *The Grothendieck Festschrift*, vol. 1, edited by P. Cartier et al., Progr. Math. **86**, Birkhäuser, Boston, 1990. MR 92e:14002 Zbl 0744.14024

[Artin et al. 1991] M. Artin, J. Tate, and M. Van den Bergh, "Modules over regular algebras of dimension 3", *Invent. Math.* **106**:2 (1991), 335–388. MR 93e:16055 Zbl 0763.14001

[Beauville 1991] A. Beauville, "Systèmes hamiltoniens complètement intégrables associés aux surfaces $K3$", pp. 25–31 in *Problems in the theory of surfaces and their classification* (Cortona, Italy, 1988), edited by F. Catanese et al., Sympos. Math. **32**, Academic, London, 1991. MR 95c:14052 Zbl 0827.58022

[Beauville 2000] A. Beauville, "Determinantal hypersurfaces", *Michigan Math. J.* **48** (2000), 39–64. MR 2002b:14060 Zbl 1076.14534

[Bottacin 1995] F. Bottacin, "Poisson structures on moduli spaces of sheaves over Poisson surfaces", *Invent. Math.* **121**:2 (1995), 421–436. MR 96i:14009 Zbl 0829.14019

[Conte 1999] R. Conte (editor), *The Painlevé property: one century later*, Springer, New York, 1999. MR 2000e:34001 Zbl 0989.00036

[Hurtubise and Markman 2002] J. C. Hurtubise and E. Markman, "Elliptic Sklyanin integrable systems for arbitrary reductive groups", *Adv. Theor. Math. Phys.* **6**:5 (2002), 873–978. MR 2005c:37104

[Huybrechts and Lehn 2010] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, 2nd ed., Cambridge Univ., 2010. MR 2011e:14017 Zbl 1206.14027

[Kajiwara et al. 2006] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta, and Y. Yamada, "Point configurations, Cremona transformations and the elliptic difference Painlevé equation", pp. 169–198 in *Théories asymptotiques et équations de Painlevé* (Angers, France, 2004), edited by É. Delabaere and M. Loday-Richaud, Sémin. Congr. **14**, Soc. Math. France, Paris, 2006. MR 2009a:14019 Zbl 1137.39011

[Kenyon 2009] R. Kenyon, "Lectures on dimers", pp. 191–230 in *Statistical mechanics* (Park City, UT, 2007), edited by S. Sheffield and T. Spencer, IAS/Park City Math. Ser. **16**, Amer. Math. Soc., Providence, RI, 2009. MR 2010j:82023 Zbl 1180.82001

[Mitchell 1965] B. Mitchell, *Theory of categories*, Pure and Applied Math. **17**, Academic, New York, 1965. MR 34 #2647 Zbl 0136.00604

[de Naeghel and Van den Bergh 2004] K. de Naeghel and M. Van den Bergh, "Ideal classes of three-dimensional Sklyanin algebras", *J. Algebra* **276**:2 (2004), 515–551. MR 2005k:16060 Zbl 1064.16028

[Nevins and Stafford 2007] T. A. Nevins and J. T. Stafford, "Sklyanin algebras and Hilbert schemes of points", *Adv. Math.* **210**:2 (2007), 405–478. MR 2008f:14002 Zbl 1116.14003

[Okounkov 2009] A. Okounkov, "Noncommutative geometry of random surfaces", preprint, 2009. arXiv 0907.2322v1

[Okounkov 2010a] A. Okounkov, "Applied noncommutative geometry, I: Algebra and geometry of random surfaces", lecture slides, Massachusetts Institute of Technology, 2010, Available at http://math.columbia.edu/~okounkov/Simons1.pdf.

[Okounkov 2010b] A. Okounkov, "Applied noncommutative geometry, II: Noncommutative geometry and Painlevé equations", lecture slides, Massachusetts Institute of Technology, 2010, Available at http://math.columbia.edu/~okounkov/Simons2.pdf.

[Okounkov 2010c] A. Okounkov, "Applied noncommutative geometry, III: Towards probabilistic mirror symmetry", lecture slides, Massachusetts Institute of Technology, 2010, Available at http://math.columbia.edu/~okounkov/Simons3.pdf.

[Polishchuk 1998] A. Polishchuk, "Poisson structures and birational morphisms associated with bundles on elliptic curves", *Internat. Math. Res. Notices* **1998**:13 (1998), 683–703. MR 99f:14014 Zbl 0933.14016

[Rains 2013a] E. M. Rains, "Birational morphisms and Poisson moduli spaces", preprint, 2013. arXiv 1307.4032v1

[Rains 2013b] E. M. Rains, "Generalized Hitchin systems on rational surfaces", preprint, 2013. arXiv 1307.4033v1

[Rains ≥ 2015] E. M. Rains, "The noncommutative geometry of difference equations", in preparation.

[Sakai 2001] H. Sakai, "Rational surfaces associated with affine root systems and geometry of the Painlevé equations", *Comm. Math. Phys.* **220**:1 (2001), 165–229. MR 2003c:14030 Zbl 1010.34083

[Stafford and Van den Bergh 2001] J. T. Stafford and M. Van den Bergh, "Noncommutative curves and noncommutative surfaces", *Bull. Amer. Math. Soc.* (*N.S.*) **38**:2 (2001), 171–216. MR 2002d:16036 Zbl 1042.16016

[Tyurin 1988] A. N. Tyurin, "Symplectic structures on the varieties of moduli of vector bundles on algebraic surfaces with $p_g > 0$", *Izv. Akad. Nauk SSSR Ser. Mat.* **52**:4 (1988), 813–852. In Russian; translated in *Math. USSR Izv.* **33**:1 (1989), 139–177. MR 90a:14020 Zbl 0673.14021

Communicated by J. Toby Stafford
Received 2014-05-15 Revised 2015-04-02 Accepted 2015-05-17

okounkov@math.columbia.edu *Columbia University, Department of Mathematics, 2990 Broadway, New York, NY 10027, United States*

rains@caltech.edu *California Institute of Technology, Division of Physics, Mathematics and Astronomy, 1200 East California Boulevard, Pasadena, CA 91125, United States*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2015 is US $255/year for the electronic version, and $440/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 9 No. 6 2015

1937-0652(2015)9:6;1-3