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Adams operations and Galois structure

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We present a new method for determining the Galois module structure of the cohomology of coherent sheaves on varieties over the integers with a tame action of a finite group. This uses a novel Adams–Riemann–Roch-type theorem obtained by combining the Künneth formula with localization in equivariant K-theory and classical results about cyclotomic fields. As an application, we show two conjectures of Chinburg, Pappas, and Taylor in the case of curves.

1. Introduction

We consider G-covers $\pi : X \to Y$ of a projective flat scheme $Y \to \mathrm{Spec}(\mathbb{Z})$ where G is a finite group. Let $\mathscr{F}$ be a G-equivariant coherent sheaf of $\mathbb{O}_X$-modules on X, i.e., a coherent $\mathbb{O}_X$-module equipped with a G-action compatible with the G-action on the scheme X. Our main objects of study are the Zariski cohomology groups $\mathrm{H}^i(X, \mathscr{F})$. These are finitely generated modules for the integral group ring $\mathbb{Z}[G]$.

Let us consider the total cohomology $\mathrm{R}\Gamma(X, \mathscr{F})$ in the derived category of complexes of $\mathbb{Z}[G]$-modules that are bounded below. If π is tamely ramified, then the complex $\mathrm{R}\Gamma(X, \mathscr{F})$ is "perfect", i.e., isomorphic to a bounded complex $(P^\bullet)$ of finitely generated *projective* $\mathbb{Z}[G]$-modules [Chinburg 1994; Chinburg and Erez 1992; Pappas 2008]. This observation goes back to E. Noether when X is the

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spectrum of the ring of integers of a number field; in this general setup, it is due to T. Chinburg.

Definition 1.0.1. We say that the cohomology of $\mathscr{F}$ has a normal integral basis if there exists a bounded complex $(F^\bullet)$ of finitely generated *free* $\mathbb{Z}[G]$-modules that is isomorphic to $\mathrm{R}\Gamma(X, \mathscr{F})$.

To measure the obstruction to the existence of a normal integral basis, we use the Grothendieck group $\mathrm{K}_0(\mathbb{Z}[G])$ of finitely generated projective $\mathbb{Z}[G]$-modules. We consider the projective class group $\mathrm{Cl}(\mathbb{Z}[G])$, which is defined as the quotient of $\mathrm{K}_0(\mathbb{Z}[G])$ by the subgroup generated by the class of the free module $\mathbb{Z}[G]$. The obstruction to the existence of a normal integral basis for the cohomology of $\mathscr{F}$ is given by the (stable) projective Euler characteristic

$$\overline{\chi}(X, \mathscr{F}) = \textstyle\sum_i (-1)^i [P^i] \in \mathrm{Cl}(\mathbb{Z}[G]),$$

which is independent of the choice of $(P^\bullet)$.

The main problem in the theory of geometric Galois structure is to understand such Euler characteristics. Often there are interesting connections with other invariants of X. For example, it was shown in [Chinburg et al. 1997a] that, under some additional hypotheses, the projective Euler characteristic of a version of the de Rham complex of X can be calculated using ϵ-factors of Hasse–Weil–Artin L-functions for the cover $X \to X/G$. Also, when X is a curve over $\mathbb{Z}$, the obstruction $\overline{\chi}(X, \mathbb{O}_X)$ is related, via a suitable equivariant version of the Birch and Swinnerton-Dyer conjecture, to the G-module structure of the Mordell–Weil and Tate–Shafarevich groups of the Jacobian of the generic fiber $X_{\mathbb{Q}}$ [Chinburg et al. 2009].

We will first discuss the case when $\pi : X \to Y$ is unramified. Let us denote by d the relative dimension of $Y \to \mathrm{Spec}(\mathbb{Z})$.

When $d = 0$, the problem of the existence of a normal integral basis reduces to a classical question. Suppose that N/K is an unramified Galois extension of number fields with Galois group G, and consider the ring of integers $\mathbb{O}_N$ that is then a projective $\mathbb{Z}[G]$-module. Take $X = \mathrm{Spec}(\mathbb{O}_N)$, $Y = \mathrm{Spec}(\mathbb{O}_K)$, and $\mathscr{F} = \mathbb{O}_X$; then the Euler characteristic $\overline{\chi}(X, \mathbb{O}_X)$ is the class $[\mathbb{O}_N]$ in $\mathrm{Cl}(\mathbb{Z}[G])$. We are then asking if $\mathbb{O}_N$ is "stably free", i.e., if there are integers n and m such that $\mathbb{O}_N \oplus \mathbb{Z}[G]^n \simeq \mathbb{Z}[G]^m$. Results of A. Fröhlich and M. Taylor imply that $[\mathbb{O}_N]$ is always 2-torsion in $\mathrm{Cl}(\mathbb{Z}[G])$ and is trivial when the group G has no irreducible symplectic representations; this result also holds when, more generally, N/K is tamely ramified. Indeed, if N/K is at most tamely ramified, Fröhlich's conjecture (shown by Taylor [1981]) explains how to determine the class $[\mathbb{O}_N]$ from the root numbers of Artin L-functions for irreducible symplectic representations of $\mathrm{Gal}(N/K)$. In particular, $\gcd(2, \#G) \cdot [\mathbb{O}_N] = 0$. When G is of odd order, $[\mathbb{O}_N] = 0$ and $\mathbb{O}_N$ is stably free; when G is of odd order, it then follows that $\mathbb{O}_N$ is actually a free $\mathbb{Z}[G]$-module. In general, the class of a projective module in the class group $\mathrm{Cl}(\mathbb{Z}[G])$ contains a lot of information about the isomorphism class of the module. Hence, the projective Euler characteristics that we consider in this paper also contain a lot of information about the Galois modules given by cohomology. This is not necessarily the case for the "naive" Euler characteristics that can be easily defined as classes in the weaker Grothendieck group of all finitely generated G-modules.

When $d > 0$, some progress towards calculating $\bar{\chi}(X, \mathscr{F})$ for general $\mathscr{F}$ was achieved after the introduction of the technique of cubic structures (see [Pappas 1998; 2008; Chinburg et al. 2009]; [Chinburg et al. 1997a] only dealt with the de Rham complex). This technique is very effective when *all* the Sylow subgroups of G are abelian. In particular, it allowed us to show that, under this hypothesis, $\gcd(2, \#G) \cdot \bar{\chi}(X, \mathscr{F}) = 0$ if $d = 1$ [Pappas 1998]. Some general results were obtained in [Pappas 2008] when $d > 1$, but the problem appears to be quite hard. In fact, as is explained in [loc. cit.], it is plausible that the statement that $\bar{\chi}(X, \mathscr{F}) = 0$ for G abelian and all X of dimension $< \#G$ is equivalent to the truth of Vandiver's conjecture for all prime divisors of the order $\#G$. In [Pappas 2008; Chinburg et al. 2009], it was conjectured that, for all G, there are integers N (that depends only on d) and δ (that depends only on $\#G$) such that $\gcd(N, \#G)^{\delta} \cdot \bar{\chi}(X, \mathscr{F}) = 0$; this was shown for G with abelian Sylows.

In this paper, we introduce a new method that allows us to handle nonabelian groups. We obtain strong results and, in the case $d = 1$ of curves over $\mathbb{Z}$, a proof of the above conjecture. This is based on Adams–Riemann–Roch-type identities that are proven using the Künneth formula and "localization" or "concentration" theorems (as given for example by Thomason) in equivariant K-theory. We combine these with a study of the action of the Adams–Cassou-Noguès–Taylor operations on $\mathrm{Cl}(\mathbb{Z}[G])$ and some classical algebraic number theory.

Write $\#G = 2^s 3^t m$, with m relatively prime to 6, and define a and b as follows. If the 2-Sylow subgroups of G have order ≤ 4, are cyclic of order 8, or are dihedral groups, set $a = 0$. If the 2-Sylow subgroups of G are abelian (but not as in the case above) or generalized quaternion or semidihedral groups, set $a = 1$. In all other cases, set $a = s + 1$. If the 3-Sylows subgroups of G are abelian, set $b = 0$. In all other cases, set $b = t - 1$.

Theorem 1.0.2. *Let $X \to Y$ be an unramified G-cover with $Y \to \mathrm{Spec}(\mathbb{Z})$ projective and flat of relative dimension* 1. *Let $\mathscr{F}$ be a G-equivariant coherent $\mathcal{O}_X$-module. For a and b as above, we have $2^a 3^b \cdot \bar{\chi}(X, \mathscr{F}) = 0$. In particular, if G has order prime to* 6, *then $\bar{\chi}(X, \mathscr{F}) = 0$.*

In particular, this implies, in this case of free G-action, the "localization conjecture" of [Chinburg et al. 2009] for $d = 1$ with $N = 6$ and $\delta = \max\{s, t\} + 1$ and significantly extends the results of [Pappas 1998]. For example, the hypothesis “all Sylows are abelian” in [Pappas 1998, Theorem 5.2(a), Corollary 5.3, Theorem 5.5(a)] can be relaxed. Hence, we obtain a “projective normal integral basis theorem” when $d = 1$ and the action is free: if X is normal and G is such that $a = b = 0$, then $X \cong \mathbf{Proj}\big(\bigoplus_{n\geq 0} A_n\big)$ with A_n, for all $n > 0$, free $\mathbb{Z}[G]$-modules. We can also give a result for the G-module of regular differentials of X: suppose that X is normal; then X is Cohen–Macaulay and we can consider the relative dualizing sheaf ω_X of $X \to \mathrm{Spec}(\mathbb{Z})$. Suppose that $\{g_1, \dots, g_r\}$ is a set of generators of G. Then $\{g_1 - 1, \dots, g_r - 1\}$ is a set of generators of the augmentation ideal $\mathfrak{g} \subset \mathbb{Z}[G]$; this gives a surjective $\phi : \mathbb{Z}[G]^{r_G} \to \mathfrak{g}$, where r_G is the minimal number of generators of G. Denote by $\Omega^2(\mathbb{Z})$ the kernel of ϕ. By Schanuel’s lemma, the stable isomorphism class of $\Omega^2(\mathbb{Z})$ is independent of the choice of generators. As in [Pappas 1998, Theorem 5.5(a)], we will see that Theorem 1.0.2 implies:

Theorem 1.0.3. *Let $X \to Y$ be an unramified G-cover with $Y \to \mathrm{Spec}(\mathbb{Z})$ projective and flat of relative dimension 1. Assume that X is normal, $X_{\mathbb{Q}}$ is smooth, and G acts trivially on $\mathrm{H}^0(X_{\mathbb{Q}}, \mathbb{O}_{X_{\mathbb{Q}}})$. Assume also that $\mathrm{H}^1(X, \omega_X)$ is torsion-free. Set $h = \dim_{\mathbb{Q}} \mathrm{H}^0(X_{\mathbb{Q}}, \mathbb{O}_{X_{\mathbb{Q}}})$ and $g_Y = \mathrm{genus}(Y_{\mathbb{Q}}) = \dim_{\mathbb{Q}} \mathrm{H}^0(Y_{\mathbb{Q}}, \Omega^1_{Y_{\mathbb{Q}}})$. If G is such that $a = b = 0$, then the $\mathbb{Z}[G]$-module $\mathrm{H}^0(X, \omega_X)$ is stably isomorphic to $\Omega^2(\mathbb{Z})^{\oplus h}$. If in addition G has odd order and $g_Y > h \cdot r_G$, then we have*

$$\mathrm{H}^0(X, \omega_X) \cong \Omega^2(\mathbb{Z})^{\oplus h} \oplus \mathbb{Z}[G]^{\oplus(g_Y - h\cdot r_G)}.$$

Now, let us give some more details. Our technique is K-theoretic and was inspired by work of M. Nori [2000] and especially of B. Köck [2000]. The beginning point is that the Adams–Riemann–Roch theorem for a smooth variety X can be obtained by combining a fixed-point theorem for the permutation action of the cyclic group $C_\ell = \mathbb{Z}/\ell\mathbb{Z}$ on the product $X^\ell = X \times \cdots \times X$ with the Künneth formula (see [Nori 2000; Köck 2000]; here ℓ is a prime number). Very roughly, this goes as follows. If $\mathscr{E}$ is a vector bundle over X, the Künneth formula gives an isomorphism $\mathrm{R}\Gamma(X, \mathscr{E})^{\otimes \ell} \cong \mathrm{R}\Gamma(X^\ell, \mathscr{E}^{\boxtimes \ell})$. Using localization and the Lefschetz–Riemann–Roch theorem, we can relate the cohomology of the exterior tensor product sheaf $\mathscr{E}^{\boxtimes \ell}$ on X^ℓ to the cohomology of the restriction $\mathscr{E}^{\otimes \ell} = \mathscr{E}^{\boxtimes \ell}|_X$ on the C_ℓ-fixed-point locus, which here is the diagonal $X = \Delta(X) \subset X^\ell$. Of course, there is a correction that involves the conormal bundle $\mathscr{N}_{X|X^\ell}$ of X in X^ℓ. This correction amounts to multiplying $\mathscr{E}^{\otimes \ell}$ by $\lambda_{-1}(\mathscr{N}_{X|X^\ell})^{-1}$. The multiplier $\lambda_{-1}(\mathscr{N}_{X|X^\ell})^{-1}$ is familiar in Lefschetz–Riemann–Roch-type theorems and in this case is given by the inverse $\theta^\ell(\Omega^1_X)^{-1}$ of the ℓ-th Bott class of the differentials; the resulting identity eventually gives the Adams–Riemann–Roch theorem for the Adams operation ψ^ℓ.

In our situation, we have to take great care to explain how enough of this can be done G-equivariantly and also for projective G-modules without losing much information. It is also important to connect the ℓ-th tensor powers used above to the versions of the ℓ-th Adams operators on the projective class group $\mathrm{Cl}(\mathbb{Z}[G])$ as defined by Cassou-Noguès and Taylor. This is all quite subtle and eventually needs to be applied for a carefully selected set of primes ℓ. Most of our arguments are valid in any dimension d, and we show various general Adams–Riemann–Roch-type results. Combining these with a Chebotarev density argument, we obtain that, for a p-group G with p an odd prime, the obstruction $\bar{\chi}(X, \mathscr{F})$ lies in a specific sum of eigenspaces for the action of the Adams–Cassou-Noguès–Taylor operators on the p-power part of the class group $\mathrm{Cl}(\mathbb{Z}[G])$ (Theorem 6.3.2). When $d = 1$, there is only one eigenspace in this sum. We can then see, using classical results of the theory of cyclotomic fields, that this eigenspace is trivial when $p \geq 5$. When $p = 3$, results of R. Oliver [1983] imply that the eigenspace is annihilated by the Artin exponent of G. The crucial number-theoretic ingredients are classical results of Iwasawa on cyclotomic class groups and units and the following fact: for any $p > 2$, both the second and the $(p-2)$-th Teichmüller eigenspaces of the p-part of the class group of $\mathbb{Q}(\zeta_p)$ are trivial. An argument that uses again localization implies that we can reduce the calculation to p-groups. Therefore, by the above, for general G, the class $\bar{\chi}(X, \mathscr{F})$ in $\mathrm{Cl}(\mathbb{Z}[G])$ is determined by the classes $c_2 = \bar{\chi}(X, \mathscr{F})$ in $\mathrm{Cl}(\mathbb{Z}[G_2])$ for the cover $X \to X/G_2$ and $c_3 = \bar{\chi}(X, \mathscr{F})$ in $\mathrm{Cl}(\mathbb{Z}[G_3])$ for $X \to X/G_3$. Here G_2 and G_3 are 2-Sylow and 3-Sylow subgroups of G, respectively. This then eventually leads to Theorem 1.0.2.

It is also important to treat (tamely) ramified covers $X \to Y$ over $\mathbb{Z}$. However, usually the unramified-over-$\mathbb{Z}$ case is the hardest one to deal with initially. After this is done, ramified covers can be examined by localizing at the branch locus (see for example [Chinburg et al. 2009]). This also occurs here. In fact, our method applies when $\pi : X \to Y$ is tamely ramified. In our situation, the assumption that the ramification is tame implies that the corresponding G-cover $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ given by the generic fibers is unramified. Under certain regularity assumptions, we obtain an Adams–Riemann–Roch formula for $\bar{\chi}(X, \mathscr{F})$ (Theorem 5.5.3) that generalizes formulas of Burns and Chinburg [1996] and Köck [1999] to all dimensions. In this case, one cannot expect that $\bar{\chi}(X, \mathscr{F})$ is annihilated by small integers. Nevertheless, when $d = 1$, we can obtain (almost full) information about the class $\bar{\chi}(X, \mathscr{F})$ from the ramification locus of the cover $X \to Y$. In particular, under some further conditions on X and Y, we show that the class $\gcd(2, \#G)^{v_2(\#G)+2} \gcd(3, \#G)^{v_3(\#G)-1} \cdot \bar{\chi}(X, \mathscr{F})$ only depends on the restrictions of $\mathscr{F}$ on the local curves $X_{\mathbb{Z}_p}$, where p runs over the primes that contain the support of the ramification locus of $X \to Y$. This proves a slightly weakened variant of the "input localization conjecture" of [Chinburg et al. 2009] for $d = 1$ (see Theorem 7.2.1). Dealing with wildly ramified covers presents a host of additional difficulties that are still not resolved, not even in the case $d = 0$ of number fields.

2. Grothendieck groups and Euler characteristics

2.1. *G-modules, sheaves, and Grothendieck groups.*

2.1.1. In this paper, modules are left modules while groups act on schemes on the right. Sometimes we will use the term "G-module" instead of "$\mathbb{Z}[G]$-module". If we fix a prime number ℓ, we set $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$, and in general, we denote inverting ℓ or localizing at the multiplicative set of powers of ℓ by a prime $'$. We reserve the notation $\mathbb{Z}_\ell$ for the ℓ-adic integers. Set $C_\ell = \mathbb{Z}/\ell\mathbb{Z}$, and denote by S_ℓ the symmetric group in ℓ letters. We regard C_ℓ as the subgroup of S_ℓ generated by the cycle $(12\cdots\ell)$.

2.1.2. If $R = \mathbb{Z}$ or $\mathbb{Z}'$, then the Grothendieck group $\mathrm{G}_0(R[G])$ of finitely generated $R[G]$-modules can be identified with the Grothendieck ring of finitely generated $R[G]$-modules that are free as R-modules ("$R[G]$-lattices") with multiplication given by the tensor product. The ring $\mathrm{G}_0(R[G])$ is a finite and flat $\mathbb{Z}$-algebra. The natural homomorphism $\mathrm{G}_0(R[G]) \to \mathscr{R}_{\mathbb{Q}}(G) := \mathrm{G}_0(\mathbb{Q}[G])$, $[M] \mapsto [M \otimes_R \mathbb{Q}]$, is surjective with nilpotent kernel [Swan 1963]. Therefore, the prime ideals of $\mathrm{G}_0(R[G])$ can be identified with the prime ideals of the representation ring $\mathscr{R}_{\mathbb{Q}}(G)$; these can be described following Segal and Serre (see for example [Chinburg et al. 1997b, §4]). Suppose that $\mathfrak{p}$ is a prime ideal of R. An element g in G is called $\mathfrak{p}$-regular if it is of order prime to the characteristic of $\mathfrak{p}$; the set of prime ideals of $\mathscr{R}_{\mathbb{Q}}(G)$ that lie above $\mathfrak{p}$ are in 1-1 correspondence with the set of "$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugacy classes" of $\mathfrak{p}$-regular elements. Here g and g' are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate, when there is $t \in \mathbb{Z}$ prime to the exponent $\exp(G)$ of G, such that g' is conjugate to g^t. The prime ideal $\rho = \rho_{(g,\mathfrak{p})}$, where g is $\mathfrak{p}$-regular, is

$$\rho_{(g,\mathfrak{p})} = \{\phi \in \mathscr{R}_{\mathbb{Q}}(G) \mid \mathrm{Tr}(g \mid \phi) \in \mathfrak{p}\}.$$

2.1.3. We say that a G-module M is *G-cohomologically trivial* (G-c.t.), or simply *cohomologically trivial* (c.t.) when G is clear from the context, if for all subgroups $H \subset G$ the cohomology groups $\mathrm{H}^i(H, M)$ are trivial when $i > 0$. If the $\mathbb{Z}[G]$-module M has finite projective dimension, it is G-c.t. In fact, by [Atiyah and Wall 1967, Theorem 9], the converse is true and M is G-c.t. if and only if it has a resolution

$$0 \to P \to Q \to M \to 0$$

where P and Q are projective $\mathbb{Z}[G]$-modules. If M is in addition finitely generated, we can take P and Q to also be finitely generated. Hence, we can identify

$$\mathrm{G}_0^{\mathrm{ct}}(G, \mathbb{Z}) = \mathrm{K}_0(\mathbb{Z}[G])$$

where $\mathrm{G}_0^{\mathrm{ct}}(G, \mathbb{Z})$ is the Grothendieck group of finitely generated G-c.t. modules. The following facts can be found in [Swan 1960; Taylor 1984]. $\mathrm{K}_0(\mathbb{Z}[G])$ is a finitely generated $\mathrm{G}_0(\mathbb{Z}[G])$-module by action given by the tensor product by $\mathbb{Z}[G]$-lattices. The subgroup $\langle\mathbb{Z}[G]\rangle$ of free classes is a $\mathrm{G}_0(\mathbb{Z}[G])$-submodule, and so the quotient $\mathrm{Cl}(\mathbb{Z}[G])$ is also a $\mathrm{G}_0(\mathbb{Z}[G])$-module. The $\mathrm{G}_0(\mathbb{Z}[G])$-module structure on $\mathrm{Cl}(\mathbb{Z}[G])$ factors through the quotient $\mathrm{G}_0(\mathbb{Z}[G]) \to \mathscr{R}_{\mathbb{Q}}(G)$.

2.1.4. For S a scheme (with trivial G-action), we will consider quasicoherent sheaves of $\mathbb{O}_S[G]$-modules on S and simply call these quasicoherent $\mathbb{O}_S[G]$-modules. We say that a quasicoherent $\mathbb{O}_S[G]$-module $\mathscr{F}$ is G-cohomologically trivial (G-c.t.) if, for all $s \in S$, the stalk $\mathscr{F}_s$ is a G-c.t. module. As in [Chinburg 1994, p. 448], we can easily see that a quasicoherent $\mathbb{O}_S[G]$-module $\mathscr{F}$ is G-c.t. if and only if, for every open affine subscheme U of S, $\mathscr{F}(U)$ is a G-c.t. module.

2.2. ***Grothendieck groups and Euler characteristics.***

2.2.1. We refer to [Thomason 1987] for general facts about the equivariant K-theory of schemes with group action and for the notations $\mathrm{G}_i(G,-)$ and $\mathrm{K}_i(G,-)$ of G-groups and K-groups for coherent and coherent locally free, G-equivariant sheaves, respectively. If S is as above, we denote by $\mathrm{G}_0^{\mathrm{ct}}(G,S)$ the Grothendieck group of G-c.t. coherent $\mathbb{O}_S[G]$-modules.

If the structure morphism $g : S \to \mathrm{Spec}(\mathbb{Z})$ is projective, there is (see [Chinburg 1994]; see also below) a group homomorphism (the "projective equivariant Euler characteristic")

$$g_*^{\mathrm{ct}} : \mathrm{G}_0^{\mathrm{ct}}(G,S) \to \mathrm{G}_0^{\mathrm{ct}}(G,\mathrm{Spec}(\mathbb{Z})) = \mathrm{G}_0^{\mathrm{ct}}(G,\mathbb{Z}) = \mathrm{K}_0(\mathbb{Z}[G])$$

where in the target we identify the class of a sheaf with the class of the module of its global sections. We will sometimes abuse notation and still write g_*^{ct} for the composition of g_*^{ct} with $\mathrm{K}_0(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}[G]) = \mathrm{K}_0(\mathbb{Z}[G])/\langle\mathbb{Z}[G]\rangle$.

Here is a review of the construction of g_*^{ct}. Suppose that $\mathscr{E}$ is a coherent $\mathbb{O}_S[G]$-module with G-c.t. stalks. Following [Chinburg 1994], we first give a bounded complex $(M^\bullet, d^\bullet)$ of finitely generated projective $\mathbb{Z}[G]$-modules that is isomorphic to $\mathrm{R}\Gamma(S,\mathscr{E})$ in the derived category of complexes of $\mathbb{Z}[G]$-modules bounded below. Choose a finite open cover $\mathscr{U} = \{U_j\}_j$ of S by affine subschemes $U_j = \mathrm{Spec}(R_j)$. Then all intersections $U_{j_1\cdots j_m} := U_{j_1} \cap \cdots \cap U_{j_m}$ are also affine. The sections $\mathscr{E}(U_{j_1\cdots j_m})$ are cohomologically trivial G-modules, and the usual Čech complex $C^\bullet(\mathscr{U},\mathscr{E})$ is a bounded complex of cohomologically trivial G-modules (see the proof of Theorem 1.1 in [Chinburg 1994]). The complex $C^\bullet(\mathscr{U},\mathscr{E})$ is isomorphic to $\mathrm{R}\Gamma(S,\mathscr{E})$, and since $S \to \mathrm{Spec}(\mathbb{Z})$ is projective, it has finitely generated homology groups. Now the usual procedure, described for example in [loc. cit.], produces a perfect complex $M^\bullet$ of $\mathbb{Z}[G]$-modules (i.e., a bounded complex of finitely generated projective $\mathbb{Z}[G]$-modules) together with a morphism of complexes $M^\bullet \to C^\bullet(\mathscr{U},\mathscr{E})$ that is a quasi-isomorphism. Then $M^\bullet$ is also isomorphic to $\mathrm{R}\Gamma(S,\mathscr{E})$ in the derived category, and we define

$$g_*^{\mathrm{ct}}(\mathscr{E}) = \textstyle\sum_i (-1)^i [M^i] \tag{2.2.2}$$

in $\mathrm{G}_0^{\mathrm{ct}}(G, \mathbb{Z}) = \mathrm{K}_0(\mathbb{Z}[G])$; this is independent of our choices.

2.2.3. When in addition the symmetric group S_ℓ acts on S, we can also consider the Grothendieck groups $\mathrm{G}_0^{\mathrm{ct}}(S_\ell; G, S)$ of S_ℓ-equivariant coherent $\mathbb{O}_S[G]$-modules that are G-cohomologically trivial. If $g : S \to \mathrm{Spec}(\mathbb{Z})$ is projective and in addition S_ℓ acts on S, a construction as above gives an Euler characteristic homomorphism

$$g_*^{\mathrm{ct}} : \mathrm{G}_0^{\mathrm{ct}}(S_\ell; G, S) \to \mathrm{G}_0^{\mathrm{ct}}(S_\ell; G, \mathrm{Spec}(\mathbb{Z}))$$

and similarly for S_ℓ replaced by the cyclic group C_ℓ. For simplicity, we will sometimes omit the superscript ct from g_*^{ct} when it is clear from the context.

2.2.4. We now assume that the group G acts on S on the right. We say that G acts tamely on S if, for every point $s \in S$, the inertia subgroup $I_s \subset G$, which is, by definition, the largest subgroup of G that fixes s and acts trivially on the residue field $k(s)$, has order prime to $\mathrm{char}(k(s))$. Suppose that the quotient scheme S/G exists and the map $\pi : S \to T = S/G$ is finite. Then, by [Raynaud 1970, Chapitre XI, Lemme 1], the G-cover π is, étale locally around $\pi(s)$ on T, induced from an I_s-cover. Hence, if $\mathscr{F}$ is a G-equivariant $\mathbb{O}_S$-module, then $\pi_*\mathscr{F}$ is a G-c.t. coherent $\mathbb{O}_T[G]$-module [Chinburg 1994; Chinburg and Erez 1992]. We then obtain $\pi_*^{\mathrm{ct}} : \mathrm{G}_0(G, S) \to \mathrm{G}_0^{\mathrm{ct}}(G, T)$ given by $[\mathscr{F}] \mapsto [\pi_*\mathscr{F}]$. If $f : S \to \mathrm{Spec}(\mathbb{Z})$ is projective, then S/G exists, π is finite, and $g : T \to \mathrm{Spec}(\mathbb{Z})$ is projective [Mumford 1970, Chapter III, Theorem 1]. Then the composition $f_*^{\mathrm{ct}} = g_*^{\mathrm{ct}} \cdot \pi_*^{\mathrm{ct}}$ gives the projective equivariant Euler characteristic

$$f_*^{\mathrm{ct}} : \mathrm{G}_0(G, S) \to \mathrm{G}_0^{\mathrm{ct}}(G, \mathrm{Spec}(\mathbb{Z})) = \mathrm{K}_0(\mathbb{Z}[G]). \tag{2.2.5}$$

The Grothendieck groups $\mathrm{G}_0(G, S)$ and $\mathrm{K}_0(\mathbb{Z}[G])$ are $\mathrm{G}_0(\mathbb{Z}[G])$-modules, and the map f_*^{ct} is a $\mathrm{G}_0(\mathbb{Z}[G])$-module homomorphism and similarly for $\mathbb{Z}$ replaced by $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$. Sometimes, we will denote $f_*^{\mathrm{ct}}(\mathscr{F})$ by $\chi(X, \mathscr{F})$; then $\bar{\chi}(X, \mathscr{F})$ is the image of $\chi(X, \mathscr{F})$ in $\mathrm{Cl}(\mathbb{Z}[G]) = \mathrm{K}_0(\mathbb{Z}[G])/\langle \mathbb{Z}[G] \rangle$.

Similarly, if $S_\ell \times G$ acts on the projective $f : S \to \mathrm{Spec}(\mathbb{Z})$ with the subgroup $G = 1 \times G$ acting tamely, we have

$$f_*^{\mathrm{ct}} : \mathrm{G}_0(S_\ell \times G, S) \to \mathrm{G}_0^{\mathrm{ct}}(S_\ell; G, \mathrm{Spec}(\mathbb{Z})) \tag{2.2.6}$$

given as $f_*^{\mathrm{ct}} = g_*^{\mathrm{ct}} \cdot \pi_*^{\mathrm{ct}}$. Here f_*^{ct} is a $\mathrm{G}_0(\mathbb{Z}[S_\ell \times G])$-module homomorphism. These constructions also work with $\mathbb{Z}$ replaced by $\mathbb{Z}'$ and with S_ℓ replaced by C_ℓ.

3. The Künneth formula

3.1. *Tensor powers.*

3.1.1. Let R be a commutative Noetherian ring with 1. If $(M^\bullet, d^\bullet)$ is a bounded chain complex of $R[G]$-modules that are flat as R-modules, we can consider the total tensor product complex

$$(M^{\bullet\otimes\ell}, \partial^\bullet)$$

whose term of degree n is

$$(M^{\bullet\otimes\ell})^n = \bigoplus_{\substack{(i_1,\dots,i_\ell)\in\mathbb{Z}^\ell\\ i_1+\cdots+i_\ell=n}} (M^{i_1}\otimes_R\cdots\otimes_R M^{i_\ell})$$

with diagonal G-action and differential ∂^n is given by

$$\partial^n(m_{i_1}\otimes\cdots\otimes m_{i_\ell}) = \sum_{a=1}^{\ell}(-1)^{i_1+\cdots+i_{a-1}} m_{i_1}\otimes\cdots\otimes m_{i_{a-1}}\otimes d^{i_a}(m_{i_a})\otimes\cdots\otimes m_{i_\ell}.$$

Since the modules M^i are R-flat, the complex $(M^\bullet)^{\otimes\ell}$ is isomorphic to the ℓ-fold derived tensor

$$M^\bullet \overset{\mathrm{L}}{\otimes}_R M^\bullet \overset{\mathrm{L}}{\otimes}_R \cdots \overset{\mathrm{L}}{\otimes}_R M^\bullet$$

in the derived category of complexes of $R[G]$-modules that are bounded above.

Lemma 3.1.2. (a) *Let M_1 and M_2 be projective $R[G]$-modules. Then $M_1\otimes_R M_2$ with the diagonal G-action is also a projective $R[G]$-module.*

(b) *Let M_1 and M_2 be cohomologically trivial G-modules that are $\mathbb{Z}$-flat. Then $M_1\otimes_{\mathbb{Z}} M_2$ with the diagonal G-action is also a cohomologically trivial G-module.*

Proof. (a) This follows easily from the fact that the tensor product $R[G]\otimes_R R[G]$ with diagonal G-action is $R[G]$-free.

(b) By [Atiyah and Wall 1967, Theorem 9], we have resolutions $0\to Q_i\to P_i\to M_i\to 0$ with P_i and Q_i projective $\mathbb{Z}[G]$-modules. Using this and (a), we can see that $M_1\otimes_{\mathbb{Z}} M_2$ has finite projective dimension; hence, it is cohomologically trivial. □

3.1.3. We define an action of the symmetric group S_ℓ on the complex $(M^{\bullet\otimes\ell}, \partial^\bullet)$ as follows [Atiyah 1966, p. 176]: $\sigma\in S_\ell$ acts on $(M^{\bullet\otimes\ell})^n = \bigoplus(M^{i_1}\otimes_R\cdots\otimes_R M^{i_\ell})$ by permuting the factors and with the *appropriate sign changes* so that a transposition of two terms $m_i\otimes m_j$ (where $m_i\in M^i$ and $m_j\in M^j$) comes with the sign $(-1)^{ij}$. (We see that the action of σ commutes with the differentials ∂^n.)

Let M^0 and M^1 be finitely generated projective $R[G]$-modules, and let us consider the complex $M^\bullet := [M^0 \xrightarrow{0} M^1]$ (in degrees 0 and 1). Using Lemma 3.1.2(a), we form the Euler characteristic

$$\chi((M^\bullet)^{\otimes\ell}) = \textstyle\sum_n (-1)^n \cdot [(M^{\bullet\otimes\ell})^n] \in \mathrm{K}_0(S_\ell;\, G,\, R),$$

where $\mathrm{K}_0(S_\ell;\, G,\, R)$ is the Grothendieck group of finitely generated $R[S_\ell \times G]$-modules that are $R[G]$-projective. As in [Atiyah 1966, Proposition 2.2], we can see that $\chi((M^\bullet)^{\otimes\ell})$ only depends on the class $\chi(M^\bullet) = [M^0] - [M^1]$ in $\mathrm{K}_0(R[G])$ and gives a well-defined map, the "tensor power operation"

$$\tau^\ell : \mathrm{K}_0(R[G]) \to \mathrm{K}_0(S_\ell;\, G,\, R), \quad \tau^\ell([M^0] - [M^1]) = \chi((M^\bullet)^{\otimes\ell}).$$

(This statement also follows from [Grayson 1989]; see for example [Köck 2000, §1].) In general, if $M^\bullet$ is a perfect complex of $R[G]$-modules, then as in [Atiyah 1966],

$$\tau^\ell\big(\textstyle\sum_i (-1)^i [M^i]\big) = \textstyle\sum_n (-1)^n [(M^{\bullet\otimes\ell})^n].$$

3.2. ***The formula.*** Suppose that $g : Y \to \mathrm{Spec}(\mathbb{Z})$ is projective and flat and that $\mathscr{E}$ is a coherent $\mathbb{O}_Y[G]$-module that is G-c.t. and $\mathbb{O}_Y$-locally free. Consider the exterior tensor product $\mathscr{E}^{\boxtimes\ell} = \bigotimes_{i=1}^{\ell} p_i^*\mathscr{E}$ of $\mathscr{E}$ on Y^ℓ with $p_i : Y^\ell \to Y$ the i-th projection. Then $\mathscr{E}^{\boxtimes\ell}$ is an $S_\ell \times G$-equivariant coherent $\mathbb{O}_{Y^\ell}[G]$-module that is $\mathbb{O}_{Y^\ell}$-locally free and by Lemma 3.1.2(b) G-cohomologically trivial. Denote by $g^\ell : Y^\ell \to \mathrm{Spec}(\mathbb{Z})$ the structure morphism.

Theorem 3.2.1 (Künneth formula). *We have*

$$\tau^\ell(g_*^{\mathrm{ct}}(\mathscr{E})) = (g^\ell)_*^{\mathrm{ct}}(\mathscr{E}^{\boxtimes\ell}) \tag{3.2.2}$$

in $\mathrm{K}_0(S_\ell;\, G,\, \mathbb{Z}) = \mathrm{G}_0^{\mathrm{ct}}(S_\ell;\, G,\, \mathbb{Z})$.

Proof. This follows [Kempf 1980, proof of Theorem 14]. Note that [Köck 2000, Theorem A] gives a similar result for $G = \{1\}$. Recall the construction of a bounded complex $(M^\bullet, d^\bullet)$ of finitely generated projective $\mathbb{Z}[G]$-modules that is quasi-isomorphic to $\mathrm{R}\Gamma(Y, \mathscr{E})$ described in Section 2.2.1 that uses the Čech complex $C^\bullet(\mathcal{U}, \mathscr{E})$. In this case, all the terms of $C^\bullet(\mathcal{U}, \mathscr{E})$ are $\mathbb{Z}$-flat. The complex $C^\bullet(\mathcal{U}, \mathscr{E})$ is given as the global sections of a corresponding complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$ of quasicoherent $\mathbb{O}_Y[G]$-modules whose terms are direct sums of sheaves of the form $j_*\mathscr{E}$, where $j : U_{j_1\cdots j_m} \hookrightarrow Y$ is the open immersion. Since all the intersections $U_{j_1\cdots j_m}$ are affine, the complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$ gives an acyclic resolution

$$0 \to \mathscr{E} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$$

of the $\mathbb{O}_Y[G]$-module $\mathscr{E}$. Consider the (exterior) tensor product

$$\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})^{\boxtimes\ell} = \bigotimes_{i=1}^{\ell} p_i^*\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$$

with S_ℓ-action defined following the rule of signs as before. Notice that all the terms of $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$ have $\mathbb{Z}$-flat stalks. We can also see that $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})^{\boxtimes \ell}$ is acyclic; thus, it gives an acyclic resolution

$$0 \to \mathscr{E}^{\boxtimes \ell} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})^{\boxtimes \ell}$$

that respects the S_ℓ-action. It follows from the definition that the global sections of $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})^{\boxtimes \ell}$ are $C^\bullet(\mathcal{U}, \mathscr{E})^{\otimes \ell}$. Hence, we obtain an isomorphism in the derived category of complexes of $\mathbb{Z}[S_\ell \times G]$-modules

$$\mathrm{R}\Gamma(Y^\ell, \mathscr{E}^{\boxtimes \ell}) \xrightarrow{\sim} C^\bullet(\mathcal{U}, \mathscr{E})^{\otimes \ell}.$$

Using $\phi : M^\bullet \to C^\bullet(\mathcal{U}, \mathscr{E})$, we also obtain a $\mathbb{Z}[S_\ell \times G]$-morphism of complexes

$$\phi^{\otimes \ell} : (M^\bullet)^{\otimes \ell} \to C^\bullet(\mathcal{U}, \mathscr{E})^{\otimes \ell}.$$

Since ϕ is a $\mathbb{Z}[G]$-quasi-isomorphism and the terms of $M^\bullet$ and $C^\bullet(\mathcal{U}, \mathscr{E})$ are $\mathbb{Z}$-flat, $\phi^{\otimes \ell}$ is a quasi-isomorphism. Combining these, we get an isomorphism in the derived category of complexes of $\mathbb{Z}[S_\ell \times G]$-modules

$$(M^\bullet)^{\otimes \ell} \xrightarrow{\sim} \mathrm{R}\Gamma(Y^\ell, \mathscr{E}^{\boxtimes \ell}), \tag{3.2.3}$$

and by Lemma 3.1.2(a), $(M^\bullet)^{\otimes \ell}$ is perfect as a complex of $\mathbb{Z}[G]$-modules. By taking the Euler characteristics of both sides, we obtain the result. □

4. Adams operations

4.1. *Cyclic powers.*

4.1.1. Again ℓ is a prime and $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$. By [Köck 1997], there is an "Adams operator" homomorphism

$$\psi^\ell : \mathrm{K}_0(\mathbb{Z}'[G]) \to \mathrm{K}_0(\mathbb{Z}'[G])$$

defined using "cyclic power operations" (following constructions of Kervaire and of Atiyah) as follows. Set $\Delta = \mathrm{Aut}(C_\ell) = (\mathbb{Z}/\ell\mathbb{Z})^*$, and consider the semidirect product $C_\ell \rtimes \Delta$ given by the tautological Δ-action on C_ℓ. We view $C_\ell \rtimes \Delta$ as a subgroup of $S_\ell = \mathrm{Perm}(C_\ell)$ in the natural way. Denote by σ the generator 1 of $\mathbb{Z}/\ell\mathbb{Z} = C_\ell$. If P is a finitely generated projective $\mathbb{Z}'[G]$-module, the tensor product $P^{\otimes \ell}$ is naturally an $S_\ell \times G$-module that is $\mathbb{Z}'[G]$-projective by Lemma 3.1.2. Let $S = \mathbb{Z}'[X]/(X^{\ell-1} + X^{\ell-2} + \cdots + 1)$, and set z for the image of X in S. We have $\mathbb{Z}'[C_\ell] = \mathbb{Z}' \times S$ by $\sigma \mapsto z$. Then $z^\ell = 1$. The group Δ acts on S via automorphisms given by $\delta(z) = z^\delta$ for $\delta \in \Delta = (\mathbb{Z}/\ell\mathbb{Z})^*$. For $a \in \mathbb{Z}/\ell\mathbb{Z}$, we set

$$F_a(P^{\otimes \ell}) := ((S \otimes_{\mathbb{Z}'} P^{\otimes \ell})_a)^\Delta,$$

where by definition

$$(S \otimes_{\mathbb{Z}'} P^{\otimes \ell})_a = \{x \in S \otimes_{\mathbb{Z}'} P^{\otimes \ell} \mid \sigma(x) = z^a \cdot x\}.$$

(Notice that Δ acts on $(S \otimes_{\mathbb{Z}'} P^{\otimes \ell})_a \subset S \otimes_{\mathbb{Z}'} P^{\otimes \ell}$ by the diagonal action.) By [Köck 1997, Corollary 1.4], $F_a(P^{\otimes \ell})$ are projective $\mathbb{Z}'[G]$-modules, and we have by definition

$$\psi^\ell([P]) := [F_0(P^{\otimes \ell})] - [F_1(P^{\otimes \ell})]$$

in $\mathrm{K}_0(\mathbb{Z}'[G])$.

4.1.2. Consider now the Grothendieck rings $\mathrm{G}_0(\mathbb{Z}'[C_\ell])$, $\mathrm{G}_0(\mathbb{Z}'[C_\ell \times G])$, etc., where ℓ is a prime that does not divide the order $\#G$. Inflation gives homomorphisms $\mathrm{G}_0(\mathbb{Z}'[C_\ell]) \to \mathrm{G}_0(\mathbb{Z}'[C_\ell \times G])$ and $\mathrm{G}_0(\mathbb{Z}'[G]) \to \mathrm{G}_0(\mathbb{Z}'[C_\ell \times G])$ that we will suppress in the notation. Set $v = [\mathbb{Z}'[C_\ell]]$ for the class of the free module in $\mathrm{G}_0(\mathbb{Z}'[C_\ell])$, and let $\alpha := v - 1$ be the class of the augmentation ideal.

4.1.3. Write $\mathbb{Z}'[C_\ell \times G] = \mathbb{Z}'[G] \times S[G]$. Suppose that R is a $\mathbb{Z}'$-algebra. If N is an $R[C_\ell \times G]$-module, then the C_ℓ-invariants N^{C_ℓ} give an $R[G]$-module that is a direct summand of N. Hence, if N is projective as an $R[G]$-module or is G-c.t., then N^{C_ℓ} is projective as an $R[G]$-module or is G-c.t., respectively. The functor $N \mapsto N^{C_\ell}$ from $R[C_\ell \times G]$-modules to $R[G]$-modules is exact. Let us consider the homomorphism [Köck 2000, Lemma 4.3, Corollary 4.4]

$$\zeta : \mathrm{K}_0(R[C_\ell \times G]) \to \mathrm{K}_0(R[G]), \quad \zeta([N]) = \ell \cdot [N^{C_\ell}] - [N], \tag{4.1.4}$$

where we subtract the class of N as an $R[G]$-module by forgetting the C_ℓ-action. This is a $\mathrm{G}_0(\mathbb{Z}'[G])$-module homomorphism. We can see that ζ vanishes on the subgroup $(v) \cdot \mathrm{K}_0(R[C_\ell \times G])$. Indeed, suppose that Q is a finitely generated projective $R[C_\ell \times G]$-module, and let us consider the $R[C_\ell \times G]$-module $R[C_\ell] \otimes_R Q \simeq \mathbb{Z}'[C_\ell] \otimes_{\mathbb{Z}'} Q$. Then there is an isomorphism (Frobenius reciprocity)

$$R[C_\ell] \otimes_R Q \simeq R[C_\ell] \otimes_R P = \mathrm{Ind}_G^{C_\ell \times G}(P), \tag{4.1.5}$$

where $P = \mathrm{Res}_G^{C_\ell \times G}(Q)$ is a projective G-module with trivial C_ℓ-action. We have

$$(R[C_\ell] \otimes_R Q)^{C_\ell} \simeq (R[C_\ell] \otimes_R P)^{C_\ell} \simeq P$$

and so $\zeta(R[C_\ell] \otimes_R Q) = \ell \cdot [P] - [P^{\oplus \ell}] = 0$.

4.1.6. Here R is still a $\mathbb{Z}'$-algebra. Consider also the map

$$\xi : \mathrm{K}_0(R[G]) \to \mathrm{K}_0(R[C_\ell \times G])$$

obtained by inflation (i.e., by considering a G-module as a $C_\ell \times G$-module with C_ℓ acting trivially). This is a $\mathrm{G}_0(\mathbb{Z}'[G])$-module homomorphism, and we can see from the definitions that

$$\zeta \circ \xi = (\ell - 1) \cdot \mathrm{id} \tag{4.1.7}$$

as maps $\mathrm{K}_0(R[G]) \to \mathrm{K}_0(R[G])$.

4.2. ***Cyclic and tensor powers.***

4.2.1. Recall the tensor power operation

$$\tau^\ell : \mathrm{K}_0(\mathbb{Z}'[G]) = \mathrm{G}_0^{\mathrm{pr}}(G, \mathbb{Z}') \to \mathrm{G}_0^{\mathrm{ct}}(C_\ell; G, \mathbb{Z}') = \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])$$

given by the construction in Section 3.1.3 applied to $R = \mathbb{Z}'$ followed by restriction from $S_\ell \times G$ to $C_\ell \times G$.

Proposition 4.2.2. *For each x in* $\mathrm{K}_0(\mathbb{Z}'[G])$, *we have*

$$\tau^\ell(x) = \psi^\ell(x) \quad \textit{in } \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])/(v) \cdot \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G]).$$

Proof. This follows the lines of the proof of [Köck 2000, Proposition 1.13]. First observe that the argument in [loc. cit.] gives that the map

$$\bar{\tau}^\ell : \mathrm{K}_0(\mathbb{Z}'[G]) \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])/(v) \cdot \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])$$

obtained from τ^ℓ is a homomorphism. It is then enough to show the identity for $x = [P]$, the class of a finitely generated projective $\mathbb{Z}'[G]$-module P. As above, consider $P^{\otimes\ell}$. Let $e = \ell^{-1} \cdot \left(\sum_{i=0}^{\ell-1} \sigma^i\right)$ be the idempotent in $\mathbb{Z}'[C_\ell]$ so that $e \cdot P^{\otimes\ell} = (P^{\otimes\ell})^{C_\ell}$. Write $P^{\otimes\ell} = (P^{\otimes\ell})^{C_\ell} \oplus Q_1$ with $Q_1 := (1-e) \cdot P^{\otimes\ell}$ an $S[G]$-module (via $\mathbb{Z}'[G] \to S[G]$). As in [Köck 1997, Examples 1.5], we see that

$$F_0(P^{\otimes\ell}) = e \cdot P^{\otimes\ell} = (P^{\otimes\ell})^{C_\ell}, \qquad F_1(P^{\otimes\ell}) = Q_1^\Delta.$$

There is a short exact sequence

$$0 \to Q_1 \to Q_1^\Delta \otimes_{\mathbb{Z}'} \mathbb{Z}'[C_\ell] \to Q_1^\Delta \to 0$$

of $C_\ell \times G$-modules where the first map is given by

$$q \mapsto \sum_{i=0}^{\ell-1} \left(\sum_{a \in \Delta} a\sigma^{-i} \cdot q \right) \otimes \sigma^i$$

and the second map is obtained by tensoring the augmentation map $\mathbb{Z}[C_\ell] \to \mathbb{Z}$; here Q_1^Δ is viewed as having trivial C_ℓ-action. This gives $[Q_1] = \alpha \cdot [Q_1^\Delta]$ in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])$. This now implies $[P^{\otimes\ell}] = [F_0(P^{\otimes\ell})] + \alpha \cdot [F_1(P^{\otimes\ell})]$ in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])$, where we regard $F_0(P^{\otimes\ell})$ and $F_1(P^{\otimes\ell})$ as having trivial C_ℓ-action. Since $\alpha = v - 1$,

$$[P^{\otimes l}] = [F_0(P^{\otimes\ell})] - [F_1(P^{\otimes\ell})] \quad \text{in } \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])/(v) \cdot \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G]),$$

and the result follows. □

Proposition 4.2.3. *Let $x_0 = [\mathbb{Z}'[G]]$ be the class of the free module in* $\mathrm{K}_0(\mathbb{Z}'[G])$. *We have $\psi^\ell(x_0) = x_0$ in* $\mathrm{K}_0(\mathbb{Z}'[G])$.

Proof. Note that [Köck 1999, Theorem 1.6(e)] gives a corresponding result for the (a priori different) Adams operators defined via exterior powers. Set $\Gamma = C_\ell \rtimes (\mathbb{Z}/\ell\mathbb{Z})^* \subset S_\ell = \mathrm{Perm}(\mathbb{F}_\ell)$; the element $\gamma = (\sigma^a, b)$ is then the (affine) map $\mathbb{F}_\ell \to \mathbb{F}_\ell$ given by $\gamma(x) = bx + a$. Consider the set $G^\ell = \mathrm{Maps}(\mathbb{F}_\ell, G)$ with $G \times \Gamma$-action given by $(g, \gamma) \cdot (g_x)_{x\in\mathbb{F}_\ell} = (gg_{\gamma^{-1}(x)})_{x\in\mathbb{F}_\ell}$. Suppose (g, γ) stabilizes $(g_x)_{x\in\mathbb{F}_\ell}$ so that $gg_x = g_{\gamma(x)}$ for all $x \in \mathbb{F}_\ell$. If $b \neq 1$, there is $y \in \mathbb{F}_\ell$ such that $\gamma(y) = y$. Then $gg_y = g_y$ and so $g = 1$. If $b = 1$, we have $gg_x = g_{x+a}$ for all $x \in \mathbb{F}_\ell$; this gives $g^\ell = 1$ and so again $g = 1$ since ℓ is prime to $\#G$. We conclude that the stabilizer in $G \times \Gamma$ of any $\underline{g} = (g_x)_{x\in\mathbb{F}_\ell} \in G^\ell$ lies in $1 \times \Gamma$. Therefore, G^ℓ is in $G \times \Gamma$-equivariant bijection with a disjoint union of sets of the form $G \times (\Gamma/\Gamma_{\underline{g}})$ with $\Gamma_{\underline{g}}$ the stabilizer subgroup in Γ. Hence, the $\mathbb{Z}'[G \times \Gamma]$-module $\mathbb{Z}'[G]^{\otimes\ell} = \mathbb{Z}'[G^\ell]$ is isomorphic to a direct sum of modules of the form $\mathbb{Z}'[G] \otimes_{\mathbb{Z}'} \mathbb{Z}'[\Gamma/\Gamma_{\underline{g}}]$. By the definition of F_a (see [Köck 1997, §1] or Section 4.1.1), we see that $F_a(\mathbb{Z}'[G] \otimes_{\mathbb{Z}'} \mathbb{Z}'[\Gamma/\Gamma_{\underline{g}}]) \simeq \mathbb{Z}'[G] \otimes_{\mathbb{Z}'} F_a(\mathbb{Z}'[\Gamma/\Gamma_{\underline{g}}])$. We conclude that $F_a(\mathbb{Z}'[G]^{\otimes\ell})$ are free $\mathbb{Z}'[G]$-modules. Therefore, $\psi^\ell(x_0) = m \cdot x_0$ in $\mathrm{K}_0(\mathbb{Z}'[G])$ for some $m \in \mathbb{Z}$, and by comparing $\mathbb{Z}'$-ranks (for this we may assume $G = \{1\}$), we can easily see that $m = 1$. □

4.2.4. If Z is a projective flat G-scheme over $\mathrm{Spec}(\mathbb{Z}')$, there is an Adams operation $\psi^\ell : \mathrm{K}_0(G, Z) \to \mathrm{K}_0(G, Z)$ [Köck 1998]. By an argument as above, or by using the equivariant splitting principle as in [Köck 2000], we can see that

$$\psi^\ell(\mathscr{F}) = [\mathscr{F}^{\otimes\ell}] \tag{4.2.5}$$

in the quotient $\mathrm{K}_0(C_\ell \times G, Z)/(v) \cdot \mathrm{K}_0(C_\ell \times G, Z)$.

4.3. *The Cassou-Noguès–Taylor Adams operations.*

4.3.1. We continue to assume that ℓ is prime to the order $\#G$. Then by [Swan 1960], finitely generated projective $\mathbb{Z}'[G]$-modules are locally free.

If $(n, \#G) = 1$, we denote by $\psi_n^{\mathrm{CNT}} : \mathrm{Cl}(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}[G])$ the Adams operator homomorphism defined by Cassou-Noguès and Taylor [1985; Taylor 1984]. (Roughly speaking, this is given, via the Fröhlich description, as the dual of the Adams operation $\psi^n(\chi)(g) := \chi(g^n)$ on the character group.) The operators ψ_n^{CNT} also restrict to operators on $\mathrm{Cl}(\mathbb{Z}'[G])$; we will denote these by the same symbol. Note that we can identify $\mathrm{Cl}(\mathbb{Z}'[G])$ with the subgroup of $\mathrm{K}_0(\mathbb{Z}'[G])$ of elements of rank 0. Denote by $r : \mathrm{K}_0(\mathbb{Z}'[G]) \to \mathbb{Z}$ the rank homomorphism.

Köck has shown that the Cassou-Noguès–Taylor Adams operators can be described in terms of (arguably more natural) Adams operators ψ_{ext}^ℓ defined via exterior powers and the Newton polynomial [Köck 1999, Theorem 3.7]. Here, we use his arguments to obtain a similar relation with the Adams operators ψ^ℓ of [Köck 1997] defined via cyclic powers, which are better suited to our application.

Proposition 4.3.2. *Let ℓ' be a prime with $\ell\ell' \equiv 1 \bmod \exp(G)$, and set $x_0 = [\mathbb{Z}'[G]]$ for the class of the free module of rank* 1. *Then we have*

$$\psi^\ell(x - r(x)\cdot x_0) = \ell \cdot \psi_{\ell'}^{\mathrm{CNT}}(x - r(x)\cdot x_0) \tag{4.3.3}$$

for all $x \in \mathrm{K}_0(\mathbb{Z}'[G])$.

Proof. Let us explain how we can deduce this by combining results and arguments from [Köck 1996; 1997; 1999]. Since $\psi^\ell : \mathrm{K}_0(\mathbb{Z}'[G]) \to \mathrm{K}_0(\mathbb{Z}'[G])$ is a group homomorphism [Köck 1997, Proposition 2.5] that preserves the rank, ψ^ℓ restricts to an (additive) operation on the subgroup $\mathrm{Cl}(\mathbb{Z}'[G])$ of rank-0 elements. The group $\mathrm{K}_0(\mathbb{Z}'[G])$ is generated by the classes of locally free left ideals in $\mathbb{Z}'[G]$; hence, we can assume $x = [P]$ where P is such a ideal. We may assume that $P \otimes_{\mathbb{Z}'} \mathbb{Z}_v = \mathbb{Z}_v[G]\cdot\lambda_v$, with $\lambda_v \in \mathbb{Q}_v[G]^* \cap \mathbb{Z}_v[G]$, so that $P = \bigcap_{v\neq l}(\mathbb{Z}_v[G]\cdot\lambda_v \cap \mathbb{Q}[G])$. Then a Fröhlich representative of $x - x_0$ is given via the classes ("reduced norms") $[\lambda_v] \in \mathrm{K}_1(\mathbb{Q}_v[G])$ of $\lambda_v \in \mathbb{Q}_v[G]^*$. Note that we have [Taylor 1984]

$$\mathrm{K}_1(\mathbb{Q}_v[G]) \simeq \mathrm{Hom}_{\mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v)}(\mathrm{K}_0(\bar{\mathbb{Q}}_v[G]), \bar{\mathbb{Q}}_v^*). \tag{4.3.4}$$

By [Köck 1997], cyclic power Adams operators ψ^ℓ can also be defined on the higher K-groups $\mathrm{K}_i(R[G])$, $i \geq 1$, for every commutative $\mathbb{Z}'$-algebra. In particular, we have ψ^ℓ on $\mathrm{K}_1(\mathbb{Z}_v[G])$ and $\mathrm{K}_1(\mathbb{Q}_v[G])$, for $v \neq (\ell)$, and on $\mathrm{K}_1(\mathbb{Q}[G])$. Using [Köck 1997, Corollary 1.4(c)], we see that the base change homomorphism

$$\mathrm{K}_1(\mathbb{Z}_v[G]) \to \mathrm{K}_1(\mathbb{Q}_v[G])$$

commutes with ψ^ℓ. In fact, by [Köck 1997, §3], the operators ψ^ℓ are defined via the cyclic operations $[a]_\ell$ of [loc. cit.] as $\psi^\ell = [0]_\ell - [1]_\ell$. Moreover, the operations $[a]_\ell$ are given via continuous maps on the level of spaces that give K-theory in the style of Gillet and Grayson [loc. cit.]. Using this, we can see that the topological argument of the proof of [Köck 1999, Proposition 3.1] also applies to the operators $[a]_\ell$ and ψ^ℓ, and so we obtain the commutative diagram of [Köck 1999, Proposition 3.1] for $K = \mathbb{Q}$, $\mathfrak{p} = (v)$, and $\gamma = [a]_\ell$ or ψ^ℓ, i.e., that $[a]_\ell$ and ψ^ℓ commute with the connecting homomorphism $\Phi : \mathrm{K}_1(\mathbb{Q}_v[G]) \to \mathrm{K}_0(\mathbb{Z}'[G])$. As in the proof of [Köck 1999, Corollary 3.5], this, together with the localization sequence, implies that the element $\psi^\ell(x - x_0) = \psi^\ell(x) - \psi^\ell(x_0)$ has Fröhlich representative given by $(\psi^\ell([\lambda_v]))_v$ with $\psi^\ell : \mathrm{K}_1(\mathbb{Q}_v[G]) \to \mathrm{K}_1(\mathbb{Q}_v[G])$ as above (see [Köck 1999] for more details).

Now by [Köck 1996, Corollary (c) of Proposition 1], the operator ψ^ℓ on $\mathrm{K}_1(\mathbb{Q}_v[G])$ and $\mathrm{K}_1(\mathbb{Q}[G])$ agrees with the (more standard) Adams operator ψ^ℓ_{ext} defined using exterior powers and the Newton polynomial (as for example in [Köck

1999]). In fact, then by [Köck 1996, Corollary 1 of Theorem 1] (or the proof of [Köck 1999, Theorem 3.7]), $\psi^\ell([\lambda_v])$ is given via (4.3.4) by $\chi \mapsto ([\lambda_v](\psi^{\ell'}(\chi)))^\ell$. The result now follows from the Fröhlich description of $\mathrm{Cl}(\mathbb{Z}'[G])$ [Taylor 1984] and the definition of the Cassou-Noguès–Taylor Adams operator. □

Remark 4.3.5. Since ψ^ℓ is only defined for $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$-algebras, the above proposition does not give an expression for the Cassou-Noguès–Taylor Adams operators on $\mathrm{Cl}(\mathbb{Z}[G])$. Still, this weaker result is enough for our purposes.

5. Localization and Adams–Riemann–Roch identities

5.1. *Localization for C_ℓ- and $C_\ell \times G$-modules.*

5.1.1. For simplicity, we set $\mathscr{R}(C_\ell) = \mathrm{G}_0(\mathbb{Z}'[C_\ell])' = \mathrm{G}_0(\mathbb{Z}'[C_\ell])[\ell^{-1}]$, $\mathscr{R}(C_\ell \times G) = \mathrm{G}_0(\mathbb{Z}'[C_\ell \times G])'$, etc., where ℓ is a prime that does not divide the order $\#G$. Inflation gives homomorphisms $\mathscr{R}(C_\ell) \to \mathscr{R}(C_\ell \times G)$ and $\mathscr{R}(G) \to \mathscr{R}(C_\ell \times G)$ that we will suppress in the notation. Denote by α the class in $\mathscr{R}(C_\ell)$ of the augmentation ideal of $\mathbb{Z}'[C_\ell]$, and set $v = 1 + \alpha = [\mathbb{Z}'[C_\ell]]$. The ring structure in $\mathscr{R}(C_\ell)$ is such that $v^2 = \ell \cdot v$. Also denote by I_G the ideal of $\mathscr{R}(G)$ given as the kernel of the rank homomorphism. Set

$$\mathscr{R}(C_\ell \times G)^\wedge := \varprojlim_n \mathscr{R}(C_\ell \times G)/(I_G^n \mathscr{R}(C_\ell \times G) + v\mathscr{R}(C_\ell \times G)).$$

In general, if M is an $\mathscr{R}(C_\ell \times G)$-module, we set $M^\wedge := \varprojlim_n M/(I_G^n M + vM)$, which is an $\mathscr{R}(C_\ell \times G)^\wedge$-module. The maximal ideals of $\mathscr{R}(C_\ell \times G)^\wedge$ correspond to maximal ideals of $\mathscr{R}(C_\ell \times G)$ that contain $I_G \mathscr{R}(C_\ell \times G) + v\mathscr{R}(C_\ell \times G)$; we can see (see Section 2.1.2) that these are the maximal ideals of the form $\rho_{((\sigma,1),(q))}$ with $\ell \neq q$ and σ is a generator of C_ℓ. (The ideal $\rho_{((\sigma,1),(q))}$ is independent of the choice of the generator σ; there is exactly one ideal for each prime $q \neq \ell$.) If $\phi : M \to N$ is an $\mathscr{R}(C_\ell \times G)$-module homomorphism such that the localization $\phi_\rho : M_\rho \to N_\rho$ is an isomorphism for every ρ of the form $\rho_{((\sigma,1),(q))}$ with $q \neq \ell$ as above, then the induced $\phi^\wedge : M^\wedge \to N^\wedge$ is an isomorphism of $\mathscr{R}(C_\ell \times G)^\wedge$-modules.

5.2. *Cyclic localization on products.*

5.2.1. Fix a prime ℓ that does not divide $\#G$, and as before, set $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$. Suppose that $Z \to \mathrm{Spec}(\mathbb{Z}')$ is a quasiprojective scheme equipped with an action of G. We will consider "localization" on the fixed points for the action of the cyclic group C_ℓ on the ℓ-fold fiber product Z^ℓ over $\mathrm{Spec}(\mathbb{Z}')$. In fact, the product $C_\ell \times G$ acts on Z^ℓ; C_ℓ acts by permutation of the factors and G acts diagonally.

5.2.2. Consider a maximal ideal ρ of $\mathscr{R}(C_\ell \times G)$ of the form $\rho_{((\sigma,1),(q))}$ with $q \neq \ell$ and σ a generator of C_ℓ as before. The corresponding fixed-point subscheme Z^ρ of Z^ℓ is by definition the reduced union of the translates of the fixed subscheme $Z^{(\sigma,1)}$

of the element $(\sigma, 1)$. In our case, this is the diagonal:

$$(Z^\ell)^\rho = (Z^\ell)^{(\sigma,1)} \cdot (C_\ell \times G) = \Delta(Z) = Z \hookrightarrow Z^\ell. \tag{5.2.3}$$

Consider the homomorphism obtained by push-forward of coherent sheaves

$$\Delta_* : \mathrm{G}_0(C_\ell \times G, Z) \to \mathrm{G}_0(C_\ell \times G, Z^\ell).$$

The "concentration" theorem [Chinburg et al. 1997b, Theorem 6.1] implies that, after localizing at any ρ as above, we obtain an $\mathscr{R}(C_\ell \times G)_\rho$-module isomorphism

$$(\Delta_*)_\rho : \mathrm{G}_0(C_\ell \times G, Z)_\rho \xrightarrow{\sim} \mathrm{G}_0(C_\ell \times G, Z^\ell)_\rho.$$

It now follows as above that Δ_* gives an $\mathscr{R}(C_\ell \times G)^\wedge$-module isomorphism

$$\Delta_*^\wedge : \mathrm{G}_0(C_\ell \times G, Z)^\wedge \xrightarrow{\sim} \mathrm{G}_0(C_\ell \times G, Z^\ell)^\wedge. \tag{5.2.4}$$

Here the completions are given as above for the $\mathscr{R}(C_\ell \times G)$-modules $\mathrm{G}_0(C_\ell \times G, Z)'$ and $\mathrm{G}_0(C_\ell \times G, Z^\ell)'$.

Definition 5.2.5. Let $L_\bullet : \mathrm{G}_0(C_\ell \times G, Z^\ell) \to \mathrm{G}_0(C_\ell \times G, Z)^\wedge$ be the $\mathscr{R}(C_\ell \times G)$-homomorphism defined as the composition of $\mathrm{G}_0(C_\ell \times G, Z^\ell) \to \mathrm{G}_0(C_\ell \times G, Z^\ell)^\wedge$ with the inverse of $\Delta_*^\wedge$. Set $\vartheta^\ell := L_\bullet(1) \in \mathrm{G}_0(C_\ell \times G, Z)^\wedge$. Then

$$\Delta_*^\wedge(\vartheta^\ell) = 1 = [\mathscr{O}_{Z^\ell}]. \tag{5.2.6}$$

5.2.7. If $\mathscr{F}$ is a G-equivariant locally free coherent $\mathscr{O}_Z$-module on Z, then $\mathscr{F}^{\boxtimes \ell}$ is a $C_\ell \times G$-equivariant locally free coherent $\mathscr{O}_{Z^\ell}$-module on Z^ℓ and $\mathscr{F}^{\otimes \ell} \simeq \Delta^*(\mathscr{F}^{\boxtimes \ell})$ as $C_\ell \times G$-sheaves on Z. Using (5.2.6) and the projection formula, we obtain

$$\mathscr{F}^{\boxtimes \ell} = \Delta_*^\wedge(\vartheta^\ell) \otimes \mathscr{F}^{\boxtimes \ell} = \Delta_*^\wedge(\vartheta^\ell \otimes \Delta^*(\mathscr{F}^{\boxtimes \ell})) = \Delta_*^\wedge(\vartheta^\ell \otimes \mathscr{F}^{\otimes \ell}).$$

Therefore,

$$\mathscr{F}^{\boxtimes \ell} = \Delta_*^\wedge(\vartheta^\ell \otimes \mathscr{F}^{\otimes \ell}) \tag{5.2.8}$$

in $\mathrm{G}_0(C_\ell \times G, Z^\ell)^\wedge$.

5.2.9. Suppose that Z is smooth over $\mathrm{Spec}(\mathbb{Z}')$; then Z^ℓ is also smooth. Denote by $\mathscr{N}_{Z|Z^\ell}$ the locally free conormal bundle $\mathscr{I}_Z/\mathscr{I}_Z^2$ of $Z \subset Z^\ell$ that gives a class in $\mathrm{K}_0(C_\ell \times G, Z)$. (Here $\mathscr{I}_Z$ is the ideal sheaf of $Z \subset Z^\ell$.) As in the proof of the Lefschetz–Riemann–Roch theorem [Thomason 1992; Chinburg et al. 1997b], we can see using the self-intersection formula that the homomorphism $L_\bullet$ is given as the composition of the restriction

$$\mathrm{G}_0(C_\ell \times G, Z^\ell) \cong \mathrm{K}_0(C_\ell \times G, Z^\ell) \xrightarrow{\Delta^*} \mathrm{K}_0(C_\ell \times G, Z) \cong \mathrm{G}_0(C_\ell \times G, Z)$$

followed by multiplication by $\lambda_{-1}(\mathcal{N}_{Z|Z^\ell})^{-1} \in \mathrm{K}_0(C_\ell \times G, Z)^\wedge$, and so ϑ^ℓ is the image of $\lambda_{-1}(\mathcal{N}_{Z|Z^\ell})^{-1}$ in $\mathrm{G}_0(C_\ell \times G, Z)^\wedge$. By [Chinburg et al. 1997b],

$$\lambda_{-1}(\mathcal{N}_{Z|Z^\ell}) := \sum_{i=0}^{\mathrm{top}} (-1)^i [\wedge^i \mathcal{N}_{Z|Z^\ell}]$$

is invertible in all the localizations $\mathrm{K}_0(C_\ell \times G, Z)_\rho$, with ρ as above, so it is invertible in $\mathrm{K}_0(C_\ell \times G, Z)^\wedge$.

5.2.10. If Z is projective but not necessarily smooth, we can describe $L_\bullet$ by following [Baum et al. 1979; Quart 1979]. Embed Z as a closed G-subscheme of a smooth projective bundle $P = \mathbb{P}(\mathcal{G}) \to \mathrm{Spec}(\mathbb{Z}')$ with G-action [Köck 1998]. Then P^ℓ is also smooth over $\mathrm{Spec}(\mathbb{Z}')$. Let us write $i : Z \hookrightarrow P$, $i^\ell : Z^\ell \hookrightarrow P^\ell$, for the corresponding closed immersions. Starting with $x \in \mathrm{G}_0(C_\ell \times G, Z^\ell)$, we first "resolve x on P^ℓ", i.e., represent the push-forward $(i^\ell)_*(x)$ by a bounded complex $\mathcal{E}^\bullet(x)$ of $C_\ell \times G$-equivariant locally free coherent sheaves on P^ℓ that is exact off Z^ℓ (so that the homology of $\mathcal{E}^\bullet(x)$ gives back x). Next, we restrict the complex $\mathcal{E}^\bullet(x)$ to P to obtain $\mathcal{E}^\bullet(x)_{|P}$, a complex exact off $Z = P \cap Z^\ell$. Finally, we take the class $[h(\mathcal{E}^\bullet(x)_{|P})]$ of the homology of $\mathcal{E}^\bullet(x)_{|P}$ to obtain an element of $\mathrm{G}_0(C_\ell \times G, Z)$ [Soulé et al. 1992]. For simplicity, write $T_\ell = C_\ell \times G$. Then $x \mapsto [h(\mathcal{E}^\bullet(x)_{|P})]$ is the composition

$$\mathrm{G}_0(T_\ell, Z^\ell) \xrightarrow[\sim]{h^{-1}} \mathrm{K}_0^{Z^\ell}(T_\ell, P^\ell) \xrightarrow{|P} \mathrm{K}_0^{P\cap Z^\ell}(T_\ell, P) \xrightarrow{h} \mathrm{G}_0(T_\ell, Z), \tag{5.2.11}$$

where in the middle we have the relative K-groups of complexes of T_ℓ-equivariant locally free sheaves exact off Z^ℓ and $Z^\ell \cap P = Z$, respectively (see [Soulé et al. 1992, §3; Baum et al. 1979, Definition 2.1] for more details). Finally, we multiply $[h(\mathcal{E}^\bullet(x)_{|P})]$ by the restriction $\lambda_{-1}(\mathcal{N}_{P|P^\ell})^{-1}_{|Z}$ in $\mathrm{K}_0(C_\ell \times G, Z)^\wedge$ of the inverse $\lambda_{-1}(\mathcal{N}_{P|P^\ell})^{-1} \in \mathrm{K}_0(C_\ell \times G, P)^\wedge$. We claim that

$$L_\bullet(x) = [h(\mathcal{E}^\bullet(x)_{|P})] \cdot \lambda_{-1}(\mathcal{N}_{P|P^\ell})^{-1}_{|Z}. \tag{5.2.12}$$

To verify (5.2.12), we can follow the argument in the proof of Lemma 1 in [Quart 1979], which applies in this situation. The reader is referred there for more details.

5.3. *Adams–Riemann–Roch identities.*

5.3.1. Let $f : X \to \mathrm{Spec}(\mathbb{Z})$ be projective and flat with a tame action of G. Then the quotient scheme $\pi : X \to Y = X/G$ exists and π is finite. Choose a prime ℓ with $(\ell, \#G) = 1$. We will apply the setup of the previous section to $Z = X' = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}')$.

We have the projective equivariant Euler characteristics

$$f_* : \mathrm{G}_0(C_\ell \times G, X')' \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])'$$

and similarly $f^\ell_* : \mathrm{G}_0(C_\ell \times G, (X')^\ell)' \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])'$. (Here, we omit the superscript ct from f^{ct}_* and $(f^\ell_*)^{\mathrm{ct}}$.) These are both $\mathscr{R}(C_\ell \times G) = \mathrm{G}_0(\mathbb{Z}'[C_\ell \times G])'$-module homomorphisms and induce $\mathscr{R}(C_\ell \times G)^\wedge$-homomorphisms

$$f^\wedge_* : \mathrm{G}_0(C_\ell \times G, X')^\wedge \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge,$$
$$(f^\ell_*)^\wedge : \mathrm{G}_0(C_\ell \times G, (X')^\ell)^\wedge \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge.$$

Now suppose $\mathscr{F}$ is a G-equivariant coherent locally free $\mathbb{O}_X$-module. We apply $(f^\ell_*)^\wedge$ on both sides of (5.2.8). Using $f^\ell_* \cdot \Delta_* = f_*$, we obtain

$$(f^\ell)^\wedge_*(\mathscr{F}^{\boxtimes\ell}) = f^\wedge_*(\vartheta^\ell \otimes \mathscr{F}^{\otimes\ell}) \tag{5.3.2}$$

in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge$.

5.3.3. Assume in addition that π is flat; then so is $\pi^\ell : X^\ell \to Y^\ell$. Set $\mathscr{E} = \pi_*\mathscr{F}$. This is a G-c.t. coherent $\mathbb{O}_Y[G]$-module that is $\mathbb{O}_Y$-locally free. We have $\pi^\ell_*(\mathscr{F}^{\boxtimes\ell}) = \mathscr{E}^{\boxtimes\ell}$ on Y^ℓ. The Künneth formula (3.2.2) now gives

$$f^\ell_*(\mathscr{F}^{\boxtimes\ell}) = g^\ell_*(\pi^\ell_*(\mathscr{F}^{\boxtimes\ell})) = g^\ell_*(\mathscr{E}^{\boxtimes\ell}) = \tau^\ell(g^{\mathrm{ct}}_*(\mathscr{E})) = \tau^\ell(f^{\mathrm{ct}}_*(\mathscr{F})).$$

Combining this with (5.3.2) gives

$$\tau^\ell(f^{\mathrm{ct}}_*(\mathscr{F})) = f^\wedge_*(\vartheta^\ell \otimes \mathscr{F}^{\otimes\ell})$$

in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge$. Using Proposition 4.2.2, we obtain

$$\psi^\ell(f^{\mathrm{ct}}_*(\mathscr{F})) = f^\wedge_*(\vartheta^\ell \otimes \mathscr{F}^{\otimes\ell}). \tag{5.3.4}$$

Finally, using (4.2.5), this gives the Adams–Riemann–Roch identity

$$\psi^\ell(f^{\mathrm{ct}}_*(\mathscr{F})) = f^\wedge_*(\vartheta^\ell \otimes \psi^\ell(\mathscr{F})) \tag{5.3.5}$$

in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge$.

5.3.6. Now consider $\zeta : \mathrm{G}_0(C_\ell \times G, Z)' \to \mathrm{G}_0(G, Z)'$ given by $\mathscr{F} \mapsto \ell \cdot [\mathscr{F}^{C_\ell}] - [\mathscr{F}]$ (see Section 4.1.3). This is an $\mathscr{R}(G) = \mathrm{G}_0(\mathbb{Z}'[G])'$-module homomorphism. As in Section 4.1.3, we can see that ζ vanishes on $(v) \cdot \mathrm{G}_0(C_\ell \times G, Z)'$ using Frobenius reciprocity. Therefore, it also gives

$$\zeta^\wedge : \mathrm{G}_0(C_\ell \times G, Z)^\wedge \to \mathrm{G}_0(G, Z)^\wedge = \varprojlim_n \mathrm{G}_0(G, Z)'/I^n_G \cdot \mathrm{G}_0(G, Z)'.$$

Theorem 5.3.7. *Under the above assumptions, we have*

$$(\ell - 1)\psi^\ell(f^{\mathrm{ct}}_*(\mathscr{F})) = f^\wedge_*(\zeta^\wedge(\vartheta^\ell) \otimes \psi^\ell(\mathscr{F})) \tag{5.3.8}$$

in $\mathrm{K}_0(\mathbb{Z}'[G])^\wedge = \varprojlim_n \mathrm{K}_0(\mathbb{Z}'[G])'/I^n_G \cdot \mathrm{K}_0(\mathbb{Z}'[G])'$.

Proof. Apply to the identity (5.3.5) the natural map induced on the completions by the map ζ of Section 4.1.3. The result then follows by using (4.1.7). $\square$

Under some additional assumptions, we will see in the next section that $\vartheta^\ell = L_\bullet(1)$ is given as the inverse of a Bott element. This justifies calling (5.3.5) and (5.3.8) Adams–Riemann–Roch identities.

5.4. *Localization and Bott classes.*

5.4.1. We return to the more general setup of Section 5.2. Suppose in addition that $f : Z \to \mathrm{Spec}(\mathbb{Z}')$ is a local complete intersection. Then, we can find a $\mathbb{Z}'[G]$-lattice E such that f factors G-equivariantly $Z \hookrightarrow \mathbb{P}(E) \to \mathrm{Spec}(\mathbb{Z}')$, where $P = \mathbb{P}(E)$ is the projective space with linear G-action determined by E. In this, the first morphism $i : Z \to \mathbb{P}(E)$ is a closed immersion, and the conormal sheaf $\mathscr{N}_{Z|P} := \mathscr{I}_Z/\mathscr{I}_Z^2$ is a G-equivariant sheaf locally free over Z of rank equal to the codimension c of Z in P [Köck 1998, §3]. By definition, the cotangent element of Z is $T_Z^\vee := [i^*\Omega_P^1] - [\mathscr{N}_{Z|P}]$ in $\mathrm{K}_0(G, Z)$; it is independent of the choice of such an embedding. The following result appears in [Köck 2000] if Z is smooth and G acts trivially on Z; it is inspired by an observation of Nori [2000]. (The case that Z is smooth and G acts trivially is enough for the proof of our main result in the unramified case, Theorem 1.0.2.)

Theorem 5.4.2. *Suppose that $Z \to \mathrm{Spec}(\mathbb{Z}')$ is a projective scheme with G-action that is a local complete intersection. Then the element $\vartheta^\ell = L_\bullet(1) \in \mathrm{G}_0(C_\ell \times G, Z)^\wedge$ is the image of the inverse $\theta^\ell(T_Z^\vee)^{-1} \in \mathrm{K}_0(G, Z)^\wedge$ of the Bott class under the natural homomorphism $\mathrm{K}_0(G, Z)^\wedge \to \mathrm{G}_0(C_\ell \times G, Z)^\wedge$.*

Here the inverse of the Bott class $\theta^\ell(T_Z^\vee)^{-1} = \theta^\ell(i^*\Omega_P^1)^{-1} \cdot \theta^\ell(\mathscr{N}_{Z|P})$ is defined in the I_G-adic completion $\mathrm{K}_0(G, Z)^\wedge$ as in [Köck 1998, p. 432]. As is remarked in [loc. cit.], if G acts trivially on Z, then the completion is not needed: the Bott class $\theta^\ell(T_Z^\vee)$ is then defined and is invertible in $\mathrm{K}_0(Z)' = \mathrm{K}_0(Z)[\ell^{-1}]$.

Proof. Starting with $i : Z \hookrightarrow P$ as above, we obtain a similar embedding of the fibered product $i^\ell : Z^\ell \hookrightarrow P^\ell$ that also makes Z^ℓ a local complete intersection in the smooth P^ℓ. We now use Section 5.2.10 to calculate $\vartheta^\ell = L_\bullet(1)$. Since $P \to \mathrm{Spec}(\mathbb{Z}')$ is smooth and projective, we can find a G-equivariant resolution $\mathscr{E}^\bullet \to i_*\mathbb{O}_Z$ of $i_*\mathbb{O}_Z$ by a bounded complex of G-equivariant locally free coherent $\mathbb{O}_P$-sheaves on P. Then, the total exterior tensor product $(\mathscr{E}^\bullet)^{\boxtimes\ell} = \bigotimes_{j=1}^\ell p_j^*\mathscr{E}^\bullet$ with its natural C_ℓ-action (with the rule of signs as in Section 3.1.1) provides a $C_\ell \times G$-equivariant locally free resolution of the $C_\ell \times G$-coherent sheaf $(i^\ell)_*(\mathbb{O}_{Z^\ell}) \simeq (i_*\mathbb{O}_Z)^{\boxtimes\ell} = \bigotimes_{j=1}^\ell p_j^*(i_*\mathbb{O}_Z)$ on P^ℓ. Let us restrict to P, i.e., pull back the complex $(\mathscr{E}^\bullet)^{\boxtimes\ell}$ via the diagonal $\Delta_P : P \hookrightarrow P^\ell$. We obtain the total ℓ-th tensor product

$$(\mathscr{E}^\bullet)^{\otimes\ell} = (\mathscr{E}^\bullet)^{\boxtimes\ell}|_P = \Delta_P^*((\mathscr{E}^\bullet)^{\boxtimes\ell}) \tag{5.4.3}$$

with its natural $C_\ell \times G$-action. Since $\mathscr{E}^\bullet$ is exact off Z, so is $(\mathscr{E}^\bullet)^{\otimes \ell}$; consider the element $[h((\mathscr{E}^\bullet)^{\otimes \ell})]$ of $\mathrm{G}_0(C_\ell \times G, Z)$ obtained from its total homology:

$$[h((\mathscr{E}^\bullet)^{\otimes \ell})] := \textstyle\sum_j (-1)^j [\mathrm{H}^j((\mathscr{E}^\bullet)^{\otimes \ell})] \tag{5.4.4}$$

as in [Soulé et al. 1992]. By Section 5.2.10, we have

$$\vartheta^\ell = L_\bullet(1) = [h((\mathscr{E}^\bullet)^{\otimes \ell})] \cdot \lambda_{-1}(\mathscr{N}_{P|P^\ell})^{-1}_{|Z}. \tag{5.4.5}$$

We now use two results of Köck. By [Köck 2001, Theorem 5.1], we have canonical $C_\ell \times G$-isomorphisms $\mathrm{H}^j((\mathscr{E}^\bullet)^{\otimes \ell}) \simeq \wedge^j(\mathscr{N}_{Z|P} \otimes \alpha)$, where α again is the augmentation ideal. Therefore,

$$[h((\mathscr{E}^\bullet)^{\otimes \ell})] = \textstyle\sum_j (-1)^j [\wedge^j(\mathscr{N}_{Z|P} \cdot \alpha)] = \lambda_{-1}(\mathscr{N}_{Z|P} \cdot \alpha). \tag{5.4.6}$$

On the other hand, [Köck 2000, Lemma 3.5], gives a C_ℓ-isomorphism

$$\Omega^1_P \otimes \alpha \xrightarrow{\sim} \mathscr{N}_{P|P^\ell},$$

which, as we can easily see, is also G-equivariant. Therefore,

$$\lambda_{-1}(\mathscr{N}_{P|P^\ell})_{|Z} = \lambda_{-1}(i^*\Omega^1_P \cdot \alpha)$$

in $\mathrm{K}_0(C_\ell \times G, Z)$.

Now, as in [Köck 2000, Proposition 3.2], we see using the G-equivariant splitting principle (e.g., [Köck 1998]) that, if $\mathscr{F}$ is a G-equivariant locally free coherent sheaf over Z, then

$$\theta^\ell(\mathscr{F}) = \lambda_{-1}(\mathscr{F} \cdot \alpha)$$

in $\mathrm{K}_0(C_\ell \times G, Z)'/(v) \cdot \mathrm{K}_0(C_\ell \times G, Z)'$.

Combining (5.4.5), (5.4.6), and the above, we obtain

$$\begin{aligned}\vartheta^\ell = L_\bullet(1) &= [h((\mathscr{E}^\bullet)^{\otimes \ell})] \cdot \lambda_{-1}(\mathscr{N}_{P|P^\ell})^{-1}_{|Z} \\ &= \lambda_{-1}(\mathscr{N}_{Z|P} \cdot \alpha) \cdot \lambda_{-1}(i^*\Omega^1_P \cdot \alpha)^{-1} = \theta^\ell(\mathscr{N}_{Z|P}) \cdot \theta^\ell(i^*\Omega^1_P)^{-1}\end{aligned}$$

and therefore

$$\vartheta^\ell = L_\bullet(1) = \theta^\ell(T^\vee_Z)^{-1}$$

in $\mathrm{G}_0(C_\ell \times G, Z)^\wedge$. This completes the proof. □

5.5. *A general Adams–Riemann–Roch formula.*

5.5.1. In this section, we assume that G acts tamely on $f : X \to \mathrm{Spec}(\mathbb{Z})$, which is always projective and flat of relative dimension d. We also assume that f is a local complete intersection and that $\pi : X \to Y = X/G$ is flat.

5.5.2. We now see that our results imply:

Theorem 5.5.3. *Let $\mathscr{F}$ be a G-equivariant coherent locally free $\mathbb{O}_X$-module. Under the above assumptions on X, if ℓ is a prime with $(\ell, \#G) = 1$ and ℓ' is another prime with $\ell\ell' \equiv 1 \bmod \exp(G)$, we have*

$$\begin{aligned}(\ell-1)\cdot\psi^\ell(\bar\chi(X,\mathscr{F}))\\ = \ell(\ell-1)\cdot\psi^{\mathrm{CNT}}_{\ell'}(\bar\chi(X,\mathscr{F})) = (\ell-1)\cdot f^\wedge_*(\theta^\ell(T^\vee_X)^{-1}\otimes\psi^\ell(\mathscr{F}))\end{aligned} \tag{5.5.4}$$

in $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$. *Here* $\theta^\ell(T^\vee_X)^{-1}$ *belongs to* $\mathrm{K}_0(G, X)^\wedge$.

Proof. In this, we also denote by ψ^ℓ the action of this operator on the completion $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ of the quotient $\mathrm{Cl}(\mathbb{Z}'[G])' = \mathrm{K}_0(\mathbb{Z}'[G])'/\langle[\mathbb{Z}'[G]]\rangle$; this action is well-defined by Proposition 4.2.3 and [Köck 1997, Proposition 2.10]. The statement then follows from the Adams–Riemann–Roch identity (5.3.8) and Theorem 5.4.2 by using also $\zeta^\wedge(\vartheta^\ell) = \zeta^\wedge(\theta^\ell(T^\vee_X)^{-1}) = (\ell-1)\cdot\theta^\ell(T^\vee_X)^{-1}$ as in (4.1.7) and Proposition 4.3.2. □

5.5.5. The result in this paragraph will be used in Section 7. Suppose in addition that $Y \to \mathrm{Spec}(\mathbb{Z})$ is regular (then the condition that π is flat is implied by the rest of our assumptions; see, e.g., [Matsumura 1980, (18.H)]). Recall we have classes $T^\vee_X$, $\pi^* T^\vee_Y$, and $T^\vee_{X/Y} := T^\vee_X - \pi^* T^\vee_Y$ in the Grothendieck group $\mathrm{K}_0(G, X)$. Notice that $\theta^\ell(T^\vee_Y)$ and $\theta^\ell(T^\vee_Y)^{-1}$ are defined in $\mathrm{K}_0(Y)'$ and $\theta^\ell(T^\vee_X)^{-1}$ in $\mathrm{K}_0(G, X)^\wedge$ and we have

$$\theta^\ell(T^\vee_{X/Y})^{-1} = \theta^\ell(T^\vee_X)^{-1}\pi^*(\theta^\ell(T^\vee_Y))$$

in $\mathrm{K}_0(G, X)^\wedge$. Then also

$$\theta^\ell(T^\vee_X)^{-1} = \theta^\ell(T^\vee_{X/Y})^{-1}\pi^*(\theta^\ell(T^\vee_Y))^{-1} \tag{5.5.6}$$

in $\mathrm{K}_0(G, X)^\wedge$. We can now write

$$\theta^\ell(T^\vee_Y)^{-1} = \ell^{-d} + c_\ell, \qquad \theta^\ell(T^\vee_{X/Y})^{-1} = 1 + r^\ell_{X/Y}$$

with $c_\ell \in \mathrm{K}_0(Y)' = \mathrm{G}_0(Y)'$ supported on a proper closed subset of Y and $r^\ell_{X/Y} \in \mathrm{K}_0(G, X)^\wedge$. (Here $1 = [\mathbb{O}_X]$.) From Theorem 5.5.3 and (5.5.6), we obtain

$$(\ell-1)\cdot\psi^\ell(\bar\chi(X,\mathbb{O}_X)) = (\ell-1)\cdot f^\wedge_*((1+r^\ell_{X/Y})\cdot(\ell^{-d}+\pi^* c_\ell)). \tag{5.5.7}$$

This gives the identity

$$\begin{aligned}(\ell-1)(\psi^\ell - \ell^{-d})\cdot(\bar\chi(X,\mathbb{O}_X))\\ = \ell^{-d}(\ell-1) f^\wedge_*(r^\ell_{X/Y}) + (\ell-1) f^\wedge_*(\pi^*(c_\ell)\cdot\theta^\ell(T^\vee_{X/Y})^{-1})\end{aligned} \tag{5.5.8}$$

in $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$. In this situation, since X and Y are both local complete intersections over $\mathbb{Z}$, the cotangent complexes $L_{X/\mathbb{Z}}$, $L_{Y/\mathbb{Z}}$, and hence $\pi^* L_{Y/\mathbb{Z}}$ are all perfect of $\mathbb{O}_X$-tor amplitude in $[-1, 0]$. (Here $L_{X/\mathbb{Z}}$ and $\pi^* L_{Y/\mathbb{Z}}$ are complexes of G-equivariant $\mathbb{O}_X$-modules; see [Illusie 1971] for the definition and properties of the cotangent complex.) If $i : Y \hookrightarrow P$ is an embedding in a smooth scheme as before, then i is a regular immersion and there is a quasi-isomorphism $L_{Y/\mathbb{Z}} \simeq [\mathscr{N}_{Y|P} \to i^*\Omega^1_{P/\mathbb{Z}}]$ and similarly for $L_{X/\mathbb{Z}}$ after choosing a G-equivariant embedding. There is a canonical distinguished triangle

$$\pi^* L_{Y/\mathbb{Z}} \to L_{X/\mathbb{Z}} \to L_{X/Y} \to \pi^* L_{Y/\mathbb{Z}}[1].$$

The cotangent complex $L_{X/Y}$ is then also perfect and gives the class $T^\vee_{X/Y}$ in $\mathrm{K}_0(G, X)$. The morphism $\pi^* L_{Y/\mathbb{Z}} \to L_{X/\mathbb{Z}}$ is an isomorphism over the largest open subscheme U of X such that $\pi : U \to V = U/G$ is étale.

6. Unramified covers

6.1. *The main identity.*

6.1.1. Here we suppose in addition that $\pi : X \to Y$ is unramified, i.e., that the cover π is étale. In this case, by étale descent, pull-back by π gives isomorphisms $\pi^* : \mathrm{G}_0(Y) \xrightarrow{\sim} \mathrm{G}_0(G, X)$ and $\pi^* : \mathrm{G}_0(C_\ell, Y) \xrightarrow{\sim} \mathrm{G}_0(C_\ell \times G, X)$. We can see that, by using such isomorphisms, the map

$$\Delta_* : \mathrm{G}_0(C_\ell, Y') \to \mathrm{G}_0(C_\ell, (Y')^\ell)$$

can be identified with the map Δ_* of Section 5.2.2. In this case, we can show more directly, using localization for the C_ℓ-action on $(Y')^\ell$, that Δ_* above gives an isomorphism after localizing at each maximal ideal $\rho_{(\sigma,(q))}$, for $q \neq \ell$, of $\mathscr{R}(C_\ell) = \mathrm{G}_0(\mathbb{Z}'[C_\ell])'$ that contains the element v. Set $\mathscr{R}(C_\ell)^\flat = \mathscr{R}(C_\ell)/(v)$, and use $^\flat$ to denote base change via $\mathscr{R}(C_\ell) \to \mathscr{R}(C_\ell)^\flat$. We see that $\Delta^\flat_*$ is an isomorphism and there is $\eta^\ell \in \mathrm{G}_0(C_\ell, Y')^\flat$ whose pull-back by π maps to $\vartheta^\ell \in \mathrm{G}_0(C_\ell \times G, X')^\wedge$. In particular, $\zeta(\eta^\ell)$ makes sense as an element in $\mathrm{G}_0(Y')$ and pulls back to an element of $\mathrm{G}_0(G, X')$ that is equal to $\zeta^\wedge(\vartheta^\ell)$ in $\mathrm{G}_0(G, X')^\wedge$. Given the above, the proof of the identity (5.3.8) now goes through without having to take completions and we obtain

$$(\ell - 1) \cdot \psi^\ell(f^{\mathrm{ct}}_*(\mathbb{O}_X)) = f^{\mathrm{ct}}_*(\pi^*\zeta(\eta^\ell)) \tag{6.1.2}$$

in $\mathrm{K}_0(\mathbb{Z}'[G])'$.

6.1.3. We will now show, by Noetherian induction, the following result:

Theorem 6.1.4. *Suppose $\pi : X \to Y$ is a G-torsor with $Y \to$ Spec($\mathbb{Z}$) projective and flat of relative dimension d, and let $\mathscr{F}$ be a G-equivariant coherent $\mathbb{O}_X$-module.*

Let ℓ be a prime such that $(\ell, \#G) = 1$. Then

$$(\ell-1)^{d+1} \cdot \prod_{i=0}^{d} (\psi^\ell - \ell^{-i}) \cdot f_*^{\mathrm{ct}}(\mathscr{F}) = 0 \tag{6.1.5}$$

in $\mathrm{Cl}(\mathbb{Z}'[G])' = \mathrm{Cl}(\mathbb{Z}'[G])[\ell^{-1}]$.

Proof. Set $\Psi(\ell, d) = (\ell-1)^{d+1} \cdot \prod_{i=0}^{d}(\psi^\ell - \ell^{-i})$ for the endomorphism of $\mathrm{Cl}(\mathbb{Z}'[G])' = \mathrm{K}_0(\mathbb{Z}'[G])'/\langle[\mathbb{Z}'[G]]\rangle$; this is well-defined by Proposition 4.2.3. We start by recalling that since $\pi : X \to Y$ is a G-torsor, by descent, all G-equivariant coherent $\mathbb{O}_X$-modules $\mathscr{F}$ are obtained by pulling back along π, i.e., are of the form $\mathscr{F} \simeq \pi^*\mathscr{G}$ for $\mathscr{G}$ a coherent $\mathbb{O}_Y$-module. By [Chinburg et al. 1997b, Theorem 6.1], the image of the class $f_*^{\mathrm{ct}}(\pi^*\mathscr{G})$ in $\mathrm{Cl}(\mathbb{Z}'[G])_\rho$ is trivial for any prime ρ of $\mathscr{R}(G)$ that does not contain the ideal I_G (see the proof of Proposition 4.5 in [Pappas 1998]). Therefore, it is enough to consider (6.1.5) for the image of $f_*^{\mathrm{ct}}(\pi^*\mathscr{G})$ in $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$. We will argue by induction on d. The map $\mathscr{G} \mapsto f_*^{\mathrm{ct}}(\pi^*\mathscr{G})$ induces a group homomorphism

$$\chi : \mathrm{G}_0(Y) \to \mathrm{Cl}(\mathbb{Z}'[G])'.$$

Now note that $\mathrm{G}_0(Y)$ is generated by classes of the form $i_*(\mathbb{O}_T)$ where $i : T \hookrightarrow Y$ is an integral closed subscheme of Y. Let us consider the G-torsor $\pi_{|T} : \pi^{-1}(T) = X \times_Y T \to T$ obtained by restriction, and denote by $h : \pi^{-1}(T) \to \mathrm{Spec}(\mathbb{Z})$ the structure morphism. Observe that, by the definitions, we have $f_*^{\mathrm{ct}} \cdot \pi^* \cdot i_* = h_*^{\mathrm{ct}} \cdot \pi_{|T}^*$. If T is fibral, i.e., if $T \to \mathrm{Spec}(\mathbb{Z})$ factors through $\mathrm{Spec}(\mathbb{F}_p)$ for some prime p, then $\chi(i_*(\mathbb{O}_T)) = 0$ by a result of Nakajima [1984]; see [Chinburg et al. 1997a, Theorem 1.3.2]. It remains to deal with T that are integral and flat over $\mathrm{Spec}(\mathbb{Z})$; the above allows us to reduce to the case when Y is integral and $\mathscr{G} = \mathbb{O}_Y$. We first show that, for Y integral, projective, and flat over $\mathrm{Spec}(\mathbb{Z})$ of dimension $d+1$, there is a proper reduced closed subscheme $i : W \hookrightarrow Y$ (therefore of smaller dimension) and a class $c' \in \mathrm{G}_0(C_\ell, W')^\flat$ such that

$$\eta^\ell - \ell^{-d} = i_*(c'). \tag{6.1.6}$$

To see this, take W to be given by the complement $Y - U$ of an open subset U of Y where $(\Omega^1_{Y/\mathbb{Z}})_{|U} \simeq \mathbb{O}_U^d$. Indeed, then since U is smooth, by the easy case of Theorem 5.4.2 [Köck 2000], the element $\eta^\ell_{U'}$ for U' is $\theta^\ell(\mathbb{O}^d_{U'})^{-1} = \ell^{-d}$. Since Δ_* and therefore $L_\bullet$ commutes with restriction to open subschemes, we obtain that the restriction of η^ℓ to U is ℓ^{-d} and so there is c' as above. Applying ζ to (6.1.6), we obtain

$$\zeta(\eta^\ell) - (\ell-1)\ell^{-d} = i_*(\zeta(c')) \tag{6.1.7}$$

with $\zeta(c') \in \mathrm{G}_0(W')'$. The identity (6.1.2) now implies

$$(\ell-1)\cdot(\psi^\ell-\ell^{-d})(f_*^{\mathrm{ct}}(\mathcal{O}_X)) = f_*^{\mathrm{ct}}(\pi^* i_*(\zeta(c'))) \tag{6.1.8}$$

in $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$. Consider the G-torsor $\pi_{|W} : \pi^{-1}(W) = X \times_Y W \to W$ obtained by restriction, and denote by $h : \pi^{-1}(W) \to \mathrm{Spec}(\mathbb{Z})$ the structure morphism. Again, we have $f_*^{\mathrm{ct}} \cdot \pi^* \cdot i_* = h_*^{\mathrm{ct}} \cdot \pi_{|W}^*$. We can extend $\zeta(c') \in \mathrm{G}_0(W')'$ to a class $z \in \mathrm{G}_0(W)'$. Then

$$f_*^{\mathrm{ct}}(\pi^* i_*(\zeta(c'))) = h_*^{\mathrm{ct}}(\pi_{|W}^* z).$$

This identity allows us to reduce to considering the G-torsor $\pi_{|W}$. If $d=0$, then W is fibral and $h_*^{\mathrm{ct}}(\pi_{|W}^* z) = 0$ by the result of Nakajima (in this case, the normal basis theorem and dévissage is enough). For $d > 0$, we can use the induction hypothesis on h and the G-torsor $\pi^{-1}(W) \to W$. This implies that $\Psi(\ell, d-1)$ annihilates $h_*^{\mathrm{ct}}(\pi_{|W}^* z)$, and the result follows from (6.1.8) and the above. □

Remark 6.1.9. Taylor's [1981] proof of the Fröhlich conjecture easily implies that $2 \cdot f_*^{\mathrm{ct}}(\mathcal{F}) = 0$ in $\mathrm{Cl}(\mathbb{Z}[G])$ if $d = 0$ [Pappas 1998, Theorem 4.1]. By using this input at the step $d=0$ of the inductive proof above, we can improve Theorem 6.1.4 and obtain the following: if $d \geq 1$ and ℓ is odd, then $f_*^{\mathrm{ct}}(\mathcal{F})$ is actually annihilated by $(\ell-1)^{d+1} \cdot \prod_{i=1}^{d}(\psi^\ell - \ell^{-i})$ in $\mathrm{Cl}(\mathbb{Z}'[G])'$, i.e., we can omit the factor $\psi^\ell - 1$ that corresponds to $i = 0$.

6.2. ***Class groups of p-groups and class field theory.*** In this section, G is a p-group of exponent p^N with p an odd prime.

6.2.1. By [Roquette 1958], we can write the group algebra $\mathbb{Q}[G]$ as a direct product of matrix rings $\mathbb{Q}[G] = \prod_i \mathrm{Mat}_{n_i \times n_i}(\mathbb{Q}(\zeta_{p^{s_i}}))$ with center $Z = \prod_i K_i$. Here $K_i = \mathbb{Q}(\zeta_{p^{s_i}})$ is the cyclotomic field, $s_i \leq N$. Denote by $\mathcal{O}_i$ the ring of integers of K_i.

6.2.2. Denote by $\mathrm{Cl}(\mathbb{Z}[G])_p$ and $\mathrm{Cl}(\mathbb{Z}'[G])_p$ the p-power torsion parts of $\mathrm{Cl}(\mathbb{Z}[G])$ and $\mathrm{Cl}(\mathbb{Z}'[G])$. We now show:

Proposition 6.2.3. *There exists an infinite set of primes $\ell \neq p$ with the following properties*:

(a) $\ell \bmod p$ *generates* $(\mathbb{Z}/p\mathbb{Z})^*$,

(b) $\ell^{p-1} \equiv 1 \bmod p^N$, *and*

(c) *for* $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$ *the restriction* $\mathrm{Cl}(\mathbb{Z}[G])_p \to \mathrm{Cl}(\mathbb{Z}'[G])_p$ *is an isomorphism.*

Proof. In what follows, we will write $\mathbb{A}(K)$ and $\mathbb{A}(K)^*$ for the adeles and ideles, respectively, of the number field K and $\mathbb{A}(\mathcal{O}_K)^* = \prod_v \mathcal{O}_{K,v}^*$ for the integral ideles. Using the Fröhlich description of the class group [Taylor 1984], we can write

$$\mathrm{Cl}(\mathbb{Z}[G]) = (\mathbb{A}(Z)^*/Z^*)/\mathcal{U} = \big(\textstyle\prod_i \mathbb{A}(K_i)^*/K_i^*\big)/\mathcal{U} \tag{6.2.4}$$

where $\mathcal{U} = \prod_v \mathrm{Det}(\mathbb{Z}_v[G]) \subset \mathbb{A}(\mathcal{O}_Z)^* = \prod_i \mathbb{A}(\mathcal{O}_i)^*$. The subgroup $\mathcal{U}$ is open of finite index in $\mathbb{A}(\mathcal{O}_Z)^*$. We can find finite index subgroups $\mathcal{U}_i = \prod_v \mathcal{U}_{i,v} \subset \mathbb{A}(\mathcal{O}_i)^*$, with $\mathcal{U}_{i,v} = (\mathcal{O}_i)_v^*$ if v does not divide p, such that

$$\prod_i \mathcal{U}_i \subset \mathcal{U} \subset \mathbb{A}(\mathcal{O}_Z)^*. \tag{6.2.5}$$

We can also assume $\mathcal{U}_{i,p}$ is stable for the action of the Galois group $\mathrm{Gal}(K_i/\mathbb{Q}) = \mathrm{Gal}(K_{i,p}/\mathbb{Q}_p)$. It follows now from (6.2.4) and (6.2.5) that we can write $\mathrm{Cl}(\mathbb{Z}[G])$ as a quotient

$$\prod_i \mathbb{A}(K_i)^*/(K_i^* \cdot \mathcal{U}_i) \twoheadrightarrow \mathrm{Cl}(\mathbb{Z}[G]). \tag{6.2.6}$$

By class field theory, the source can be identified with the product of the Galois groups of ray class field extensions of K_i that are at most ramified at the unique prime over p. In fact, this also implies that the p-Sylow $\mathrm{Cl}(\mathbb{Z}[G])_p$ can also be written as a quotient

$$\prod_i \mathrm{Gal}(L_i/K_i) \twoheadrightarrow \mathrm{Cl}(\mathbb{Z}[G])_p$$

where L_i/K_i is a p-power ray class field of K_i that is ramified at most at the unique prime of K_i over p such that $\mathrm{Gal}(L_i/K_i)$ is identified with the p-power quotient of $\mathbb{A}(K_i)^*/(K_i^* \cdot \mathcal{U}_i)$. Let us consider $\mathrm{Cl}(\mathbb{Z}'[G])$ for $\ell \neq p$. The discussion above applies again, and as before, we can write

$$\prod_i \mathbb{A}^\ell(K_i)^*/(K_i^* \cdot \mathcal{U}_i^\ell) \twoheadrightarrow \mathrm{Cl}(\mathbb{Z}'[G])$$

where the superscript ℓ, as in $\mathbb{A}^\ell$, means adeles away from ℓ. (Also, as usual, $\mathbb{A}_\ell$ will denote adeles over ℓ.) We can easily see that $\mathrm{Cl}(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}'[G])$ and hence $\mathrm{Cl}(\mathbb{Z}[G])_p \to \mathrm{Cl}(\mathbb{Z}'[G])_p$ is surjective.

Now choose n such that $n \geq N \geq \max_i\{s_i\}$ and such that $\mathbb{Q}(\zeta_{p^n})$ contains all the (cyclotomic) p-power ray class fields L_i, for i with $s_i = 0$, of $K_i = \mathbb{Q}$. Also set M_n to be a p-ray class field of $\mathbb{Q}(\zeta_{p^n})$ ramified only above p that contains all the fields L_i, for $s_i \geq 1$, and corresponds to a $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$-stable subgroup of $\mathbb{A}(\mathbb{Q}(\zeta_{p^n}))^*$. The extension $M_n/\mathbb{Q}$ is Galois; set $G_n = \mathrm{Gal}(M_n/\mathbb{Q})$. This is an extension

$$1 \to \mathrm{Gal}(M_n/\mathbb{Q}(\zeta_{p^n})) \to G_n \to \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) = \Delta \times \Gamma_n \to 1 \tag{6.2.7}$$

with $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^*$ and $\Gamma_n = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\zeta_p)) \simeq (\mathbb{Z}/p^{n-1}\mathbb{Z})$. We can find an element $\tau \in G_n$ of order $p-1$ as follows: lift the generator of Δ to an element τ_0; then a suitable power $\tau = \tau_0^{p^m}$ has order $p-1$. By the Chebotarev density theorem, there is an infinite set of primes ℓ so that Frob_ℓ lies in the conjugacy class $\langle\tau\rangle$ of τ in G_n. Then Frob_ℓ generates the group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Hence, the ideal (ℓ) remains prime in $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and ℓ generates $(\mathbb{Z}/p\mathbb{Z})^*$. Write $\mathfrak{L} = (\ell)$ in $\mathbb{Q}(\zeta_p)$, and consider $\mathrm{Frob}_\mathfrak{L}$ in $\mathrm{Gal}(M_n/\mathbb{Q}(\zeta_p))$. We have $\mathrm{Frob}_\mathfrak{L} = \mathrm{Frob}_\ell^{p-1} = \langle\tau^{p-1}\rangle = \langle 1\rangle$, and so $\mathfrak{L}$ splits completely in M_n.

By class field theory and the assumption $\mathcal{U}_{i,v} = \mathbb{O}_{i,v}^*$ for $v \neq p$, we see that the above implies that there is an infinite set of primes $\ell \neq p$ that generate $(\mathbb{Z}/p\mathbb{Z})^*$ and satisfy $\ell^{p-1} \equiv 1 \bmod p^N$ and additionally such that, for all i, the image of the subgroup

$$\mathbb{A}_\ell(K_i)^* = \prod_{\mathfrak{Q}|\ell} K_{i,\mathfrak{Q}}^* \subset \mathbb{A}(K_i)^*$$

in $\mathbb{A}(K_i)^*/K_i^* \cdot \mathcal{U}_i$ has order prime to p. Let us denote by $Q_i = Q_i(\ell)$ this order and set $Q = \prod_i Q_i$. For such an ℓ, suppose c is an element in the kernel of $\mathrm{Cl}(\mathbb{Z}[G])_p \to \mathrm{Cl}(\mathbb{Z}'[G])_p$ that is given by an idele $(a_i) \in \prod_i \mathbb{A}(K_i)^*$. Then

$$(a_i^\ell)_i = (\gamma_i)_i \cdot u^\ell$$

with $\gamma_i \in K_i^*$ (diagonally embedded in the prime to ℓ-ideles) and $u^\ell \in \mathcal{U}^\ell$. Here $(a_i)_i = (a_i^\ell)_i \cdot (a_{i,\ell})_i$ with $(a_{i,\ell})$ the ℓ-component of (a_i) (considered as an idele with 1 at all places away from ℓ). The product idele $(b_i)_i = (\gamma_i^{-1} \cdot a_i)_i$ also produces the class c in $\mathrm{Cl}(\mathbb{Z}[G])_p$. We can write $(b_i)_i = (b_i^\ell)_i \cdot (b_{i,\ell})_i$. The component $(b_{i,v})_i$ at a place v away from ℓ is equal to the corresponding component of u^ℓ, and so it is in $\mathcal{U}_v$. Using our assumption on ℓ, we can write

$$(b_{i,\ell}^Q) = (\delta_i \cdot u_i)$$

with $u_i \in \mathcal{U}_i$ and $\delta_i \in K_i^* \subset \mathbb{A}(K_i)^*$ (embedded diagonally). Combining these gives

$$(b_i)_i^Q = (\delta_i \cdot u_i)_i \cdot (b_i^\ell)_i^Q,$$

which is in $\left(\prod_i K_i^*\right) \cdot \mathcal{U}$. Therefore, $Q \cdot c$ is trivial in $\mathrm{Cl}(\mathbb{Z}[G])$; hence, c is trivial. $\square$

6.3. *Adams eigenspaces and the proof of the main result.*

6.3.1. Recall that G is a p-group of exponent p^N for an odd prime p. The group $(\mathbb{Z}/p^N\mathbb{Z})^* = (\mathbb{Z}/p\mathbb{Z})^* \times \mathbb{Z}/p^{N-1}\mathbb{Z}$ acts on the p-power torsion $\mathrm{Cl}(\mathbb{Z}[G])_p$ via the Cassou-Noguès–Taylor Adams operations: indeed, these operators are periodic, and this is essential for our argument. The element $a \in (\mathbb{Z}/p^N\mathbb{Z})^*$ acts via ψ_a^{CNT}; we will simply denote this by ψ_a in what follows. We will restrict this action to the subgroup $(\mathbb{Z}/p\mathbb{Z})^*$. This gives a decomposition into eigenspaces

$$\mathrm{Cl}(\mathbb{Z}[G])_p = \bigoplus_{\chi:(\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{Z}_p^*} \mathrm{Cl}(\mathbb{Z}[G])_p^\chi = \bigoplus_{i=0}^{p-2} \mathrm{Cl}(\mathbb{Z}[G])_p^{(i)}$$

where $\mathrm{Cl}(\mathbb{Z}[G])_p^{(i)} = \{c \in \mathrm{Cl}(\mathbb{Z}[G])_p \mid \psi_a(c) = \omega(a)^i \cdot c \text{ for all } a \in (\mathbb{Z}/p\mathbb{Z})^*\}$. Here $\omega : (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{Z}_p^*$ is the Teichmüller character. There is a similar result for $\mathrm{Cl}(\mathbb{Z}'[G])_p$ after we choose a prime $\ell \neq p$.

Theorem 6.3.2. *Suppose $\pi : X \to Y$ is a G-torsor with $Y \to \mathrm{Spec}(\mathbb{Z})$ projective and flat of relative dimension d, and assume that G is a p-group for an odd prime p. Let $\mathcal{F}$ be a G-equivariant coherent $\mathcal{O}_X$-module. Then the class $\bar{\chi}(X, \mathcal{F}) \in \mathrm{Cl}(\mathbb{Z}[G])$ is p-power torsion. If $p > d$, the class lies in $\bigoplus_{i=2}^{d+1} \mathrm{Cl}(\mathbb{Z}[G])_p^{(i)}$. In particular, if $d = 1$, then $\bar{\chi}(X, \mathcal{F})$ lies in the eigenspace $\mathrm{Cl}(\mathbb{Z}[G])_p^{(2)}$.*

Proof. The fact that $\bar{\chi}(X, \mathcal{F})$ is p-power torsion follows from the localization theorem of [Chinburg et al. 1997b] as in [Pappas 1998, Proposition 4.5]. Choose an odd prime ℓ as in Proposition 6.2.3; in particular, $\ell - 1$ is prime to p. Propositions 4.2.3 and 4.3.2 together imply that, for $x \in \mathrm{Cl}(\mathbb{Z}'[G])_p$, we have $\psi^\ell(x) = \ell \cdot \psi_{\ell'}(x)$ for $\ell'\ell \equiv 1 \bmod p^N$. From Theorem 6.1.4 and Remark 6.1.9, we then obtain that $f_*^{\mathrm{ct}}(\mathcal{F})$ lies in $\bigoplus_{i=2}^{d+1} \mathrm{Cl}(\mathbb{Z}'[G])_p^{(i)}$. However, for our choice of ℓ, the restriction $\mathrm{Cl}(\mathbb{Z}[G])_p \to \mathrm{Cl}(\mathbb{Z}'[G])_p$ is an isomorphism, and this gives the result. □

6.3.3. We continue to assume that G is a p-group, p an odd prime. Let $\#G = p^n$. We show:

Proposition 6.3.4. *If $p \geq 5$, then $\mathrm{Cl}(\mathbb{Z}[G])_p^{(2)} = (0)$. If $p = 3$, then $\mathrm{Cl}(\mathbb{Z}[G])_p^{(2)} = \mathrm{Cl}(\mathbb{Z}[G])_p^{(0)}$ is annihilated by the Artin exponent $A(G) = p^{n-1}$ of G.*

Proof. As above, we can write

$$\mathrm{Cl}(\mathbb{Z}[G]) = \textstyle\prod_i \mathbb{A}(K_i)^* / \big(\prod_i K_i^*\big) \cdot \mathcal{U},$$

where $K_i = \mathbb{Q}(\zeta_{p^{s_i}})$ and $\mathcal{U}$ is an open subgroup of the product $\prod_i \mathbb{A}(\mathcal{O}_i)^*$, which is maximal at all $v \neq p$. For $b \in (\mathbb{Z}/p\mathbb{Z})^*$, denote by σ_b the Galois automorphism of $\mathbb{Q}(\zeta_{p^\infty}) = \bigcup_m \mathbb{Q}(\zeta_{p^m})$ given by $\sigma(\zeta) = \zeta^{\omega(b)}$. We can see [Cassou-Noguès and Taylor 1985] that the operator ψ_a, for $a \in (\mathbb{Z}/p\mathbb{Z})^* \subset (\mathbb{Z}/p^N\mathbb{Z})^*$, is induced by the action of σ_a on the product $\prod_i \mathbb{A}(K_i)^*$. Let $\mathcal{M}_G \simeq \bigoplus_i \mathrm{Mat}_{n_i \times n_i}(\mathbb{Z}[\zeta_{p^{s_i}}])$ be a maximal $\mathbb{Z}[G]$-order in $\mathbb{Q}[G]$. Denote by $D(\mathbb{Z}[G])$ the kernel of the natural group homomorphism $\mathrm{Cl}(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathcal{M}_G) = \prod_i \mathrm{Cl}(\mathbb{Q}(\zeta_{p^{s_i}}))$. The kernel group $D(\mathbb{Z}[G])$ has p-power order [Taylor 1984, p. 37]. Since $\mathcal{U}$ is maximal at $v \neq p$, we can write

$$D(\mathbb{Z}[G]) = \frac{\prod_i \mathbb{Z}_p[\zeta_{p^{s_i}}]^*}{\big(\prod_i \mathbb{Z}[\zeta_{p^{s_i}}]^*\big) \cdot \mathcal{U}_p}.$$

For $x \in \mathrm{Cl}(\mathbb{Z}[G])_p^{(2)}$, let $(x_i)_i$ be the image of x in the class group $\mathrm{Cl}(\mathcal{M}_G)$. Then x_i is a p-power torsion element in $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^{s_i}}))$ that satisfies $\sigma_a(x_i) = \omega(a)^2 \cdot x_i$ for all $a \in (\mathbb{Z}/p\mathbb{Z})^*$. The second eigenspace of the p-part of the class group of $\mathbb{Q}(\zeta_{p^m})$ is trivial. (Combine $B_2 = \frac{1}{6}$ with Herbrand's theorem and the "reflection theorems"; see [Washington 1997, Theorems 6.17 and 10.9] to see this for $m = 1$; the result then follows.) It follows that $(x_i)_i = 0$, and so x is in $D(\mathbb{Z}[G])$. Such an x is then represented by $(u_i)_i$ with $u_i \in (\mathbb{Z}_p[\zeta_{p^{s_i}}]^*)^{(2)}$. For $m \geq 0$, consider the pro-p-Sylow subgroup $(\mathbb{Z}_p[\zeta_{p^m}]^*)_p$ of $\mathbb{Z}_p[\zeta_{p^m}]^*$. Denote by $(\widehat{\mathbb{Z}[\zeta_{p^m}]^*})_p$ the intersection $\widehat{\mathbb{Z}[\zeta_{p^m}]^*} \cap (\mathbb{Z}_p[\zeta_{p^m}]^*)_p$ of the p-adic closure of the global units $\mathbb{Z}[\zeta_{p^m}]^*$ in $\mathbb{Z}_p[\zeta_{p^m}]^*$

with $(\mathbb{Z}_p[\zeta_{p^m}]^*)_p$. If $p \geq 5$, then since the second Bernoulli number $B_2 = \frac{1}{6}$ is not divisible by p, we have

$$(\mathbb{Z}_p[\zeta_{p^m}]^*)_p^{(2)} = (\widehat{\mathbb{Z}[\zeta_{p^m}]}^*)_p^{(2)}$$

by a classical result of Iwasawa (see for example [Washington 1997, Theorem 13.56; Oliver 1983, p. 296]). This shows that x is trivial in $D(\mathbb{Z}[G])$. If $p = 3$, then since 3 is regular, we have as above $\mathrm{Cl}(\mathbb{Z}[G])_p^{(2)} = \mathrm{Cl}(\mathbb{Z}[G])_p^{(0)} = D(\mathbb{Z}[G])_p^{(0)}$. This group is annihilated by $A(G)$ by [Oliver 1983, Theorem 9]; in this case, $A(G) = p^{n-1} = 3^{n-1}$ by [Lam 1968]. □

6.3.5. We can now show Theorems 1.0.2 and 1.0.3 of the introduction. For this, we allow G to stand for an arbitrary finite group.

Proof of Theorem 1.0.2. Using Noetherian induction and the 0-dimensional result of Taylor exactly as in [Pappas 1998, Proposition 4.4], we see that

$$\gcd(2, \#G) \cdot \#G \cdot f_*^{\mathrm{ct}}(\mathscr{F}) = 0. \tag{6.3.6}$$

Using localization as in [Pappas 1998, Proposition 4.5], we see that the p-power torsion part of the class $f_*^{\mathrm{ct}}(\mathscr{F})$ is annihilated by any power of p that annihilates its restriction $\mathrm{Res}^G_{G_p}(f_*^{\mathrm{ct}}(\pi^*\mathscr{G}))$ in the class group $\mathrm{Cl}(\mathbb{Z}[G_p])$ of a p-Sylow G_p. By definition, this restriction is the Euler characteristic class for the G_p-cover $X \to X/G_p$. If G is a p-group of order $p \geq 5$, we have $f_*^{\mathrm{ct}}(\mathscr{F}) = 0$ in $\mathrm{Cl}(\mathbb{Z}[G])$ by Theorem 6.3.2 and Proposition 6.3.4. We can apply this to a p-Sylow G_p of G and the G_p-torsor $X \to X/G_p$. By the above, we obtain that the prime to 6 part of $f_*^{\mathrm{ct}}(\mathscr{F})$ is trivial. By (6.3.6), the 2-part is always annihilated by $\gcd(2, \#G)^{v_2(\#G)+1}$. When the 2-Sylow G_2 of G is abelian, [Pappas 1998, Theorem 1.1], applied to the cover $X \to X/G_2$, shows that the restriction of $f_*^{\mathrm{ct}}(\mathscr{F})$ to $\mathrm{Cl}(\mathbb{Z}[G_2])$ is 2-torsion. In general, by [Pappas 1998, Theorem 1.1], the restriction of $f_*^{\mathrm{ct}}(\mathscr{F})$ to $\mathrm{Cl}(\mathbb{Z}[G_2])$ lies in the kernel subgroup $D(\mathbb{Z}[G_2])$. The kernel subgroup $D(\mathbb{Z}[G_2])$ is trivial when the 2-group G_2 has order ≤ 4, is cyclic of order 8, or is dihedral; it has order 2 when G_2 is generalized quaternion or semidihedral [Curtis and Reiner 1987, p. 129, line 30; Taylor 1984, Theorem 2.1, p. 79]. Also, by Theorem 6.3.2 and Proposition 6.3.4, when $p = 3$, the restriction of $f_*^{\mathrm{ct}}(\mathscr{F})$ to $\mathrm{Cl}(\mathbb{Z}[G_3])$ is annihilated by the Artin exponent $A(G_3)$. If G_3 is abelian, this restriction is trivial, again by [Pappas 1998, Theorem 1.1]. Theorem 1.0.2 now follows. □

Proof of Theorem 1.0.3. Let us note that, in this situation, we have $\mathrm{H}^0(X_{\mathbb{Q}}, \mathbb{O}_{X_{\mathbb{Q}}}) = \mathrm{H}^0(Y_{\mathbb{Q}}, \mathbb{O}_{Y_{\mathbb{Q}}})$, and since the G-cover $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is unramified, the Hurwitz formula gives $g_X - h = \#G \cdot (g_Y - h)$. The result then follows from Theorem 1.0.2 exactly as in the proof of [Pappas 1998, Theorem 5.5] provided we show that $\mathrm{H}^1(X, \omega_X) \simeq \mathbb{Z}^{\oplus h}$. Since G acts trivially on $\mathrm{H}^0(X_{\mathbb{Q}}, \mathbb{O}_{X_{\mathbb{Q}}})$, we have $\mathrm{H}^0(X, \mathbb{O}_X) \simeq \mathbb{Z}^{\oplus h}$ with trivial G-action. Under the rest of our assumptions, duality implies that there is a G-equivariant isomorphism $\mathrm{H}^1(X, \omega_X) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}^0(X, \mathbb{O}_X), \mathbb{Z})$, and the result then follows. □

7. Tamely ramified covers of curves

7.1. ***Curves over*** $\mathbb{Z}$**.** We assume that G acts tamely on $f : X \to \mathrm{Spec}(\mathbb{Z})$, which is projective, flat, and a local complete intersection of relative dimension 1. We also assume that $Y = X/G$ is irreducible and regular; then $\pi : X \to Y$ is finite and flat. Let U be the largest open subscheme of X such that $\pi : U \to V = U/G$ is étale. The complements $R(X/Y) = X - U$ and $B(X/Y) = Y - V$ are respectively the ramification and branch loci of π. The ramification locus is the closed subset of X defined by the annihilator $\mathrm{Ann}(\Omega^1_{X/Y})$. Our assumption of tameness implies that both the ramification and branch loci are fibral, i.e., are subsets of the union of fibers of $X \to \mathrm{Spec}(\mathbb{Z})$ and $Y \to \mathrm{Spec}(\mathbb{Z})$, respectively, over a finite set S of primes (p) [Chinburg et al. 1997a, §1.2]. Fix a prime $\ell \neq 2$ that does not divide $\#G$.

7.1.1. Denote by $F_i\mathrm{G}_0(Y)$ the subgroup of elements of $\mathrm{K}_0(Y) = \mathrm{G}_0(Y)$ represented as linear combinations of coherent sheaves supported on subschemes of Y of dimension $\leq i$. Consider the homomorphisms

$$\begin{aligned} \overline{\chi} : F_1\mathrm{G}_0(Y) &\to \mathrm{Cl}(\mathbb{Z}[G]), & \overline{\chi}(c) &= f_*^{\mathrm{ct}}(\pi^*(c)), \\ \mathrm{cl}_{X/Y} : F_1\mathrm{G}_0(Y) &\to \mathrm{Cl}(\mathbb{Z}'[G])^\wedge, & \mathrm{cl}_{X/Y}(c) &:= f_*^\wedge(\pi^*(c) \cdot \theta^\ell(T^\vee_{X/Y})^{-1}), \end{aligned}$$

where $\theta^\ell(T^\vee_{X/Y})^{-1}$ is as in Section 5.5.5. Note that a value of $\mathrm{cl}_{X/Y}$ appears in the right-hand side of (5.5.8).

Proposition 7.1.2. *Under the above assumptions,*

(1) *the image of* $\overline{\chi}$ *is* $\gcd(2, \#G)$*-torsion and*

(2) *the image of* $\mathrm{cl}_{X/Y}$ *is* $(\ell - 1)$*-torsion.*

Proof. We note that, under our assumptions, we have isomorphisms $\mathrm{Pic}(Y) = \mathrm{CH}_1(Y) \xrightarrow{\sim} F_1\mathrm{G}_0(Y)/F_0\mathrm{G}_0(Y)$ and $\mathrm{CH}_0(Y) \xrightarrow{\sim} F_0\mathrm{G}_0(Y)$. (This follows from "Riemann–Roch without denominators" as in [Soulé 1985]; see also [Fulton 1998, Example 15.3.6]). Here $\mathrm{CH}_i(Y)$ is the Chow group of dimension-i cycles modulo rational equivalence on Y, and both maps are given by sending the class $[V]$ of a dimension-1 or -0, respectively, integral subscheme V of Y to $i_*([\mathbb{O}_V])$ where $i : V \to Y$ is the corresponding morphism.

Now observe:

(a) Both $\mathrm{cl}_{X/Y}$ and $\overline{\chi}$ are trivial on $F_0\mathrm{G}_0(Y)$. By 2-dimensional class field theory [Kato and Saito 1983, Theorem 2], $\mathrm{CH}_0(Y)$ is a finite abelian group and there is a reciprocity isomorphism $\mathrm{CH}_0(Y) \xrightarrow{\sim} \tilde{\pi}_1^{\mathrm{ab}}(Y)$, where $\tilde{\pi}_1^{\mathrm{ab}}(Y)$ classifies unramified abelian covers of Y that split completely over all real-valued points of Y. Suppose $Y' \to Y$ is an irreducible unramified abelian Galois cover. A standard argument using the classical description of unramified abelian covers of curves via isogenies of their Jacobians (or, alternatively, smooth base change for étale cohomology) shows that there is an infinite set of primes q such that the base change $Y'_{\mathbb{F}_q} \to Y_{\mathbb{F}_q}$ is nonsplit, i.e., such that $Y'_{\mathbb{F}_q}$ is irreducible. By applying this to the universal cover $Y^{\mathrm{uni}} \to Y$ with Galois group $\mathrm{CH}_0(Y)$, we obtain that there is a prime q not in $\{\ell\} \cup S$ such that $Y_{\mathbb{F}_q}$ is smooth and with the property that $Y^{\mathrm{uni}}_{\mathbb{F}_q}$ is irreducible. This implies that the Frobenius elements of the closed points of the smooth projective curve $Y_{\mathbb{F}_q}$ generate the Galois group or, in other words, that the group $\mathrm{CH}_0(Y)$ is generated by the classes of points that are supported on $Y_{\mathbb{F}_q} \subset Y$. Now $\theta^\ell(T^\vee_{X/Y})^{-1}_{|U} = 1$, and since $X_{\mathbb{F}_q} \subset U$, if c corresponds to a point on $Y_{\mathbb{F}_q}$, we obtain $\mathrm{cl}_{X/Y}(c) = f^{\mathrm{ct}}_*(\pi^*(c)) = 0$ by the normal basis theorem for the G-Galois algebra that corresponds to $\pi^{-1}(c)$. (See also [Chinburg et al. 1997a, Theorem 1.3.2]).

(b) By (a) above, $\mathrm{cl}_{X/Y}$ and $\bar{\chi}$ both factor through $F_1\mathrm{G}_0(Y)/F_0\mathrm{G}_0(Y) \simeq \mathrm{Pic}(Y)$. Suppose now that $\delta \in \mathrm{Pic}(Y)$. By [Chinburg et al. 1997a, Proposition 9.1.3] (the assumption that the special fibers are divisors with normal crossings is not needed for this), there is a "harmless" base extension given by a number field $N/\mathbb{Q}$, unramified at all primes over S, of degree $[N:\mathbb{Q}]$ a power of a prime number $\neq \ell$, and $[N:\mathbb{Q}] \equiv 1 \bmod \#\mathrm{Cl}(\mathbb{Z}[G])$ such that the following is true: we can write the base change $\delta_{\mathbb{O}_N} \in \mathrm{Pic}(Y_{\mathbb{O}_N})$ as a sum $\delta_{\mathbb{O}_N} = \sum_i m_i[D_i]$ with $m_i = \pm 1$, where D_i are horizontal divisors in $Y_{\mathbb{O}_N}$, which at most intersect each irreducible component of $(Y_{\mathbb{O}_N})_p$, $p \in S$, transversely at closed points that are away from the singular locus of the reduced special fiber $(Y_{\mathbb{O}_N})^{\mathrm{red}}_p$. Denote by $\iota_i : D_i \hookrightarrow Y_{\mathbb{O}_N}$ the closed immersion and by $\mathfrak{D}_i$ the normalization of D_i. Then, we can see as in [loc. cit.] that, for each i, the morphism $\widetilde{\mathfrak{D}}_i = \pi^{-1}(\mathfrak{D}_i) = \mathfrak{D}_i \times_Y X \to \mathfrak{D}_i$ is a tame G-cover of regular affine schemes of dimension 1 flat over $\mathbb{Z}$, which is unramified away from S. The normalization morphism $q_i : \mathfrak{D}_i \to D_i$ is an isomorphism over an open subset of D_i that contains all primes over S. As in [loc. cit.], we can now see that our conditions on the field N together with $\delta_{\mathbb{O}_N} = \sum_i m_i[D_i]$ imply that

$$\bar{\chi}(\delta) = \textstyle\sum_i m_i \cdot \bar{\chi}(X_{\mathbb{O}_N}, (\iota_i)_*\mathbb{O}_{D_i}) = \sum_i m_i \cdot [\Gamma(\widetilde{\mathfrak{D}}_i, \mathbb{O}_{\widetilde{\mathfrak{D}}_i})]$$

in $\mathrm{Cl}(\mathbb{Z}[G])$. Observe that the classes $[\Gamma(\widetilde{\mathfrak{D}}_i, \mathbb{O}_{\widetilde{\mathfrak{D}}_i})]$ are $\gcd(2, \#G)$-torsion by Taylor's theorem and part (1) follows.

The proof of part (2) is similar. For simplicity, write $D = D_i$, $\mathfrak{D} = \mathfrak{D}_i$, $\widetilde{\mathfrak{D}} = \widetilde{\mathfrak{D}}_i$, and $\iota : D \hookrightarrow Y_{\mathbb{O}_N}$ and denote by $h : \widetilde{\mathfrak{D}} \to \mathrm{Spec}(\mathbb{Z})$ the structure morphism. By Taylor's theorem, $\gcd(2, \#G) \cdot h^{\mathrm{ct}}_*(\mathbb{O}_{\widetilde{\mathfrak{D}}}) = 0$ in $\mathrm{Cl}(\mathbb{Z}'[G])$. Therefore, since $\ell - 1$ is even, by applying (5.5.8) (for $d = 0$) to the cover $X = \widetilde{\mathfrak{D}} \to Y = \mathfrak{D}$, we obtain that

$$(\ell - 1) \cdot \big[h^\wedge_*(\theta^\ell(T^\vee_{\widetilde{\mathfrak{D}}/\mathfrak{D}})^{-1}) + h^\wedge_*(\pi^*(c') \cdot \theta^\ell(T^\vee_{\widetilde{\mathfrak{D}}/\mathfrak{D}})^{-1})\big] = 0$$

in $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ where $c' \in \mathrm{K}_0(\mathscr{D})' = \mathrm{G}_0(\mathscr{D})'$ is supported on a proper closed subset of $\mathscr{D}$. We can assume that c' is supported away from primes in S. Then, as in (a) above, we obtain that the second term in the above sum vanishes. Therefore,

$$(\ell-1)\cdot(h_*^\wedge(\theta^\ell(T^\vee_{\widetilde{\mathscr{D}}/\mathscr{D}})^{-1}))=0$$

also. Observe that $q:\mathscr{D}\to D$ is an isomorphism over an open subset of $\mathscr{D}$ whose complement has image in X disjoint from the support of $T^\vee_{X/Y}$. Also the formation of cotangent complexes of flat morphisms commutes with base change [Illusie 1971]: we can see that the base change of $\theta^\ell(T^\vee_{X/Y})^{-1}\in \mathrm{K}_0(G,X)^\wedge$ to $\widetilde{\mathscr{D}}$ is equal to $\theta^\ell(T^\vee_{\widetilde{\mathscr{D}}/\mathscr{D}})^{-1}\in \mathrm{K}_0(G,\widetilde{\mathscr{D}})^\wedge$. Using these two facts, and the projection formula, we now obtain

$$\begin{aligned}(\ell-1)\cdot \mathrm{cl}_{X_{\mathbb{O}_N}/Y_{\mathbb{O}_N}}(\iota_*[\mathbb{O}_D]) &= (\ell-1)\cdot f_*^\wedge(\pi^*(\iota_*[\mathbb{O}_D])\cdot\theta^\ell(T^\vee_{X_{\mathbb{O}_N}/Y_{\mathbb{O}_N}})^{-1})\\ &= (\ell-1)\cdot f_*^\wedge(\pi^*(\iota_*[q_*\mathbb{O}_{\mathscr{D}}])\cdot\theta^\ell(T^\vee_{X_{\mathbb{O}_N}/Y_{\mathbb{O}_N}})^{-1})\\ &= (\ell-1)\cdot h_*^\wedge(\theta^\ell(T^\vee_{\widetilde{\mathscr{D}}/\mathscr{D}})^{-1})=0.\end{aligned}$$

Here, for simplicity, we also write π for the cover $X_{\mathbb{O}_N}\to Y_{\mathbb{O}_N}$ and denote by f the structure morphism $X_{\mathbb{O}_N}\to \mathrm{Spec}(\mathbb{Z})$. As above, we can now see that we have

$$\mathrm{cl}_{X/Y}(\delta)=\textstyle\sum_i m_i\cdot \mathrm{cl}_{X_{\mathbb{O}_N}/Y_{\mathbb{O}_N}}((\iota_i)_*[\mathbb{O}_{D_i}]),$$

and this, together with the above, concludes the proof of part (2). $\square$

Corollary 7.1.3. *Under the above assumptions, if $\mathscr{G}$ is a locally free coherent $\mathbb{O}_Y$-module of rank r, then*

$$\gcd(2,\#G)\cdot(\overline{\chi}(X,\pi^*\mathscr{G})-r\cdot\overline{\chi}(X,\mathbb{O}_X))=0$$

in $\mathrm{Cl}(\mathbb{Z}[G])$.

Proof. This follows from Proposition 7.1.2(1) since $[\mathscr{G}]-r\cdot[\mathbb{O}_Y]\in F_1\mathrm{G}_0(Y)$. $\square$

7.2. *The input localization theorem.* Here we let S be the smallest finite set of rational primes that contains the support of the branch locus of $\pi:X\to Y$. For simplicity, let us set $X_S=\bigcup_{p\in S}X_{\mathbb{F}_p}$ and $\widehat{X}_S=\bigcup_{p\in S}X_{\mathbb{Z}_p}$.

Theorem 7.2.1. *Let $\pi:X\to Y$ be a tamely ramified G-cover of schemes that are projective and flat over* $\mathrm{Spec}(\mathbb{Z})$ *of relative dimension* 1. *Suppose that Y is regular and that X is a local complete intersection. Let $\mathscr{F}$ be a G-equivariant coherent $\mathbb{O}_X$-module. Then*

$$\gcd(2,\#G)^{v_2(\#G)+2}\gcd(3,\#G)^{v_3(\#G)-1}\cdot\overline{\chi}(X,\mathscr{F})$$

in $\mathrm{Cl}(\mathbb{Z}[G])$ *depends only on the pair $(\widehat{X}_S,\mathscr{F}|_{\widehat{X}_S})$ where $\mathscr{F}|_{\widehat{X}_S}$ denotes the pull-back of $\mathscr{F}$ from X to $\widehat{X}_S$.*

Proof. We consider the projections $\overline{\chi}(X, \mathscr{F})_\rho$ on the localizations $\mathrm{Cl}(\mathbb{Z}[G])_\rho$ of the finite $\mathrm{G}_0(\mathbb{Z}[G])$-module $\mathrm{Cl}(\mathbb{Z}[G])$ at the maximal ideals $\rho \subset \mathrm{G}_0(\mathbb{Z}[G])$. Recall

$$\mathrm{Cl}(\mathbb{Z}[G]) = \bigoplus\nolimits_\rho \mathrm{Cl}(\mathbb{Z}[G])_\rho.$$

Consider ρ that do not contain the kernel I_G of the rank map. The projection $\overline{\chi}(X, \mathscr{F})_\rho$ depends only on the inverse image $(\iota_*)_\rho^{-1}([\mathscr{F}])$ of the class of $\mathscr{F}$ under the isomorphism [Chinburg et al. 1997b, Theorem 6.1]

$$(\iota_*)_\rho : \mathrm{G}_0(G, X^\rho)_\rho \xrightarrow{\sim} \mathrm{G}_0(G, X)_\rho \tag{7.2.2}$$

where $\iota : X^\rho \subset X$. For such ρ, the fixed-point subscheme X^ρ is contained in the ramification locus $R = R(X/Y) \subset X_S$ and there is a similar isomorphism

$$(\hat{\iota}_*)_\rho : \mathrm{G}_0(G, X^\rho)_\rho \xrightarrow{\sim} \mathrm{G}_0(G, \widehat{X}_S)_\rho \tag{7.2.3}$$

with the property that $(\iota_*)_\rho$ and $(\hat{\iota}_*)_\rho$ commute with the base change homomorphism $\mathrm{G}_0(G, X) \to \mathrm{G}_0(G, \widehat{X}_S)$, $\mathscr{F} \mapsto \mathscr{F}|_{\widehat{X}_S}$. This shows that $(\iota_*)_\rho^{-1}([\mathscr{F}])$ and therefore also $\overline{\chi}(X, \mathscr{F})_\rho$, for $I_G \not\subset \rho$, only depend on $(\widehat{X}_S, \mathscr{F}|_{\widehat{X}_S})$. It remains to deal with maximal ρ such that $I_G \subset \rho$. These are of the form $\rho = \rho_{(1,p)}$ for some prime p that divides $\#G$. The argument in the proof of [Pappas 1998, Proposition 4.5] shows that, for such ρ, the component $\overline{\chi}(X, \mathscr{F})_\rho$ depends only on the p-power part $\overline{\chi}(X, \mathscr{F})_p$ of the restriction of $\overline{\chi}(X, \mathscr{F})$ to the p-Sylow G_p. To avoid a conflict in the notation, we will use in this proof the symbol q to denote a prime in the set S.

Note that, under our assumptions, $\pi : X \to Y$ is finite and flat. If $\mathscr{G} = (\pi_*(\mathscr{F}))^G$, there is a canonical short exact sequence of G-equivariant coherent $\mathbb{O}_X$-modules

$$0 \to \pi^*\mathscr{G} \to \mathscr{F} \to \mathscr{Y} \to 0$$

with $\mathscr{Y}$ supported on the ramification locus $R(X/Y)$. In fact, $\mathscr{Y}$ is canonically isomorphic to the cokernel of $\pi^*\mathscr{G}|_{\widehat{X}_S} \to \mathscr{F}|_{\widehat{X}_S}$ and $\mathscr{G}|_{\widehat{X}_S} = (\pi_*(\mathscr{F}|_{\widehat{X}_S}))^G$. This shows that $\mathscr{Y}$ is determined from $\mathscr{F}|_{\widehat{X}_S}$. Therefore, it is enough to show the statement for sheaves of the form $\mathscr{F} = \pi^*\mathscr{G}$. In view of Corollary 7.1.3, we first consider the case $\mathscr{F} = \mathbb{O}_X$. Notice that $\pi_*\mathbb{O}_X$ is $\mathbb{O}_Y$-locally free of rank $\#G$ on Y and hence, again by Corollary 7.1.3, the difference

$$\overline{\chi}(X, \pi^*(\pi_*(\mathbb{O}_X))) - \#G \cdot \overline{\chi}(X, \mathbb{O}_X)$$

is $\gcd(2, \#G)$-torsion. On the other hand, the G-action morphism, $m : X \times G \to X \times_Y X$, $(x, g) \mapsto (x, x \cdot g)$, restricts to an isomorphism over U. This gives a G-equivariant homomorphism

$$\pi^*(\pi_*(\mathbb{O}_X)) = \pi_*(\mathbb{O}_X) \otimes_{\mathbb{O}_Y} \mathbb{O}_X \xrightarrow{m^*} \bigoplus_{g \in G} \mathbb{O}_X = \mathrm{Maps}(G, \mathbb{O}_X)$$

of G-equivariant coherent $\mathbb{O}_X$-modules that is injective. The cokernel of m^* is supported on $R(X/Y)$ and, as above, is determined from $\widehat{X}_S$. Since we have $\overline{\chi}(X, \mathrm{Maps}(G, \mathbb{O}_X)) = 0$, this shows that $\gcd(2, \#G) \cdot (\#G) \cdot \overline{\chi}(X, \mathbb{O}_X)$ depends only on $\widehat{X}_S$. Therefore, $\gcd(2, \#G)^{v_2(\#G)+1} \cdot \overline{\chi}(X, \mathbb{O}_X)_2$ depends only on $\widehat{X}_S$. By Corollary 7.1.3 and the above, we see that $\gcd(2, \#G)^{v_2(\#G)+2} \cdot \overline{\chi}(X, \mathcal{F})_2$ depends only on $(\widehat{X}_S, \mathcal{F}|_{\widehat{X}_S})$.

We now deal with the p-power part $\overline{\chi}(X, \mathbb{O}_X)_p$ for p odd. By the above, it is enough to consider the case that G is a p-group. We claim that, in this case, the I_G-adic completion $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ is the p-power part $\mathrm{Cl}(\mathbb{Z}'[G])_p$. Indeed, we observe that the class $[\mathbb{Z}'[G]] \in \mathrm{G}_0(\mathbb{Z}[G])$ annihilates $\mathrm{Cl}(\mathbb{Z}'[G])$, but since it has rank $\#G$, it is invertible in the localizations of $\mathrm{G}_0(\mathbb{Z}[G])$ at $\rho = I_G + (q)$ for all $q \neq p$. Hence, the completion $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ is supported at p and the claim follows since, by Section 2.1.2, the only prime ideal of $\mathrm{G}_0(\mathbb{Z}[G])$ supported over p is $I_G + (p)$ (see the proof of [Pappas 1998, Proposition 4.5]). Combining Proposition 7.1.2(2) and (5.5.8), we obtain

$$(\ell - 1)(\psi^\ell - \ell^{-1}) \cdot \overline{\chi}(X, \mathbb{O}_X) = \ell^{-1}(\ell - 1) \cdot f_*^\wedge(r^\ell_{X/Y}) \tag{7.2.4}$$

in $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$. Now apply (7.2.4) to a prime ℓ as in Proposition 6.2.3. We see that Proposition 6.3.4 implies that the multiple $\gcd(3, \#G)^{v_3(\#G)-1} \cdot \overline{\chi}(X, \mathbb{O}_X)_p$ is determined by $f_*^\wedge(r^\ell_{X/Y}) \in \mathrm{Cl}(\mathbb{Z}'[G])^\wedge$.

We will show that $f_*^\wedge(r^\ell_{X/Y})$ depends only on the G-cover $\widehat{X}_S \to \widehat{Y}_S$. Set $U_S = X - X_S$. Recall

$$r^\ell_{X/Y} = \theta^\ell(T_X^\vee)^{-1} \cdot \theta^\ell(\pi^* T_Y^\vee) - 1$$

is in $\mathrm{K}_0(G, X)^\wedge$ with trivial image in $\mathrm{K}_0(G, U_S)^\wedge$ under restriction. Observe that $f_*^\wedge(r^\ell_{X/Y})$ only depends on the image of $r^\ell_{X/Y}$ in $\mathrm{G}_0(G, X)^\wedge$. We have a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{G}_1(G, U_S)^\wedge & \longrightarrow & \mathrm{G}_0(G, X_S)^\wedge & \longrightarrow & \mathrm{G}_0(G, X)^\wedge & \longrightarrow & \mathrm{G}_0(G, U_S)^\wedge & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\
\mathrm{G}_1(G, \widehat{U}_S)^\wedge & \longrightarrow & \mathrm{G}_0(G, X_S)^\wedge & \longrightarrow & \mathrm{G}_0(G, \widehat{X}_S)^\wedge & \longrightarrow & \mathrm{G}_0(G, \widehat{U}_S)^\wedge & \longrightarrow & 0
\end{array}$$

where the rows are exact and are obtained by completing the standard equivariant localization sequences [Thomason 1987]. Here $\widehat{U}_S = \bigcup_{q \in S} X_{\mathbb{Q}_q}$ and the vertical maps are given by base change; the second vertical map is the identity. Since the action of G on $\widehat{U}_S$ is free, we have by étale descent $\mathrm{G}_1(G, \widehat{U}_S) = \mathrm{G}_1(\widehat{U}_S/G) = \bigoplus_{q \in S} \mathrm{G}_1(Y_{\mathbb{Q}_q})$. (In particular, this also implies that $I_G^m \cdot \mathrm{G}_1(G, \widehat{U}_S) = (0)$ for $m > 1$ and so $\mathrm{G}_1(G, \widehat{U}_S)^\wedge = \mathrm{G}_1(G, \widehat{U}_S)$.) As in [Chinburg 1994; Chinburg et al. 1997a], we can see that $f_*^\wedge \cdot (i_S)_*^\wedge : \mathrm{G}_0(G, X_S)^\wedge \to \mathrm{G}_0(G, X)^\wedge \to \mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ can be written as a composition $(f_S)_*: \mathrm{G}_0(G, X_S)^\wedge \to \bigoplus_{q\in S} \mathrm{K}_0(\mathbb{F}_q[G])^\wedge \to \mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ where the first arrow is given by the sum of the projective equivariant Euler characteristics for the G-schemes $f_q : X_{\mathbb{F}_q} \to \mathrm{Spec}(\mathbb{F}_q)$.

Proposition 7.2.5. *The homomorphism $(f_S)_*$ vanishes on the image of the map $\mathrm{G}_1(G, \widehat{U}_S)^\wedge \to \mathrm{G}_0(G, X_S)^\wedge$.*

Proof. For this, we need:

Lemma 7.2.6. *The maps above give a commutative diagram*

$$\begin{tikzcd} \mathrm{G}_1(Y_{\mathbb{Q}_q}) \arrow[r, "\sim"] \arrow[d, "(g_{\mathbb{Q}_q})_*"'] & \mathrm{G}_1(G, X_{\mathbb{Q}_q}) \arrow[r] & \mathrm{G}_0(G, X_{\mathbb{F}_q}) \arrow[d, "(f_q)_*"'] \\ \mathbb{Q}_q^* \arrow[r] & \mathrm{G}_1(\mathbb{Q}_q[G]) \arrow[r] & \mathrm{G}_0(\mathbb{F}_q[G]) \end{tikzcd}$$

where the bottom left horizontal arrow

$$\mathbb{Q}_q^* \to \mathrm{G}_1(\mathbb{Q}_q[G]) = \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}(\mathscr{R}_{\overline{\mathbb{Q}}_q}(G), \overline{\mathbb{Q}}_q^*)$$

sends $\mu \in \mathbb{Q}_q^*$ to the character function $\chi \mapsto \mu^{\deg(\chi)}$.

Proof. Using the Quillen–Gersten spectral sequence, we can see that $\mathrm{G}_1(Y_{\mathbb{Q}_q})$ is generated by the following two types of elements: constants $c \in \mathbb{Q}_q^*$ considered as giving automorphisms of the structure sheaf of $Y_{\mathbb{Q}_q}$ (i.e., elements in the image of $(f_{\mathbb{Q}_q})^* : \mathbb{Q}_q^* = \mathrm{G}_1(\mathrm{Spec}(\mathbb{Q}_q))^* \to \mathrm{G}_1(Y_{\mathbb{Q}_q})$) and elements in the image of $k(y)^* = \mathrm{G}_1(\mathrm{Spec}(k(y))) \to \mathrm{G}_1(Y_{\mathbb{Q}_q})$ where $k(y)$ is the residue field of some closed point y of $Y_{\mathbb{Q}_q}$ [Gillet 1981, Example 4.6]. It is enough to check the commutativity on these two types of elements. The statement for the first type follows easily from the fact (a consequence of Lefschetz–Riemann–Roch or of the main result of [Nakajima 1984]) that the G-character $[\mathrm{H}^0(X_{\mathbb{Q}_q}, \mathcal{O}_{X_{\mathbb{Q}_q}})] - [\mathrm{H}^1(X_{\mathbb{Q}_q}, \mathcal{O}_{X_{\mathbb{Q}_q}})]$ is a multiple of the regular character. The statement for the second type follows from an explicit calculation by using the normal basis theorem. □

The lemma implies that the values of $(f_S)_*$ on the image of $\mathrm{G}_1(G, \widehat{U}_S)^\wedge \to \mathrm{G}_0(G, X_S)^\wedge$ are in the subgroup generated by the sums of images of the character functions $\chi \mapsto \mu^{\deg(\chi)}$. These values are all classes of virtually free $\mathbb{F}_q[G]$-modules in $\mathrm{K}_0(\mathbb{F}_q[G]) \subset \mathrm{G}_0(\mathbb{F}_q[G])$, and this shows the desired vanishing. □

Proposition 7.2.5, combined with a chase on the commutative diagram on page 1510, now implies that $f_*^\wedge(r^\ell_{X/Y})$ only depends on the image of $r^\ell_{X/Y}$ under the base change $\beta : \mathrm{G}_0(G, X)^\wedge \to \mathrm{G}_0(G, \widehat{X}_S)^\wedge$. By flat base change for the cotangent complex [Illusie 1971], this image is the class of

$$r^\ell_{\widehat{X}_S/\widehat{Y}_S} := \theta^\ell(T^\vee_{\widehat{X}_S})^{-1} \cdot \theta^\ell(\pi^* T^\vee_{\widehat{Y}_S}) - 1.$$

Therefore, $f_*^\wedge(r^\ell_{X/Y})$ only depends on $\widehat{X}_S \to \widehat{Y}_S$, which in turn, since $\widehat{Y}_S = \widehat{X}_S/G$, only depends on the G-scheme $\widehat{X}_S$.

Combining all these, we now obtain the statement of Theorem 7.2.1. $\square$

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