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# Bivariant algebraic cobordism

José Luis González and Kalle Karu

We associate a bivariant theory to any suitable oriented Borel–Moore homology theory on the category of algebraic schemes or the category of algebraic $G$-schemes. Applying this to the theory of algebraic cobordism yields operational cobordism rings and operational $G$-equivariant cobordism rings associated to all schemes in these categories. In the case of toric varieties, the operational $T$-equivariant cobordism ring may be described as the ring of piecewise graded power series on the fan with coefficients in the Lazard ring.

## 1. Introduction

The purpose of this article is to study the *operational bivariant theory* $B^*$ associated to a *refined oriented Borel–Moore prehomology theory* $B_*$, and the equivariant versions of these theories. We apply this to the algebraic cobordism theory $\Omega_*$ of Levine and Morel [2007] to construct the *operational bivariant cobordism theory* $\Omega^*$. As an application, we describe the *operational $T$-equivariant cobordism* $\Omega^*_T(X_\Delta)$ for a quasiprojective toric variety $X_\Delta$.

Bivariant theories were defined in [Fulton and MacPherson 1981; Fulton 1998]. A bivariant theory assigns a group $B^*(X \to Y)$ to every morphism $X \to Y$ of schemes. The theory contains both a covariant homology theory $B_*(X) = B^*(X \to \text{pt})$ and a contravariant cohomology theory $B^*(X) = B^*(\text{Id}_X : X \to X)$, but the bivariant

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theory can be more general in the sense that there may be invariants of the theory that are not determined by homology and cohomology alone [Fulton and MacPherson 1981]. It was also shown in [Fulton and MacPherson 1981; Fulton 1998] how to extend a given homology theory $B_*$ to a bivariant theory. Bivariant classes in this theory are certain compatible operators on the homology groups $B_*$; hence the bivariant theory is called the operational bivariant theory. The only bivariant theories we consider in this paper are the operational ones. An operational bivariant theory can be viewed as a method of constructing a cohomology theory $B^*$ out of a homology theory $B_*$. The cohomology theory takes values in rings, hence there is a well-defined intersection product in this theory. The Chern class operators naturally lie in the cohomology $B^*$.

The operational cohomology theory $B^*$ at first seems very intractable. A single element of $B^*(X)$ is defined by an infinite set of homomorphisms. However, Kimura [1992] has shown that, in case of Chow theory $A_*$, the bivariant cohomology groups $A^*(X)$ for an arbitrary variety $X$ can often be computed if one knows the homology groups $A_*(Y)$ for smooth varieties $Y$. Payne [2006] carried out this computation for the equivariant Chow cohomology $A^*_T$ of toric varieties. By a result of Brion [1997], the $T$-equivariant Chow ring $A^*_T(X_\Delta)$ of a smooth toric variety $X_\Delta$ can be identified with the group of integral piecewise polynomial functions on the fan $\Delta$. Payne showed that the ring of such functions on an arbitrary fan $\Delta$ gives the operational $T$-equivariant Chow cohomology $A^*_T(X_\Delta)$. A similar computation in the case of $K$-theory is done by Anderson and Payne [2015]. Brion and Vergne [1997] (see also [Vezzosi and Vistoli 2003]) proved that the $T$-equivariant $K$-theory ring of a smooth toric variety $X_\Delta$ is isomorphic to the ring of integral piecewise exponential functions on the fan $\Delta$. Anderson and Payne show that for an arbitrary fan $\Delta$ this ring gives the operational $T$-equivariant $K$-cohomology of the variety $X_\Delta$.

One of the goals of this article is to extend Anderson and Payne's results to the case of algebraic cobordism. The $T$-equivariant algebraic cobordism of smooth toric varieties was computed by Krishna and Uma [2013]. Using the same terminology as in the case of Chow theory and $K$-theory, the equivariant cobordism ring $\Omega^*_T(X_\Delta)$ of a smooth quasiprojective toric variety $X_\Delta$ can be identified with the ring of piecewise graded power series on the fan $\Delta$, with coefficients in the Lazard ring $\mathbb{L}$. We prove below that, for any quasiprojective toric variety $X_\Delta$, the ring of piecewise graded power series on $\Delta$ gives the operational $T$-equivariant cobordism ring $\Omega^*_T(X_\Delta)$.

We start by constructing the operational bivariant theory $B^*$ for a suitably general class of homology theories $B_*$. To carry out the construction of the operational bivariant theory, it suffices to assume that $B_*$ is a *refined oriented Borel–Moore prehomology theory* (ROBM prehomology theory). This is a weakening of the notion of oriented Borel–Moore homology theory [Levine and Morel 2007, Definition 5.3.1] with refined Gysin homomorphisms, where we do not require the projective bundle, extended homotopy and the cellular decomposition properties. The various constructions can be summarized by a diagram as follows:

$$\begin{tikzcd}
B_*^G \arrow[r] & B_G^* \\
B_* \arrow[r] \arrow[u] & B^* \arrow[u]
\end{tikzcd}$$

Each horizontal arrow associates to an ROBM prehomology theory its operational bivariant theory. This step can be applied to an arbitrary ROBM prehomology theory $B_*$, including its equivariant version $B_*^G$ for a linear algebraic group $G$. The vertical arrows associate to a theory its $G$-equivariant version using Totaro's [1999] algebraic approximation of the Borel construction from topology. For these constructions to be well-defined, we need to assume that the ROBM prehomology theory $B_*$ has the localization and homotopy properties. The construction of $B_*^G$ is a direct generalization of similar constructions in Chow theory by Totaro [1999] and by Edidin and Graham [1998], and in algebraic cobordism by Deshpande [2009], Krishna [2012] and Heller and Malagón-López [2013]. Note that, unlike equivariant Chow and algebraic cobordism theories, the equivariant $K$-theory is constructed not by the Borel construction, but using equivariant sheaves.

We will prove that the above square commutes; more precisely, the two ways to construct $B_G^*$ agree if we assume that the original theory $B_*$ has certain exact descent sequences for envelopes. Such sequences were first proved by Gillet [1984] in $K$-theory and Chow theory, and they were used by Edidin and Graham [1998] to prove the commutativity of the square above for Chow theory. The Edidin–Graham proof can be generalized to an arbitrary ROBM prehomology theory, but the descent property depends on the theory. The descent property for the algebraic cobordism theory was proved in [González and Karu 2015].

The descent property in the Chow theory was used by Kimura [1992] to give an inductive construction of operational Chow cohomology classes. We will generalize Kimura's proofs to arbitrary ROBM prehomology theories that satisfy the descent property.

Levine and Morel [2007] showed that the algebraic cobordism theory is universal among all oriented Borel–Moore homology theories. We do not know any similar universality statement for the operational cobordism theory. In any bivariant theory, multiplication with a bivariant class defines an operation on homology. This gives a functor from the bivariant theory to the operational theory constructed from its homology. Hence the operational theory is the universal target of all bivariant theories with a fixed homology theory. Yokura [2009] has proposed a geometric method for constructing a bivariant algebraic cobordism theory $\widetilde{\Omega}^*$, which would be universal among a class of oriented bivariant theories. The homology of this bivariant theory $\widetilde{\Omega}^*$ is expected to be the algebraic cobordism $\Omega_*$. By universality, there should exist a natural transformation from Yokura's bivariant $\widetilde{\Omega}^*$ to the operational $\Omega^*$, restricting to an isomorphism between the homology theories. To relate these two theories would then be an interesting problem.

We state our result describing the operational $T$-equivariant cobordism of a toric variety only in the quasiprojective case since this assumption is used in the development of equivariant cobordism in our references [Deshpande 2009; Krishna 2012; Heller and Malagón-López 2013]. However, for torus actions one could define a version of equivariant cobordism without the quasiprojectivity assumption, and our description would still hold in that setting. More precisely, the required GIT quotients for the Borel construction exist for torus actions even in the nonquasiprojective case because $T$ is a special group (see Proposition 23 in [Edidin and Graham 1998]); in addition in the case of $T$-equivariant cobordism one has induced pushforwards for proper morphisms, and then the argument provided in Section 7 goes through without the quasiprojectivity assumption.

***Dependence on the refined Gysin homomorphisms for cobordism.*** Following Fulton and MacPherson, we associate an operational bivariant theory to any ROBM prehomology theory, which in particular must have refined Gysin homomorphisms as in Definition 2.5. Constructing the refined Gysin homomorphisms in the theory of algebraic cobordism is the most delicate part of [Levine and Morel 2007]. A reader with the case of algebraic cobordism in mind may therefore wonder how much our results depend on these homomorphisms. Refined Gysin homomorphisms appear in our definition of operational theories $B^*$ in Section 3C (in axiom ($C_3$)), in the definition of equivariant theories $B_*^G$ in Sections 4B–4C (as the maps in a directed system used to define $B_*^G$), in the definition of the ring structure on $B_*$ in the smooth case in Section 2C (as a pullback along the diagonal), in the proof of the Poincaré duality isomorphism Proposition 3.2 (as a pullback along a graph morphism), and in the proof of the commutativity of the square above in Proposition 5.2 using Poincaré duality. Finally, Gysin homomorphisms appear in the theorem of Krishna and Uma, Theorem 7.2, that describes the cobordism ring of a smooth toric variety via pullback to the fixed-point set.

The paper is organized as follows. We define refined oriented Borel–Moore prehomology theories in Section 2. In Section 3 we define bivariant theories and associate the operational bivariant theory $B^*$ to any ROBM prehomology theory $B_*$ in the categories $\mathrm{Sch}_k$ and $G$-$\mathrm{Sch}_k$, which among other properties has the original theory $B_*$ as its associated homology theory (see Proposition 3.1) and has a Poincaré duality isomorphism between homology and cohomology in the nonsingular case (see Proposition 3.2). In Section 4 we start from any ROBM prehomology theory $B_*$ on $\mathrm{Sch}_k$ that satisfies the localization and homotopy properties and construct the $G$-equivariant ROBM prehomology theory $B_*^G$ on $G$-$\mathrm{Sch}_k$ by taking a limit over successively better approximations of the Borel construction. In Section 5 we show that if $B_*$ has exact descent sequences (5-1), then the computation of bivariant classes can be inductively reduced to the nonsingular case (see Theorem 5.6 and Theorem 5.3), and that furthermore the operational equivariant theory $B_G^*$ can alternatively be computed by applying the limit construction directly to the operational theory $B^*$ (see Proposition 5.2).

In Section 6 we overview the theory of algebraic cobordism $\Omega_*$. We conclude this article in Section 7 by showing in Theorem 7.3 that the operational $T$-equivariant cobordism ring of a quasiprojective toric variety can be described as the ring of piecewise graded power series on the fan with coefficients in the Lazard ring.

## 2. Refined oriented Borel–Moore prehomology theories

### 2A. *Notation and conventions.*

**2A.1.** Throughout this article all of our schemes will be defined over a fixed field $k$. We denote by $\mathrm{Sch}_k$ the category of separated finite-type schemes over $\operatorname{Spec} k$ and by $\mathrm{Sch}'_k$ the subcategory of $\mathrm{Sch}_k$ with the same objects but whose morphisms are the projective morphisms. We denote by $\mathrm{Sm}_k$ the full subcategory of $\mathrm{Sch}_k$ of smooth and quasiprojective schemes. By a smooth morphism we always mean a smooth and quasiprojective morphism. $\mathrm{Ab}_*$ will denote the category of graded abelian groups.

**2A.2.** Let $G$ be a linear algebraic group. A $G$-linearization of a line bundle $f : L \to X$ over the $G$-scheme $X$ is a $G$-action $\Phi : G \times L \to L$ on $L$ such that $f$ is $G$-equivariant and for every $x \in X$ and $g \in G$ the action map $\Phi_g : L_x \to L_{gx}$ is linear. We denote by $G$-$\mathrm{Sch}_k$ the category whose objects are the separated finite-type $G$-schemes over $\operatorname{Spec} k$ that admit an ample $G$-linearizable line bundle and whose morphisms are $G$-equivariant morphisms. We denote by $G$-$\mathrm{Sch}'_k$ the subcategory of $G$-$\mathrm{Sch}_k$ with the same objects but whose morphisms are the projective $G$-equivariant morphisms. Note that all schemes in $G$-$\mathrm{Sch}_k$ are assumed to be quasiprojective; this is needed in the construction of equivariant theories using the GIT quotients.

**2A.3.** In Sections 5–7 we will assume that $k$ has characteristic zero and in Sections 4B–4D we will assume that $k$ is infinite. The assumption on the characteristic of $k$ is only meant to guarantee the existence of smooth projective envelopes in the categories $\mathrm{Sch}_k$ and $G$-$\mathrm{Sch}_k$ and to provide the setting for the use of the Levine–Pandharipande version of algebraic cobordism, which requires resolution of singularities by projective morphisms, weak factorization for birational maps and some Bertini-type theorems that hold in characteristic zero. The assumption on the cardinality of $k$ is only used explicitly in the proof of Proposition 4.3.

**2A.4.** We call a morphism $f : Z \to X$ in one of the categories $\mathscr{C} = \mathrm{Sch}_k$ or $\mathscr{C} = G\text{-}\mathrm{Sch}_k$ a locally complete intersection morphism in $\mathscr{C}$, or simply an lci morphism in $\mathscr{C}$, if there exist a regular embedding $i : Z \to Y$ and a smooth morphism $g : Y \to X$, with $g$ and $i$ in $\mathscr{C}$ such that $f = gi$. When we work in the category $G\text{-}\mathrm{Sch}_k$ and we say that a morphism $f$ is an equivariant lci morphism, or simply an lci morphism, we mean that $f$ is an lci morphism in $G\text{-}\mathrm{Sch}_k$. We follow the convention that smooth morphisms, and more generally lci morphisms, are assumed to have a relative dimension. If $f : X \to Y$ is an lci morphism of relative dimension $d$ (or relative codimension $-d$) and $Y$ is irreducible, then $X$ is a scheme of pure dimension equal to $\dim Y + d$.

## 2B. *ROBM prehomology theories.*

**2B.1.** For simplicity, we unify the treatment of the cases when the ambient category is $\mathrm{Sch}_k$ or $G\text{-}\mathrm{Sch}_k$ for some algebraic group $G$. Therefore, through the rest of this section we fix the category $\mathscr{C}$, which is either $\mathrm{Sch}_k$ or $G\text{-}\mathrm{Sch}_k$, and we assume that all the schemes and morphisms are in $\mathscr{C}$ (e.g., the statement *for any morphism* should be interpreted as *for any morphism in* $\mathscr{C}$). Likewise, when $\mathscr{C} = G\text{-}\mathrm{Sch}_k$, by an lci morphism we mean an equivariant lci morphism. The category $\mathscr{C}'$ is defined to be $\mathrm{Sch}'_k$ or $G\text{-}\mathrm{Sch}'_k$, depending on whether $\mathscr{C}$ is equal to $\mathrm{Sch}_k$ or $G\text{-}\mathrm{Sch}_k$, respectively.

Let us start by recalling the definition of a Borel–Moore functor on $\mathscr{C}$, and several extra structures on it, from [Levine and Morel 2007].

**Definition 2.1.** A *Borel–Moore functor* on $\mathscr{C}$ is given by:

- ($\mathrm{D}_1$) An *additive* functor $H_* : \mathscr{C}' \to \mathrm{Ab}_*$, i.e., a functor $H_* : \mathscr{C}' \to \mathrm{Ab}_*$ such that, for any finite family $(X_1, \dots, X_r)$ of schemes in $\mathscr{C}'$, the morphism

$$\bigoplus_{i=1}^{r} H_*(X_i) \to H_*\left(\coprod_{i=1}^{r} X_i\right)$$

  induced by the projective morphisms $X_i \subseteq \coprod_{i=1}^{r} X_i$ is an isomorphism.

- ($\mathrm{D}_2$) For each smooth equidimensional morphism $f : Y \to X$ of relative dimension $d$ in $\mathscr{C}$, a homomorphism of graded groups

$$f^* : H_*(X) \to H_{*+d}(Y).$$

These data satisfy the following axioms:

- ($\mathrm{A}_1$) For any pair of composable smooth equidimensional morphisms ($f : Y \to X$, $g : Z \to Y$) of relative dimensions $d$ and $e$ respectively, one has

$$(f \circ g)^* = g^* \circ f^* : H_*(X) \to H_{*+d+e}(Z).$$

  In addition, $\mathrm{Id}_X^* = \mathrm{Id}_{H_*(X)}$ for any $X \in \mathscr{C}$.

- ($A_2$) For any projective morphism $f : X \to Z$ and any smooth equidimensional morphism $g : Y \to Z$, if one forms the fiber diagram

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

  then

$$g^* f_* = f'_* g'^*.$$

**Notation 2.2.** For each projective morphism $f$ the homomorphism $H_*(f)$ is denoted by $f_*$ and is called the *pushforward along* $f$. For each smooth equidimensional morphism $g$ the homomorphism $g^*$ is called the *pullback along* $g$.

**Definition 2.3.** A *Borel–Moore functor with exterior product* on $\mathscr{C}$ consists of a Borel–Moore functor $H_*$ on $\mathscr{C}$, together with:

- ($D_3$) An element $1 \in H_0(\operatorname{Spec} k)$, and for each pair $(X, Y)$ of schemes in $\mathscr{C}$ a bilinear graded pairing (called the *exterior product*)

$$\times : H_*(X) \times H_*(Y) \to H_*(X \times Y),$$
$$(\alpha, \beta) \mapsto \alpha \times \beta$$

  which is (strictly) commutative, associative, and admits 1 as unit.

These satisfy:

- ($A_3$) Given projective morphisms $f : X \to X'$ and $g : Y \to Y'$, one has that for any classes $\alpha \in H_*(X)$ and $\beta \in H_*(Y)$

$$(f \times g)_*(\alpha \times \beta) = f_*(\alpha) \times g_*(\beta) \in H_*(X' \times Y').$$

- ($A_4$) Given smooth equidimensional morphisms $f : X \to X'$ and $g : Y \to Y'$, one has that for any classes $\alpha \in H_*(X')$ and $\beta \in H_*(Y')$

$$(f \times g)^*(\alpha \times \beta) = f^*(\alpha) \times g^*(\beta) \in H_*(X \times Y).$$

**Remark 2.4.** Given a Borel–Moore functor with exterior product $H_*$, the axioms give $H_*(\operatorname{Spec} k)$ a commutative, graded ring structure, give to each $H_*(X)$ the structure of $H_*(\operatorname{Spec} k)$-module, and imply that the operations $f_*$ and $f^*$ preserve the $H_*(\operatorname{Spec} k)$-module structure.

**Definition 2.5.** A *Borel–Moore functor with intersection products* on $\mathscr{C}$ is a Borel–Moore functor $H_*$ on $\mathscr{C}$, together with:

- (D$_4$) For each lci morphism $f : Z \to X$ of relative codimension $d$ and any morphism $g : Y \to X$ giving the fiber diagram

$$\begin{tikzcd}
W \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
Z \arrow[r, "f"] & X
\end{tikzcd}$$

a homomorphism of graded groups

$$f_g^! : H_*(Y) \to H_{*-d}(W).$$

These satisfy:

- (A$_5$) If $f_1 : Z_1 \to X$ and $f_2 : Z_2 \to Z_1$ are lci morphisms and $g : Y \to X$ any morphism giving the fiber diagram

$$\begin{tikzcd}
W_2 \arrow[r, "f_2'"] \arrow[d] & W_1 \arrow[r, "f_1'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
Z_2 \arrow[r, "f_2"] & Z_1 \arrow[r, "f_1"] & X
\end{tikzcd}$$

one has $(f_1 \circ f_2)_g^! = (f_2)_{g'}^! \circ (f_1)_g^!$.

- (A$_6$) If $f_1 : Z_1 \to X_1$ and $f_2 : Z_2 \to X_2$ are lci morphisms of relative codimensions $d$ and $e$, respectively, and $h_1 : Y \to X_1$ and $h_2 : Y \to X_2$ are arbitrary morphisms giving the fiber diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & W_2 \arrow[r] \arrow[d, "f_2'"] & Z_2 \arrow[d, "f_2"] \\
W_1 \arrow[r, "f_1'"] \arrow[d] & Y \arrow[r, "h_2"] \arrow[d, "h_1"] & X_2 \\
Z_1 \arrow[r, "f_1"] & X_1 &
\end{tikzcd}$$

one has $(f_1)_{h_1 f_2'}^! \circ (f_2)_{h_2}^! = (f_2)_{h_2 f_1'}^! \circ (f_1)_{h_1}^! : B_*Y \to B_{*-d-e}W$.

- (A$_7$) For any smooth morphism $f : Y \to X$ one has $f_{\mathrm{Id}_X}^! = f^*$.

For any lci morphism $f : Z \to X$, any morphism $g : Y \to X$ and any morphism $h : Y' \to Y$, if one forms the fiber diagram

$$\begin{tikzcd}
W' \arrow[r, "f''"] \arrow[d, "h'"] & Y' \arrow[d, "h"] \\
W \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
Z \arrow[r, "f"] & X
\end{tikzcd}$$

then:

- ($A_8$) If $g$ and $f$ are *Tor-independent* in $\mathrm{Sch}_k$ (i.e., if $\mathrm{Tor}_j^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for all $j > 0$) then
$$f^!_{gh} = (f')^!_h.$$
- ($A_9$) If $h$ is projective then $f^!_g \circ h_* = h'_* \circ f^!_{gh}$.
- ($A_{10}$) If $h$ is smooth equidimensional then $f^!_{gh} \circ h^* = h'^* \circ f^!_g$.

**Notation 2.6.** Given a Borel–Moore functor with intersection products $H_*$, for any lci morphism $f : Z \to X$ of relative codimension $d$, the map $f^!_{\mathrm{Id}_X} : H_*(X) \to H_{*-d}(Z)$ is called the *lci pullback along* $f$ and denoted by $f^*$. For each lci morphism $f : Z \to X$ and each morphism $g : Y \to X$ we call the morphism $f^!_g$ *the refined lci pullback along* $f$ *associated to* $g$. We will usually denote $f^!_g$ simply by $f^!$ with an indication of where it acts. When the lci morphism $f$ is a regular embedding then $f^!_g$ is called a refined Gysin homomorphism. Refined Gysin homomorphisms and smooth pullbacks can be composed to construct all refined lci pullbacks.

**Definition 2.7.** A *Borel–Moore functor with compatible exterior and intersection products* on $\mathcal{C}$ consists of a Borel–Moore functor $H_*$ on $\mathcal{C}$ endowed with exterior products and intersection products that in addition satisfy:

- ($A_{11}$) If, for $i = 1$ and $i = 2$, $f_i : Z_i \to X_i$ is an lci morphism and $g_i : Y_i \to X_i$ is an arbitrary morphism, and one forms the fiber diagram

$$\begin{tikzcd}
W_i \arrow[r, "f_i'"] \arrow[d, "g_i'"] & Y_i \arrow[d, "g_i"] \\
Z_i \arrow[r, "f_i"] & X_i
\end{tikzcd}$$

one has that for any classes $\alpha_1 \in H_*(Y_1)$ and $\alpha_2 \in H_*(Y_2)$

$$(f_1 \times f_2)^!_{g_1 \times g_2}(\alpha_1 \times \alpha_2) = (f_1)^!_{g_1}(\alpha_1) \times (f_2)^!_{g_2}(\alpha_2) \in H_*(W_1 \times W_2).$$

**Notation 2.8.** We will call a Borel–Moore functor with compatible exterior and intersection products a *refined oriented Borel–Moore prehomology theory* (ROBM prehomology theory, for short).

Examples of ROBM prehomology theories are Chow theory $A_*$ (see [Fulton 1998]), $K$-theory (i.e., the Grothendieck $K$-group functor $\mathrm{G}_0$ of the category of coherent $\mathscr{O}_X$-modules, graded by $\mathrm{G}_0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$ — see [Levine and Morel 2007, Example 2.2.5]), and algebraic cobordism $\Omega_*$ (see [Levine and Morel 2007]) on the category $\mathrm{Sch}_k$; and equivariant Chow theory $A_*^G$ and equivariant algebraic cobordism $\Omega_*^G$ on the category $G$-$\mathrm{Sch}_k$ constructed as in Section 4 (see [Edidin and Graham 1998; Krishna 2012; Heller and Malagón-López 2013]).

Levine and Morel [2007] considered the notion of an *oriented Borel–Moore homology theory*, which is an ROBM prehomology theory but with lci pullbacks only instead of refined lci pullbacks, and with additional axioms called projective bundle, extended homotopy and cellular decomposition properties. Because of the refined lci pullbacks, an oriented Borel–Moore homology theory is not necessarily an ROBM prehomology theory. However, one can construct refined lci pullbacks from ordinary lci pullbacks by the deformation to the normal cone argument of Fulton and MacPherson, provided that the theory additionally satisfies the homotopy and localization properties (see Section 4 for these properties). We will need the homotopy and localization properties when working with equivariant theories; hence an alternative theory that is sufficient for the constructions below would be a Borel–Moore functor with compatible lci pullbacks and exterior products, which additionally satisfies the homotopy and localization properties.

**Definition 2.9.** If $H_*$ is an ROBM prehomology theory, for any line bundle $L \to Y$ in $\mathscr{C}$ with zero section $s : Y \to L$ one defines the operator $\tilde{\mathrm{c}}_1(L) : H_*(Y) \to H_{*-1}(Y)$ by $\tilde{\mathrm{c}}_1(L) = s^* s_*$, and calls it the *first Chern class operator of* $L$.

### 2C. *Cohomology theory.*

Let $H_*$ be an ROBM prehomology theory. For a smooth scheme $X$ of pure dimension $n$, define

$$H^*(X) = H_{n-*}(X).$$

For an arbitrary smooth scheme we extend this notion by taking the direct sum over pure-dimensional parts of $X$.

The groups $H^*(X)$ are commutative graded rings with unit, with product defined by

$$\begin{aligned} H^*(X) \times H^*(X) &\to H^*(X), \\ (a, b) &\mapsto \Delta_X^*(a \times b), \end{aligned}$$

where $\Delta_X : X \to X \times X$ is the diagonal map and $\Delta_X^*$ is the lci pullback. Associativity of the product follows from ($A_5$) and ($A_{11}$) applied to two different ways to construct the diagonal $X \to X \times X \times X$ by composing $\Delta_X$.

Let $\pi : X \to \operatorname{Spec} k$ be the structure morphism, and define $1_X = \pi^*(1) \in H^0(X)$, where $1 = 1_{\operatorname{Spec} k} \in H^0(\operatorname{Spec} k)$ is the element specified in ($D_3$). Then $1_X$ is the multiplicative identity in the ring $H^*(X)$.

Axioms ($A_5$) and ($A_{11}$) imply that, if $f : X \to Y$ is an lci morphism between smooth schemes, then $f^* : H^*(Y) \to H^*(X)$ is a homomorphism of graded rings with unit. Thus, we may view $H^*$ as a contravariant functor from the category of smooth schemes and lci morphisms in $\mathscr{C}$ to the category of commutative graded rings with unit. In the next section we extend this functor to the whole category $\mathscr{C}$.

## 3. Operational bivariant theories

In this section we consider a refined oriented Borel–Moore prehomology theory $B_*$ on one of the categories $\mathscr{C} = \mathrm{Sch}_k$ or $\mathscr{C} = G\text{-}\mathrm{Sch}_k$, and associate to it a bivariant theory $B^*$ on $\mathscr{C}$. We present a unified treatment of these two cases. Therefore throughout this section we fix one of these two categories and denote it by $\mathscr{C}$, and we assume that all the schemes and morphisms are in $\mathscr{C}$ following the conventions described in Section 2B.1.

Operational theories were defined for general homology theories by Fulton and MacPherson [1981]. The constructions that we present in this section follow [Fulton 1998, Chapter 17] where the operational theory $A^*$ is constructed for the Chow theory $A_*$. Some definitions and proofs have been modified to adapt them to our setting.

**3A.** ***Bivariant theories.*** A *bivariant theory* $B^*$ on $\mathscr{C}$ assigns to each morphism $f : X \to Y$ in $\mathscr{C}$ a graded abelian group $B^*(X \to Y)$. The groups $B^*(X \to Y)$ are endowed with three operations called product, pushforward and pullback, which are mutually compatible and admit units:

- ($P_1$) *Product.* For all morphisms $f : X \to Y$ and $g : Y \to Z$, and all integers $p$ and $q$, there is a homomorphism

$$B^p\big(X \xrightarrow{f} Y\big) \otimes B^q\big(Y \xrightarrow{g} Z\big) \xrightarrow{\cdot} B^{p+q}\big(X \xrightarrow{gf} Z\big).$$

  The image of $c \otimes d$ is denoted $c \cdot d$.

- ($P_2$) *Pushforward.* If $f : X \to Y$ is a projective morphism, $g : Y \to Z$ is any morphism and $p$ is an integer, there is a homomorphism

$$f_* : B^p\big(X \xrightarrow{gf} Z\big) \longrightarrow B^p\big(Y \xrightarrow{g} Z\big).$$

- (P$_3$) *Pullback.* If $f : X \to Y$ and $g : Y' \to Y$ are arbitrary morphisms, $f' : X' = X \times_Y Y' \to Y'$ is the projection and $p$ is an integer, there is a homomorphism
$$g^* : B^p\big(X \xrightarrow{f} Y\big) \longrightarrow B^p\big(X' \xrightarrow{f'} Y'\big).$$
- (U) *Units.* For each $X$ there is an element $1_X \in B^0\big(X \xrightarrow{\mathrm{Id}} X\big)$ such that $\alpha \cdot 1_X = \alpha$ and $1_X \cdot \beta = \beta$ for all morphisms $W \to X$ and $X \to Y$ and all classes $\alpha \in B^*(W \to X)$ and $\beta \in B^*(X \to Y)$. These unit elements are compatible with pullbacks, i.e., $g^*(1_X) = 1_Z$ for all morphisms $g : Z \to X$.

These operations are required to satisfy seven compatibility properties:

- (B$_1$) *Associativity of products.* If $c \in B^*(X \to Y)$, $d \in B^*(Y \to Z)$ and $e \in B^*(Z \to W)$, then
$$(c \cdot d) \cdot e = c \cdot (d \cdot e) \in B^*(X \to W).$$
- (B$_2$) *Functoriality of pushforwards.* If $c \in B^*(X \to Y)$, then $\mathrm{Id}_{X*}c = c \in B^*(X \to Y)$. Moreover, if $f : X \to Y$ and $g : Y \to Z$ are projective morphisms, $Z \to W$ is arbitrary, and $d \in B^*(X \to W)$, then
$$(gf)_*d = g_*(f_*d) \in B^*(Z \to W).$$
- (B$_3$) *Functoriality of pullbacks.* If $c \in B^*(X \to Y)$, then $\mathrm{Id}_Y^*\, c = c \in B^*(X \to Y)$. Moreover, if $f : X \to Y$, $g : Y' \to Y$ and $h : Y'' \to Y'$ are arbitrary morphisms, $X'' = X \times_Y Y'' \to Y''$ is the projection, and $d \in B^*(X \to Y)$, then
$$(gh)^*d = h^*(g^*d) \in B^*(X'' \to Y'').$$
- (B$_{12}$) *Product and pushforward commute.* If $f : X \to Y$ is projective, $Y \to Z$ and $Z \to W$ are arbitrary, and $c \in B^*(X \to Z)$ and $d \in B^*(Z \to W)$, then
$$f_*(c) \cdot d = f_*(c \cdot d) \in B^*(Y \to W).$$
- (B$_{13}$) *Product and pullback commute.* If $c \in B^*\big(X \xrightarrow{f} Y\big)$ and $d \in B^*\big(Y \xrightarrow{g} Z\big)$, and $h : Z' \to Z$ is arbitrary, and one forms the fiber diagram
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h''"] & Y' \arrow[r, "g'"] \arrow[d, "h'"] & Z' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$
then
$$(h)^*(c \cdot d) = h'^*(c) \cdot h^*(d) \in B^*(X' \to Z').$$

- ($\mathrm{B}_{23}$) *Pushforward and pullback commute.* If $f : X \to Y$ is projective, $g : Y \to Z$ and $h : Z' \to Z$ are arbitrary morphisms, and $c \in B^*(X \to Z)$, and one forms the fiber diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h''"] & Y' \arrow[r, "g'"] \arrow[d, "h'"] & Z' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

then

$$h^*(f_* c) = f'_*(h^* c) \in B^*(Y' \to Z').$$

- ($\mathrm{B}_{123}$) *Projection formula.* If $f : X \to Y$ and $g : Y \to Z$ are arbitrary morphisms, $h' : Y' \to Y$ is projective and $c \in B^*(X \to Y)$ and $d \in B^*(Y' \to Z)$, and one forms the fiber diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h''"] & Y' \arrow[d, "h'"] & \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

then

$$c \cdot h'_*(d) = h''_*(h'^*(c) \cdot d) \in B^*(X \to Z).$$

The group $B^p\big(X \xrightarrow{f} Y\big)$ may be denoted by simply by $B^p(X \to Y)$ or $B^p(f)$. We will denote by $B^*\big(X \xrightarrow[f]{} Y\big)$, $B^*(X \to Y)$ or $B^*(f)$ the direct sum of all $B^p\big(X \xrightarrow[f]{} Y\big)$, for $p \in \mathbb{Z}$.

**3B.** ***Homology and cohomology.*** A bivariant theory $B^*(X \to Y)$ contains both a covariant homology theory $B_*(X)$ and a contravariant cohomology theory $B^*(X)$.

The homology is defined by $B_p(X) = B^{-p}(X \to \operatorname{Spec} k)$. For any projective morphism $f : X \to Y$, the pushforward in the bivariant theory defines the functorial pushforward map in homology $f_* : B_*(X) \to B_*(Y)$.

The cohomology is defined by $B^p(X) = B^p(\mathrm{Id}_X : X \to X)$. The product operation in the bivariant theory turns $B^*(X)$ into a graded ring with unit $1_X$ and turns $B_*(X)$ into a graded left module over $B^*(X)$. The product operation $B^*(X) \times B_*(X) \to B_*(X)$ is called the cap product and denoted $(\alpha, \beta) \mapsto \alpha \cap \beta$. For any morphism $f : X \to Y$, the pullback in the bivariant theory defines a functorial pullback $f^* : B^*(Y) \to B^*(X)$. The pullback map is a homomorphism of graded rings. When $f$ is a projective morphism, the projection formula relates the pullback, pushforward, and cap product as follows:

$$f_*(f^*(\alpha) \cap \beta) = \alpha \cap f_*(\beta).$$

**3C.** ***Operational bivariant theories.*** We now fix an ROBM prehomology theory $B_*$ on $\mathscr{C}$, and associate a bivariant theory $B^*$ to it.

Let $f : X \to Y$ be any morphism. For each morphism $g : Y' \to Y$, form the fiber square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

with induced morphisms as labeled. An element $c$ in $B^p\bigl(X \xrightarrow{f} Y\bigr)$, called a *bivariant class*, is a collection of homomorphisms

$$c_g^{(m)} : B_m Y' \to B_{m-p} X'$$

for all $g : Y' \to Y$ and all $m \in \mathbb{Z}$, compatible with projective pushforwards, smooth pullbacks, intersection products and exterior products, that is:

- (C$_1$) If $h : Y'' \to Y'$ is projective and $g : Y' \to Y$ is arbitrary, and one forms the fiber diagram

$$\begin{tikzcd}
X'' \arrow[r, "f''"] \arrow[d, "h'"] & Y'' \arrow[d, "h"] \\
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

  then, for all $\alpha \in B_m(Y'')$,

$$c_g^{(m)}(h_*\alpha) = h'_* c_{gh}^{(m)}(\alpha)$$

  in $B_{m-p}(X')$.

- (C$_2$) If $h : Y'' \to Y'$ is smooth of relative dimension $n$ and $g : Y' \to Y$ is arbitrary, and one forms the fiber diagram

$$\begin{tikzcd}
X'' \arrow[r, "f''"] \arrow[d, "h'"] & Y'' \arrow[d, "h"] \\
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

then, for all $\alpha \in B_m(Y')$,

$$c_{gh}^{(m+n)}(h^*\alpha) = h'^* c_g^{(m)}(\alpha)$$

in $B_{m+n-p}(X'')$.

- (C$_3$) If $g : Y' \to Y$ and $h : Y' \to Z'$ are morphisms, and $i : Z'' \to Z'$ is an lci morphism of codimension $e$, and one forms the fiber diagram

$$\begin{tikzcd} X'' \arrow[r, "f''"] \arrow[d, "i''"] & Y'' \arrow[r, "h'"] \arrow[d, "i'"] & Z'' \arrow[d, "i"] \\ X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[r, "h"] \arrow[d, "g"] & Z' \\ X \arrow[r, "f"] & Y & \end{tikzcd}$$

then for all $\alpha \in B_m(Y')$,

$$c_{gi'}^{(m-e)}(i^!\alpha) = i^! c_g^{(m)}(\alpha)$$

in $B_{m-e-p}(X'')$.

- (C$_4$) If $g : Y' \to Y$ is arbitrary, and $h : Y' \times Z \to Y'$ and $h' : X' \times Z \to X'$ are the projections, and one forms the fiber diagram

$$\begin{tikzcd} X' \times Z \arrow[r, "f''"] \arrow[d, "h'"] & Y' \times Z \arrow[d, "h"] \\ X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

then, for all $\alpha \in B_m(Y')$ and $\beta \in B_l(Z)$,

$$c_{gh}^{(m+l)}(\alpha \times \beta) = c_g^{(m)}(\alpha) \times \beta$$

in $B_{m+l-p}(X' \times Z)$.

The three operations are defined as follows:

- Product: Let $c \in B^p\big(X \xrightarrow{f} Y\big)$ and $d \in B^q\big(Y \xrightarrow{g} Z\big)$. Given any morphism $h : Z' \to Z$, form the fiber diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h''"] & Y' \arrow[r, "g'"] \arrow[d, "h'"] & Z' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

and for each integer $m$ define $(c \cdot d)_h^{(m)} = c_{h'}^{(m-q)} \circ d_h^{(m)} : B_m Z' \to B_{m-p-q} X'$.

- Pushforward: Given $c \in B^p\big(X \xrightarrow{f} Y \xrightarrow{g} Z\big)$ and any morphism $h : Z' \to Z$, form the fiber diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h''"] & Y' \arrow[r, "g'"] \arrow[d, "h'"] & Z' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

and for each integer $m$ define $(f_* c)_h^{(m)} = f'_* \circ c_h^{(m)} : B_m Z' \to B_{m-p} Y'$.

- Pullback: Given $c \in B^p\big(X \xrightarrow{f} Y\big)$ and morphisms $g : Y' \to Y$ and $h : Y'' \to Y'$, form the fiber diagram

$$\begin{tikzcd}
X'' \arrow[r, "f''"] \arrow[d, "h'"] & Y'' \arrow[d, "h"] \\
X' \arrow[r, "f'"] \arrow[d, "g'"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and for each integer $m$ define $(g^* c)_h^{(m)} = c_{gh}^{(m)} : B_m Y'' \to B_{m-p} X''$.

It is straightforward to verify that these three operations are well-defined (i.e., that $c \cdot d$, $f_* c$ and $g^* c$ satisfy $(C_1)$–$(C_4)$, so that they define classes in the appropriate bivariant groups). For each $X$, unit elements $1_X \in B^0(X \to X)$ satisfying the property (U) are defined by letting them act by identity homomorphisms. It is also straightforward to check that the three operations satisfy properties $(B_1)$–$(B_{123})$. In conclusion, the operational theory is a bivariant theory.

The only bivariant theories we will consider are the operational ones. By a bivariant theory we will mean an operational bivariant theory associated to an ROBM prehomology theory.

**3D.** ***Homology and cohomology for operational bivariant theories.*** Recall that any bivariant theory $B^*(X \to Y)$ contains both a covariant homology theory $B^{-*}(X \to \operatorname{Spec} k)$ and a contravariant cohomology theory $B^*(\mathrm{Id}_X : X \to X)$. We claim that if the bivariant theory $B^*(X \to Y)$ is the operational theory associated to an ROBM prehomology theory $B_*$, then the homology theory $B^{-*}(X \to \operatorname{Spec} k)$ is isomorphic to the original theory $B_*(X)$. Similarly, the cohomology theory $B^*(X \to X)$ agrees with the cohomology theory $B^*(X)$ constructed in Section 2C for smooth schemes $X$. The proofs in this section are adapted from the proofs in [Fulton 1998] for the Chow theory.

**Proposition 3.1.** *For any $X$ and each integer $p$, the homomorphism*

$$\varphi : B^{-p}(X \to \operatorname{Spec} k) \to B_p(X)$$

*taking a bivariant class $c$ to $c(1)$ is an isomorphism. Here $1 = 1_{\operatorname{Spec} k} \in B_0(\operatorname{Spec} k)$ is the element specified by* ($D_3$) *in Definition 2.3. The isomorphism $\varphi$ is natural with respect to pushforwards along projective morphisms.*

*Proof.* Define a homomorphism $\psi : B_p(X) \to B^{-p}(X \to \operatorname{Spec} k)$ as follows: given any $a \in B_p(X)$, any morphism $f : Y \to \operatorname{Spec} k$ and a class $\alpha \in B_m(Y)$, we set $\psi(a)(\alpha) = a \times \alpha \in B_{m+p}(X \times Y)$. It follows at once that $\psi(a)$ satisfies ($C_1$)–($C_4$), and $\psi$ is a well-defined homomorphism.

For each $a \in B_p(X)$, one has $\varphi(\psi(a)) = \psi(a)(1) = a \times 1 = a \in B_p(X)$, so $\varphi \circ \psi$ is the identity. Given any $c \in B^{-p}(X \to \operatorname{Spec} k)$, any morphism $Y \to \operatorname{Spec} k$ and any class $\alpha \in B_m(Y)$, one has $\psi(\varphi(c))(\alpha) = \psi(c(1))(\alpha) = c(1) \times \alpha = c(1 \times \alpha) = c(\alpha) \in B^{m+p}(X \times Y)$, so $\psi \circ \varphi$ is also the identity.

Naturality of $\varphi$ with respect to projective pushforwards follows from the definition of pushforward in the operational bivariant theory. □

Let us now consider the cohomology theory $B^*(X \to X)$. Note that an element $c \in B^p(X \to X)$ is a collection of homomorphisms $c_f^{(m)} : B_m X' \to B_{m-p} X'$, for all morphisms $f : X' \to X$ and all integers $m$, that are compatible with projective pushforwards, smooth pullbacks, and exterior and intersection products. Using the previous proposition, we identify $B_*(X)$ with $B^{-*}(X \to \operatorname{Spec} k)$, and thus give $B_*(X)$ the structure of a module over $B^*(X \to X)$.

**Proposition 3.2** (Poincaré duality). *Let $X$ be a smooth purely $n$-dimensional scheme and let $B^*(X) = B_{n-*}(X)$ be the cohomology theory defined in Section 2C. The homomorphism defined by cap product with $1_X \in B^0(X)$,*

$$\varphi : B^*(X \xrightarrow{\mathrm{Id}_X} X) \xrightarrow{\cap 1_X} B^*(X),$$

*is an isomorphism of graded rings that takes $1_X \in B^0(X \to X)$ to $1_X \in B^0(X)$. The isomorphism $\varphi$ is natural with respect to pullbacks by lci morphisms.*

*Proof.* Let us fix an integer $p$ and define a homomorphism $\psi : B^p(X) = B_{n-p}(X) \to B^p(X \to X)$ as follows: given any $a \in B_{n-p}(X)$, any morphism $f : Y \to X$ and a class $\alpha \in B_m(Y)$, we set $\psi(a)(\alpha) = \psi(a)_f^{(m)}(\alpha) = \gamma_f^*(a \times \alpha) \in B_{m-p}Y$, where $\gamma_f = (f, \mathrm{Id}_Y) : Y \to X \times Y$ is the transpose of the graph of $f$, which in this case is a regular embedding of codimension $n$. It is straightforward to check that $\psi(a)$ is a bivariant class and $\psi$ is a group homomorphism.

For each $a \in B_{n-p}X$, one has $\varphi(\psi(a)) = \psi(a)(1_X) = \gamma_{\mathrm{Id}_X}^*(a \times 1_X) = \mathrm{Id}_X^* a = a \in B_{n-p}X$, so $\varphi \circ \psi$ is the identity. Given any $c \in B^p(X \to X)$, any morphism $f : Y \to X$ and any class $\alpha \in B_mY$, one has $\psi(\varphi(c))(\alpha) = \psi(c(1_X))(\alpha) = \gamma_f^*(c(1_X) \times \alpha) = \gamma_f^*(c(1_X \times \alpha)) = c(\gamma_f^*(1_X \times \alpha)) = c(\alpha) \in B_{m-p}Y$, so $\psi \circ \varphi$ is also the identity.

To verify the compatibility with multiplication, we show that for arbitrary classes $a \in B_{n-p}(X)$ and $b \in B_{n-q}(X)$ we have $\psi(a \cdot b) = \psi(a) \cdot \psi(b) \in B^{p+q}(X \to X)$. Indeed, given any morphism $f : Y \to X$ and any class $\theta \in B_mY$, we have

$$\begin{aligned}\psi(a \cdot b)(\theta) &= \gamma_f^*(\gamma_{\mathrm{Id}_X}^*(a \times b) \times \theta) = (f, f, \mathrm{Id}_Y)^*(a \times b \times \theta)\\ &= \gamma_f^*((\mathrm{Id}_X, \gamma_f)^*(a \times b \times \theta)) = \gamma_f^*(a \times \gamma_f^*(b \times \theta))\\ &= \psi(a)(\psi(b)(\theta)) = (\psi(a) \cdot \psi(b))(\theta) \in B_{m-p-q}Y.\end{aligned}$$

The element $1_X \in B^0(X \to X)$ acts as multiplicative identity, hence the cap product with it maps $1_X \in B^0(X)$ to itself.

Let $f : X \to Y$ be an lci morphism between pure-dimensional smooth schemes. Naturality of $\varphi$ with respect to pullback by $f$ is equivalent to the identity

$$f^*(c) \cap 1_X = f^*(c \cap 1_Y)$$

for any $c \in B^*(Y \to Y)$, which holds by ($C_3$) since $1_X = f^*(1_Y)$. □

**Notation 3.3.** We will set $B^*(X) = B^*(\mathrm{Id}_X : X \to X)$ for an arbitrary scheme $X$ in $\mathscr{C}$. Proposition 3.2 shows that this contravariant functor on $\mathscr{C}$, when restricted to the category of smooth schemes and lci morphisms, is isomorphic to the functor $B^*$ defined in Section 2C.

**Remark 3.4.** In his construction of a bivariant theory associated to Chow theory, Fulton [1998, Chapter 17] used the Chow theory versions of our axioms ($C_1$), ($C_2$) and ($C_3$), namely, compatibility of the bivariant classes with proper pushforwards, flat pullbacks and refined Gysin homomorphisms, but there is no explicit requirement of our axiom ($C_4$), compatibility with exterior products. This axiom ($C_4$) does not appear explicitly in Fulton's construction because, in the case of Chow theory $A_*$, axioms ($C_1$) and ($C_2$) imply axiom ($C_4$). More generally, this is true for any ROBM prehomology theory $B_*$ for which, for every $X$ in $\mathscr{C}$, the group $B_*(X)$ is generated by the projective pushforward images of the classes $1_Y$ for all smooth varieties $Y$

with a projective map to $X$. Indeed, using ($C_1$), one reduces ($C_4$) to the case where both $Y'$ and $Z$ are smooth varieties, and $\alpha = 1_{Y'}$ and $\beta = 1_Z$. In that case, from ($C_2$) one obtains that $c_{gh}^{(m+l)}(1_{Y'} \times 1_Z) = c_{gh}^{(m+l)}(1_{Y' \times Z}) = c_{gh}^{(m+l)}(h^* 1_{Y'}) = h'^* c_g^{(m)}(1_{Y'}) = c_g^{(m)}(1_{Y'}) \times 1_Z$, and then ($C_4$) holds in general for $B_*$.

## 4. The equivariant version of an ROBM prehomology theory

In this section we fix an ROBM prehomology theory $B_*$ on the category $\mathrm{Sch}_k$, and construct its *equivariant version* $B_*^G$, which is an ROBM prehomology theory on $G$-$\mathrm{Sch}_k$. The construction of $B_*^G$ generalizes to arbitrary ROBM prehomology theories similar constructions in Chow theory by Totaro [1999] and Edidin and Graham [1998], and in algebraic cobordism by Krishna [2012] and Heller and Malagón-López [2013].

We will need to assume throughout this section that the field $k$ is infinite and the theory $B_*$ satisfies the homotopy property (H) and the localization property (L):

- (H) Let $p : E \to X$ be a vector bundle of rank $r$ over $X$ in $\mathrm{Sch}_k$. Then $p^* : B_*(X) \to B_{*+r}(E)$ is an isomorphism.
- (L) For any closed immersion $i : Z \to X$ with open complement $j : U = X \setminus Z \to X$, the following sequence is exact:

$$B_*(Z) \xrightarrow{i_*} B_*(X) \xrightarrow{j^*} B_*(U) \longrightarrow 0.$$

**4A.** ***Algebraic groups, quotients and good systems of representations.*** Let $G$ be a linear algebraic group. If $X$ is a scheme with a $G$-action $\sigma : G \times X \to X$ and the geometric quotient of $X$ by the action of $G$ exists, it will be denoted by $X \to X/G$. When the geometric quotient $\pi : X \to X/G$ exists, it is called a *principal $G$-bundle* if the morphism $\pi$ is faithfully flat and the morphism $\psi = (\sigma, \mathrm{pr}_X) : G \times X \to X \times_{X/G} X$ is an isomorphism. By [Mumford et al. 1994, Proposition 0.9], if $G$ acts freely on $U \in G$-$\mathrm{Sch}_k$ and the geometric quotient $\pi : U \to U/G$ exists in $\mathrm{Sch}_k$, for some quasiprojective scheme $U/G$, then $\pi : U \to U/G$ is a principal $G$-bundle. Moreover, by [Mumford et al. 1994, Proposition 7.1], for any $X \in G$-$\mathrm{Sch}_k$ the geometric quotient $\pi' : X \times U \to (X \times U)/G$ also exists in $\mathrm{Sch}_k$, it is a principal $G$-bundle and $(X \times U)/G$ is quasiprojective. In this case we denote the scheme $(X \times U)/G$ by $X \times^G U$.

**Definition 4.1.** We say that $\{(V_i, U_i)\}_{i \in \mathbb{Z}^+}$ is a *good system of representations* of $G$ if each $V_i$ is a $G$-representation, $U_i \subseteq V_i$ is a $G$-invariant open subset and they satisfy the following conditions:

(1) $G$ acts freely on $U_i$ and $U_i/G$ exists as a quasiprojective scheme in $\mathrm{Sch}_k$.

(2) For each $i$ there is a $G$-representation $W_i$ so that $V_{i+1} = V_i \oplus W_i$.

(3) $U_i \oplus \{0\} \subseteq U_{i+1}$ and the inclusion factors as $U_i = U_i \oplus \{0\} \subseteq U_i \oplus W_i \subseteq U_{i+1}$.

(4) $\operatorname{codim}_{V_i}(V_i \setminus U_i) < \operatorname{codim}_{V_j}(V_j \setminus U_j)$ for $i < j$.

For any algebraic group $G$ there exist good systems of representations (see [Totaro 1999, Remark 1.4]). The following lemma lists some basic facts regarding the properties of morphisms induced on geometric quotients:

**Lemma 4.2.** *Let $f : X \to Y$ be a $G$-equivariant morphism in $G$-$\mathrm{Sch}_k$, and let $\{(V_i, U_i)\}$ be a good system of representations of $G$.*

(1) *For each $i$ the quotient $X \times^G U_i = (X \times U_i)/G$ exists in $\mathrm{Sch}_k$ and it is quasiprojective. The induced morphisms $\phi_{ij} : X \times^G U_i \to X \times^G U_j$ are lci morphisms for $j \geq i$. If $X$ is smooth then $X \times^G U_i$ is also smooth.*

(2) *Let $\mathbf{P}$ be one of the following properties of morphisms: open immersion, closed immersion, regular embedding, projective, smooth, lci. If $f : X \to Y$ satisfies the property $\mathbf{P}$ in the category $G$-$\mathrm{Sch}_k$, then the induced maps $f_i : X \times^G U_i \to Y \times^G U_i$ satisfy property $\mathbf{P}$ in the category $\mathrm{Sch}_k$.*

(3) *For any morphisms $g : Y \to X$ and $f : Z \to X$ in $G$-$\mathrm{Sch}_k$ and any indices $j \geq i$, the square diagrams*

$$\begin{tikzcd} W \times U_i \arrow[r] \arrow[d] & Y \times U_j \arrow[d] \\ Z \times U_i \arrow[r] & X \times U_j \end{tikzcd} \qquad \begin{tikzcd} W \times^G U_i \arrow[r] \arrow[d] & Y \times^G U_j \arrow[d, "g'"] \\ Z \times^G U_i \arrow[r, "f'"] & X \times^G U_j \end{tikzcd}$$

*induced by the Cartesian product $W = Y \times_X Z$ are fiber squares, and furthermore they are Tor-independent if $f$ and $g$ are Tor-independent.*

*Proof.* For proofs of the assertions in (1) and (2) see [Edidin and Graham 1998, Proposition 2] and [Heller and Malagón-López 2013, 2.2.2, Lemma 9]. For (3), given any $T \in \mathrm{Sch}_k$ and morphisms $g'' : T \to Y \times^G U_j$ and $f'' : T \to Z \times^G U_i$ such that $f' \circ f'' = g' \circ g''$, we let $G$ act on $\widetilde{T} = T \times_{(X \times^G U_j)} (X \times U_j)$ via the morphism $G \times \widetilde{T} \to \widetilde{T}$ induced by the product of the trivial action of $G$ on $T$ and the action of $G$ on $X \times U_j$. By [Mumford et al. 1994, Amplification 7.1], $G$-equivariant morphisms from $\widetilde{T}$ to each of $X \times U_j$, $Y \times U_j$, $Z \times U_i$ and $W \times U_i$ correspond to the morphisms induced on the quotients from $T$ to $X \times^G U_j$, $Y \times^G U_j$, $Z \times^G U_i$ and $W \times^G U_i$, respectively. The assertion that the squares in (3) are Cartesian follows easily from these observations. If $f$ and $g$ are Tor-independent, then $\mathrm{Tor}_m^{\mathbb{O}_X}(\mathbb{O}_Y, \mathbb{O}_Z) = 0$ for all $m > 0$, and clearly $\mathrm{Tor}_m^{\mathbb{O}_{U_j}}(\mathbb{O}_{U_j}, \mathbb{O}_{U_i}) = 0$ for all $m > 0$. Hence, by applying locally the spectral sequence associated to the double complex obtained as the tensor product of two complexes, we obtain that $\mathrm{Tor}_m^{\mathbb{O}_{X \times U_j}}(\mathbb{O}_{Y \times U_j}, \mathbb{O}_{Z \times U_i}) = 0$ for all $m > 0$. Since $Y \times U_j = (Y \times^G U_j) \times_{(X \times^G U_j)} (X \times U_j)$ and $Z \times U_i =$

$(Z \times^G U_i) \times_{(X \times^G U_j)} (X \times U_j)$, by faithfully flat base change for $\mathrm{Tor}_m$ we have that $\mathrm{Tor}_m^{\mathcal{O}_{X \times^G U_j}}(\mathcal{O}_{Y \times^G U_j}, \mathcal{O}_{Z \times^G U_i}) = 0$ for all $m > 0$. Therefore, in this case the given squares are also Tor-independent. □

**4B.** ***Construction of $B_*^G(X)$.*** Fix a good system of representations $\{(V_i, U_i)\}$ of $G$. For any scheme $X \in G\text{-}\mathrm{Sch}_k$, define $B_*^G(X) = \bigoplus_{n \in \mathbb{Z}} B_n^G(X)$, where

$$B_n^G(X) = \varprojlim_i B_{n+\dim U_i - \dim G}(X \times^G U_i).$$

To simplify notation, we will often write

$$B_*^G(X) = \varprojlim_i B_*(X \times^G U_i),$$

where the limit is taken in each degree separately. Equivalently, the limit is taken in the category of graded abelian groups, with $B_*(X \times^G U_i)$ having grading shifted so that the maps in the inverse system are homogeneous of degree zero.

To see that $B_*^G$ is independent of the choice of a good system of representations, one can formally follow the argument presented in the case of algebraic cobordism in [Heller and Malagón-López 2013, Proposition 15 and Theorem 16], which we outline below for the reader's convenience. The proof of the next proposition requires the field $k$ to be infinite (this is used in the proof to construct a local section of the projection $\pi|_U : U \to X$).

**Proposition 4.3.** *Assume that the ROBM prehomology theory $B_*$ satisfies properties* (H) *and* (L). *Let $\pi : E \to X$ be a vector bundle over a scheme $X$ of rank $r$. Let $U \subseteq E$ be an open subscheme with closed complement $S = E \setminus U$.*

(1) *If $X$ is affine and* $\mathrm{codim}_E\, S > \dim X$ *then $\pi|_U^* : B_m(X) \to B_{m+r}(U)$ is an isomorphism for all $m$.*

(2) *For $X$ arbitrary, there is an integer $n(X)$ depending only on $X$, such that $\pi|_U^* : B_m(X) \to B_{m+r}(U)$ is an isomorphism for all $m$ whenever* $\mathrm{codim}_E\, S > n(X)$.

*Proof.* The case when $B_*$ is algebraic cobordism is [Heller and Malagón-López 2013, Proposition 15]. The proof given there only uses the formal properties of algebraic cobordism as an ROBM prehomology theory satisfying (H) and (L), so it translates formally to the present setting. □

**Proposition 4.4.** *For any $X \in G\text{-}\mathrm{Sch}_k$, $B_*^G(X)$ is independent of the choice of a good system of representations of $G$ up to canonical isomorphism.*

*Proof.* We use Bogomolov's double filtration argument. Assume that $\{(V_i, U_i)\}$ and $\{(V_i', U_i')\}$ are good systems of representations of $G$. For a fixed index $i$, since $G$ acts freely on $U_i$, it also acts freely on $U_i \times V_j'$ and $U_i \times U_j'$ for all $j$. Hence

$X \times^G (U_i \times V_j') \to X \times^G U_i$ is a vector bundle. By Proposition 4.3(2) there is an integer $N_i$ such that the lci pullbacks induce canonical isomorphisms

$$B_*(X \times^G U_i) \cong B_*(X \times^G (U_i \times U_j'))$$

for each $j \geq N_i$. Therefore, there is a canonical isomorphism

$$\varprojlim_i B_*(X \times^G U_i) = \varprojlim_i \varprojlim_j B_*(X \times^G (U_i \times U_j')). \tag{4-1}$$

Similarly, exchanging the role of the good systems of representations we obtain a canonical isomorphism

$$\varprojlim_j B_*(X \times^G U_j') = \varprojlim_j \varprojlim_i B_*(X \times^G (U_i \times U_j')). \tag{4-2}$$

To obtain the conclusion, we only need to observe that the right-hand sides of (4-1) and (4-2) are canonically isomorphic to the inverse limit of the system $\{B_*(X \times^G (U_i \times U_j'))\}_{i,j}$, where the maps

$$B_*(X \times^G (U_i \times U_j')) \to B_*(X \times^G (U_{i'} \times U_{j'}'))$$

are the corresponding lci pullbacks for all $i \geq i'$ and $j \geq j'$. □

**4C.** ***The induced ROBM prehomology theory structure on $B_*^G$.*** We now show that the ROBM prehomology structure of the theory $B_*$ induces an ROBM prehomology structure on $B_*^G$. We use functoriality of the inverse limit to construct projective pushforwards, smooth pullbacks, and exterior and intersection products on $B_*^G$. To define a homomorphism between two inverse limits, we construct a map between the two inverse systems.

Fix a good system of representations $\{(V_i, U_i)\}$ of $G$. Given any projective $G$-equivariant morphism $f : X \to Y$, the morphisms $\{f_i : X \times^G U_i \to Y \times^G U_i\}$ are projective. The fiber square on the left in (4-3) is Tor-independent for any $j \geq i$,

$$\begin{tikzcd}
X \times^G U_i \arrow[r] \arrow[d] & Y \times^G U_i \arrow[d] & & B_*(X \times^G U_i) \arrow[r, "f_{i\,*}"] & B_*(Y \times^G U_i) \\
X \times^G U_j \arrow[r] & Y \times^G U_j & & B_*(X \times^G U_j) \arrow[r, "f_{j\,*}"] \arrow[u] & B_*(Y \times^G U_j) \arrow[u]
\end{tikzcd} \tag{4-3}$$

hence the square on the right in (4-3) is commutative for any $j \geq i$. Hence, the maps $f_{i\,*}$ induce a homomorphism between the limits $f_* : B_*^G(X) \to B_*^G(Y)$.

Smooth pullbacks are defined in a similar way. For intersection products, given a $G$-equivariant lci morphism $f : Z \to X$ of codimension $d$ and any $G$-equivariant morphism $g : Y \to X$, with $W = Z \times_X Y$, first we apply the operation $\times^G U_i$ to the whole intersection product diagram. The result is again an intersection product diagram. The two fiber squares on the first line of (4-4) are Tor-independent for any $j \geq i$:

$$\begin{tikzcd}
Z \times^G U_i \arrow[r, "f_i"] \arrow[d] & X \times^G U_i \arrow[d] & W \times^G U_i \arrow[r] \arrow[d] & Y \times^G U_i \arrow[d] \\
Z \times^G U_j \arrow[r, "f_j"] & X \times^G U_j & W \times^G U_j \arrow[r] & Y \times^G U_j \\
& B_{*-d}(W \times^G U_i) & B_*(Y \times^G U_i) \arrow[l, "f_i^!"'] & \\
& B_{*-d}(W \times^G U_j) \arrow[u] & B_*(Y \times^G U_j) \arrow[l, "f_j^!"'] \arrow[u] &
\end{tikzcd} \tag{4-4}$$

hence the square on the bottom line of (4-4) is commutative for any $j \geq i$. Hence, the maps $f_i^!$ induce a homomorphism between the limits $f^! : B_*^G(Y) \to B_{*-d}^G(W)$.

To define the exterior product, note that the morphisms

$$\phi_i : (X \times Y) \times^G (U_i \times U_i) \to (X \times^G U_i) \times (Y \times^G U_i)$$

are smooth (see [Borel 1991, Theorem 6.8]). We compose the associated smooth pullback and the exterior product of $B_*$ to get

$$\begin{aligned} \times_i : B_*(X \times^G U_i) \times B_*(Y \times^G U_i) \to B_*((X \times^G U_i) \times (Y \times^G U_i)) \\ \to B_*((X \times Y) \times^G (U_i \times U_i)). \end{aligned}$$

The morphisms $\times_i$ are compatible with maps in the inverse systems, and hence they define the exterior product map between limits:

$$\times : B_*^G X \times B_*^G Y \to B_*^G(X \times Y).$$

The elements $1_{U_i/G} \in B_*(\operatorname{Spec} k \times^G U_i)$ define $1 \in B_*^G(X)$.

It remains to prove that the theory $B_*^G$ with the operations defined above satisfies the axioms of an ROBM prehomology theory. Each axiom amounts to a statement about the commutativity of a diagram of homomorphisms. One can check that in each case the commutativity holds at each level $i$, and hence it also holds in the limit. Using the double filtration argument as before, one can also check that these projective pushforwards, smooth pullbacks, exterior and intersection products are independent of the good system of representations.

The conclusions of this section can then be summarized as:

**Theorem 4.5.** *The functor $B_*^G$ with the projective pushforwards, smooth pullbacks, and exterior and intersection products constructed above is a refined oriented Borel–Moore prehomology theory on the category $G$-*$\mathrm{Sch}_k$*. We call $B_*^G$ the* equivariant version *of $B_*$.*

**4D.** ***Operational equivariant theory.*** For a given ROBM prehomology theory $B_*$ on $\mathrm{Sch}_k$, we constructed an associated equivariant version $B_*^G$ as an ROBM prehomology theory on $G$-$\mathrm{Sch}_k$. The construction of Section 3 applied to $B_*$ and to $B_*^G$ produces operational bivariant theories $B^*$ on $\mathrm{Sch}_k$ and $(B^G)^*$ on $G$-$\mathrm{Sch}_k$ respectively. We denote $(B^G)^*$ by $B_G^*$ and call it the *operational equivariant version* of $B_*$

One can switch the order of the two steps in the construction of $B_G^*$ and define an "*equivariant operational*" theory

$$\widetilde{B}_G^*(X) = \varprojlim_i B^*(X \times^G U_i).$$

The two theories $B_G^*(X)$ and $\widetilde{B}_G^*(X)$ turn out to be isomorphic if we assume that $B_*$ satisfies the descent property (D) described in the next section. This property was first proved by Gillet [1984] in the case of Chow groups and $K$-theory. It has several other consequences for ROBM prehomology theories that are studied in the next section.

## 5. Descent sequences

We assume in this section that the field $k$ has characteristic zero, or, more generally, we assume that every scheme $X$ in $\mathrm{Sch}_k$ or in $G$-$\mathrm{Sch}_k$ has a smooth projective (equivariant) envelope $\pi : \widetilde{X} \to X$ as defined in Section 5A. We fix an ROBM prehomology theory $B_*$ on one of the categories $\mathrm{Sch}_k$ or $G$-$\mathrm{Sch}_k$ and consider the following property (D):

- (D) For any envelope $\pi : \widetilde{X} \to X$, with $\pi$ projective, the sequence

$$B_*(\widetilde{X} \times_X \widetilde{X}) \xrightarrow{p_{1*} - p_{2*}} B_*(\widetilde{X}) \xrightarrow{\pi_*} B_*(X) \longrightarrow 0, \qquad (5\text{-}1)$$

is exact, where $p_i : \widetilde{X} \times_X \widetilde{X} \to \widetilde{X}$ is the projection on the $i$-th factor for $i = 1, 2$.

**5A.** ***Envelopes.*** An *envelope* of a scheme $X$ in $\mathrm{Sch}_k$ is a proper morphism $\pi : \widetilde{X} \to X$ such that for every subvariety $V$ of $X$ there is a subvariety $\widetilde{V}$ of $\widetilde{X}$ that is mapped birationally onto $V$ by $\pi$. If $G$ is an algebraic group, a $G$-*equivariant envelope* of a scheme $X$ in $G$-$\mathrm{Sch}_k$ is a proper $G$-equivariant morphism $\pi : \widetilde{X} \to X$ such that for every $G$-invariant subvariety $V$ of $X$ there is a $G$-invariant subvariety $\widetilde{V}$ of $\widetilde{X}$ that is mapped birationally onto $V$ by $\pi$.

In the following, an envelope in the category $\mathrm{Sch}_k$ means an ordinary envelope and an envelope in the category $G$-$\mathrm{Sch}_k$ means a $G$-equivariant envelope. If $\pi : \widetilde{X} \to X$ is an envelope, we say that it is a *smooth envelope* if $\widetilde{X}$ is smooth, and we say that it is a *projective envelope* if $\pi$ is a projective morphism. Likewise, we say that the envelope $\pi : \widetilde{X} \to X$ is *birational* if, for some dense open subset $U$ of $X$, $\pi$ induces an isomorphism $\pi| : \pi^{-1}(U) \to U$. The composition of envelopes is again an envelope and the fiber product of an envelope by any morphism is again an envelope.

The domain $\widetilde{X}$ of an envelope is not required to be connected; hence, if we assume that varieties over $k$ admit (equivariant) resolutions of singularities via a projective morphism, then by induction on the dimension it follows easily that for every scheme $X$ in $\mathrm{Sch}_k$ (respectively in $G$-$\mathrm{Sch}_k$) there exists a smooth projective birational (equivariant) envelope $\pi : \widetilde{X} \to X$.

Notice that if $\pi : \widetilde{X} \to X$ is an envelope in $G$-$\mathrm{Sch}_k$ and if $\widetilde{U} \in G$-$\mathrm{Sch}_k$ has a free $G$-action such that $\widetilde{U}/G$ exists as a quasiprojective scheme in $\mathrm{Sch}_k$, then the induced morphism $\pi_G : \widetilde{X} \times^G \widetilde{U} \to X \times^G \widetilde{U}$ is an envelope in $\mathrm{Sch}_k$. Furthermore if $\pi$ is either a smooth, projective or birational envelope, then $\pi_G$ is a smooth, projective or birational envelope, respectively. Moreover, if $\pi| : \pi^{-1}(U) \to U$ is an isomorphism for some $G$-invariant open subset $U$ of $X$ and $\{Z_i\}$ are $G$-invariant closed subschemes of $X$ such that $X \setminus U = \bigcup Z_i$, then $\pi_G$ maps the open subset $\pi_G^{-1}(U \times^G \widetilde{U}) = \pi^{-1}(U) \times^G \widetilde{U}$ of $\widetilde{X} \times^G \widetilde{U}$ isomorphically onto the open subset $U \times^G \widetilde{U}$ of $X \times^G \widetilde{U}$, and the closed subschemes $\{Z_i \times^G \widetilde{U}\}$ of $X \times^G \widetilde{U}$ satisfy that $(X \setminus U) \times^G \widetilde{U} = (X \times^G \widetilde{U}) \setminus (U \times^G \widetilde{U}) = \bigcup (Z_i \times^G \widetilde{U})$.

**5B.** ***Operational equivariant versus equivariant operational.*** Assume now that the theory $B_*$ on $\mathrm{Sch}_k$ satisfies properties (H) and (L), and we can thus define the operational equivariant theory $B_G^*$ as well as the “equivariant operational” theory $\widetilde{B}_G^*$ as in Section 4D. We show that if $B_*$ also satisfies property (D), then these two bivariant theories are isomorphic. For simplicity, we prove this isomorphism only for the bivariant cohomology theory $B_G^*(X)$.

**Lemma 5.1.** *Let $B_*$ be an ROBM prehomology theory on* $\mathrm{Sch}_k$ *that satisfies* (H), (L) *and* (D). *Then, for any scheme $X \in G$-*$\mathrm{Sch}_k$ *and any projective envelope $\pi : \widetilde{X} \to X$, the pushforward homomorphism $\pi_* : B_*^G(\widetilde{X}) \to B_*^G(X)$ is surjective.*

*Proof.* Recall that the inverse limit $\varprojlim$ is a left exact functor from the category of inverse systems of (graded) abelian groups. Given a short exact sequence of inverse systems

$$0 \longrightarrow (E_i) \longrightarrow (F_i) \longrightarrow (G_i) \longrightarrow 0,$$

the sequence of limits

$$0 \longrightarrow \varprojlim E_i \longrightarrow \varprojlim F_i \longrightarrow \varprojlim G_i \longrightarrow 0$$

is exact if the system $(E_i)$ satisfies the Mittag-Leffler condition. This condition is satisfied, for example, if the maps $E_i \to E_j$ for $i > j$ in the inverse system are all surjective.

Given an exact sequence

$$(E_i) \longrightarrow (F_i) \longrightarrow (G_i) \longrightarrow 0,$$

one may replace the system $(E_i)$ with its image $(I_i)$ in $(F_i)$ to get a short exact sequence. If all maps in the inverse system $(E_i)$ are surjective, then they are also surjective in $(I_i)$, and it follows that the map of limits $\varprojlim F_i \to \varprojlim G_i$ is surjective.

We apply the previous discussion to the sequence of inverse systems

$$B_*((\tilde{X} \times_X \tilde{X}) \times^G U_i) \longrightarrow B_*(\tilde{X} \times^G U_i) \longrightarrow B_*(X \times^G U_i) \longrightarrow 0.$$

This sequence is exact by property (D): the map $\tilde{X} \times^G U_i \to X \times^G U_i$ is an envelope and

$$(\tilde{X} \times_X \tilde{X}) \times^G U_i \cong (\tilde{X} \times^G U_i) \times_{X \times^G U_i} (\tilde{X} \times^G U_i).$$

For any scheme $Y$ in $G$-Sch$_k$, the lci pullbacks

$$B_*(Y \times^G U_i) \to B_*(Y \times^G U_j)$$

are surjective for all $i > j$. This follows from properties (H) and (L) because the inclusion $Y \times^G U_j \to Y \times^G U_i$ can be factored as the inclusion of the zero section of a vector bundle followed by an open immersion. Applying this to the case $Y = \tilde{X} \times_X \tilde{X}$ gives the statement of the lemma. □

**Proposition 5.2.** *If the ROBM prehomology theory $B_*$ on* Sch$_k$ *satisfies properties* (H), (L) *and* (D), *then for any $X$ in $G$-*Sch$_k$ *there exists an isomorphism*

$$B_G^*(X) \cong \varprojlim_i B^*(X \times^G U_i).$$

*The isomorphism is natural with respect to pullback by any morphism $f : X' \to X$ in $G$-*Sch$_k$.

*Proof.* Given $X$ in $G$-Sch$_k$ we define a homomorphism

$$\varphi_X : \varprojlim_i B^*(X \times^G U_i) \to B_G^*(X)$$

as follows: given $c = (c_i) \in \varprojlim_i B^*(X \times^G U_i)$, where $c_i \in B^*(X \times^G U_i)$ for each $i$, and given any $G$-equivariant morphism $f : Y \to X$ and a class $\alpha = (\alpha_i) \in B_*^G(Y) = \varprojlim_i B_*(Y \times^G U_i)$, where $\alpha_i \in B_*(Y \times^G U_i)$ for each $i$, one sets $\varphi_X(c)(\alpha) = (c_i(\alpha_i)) \in \varprojlim_i B_*(Y \times^G U_i) = B_*^G(Y)$. It is straightforward to verify that $\varphi_X(c)$ is well-defined and is indeed a bivariant class in $B_*^G(X)$, so that $\varphi_X$ is a well-defined homomorphism.

It is clear from the definitions that $\varphi_X$ is natural with respect to pullbacks.

We prove that $\varphi_X$ is an isomorphism. First, we consider the case when $X$ is smooth. In this case, each $X \times^G U_i$ is smooth as well. By Poincaré duality (Proposition 3.2) we have isomorphisms

$$B_G^*(X) \xrightarrow{\cap 1_X} B_*^G(X) \quad \text{and} \quad B^*(X \times^G U_i) \xrightarrow{\cap 1_{X \times^G U_i}} B_*(X \times^G U_i)$$

for each $i$. Passing to the component-wise inverse limit and composing appropriately, one obtains the isomorphism

$$\begin{tikzcd}
\varprojlim_i B^*(X \times^G U_i) \arrow[r, "(\cap 1_{X \times^G U_i})_i"] \arrow[rr, bend right, "\varphi_X"'] & \varprojlim_i B_*(X \times^G U_i) = B_*^G(X) \arrow[r, "(\cap 1_X)^{-1}"] & B_G^*(X)
\end{tikzcd}$$

which can be verified directly to be $\varphi_X$.

For the general case, given $X$ we choose a $G$-equivariant envelope $\pi : \widetilde{X} \to X$ so that $\pi$ is projective and $\widetilde{X}$ is smooth. Let $\pi_i : \widetilde{X} \times^G U_i \to X \times^G U_i$ be the induced morphisms. We get a commutative diagram

$$\begin{tikzcd}
\varprojlim_i B^*(X \times^G U_i) \arrow[r, hook, "(\pi_i^*)"] \arrow[d, "\varphi_X"] & \varprojlim_i B^*(\widetilde{X} \times^G U_i) \arrow[d, "\varphi_{\widetilde{X}}", "\cong"'] \\
B_G^*(X) \arrow[r, hook, "\pi^*"] & B_G^*(\widetilde{X})
\end{tikzcd}$$

We claim that $\pi^*$ and $(\pi_i^*)$ are injective. To see this, assume that $\pi^* c = 0$ for some $c \in B_G^*(X)$. Given a $G$-equivariant map $f : Y \to X$ and a class $\alpha \in B_*^G(Y)$, form the fiber product

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\pi'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\
\widetilde{X} \arrow[r, "\pi"] & X
\end{tikzcd}$$

with morphisms as indicated. Since $\pi'$ is an envelope, by Lemma 5.1 there exists $\tilde{\alpha} \in B_*^G(\widetilde{Y})$ such that $\pi'_*(\tilde{\alpha}) = \alpha$. We have $c(\alpha) = c(\pi'_*(\tilde{\alpha})) = \pi'_*(c(\tilde{\alpha})) = \pi'_*((\pi^* c)(\tilde{\alpha})) = 0$, and it follows that $\pi^*$ is injective. Since each $\pi_i$ is also an envelope, the same argument proves that the $\pi_i^*$ are injective, and then so is the component-wise inverse limit homomorphism $(\pi_i^*)$. In particular, it follows that $\varphi_X$ is injective, as $\varphi_{\widetilde{X}}$ is an isomorphism by the smooth case.

To prove the surjectivity of $\varphi_X$, we consider $c \in B_G^*(X)$, and construct an element mapping to $\varphi_{\widetilde{X}}^{-1}(\pi^* c)$ by $(\pi_i^*)$. Let

$$p_j : \widetilde{X} \times_X \widetilde{X} \to \widetilde{X} \quad \text{and} \quad p'_j : (\widetilde{X} \times^G U_i) \times_{X \times^G U_i} (\widetilde{X} \times^G U_i) \to \widetilde{X} \times^G U_i$$

be the projections on the corresponding $j$-th factor for $j = 1, 2$. Let $(\tilde{c}_i) \in \varprojlim_i B^*(\widetilde{X} \times^G U_i)$ be the image of $c$ under $\varphi_{\widetilde{X}}^{-1} \circ \pi^*$, where $\tilde{c}_i \in B^*(\widetilde{X} \times^G U_i)$ for all $i$. Now, using that $\varphi_{\widetilde{X}\times_X \widetilde{X}}$ is injective and that $(p_1^* - p_2^*)(\pi^* c) = 0$, it follows that $({p'_1}^* - {p'_2}^*)(\tilde{c}_i) = 0$ for each $i$. We define a class $(c_i) \in \varprojlim_i B^*(X \times^G U_i)$ as follows: given a morphism $f : Y \to X \times^G U_i$ and a class $\alpha \in B_*(Y)$, form the fiber product

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\pi'_i"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\
\widetilde{X} \times^G U_i \arrow[r, "\pi_i"] & X \times^G U_i
\end{tikzcd}$$

with morphisms as indicated, and set $c_i(\alpha) = {\pi'_i}_*(\tilde{c}_i(\tilde{\alpha}))$, where $\tilde{\alpha} \in B_*(\widetilde{Y})$ is any class satisfying ${\pi'_i}_*(\tilde{\alpha}) = \alpha$ (which exists since $\pi'_i$ is an envelope). To see that $c_i(\alpha)$ is independent of the choice of $\tilde{\alpha}$, it is enough to see that ${\pi'_i}_*(\tilde{c}_i(\beta)) = 0$ for each class $\beta \in B_*(\widetilde{Y})$ such that ${\pi'_i}_*\beta = 0$. By property (D), given such a class $\beta$ there exists a class $\gamma \in B_*(\widetilde{Y} \times_Y \widetilde{Y})$ such that $\beta = g_{1*}(\gamma) - g_{2*}(\gamma)$, where $g_j : \widetilde{Y} \times_Y \widetilde{Y} \to \widetilde{Y}$ are the projections for $j = 1, 2$. Since $\pi'_i \circ g_1 = \pi'_i \circ g_2$, it follows that ${\pi'_i}_*(\tilde{c}_i(\beta)) = {\pi'_i}_*\big(\tilde{c}_i(g_{1*}(\gamma) - g_{2*}(\gamma))\big) = ((\pi'_i \circ g_1)_* - (\pi'_i \circ g_2)_*)(\tilde{c}_i(\gamma)) = 0$, and then $c_i$ is well-defined. It is straightforward to verify that each $c_i$ satisfies conditions (C$_1$)–(C$_4$), so they define bivariant classes $c_i \in B^*(X \times^G U_i)$. Moreover, it is clear that the classes $c_i$ agree under the bivariant pullbacks $B^*(X \times^G U_j) \to B^*(X \times^G U_i)$ for each $j \geq i$, so they define a class $(c_i) \in \varprojlim_i B^*(X \times^G U_i)$. To prove that $\varphi_X$ is surjective, it is enough to see that $\pi_i^* c_i = \tilde{c}_i$ for each $i$. For this, let $f : Y \to \widetilde{X} \times^G U_i$ be any morphism and form the fiber diagram

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\pi'_i"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\
(\widetilde{X} \times^G U_i) \times_{X \times^G U_i} (\widetilde{X} \times^G U_i) \arrow[r, "p'_2"] \arrow[d, "p'_1"] & \widetilde{X} \times^G U_i \arrow[d, "\pi_i"] \\
\widetilde{X} \times^G U_i \arrow[r, "\pi_i"] & X \times^G U_i
\end{tikzcd}$$

with morphisms as indicated. Consider a class $\alpha \in B_*(Y)$ and any class $\tilde{\alpha} \in B_*(\widetilde{Y})$ such that ${\pi'_i}_*(\tilde{\alpha}) = \alpha$. Since $p_1'^*(\tilde{c}_i) = p_2'^*(\tilde{c}_i)$, we have that

$$\begin{aligned}
(\pi_i^* c_i)_f(\alpha) &= {\pi'_i}_*((\tilde{c}_i)_{p'_1 \circ f'}(\tilde{\alpha})) = {\pi'_i}_*((f'^* p_1'^* \tilde{c}_i)(\tilde{\alpha})) \\
&= {\pi'_i}_*((f'^* p_2'^* \tilde{c}_i)(\tilde{\alpha})) = {\pi'_i}_*((\tilde{c}_i)_{f \circ \pi'_i}(\tilde{\alpha})) = (\tilde{c}_i)_f(\alpha).
\end{aligned}$$

Hence $\pi_i^* c_i = \tilde{c}_i$ for each $i$, and the proof is complete. □

**5C.** ***Kimura-type descent sequence for bivariant theories and inductive computation of bivariant groups.*** Theorem 5.3 and Theorem 5.6 below were proved by Kimura [1992] in the case of Chow theory $A_*$. We generalize his proofs to arbitrary ROBM prehomology theories.

**Theorem 5.3.** *Let $B_*$ be an ROBM prehomology theory on $\mathscr{C} = \mathrm{Sch}_k$ or $\mathscr{C} = G\text{-}\mathrm{Sch}_k$ that satisfies property* (D). *Let $\pi : \widetilde{X} \to X$ be a projective envelope in $\mathscr{C}$, $Y \to X$ a morphism in $\mathscr{C}$, and $\widetilde{Y} = \widetilde{X} \times_X Y$. Then the following sequence is exact*:

$$0 \longrightarrow B^*(Y \to X) \xrightarrow{\pi^*} B^*(\widetilde{Y} \to \widetilde{X}) \xrightarrow{p_1^* - p_2^*} B^*(\widetilde{Y} \times_Y \widetilde{Y} \to \widetilde{X} \times_X \widetilde{X}). \tag{5-2}$$

*Proof.* Assume that $\pi^* c = 0$ for some $c \in B^*(Y \to X)$. Given a morphism $f : X' \to X$ and a class $\alpha \in B_*(X')$, form the fiber diagram on the left in (5-3):

$$\begin{tikzcd}
& \widetilde{Y}' \arrow[rr] \arrow[dl, "\pi''"'] \arrow[dd] & & \widetilde{X}' \arrow[dl, "\pi'"] \arrow[dd] \\
Y' \arrow[rr] \arrow[dd] & & X' \arrow[dd] & \\
& \widetilde{Y} \arrow[rr] \arrow[dl] & & \widetilde{X} \arrow[dl, "\pi"] \\
Y \arrow[rr] & & X &
\end{tikzcd}$$

$$\begin{tikzcd}
& \widetilde{Y}' \arrow[rr] \arrow[dl, "\pi''"'] \arrow[dd] & & \widetilde{X}' \arrow[dl, "\pi'"] \arrow[dd, "f'"] \\
Y' \arrow[rr] \arrow[dd] & & X' \arrow[dd, "f"] & \\
& \widetilde{Y} \times_Y \widetilde{Y} \arrow[rr] \arrow[dl] \arrow[dd] & & \widetilde{X} \times_X \widetilde{X} \arrow[dl, "p_2"] \arrow[dd, "p_1"] \\
\widetilde{Y} \arrow[rr] \arrow[dd] & & \widetilde{X} \arrow[dd, "\pi"] & \\
& \widetilde{Y} \arrow[rr] \arrow[dl] & & \widetilde{X} \arrow[dl, "\pi"] \\
Y \arrow[rr] & & X &
\end{tikzcd}$$

(5-3)

with morphisms labeled as indicated. Since $\pi'$ is an envelope, there exists $\tilde{\alpha} \in B_*(\widetilde{X}')$ such that $\pi'_*(\tilde{\alpha}) = \alpha$. We have that $c(\alpha) = c(\pi'_*(\tilde{\alpha})) = \pi''_*(c(\tilde{\alpha})) = \pi''_*((\pi^* c)(\tilde{\alpha})) = 0$, and it follows that $\pi^*$ is injective. By the functoriality of pullbacks it follows that $(p_1^* - p_2^*) \circ \pi^* = 0$.

Now, let $\tilde{c} \in B^*(\widetilde{Y} \to \widetilde{X})$ be a bivariant class such that $(p_1^* - p_2^*)(\tilde{c}) = 0$. We define a class $c \in B^*(Y \to X)$ as follows: given a morphism $f : X' \to X$ and a class $\alpha \in B_*(X')$, form once again the fiber diagram on the left in (5-3), with morphisms as indicated, and set $c(\alpha) = \pi''_*(\tilde{c}(\tilde{\alpha}))$, where $\tilde{\alpha} \in B_*(\widetilde{X}')$ is any class satisfying $\pi'_*(\tilde{\alpha}) = \alpha$ (which exists since $\pi'$ is an envelope). To see that $c(\alpha)$ is independent of the choice of $\tilde{\alpha}$, it is enough to see that $\pi''_*(\tilde{c}(\beta)) = 0$ for each class $\beta \in B_*(\widetilde{X}')$ such that $\pi'_* \beta = 0$. Let $g_j : \widetilde{X}' \times_{X'} \widetilde{X}' \to \widetilde{X}'$ and $g'_j : \widetilde{Y}' \times_{Y'} \widetilde{Y}' \to \widetilde{Y}'$ be the projections for $j = 1, 2$. By property (D), given such a class $\beta$ there exists a class $\gamma \in B_*(\widetilde{X}' \times_{X'} \widetilde{X}')$ such that $\beta = g_{1*}(\gamma) - g_{2*}(\gamma)$. Since $\pi'' \circ g'_1 = \pi'' \circ g'_2$, it follows that $\pi''_*(\tilde{c}(\beta)) = \pi''_*\big(\tilde{c}(g_{1*}(\gamma) - g_{2*}(\gamma))\big) = ((\pi'' \circ g'_1)_* - (\pi'' \circ g'_2)_*)(\tilde{c}(\gamma)) = 0$, and then $c$ is well-defined. It is straightforward to verify that $c$ satisfies the conditions ($C_1$)–($C_4$), so this construction yields a bivariant class $c \in B_*(Y \to X)$. To finish the proof we show that $\pi^* c = \tilde{c}$. For this, let $f : X' \to \widetilde{X}$ be any morphism and consider a class $\alpha \in B_*(X')$ and any class $\tilde{\alpha} \in B_*(\widetilde{X}')$ such that $\pi'_*(\tilde{\alpha}) = \alpha$. Form the fiber diagram on the right in (5-3) with morphisms as indicated. Since $p_1^*(\tilde{c}) = p_2^*(\tilde{c})$, we have that

$$\begin{aligned}(\pi^* c)_f(\alpha) = \pi''_*(\tilde{c}_{p_1 \circ f'}(\tilde{\alpha})) &= \pi''_*((f'^* p_1^* \tilde{c})(\tilde{\alpha})) \\ &= \pi''_*((f'^* p_2^* \tilde{c})(\tilde{\alpha})) = \pi''_*(\tilde{c}_{f \circ \pi'}(\tilde{\alpha})) = \tilde{c}_f(\alpha).\end{aligned}$$

Hence $\pi^* c = \tilde{c}$, and the proof is complete. □

**Corollary 5.4.** *Let $B_*$ be an ROBM prehomology theory on $\mathscr{C} = \mathrm{Sch}_k$ or $\mathscr{C} = G\text{-}\mathrm{Sch}_k$ that satisfies property* (D). *Then for any projective envelope $\pi : \widetilde{X} \to X$ in $\mathscr{C}$ the following sequence is exact*:

$$0 \longrightarrow B^*(X) \xrightarrow{\pi^*} B^*(\widetilde{X}) \xrightarrow{p_1^* - p_2^*} B^*(\widetilde{X} \times_X \widetilde{X}).$$

The following lemma is a simple consequence of the localization property (L) or of the property (D).

**Lemma 5.5.** *Let $B_*$ be an ROBM prehomology theory on $\mathscr{C} = \mathrm{Sch}_k$ or $\mathscr{C} = G\text{-}\mathrm{Sch}_k$ that satisfies either property* (L) *or property* (D). *If $X = \bigcup_{i=1}^r Z_i$, where each $f_i : Z_i \to X$ is a closed subscheme of $X$, then*

$$B_*(X) = \sum_{i=1}^{r} f_{i\,*}(B_*(Z_i))$$

*Proof.* The general case follows at once from the case $r = 2$, so we assume that $X = Z_1 \cup Z_2$. If $B_*$ satisfies (L), we have that the sequence

$$B_*(Z_1) \xrightarrow{f_{1\,*}} B_*(X) \xrightarrow{f_2|^*} B_*(X \setminus Z_1) \longrightarrow 0.$$

is exact. By the compatibility of pullbacks and pushforwards and using localization, it is clear that $f_2|^*$ maps $f_{2\,*}(B_*(Z_2))$ onto $B_*(X \setminus Z_1)$. It is clear now that $B_*(X) = f_{1\,*}(B_*(Z_1)) + f_{2\,*}(B_*(Z_2))$, as desired. If $B_*$ satisfies (D), the result follows since $B_*$ is additive and the projective morphism from the disjoint union $Z_1 \coprod Z_2 \to X$ induced by the inclusions is an envelope. □

The following result can be proved as a corollary of Theorem 5.3. It gives an inductive method for computing bivariant groups:

**Theorem 5.6.** *Let $B_*$ be an ROBM prehomology theory on $\mathscr{C} = \mathrm{Sch}_k$ or $\mathscr{C} = G\text{-}\mathrm{Sch}_k$. Let $\pi : \widetilde{X} \to X$ be a projective and birational envelope in $\mathscr{C}$. Let $Y \to X$ be a morphism in $\mathscr{C}$ and $\widetilde{Y} = \widetilde{X} \times_X Y$. Assume that $B_*$ satisfies the conclusion of Lemma 5.5 and that the sequence (5-2) in Theorem 5.3 is exact* (*e.g., it is enough to assume that $B_*$ satisfies* (D)). *Let $\pi : \pi^{-1}(U) \xrightarrow{\sim} U$ for some open dense $U \subset X$. Let $S_i \subset X$ be closed subschemes such that $X \setminus U = \bigcup S_i$. Let $E_i = \pi^{-1}(S_i)$, and let $\pi_i : E_i \to S_i$ be the induced morphism. Then for a class $\tilde{c} \in B^*(\widetilde{Y} \to \widetilde{X})$ the following are equivalent*:

(1) *$\tilde{c} = \pi^*(c)$ for some $c \in B^*(Y \to X)$.*

(2) *For all $i$, $\tilde{c}|_{E_i} = \pi_i^*(c_i)$ for some $c_i \in B^*(Y \times_X S_i \to S_i)$.*

*Proof.* If $\tilde{c} = \pi^*(c)$ for some $c \in B^*(Y \to X)$, then by the functoriality of pullbacks $\tilde{c}|_{E_i} = \pi_i^*(c|_{S_i})$ for all $i$, and then (2) holds if we take $c_i = c|_{S_i} \in B^*(Y \times_X S_i \to S_i)$. Reciprocally, assume that there are classes $c_i$ as in (2). Let $p_1, p_2 : \widetilde{X} \times_X \widetilde{X} \to \widetilde{X}$ be the projections. By Theorem 5.3 it is enough to show that $p_1^*\tilde{c} = p_2^*\tilde{c}$, i.e., that for any morphism $f : Z \to \widetilde{X} \times_X \widetilde{X}$ and for any class $\alpha \in B_*Z$ we have that $(p_1^*\tilde{c})(\alpha) = (p_2^*\tilde{c})(\alpha)$. Let $f_i' : E_i \times_X E_i \to \widetilde{X} \times_X \widetilde{X}$ be the corresponding closed embeddings and let $\Delta : \widetilde{X} \to \widetilde{X} \times_X \widetilde{X}$ be the diagonal morphism which is also a closed embedding. Notice that $\widetilde{X} \times_X \widetilde{X}$ is the union the closed subschemes $\Delta(\widetilde{X})$ and $f_i'(E_i \times_X E_i)$ for all $i$. Let $Z_0 = f^{-1}(\Delta(\widetilde{X}))$ and $Z_i = f^{-1}(f_i'(E_i \times_X E_i))$, with inclusions $\Delta' : Z_0 \to Z$ and $f_i'' : Z_i \to Z$ for each $i$. Then, by Lemma 5.5, in order to prove that $(p_1^*\tilde{c})(\alpha) = (p_2^*\tilde{c})(\alpha)$, we can assume that either $\alpha = f''_{i\,*}(\alpha_i)$ for some $i$ and some $\alpha_i \in B_*(Z_i)$ or $\alpha = \Delta'_*(\alpha_0)$ for some $\alpha_0 \in B_*(Z_0)$. In the first case, for $j = 1$ and $j = 2$ consider the fiber diagram

$$\begin{tikzcd}
E_i \times_X E_i \arrow[r, "f_i'"] \arrow[d, "p_j|"] & \widetilde{X} \times_X \widetilde{X} \arrow[d, "p_j"] \\
E_i \arrow[r, "\tilde{f}_i"] \arrow[d, "\pi_i"] & \widetilde{X} \arrow[d, "\pi"] \\
S_i \arrow[r, "f_i"] & X
\end{tikzcd}$$

with morphisms as labeled. Let $\Delta'' : Z_0 \times_X Y \to Z \times_X Y$ and $g_i'' : Z_i \times_X Y \to Z \times_X Y$ be the morphisms obtained from $\Delta'$ and $f_i''$ by base change. We have

$$\begin{aligned}
(p_j^*\tilde{c})(\alpha) &= (p_j^*\tilde{c})(f''_{i\,*}(\alpha_i)) = g''_{i\,*}((p_j^*\tilde{c})(\alpha_i)) = g''_{i\,*}(\tilde{c}(\alpha_i)) \\
&= g''_{i\,*}((\tilde{f}_i{}^*\tilde{c})(\alpha_i)) = g''_{i\,*}((\pi_i{}^*c_i)(\alpha_i)) = g''_{i\,*}((p_j|^*\pi_i{}^*c_i)(\alpha_i)) \\
&= g''_{i\,*}\big(((\pi_i \circ p_j|)^*c_i)(\alpha_i)\big).
\end{aligned}$$

Since $\pi_i \circ p_1| = \pi_i \circ p_2|$, it follows that in the first case $(p_1^*\tilde{c})(\alpha) = (p_2^*\tilde{c})(\alpha)$. In the second case, for $j = 1$ and $j = 2$ we have

$$\begin{aligned}(p_j^*\tilde{c})(\alpha) = (p_j^*\tilde{c})(\Delta'_*(\alpha_0)) = \Delta''_*((p_j^*\tilde{c})(\alpha_0)) = \Delta''_*((\Delta^* p_j^*\tilde{c})(\alpha_0))\\ = \Delta''_*\big(((p_j \circ \Delta)^*\tilde{c})(\alpha_0)\big) = \Delta''_*((\mathrm{Id}_{\tilde{X}}{}^*\tilde{c})(\alpha_0)) = \Delta''_*(\tilde{c}(\alpha_0)).\end{aligned}$$

Therefore, in the second case $(p_1^*\tilde{c})(\alpha) = (p_2^*\tilde{c})(\alpha)$, and the proof is complete. □

**5D.** ***Kimura-type descent sequence for bivariant equivariant theories and inductive computation of bivariant equivariant groups.*** Theorem 5.3 and Theorem 5.6 proved above can be applied to the equivariant theory $B_*^G$, provided that it satisfies property (D). We cannot prove property (D) for $B_*^G$ assuming that it holds for $B_*$. We will therefore give a different proof of the statements of Theorem 5.3 and Theorem 5.6 for $B_*^G$ that depends only on $B_*$ satisfying property (D). We give proofs for the bivariant cohomology groups $B_G^*(X)$ only.

**Theorem 5.7.** *Let $B_*$ be an ROBM prehomology theory on* Sch$_k$ *that satisfies properties* (H), (L) *and* (D). *Let $\pi : \tilde{X} \to X$ be a projective envelope in $G$-*Sch$_k$, *and let the terminology be as in Theorem 5.6.*

(a) *The following sequence is exact*:

$$0 \longrightarrow B_G^*(X) \xrightarrow{\pi^*} B_G^*(\tilde{X}) \xrightarrow{p_1^* - p_2^*} B_G^*(\tilde{X} \times_X \tilde{X}).$$

(b) *If $\pi$ is also birational, then for a class $\tilde{c} \in B_G^*(\tilde{X})$ the following are equivalent*:

(1) *$\tilde{c} = \pi^*(c)$ for some $c \in B_G^*(X)$.*

(2) *For all $i$, $\tilde{c}|_{E_i} = \pi_i^*(c_i)$ for some $c_i \in B_G^*(S_i)$.*

*Proof.* (a) Since for each $i$ the map $\tilde{X} \times^G U_i \to X \times^G U_i$ is an envelope, by Corollary 5.4 the sequence

$$0 \longrightarrow B^*(X \times^G U_i) \longrightarrow B^*(\tilde{X} \times^G U_i) \longrightarrow B^*((\tilde{X} \times_X \tilde{X}) \times^G U_i)$$

is exact. Applying the left exact functor $\varprojlim$ and using Proposition 5.2 gives the desired result.

(b) In view of part (a), the conclusion follows from Theorem 5.6 if we show that the ROBM prehomology theory $B_*^G$ satisfies the conclusion of Lemma 5.5.

For this, it is enough to consider the case $r = 2$, so we let $X = Z_1 \cup Z_2$ and let $Z = Z_1 \sqcup Z_2$. The projective morphism $Z \to X$ induced by the inclusions is an envelope; hence $B_*^G(Z) \to B_*^G(X)$ is surjective by Lemma 5.1 □

## 6. An overview of algebraic cobordism theory

In this section we recall the definition and main properties of *algebraic cobordism* $\Omega_*$. This theory was constructed by Levine and Morel [2007]. Later, Levine and Pandharipande [2009] found a geometric presentation of the cobordism groups. We will use the construction of [Levine and Pandharipande 2009] as the definition, but refer to [Levine and Morel 2007] for its properties. This construction and the proofs of some of the facts stated below use resolution of singularities, factorization of birational maps and some Bertini-type theorems; thus, we will assume for the remainder of this article that the field $k$ has characteristic zero.

The *equivariant algebraic cobordism* $\Omega_*^G$ was constructed first by Krishna [2012] and by Heller and Malagón-López [2013]. Krishna and Uma [2013] showed how to compute the equivariant and ordinary cobordism groups of smooth toric varieties; we will recall their result in Section 7.

For $X$ in $\mathrm{Sch}_k$, let $\mathcal{M}(X)$ be the set of isomorphism classes of projective morphisms $f : Y \to X$ for $Y \in \mathrm{Sm}_k$. This set is a monoid under disjoint union of the domains; let $\mathcal{M}^+(X)$ be its group completion. The elements of $\mathcal{M}^+(X)$ are called cycles. The class of $f : Y \to X$ in $\mathcal{M}^+(X)$ is denoted $[f : Y \to X]$. The group $\mathcal{M}^+(X)$ is free abelian, generated by the cycles $[f : Y \to X]$ where $Y$ is irreducible.

A double point degeneration is a morphism $\pi : Y \to \mathbb{P}^1$, with $Y \in \mathrm{Sm}_k$ of pure dimension, such that $Y_\infty = \pi^{-1}(\infty)$ is a smooth divisor on $Y$ and $Y_0 = \pi^{-1}(0)$ is a union $A \cup B$ of smooth divisors intersecting transversely along $D = A \cap B$. Define

$$\mathbb{P}_D = \mathbb{P}(\mathcal{O}_D(A) \oplus \mathcal{O}_D) = \mathrm{Proj}_{\mathcal{O}_D}(\mathrm{Sym}^*_{\mathcal{O}_D}(\mathcal{O}_D(A) \oplus \mathcal{O}_D)),$$

where $\mathcal{O}_D(A)$ denotes $\mathcal{O}_Y(A)|_D$. (Notice that $\mathbb{P}(\mathcal{O}_D(A) \oplus \mathcal{O}_D) \cong \mathbb{P}(\mathcal{O}_D(B) \oplus \mathcal{O}_D)$ because $\mathcal{O}_D(A+B) \cong \mathcal{O}_D$.)

Let $X \in \mathrm{Sch}_k$, and let $Y \in \mathrm{Sm}_k$ have pure dimension. Let $p_1, p_2$ be the two projections of $X \times \mathbb{P}^1$. A double point relation is defined by a projective morphism $\pi : Y \to X \times \mathbb{P}^1$ such that $p_2 \circ \pi : Y \to \mathbb{P}^1$ is a double point degeneration. Let

$$[Y_\infty \to X], \quad [A \to X], \quad [B \to X], \quad [\mathbb{P}_D \to X]$$

be the cycles obtained by composing with $p_1$. The double point relation is

$$[Y_\infty \to X] - [A \to X] - [B \to X] + [\mathbb{P}_D \to X] \in \mathcal{M}^+(X).$$

Let $\mathcal{R}(X)$ be the subgroup of $\mathcal{M}^+(X)$ generated by all the double point relations. The cobordism group of $X$ is defined to be

$$\Omega_*(X) = \mathcal{M}^+(X)/\mathcal{R}(X).$$

The group $\mathcal{M}^+(X)$ is graded so that $[f : Y \to X]$ lies in degree $\dim Y$ when $Y$ has pure dimension. Since double point relations are homogeneous, this grading gives a grading on $\Omega_*(X)$. We write $\Omega_n(X)$ for the degree-$n$ part of $\Omega_*(X)$.

There is a functorial pushforward homomorphism $f_* : \Omega_*(X) \to \Omega_*(Z)$ for $f : X \to Z$ projective, and a functorial pullback homomorphism $g^* : \Omega_*(Z) \to \Omega_{*+d}(X)$ for $g : X \to Z$ a smooth morphism of relative dimension $d$. These homomorphisms are both defined on the cycle level; the pullback does not preserve grading. The exterior product on $\Omega_*(X)$ is also defined on the cycle level:

$$[Y \to X] \times [Z \to W] = [Y \times Z \to X \times W].$$

Levine and Morel [2007] presented a construction of functorial pullbacks $g^*$ along lci morphisms $g$, and, more generally, of refined lci pullbacks.

The groups $\Omega_*(X)$ with these projective pushforward, lci pullback and exterior products form an oriented Borel–Moore homology theory (see [Levine and Morel 2007]). Moreover, with those refined lci pullbacks it is also an ROBM prehomology theory (see [Levine and Morel 2007]).

As in the case of a general ROBM prehomology theory, $\Omega_*(\operatorname{Spec} k)$ is a ring, $\Omega_*(X)$ is a module over $\Omega_*(\operatorname{Spec} k)$ for general $X$, and $\Omega_*(X)$ is an algebra over $\Omega_*(\operatorname{Spec} k)$ for smooth $X$. When $X$ is smooth and has pure dimension, we also use the cohomological notation

$$\Omega^*(X) = \Omega_{\dim X - *}(X).$$

Then $\Omega^*(X)$ is a graded algebra over the graded ring $\Omega^*(\operatorname{Spec} k)$. The class $1_X = [\mathrm{Id}_X : X \to X]$ is the identity of the algebra. Similar conventions are used for the equivariant cobordism groups.

**Remark 6.1.** Algebraic cobordism satisfies the homotopy property (H) and the localization property (L) [Levine and Morel 2007], as well as the descent property (D) [González and Karu 2015]. It follows that *everything* proved in the previous sections for general ROBM prehomology theories can be applied to the algebraic cobordism theory. This includes the construction of the equivariant cobordism theory $\Omega_*^G$, the operational cobordism theory $\Omega^*$ and the operational equivariant cobordism theory $\Omega_G^*$. This also includes the descent exact sequences for the operational theories $\Omega^*$ and $\Omega_G^*$ and the inductive method for their computation using envelopes.

The following part of Theorem 5.7 applied to $\Omega_G^*(X)$ will be used in the next section:

**Theorem 6.2.** *Assume that $\pi : \widetilde{X} \to X$ is a projective birational envelope in $G$-$\mathrm{Sch}_k$, with $\pi : \pi^{-1}(U) \xrightarrow{\sim} U$ for some open nonempty $G$-equivariant $U \subset X$. Let $S_i \subset X$ be closed $G$-equivariant subschemes, such that $X \setminus U = \bigcup S_i$. Let $E_i = \pi^{-1}(S_i)$,* *and let* $\pi_i : E_i \to S_i$ *be the induced morphism. Then* $\pi^* : \Omega^*_G(X) \to \Omega^*_G(\widetilde{X})$ *is injective, and for a class* $\tilde{c} \in \Omega^*_G(\widetilde{X})$ *the following are equivalent*:

(1) $\tilde{c} = \pi^*(c)$ *for some* $c \in \Omega^*_G(X)$.

(2) *For all* $i$, $\tilde{c}|_{E_i} = \pi_i^*(c_i)$ *for some* $c_i \in \Omega^*_G(S_i)$.

**6A.** ***Formal group law.*** Algebraic cobordism is endowed with first Chern class operators associated to line bundles, whose definition agrees with the one in Definition 2.9. We recall the formal group law satisfied by these operators.

A *formal group law* on a commutative ring $R$ is a power series $F_R(u, v) \in R[[u, v]]$ satisfying:

(a) $F_R(u, 0) = F_R(0, u) = u$.

(b) $F_R(u, v) = F_R(v, u)$.

(c) $F_R(F_R(u, v), w) = F_R(u, F_R(v, w))$.

Thus

$$F_R(u, v) = u + v + \sum_{i,j>0} a_{i,j} u^i v^j,$$

where the $a_{i,j} \in R$ satisfy $a_{i,j} = a_{j,i}$ and some additional relations coming from property (c). We think of $F_R$ as giving a formal addition

$$u +_{F_R} v = F_R(u, v).$$

There exists a unique power series $\chi(u) \in R[[u]]$ such that $F_R(u, \chi(u)) = 0$. Set $[-1]_{F_R} u = \chi(u)$. Composing $F_R$ and $\chi$, we can form linear combinations

$$[n_1]_{F_R} u_1 +_{F_R} [n_2]_{F_R} u_2 +_{F_R} \cdots +_{F_R} [n_r]_{F_R} u_r \in R[[u_1, \ldots, u_r]]$$

for $n_i \in \mathbb{Z}$ and $u_i$ variables.

There exists a universal formal group law $F_{\mathbb{L}}$, and its coefficient ring $\mathbb{L}$ is called the *Lazard ring*. This ring can be constructed as the quotient of the polynomial ring $\mathbb{Z}[A_{i,j}]_{i,j>0}$ by the relations imposed by the three axioms above. The images of the variables $A_{i,j}$ in the quotient ring are the coefficients $a_{i,j}$ of the formal group law $F_{\mathbb{L}}$. It is shown in [Levine and Morel 2007] that $\Omega^*(\operatorname{Spec} k)$ is isomorphic as a graded ring to $\mathbb{L}$ with grading induced by letting $A_{i,j}$ have degree $-i - j + 1$. The power series $F_{\mathbb{L}}(u, v)$ is then homogeneous of degree 1 if $u$ and $v$ both have degree 1.

The formal group law on $\mathbb{L}$ describes the first Chern class operators of tensor products of line bundles:

$$\tilde{c}_1(L \otimes M) = F_{\mathbb{L}}(\tilde{c}_1(L), \tilde{c}_1(M))$$

for any line bundles $L$ and $M$ on any scheme $X$ in $\mathrm{Sch}_k$.

## 7. Operational equivariant cobordism of toric varieties

Let $X_\Delta$ be a smooth quasiprojective toric variety corresponding to a fan $\Delta$. Krishna and Uma [2013] showed that the $T$-equivariant cobordism ring of $X_\Delta$ is isomorphic to the ring of piecewise graded power series on the fan $\Delta$. Our goal here is to show that, for any fan $\Delta$, the ring of piecewise graded power series on $\Delta$ is isomorphic to the operational $T$-equivariant cobordism ring of $X_\Delta$. When $X_\Delta$ is smooth, this follows from the result of Krishna and Uma by Poincaré duality. For singular $X_\Delta$ we follow the argument of [Payne 2006] in the case of Chow theory to reduce to the smooth case.

We use the standard notation for toric varieties [Fulton 1993]. Let $N \cong \mathbb{Z}^n$ be a lattice. It determines a split torus $T$ with character lattice $M = \operatorname{Hom}(N, \mathbb{Z})$. A toric variety $X_\Delta$ is defined by a fan $\Delta$ in $N$.

Every quasiprojective toric variety $X_\Delta$ with torus $T$ is in the category $T$-$\mathrm{Sch}_k$, since each line bundle on such variety admits a $T$-linearization. We will write $\Omega_*^T(X_\Delta)$ for the $T$-equivariant cobordism group of $X_\Delta$. For a smooth $X_\Delta$ we also use the cohomological notation $\Omega_T^*(X_\Delta) = \Omega^T_{\dim X_\Delta - *}(X_\Delta)$. The operational $T$-equivariant cobordism ring is denoted $\Omega_T^*(X_\Delta)$. For smooth $X_\Delta$, the two definitions of $\Omega_T^*(X_\Delta)$ are identified via Poincaré duality.

**7A.** ***Graded power series rings.*** We start by recalling some notions from [Krishna and Uma 2013].

Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a commutative graded ring. The *graded power series ring* is

$$A[\![t_1, t_2, \dots, t_n]\!]_{\mathrm{gr}} = \bigoplus_{d \in \mathbb{Z}} S_d.$$

Here $S_d$ is the group of degree-$d$ homogeneous power series $\sum_I a_I t^I$, where the sum runs over multi-indices $I = (i_1, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$, $t^I = t_1^{i_1} \cdots t_n^{i_n}$, and $a_I \in A_{d - i_1 - \cdots - i_n}$. This ring can be viewed as the inverse limit in the category of commutative graded rings

$$A[\![t_1, t_2, \dots, t_n]\!]_{\mathrm{gr}} \cong \varprojlim_j A[t_1, \dots, t_n]/(t_1^j, \dots, t_n^j).$$

Let us also recall the topological tensor product. Given two inverse limits of graded $A$-modules $B = \varprojlim_j B_j$ and $C = \varprojlim_j C_j$, define

$$B \mathbin{\hat{\otimes}_A} C = \varprojlim_j (B_j \otimes_A C_j).$$

(All limits are in the category of graded $A$-modules, hence the tensor product is again a graded $A$-module.) Consider now the case where $C = A[\![t_1, t_2, \dots, t_n]\!]_{\mathrm{gr}}$ and $B$ is a limit of graded $A$-algebras. We may then identify

$$B \mathbin{\hat{\otimes}_A} A[\![t_1, t_2, \dots, t_n]\!]_{\mathrm{gr}} = B[\![t_1, t_2, \dots, t_n]\!]_{\mathrm{gr}}.$$

Let $T$ be a torus determined by a lattice $N$, and let $\chi_1, \dots, \chi_n$ be a basis for the dual lattice $M$. It is shown in [Krishna 2012] that the equivariant cobordism ring of a point with trivial $T$-action is isomorphic to

$$\Omega_T^*(\operatorname{Spec} k) \cong \mathbb{L}[\![t_1, \dots, t_n]\!]_{\mathrm{gr}}, \tag{7-1}$$

with $t_i$ corresponding to the first Chern class transformation $\tilde{c}_1^T(L_{\chi_i})$ of the equivariant line bundle $L_{\chi_i}$. Since this ring depends on the lattice $M$ only, we will write it as $\mathbb{L}[\![M]\!]_{\mathrm{gr}}$ (see Section 7D for a presentation of $\Omega_T^*(\operatorname{Spec} k) \cong \mathbb{L}[\![M]\!]_{\mathrm{gr}}$ functorial in $M$ and independent of the choice of a basis for $M$).

We will need a slight generalization of the isomorphism (7-1). First recall that, given a closed subgroup $H \subset G$, there is a natural *change of groups* homomorphism $\tau : \Omega_G^*(X) \to \Omega_H^*(X)$ for any smooth $X$ in $G$-$\mathrm{Sch}_k$.

**Lemma 7.1.** *Let $X$ be a smooth scheme in $G$-$\mathrm{Sch}_k$, and let $T$ act trivially on $X$. Then there is an isomorphism*

$$\Omega_{G\times T}^*(X) \cong \Omega_G^*(X) \mathbin{\widehat{\otimes}_{\mathbb{L}}} \mathbb{L}[\![M]\!]_{\mathrm{gr}} = \Omega_G^*(X)[\![M]\!]_{\mathrm{gr}}.$$

*This isomorphism is compatible with lci pullbacks and change of groups in the following sense. Let $f : Y \to X$ be an lci morphism between smooth schemes in $G$-$\mathrm{Sch}_k$ and let $H \subset G$ be a closed subgroup. Then the following diagrams commute*:

$$\begin{tikzcd}
\Omega_{G\times T}^*(X) \arrow[r, "f^*"] \arrow[d, "\cong"] & \Omega_{G\times T}^*(Y) \arrow[d, "\cong"] \\
\Omega_G^*(X)[\![M]\!]_{\mathrm{gr}} \arrow[r] & \Omega_G^*(Y)[\![M]\!]_{\mathrm{gr}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_{G\times T}^*(X) \arrow[r, "\tau"] \arrow[d, "\cong"] & \Omega_{H\times T}^*(X) \arrow[d, "\cong"] \\
\Omega_G^*(X)[\![M]\!]_{\mathrm{gr}} \arrow[r] & \Omega_H^*(X)[\![M]\!]_{\mathrm{gr}}
\end{tikzcd}$$

*Here the lower horizontal maps are the maps induced on the tensor product by the identity on $\mathbb{L}[\![M]\!]_{\mathrm{gr}}$ and the maps $f^*$, $\tau$ on the other factor.* (*Equivalently, the maps are $f^*$ and $\tau$ applied to the coefficients of power series.*)

*Proof.* We may assume that $\dim T = 1$. Let

$$\Omega_G^*(X) = \varprojlim_j \Omega^*(X \times^G U_j),$$

where $\{(V_j, U_j)\}$ is a good system of representations of $G$. Let $T$ act on $\mathbb{A}^j = k^j$ diagonally. Then we may choose $\{(\mathbb{A}^j, \mathbb{A}^j \setminus \{0\})\}$ as a good system of representations for $T$. Now,

$$\begin{aligned}
\Omega_{G\times T}^*(X) &= \varprojlim_j \Omega^*(X \times^{G\times T} (U_j \times \mathbb{A}^j \setminus \{0\})) \\
&= \varprojlim_j \Omega^*((X \times^G U_j) \times \mathbb{P}^{j-1}) \\
&= \varprojlim_j \Omega^*(X \times^G U_j) \otimes_{\mathbb{L}} \mathbb{L}[t]/(t^j) \\
&= \Omega_G^*(X) \mathbin{\widehat{\otimes}_{\mathbb{L}}} \mathbb{L}[\![t]\!]_{\mathrm{gr}}.
\end{aligned}$$

In the third equality we used the projective bundle formula [Levine and Morel 2007]

$$\Omega^*(Z \times \mathbb{P}^n) \cong \Omega^*(Z)[t]/(t^{n+1}),$$

for any smooth scheme $Z$. The compatibility of each step with the change of group homomorphisms and the compatibility of the projective bundle isomorphism with lci pullbacks $W \to Z$ give the compatibility statement of the lemma. □

**7B.** ***Piecewise graded power series on*** $\boldsymbol{\Delta}$**.** Consider now a toric variety $X_\Delta$. There is a one-to-one correspondence between cones $\sigma \in \Delta$ and $T$-orbits in $X_\Delta$. Let $O_\sigma$ be the orbit corresponding to a cone $\sigma$. Then the stabilizer (of a point) of $O_\sigma$ is the subtorus $T_\sigma \subset T$ corresponding to the sublattice $N_\sigma = \operatorname{Span} \sigma \cap N \subset N$. The Morita isomorphism [Krishna and Uma 2013] then gives

$$\Omega_T^*(O_\sigma) = \Omega_T^*(T \times^{T_\sigma} \operatorname{Spec} k) \cong \Omega_{T_\sigma}^*(\operatorname{Spec} k).$$

We set $S_\sigma = \Omega_T^*(O_\sigma) \cong \Omega_{T_\sigma}^*(\operatorname{Spec} k)$. When $\tau$ is a face of $\sigma$, we have the inclusion of lattices $N_\tau \subset N_\sigma$, giving rise to the inclusion of tori $T_\tau \subset T_\sigma$; the change of groups homomorphism then defines the restriction map $S_\sigma \to S_\tau$. Let $S = \Omega_T^*(\operatorname{Spec} k)$. The rings $S_\sigma$ and $S_\tau$ are graded $S$-algebras and the restriction map $S_\sigma \to S_\tau$ is a morphism of graded $S$-algebras.

A piecewise graded power series on $\Delta$ is a collection $(a_\sigma \in S_\sigma)_{\sigma \in \Delta}$ such that $a_\sigma$ restricts to $a_\tau$ whenever $\sigma \geq \tau$. Let $\operatorname{PPS}(\Delta)$ be the graded $S$-algebra of all piecewise graded power series on $\Delta$. Similarly, let $\operatorname{PPS}(\operatorname{St} \rho)$ be the graded $S$-algebra of piecewise graded power series on $\operatorname{St} \rho = \{\sigma \in \Delta \mid \rho \leq \sigma\}$, that is, collections $(a_\sigma \in S_\sigma)_{\sigma \in \operatorname{St} \rho}$ such that $a_\sigma$ restricts to $a_\tau$ for $\sigma \geq \tau \in \operatorname{St} \rho$.

**7C.** ***Operational equivariant cobordism of toric varieties.*** The inclusion map $i_\sigma : O_\sigma \hookrightarrow X_\Delta$ is an lci morphism when $X_\Delta$ is smooth; hence there exists a pullback map

$$i_\sigma^* : \Omega_T^*(X_\Delta) \to \Omega_T^*(O_\sigma) = S_\sigma.$$

**Theorem 7.2** [Krishna and Uma 2013]. *Let $X_\Delta$ be a smooth quasiprojective toric variety. Then the morphism of $S$-algebras*

$$\Omega_T^*(X_\Delta) \xrightarrow{(i_\sigma^*)} \prod_{\sigma \in \Delta} S_\sigma$$

*is injective and the image is equal to the $S$-algebra* $\operatorname{PPS}(\Delta)$ *of piecewise graded power series on $\Delta$.*

In the proof of Krishna and Uma, the group $\Omega_T^*(X_\Delta)$ stands for the cohomological notation of $\Omega^T_{\dim X_\Delta - *}(X_\Delta)$ and the maps $i_\sigma^*$ are lci pullbacks. We claim that the same statement is true for general $X_\Delta$ when $\Omega_T^*(X_\Delta)$ stands for the operational cobordism ring and $i_\sigma^*$ is the pullback morphism in the operational theory.

**Theorem 7.3.** *Let $X_\Delta$ be a quasiprojective toric variety. Then the morphism of $S$-algebras*

$$\Omega^*_T(X_\Delta) \xrightarrow{(i^*_\sigma)} \prod_{\sigma\in\Delta} S_\sigma$$

*is injective and the image is equal to the $S$-algebra* $\mathrm{PPS}(\Delta)$ *of piecewise graded power series on* $\Delta$.

*Proof.* We prove the theorem by induction on $\dim X_\Delta$. In the inductive proof we will need a slightly stronger statement. Let $\widetilde{T}$ be another torus and $\widetilde{T} \to T$ a split surjective group homomorphism $\widetilde{T} \cong T \times T'$ from some torus $T'$. Let $\widetilde{T}$ act on $X_\Delta$ via the homomorphism $\widetilde{T} \to T$. Replacing $T$ with $\widetilde{T}$, we define as above $\widetilde{S} = \Omega^*_{\widetilde{T}}(\operatorname{Spec} k)$, $\widetilde{S}_\sigma = \Omega^*_{\widetilde{T}}(O_\sigma)$, and the restriction maps $\widetilde{S}_\sigma \to \widetilde{S}_\tau$ for $\sigma \geq \tau$ (note that $T$ and $\widetilde{T}$ have the same orbits). We let $\widetilde{\mathrm{PPS}}(\Delta)$ denote the $\widetilde{S}$ algebra of piecewise graded power series on $\Delta$ defined using the rings $\widetilde{S}_\sigma$. The $\widetilde{S}$ algebra $\widetilde{\mathrm{PPS}}(\mathrm{St}\,\rho)$ is defined similarly.

Instead of the theorem, we prove the following stronger statement:

**Claim.** *The morphism*

$$\Omega^*_{\widetilde{T}}(X_\Delta) \xrightarrow{(i^*_\sigma)} \prod_{\sigma\in\Delta} \widetilde{S}_\sigma$$

*is injective and the image is equal to the graded $\widetilde{S}$-algebra* $\widetilde{\mathrm{PPS}}(\Delta)$.

Let us first check the claim for smooth $X_\Delta$. The statement of Theorem 7.2 can be restated as saying that the following sequence is exact:

$$0 \longrightarrow \Omega^*_T(X_\Delta) \xrightarrow{(i^*_\sigma)} \prod_{\sigma\in\Delta} S_\sigma \rightrightarrows \prod_{\sigma,\tau\in\Delta} S_{\sigma\cap\tau}.$$

Here the last two maps are constructed from restrictions $S_\sigma \to S_{\sigma\cap\tau}$ and $S_\sigma \to S_{\tau\cap\sigma}$. Let us write $\widetilde{T} \cong T \times T'$, where $T'$ has character lattice $M'$. Tensoring the sequence with $\mathbb{L}[\![M']\!]_{\mathrm{gr}}$, we get the exact sequence

$$0 \to \Omega^*_T(X_\Delta)[\![M']\!]_{\mathrm{gr}} \xrightarrow{(i^*_\sigma)} \prod_{\sigma\in\Delta} S_\sigma[\![M']\!]_{\mathrm{gr}} \rightrightarrows \prod_{\sigma,\tau\in\Delta} S_{\sigma\cap\tau}[\![M']\!]_{\mathrm{gr}}.$$

The maps in this sequence are the old maps applied to coefficients of power series. From Lemma 7.1 we know that this sequence is isomorphic to the sequence

$$0 \to \Omega^*_{\widetilde{T}}(X_\Delta) \xrightarrow{(i^*_\sigma)} \prod_{\sigma\in\Delta} \widetilde{S}_\sigma \rightrightarrows \prod_{\sigma,\tau\in\Delta} \widetilde{S}_{\sigma\cap\tau},$$

where the maps are again lci pullbacks and change of group homomorphisms. This proves the claim in the case of smooth $X_\Delta$.

When $X_\Delta$ is singular, we resolve its singularities by a sequence of star subdivisions of $\Delta$:

$$X_\Delta \longleftarrow X_{\Delta'} \longleftarrow \cdots \longleftarrow X_{\Delta''}.$$

We may assume by induction on the number of star subdivisions that the claim holds for $X_{\Delta'}$. The morphism $f : X_{\Delta'} \to X_\Delta$ is the blowup of $X_\Delta$ along a closed subscheme $C \subset X_\Delta$ with support $|C|$ equal to the orbit closure $V_\pi = \overline{O}_\pi$, where $\pi \in \Delta$ is the cone containing the subdivision ray in its relative interior. Let $\rho \in \Delta'$ be the new ray. Then the exceptional divisor $E = f^{-1}(C)$ has support $|E| = V_\rho$. The morphism $f$ is a birational envelope. In order to use Theorem 6.2, we need to identify $\Omega^*_{\widetilde{T}}(C), \Omega^*_{\widetilde{T}}(E)$ and the pullback map between them.

**Lemma 7.4.** *For any* $0 \neq \pi \in \Delta$, *the map*

$$\Omega^*_{\widetilde{T}}(V_\pi) \xrightarrow{(i^*_\sigma)} \prod_{\sigma \in \mathrm{St}(\pi)} \widetilde{S}_\sigma$$

*is injective and the image is equal to the graded* $\widetilde{S}$*-algebra* $\widetilde{\mathrm{PPS}}(\mathrm{St}\,\pi)$.

*Proof.* The orbit closure $V_\pi$ is again a toric variety corresponding to the fan $\Delta_\pi$ that is the image of $\mathrm{St}\,\pi$ in $N/(\mathrm{Span}\,\pi \cap N)$. There is a split surjection from the torus $T$ (and hence also from $\widetilde{T}$) to the big torus in $V_\pi$. The result now follows by induction on the dimension of the toric variety. ☐

We can also apply the previous lemma to $\rho \in \Delta'$, to get that $\Omega^*_{\widetilde{T}}(V_\rho)$ is isomorphic to $\widetilde{\mathrm{PPS}}(\mathrm{St}\,\rho)$. Moreover, since by assumption we know the claim for $X_{\Delta'}$, the pullback map $\Omega^*_{\widetilde{T}}(X_{\Delta'}) \to \Omega^*_{\widetilde{T}}(V_\rho)$ is the restriction of piecewise power series $\widetilde{\mathrm{PPS}}(\Delta') \to \widetilde{\mathrm{PPS}}(\mathrm{St}\,\rho)$.

Next we describe the pullback morphism $\Omega^*_{\widetilde{T}}(V_\pi) \to \Omega^*_{\widetilde{T}}(V_\rho)$. Note that every cone $\sigma \in \mathrm{St}\,\rho$ lies in some cone of $\tau \in \mathrm{St}\,\pi$, and hence we have the restriction map $\widetilde{S}_\tau \to \widetilde{S}_\sigma$. These maps combine to give a well-defined pullback map of piecewise graded power series $\widetilde{\mathrm{PPS}}(\mathrm{St}\,\pi) \to \widetilde{\mathrm{PPS}}(\mathrm{St}\,\rho)$.

**Lemma 7.5.** *The pullback morphism*

$$\Omega^*_{\widetilde{T}}(V_\pi) \cong \widetilde{\mathrm{PPS}}(\mathrm{St}\,\pi) \to \Omega^*_{\widetilde{T}}(V_\rho) \cong \widetilde{\mathrm{PPS}}(\mathrm{St}\,\rho)$$

*is the pullback of piecewise graded power series.*

*Proof.* Define a map $\phi : \Delta' \to \Delta$ so that $\phi(\sigma)$ is the smallest cone in $\Delta$ containing $\sigma$. Then the map $f : X_{\Delta'} \to X_\Delta$ takes $O_\sigma$ onto $O_{\phi(\sigma)}$. The pullback morphism

$$\Omega^*_{\widetilde{T}}(O_{\phi(\sigma)}) = \widetilde{S}_{\phi(\sigma)} \to \Omega^*_{\widetilde{T}}(O_\sigma) = \widetilde{S}_\sigma$$

is the restriction of power series.

Consider the commutative diagram

$$\begin{tikzcd}
\Omega^*_{\tilde{T}}(V_\rho) \arrow[r] & \prod_{\sigma\in\operatorname{St}\rho}\Omega^*_{\tilde{T}}(O_\sigma) \\
\Omega^*_{\tilde{T}}(V_\pi) \arrow[r] \arrow[u] & \prod_{\tau\in\operatorname{St}\pi}\Omega^*_{\tilde{T}}(O_\tau) \arrow[u]
\end{tikzcd}$$

where all maps are pullback morphisms. The right vertical map sends $(a_\tau)$ to $(b_\sigma)$ such that $b_\sigma$ is the restriction of $a_{\phi(\sigma)}$; this map restricts to the pullback of piecewise power series $\widetilde{\mathrm{PPS}}(\operatorname{St}\pi) \to \widetilde{\mathrm{PPS}}(\operatorname{St}\rho)$, which proves the lemma. □

To finish the proof of the claim, we apply Theorem 6.2 with $S_1 = C^{\mathrm{red}} = V_\pi$ and $E_1 = E^{\mathrm{red}} = V_\rho$. Since the pullback map $\Omega^*_{\tilde{T}}(V_\pi) \to \Omega^*_{\tilde{T}}(V_\rho)$ is injective, Theorem 6.2 implies that we have a Cartesian diagram, with all maps pullbacks:

$$\begin{tikzcd}
\Omega^*_{\tilde{T}}(X_\Delta) \arrow[r] \arrow[d] & \Omega^*_{\tilde{T}}(V_\pi) \arrow[d] \\
\Omega^*_{\tilde{T}}(X_{\Delta'}) \arrow[r] & \Omega^*_{\tilde{T}}(V_\rho)
\end{tikzcd}$$

The following diagram of piecewise power series and pullback maps is clearly Cartesian:

$$\begin{tikzcd}
\widetilde{\mathrm{PPS}}(\Delta) \arrow[r] \arrow[d] & \widetilde{\mathrm{PPS}}(\operatorname{St}\pi) \arrow[d] \\
\widetilde{\mathrm{PPS}}(\Delta') \arrow[r] & \widetilde{\mathrm{PPS}}(\operatorname{St}\rho)
\end{tikzcd}$$

The first diagram maps to the second one by the pullback maps $i^*_\sigma$. This implies that $\Omega^*_{\tilde{T}}(X_\Delta) \cong \widetilde{\mathrm{PPS}}(\Delta)$. □

**7D.** ***Piecewise graded exponential power series.*** We give in this subsection a canonical presentation of the ring

$$\Omega^*_T(\operatorname{Spec} k) \cong \mathbb{L}[\![M]\!]_{\mathrm{gr}}$$

that is functorial in $M$ and independent of the choice of a basis for $M$. This leads to the description of $\Omega^*_T(X_\Delta)$ as the algebra of piecewise graded exponential power series, similar to the case of equivariant $K$-theory [Anderson and Payne 2015]. An even more general construction of the *formal group ring* $R[\![M]\!]_F$ was given in [Calmès et al. 2013] for an arbitrary ring $R$ with a formal group law $F$.

Let $M$ be the character lattice of a torus $T$, and define

$$\mathbb{L}[\![M]\!]_{\mathrm{gr}} = \bigoplus_d \prod_k \mathbb{L}_{d-k} \otimes \operatorname{Sym}^k M.$$

This ring is (noncanonically) isomorphic to the graded power series ring in $\operatorname{rank}(M)$ variables.

For any formal group law $F$ on a ring $R$ there exists a unique power series

$$e_F(x) = x + b_2x^2 + b_3x^3 + \cdots \in R[\![x]\!]_{\mathrm{gr}} \otimes \mathbb{Q},$$

called the exponential series, such that

$$F(e_F(u), e_F(v)) = e_F(u+v).$$

The series $e_F(x)$ is homogeneous of degree 1. (See [Levine and Morel 2007, Lemma 4.1.29] for the construction of the inverse power series $l_F(x)$.) For the additive group law $F(u,v) = u+v$, we have $e_F(x) = x$. For the multiplicative group law $F(u,v) = u+v+buv$,

$$e_F(x) = x + b\frac{x^2}{2!} + b^2\frac{x^3}{3!} + \cdots .$$

We consider the exponential power series $e(x)$ for the formal group law $F_{\mathbb{L}}$ on $\mathbb{L}$. The map

$$e : M \to \mathbb{L}[\![M]\!]_{\mathrm{gr}} \otimes \mathbb{Q}$$

that sends $u$ to $e(u)$ satisfies the equality $F_{\mathbb{L}}(e(u), e(v)) = e(u+v)$. We get a canonical isomorphism

$$\mathbb{L}[\![M]\!]_{\mathrm{gr}} \otimes \mathbb{Q} \cong \Omega_T(\operatorname{Spec} k) \otimes \mathbb{Q},$$

identifying $e(u)$ with the first Chern class transformation $\tilde{c}_1^T(L_u)$. The integral cobordism ring $\Omega_T(\operatorname{Spec} k)$ is then canonically isomorphic to the subring of $\mathbb{L}[\![M]\!]_{\mathrm{gr}} \otimes \mathbb{Q}$ consisting of graded power series in $e(u)$ for $u \in M$ and coefficients in $\mathbb{L}$. (Here we need the fact that the additive group $\mathbb{L}$ is a free abelian group and hence embeds in $\mathbb{L} \otimes \mathbb{Q}$.)

The construction of the ring of graded exponential power series is functorial in $M$. Indeed, a homomorphism of lattices $M \to M'$ gives rise to the ring homomorphism $\mathbb{L}[\![M]\!]_{\mathrm{gr}} \to \mathbb{L}[\![M']\!]_{\mathrm{gr}}$, such that the diagram

$$\begin{tikzcd}
M \arrow[r, "e"] \arrow[d] & \mathbb{L}[\![M]\!]_{\mathrm{gr}} \otimes \mathbb{Q} \arrow[d] \\
M' \arrow[r, "e"] & \mathbb{L}[\![M']\!]_{\mathrm{gr}} \otimes \mathbb{Q}
\end{tikzcd}$$

is commutative. It follows that graded exponential power series are mapped to graded exponential power series. This corresponds to the pullback map

$$\Omega_T(\operatorname{Spec} k) \to \Omega_{T'}(\operatorname{Spec} k).$$

Theorem 7.3 now states that the equivariant operational cobordism ring of $X_\Delta$ is canonically isomorphic to the ring of piecewise graded exponential power series on $\Delta$.

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# Schubert decompositions for quiver Grassmannians of tree modules

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Appendix by Thorsten Weist

Let $Q$ be a quiver, $M$ a representation of $Q$ with an ordered basis $\mathcal{B}$ and $\underline{e}$ a dimension vector for $Q$. In this note we extend the methods of Lorscheid (2014) to establish Schubert decompositions of quiver Grassmannians $\mathrm{Gr}_{\underline{e}}(M)$ into affine spaces to the ramified case, i.e., the canonical morphism $F : T \to Q$ from the coefficient quiver $T$ of $M$ w.r.t. $\mathcal{B}$ is not necessarily unramified.

In particular, we determine the Euler characteristic of $\mathrm{Gr}_{\underline{e}}(M)$ as the number of *extremal successor closed subsets of* $T_0$, which extends the results of Cerulli Irelli (2011) and Haupt (2012) (under certain additional assumptions on $\mathcal{B}$).

## Introduction

The recent interest in quiver Grassmannians stems from a formula of Caldero and Chapoton [2006] that relates cluster variables of a quiver $Q$ with the Euler characteristics of the quiver Grassmannians of exceptional modules of $Q$. Formulas for the Euler characteristics for a given quiver yield a description of the associated cluster algebra in terms of generators and relations. This opened a way to understand cluster algebras, which are defined by an infinite recursive procedure, in terms of closed formulas—provided one knows the Euler characteristics of the associated quiver Grassmannians.

***Torus actions and cluster algebras associated with string algebras.*** While the classification of all cluster algebras seems to be as much out of reach as a classification of wild algebras, their is some hope to understand and classify cluster algebras that are associated with tame algebras. A first step in this direction has been realized by Cerulli Irelli [2011] and Haupt [2012] who established a formula for the Euler characteristics of quiver Grassmannians in the so-called unramified case. These results sufficed to understand all cluster algebras associated path with string algebras.

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We review the method of Cerulli Irelli and Haupt in brevity: following Ringel [1998], we note that every exceptional representation $M$ of a quiver $Q$ has a tree basis $\mathscr{B}$, meaning that the coefficient quiver $T = \Gamma(M, \mathscr{B})$ is a tree. A subset $\beta$ of $T_0 = \mathscr{B}$ is *successor closed* if for all $i \in \beta$ and all arrows $\alpha : i \to j$ in $T$, also $j \in \beta$. A subset $\beta$ of $T_0 = \mathscr{B}$ is *of type* $\underline{e} = (e_p)_{p \in Q_0}$ if $\#\beta \cap M_p = e_p$ for all $p \in Q_0$.

If the canonical morphism $F : T \to Q$ is *unramified*, i.e., the morphism of the underlying CW-complexes is locally injective, then one can define a (piecewise continuous) action of the torus $\mathbb{G}_m$ on $\mathrm{Gr}_{\underline{e}}(M)$ that has only finitely many fixed points. This yields the formula

$$\chi(\mathrm{Gr}_{\underline{e}}(M)) = \#\{\text{fixed points}\} = \#\{\text{successor closed } \beta \subset T_0 \text{ of type } \underline{e}\}.$$

For other types of cluster algebras, the exceptional modules are in general not unramified tree modules. This is, for instance, the case for cluster algebras associated with exceptional Dynkin quivers of types $\widetilde{D}$ and $\widetilde{E}$, or, more general, for cluster algebras associated with clannish algebras or exceptional tame algebras. Therefore other methods are required to treat ramified tree modules.

***Cluster algebras from marked surfaces.*** Fomin, Shapiro and Thurston explore in [Fomin et al. 2008] the connection between cluster algebras and marked surfaces. Namely, to each surface with boundary and finitely many marked points such that each boundary component contains at least one marked point, one can associate a cluster algebra. It is shown in that paper that all cluster algebras associated with quivers of extended Dynkin types $\widetilde{A}$ and $\widetilde{D}$ come from marked surfaces.

This connection with marked surfaces yields a description of the cluster variables in terms of triangulations of the surface, which leads to a combinatorial description of the algebra. For unpunctured surfaces, i.e., all marked points are contained in the boundary, Musiker, Schiffler and Williams [Musiker et al. 2013] construct a basis for the associated cluster algebra.

Cluster algebras of punctured surfaces, which includes type $D$ algebras, are more difficult to treat since not all mutations of clusters come from flips of triangulations. For recent results in this direction, see [Qiu and Zhou 2013]. However, these methods do not suffice yet for a complete understanding of the associated cluster algebras.

***Schubert decompositions and ramified tree modules.*** Caldero and Reineke [2008] show that $\mathrm{Gr}_{\underline{e}}(M)$ is smooth projective if $M$ is exceptional. If $M$ is an equi-oriented string module, i.e., the coefficient quiver $T$ is an equi-oriented Dynkin quiver of type $A_n$, then $\mathrm{Gr}_{\underline{e}}(M)$ has a continuous torus action with finitely many fixed points, see [Cerulli Irelli 2011]. Thus if $M$ is an exceptional equi-oriented string module, then the Białynicki–Birula decomposition [1973, Theorem 4.3] yields a decomposition of $\mathrm{Gr}_{\underline{e}}(M)$ into affine spaces.

While a torus action with finitely many fixed points determines the Euler characteristic, a decomposition of $\mathrm{Gr}_{\underline{e}}(M)$ into affine spaces determines the (additive structure of the) cohomology of $\mathrm{Gr}_{\underline{e}}(M)$, which is a much stronger result. In particular, we re-obtain the Euler characteristic as the number of affine spaces occurring in the decomposition. However, the class of exceptional equi-oriented string modules is very limited. In particular, most exceptional modules of affine type $D$ are not of this kind.

In [Lorscheid 2014], we extend decompositions of $\mathrm{Gr}_{\underline{e}}(M)$ into affine spaces to a larger class of quiver Grassmannians by a different method. Namely, the choice of an ordered basis $\mathscr{B}$ of $M$ defines a decomposition of $\mathrm{Gr}_{\underline{e}}(M)$ into Schubert cells, which are, in general, merely closed subsets of affine spaces. In certain cases, however, these Schubert cells are affine spaces themselves. The method of proof is to exhibit explicit presentations of Schubert cells in terms of generators and relations.

One requirement of [Lorscheid 2014] is that the morphism $F : T \to Q$ is unramified. It is the purpose of this note to extend the methods of that paper to ramified $F : T \to Q$. In particular, this extends, under the given additional assumptions, the formula of Cerulli Irelli and Haupt to the ramified case.

As will be shown in joint work with Thorsten Weist, the results of this text are indeed applicable to all exceptional modules of affine type $\widetilde{D}_n$, which yields combinatorial formulas for the Euler characteristics of $\mathrm{Gr}_{\underline{e}}(M)$.

***The main result of this text.*** An arrow $\alpha$ of $T$ is *extremal* if for every other arrow $\alpha'$ with $F(\alpha') = F(\alpha)$ either $s(\alpha) < s(\alpha')$ or $t(\alpha') < t(\alpha)$. A subset $\beta$ of $T_0$ is *extremal successor closed* if for every $i \in \beta$ and every extremal arrow $\alpha : i \to j$ in $T$, we also have $j \in \beta$.

Under certain additional assumptions on $\mathscr{B}$, the quiver Grassmannian $\mathrm{Gr}_{\underline{e}}(M)$ decomposes into affine spaces (Theorem 4.1), and the parametrization of the nonempty Schubert cells yields the formula

$$\chi(\mathrm{Gr}_{\underline{e}}(M)) = \#\{\text{extremal successor closed } \beta \subset T_0 \text{ of type } \underline{e}\},$$

see Corollary 4.4.

***Content overview.*** To keep the technical complexity as low as possible, we restrict ourselves in this text to tree modules over the complex numbers, though the methods work in the more general context of modules of tree extensions over arbitrary rings as considered in [Lorscheid 2014]. The technique of proof in the ramified case is essentially the same as the one used in [Lorscheid 2014]. But since the presentation of our results is different and simplified, we include all details.

This text is organized as follows. In Section 1, we review basic facts about quiver Grassmannians, their Schubert decompositions and tree modules. In Section 2, we describe generators and relations for a Schubert cell, which are labeled by *relevant pairs* and *relevant triples*, respectively. In Section 3, we introduce *extremal successor closed subsets*, *polarizations* and *maximal relevant pairs*, and we establish preliminary facts. In Section 4, we state the main results and conclude with several remarks and examples.

The Appendix, by Thorsten Weist, shows how to establish polarizations for exceptional modules along Schofield induction.

## 1. Setup

To start with, let us explain the notation and terminology that we use. By a variety we understand the space of complex points of an underlying scheme, and we broadly ignore the schematic structure of quiver Grassmannians. For more details on the notions in this section, see Sections 1 and 2 of [Lorscheid 2014].

**1A.** ***Quiver Grassmannians.*** Let $Q = (Q_0, Q_1, s, t)$ be a quiver with (complex) representation $M = \big(\{M_i\}_{i \in Q_0}, \{M_\alpha\}_{\alpha \in Q_1}\big)$, with dimension vector $\underline{d} = \underline{\dim}\, M$, and let $\underline{e} \leq \underline{d}$ be another dimension vector for $Q$. The *quiver Grassmannian* $\mathrm{Gr}_{\underline{e}}(M)$ is the set of subrepresentations $N$ of $M$ with $\underline{\dim}\, N = \underline{e}$. A *basis* $\mathscr{B}$ *for* $M$ is the union $\bigcup_{p \in Q_0} \mathscr{B}_p$ of bases $\mathscr{B}_p$ for the vector spaces $M_p$. An *ordered basis of* $M$ is a basis $\mathscr{B}$ of $M$ whose elements $b_1, \dots, b_n$ are linearly ordered. The choice of an ordered basis yields an inclusion

$$\mathrm{Gr}_{\underline{e}}(M) \to \prod_{p \in Q_0} \mathrm{Gr}(e_p, d_p)$$

that sends $N$ to $(N_p)_{p \in Q_0}$, which endows $\mathrm{Gr}_{\underline{e}}(M)$ with the structure of a projective variety.

**1B.** ***Schubert decompositions.*** A point of the Grassmannian $\mathrm{Gr}(e, d)$ is an $e$-dimensional subspace $V$ of $\mathbb{C}^d$. Let $V$ be spanned by vectors $w_1, \dots, w_e \in \mathbb{C}^d$. We write $w = (w_{i,j})_{i=1\dots d, j=1\dots,e}$ for the matrix of all coordinates of $w_1, \dots, w_e$. For a subset $I$ of $\{1, \dots, d\}$ of cardinality $e$, the Plücker coordinates

$$\Delta_I(V) = \det(w_{i,j})_{i \in I, j=1\dots,e}$$

define a point $(\Delta_I(V))_I$ in $\mathbb{P}\big(\bigwedge^e \mathbb{C}^d\big)$. For two ordered subsets $I = \{i_1, \dots, i_e\}$ and $J = \{j_1, \dots, j_e\}$ of $\{1, \dots, d\}$, we define $I \leq J$ if $i_l \leq j_l$ for all $l = 1 \dots, e$. The Schubert cell $C_I(d)$ of $\mathrm{Gr}(e, d)$ is defined as the locally closed subvariety of all subspaces $V$ such that $\Delta_I(V) \neq 0$ and $\Delta_J(V) = 0$ for all $J > I$.

Given a quiver $Q$, a representation $M$ with ordered basis $\mathscr{B}$ and a dimension vector $\underline{e}$, we say that a subset $\beta$ of $\mathscr{B}$ is *of type* $\underline{e}$ if $\beta_p = \beta \cap \mathscr{B}_p$ is of cardinality $e_p$ for every $p \in Q_0$. For $\underline{d} = \underline{\dim}\, M$, the Schubert cell $C_\beta(\underline{d})$ is defined as the locally closed subset $\prod_{p\in Q_0} C_{\beta_p}(d_p)$ of $\prod_{p\in Q_0}\mathrm{Gr}(e_p, d_p)$. The *Schubert cell* $C_\beta^M$ is defined as the intersection of $C_\beta(\underline{d})$ with $\mathrm{Gr}_{\underline{e}}(M)$ inside $\prod_{p\in Q_0} C_{\beta_p}(d_p)$. The *Schubert decomposition of* $\mathrm{Gr}_{\underline{e}}(M)$ *(w.r.t. the ordered basis* $\mathscr{B}$*)* is the decomposition

$$\mathrm{Gr}_{\underline{e}}(M) = \coprod_{\substack{\beta\subset\mathscr{B}\\ \text{of type } \underline{e}}} C_\beta^M$$

into locally closed subvarieties. Note that the Schubert cells $C_\beta^M$ are affine varieties, but that they are, in general, not affine spaces. In particular, a Schubert cell $C_\beta^M$ might be empty. We say that $\mathrm{Gr}_{\underline{e}}(M) = \coprod C_\beta^M$ is a *decomposition into affine spaces* if every Schubert cell $C_\beta^M$ is either an affine space or empty.

**1C.** ***Tree modules.*** Let $M$ be a representation of a quiver $Q$ with basis $\mathscr{B}$. Let $\alpha : s \to t$ be an arrow of $Q$ and $b \in \mathscr{B}_s$. Then we have the equations

$$M_\alpha(b) = \sum_{b'\in\mathscr{B}_t} \lambda_{\alpha,b,c}\, c$$

with uniquely determined coefficients $\lambda_{\alpha,b,c} \in \mathbb{C}$. The *coefficient quiver of* $M$ *w.r.t.* $\mathscr{B}$ is the quiver $T = \Gamma(M, \mathscr{B})$ with vertex set $T_0 = \mathscr{B}$ and with arrow set

$$T_1 = \bigl\{(\alpha, b, c) \in Q_1 \times \mathscr{B} \times \mathscr{B} \bigm| b \in \mathscr{B}_{s(\alpha)}, c \in \mathscr{B}_{t(\alpha)} \text{ and } \lambda_{\alpha,b,c} \neq 0\bigr\}.$$

It comes equipped with a morphism $F : T \to Q$ that sends $b \in \mathscr{B}_p$ to $p$ and $(\alpha, b, c)$ to $\alpha$, and with a thin sincere representation $N = N(M, \mathscr{B})$ of $T$ with basis $\mathscr{B}$ and $1 \times 1$-matrices $N_{(\alpha,b,c)} = (\lambda_{\alpha,b,c})$. Note that $M$ is canonically isomorphic to the pushforward $F_*N$ (see [Lorscheid 2014, Section 4]).

The representation $M$ is called a *tree module* if there exists a basis $\mathscr{B}$ of $M$ such that the coefficient quiver $T = \Gamma(M, \mathscr{B})$ is a tree. We call such a basis a *tree basis for* $M$.

If $T$ is a tree, we can replace the basis elements $b$ by certain nonzero multiples $b'$ such that all $\lambda_{\alpha,b,c}$ equal 1. We refer to this assumption by the expression $M = F_*T$ where we identify $T$, by abuse of notation, with its thin sincere representation with basis $T_0 = \mathscr{B}$ and matrices (1). In this case, $M$ and $\mathscr{B}$ are determined as the pushforward of this thin sincere representation of $T$ along $F : T \to Q$. In general, $T$ is not determined by $M$: there are examples of tree modules $M$ and bases $\mathscr{B}$ and $\mathscr{B}'$ such that $\Gamma(M, \mathscr{B})$ and $\Gamma(M, \mathscr{B}')$ are nonisomorphic trees.

## 2. Presentations of Schubert cells

Let $Q$ be a quiver and $M$ a representation with ordered basis $\mathscr{B}$ and dimension vector $\underline{d}$. Let $\underline{e}$ be another dimension vector for $Q$ and $\beta \subset \mathscr{B}$ of type $\underline{e}$. In this section, we will describe coordinates and relations for the Schubert cell $C_\beta^M$ of $\mathrm{Gr}_{\underline{e}}(M)$.

**2A.** ***Normal form for matrix representations.*** Let $N$ be a point of $C_\beta^M$. Then $N_p$ is an $e_p$-dimensional subspace of $M_p$ for every $p \in Q_0$ and has a basis $(w_j)_{j \in \beta_p}$ where $w_j = (w_{i,j})_{i \in \mathscr{B}_p}$ are column vectors in $M_p$. If we define $w_{i,j} = 0$ for $i, j \in \mathscr{B}$ whenever $j \in \beta$, or $i \in \mathscr{B}_p$ and $j \in \mathscr{B}_q$ with $p \neq q$, then we obtain a matrix $w = (w_{i,j})_{i,j \in \mathscr{B}}$. We call $w$ a *matrix representation of* $N$. Note that $N$ is determined by the matrix representation $w$, but there are in general many different matrix representations of $N$.

We say that a matrix $w = (w_{i,j})_{i,j \in \mathscr{B}}$ in $\mathrm{Mat}_{\mathscr{B} \times \mathscr{B}}$ is *in $\beta$-normal form* if

(i) $w_{i,i} = 1$ for all $i \in \beta$,

(ii) $w_{i,j} = 0$ for all $i, j \in \beta$ with $j \neq i$,

(iii) $w_{i,j} = 0$ for all $i \in \mathscr{B}$ and $j \in \beta$ with $j < i$,

(iv) $w_{i,j} = 0$ for all $i \in \mathscr{B}$ and $j \in \mathscr{B} - \beta$, and

(v) $w_{i,j} = 0$ for all $i \in \mathscr{B}_p$ and $j \in \beta_q$ with $p \neq q$.

**Lemma 2.1.** *Every $N \in C_\beta^M$ has a unique matrix representation $w = (w_{i,j})_{i,j \in \mathscr{B}}$ in $\beta$-normal form.*

*Proof.* The uniqueness follows from the fact that a matrix $w$ in $\beta$-normal form is in reduced column echelon form by (i)–(iv). The vanishing of the Plücker coordinates $\Delta_J(N_p)$ for $J > \beta_p$ and the nonvanishing of $\Delta_{\beta_p}(N_p)$ implies that we find pivot elements in the rows $i \in \beta_p$ for each $p \in Q_0$ for a matrix presentation $w$ of $N$ in reduced echelon form. This shows that there is a matrix presentation $w$ of $N$ that satisfies (i)–(iv). Since $\mathscr{B}_p \subset N_p$, the matrix $w$ is a block matrix and satisfies (v). □

**2B.** ***Defining equations.*** Lemma 2.1 identifies $C_\beta^M$ with a subset of the affine matrix space $\mathrm{Mat}_{\mathscr{B} \times \mathscr{B}}$. The following lemma determines defining equations (along with equations (i)–(v) from Section 2A) for $C_\beta^M$, which shows that $C_\beta^M$ is a closed subvariety of $\mathrm{Mat}_{\mathscr{B} \times \mathscr{B}}$.

**Lemma 2.2.** *Let $T = \Gamma(M, \mathscr{B})$ be the coefficient quiver of $M$ w.r.t. $\mathscr{B}$ and $F : T \to Q$ the canonical morphism, and recall that $T_0 = \mathscr{B}$. A matrix $w = (w_{i,j})_{i,j \in \mathscr{B}}$ in $\beta$-normal form is the matrix representation of a point $N$ of $C_\beta^M$ if and only if $w$ satisfies the equation*

$$E(\bar{\alpha}, t, s): \qquad \sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \\ \text{with } t(\alpha) = t}} w_{s(\alpha),s} = \sum_{\alpha \in F^{-1}(\bar{\alpha})} w_{t,t(\alpha)} w_{s(\alpha),s}.$$

*for all arrows $\bar{\alpha} \in Q_1$ and all vertices $s \in F^{-1}(s(\bar{\alpha}))$ and $t \in F^{-1}(t(\bar{\alpha}))$.*

*If $t \in \beta$ or $s \notin \beta$, then $E(\bar{\alpha}, t, s)$ is satisfied for any $w$ in $\beta$-normal form.*

*Proof.* Given a matrix $w = (w_{i,j})_{i,j \in \mathscr{B}}$ in $\beta$-normal form, we write $w_i$ for the column vector $(w_{i,j})_{j \in \mathscr{B}_p}$ where $p \in Q_0$ and $i \in \mathscr{B}_p$. The matrix $w$ represents a point $N$ of $\mathrm{Gr}_{\underline{e}}(M)$ if and only if for all $\bar{\alpha} \in Q_1$ and all $s \in \beta_{s(\bar{\alpha})}$, there are $\lambda_k \in \mathbb{C}$ for $k \in \beta_{t(\bar{\alpha})}$ such that

$$M_{\bar{\alpha}}\, w_s = \sum_{k \in \beta_{t(\bar{\alpha})}} \lambda_k w_k.$$

This means that for all $t \in F^{-1}(t(\bar{\alpha}))$,

$$\sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \\ \text{with } t(\alpha)=t}} w_{s(\alpha),s} = \big[M_{\bar{\alpha}}\, w_s\big]_t \quad \text{equals} \quad \sum_{k \in \beta_{t(\bar{\alpha})}} \lambda_k w_{t,k}.$$

For $t \in \beta_{t(\bar{\alpha})}$, we obtain

$$\sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \\ \text{with } t(\alpha)=t}} w_{s(\alpha),s} = \sum_{k \in \beta_{t(\bar{\alpha})}} \lambda_k w_{t,k} = \sum_{k \in \beta_{t(\bar{\alpha})}} \lambda_k \delta_{t,k} = \lambda_t$$

by (i) and (ii) for $w$ in $\beta$-normal form. Therefore, for arbitrary $t \in F^{-1}(t(\bar{\alpha}))$ we obtain that

$$\sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \\ \text{with } t(\alpha)=t}} w_{s(\alpha),s} = \sum_{k \in \beta_{t(\bar{\alpha})}} \Bigg( \sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \\ \text{with } t(\alpha)=k}} w_{s(\alpha),s} \Bigg) w_{t,k} = \sum_{\alpha \in F^{-1}(\bar{\alpha})} w_{t,t(\alpha)} w_{s(\alpha),s},$$

as claimed. If $t \in \beta$, then this equation is satisfied for all $w$ in $\beta$-normal form by the definition of the $\lambda_k$ and since $w_{t,k} = \delta_{t,k}$ for $t \in \beta$. If $s \notin \beta$, then all coefficients $w_{s(\alpha),s}$ are 0, i.e., we obtain the tautological equation $0 = 0$. This proves the latter claim of the lemma. □

**2C.** ***Relevant pairs and relevant triples.*** A *relevant pair* is an element of the set

$$\mathrm{Rel}^2 = \{(i, j) \in T_0 \times T_0 \mid F(i) = F(j) \text{ and } i \leq j\}$$

and a *relevant triple* is an element of the set

$$\mathrm{Rel}^3 = \left\{ (\bar{\alpha}, t, s) \in Q_1 \times T_0 \times T_0 \;\middle|\; \begin{array}{c} \text{there is an } \alpha' : s' \to t' \text{ in } T \text{ with } F(\alpha') = \bar{\alpha}, \\ F(s') = F(s),\; F(t') = F(t),\; s' \leq s \text{ and } t \leq t' \end{array} \right\}.$$

Given a matrix $w = (w_{i,j})$ in $\beta$-normal form, we say that $w_{i,j}$ is a *constant coefficient* (*w.r.t.* $\beta$) if it appears in one of the equations (i)–(v) from Section 2A, and otherwise we say that $w_{i,j}$ is a *free coefficient* (*w.r.t.* $\beta$), which is the case if and only if there is a $p \in Q_0$ such that $i \in \mathscr{B}_p - \beta_p$, $j \in \beta_p$ and $i < j$. The significance of $\mathrm{Rel}^2$ is that if $w_{i,j}$ is not constant equal to 0 w.r.t. $\beta$ (for any $\beta$), then $(i, j)$ is a relevant pair.

If we substitute for a given $\beta$ all constant coefficients $w_{i,j}$ with $i \neq j$ by 0, then we obtain the *$\beta$-reduced form of $E(\bar{\alpha}, t, s)$*:

$$\sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \text{ with} \\ t(\alpha)=t,\ s(\alpha)\leq s, \\ s(\alpha)\notin\beta \text{ or } s(\alpha)=s}} w_{s(\alpha),s} = \sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \text{ with} \\ t<t(\alpha),\ s(\alpha)<s, \\ t(\alpha)\in\beta,\ s(\alpha)\notin\beta}} w_{t,t(\alpha)} w_{s(\alpha),s} + \sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \text{ with} \\ s(\alpha)=s,\ t\leq t(\alpha), \\ t(\alpha)\in\beta \text{ or } t(\alpha)=t}} w_{t,t(\alpha)}. \tag{1}$$

The significance of $\mathrm{Rel}^3$ is that if $E(\bar{\alpha}, t, s)$ is a nontrivial equation in the coefficients of a matrix $w$ in $\beta$-normal form (for any $\beta$), then $(\bar{\alpha}, t, s)$ is a relevant triple.

In the following, we will associate certain values with relevant pairs and relevant triples. Since $T_0 = \mathscr{B}$ is linearly ordered, we can identify it order-preservingly with $\{1, \ldots, n\}$. We define the root of a connected component of $T$ as its smallest vertex, and we denote by $r(i)$ the root of the component that contains the vertex $i$. In particular, if $T$ is connected, then 1 is the only root and $r(i) = 1$ for all $i \in T_0$. Let $d(i, j)$ denote the graph distance of two vertices $i, j \in T_0$. We define the *root distance of a relevant pair* $(i, j)$ as

$$\delta(i, j) = \max\{d(i, r(i)), d(j, r(j))\}.$$

We define the *fiber length of a relevant pair* $(i, j)$ as

$$\epsilon(i, j) = \#\big\{k \in T_0 \mid F(k) = F(i) \text{ and } i \leq k < j\big\}.$$

We consider $\mathbb{N} \times \mathbb{N} \times T_0$ with its lexicographical order, i.e., $(i, j, k) < (i', j', k')$ if $i < i'$, or $i = i'$ and $j < j'$, or $i = i'$, $j = j'$ and $k < k'$. The inclusion

$$\begin{array}{rccc} \Psi : & \mathrm{Rel}^2 & \to & \mathbb{N} \times \mathbb{N} \times T_0 \\ & (i, j) & \mapsto & \big(\epsilon(i, j), \delta(i, j), j\big) \end{array}$$

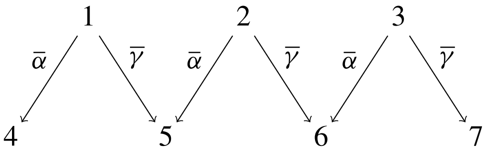
induces a linear order on $\mathrm{Rel}^2$, i.e., $(i, j) < (i', j')$ if $\Psi(i, j) < \Psi(i', j')$.

Let $(\bar{\alpha}, t, s)$ be a relevant triple. We define $\Psi(\bar{\alpha}, t, s)$ as the maximum of $\Psi(s_{\min}, s)$ and $\Psi(t, t_{\max})$, where $s_{\min}$ is the smallest vertex that is the source of an arrow $\alpha \in F^{-1}(\bar{\alpha})$ with $t \leq t(\alpha)$ and $t_{\max}$ is the largest vertex that is the target of an arrow $\alpha \in F^{-1}(\bar{\alpha})$ with $s(\alpha) \leq s$.

For a relevant triple $(\bar{\alpha}, t, s)$ with $t \notin \beta$ and $s \in \beta$, we define $\Psi_\beta(\bar{\alpha}, t, s)$ as $\Psi(i, j)$ where $(i, j)$ is the largest relevant pair that appears as an index in the $\beta$-reduced form (1) of $E(\bar{\alpha}, t, s)$. Note that $E(\bar{\alpha}, t, s)$ contains at least one nontrivial term by the definition of a relevant triple. Note further that if there is an arrow $\alpha : s \to t$ in $F^{-1}(\bar{\alpha})$ and every other arrow $\alpha' \in F^{-1}(\bar{\alpha})$ satisfies either $s < s(\alpha')$ or $t(\alpha') < t$, then the only nontrivial terms in (1) are the constant coefficients $w_{s,s}$ and $w_{t,t}$. Thus in this case $\Psi_\beta(\bar{\alpha}, t, s) = \max\{\Psi(s, s), \Psi(t, t)\}$.

Since $w_{i,j} = 0$ if $j < i$ for $w$ in $\beta$-normal form, we have $\Psi_\beta(\bar{\alpha}, t, s) \leq \Psi(\bar{\alpha}, t, s)$. In Section 3D, we consider cases in which $\Psi_\beta(\bar{\alpha}, t, s)$ and $\Psi(\bar{\alpha}, t, s)$ are equal.

**Example 2.3.** A good example to illustrate the roles of relevant pairs, relevant triples and the function $\Psi$ is the following. Let $M$ be the preinjective representation of the Kronecker quiver $Q = K(2)$ with dimension vector $(3, 4)$. Denote the two arrows of $Q$ by $\bar{\alpha}$ and $\bar{\gamma}$. Then there exists an ordered basis $\mathscr{B}$ of $M$ such that the coefficient quiver $T = \Gamma(M, \mathscr{B})$ looks like



where we label the arrows by their image under $F$. We investigate the Schubert cell $C^M_\beta$ for $\beta = \{3, 6, 7\}$. A matrix $w = (w_{i,j})_{i,j\in\mathscr{B}}$ in $\beta$-normal form has the six free coefficients $w_{1,3}$, $w_{2,3}$, $w_{4,6}$, $w_{5,6}$, $w_{4,7}$, $w_{5,7}$, and $w_{3,3} = w_{6,6} = w_{7,7} = 1$. All other coefficients vanish. The nontrivial equations on the free coefficients are labeled by the relevant triples $(\bar{\alpha}, 5, 3)$, $(\bar{\alpha}, 4, 3)$, $(\bar{\gamma}, 5, 3)$ and $(\bar{\gamma}, 4, 3)$, and their respective $\beta$-reduced forms are

$$w_{2,3} = w_{5,6}, \quad w_{1,3} = w_{4,6}, \quad w_{1,3} = w_{2,3}w_{5,6} + w_{5,7} \quad \text{and} \quad 0 = w_{2,3}w_{4,6} + w_{4,7}.$$

It is easy to see that these equations can be solved successively in linear terms. We show how these equations are organized by the ordering of $\mathrm{Rel}^2$ defined by $\Psi$. The relevant pairs that appear as indices of free coefficients are ordered as follows:

$$(5, 6) < (2, 3) < (4, 6) < (1, 3) < (5, 7) < (4, 7).$$

Ordered by size, we have

$$\Psi_\beta(\bar{\alpha}, 5, 3) = (2, 3), \quad \Psi_\beta(\bar{\alpha}, 4, 3) = (1, 3),$$
$$\Psi_\beta(\bar{\gamma}, 5, 3) = (5, 7), \quad \Psi_\beta(\bar{\gamma}, 4, 3) = (4, 7),$$

which correspond to the indices of linear terms in each of the corresponding equations. Therefore, we find a unique solution in $w_{2,3}$, $w_{1,3}$, $w_{5,7}$ and $w_{4,7}$ for every $w_{5,6}$ and $w_{4,6}$, which shows that $C^M_\beta$ is isomorphic to $\mathbb{A}^2$.

This demonstrates how the ordering of relevant pairs organizes the defining equations for $C^M_\beta$ in such a way that they are successively solvable in variables that appear in linear terms. In the next section, we will develop criteria under which this example generalizes to other representations $M$ and ordered bases $\mathscr{B}$.

**Remark 2.4.** The definition of $\Psi$ is based on heuristics with random examples of tree modules with ordered $F : T \to Q$. It is possible that different orders of $\mathrm{Rel}^2$

lead to analogues of Theorem 4.1 that include quiver Grassmannians not covered in this text. Interesting variants might include the graph distance $d(i, j)$ of $i$ and $j$ as an ordering criterion; e.g., consider the ordering of $\mathrm{Rel}^2$ given by the map $\widetilde{\Psi} : \mathrm{Rel}^2 \to \mathbb{N} \times \mathbb{N} \to T_0$ with $\widetilde{\Psi}(i, j) = (d(i, j), \epsilon(i, j), j)$. This might be of particular interest for exceptional modules that do not have an ordered tree basis such that $F : T \to Q$ is ordered. See, however, Section 4B for some limiting examples.

## 3. Preliminaries for the main theorem

In this section, we develop the terminology and establish preliminary facts to formulate and prove the main theorem in Section 4. As before, we let $Q$ be a quiver and $M$ a representation with ordered basis $\mathscr{B}$ and dimension vector $\underline{d}$. Let $\underline{e}$ be another dimension vector for $Q$ and $\beta \subset \mathscr{B}$ of type $\underline{e}$. Let $T = \Gamma(M, \mathscr{B})$ be the coefficient quiver of $M$ w.r.t. $\mathscr{B}$ and $F : T \to Q$ the canonical morphism. We identify the linearly ordered set $T_0 = \mathscr{B}$ with $\{1, \dots, n\}$.

**3A.** ***Extremal successor closed subsets.*** An arrow $\alpha : s \to t$ in $T$ is called *extremal* (*with respect to* $F$) if all other arrows $\alpha' : s' \to t'$ with $F(\alpha') = F(\alpha)$ satisfy either $s < s'$ or $t' < t$. Note that if $F$ is ordered and unramified, then every arrow of $T$ is extremal.

Recall that $T_0 = \mathscr{B}$, which allows us to consider $\beta$ as a subset of $T_0$. We say that $\beta$ is *extremal successor closed* if for all extremal arrows $\alpha : s \to t$ of $T$, either $s \notin \beta$ or $t \in \beta$. Note that if $F$ is ordered and unramified, then $\beta$ is extremal successor closed if and only if $\beta$ is successor closed in the sense of [Cerulli Irelli 2011] and [Haupt 2012].

**Lemma 3.1.** *If $\beta$ is not extremal successor closed, then $C_\beta^M$ is empty.*

*Proof.* We assume that $C_\beta^M$ is nonempty and prove the lemma by contraposition. Let $\alpha : s \to t$ be an extremal arrow in $T$ and $\bar{\alpha} = F(\alpha)$. Let $N \in C_\beta^M$ have the matrix representation $w$ in $\beta$-normal form. The $\beta$-reduced form of $E(\bar{\alpha}, t, s)$ is

$$w_{s,s} = w_{t,t} w_{s,s}$$

since $\alpha : s \to t$ is extremal and thus for every other $\alpha' : s' \to t'$ in $F^{-1}(\bar{\alpha})$ either $s < s'$, and thus $w_{s',s} = 0$, or $t' < t$, and thus $w_{t',t} = 0$. Since $w_{s,s} = 1$ if $s \in \beta$ (by (i)) and $w_{t,t} = 0$ if $t \notin \beta$ (by (iv)), equation $E(\bar{\alpha}, t, s)$ would be $1 = 0$ if $s \in \beta$ and $t \notin \beta$. This is not possible since we assumed that $C_\beta$ is nonempty. Therefore $s \notin \beta$ or $t \in \beta$, which shows that $\beta$ is extremal successor closed. □

**3B.** ***Ordered and ramified morphisms.*** The morphism $F : T \to Q$ is *ordered* if for all arrows $\alpha : s \to t$ and $\alpha' : s' \to t'$ of $T$ with $F(\alpha) = F(\alpha')$, we have $s \leq s'$ if and only if $t \leq t'$.

Consider an arrow $\bar{\alpha} \in Q_1$ and a vertex $i \in T_0$. The *ramification index* $r_{\bar{\alpha}}(i)$ *at* $i$ *in direction* $\bar{\alpha}$ is the number of arrows $\alpha \in F^{-1}(\bar{\alpha})$ with source or target $i$. If $r_{\bar{\alpha}}(i) > 1$, we say that $F$ *branches at* $i$ *in direction* $\bar{\alpha}$ and that $F$ *ramifies above* $F(i)$. The morphism $F : T \to Q$ is *unramified* or a *winding* if for all $\bar{\alpha} \in Q_1$ and all $i \in T_0$, we have $r_{\bar{\alpha}}(i) \leq 1$. In other words, $F : T \to Q$ is unramified if and only if the associated map of CW-complexes is unramified.

Note that $F$ is strictly ordered (in the sense of [Lorscheid 2014, Section 4.2]) if and only if $F$ is ordered and unramified. From this viewpoint, we can say that we extend Theorem 4.2 of [Lorscheid 2014] from unramified morphisms $F : T \to Q$ to ramified $F$ in this text.

**3C.** ***Polarizations.*** Let $I = \{i_1, \dots, i_r\}$ be a finite ordered set with $i_1 < \cdots < i_r$. A *sorting of* $I$ is a decomposition $I = I^< \amalg I^>$ such that $I^< = \{i_1, \dots, i_s\}$ and $I^> = \{i_{s+1}, \dots, i_r\}$ for some $s \in \{1, \dots, r-1\}$. A *polarization for a linear map* $M_{\bar{\alpha}} : M_p \to M_q$ (between finite dimensional complex vector spaces) are ordered bases $\mathscr{B}_p$ and $\mathscr{B}_q$ for $M_p$ and $M_q$, respectively, that admit sortings $\mathscr{B}_p = \mathscr{B}^<_{p,\bar{\alpha}} \amalg \mathscr{B}^>_{p,\bar{\alpha}}$ and $\mathscr{B}_q = \mathscr{B}^<_{q,\bar{\alpha}} \amalg \mathscr{B}^>_{q,\bar{\alpha}}$ such that $M_{\bar{\alpha}}$ restricts to a surjection $\mathscr{B}^<_{p,\bar{\alpha}} \cup \{0\} \twoheadrightarrow \mathscr{B}^<_{q,\bar{\alpha}} \cup \{0\}$ and its adjoint map $M^{\text{ad}}_{\bar{\alpha}}$ restricts to a surjection $\mathscr{B}^>_{q,\bar{\alpha}} \cup \{0\} \twoheadrightarrow \mathscr{B}^>_{p,\bar{\alpha}} \cup \{0\}$. We call these decompositions of $\mathscr{B}_p$ and $\mathscr{B}_q$ a *sorting for* $M_{\bar{\alpha}}$.

Let $M$ be a representation of $Q$. A *polarization for* $M$ is an ordered basis $\mathscr{B}$ of $M$ such that $\mathscr{B}_p$ and $\mathscr{B}_q$ are a polarization for every arrow $\bar{\alpha} : p \to q$ in $Q$. In this case, we also say that $M$ *is polarized by* $\mathscr{B}$. An *ordered polarization of* $M$ is a polarization $\mathscr{B}$ such that the canonical morphism $F : T \to Q$ from the coefficient quiver is ordered.

In other words, $M$ is polarized by $\mathscr{B}$ if and only if there are, for all arrows $\bar{\alpha} : p \to q$ in $Q$, sortings $\mathscr{B}_p = \mathscr{B}^<_{p,\bar{\alpha}} \amalg \mathscr{B}^>_{p,\bar{\alpha}}$ and $\mathscr{B}_q = \mathscr{B}^<_{q,\bar{\alpha}} \amalg \mathscr{B}^>_{q,\bar{\alpha}}$ such that $r_{\bar{\alpha}}(i) \leq 1$ for all $i \in \mathscr{B}^<_{p,\bar{\alpha}} \amalg \mathscr{B}^>_{q,\bar{\alpha}}$ and $r_{\bar{\alpha}}(i) \geq 1$ for all $i \in \mathscr{B}^<_{q,\bar{\alpha}} \amalg \mathscr{B}^>_{p,\bar{\alpha}}$. This means that the nonzero matrix coefficients of $M_{\bar{\alpha}}$ w.r.t. $\mathscr{B}_p$ and $\mathscr{B}_q$ can be covered by an upper left submatrix $M^<_{\bar{\alpha}}$ and a lower right submatrix $M^>_{\bar{\alpha}}$. Here $M^<_{\bar{\alpha}}$ has at most one nonzero entry in each column and at least one nonzero entry in each row; $M^>_{\bar{\alpha}}$ has at least one nonzero entry in each column and at most one nonzero entry in each row.

Figure 1 illustrates the typical shape of a fiber of an arrow $\bar{\alpha} : p \to q$ of $Q$ in the coefficient quiver $T = \Gamma(M, \mathscr{B})$ where $\mathscr{B}$ is an ordered polarization for $M$. We use the convention that we order the vertices from left to right in growing order. The property that $\mathscr{B}$ is a polarization is visible by the number of arrows connecting

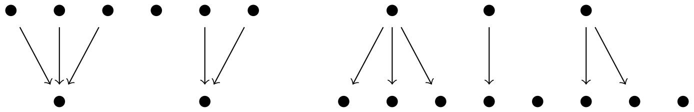


**Figure 1**

to a vertex in the upper left / lower left / upper right / lower right of the picture, and the property that $F : T \to Q$ is ordered is visible from the fact that the arrows do not cross each other.

**Lemma 3.2.** *Let $\mathscr{B}$ be an ordered polarization for $M$. Let $\bar{\alpha} : p \to q$ be an arrow in $Q$ and $\mathscr{B}_p = \mathscr{B}^{<}_{p,\bar{\alpha}} \amalg \mathscr{B}^{>}_{p,\bar{\alpha}}$ and $\mathscr{B}_q = \mathscr{B}^{<}_{q,\bar{\alpha}} \amalg \mathscr{B}^{>}_{q,\bar{\alpha}}$ a sorting for $M_{\bar{\alpha}}$. Then every $i$ in $\mathscr{B}^{<}_{q,\bar{\alpha}} \amalg \mathscr{B}^{>}_{p,\bar{\alpha}}$ connects to a unique extremal arrow.*

*Proof.* It is clear that every vertex $i$ connects to at most one extremal arrow in $F^{-1}(\bar{\alpha})$. Since $M$ is polarized by $\mathscr{B}$, we have that if $r_\alpha(i) \geq 1$, then $r_\alpha(j) = 1$ for all $j$ such that there is an arrow $\alpha : i \to j$ or $\alpha : j \to i$ in $F^{-1}(\bar{\alpha})$. In case $i = s(\alpha)$, this means that $\alpha : i \to j_0$ is extremal where $j_0$ is minimal among the targets of arrows in $F^{-1}(\bar{\alpha})$ with source $i$. In case $i = t(\alpha)$, this means that $\alpha : j_0 \to i$ is extremal where $j_0$ is maximal among the sources of arrows in $F^{-1}(\bar{\alpha})$ with target $i$. This establishes the lemma. ☐

**Remark 3.3.** Ringel [2013] develops the notion of a radiation basis in order to exhibit distinguished tree bases for exceptional modules. By Proposition 3 of that paper, a radiation basis $\mathscr{B}$ is a polarization of $M$ (w.r.t. any ordering of $\mathscr{B}$). Examples of representations with radiation basis are indecomposable representations of Dynkin quivers (with an exception for $E_8$) and the pullback of preinjective or preprojective modules of the Kronecker quiver $K(n)$ with $n$ arrows to its universal covering graph. Since the coefficient quiver of a pullback is the same as the coefficient quiver of the original representation, it follows that every preinjective or preprojective representation of the Kronecker quiver $K(n)$ is polarized by some ordered basis.

The Appendix gives a general strategy to establish polarizations of exceptional modules along Schofield induction. In [Lorscheid and Weist $\geq$ 2015], we will show that every exceptional representation $M$ of a quiver of affine Dynkin type $\widetilde{D}_n$ has a polarization which yields a Schubert decomposition of $\mathrm{Gr}_{\underline{e}}(M)$ into affine spaces.

**3D.** ***Maximal relevant pairs.*** Let $\bar{\alpha} \in Q_1$. A relevant pair $(i, j)$ is *maximal for* $\bar{\alpha}$ if there exists a relevant triple $(\bar{\alpha}, t, s)$ such that $\Psi(i, j) = \Psi(\bar{\alpha}, t, s)$.

**Lemma 3.4.** *Assume that $M$ is polarized by $\mathscr{B}$ and that $\beta \subset \mathscr{B}$ is extremal successor closed. Let $(\bar{\alpha}, t, s)$ be a relevant triple with $s \in \beta$ and $t \notin \beta$. Then one of the following holds.*

(i) *There is an extremal arrow $\alpha' : s' \to t$ in $F^{-1}(\bar{\alpha})$ such that $s' \notin \beta$ and*

$$\Psi_\beta(\bar{\alpha}, t, s) = \Psi(s', s) = \Psi(\bar{\alpha}, t, s).$$

*In this case, the $\beta$-reduced form of $E(\bar{\alpha}, t, s)$ is*

$$w_{s',s} = -\sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \text{ with} \\ t(\alpha)=t,\ s(\alpha)\notin\beta}} w_{s(\alpha),s} + \sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \text{ with} \\ s'<s(\alpha)<s, \\ s(\alpha)\notin\beta,\ t(\alpha)\in\beta}} w_{t,t(\alpha)} w_{s(\alpha),s} + \sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \\ \text{with } s(\alpha)=s, \text{ and} \\ t(\alpha)\in\beta \text{ or } t(\alpha)=t}} w_{t,t(\alpha)}.$$

(ii) *There is an extremal arrow* $\alpha' : s \to t'$ *in* $F^{-1}(\bar{\alpha})$ *such that* $t' \in \beta$ *and*

$$\Psi_\beta(\bar{\alpha}, t, s) = \Psi(t, t') = \Psi(\bar{\alpha}, t, s).$$

*In this case, the* $\beta$*-reduced form of* $E(\bar{\alpha}, t, s)$ *is*

$$w_{t,t'} = \sum_{\substack{\alpha \in F^{-1}(\bar{\alpha}) \\ \text{with } t(\alpha)=t, \text{ and} \\ s(\alpha)\notin\beta \text{ or } s(\alpha)=s}} w_{s(\alpha),s} - \sum_{\substack{\alpha\in F^{-1}(\bar{\alpha}) \text{ with} \\ t<t(\alpha)<t', \\ s(\alpha)\notin\beta,\ t(\alpha)\in\beta}} w_{t,t(\alpha)} w_{s(\alpha),s} - \sum_{\substack{\alpha\in F^{-1}(\bar{\alpha}) \text{ with} \\ s(\alpha)=s, t(\alpha)\in\beta}} w_{t,t(\alpha)}.$$

*Proof.* Once we know that there is an extremal arrow $\alpha' : s' \to t$ (or $\alpha' : s \to t'$), it is clear that $s' \notin \beta$ (or $t \in \beta$), that $w_{s',s}$ (or $w_{t,t'}$) is a free coefficient and that the $\beta$-reduced form of $E(\bar{\alpha}, t, s)$ looks as described in (i) (or (ii)).

If there are extremal arrows $\alpha' : s' \to t$ and $\alpha'' : s \to t''$, then $s'$ is minimal among the sources of arrows in $F^{-1}(\bar{\alpha})$ with target $t$, and $t''$ is maximal among the targets of arrows in $F^{-1}(\bar{\alpha})$ with source $s$. Clearly, we have $\Psi(\bar{\alpha}, t, s) = \max\{\Psi(s', s), \Psi(t, t'')\}$. By the definition of a relevant triple, we have $s' \leq s$ and $t \leq t''$. Since $\beta$ is extremal successor closed, $s' \notin \beta$ and $t'' \in \beta$. In particular, this means that $s' \neq s$ and $t \neq t''$, and thus $w_{s',s}$ and $w_{t,t''}$ are free coefficients. By the minimality of $s'$ and the maximality of $t''$, every other free coefficient $w_{i,j}$ in the $\beta$-reduced form of $E(\bar{\alpha}, t, s)$ must satisfy $\epsilon(i, j) < \max\{\epsilon(s', s), \epsilon(t, t'')\}$. Therefore also $\Psi_\beta(\bar{\alpha}, t, s) = \max\{\Psi(s', s), \Psi(t, t'')\}$, which establishes the proposition in the case that both $s$ and $t$ connect to extremal arrows in the fiber of $\bar{\alpha}$.

Let $p = s(\bar{\alpha})$ and $q = t(\bar{\alpha})$. Let $\mathscr{B}_p = \mathscr{B}^{<}_{p,\bar{\alpha}} \amalg \mathscr{B}^{>}_{p,\bar{\alpha}}$ and $\mathscr{B}_q = \mathscr{B}^{<}_{q,\bar{\alpha}} \amalg \mathscr{B}^{>}_{q,\bar{\alpha}}$ be sortings for $M_{\bar{\alpha}}$. If $s$ is not the source of any extremal arrow in the fiber of $\bar{\alpha}$, then Lemma 3.2 implies that $s \in \mathscr{B}^{<}_{p,\bar{\alpha}}$. By the definition of a relevant triple, there is an arrow $\alpha \in F^{-1}(\bar{\alpha})$ with $s(\alpha) \leq s$ and $t(\alpha) \leq t$. This implies that $t \in \mathscr{B}^{<}_{q,\bar{\alpha}}$ and, by Lemma 3.2, that there is an extremal arrow $\alpha' : s' \to t$. Since $\beta$ is extremely successor closed, $s' \notin \beta$ and $w_{s,s'}$ is a free coefficient.

We claim that in this situation $\Psi_\beta(\bar{\alpha}, t, s) = \Psi(s', s) = \Psi(\bar{\alpha}, t, s)$. Since $\alpha'$ is extremal, all $s'' \in F^{-1}(p)$ appearing in an index of the $\beta$-reduced form of $E(\bar{\alpha}, t, s)$ must lie between $s'$ and $s$. This means that $\epsilon(s', s)$ is larger than $\epsilon(s', s'')$ and $\epsilon(s'', s)$ if $s''$ is different from both $s$ and $s'$. Similarly, the largest relevant pair $(t'', t')$ with $F(t'') = F(t') = q$ satisfies $t'' = t$ and that $t'$ is maximal among the targets of arrows in $F^{-1}(\bar{\alpha})$ whose source is less or equal to $s$. Since $t, t' \in \mathscr{B}^{<}_{q,\bar{\alpha}}$, we have $\epsilon(t, t') \leq \epsilon(s', s)$. Equality can only hold if every $s''$ between $s'$ and $s$ is the source of precisely one arrow in $F^{-1}(\bar{\alpha})$. But then there would be such a unique arrow with source $s$, which is necessarily extremal. Since this contradicts the assumption that there is no extremal arrow with source $s$ in $F^{-1}(\bar{\alpha})$, we see that $\Psi(s', s) > \Psi(t, t')$. This shows that $\Psi_\beta(\bar{\alpha}, t, s) = \Psi(s', s) = \Psi(\bar{\alpha}, t, s) = \Psi(\bar{\alpha}, t, s)$, which means that (i) is satisfied.

If $t$ is not the target of any extremal arrow in the fiber of $\bar{\alpha}$, then we conclude analogously to the previous case that there is an extremal arrow $\alpha' : s \to t'$ with $t' \in \beta$ such that $\Psi_\beta(\bar{\alpha}, t, s) = \Psi(t, t')$. Thus in this case, (ii) is satisfied. $\square$

**Lemma 3.5.** *Let* $\bar{\alpha} \in Q_1$ *and* $(i, j) \in \mathrm{Rel}^2$. *Assume that* $M$ *is polarized by* $\mathscr{B}$.

(i) *If* $F(i) = s(\bar{\alpha})$, *then there is at most one* $\alpha : i \to t$ *in* $F^{-1}(\bar{\alpha})$ *such that* $\Psi(i, j) = \Psi(\bar{\alpha}, t, j)$.

(ii) *If* $F(j) = t(\bar{\alpha})$, *then there is at most one* $\alpha : s \to j$ *in* $F^{-1}(\bar{\alpha})$ *such that* $\Psi(i, j) = \Psi(\bar{\alpha}, i, s)$.

*Proof.* We prove (i). If there is only one arrow $\alpha$ in $F^{-1}(\bar{\alpha})$ with source $i$, then (i) is clear. Assume that there are two different arrows $\alpha : i \to t$ and $\alpha' : i \to t'$ in $F^{-1}(\bar{\alpha})$ with $t' < t$. Since $M$ is polarized, we have $r_{\bar{\alpha}}(k) \leq 1$ for all $k \geq i$ and $r_{\bar{\alpha}}(l) \geq 1$ for all $l \geq t'$. This means that there is an arrow $\alpha'' : j \to t''$ and that $\epsilon(t', t'') > \epsilon(t, t'') \geq \epsilon(i, j)$. An equality $\epsilon(t, t'') = \epsilon(i, j)$ is only possible if $t$ is maximal among the targets of arrows in $F^{-1}(\bar{\alpha})$ with source $i$.

This shows (i). The proof of (ii) is analogous. $\square$

**Corollary 3.6.** *Assume that* $M$ *is polarized by* $\mathscr{B}$ *and that* $\beta \subset \mathscr{B}$ *is extremal successor closed. Let* $\bar{\alpha} \in Q_1$. *If* $(i, j)$ *is maximal for* $\bar{\alpha}$, *then there is a unique* $(\bar{\alpha}, t, s) \in \mathrm{Rel}^3$ *such that* $\Psi_\beta(\bar{\alpha}, t, s) = \Psi(i, j) = \Psi(\bar{\alpha}, t, s)$. *If* $(i, j)$ *is not maximal for* $\bar{\alpha}$, *then there is no relevant triple* $(\bar{\alpha}, t, s)$ *with* $\Psi(i, j) = \Psi(\bar{\alpha}, t, s)$.

*Proof.* This is an immediate consequence of Lemmas 3.4 and 3.5. $\square$

## 4. Schubert decompositions for tree modules

**Theorem 4.1.** *Let* $M$ *be a representation of* $Q$ *and* $\mathscr{B}$ *an ordered polarization for* $M$. *Let* $\underline{e}$ *be a dimension vector for* $Q$. *Assume that every* $(i, j) \in \mathrm{Rel}^2$ *is maximal for at most one* $\bar{\alpha} \in Q_1$. *Then*

$$\mathrm{Gr}_{\underline{e}}(M) = \coprod_{\substack{\beta \subset \mathscr{B} \\ \text{of type } \underline{e}}} C_\beta^M$$

*is a decomposition into affine spaces. Moreover,* $C_\beta^M$ *is not empty if and only if* $\beta$ *is extremal successor closed.*

*Proof.* By Lemma 3.1, $C_\beta^M$ is empty if $\beta$ is not extremal successor closed. Let $\beta$ be extremal successor closed. The theorem is proven once we have shown that $C_\beta^M$ is an affine space. As before, we identify $T_0$ order-preservingly with $\{1, \ldots, n\}$. For $\psi \in \mathbb{N} \times \mathbb{N} \times T_0$, we denote by $C_\beta^M(\psi)$ the solution space of all coefficients $w_{i,j}$ with $\Psi(i, j) \leq \psi$ in all equations $E(\bar{\alpha}, t, s)$ where $(\bar{\alpha}, t, s)$ is a relevant triple with $\Psi_\beta(\bar{\alpha}, t, s) < \psi$. We show by induction over $\psi \in \Psi(\mathrm{Rel}^2)$ that $C_\beta^M(\psi)$ is an affine space. Since $\Psi(\mathrm{Rel}^2)$ is finite, this implies that $C_\beta^M$ is an affine space as required.

As base case, consider $\psi = \Psi(n, n)$. By Lemma 2.2, only those relevant triples $(\bar{\alpha}, t, s)$ with $t \notin \beta$ and $s \in \beta$ lead to nontrivial equations $E(\bar{\alpha}, t, s)$. For such a relevant triple, $\Psi_\beta(\bar{\alpha}, t, s) \leq \psi$ if and only if $E(\bar{\alpha}, t, s)$ does not contain any free coefficient and thus is of the form $w_{s,s} = w_{t,t} w_{s,s}$. This is the case if and only if there is an extremal arrow $\alpha : s \to t$ in $F^{-1}(\bar{\alpha})$. Since $\beta$ is extremal successor closed, $w_{s,s} = w_{t,t} w_{s,s}$ is satisfied. This means that $C_\beta^M(\psi) = \mathbb{A}^0$ is a point.

Consider $\psi > \Psi(n, n)$ and let $\psi'$ be its predecessor in $\Psi(\mathrm{Rel}^2)$. We assume that $C_\beta^M(\psi')$ is an affine space. By the assumption of the theorem, $(i, j)$ is maximal for at most one $\bar{\alpha} \in Q_1$. If there is no such $\bar{\alpha}$, then there is no relevant triple $(\bar{\alpha}, t, s)$ with $\Psi_\beta(\bar{\alpha}, t, s) = \Psi(i, j)$, which means that $w_{i,j}$ does not appear as a maximal coefficient of an equation $E(\bar{\alpha}, t, s)$. If $i \in \beta$ or $j \notin \beta$, then $w_{i,j} = 0$ and $C_\beta^M(\psi) = C_\beta^M(\psi')$. Otherwise $w_{i,j}$ is free and $C_\beta^M(\psi) = C_\beta^M(\psi') \times \mathbb{A}^1$.

If there is an arrow $\bar{\alpha} \in Q_1$ such that $(i, j)$ is maximal for $\bar{\alpha}$, then there exists a unique relevant triple $(\bar{\alpha}, t, s)$ such that $\Psi_\beta(\bar{\alpha}, t, s) = \Psi(i, j)$ by Corollary 3.6. If $w_{i,j}$ is not free, then $i \in \beta$ or $j \notin \beta$. By Lemma 3.4, either $t = i$ and there is an extremal arrow $\alpha : s \to j$ in $F^{-1}(\bar{\alpha})$ or $s = j$ and there is an extremal arrow $\alpha : i \to t$ in $F^{-1}(\bar{\alpha})$. In either case, if $i \in \beta$ or $j \notin \beta$, then $t \in \beta$ or $s \notin \beta$ since $\beta$ is extremal successor closed. This means that $E(\bar{\alpha}, t, s)$ is trivial and thus $C_\beta^M(\psi) = C_\beta^M(\psi')$. If $w_{i,j}$ is free, but $E(\bar{\alpha}, t, s)$ is trivial, then $C_\beta^M(\psi) = C_\beta^M(\psi') \times \mathbb{A}^1$. If finally $w_{i,j}$ is free and $E(\bar{\alpha}, t, s)$ is nontrivial, then $w_{i,j}$ is determined by all coefficients $w_{i',j'}$ with $\Psi(i', j') < \Psi(i, j)$ by one of the formulas in Lemma 3.4. This means that $C_\beta^M(\psi) = C_\beta^M(\psi')$.

Thus we have shown that in all possible cases, $C_\beta^M(\psi)$ equals either $C_\beta^M(\psi')$ or $C_\beta^M(\psi') \times \mathbb{A}^1$, which are both affine spaces by the inductive hypothesis. This finishes the proof of the theorem. □

**Remark 4.2.** Though the assumptions of Theorem 4.1 come in a different shape than Hypothesis (H) in Section 4.5 of [Lorscheid 2014], they are indeed equivalent to Hypothesis (H) if $F : Q \to T$ is unramified.

**Remark 4.3.** Though we do not explicitly require that $\mathscr{B}$ is a tree basis, it follows from the other assumptions of the theorem that $M$ is a tree module. Indeed, if the coefficient quiver $T$ had a loop and $i$ was the largest vertex of this loop at maximal distance to 1, then the relevant pair $(i, i)$ would be maximal for the two connecting arrows of the loop. Note that if $M$ is not indecomposable, then $T = \Gamma(M, \mathscr{B})$ is not necessarily connected (cf. Example 4.7).

By [Ringel 1998], every exceptional module is a tree module. But it is clear that not every exceptional module admits an ordered tree basis such that the canonical morphism $F : T \to Q$ from the coefficient quiver is ordered. For instance, there are exceptional representations of the Kronecker quiver $K(3)$ with three arrows that attest to this fact: see the example $P(x, 3)$ in [Ringel 2013, page 15].

However, if $M$ has a radiation basis $\mathscr{B}$, then we can order $\mathscr{B}$ inductively along the construction of $M$ by smaller radiation modules in such a way that $\mathscr{B}$ satisfies the assumptions of the theorem. In particular, this includes all exceptional representations of Dynkin type, with an exception for $E_8$. We see that the class of modules that admit an ordered basis to that we can apply the theorem lies somewhere between radiation modules and tree modules.

**Corollary 4.4.** *Under the assumptions of Theorem 4.1, the Euler characteristic of* $\mathrm{Gr}_{\underline{e}}(M)$ *equals the number of extremal successor closed subsets* $\beta \subset \mathscr{B}$ *of type* $\underline{e}$.

*Proof.* Since the Euler characteristic is additive under decompositions into locally closed subsets,

$$\chi(\mathrm{Gr}_{\underline{e}}(M)) = \sum_{\substack{\beta\subset\mathscr{B} \\ \text{of type } \underline{e}}} \chi(C_\beta^M).$$

The Euler characteristic of an affine space is 1 and the Euler characteristic of the empty set is 0. Therefore the corollary follows immediately from Theorem 4.1. □

**Corollary 4.5.** *If* $\mathrm{Gr}_{\underline{e}}(M)$ *is smooth and the assumptions of Theorem 4.1 are satisfied, then the closures of the nonempty Schubert cells* $C_\beta^M$ *of* $\mathrm{Gr}_{\underline{e}}(M)$ *represent an additive basis for the cohomology ring* $H^*(\mathrm{Gr}_{\underline{e}}(M))$. *If* $n = \dim \mathrm{Gr}_{\underline{e}}(M)$ *and* $d = \dim C_\beta^M$, *then the class of the closure of* $C_\beta^M$ *is in* $H^{n-2d}(\mathrm{Gr}_{\underline{e}}(M))$.

*Proof.* This follows immediately from [Lorscheid 2014, Cor. 6.2]. □

### 4A. *Two examples for type $D_4$.*

**Example 4.6** (quiver Grassmannian of a ramified tree module). We give an instance of a ramified tree module to which the methods of this text apply. Let $Q$ be the quiver

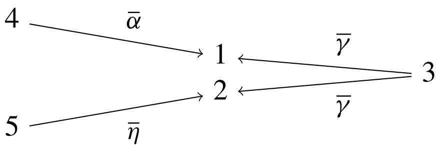
$$\begin{tikzcd}
x \arrow[dr, "\bar{\alpha}"] & & \\
& t & y \arrow[l, "\bar{\gamma}"'] \\
z \arrow[ur, "\bar{\eta}"'] & &
\end{tikzcd}$$

of type $D_4$ and let $M$ be the exceptional module

$$\begin{tikzcd}
\mathbb{C}^1 \arrow[dr, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & & \\
& \mathbb{C}^2 & \mathbb{C}^1 \arrow[l, "\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)"'] \\
\mathbb{C}^1 \arrow[ur, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] & &
\end{tikzcd}$$

of $Q$. We can order the obvious basis $\mathscr{B}$ in such a way that the coefficient quiver $T$

looks like



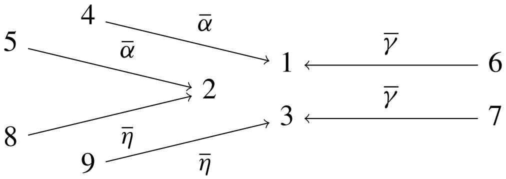
where we label the arrows by their images under $F$. For the dimension vector $\underline{e}$ with $e_x = e_z = 0$ and $e_y = e_t = 1$, we obtain precisely one subrepresentation $N$ of $M$ with $\underline{\dim}\, N = \underline{e}$. This means that $\mathrm{Gr}_{\underline{e}}(M)$ is a point. Therefore, the Euler characteristic of $\mathrm{Gr}_{\underline{e}}(M)$ equals 1.

There is precisely one extremal successor closed subset of type $\underline{e}$, namely $\beta = \{2, 3\}$, which accounts for the Euler characteristic. It is indeed easily verified that the assumptions of Theorem 4.1 are satisfied. Note that $\beta$ is not successor closed, which shows that the number of successor closed subsets does not coincide with the Euler characteristic in this example.

**Example 4.7** (a del Pezzo surface of degree 6). The previous representation appears as a subrepresentation of the following unramified representation. This example arose from discussions with Markus Reineke. Let $Q$ be the same quiver as in the previous example and $M$ the representation

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[dr, "\left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right)"] & & \\
& \mathbb{C}^3 & & \mathbb{C}^2 \arrow[ll, "\left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)"'] \\
\mathbb{C}^2 \arrow[ur, "\left(\begin{smallmatrix}0&0\\1&0\\0&1\end{smallmatrix}\right)"'] & & &
\end{tikzcd}$$

of $Q$. We can order the obvious basis $\mathscr{B}$ in such a way that the coefficient quiver $T$ is



where we label the arrows by their images under $F$. It is clear from this picture that $\mathscr{B}$ is an ordered polarization, and it is easily verified that every relevant pair is maximal for at most one arrow. Thus Theorem 4.1 implies that the nonempty Schubert cells are affine spaces and that they are indexed by the extremal successor closed subsets $\beta$ of $T_0$. For type $\underline{e} = (2, 1, 1, 1)$, we obtain the nonempty Schubert cells

$$C^M_{\{1,2,4,6,8\}} \simeq \mathbb{A}^0, \quad C^M_{\{1,2,5,6,8\}} \simeq \mathbb{A}^1, \quad C^M_{\{1,3,4,6,9\}} \simeq \mathbb{A}^1,$$
$$C^M_{\{1,3,4,7,9\}} \simeq \mathbb{A}^1, \quad C^M_{\{2,3,5,7,8\}} \simeq \mathbb{A}^1, \quad C^M_{\{2,3,5,7,9\}} \simeq \mathbb{A}^2.$$

Therefore the Euler characteristic of $X = \mathrm{Gr}_{\underline{e}}(M)$ is 6 and since $X$ is smooth (as we will see in a moment), Corollary 4.5 tells us that $H^0(X) = \mathbb{Z}$, $H^1(X) = \mathbb{Z}^4$ and $H^2(X) = \mathbb{Z}$ are additively generated by the closures of the Schubert cells.

To show that $X$ is smooth, we consider $X$ as a closed subvariety of $\mathrm{Gr}(2, 3) \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Note that for a subrepresentation $N$ of $M$ with dimension vector $\underline{e}$, the 1-dimensional subspaces $N_x$, $N_y$ and $N_z$ of $M_x$, $M_y$ and $M_z$, respectively, determine the 2-dimensional subspace $N_t$ of $M_t$ uniquely. The images of $N_x = \langle\binom{x_0}{x_1}\rangle$, $N_y = \langle\binom{y_0}{y_1}\rangle$ and $N_z = \langle\binom{z_0}{z_1}\rangle$ in $M_t$ lie in a plane if and only if

$$\det\begin{bmatrix} x_0 & y_0 & 0 \\ x_1 & 0 & z_0 \\ 0 & y_1 & z_1 \end{bmatrix} = -x_0\, y_1\, z_0 - x_1\, y_0\, z_1 = 0.$$

Therefore the projection $\mathrm{Gr}(2, 3) \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ yields an isomorphism

$$\mathrm{Gr}_{\underline{e}}(M) \xrightarrow{\sim} \big\{[\, x_0 : x_1 \,|\, y_0 : y_1 \,|\, z_0 : z_1] \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \,|\, x_0\, y_1\, z_0 \,+\, x_1 y_0\, z_1 = 0\big\}.$$

Since there is no point in $\mathrm{Gr}_{\underline{e}}(M)$ that vanishes for all derivatives of the defining equation, $\mathrm{Gr}_{\underline{e}}(M)$ is smooth.

The projection $\pi_{1,3} : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ to the first and third coordinate restricts to a surjective morphism $\pi_{1,3} : \mathrm{Gr}_{\underline{e}}(M) \to \mathbb{P}^1 \times \mathbb{P}^1$. It is bijective outside the fibers of $[1 : 0 \,|\, 0 : 1]$ and $[0 : 1 \,|\, 1 : 0]$, and these two fibers are

$$\pi_{1,3}^{-1}([1 : 0 \,|\, 0 : 1]) = \{[1 : 0 \,|\, y_0 : y_1 \,|\, 0 : 1]\} \simeq \mathbb{P}^1$$

and

$$\pi_{1,3}^{-1}([\, 0 : 1 \,|\, 1 : 0]) = \{[0 : 1 \,|\, y_0 : y_1 \,|\, 1 : 0]\} \simeq \mathbb{P}^1.$$

This shows that $\mathrm{Gr}_{\underline{e}}(M)$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in two points, which is a del Pezzo surface of degree 6. Note that the closure of the Schubert cells $C^M_{\{1,2,5,6,8\}}$, $C^M_{\{1,3,4,6,9\}}$, $C^M_{\{1,3,4,7,9\}}$ and $C^M_{\{2,3,5,7,8\}}$ are four of the six curves on $\mathrm{Gr}_{\underline{e}}(M)$ with self-intersection $-1$. In particular, the closures of the latter two cells are the two connected components of the exceptional divisor w.r.t. the blow-up $\pi_{1,3} : \mathrm{Gr}_{\underline{e}}(M) \to \mathbb{P}^1 \times \mathbb{P}^1$.

To return to the opening remark of this example, we see that every point of $\mathrm{Gr}_{\underline{e}}(M)$, apart from the intersection points of pairs of $(-1)$-curves, is a subrepresentation of $M$ that is isomorphic to the representation of Example 4.6. There are six intersection points of pairs of $(-1)$-curves on $\mathrm{Gr}_{\underline{e}}(M)$, whose coordinates in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are

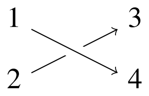
$$[1 : 0 \,|\, 1 : 0 \,|\, 1 : 0], \quad [0 : 1 \,|\, 1 : 0 \,|\, 1 : 0], \quad [0 : 1 \,|\, 0 : 1 \,|\, 1 : 0],$$
$$[0 : 1 \,|\, 0 : 1 \,|\, 0 : 1], \quad [1 : 0 \,|\, 0 : 1 \,|\, 0 : 1], \quad [1 : 0 \,|\, 1 : 0 \,|\, 0 : 1].$$

Note that each Schubert cell contains precisely one of these points, and that these points coincide with the subrepresentations $N$ of $M$ that are spanned by the successor closed subsets $\beta$ of $\mathscr{B}$.

This exemplifies the idea that the Euler characteristic of a projective variety should equal the number of $\mathbb{F}_1$-points. The naive definition of the $\mathbb{F}_1$-points as the points with coordinates in $\mathbb{F}_1 = \{0, 1\}$ yields the right outcome in this case. The more elaborate definition of the $\mathbb{F}_1$-points as the Weyl extension $\mathscr{W}(X_{\mathbb{F}_1})$ of the blue scheme $X_{\mathbb{F}_1}$ associated with $X = \mathrm{Gr}_{\underline{e}}(M)$ and $\mathscr{B}$ yields a intrinsic bijection between the elements of $\mathscr{W}(X_{\mathbb{F}_1})$ and the above points. This definition of $\mathbb{F}_1$-points generalizes the connection between Euler characteristics and $\mathbb{F}_1$-points to a larger class of quiver Grassmannians than the naive definition. See [Lorscheid 2013, Section 4] for more details.

**4B.** ***Limiting examples.*** As already mentioned in Remark 2.4, there are different possible choices to order $\mathrm{Rel}^2$, which might lead to different generalities of analogues of Theorem 4.1. The following examples show, however, that we cannot simply drop an assumption in Theorem 4.1.

**Example 4.8** (nonordered $F$). Consider the representation $M = \left[\begin{pmatrix}0&1\\1&0\end{pmatrix} : \mathbb{C}^2 \to \mathbb{C}^2\right]$ of the quiver $Q = [\bullet \to \bullet]$. With the obvious choice of the ordered basis $\mathscr{B} = \{1, 2, 3, 4\}$ of $M$, the coefficient quiver $T = \Gamma(M, \mathscr{B})$ looks as follows:



The Schubert cells in the decomposition

$$\mathrm{Gr}_{(1,1)}(M) = C^M_{\{1,3\}} \amalg C^M_{\{1,4\}} \amalg C^M_{\{2,3\}} \amalg C^M_{\{2,4\}}$$

are easily determined to be

$$C^M_{\{1,3\}} = \varnothing, \quad C^M_{\{1,4\}} \simeq \mathbb{A}^0, \quad C^M_{\{2,3\}} \simeq \mathbb{A}^0 \quad \text{and} \quad C^M_{\{2,4\}} \simeq \mathbb{G}_m.$$

In this example, we come across a Schubert cell that is isomorphic to $\mathbb{G}_m = \mathbb{A}^1 - \mathbb{A}^0$. Theorem 4.1 does indeed not apply since $F : T \to Q$ is not ordered. However, the other conditions of Theorem 4.1 are satisfied: $\mathscr{B}$ is a polarization and every relevant pair is maximal for at most one arrow (since $Q$ has only one arrow).

Note that the indices of the nonempty Schubert cells are precisely the extremely successor closed subsets $\beta \subset \mathscr{B}$ of type $\underline{e}$. However, only $\{1, 4\}$ and $\{2, 3\}$ contribute to the Euler characteristic of $\mathrm{Gr}_{\underline{e}}(M) \simeq \mathbb{P}^1$, which is 2. These two subsets are precisely the successor closed subsets of $\mathscr{B}$, in coherence with the methods of [Cerulli Irelli 2011] and [Haupt 2012], which apply to this example.

**Example 4.9** (nonpolarized basis). Consider the representation $M = \left[\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) : \mathbb{C}^2 \to \mathbb{C}^2\right]$ of the quiver $Q = [\bullet \to \bullet]$. With the obvious choice of ordered basis $\mathscr{B} = \{1, 2, 3, 4\}$ of $M$, the coefficient quiver $T = \Gamma(M, \mathscr{B})$ looks as follows:

$$\begin{tikzcd}
1 \arrow[r] \arrow[dr] & 3 \\
2 \arrow[r] & 4
\end{tikzcd}$$

The Schubert cells in the decomposition

$$\mathrm{Gr}_{(1,1)}(M) = C^M_{\{1,3\}} \amalg C^M_{\{1,4\}} \amalg C^M_{\{2,3\}} \amalg C^M_{\{2,4\}}$$

are easily determined to be

$$C^M_{\{1,3\}} = \varnothing, \quad C^M_{\{1,4\}} \simeq \mathbb{A}^0, \quad C^M_{\{2,3\}} \simeq \mathbb{A}^0 \quad \text{and} \quad C^M_{\{2,4\}} \simeq \mathbb{G}_m.$$

The Schubert cell $C^M_{\{2,4\}} \simeq \mathbb{G}_m$ does not contradict Theorem 4.1 since $\mathscr{B}$ is not a polarization, though the canonical morphism $F : T \to Q$ is ordered and every relevant pair is maximal for at most one arrow (as $Q$ has only one arrow).

***Acknowledgements.*** I would like to thank Dave Anderson, Giovanni Cerulli Irelli, Markus Reineke, Cecília Salgado and Jan Schroer for helpful discussions. I would like to thank Thorsten Weist for including his ideas on polarizations as an appendix to this text.

## Appendix: Tree modules with polarizations

### by Thorsten Weist

Let $Q$ be a quiver without loops and oriented cycles. The aim of this appendix is to investigate under which conditions we can construct indecomposable tree modules $X$ such that the basis $\mathfrak{B}$ of the respective coefficient quiver $T_X := \Gamma(X, \mathfrak{B})$ is a polarization for $X$. In many cases, the question whether there exists a polarization for $X$ is closely related to the question whether there exists a coefficient quiver without a subdiagram of the form

$$s_1 \xrightarrow{a} t_1 \xleftarrow{a} s_2 \xrightarrow{a} t_2.$$

We call a coefficient quiver without such a subdiagram a weak polarization for $X$. Clearly, a polarization does not have such a subdiagram. But we will see that in many cases these two conditions are already equivalent, for instance for exceptional representations. In the following, we will not always distinguish between an arrow $a$ of the coefficient quiver and its color $F(a)$. Moreover, we will often label the arrows of the coefficient quiver by its color.

One of the main tools which can be used to construct tree modules is Schofield induction, see [Schofield 1991] and [Weist 2012] for an application to tree modules. A direct consequence is that, fixing an exceptional sequence $(Y, X)$ with $\mathrm{Hom}(X, Y) = 0$ and a basis $(e_1, \dots, e_m)$ of $\mathrm{Ext}(X, Y)$, representations appearing as the middle terms of exact sequences

$$0 \to Y^e \to Z \to X^d \to 0$$

give rise to a full subcategory $\mathcal{F}(X, Y)$ of $\mathrm{Rep}(Q)$, the category of representations of $Q$. Moreover, we obtain that $\mathcal{F}(X, Y)$ is equivalent to the category of representations of the generalized Kronecker quiver $K(m)$ with $K(m)_0 = \{q_0, q_1\}$ and $K(m)_1 = \{\rho_i : q_0 \to q_1 \mid i \in \{1, \dots, m\}\}$ where $m = \dim \mathrm{Ext}(X, Y)$. Fixing a real root $\alpha$ of $Q$, we denote by $X_\alpha$ the indecomposable representation of dimension $\alpha$, which is unique up to isomorphism. By Schofield induction, we also know that, if $\alpha$ is an exceptional root of $Q$, there already exist exceptional roots $\beta$ and $\gamma$ such that $X_\beta \in X_\gamma^\perp$, $\mathrm{Hom}(X_\beta, X_\gamma) = 0$ and $\alpha = \beta^d + \gamma^e$, where $(d, e)$ is a real root of the generalized Kronecker quiver $K(\dim \mathrm{Ext}(X_\beta, X_\gamma))$.

Let $X$ and $Y$ be two representations of a quiver $Q$. Then we can consider the linear map

$$\gamma_{X,Y} : \bigoplus_{q \in Q_0} \mathrm{Hom}_k(X_q, Y_q) \to \bigoplus_{a:s \to t \in Q_1} \mathrm{Hom}_k(X_s, Y_t)$$

defined by $\gamma_{X,Y}((f_q)_{q \in Q_0}) = (Y_a f_s - f_t X_a)_{a:s \to t \in Q_1}$.

It is well-known that we have $\ker(\gamma_{X,Y}) = \mathrm{Hom}(X, Y)$ and $\mathrm{coker}(\gamma_{X,Y}) = \mathrm{Ext}(X, Y)$. The first statement is straightforward. The second statement follows because every morphism $f \in \bigoplus_{a:s \to t \in Q_1} \mathrm{Hom}_k(X_s, Y_t)$ defines an exact sequence

$$0 \to Y \to \left( (Y_q \oplus X_q)_{q \in Q_0}, \left( \begin{pmatrix} Y_a & f_a \\ 0 & X_a \end{pmatrix} \right)_{a \in Q_1} \right) \to X \to 0$$

with the canonical inclusion on the left hand side and the canonical projection on the right hand side.

Assume that the representations $X$ and $Y$ are tree modules and let $T_X = \Gamma(X, \mathcal{B}_X)$ and $T_Y = \Gamma(Y, \mathcal{B}_Y)$ be the corresponding coefficient quivers. Let $x = \underline{\dim} X$, $y = \underline{\dim} Y$. Fixing a vertex $q$, from now on we will denote the corresponding vertices of the coefficient quivers by $(\mathcal{B}_X)_q = \{b_1^q, \dots, b_{x_q}^q\}$ and $(\mathcal{B}_Y)_q = \{c_1^q, \dots, c_{y_q}^q\}$. Let $e_{k,l}^a$, where $a : s \to t \in Q_1$, $k = 1, \dots, x_s$ and $l = 1, \dots, y_t$, be the canonical basis of $\bigoplus_{a:s \to t} \mathrm{Hom}_k(X_s, Y_t)$ with respect to $\mathcal{B}_X$ and $\mathcal{B}_Y$, i.e., $e_{k,l}^a(b_i^s) = \delta_{i,k}\delta_{j,l}c_j^t$.

This means that the coefficient quiver $\Gamma(Z, \mathcal{B}_X \cup \mathcal{B}_Y)$ of the middle term of the exact sequence

$$E(e_{k,l}^a) : 0 \to Y \to Z \to X \to 0$$

is obtained by adding an extra arrow with color $a$ from $b_k^s$ to $c_l^t$ to $T_X \cup T_Y$.

Following [Weist 2012] we call a basis of $\mathscr{E}(X, Y)$ of $\operatorname{Ext}(X, Y)$, which solely consists of elements of the form $\overline{e^{a_k}_{i_k, j_k}}$ with $k = 1, \dots, \dim \operatorname{Ext}(X, Y)$, $a_k \in Q_1$, $1 \le i_k \le x_s$ and $1 \le j_k \le y_t$, tree-shaped. In abuse of notation, we will not always distinguish between $\overline{e^{a_k}_{i_k, j_k}}$ and $e^{a_k}_{i_k, j_k}$.

Let $X$ be a tree module. For a vertex $b^s_i$ and an arrow $a : s \to t \in Q_1$ we define

$$N(a, b^s_i) := \{b^t_j \in (T_X)_0 \mid b^s_i \xrightarrow{a} b^t_j \in (T_X)_1\}.$$

Analogously, we define $N(a, b^t_i)$. If $T_X$ is a weak polarization for $X$, we say that it is strict if we have, for all arrows $s \xrightarrow{a} t \in Q_1$, that $|N(a, b^s_i)| \le 1$ for all $1 \le i \le x_s$ or $|N(a, b^t_i)| \le 1$ for all $1 \le i \le x_t$. Clearly, a weak polarization which is strict is a polarization as defined in Section 3C. Note that we can always assume that $\mathscr{B}$ is ordered.

For a vertex $q$ of $T_X$ let $S(q) = \{F(a) \in Q_1 \mid a \in (T_X)_1,\ s(a) = q\}$ and $T(q) = \{F(a) \in Q_1 \mid a \in (T_X)_1,\ t(a) = q\}$.

**Lemma A.1.** *Let $X$ be a tree module with coefficient quiver $T_X$ such that for every $a \in Q_1$, the map $X_a$ is of maximal rank. Then $T_X$ is a polarization if and only if $T_X$ is a weak polarization.*

*Proof.* Since $X_a$ is of maximal rank, $X_a$ is either surjective or injective. Thus if, in addition, $T_X$ is a weak polarization, this means that $|N(a, b^s_i)| \le 1$ for all $1 \le i \le x_s$ or $|N(a, b^t_i)| \le 1$ for all $1 \le i \le x_t$. It follows that $T_X$ is a polarization. ☐

**Remark A.2.** For general representations of a fixed dimension, and thus in particular for exceptional representations, it is true that all linear maps appearing are of maximal rank.

Using the notation from above we introduce the following definition:

**Definition A.3.** (i) Let $X$ and $Y$ be two tree modules with coefficient quivers $T_X$ and $T_Y$. Moreover, let $\mathscr{E}(X, Y) = (e^{a_k}_{i_k, j_k})_k$ with $s_k \xrightarrow{a_k} t_k \in Q_1$ be a tree-shaped basis of $\operatorname{Ext}(X, Y)$, i.e., $e^{a_k}_{i_k, j_k}(b^{s_k}_{i_k}) = c^{t_k}_{j_k}$. Then we call $\mathscr{E}(X, Y)$ a polarization if

- (a) we have that $a_k \notin S(b^{s_k}_{i_k})$ or $a_k \notin T(c^{t_k}_{j_k})$ for all $k$,
- (b) if $a_k = a_l$ and $b^{s_k}_{i_k} = b^{s_l}_{i_l}$ (resp. $c^{t_k}_{j_k} = c^{t_l}_{j_l}$) for $k \neq l$, we have $a_k \notin T(c^{t_k}_{j_k})$ (resp. $a_k \notin S(b^{s_k}_{i_k})$),
- (c) for all $b^{s_k}_{i_k} \xrightarrow{a_k} b^t_j \in (T_X)_1$ we have $|N(a_k, b^t_j)| = 1$ and for all $c^s_i \xrightarrow{a_k} c^{t_k}_{j_k} \in (T_Y)_1$ we have $|N(a_k, c^s_i)| = 1$.

(ii) If we have $a_k \notin S(b^{s_k}_{i_k})$ <u>and</u> $a_k \notin T(c^{t_k}_{j_k})$ for all $k$ in the first condition and if we also have $a_k \neq a_l$ if $k \neq l$, we say that the basis is a strong polarization.

**Remark A.4.** Roughly speaking condition (c) ensures that $b^{s_k}_{i_k}$ is the only neighbor which is connected to $b^t_j$ by an arrow with color $a_k$.

Condition (a) means that either $b_{i_k}^{s_k}$ is not the source of an arrow with color $a_k$ (when only the coefficient quiver $T_X$ is considered) or $c_{j_k}^{t_k}$ is not the target of an arrow with color $a_k$ (when only the coefficient quiver $T_Y$ is considered). In particular, if we have $a_k \notin S(b_{i_k}^{s_k})$ <u>and</u> $a_k \notin T(c_{j_k}^{t_k})$ for all $k$ in the first condition, the second and third conditions are clearly satisfied.

Now we are in a position to state under which conditions an exceptional sequence together with a tree-shaped basis of the Ext-group gives rise to indecomposable representations such that, in addition, there exists a coefficient quiver which is a (weak) polarization:

**Theorem A.5.** *Let* $(Y, X)$ *be an exceptional sequence* (*of tree modules*) *such that the coefficient quivers* $T_X$ *and* $T_Y$ *are weak polarizations. Moreover, let* $\mathscr{E}(X, Y) = (e_{i_1,j_1}^{a_1}, \dots, e_{i_m,j_m}^{a_m})$ *be a basis of* $\operatorname{Ext}(X, Y)$ *which is a polarization and let* $M$ *be an indecomposable tree module of* $K(m)$.

(i) *If* $T_M$ *is unramified, then the induced coefficient quiver* $T_Z$ *of the middle term* $Z$ *of the corresponding exact sequence*

$$e_M : 0 \to Y^e \to Z \to X^d \to 0$$

*is a weak polarization for* $Z$. *Moreover,* $Z$ *is indecomposable.*

(ii) *If the polarization of the basis is strong and* $T_M$ *is a weak polarization, then the induced coefficient quiver* $T_Z$ *of the middle term* $Z$ *of the corresponding exact sequence*

$$e_M : 0 \to Y^e \to Z \to X^d \to 0$$

*is a weak polarization for* $Z$. *Moreover,* $Z$ *is indecomposable.*

(iii) *If* $X_a$ *is injective* (*resp. surjective*) *if and only if* $Y_a$ *is injective* (*resp. surjective*) *for all arrows* $a \in Q_1$, *then* $T_Z$ *is a weak polarization if and only if* $T_Z$ *is a polarization.*

(iv) *If* $M$, *and thus also* $Z$, *is exceptional, the polarization is strict and thus* $T_Z$ *is a polarization for* $Z$.

*Proof.* By simply counting arrows and vertices of the induced coefficient quiver $T_Z$ it follows that $Z$ is a tree module, see also [Weist 2012, Proposition 3.9]. Moreover, since $M$ is indecomposable, by Schofield induction we know that $Z$ is indecomposable.

Thus we only need to check that $T_Z$ is a weak polarization for $Z$. We first consider the case when $T_M$ is unramified and $\mathscr{E}(X, Y)$ not necessarily a strong polarization. Clearly, in this case $T_M$ is a weak polarization for $M$. Moreover, note that, since $\mathscr{E}(X, Y)$ is a basis, if $a_k = a_l$ for $k \neq l$, we either have $j_k \neq j_l$ or $i_k \neq i_l$.

The coefficient quiver could contradict the polarization property if $b_{i_k}^{s_k} = b_{i_l}^{s_l}$ (resp. $c_{j_k}^{t_k} = c_{j_l}^{t_l}$), $a_k = a_l$ for $l \neq k$ and $a_k \in T(c_{j_k}^{t_k}) \cap (T_Y)_1$ (resp. $a_k \in S(b_{i_k}^{s_k}) \cap (T_X)_1$). But this is not possible because $\mathscr{E}(X, Y)$ is a polarization. Indeed, this would contradict condition (b).

Another possibility for $T_Z$ not being a weak polarization is if $T_M$ had a subdiagram

$$b_j^{q_0} \xrightarrow{a_i} b_k^{q_1} \xleftarrow{a_i} b_l^{q_0} \text{ or } b_j^{q_1} \xleftarrow{a_i} b_k^{q_0} \xrightarrow{a_i} b_l^{q_1}$$

for some $i \in \{1, \dots, m\}$. But since $T_M$ is unramified, this is not possible.

The last possibility for $T_Z$ not being a weak polarization is if the basis contradicts condition (c) of Definition A.3.

Next we consider the case that the polarization is strong, the representation $M$ is a weak polarization and the representation is not forced to be unramified. But in this case it is straightforward to check that the induced coefficient quiver is a weak polarization. Indeed, for two basis elements $a_k : b_{i_k}^{s_k} \to c_{j_k}^{t_k}$ and $a_l : b_{i_l}^{s_l} \to c_{j_l}^{t_l}$ with $k \neq l$, we have $a_k \neq a_l$ and, moreover, considering the original coefficient quiver $T_X$ and $T_Y$, we have $|N(a_r, q)| = 0$ for $q \in \{b_{i_k}^{s_k}, c_{j_k}^{t_k}, b_{i_l}^{s_l}, c_{j_l}^{t_l}\}$ and $r \in \{k, l\}$. Thus all subdiagrams which could prevent $T_Z$ from being a weak polarization are forced to be induced from $T_M$. But since $T_M$ is a weak polarization, this cannot happen.

The third claim is straightforward because, in general, for an exact sequence $e \in \mathrm{Ext}(X, Y)$ with middle term $Z$, the matrix $Z_a$ is a block matrix with diagonal blocks $X_a$ and $Y_a$ for every arrow.

The last claim follows from Lemma A.1, see also Remark A.2. □

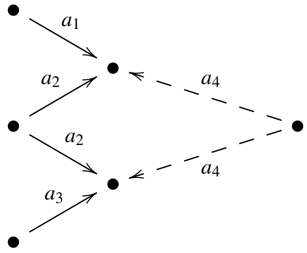
**Remark A.6.** (i) If we are only interested in (weak) polarizations, we can drop the condition that $X$ and $Y$ are exceptional. But in this case it is far more complicated or even impossible to say anything concerning the indecomposability of $Z$.

(ii) If $Q$ is of extended Dynkin type and, moreover, $(Y, X)$ is an exceptional sequence, we have $\dim \mathrm{Ext}(X, Y) \leq 2$ because otherwise there would be a root $d$ of $Q$ having an $n$-parameter family of indecomposables for $n \geq 2$. Then things become easier because every indecomposable tree module of $K(2)$ is unramified.

Let $S(n)$ be the $n$-subspace quiver with vertices $S(n)_0 = \{q_0, q_1, \dots, q_n\}$ and arrows $S(n)_1 = \{q_i \xrightarrow{a_i} q_0 \mid i = 1, \dots, n\}$. Let us consider two examples:

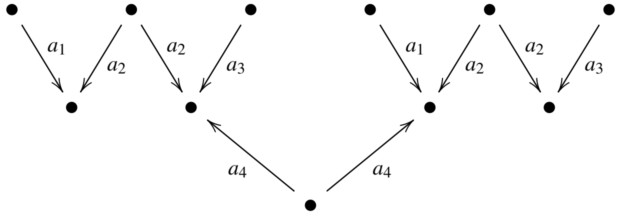
**Example A.7.** First let $n = 4$ and consider the exceptional sequence induced by the roots $\alpha = (2, 1, 1, 1, 0)$ and $\beta = (0, 0, 0, 0, 1)$. Then coefficient quivers of $X_\alpha$,

$X_\beta$ and a basis of $\mathrm{Ext}(X_\beta, X_\alpha)$ are for instance given by



Here the dotted arrows correspond to the tree-shaped basis of $\mathrm{Ext}(X_\beta, X_\alpha)$ under consideration, whence the remaining vertices and arrows correspond to the two coefficient quivers.

Since the basis of $\mathrm{Ext}(X_\beta, X_\alpha)$ is a polarization, which is not strong, and since we have $\dim \mathrm{Ext}(X_\beta, X_\alpha) \le 2$, the first part of Theorem A.5 applies. For instance, considering the exceptional representation of dimension $(1, 2)$ of $K(2)$, we obtain



on the $S(4)$-side. This is obviously a (strict) polarization.

**Example A.8.** An example for a basis which is a strong polarization can be obtained when considering $S(n)$ with $n \ge 3$ and the exceptional sequence induced by the roots $\alpha = (1, 1, 0, \dots, 0)$ and $\beta = (1, 0, 1, \dots, 1)$. In this case such a basis of $\mathrm{Ext}(X_\beta, X_\alpha)$ is given by choosing $n - 2$ out of the $n - 1$ maps mapping the one-dimensional subspace $(X_\beta)_{q_i}$ to $(X_\alpha)_{q_0}$ for $i = 2, \dots, n$.

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# Noncommutative geometry and Painlevé equations

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We construct the elliptic Painlevé equation and its higher dimensional analogs as the action of line bundles on 1-dimensional sheaves on noncommutative surfaces.

## 1. Introduction

**1.1.** The classical Painlevé equations are very special 2-dimensional dynamical systems; they and their generalizations (including discretizations) appear in many applications. Their theory is very well developed, in fact, from many different angles; see for example [Conte 1999] for an introduction. Many of these approaches are very geometric, and some can be interpreted in terms of noncommutative geometry.[1] A full discussion of the relation between the two topics in the title is outside of the scope of the present paper.

Our goals here are very practical. The dynamical systems we discuss appear in a very simple, yet challenging, problem of probability theory and mathematical physics: planar dimer (or lattice fermion) with a changing boundary; see [Kenyon 2009] for an introduction and [Okounkov 2009; 2010a; 2010b; 2010c] for the developments that lead to the present paper. The link to Artin-style noncommutative geometry, which is the subject of this paper, turns out to be very useful for dynamical and probabilistic applications.

Our hope is to promote further interaction between the two fields, and with that goal in mind, we state most of our results in the minimal interesting generality, with only a hint of the bigger picture. We also emphasize explicit examples.

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[1] In particular, Arinkin and Borodin [2006] gave an algebrogeometric interpretation of a degenerate discrete Painlevé equation. Their dynamics takes place not on the moduli spaces of sheaves but rather on moduli of discrete analogs of connections. Their construction may, in fact, be interpreted in terms of ours, as will be shown in [Rains $\geq$ 2015].

**1.2.** In algebraic geometry, there is an abundance of group actions of the following kind. Let $S \subset \mathbb{P}^N$ be a projective algebraic variety (it will be a surface in what follows, whence the choice of notation), and let

$$\mathsf{A} = \mathbb{C}[x_0, \dots, x_N]/(\text{equations of } S)$$

be its homogeneous coordinate ring. Coherent sheaves $\mathscr{M}$ on $S$ may be described as

$$\operatorname{Coh} S = \frac{\text{finitely generated graded A-modules M}}{\text{those of finite dimension}}. \tag{1}$$

They depend on discrete as well as continuous parameters so that

$$X = \text{moduli space of } \mathscr{M}$$

is a countable union of algebraic varieties. While there is a very developed general theory of such moduli spaces (see, e.g., [Huybrechts and Lehn 2010]), one can get a very concrete sense of $X$ by giving generators and relations for $\mathsf{M}$, as we will do below.

The group $\operatorname{Pic} S$ of line bundles $\mathscr{L}$ on $S$ acts on $X$ by

$$\mathscr{M} \mapsto \mathscr{L} \otimes \mathscr{M}. \tag{2}$$

If $\mathscr{L}$ is topologically nontrivial, this permutes connected components of $X$. A great many *integrable* actions of abelian groups can be understood from this perspective, an obvious invariant of the dynamics being the cycle in $S$ given by the support of $\mathscr{M}$.

**1.3.** Our point of departure in this paper is the observation that $\mathsf{A}$ need not be commutative for the constructions of Section 1.2. In fact, noncommutative projective geometry in the sense of M. Artin [Stafford and Van den Bergh 2001] is precisely the study of graded algebras with a good category (1).

The key new feature of the noncommutative situation is that for tensor products like (2) one needs a *right* $\mathsf{A}$-module $\mathsf{L}$ and then

$$\mathsf{L} \otimes_{\mathsf{A}} \mathsf{M} \in \operatorname{Mod} \mathsf{A}', \quad \mathsf{A}' = \operatorname{End}_{\mathsf{A}} \mathsf{L}.$$

If $\mathsf{L}$ is a deformation of a line bundle, then $\mathsf{A}'$ is closely related to $\mathsf{A}$, but in general,

$$\mathsf{A}' \not\cong \mathsf{A},$$

as can be already seen in very simple examples; see Section 2.3. As a result, we have

$$X \xrightarrow{\mathsf{L}\otimes} X' \tag{3}$$

where $X'$ is the corresponding moduli space for $\mathsf{A}'$.

While this sounds very abstract, we will be talking about a very concrete special case in which $S$ is a blowup of another surface $S_0$,

$$S = \text{blowup of } S_0 \text{ at } p \in S_0,$$

and $\mathsf{L}$ is the exceptional divisor. In the noncommutative case, tensoring with $\mathsf{L}$ will make the point $p$ move in $S_0$ by an amount proportional to the strength of noncommutativity; see Section 2.3.

**1.4.** Noncommutativity deforms the dynamics in two ways. First, the action (3) happens on a larger space that parametrizes both the module $\mathsf{M}$ and the algebra $\mathsf{A}$, with an invariant fibration given by forgetting $\mathsf{M}$. Specifically, we will be talking about sheaves on blowups of $\mathbb{P}^2$, where the centers $p_1, \dots, p_n$ of the blowup are allowed to move on a fixed cubic curve $E \subset \mathbb{P}^2$. There will be a $\mathbb{Z}^n$-action on these that covers a $\mathbb{Z}^n$-action on $E^n$ by translations.

Second, the notion of a support of a sheaf is lost in noncommutative geometry, so noncommutative deformation destroys whatever algebraic integrability that the action (2) may have. It is sometimes replaced by local analytic integrals (given, e.g., by monodromy of certain linear difference equations), but even then the orbits of the dynamics are typically dense; see also Section 4.8 below.

**1.5.** In noncommutative projective geometry, the 3-generator Sklyanin algebra, or the elliptic quantum $\mathbb{P}^2$, occupies a special place. In this paper, we focus on this key special case and discuss the corresponding dynamics from several points of view, including an explicit linear algebra description of it; see Section 5. This explicit description may be reformulated as addition on a moving Jacobian, generalizing the dynamics of [Kajiwara et al. 2006, §7].

In the first nontrivial case, we find the elliptic difference Painlevé equation of [Sakai 2001], the one that gives all other Painlevé equations by degenerations and continuous limits. A particularly detailed discussion of this example may be found in Section 6. In particular, we will see that, in this case, our system of isomorphisms between moduli spaces agrees (for sufficiently general parameters) with the corresponding system of isomorphisms between rational surfaces considered by Sakai.

In the semiclassical limit, the elliptic quantum $\mathbb{P}^2$ degenerates to a Poisson structure on a commutative $\mathbb{P}^2$, which induces a Poisson structure on suitable moduli spaces of sheaves [Tyurin 1988; Bottacin 1995; Hurtubise and Markman 2002], and the moduli spaces we consider in the commutative case are particularly simple instances of symplectic leaves in these Poisson spaces. In Section 7, we show that these Poisson structures on moduli spaces carry over to the noncommutative setting.

## 2. Blowups and Hecke modifications

**2.1.1.** In this paper, we work with 1-dimensional sheaves on noncommutative projective planes. They closely resemble their commutative ancestors, which we briefly review now.

A coherent sheaf $\mathscr{M}$ on $\mathbb{P}^2$ is an object in the category (1) for $\mathsf{A} = \mathbb{C}[x_0, x_1, x_2]$. A basic invariant of $\mathscr{M}$ is its Hilbert polynomial

$$h_{\mathscr{M}}(n) = \dim \mathsf{M}_n, \quad n \gg 0.$$

The dimension of $\mathscr{M}$ is the degree of this polynomial, so for 1-dimensional sheaves,

$$h_{\mathscr{M}}(n) = dn + \chi$$

where $d$ is the degree of the scheme-theoretic support of $\mathscr{M}$ and $\chi$ is the Euler characteristic of $\mathscr{M}$. The ratio $\chi/d$ is called the slope of $\mathscr{M}$. Sheaves with

$$\operatorname{slope} \mathscr{M}' < \operatorname{slope} \mathscr{M}$$

for all proper subsheaves $\mathscr{M}'$ are called *stable*; the moduli spaces of stable sheaves are particularly nice.

**2.1.2.** We will be content with *birational* group actions; hence, it will be enough for us to consider open dense subsets of the moduli spaces formed by sheaves of the form

$$\mathscr{M} = \iota_* L$$

where $\iota : C \hookrightarrow S$ is an inclusion of a smooth curve of degree $d$ and $L$ is a line bundle on $C$. All such sheaves are stable with

$$\chi = \deg L + 1 - g.$$

Here $g = (d-1)(d-2)/2$ is the genus of $C$. Their moduli space is a fibration over the base

$$B = \mathbb{P}^{d(d+3)/2} \setminus \{\text{singular curves}\}$$

of nonsingular curves $C$ with the fiber $\operatorname{Jac}_{\deg L} C$, the Jacobian of line bundles of degree $\deg L$. In particular, this moduli space has dimension

$$\dim X = d^2 + 1.$$

**2.1.3.** Curves $C$ meeting a point $p \in \mathbb{P}^2$ form a hyperplane in $B$. Incidence to $p$ may be rephrased in terms of the blowup

$$\mathrm{Bl} : S \to \mathbb{P}^2$$

with center $p$. Namely, $C$ meets $p$ if and only if

$$C = \mathrm{Bl}\,\widetilde{C}$$

where $\widetilde{C} \subset S$ is a curve of degree

$$[\widetilde{C}] = d \cdot [\text{line}] - [\mathscr{E}] \in H_2(S, \mathbb{Z}).$$

Here $\mathscr{E} = \mathrm{Bl}^{-1}\, p$ is the exceptional divisor of the blowup.

Line bundles $\widetilde{L}$ on $\widetilde{C}$ may be pushed forward to $\mathbb{P}^2$ to give sheaves that surject to the structure sheaf $\mathscr{O}_p$ of $p$. If $\widetilde{\mathscr{M}}$ is such a line bundle viewed as a sheaf on $S$, then the sheaves

$$\mathscr{M} = \mathrm{Bl}_*\,\widetilde{\mathscr{M}}, \qquad \mathscr{M}' = \mathrm{Bl}_*\,\widetilde{\mathscr{M}}(-\mathscr{E}) \tag{4}$$

fit into an exact sequence of the form

$$0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{O}_p \to 0. \tag{5}$$

When two sheaves $\mathscr{M}$ and $\mathscr{M}'$ differ by (5), one says that one is a *Hecke modification* of another. Thus, Hecke modifications at $p$ correspond to twists by the exceptional divisor on the blowup with center $p$. For noncommutative algebras, the language of Hecke modifications will be more convenient.

**2.2.** A fundamental fact that goes back to Mukai and Tyurin is that a Poisson structure $\omega^{-1}$ on a surface induces a Poisson structure on moduli of sheaves; see for example Chapter 10 of [Huybrechts and Lehn 2010]. Let $(\omega^{-1})$ denote the divisor of the Poisson structure. For $\mathbb{P}^2$, this is a curve $E$ of degree 3. Fix

$$p_1, \ldots, p_{3d} \in E \tag{6}$$

that lie on a curve $d$; that is,

$$\sum_{i=1}^{3d} [p_i] = \mathscr{O}_{\mathbb{P}^2}(d)|_E \in \mathrm{Jac}_{3d}\, E.$$

Let

$$B' = \mathbb{P}^{(d-1)(d-2)/2} \setminus \{\text{singular curves}\} \subset B$$

parametrize curves $C$ meeting (6) or, equivalently, curves $\widetilde{C}$ in

$$S = \mathrm{Bl}_{p_1,\ldots,p_{3d}}\, \mathbb{P}^2$$

of degree $d \cdot [\text{line}] - \sum [\mathscr{E}_i]$. It is by now a classical fact [Beauville 1991] that the fibration

$$\begin{tikzcd} \mathrm{Pic}\, C \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ {[C]} \arrow[r, hook] & B' \end{tikzcd} \tag{7}$$

is Lagrangian and that these are the symplectic leaves of the Poisson structure on $X$. Further, the group Pic $S$ acts on (7) preserving the fibers and the symplectic form. Here Pic $C$ is a countable union of algebraic varieties that parametrize line bundles on $C$ of arbitrary degree.

The noncommutative deformation will perturb this discrete integrable system. In particular, the points $p_i$ will have to move on the cubic curve $E$. The following model example illustrates this phenomenon.

**2.3.1.** The effect of noncommutativity may already be seen in the affine situation. Let $R$ be a noncommutative deformation of $\mathbb{C}[x, y]$, and let us examine the effect of Hecke modifications (5) on $R$-modules.

The point modules for $R$ are 0-dimensional modules. These are annihilated by the 2-sided ideal generated by commutators in $R$, so this ideal has to be nontrivial for point modules to exist. We consider

$$R = \mathbb{C}\langle x, y\rangle/(xy - yx = \hbar y). \tag{8}$$

Here $\hbar$ is a parameter that measures the strength of the noncommutativity. Setting $\hbar = 0$, one recovers the commutative ring $\mathbb{C}[x, y]$ with the Poisson bracket

$$\{x, y\} = \lim_{\hbar \to 0} \frac{xy - yx}{\hbar} = y. \tag{9}$$

The line $\{y = 0\}$ is formed by 0-dimensional leaves of this Poisson bracket; they correspond to point modules

$$\mathbb{O}_s = R/\mathfrak{m}_s, \quad \mathfrak{m}_s = (y, x - s),\ s \in \mathbb{C},$$

for $R$. All of them are annihilated by $y$, which generates the commutator ideal of $R$.

Let $M$ be an $R$-module of the form

$$M = R/f, \quad f = f_0(x) + f_1(x, y) \cdot y \in R,\ f_0 \neq 0.$$

The maps $M \to \mathbb{O}_s$ factor through the map

$$M \to M/yM = \mathbb{C}[x]/f_0(x)$$

and hence correspond to the roots of $f_0(x)$. So far, this is entirely parallel to the commutative case, except there are a lot fewer points — those are confined to the divisor of the Poisson bracket (9).

**2.3.2.** The following simple lemma shows Hecke correspondences move the points of intersection with this divisor.

**Lemma 1.** *Let* $s_1, s_2, \ldots$ *be the roots of* $f_0(x)$, *and let* $M' = \mathfrak{m}_s M$ *be the kernel in the exact sequence*

$$0 \to M' \to M \to \mathbb{O}_{s_1} \to 0.$$

*Then* $M' = R/f'$ *where the roots of* $f_0'(x)$ *are* $s_1 - \hbar, s_2, s_3, \ldots$.

In particular, the iteration of Hecke correspondences gives a chain of submodules of the form

$$M \supset \mathfrak{m}_s M \supset \mathfrak{m}_{s-\hbar}\mathfrak{m}_s M \supset \mathfrak{m}_{s-2\hbar}\mathfrak{m}_{s-\hbar}\mathfrak{m}_s M \supset \cdots .$$

A more general statement will be shown in Proposition 1.

**2.3.3.** For a noncommutative analog of the correspondence between Hecke modifications (5) and twists on the blowup (4), we need to retrace geometric constructions in module-theoretic terms.

Let $S$ be the blowup of an affine surface $S_0 = \operatorname{Spec} R$ with center in an ideal $I \subset R$. Sheaves on $S$ correspond to the quotient category (1) for

$$\mathsf{A} = \bigoplus_{n=0}^{\infty} \mathsf{A}_n, \quad \mathsf{A}_0 = R, \tag{10}$$

and $\mathsf{A}_n = I^n \subset R$. This quotient category

$$\operatorname{Coh} S = \operatorname{Tails} \mathsf{A}$$

is informally known as the category of tails; the morphisms in it are

$$\operatorname{Hom}_{\text{tails}}(\mathsf{M}, \mathsf{M}') = \varinjlim \operatorname{Hom}_{\text{graded } \mathsf{A}\text{-modules}}(\mathsf{M}_{\geq k}, \mathsf{M}')$$

where $\mathsf{M}_{\geq k} \subset \mathsf{M}$ is the submodule of elements of degree $k$ and higher. The pushforward $\mathrm{Bl}_*$ of sheaves is the functor

$$\mathsf{M} \mapsto \mathrm{Bl}_* \mathsf{M} = \operatorname{Hom}_{\text{tails}}(\mathsf{A}, \mathsf{M}) \in \operatorname{Mod} R. \tag{11}$$

In the opposite direction, we have the pullback $\mathrm{Bl}^* M = \mathsf{A} \otimes_R M$ of modules as well as their proper transform

$$\mathrm{Bl}^{-1} M = \bigoplus I^n M \in \operatorname{Coh} S.$$

**2.3.4.** Now for a noncommutative ring $R$ as in (8), we look for a graded module $\mathsf{M}$ over a graded algebra $\mathsf{A}$ such that

$$\mathrm{Bl}_* \mathsf{M}(n) = \mathfrak{m}_{s-(n-1)\hbar} \cdots \mathfrak{m}_{s-\hbar}\mathfrak{m}_s M \tag{12}$$

where $\mathsf{M}(n)_k = \mathsf{M}_{n+k}$ is the shift of the grading and the pushforward is defined as in (11). Here $r \in R$ acts on $\phi \in \operatorname{Hom}_{\text{tails}}(\mathsf{A}, \mathsf{M})$ by

$$[r \cdot \phi](a) = \phi(ar).$$

The algebra $\mathsf{A}$, known as Van den Bergh's noncommutative blowup [Artin 1997], is constructed as

$$\mathsf{A} = \bigoplus_{n \geq 0} (T\mathfrak{m}_s)^n$$

where $T$ is a new generator subject to

$$T^{-1}rT = yry^{-1} \quad \text{for all } r \in R,$$

which means that

$$xT = T(x-\hbar), \qquad yT = Ty,$$

and hence

$$(T\mathfrak{m}_s)^n = T^n \mathfrak{m}_{s-(n-1)\hbar} \cdots \mathfrak{m}_{s-\hbar}\mathfrak{m}_s.$$

It is easy to see that the $\mathsf{A}$-module

$$\mathsf{M} = \mathrm{Bl}^{-1} M = \bigoplus T^n \mathfrak{m}_{s-(n-1)\hbar} \cdots \mathfrak{m}_{s-\hbar}\mathfrak{m}_s M$$

satisfies (12) provided $M$ has no 0-dimensional submodules supported on $s$, $s-\hbar, \dots$.

**2.3.5.** To relate Hecke modifications to tensor products, we note that

$$\mathrm{Bl}^{-1}_{s-\hbar}(\mathfrak{m}_s M) = \mathsf{L} \otimes_{\mathsf{A}_s} (\mathrm{Bl}^{-1}_s M)$$

where

$$\mathsf{L} = T^{-1}\mathsf{A}_s(1) \in \mathrm{Bimod}(\mathsf{A}_{s-\hbar}, \mathsf{A}_s).$$

Here we indicated the centers of the blowup by subscripts $s$ and $s-\hbar$, respectively.

The functor $\mathsf{L} \otimes_{\mathsf{A}_s}$ is the noncommutative version of $\mathscr{O}_S(-\mathscr{E})\otimes$, and we see that it moves the center of the blowup by minus (to match the minus in $\mathscr{O}_S(-\mathscr{E})$) the noncommutativity parameter $\hbar$.

## 3. Sheaves on quantum planes

**3.1.** One of the most interesting noncommutative surfaces is associated with the 3-dimensional *Sklyanin algebra* $\mathsf{A}$, which is a graded algebra, generated over $\mathsf{A}_0 = \mathbb{C}$ by three generators $x_1, x_2, x_3 \in \mathsf{A}_1$ subject to three quadratic relations.

The relations in $\mathsf{A}$ may be written in the superpotential form

$$\frac{\partial}{\partial x_i}\mathscr{W} = 0$$

where

$$\mathscr{W} = ax_1x_2x_3 + bx_3x_2x_1 + \frac{c}{3}\sum x_i^3$$

and the derivative is applied cyclically, that is,

$$\frac{\partial}{\partial x_1} x_{i_0} \cdots x_{i_{p-1}} = \sum_{k=0}^{p-1} \delta_{1,i_k} x_{i_{k+1}} \cdots x_{i_{p-1+k}},$$

where the subsubscripts are taken modulo $p$. The parameters $a$, $b$, and $c$ will be assumed generic in what follows.

**3.2.** The structure of $\mathsf{A}$ has been much studied; see for example [Stafford and Van den Bergh 2001]. In particular, it is a Noetherian domain and

$$\sum_n \dim \mathsf{A}_n t^n = (1-t)^{-3}.$$

By definition, the category $\mathrm{Tails}\,\mathsf{A}$ is the category of coherent sheaves on a quantum $\mathbb{P}^2$. The Grothendieck group of this category is the same as the $K$-theory of $\mathbb{P}^2$; that is,

$$K(\mathrm{Tails}\,\mathsf{A}) = \mathbb{Z}^3,$$

corresponding to the 3 coefficients in the Hilbert polynomial. In particular, for 1-dimensional sheaves, we have

$$\dim \mathsf{M}_n = n \deg \mathsf{M} + \chi(\mathsf{M}), \quad n \gg 0. \tag{13}$$

**3.3.** Modules $\mathsf{M}$ such that $\dim \mathsf{M}_n = 1$, $n \gg 1$, are called *point modules* and play a very special role. Choosing a nonzero $v_n \in \mathsf{M}_n$, we get a sequence of points

$$p_n = (p_{1,n} : p_{2,n} : p_{3,n}) \in \mathbb{P}^2 = \mathbb{P}(\mathsf{A}_1)^*$$

such that

$$x_i v_n = p_{i,n} v_{n+1}.$$

The relations in $\mathsf{A}$ then imply that the locus

$$\{(p_n, p_{n+1})\} \in \mathbb{P}^2 \times \mathbb{P}^2$$

is a graph of an automorphism $p_{n+1} = \tau(p_n)$ of a plane cubic curve $E \subset \mathbb{P}^2$ [Artin et al. 1990]. The assignment

$$\mathsf{M} \mapsto p_0 \in E \tag{14}$$

identifies $E$ with the moduli space of point modules $\mathsf{M}$ and $\tau$ with the automorphism[2]

$$\tau : \mathsf{M} \mapsto \mathsf{M}(1)$$

of the shift of grading $\mathsf{M}(1)_n = \mathsf{M}_{n+1}$. The inverse of (14) is given by

$$p \mapsto \mathsf{A}/\mathsf{A}p^{\perp},$$

where $p^{\perp} \subset \mathsf{A}_1$ is the kernel of $p \in \mathbb{P}(\mathsf{A}_1)^*$.

---

[2] Note that, if $\tau'$ is any other automorphism of $E$ such that $\tau'^3 = \tau^3$, then the Sklyanin algebra associated with the pair $(E, \tau')$ has an equivalent category of coherent sheaves. Indeed, one has a natural isomorphism

$$\bigoplus_n \mathsf{A}_{3n} \cong \bigoplus_n \mathsf{A}'_{3n},$$

though this does not extend to an isomorphism $\mathsf{A} \cong \mathsf{A}'$. (Here 3 is the degree of the anticanonical bundle on $\mathbb{P}^2$.) This is why all key formulas below depend only on $\tau^3$.

**3.4.** The action of $\mathsf{A}$ on point modules factors through the surjection in

$$0 \to (\mathsf{E}) \to \mathsf{A} \to \mathsf{B} \to 0, \tag{15}$$

where $\mathsf{E} \in \mathsf{A}_3$ is a distinguished normal (in fact, central) element and $\mathsf{B}$ is the twisted homogeneous coordinate ring of $E$. By definition,

$$\mathsf{B} = B(E, \mathscr{O}(1), \tau) = \bigoplus_{n \geq 0} H^0(E, \mathscr{L}_0 \otimes \cdots \otimes \mathscr{L}_{n-1}),$$

where $\mathscr{L}_k = (\tau^{-k})^* \mathscr{O}(1)$. The multiplication

$$\mathrm{mult}_\tau : \mathsf{B}_n \otimes \mathsf{B}_m \to \mathsf{B}_{n+m}$$

is the usual multiplication precomposed with $\tau^{-m} \otimes 1$. See for example [Stafford and Van den Bergh 2001] for a general discussion of such algebras.

The map

$$\mathrm{Coh}\, E \ni \mathscr{F} \mapsto \Gamma(\mathscr{F}) = \bigoplus_n H^0(E, \mathscr{F} \otimes \mathscr{L}_0 \otimes \cdots \otimes \mathscr{L}_{n-1})$$

induces an equivalence between the category of coherent sheaves on $E$ and finitely generated graded $\mathsf{B}$-modules up to torsion; see Theorem 2.1.5 in [Stafford and Van den Bergh 2001]. Note in particular that

$$\mathsf{B}(k) \cong \begin{cases} \Gamma(\mathscr{L}_{-k} \otimes \cdots \otimes \mathscr{L}_{-1}), & k \geq 0, \\ \Gamma(\mathscr{L}_0^{-1} \otimes \cdots \otimes \mathscr{L}_{-k-1}^{-1}), & k < 0. \end{cases} \tag{16}$$

**3.5.** It is shown in [Artin et al. 1991, Theorem 7.3] that the algebra $\mathsf{A}[\mathsf{E}^{-1}]_0$ is simple. If $\mathsf{M}$ is a 0-dimensional $\mathsf{A}$-module, then $\mathsf{M}[\mathsf{E}^{-1}]_0$ is a finite-dimensional $\mathsf{A}[\mathsf{E}^{-1}]_0$-module, hence zero. It follows that any 0-dimensional $\mathsf{A}$-module has a filtration with point quotients.

**3.6.** Moduli spaces of stable $\mathsf{M} \in \mathrm{Tails}\, \mathsf{A}$ may be constructed using the standard tools of geometric invariant theory, as in, e.g., [Nevins and Stafford 2007], or using the existence of an exceptional collection

$$\mathsf{A}, \mathsf{A}(1), \mathsf{A}(2) \in \mathrm{Tails}\, \mathsf{A},$$

as in [de Naeghel and Van den Bergh 2004]. In any event, at least for generic parameters of $\mathsf{A}$, the moduli space $\mathfrak{M}(d, \chi)$ of 1-dimensional sheaves of degree $d$ and Euler characteristic $\chi$ is irreducible of dimension

$$\dim \mathfrak{M}(d, \chi) = d^2 + 1.$$

It is enough to see this in the commutative case, where a generic $\mathsf{M}$ has a presentation of the form

$$0 \to \mathsf{A}(-2)^{d-\chi} \xrightarrow{L} \mathsf{A}(-1)^{d-2\chi} \oplus \mathsf{A}^{\chi} \to \mathsf{M} \to 0, \tag{17}$$

assuming

$$0 \le \chi \le \tfrac{1}{2}d;$$

see in particular [Beauville 2000]. When $\frac{1}{2}d < \chi \le d$, the $\mathsf{A}(-1)$ term moves from the generators to syzygies. The values of $\chi$ outside $[0, d]$ are obtained by a shift of grading.

It follows that (17) also gives a presentation of a generic stable 1-dimensional $\mathsf{M}$ for Sklyanin algebras.

**3.7.** The letter $L$ is chosen in (17) to connect with the so-called $L$-operators in theory of integrable systems. In (17), $L$ is a just a matrix with linear and quadratic entries in the generators $x_1, x_2, x_3 \in \mathsf{A}_1$. The space of possible $L$'s, therefore, is just a linear space that needs to be divided by the action of

$$\operatorname{Aut}\operatorname{Source} L \times \operatorname{Aut}\operatorname{Target} L \cong GL(d-\chi) \times GL(d-2\chi) \times GL(\chi) \ltimes \mathbb{C}^{3\chi(d-2\chi)}.$$

In particular, we have a birational map

$$\operatorname{Mat}(d \times d)^3 / GL(d) \times GL(d) \dashrightarrow \mathfrak{M}(d, 0), \tag{18}$$

which is literally unchanged from the commutative situation.

## 4. Weyl group action on parabolic sheaves

**4.1.** Our goal in this section is to examine the action of Hecke correspondences (5) on 1-dimensional sheaves $\mathsf{M}$ on quantum planes. For this, the language of parabolic sheaves will be convenient.

In what follows, we assume $\mathsf{M} \in \operatorname{Tails}\mathsf{A}$ is stable 1-dimensional without $\mathsf{E}$-torsion. This means the sequence

$$0 \to \mathsf{M}(-3) \xrightarrow{\mathsf{E}} \mathsf{M} \to \mathsf{M}/\mathsf{E}\mathsf{M} \to 0 \tag{19}$$

is exact, and comparing the Hilbert polynomials, we see $\mathsf{M}/\mathsf{E}\mathsf{M}$ has a filtration with $3 \deg \mathsf{M}$ point quotients. A choice of such filtration

$$\mathsf{M} = \mathsf{M}_0 \supset \mathsf{M}_1 \supset \mathsf{M}_2 \supset \cdots \supset \mathsf{M}_{3\deg \mathsf{M}} = \mathsf{E}\mathsf{M} \cong \mathsf{M}(-3)$$

is called a *parabolic* structure on $\mathsf{M}$.

**4.2.** Moduli spaces $\mathfrak{P}(d, \chi)$ of parabolic sheaves may be constructed as in the commutative situation. The forgetful map

$$\mathfrak{P}(d, \chi) \dashrightarrow \mathfrak{M}(d, \chi)$$

is generically finite of degree $(3 \deg \mathsf{M})!$ corresponding to the generic module $\mathsf{M}/\mathsf{E}\mathsf{M}$ being a direct sum of nonisomorphic point modules.

**4.3.** Given a parabolic module $\mathsf{M}$, we denote by

$$\partial \mathsf{M} = (\mathsf{M}_0/\mathsf{M}_1, \dots, \mathsf{M}_{3d-1}/\mathsf{M}_{3d}) \in E^{3d}$$

the isomorphism class of its point factors.

In the commutative case, the sum of $\partial\mathsf{M}$ in Pic $E$ equals $\mathscr{O}(\deg M)$. The analogous noncommutative statement reads:

**Proposition 1.** *Let* $\mathsf{M}$ *be* 1*-dimensional and have no* $\mathsf{E}$*-torsion. Then*

$$\sum_{p \in \partial\mathsf{M}} p = \mathscr{O}(\deg \mathsf{M}) + 3(\chi(\mathsf{M}) - \deg \mathsf{M})\tau \in \operatorname{Pic}_{3d} E. \tag{20}$$

Here we identify the automorphism $\tau$ with the element $\tau(p) - p \in \operatorname{Pic}_0 E$. This does not depend on the choice of $p \in E$.

*Proof.* Let

$$\cdots \to F_1 \to F_0 \to \mathsf{M} \to 0$$

be a graded free resolution of a module $\mathsf{M}$. The cohomology groups of

$$\cdots \to \mathsf{B} \otimes F_1 \to \mathsf{B} \otimes F_0 \to 0$$

are, by definition, the groups $\operatorname{Tor}^i(\mathsf{B}, \mathsf{M})$. The class of the Euler characteristic

$$[\operatorname{Tor}(\mathsf{B}, \mathsf{M})] = \sum (-1)^i [\operatorname{Tor}^i(\mathsf{B}, \mathsf{M})] \in K(\mathsf{B}) \cong K(E)$$

may be computed using only the $K$-theory class of $\mathsf{M}$. In fact,

$$c_1([\operatorname{Tor}(\mathsf{B}, \mathsf{M})]) = \mathscr{O}(\deg \mathsf{M}) + 3(\chi(\mathsf{M}) - \deg \mathsf{M} - \operatorname{rk} \mathsf{M})\tau.$$

It is enough to check this for $\mathsf{M} = \mathsf{A}(k)$, which follows from (16).

Alternatively, the groups $\operatorname{Tor}^i(\mathsf{B}, \mathsf{M})$ may be computed from a free resolution of $\mathsf{B}$. From (19), we find

$$\operatorname{Tor}^0(\mathsf{B}, \mathsf{M}) = \mathsf{M}/\mathsf{E}\mathsf{M},$$

while all higher ones vanish. The proposition follows. □

**4.4.** Let

$$\mathsf{W} = S(3d) \ltimes \mathbb{Z}^{3d}$$

be the extended affine Weyl group of $GL(3d)$. Weyl group actions on moduli of parabolic objects is a classic of geometric representation theory. In our context, the lattice subgroup may be interpreted as

$$\mathbb{Z}^{3d} \cong \operatorname{Pic} \mathrm{Bl}_{p_1,\dots,p_{3d}} \mathbb{P}^2 / \operatorname{Pic} \mathbb{P}^2,$$

while $S(3d)$ acts on it by monodromy as the centers of the blowup move around.

The group $\mathsf{W}$ is generated by reflections $s_0, \dots, s_{3d-1}$ in the hyperplanes

$$\{a_{3d} = a_1 - 1\}, \{a_1 = a_2\}, \dots, \{a_{3d-1} = a_{3d}\}$$

together with the transformation

$$g \cdot (a_1, \dots, a_{3d}) \mapsto (a_2, \dots, a_{3d}, a_1 - 1).$$

The involutions $s_i$ satisfy the Coxeter relations

$$(s_i s_{i+1})^3 = 1$$

of the affine Weyl group of $GL(3d)$ while $g$ acts on them as the Dynkin diagram automorphism

$$g s_i g^{-1} = s_{i-1}.$$

Here and above the indices are taken modulo $3d$.

**4.5.** On the open locus where

$$\mathsf{M}/\mathsf{E}\mathsf{M} = \bigoplus_{i=1}^{3d} \mathbb{O}_{p_i}$$

and all $p_i$ are distinct, the symmetric group $S(3d)$ acts on parabolic structures by permuting the factors.

This extends to a birational action of $S(3d)$ on $\mathfrak{P}(d, \chi)$. The closure of the graph of $s_k$ may be described as the nondiagonal component of the correspondence

$$\{(\mathsf{M}, \mathsf{M}') \mid \mathsf{M}_i = \mathsf{M}'_i, \ i \neq k\} \subset \mathfrak{P} \times \mathfrak{P}.$$

**4.6.** We define

$$g \cdot \mathsf{M} = \mathsf{M}_1$$

with the parabolic structure

$$\mathsf{M}_1 \supset \cdots \supset \mathsf{M}_{3d} \supset \mathsf{E}\mathsf{M}_1,$$

where, as before, we assume that $\mathsf{M}$ has no $\mathsf{E}$-torsion. This gives a birational map

$$g : \mathfrak{P}(d, \chi) \to \mathfrak{P}(d, \chi - 1).$$

**4.7.** We make $\mathsf{W}$ act on $E^{3d}$ by

$$\mathbb{Z}^{3d} \ni (a_1, a_2, \dots) \mapsto (\tau^{3a_1}, \tau^{3a_2}, \dots) \in \operatorname{Aut} E^{3d} \tag{21}$$

while $S(3d)$ permutes the factors. Then we have

$$\partial \mathsf{E}\mathsf{M} = \partial \mathsf{M}(-3) = (-1, \dots, -1) \cdot \partial \mathsf{M}. \tag{22}$$

**Theorem 1.** *The transformations* $s_1, \dots, s_{3d-1}$ *and* $g$ *generate an action of* $\mathsf{W}$ *by birational transformations of* $\bigsqcup_\chi \mathfrak{P}(d, \chi)$. *The map*

$$\partial : \mathfrak{P}(d, \chi) \to E^{3d}$$

*is equivariant with respect to this action.*

*Proof.* Clearly, $s_1, \dots, s_{3d-1}$ generate the symmetric group $S(3d)$, as do their conjugates under the action of $g$. Setting

$$s_0 = g s_1 g^{-1},$$

one sees that $g$ permutes $s_0, \dots, s_{3d-1}$ cyclically, verifying all relations in $\Lambda$. Equivariance of $\partial$ follows from (22). □

**4.8.** Evidently, the $\mathsf{A}[\mathsf{E}^{-1}]$ module $\mathsf{M}[\mathsf{E}^{-1}]$ is not changed by the dynamics; that is to say, its isomorphism class is an invariant of the dynamics. From a dynamical viewpoint, however, this is not very useful information since no reasonable moduli space for $\mathsf{A}[\mathsf{E}^{-1}]$-modules exists, which is just another way of stating the fact that generic orbits of our dynamical system are dense in the analytic topology.

Local analytic integrals of the dynamics may be constructed in this setting if a representation of the noncommutative algebra by linear difference operators is given. (This will be done in [Rains ≥ 2015].) The monodromy of the difference equation corresponding to a module is the required invariant. An important virtue of such local invariants is their convergence to algebraic invariants as the noncommutative deformation is removed, which is very useful, for example, for the study of averaging of perturbations.

## 5. A concrete description of the action

**5.1.** The goal of this section to make the action in Theorem 1 as explicit as possible. Consider the exact sequence

$$1 \to \mathsf{W}_0 \to \mathsf{W} \xrightarrow{\chi} \mathbb{Z} \to 1$$

where $\chi$ is the sum of entries on $\mathbb{Z}^{3d}$ and 0 on $S(3d)$. We have

$$\chi(w \cdot \mathsf{M}) = \chi(\mathsf{M}) + \chi(w),$$

so the subgroup $\mathsf{W}_0$ acts on $\mathfrak{P}(d, \chi)$ for any $\chi \in \mathbb{Z}$. Since all of them are birational, we can focus on one, for example

$$X = \mathfrak{P}(d, d+1).$$

**5.2.** We will see that there is a diagram of maps, with birational top row,

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dr, "\partial"'] & S^g\mathbb{P}^2 \times E^{3d-1} \arrow[d] \\ & E^{3d-1} \end{tikzcd} \tag{23}$$

where $g = \binom{d-1}{2}$ is the genus of a smooth curve of degree $d$ and $S^g\mathbb{P}^2$ parametrizes unordered collections $D \subset \mathbb{P}^2$ of $g$ points. We view the product $E^{3d-1}$ as embedded in $E^{3d}$ via

$$E^{3d-1} = \left\{ \sum_{i=1}^{3d} p_i = \mathcal{O}(d) + 3\tau \right\} \subset E^{3d}. \tag{24}$$

This subset is $\mathsf{W}_0$-invariant.

**5.3.** The action of $\mathsf{W}_0$ has a particularly nice description in terms of (23), and it agrees with the action on $E^d$ already defined in Section 4.7.

The symmetric group $S(3d)$ permutes the points $p_i \in E$ and does nothing to $D \subset \mathbb{P}^2$. It remains to define the action of the lattice generators

$$\alpha_{ij} = \delta_i - \delta_j \in \mathbb{Z}^{3d},$$

where $\delta_i = (0, \dots, 0, 1, 0, \dots, 0)$ form the standard basis of $\mathbb{Z}^n$. We claim

$$\alpha_{ij}(D, P) = (D', P')$$

where $P' = \alpha_{ij} P$ as in Section 4.7, while the points $D'$ are found from the following construction.

Let $C \subset \mathbb{P}^2$ be the degree-$d$ curve that meets $D$ and $(-\delta_j) \cdot P$. Because of the condition (24), such a curve exists and is generically unique. The divisor $D' \subset C$ is found from

$$D + p_i = D' + p'_j \in \operatorname{Pic} C.$$

Again, since $D'$ is of degree $g(C)$, generically, there is a unique effective divisor satisfying this equation.

**Theorem 2.** *This defines a birational action of* $\mathsf{W}_0$ *that is birationally isomorphic to the action from Theorem 1.*

**Remark.** The case $d = 3$ of the above dynamical system was considered in [Kajiwara et al. 2006, §7] as a description of the elliptic Painlevé equation in terms of the arithmetic on a moving elliptic curve. Since we only consider this in terms of birational maps, this only shows that our dynamics is birationally equivalent to elliptic Painlevé; we will see in Section 6 that (for generic parameters) the description in terms of sheaves agrees *holomorphically* with elliptic Painlevé.

We break up the proof into a sequence of propositions.

**5.4.** A general point $\mathsf{M} \in X$ is of the form $\mathsf{M} = \operatorname{Coker} L$ where

$$L : \mathsf{A}(-1)^{d-1} \to \mathsf{A}(1) \oplus \mathsf{A}^{d-2};$$

see Section 3.6. Consider the submatrix

$$\bar{L} : \mathsf{A}(-1)^{d-1} \to \mathsf{A}^{d-2},$$

and let $\operatorname{div} \mathsf{M} \subset \mathbb{P}^2$ be the subscheme cut out by the maximal minors of $\bar{L}$. Generically,

$$\operatorname{div} \mathsf{M} = \binom{d-1}{2} \text{ distinct points.}$$

**Proposition 2.** *The map*

$$X \ni \mathsf{M} \mapsto (\operatorname{div} \mathsf{M}, \partial \mathsf{M}) \in S^g \mathbb{P}^2 \times E^{3d-1},$$

*where* $E^{3d-1} \subset E^{3d}$ *as in* (24), *is birational.*

*Proof.* The statement about $\sum p$ follows from (20). Since the source and the target have the same dimension, it is enough to show that the map has degree 1. For this, we may assume that $\mathsf{A}$ is commutative, in which case the claim is classically known; see, e.g., [Beauville 2000]. ☐

**5.5.** In fact, in the commutative case, $\mathsf{M}$ is generically a line bundle $\mathscr{L}$ on a smooth curve $C = \operatorname{supp} \mathsf{M}$. From its resolution, we see that $\mathscr{L}(-1)$ has a unique section. The divisor of this section on $C$ is $\operatorname{div} \mathsf{M}$.

**5.6.** Similarly, a general point $\mathsf{M} \subset \mathfrak{P}(d, d)$ is of the form $\mathsf{M} = \operatorname{Coker} L$, where

$$L : \mathsf{A}(-1)^d \to \mathsf{A}^d$$

is a $d \times d$ matrix $L$ of linear forms. The matrix $L$ is unique up to left and right multiplication by elements of $GL(d)$. The description of this $GL(d) \times GL(d)$ quotient is classically known [Beauville 2000] and given by

$$L \mapsto (C, \mathscr{L})$$

where

$$C = \{\det L = 0\} \subset \mathbb{P}^2$$

is a degree-$d$ curve cut out by the usual, commutative determinant and $\mathscr{L}$ is the cokernel of the commutative morphism, viewed as a sheaf on $C$. Generically, $C$ is smooth, $\mathscr{L}$ is a line bundle, and

$$g(C) = \binom{d-1}{2}, \qquad \deg \mathscr{L} = g(C) - 1 + d.$$

**Proposition 3.** *The point modules $\mathcal{O}_p$ in $\partial\mathsf{M}$ correspond to points $p \in C \cap E$.*

*Proof.* Let $f_1, \dots, f_d$ be the images in $\mathsf{M}$ of the generators of $\mathsf{A}^d$. We may assume that $f_2, \dots, f_d$ are in the kernel of $\mathsf{M} \to \mathcal{O}_p$. The coefficient of $f_1$ in any relation among the $f_i$ must belong to $\mathsf{A}p^\perp$, whence the claim. □

**5.7.** Now suppose $\mathsf{M} \in \mathfrak{P}(d, d+1)$ and $p \in \partial\mathsf{M}$, and define

$$\mathsf{M}_p = \operatorname{Ker}(\mathsf{M} \to \mathcal{O}_p) = \mathfrak{m}_p\mathsf{M}.$$

Then $\mathsf{M}_p \in \mathfrak{P}(d, d)$, and we denote by $(C_p, \mathscr{L}_p)$ the curve and the line bundle that correspond to $\mathsf{M}_p$.

**Proposition 4.** *The curve $C_p$ meets* div $\mathsf{M}$.

*Proof.* Let $f$ and $g_1, \dots, g_{d-2}$ be the images in $\mathsf{M}$ of the generators of $\mathsf{A}(1)$ and $\mathsf{A}^{d-2}$, respectively. We may chose them so that all $g_i$ are mapped to 0 in $\mathcal{O}_p$. Let

$$0 \to \mathsf{A}(-1) \xrightarrow{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} \mathsf{A}^{\oplus 2} \xrightarrow{\begin{bmatrix} l_1 & l_2 \end{bmatrix}} \mathsf{A}(1) \to \mathcal{O}_p \to 0$$

be a free resolution. All relations in $\mathsf{M}$ must be of the form

$$(r_1 l_1 + r_2 l_2) f + \sum s_i g_i = 0, \quad r_1, r_2, s_i \in \mathsf{A}_1.$$

There are $d-1$ linearly independent relations like this, and the coefficients $s_i$ in them form the matrix $\bar{L}$.

We observe that $l_1 f, l_2 f, g_1, \dots, g_{d-2}$ generate $\mathsf{M}_p$, and for this presentation, the matrix $L_p$ has the block form

$$L_p = \begin{bmatrix} u & r \\ 0 & \bar{L} \end{bmatrix}. \tag{25}$$

The proposition follows. □

**5.8.** Denote

$$\partial\mathsf{M} = (p, p_2, \dots, p_{3d}).$$

Since $\mathsf{M}_p = \mathfrak{m}_p\mathsf{M}$ still surjects to $\mathcal{O}_{p_i}$, $i \geq 2$, we have

$$\partial\mathsf{M}_p = (p', p_2, \dots, p_{3d})$$

for some $p' \in E$ and from (20) we see that

$$p' = \tau^{-3}(p).$$

From the proof of Proposition 4, we note that

$$p' = \{u_1 = u_2 = 0\}.$$

**5.9.** The following proposition concludes the proof of Theorem 2.

**Proposition 5.** *We have* $\mathscr{L}_p = \operatorname{div} \mathsf{M} + \mathcal{O}(1) - p'$ *in* $\operatorname{Pic} C_p$.

*Proof.* This is a purely commutative statement, in fact, just a restatement of the remark in Section 5.5. ☐

## 6. The elliptic Painlevé equation

Let us consider the case $d = 3$ in more detail. In this case, we can be fairly explicit about the moduli space, at least for sufficiently general parameters. We suppose that the $3d = 9$ points of $\partial\mathsf{M}$ are distinct (and ordered) so that specifying a point in $\mathfrak{P}(3, \chi)$ is equivalent to specifying the corresponding point in $\mathfrak{M}(3, \chi)$. We consider the case $\chi = 1$ as in that case we need to consider only one shape of presentation. (Of course, this also holds for $\chi = -1$ by duality; the case $\chi = 0$ is somewhat trickier, though the calculation below of the action of the Hecke modifications implies a similar description for that moduli space.)

We naturally restrict our attention to semistable sheaves and note that a sheaf in $\mathfrak{M}(3, 1)$ is semistable if and only if it is stable, which holds if and only if it has no proper subsheaf with positive Euler characteristic.

**Lemma 2.** *Suppose the sheaf* $\mathsf{M} \in \mathfrak{M}(3, 1)$ *has a free resolution of the form*

$$0 \to \mathsf{A}(-2)^2 \to \mathsf{A}(-1) \oplus \mathsf{A} \to \mathsf{M} \to 0$$

*or in other words is generated by elements* $f$ *and* $g$ *of degrees* $1$ *and* $0$ *satisfying relations*

$$v_1 f + w_1 g = v_2 f + w_2 g = 0,$$

*with* $v_1, v_2 \in A_1$ *and* $w_1, w_2 \in A_2$. *If* $\mathsf{M}$ *is stable, then* $v_1$ *and* $v_2$ *are linearly independent and there is no element* $x \in A_1$ *such that* $v_1 x = w_1$ *and* $v_2 x = w_2$. *Conversely, the cokernel of any morphism* $\mathsf{A}(-2)^2 \to \mathsf{A}(-1) \oplus \mathsf{A}$ *satisfying these conditions is a stable sheaf* $\mathsf{M} \in \mathfrak{M}(3, 1)$.

*Proof.* If $v_1$ and $v_2$ are not linearly independent, then without loss of generality we may assume $v_2 = 0$. But then $w_2 \neq 0$ (by injectivity) and thus the submodule generated by the image of $\mathsf{A}$ is the cokernel of the map $w_2 : \mathsf{A}(-2) \to \mathsf{A}$ and has Euler characteristic 1, violating stability.

Similarly, if the second condition is violated, then we may replace $f$ by $f - xg$ and thus eliminate the dependence of the relations on $g$. But then $\mathsf{A}$ is a direct summand of $\mathsf{M}$, violating the condition that $\mathsf{M}$ have rank 0.

For the converse, note first that, if the map is not injective, then there is in particular a nonzero morphism $\mathsf{A}(-d) \to \mathsf{A}(-2)^2$ exhibiting the failure of injectivity. In particular, the composition

$$0 \to \mathsf{A}(-d) \to \mathsf{A}(-2)^2 \to \mathsf{A}(-1)$$

must be 0. Since by assumption $v_1$ and $v_2$ are linearly independent, the existence of a kernel implies that $v_1$ and $v_2$ have a single point in common. The kernel of the map $\begin{bmatrix} v_1 & v_2 \end{bmatrix} : \mathsf{A}(-2)^2 \to \mathsf{A}(-1)$ is thus isomorphic to $\mathsf{A}(-3)$ (since point sheaves have resolutions of this form). As the map from $\mathsf{A}(-d)$ factors through this kernel, we may as well take $d = 3$. But then the dual argument shows that the map $\begin{bmatrix} w_1 & w_2 \end{bmatrix}$ must factor through $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$, contradicting our second condition.

Thus, in particular, the conditions ensure that the cokernel is in $\mathfrak{M}(3, 1)$, and it remains to show stability. Note that any destabilizing subsheaf has positive Euler characteristic and thus has a map from $\mathsf{A}$. Moreover, since it has degree $< 3$, it must be globally generated so that, if $\mathsf{M}$ is unstable, the subsheaf generated by $g$ is destabilizing. Furthermore, to have degree $< 3$, $g$ must in particular satisfy a relation $wg = 0$, implying that $v_1$ and $v_2$ are linearly dependent, giving a contradiction. □

**Remark.** One can show that every stable sheaf in $\mathfrak{M}(3, 1)$ must have a presentation of the above form, but for present purposes, we simply restrict our attention to the corresponding open subset of the stable moduli space, which by the following proof is projective.

**Theorem 3.** *Suppose $p_1, \dots, p_9$ is a sequence of 9 distinct points of $E$ such that $p_1 + \cdots + p_9 = \mathcal{O}(3) - 6\tau$. Then the moduli space of stable sheaves in $\mathfrak{M}(3, 1)$ with $\mathsf{M}|_E \supset (p_1, \dots, p_9)$ is canonically isomorphic to the blowup of $\mathbb{P}^2$ in the images of $p_1, \dots, p_9$ under the embedding $E \to \mathbb{P}^2$ coming from $\mathcal{L}_1 \sim \mathcal{O}(1) - 3\tau$.*

*Proof.* We may view the coefficients $v_1$, $v_2$, $w_1$, and $w_2$ as global sections of line bundles on $E$; to be precise,

$$v_1, v_2 \in \operatorname{Hom}(\mathsf{A}(-2), \mathsf{A}(-1)) \cong \operatorname{Hom}(\mathsf{B}(-2), \mathsf{B}(-1)) \cong H^0(\mathcal{L}_1)$$

and

$$w_1, w_2 \in \operatorname{Hom}(\mathsf{A}(-2), \mathsf{A}) \cong \operatorname{Hom}(\mathsf{B}(-2), \mathsf{B}) \cong H^0(\mathcal{L}_0 \otimes \mathcal{L}_1).$$

Now, $H^0(\mathcal{L}_1)$ is 3-dimensional, and $v_1$ and $v_2$ are linearly independent by stability, and we thus obtain a morphism from the moduli space of stable sheaves with the above presentation to $\mathbb{P}^2$. Note also that, in this identification, the constraint on $\partial\mathsf{M}$ reduces to a requirement that

$$w_1 v_2 - w_2 v_1 \in H^0((\mathcal{L}_0 \otimes \mathcal{L}_1) \otimes \mathcal{L}_1)$$

vanish at $p_1, \dots, p_9$.

We need to show that this morphism has 0-dimensional fibers except over the points $p_1, \dots, p_9$, where the fiber is $\mathbb{P}^1$; this together with smoothness will imply the identification with the blowup.

Note that a point $p \in E$ corresponds to the subspace

$$H^0(\mathcal{L}_1(-p)) \subset H^0(\mathcal{L}_1)$$

and thus the cases to consider are those in which $v_1$ and $v_2$ have no common zero, those in which they have a single common zero not of the form $p_i$, and those in which they have a common zero at $p_i$. The key fact is the following statement about global sections of line bundles on elliptic curves.

**Lemma 3.** *Let $v_1$ and $v_2$ be linearly independent global sections of $\mathscr{L}_1$. Then the map*

$$H^0(\mathscr{L}_0 \otimes \mathscr{L}_1)^2 \to H^0(\mathscr{L}_0 \otimes \mathscr{L}_1^2)$$

*given by $(w_1, w_2) \mapsto v_2 w_1 - v_1 w_2$ is surjective if $v_1$ and $v_2$ have no common zero and otherwise has image of codimension $1$, consisting of those global sections vanishing at said common zero.*

*Proof.* Consider the complex

$$0 \to H^0(\mathscr{L}_0) \to H^0(\mathscr{L}_0 \otimes \mathscr{L}_1)^2 \to H^0(\mathscr{L}_0 \otimes \mathscr{L}_1^2) \to 0,$$

with the left map being $x \mapsto (v_1 x, v_2 x)$. This has Euler characteristic $3-6-6+9=0$ and is exact on the left, so it will suffice to understand the middle cohomology. Now, the kernel of the above determinant map consists of pairs $(w_1, w_2)$ with $v_2 w_1 = v_1 w_2$. Assuming neither $w_1$ nor $w_2$ is 0 (which would clearly imply $w_1 = w_2 = 0$), we find that

$$\operatorname{div} v_2 + \operatorname{div} w_1 = \operatorname{div} v_1 + \operatorname{div} w_2.$$

If $v_1$ and $v_2$ have no common zero, we conclude that

$$\operatorname{div} w_1 - \operatorname{div} v_1 = \operatorname{div} w_2 - \operatorname{div} v_2$$

is an effective divisor, and thus,

$$w_1/v_1 = w_2/v_2 \in H^0(\mathscr{L}_0).$$

But this gives exactness in the middle.

Similarly, if $v_1$ and $v_2$ both vanish at $p$, then the same reasoning shows that

$$w_1/v_1 = w_2/v_2 \in H^0(\mathscr{L}_0(p)).$$

In particular, we find that the middle cohomology has dimension at most 1; since the right map is clearly not surjective in this case, its image must therefore have codimension 1 as required. □

In particular, if $v_1$ and $v_2$ have no common zero, the map from pairs $(w_1, w_2)$ to the corresponding determinant is surjective, and thus, there is a unique pair $(w_1, w_2)$ up to equivalence compatible with the constraints on $\partial \mathsf{M}$. If $v_1$ and $v_2$ have a common zero not of the form $p_i$, then the map fails to be surjective, and the only allowed determinant is 0. We thus obtain a 1-dimensional space of possible pairs (modulo multiples of $(v_1, v_2)$), giving rise to a single equivalence class of stable sheaves. Finally, if $v_1$ and $v_2$ have a common zero at $p_i$, then the 1-dimensional space of allowed determinants pulls back to a 2-dimensional space of pairs $(w_1, w_2)$ modulo multiples of $(v_1, v_2)$ and thus gives rise to a $\mathbb{P}^1$-worth of stable sheaves.

It remains to show smoothness. The tangent space to a sheaf with presentation $v_1 f + w_1 g = v_2 f + w_2 g = 0$ consists of the set of quadruples $(v_1', v_2', w_1', w_2')$ such that

$$v_1' w_2 - v_2' w_1 + v_1 w_2' - v_2 w_1'$$

vanishes at $p_1, \dots, p_9$. (More precisely, it is the quotient of this space by the space of trivial deformations, induced by infinitesimal automorphisms of $\mathsf{A}(-2)^2$ and $\mathsf{A}(-1) \oplus \mathsf{A}$; stability implies that the trivial subspace has dimension $4+5-1=8$, independent of $\mathsf{M}$.) It will thus suffice to show that this surjects onto $H^0(\mathscr{L}_0 \otimes \mathscr{L}_1^2)$ since then the dimension will be independent of $\mathsf{M}$ (and equal to $18-8-(9-1)=2$). If $v_1$ and $v_2$ have no common zero, this follows directly from the lemma. If they have a common zero at $p$, but $v_1 w_2 - v_2 w_1 \neq 0$, then since $p$ is equal to at most one $p_i$, we conclude that one of $w_1$ or $w_2$ must not vanish at $p$, so again the lemma gives surjectivity.

Finally, if $v_1$, $v_2$, $w_1$, and $w_2$ all vanish at $p$, then $v_1 w_2 - v_2 w_1 = 0$. But then we again find as in the lemma that

$$w_1/v_1 = w_2/v_2 \in H^0(\mathscr{L}_0),$$

and this violates stability. ☐

**Remark.** Presumably this result could be extended to remove the constraint that the base points are distinct, except that the blowup will no longer be smooth since the base locus of the blowup is then singular.

We also wish to understand how the action of the affine Weyl group $\Lambda_0 \cong \tilde{A}_8$ translates to this explicit description of the moduli space. The action of $S_9$ is essentially trivial as this simply permutes the points $p_1, \dots, p_9$. It remains to consider the generator $s_0$. This can be performed in two steps: first shift down $p_1$ to obtain a sheaf with Euler characteristic 0 and $\partial \mathsf{M} = (p_2, \dots, p_9, \tau^{-3}(p_1))$, and then shift up $p_9$ to obtain a sheaf with Euler characteristic 1 and $\partial \mathsf{M} = (\tau^3(p_9), p_2, \dots, p_8, \tau^{-3}(p_1))$ as required.

Let $\mathbb{O}_{p_1}$ have the free resolution

$$0 \to \mathsf{A} \xrightarrow{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} \mathsf{A}(-1)^{\oplus 2} \xrightarrow{\begin{bmatrix} l_1 & l_2 \end{bmatrix}} \mathsf{A} \to \mathbb{O}_{p_1} \to 0.$$

Then there are two cases to consider. If $\langle v_1, v_2 \rangle \neq \langle l_1, l_2 \rangle$, then we may choose our generators $f$ and $g$ of $\mathsf{M}$ in such a way that $g$ maps to 0 in $\mathbb{O}_{p_1}$. Then we have relations of the form

$$(r_{11}l_1+r_{21}l_2)f+v_1g=(r_{21}l_1+r_{22}l_2)f+v_2g=0,$$

and the submodule generated by $l_1f$, $l_2f$, and $g$ has a presentation of the form

$$0\to \mathsf{A}(-2)^3\xrightarrow{L}\mathsf{A}(-1)^3\to \mathsf{M}_{p_1}$$

with

$$L=\begin{bmatrix} u_1 & r_{11} & r_{12}\\ u_2 & r_{21} & r_{22}\\ 0 & v_1 & v_2\end{bmatrix}.$$

Generically, $\det L$ has divisor $\tau^{-3}(p_1)+p_2+\cdots+p_9$ and thus has rank 2 at $p_9$. Then suitable row and column operations recover a new matrix of the above form except with $u_1'$ and $u_2'$ vanishing at $p_9$; thus, $\mathsf{M}_{p_1}\cong \mathsf{M}'_{\tau^3(p_9)}$ for suitable $\mathsf{M}'$ as required.

This can fail in only two ways: either the rank of $L(p_9)$ could be smaller than 2, or the corresponding column could have rank only 1. Suppose first that $\det L\neq 0$. As long as $\tau^{-3}(p_1)\neq p_9$, we find that $p_9$ is a simple zero of $\det L$, so the rank cannot drop below 2. If after the row and column operations we find that $u_1'$ and $u_2'$ are linearly dependent, then we find that $\det L$ must factor as a product $uv$ with $u\in H^0(\mathscr{L}_1(-p_9))$ and $v\in H^0(\mathscr{L}_1^2)$. In particular, this can only happen if we have

$$\mathscr{L}_1(p_i+p_j+p_9)\cong \mathcal{O}_E$$

for some $2\le i<j\le 8$ or

$$\mathscr{L}_1(\tau^{-3}p_1+p_i+p_9)\cong \mathcal{O}_E$$

for some $2\le i\le 8$.

If $\det L=0$ on $E$, we may consider the cubic polynomial obtained by viewing $L$ as a matrix over $\mathbb{P}^2$, and observe that this must be the equation of $E$. In particular, since $E$ is smooth, we still cannot have rank $<2$ at any point, and since $E$ is irreducible, $L$ cannot be made block-upper-triangular.

We thus conclude that, as long as the points

$$\tau^{-3}(p_1),\, p_1,\, p_2,\, \dots,\, p_9,\, \tau^3(p_9)$$

are all distinct and no three add to a divisor representing $\mathscr{L}_1$, then $s_0$ induces a morphism between the complement of the fiber over $p_1$ in the original moduli space and the complement of the fiber over $\tau^3(p_9)$ in the new moduli space.

It remains to consider the fiber over $p_1$. In this case, we note that $f$ maps to 0 in $\mathcal{O}_{p_1}$, and thus, we can no longer proceed as above. In a suitable basis, the relations of $\mathsf{M}$ now read

$$r_1f+l_1g=r_2f+l_2g=0,$$

and thus, $\mathsf{M}_{p_1}$ is generated by $f$, with the single relation

$$(u_1r_1 + u_2r_2)f = 0.$$

In particular, we obtain a sheaf with presentation of the form

$$0 \to \mathsf{A}(-3) \to \mathsf{A} \to \mathsf{M}_{p_1} \to 0.$$

It remains only to show that every sheaf with such a presentation arises in this way and that we can recover the original sheaf from the presentation.

**Lemma 4.** *Let* $p \in E$ *be any point, cut out by linear equations* $l_1 = l_2 = 0$*. Then the central element* $\mathsf{E} \in \mathsf{A}_3$ *can be expressed in the form*

$$\mathsf{E} = l_1 f_1 + l_2 f_2$$

*with* $f_1, f_2 \in \mathsf{A}_2$*, and this expression is unique up to adding a pair*

$$(v_1 g, v_2 g)$$

*to* $(f_1, f_2)$*, where* $v_1, v_2, g \in \mathsf{A}_1$ *satisfy* $l_1 v_2 + l_2 v_2 = 0$*.*

*Proof.* Consider the map

$$\mathsf{A}_2^2 \to \mathsf{A}_3$$

given by $(f_1, f_2) \mapsto l_1 f_1 + l_2 f_2$. Since $(l_1, l_2)$ cuts out a point module, the image of this map must be codimension-1, and since $\mathsf{E}$ annihilates every point module, the image contains $\mathsf{E}$. Uniqueness follows by dimension-counting since the specified kernel is 3-dimensional. □

We conclude that, for any element of $\Lambda_0$, there is a finite collection of linear inequalities on the base points, which guarantee that both the domain and range are blowups of $\mathbb{P}^2$, and the element of $\Lambda_0$ acts as a morphism. In particular, we see that it suffices to have

$$\tau^{3k} p_i \neq p_j$$

for $i < j$, $k \in \mathbb{Z}$, and

$$\mathcal{L}_{3l+1} \not\cong \mathcal{O}_E(p_i + p_j + p_k)$$

for $i < j < k$, $l \in \mathbb{Z}$, in order for the entire group $\Lambda_0$ to act as isomorphisms between blowups of $\mathbb{P}^2$.

In particular, we find that the translation subgroup of $\Lambda_0$ acts in the same way as the translation subgroup of $\widetilde{E}_8$ in Sakai's description of the elliptic Painlevé equation. Note that both groups have the same rank, and by comparing determinants under the intersection form in the commutative limit, we conclude that the translation subgroup of $\Lambda_0$ has index 3 in the translation subgroup of $\widetilde{E}_8$. It is straightforward to see that we can generate the entire lattice by including the operation

$$\mathsf{M} \mapsto g^3 \mathsf{M}(1),$$

though it is more difficult to see how this acts in terms of presentations of sheaves.

We note in passing that the above calculation of $\mathsf{M}_{p_i}$ shows that the $-1$-curve corresponding to the fiber over $p_i \in \mathbb{P}^2$ can be described as the subscheme of moduli space where the sheaf $\mathsf{M}_{p_i}$ of Euler characteristic 0 has a global section; this should be compared to the cohomological description of $\tau$-divisors in [Arinkin and Borodin 2009]. In fact, every $-1$-curve on the moduli space has a similar description: act by a suitable element of $\Lambda_{E_8}$, and then ask for the Hecke modification at $p_1$ to have a global section.

We also note that the results of [Rains 2013b, Theorem 7.1] suggest that one should consider the moduli space of stable sheaves $\mathsf{M}$ on $\mathsf{A}$ such that $h(\mathsf{M}) = 3rt + r$ and

$$\mathsf{M}|_E = (\mathbb{O}_{p_1} \oplus \cdots \oplus \mathbb{O}_{p_9})^r,$$

where now $p_1 + \cdots + p_9 - \mathbb{O}(3) + 6\tau$ is a torsion point of order $r$. This moduli space remains 2-dimensional and is expected to again be a 9-point blowup of $\mathbb{P}^2$. (A variant of this will be considered in [Rains $\geq$ 2015].)

## 7. Poisson structures

**7.1.** The Poisson structure on the moduli space of sheaves on a commutative Poisson surface has a purely categorical definition (originally constructed by [Tyurin 1988], shown to satisfy the Jacobi identity for vector bundles in [Bottacin 1995], and extended to general sheaves of homological dimension 1 in [Hurtubise and Markman 2002]). This definition can be carried over to the noncommutative case, and we will see that the analogous bivector again gives a Poisson structure, and the Hecke modifications again act as symplectomorphisms. The main qualitative difference in the noncommutative case is (as we have seen) that the Hecke modifications are no longer automorphisms of a given symplectic leaf but rather give maps between related symplectic leaves.

Tyurin's construction in the commutative setting relies on the observation that the tangent space at a sheaf $M$ on a Poisson surface $X$, or equivalently the space of infinitesimal deformations, is given by the self-Ext group

$$\operatorname{Ext}^1(M, M).$$

By Serre duality, the cotangent space is given by

$$\operatorname{Ext}^1(M, M \otimes \omega_X).$$

This globalizes to general sheaves such that $\dim \operatorname{End} M = 1$; the cotangent sheaf on the moduli space is given by $\operatorname{Ext}^1(M, M \otimes \omega_X)$, where $M$ is the universal sheaf on the moduli space. A nontrivial Poisson structure on $X$ corresponds to a nonzero morphism $\wedge^2 \Omega_X \to \mathcal{O}_X$ or equivalently to a nonzero morphism $\alpha : \omega_X \to \mathcal{O}_X$. We thus obtain a map

$$\operatorname{Ext}^1(M, M \otimes \omega_X) \xrightarrow{1 \otimes \alpha} \operatorname{Ext}^1(M, M)$$

and a bilinear form on $\operatorname{Ext}^1(M, M \otimes \omega_X)$. One can then show [Bottacin 1995; Hurtubise and Markman 2002] that this bilinear form induces a Poisson structure. In addition, the resulting Poisson variety has a natural foliation by *algebraic* symplectic leaves: if $C_\alpha$ is the curve $\alpha = 0$, then for any sheaf $M_\alpha$ on $C_\alpha$, the (Poisson) subspace of sheaves $M$ with $M \otimes \mathcal{O}_{C_\alpha} \cong M_\alpha$ and $\operatorname{Tor}_1(M, \mathcal{O}_{C_\alpha}) = 0$ is a smooth symplectic leaf.

In our noncommutative setting, there is again an analogue of Serre duality; one finds that $H^2(\mathsf{A}(-3)) \cong \mathbb{C}$ (just as in the commutative case), and for any $\mathsf{M}$ and $\mathsf{M}'$, we have canonical pairings

$$\langle -, - \rangle : \operatorname{Ext}^i(\mathsf{M}', \mathsf{M}(-3)) \otimes \operatorname{Ext}^{2-i}(\mathsf{M}, \mathsf{M}') \to H^2(\mathsf{A}(-3)).$$

Moreover, these pairings are (super)symmetric in the following sense. If $\alpha \in \operatorname{Ext}^i(\mathsf{M}', \mathsf{M}(-3))$ and $\beta \in \operatorname{Ext}^{2-i}(\mathsf{M}, \mathsf{M}')$, then

$$\langle \alpha, \beta \rangle = (-1)^i \langle \beta(-3), \alpha \rangle = (-1)^i \langle \beta, \alpha(3) \rangle.$$

In addition, the pairing factors through the Yoneda product in that

$$\langle \alpha, \beta \rangle = \langle \alpha \cup \beta, 1 \rangle =: \operatorname{tr}(\alpha \cup \beta).$$

As in the commutative case, infinitesimal deformations of $\mathsf{M}$ are classified by $\operatorname{Ext}^1(\mathsf{M}, \mathsf{M})$, and the map

$$\operatorname{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \xrightarrow{E \cup -} \operatorname{Ext}^1(\mathsf{M}, \mathsf{M})$$

induces a skew-symmetric pairing

$$\operatorname{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \otimes \operatorname{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \xrightarrow{\operatorname{tr}(- \cup E \cup -)} H^2(\mathsf{A}(-3)) \cong \mathbb{C},$$

and this should be our desired Poisson structure.

Note here that the Poisson structure depends on the choice of $E$ and the choice of automorphism $H^2(\mathsf{A}(-3)) \cong \mathbb{C}$; both are unique up to a scalar, and only the product of the scalars matters. The cohomology long exact sequence associated with

$$0 \to \mathsf{A}(-3) \xrightarrow{E} \mathsf{A} \to \mathsf{B} \to 0$$

induces (since $H^1(\mathsf{A}) = H^2(\mathsf{A}) = 0$) an isomorphism

$$H^1(\mathsf{B}) \cong H^2(\mathsf{A}(-3)),$$

depending linearly on $E$, so that the composition

$$H^1(\mathsf{B}) \cong H^2(\mathsf{A}(-3)) \cong \mathbb{C}$$

scales in the same way as the Poisson structure. In other words, the scalar freedom in the Poisson structure corresponds to a choice of isomorphism $H^1(\mathsf{B}) \cong \mathbb{C}$; the canonical equivalence $\operatorname{Tails} \mathsf{B} \cong \operatorname{Coh} E$ turns this into an isomorphism $H^1(\mathbb{O}_E) \cong \mathbb{C}$ or equivalently a choice of nonzero holomorphic differential on $E$.

We will see that this construction remains Poisson in the noncommutative setting and that the description of the symplectic leaves carries over mutatis mutandis. Note that, in this section, we will refer to the "moduli space of simple sheaves on $\mathsf{A}$", where a sheaf is *simple* if $\operatorname{End} \mathsf{M} = \mathbb{C}$ (in the commutative setting, this is a weakened form of the constraint that a sheaf is stable). One expects following [Altman and Kleiman 1980] that this should be a quasiseparated algebraic space $\mathfrak{M}_{\mathsf{A}}$. Per [Rains 2013a], a Poisson structure on such a space is just a compatible system of Poisson structures on the domains of étale morphisms to the space; in the moduli space setting, we must thus assign a Poisson structure to every formally universal family of simple sheaves on $\mathsf{A}$. The above bivector is clearly compatible so will be Poisson if and only if it is Poisson on every formally universal family. Any statement below about $\mathfrak{M}_{\mathsf{A}}$ should be interpreted as a statement about formally universal families in this way.

We will sketch two proofs of the following result below.

**Theorem 4.** *The above construction defines a Poisson structure on* $\mathfrak{M}_{\mathsf{A}}$, *and on the open subspace of sheaves transverse to* $E$ (*i.e.*, *such that* $\operatorname{Tor}_1(\mathsf{M}, \mathsf{B}) = 0$), *the fibers of the map* $\mathsf{M} \to \mathsf{M} \otimes \mathsf{B} \in \operatorname{Coh} E$ *are unions of* (*smooth*) *symplectic leaves of this Poisson structure.*

**Remark.** One expects that, as in [Rains 2013a], one should have a covering by algebraic symplectic leaves even without the transversality assumption; in general, the symplectic leaves should be the preimages of the *derived* restriction $\mathsf{M} \to \mathsf{M} \otimes^{\mathsf{L}} \mathsf{B}$, taking sheaves on $\mathsf{A}$ to the derived category of $\operatorname{Coh} E$.

We should note here that, in the case of $\mathsf{M}$ torsion-free (and stable), an alternate construction of a Poisson structure was given in [Nevins and Stafford 2007]; their Poisson structure is presumably a constant multiple of the Tyurin-style Poisson structure.

**7.2.** Although the above construction is somewhat difficult to deal with computationally (but see below), it has significant advantages in terms of functoriality.

In particular, it is quite straightforward to show that Hecke modifications give symplectomorphisms on the relevant symplectic leaves. Curiously, the argument ends up depending crucially on noncommutativity!

With an eye to future applications, we consider a generalization of Hecke modifications as follows. Let $\mathsf{M}$ be a simple 1-dimensional sheaf on $\mathsf{A}$. We define the "downward pseudotwist" at $p \in E$ of $\mathsf{M}$ to be the kernel of the natural map $\mathsf{M} \to \mathsf{M} \otimes \mathcal{O}_p$; similarly, the "upward pseudotwist" is the universal extension of $\mathcal{O}_p \otimes \operatorname{Ext}^1(\mathcal{O}_p, \mathsf{M})$ by $\mathsf{M}$. If the restriction $\mathsf{M}|_E$ of $\mathsf{M}$ to $E$ (i.e., $\mathsf{M} \otimes \mathsf{B}$, viewed as a sheaf on $E$) is not equal to the sum over $p$ of $\mathsf{M} \otimes \mathcal{O}_p$, then one could consider some other natural modifications along these lines; in the commutative case, these correspond to twists by line bundles on iterated blowups in which we have blown up the same point on $E$ multiple times. These will always be limits of the above operations so will again be symplectic by the limiting argument considered below.

**Proposition 6.** *The two pseudotwists define* (*inverse*) *birational maps between symplectic leaves of the open subspace of* $\mathfrak{M}_{\mathsf{A}}$ *classifying* 1*-dimensional sheaves transverse to* $E$. *Where the maps are defined*, *they are symplectic.*

**Remark.** Note that we need to merely prove that the morphisms preserve the above bivector; this can be verified independently of whether the bivector satisfies the Jacobi identity. In addition, it suffices to prove that the pseudotwists are Poisson on suitable open subsets of the moduli space.

*Proof.* We first consider the downward pseudotwist $\mathsf{M}'$ of $\mathsf{M}$, corresponding to the short exact sequence

$$0 \to \mathsf{M}' \to \mathsf{M} \to \mathsf{M} \otimes \mathcal{O}_p \to 0.$$

We impose the additional conditions that

$$\operatorname{Hom}(\mathsf{M}', \mathcal{O}_p) = \operatorname{Hom}(\mathsf{M}, \mathcal{O}_p(-3)) = 0.$$

Observe that this is really just a condition on the commutative sheaf $\mathsf{M}|_E$, stating that it is 0 near $\tau^{-3}(p)$ and near $p$ is a sum of copies of $\mathcal{O}_p$. Indeed, the first condition is precisely that $\operatorname{Hom}(\mathsf{M}, \mathcal{O}_p(-3)) = 0$ and implies $\operatorname{Tor}_1(\mathsf{M}, \mathsf{B}) = 0$ while the second condition follows from the 4-term exact sequence

$$0 \to (\mathsf{M} \otimes \mathcal{O}_p)(-3) \to \mathsf{M}'|_E \to \mathsf{M}|_E \to \mathsf{M} \otimes \mathcal{O}_p \to 0.$$

Note that, if $\tau^{-3}(p) = p$, then the above conditions imply $\mathsf{M}' \cong \mathsf{M}$ and thus eliminate any interesting examples of pseudotwists. Of course, since $E$ is smooth, $\tau^{-3}(p) = p$ if and only if $\tau^3 = 1$, and this is equivalent to the existence of an equivalence $\operatorname{Tails} \mathsf{A} \cong \operatorname{Coh} \mathbb{P}^2$. Away from the commutative case, the conditions are not particularly hard to satisfy; in particular, the generic sheaf in any component of the moduli space of 1-dimensional sheaves will satisfy this condition at every point of $E$.

By Serre duality, we have $\mathrm{Ext}^2(\mathsf{M}, \mathbb{O}_p(-3)) \cong \mathrm{Hom}(\mathbb{O}_p, \mathsf{M}) = 0$ and similarly $\mathrm{Ext}^2(\mathsf{M}', \mathbb{O}_p) = 0$. It then follows by an Euler characteristic calculation that $\mathrm{Ext}^1(\mathsf{M}, \mathbb{O}_p(-3)) = \mathrm{Ext}^1(\mathsf{M}', \mathbb{O}_p) = 0$ as well. Since $\mathsf{M} \otimes \mathbb{O}_p$ is a sum of copies of $\mathbb{O}_p$, we find that the natural maps

$$\mathrm{Ext}^i(\mathsf{M}', \mathsf{M}') \to \mathrm{Ext}^i(\mathsf{M}', \mathsf{M}),$$
$$\mathrm{Ext}^i(\mathsf{M}, \mathsf{M}'(-3)) \to \mathrm{Ext}^i(\mathsf{M}, \mathsf{M}(-3))$$

are isomorphisms. (In particular, $\mathsf{M}'$ is simple if and only if $\mathsf{M}$ is simple.) By Serre duality, the same applies to

$$\mathrm{Ext}^i(\mathsf{M}, \mathsf{M}) \to \mathrm{Ext}^i(\mathsf{M}', \mathsf{M}),$$
$$\mathrm{Ext}^i(\mathsf{M}, \mathsf{M}'(-3)) \to \mathrm{Ext}^i(\mathsf{M}', \mathsf{M}'(-3)).$$

By the functoriality of Ext, we find that the compositions

$$\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}'(-3)) \cong \mathrm{Ext}^1(\mathsf{M}', \mathsf{M}'(-3)) \xrightarrow{E} \mathrm{Ext}^1(\mathsf{M}', \mathsf{M}') \cong \mathrm{Ext}^1(\mathsf{M}', \mathsf{M})$$

and

$$\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}'(-3)) \cong \mathrm{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \xrightarrow{E} \mathrm{Ext}^1(\mathsf{M}, \mathsf{M}) \cong \mathrm{Ext}^1(\mathsf{M}', \mathsf{M})$$

agree, and thus, the induced isomorphism

$$\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}) \cong \mathrm{Ext}^1(\mathsf{M}', \mathsf{M}')$$

respects the Poisson structure.

It remains only to show that this isomorphism is the differential of the pseudotwist. Note that the pseudotwist is only a morphism on the strata of the moduli space with fixed $\dim \mathrm{Hom}(\mathsf{M}, \mathbb{O}_p)$. Thus, we only need to consider those classes in $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M})$ that preserve this dimension. In other words, we must consider extensions

$$0 \to \mathsf{M} \to \mathsf{N} \to \mathsf{M} \to 0$$

that remain exact when tensored with $\mathbb{O}_p$. Then the corresponding extension $\mathsf{N}'$ of $\mathsf{M}'$ by $\mathsf{M}'$ is the kernel of the natural map $\mathsf{N} \to \mathsf{N} \otimes \mathbb{O}_p$. That both extensions have the same image in $\mathrm{Ext}^1(\mathsf{M}', \mathsf{M})$ follows from exactness of the complex

$$0 \to \mathsf{M}' \to \mathsf{M} \oplus \mathsf{N}' \to \mathsf{N} \otimes \mathsf{M}' \to \mathsf{M} \to 0$$

(the two extensions are the cokernel of the map from $\mathsf{M}'$ and the kernel of the map to $\mathsf{M}$), and this is the total complex of a double complex with exact rows.

Note that we also have $\mathrm{Ext}^*(\mathbb{O}_p, \mathsf{M}) = 0$, and thus, the connecting map

$$\mathrm{Hom}(\mathbb{O}_p, \mathsf{M} \otimes \mathbb{O}_p)) \to \mathrm{Ext}^1(\mathbb{O}_p, \mathsf{M}')$$

is an isomorphism, implying that $\mathsf{M}$ is the upward pseudotwist of $\mathsf{M}'$. Since we can restate the conditions on $\mathsf{M}$ and $\mathsf{M}'$ in terms of $\mathsf{M}'|_C$, we find that the upward pseudotwist is also Poisson.

In fact, the hypotheses on $\mathsf{M}$ and $\mathsf{M}'$ are significantly stronger than necessary. The point is that, once we constrain $\dim \operatorname{Hom}(\mathsf{M}, \mathbb{O}_p)$, the further constraints in the above argument are dense open conditions. If we replace this by the weaker open condition that $\mathsf{M}'$ is simple, we still obtain a morphism between Poisson spaces. The failure of such a morphism to be Poisson is measured by a form on the cotangent sheaf, which by the above argument vanishes on a dense open subset and thus vanishes identically. □

**Remark.** This limiting argument also lets us deduce the commutative case from the noncommutative case, though in the commutative setting we can also use an interpretation involving twists on blowups [Rains 2013a]; this actually works for arbitrary sheaves of homological dimension 1, and presumably the same holds in the noncommutative setting. The above argument fails for torsion-free sheaves, however, as does the fact that the upward and downward pseudotwists are inverse to each other.

**7.3.** We now turn our attention to showing that the above actually defines a Poisson structure, i.e., that the corresponding biderivation on the structure sheaf satisfies the Jacobi identity. Unfortunately, the existing arguments in the commutative setting involve working with explicit Čech cocycles for extensions of vector bundles; while both Čech cocycles and vector bundles have noncommutative analogues, neither is particularly easy to compute with. It turns out, however, that in many cases we can reduce the computation of the pairing to a computation on the commutative curve $E$. (In fact, combined with the construction of [Hurtubise and Markman 2002], this is enough to verify the Jacobi identity in general.)

We assume here that $\mathsf{M}$ is a simple sheaf transverse to $E$; we also assume $\mathsf{M}/E\mathsf{M} \neq 0$. (In our case, we could equivalently just assume $\mathsf{M} \neq 0$, but this is the form in which the condition appears below; for commutative surfaces, the two conditions are not equivalent, and the conditions are likely to deviate from each other for more general noncommutative surfaces as well.) The map giving the Poisson structure then fits into a long exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(\mathsf{M}, \mathsf{M}) \to \operatorname{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \to \operatorname{Ext}^1(\mathsf{M}, \mathsf{M}(-3)) \\ \to \operatorname{Ext}^1(\mathsf{M}, \mathsf{M}) \to \operatorname{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \to \operatorname{Ext}^2(\mathsf{M}, \mathsf{M}(-3)) \to 0, \end{aligned}$$

where $\operatorname{Hom}(\mathsf{M}, \mathsf{M}(-3)) \subset \operatorname{Hom}(\mathsf{M}, \mathsf{M})$ is trivial since $\operatorname{Hom}(\mathsf{M}, \mathsf{M}) \cong \mathbb{C}$ injects in $\operatorname{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M})$, and $\operatorname{Ext}^2(\mathsf{M}, \mathsf{M}) = 0$ by duality. Now,

$$\operatorname{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \cong \operatorname{Hom}_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M})$$

and may thus be computed entirely inside $\operatorname{Coh} E$ so via commutative geometry. Since the sequence is essentially self-dual, we should also expect to have $\operatorname{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \cong \operatorname{Ext}^1_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M})$. We can make this explicit as follows: a class in $\operatorname{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M})$ is represented by an extension

$$0 \to \mathsf{M}/E\mathsf{M} \to \mathsf{N} \to \mathsf{M} \to 0,$$

and since $\mathsf{M}$ is $\mathsf{B}$-flat, this induces an extension

$$0 \to \mathsf{M}/E\mathsf{M} \to \mathsf{N}/E\mathsf{N} \to \mathsf{M}/E\mathsf{M} \to 0,$$

and pulling back recovers the original extension. Conversely, any extension of $\mathsf{M}/E\mathsf{M}$ by $\mathsf{M}/E\mathsf{M}$ over $\mathsf{B}$ can be viewed as an extension of $\mathsf{M}/E\mathsf{M}$ by $\mathsf{M}/E\mathsf{M}$ in $\operatorname{Tails} \mathsf{A}$ and pulled back to an extension of $\mathsf{M}$ by $\mathsf{M}/E\mathsf{M}$ that restricts back to the original extension. In other words, "tensor with $\mathsf{B}$" and "pull back" give inverse maps as required.

Since the map $R\operatorname{Hom}(\mathsf{M}, \mathsf{M}(-3)) \to R\operatorname{Hom}(\mathsf{M}, \mathsf{M})$ in the derived category is self-dual (subject to our choice of isomorphism $H^2(\mathsf{A}(-3)) \cong \mathbb{C}$), it follows that the corresponding exact triangle is self-dual and thus that the remaining maps

$$R\operatorname{Hom}(\mathsf{M}, \mathsf{M}) \to R\operatorname{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \cong R\operatorname{Hom}_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M})$$

and

$$R\operatorname{Hom}_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M}) \cong R\operatorname{Hom}(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \to R\operatorname{Hom}(\mathsf{M}, \mathsf{M}(-3))[1]$$

in the exact triangle are dual. In particular, it follows that we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M}) \arrow[r, "\sim"] \arrow[d] & \operatorname{Ext}^1_{\mathsf{B}}(\mathsf{M}/E\mathsf{M}, \mathsf{M}/E\mathsf{M}) \arrow[r, "\mathrm{tr}"] & H^1(\mathcal{O}_E) \arrow[d, "\wr"] \\
\operatorname{Ext}^2(\mathsf{M}, \mathsf{M}(-3)) \arrow[r, "\mathrm{tr}"] & H^2(\mathsf{A}(-3)) \arrow[r, "\sim"] & \mathbb{C}
\end{tikzcd}$$

Since the map from $\operatorname{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M})$ to $\operatorname{Ext}^2(\mathsf{M}, \mathsf{M}(-3))$ is surjective, to compute the trace of any class in $\operatorname{Ext}^2(\mathsf{M}, \mathsf{M}(-3))$, we need to simply choose a preimage in $\operatorname{Ext}^1(\mathsf{M}, \mathsf{M}/E\mathsf{M})$, interpret it as an extension of sheaves on the commutative curve $E$, and take the trace there.

Since we only need to consider classes in $\operatorname{Ext}^2(\mathsf{M}, \mathsf{M}(-3))$ that arise via the Yoneda product, it will be particularly convenient to use the Yoneda interpretation of such classes via 2-extensions. If $\mathsf{N}'$ is an extension of $\mathsf{M}$ by $\mathsf{M}$ and $\mathsf{N}$ is an extension of $\mathsf{M}$ by $\mathsf{M}(-3)$, then $\mathsf{N} \cup \mathsf{N}'$ is represented by the 2-extension

$$0 \to \mathsf{M}(-3) \to \mathsf{N} \to \mathsf{N}' \to \mathsf{M} \to 0, \tag{26}$$

where $\mathsf{N} \to \mathsf{N}'$ is the composition $\mathsf{N} \to \mathsf{M} \to \mathsf{N}'$. Recall that two 2-extensions are equivalent if and only if the complexes $\mathsf{N} \to \mathsf{N}'$ are quasi-isomorphic. The functoriality of $\operatorname{Ext}^2(-,-)$ is expressed via pullback and pushforward, as appropriate; the connecting maps are more complicated but are again amenable to explicit description [Mitchell 1965].

In our case, we have the following. The pushforward of (26) under the map $\mathsf{M}(-3) \xrightarrow{\mathsf{E}} \mathsf{M}$ has the form

$$0 \to \mathsf{M} \to \mathsf{N}'' \to \mathsf{N}' \to \mathsf{M} \to 0, \tag{27}$$

where $\mathsf{N}'' \cong (\mathsf{N} \oplus \mathsf{M})/\mathsf{M}(-3)$. Since $\operatorname{Ext}^2(\mathsf{M},\mathsf{M}) = 0$, this 2-extension is trivial, and thus, there exists a sheaf $\mathsf{Z}$ and a filtration

$$0 \subset \mathsf{Z}_1 \subset \mathsf{Z}_2 \subset \mathsf{Z}$$

such that the sequence

$$0 \to \mathsf{Z}_1 \to \mathsf{Z}_2 \to \mathsf{Z}/\mathsf{Z}_1 \to \mathsf{Z}/\mathsf{Z}_2 \to 0$$

agrees with (27) or equivalently such that

$$0 \to \mathsf{M} \to \mathsf{N}' \to \mathsf{M} \to 0$$

is the pushforward under $\mathsf{N}'' \to \mathsf{M}$ of an extension

$$0 \to \mathsf{N}'' \to \mathsf{Z} \to \mathsf{M} \to 0.$$

It follows that the 2-extension (26) is equivalent to

$$0 \to \mathsf{M}(-3) \to \mathsf{Z}' \to \mathsf{Z} \to \mathsf{M} \to 0,$$

where $\mathsf{Z}'$ is the pullback of $\mathsf{N}$ under $\mathsf{N}'' \to \mathsf{M}$. Now, since $\mathsf{N}''$ was itself obtained by pushing $\mathsf{N}$ forward, we have $\mathsf{N} \subset \mathsf{N}''$ in a natural way, giving $\mathsf{N} \subset \mathsf{Z}', \mathsf{Z}$ in compatible ways. Quotienting by this gives an equivalent 2-extension

$$0 \to \mathsf{M}(-3) \xrightarrow{\mathsf{E}} \mathsf{M} \to \mathsf{Z}/\mathsf{N} \to \mathsf{M} \to 0,$$

expressing (26) as the image under the connecting map of

$$0 \to \mathsf{M}/\mathsf{E}\mathsf{M} \to \mathsf{Z}/\mathsf{N} \to \mathsf{M} \to 0.$$

The corresponding class in $\operatorname{Ext}^1_{\mathsf{B}}(\mathsf{M}/\mathsf{E}\mathsf{M}, \mathsf{M}/\mathsf{E}\mathsf{M})$ is then obtained by tensoring with $\mathsf{B}$:

$$0 \to \mathsf{M}/\mathsf{E}\mathsf{M} \to \mathsf{Z}/(\mathsf{E}\mathsf{Z} + \mathsf{N}) \to \mathsf{M}/\mathsf{E}\mathsf{M} \to 0.$$

It will be helpful to think of this last extension in a slightly different way. Since $\operatorname{Tor}_1(\mathsf{B},\mathsf{M}) = 0$, the quotient $\mathsf{Z}/\mathsf{E}\mathsf{Z}$ inherits a filtration

$$0 \subset \mathsf{M}/\mathsf{E}\mathsf{M} \subset \mathsf{N}''/\mathsf{E}\mathsf{N}'' \subset \mathsf{Z}/\mathsf{E}\mathsf{Z} \to 0,$$

so to compute $\mathsf{Z}/(\mathsf{E}\mathsf{Z}+\mathsf{N})$, we only need to understand the map $\mathsf{N} \to \mathsf{N}''/\mathsf{E}\mathsf{N}''$. Since $\mathsf{M}(-3) \subset \mathsf{N}$ maps to $\mathsf{E}\mathsf{M} \subset \mathsf{E}\mathsf{N}''$, the map $\mathsf{N} \to \mathsf{N}''/\mathsf{E}\mathsf{N}''$ factors through the natural map $\mathsf{N} \to \mathsf{M}$ and then through the quotient map $\mathsf{M} \to \mathsf{M}/\mathsf{E}\mathsf{M}$. In other words, the map $\mathsf{N} \to \mathsf{N}''/\mathsf{E}\mathsf{N}''$ precisely gives a splitting of the short exact sequence

$$0 \to \mathsf{M}/\mathsf{E}\mathsf{M} \to \mathsf{N}''/\mathsf{E}\mathsf{N}'' \to \mathsf{M}/\mathsf{E}\mathsf{M} \to 0.$$

Note in particular that the pairing of $\mathsf{N}$ and $\mathsf{N}'$ depends only on the two extensions $\mathsf{N}'', \mathsf{N}' \in \mathrm{Ext}^1(\mathsf{M}, \mathsf{M})$ and a splitting of $\mathsf{N}''/\mathsf{E}\mathsf{N}''$. We need to simply combine $\mathsf{N}''$ and $\mathsf{N}'$ into a filtered sheaf, quotient by $\mathsf{E}$, then mod out by the submodule $\mathsf{M}/\mathsf{E}\mathsf{M}$ coming from the splitting to obtain the desired extension. Finally, given this resulting extension, we simply compute the trace in the usual commutative algebraic geometry sense. If we were given a splitting of $\mathsf{N}'/\mathsf{E}\mathsf{N}'$, we could instead take the kernel of the resulting map $\mathsf{Z}/\mathsf{E}\mathsf{Z} \to \mathsf{M}/\mathsf{E}\mathsf{M}$; a splitting of both makes $\mathsf{M}/\mathsf{E}\mathsf{M}$ a direct summand.

**7.4.** At this point, we can understand the Poisson structure entirely in terms of extensions of $\mathsf{M}$ by $\mathsf{M}$ together with commutative data; to proceed further, we will need a more explicit description of self-extensions of $\mathsf{M}$. Suppose that $\mathsf{M}$ is given by a presentation

$$0 \to V \xrightarrow{L} W \to \mathsf{M} \to 0,$$

and consider an extension $0 \to \mathsf{M} \to \mathsf{N} \to \mathsf{M} \to 0$.

We first note that, if $\mathrm{Ext}^2(W, V) = 0$, then there exists a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & V \arrow[r, "L"] \arrow[d] & W \arrow[r] \arrow[d] & \mathsf{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V' \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & \mathsf{N} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V \arrow[r, "L"] \arrow[d] & W \arrow[r] \arrow[d] & \mathsf{M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with short exact rows and columns. Indeed, we may pull $\mathsf{N}$ back to an extension of $W$ by $\mathsf{M}$, which is in the kernel of the connecting map $\mathrm{Ext}^1(W, \mathsf{M}) \to \mathrm{Ext}^2(W, V) = 0$ and thus is the pushforward of an extension $W'$, giving a surjective map of short exact sequences, the kernel of which is as required.

If we further have $\mathrm{Ext}^1(V, V) = \mathrm{Ext}^1(W, W) = 0$, then both $V'$ and $W'$ are trivial extensions, and we find that $\mathsf{N}$ has a presentation

$$0 \to V \oplus V \xrightarrow{\begin{bmatrix} L & L' \\ 0 & L \end{bmatrix}} W \oplus W \to \mathsf{N} \to 0.$$

(This corresponds to the deformation $\mathrm{Coker}(L + \epsilon L')$ over $\mathbb{C}[\epsilon]/\epsilon^2$.)

With this in mind, we assume

$$\mathrm{Ext}^2(W, V) = \mathrm{Ext}^1(V, V) = \mathrm{Ext}^1(W, W) = 0$$

so that extensions of $\mathsf{M}$ by $\mathsf{M}$ are represented by maps $L' : V \to W$. (Of course, this representation is by no means unique!) Given two such extensions, it is trivial to construct the desired filtered sheaf: $\mathsf{Z}$ is simply the kernel of the morphism

$$\begin{bmatrix} L & L' & 0 \\ 0 & L & L'' \\ 0 & 0 & L \end{bmatrix} : V^3 \to W^3.$$

(We could equally well take the $(1, 3)$ entry to be an arbitrary map $L''' : V \to W$; this corresponds to the fact that the class in $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M}/\mathsf{E}\mathsf{M})$ we obtain is only determined modulo the image of $\mathrm{Ext}^1(\mathsf{M}, \mathsf{M})$.)

If we further assume that $\mathrm{Tor}_1(\mathsf{B}, W) = 0$, so $\mathrm{Tor}_1(\mathsf{B}, V) = 0$ (and recall we have already assumed $\mathrm{Tor}_1(\mathsf{B}, \mathsf{M}) = 0$), then we have an exact sequence

$$0 \to \mathrm{Hom}(V, W(-3)) \to \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W/\mathsf{E}W)$$

and $\mathrm{Hom}(V, W(-3)) \cong \mathrm{Ext}^2(W, V)^* = 0$, and thus, the extension $L'$ only depends on its restriction to $\mathrm{Hom}(V, W/\mathsf{E}W) \cong \mathrm{Hom}_{\mathsf{B}}(V/\mathsf{E}V, W/\mathsf{E}W)$. (Note that, if we also assumed $\mathrm{Ext}^1(W, V) = 0$, every map in $\mathrm{Hom}_{\mathsf{B}}(V/\mathsf{E}V, W/\mathsf{E}W)$ would come from a deformation, but we will not need this assumption.)

We thus obtain the following, purely commutative construction. Given sheaves (which for our purposes will always be locally free) $V_E$ and $W_E$ on $E$ and an injective morphism $L_E : V_E \to W_E$, say that $L'_E : V_E \to W_E$ is isotrivial if the corresponding deformation of the cokernel is trivial or in other words if the extension

$$\mathrm{Coker} \begin{bmatrix} L_E & L'_E \\ 0 & L_E \end{bmatrix}$$

of $\mathrm{Coker}\, L_E$ by $\mathrm{Coker}\, L_E$ splits. Then we may define a bilinear form on the space of isotrivial morphisms (or between the space of isotrivial morphisms and the space of all morphisms) by combining the two morphisms into a triangular matrix

$$\begin{bmatrix} L_E & L'_E & 0 \\ 0 & L_E & L''_E \\ 0 & 0 & L \end{bmatrix},$$

splitting off $\operatorname{Coker} L_E$ as a direct summand of the cokernel, and then taking the trace of the class of the corresponding extension.

It turns out this is already enough to let us prove Poissonness in several important cases. Suppose, for instance, that $V \cong \mathsf{A}^n$ and $W \cong \mathsf{A}[1]^m$; this implies the various vanishing statements we require. Then $V_E \cong \mathcal{O}_E^n$ is independent of $\tau$ while $W_E \cong \mathscr{L}^m$ for a degree-3 line bundle $\mathscr{L}$; the latter depends on $\tau$, but any two such bundles are related under pulling back through a translation of $E$. Moreover, a given map $L_E : V_E \to W_E$ lifts to a unique morphism $L : V \to W$, and $L$ is injective if and only if $L_E$ is injective. (Even the condition that $\operatorname{Coker} L$ is simple turns out to be reducible to a question on $L_E$, but in any case, this is an open condition.) In particular, given any value of $\tau$, we have an open subspace of the moduli space parametrizing sheaves with such a presentation, and for any other value $\tau'$, the corresponding open subspace is birational in a way preserving the Poisson structure. In particular, we may take $\tau' = 1_E$, at which point the corresponding moduli space is just a moduli space of sheaves on $\mathbb{P}^2$. Since the Jacobi identity is known to hold there, it holds on an open subspace for any $\tau$ and thus (since the failure of the Jacobi identity is measured by a morphism $\wedge^3\Omega \to \mathcal{O}$) on the closure of that open subspace so for any sheaf with a presentation of the given form.

In fact, with a bit more work, we can extend Poissonness to *any* simple sheaf (apart from point sheaves). The point is that, if $\mathsf{M}(d)$ is acyclic for $d \geq -3$, then $\mathsf{M}$ has a resolution

$$0 \to \mathsf{A}(-2)^{n_2} \to \mathsf{A}(-1)^{n_1} \to \mathsf{A}^{n_0} \to \mathsf{M} \to 0,$$

and as in [Hurtubise and Markman 2002], we can recover $\mathsf{M}$ from the cokernel of the map $\mathsf{A}(-2)^{n_2} \to \mathsf{A}(-1)^{n_1}$. The Poisson structure satisfies the Jacobi identity in the neighborhood of the latter sheaf (since this is just a twist of the kind of presentation we have already considered), and the calculation of [Hurtubise and Markman 2002] shows that the map from a neighborhood of $\mathsf{M}$ to this neighborhood simply negates the Poisson structure.

Note that it follows from this construction that we do not obtain any new symplectic varieties; every symplectic leaf in the noncommutative setting is mapped in this way to an open subset of a symplectic leaf in the moduli space of vector bundles on $\mathbb{P}^2$.

**7.5.** The above argument is somewhat unsatisfactory, as it depends on a somewhat delicate reduction to the commutative case, so is likely to be difficult to generalize to other noncommutative surfaces (e.g., deformations of del Pezzo surfaces). We thus continue our investigation of the pairing.

Since we are now in a completely commutative setting, we may use Čech cocycles to perform computations. In particular, a splitting of the extension $N'_E$ corresponding to $L'_E$ is a cocycle for $\operatorname{Hom}(N'_E, M_E)$ while the filtered sheaf $Z_E$ is represented by a cocycle for $\operatorname{Ext}^1(M_E, N'_E)$. The desired trace is then simply the trace pairing of these two classes, which reduces to the trace pairing on matrices.

By the structure of $Z_E$, we find that the cocycle representing $Z_E$ is simply the (global) morphism

$$\begin{bmatrix} L''_E & 0 \end{bmatrix} \in \operatorname{Hom}(V_E, W_E^2).$$

The splitting of $N'_E$ is slightly more complicated. If we write $E = U_1 \cup U_2$ with $U_1$ and $U_2$ affine opens, then the relevant map $N'_E \to M_E$ is represented over $U_i$ by

$$\begin{bmatrix} B'_i \\ 0 \end{bmatrix} \in \operatorname{Hom}_{U_i}(W_E^2, W_E), \qquad \begin{bmatrix} A'_i \\ 0 \end{bmatrix} \in \operatorname{Hom}_{U_i}(V_E^2, V_E)$$

such that

$$L'_E = B'_i L_E - L_E A'_i,$$

and there exists

$$\begin{bmatrix} \Phi'_{12} \\ 0 \end{bmatrix} \in \operatorname{Hom}_{U_1 \cap U_2}(W_E^2, V_E)$$

such that

$$B'_2 - B'_1 = L_E \Phi'_{12}, \qquad A'_2 - A'_1 = \Phi'_{12} L_E.$$

Note that, since $L_E$ is assumed injective, $\Phi'_{12}$ is uniquely determined. Combining this, we find that the trace pairing is given by

$$-\operatorname{Tr}(L''_E \Phi'_{12}) \in \Gamma(U_1 \cap U_2, \mathbb{O}_E),$$

viewed as a cocycle for $H^1(\mathbb{O}_E)$.

Essentially the same formula (possibly up to sign) was given by Polishchuk [1998], who constructed a Poisson structure on the moduli space of *stable* morphisms between vector bundles on $E$. Although he allows the vector bundles to vary, it is easy to check that any deformation in the image of the cotangent space induces the trivial deformation of the two bundles. As a result, Polishchuk's proof of the Jacobi identity carries over to our case. (Note that he imposes a stability condition, which is typically stronger than the natural stability condition in Tails $A$. However, all he really uses is that $\operatorname{Hom}(W, V) = 0$ and that the complex has no nonscalar automorphisms, i.e., the natural analogue of "simple".) Note that the interpretation of Polishchuk's Poisson structure coming from our calculation makes it straightforward to identify the symplectic leaves: each symplectic leaf classifies the ways of representing a particular sheaf as the cokernel of a map $V \to W$ with $V$ and $W$ fixed.

In the 1-dimensional case, the bundles $V_E$ and $W_E$ have the same rank, and thus, $L_E$ is generically invertible. If we choose $U_1$ such that $L_E$ is invertible on $U_1$, then we can arrange that

$$A'_1 = L_E^{-1} L'_E, \qquad B'_1 = 0,$$

at which point

$$\Phi'_{12} = L_E^{-1} B'_2,$$

so the pairing is given by the cocycle

$$-\operatorname{Tr}(L''_E L_E^{-1} B'_2).$$

Given a holomorphic differential $\omega$, the corresponding map to $\mathbb{C}$ is given by

$$\sum_{x \in U_2} \operatorname{Res}_x \operatorname{Tr}(L''_E L_E^{-1} B'_2)\omega.$$

The contributions come only from those points where $L_E$ fails to be invertible, i.e., from the support of $M_E$. Moreover, we readily see that the local contribution at $x$ will not change if we replace $(A'_2, B'_2)$ by any other splitting holomorphic at $x$.

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# Electrical networks and Lie theory

Thomas Lam and Pavlo Pylyavskyy

We introduce a new class of “electrical” Lie groups. These Lie groups, or more precisely their nonnegative parts, act on the space of planar electrical networks via combinatorial operations previously studied by Curtis, Ingerman and Morrow. The corresponding electrical Lie algebras are obtained by deforming the Serre relations of a semisimple Lie algebra in a way suggested by the star-triangle transformation of electrical networks. Rather surprisingly, we show that the type $A$ electrical Lie group is isomorphic to the symplectic group. The electrically nonnegative part $(\mathrm{EL}_{2n})_{\geq 0}$ of the electrical Lie group is an analogue of the totally nonnegative subsemigroup $(U_n)_{\geq 0}$ of the unipotent subgroup of $\mathrm{SL}_n$. We establish decomposition and parametrization results for $(\mathrm{EL}_{2n})_{\geq 0}$, paralleling Lusztig’s work in total nonnegativity, and work of Curtis, Ingerman and Morrow and of Colin de Verdière, Gitler and Vertigan for networks. Finally, we suggest a generalization of electrical Lie algebras to all Dynkin types.

## 1. Introduction

We consider the simplest of electrical networks, namely those that consist of only resistors. The electrical properties of such a network $N$ are completely described by the *response matrix* $L(N)$, which computes the current that flows through the network when certain voltages are fixed at the boundary vertices of $N$. The study of the response matrices of *planar* electrical networks has led to a robust theory; see Curtis, Ingerman and Morrow [CIM 1998], or Colin de Verdière, Gitler and Vertigan [CdVGV 1996]. Kennelly [1899] described a local transformation (see Figure 3) of a network, called the *star-triangle* or $Y-\Delta$ transformation, which preserves the response matrix of a network. This transformation is one of the many places where a Yang–Baxter-style transformation occurs in mathematics or physics.

Curtis, Ingerman and Morrow [CIM 1998] studied the operations of *adjoining a boundary spike* and *adjoining a boundary edge* to (planar) electrical networks. Our point of departure is to consider these operations as one-parameter subgroups of a

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Lie group action. The star-triangle transformation then leads to an "electrical Serre relation" in the corresponding Lie algebra, which turns out to be a deformation of the Chevalley–Serre relation for $\mathfrak{sl}_n$:

$$\text{Serre relation: } [e, [e, e']] = 0; \quad \text{electrical Serre relation: } [e, [e, e']] = -2e.$$

The corresponding one-parameter subgroups satisfy a Yang–Baxter style relation which is a deformation of Lusztig's relation in total positivity:

Lusztig's relation:

$$u_i(a)u_j(b)u_i(c) = u_j\Big(\frac{bc}{a+c}\Big)u_i(a+c)u_j\Big(\frac{ab}{a+c}\Big);$$

electrical relation:

$$u_i(a)u_j(b)u_i(c) = u_j\Big(\frac{bc}{a+c+abc}\Big)u_i(a+c+abc)u_j\Big(\frac{ab}{a+c+abc}\Big).$$

We study in detail the Lie algebra $\mathfrak{el}_{2n}$ with $2n$ generators satisfying the electrical Serre relation. An (electrically) nonnegative part $(\mathrm{EL}_{2n})_{\geq 0}$ of the corresponding Lie group $\mathrm{EL}_{2n}$ acts on planar electrical networks with $n+1$ boundary vertices, and one obtains a dense subset of all response matrices of planar networks in this way. This nonnegative part is a rather precise analogue of the totally nonnegative subsemigroup $(U_{2n+1})_{\geq 0}$ of the unipotent subgroup of $\mathrm{SL}_{2n+1}$, studied in Lie-theoretic terms by Lusztig [1994]. We show (Proposition 4.2) that $(\mathrm{EL}_{2n})_{\geq 0}$ has a cell decomposition labeled by permutations $w \in S_{2n+1}$, precisely paralleling one of Lusztig's results for $(U_{2n})_{\geq 0}$ and reminiscent of the Bruhat decomposition. This can be considered an algebraic analogue of parametrization results in the theory of electrical networks [CIM 1998; CdVGV 1996]. Surprisingly, $\mathrm{EL}_{2n}$ itself is isomorphic to the symplectic Lie group $\mathrm{Sp}_{2n}(\mathbb{R})$ (Theorem 3.1). This semisimplicity does not in general hold for electrical Lie groups: $\mathrm{EL}_{2n+1}$ is not semisimple. We also describe (Theorem 4.10) the stabilizer Lie algebra of the infinitesimal action of $\mathfrak{el}_{2n}$ on electrical networks. We caution that the nonnegative part $(\mathrm{EL}_{2n})_{\geq 0}$ is distinct from Lusztig's totally nonnegative part $\mathrm{Sp}_{2n}(\mathbb{R})_{\geq 0}$ of the symplectic group.

While we focus on the planar case in this paper, we shall connect the results of this paper to the inverse problem for electrical networks on cylinders in a future paper. There we borrow ideas from representation theory, such as those of crystals and $R$-matrices.

One obtains the types $B$ and $G$ electrical Serre relations by the standard technique of "folding" the type $A$ electrical Serre relation. These lead to a new species of electrical Lie algebras $\mathfrak{e}_D$ defined for any Dynkin diagram $D$. Besides the above results for type $A$, there appear to be other interesting relations between these $\mathfrak{e}_D$ and simple Lie algebras. For example, $\mathfrak{eb}_2 := \mathfrak{e}_{B_2}$ is isomorphic to $\mathfrak{gl}_2$. We conjecture (Conjecture 5.2) that, for a finite-type diagram $D$, the dimension $\dim(\mathfrak{e}_D)$ is always equal to the dimension of the maximal nilpotent subalgebra of the semisimple Lie algebra $\mathfrak{g}_D$ with Dynkin diagram $D$, and furthermore (Conjecture 5.3) that $\mathfrak{e}_D$ is finite-dimensional if and only if $\mathfrak{g}_D$ is finite-dimensional. We give the electrical analogue of Lusztig's relation for type $B$ (see [Berenstein and Zelevinsky 1997]) where we observe similar positivity to the totally nonnegative case.

The most interesting case beyond finite Dynkin types is affine type $A$. The corresponding electrical Lie (semi-)group action on planar electrical networks is perhaps even more natural than that of $\mathrm{EL}_{2n}$, since one can obtain *all* (rather than just a dense subset of; see Corollary 4.7) response matrices of planar networks in this way [CdVGV 1996; CIM 1998], and furthermore the circular symmetry of the planar networks is preserved. We do not, however, attempt to address this case in the current paper.

## 2. Electrical networks

For more background on electrical networks, we refer the reader to [CIM 1998; CdVGV 1996].

***Response matrix.*** For our purposes, an electrical network is a finite weighted undirected graph $N$, where the vertex set is divided into the *boundary* vertices and the *interior* vertices. The weight $w(e)$ of an edge is to be thought of as the conductance of the corresponding resistor, and is generally taken to be a positive real number. A 0-weighted edge would be the same as having no edge, and an edge with infinite weight would be the same as identifying the endpoint vertices.

We define the *Kirchhoff matrix* $K(N)$ to be the square matrix with rows and columns labeled by the vertices of $N$, defined by

$$K_{ij} = \begin{cases} -\sum_{e \text{ joins } i \text{ and } j} w(e) & \text{for } i \neq j, \\ \sum_{e \text{ incident to } i} w(e) & \text{for } i = j. \end{cases}$$

We define the *response matrix* $L(N)$ to be the square matrix with rows and columns labeled by the boundary vertices of $N$, given by the Schur complement

$$L(N) = K/K_I$$

where $K_I$ denotes the submatrix of $K$ indexed by the interior vertices. The response matrix encodes all the electrical properties of $N$ that can be measured from the boundary vertices. For example, the Kirchhoff matrix and response matrix of the "Y"-graph in Figure 3 (with the surrounding edges removed and the central vertex made interior), are given by

$$K(N) = \begin{bmatrix} a & 0 & 0 & -a \\ 0 & b & 0 & -b \\ 0 & 0 & c & -c \\ a & b & c & a+b+c \end{bmatrix} \quad \text{and} \quad L(N) = \begin{bmatrix} \frac{a(b+c)}{a+b+c} & -\frac{ab}{a+b+c} & -\frac{ac}{a+b+c} \\ -\frac{ab}{a+b+c} & \frac{b(a+c)}{a+b+c} & -\frac{bc}{a+b+c} \\ -\frac{ac}{a+b+c} & -\frac{bc}{a+b+c} & \frac{(a+b)c}{a+b+c} \end{bmatrix}.$$

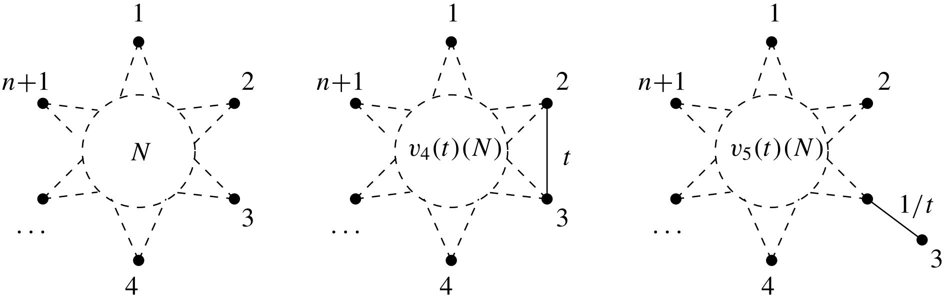

**Figure 1.** The operations $v_i(t)$ acting on a network $N$.

***Planar electrical networks.*** We shall usually consider electrical networks $N$ embedded in a disk, so that the boundary vertices, numbered $1, 2, \dots, n+1$, lie on the boundary of the disk.

Given an odd integer $2k-1$, for $k = 1, 2, \dots, n+1$, and a nonnegative real number $t$, we define $v_{2k-1}(t)(N)$ to be the electrical network obtained from $N$ by adding a new edge from a new vertex $v$ to $k$, with weight $1/t$, followed by treating $k$ as an interior vertex and the new vertex $v$ as a boundary vertex (now named $k$).

Given an even integer $2k$, for $k=1, 2, \dots, n+1$, and a nonnegative real number $t$, we define $v_{2k}(t)(N)$ to be the electrical network obtained from $N$ by adding a new edge from $k$ to $k+1$ (indices taken modulo $n+1$), with weight $t$.

The two operations are shown in Figure 1. In [CIM 1998], these operations are called *adjoining a boundary spike*, and *adjoining a boundary edge* respectively. Our notation suggests, and we shall establish, that there is some symmetry between these two types of operations.

In [CIM 1998, §8], it is shown that $L(v_i(a) \cdot N)$ depends only on $L(N)$, giving an operation $v_i(t)$ on response matrices. Denote by $x_{ij}$ the entries of the response matrix, where $1 \le i, j \le n+1$. In particular, we have $x_{ij} = x_{ji}$. Then, if $\delta_{ij}$ denotes the Kronecker delta, we have

$$v_{2k-1}(t) : x_{ij} \mapsto x_{ij} - \frac{t x_{ik} x_{kj}}{t x_{kk} + 1}, \quad v_{2k}(t) : x_{ij} \mapsto x_{ij} + (\delta_{ik} - \delta_{(i+1)k})(\delta_{jk} - \delta_{(j+1)k})\, t. \tag{1}$$

We caution that the parameter $\xi$ in [CIM 1998] is the inverse of our $t$ in the odd case.

***Electrically equivalent transformations of networks.*** Series-parallel transformations are shown in Figure 2. The following proposition is well-known and can be found for example in [CdVGV 1996].

**Figure 2.** Series-parallel transformations.

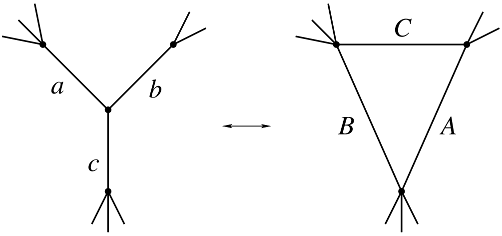

**Figure 3.** The $Y-\Delta$, or star-triangle, transformation.

**Proposition 2.1.** *Series-parallel transformations, removing loops, and removing interior degree* 1 *vertices do not change the response matrix of a network.*

The following theorem is attributed to Kennelly [1899].

**Proposition 2.2** ($Y-\Delta$ transformation). *Assume that the parameters* $a, b, c$ *and* $A, B, C$ *are related by*

$$A = \frac{bc}{a+b+c}, \quad B = \frac{ac}{a+b+c}, \quad C = \frac{ab}{a+b+c},$$

*or equivalently by*

$$a = \frac{AB+AC+BC}{A}, \quad b = \frac{AB+AC+BC}{B}, \quad c = \frac{AB+AC+BC}{C}.$$

*Then switching a local part of an electrical network between the two options shown in Figure 3 does not change the response matrix of the whole network.*

## 3. The electrical Lie algebra $\mathfrak{el}_{2n}$

Let $\mathfrak{el}_{2n}$ be the Lie algebra generated over $\mathbb{R}$ by $e_i$, $i = 1, \dots, 2n$, subject to the relations

- $[e_i, e_j] = 0$ for $|i-j| > 1$,
- $[e_i, [e_i, e_j]] = -2e_i$, $|i-j| = 1$.

**Theorem 3.1.** *The electrical Lie algebra* $\mathfrak{el}_{2n}$ *is isomorphic to the real symplectic Lie algebra* $\mathfrak{sp}_{2n}$.

*Proof.* We identify the symplectic algebra $\mathfrak{sp}_{2n}$ with the space of $2n \times 2n$ matrices which in block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

satisfy $A = -D^T$, $B = B^T$ and $C = C^T$ (see [Fulton and Harris 1991, Lecture 16]). Note that the Lie subalgebra consisting of the matrices where $B = C = 0$ is naturally isomorphic to $\mathfrak{gl}_n$.

Let $\varepsilon_i \in \mathbb{R}^n$ denote the standard basis column vector with a 1 in the $i$-th position. Let

$$a_1 = \varepsilon_1, \quad a_2 = \varepsilon_1 + \varepsilon_2, \quad a_3 = \varepsilon_2 + \varepsilon_3, \quad \ldots, \quad a_n = \varepsilon_{n-1} + \varepsilon_n,$$

and let

$$b_1 = \varepsilon_1, \quad b_2 = \varepsilon_2, \quad \ldots, \quad b_n = \varepsilon_n.$$

For $1 \le i \le n$, define elements of $\mathfrak{sp}_{2n}$ by the formulae

$$e_{2i-1} = \begin{pmatrix} 0 & a_i \cdot a_i^T \\ 0 & 0 \end{pmatrix}, \quad e_{2i} = \begin{pmatrix} 0 & 0 \\ b_i \cdot b_i^T & 0 \end{pmatrix}.$$

We claim that this gives a symplectic representation $\phi$ of $\mathfrak{el}_{2n}$ which is an isomorphism of $\mathfrak{el}_{2n}$ with $\mathfrak{sp}_{2n}$. We first check that the relations of $\mathfrak{el}_{2n}$ are satisfied. It is clear from the block matrix form that $[e_{2i-1}, e_{2j-1}] = 0 = [e_{2i}, e_{2j}]$ for any $i$, $j$. Now,

$$[e_{2i-1}, e_{2j}] = \begin{pmatrix} (a_i \cdot a_i^T)(b_j \cdot b_j^T) & 0 \\ 0 & -(b_j \cdot b_j^T)(a_i \cdot a_i^T) \end{pmatrix}.$$

But, by construction, $b_j^T \cdot a_i = 0 = a_i^T \cdot b_j$ unless $j = i$ or $j = i - 1$. Thus $[e_k, e_l] = 0$ unless $|k - l| = 1$. Finally,

$$\begin{aligned}[e_{2i-1}, [e_{2i-1}, e_{2i}]] &= \begin{pmatrix} 0 & -2(a_i \cdot a_i^T)(b_i \cdot b_i^T)(a_i \cdot a_i^T) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2a_i \cdot a_i^T \\ 0 & 0 \end{pmatrix} = -2e_{2i-1}\end{aligned}$$

using the fact that $a_i^T \cdot b_i = 1 = b_i^T \cdot a_i$. Similarly, one obtains $[e_k, [e_k, e_{k\pm1}]] = -2e_k$. Thus we have a symplectic representation $\phi : \mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$.

Now we show that this map is surjective. First we verify that $\mathfrak{gl}_n \subset \phi(\mathfrak{el}_{2n})$. The nonzero commutators $[e_i, e_j]$ give matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & -A^T \end{smallmatrix}\right)$, where $A$ is a scalar multiple of one of the matrices $E_{1,1}$, $E_{i,i} + E_{i+1,i}$ or $E_{i+1,i} + E_{i+1,i+1}$. Here $E_{i,j}$ denotes the $n \times n$ matrix with a single nonzero entry equal to 1 in the $(i, j)$-th position. All the matrices of the above form occur. It is easy to see that $\phi(\mathfrak{el}_{2n})$ must then contain the matrices where $A = E_{i,i}$, $E_{i,i+1}$ or $E_{i+1,i}$ for each $i$. But these matrices generate $\mathfrak{gl}_n$ as a Lie algebra. However, it is known [Humphreys 1990, Proposition 8.4(d)] that for a semisimple Lie algebra $L$, one has $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ when $\alpha, \beta, \alpha + \beta$ are all roots and $L_\alpha$ denotes a root subspace. It follows easily from the explicit description of the root system of $\mathfrak{sp}_{2n}$ and the definition of $e_i$ that every root subspace of $\mathfrak{sp}_{2n}$ is contained in $\phi(\mathfrak{el}_{2n})$, completing the proof.

To see that the map $\phi : \mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$ is injective, we note:

**Lemma 3.2.** *The dimension of* $\mathfrak{el}_{2n}$ *is* $n(2n + 1)$.

*Proof.* According to Lemma 5.1, the dimension of $\mathfrak{el}_{2n}$ is at most $n(2n+1)$. On the other hand, we just saw that the map $\mathfrak{el}_{2n} \to \mathfrak{sp}_{2n}$ is surjective. The statement follows. $\square$

## 4. The electrical Lie group $\mathrm{EL}_{2n}$

Let $\mathrm{EL}_{2n}$ be a split real Lie group with Lie algebra $\mathfrak{el}_{2n}$. For concreteness, we shall choose $\mathrm{EL}_{2n}$ to be the real symplectic group, but we will use the notation $\mathrm{EL}_{2n}$ to remind the reader that the generators we consider are associated to the presentation of $\mathfrak{el}_{2n}$, rather than the usual presentation of $\mathfrak{sp}_{2n}$. Let $u_i(a) = \exp(ae_i)$ denote the one-parameter subgroups corresponding to the generators of $\mathfrak{el}_{2n}$.

**Theorem 4.1.** *The elements $u_i(a)$ satisfy the relations*

1. $u_i(a)u_i(b) = u_i(a+b)$,
2. $u_i(a)u_j(b) = u_j(b)u_i(a)$ *if* $|i-j| > 1$, *and*
3. $u_i(a)u_j(b)u_i(c) = u_j(bc/(a+c+abc))u_i(a+c+abc)u_j(ab/(a+c+abc))$ *if* $|i-j| = 1$.

*Proof.* The first two relations are clear. For the third, observe that, for each $1 \le i \le 2i-1$, the elements $e_i$, $e_{i+1}$, and $[e_i, e_{i+1}]$ are the usual Chevalley generators of a Lie subalgebra isomorphic to $\mathfrak{sl}_2$. Thus we can verify the relation inside $\mathrm{SL}_2(\mathbb{R})$:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+ab & a+c+abc \\ b & 1+bc \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{bc}{a+c+abc} & 1 \end{pmatrix}\begin{pmatrix} 1 & a+c+abc \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{ab}{a+c+abc} & 1 \end{pmatrix}. \quad \square$$

**Remark.** There is a one-parameter deformation which connects the relation in Theorem 4.1(3) with Lusztig's relation [1994] in total positivity:

$$u_i(a)u_j(b)u_i(c) = u_j\Big(\frac{bc}{a+c}\Big)u_i(a+c)u_j\Big(\frac{ab}{a+c}\Big). \tag{2}$$

Consider the associative algebra $U_\tau(\mathfrak{el}_{2n})$ where the generators $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n}$ satisfy the following deformed Serre relations:

- $\varepsilon_i\varepsilon_j = \varepsilon_j\varepsilon_i$ if $|i-j| > 1$,
- $\varepsilon_i\varepsilon_j\varepsilon_i = \tau\varepsilon_i + \frac{1}{2}(\varepsilon_i^2\varepsilon_j + \varepsilon_j\varepsilon_i^2)$ if $|i-j| = 1$.

This is a one-parameter family of algebras which at $\tau = 0$ reduces to $U(\mathfrak{n}^+)$, where $\mathfrak{sl}_{2n+1} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is the Cartan decomposition, while at $\tau = 1$ it gives the universal enveloping algebra $U(\mathfrak{el}_{2n})$ of the electrical Lie algebra. For $U_\tau(\mathfrak{el}_{2n})$, the "braid move" for the elements $\exp(a\varepsilon_i)$ then takes the form

$$u_i(a)u_j(b)u_i(c) = u_j\Big(\frac{bc}{a+c+\tau abc}\Big)u_i(a+c+\tau abc)u_j\Big(\frac{ab}{a+c+\tau abc}\Big)$$

if $|i-j|=1$. At $\tau=0$ this reduces to (2), and at $\tau=1$ it is the relation Theorem 4.1(3).

***Nonnegative part of* $\mathbf{EL_{2n}}$.** Define the *nonnegative* part $(\mathrm{EL}_{2n})_{\geq 0}$ of $\mathrm{EL}_{2n}$, or equivalently the *electrically nonnegative* part of $\mathrm{Sp}_{2n}$, to be the subsemigroup generated by the $u_i(a)$ with nonnegative parameters $a$.

For a reduced word $\boldsymbol{i}=i_1\cdots i_\ell$ of $w\in S_{2n+1}$, denote by $E(w)\subset(\mathrm{EL}_{2n})_{\geq 0}$ the image of the map

$$\psi_{\boldsymbol{i}}:(a_1,\ldots,a_\ell)\in\mathbb{R}^\ell_{>0}\mapsto u_{i_1}(a_1)\ldots u_{i_\ell}(a_\ell).$$

It follows from the relations in Theorem 4.1 that the set $E(w)$ depends only on $w$, and not on the chosen reduced word. The following proposition is similar to [Lusztig 1994, Proposition 2.7], which gives a decomposition $U_{\geq 0}=\bigsqcup_w U^w_{\geq 0}$ of the totally nonnegative part of the unipotent group.

**Proposition 4.2.** *The sets $E(w)$ are disjoint and cover the whole* $(\mathrm{EL}_{2n})_{\geq 0}$. *Each of the maps $\psi_{\boldsymbol{i}}:\mathbb{R}^\ell_{>0}\to E(w)$ is a homeomorphism.*

*Proof.* Using Theorem 4.1, we can rewrite any product of generators $u_i(a)$ by performing braid moves similar to those in the symmetric group $S_{2n+1}$. Any product of the generators can be transformed into a product that corresponds to a reduced word in $S_{2n+1}$, and thus it belongs to one of the $E(w)$.

If the map $\psi_{\boldsymbol{i}}$ is not injective for some reduced word $\boldsymbol{i}$, then we can find two reduced products

$$u_{i_1}(a_1)\cdots u_{i_\ell}(a_\ell)=u_{i_1}(b_1)\cdots u_{i_\ell}(b_\ell)$$

for two $\ell$-tuples of positive numbers such that $a_1\neq b_1$. Without loss of generality we can assume $a_1>b_1$, and thus

$$u_{i_1}(a_1-b_1)u_{i_2}(a_2)\cdots u_{i_\ell}(a_\ell)=u_{i_2}(b_2)\cdots u_{i_\ell}(b_\ell).$$

This shows that two different $E(w)$'s have nonempty intersection. Thus it suffices to show that the latter is impossible.

Furthermore, it suffices to prove that the top cell corresponding to the longest element $w_0$ does not intersect any other cell. Indeed, if any two cells intersect, by adding some extra factors to both we can lift it to the top cell intersecting one of the other cells. Assume we have a product of the form

$$u=[u_1(a_1)][u_2(a_2)u_1(a_3)]\cdots[u_k(a_{\ell-k+1})\cdots u_1(a_\ell)],\tag{3}$$

where $\ell=\binom{k}{2}$. Let $\Phi(u)\in\mathrm{Sp}_{2n}$ be the image of $u$ in $\mathrm{Sp}_{2n}$ under the natural map $\mathrm{EL}_{2n}\to\mathrm{Sp}_{2n}$ induced by the map $\phi:\mathfrak{el}_{2n}\to\mathfrak{sp}_{2n}$ of Theorem 3.1. We argue that the positive parameters $a_{\ell-k+1},\ldots,a_\ell$ can be recovered uniquely from just looking at the $(n+k/2)$-th row of $\Phi(u)$ for $k$ even, or the $(k+1)/2$-th row for $k$ odd. Furthermore, the same calculation will tell us if exactly one of them is equal to zero.

Indeed, assume $k$ is odd. Then the $n+(k+1)/2$-th entry in the $(k+1)/2$-th row is just equal to $a_{\ell-k+1}$. Next, once we know $a_{\ell-k+1}$, we can use $(k-1)/2$-th entry of the same row to recover $a_{\ell-k+2}$, after which we can use $n+(k-1)/2$-th entry to recover $a_{\ell-k+3}$, and so on. For each step we solve a linear equation where the parameter we divide by is strictly positive as long as all previous parameters $a_i$ recovered are positive. For example, let $n=2$ and take the product

$$u_1(a_1)u_2(a_2)u_1(a_3)u_3(a_4)u_2(a_5)u_1(a_6).$$

Then the second row of this product is $(a_4a_5, 1, a_4+a_4a_5a_6, a_4)$. The last entry tells us $a_4$, then from the first entry we solve for $a_5$, then from the third entry we solve for $a_6$. The case of even $k$ is similar; the order in which we have to read the entries of the $(n+k/2)$-th row in this case is $k/2$-th, $(n+k/2)$-th, $(k/2-1)$-th, $(n+k/2-1)$-th, etc. For example, in the product

$$u_1(a_1)u_2(a_2)u_1(a_3)u_3(a_4)u_2(a_5)u_1(a_6)u_4(a_7)u_3(a_8)u_2(a_9)u_1(a_{10}),$$

the last row is $(a_7a_8a_9, a_7, a_7a_8+a_7a_8a_9a_{10}, 1+a_7a_8)$. By looking at second, last, first and third entries we can solve for $a_7$, $a_8$, $a_9$ and $a_{10}$ one after the other.

Now we are ready to complete the argument. Assume that $E(w_0)$ intersects some other $E(v)$. For any reduced word of $w_0$, one can find a subword which is a reduced word for $v$, so that we have

$$\begin{aligned} u &= [u_1(a_1)][u_2(a_2)u_1(a_3)]\cdots[u_k(a_{\ell-k+1})\cdots u_1(a_\ell)] \\ &= [u_1(b_1)][u_2(b_2)u_1(b_3)]\cdots[u_k(b_{\ell-k+1})\cdots u_1(b_\ell)], \end{aligned}$$

where the $a_i$ are all positive, but the $b_i$ are nonnegative, with at least one zero. The above algorithm of recovering the $a_i$ will arrive at a contradiction. Thus $E(w_0)$ is disjoint from all other $E(v)$.

It remains to show that $\phi_{\boldsymbol{i}}^{-1}$ is continuous for any reduced word. If $\boldsymbol{i}$ is the reduced word of $w_0$ used in (3), this follows from the algorithm above: the $a_i$ depend continuously on the matrix entries of $\Phi(u)$. But any two reduced words are connected by braid and commutation moves, so it follows from the (continuously invertible) formulae in Theorem 4.1 that $\phi_{\boldsymbol{i}}^{-1}$ is continuous for any reduced word of $w_0$. But for any other $v \in S_{2n+1}$, a reduced word $\boldsymbol{j}$ for $v$ can be found as an initial subword of some reduced word $\boldsymbol{i}$ for $w_0$. The map $\phi_{\boldsymbol{j}}^{-1}$ can then be expressed as a composition of right multiplication by a fixed element of $\mathrm{EL}_{2n}$, the map $\phi_{\boldsymbol{i}}^{-1}$ and the projection from $\mathbb{R}_{>0}^{\ell(w_0)}$ to $\mathbb{R}_{>0}^{\ell(v)}$, all of which are continuous. $\square$

***Action on electrical networks.*** Let $\mathcal{P}(n+1) \subset \mathbb{R}^{(n+1)^2}$ denote the set of response matrices of planar electrical networks with $n+1$ boundary vertices. In [CIM 1998],

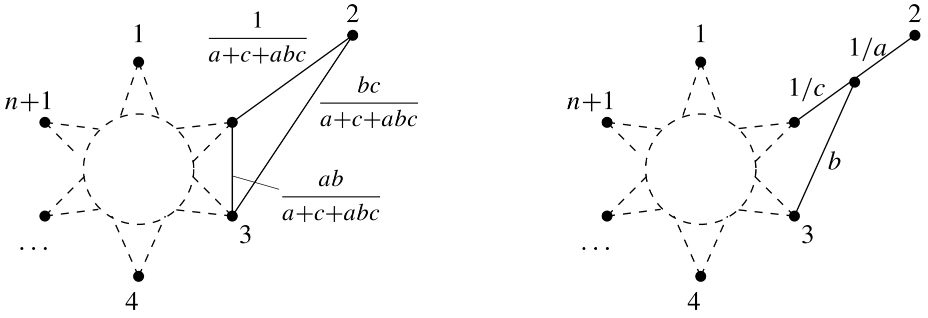

**Figure 4.** Action of two products on an electrical network.

it is shown that $\mathcal{P}(n+1)$ is exactly the set of symmetric, circular totally nonnegative $(n+1)\times(n+1)$ matrices with row sums equal to 0. A matrix $M$ is circular totally nonnegative if the signed minors $(-1)^k A_{(p_1,p_2,\dots,p_k),(q_1,q_2,\dots,q_k)}$ are all nonnegative whenever $(p_1, p_2, \dots, p_k, q_k, q_{k-1}, \dots, q_1)$ is in circular order. In [CdVGV 1996, Théorème 4], it is shown that $\mathcal{P}(n+1)$ can be identified with the set of planar electrical networks with $n+1$ boundary vertices modulo the local transformations described at the end of Section 2. Let $N_0$ denote the empty network (with $n+1$ boundary vertices) and let $L_0 = L(N_0)$ denote the zero matrix.

**Theorem 4.3.** *The nonnegative part* $(\mathrm{EL}_{2n})_{\geq 0}$ *of the electrical group acts on* $\mathcal{P}(n+1)$ *via*

$$u_i(a)\cdot L(N) = L(v_i(a)(N)).$$

*Proof.* For a single generator $u_i(a)$, the stated action is well-defined because it can be described explicitly on the level of response matrices. The formula for $L(v_i(a)(N))$ in terms of $L(N)$ is given by (1).

We first show that the relations of Theorem 4.1 hold for this action. Relation (1) follows from the series-parallel relation for networks. Relation (2) is immediate: the corresponding networks are identical without any transformations. Relation (3) follows from the $Y-\Delta$ transformation (see Example 4.4).

But now suppose we have two different expressions for $u \in (\mathrm{EL}_{2n})_{\geq 0}$ in terms of generators. Then using relations (1)–(3) of Theorem 4.1, we may assume that both expressions are products corresponding to a reduced word. By Proposition 4.2, the two products come from reduced words for the same $w \in S_{2n+1}$. It follows that they are related by relations (1)–(3). □

**Example 4.4.** The products

$$u_3(a)u_4(b)u_3(c) \quad \text{and} \quad u_4\Big(\frac{bc}{a+c+abc}\Big)u_3(a+c+abc)u_4\Big(\frac{ab}{a+c+abc}\Big)$$

act on a network in exactly the same way, as shown in Figure 4.

A permutation $w = w(1)w(2)\cdots w(2n+1) \in S_{2n+1}$ is *efficient* if

(1) $w(1) < w(3) < \cdots < w(2n+1)$ and $w(2) < w(4) < \cdots < w(2n)$, and

(2) $w(1) < w(2), w(3) < w(4), \ldots, w(2n-1) < w(2n)$.

Recall the *left weak order* of permutations is given by $w \preceq v$ if and only if there is a $u$ so that $uw = v$ and $\ell(v) = \ell(u) + \ell(w)$. It is a standard fact that $w \preceq v$ if and only if $v(i) > v(j)$ whenever $w(i) > w(j)$ and $i < j$. Thus the set of efficient permutations has a maximum in left weak order, namely $w = 1(n+2)2(n+3)\cdots(n)(2n+1)(n+1)$ with length $\frac{1}{2}n(n-1)$.

**Theorem 4.5.** *Let $w \in S_{2n+1}$. The map $\Theta_w : E(w) \to \mathcal{P}(n+1)$ given by $\Theta_w(u) = u \cdot L_0$ is injective if and only if $w$ is efficient. We have* $\operatorname{image}(\Theta_w) \cap \operatorname{image}(\Theta_v) = \varnothing$ *for $w \neq v$ both efficient. If $w$ is not efficient, there is a unique efficient $v$ such that* $\operatorname{image}(\Theta_w) = \operatorname{image}(\Theta_v)$.

*Proof.* Let $w^* \in S_{2n+1}$ be the efficient permutation of maximal length. Then a possible reduced word for $w^*$ is

$$(n+1)\cdots\bigl(46\ldots(2n-2)\bigr)\bigl(35\ldots(2n-1)\bigr)\bigl(246\ldots(2n)\bigr).$$

The graph obtained by taking the corresponding $u_i(a)$ and acting on the empty network is exactly the "standard graph" of [CIM 1998, §7] or the graph $C_N$ of [CdVGV 1996]. In particular, $\Theta_{w^*}$ is injective by [CIM 1998, Theorem 2] or [CdVGV 1996, Théorème 3]. Suppose $w$ is an arbitrary efficient permutation. Then since $w^* = uw$ for some $u \in S_{2n+1}$, if $\Theta_w$ is not injective then $\Theta_{w^*}$ is not injective as well, so we conclude that $\Theta_w$ is injective.

For a pair $(i, j)$ with $1 \le i < j \le n+1$, let us say that a network $N$ is $(i, j)$-*connected* if we can find a disjoint set of paths $p_1, p_2, \ldots, p_{\lfloor (j-i+1)/2 \rfloor}$ so that for each $k$, the path $p_k$ connects boundary vertex $i+k-1$ to boundary vertex $j-k+1$ without passing through any other boundary vertex. This is a special case of the connections of circular pairs $(P, Q)$ of [CIM 1998]. Let $N_w$ be a graph constructed from some reduced word of an efficient $w$. We observe that $N_w$ is $(i, j)$-connected if and only if $w(2i) > w(2j-1)$.

For example, the first network in Figure 5 corresponds to $w = s_5s_3s_6s_4s_2 = (1, 3, 5, 2, 7, 4, 6)$. We see that it is $(2, 4)$-connected, which agrees with the inequality $w(4) = 6 > 5 = w(7)$. If the dashed edge is not there, we have $w = s_3s_6s_4s_2 = (1, 3, 5, 2, 4, 7, 6)$ and $w(4) = 5 < 6 = w(7)$, in agreement with the network not being $(2, 4)$-connected. Similarly, the second network in Figure 5 corresponds to $w = s_4s_5s_3s_6s_4s_2 = (1, 3, 5, 7, 2, 4, 6)$ with the dashed edge and to $w = s_5s_3s_6s_4s_2 = (1, 3, 5, 2, 7, 4, 6)$ without. In the first case it is $(1, 4)$-connected, in the second it is not, in agreement with relative order of $w(2)$ and $w(7)$.

Note that the inversions $w(2i) > w(2j-1)$ are exactly the possible inversions of an efficient permutation. Since $w$ is determined by its inversions, it follows that the

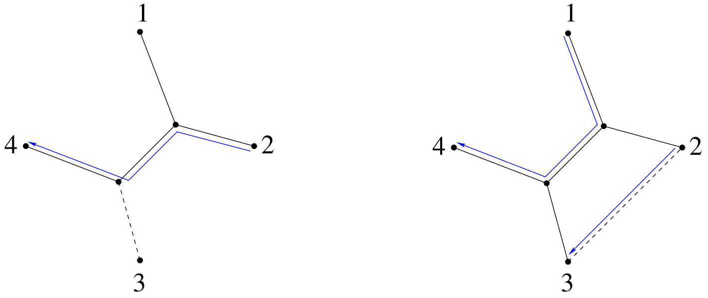

**Figure 5.** Networks and connections on them.

set of $(i, j)$-connections of $N_w$ determines $w$, and that $\mathrm{image}(\Theta_w) \cap \mathrm{image}(\Theta_v) = \varnothing$ for $w \neq v$ both efficient.

Suppose $w$ is not efficient. Then it is easy to see that $w$ has a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_\ell}$, where either (1) $i_\ell$ is odd, or (2) $i_\ell$ is even and $i_\ell = i_{\ell-1} \pm 1$. But $u_i(a) \cdot L_0 = L_0$ for odd $i$ since all the boundary vertices are still disconnected in $u_i(a) \cdot N_0$, and for even $i$ we have $u_{i\pm1}(a) u_i(b) \cdot L_0 = u_i(1/a + b) L_0$, using the series-transformation (Proposition 2.1). It follows that $\Theta_w$ is not injective. Furthermore, $\mathrm{image}(\Theta_w) = \mathrm{image}(\Theta_v)$, where $v$ is obtained from $w$ by recursively (1) removing $i_\ell$ from a reduced word of $w$ if $i_\ell$ is odd, or (2) changing the last two letters $(i_\ell \pm 1) i_\ell$ to $i_\ell$ when $i_\ell$ is even. An efficient $v$ obtained in this way must be unique, since $\mathrm{image}(\Theta_v) \cap \mathrm{image}(\Theta_{v'}) = \varnothing$ for efficient $v \neq v'$. □

**Corollary 4.6.** *The number of efficient $w \in S_{2n+1}$ is equal to the Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$.*

*Proof.* It is clear that $(2i, 2j+1)$ is an inversion of $w$ only if $(2i, 2j-1)$ and $(2i+2, 2j+1)$ are also inversions, and this characterizes inversion sets of efficient $w$. Thus the set of efficient $w \in S_{2n+1}$ is in bijection with the lower order ideals of the positive root poset of $\mathfrak{sl}_{n+1}$, which is well known to be enumerated by the Catalan number [Fomin and Reading 2007]. □

**Corollary 4.7.** $(\mathrm{EL}_{2n})_{\geq 0} \cdot L_0$ *is dense in* $\mathcal{P}(n+1)$.

*Proof.* Follows from Theorem 4.5 and [CdVGV 1996, Théorème 5]. □

***Stabilizer.***

**Lemma 4.8.** *The stabilizer subsemigroup of* $(\mathrm{EL}_{2n})_{\geq 0}$ *acting on the zero matrix $L_0$ is the subsemigroup generated by $u_{2i+1}(a)$ for all $a \in \mathbb{R}_{\geq 0}$ and all $i$.*

*Proof.* It is clear that $u_{2i+1}(a)$ lies in the stabilizer. But the action of any $u_{2i}(a)$ will change the connectivity of the network, and it is impossible to return to trivial connectivity by adding more edges, or by relations. □

The semigroup stabilizer is too small in the sense that it does not detect the relations $u_{i\pm 1}(a)u_i(b)\cdot L_0 = u_i(1/a+b)L_0$ used in the proof of Theorem 4.5. We shall calculate the stabilizer subalgebra of the corresponding infinitesimal action of the Lie algebra $\mathfrak{el}_{2n}$, which will in particular give an algebraic explanation of these relations. The reason we do not work with the whole Lie group $\mathrm{EL}_{2n}$ is threefold: (1) nonpositive elements of $\mathrm{EL}_{2n}$ will produce networks that are "virtual", that is, that have negative edge weights; (2) the topology of $\mathrm{EL}_{2n}$ means that to obtain an action one cannot just check the relations of Theorem 4.1; (3) when the parameters are nonpositive, the relation Theorem 4.1(3) develops singularities.

To describe the infinitesimal action of $\mathfrak{el}_{2n}$, we give derivations of $\mathbb{R}[x_{ij}]$, the polynomial ring in $\frac{1}{2}(n+1)(n+2)$ variables $x_{ij}$ where $1 \le i, j \le n+1$ and we set $x_{ij} = x_{ji}$.

**Proposition 4.9.** *The electrical Lie algebra* $\mathfrak{el}_{2n}$ *acts on* $\mathbb{R}[x_{ij}]$ *via the derivations*

$$e_{2i} \mapsto \partial_{ii} + \partial_{i+1,i+1} - \partial_{i,i+1}, \tag{4}$$

$$e_{2i-1} \mapsto -\sum_{1\le p\le q\le n+1} x_{ip}x_{iq}.\partial_{pq} \tag{5}$$

*Proof.* These formulae can be checked by directly verifying the defining relations of $\mathfrak{el}_{2n}$. Alternatively, they can be deduced by differentiating the formulae (1). ☐

We calculate that

$$[e_{2i}, e_{2i-1}] \mapsto -x_{ii}\partial_{ii} + x_{i,i+1}\partial_{i+1,i+1} + \sum_{1\le p\le n+1} (x_{i+1,p}\partial_{i+1,p} - x_{ip}\partial_{ip}). \tag{6}$$

**Theorem 4.10.** *The stabilizer subalgebra* $\mathfrak{el}^0_{2n}$*, at the zero matrix* $L_0$*, of the infinitesimal action of* $\mathfrak{el}_{2n}$ *on the space of response matrices is generated by* $e_i$ *for* $i$ *odd, and* $[e_{2i-1}, e_{2i}]$ *for* $i = 1, 2, \ldots, n$.

*Proof.* The fact the stated elements lie in $\mathfrak{el}^0_{2n}$ follows from (5) and (6), since $x_{ij} = 0$ at the zero matrix. By Lemma 3.2, the elements $e_\alpha$ in the proof of Lemma 5.1 form a basis of $\mathfrak{el}_{2n}$. Write $\alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \le i \le j \le 2n$ to denote the positive roots of $\mathfrak{sl}_{2n+1}$. Then we know that $e_{\alpha_{i,i}} \in \mathfrak{el}^0_{2n}$ for $i$ odd, and $e_{\alpha_{i,i+1}} \in \mathfrak{el}^0_{2n}$ for each $i$. It follows easily that $e_{\alpha_{i,j}} \in \mathfrak{el}^0_{2n}$ for every pair $1 \le i \le j \le 2n$ where at least one of $i$ and $j$ are odd. It follows that

$$\dim_{\mathbb{R}}(\mathfrak{el}_{2n}) - \dim_{\mathbb{R}}(\mathfrak{el}^0_{2n}) \le \#\{(i, j) \mid 1 \le i \le j \le 2n \text{ and } i, j \text{ are even}\} = \tfrac{1}{2}n(n+1).$$

By Theorem 4.5, the action of $(\mathrm{EL}_{2n})_{\ge 0}$ on the zero matrix $L_0$ gives a space of dimension $\frac{1}{2}n(n+1)$. It follows that the above inequality is an equality, and that $\mathfrak{el}^0_{2n}$ is generated by the stated elements. ☐

Note that a basis for $\mathfrak{el}_{2n}/\mathfrak{el}^0_{2n}$ is given by the $e_{\alpha_{i,j}}$ where $i$ and $j$ are both even.

These $\alpha_{i,j}$ are exactly the inversions of efficient permutations (see the proof of Theorem 4.5).

## 5. Electrical Lie algebras of finite type

***Dimension.*** Let $D$ be a Dynkin diagram of finite type and let $A = (a_{ij})$ be the associated Cartan matrix. To each node $i$ of $D$ associate a generator $e_i$, and define $\mathfrak{e}_D$ to be the Lie algebra generated over $\mathbb{R}$ by the $e_i$, subject to the relations, for each $i \neq j$,

$$\begin{cases} \mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 & \text{if } a_{ij} \neq -1, \\ \mathrm{ad}(e_i)^2(e_j) = -2e_i & \text{if } a_{ij} = -1. \end{cases}$$

These “electrical Serre relations” can be deduced from the type $A$ electrical Serre relations of Section 3 by *folding*. Namely, the relation for an edge of multiplicity two ($a_{ij} = -2$) can be obtained by finding the relation for the elements $e_2$ and $e_1 + e_3$ inside $\mathfrak{el}_3$. Similarly, the $\mathfrak{e}_{G_2}$ relation ($a_{ij} = -3$) can be obtained by finding the relation for the elements $e_1 + e_2 + e_3$ and $e_4$ inside $\mathfrak{e}_{D_4}$, where $e_4$ corresponds to the node of valency three in $D_4$.

Let $\mathfrak{u}_D$ denote the nilpotent Lie subalgebra of the simple Lie algebra of type $D$. It is well known that $\mathfrak{u}_D$ is generated by Chevalley generators $\tilde{e}_i$ subject to the Serre relations $\mathrm{ad}(\tilde{e}_i)^{1-a_{ij}}(\tilde{e}_j) = 0$ for each $i \neq j$. Thus the electrical Lie algebra $\mathfrak{e}_D$ is a deformation of $\mathfrak{u}_D$, and it follows from general results that $\dim \mathfrak{e}_D \leq \dim \mathfrak{u}_D$. For future reference, we include a direct elementary proof of this fact.

**Lemma 5.1.** *The dimension of the Lie algebra $\mathfrak{e}_D$ does not exceed the number of positive roots in the finite root system associated to $D$.*

*Proof.* The Lie algebra $\mathfrak{u}_D$ has a basis $\tilde{e}_\alpha$ labeled by positive roots $\alpha \in R^+$ of the root system, where the $\tilde{e}_i$ correspond to simple roots. Let us fix an expression

$$\tilde{e}_\alpha = [\tilde{e}_{i_1}, [\tilde{e}_{i_2}, [\ldots, [\tilde{e}_{i_{l-1}}, \tilde{e}_{i_l}] \ldots]]]$$

of shortest possible length for each $\tilde{e}_\alpha$, and define the corresponding

$$e_\alpha = [e_{i_1}, [e_{i_2}, [\ldots, [e_{i_{l-1}}, e_{i_l}] \ldots]]] \tag{7}$$

in $\mathfrak{e}_D$. We claim that the $e_\alpha$ span $\mathfrak{e}_D$. Indeed, it is enough to show that any expression $[e_{j_1}, [e_{j_2}, [\ldots, [e_{j_{\ell-1}}, e_{j_\ell}] \ldots]]]$ lies in the linear span of the $e_\alpha$. Assume otherwise, and take a counterexample of smallest possible total length $\ell$. Let us define $\hat{e}_\alpha$ in the free Lie algebra $\mathfrak{f}_D$ with generators $\hat{e}_i$ using (7). Then in $\mathfrak{f}_D$ we have a relation of the form

$$[\hat{e}_{j_1}, [\hat{e}_{j_2}, [\ldots, [\hat{e}_{j_{\ell-1}}, \hat{e}_{j_\ell}] \ldots]]] - \sum_\alpha c_\alpha \hat{e}_\alpha = \hat{x} \in I$$

where $I$ denotes the ideal generated by the Serre relations, so that $\mathfrak{u}_D = \mathfrak{f}_D/I$. Now $\mathfrak{f}_D$ is naturally $\mathbb{Z}$-graded, and $I$ is a graded ideal, so we may assume all terms in the relation are homogeneous with the same degree. Now replacing each instance of the Serre relation in $x$ with the corresponding electrical Serre relation gives us a relation

$$[e_{j_1}, [e_{j_2}, [\dots, [e_{j_{\ell-1}}, e_{j_\ell}] \dots]]] - \sum_{\alpha} c_\alpha e_\alpha = x$$

in $\mathfrak{e}_D$, where $x$ is a sum of terms of the form $[e_{k_1}, [e_{k_2}, [\dots, [e_{k_{\ell-1}}, e_{k_{\ell'}}] \dots]]]$ with $\ell' < \ell$, and thus by assumption lies in the span of the $e_\alpha$. The statement of the lemma follows. □

**Conjecture 5.2.** *The dimension of $\mathfrak{e}_D$ coincides with the number of positive roots in the root system of $D$.*

A proof of Conjecture 5.2 for Dynkin diagrams of classical type has been announced by Yi Su. One can also define the Lie algebras $\mathfrak{e}_D$ for any Dynkin diagram $D$, of finite type or not.

**Conjecture 5.3.** *The Lie algebra $\mathfrak{e}_D$ is finite-dimensional if and only if $D$ is of finite type.*

***Examples.*** We illustrate Conjecture 5.2 with some examples. For electrical type $A_{2n}$ it has already been verified in Theorem 3.1.

*Electrical $B_2$.* Consider the case when $D$ has two nodes connected by a double edge. In that case we denote by $\mathfrak{e}_D = \mathfrak{eb}_2$ the Lie algebra generated by two generators $e$ and $f$ subject to the relations

$$[e, [e, [e, f]]] = 0, \quad [f, [f, e]] = -2f.$$

**Lemma 5.4.** *The Lie algebras $\mathfrak{eb}_2$ and $\mathfrak{gl}_2$ are isomorphic.*

*Proof.* Consider the map

$$e \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

One easily checks that it is a Lie algebra homomorphism, and that it is surjective. By Lemma 5.1 we know the dimension of $\mathfrak{eb}_2$ is at most four, so the map must be an isomorphism. □

*Electrical $G_2$.* Consider the Lie algebra $\mathfrak{eg}_2$ generated by two generators $e$ and $f$ subject to the relations

$$[e, [e, [e, [e, f]]]] = 0, \quad [f, [f, e]] = -2f.$$

**Lemma 5.5.** *The Lie algebra $\mathfrak{eg}_2$ is six-dimensional.*

*Proof.* According to Lemma 5.1 the elements

$$e, \quad f, \quad [ef], \quad [e[ef]], \quad [e[e[ef]]], \quad [f[e[e[ef]]]]$$

form a spanning set for $\mathfrak{eg}_2$. Thus it remains to check that they are linearly independent. This is easily done inside the following faithful representation of $\mathfrak{eg}_2$ in $\mathfrak{gl}_4$:

$$e \mapsto \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \qquad \square$$

*Electrical* $C_3$. Consider the Lie algebra $\mathfrak{ec}_3$ generated by three generators $e$, $f$ and $g$ subject to the relations

$$[e,[e,[e,f]]] = 0, \quad [f,[f,e]] = -2f,$$
$$[f,[f,g]] = -2f, \quad [g,[g,f]] = -2g, \quad [e,g] = 0.$$

**Lemma 5.6.** *The Lie algebra* $\mathfrak{ec}_3$ *is nine-dimensional.*

*Proof.* We consider two representations of $\mathfrak{ec}_3$: one inside $\mathfrak{sl}_9$ and one inside $\mathfrak{sl}_2$. Let $E_{ij}$ denote a matrix with a 1 in the $(i, j)$-th position and 0s elsewhere. For the first representation, we define

$$\begin{aligned} e &\mapsto E_{42} + E_{54} + E_{76} + E_{87} + E_{89}, \\ f &\mapsto 2E_{24} - 2E_{26} + 2E_{41} + 2E_{45} - E_{47} - 2E_{63} + E_{67} + E_{98}, \\ g &\mapsto -E_{36} - E_{62} - E_{74} - E_{85} + E_{89}. \end{aligned}$$

For the second representation we define

$$e \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One verifies directly that these are indeed representations of $\mathfrak{ec}_3$, and the direct sum of these two representations is faithful, from which the dimension is easily calculated. In fact, the first representation is simply the adjoint representation of $\mathfrak{ec}_3$, in the basis

$$e, \quad f, \quad g, \quad [ef], \quad [e[ef]], \quad [fg], \quad [e[fg]], \quad [e[e[fg]]], \quad [f[e[e[fg]]]],$$

and $\mathfrak{ec}_3$ has a one-dimensional center which acts nontrivially in the second representation. $\square$

***$Y-\Delta$ transformation of type $B$.*** Let $e$ and $f$ be the generators of $\mathfrak{eb}_2$ as before, and denote by $u(t)=\exp(te)$ and $v(t)=\exp(tf)$ the corresponding one-parameter subgroups. The following proposition is a type $B$ analog of the star-triangle transformation (Proposition 2.2).

**Proposition 5.7.** *We have*

$$u(t_1)v(t_2)u(t_3)v(t_4)=v(p_1)u(p_2)v(p_3)u(p_4),$$

*where*

$$p_1=\frac{t_2t_3^2t_4}{\pi_2},\quad p_2=\frac{\pi_2}{\pi_1},\quad p_3=\frac{\pi_1^2}{\pi_2},\quad p_4=\frac{t_1t_2t_3}{\pi_1},$$

*where*

$$\pi_1=t_1t_2+(t_1+t_3)t_4+t_1t_2t_3t_4,\quad \pi_2=t_1^2t_2+(t_1+t_3)^2t_4+t_1t_2t_3t_4(t_1+t_3).$$

*Proof.* Direct calculation inside $\mathrm{GL}_2$, using Lemma 5.4. We have

$$u(t)=\begin{pmatrix} e^t & e^t t \\ 0 & e^t \end{pmatrix},\quad v(t)=\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

and both sides of the equality are equal to

$$\begin{pmatrix} \mathfrak{e}^{t_1+t_3}(1+t_3t_4+t_1(t_2+t_4+t_2t_3t_4)) & e^{t_1+t_3}(t_1+t_3+t_1t_2t_3) \\ e^{t_1+t_3}(t_2+t_4+t_2t_3t_4) & e^{t_1+t_3}(1+t_2t_3) \end{pmatrix}. \qquad \square$$

**Remark.** One can consider a one-parameter family of deformations of the above formulae by taking

$$\pi_1=t_1t_2+(t_1+t_3)t_4+\tau t_1t_2t_3t_4,\quad \pi_2=t_1^2t_2+(t_1+t_3)^2t_4+\tau t_1t_2t_3t_4(t_1+t_3).$$

At $\tau=0$ this specializes to the transformation found in [Berenstein and Zelevinsky 1997, Theorem 3.1]. Furthermore, just like that transformation, this deformation is given by *positive* rational formulae.

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# The Kac–Wakimoto character formula for the general linear Lie superalgebra

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We prove the Kac–Wakimoto character formula for the general linear Lie superalgebra $\mathfrak{gl}(m|n)$, which was conjectured by Kac and Wakimoto in 1994. This formula specializes to the well-known Kac–Weyl character formula when the modules are typical and to the Weyl denominator identity when the module is trivial. We also prove a determinantal character formula for KW-modules.

In our proof, we demonstrate how to use odd reflections to move character formulas between the different sets of simple roots of a Lie superalgebra. As a consequence, we show that KW-modules are precisely Kostant modules, which were studied by Brundan and Stroppel, thus yielding a simple combinatorial defining condition for KW-modules and a classification of these modules.

## 1. Introduction

It is well known that character theory for Lie superalgebras is a nontrivial generalization of the classical theory. The search for a Kac–Weyl-type character formula has been a driving force in the field. The heart of the problem lies in the existence of the so-called "atypical roots". In their absence, a formula similar to the classical Weyl character formula was proven for finite-dimensional typical highest weight modules by Kac [1977a; 1977b; 1978]. For the singly atypical weights (those with only one atypical root), a closed formula was proven by Bernstein and Leites [1980] for $\mathfrak{gl}(m|n)$ using geometrical methods. It was a long-standing question to generalize this formula, and many people contributed, including Van der Jeugt, Hughes, King and Thierry-Mieg [Van der Jeugt et al. 1990], who proposed a conjectural character formula for $\mathfrak{gl}(m|n)$, which was later proven by Su and Zhang [2007].

Serganova [1996; 1998] introduced the generalized Kazhdan–Lusztig polynomials for $\mathfrak{gl}(m|n)$, and used them to give a general character formula for finite-dimensional irreducible representations of $\mathfrak{gl}(m|n)$. For each $\lambda$ and $\mu$ dominant

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integral weights, evaluating the Kazhdan–Lusztig polynomial $K_{\lambda,\mu}(q)$ at $q=-1$ was shown to yield the multiplicity of the Kac module $\overline{L}(\mu)$ inside the finite-dimensional simple module $L(\lambda)$.

Brundan [2003] gave a new algorithm for computing the generalized Kazhdan–Lusztig polynomials for $\mathfrak{gl}(m|n)$. This algorithm was proven to be combinatorially equivalent to Serganova's original algorithm by Musson and Serganova [2011]. Cheng, Wang and Zhang [Cheng et al. 2008b] showed that the generalized Kazhdan–Lusztig polynomials for $\mathfrak{gl}(m|n)$ correspond to the usual parabolic Kazhdan–Lusztig polynomials of type $A$.

Su and Zhang [2007, 4.43] used Brundan's algorithm to compute the generalized Kazhdan–Lusztig polynomials and to prove a Kac–Weyl type character formula for finite-dimensional simple modules of $\mathfrak{gl}(m|n)$ in the standard choice of simple roots, which we refer to as the Su–Zhang character formula. When the highest weight $\lambda$ is "totally connected" (see Definition 18), every nonzero Kazhdan–Lusztig polynomial is a monomial with coefficient 1, which drastically simplifies the Su–Zhang character formula (4-1) [2007, 4.46].

We use the Su–Zhang character formula to prove the Kac–Wakimoto character formula for $\mathfrak{gl}(m|n)$, which was conjectured in [Kac and Wakimoto 1994, Theorem 3.1]. This conjecture was recently restated in [Kac and Wakimoto 2014, Conjecture 3.6], where the affine version of the formula was shown to yield a new class of modular invariant functions.

**Theorem 1.** *Let $L$ be a finite-dimensional simple module. For any choice of simple roots $\pi$ and weight $\lambda$ such that $L=L_\pi(\lambda)$ and $\pi$ contains a $\lambda+\rho$-maximal isotropic subset $S_\lambda$, we have*

$$e^\rho R\cdot \operatorname{ch} L_\pi(\lambda)=\frac{1}{r!}\sum_{w\in W}(-1)^{l(w)}w\left(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S_\lambda}(1+e^{-\beta})}\right), \tag{1-1}$$

*where $r=|S_\lambda|$.*

We call a finite-dimensional simple module $L$ a *KW-module* if there exists a set of simple roots $\pi$ that satisfies the hypothesis of Theorem 1, and we call such a $\pi$ an *admissible* choice of simple roots for $L$.

Our proof goes as follows. We prove that a finite-dimensional simple module is a KW-module if and only if its highest weight with respect to the standard choice of simple roots is totally connected (Theorem 21), by presenting an algorithm to move between these different bases. We generalize the arc diagrams defined in [Gorelik et al. 2012], and we use these diagrams to define the steps of our algorithm, which are composed of a specified sequence of odd reflections. We show that each step of the algorithm preserves a generalized version of the Su–Zhang character formula, and that this generalized formula specializes to the Kac–Wakimoto character formula when the set of simple roots is admissible, thus proving that KW-modules for $\mathfrak{gl}(m|n)$ are tame.

In this way, we obtain a character formula for each admissible set of simple roots $\pi$ and $(\lambda+\rho)$-maximal isotropic subset $S_\lambda \subset \pi$. We also obtain character formulas for the sets of simple roots encountered when applying the "shortening algorithm" to a totally connected weight, but not for all sets of simple roots. When the module is trivial, these character formulas specialize to the denominator identities obtained by Gorelik, Kac, Möseneder Frajria and Papi [Gorelik 2012; Gorelik et al. 2012].

We use the Kac–Wakimoto character formula to prove a determinantal character formula for KW-modules (Theorem 44), which is motivated by the determinantal character formula proven in [Moens and Van der Jeugt 2004] for critical modules labeled by nonintersecting composite partitions. Our determinantal character formula can be expressed using the data of the "special arc diagram" for a KW-module $L$ (see Definition 24), and is useful for computer computations.

To make the paper self-contained, we give a simplified version of the proof of the Su–Zhang character formula in the special case that the highest weight of the module is totally connected, i.e., a KW-module. Along the way, we obtain another characterization of KW-modules in terms of their Kazhdan–Lusztig polynomials (see Corollary 31), and we obtain the character formula of Theorem 37, which is motivated by the denominator identity given in [Gorelik et al. 2012, (1.10)] for the standard choice of simple roots.

An important class of KW-modules are the covariant modules. The tensor module $T(V)$ of the $(m+n)$-dimensional natural representation $V$ of $\mathfrak{gl}(m|n)$ is completely reducible, and its irreducible components are called *covariant modules*. These modules and their characters (super-Schur functions) were studied by Berele and Regev [1987], and Sergeev [1984], who in particular gave a necessary and sufficient condition on a weight $\lambda$ in terms of Young diagrams to be the highest weight of a covariant module for $\mathfrak{gl}(m|n)$ (with respect to the standard set of simple roots). Specifically, the irreducible components of the $k$-th tensor power of $V$ are parametrized by Young diagrams of size $k$ contained in the $(m, n)$-hook. Using this description, it is easy to show that the highest weight of a covariant module is totally connected (see Example 20). Hence, it follows from Theorem 21 that covariant modules are KW-modules, which was originally proven in [Moens and Van der Jeugt 2003].

The KW-modules studied in this paper turn out to be the same as what are called Kostant modules in [Brundan and Stroppel 2010; Brundan and Stroppel 2012] (see Remark 33). Brundan and Stroppel proved that all Kostant modules possess a BGG-type resolution, generalizing the BGG-type resolutions for covariant modules previously constructed by Cheng, Kwon and Lam [Cheng et al. 2008a]. In this way, the character formula in (4-4) can be realized as the Euler characteristic of this resolution.

## 2. Preliminaries

**2.1.** ***The general linear Lie superalgebra.*** In this paper, $\mathfrak{g}$ will always denote the general linear Lie superalgebra $\mathfrak{gl}(m|n)$ over the complex field $\mathbb{C}$. As a vector space, $\mathfrak{g}$ can be identified with the endomorphism algebra $\mathrm{End}(V_{\bar{0}} \oplus V_{\bar{1}})$ of a $\mathbb{Z}_2$-graded vector space $V_{\bar{0}} \oplus V_{\bar{1}}$ with $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. Then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where

$$\mathfrak{g}_{\bar{0}} = \mathrm{End}(V_{\bar{0}}) \oplus \mathrm{End}(V_{\bar{1}}) \quad \text{and} \quad \mathfrak{g}_{\bar{1}} = \mathrm{Hom}(V_{\bar{0}}, V_{\bar{1}}) \oplus \mathrm{Hom}(V_{\bar{1}}, V_{\bar{0}}).$$

A homogeneous element $x \in \mathfrak{g}_{\bar{0}}$ has degree 0, denoted $\deg(x) = 0$, while $x \in \mathfrak{g}_{\bar{1}}$ has degree 1, denoted $\deg(x) = 1$. We define a bilinear operation on $\mathfrak{g}$ by letting

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)} yx$$

on homogeneous elements and then extending linearly to all of $\mathfrak{g}$.

By fixing a basis of $V_{\bar{0}}$ and $V_{\bar{1}}$, we can realize $\mathfrak{g}$ as the set of $(m+n) \times (m+n)$ matrices, where

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \;\middle|\; A \in M_{m,m},\, B \in M_{n,n} \right\},$$

$$\mathfrak{g}_{\bar{1}} = \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \;\middle|\; C \in M_{m,n},\, D \in M_{n,m} \right\},$$

and $M_{r,s}$ denotes the set of $r \times s$ matrices.

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the set of diagonal matrices, and it has a natural basis

$$\{E_{1,1}, \dots, E_{m,m}; E_{m+1,m+1}, \dots, E_{m+n,m+n}\},$$

where $E_{ij}$ denotes the matrix whose $ij$-entry is 1 and there are 0s elsewhere. Fix the dual basis $\{\varepsilon_1, \dots, \varepsilon_m; \delta_1, \dots, \delta_n\}$ for $\mathfrak{h}^*$. We define a bilinear form on $\mathfrak{h}^*$ by $(\varepsilon_i, \varepsilon_j) = \delta_{ij} = -(\delta_i, \delta_j)$ and $(\varepsilon_i, \delta_j) = 0$, and use it to identify $\mathfrak{h}$ with $\mathfrak{h}^*$.

Then $\mathfrak{g}$ has a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \big(\bigoplus_{\alpha \in \Delta_{\bar{0}}} \mathfrak{g}_\alpha\big) \oplus \big(\bigoplus_{\alpha \in \Delta_{\bar{1}}} \mathfrak{g}_\alpha\big)$, where the set of roots of $\mathfrak{g}$ is $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$, with

$$\Delta_{\bar{0}} = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le m\} \cup \{\delta_k - \delta_l \mid 1 \le k \ne l \le n\},$$
$$\Delta_{\bar{1}} = \{\pm(\varepsilon_i - \delta_k) \mid 1 \le i \le m,\, 1 \le k \le n\},$$

and

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}E_{ij}, \quad \mathfrak{g}_{\delta_k - \delta_l} = \mathbb{C}E_{m+k,m+l}, \quad \mathfrak{g}_{\varepsilon_i - \delta_k} = \mathbb{C}E_{i,m+k}, \quad \mathfrak{g}_{\delta_k - \varepsilon_i} = \mathbb{C}E_{m+k,i}.$$

A set of simple roots $\pi \subset \Delta$ determines a decomposition of $\Delta$ into positive and negative roots $\Delta = \Delta^+ \cup \Delta^-$. There is a corresponding triangular decomposition of $\mathfrak{g}$ given by $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$. Let $\Delta^+_{\bar{d}} = \Delta_{\bar{d}} \cap \Delta^+$ for $d \in \{0, 1\}$, and define $\rho_\pi = \frac{1}{2}\sum_{\alpha \in \Delta^+_{\bar{0}}} \alpha - \frac{1}{2}\sum_{\alpha \in \Delta^+_{\bar{1}}} \alpha$. Then for $\alpha \in \pi$ we have $(\rho_\pi, \alpha) = (\alpha, \alpha)/2$.

The Weyl group of $\mathfrak{g}$ is $W = \mathrm{Sym}_m \times \mathrm{Sym}_n$, and $W$ acts on $\mathfrak{h}^*$ by permuting separately indices on $\varepsilon$ and $\delta$. In particular, the even reflection $s_{\varepsilon_i - \varepsilon_j}$ interchanges indices $i$ and $j$ on $\varepsilon$ and fixes all the others, while $s_{\delta_k - \delta_l}$ interchanges indices $k$ and $l$ on $\delta$ and fixes all the others.

A proof of the following lemma can be found in [Gorelik 2012, 4.1.1]:

**Lemma 2.** *For any $\mu \in \mathfrak{h}^*_{\mathbb{R}}$ the stabilizer of $\mu$ in $W$ is either trivial or contains a reflection.*

Suppose $\beta \in \pi$ is an odd (isotropic) root. An odd reflection $r_\beta$ of each $\alpha \in \pi$ is defined by

$$r_\beta(\alpha) = \begin{cases} cl - \alpha & \text{if } \beta = \alpha, \\ \alpha & \text{if } (\alpha, \beta) = 0, \\ \alpha + \beta & \text{if } (\alpha, \beta) \neq 0. \end{cases}$$

Then $r_\beta \pi := \{r_\beta(\alpha) \mid \alpha \in \pi\}$ is also a set of simple roots for $\mathfrak{g}$ [Serganova 2011]. The corresponding root decomposition is $\Delta = \Delta'^{+} \cup \Delta'^{-}$, where $\Delta'^{+}_{\bar{0}} = \Delta^{+}_{\bar{0}}$ is unchanged and $\Delta'^{+}_{\bar{1}} = (\Delta^{+}_{\bar{1}} \setminus \{\beta\}) \cup \{-\beta\}$. Using a sequence of even and odd reflections, one can move between any two sets of simple roots for $\mathfrak{g}$. Moreover, if $\pi$ and $\pi''$ have the property that $\Delta''^{+}_{\bar{0}} = \Delta^{+}_{\bar{0}}$, then there exists a sequence of odd reflections from $\pi$ to $\pi''$.

We denote by $\pi_{\mathrm{st}}$ the standard choice of simple roots

$$\pi_{\mathrm{st}} = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \delta_1 - \delta_2 \ldots, \delta_{n-1} - \delta_n\}.$$

The corresponding decomposition $\Delta = \Delta^+ \cup \Delta^-$ is given by

$$\Delta^{+}_{\bar{0}} = \{\varepsilon_i - \varepsilon_j\}_{1 \le i < j \le m} \cup \{\delta_k - \delta_l\}_{1 \le k < l \le n} \quad \text{and} \quad \Delta^{+}_{\bar{1}} = \{\varepsilon_i - \delta_k\}_{1 \le i \le m, 1 \le k \le n}. \tag{2-1}$$

The standard choice $\pi_{\mathrm{st}}$ has the unique property that $W$ fixes $\Delta^{+}_{\bar{1}}$. Moreover, $\pi_{\mathrm{st}}$ contains a basis for $\Delta^{+}_{\bar{0}}$, which is denoted by $\pi_{\bar{0}} := \pi_{\mathrm{st}} \cap \Delta^{+}_{\bar{0}}$.

The root lattice $Q = \sum_{\alpha \in \pi} \mathbb{Z}\alpha$ is independent of the choice of $\pi$. Let $Q^{+}_{\pi} = \sum_{\alpha \in \pi} \mathbb{N}\alpha$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, and define a partial order on $\mathfrak{h}^*$ by $\mu > \nu$ when $\mu - \nu \in Q^{+}_{\mathrm{st}}$.

**Remark 3.** For convenience, we fix $\Delta^{+}_{\bar{0}}$ as in (2-1). This choice is arbitrary since we can relabel the indices of the $\varepsilon$'s and $\delta$'s. We let $l(w)$ denote the length of $w \in W$ with respect to the set of simple reflections $s_{\varepsilon_1 - \varepsilon_2}, \ldots, s_{\varepsilon_{m-1} - \varepsilon_m}; s_{\delta_1 - \delta_2}, \ldots, s_{\delta_{n-1} - \delta_n}$ generating $W$.

### 2.2. *Finite-dimensional modules for* $\mathfrak{gl}(m|n)$.

For each set of simple roots $\pi$ and weight $\lambda \in \mathfrak{h}^*$, the *Verma module* of highest weight $\lambda$ is the induced module

$$M_\pi(\lambda) := \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{n}^+ \oplus \mathfrak{h}} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional module such that $h \in \mathfrak{h}$ acts by scalar multiplication of $\lambda(h)$ and $\mathfrak{n}^+$ acts trivially. The Verma module $M_\pi(\lambda)$ has a unique simple quotient, which we denote by $L_\pi(\lambda)$ or simply by $L(\lambda)$. Given $\pi$ and $\lambda$, we define

$$\lambda_\pi^\rho := \lambda + \rho_\pi,$$

which we also denote by $\lambda^\rho$. If $L_\pi(\lambda) \cong L_{\pi'}(\lambda')$ for some $\lambda, \lambda' \in \mathfrak{h}^*$ and $\pi' = r_\beta \pi$ for an odd reflection $r_\beta$ with $\beta \in \pi$, then

$$(\lambda')^\rho = \begin{cases} \lambda^\rho & \text{if } (\lambda, \beta) \neq 0, \\ \lambda^\rho + \beta & \text{if } (\lambda, \beta) = 0. \end{cases}$$

For each $\lambda \in \mathfrak{h}^*$, let $L_{\bar{0}}(\lambda)$ denote the simple highest weight $\mathfrak{g}_{\bar{0}}$-module with respect to $\pi_{\bar{0}}$. The *Kac module* of highest weight $\lambda$ with respect to $\pi_{\text{st}}$ is the induced module

$$\overline{L}(\lambda) := \operatorname{Ind}_{\mathfrak{g}_{\bar{0}} \oplus \mathfrak{n}_{\bar{1}}^+}^{\mathfrak{g}} L_{\bar{0}}(\lambda)$$

defined by letting $\mathfrak{n}_{\bar{1}}^+ := \bigoplus_{\alpha \in \Delta_{\bar{1}}^+} \mathfrak{g}_\alpha$ act trivially on the $\mathfrak{g}_{\bar{0}}$-module $L_{\bar{0}}(\lambda)$. The unique simple quotient of $\overline{L}(\lambda)$ is $L_{\pi_{\text{st}}}(\lambda)$.

A weight $\lambda \in \mathfrak{h}^*$ is called *dominant* if $2(\lambda, \alpha)/(\alpha, \alpha) \geq 0$ for all $\alpha \in \Delta_{\bar{0}}^+$, *strictly dominant* if $2(\lambda, \alpha)/(\alpha, \alpha) > 0$ for all $\alpha \in \Delta_{\bar{0}}^+$ and *integral* if $2(\lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta_{\bar{0}}^+$. It is sufficient to check dominance and integrality on the set $\pi_{\bar{0}}$ of simple roots of $\Delta_{\bar{0}}^+$.

The proof of the following lemma is straightforward when viewing $L_\pi(\lambda)$ as a $\mathfrak{g}_{\bar{0}}$-module.

**Lemma 4.** *If the simple module $L_\pi(\lambda)$ is finite-dimensional, then $\lambda$ is a dominant integral weight.*

For a proof of the following proposition, see for example [Musson 2012, 14.1.1]:

**Proposition 5.** *For $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\lambda \in \mathfrak{h}^*$, the following are equivalent*:

(1) *The simple highest weight $\mathfrak{g}$-module $L_{\pi_{\text{st}}}(\lambda)$ is finite-dimensional.*

(2) *The Kac module $\overline{L}(\lambda)$ is finite-dimensional.*

(3) *The simple highest weight $\mathfrak{g}_{\bar{0}}$-module $L_{\bar{0}}(\lambda)$ is finite-dimensional.*

(4) *$\lambda$ is a dominant integral weight.*

(5) *$\lambda_{\text{st}}^\rho$ is a strictly dominant integral weight.*

**2.3.** ***Atypical modules.*** The *atypicality* of $L_\pi(\lambda)$ is the maximal number of linearly independent positive roots $\beta_1, \ldots, \beta_r$ such that $(\beta_i, \beta_j) = 0$ and $(\lambda^\rho, \beta_i) = 0$ for $i, j = 1, \ldots, r$. Such a set $S = \{\beta_1, \ldots, \beta_r\}$ is called a *$\lambda^\rho$-maximal isotropic set*. The module $L_\pi(\lambda)$ is called *typical* if this set is empty, and *atypical* otherwise. The atypicality of a simple finite-dimensional module is independent of the choice of simple roots.

**Definition 6.** We call a simple finite-dimensional module $L$ a *KW-module* if $L \cong L_\pi(\lambda)$ for some choice of simple roots $\pi$ which contains a $\lambda^\rho$-maximal isotropic set $S \subset \pi$. In this case, we call such a $\pi$ an *admissible* choice of simple roots for $L$.

**Remark 7.** A typical module is a KW-module with $S = \varnothing$.

Let $P$ denote the set of integral weights, $P^+$ the set of dominant integral weights, and define

$$\mathbb{P}^+ = \{\mu \in P^+ \mid (\mu_\pi^\rho, \varepsilon_i) \in \mathbb{Z}, (\mu_\pi^\rho, \delta_j) \in \mathbb{Z}\}.$$

Note that the definition of $\mathbb{P}^+$ is independent of the choice of $\pi$, since changing the set of simple roots by an odd reflection only changes the entries of $\lambda_\pi^\rho$ by integer values.

**Remark 8.** When studying the character of a simple finite-dimensional atypical module, we may restrict to the case that $\lambda \in \mathbb{P}^+$. Indeed, let $\lambda \in P^+$; then the module $L_\pi(\lambda)$ is atypical if and only if $(\lambda_\pi^\rho, \varepsilon_i) = (\lambda_\pi^\rho, \delta_j)$ for some $(\varepsilon_i - \delta_j) \in \Delta_{\bar{1}}$. So, by tensoring $L_\pi(\lambda)$ with a one-dimensional module with character $e^{c(\sum_{i=1}^m \varepsilon_i - \sum_{j=1}^n \delta_j)}$ for appropriate $c \in \mathbb{C}$, we obtain a module $L_\pi(\lambda')$ with $\lambda' \in \mathbb{P}^+$.

Fix a set of simple roots $\pi$, a weight $\lambda \in \mathbb{P}^+$ and a $\lambda^\rho$-maximal isotropic set $S_\lambda \subset \Delta_{\bar{1}}^+$. Write

$$\lambda_\pi^\rho = \sum_{i=1}^m a_i \varepsilon_i - \sum_{j=1}^n b_j \delta_j. \tag{2-2}$$

We refer to the coefficients $a_i$ and $b_j$ as the *$\varepsilon_i$-entry* and *$\delta_j$-entry*, respectively. If one of $\pm(\varepsilon_k - \delta_l)$ is in $S_\lambda$, then we call the $\varepsilon_k$- and $\delta_l$-entries *atypical*. Otherwise, an entry is called *typical*.

We denote by $(\lambda_\pi^\rho)^\Uparrow$ the element obtained from $\lambda_\pi^\rho$ by replacing all its atypical entries by the maximal atypical entry. Note that this can depend on the choice of $S_\lambda$, and not only on $\lambda_\pi^\rho$. However, for $\pi_{\mathrm{st}}$ and $\lambda \in \mathbb{P}^+$, there is a unique $S_\lambda \subset \Delta^+$. If $\nu \in \mathfrak{h}^*$ can be written as $\nu = \sum_{\alpha \in S_\lambda} k_\alpha \alpha$, then we define

$$|\nu|_{S_\lambda} := \sum_{\alpha \in S_\lambda} k_\alpha.$$

Observe that $|(\lambda_\pi^\rho)^\Uparrow - \lambda_\pi^\rho|_{S_\lambda}$ is nonnegative integer.

**2.4.** ***Arc diagrams.*** We generalize the arc diagrams defined in [Gorelik et al. 2012]. Let $L$ be a finite-dimensional atypical module. For each set of simple roots $\pi$, weight $\lambda \in \mathbb{P}^+$ such that $L = L_\pi(\lambda)$ and $\lambda^\rho$-maximal isotropic set $S_\lambda \subset \Delta_{\bar{1}}^+$, there is an arc diagram that encodes the data $\pi$, $\lambda_\pi^\rho$ and $S_\lambda$.

In order to define the arc diagram corresponding to the data $(\pi, \lambda^{\rho}_{\pi}, S_{\lambda})$, we first define a total order on the set $\{\varepsilon_1, \dots, \varepsilon_m\} \cup \{\delta_1, \dots, \delta_n\}$ determined by $\Delta^+$. In particular, $\varepsilon_i < \varepsilon_{i+1}$, $\delta_j < \delta_{j+1}$, and for each $i$ and $j$ we let

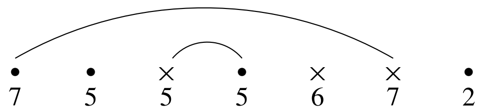
$$\begin{aligned} \delta_j < \varepsilon_i \quad &\text{if } (\delta_j - \varepsilon_i) \in \Delta^+, \\ \varepsilon_i < \delta_j \quad &\text{if } (\varepsilon_i - \delta_j) \in \Delta^+. \end{aligned}$$

Let $T = \{\gamma_1, \dots, \gamma_{m+n}\}$ be this totally ordered set, and express $\lambda^{\rho}_{\pi}$ as in (2-2). The *nodes* and *entries* of the diagram are determined from left to right by the elements $\gamma_k \in T$, $k = 1, \dots, m+n$ by putting a node $\bullet$ labeled with the entry $a_i$ if $\gamma_k = \varepsilon_i$ and a node $\times$ labeled with the entry $b_j$ if $\gamma_k = \delta_j$. The set $S_\lambda$ determines an *arc arrangement* as follows: The $\varepsilon_i$-node $\bullet$ and the $\delta_j$-node $\times$ are connected by an arc when one of $\pm(\varepsilon_i - \delta_j)$ is in $S_\lambda$, and in this case $a_i = b_j$. A node and its entry are called *atypical* if the node is connected by some arc, and *typical* otherwise.

**Example 9.** Suppose

$$\begin{aligned} \pi &= \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3, \varepsilon_3 - \delta_2, \delta_2 - \delta_3, \delta_3 - \varepsilon_4\}, \\ \lambda^{\rho}_{\pi} &= 7\varepsilon_1 + 5\varepsilon_2 + 5\varepsilon_3 + 2\varepsilon_4 - 5\delta_1 - 6\delta_2 - 7\delta_3, \\ S_\lambda &= \{\varepsilon_1 - \delta_3, \delta_1 - \varepsilon_3\}. \end{aligned}$$

Then the corresponding arc diagram will be:



We can recover the data $(\pi, \lambda^{\rho}_{\pi}, S_\lambda)$ from an arc diagram as follows. Label the $\bullet$-nodes from left to right by $\varepsilon_1, \dots, \varepsilon_m$, and the $\times$-nodes by $\delta_1, \dots, \delta_n$. Let $T = \{\gamma_1, \dots, \gamma_{m+n}\}$ be the ordered set determined by this labeling. Then $\pi = \{\gamma_1 - \gamma_2, \gamma_2 - \gamma_3, \dots, \gamma_{m+n-1} - \gamma_{m+n}\}$, $\lambda^{\rho}_{\pi}$ is given by (2-2), where $a_i$ is the $\varepsilon_i$-entry and $b_j$ is the $\delta_j$-entry, and $S_\lambda = \{\gamma_i - \gamma_j \mid i < j \text{ and } \gamma_i \text{ is connected by an arc to } \gamma_j\}$.

**Remark 10.** All entries of an arc diagram are integers, and adjacent $\bullet$-entries are strictly decreasing, while adjacent $\times$-entries are strictly increasing, because $\lambda \in \mathbb{P}^+$ and $\lambda^{\rho}_{\pi} = \lambda + \rho_\pi$ (see Remark 8).

**Remark 11.** We call the arc diagram for the standard choice of simple roots the *standard arc diagram*. For the standard arc diagram, there is only one possible arc arrangement and all arcs are “nested”. See for example diagram (3-3).

Since $S_\lambda \subset \Delta^+_{\bar{1}}$, an arc is always connected to a $\bullet$-node and a $\times$-node. We call an arc *short* if the $\bullet$-node and $\times$-node are adjacent. We say that an arc has $\bullet$-$\times$ type if the $\bullet$ precedes the $\times$, and $\times$-$\bullet$ type if the $\times$ precedes the $\bullet$. Note that no two arcs can share an endpoint, since $S_\lambda$ is a $\lambda^\rho$-maximal isotropic set. Moreover, an arc diagram by definition has a maximal arc arrangement, that is, it is not possible to add an arc to the diagram between typical $\bullet$- and $\times$-nodes with equal entries.

Adjacent $\bullet$- and $\times$-nodes correspond to an odd simple root $\beta$, and applying the odd reflection $r_\beta$ swaps these nodes. In terms of the diagram, this means that if $a \neq b$, then

$$\begin{array}{ccccccccc} \cdots & \underset{a}{\bullet} & \underset{b}{\times} & \cdots & \xleftrightarrow{r_\beta} & \cdots & \underset{b}{\times} & \underset{a}{\bullet} & \cdots \\ \cdots & \underset{a}{\bullet} & \underset{a}{\times} & \cdots & \xleftrightarrow{r_\beta} & \cdots & \underset{a+1}{\times} & \underset{a+1}{\bullet} & \cdots \end{array} \tag{2-3}$$

while all other nodes and entries are unchanged.

**2.5.** ***Weight diagrams.*** Weight diagrams are a convenient way to work with the highest weight of a module with respect to the standard choice of simple roots. They were introduced by Brundan and Stroppel [2011] and were used to give algorithmic character formulas for basic classical Lie superalgebras in [Gruson and Serganova 2010] (see also [Su and Zhang 2012, 5.1]).

Let $\lambda \in \mathbb{P}^+$, and write $\lambda^\rho_{\text{st}}$ as in (2-2). On the $\mathbb{Z}$-lattice, put $\times$ above $t$ if $t \in \{a_i\} \cap \{b_j\}$, put $>$ above $t$ if $t \in \{a_i\} \backslash \{b_j\}$, and put $<$ above $t$ if $t \in \{b_i\} \backslash \{a_j\}$. If $t \notin \{a_i\} \cup \{b_j\}$, then we refer to the placeholder above $t$ as an *empty spot*.

**Example 12.** If

$$\lambda^\rho_{\text{st}} = 10\varepsilon_1 + 9\varepsilon_2 + 7\varepsilon_3 + 5\varepsilon_4 + 4\varepsilon_5 - \delta_1 - 4\delta_2 - 6\delta_3 - 7\delta_4,$$

then the corresponding weight diagram $D_\lambda$ is

$$\begin{array}{cccccccccccccc} -1 & 0 & \overset{<}{1} & 2 & 3 & \overset{\times}{4} & \overset{>}{5} & \overset{<}{6} & \overset{\times}{7} & 8 & \overset{>}{9} & \overset{>}{10} & 11 & 12 \end{array} \tag{2-4}$$

**2.6.** ***Characters for category*** $\mathcal{O}$. Let $M$ be a module from the BGG category $\mathcal{O}$ [Musson 2012, 8.2.3]. Then $M$ has a weight-space decomposition $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$, where $M_\mu = \{x \in M \mid h.x = \mu(h)x \text{ for all } h \in \mathfrak{h}^*\}$, and the *character of* $M$ is by definition $\operatorname{ch} M = \sum_{\mu \in \mathfrak{h}^*} \dim M_\mu e^\mu$.

Denote by $\mathscr{E}$ the algebra of rational functions $\mathbb{Q}(e^\nu, \nu \in \mathfrak{h}^*)$. The group $W$ acts on $\mathscr{E}$ by mapping $e^\nu$ to $e^{w(\nu)}$. Corresponding to a choice of positive roots $\Delta^+$, the *Weyl denominator of* $\mathfrak{g}$ is defined to be

$$R = \frac{\prod_{\alpha \in \Delta^+_{\bar{0}}}(1 - e^{-\alpha})}{\prod_{\alpha \in \Delta^+_{\bar{1}}}(1 + e^{-\alpha})}.$$

Then $e^\rho R$ is $W$-skew-invariant, i.e., $w(e^\rho R) = (-1)^{l(w)} e^\rho R$, and $\operatorname{ch} L(\lambda)$ is $W$-invariant for $\lambda \in P^+$. The character of a Verma module $M(\lambda)$ with $\lambda \in \mathfrak{h}^*$ is $\operatorname{ch} M(\lambda) = e^\lambda R^{-1}$. The character of the Kac module $\overline{L}(\lambda)$ with $\lambda \in P^+$ is

$$\operatorname{ch} \overline{L}(\lambda) = \frac{1}{e^\rho R} \sum_{w \in W} (-1)^{l(w)} w(e^{\lambda^\rho}). \tag{2-5}$$

For $X \in \mathscr{E}$, we define

$$\mathscr{F}_W(X) := \sum_{w \in W} (-1)^{l(w)} w(X).$$

The proof of the following lemma is immediate:

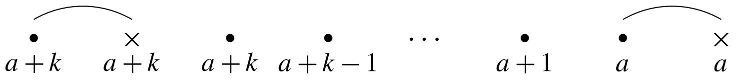
**Lemma 13.** *Let $X \in \mathscr{E}$. If $\sigma(X) = X$ for some reflection $\sigma \in W$, then $F_W(X) = 0$.*

## 3. Highest weights of KW-modules

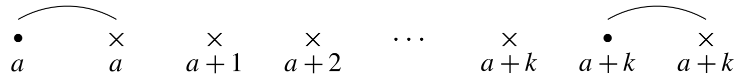
We describe the highest weights and arc diagrams of KW-modules with respect to different choices of simple roots.

**3.1.** ***An admissible choice of simple roots.*** Let us characterize the highest weight $\lambda$ of a KW-module $L$ with respect to an admissible choice of simple roots $\pi$ and a $\lambda^\rho$-maximal isotropic subset $S \subset \pi$.

**Lemma 14.** *Consider two arcs in the arc diagram of $\lambda^\rho$ with no arcs between them. Up to a reflection along these arcs, the corresponding subdiagram has one of the following forms:*



(3-1)



(3-2)

*Equivalently, the neighboring $\bullet$- and $\times$-entries are equal, and the entries between the two arcs are either all of $\bullet$-type and are increasing consecutive integers or are all of $\times$-type and are decreasing consecutive integers. Moreover, diagrams (3-1) and (3-2) cannot both be a subdiagram of the same admissible diagram.*

*Proof.* We shall analyze the entries of $\lambda$ and $\rho$ separately.

Let us show that all the entries of $\lambda$ in the range of the subdiagram are equal. We write $\lambda = \sum_{i=1}^m a_i \varepsilon_i - \sum_{i=1}^n b_i \delta_i$ and suppose that the arcs in diagram correspond to the simple isotropic roots $\alpha = \varepsilon_i - \delta_{i'}$ and $\beta = \varepsilon_j - \delta_{j'}$, with $i < j$, $i' < j'$. Since $\alpha$ and $\beta$ are simple and isotropic, they are orthogonal to $\rho$ and hence to $\lambda$. This implies that $a_i = b_{i'}$ and $a_j = b_{j'}$. Since $\lambda$ is dominant, $a_i \geq a_{i+1} \geq \cdots \geq a_j$ and $b_{i'} \leq b_{i'+1} \leq \cdots \leq b_{j'}$. Hence, $a_i = \cdots = a_j = b_{i'} = \cdots = b_{j'}$.

It follows that two entries in the subdiagram are equal if and only if they are equal in $\rho$. Since $\rho$ is orthogonal to all simple isotropic roots, the $\bullet$- and $\times$-entries are equal when adjacent. Hence, at least one of them must be connected with an arc, since otherwise we contradict the maximality property of the arc arrangement. Therefore, the entries between the two arcs are either all of $\bullet$-type or all of $\times$-type.

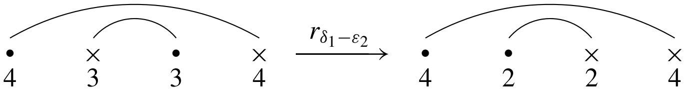
The difference between two consecutive entries of the subdiagram of $\lambda^\rho$ will be as in $\rho$, since all such entries of $\lambda$ are equal. In particular, they should decrease (resp. increase) by 1 whenever we have consecutive $\bullet$-entries (resp. $\times$-entries). $\square$

The following definition was introduced in [Moens and Van der Jeugt 2004, Proposition 1] for the standard choice of simple roots.

**Definition 15.** Let $\pi$ be any set of simple roots and let $\lambda \in \mathfrak{h}^*$. Write $\lambda^\rho = \sum_{i=1}^m a_i \varepsilon_i - \sum_{i=1}^n b_i \delta_i$. Suppose $S_\lambda = \{\varepsilon_{m_i} - \delta_{n_i}\}_{i=1}^r$ is a $\lambda^\rho$-maximal isotropic set ordered so that $a_{m_1} \leq a_{m_2} \leq \cdots \leq a_{m_r}$. We say that $\lambda^\rho$ satisfies the *interval property* if all the integers between $a_{m_1}$ and $a_{m_r}$ (equivalently, between $b_{n_1}$ and $b_{n_r}$) are contained in the set $\{a_i, b_j\}_{i=1,\dots,m;\, j=1,\dots,n}$.

**Corollary 16.** *Let $L$ be a KW-module, and let $\lambda$ be its highest weight with respect to an admissible choice of simple roots. Then $\lambda^\rho$ satisfies the interval property.*

**Remark 17.** Note that the interval property is not a property of a module. In particular, if $\pi_1, \pi_2$ are two choices of simple roots and $\lambda_1, \lambda_2 \in \mathfrak{h}^*$ are such that $= L_{\pi_1}(\lambda_{\pi_1}) = L_{\pi_2}(\lambda_{\pi_2})$, we can have that $\lambda_{\pi_1}^\rho$ satisfies the interval property but $\lambda_{\pi_2}^\rho$ doesn't. For example,



**3.2.** ***Totally connected weights in the standard choice.*** We prove a criterion for a module to be a KW-module given its highest weight $\lambda_{\mathrm{st}}$ with respect to the standard choice of simple roots $\pi_{\mathrm{st}}$. The following definition is equivalent to the one given in [Su and Zhang 2007], which was first observed in [Moens and Van der Jeugt 2004].

**Definition 18.** Let $\lambda_{\mathrm{st}} \in \mathbb{P}^+$. We say that $\lambda_{\mathrm{st}}$ is *totally connected* if $\lambda_{\mathrm{st}}^\rho$ satisfies the interval property with respect to $\pi_{\mathrm{st}}$.

**Remark 19.** In terms of weight diagrams, this is equivalent to the condition that there are no empty spots between the $\times$'s. For example, diagram (2-4) is totally connected, whereas diagram (4-3) is not.

**Example 20.** The highest weight of a covariant module is totally connected. Indeed, such a module corresponds to a partition $\mu$ of $k$ that lies in the $(m, n)$-hook, i.e., $\mu_{m+1} \leq n$ [Berele and Regev 1987; Sergeev 1984]. The corresponding covariant module is $L_{\pi_{\mathrm{st}}}(\lambda)$, where

$$\lambda = \mu_1 \varepsilon_1 + \cdots + \mu_m \varepsilon_m + \tau_1 \delta_1 + \cdots + \tau_n \delta_n,$$

$\tau_j = \max\{0, \mu'_j - m\}$ for $j = 1, \ldots, n$, and $\mu'_j$ is the length of the $j$-th column (see for example [Van der Jeugt et al. 1990, Section V]). Then $\tau_j = 0$ for $j > \mu_m$ and the arc diagram of $\lambda^\rho$ is

$$\begin{array}{cccccccccc} \bullet & \cdots & \bullet & \times & \cdots & \times & \times & \times & \cdots & \times \\ \mu_1+m & & \mu_m+1 & -\tau_1+1 & & -\tau_{\mu_m}+\mu_m & \mu_m+1 & \mu_m+2 & & n \end{array}$$

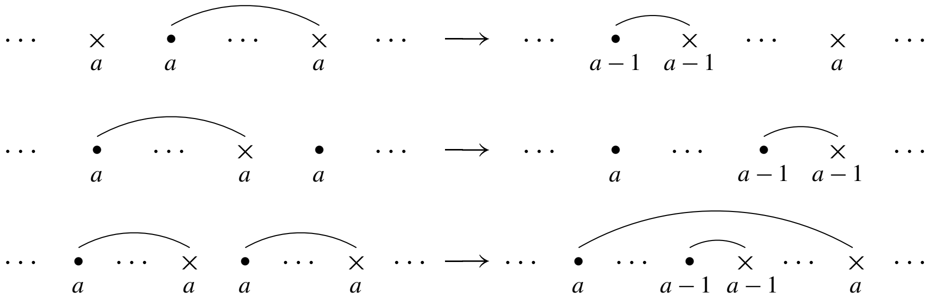
Since $\mu_m + 1 > -\tau_{\mu_m} + \mu_m$, there are no arcs connected to the first $\mu_m$ $\times$-nodes. Since the rest of the $\times$-entries are consecutive integers, $\lambda^\rho$ satisfies the interval property and hence $\lambda$ is totally connected. The covariant module $L_{\pi_{\mathrm{st}}}(\lambda)$ is typical if and only if $\mu_m \geq n$.

**Theorem 21.** *The finite-dimensional simple module L is a KW-module if and only if its highest weight with respect to the standard choice of simple roots is totally connected.*

*Proof of Theorem 21* "$\Rightarrow$". We start with an arc diagram that corresponds to an admissible choice of simple roots, that is, all arcs are short, and we move to the standard arc diagram by applying a sequence of odd reflections which preserve the interval property. We begin with reflecting along all the arcs which are of $\times$ - $\bullet$ type, and get another admissible choice of simple roots, for which the interval property holds by Corollary 16.

Now we push all the $\times$-entries to the right one at a time, starting with the rightmost $\times$-entry. All of our reflections are along consecutive $\times$- and $\bullet$-entries, and at each reflection there are several cases. If the entries below the $\times$ and the $\bullet$ are not equal, then the reflection does not change them and the interval property is clearly preserved.

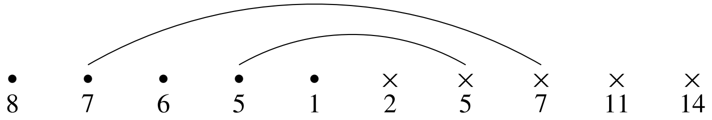
If the $\times$- and $\bullet$-entries are equal then at least one of them is connected to an arc, since otherwise the number of arcs could be increased, contradicting the maximality property of the arc arrangement. So there are three possibilities, namely, either the $\bullet$-entry or the $\times$-entry is connected to an arc, or both. In each case, we reflect at the consecutive $\times$ - $\bullet$ entries and then arrange the arcs to be of $\bullet$ - $\times$ type as follows:



In each case, an atypical entry of $a-1$ was added, but the interval property is preserved since we still have an $a$ in the diagram. The maximality property is also preserved since the new diagram has the same number of arcs as the old diagram. Moreover, the arcs of the new diagram are again nonintersecting and of $\bullet$ - $\times$ type. We proceed to the next $\times$, and continue until each $\times$ is to the right of every $\bullet$. □

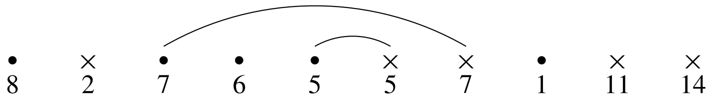
To prove the other direction, we introduce an algorithm to move from a totally connected weight in the standard choice of simple roots to an admissible choice of simple roots, which we call the *shortening algorithm*. Let's first illustrate our algorithm with an example.

**Example 22.** Let us take the following totally connected weight, and show a sequence of odd reflections that brings it to an admissible choice of simple roots:

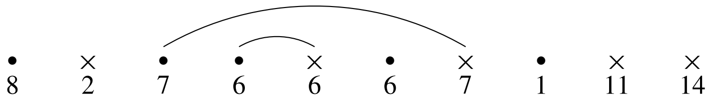


(3-3)

Since the innermost entries 1 and 2 are different from all the other entries, using the odd reflection defined in (2-3) we can move them outside the arcs to the right and left, respectively:

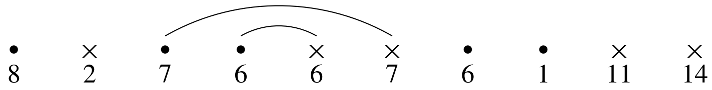


Next, we apply the odd reflection $s_{\varepsilon_4-\delta_2}$, and then choose the arc arrangement to be of $\bullet$ - $\times$ type so that we can move the extra 6 outside of the arcs:



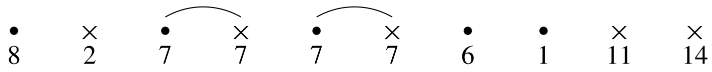
(3-4)

Then we move the 6 •-entry to the right outside of the arcs:



(3-5)

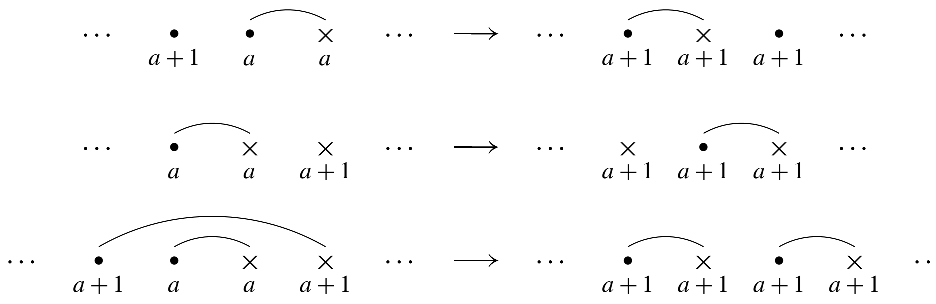
Finally, we apply $s_{\varepsilon_3-\delta_2}$, and then arrange the arcs to be short, obtaining an admissible choice of simple roots:



*Proof of Theorem 21* "⇐". We give an algorithm to move from a totally connected weight $\lambda_{\mathrm{st}}$ of a finite-dimensional simple module to an admissible choice of simple roots using a sequence of odd reflections. The main idea is to push all the typical entries which are below the arcs to the side, making all the atypical entries the same. This will allow us to choose an arc arrangement with only short arcs.

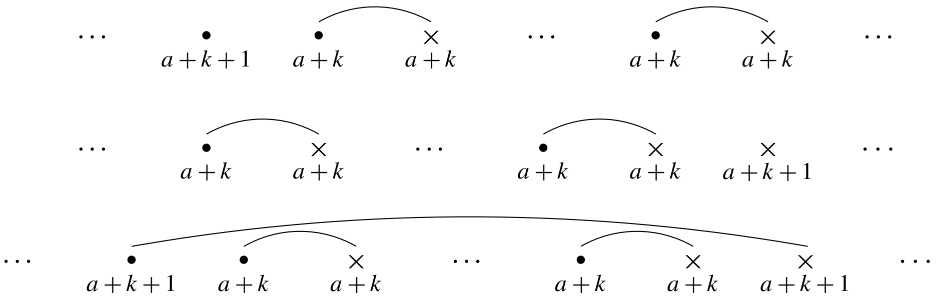
We have from Remark 10 that adjacent •-entries of $\lambda_\pi^\rho$ are strictly decreasing, while adjacent ×-entries of $\lambda_\pi^\rho$ are strictly increasing. Hence, in the standard arc diagram all equalities between entries correspond to arcs, and the arcs are nested in each other. Moreover, all of the entries below the innermost arc are typical and are different from the rest of the entries. Before applying the algorithm, we move these entries outside of the arcs by pushing the ×'s to the left and the •'s to the right, which makes the innermost arc short and of • - × type.

We begin the algorithm by reflecting along the innermost arc and then we arrange the arcs to be of • - × type. Due to the interval property there are three possibilities: either there is an $a+1$ •-entry on the left, an $a+1$ ×-entry on the right, or both, in which case they must be connected by an arc:

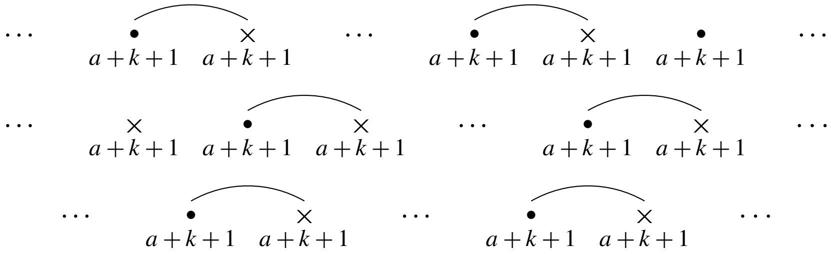


After the reflection, in the first case we push the unmatched $a+1$ •-entry to the right outside of the arcs, and in the second case we push the unmatched $a+1$ ×-entry to the left outside of the arcs.

This will be the base of our induction. Suppose that after $k$ steps all the atypical entries below $a+k+1$ are now equal to $a+k$ and are paired by short arcs, and all other entries which are below an arc remained as in the original diagram of $\lambda^{\rho}_{\text{st}}$. Due to the interval property there are now three possibilities, namely, either there is an $a+k+1$ $\bullet$-entry on the left, or an $a+k+1$ $\times$-entry on the right, or both, in which case they must be connected by an arc:



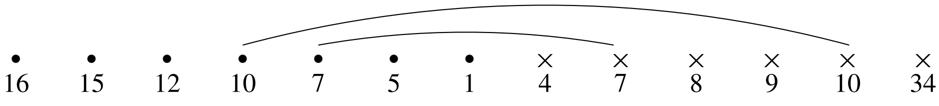
We reflect along the short arcs and arrange the arcs to get the following three diagrams, respectively:



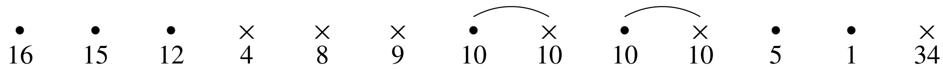
In the first case we push the unmatched $a+k+1$ $\bullet$-entry to the right outside of the arcs, and in the second case we push the unmatched $a+k+1$ $\times$-entry to the left outside of the arcs.

We continue this procedure until we reach the outermost arc. After doing the last step, all of the arcs become short and we get an admissible choice of simple roots. Moreover, all the atypical entries are now adjacent and equal to the largest atypical entry of $\lambda^{\rho}_{\text{st}}$. The typical $\bullet$-entries (resp. $\times$-entries) which were under an arc of $\lambda^{\rho}_{\text{st}}$ were pushed to the right (resp. left). $\square$

**Remark 23.** For each $\lambda^{\rho}_{\text{st}}$ which corresponds to a totally connected $\lambda$ in $\pi_{\text{st}}$, one can immediately determine the arc diagram given by shortening algorithm. For example, if $\lambda^{\rho}_{\text{st}}$ corresponds to the arc diagram



then the shortening algorithm gives



where the typical •-entries under an arc of $\lambda^{\rho}_{\mathrm{st}}$ are pushed to the right outside the arcs, the typical ×-entries under an arc of $\lambda^{\rho}_{\mathrm{st}}$ are pushed to left outside the arcs, the atypical entries are set equal to the maximal atypical entry of $\lambda^{\rho}_{\mathrm{st}}$, and then all arcs are chosen to be short.

**Definition 24.** We call an arc diagram for $L$ the *special arc diagram* if:

(1) All arcs are short, of •-× type and are adjacent.

(2) All atypical entries are equal.

(3) The typical nodes at each end of the diagram are organized so that the •'s precede the ×'s.

(4) The •-entries are strictly decreasing left to right, except for atypical entries which are all equal.

(5) The ×-entries are strictly increasing left to right, except for atypical entries which are all equal.

**Remark 25.** The arc diagram obtained in the last step of the shortening algorithm is a special arc diagram for $L$, since it satisfies (1)–(5). Hence, every KW-module has a special set of simple roots, since the highest weight of a KW-module with respect to $\pi_{\mathrm{st}}$ is totally connected. Moreover, it is unique since we can apply the reverse of the shortening algorithm to a special arc diagram to obtain the standard arc diagram for a totally connected weight $\lambda$ of a finite-dimensional module.

## 4. The Su–Zhang character formula for the totally connected case

We use Brundan's algorithm to characterize KW-modules in terms of Kazhdan–Lusztig polynomials and to prove the Su–Zhang character formula for finite-dimensional simple modules with a totally connected highest weight in the standard choice of simple roots $\pi_{\mathrm{st}}$. Recall the notation $(\lambda^{\rho}_{\mathrm{st}})^{\Uparrow}$ and $|\nu|_{S_\lambda}$ from Section 2.3.

**Theorem 26** [Su and Zhang 2007, 4.13]. *Let $\lambda_{\mathrm{st}}$ be a totally connected weight with a $\lambda^{\rho}$-maximal isotropic set $S_\lambda$ such that $|S_\lambda| = r$. Then*

$$e^{\rho}R\cdot \mathrm{ch}\, L_{\pi_{\mathrm{st}}}(\lambda_{\mathrm{st}}) = \frac{(-1)^{|(\lambda^{\rho}_{\mathrm{st}})^{\Uparrow}-\lambda^{\rho}_{\mathrm{st}}|_{S_\lambda}}}{r!}\mathcal{F}_W\left(\frac{e^{(\lambda^{\rho}_{\mathrm{st}})^{\Uparrow}}}{\prod_{\beta\in S_\lambda}(1+e^{-\beta})}\right). \tag{4-1}$$

To prove this theorem, we extend the ring $\mathcal{E}$ by adding expansions of the elements $1/(1+e^{-\beta})$ for $\beta\in\Delta^{+}_{\bar{1}}$ with respect to $\pi_{\mathrm{st}}$ as geometric series in the domain $|e^{-\beta}|<1$. Since $\Delta^{+}_{\bar{1}}$ is fixed by $W$, expanding commutes with the action of $W$.

**4.1.** ***Brundan's algorithm.*** Serganova [1996] introduced the generalized Kazhdan–Lusztig polynomials to give a character formula for finite-dimensional irreducible representations of $\mathfrak{gl}(m|n)$. For each $\lambda$ and $\mu$ dominant integral, the Kazhdan–Lusztig polynomial $K_{\lambda,\mu}(q)$ was shown to yield the multiplicity of Kac module $\overline{L}(\mu)$ inside the simple module $L_{\mathrm{st}}(\lambda)$ in the following sense:

$$\operatorname{ch} L_{\pi_{\mathrm{st}}}(\lambda) = \sum_{\mu\in\mathfrak{h}^*} K_{\lambda,\mu}(-1) \operatorname{ch} \overline{L}(\mu). \tag{4-2}$$

In this section we recall the algorithm of [Brundan 2003] computing $K_{\lambda,\mu}(q)$ in terms of weight diagrams.

We define a *right move* map from the set of (labeled) weight diagrams to itself in two steps: Let $D_\mu$ be a weight diagram for $\mu \in \mathbb{P}^+$, and choose a labeling of the $\times$'s with indexing set $\{1, \dots, r\}$. Then, for each $\times$, starting with the rightmost $\times$, "mark" the next empty spot to the right of it (which is unmarked). The right move $R_i$ is then defined by moving $\times_i$ to the empty spot it marked.

**Example 27.** Let $D_\mu$ be

$$\cdots \quad -1 \quad 0 \quad 1 \quad \overset{<}{2} \quad 3 \quad \overset{\times}{4} \quad \overset{>}{5} \quad \overset{\times}{6} \quad 7 \quad \overset{\times}{8} \quad \overset{>}{9} \quad \overset{>}{10} \quad 11 \quad 12 \quad \cdots \tag{4-3}$$

The rightmost $\times$ is at 8 and we mark 11 for it. The next $\times$ is at 6 and so we mark 7. Finally for the leftmost $\times$ we mark 12. Then

$$\begin{aligned}
R_1(D_\mu) &= \cdots \quad -1 \quad 0 \quad 1 \quad \overset{<}{2} \quad 3 \quad 4 \quad \overset{>}{5} \quad \overset{\times}{6} \quad 7 \quad \overset{\times}{8} \quad \overset{>}{9} \quad \overset{>}{10} \quad 11 \quad \overset{\times}{12} \quad \cdots\\
R_2(D_\mu) &= \cdots \quad -1 \quad 0 \quad 1 \quad \overset{<}{2} \quad 3 \quad \overset{\times}{4} \quad \overset{>}{5} \quad 6 \quad \overset{\times}{7} \quad \overset{\times}{8} \quad \overset{>}{9} \quad \overset{>}{10} \quad 11 \quad 12 \quad \cdots\\
R_3(D_\mu) &= \cdots \quad -1 \quad 0 \quad 1 \quad \overset{<}{2} \quad 3 \quad \overset{\times}{4} \quad \overset{>}{5} \quad \overset{\times}{6} \quad 7 \quad 8 \quad \overset{>}{9} \quad \overset{>}{10} \quad \overset{\times}{11} \quad 12 \quad \cdots
\end{aligned}$$

Note that the weight $\mu_i^\rho$ corresponding to $R_i(D_\mu)$ does not only differ from $\mu^\rho$ by atypical roots. It also has a different atypical set. In the previous example

$$\begin{aligned}
S_\mu &= \{\varepsilon_3 - \delta_2, \varepsilon_4 - \delta_3, \varepsilon_6 - \delta_4\},\\
S_{\mu_1} &= \{\varepsilon_1 - \delta_2, \varepsilon_4 - \delta_3, \varepsilon_5 - \delta_4\}.
\end{aligned}$$

**Definition 28.** Let $\mu, \lambda \in \mathbb{P}^+$, and label the $\times$'s in the diagram $D_\mu$ from left to right with $1, \dots, r$. A *right path* from $D_\mu$ to $D_\lambda$ is a sequence of right moves $\theta = R_{i_1} \circ \cdots \circ R_{i_k}$, where $i_1 \le \ldots \le i_k$ and $\theta(D_\mu) = D_\lambda$. The length of the path is $l(\theta) := k$.

**Example 29.** Let $D_\mu$ be

$$\cdots \quad -1 \quad 0 \quad \overset{\times}{1} \quad \overset{<}{2} \quad 3 \quad \overset{\boxed{\times}}{4} \quad \overset{>}{5} \quad \overset{\boxed{\times}}{6} \quad 7 \quad \overset{\boxed{\phantom{\times}}}{8} \quad \overset{>}{9} \quad \overset{>}{10} \quad 11 \quad 12 \quad \cdots$$

and $D_\lambda$ be as in Example 27. The boxes in the diagram represent the locations of the ×'s in $\lambda$. Then there are two paths from $D_\mu$ to $D_\lambda$, namely

$$D_\lambda = R_1 \circ R_1 \circ R_2 \circ R_3 \circ R_3(D_\mu),$$
$$D_\lambda = R_1 \circ R_1 \circ R_2(D_\mu).$$

The first path sends the $i$-th × of $\mu$ to the $i$-th box, whereas the second path permutes the order. Not all such permutations are valid, for example the third × in $\mu$ cannot be moved to the left. Also suppose the second and third ×'s will remain in the first and second boxes, respectively. Then the first × will never reach the box at 8 since it is marked by the × at 4, and hence such a path would be invalid.

If there exist paths from $D_\mu$ to $D_\lambda$, then one of them sends the $i$-th × of $\mu^\rho$ to the location of the $i$-th × of $\lambda^\rho$. This path is unique because the ×'s are moved in order. We call it the *trivial path* from $D_\mu$ to $D_\lambda$ and denote its length by $l_{\lambda,\mu}$ (this was denoted as $l(\lambda, \mu)$ in [Su and Zhang 2007, (3.15)]).

The trivial path is strictly longer than the rest of the paths. Indeed, in other paths, there is at least one × which is not moved as far as possible. This implies that another × will jump over it, making the move longer. So in this case one needs less moves to fill all of the boxes.

By [Brundan 2003, Corollary 3.39], we have

$$K_{\lambda,\mu}(q) = \sum_{\theta \in P} q^{l(\theta)},$$

where $P$ is the set of paths from $D_\mu$ to $D_\lambda$. In the previous example $K_{\lambda,\mu} = q^5 + q^3$.

**4.2.** ***Kazhdan–Lusztig polynomials and the Su–Zhang character formula.*** First we show that $\lambda$ being totally connected is equivalent to all paths to $D_\lambda$ being trivial. This yields a new characterization of KW-modules in terms of the Kazhdan–Lusztig polynomials. Then we use Brundan's algorithm to give a closed formula for $e^\rho R \cdot \mathrm{ch}\, L_{\pi_{\mathrm{st}}}(\lambda)$, which was originally proven in [Su and Zhang 2007, 4.13].

**Lemma 30.** *Let $\lambda \in \mathbb{P}^+$. Then $\lambda$ is totally connected if and only if for every $\mu \in \mathbb{P}^+$ there is at most one path from $D_\mu$ to $D_\lambda$.*

*Proof.* We will refer to the locations of the ×'s of the weight diagram $D_\lambda$ as boxes, and the paths will send the ×'s in $D_\mu$ to boxes.

The weight $\lambda$ is totally connected when there are no empty spots between the boxes. This implies that a path from $D_\mu$ to $D_\lambda$ must send the $i$-th × of $\mu$ to the $i$-th box. Indeed, if the $i$-th × of $\mu$ is sent to the $j$-th box, $j < i$, then the next empty box to the right of the $j$-th box will be marked by the $i$-th ×, and so no other × can be sent there. This implies that the path must be unique.

Suppose that there exists $\mu$ for which there is a nontrivial path to $D_\lambda$. In this path there is an $\times$ of $\mu$, say the $i$-th, which is sent to the $j$-th box where $j < i$. Then the next empty spot after this box cannot have a box in it. So $\lambda$ is not totally connected. □

**Corollary 31.** *A module $L_{\pi_{\text{st}}}(\lambda)$ is a KW-module if and only if all its Kazhdan–Lusztig polynomials are monomials. In this case, $K_{\lambda,\mu}(q) = q^{l_{\lambda,\mu}}$:*

From (4-2) and (2-5) we also obtain the following, which is a special case of [Su and Zhang 2007, Theorem 4.1].

**Corollary 32.** *Suppose $\lambda$ is totally connected. Since the unique path from $D_\mu$ to $D_\lambda$ is the trivial one, we get*

$$e^\rho R \cdot \operatorname{ch} L_{\pi_{\text{st}}}(\lambda) = \sum_{\mu \in P_\lambda} (-1)^{l_{\lambda,\mu}} \mathcal{F}_W(e^{\mu^\rho}), \tag{4-4}$$

*where $P_\lambda \subset \mathbb{P}^+$ is the set of $\mu \in \mathbb{P}^+$ for which there is a path from $D_\mu$ to $D_\lambda$.*

**Remark 33.** By Corollary 31, KW-modules are the same as what are called Kostant modules in [Brundan and Stroppel 2012]. Brundan and Stroppel [2012] showed that Kostant modules are parametrized in their notation by the weights in which no two vertices labeled $\vee$ have a vertex labeled $\wedge$ in between. This is equivalent to the combinatorial condition given in Remark 19 of this paper. They also proved that all Kostant modules possess a BGG-type resolution, so (4-4) can be realized as the Euler characteristic of this resolution.

Note that for each $\mu \in P_\lambda$ the $W$ orbit of $\mu^\rho$ intersects $(\lambda^\rho - \mathbb{N}S_\lambda)$. We denote by $\bar{\mu}$ the unique maximal element of this intersection with respect to the standard order on $\mathfrak{h}^*$. We define

$$C^{\text{Lexi}}_{\lambda,\text{reg}} := \{\bar{\mu} \mid \mu \in P_\lambda\}.$$

Since $P_\lambda \subset \mathbb{P}^+$, this defines a bijection between the sets $P_\lambda$ and $C^{\text{Lexi}}_{\lambda,\text{reg}}$.

For $\mu \in P_\lambda$, we can realize $\bar{\mu}$ more explicitly as follows. If $\varepsilon_i$ and $\delta_j$ are typical nodes of $\lambda^\rho$, then $(\bar{\mu}, \varepsilon_i) := (\lambda^\rho, \varepsilon_i)$ and $(\bar{\mu}, \delta_j) := (\lambda^\rho, \delta_j)$. The location of the atypical entries for $\bar{\mu}$ is determined by locations in $\lambda^\rho$. The set of atypical entries of $\bar{\mu}$ correspond to the set of atypical entries of $\mu^\rho$, ordered such that the $\varepsilon$-atypical entries are decreasing and the $\delta$-atypical entries are increasing. In particular,

$$C^{\text{Lexi}}_{\lambda,\text{reg}} = \left\{ \nu \in \lambda^\rho - \sum_{i=1}^{r} \mathbb{N}\beta_i \;\middle|\; \nu_{\beta_1} < \nu_{\beta_2} < \cdots < \nu_{\beta_r} \text{ and } w(\nu) \in \mathbb{P}^+ \text{ for some } w \in W \right\}.$$

**Example 34.** Consider

$$\begin{aligned}\lambda^\rho &= 10\varepsilon_1+9\varepsilon_2+\underline{8\varepsilon_3}+\underline{6\varepsilon_4}+5\varepsilon_5+\underline{4\varepsilon_6}-2\delta_1-\underline{4\delta_2}-\underline{6\delta_3}-\underline{8\delta_4},\\ \mu^\rho &= 10\varepsilon_1+9\varepsilon_2+\underline{6\varepsilon_3}+5\varepsilon_4+\underline{4\varepsilon_5}+\underline{\varepsilon_6}-\underline{\delta_1}-2\delta_2-\underline{4\delta_3}-\underline{6\delta_4},\\ \bar{\mu} &= 10\varepsilon_1+9\varepsilon_2+\underline{6\varepsilon_3}+\underline{4\varepsilon_4}+5\varepsilon_5+\underline{\varepsilon_6}-2\delta_1-\underline{\delta_2}-\underline{4\delta_3}-\underline{6\delta_4}.\end{aligned}$$

Here $S_\lambda=\{\varepsilon_6-\delta_2,\, \varepsilon_4-\delta_3,\, \varepsilon_3-\delta_4\}$ and $\lambda^\rho=\bar{\mu}+3(\varepsilon_6-\delta_2)+2(\varepsilon_4-\delta_3)+2(\varepsilon_3-\delta_4)$.

**Remark 35.** The element $w\in W$ for which $w(\mu^\rho)=\bar{\mu}$ can be described explicitly in terms of the trivial path $\theta$. Define $\theta=R_{i_1}\circ\cdots\circ R_{i_N}$; then $w=w_1\cdot\cdots\cdot w_N$, where each $w_j$ is defined as follows: Suppose that the move $R_{i_j}$ moved the × at $n_j$ to an empty spot at $n_j+k_j$, namely, it skipped over $k_j-1$ spots with $>$'s and $<$'s. Then $w_j=s_1\cdot\cdots\cdot s_{k_j-1}$, where $s_i$ is of the form $s_{\varepsilon_l-\varepsilon_{l+1}}$ if the $i$-th skip is over the $>$ of $\varepsilon_l$ and is of the form $s_{\delta_l-\delta_{l+1}}$ if it is over the $<$ of $\delta_l$. In particular, $l(w_j)=k_j-1$. Moreover $l(w)=\sum l(w_i)$.

The following lemma is the main step of the proof, in which we move from an algorithmic formula to a closed one, and it is a special case of [Su and Zhang 2007, Theorem 4.2].

**Lemma 36.**
$$e^\rho R\cdot \operatorname{ch} L_{\pi_{\mathrm{st}}}(\lambda)=\sum_{\bar{\mu}\in C^{\mathrm{Lexi}}_{\lambda,\mathrm{reg}}}(-1)^{|\lambda^\rho-\bar{\mu}|_{S_\lambda}}\mathcal{F}_W(e^{\bar{\mu}}).$$

*Proof.* Let us show that for each $\mu\in P_\lambda$

$$(-1)^{l_{\lambda,\mu}}\mathcal{F}_W(e^{\mu^\rho})=(-1)^{|\lambda^\rho-\bar{\mu}|_{S_\lambda}}\mathcal{F}_W(e^{\bar{\mu}}). \tag{4-5}$$

Let $w\in W$ such that $w(\mu^\rho)=\bar{\mu}$. We claim that $|\lambda^\rho-\bar{\mu}|_{S_\lambda}=l_{\lambda,\mu}+l(w)$, which proves (4-5). Indeed, the number $|\lambda^\rho-\bar{\mu}|_{S_\lambda}$ is the sum of the differences between the atypical entries of $\lambda^\rho$ and $\bar{\mu}$. This is equal to the number of moves in the trivial path $l_{\lambda,\mu}$ plus the total number of spots being skipped. By Remark 35, $l(w)$ is exactly the number of spots skipped in the trivial path. □

*Proof of Theorem 26.* Our proof goes as follows. First, we enlarge the set $C^{\mathrm{Lexi}}_{\lambda,\mathrm{reg}}$ by adding elements which are annihilated by $\mathcal{F}_W$; we call this new set $C^{\mathrm{Lexi}}_{\lambda}$. Then we express $C^{\mathrm{Lexi}}_{\lambda}$ in terms of $(\lambda^\rho_{\mathrm{st}})^{\Uparrow}$. Finally, we add more summands to the expression

$$\sum_{\bar{\mu}\in C^{\mathrm{Lexi}}_{\lambda}}(-1)^{|\lambda^\rho-\bar{\mu}|_{S_\lambda}}\mathcal{F}_W(e^{\bar{\mu}})$$

which are annihilated by $\mathcal{F}_W$ but that allow us to write the sum in a nicer way.

Denote the $\lambda^\rho$-maximal atypical set by $S_\lambda=\{\beta_1,\dots,\beta_r\}$, where the elements $\beta_i=\varepsilon_{s_i}-\delta_{t_i}$ are ordered such that $t_i<t_{i+1}$. Let

$$C^{\mathrm{Lexi}}_{\lambda}=\left\{\nu\in\lambda^\rho-\sum_{i=1}^{r}\mathbb{N}\beta_i \;\middle|\; \nu_{\beta_1}<\nu_{\beta_2}<\cdots<\nu_{\beta_r}\right\}.$$

Then $C_\lambda^{\mathrm{Lexi}}$ can be expressed as follows. Let $k_1, \dots, k_r \in \mathbb{N}$ be such that $\lambda^\rho = (\lambda^\rho)^{\Uparrow} - \sum_{i=1}^r k_i \beta_i$. Then $k_1 > \cdots > k_r = 0$. Let $a_1 < \cdots < a_r$ be the atypical entries of $\lambda^\rho$. Then $a_i + k_i = a_j + k_j$, and we have

$$\begin{aligned} C_\lambda^{\mathrm{Lexi}} &= \left\{ \lambda^\rho - \sum_{i=1}^r n_i \beta_i \;\middle|\; n_i \in \mathbb{N}, a_1 - n_1 < a_2 - n_2 < \cdots < a_r - n_r \right\} \\ &= \left\{ \lambda^\rho - \sum_{i=1}^r n_i \beta_i \;\middle|\; n_i \in \mathbb{N}, k_1 + n_1 > k_2 + n_2 > \cdots > k_r + n_r \right\} \\ &= \left\{ (\lambda^\rho)^{\Uparrow} - \sum_{i=1}^r (k_i + n_i) \beta_i \;\middle|\; n_i \in \mathbb{N}, k_1 + n_1 > k_2 + n_2 > \cdots > k_r + n_r \right\} \\ &= \left\{ (\lambda^\rho)^{\Uparrow} - \sum_{i=1}^r m_i \beta_i \;\middle|\; m_i \in \mathbb{Z}_{\geq k_i}, m_1 > m_2 > \cdots > m_r \right\}. \end{aligned}$$

Now let us enlarge $C_\lambda^{\mathrm{Lexi}}$; namely, we define

$$\overline{C_\lambda^{\mathrm{Lexi}}} = \left\{ (\lambda^\rho)^{\Uparrow} - \sum_{i=1}^r m_i \beta_i \;\middle|\; m_i \in \mathbb{N}, m_1 > m_2 > \cdots > m_r \right\}.$$

We claim that

$$\sum_{\bar{\mu} \in C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}} (-1)^{|\lambda^\rho - \bar{\mu}|_{S_\lambda}} \mathcal{F}_W(e^{\bar{\mu}}) = \sum_{\nu \in \overline{C_\lambda^{\mathrm{Lexi}}}} (-1)^{|\lambda^\rho - \nu|_{S_\lambda}} \mathcal{F}_W(e^\nu). \tag{4-6}$$

Indeed, if $\nu \in C_\lambda^{\mathrm{Lexi}} \setminus C_{\lambda,\mathrm{reg}}^{\mathrm{Lexi}}$ then the $\varepsilon$ or $\delta$ entries of $\nu$ are not distinct, so $\mathcal{F}_W(e^\nu) = 0$. Thus, we are left to show that $\mathcal{F}_W(e^\nu) = 0$ if $\nu$ is of the form $\nu = (\lambda^\rho)^{\Uparrow} - \sum_{i=1}^r m_i \beta_i \in \overline{C_\lambda^{\mathrm{Lexi}}}$ and $m_i < k_i$ for some $1 \leq i \leq r$. Indeed, let $j$ be such that $m_j < k_j$ and $m_i \geq k_i$ for all $i > j$. Note that since $\lambda$ is totally connected, all the integers between $a_r$ and $a_j + 1$ are entries of $\lambda^\rho$. The typical entries of $\nu$ are the same as of $\lambda^\rho$ and there are $r - j + 1$ atypical entries which are strictly greater than $a_j$. This implies that there must be equal entries of the same type, that is, $\nu$ has a stabilizer in $W$. Hence, by Lemma 13 and Lemma 2 we conclude that $\mathcal{F}_W(e^\nu) = 0$.

Let $W_r$ be the subgroup of $W$ that permutes $S_\lambda$. This subgroup is generated by elements of the form $s_{\varepsilon_i - \varepsilon_j} s_{\delta_{i'} - \delta_{j'}}$, where $\varepsilon_i - \delta_{i'}, \varepsilon_j - \delta_{j'} \in S_\lambda$, so $|W_r| = r!$ and all $w \in W_r$ have positive sign. Hence

$$\mathcal{F}_W\left( \sum_{w \in W_r} w e^\nu \right) = r! \mathcal{F}_W(e^\nu)$$

for any $\nu \in \mathfrak{h}^*$. Let $W_r(\overline{C_\lambda^{\mathrm{Lexi}}}) = \{w(\nu) \mid w \in W_r,\, \nu \in \overline{C_\lambda^{\mathrm{Lexi}}}\}$. Then

$$W_r(\overline{C_\lambda^{\mathrm{Lexi}}}) = \left\{(\lambda^\rho)^{\Uparrow} - \sum_{i=1}^{r} m_i \beta_i \;\middle|\; m_i \in \mathbb{N},\, m_i \neq m_j \text{ for } i \neq j\right\},$$

and so elements from $((\lambda^\rho)^{\Uparrow} - \mathbb{N}S_\lambda) \setminus W_r(\overline{C_\lambda^{\mathrm{Lexi}}})$ have a stabilizer in $W$. Thus,

$$\begin{aligned}
r!(-1)^{|(\lambda^\rho)^{\Uparrow}-\lambda^\rho|_{S_\lambda}} \sum_{\nu \in \overline{C_\lambda^{\mathrm{Lexi}}}} (-1)^{|\lambda^\rho-\nu|_{S_\lambda}} \mathscr{F}_W(e^\nu)&\\
&= \sum_{\nu \in \overline{C_\lambda^{\mathrm{Lexi}}}} (-1)^{|(\lambda^\rho)^{\Uparrow}-\nu|_{S_\lambda}} \mathscr{F}_W\Bigl(\sum_{w \in W_r} e^{w(\nu)}\Bigr)\\
&= \sum_{\nu \in (\lambda^\rho)^{\Uparrow}-\mathbb{N}S_\lambda} (-1)^{|(\lambda^\rho)^{\Uparrow}-\nu|_{S_\lambda}} \mathscr{F}_W(e^\nu)\\
&= \mathscr{F}_W\Bigl(\sum_{\nu \in (\lambda^\rho)^{\Uparrow}-\mathbb{N}S_\lambda} (-1)^{|(\lambda^\rho)^{\Uparrow}-\nu|_{S_\lambda}} e^\nu\Bigr)\\
&= \mathscr{F}_W\Bigl(\frac{e^{(\lambda^\rho)^{\Uparrow}}}{\prod_{\beta \in S_\lambda}(1+e^{-\beta})}\Bigr). \qquad \square
\end{aligned}$$

The character formula in the following theorem is motivated by the denominator identity given in [Gorelik et al. 2012, (1.10)] for $\pi_{\mathrm{st}}$, and can be proven using Lemma 36, formula (4-6) and the methods above.

**Theorem 37.** *Let $\lambda_{\mathrm{st}}$ be a totally connected weight with a $\lambda^\rho$-maximal isotropic set $\beta_1, \ldots, \beta_r$ ordered such that $\beta_i < \beta_{i+1}$ for $i = 1, \ldots, r-1$. Then*

$$e^\rho R \cdot \operatorname{ch} L_{\pi_{\mathrm{st}}}(\lambda) = \mathscr{F}_W\left(\frac{e^{(\lambda^\rho)^{\Uparrow}}}{(1+e^{-\beta_1})(1-e^{-\beta_1-\beta_2})\cdots(1-(-1)^r e^{-\sum_{i=1}^r \beta_i})}\right).$$

## 5. Kac–Wakimoto character formula for KW-modules

**5.1.** ***The special case.*** Let us show that Theorem 26 generalizes to other sets of simple roots by proving that the character formula is preserved under the steps of the shortening algorithm given in the proof of Theorem 21. This will prove the Kac–Wakimoto character formula for the special set of simple roots.

For a totally connected highest weight $\lambda$ of a finite-dimensional simple module $L_{\pi_{\mathrm{st}}}(\lambda)$, we let $\pi_k$ denote the set of simple roots obtained after $k$ steps of the shortening algorithm applied to $\lambda^\rho$. Let $\lambda_{\pi_k} \in \mathfrak{h}^*$ be such that $L = L_{\pi_k}(\lambda_{\pi_k})$. Set $S_0 = S_\lambda$ and let $S_k$ be the $\lambda_{\pi_k}^\rho$-maximal isotropic set corresponding to the arc arrangement obtained by the $k$-th step of the algorithm. Then we have the following:

**Theorem 38.** *Let $\lambda$ be a totally connected weight and let $L = L_{\pi_{\mathrm{st}}}(\lambda)$. Then*

$$e^{\rho} R \cdot \operatorname{ch} L = \frac{(-1)^{|(\lambda^{\rho}_{\pi_k})^{\Uparrow} - \lambda^{\rho}_{\pi_k}|_{S_k}}}{r!} \mathcal{F}_W \left( \frac{e^{(\lambda^{\rho}_{\pi_k})^{\Uparrow}}}{\prod_{\beta \in S_k} (1 + e^{-\beta})} \right). \tag{5-1}$$

Our proof is by induction on the steps of the shortening algorithm. Let us first see an example:

**Example 39.** Given $\lambda^{\rho}_{\mathrm{st}}$ corresponding to diagram (3-3) from Example 22, we show that formula (5-1) holds after one step of the shortening algorithm.

To obtain $(\lambda^{\rho}_{\mathrm{st}})^{\Uparrow} = \lambda^{\rho}_{\mathrm{st}} + 2(\varepsilon_4 - \delta_2)$ from the $\lambda^{\rho}_{\mathrm{st}}$ diagram, each entry labeled with a 5 should be replaced by a 7. To start the algorithm, we first push the entries 1 and 2 outside of the arcs, which does not change $\lambda^{\rho}_{\mathrm{st}}$ and $S_{\lambda_{\mathrm{st}}}$ so the formula is clearly preserved.

Next we apply $r_{\varepsilon_4 - \delta_2}$ and obtain $\lambda^{\rho}_{\pi_1}$ corresponding to diagram (3-4). Then $\lambda^{\rho}_{\pi_1} = \lambda^{\rho}_{\mathrm{st}} + (\varepsilon_4 - \delta_2)$ and $S_1 = s_{\varepsilon_3 - \varepsilon_4} S_{\lambda_{\mathrm{st}}}$. So $(\lambda^{\rho}_{\pi_1})^{\Uparrow} = \lambda^{\rho}_{\pi_1} + (\varepsilon_3 - \delta_2)$ and $(\lambda^{\rho}_{\pi_1})^{\Uparrow} = s_{\varepsilon_3 - \varepsilon_4} (\lambda^{\rho}_{\mathrm{st}})^{\Uparrow}$. Hence

$$\begin{aligned} e^{\rho} R \cdot \operatorname{ch} L &= \frac{(-1)^2}{2} \mathcal{F}_W \left( (-1) \cdot s_{\varepsilon_3 - \varepsilon_4} \frac{e^{(\lambda^{\rho}_{\mathrm{st}})^{\Uparrow}}}{\prod_{\beta \in S_{\lambda_{\mathrm{st}}}} (1 + e^{-\beta})} \right) \\ &= \frac{(-1)^{|(\lambda^{\rho}_{\pi_1})^{\Uparrow} - \lambda^{\rho}_{\pi_1}|_{S_1}}}{2} \mathcal{F}_W \left( \frac{e^{(\lambda^{\rho}_{\pi_1})^{\Uparrow}}}{\prod_{\beta \in S_1} (1 + e^{-\beta})} \right), \end{aligned}$$

and the formula is preserved. Finally, we move the 6 out to obtain diagram (3-5). Since this does not change $\lambda^{\rho}_{\pi_1}$ and $S_{\pi_1}$ the formula is preserved.

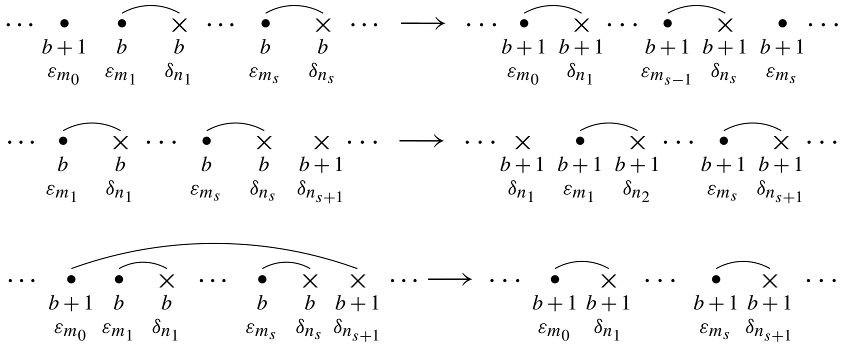
*Proof of Theorem 38.* Our proof is by induction on the steps of the shortening algorithm from the proof of Theorem 21. After $k$ steps, we obtain new data $\pi_k$ $\lambda^{\rho}_{\pi_k}$, $\lambda_{\pi_k}$, $S_k$. We express the right-hand side of formula (5-1) in terms of this new data, and prove that it is equal to $e^{\rho} R \cdot \operatorname{ch} L$ using the formula obtained after $k - 1$ steps.

Before applying the algorithm we start by pushing the entries located below the innermost arc outside of the arcs. Since this corresponds to reflections with respect to roots which are not orthogonal to $\lambda^{\rho}_{\pi_{\mathrm{st}}}$, this changes $\pi_{\mathrm{st}}$ but not $\lambda^{\rho}_{\pi_{\mathrm{st}}}$ or $S_{\lambda_{\mathrm{st}}}$. So formula (5-1) is unchanged.

Now suppose that the formula holds after $k - 1$ steps of the algorithm, that is,

$$e^{\rho} R \cdot \operatorname{ch} L = \frac{(-1)^{|(\lambda^{\rho}_{\pi_{k-1}})^{\Uparrow} - \lambda^{\rho}_{\pi_{k-1}}|_{S_{k-1}}}}{r!} \mathcal{F}_W \left( \frac{e^{(\lambda^{\rho}_{\pi_{k-1}})^{\Uparrow}}}{\prod_{\beta \in S_{k-1}} (1 + e^{-\beta})} \right). \tag{5-2}$$

Let us apply one more step and show that (5-2) implies (5-1). There are three cases, depending on the location of the $b + 1$ entry (see the proof of the "$\Leftarrow$" direction of Theorem 21). In each case, we reflect at all of the short arcs, and then arrange the arcs to be of •-× type:



(5-3)

This does not change the typical entries or the maximal atypical entry, so $(\lambda^{\rho}_{\pi_{k-1}})^{\Uparrow}$ only differs $(\lambda^{\rho}_{\pi_k})^{\Uparrow}$ by a permutation of the entries. Moreover, $\lambda^{\rho}_{\pi_k}$ is "closer" to $(\lambda^{\rho}_{\mathrm{st}})^{\Uparrow}$ than in the previous step because some entries were increased. Indeed, the sequence of odd reflections is with respect to the simple atypical odd roots $S_{k-1} \cap \pi_{k-1}$, so we have

$$\lambda^{\rho}_{\pi_k} = \lambda^{\rho}_{\pi_{k-1}} + \sum_{\beta \in S_{k-1} \cap \pi_{k-1}} \beta.$$

For each diagram from (5-3), there exists $w \in W$ such that

$$wS_{k-1} = S_k, \quad w(\lambda^{\rho}_{\pi_{k-1}})^{\Uparrow} = (\lambda^{\rho}_{\pi_k})^{\Uparrow}, \quad \text{and} \quad l(w) = |S_{k-1} \cap \pi_{k-1}|.$$

Indeed, we have that $S_{k-1} \cap \pi_{k-1} = \{\beta_1, \ldots, \beta_s\}$, where each $\beta_i = \varepsilon_{m_i} - \delta_{n_i}$ is a simple atypical root corresponding to a diagram of (5-3). For the first and third case, we take $w \in \mathrm{Sym}_m$ such that $w(\varepsilon_{m_i}) = \varepsilon_{m_{i-1}}$, $w(\varepsilon_{m_0}) = \varepsilon_{m_s}$ and all other elements are fixed, and, in the second case, we take $w \in \mathrm{Sym}_n$ such that $w(\delta_i) = \delta_{i+1}$, $w(\delta_{s+1}) = \delta_1$ and all other elements are fixed. Then

$$\begin{aligned} e^{\rho} R \cdot \mathrm{ch} L &= \frac{(-1)^{|(\lambda^{\rho}_{\pi_{k-1}})^{\Uparrow} - \lambda^{\rho}_{\pi_{k-1}}|_{S_{k-1}}}}{r!} \mathcal{F}_W \left( \frac{e^{(\lambda^{\rho}_{\pi_{k-1}})^{\Uparrow}}}{\prod_{\beta \in S_{k-1}} (1+e^{-\beta})} \right) \\ &= \frac{(-1)^{|(\lambda^{\rho}_{\pi_{k-1}})^{\Uparrow} - \lambda^{\rho}_{\pi_{k-1}}|_{S_{k-1}} + |S_{\lambda_{k-1}} \cap \pi_{k-1}|}}{r!} \mathcal{F}_W \left( (-1)^{l(w)} w \frac{e^{(\lambda^{\rho}_{\pi_{k-1}})^{\Uparrow}}}{\prod_{\beta \in S_{k-1}} (1+e^{-\beta})} \right) \\ &= \frac{(-1)^{|(\lambda^{\rho}_{\pi_k})^{\Uparrow} - \lambda^{\rho}_{\pi_k}|_{S_k}}}{r!} \mathcal{F}_W \left( \frac{e^{(\lambda^{\rho}_{\pi_k})^{\Uparrow}}}{\prod_{\beta \in S_k} (1+e^{-\beta})} \right). \end{aligned}$$

Finally, we push the smallest unmatched entry outside of the arcs. Since this corresponds to reflections with respect to roots which are not orthogonal to $\lambda^{\rho}_{\pi_{k-1}}$, this does not change formula (5-1). □

Since for the special set of simple roots all atypical entries are equal, we conclude the following:

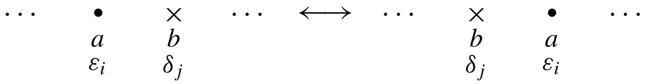
**Corollary 40.** *Let $L$ be a KW-module with an admissible choice of simple roots $\pi$, a corresponding highest weight $\lambda$ and a $\lambda^\rho$-maximal isotropic subset $S_\lambda \subset \pi$, and let $r = |S_\lambda|$. If $\pi$ is the special set, then*

$$e^\rho R \cdot \operatorname{ch} L_\pi(\lambda) = \frac{1}{r!} \mathcal{F}_W \left( \frac{e^{\lambda^\rho}}{\prod_{\beta \in S_\lambda} (1 + e^{-\beta})} \right). \tag{5-4}$$

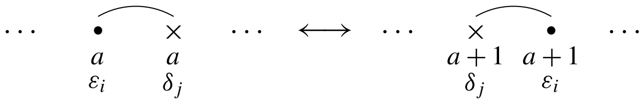
**5.2.** ***The general case.*** Let us prove the Kac–Wakimoto character formula for an arbitrary admissible choice of simple roots by showing that we can move from any admissible choice of simple roots to the special set in a way that preserves the formula in (5-4).

We will use the following four moves on arc diagrams.

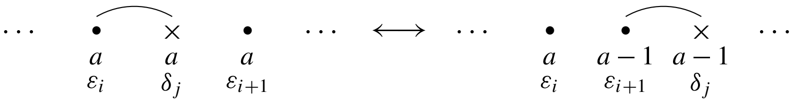
(a)



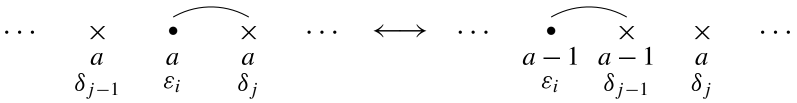
(b)



(c)



(d)



Each move is achieved by applying an odd reflection to the set of simple roots $\pi \leftrightarrow \pi'$ followed by a choice of the arc arrangement. Let us first show that each of these moves preserves the Kac–Wakimoto character formula.

**Proposition 41.** *Suppose that $(\pi, \lambda, S_\lambda)$ and $(\pi', \lambda', S_{\lambda'})$ differ by one of the above moves. Then*

$$\mathcal{F}_W \left( \frac{e^{\lambda^\rho}}{\prod_{\beta \in S_\lambda} (1 + e^{-\beta})} \right) = \mathcal{F}_W \left( \frac{e^{\lambda^{\rho'}}}{\prod_{\beta \in S_{\lambda'}} (1 + e^{-\beta})} \right).$$

*Proof.* We prove the claim for each move defined above. We take $\lambda^\rho$ to correspond to the left-hand side diagram and $\lambda^{\rho'}$ to correspond to the right-hand side diagram.

The claim is obvious for move (a), since in this case $\lambda^{\rho'} = \lambda^\rho$ and $S_{\lambda'} = S_\lambda$.

In move (b), we have $\lambda^{\rho'} = \lambda^\rho + (\varepsilon_i - \delta_j)$, and

$$S_{\lambda'} = (S_\lambda \setminus \{\varepsilon_i - \delta_j\}) \cup \{\delta_j - \varepsilon_i\},$$

so

$$\begin{aligned}\mathscr{F}_W\left(\frac{e^{\lambda^{\rho'}}}{\prod_{\beta\in S_{\lambda'}}(1+e^{-\beta})}\right)&=\mathscr{F}_W\left(\frac{e^{\lambda^{\rho}+\varepsilon_i-\delta_j}}{\prod_{\beta\in S_{\lambda\setminus(\varepsilon_i-\delta_j)}}(1+e^{-\beta})(1+e^{-(\delta_j-\varepsilon_i)})}\right)\\&=\mathscr{F}_W\left(\frac{e^{\lambda^{\rho}}}{\prod_{\beta\in S_{\lambda}}(1+e^{-\beta})}\right),\end{aligned}$$

as desired. □

In move (c), we have $\lambda^{\rho'}=\lambda^{\rho}+(\delta_j-\varepsilon_{i+1})$, and

$$S_{\lambda'}=(S_\lambda\setminus\{\varepsilon_i-\delta_j\})\cup\{\varepsilon_{i+1}-\delta_j\},$$

so

$$\begin{aligned}\mathscr{F}_W\left(\frac{e^{\lambda^{\rho'}}}{\prod_{\beta\in S_{\lambda'}}(1+e^{-\beta})}\right)&=\mathscr{F}_W\left(\frac{e^{\lambda^{\rho}-\varepsilon_{i+1}+\delta_j}}{\prod_{\beta\in S_{\lambda\setminus(\varepsilon_i-\delta_j)}}(1+e^{-\beta})(1+e^{-(\varepsilon_{i+1}-\delta_j)})}\right)\\&=\mathscr{F}_W\left(s_{\varepsilon_i-\varepsilon_{i+1}}\left(\frac{e^{\lambda^{\rho}-\varepsilon_i+\delta_j}}{\prod_{\beta\in S_{\lambda}}(1+e^{-\beta})}\right)\right)\\&=-\mathscr{F}_W\left(\frac{e^{\lambda^{\rho}-\varepsilon_i+\delta_j}}{\prod_{\beta\in S_{\lambda}}(1+e^{-\beta})}\right).\end{aligned}$$

It remains to show that

$$-\mathscr{F}_W\left(\frac{e^{\lambda^{\rho}-\varepsilon_i+\delta_j}}{\prod_{\beta\in S_{\lambda}}(1+e^{-\beta})}\right)=\mathscr{F}_W\left(\frac{e^{\lambda^{\rho}}}{\prod_{\beta\in S_{\lambda}}(1+e^{-\beta})}\right),$$

which we obtain from the following:

$$\begin{aligned}\mathscr{F}_W\left(\frac{e^{\lambda^{\rho}}}{\prod_{\beta\in S_{\lambda}}(1+e^{-\beta})}+\frac{e^{\lambda^{\rho}-\varepsilon_i+\delta_j}}{\prod_{\beta\in S_{\lambda}}(1+e^{-\beta})}\right)&=\mathscr{F}_W\left(e^{\lambda^{\rho}}\frac{1+e^{-(\varepsilon_i-\delta_j)}}{\prod_{\beta\in S_{\lambda}}(1+e^{-\beta})}\right)\\&=\mathscr{F}_W\left(\frac{e^{\lambda^{\rho}}}{\prod_{\beta\in S_{\lambda}\setminus\{\varepsilon_i-\delta_j\}}(1+e^{-\beta})}\right)\\&=0.\end{aligned}$$

The last equality holds by Lemma 13, since the argument of $\mathscr{F}_W$ is preserved by the simple reflection $s_{\varepsilon_i-\varepsilon_{i-1}}$, which completes the proof for the third move.

In move (d), we have $\lambda^{\rho'}=\lambda^{\rho}+(\delta_{j-1}-\varepsilon_i)$, and

$$S_{\lambda'}=(S_\lambda\setminus\{\varepsilon_i-\delta_j\})\cup\{\varepsilon_i-\delta_{j-1}\}.$$

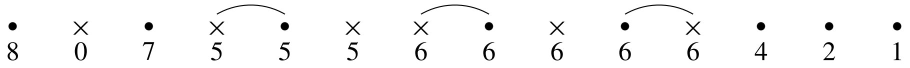
The proof in this case is analogous to the third move, except that the roles of the $\varepsilon$'s and $\delta$'s are interchanged.

Recall the definition of the special set of simple roots given in Definition 24.

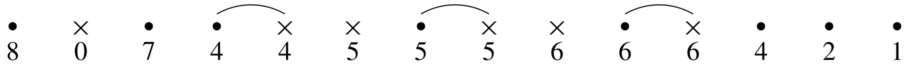
**Proposition 42.** *One can move from any admissible choice of simple roots to the special set of simple roots in a way that preserves the character formula* (5-4).

First we give an example:

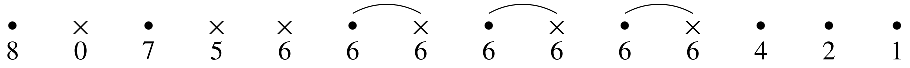
**Example 43.** Start with the following arc diagram corresponding to an admissible choice of simple roots:



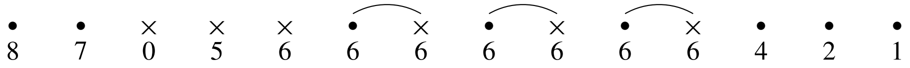
Use move (b) twice to arrange the arcs to be of •-× type:



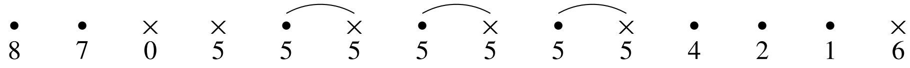
Use move (d) three times to take all the intermediate ×'s outside the arcs:



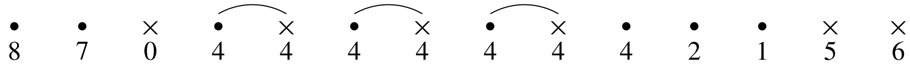
Use move (a) on each side of the arcs to organize the typical entries to have the •'s precede the ×'s:



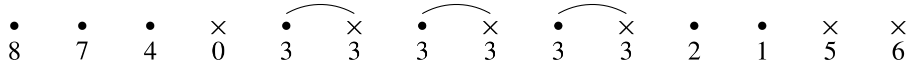
Use moves (d) and (a) to move the typical ×-entry 6 to the right of the arcs:



Use moves (d) and (a) to move the typical ×-entry 5 to the right of the arcs:



Use moves (c) and (a) to move the typical ×-entry 4 to the left of the arcs:



The resulting diagram is the special arc diagram.

*Proof of Proposition 42.* We show that we can move from an arbitrary admissible choice of simple roots to the special set in a way that uses only moves (a)–(d) defined above. Since each of these moves was shown to preserve the character formula in Proposition 41, the result then follows.

Consider an arc diagram corresponding to an arbitrary admissible choice of simple roots. Since all arcs are short, we can use move (b) to arrange all of the arcs to be of $\bullet$-$\times$ type. Then we use move (c) or move (d) to place all the arcs next to each other, which is possible due to Lemma 14. Then, by Lemma 14, all of the atypical entries are equal, and we denote them by $a$. Then on each side of the arcs we use move (a) to organize the typical entries so that the $\bullet$'s precede the $\times$'s.

Since the diagram corresponds to a finite-dimensional module, we have at each end that the typical $\times$-entries are strictly increasing, while the typical $\bullet$-entries are strictly decreasing (see Remark 10). Moreover, we can show that, on the left end, the $\times$-entries are $\leqslant a$ while the $\bullet$-entries are $> a$. Indeed, if the biggest $\times$-entry is greater than $a$, then the node can be moved inside the first arc by using move (a), and we reach a contradiction on the value of the adjacent $\times$ entries. Also, if the smallest $\bullet$-entry is not smaller than $a$, the node can be moved next to the first arc by using move (a), and the same problem arises. Similarly, on the right end the $\times$-entries are $> a$ while the $\bullet$-entries are $\leqslant a$. Together with the fact that no additional arcs are possible by the maximality property, this implies that all typical entries are distinct.

The only way that the resulting diagram is not the special arc diagram is if one of the two inequalities above is an equality, and, in particular, if one of the typical entries equals $a$. If it is the first one, then we use move (d) to transfer the relevant $\times$ to the right of the arcs and then move (a) to transfer it to the right of all $\bullet$'s. If it is the second one, then we use move (c) to transfer the relevant $\bullet$ to the left of the arcs and then move (a) to transfer it to the left of all $\times$'s. The resulting diagram satisfies the properties of the previous paragraph with the atypical entries now labeled by $a-1$.

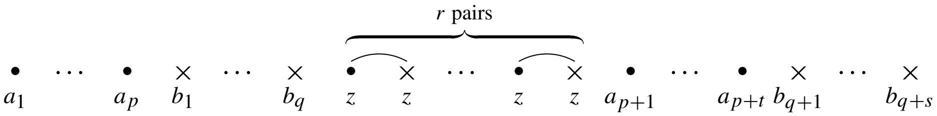
We repeat the above step until we cannot do it anymore. After each step the atypical entries are decreased by 1, while the labeling set for the typical entries remains the same. Hence the process must terminate, and the resulting diagram will be the special arc diagram. □

When combined with Corollary 40, this concludes the proof of the Kac–Wakimoto character formula for $\mathfrak{gl}(m|n)$ (Theorem 1).

## 6. A determinantal character formula for KW-modules of $\mathfrak{gl}(m|n)$

In this section, we use the Kac–Wakimoto character formula to prove a determinantal character formula for KW-modules for $\mathfrak{gl}(m|n)$, which is motivated by the determinantal character formula proven in [Moens and Van der Jeugt 2004] for critical modules labeled by nonintersecting composite partitions. Our determinantal character formula can be expressed using the data of the special arc diagram for a KW-module $L$ (recall Definition 24).

**6.1.** ***A determinantal character formula.*** Consider the special arc diagram of a KW-module $L$.



(6-1)

Note that $p+t=m-r$ and $q+s=n-r$. We set $x_i=e^{\varepsilon_i}$ and $y_j=e^{-\delta_j}$, and we let

$$X=(x_i^{a_j})_{\substack{1\le i\le m\\ 1\le j\le m-r}},\qquad Y=(y_j^{b_i})_{\substack{1\le i\le n-r\\ 1\le j\le n}},\qquad Z=\left(\frac{x_i^z y_j^z}{1+(x_iy_j)^{-1}}\right)_{\substack{1\le i\le m\\ 1\le j\le n}}.$$

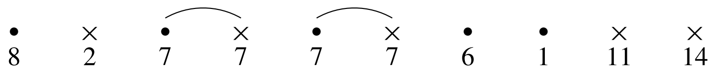
Then, $X$ encodes the typical $\bullet$-entries, $Y$ encodes the typical $\times$-entries, and $Z$ encodes atypical entries.

**Theorem 44.** *Let $L$ be a KW-module with special arc diagram* (6-1)*. Then one has*

$$e^{\rho}R\cdot\operatorname{ch}L=(-1)^{r(t+q)}\begin{vmatrix}X & Z\\ 0 & Y\end{vmatrix}. \tag{6-2}$$

**Remark 45.** The data defining (6-2) can also be determined from the standard arc diagram of a KW-module $L$. In particular, the exponents $a_j$ are the typical $\bullet$-entries, the exponents $b_i$ are the typical $\times$-entries, the exponent $z$ is the maximal atypical entry, $r$ is the number of arcs, $t$ is the number of typical $\bullet$-nodes below an arc, and $q$ is the number of typical $\times$-nodes below an arc. See Remark 23 to recall the relationship between the standard arc diagram and the special arc diagram.

**Example 46.** If the special arc diagram of $L$ is



then

$$e^{\rho}R\cdot\operatorname{ch}L$$
$$=-\begin{vmatrix}
x_1^8 & x_1^6 & x_1 & \frac{x_1^7y_1^7}{1+(x_1y_1)^{-1}} & \frac{x_1^7y_2^7}{1+(x_1y_2)^{-1}} & \frac{x_1^7y_3^7}{1+(x_1y_3)^{-1}} & \frac{x_1^7y_4^7}{1+(x_1y_4)^{-1}} & \frac{x_1^7y_5^7}{1+(x_1y_5)^{-1}}\\
x_2^8 & x_2^6 & x_2 & \frac{x_2^7y_1^7}{1+(x_2y_1)^{-1}} & \frac{x_2^7y_2^7}{1+(x_2y_2)^{-1}} & \frac{x_2^7y_3^7}{1+(x_2y_3)^{-1}} & \frac{x_2^7y_4^7}{1+(x_2y_4)^{-1}} & \frac{x_2^7y_5^7}{1+(x_2y_5)^{-1}}\\
x_3^8 & x_3^6 & x_3 & \frac{x_3^7y_1^7}{1+(x_3y_1)^{-1}} & \frac{x_3^7y_2^7}{1+(x_3y_2)^{-1}} & \frac{x_3^7y_3^7}{1+(x_3y_3)^{-1}} & \frac{x_3^7y_4^7}{1+(x_3y_4)^{-1}} & \frac{x_3^7y_5^7}{1+(x_3y_5)^{-1}}\\
x_4^8 & x_4^6 & x_4 & \frac{x_4^7y_1^7}{1+(x_4y_1)^{-1}} & \frac{x_4^7y_2^7}{1+(x_4y_2)^{-1}} & \frac{x_4^7y_3^7}{1+(x_4y_3)^{-1}} & \frac{x_4^7y_4^7}{1+(x_4y_4)^{-1}} & \frac{x_4^7y_5^7}{1+(x_4y_5)^{-1}}\\
x_5^8 & x_5^6 & x_5 & \frac{x_1^7y_1^7}{1+(x_5y_1)^{-1}} & \frac{x_1^7y_2^7}{1+(x_5y_2)^{-1}} & \frac{x_5^7y_3^7}{1+(x_5y_3)^{-1}} & \frac{x_5^7y_4^7}{1+(x_5y_4)^{-1}} & \frac{x_6^7y_5^7}{1+(x_6y_5)^{-1}}\\
0 & 0 & 0 & y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2\\
0 & 0 & 0 & y_1^{11} & y_2^{11} & y_3^{11} & y_4^{11} & y_5^{11}\\
0 & 0 & 0 & y_1^{14} & y_2^{14} & y_3^{14} & y_4^{14} & y_5^{14}
\end{vmatrix}.$$

**6.2.** ***Two linear-algebraic lemmas.*** For the proof of Theorem 44, we will need the following lemmas. Suppose $m, n, r \in \mathbb{N}$ and $m, n \geqslant r$, and let $\mathcal{M}$ be the set of matrices of the form

$$\left(\begin{array}{ccc|ccc|ccc}
z_{11} & \cdots & z_{1,m-r} & z_{1,m-r+1} & \cdots & z_{1,m} & z_{1,m+1} & \cdots & z_{1,m+n-r}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
z_{m-r,1} & \cdots & z_{m-r,m-r} & z_{m-r,m-r+1} & \cdots & z_{m-r,m} & z_{m-r,m+1} & \cdots & z_{m-r,m+n-r}\\
\hline
z_{m-r+1,1} & \cdots & z_{m-r+1,m-r} & z_{m+1-r,m-r+1} & \cdots & z_{m-r+1,m} & z_{m-r+1,m+1} & \cdots & z_{m-r+1,m+n-r}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
z_{m,1} & \cdots & z_{m,m-r} & z_{m,m-r+1} & \cdots & z_{m,m} & z_{m,m+1} & \cdots & z_{m,m+n-r}\\
\hline
 & & & z_{m+1,m-r+1} & \cdots & z_{m+1,m} & z_{m+1,m+1} & \cdots & z_{m+1,m+n-r}\\
 & 0 & & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
 & & & z_{m+n-r,m-r+1} & \cdots & z_{m+n-r,m} & z_{m+n-r,m+1} & \cdots & z_{m+n-r,m+n-r}
\end{array}\right). \tag{6-3}$$

**Lemma 47.** *There exists a unique polynomial $d(z_{ij})$, $d : \mathcal{M} \to \mathbb{C}$ such that:*

(1) *$d$ is antisymmetric with respect to interchanges of the* first *$m$ rows and of the* last *$n$ columns.*

(2) *$d$ is linear with respect to row operations on the first $m$ rows and column operations on the last $n$ columns.*

(3) *$d$ specializes to* 1 *on all matrices of the form*

$$\left(\begin{array}{c|c|c}
I_{m-r} & 0 & Z\\
\hline
0 & I_r & 0\\
\hline
0 & 0 & I_{n-r}
\end{array}\right),$$

*where $I$ denotes the identity matrix and $Z$ is arbitrary.*

*Proof.* The determinant satisfies all of the required conditions, proving existence. The proof of uniqueness is the same as the standard proof of the uniqueness of the determinant. Using row and column operations we can reduce any matrix in $\mathcal{M}$ to the form in condition (3). $\square$

**Remark 48.** Observe that a permutation of the first $m$ rows of a matrix $M \in \mathcal{M}$ corresponds to a permutation of the subset $\{1, \dots, m\}$ of the first indices of the elements $z_{i,j}$, and a permutation of the last $n$ columns of a matrix $M \in \mathcal{M}$ corresponds to a permutation of the subset $\{m+1-r, \dots, m+n-r\}$ of the second indices of the elements $z_{i,j}$.

**Lemma 49.** *Let $M \in \mathcal{M}$ be a matrix of the form (6-3). Then*

$$\det M = \frac{1}{r!} \sum_{w \in \mathrm{Sym}_m \times \mathrm{Sym}_n} (-1)^{l(w)} w(z_{1,1} z_{2,2} \dots z_{m+n-r,m+n-r}),$$

*where* $\mathrm{Sym}_m$ *permutes the subset* $\{1, \dots, m\}$ *of the first indices of the elements* $z_{i,j}$ *and* $\mathrm{Sym}_n$ *permutes the subset* $\{m+1-r, \dots, m+n-r\}$ *of the second indices of the elements* $z_{i,j}$.

*Proof.* By Lemma 47, we only need to check that the polynomial on the right-hand side satisfies the three conditions. The first two are clear. For the third, we need to count the number of elements $(u, v) \in \mathrm{Sym}_m \times \mathrm{Sym}_n$ that give a nonzero contribution to the alternating sum. It is easily seen that $u|_{\{1,\dots,m-r\}}$ and $v|_{\{m+1,\dots,m+n-r\}}$ must be identities. So to get a nonzero contribution (which must be 1) from the central block we need that $u|_{\{m+1-r,\dots,m\}} = v|_{\{m+1-r,\dots,m\}}$. These elements are all even, and there are $r!$ of them. $\square$

**6.3.** ***Proof of Theorem 44.*** Let $L$ be a KW-module with special arc diagram (6-1). We define

$$\begin{aligned} t_\lambda &= \frac{e^{\lambda^\rho}}{\prod_{\beta \in S_\lambda} (1 + e^{-\beta})} \\ &= \prod_{i=1}^{p} x_i^{a_i} \prod_{i=p+1}^{m-r} x_{r+i}^{a_i} \prod_{k=1}^{r} \frac{(x_{p+k} y_{q+k})^z}{1 + (x_{p+k} y_{q+k})^{-1}} \prod_{j=1}^{q} y_j^{b_j} \prod_{j=q+1}^{n-r} y_{r+j}^{b_j}. \end{aligned}$$

Then the Kac–Wakimoto character formula implies that

$$e^\rho R \cdot \mathrm{ch}\, L = \frac{1}{r!} \sum_{w \in \mathrm{Sym}_m \times \mathrm{Sym}_n} (-1)^{l(w)} w(t_\lambda). \tag{6-4}$$

Now by applying Lemma 49 to the matrix in question we obtain

$$\begin{aligned}\begin{vmatrix} X & Z \\ 0 & Y \end{vmatrix} &= \frac{1}{r!} \sum_{w \in \mathrm{Sym}_m \times \mathrm{Sym}_n} (-1)^{l(w)} w\left( \prod_{i=1}^{m-r} {x_i}^{a_i} \prod_{k=1}^{r} \frac{(x_{m-r+k} y_k)^z}{1+(x_{m-r+k} y_k)^{-1}} \prod_{j=1}^{n-r} {y_{r+j}}^{b_{r+j}} \right) \\ &= \frac{1}{r!} \sum_{w \in \mathrm{Sym}_m \times \mathrm{Sym}_n} (-1)^{l(w)} w((u^{-1}, v^{-1}) t_\lambda),\end{aligned}$$

where $u$ is the permutation sending $(x_1, \dots, x_m)$ to

$$(x_1, \dots, x_p, x_{p+r+1}, \dots, x_m, x_{p+1}, \dots, x_{p+r})$$

and $v$ is the permutation sending $(y_1, \dots, y_n)$ to

$$(y_{q+1}, \dots, y_{q+r}, y_1, \dots, y_q, y_{q+r+1}, \dots, y_n).$$

Then since $l(u) = rt$ and $l(v) = rq$, we have

$$\begin{vmatrix} X & Z \\ 0 & Y \end{vmatrix} = \frac{(-1)^{r(t+q)}}{r!} \sum_{w \in \mathrm{Sym}_m \times \mathrm{Sym}_n} (-1)^{l(w)} w(t_\lambda). \tag{6-5}$$

Combining (6-4) with (6-5) concludes the proof.

## Acknowledgments

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# Effective Matsusaka's theorem for surfaces in characteristic $p$

Gabriele Di Cerbo and Andrea Fanelli

We obtain an effective version of Matsusaka's theorem for arbitrary smooth algebraic surfaces in positive characteristic, which provides an effective bound on the multiple that makes an ample line bundle $D$ very ample. The proof for pathological surfaces is based on a Reider-type theorem. As a consequence, a Kawamata–Viehweg-type vanishing theorem is proved for arbitrary smooth algebraic surfaces in positive characteristic.

## 1. Introduction

A celebrated theorem of Matsusaka [1972] states that for a smooth $n$-dimensional complex projective variety $X$ and an ample divisor $D$ on it, there exists a positive integer $M$, depending only on the Hilbert polynomial $\chi(X, \mathbb{O}_X(kD))$, such that $mD$ is very ample for all $m \geq M$. Kollár and Matsusaka [1983] improved the result, showing that the integer $M$ only depends on the intersection numbers $(D^n)$ and $(K_X \cdot D^{n-1})$.

The first effective versions of this result are due to Siu [2002a; 2002b] and Demailly [1996a; 1996b]; their methods are cohomological and rely on vanishing theorems. See also [Lazarsfeld 2004b] for a full account of this approach.

Although the minimal model program for surfaces in positive characteristic has recently been established, thanks to the work of Tanaka [2014; 2012], some

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interesting effectivity questions remain open in this setting, after the influential papers [Ekedahl 1988] and [Shepherd-Barron 1991a].

The purpose of this paper is to present a complete solution for the following problem:

**Question 1.1.** Let $X$ be a smooth surface over an algebraically closed field of positive characteristic, and let $D$ and $B$ be an ample and a nef divisor on $X$ respectively. Then there exists an integer $M$ depending only on $(D^2)$, $(K_X \cdot D)$ and $(D \cdot B)$ such that

$$mD - B$$

is very ample for all $m \geq M$.

The analogous question in characteristic zero with $B = 0$ was totally solved in [Fernández del Busto 1996], and a modified technique allows one to partially extend the result in positive characteristic [Ballico 1996].

The main result of this paper is the following:

**Theorem 1.2.** *Let $D$ and $B$ be respectively an ample divisor and a nef divisor on a smooth surface $X$ over an algebraically closed field $k$, with* $\operatorname{char} k = p > 0$. *Then $mD - B$ is very ample for any*

$$m > \frac{2D \cdot (H + B)}{D^2}((K_X + 2D) \cdot D + 1),$$

*where*:

- *$H := K_X + 4D$ if $X$ is neither quasielliptic with $\kappa(X) = 1$ nor of general type.*
- *$H := K_X + 8D$ if $X$ is quasielliptic with $\kappa(X) = 1$ and $p = 3$.*
- *$H := K_X + 19D$ if $X$ is quasielliptic with $\kappa(X) = 1$ and $p = 2$.*
- *$H := 2K_X + 4D$ if $X$ is of general type and $p \geq 3$.*
- *$H := 2K_X + 19D$ if $X$ is of general type and $p = 2$.*

The effective bound obtained with $H = K_X + 4D$ is expected to hold for all surfaces. Note that this bound is not far from being sharp even in characteristic zero [Fernández del Busto 1996].

The proof of Theorem 1.2 does not rely directly on vanishing theorems, but rather on Fujita's conjecture on basepoint-freeness and very-ampleness of adjoint divisors, which is known to hold for smooth surfaces in characteristic zero [Reider 1988] and for smooth surfaces in positive characteristic which are neither quasielliptic with $\kappa(X) = 1$ nor of general type [Shepherd-Barron 1991a; Terakawa 1999].

**Conjecture 1.3** (Fujita). *Let $X$ be a smooth $n$-dimensional projective variety and let $D$ be an ample divisor on it. Then $K_X + kD$ is basepoint free for $k \geq n + 1$ and very ample for $k \geq n + 2$.*

If Fujita's conjecture on very-ampleness holds then the bound of Theorem 1.2 with $H = K_X + 4D$ would work for arbitrary smooth surfaces in positive characteristic.

For surfaces which are quasielliptic with $\kappa(X) = 1$ or of general type we can prove the following effective result in the spirit of Fujita's conjecture (see Section 4):

**Theorem 1.4.** *Let $X$ a smooth surface over an algebraically closed field of characteristic $p > 0$, let $D$ an ample Cartier divisor on $X$ and let $L(a, b) := aK_X + bD$ for positive integers $a$ and $b$. Then $L(a, b)$ is very ample for the following values of $a$ and $b$:*

(1) *If $X$ is quasielliptic with $\kappa(X) = 1$ and $p = 3$, $a = 1$ and $b \geq 8$.*

(2) *If $X$ is quasielliptic with $\kappa(X) = 1$ and $p = 2$, $a = 1$ and $b \geq 19$.*

(3) *If $X$ is of general type with $p \geq 3$, $a = 2$ and $b \geq 4$.*

(4) *If $X$ is of general type with $p = 2$, $a = 2$ and $b \geq 19$.*

The key ingredient of Theorem 1.4 is a combination of a Reider-type result due to Shepherd-Barron and bend-and-break techniques.

For other results on the geography of pathological surfaces of Kodaira dimension smaller than two, see [Langer 2014].

In Section 5, a Kawamata–Viehweg-type vanishing theorem is proved for surfaces that are quasielliptic with $\kappa(X) = 1$ or of general type (see Theorem 5.7 and Corollary 5.9); this generalizes the vanishing result in [Terakawa 1999].

The core of our approach is a beautiful construction first introduced by Tango [1972] for the case of curves and Ekedahl [1988] and Shepherd-Barron [1991a] for surfaces. The same strategy was generalized by Kollár [1996] in order to investigate the geography of varieties where Kodaira-type vanishing theorems fail, via bend-and-break techniques.

## 2. Preliminary results

In this section we recall some techniques we will need later in this paper.

**2A.** ***Volume of divisors.*** Let $D$ be a Cartier divisor on a normal variety $X$, not necessarily a surface. The volume of $D$ measures the asymptotic growth of the space of global sections of multiples of $D$. We will recall here few properties of the volume, and we refer to [Lazarsfeld 2004b] for more details.

**Definition 2.1.** Let $D$ be a Cartier divisor on $X$, with $\dim(X) = n$. The volume of $D$ is defined by

$$\operatorname{vol}(D) := \limsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

The volume of $X$ is defined as $\operatorname{vol}(X) := \operatorname{vol}(K_X)$.

It is easy to show that if $D$ is big and nef then $\operatorname{vol}(D) = D^n$. In general it is a hard invariant to compute, but thanks to Fujita's approximation theorem some of its properties can be deduced from the case where $D$ is ample. For a proof of the theorem in characteristic zero we refer to [Lazarsfeld 2004b]. More recently, Takagi [2007] gave a proof of the same theorem in positive characteristic. In particular, we can deduce the log-concavity of the volume function even in positive characteristic. The proof is exactly the same as Theorem 11.4.9 in [Lazarsfeld 2004b].

**Theorem 2.2.** *Let $D$ and $D'$ be big Cartier divisors on a normal variety $X$ defined over an algebraically closed field. Then*

$$\operatorname{vol}(D + D')^{1/n} \geq \operatorname{vol}(D)^{1/n} + \operatorname{vol}(D')^{1/n}.$$

**2B.** ***Bogomolov's inequality and Sakai's theorems.*** We start with the notion of semistability for rank-two vector bundles on surfaces. Let $X$ be a smooth surface defined over an algebraically closed field.

**Definition 2.3.** A rank-two vector bundle $\mathscr{E}$ on $X$ is *unstable* if it fits in a short exact sequence

$$0 \longrightarrow \mathscr{O}_X(D_1) \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_Z \cdot \mathscr{O}_X(D_2) \longrightarrow 0,$$

where $D_1$ and $D_2$ are Cartier divisors such that $D' := D_1 - D_2$ is big with $(D'^2) > 0$ and $Z$ is an effective 0-cycle on $X$.

The vector bundle $\mathscr{E}$ is *semistable* if it is not unstable.

In characteristic zero, the following celebrated result holds:

**Theorem 2.4** [Bogomolov 1978]. *Let $X$ be defined over a field of characteristic zero. Then every rank-two vector bundle $\mathscr{E}$ for which $c_1^2(\mathscr{E}) > 4c_2(\mathscr{E})$ is unstable.*

As a consequence, one can deduce the following theorem, due to Sakai [1990, Proposition 1], which turns out to be equivalent to Theorem 2.4. This equivalence was shown in [Di Cerbo 2013].

**Theorem 2.5.** *Let $D$ be a nonzero big divisor with $D^2 > 0$ on a smooth projective surface $X$ over a field of characteristic zero. If $H^1(X, \mathscr{O}_X(K_X + D)) \neq 0$, then there exists a nonzero effective divisor $E$ such that*:

- *$D - 2E$ is big.*
- *$(D - E) \cdot E \leq 0$.*

The previous result easily implies a weaker version of Reider's theorem:

**Theorem 2.6.** *Let $D$ be a nef divisor with $D^2 > 4$ on a smooth projective surface $X$ over a field of characteristic zero. Then $K_X + D$ has no basepoint unless there exists a nonzero effective divisor $E$ such that $D \cdot E = 0$ and $(E^2) = -1$ or $D \cdot E = 1$ and $(E^2) = 0$.*

The following result, conjectured by Fujita, can be deduced for smooth surfaces in characteristic zero:

**Corollary 2.7** (Fujita conjectures for surfaces in char 0). *Let $D_1, \dots, D_k$ be ample divisors on a smooth surface $X$ over a field of characteristic zero. Then $K_X + D_1 + \cdots + D_k$ is basepoint free if $k \geq 3$ and very ample if $k \geq 4$.*

We remark that Theorem 2.5 is not known in general for smooth surfaces in positive characteristic, although Fujita's conjectures are expected to hold.

**2C.** ***Ekedahl's construction and Shepherd-Barron's theorem.*** In this section we recall some classical results on the geography of smooth surfaces in positive characteristic (see [Ekedahl 1988; Shepherd-Barron 1991a; 1991b]).

For a good overview on the geography of surfaces in positive characteristic, see [Liedtke 2013].

We discuss here a construction which is due to Tango [1972] for the case of curves and Ekedahl [1988] for surfaces. There are many variations on the same theme, but we will focus on the one which is more related to the stability of vector bundles. We need this fundamental result:

**Theorem 2.8** (Bogomolov). *Let $\mathscr{E}$ be a rank-two vector bundle on a smooth projective surface $X$ over a field of positive characteristic such that Bogomolov's inequality does not hold (i.e., such that $c_1^2(\mathscr{E}) > 4c_2(\mathscr{E})$). Then there exists a reduced and irreducible surface $Y$ contained in the ruled threefold $\mathbb{P}(\mathscr{E})$ such that*:

- *The restriction $\rho : Y \to X$ is $p^e$-purely inseparable for some $e > 0$.*
- *$(F^*)^e(\mathscr{E})$ is unstable.*

*Proof.* See [Shepherd-Barron 1991a, Theorem 1] ☐

The previous result also provides an explicit construction of the purely inseparable cover (see [Shepherd-Barron 1991a]).

**Construction 2.9.** Take a rank-two vector bundle $\mathscr{E}$ such that Bogomolov's inequality does not hold, and let $e$ be an integer such that $F^{e*}\mathscr{E}$ is unstable. We have the commutative diagram

$$\begin{tikzcd}
\mathbb{P}(F^{e*}\mathscr{E}) \arrow[r, "G"] \arrow[d, "p'"'] & \mathbb{P}(\mathscr{E}) \arrow[d, "p"] \\
X \arrow[r, "F^e"] & X
\end{tikzcd}.$$

The fact that $F^{e*}\mathscr{E}$ is unstable gives an exact sequence

$$0 \longrightarrow \mathscr{O}_X(D_1) \longrightarrow F^{e*}\mathscr{E} \longrightarrow \mathscr{I}_Z \cdot \mathscr{O}_X(D_2) \longrightarrow 0$$

and a quasisection $X_0$ of $\mathbb{P}(F^{e*}\mathscr{E})$ (i.e., $p'_{|X_0} : X_0 \to X$ is birational). Let $Y$ be the image of $X_0$ via $G$. One can show that the induced morphism

$$\rho : Y \to X$$

is $p^e$-purely inseparable. Let us define $D' := D_1 - D_2$. One can show (see [Shepherd-Barron 1991a, Corollary 5]) that

$$K_Y \equiv \rho^*\left(K_X - \frac{p^e-1}{p^e} D'\right).$$

**Remark 2.10.** We will be particularly interested in the case when the rank-two vector bundle $\mathscr{E}$ comes as a nontrivial extension

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_X(D) \longrightarrow 0$$

associated to a nonzero element $\gamma \in H^1(X, \mathscr{O}_X(-D))$, where $D$ is a big Cartier divisor such that $(D^2) > 0$. Indeed, the instability of $F^{e*}\mathscr{E}$ guarantees the existence of a diagram (keeping the notation as in Definition 2.3)

$$\begin{tikzcd}
 & & 0 \arrow[d] & & . \\
 & & \mathscr{O}_X(D' + D_2) \arrow[d, "f_1"] & & \\
0 \arrow[r] & \mathscr{O}_X \arrow[r, "g_1"] \arrow[dr, "\sigma"'] & F^{e*}\mathscr{E} \arrow[r, "g_2"] \arrow[d, "f_2"] & \mathscr{O}_X(p^e D) \arrow[r] & 0 \\
 & & \mathscr{I}_Z \cdot \mathscr{O}_X(D_2) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

First, we claim that the composition map $\sigma$ is nonzero. Assume for a contradiction that $\sigma \equiv 0$. This gives a nonzero section $\sigma' : \mathscr{O}_X \to \mathscr{O}_X(D' + D_2)$. This forces the composition $\tau := g_2 \circ f_1$ to be zero. But this implies that $D' + D_2 \leq 0$. This is a contradiction (see the proof of [Sakai 1990, Proposition 1] and [Shepherd-Barron 1991a, Lemma 16]).

This implies that $D_2 \simeq E \geq 0$; one can then rewrite the vertical exact sequence as

$$0 \longrightarrow \mathscr{O}_X(p^e D - E) \longrightarrow F^{e*}\mathscr{E} \longrightarrow \mathscr{I}_Z \cdot \mathscr{O}_X(E) \longrightarrow 0.$$

Since [Shepherd-Barron 1991a, Corollary 8] guarantees that Corollary 2.7 holds true for smooth surfaces in positive characteristic which are neither quasielliptic with $\kappa(X) = 1$ nor of general type, we need to deduce effective basepoint-freeness and very-ampleness results only for these two classes of surfaces.

We recall here the following key result from [Shepherd-Barron 1991a]:

**Theorem 2.11.** *Let $\mathscr{E}$ be a rank-two vector bundle on a smooth projective surface $X$ over an algebraically closed field of positive characteristic such that Bogomolov's inequality does not hold and $\mathscr{E}$ is semistable.*

- *If $X$ is not of general type, then $X$ is quasielliptic with $\kappa(X) = 1$.*
- *If $X$ is of general type and*

$$c_1^2(\mathscr{E}) - 4c_2(\mathscr{E}) > \frac{\operatorname{vol}(X)}{(p-1)^2},$$

  *then $X$ is purely inseparably uniruled. More precisely, in the notation of Theorem 2.8, $Y$ is uniruled.*

*Proof.* This is [Shepherd-Barron 1991a, Theorem 7], since the volume of a surface $X$ with minimal model $X'$ equals $(K_{X'}^2)$. ☐

**Corollary 2.12** [Shepherd-Barron 1991a, Corollary 8]. *Corollary 2.7 holds in positive characteristic if $X$ is neither of general type nor quasielliptic.*

**2D. *Bend-and-break lemmas.*** We recall here a well-known result in birational geometry, based on a celebrated method due to Mori (see [Kollár 1996] for an insight into these techniques).

First we need to recall some notation. Mori theory deals with effective 1-cycles in a variety $X$; more specifically, we will consider nonconstant morphisms $h : C \to X$, where $C$ is a smooth curve. In particular, these techniques allow us to deform curves for which

$$(K_X \cdot C) := \deg_C h^* K_X < 0.$$

In what follows, we will denote by $\overset{e}{\approx}$ the *effective algebraic equivalence* defined on the space of effective 1-cycles $Z_1(X)$ (see [Kollár 1996, Definition II.4.1]).

**Theorem 2.13** (bend-and-break). *Let $X$ be a variety over an algebraically closed field, and let $C$ be a smooth, projective and irreducible curve with a morphism $h : C \to X$ such that $X$ has local complete intersection singularities along $h(C)$ and $h(C)$ intersects the smooth locus of $X$. Assume the numerical condition*

$$(K_X \cdot C) < 0$$

*holds. Then, for every point $x \in C$, there exists a rational curve $C_x$ in $X$ passing through $x$ such that*

$$h_*[C] \overset{e}{\approx} k_0[C_x] + \sum_{i \neq 0} k_i [C_i] \tag{1}$$

(*as algebraic cycles*), *with* $k_i \geq 0$ *for all* $i$ *and*

$$-(K_X \cdot C_x) \leq \dim X + 1.$$

*Proof.* See [Kollár 1996, Theorem II.5.14 and Remark II.5.15]. The relation (1) can be deduced by looking directly at the proofs of the bend-and-break lemmas [Kollár 1996, Corollary II.5.6 and Theorem II.5.7]; our notation is slightly different, since in (1) we have isolated a rational curve with the required intersection properties. ☐

In this paper we will need the following consequence of the previous theorem:

**Corollary 2.14.** *Let* $X$ *be a surface which fibers over a curve* $C$ *via* $f : X \to C$ *and let* $F$ *be the general fiber of* $f$. *Assume that* $X$ *has only local complete intersection singularities along* $F$ *and that* $F$ *is a* (*possibly singular*) *rational curve such that*

$$(K_X \cdot F) < 0.$$

*Then*

$$-(K_X \cdot F) \leq 3.$$

*Proof.* The hypotheses of Theorem 2.13 hold here, so we can take a point $x$ in the smooth locus of $X$ and deduce the existence of a rational curve $C'$ passing through $x$ such that

$$-(K_X \cdot C') \leq 3 \quad \text{and} \quad [F] \overset{e}{\approx} k_0[C'] + \sum_{i \neq 0} k_i [C_i].$$

By Exercise II.4.1.10 in [Kollár 1996], the curves appearing on the right hand side of the previous equation must be contained in the fibers of $f$. Since $F$ is the general fiber, the second relation implies that $k_0 = 1$ and $k_i = 0$ for all $i \neq 0$, and so $C' = F$. ☐

## 3. An effective Matsusaka's theorem

In this section we prove Theorem 1.2 assuming the results on effective very-ampleness that we will prove in the next section. If not specified, $X$ will denote a smooth surface over an algebraically closed field of arbitrary characteristic.

First we recall the following numerical criterion for bigness, whose characteristic-free proof is based on Riemann–Roch [Lazarsfeld 2004a, Theorem 2.2.15].

**Theorem 3.1.** *Let* $D$ *and* $E$ *be nef* $\mathbb{Q}$-*divisors on* $X$ *and assume that*

$$D^2 > 2(D \cdot E).$$

*Then* $D - E$ *is big.*

Before proving Theorem 1.2, we need some lemmas.

**Lemma 3.2.** *Let $D$ be an ample divisor on $X$. Then $K_X + 2D + C$ is nef for any irreducible curve $C \subset X$.*

*Proof.* If $X = \mathbb{P}^2$ then the lemma is trivial. By the cone theorem and the classification of surfaces with extremal rays of maximal length, we have that $K_X + 2D$ is always a nef divisor. This implies that $K_X + 2D + C$ may have negative intersection number only when intersected with $C$. On the other hand, by adjunction, $(K_X + C) \cdot C = 2g - 2 \geq -2$, where $g$ is the arithmetic genus of $C$. Since $D$ is ample, the result follows. □

We can now prove one of the main results of this section (see [Lazarsfeld 2004b, Theorem 10.2.4]).

**Theorem 3.3.** *Let $D$ be an ample divisor and let $B$ be a nef divisor on $X$. Then $mD - B$ is nef for any*

$$m \geq \frac{2D \cdot B}{D^2}((K_X + 2D) \cdot D + 1) + 1.$$

*Proof.* To simplify the notation in the proof let us define the following numbers:

$$\eta = \eta(D, B) := \inf\{t \in \mathbb{R}_{>0} \mid tD - B \text{ is nef}\},$$
$$\gamma = \gamma(D, B) := \inf\{t \in \mathbb{R}_{>0} \mid tD - B \text{ is pseudoeffective}\}.$$

The theorem will follow if we find an upper bound on $\eta$. Note that $\gamma \leq \eta$ since a nef divisor is also pseudoeffective.

By definition $\eta D - B$ is in the boundary of the nef cone and by Nakai's theorem we have two possible cases: either

- $(\eta D - B)^2 = 0$, or
- there exists an irreducible curve $C$ such that $(\eta D - B) \cdot C = 0$.

If $(\eta D - B)^2 = 0$, then it is easy to see that

$$\eta \leq 2\frac{D \cdot B}{D^2}.$$

So we can assume that there exists an irreducible curve $C$ such that $\eta D \cdot C = B \cdot C$. Let us define $G := \gamma D - B$. Then

$$G \cdot C = (\gamma - \eta) D \cdot C \leq (\gamma - \eta).$$

Let us define $A := K_X + 2D$. By Lemma 3.2 and the definition of $G$, we have that $(A + C) \cdot G \geq 0$. Combining with the previous inequality we get

$$(\eta - \gamma) \leq -G \cdot C \leq A \cdot G = \gamma A \cdot D - A \cdot B.$$

In particular,

$$\eta \leq \gamma(A \cdot D + 1) - A \cdot B \leq \gamma(A \cdot D + 1).$$

The statement of our result follows from Theorem 3.1, which guarantees that $\gamma < (2D \cdot B)/(D^2)$. ☐

**Remark 3.4.** The previous proof is characteristic-free, although the new result is for surfaces in positive characteristic.

We can now prove our main theorem, assuming the results in the next section.

*Proof of Theorem 1.2.* By Corollary 2.12, if $X$ is neither of general type nor quasielliptic and $H = K_X + 4D$ then $H + N$ is very ample for any nef divisor $N$. By Theorem 3.3, $mD - (H + B)$ is nef for any $m$ as in the statement. Then $K_X + 4D + (mD - K_X - 4D - B)$ is very ample. For surfaces in the other classes use Theorem 4.10 and Theorem 4.12 to obtain the desired very ample divisor $H$. ☐

## 4. Effective very-ampleness in positive characteristic

The aim of this section is to complete the proof of Theorem 1.2 for quasielliptic surfaces of Kodaira dimension one and for surfaces of general type. Our ultimate goal is to prove Theorem 1.4 via a case-by-case analysis.

First we need some notation (cf. Theorem 2.5).

**Definition 4.1.** A big divisor $D$ on a smooth surface $X$ with $(D^2) > 0$ is $m$-unstable for a positive integer $m$ if either:

- $H^1(X, \mathcal{O}_X(-D)) = 0$.
- $H^1(X, \mathcal{O}_X(-D)) \neq 0$ and there exists a nonzero effective divisor $E$ such that:
  - $mD - 2E$ is big.
  - $(mD - E) \cdot E \leq 0$.

**Remark 4.2.** Theorem 2.5 tells us that in characteristic zero every big divisor $D$ on a smooth surface $X$ with $(D^2) > 0$ is 1-unstable. The same holds in positive characteristic, if we assume that the surface is neither of general type nor quasielliptic of maximal Kodaira dimension; this is a consequence of Corollary 2.12. Our goal here is to clarify the picture in the remaining cases.

We start our analysis with quasielliptic surfaces of maximal Kodaira dimension.

**Proposition 4.3.** *Let $X$ be a quasielliptic surface with $\kappa(X) = 1$ and let $D$ be a big divisor on $X$ with $(D^2) > 0$.*

(1) *if $p = 3$, then $D$ is 3-unstable.*

(2) *if $p = 2$, then $D$ is 4-unstable.*

*Proof.* Assume that $p = 3$ and $H^1(X, \mathcal{O}_X(-D)) \neq 0$. This nonzero element gives a nonsplit extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X(D) \to 0.$$

Theorem 2.8 implies that $(F^*)^e\mathscr{E}$ is unstable for $e$ sufficiently large. To prove the proposition in this case we need to show that $e = 1$. Assume $e \geq 2$ and let $F$ be the general element of the pencil which gives the fibration in cuspidal curves $f : X \to B$. Let $\rho : Y \to X$ be the $p^e$-purely inseparable morphism of Construction 2.9. Then $\{C_i := \rho^* F\}$ is a family of movable rational curves in $Y$. Let us define $g := f \circ \rho$ and consider its Stein factorization:

$$\begin{tikzcd}
Y \arrow[d, "\rho"'] \arrow[dr, "h"] \arrow[dd, bend right=60, "g"'] & & . \\
X \arrow[d, "f"'] & B' \arrow[dl, "c"] & \\
B & &
\end{tikzcd}$$

Since the curves in the family $\{C_i\}$ are precisely the fibers of $h$, we can use Corollary 2.14 on $h : Y \to B'$ (since $Y$ is defined via a quasisection in a $\mathbb{P}^1$-bundle over $X$, it has hypersurface singularities along the general element of $\{C_i\}$) and deduce that

$$0 < -(K_Y \cdot C_i) \leq 3.$$

This gives a contradiction, since

$$\begin{aligned}
3 \geq -(K_Y \cdot C_i) &= \left(\rho^*\left(\frac{p^e-1}{p^e}(p^e D - 2E) - K_X\right) \cdot C_i\right) \\
&= p^e\left(\left(\frac{p^e-1}{p^e}(p^e D - 2E) - K_X\right) \cdot F\right) \\
&= ((p^e-1)(p^e D - 2E) \cdot F) \\
&\geq p^e - 1 \geq 8,
\end{aligned}$$

where $E$ is the divisor appearing in Remark 2.10.

The same proof works for $p = 2$, although in this case we can only prove that $e \leq 2$. □

We can now focus on the general type case. We need the following theorem of Shepherd-Barron [1991a, Theorem 12].

**Theorem 4.4.** *Let $D$ be a big Cartier divisor on a smooth surface $X$ of general type which satisfies one of the following hypotheses*:

- $p \geq 3$ *and* $(D^2) > \mathrm{vol}(X)$.
- $p = 2$ *and* $(D^2) > \max\{\mathrm{vol}(X), \mathrm{vol}(X) - 3\chi(\mathbb{O}_X) + 2\}$.

*Then $D$ is* 1*-unstable.*

Since the bound of the previous theorem depends on $\chi(\mathcal{O}_X)$ if $p = 2$, we need an additional result for this case. First we recall a result by Shepherd-Barron [1991b, Theorem 8].

**Theorem 4.5.** *Let $X$ be a surface in characteristic $p = 2$ of general type with $\chi(\mathcal{O}_X) < 0$. Then there is a fibration $f : X \to C$ over a smooth curve $C$ whose generic fiber is a singular rational curve with arithmetic genus $2 \leq g \leq 4$.*

We can prove now our result.

**Proposition 4.6.** *Let $D$ be a big Cartier divisor on a surface in characteristic $p = 2$ of general type with $\chi(\mathcal{O}_X) < 0$ such that $(D^2) > \operatorname{vol}(X)$. Then $D$ is 4-unstable.*

*Proof.* Assume that $H^1(X, \mathcal{O}_X(-D)) \neq 0$. As in the proof of Proposition 4.3 we have a nonsplit extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X(D) \to 0.$$

Using Theorem 2.8 we deduce the instability of $(F^*)^e\mathcal{E}$ for $e$ sufficiently large. Let $F$ be the general element of the pencil which gives the fibration in singular rational curves given by Theorem 4.5. Let $\rho : Y \to X$ be the $p^e$-purely inseparable morphism of Construction 2.9. As in the proof of Proposition 4.3, we use Corollary 2.14 on $Y$ and deduce that $0 < -(K_Y \cdot C_i) \leq 3$. This gives

$$\begin{aligned} 3 \geq -(K_Y \cdot C_i) &= \left(\rho^*\left(\frac{2^e-1}{2^e}(2^e D - 2E) - K_X\right) \cdot C_i\right) \\ &= 2^e\left(\left(\frac{2^e-1}{2^e}(2^e D - 2E) - K_X\right) \cdot F\right) \\ &= (((2^e-1)(2^e D - 2E) - 2^e K_X) \cdot F) \geq 1. \end{aligned}$$

This implies that

$$(((2^e-1)(2^{e-1} D - E) - 2^{e-1} K_X) \cdot F) = 1.$$

As a consequence, we apply Theorem 4.5 to bound the intersection $(K_X \cdot F)$:

$$(2^e-1)((2^{e-1} D - E) \cdot F) = 2^e(g-1) + 1,$$

where $g$ is the arithmetic genus of $F$. By some basic arithmetic the only possibilities for the pair $(g, e)$ are $(2, 1)$, $(3, 1)$, $(3, 2)$ and $(4, 1)$. □

We will use Theorem 4.4 to prove a variant of Reider's theorem in positive characteristic. We state a technical proposition that we will need later (see [Sakai 1990, Proposition 2]).

**Proposition 4.7.** *Let $\pi : Y \to X$ be a birational morphism between two normal surfaces. Let $\widetilde{D}$ be a Cartier divisor on $Y$ such that $\widetilde{D}^2 > 0$. Assume there is a nonzero effective divisor $\widetilde{E}$ such that*

- $\widetilde{D}-2\widetilde{E}$ *is big,*
- $(\widetilde{D}-\widetilde{E})\cdot\widetilde{E}\leq 0$.

*Set* $D:=\pi_*\widetilde{D}$, $E:=\pi_*\widetilde{E}$ *and* $\alpha=D^2-\widetilde{D}^2$. *If* $D$ *is nef and* $E$ *is a nonzero effective divisor, then*

- $0\leq D\cdot E<\alpha/2$,
- $D\cdot E-\alpha/4\leq E^2\leq (D\cdot E)^2/D^2$.

The corollary we need is the following.

**Corollary 4.8.** *Let* $\pi:Y\to X$ *be a birational morphism between two smooth surfaces and let* $\widetilde{D}$ *be a big Cartier divisor on* $Y$ *such that* $(\widetilde{D}^2)>0$. *Assume that*

- $H^1(X,\mathcal{O}_X(-\widetilde{D}))\neq 0$,
- $\widetilde{D}$ *is* $m$*-unstable for some* $m>0$.

*Set* $D:=\pi_*\widetilde{D}$ *and* $\alpha=D^2-\widetilde{D}^2$. *Then if* $D$ *is nef, there exists a nonzero effective divisor* $E$ *on* $X$ *such that*

- $0\leq D\cdot E<m\alpha/2$,
- $mD\cdot E-m^2\alpha/4\leq E^2\leq (D\cdot E)^2/D^2$.

We can now derive our effective basepoint-freeness results. We will start with quasielliptic surfaces, applying Proposition 4.3 and the previous corollary.

**Proposition 4.9.** *Let* $X$ *be a quasielliptic surface with maximal Kodaira dimension. Let* $D$ *be a big and nef divisor on* $X$. *Then the following hold.*

- *For* $p=3$:
  - *If* $D^2>4$ *and* $|K_X+D|$ *has a basepoint at* $x\in X$, *there exists a curve* $C$ *such that* $D\cdot C\leq 5$.
  - *If* $D^2>9$ *and* $|K_X+D|$ *does not separate any two points* $x,y\in X$, *there exists a curve* $C$ *such that* $D\cdot C\leq 13$.
- *For* $p=2$:
  - *If* $D^2>4$ *and* $|K_X+D|$ *has a basepoint at* $x\in X$, *there exists a curve* $C$ *such that* $D\cdot C\leq 7$.
  - *If* $D^2>9$ *and* $|K_X+D|$ *does not separate any two points* $x,y\in X$, *there exists a curve* $C$ *such that* $D\cdot C\leq 17$.

*Proof.* We start with the case $p=3$. Assume that $|K_X+D|$ has a basepoint at $x\in X$. Let $\pi:Y\to X$ be the blowup at $x$. Since $x$ is a basepoint we have that $H^1(Y,\mathcal{O}_Y(K_Y+\pi^*D-2F))\neq 0$, where $F$ is the exceptional divisor of $\pi$. Let $\widetilde{D}:=\pi^*D-2F$. By assumption we have that $\widetilde{D}^2>0$. By Proposition 4.3 we can find an effective divisor $\widetilde{E}$ such that $p\widetilde{D}-2\widetilde{E}$ is big and $(p\widetilde{D}-\widetilde{E})\cdot\widetilde{E}\leq 0$. The previous inequality easily implies that $\widetilde{E}$ is not a positive multiple of the exceptional divisor and in particular $E := \pi_* \widetilde{E}$ is a nonzero effective divisor. Moreover, $D = \pi_* \widetilde{D}$ is nef by assumption, thus we can apply Corollary 4.8. Since $\alpha = (D^2 - \widetilde{D}^2) = 4$, the first inequality of the corollary implies that $D \cdot E \leq 5$.

The statement on separation of points follows in exactly the same way. Note that we allow the case $x = y$.

The bounds for the case $p = 2$ can be obtained the same way, remarking that $\widetilde{D}$ is $p^2$-unstable in this case. ☐

The previous results can be used to derive effective very-ampleness statements for quasielliptic surfaces when $D$ is an ample divisor.

**Theorem 4.10.** *Let $D$ be an ample Cartier divisor on a smooth quasielliptic surface $X$ with $\kappa(X) = 1$.*

- *If $p = 3$, the divisor $K_X + kD$ is basepoint-free for any $k \geq 4$ and it is very ample for any $k \geq 8$.*
- *If $p = 2$, the divisor $K_X + kD$ is basepoint-free for any $k \geq 5$ and it is very ample for any $k \geq 19$.*

*In particular, if $N$ is any nef divisor, $K_X + kD + N$ is always very ample for any $k \geq 8$ (resp. $k \geq 19$) in characteristic 3 (resp. 2).*

*Proof.* The proof consists of explicitly computing the minimal multiple of $D$ which contradicts the second inequality of Corollary 4.8.

Let us start with basepoint-freeness for $p = 3$. Assume that $k \geq 5$, $K_X + kD$ has a basepoint and define $D' := kD$. Then, by Proposition 4.9, we know that there exists an effective divisor $E$ such that $(D' \cdot E) \leq 5$. This implies

$$(D \cdot E) \leq 1.$$

Now use the second inequality of Corollary 4.8 on $D'$ to deduce

$$15 - 9 \leq 3(D' \cdot E) - 9 \leq \frac{(D' \cdot E)^2}{(D'^2)} \leq 1.$$

This is a contradiction.

Similar computations give the other bounds. ☐

We now deal with surfaces of general type. The analogue of Proposition 4.9 is the following.

**Proposition 4.11.** *Let $X$ be a surface of general type and let $D$ be a big and nef divisor on $X$. Then the following hold.*

- *For $p \geq 3$:*
  - *If $D^2 > \operatorname{vol}(X) + 4$ and $|K_X + D|$ has a basepoint at $x \in X$, there exists a curve $C$ such that $D \cdot C \leq 1$.*

  - *If* $D^2 > \mathrm{vol}(X) + 9$ *and* $|K_X + D|$ *does not separate any two points* $x, y \in X$, *there exists a curve* $C$ *such that* $D \cdot C \leq 2$.
- *For* $p = 2$:
  - *If* $D^2 > \mathrm{vol}(X) + 6$ *and* $|K_X + D|$ *has a basepoint at* $x \in X$, *there exists a curve* $C$ *such that* $D \cdot C \leq 7$.
  - *If* $D^2 > \mathrm{vol}(X) + 11$ *and* $|K_X + D|$ *does not separate any two points* $x, y \in X$, *there exists a curve* $C$ *such that* $D \cdot C \leq 17$.

*Proof.* The proof is basically the same as Proposition 4.9. Let $p \geq 3$ and assume that $|K_X + D|$ has a basepoint at $x \in X$. Using the same notation as Proposition 4.9, we can blow up $x$ and deduce the existence of an effective divisor $\widetilde{E}$ such that $\widetilde{D} - 2\widetilde{E}$ is big and $(\widetilde{D} - \widetilde{E}) \cdot \widetilde{E} \leq 0$ (in order to deduce 1-instability we use Theorem 4.4). Also here, the first inequality of Corollary 4.8 implies that $(D \cdot E) \leq 1$.

The statement on separation of points follows in the same way.

For the bounds in the case $p = 2$ we use the same strategy, using a combination of Theorem 4.5 and Proposition 4.6. ☐

The following effective very-ampleness statement can be deduced. Applying Proposition 4.11 directly would provide bounds that depend on the volume. It is possible to get a uniform bound if we work with linear systems of the type $|2K_X + mD|$. Note that we get sharp statements for those linear systems.

**Theorem 4.12.** *Let* $D$ *be an ample Cartier divisor on a smooth surface* $X$ *of general type.*

- *If* $p \geq 3$, *the divisor* $2K_X + kD$ *is basepoint free for any* $k \geq 3$ *and it is very ample for any* $k \geq 4$.
- *If* $p = 2$ *the divisor* $2K_X + kD$ *is basepoint free for any* $k \geq 5$ *and it is very ample for any* $k \geq 19$.

*In particular, if* $N$ *is any nef divisor,* $2K_X + kD + N$ *is always very ample for any* $k \geq 4$ (*resp.* $k \geq 19$) *in characteristic* $p \geq 3$ (*resp.* $p = 2$).

*Proof.* Since negative extremal rays of general type surfaces have length 1, if $m \geq 3$, we know that $L := K_X + mD$ is an ample divisor and $L \cdot C \geq 2$ for any irreducible curve $C \subset X$. Moreover, by log-concavity of the volume function (see Theorem 2.2) we have that

$$L^2 = \mathrm{vol}(L) \geq \mathrm{vol}(K_X) + 9D^2 > \mathrm{vol}(X) + 4.$$

Proposition 4.11 implies that $K_X + L = 2K_X + kD$ is basepoint free for any $k \geq 4$. A similar computation allows us to derive very-ampleness.

The same strategy gives the result for $p = 2$. ☐

*Proof of Theorem 1.4.* This is simply given by Theorem 4.10 and Theorem 4.12. ☐

**Remark 4.13.** In [Terakawa 1999], similar results can be found. Nonetheless our approach allows us to deduce effective basepoint-freeness and very-ampleness also on quasielliptic surfaces and arbitrary surfaces of general type.

## 5. A Kawamata–Viehweg-type vanishing theorem in positive characteristic

In this section we give an extension of the results in [Terakawa 1999]. There, Terakawa used the results in [Shepherd-Barron 1991a] to deduce a Kawamata–Viehweg-type theorem for nonpathological surfaces. Using our methods we are able to discuss pathological surfaces and obtain an effective Kawamata–Viehweg-type theorem in positive characteristic.

Let us first recall the classical Kawamata–Viehweg vanishing theorem in its general version (see [Kollár and Mori 1998] for the general notation).

**Theorem 5.1.** *Let $(X, B)$ be a klt pair over an algebraically closed field of characteristic zero and let $D$ be a Cartier divisor on $X$ such that $D - (K_X + B)$ is big and nef. Then*

$$H^i(X, \mathcal{O}_X(D)) = 0$$

*for any $i > 0$.*

In positive characteristic, even for nonpathological smooth surfaces, there are counterexamples to Theorem 5.1: Xie [2010] provided examples of relatively minimal irregular ruled surfaces in every characteristic where Theorem 5.1 fails.

Nonetheless, assuming $B = 0$, we have the following result (see [Mukai 2013]).

**Theorem 5.2.** *Let $X$ be a smooth surface in positive characteristic. Assume that there exists a big and nef Cartier divisor $D$ on $X$ such that*

$$H^1(X, \mathcal{O}_X(K_X + D)) \neq 0.$$

*Then*:

- *$X$ is either quasielliptic of Kodaira dimension one or of general type.*
- *Up to a sequence of blowups, $X$ has the structure of a fibered surface over a smooth curve such that every fiber is connected and singular.*

Furthermore, Terakawa [1999] deduced the following vanishing result using the techniques in [Shepherd-Barron 1991a].

**Theorem 5.3.** *Let $X$ be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let $D$ be a big and nef Cartier divisor on $X$. Assume that either*:

(1) *$\kappa(X) \neq 2$ and $X$ is not quasielliptic with $\kappa(X) = 1$.*

(2) *$X$ is of general type with*

- $p \geq 3$ *and* $(D^2) > \operatorname{vol}(X)$; *or*
- $p = 2$ *and* $(D^2) > \max\{\operatorname{vol}(X), \operatorname{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$.

*Then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

*for all* $i > 0$.

Our aim is to improve this theorem for arbitrary surfaces, via bend-and-break techniques.

More generally, we want to deduce some results on the injectivity of cohomological maps

$$H^1(X, \mathcal{O}_X(-D)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-pD))$$

where $D$ is a big divisor on $X$.

The following result by Kollár is an application of bend-and-break lemmas (cf. Theorem 2.13), specialized to our two-dimensional setting.

**Theorem 5.4.** *Let* $X$ *be a smooth projective variety over a field of positive characteristic and let* $D$ *be a Cartier divisor on* $X$ *such that*:

(1) $H^1(X, \mathcal{O}_X(-mD)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-pmD))$ *is not injective for some integer* $m > 0$.

(2) *There exists a curve* $C$ *on* $X$ *such that*

$$(p-1)(D \cdot C) - (K_X \cdot C) > 0.$$

*Then through every point* $x$ *of* $C$ *there is a rational curve* $C_x$ *such that*

$$[C] \overset{e}{\approx} k_0[C_x] + \sum_{i \neq 0} k_i[C_i] \tag{2}$$

(*as algebraic cycles*), *with* $k_i \geq 0$ *for all* $i$ *and*

$$(p-1)(D \cdot C_x) - (K_X \cdot C_x) \leq \dim(X) + 1.$$

*Proof.* This is essentially a slight modification of [Kollár 1996, Theorem II.6.2]. For the reader's convenience, we sketch it ab initio. Assumption (1) allows us to construct a finite morphism

$$\pi : Y \to X,$$

where $Y$ is defined as a Cartier divisor in the projectivization of a nonsplit rank-two bundle over $X$ (see [Kollár 1996, Construction II.6.1.6], which is a slight modification of Construction 2.9). Furthermore, the following property holds:

$$K_Y = \pi^*(K_X + (k(1-p)D)),$$

where $k$ is the largest integer for which $H^1(X, -kD) \neq 0$.

Now take the curve given in (2) and consider $C' := \operatorname{red} \pi^{-1}(C)$. The hypothesis on the intersection numbers and the formula for the canonical divisor of $Y$ guarantee that $(K_Y \cdot C') < 0$. Let $y \in C'$ be a preimage of $x$ in $Y$. So we can apply Theorem 2.13 and deduce the existence of a rational curve $C'_y$ passing through $y$. Using the projection formula, we obtain a curve $C_x$ on $X$ for which

$$(p-1)(D \cdot C_x) - (K_X \cdot C_x) \leq \dim(X) + 1. \qquad \square$$

If we assume the dimension to be two and the divisor $D$ to be big and nef, the asymptotic condition

$$H^1(X, \mathcal{O}_X(-mD)) = 0$$

for $m$ sufficiently large is guaranteed by [Szpiro 1979].

This remark gives us the following corollary.

**Corollary 5.5.** *Let $X$ be a smooth projective surface over a field of positive characteristic and let $D$ be a big and nef Cartier divisor on $X$ such that $H^1(X, \mathcal{O}_X(-D)) \neq 0$. Assume there exists a curve $C$ on $X$ such that*

$$(p-1)(D \cdot C) - (K_X \cdot C) > 0.$$

*Then through every point $x$ of $C$ there is a rational curve $C_x$ such that*

$$(p-1)(D \cdot C_x) - (K_X \cdot C_x) \leq 3.$$

We will show later how Corollary 5.5 can be used to deduce an effective version of Kawamata–Viehweg-type vanishing for arbitrary smooth surfaces.

In what follows, we will also need the following lemma on fibered surfaces, which explicitly gives a bound on the genus of the fiber with respect to the volume of the surface.

**Lemma 5.6.** *Let $f : X \to C$ be a fibered surface of general type and let $g$ be the arithmetic genus of the general fiber $F$. Then*

$$\operatorname{vol}(X) \geq g - 4.$$

*Proof.* We divide our analysis into cases according to the genus $b$ of the base, after having assumed the fibration is relatively minimal (i.e., that $K_{X/C}$ is nef).

$b \geq 2$: In this case we can deduce a better estimate. Indeed,

$$\operatorname{vol}(X) \geq (K_X^2) = (K_{X/C}^2) + 8(g-1)(b-1) \geq 8g - 8.$$

$b = 1$: In this case we need a more careful analysis, since in positive characteristic we cannot assume the semipositivity of $f_* K_{X/C}$. Nonetheless the following general formula holds:

$$\deg(f_* K_{X/C}) = \chi(\mathcal{O}_X) - (g-1)(b-1), \tag{3}$$

which specializes to

$$\deg(f_* K_{X/C}) = \chi(\mathcal{O}_X) \geq 0.$$

Formula (3) can be obtained via Riemann–Roch, since we know that $R^1 f_* K_{X/C} = \mathcal{O}_C$ and that $R^1 f_* n K_{X/C} = 0$ for $n \geq 2$ by relative minimality. The last inequality can be assumed by [Shepherd-Barron 1991b, Theorem 8]. Furthermore, one can apply the following formula

$$\deg(f_*(nK_{X/C})) = \deg(f_* K_{X/C}) + \frac{n(n-1)}{2}(K^2_{X/C}).$$

Since $K_{X/C}$ is big, we deduce that

$$\deg(f_*(2K_{X/C})) \geq 1.$$

As a consequence, we can apply the results of [Atiyah 1957] and deduce a decomposition of $f_*(2K_{X/C})$ into indecomposable vector bundles

$$f_*(2K_{X/C}) = \bigoplus_i E_i,$$

where we can assume that $\deg(E_1) \geq 1$. This implies that all quotient bundles of $E_1$ have positive degree. We want to show now that there exists a degree-one divisor $L_1$ on $C$ such that $h^0(C, f_*(2K_{X/C}) \otimes \mathcal{O}_C(-L_1)) \neq 0$.

But this is clear, since, for any degree-one divisor $L$ on $C$, one has that all quotient bundles of $f_*(2K_{X/C}) \otimes \mathcal{O}_C(-L)$ have degree zero and, up to a twisting by a degree-zero divisor on $C$, one can assume there exists a quotient

$$f_*(2K_{X/C} \otimes \mathcal{O}_C(-L_1)) \to \mathcal{O}_C \to 0.$$

This implies that $h^0(X, \mathcal{O}_X(2K_{X/C} - F))(= h^0(C, f_*(2K_{X/C}) \otimes \mathcal{O}_C(-L_1))) \neq 0$, where $F$ is the general fiber of $f$ and, since $K_X = K_{X/C}$ is nef, that

$$(K_X \cdot (2K_X - F)) \geq 0.$$

This gives the bound

$$\operatorname{vol}(X) \geq (K^2_X) \geq g-1.$$

$b = 0$: Also in this case we can assume that $\chi(\mathcal{O}_X) \geq 0$ and, as a consequence, that

$$\deg(f_* K_{X/\mathbb{P}^1}) = \chi(\mathcal{O}_X) + g - 1 \geq g-1.$$

If $g \geq 6$,

$$\deg(f_* K_{X/\mathbb{P}^1}) \geq 5.$$

This implies that $\deg(f_* K_X \otimes \mathcal{O}_{\mathbb{P}^1}(3)) \geq 0$ and, as a consequence of Grothendieck's theorem on vector bundles on $\mathbb{P}^1$,

$$h^0(X, \mathcal{O}_X(K_X - f^*\mathcal{O}_C(-3))) \neq 0.$$

As before, we have assumed that $K_{X/\mathbb{P}^1} = K_X + f^*\mathcal{O}_C(2)$ is nef, so

$$((K_X + f^*\mathcal{O}_C(2)) \cdot (K_X - f^*\mathcal{O}_C(-3))) \geq 0.$$

So in this case

$$\operatorname{vol}(X) \geq (K_X^2) \geq 2g - 2.$$

If $g \leq 5$, we simply use the trivial inequality $\operatorname{vol}(X) \geq 1$ to deduce

$$\operatorname{vol}(X) \geq g - 4. \qquad \square$$

Our result in this setting is an effective bound, depending only on the birational geometry of $X$, that guarantees the injectivity of the induced Frobenius map on the $H^1$s.

**Theorem 5.7.** *Let $X$ be a smooth surface in characteristic $p > 0$ and let $D$ be a big Cartier divisor $D$ on $X$. Then, for all integers*

$$m > m_0 = \frac{2\operatorname{vol}(X) + 9}{p - 1},$$

*the induced Frobenius map*

$$H^1(X, \mathcal{O}_X(-mD)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-pmD))$$

*is injective.* (*If* $\kappa(X) \neq 2$, *the volume* $\operatorname{vol}(X) = 0$).

**Remark 5.8.** The previous result is trivial if $H^1(X, \mathcal{O}_X(-D)) = 0$. Furthermore, combined with Corollary 5.5 it gives an effective version of the Kawamata–Viehweg theorem (cf. Corollary 5.9) in the case of big and nef divisors. Our hope is to generalize this strategy in order to deduce effective vanishing theorems also in higher dimension.

*Proof.* Assume, for the sake of contradiction, that

$$H^1(X, \mathcal{O}_X(-\lceil m_0 \rceil D)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-p\lceil m_0 \rceil D))$$

has a nontrivial kernel. Then, after a sequence of blowups $f : X' \to X$, we can assume the existence of a (relatively minimal) fibration (possibly with singular general fiber) of arithmetic genus $g$

$$\pi : X' \to C.$$

We remark that we can reduce to proving our result on $X'$, since $D' := f^*D$ is a big divisor and we have the following commutative diagram

$$\begin{tikzcd}
H^1(X', \mathcal{O}_{X'}(-\lceil m_0 \rceil D')) \arrow[r, "F^*"] \arrow[d, "\simeq"'] & H^1(X', \mathcal{O}_{X'}(-p\lceil m_0 \rceil D') \arrow[d, "\simeq"] , \\
H^1(X, \mathcal{O}_X(-\lceil m_0 \rceil D)) \arrow[r, "F^*"] & H^1(X, \mathcal{O}_X(-p\lceil m_0 \rceil D))
\end{tikzcd}$$

where the vertical isomorphisms hold because of $R^1 f_* \mathcal{O}_{X'} = 0$. We can now apply Theorem 5.4 to $\lceil m_0 \rceil D'$: we can choose $C$ to be a general fiber $F$ of $\pi$, which certainly intersects $D'$ positively, and we can use Lemma 5.6 to obtain

$$(p-1)\lceil m_0 \rceil (D' \cdot F) - (K_{X'} \cdot F) \geq (p-1)\lceil m_0 \rceil - (2g-2) > 3. \tag{4}$$

So we can apply Theorem 5.4: fix a point $x \in F$ and find a rational curve $C_x$ such that

$$(p-1)m_0(D \cdot C_x) - (K_X \cdot C_x) \leq 3.$$

Notice that, by construction, $F = C_x$, because of (2) in Theorem 5.4. But this is a contradiction, because of (4). ☐

We finally obtain our effective vanishing theorem.

**Corollary 5.9.** *Let $X$ be a smooth surface in characteristic $p > 0$ and let $D$ be a big and nef Cartier divisor $D$ on $X$. Then*

$$H^1(X, \mathcal{O}_X(K_X + mD)) = 0$$

*for all integers $m > m_0$, where*:

- $m_0 = 3/(p-1)$ *if $X$ is quasielliptic with $\kappa(X) = 1$.*
- $m_0 = (2\,\mathrm{vol}(X) + 9)/(p-1)$ *if $X$ is of general type.*

*Proof.* For surfaces of general type, one simply applies the previous result. For quasielliptic surfaces, a better bound can be obtained, since in this case $(K_X \cdot F) = 0$, where $F$ is the general fiber. ☐

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# Adams operations and Galois structure

Georgios Pappas

We present a new method for determining the Galois module structure of the cohomology of coherent sheaves on varieties over the integers with a tame action of a finite group. This uses a novel Adams–Riemann–Roch-type theorem obtained by combining the Künneth formula with localization in equivariant K-theory and classical results about cyclotomic fields. As an application, we show two conjectures of Chinburg, Pappas, and Taylor in the case of curves.

## 1. Introduction

We consider $G$-covers $\pi : X \to Y$ of a projective flat scheme $Y \to \mathrm{Spec}(\mathbb{Z})$ where $G$ is a finite group. Let $\mathscr{F}$ be a $G$-equivariant coherent sheaf of $\mathbb{O}_X$-modules on $X$, i.e., a coherent $\mathbb{O}_X$-module equipped with a $G$-action compatible with the $G$-action on the scheme $X$. Our main objects of study are the Zariski cohomology groups $\mathrm{H}^i(X, \mathscr{F})$. These are finitely generated modules for the integral group ring $\mathbb{Z}[G]$.

Let us consider the total cohomology $\mathrm{R}\Gamma(X, \mathscr{F})$ in the derived category of complexes of $\mathbb{Z}[G]$-modules that are bounded below. If $\pi$ is tamely ramified, then the complex $\mathrm{R}\Gamma(X, \mathscr{F})$ is "perfect", i.e., isomorphic to a bounded complex $(P^\bullet)$ of finitely generated *projective* $\mathbb{Z}[G]$-modules [Chinburg 1994; Chinburg and Erez 1992; Pappas 2008]. This observation goes back to E. Noether when $X$ is the

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spectrum of the ring of integers of a number field; in this general setup, it is due to T. Chinburg.

**Definition 1.0.1.** We say that the cohomology of $\mathscr{F}$ has a normal integral basis if there exists a bounded complex $(F^\bullet)$ of finitely generated *free* $\mathbb{Z}[G]$-modules that is isomorphic to $\mathrm{R}\Gamma(X, \mathscr{F})$.

To measure the obstruction to the existence of a normal integral basis, we use the Grothendieck group $\mathrm{K}_0(\mathbb{Z}[G])$ of finitely generated projective $\mathbb{Z}[G]$-modules. We consider the projective class group $\mathrm{Cl}(\mathbb{Z}[G])$, which is defined as the quotient of $\mathrm{K}_0(\mathbb{Z}[G])$ by the subgroup generated by the class of the free module $\mathbb{Z}[G]$. The obstruction to the existence of a normal integral basis for the cohomology of $\mathscr{F}$ is given by the (stable) projective Euler characteristic

$$\bar{\chi}(X, \mathscr{F}) = \textstyle\sum_i (-1)^i [P^i] \in \mathrm{Cl}(\mathbb{Z}[G]),$$

which is independent of the choice of $(P^\bullet)$.

The main problem in the theory of geometric Galois structure is to understand such Euler characteristics. Often there are interesting connections with other invariants of $X$. For example, it was shown in [Chinburg et al. 1997a] that, under some additional hypotheses, the projective Euler characteristic of a version of the de Rham complex of $X$ can be calculated using $\epsilon$-factors of Hasse–Weil–Artin L-functions for the cover $X \to X/G$. Also, when $X$ is a curve over $\mathbb{Z}$, the obstruction $\bar{\chi}(X, \mathbb{O}_X)$ is related, via a suitable equivariant version of the Birch and Swinnerton-Dyer conjecture, to the $G$-module structure of the Mordell–Weil and Tate–Shafarevich groups of the Jacobian of the generic fiber $X_{\mathbb{Q}}$ [Chinburg et al. 2009].

We will first discuss the case when $\pi : X \to Y$ is unramified. Let us denote by $d$ the relative dimension of $Y \to \mathrm{Spec}(\mathbb{Z})$.

When $d = 0$, the problem of the existence of a normal integral basis reduces to a classical question. Suppose that $N/K$ is an unramified Galois extension of number fields with Galois group $G$, and consider the ring of integers $\mathbb{O}_N$ that is then a projective $\mathbb{Z}[G]$-module. Take $X = \mathrm{Spec}(\mathbb{O}_N)$, $Y = \mathrm{Spec}(\mathbb{O}_K)$, and $\mathscr{F} = \mathbb{O}_X$; then the Euler characteristic $\bar{\chi}(X, \mathbb{O}_X)$ is the class $[\mathbb{O}_N]$ in $\mathrm{Cl}(\mathbb{Z}[G])$. We are then asking if $\mathbb{O}_N$ is "stably free", i.e., if there are integers $n$ and $m$ such that $\mathbb{O}_N \oplus \mathbb{Z}[G]^n \simeq \mathbb{Z}[G]^m$. Results of A. Fröhlich and M. Taylor imply that $[\mathbb{O}_N]$ is always 2-torsion in $\mathrm{Cl}(\mathbb{Z}[G])$ and is trivial when the group $G$ has no irreducible symplectic representations; this result also holds when, more generally, $N/K$ is tamely ramified. Indeed, if $N/K$ is at most tamely ramified, Fröhlich's conjecture (shown by Taylor [1981]) explains how to determine the class $[\mathbb{O}_N]$ from the root numbers of Artin L-functions for irreducible symplectic representations of $\mathrm{Gal}(N/K)$. In particular, $\gcd(2, \#G) \cdot [\mathbb{O}_N] = 0$. When $G$ is of odd order, $[\mathbb{O}_N] = 0$ and $\mathbb{O}_N$ is stably free; when $G$ is of odd order, it then follows that $\mathbb{O}_N$ is actually a free $\mathbb{Z}[G]$-module. In general, the class of a projective module in the class group $\mathrm{Cl}(\mathbb{Z}[G])$ contains a lot of information about the isomorphism class of the module. Hence, the projective Euler characteristics that we consider in this paper also contain a lot of information about the Galois modules given by cohomology. This is not necessarily the case for the "naive" Euler characteristics that can be easily defined as classes in the weaker Grothendieck group of all finitely generated $G$-modules.

When $d > 0$, some progress towards calculating $\overline{\chi}(X, \mathscr{F})$ for general $\mathscr{F}$ was achieved after the introduction of the technique of cubic structures (see [Pappas 1998; 2008; Chinburg et al. 2009]; [Chinburg et al. 1997a] only dealt with the de Rham complex). This technique is very effective when *all* the Sylow subgroups of $G$ are abelian. In particular, it allowed us to show that, under this hypothesis, $\gcd(2, \#G) \cdot \overline{\chi}(X, \mathscr{F}) = 0$ if $d = 1$ [Pappas 1998]. Some general results were obtained in [Pappas 2008] when $d > 1$, but the problem appears to be quite hard. In fact, as is explained in [loc. cit.], it is plausible that the statement that $\overline{\chi}(X, \mathscr{F}) = 0$ for $G$ abelian and all $X$ of dimension $< \#G$ is equivalent to the truth of Vandiver's conjecture for all prime divisors of the order $\#G$. In [Pappas 2008; Chinburg et al. 2009], it was conjectured that, for all $G$, there are integers $N$ (that depends only on $d$) and $\delta$ (that depends only on $\#G$) such that $\gcd(N, \#G)^{\delta} \cdot \overline{\chi}(X, \mathscr{F}) = 0$; this was shown for $G$ with abelian Sylows.

In this paper, we introduce a new method that allows us to handle nonabelian groups. We obtain strong results and, in the case $d = 1$ of curves over $\mathbb{Z}$, a proof of the above conjecture. This is based on Adams–Riemann–Roch-type identities that are proven using the Künneth formula and "localization" or "concentration" theorems (as given for example by Thomason) in equivariant K-theory. We combine these with a study of the action of the Adams–Cassou-Noguès–Taylor operations on $\mathrm{Cl}(\mathbb{Z}[G])$ and some classical algebraic number theory.

Write $\#G = 2^s 3^t m$, with $m$ relatively prime to 6, and define $a$ and $b$ as follows. If the 2-Sylow subgroups of $G$ have order $\leq 4$, are cyclic of order 8, or are dihedral groups, set $a = 0$. If the 2-Sylow subgroups of $G$ are abelian (but not as in the case above) or generalized quaternion or semidihedral groups, set $a = 1$. In all other cases, set $a = s + 1$. If the 3-Sylows subgroups of $G$ are abelian, set $b = 0$. In all other cases, set $b = t - 1$.

**Theorem 1.0.2.** *Let $X \to Y$ be an unramified $G$-cover with $Y \to \mathrm{Spec}(\mathbb{Z})$ projective and flat of relative dimension* 1. *Let $\mathscr{F}$ be a $G$-equivariant coherent $\mathbb{O}_X$-module. For $a$ and $b$ as above, we have $2^a 3^b \cdot \overline{\chi}(X, \mathscr{F}) = 0$. In particular, if $G$ has order prime to* 6, *then $\overline{\chi}(X, \mathscr{F}) = 0$.*

In particular, this implies, in this case of free $G$-action, the "localization conjecture" of [Chinburg et al. 2009] for $d = 1$ with $N = 6$ and $\delta = \max\{s, t\} + 1$ and significantly extends the results of [Pappas 1998]. For example, the hypothesis "all Sylows are abelian" in [Pappas 1998, Theorem 5.2(a), Corollary 5.3, Theorem 5.5(a)] can be relaxed. Hence, we obtain a "projective normal integral basis theorem" when $d = 1$ and the action is free: if $X$ is normal and $G$ is such that $a = b = 0$, then $X \cong \mathbf{Proj}\big(\bigoplus_{n\geq 0} A_n\big)$ with $A_n$, for all $n > 0$, free $\mathbb{Z}[G]$-modules. We can also give a result for the $G$-module of regular differentials of $X$: suppose that $X$ is normal; then $X$ is Cohen–Macaulay and we can consider the relative dualizing sheaf $\omega_X$ of $X \to \mathrm{Spec}(\mathbb{Z})$. Suppose that $\{g_1, \dots, g_r\}$ is a set of generators of $G$. Then $\{g_1 - 1, \dots, g_r - 1\}$ is a set of generators of the augmentation ideal $\mathfrak{g} \subset \mathbb{Z}[G]$; this gives a surjective $\phi : \mathbb{Z}[G]^{r_G} \to \mathfrak{g}$, where $r_G$ is the minimal number of generators of $G$. Denote by $\Omega^2(\mathbb{Z})$ the kernel of $\phi$. By Schanuel's lemma, the stable isomorphism class of $\Omega^2(\mathbb{Z})$ is independent of the choice of generators. As in [Pappas 1998, Theorem 5.5(a)], we will see that Theorem 1.0.2 implies:

**Theorem 1.0.3.** *Let $X \to Y$ be an unramified $G$-cover with $Y \to \mathrm{Spec}(\mathbb{Z})$ projective and flat of relative dimension 1. Assume that $X$ is normal, $X_{\mathbb{Q}}$ is smooth, and $G$ acts trivially on $\mathrm{H}^0(X_{\mathbb{Q}}, \mathcal{O}_{X_{\mathbb{Q}}})$. Assume also that $\mathrm{H}^1(X, \omega_X)$ is torsion-free. Set $h = \dim_{\mathbb{Q}} \mathrm{H}^0(X_{\mathbb{Q}}, \mathcal{O}_{X_{\mathbb{Q}}})$ and $g_Y = \mathrm{genus}(Y_{\mathbb{Q}}) = \dim_{\mathbb{Q}} \mathrm{H}^0(Y_{\mathbb{Q}}, \Omega^1_{Y_{\mathbb{Q}}})$. If $G$ is such that $a = b = 0$, then the $\mathbb{Z}[G]$-module $\mathrm{H}^0(X, \omega_X)$ is stably isomorphic to $\Omega^2(\mathbb{Z})^{\oplus h}$. If in addition $G$ has odd order and $g_Y > h \cdot r_G$, then we have*

$$\mathrm{H}^0(X, \omega_X) \cong \Omega^2(\mathbb{Z})^{\oplus h} \oplus \mathbb{Z}[G]^{\oplus(g_Y - h \cdot r_G)}.$$

Now, let us give some more details. Our technique is K-theoretic and was inspired by work of M. Nori [2000] and especially of B. Köck [2000]. The beginning point is that the Adams–Riemann–Roch theorem for a smooth variety $X$ can be obtained by combining a fixed-point theorem for the permutation action of the cyclic group $C_\ell = \mathbb{Z}/\ell\mathbb{Z}$ on the product $X^\ell = X \times \cdots \times X$ with the Künneth formula (see [Nori 2000; Köck 2000]; here $\ell$ is a prime number). Very roughly, this goes as follows. If $\mathscr{E}$ is a vector bundle over $X$, the Künneth formula gives an isomorphism $\mathrm{R}\Gamma(X, \mathscr{E})^{\otimes \ell} \cong \mathrm{R}\Gamma(X^\ell, \mathscr{E}^{\boxtimes \ell})$. Using localization and the Lefschetz–Riemann–Roch theorem, we can relate the cohomology of the exterior tensor product sheaf $\mathscr{E}^{\boxtimes \ell}$ on $X^\ell$ to the cohomology of the restriction $\mathscr{E}^{\otimes \ell} = \mathscr{E}^{\boxtimes \ell}|_X$ on the $C_\ell$-fixed-point locus, which here is the diagonal $X = \Delta(X) \subset X^\ell$. Of course, there is a correction that involves the conormal bundle $\mathscr{N}_{X|X^\ell}$ of $X$ in $X^\ell$. This correction amounts to multiplying $\mathscr{E}^{\otimes \ell}$ by $\lambda_{-1}(\mathscr{N}_{X|X^\ell})^{-1}$. The multiplier $\lambda_{-1}(\mathscr{N}_{X|X^\ell})^{-1}$ is familiar in Lefschetz–Riemann–Roch-type theorems and in this case is given by the inverse $\theta^\ell(\Omega^1_X)^{-1}$ of the $\ell$-th Bott class of the differentials; the resulting identity eventually gives the Adams–Riemann–Roch theorem for the Adams operation $\psi^\ell$.

In our situation, we have to take great care to explain how enough of this can be done $G$-equivariantly and also for projective $G$-modules without losing much information. It is also important to connect the $\ell$-th tensor powers used above to the versions of the $\ell$-th Adams operators on the projective class group $\mathrm{Cl}(\mathbb{Z}[G])$ as defined by Cassou-Noguès and Taylor. This is all quite subtle and eventually needs to be applied for a carefully selected set of primes $\ell$. Most of our arguments are valid in any dimension $d$, and we show various general Adams–Riemann–Roch-type results. Combining these with a Chebotarev density argument, we obtain that, for a $p$-group $G$ with $p$ an odd prime, the obstruction $\bar{\chi}(X, \mathscr{F})$ lies in a specific sum of eigenspaces for the action of the Adams–Cassou-Noguès–Taylor operators on the $p$-power part of the class group $\mathrm{Cl}(\mathbb{Z}[G])$ (Theorem 6.3.2). When $d = 1$, there is only one eigenspace in this sum. We can then see, using classical results of the theory of cyclotomic fields, that this eigenspace is trivial when $p \geq 5$. When $p = 3$, results of R. Oliver [1983] imply that the eigenspace is annihilated by the Artin exponent of $G$. The crucial number-theoretic ingredients are classical results of Iwasawa on cyclotomic class groups and units and the following fact: for any $p > 2$, both the second and the $(p-2)$-th Teichmüller eigenspaces of the $p$-part of the class group of $\mathbb{Q}(\zeta_p)$ are trivial. An argument that uses again localization implies that we can reduce the calculation to $p$-groups. Therefore, by the above, for general $G$, the class $\bar{\chi}(X, \mathscr{F})$ in $\mathrm{Cl}(\mathbb{Z}[G])$ is determined by the classes $c_2 = \bar{\chi}(X, \mathscr{F})$ in $\mathrm{Cl}(\mathbb{Z}[G_2])$ for the cover $X \to X/G_2$ and $c_3 = \bar{\chi}(X, \mathscr{F})$ in $\mathrm{Cl}(\mathbb{Z}[G_3])$ for $X \to X/G_3$. Here $G_2$ and $G_3$ are 2-Sylow and 3-Sylow subgroups of $G$, respectively. This then eventually leads to Theorem 1.0.2.

It is also important to treat (tamely) ramified covers $X \to Y$ over $\mathbb{Z}$. However, usually the unramified-over-$\mathbb{Z}$ case is the hardest one to deal with initially. After this is done, ramified covers can be examined by localizing at the branch locus (see for example [Chinburg et al. 2009]). This also occurs here. In fact, our method applies when $\pi : X \to Y$ is tamely ramified. In our situation, the assumption that the ramification is tame implies that the corresponding $G$-cover $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ given by the generic fibers is unramified. Under certain regularity assumptions, we obtain an Adams–Riemann–Roch formula for $\bar{\chi}(X, \mathscr{F})$ (Theorem 5.5.3) that generalizes formulas of Burns and Chinburg [1996] and Köck [1999] to all dimensions. In this case, one cannot expect that $\bar{\chi}(X, \mathscr{F})$ is annihilated by small integers. Nevertheless, when $d = 1$, we can obtain (almost full) information about the class $\bar{\chi}(X, \mathscr{F})$ from the ramification locus of the cover $X \to Y$. In particular, under some further conditions on $X$ and $Y$, we show that the class $\gcd(2, \#G)^{v_2(\#G)+2} \gcd(3, \#G)^{v_3(\#G)-1} \cdot \bar{\chi}(X, \mathscr{F})$ only depends on the restrictions of $\mathscr{F}$ on the local curves $X_{\mathbb{Z}_p}$, where $p$ runs over the primes that contain the support of the ramification locus of $X \to Y$. This proves a slightly weakened variant of the "input localization conjecture" of [Chinburg et al. 2009] for $d = 1$ (see Theorem 7.2.1). Dealing with wildly ramified covers presents a host of additional difficulties that are still not resolved, not even in the case $d = 0$ of number fields.

## 2. Grothendieck groups and Euler characteristics

### 2.1. *G-modules, sheaves, and Grothendieck groups.*

**2.1.1.** In this paper, modules are left modules while groups act on schemes on the right. Sometimes we will use the term "$G$-module" instead of "$\mathbb{Z}[G]$-module". If we fix a prime number $\ell$, we set $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$, and in general, we denote inverting $\ell$ or localizing at the multiplicative set of powers of $\ell$ by a prime $'$. We reserve the notation $\mathbb{Z}_\ell$ for the $\ell$-adic integers. Set $C_\ell = \mathbb{Z}/\ell\mathbb{Z}$, and denote by $S_\ell$ the symmetric group in $\ell$ letters. We regard $C_\ell$ as the subgroup of $S_\ell$ generated by the cycle $(12 \cdots \ell)$.

**2.1.2.** If $R = \mathbb{Z}$ or $\mathbb{Z}'$, then the Grothendieck group $\mathrm{G}_0(R[G])$ of finitely generated $R[G]$-modules can be identified with the Grothendieck ring of finitely generated $R[G]$-modules that are free as $R$-modules ("$R[G]$-lattices") with multiplication given by the tensor product. The ring $\mathrm{G}_0(R[G])$ is a finite and flat $\mathbb{Z}$-algebra. The natural homomorphism $\mathrm{G}_0(R[G]) \to \mathscr{R}_{\mathbb{Q}}(G) := \mathrm{G}_0(\mathbb{Q}[G])$, $[M] \mapsto [M \otimes_R \mathbb{Q}]$, is surjective with nilpotent kernel [Swan 1963]. Therefore, the prime ideals of $\mathrm{G}_0(R[G])$ can be identified with the prime ideals of the representation ring $\mathscr{R}_{\mathbb{Q}}(G)$; these can be described following Segal and Serre (see for example [Chinburg et al. 1997b, §4]). Suppose that $\mathfrak{p}$ is a prime ideal of $R$. An element $g$ in $G$ is called $\mathfrak{p}$-regular if it is of order prime to the characteristic of $\mathfrak{p}$; the set of prime ideals of $\mathscr{R}_{\mathbb{Q}}(G)$ that lie above $\mathfrak{p}$ are in 1-1 correspondence with the set of "$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugacy classes" of $\mathfrak{p}$-regular elements. Here $g$ and $g'$ are $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate, when there is $t \in \mathbb{Z}$ prime to the exponent $\exp(G)$ of $G$, such that $g'$ is conjugate to $g^t$. The prime ideal $\rho = \rho_{(g,\mathfrak{p})}$, where $g$ is $\mathfrak{p}$-regular, is

$$\rho_{(g,\mathfrak{p})} = \{\phi \in \mathscr{R}_{\mathbb{Q}}(G) \mid \mathrm{Tr}(g \mid \phi) \in \mathfrak{p}\}.$$

**2.1.3.** We say that a $G$-module $M$ is *$G$-cohomologically trivial* ($G$-c.t.), or simply *cohomologically trivial* (c.t.) when $G$ is clear from the context, if for all subgroups $H \subset G$ the cohomology groups $\mathrm{H}^i(H, M)$ are trivial when $i > 0$. If the $\mathbb{Z}[G]$-module $M$ has finite projective dimension, it is $G$-c.t. In fact, by [Atiyah and Wall 1967, Theorem 9], the converse is true and $M$ is $G$-c.t. if and only if it has a resolution

$$0 \to P \to Q \to M \to 0$$

where $P$ and $Q$ are projective $\mathbb{Z}[G]$-modules. If $M$ is in addition finitely generated, we can take $P$ and $Q$ to also be finitely generated. Hence, we can identify

$$\mathrm{G}_0^{\mathrm{ct}}(G, \mathbb{Z}) = \mathrm{K}_0(\mathbb{Z}[G])$$

where $\mathrm{G}_0^{\mathrm{ct}}(G, \mathbb{Z})$ is the Grothendieck group of finitely generated $G$-c.t. modules. The following facts can be found in [Swan 1960; Taylor 1984]. $\mathrm{K}_0(\mathbb{Z}[G])$ is a finitely generated $\mathrm{G}_0(\mathbb{Z}[G])$-module by action given by the tensor product by $\mathbb{Z}[G]$-lattices. The subgroup $\langle\mathbb{Z}[G]\rangle$ of free classes is a $\mathrm{G}_0(\mathbb{Z}[G])$-submodule, and so the quotient $\mathrm{Cl}(\mathbb{Z}[G])$ is also a $\mathrm{G}_0(\mathbb{Z}[G])$-module. The $\mathrm{G}_0(\mathbb{Z}[G])$-module structure on $\mathrm{Cl}(\mathbb{Z}[G])$ factors through the quotient $\mathrm{G}_0(\mathbb{Z}[G]) \to \mathscr{R}_{\mathbb{Q}}(G)$.

**2.1.4.** For $S$ a scheme (with trivial $G$-action), we will consider quasicoherent sheaves of $\mathbb{O}_S[G]$-modules on $S$ and simply call these quasicoherent $\mathbb{O}_S[G]$-modules. We say that a quasicoherent $\mathbb{O}_S[G]$-module $\mathscr{F}$ is $G$-cohomologically trivial ($G$-c.t.) if, for all $s \in S$, the stalk $\mathscr{F}_s$ is a $G$-c.t. module. As in [Chinburg 1994, p. 448], we can easily see that a quasicoherent $\mathbb{O}_S[G]$-module $\mathscr{F}$ is $G$-c.t. if and only if, for every open affine subscheme $U$ of $S$, $\mathscr{F}(U)$ is a $G$-c.t. module.

## 2.2. *Grothendieck groups and Euler characteristics.*

**2.2.1.** We refer to [Thomason 1987] for general facts about the equivariant K-theory of schemes with group action and for the notations $\mathrm{G}_i(G, -)$ and $\mathrm{K}_i(G, -)$ of G-groups and K-groups for coherent and coherent locally free, $G$-equivariant sheaves, respectively. If $S$ is as above, we denote by $\mathrm{G}_0^{\mathrm{ct}}(G, S)$ the Grothendieck group of $G$-c.t. coherent $\mathbb{O}_S[G]$-modules.

If the structure morphism $g : S \to \mathrm{Spec}(\mathbb{Z})$ is projective, there is (see [Chinburg 1994]; see also below) a group homomorphism (the "projective equivariant Euler characteristic")

$$g_*^{\mathrm{ct}} : \mathrm{G}_0^{\mathrm{ct}}(G, S) \to \mathrm{G}_0^{\mathrm{ct}}(G, \mathrm{Spec}(\mathbb{Z})) = \mathrm{G}_0^{\mathrm{ct}}(G, \mathbb{Z}) = \mathrm{K}_0(\mathbb{Z}[G])$$

where in the target we identify the class of a sheaf with the class of the module of its global sections. We will sometimes abuse notation and still write $g_*^{\mathrm{ct}}$ for the composition of $g_*^{\mathrm{ct}}$ with $\mathrm{K}_0(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}[G]) = \mathrm{K}_0(\mathbb{Z}[G])/\langle\mathbb{Z}[G]\rangle$.

Here is a review of the construction of $g_*^{\mathrm{ct}}$. Suppose that $\mathscr{E}$ is a coherent $\mathbb{O}_S[G]$-module with $G$-c.t. stalks. Following [Chinburg 1994], we first give a bounded complex $(M^\bullet, d^\bullet)$ of finitely generated projective $\mathbb{Z}[G]$-modules that is isomorphic to $\mathrm{R}\Gamma(S, \mathscr{E})$ in the derived category of complexes of $\mathbb{Z}[G]$-modules bounded below. Choose a finite open cover $\mathcal{U} = \{U_j\}_j$ of $S$ by affine subschemes $U_j = \mathrm{Spec}(R_j)$. Then all intersections $U_{j_1\cdots j_m} := U_{j_1} \cap \cdots \cap U_{j_m}$ are also affine. The sections $\mathscr{E}(U_{j_1\cdots j_m})$ are cohomologically trivial $G$-modules, and the usual Čech complex $C^\bullet(\mathcal{U}, \mathscr{E})$ is a bounded complex of cohomologically trivial $G$-modules (see the proof of Theorem 1.1 in [Chinburg 1994]). The complex $C^\bullet(\mathcal{U}, \mathscr{E})$ is isomorphic to $\mathrm{R}\Gamma(S, \mathscr{E})$, and since $S \to \mathrm{Spec}(\mathbb{Z})$ is projective, it has finitely generated homology groups. Now the usual procedure, described for example in [loc. cit.], produces a perfect complex $M^\bullet$ of $\mathbb{Z}[G]$-modules (i.e., a bounded complex of finitely generated projective $\mathbb{Z}[G]$-modules) together with a morphism of complexes $M^\bullet \to C^\bullet(\mathcal{U}, \mathscr{E})$ that is a quasi-isomorphism. Then $M^\bullet$ is also isomorphic to $\mathrm{R}\Gamma(S, \mathscr{E})$ in the derived category, and we define

$$g_*^{\mathrm{ct}}(\mathscr{E})=\textstyle\sum_i(-1)^i[M^i] \tag{2.2.2}$$

in $\mathrm{G}_0^{\mathrm{ct}}(G,\mathbb{Z})=\mathrm{K}_0(\mathbb{Z}[G])$; this is independent of our choices.

**2.2.3.** When in addition the symmetric group $S_\ell$ acts on $S$, we can also consider the Grothendieck groups $\mathrm{G}_0^{\mathrm{ct}}(S_\ell;G,S)$ of $S_\ell$-equivariant coherent $\mathbb{O}_S[G]$-modules that are $G$-cohomologically trivial. If $g:S\to\mathrm{Spec}(\mathbb{Z})$ is projective and in addition $S_\ell$ acts on $S$, a construction as above gives an Euler characteristic homomorphism

$$g_*^{\mathrm{ct}}:\mathrm{G}_0^{\mathrm{ct}}(S_\ell;G,S)\to\mathrm{G}_0^{\mathrm{ct}}(S_\ell;G,\mathrm{Spec}(\mathbb{Z}))$$

and similarly for $S_\ell$ replaced by the cyclic group $C_\ell$. For simplicity, we will sometimes omit the superscript ct from $g_*^{\mathrm{ct}}$ when it is clear from the context.

**2.2.4.** We now assume that the group $G$ acts on $S$ on the right. We say that $G$ acts tamely on $S$ if, for every point $s\in S$, the inertia subgroup $I_s\subset G$, which is, by definition, the largest subgroup of $G$ that fixes $s$ and acts trivially on the residue field $k(s)$, has order prime to $\mathrm{char}(k(s))$. Suppose that the quotient scheme $S/G$ exists and the map $\pi:S\to T=S/G$ is finite. Then, by [Raynaud 1970, Chapitre XI, Lemme 1], the $G$-cover $\pi$ is, étale locally around $\pi(s)$ on $T$, induced from an $I_s$-cover. Hence, if $\mathscr{F}$ is a $G$-equivariant $\mathbb{O}_S$-module, then $\pi_*\mathscr{F}$ is a $G$-c.t. coherent $\mathbb{O}_T[G]$-module [Chinburg 1994; Chinburg and Erez 1992]. We then obtain $\pi_*^{\mathrm{ct}}:\mathrm{G}_0(G,S)\to\mathrm{G}_0^{\mathrm{ct}}(G,T)$ given by $[\mathscr{F}]\mapsto[\pi_*\mathscr{F}]$. If $f:S\to\mathrm{Spec}(\mathbb{Z})$ is projective, then $S/G$ exists, $\pi$ is finite, and $g:T\to\mathrm{Spec}(\mathbb{Z})$ is projective [Mumford 1970, Chapter III, Theorem 1]. Then the composition $f_*^{\mathrm{ct}}=g_*^{\mathrm{ct}}\cdot\pi_*^{\mathrm{ct}}$ gives the projective equivariant Euler characteristic

$$f_*^{\mathrm{ct}}:\mathrm{G}_0(G,S)\to\mathrm{G}_0^{\mathrm{ct}}(G,\mathrm{Spec}(\mathbb{Z}))=\mathrm{K}_0(\mathbb{Z}[G]). \tag{2.2.5}$$

The Grothendieck groups $\mathrm{G}_0(G,S)$ and $\mathrm{K}_0(\mathbb{Z}[G])$ are $\mathrm{G}_0(\mathbb{Z}[G])$-modules, and the map $f_*^{\mathrm{ct}}$ is a $\mathrm{G}_0(\mathbb{Z}[G])$-module homomorphism and similarly for $\mathbb{Z}$ replaced by $\mathbb{Z}'=\mathbb{Z}[\ell^{-1}]$. Sometimes, we will denote $f_*^{\mathrm{ct}}(\mathscr{F})$ by $\chi(X,\mathscr{F})$; then $\overline{\chi}(X,\mathscr{F})$ is the image of $\chi(X,\mathscr{F})$ in $\mathrm{Cl}(\mathbb{Z}[G])=\mathrm{K}_0(\mathbb{Z}[G])/\langle\mathbb{Z}[G]\rangle$.

Similarly, if $S_\ell\times G$ acts on the projective $f:S\to\mathrm{Spec}(\mathbb{Z})$ with the subgroup $G=1\times G$ acting tamely, we have

$$f_*^{\mathrm{ct}}:\mathrm{G}_0(S_\ell\times G,S)\to\mathrm{G}_0^{\mathrm{ct}}(S_\ell;G,\mathrm{Spec}(\mathbb{Z})) \tag{2.2.6}$$

given as $f_*^{\mathrm{ct}}=g_*^{\mathrm{ct}}\cdot\pi_*^{\mathrm{ct}}$. Here $f_*^{\mathrm{ct}}$ is a $\mathrm{G}_0(\mathbb{Z}[S_\ell\times G])$-module homomorphism. These constructions also work with $\mathbb{Z}$ replaced by $\mathbb{Z}'$ and with $S_\ell$ replaced by $C_\ell$.

## 3. The Künneth formula

### 3.1. *Tensor powers.*

**3.1.1.** Let $R$ be a commutative Noetherian ring with 1. If $(M^\bullet, d^\bullet)$ is a bounded chain complex of $R[G]$-modules that are flat as $R$-modules, we can consider the total tensor product complex

$$(M^{\bullet\otimes\ell}, \partial^\bullet)$$

whose term of degree $n$ is

$$(M^{\bullet\otimes\ell})^n = \bigoplus_{\substack{(i_1,\dots,i_\ell)\in\mathbb{Z}^\ell \\ i_1+\cdots+i_\ell=n}} (M^{i_1} \otimes_R \cdots \otimes_R M^{i_\ell})$$

with diagonal $G$-action and differential $\partial^n$ is given by

$$\partial^n(m_{i_1} \otimes \cdots \otimes m_{i_\ell}) = \sum_{a=1}^{\ell} (-1)^{i_1+\cdots+i_{a-1}} m_{i_1} \otimes \cdots \otimes m_{i_{a-1}} \otimes d^{i_a}(m_{i_a}) \otimes \cdots \otimes m_{i_\ell}.$$

Since the modules $M^i$ are $R$-flat, the complex $(M^\bullet)^{\otimes\ell}$ is isomorphic to the $\ell$-fold derived tensor

$$M^\bullet \overset{\mathrm{L}}{\otimes}_R M^\bullet \overset{\mathrm{L}}{\otimes}_R \cdots \overset{\mathrm{L}}{\otimes}_R M^\bullet$$

in the derived category of complexes of $R[G]$-modules that are bounded above.

**Lemma 3.1.2.** (a) *Let $M_1$ and $M_2$ be projective $R[G]$-modules. Then $M_1 \otimes_R M_2$ with the diagonal $G$-action is also a projective $R[G]$-module.*

(b) *Let $M_1$ and $M_2$ be cohomologically trivial $G$-modules that are $\mathbb{Z}$-flat. Then $M_1 \otimes_{\mathbb{Z}} M_2$ with the diagonal $G$-action is also a cohomologically trivial $G$-module.*

*Proof.* (a) This follows easily from the fact that the tensor product $R[G] \otimes_R R[G]$ with diagonal $G$-action is $R[G]$-free.

(b) By [Atiyah and Wall 1967, Theorem 9], we have resolutions $0 \to Q_i \to P_i \to M_i \to 0$ with $P_i$ and $Q_i$ projective $\mathbb{Z}[G]$-modules. Using this and (a), we can see that $M_1 \otimes_{\mathbb{Z}} M_2$ has finite projective dimension; hence, it is cohomologically trivial. □

**3.1.3.** We define an action of the symmetric group $S_\ell$ on the complex $(M^{\bullet\otimes\ell}, \partial^\bullet)$ as follows [Atiyah 1966, p. 176]: $\sigma \in S_\ell$ acts on $(M^{\bullet\otimes\ell})^n = \bigoplus(M^{i_1} \otimes_R \cdots \otimes_R M^{i_\ell})$ by permuting the factors and with the *appropriate sign changes* so that a transposition of two terms $m_i \otimes m_j$ (where $m_i \in M^i$ and $m_j \in M^j$) comes with the sign $(-1)^{ij}$. (We see that the action of $\sigma$ commutes with the differentials $\partial^n$.)

Let $M^0$ and $M^1$ be finitely generated projective $R[G]$-modules, and let us consider the complex $M^\bullet := [M^0 \xrightarrow{0} M^1]$ (in degrees 0 and 1). Using Lemma 3.1.2(a), we form the Euler characteristic

$$\chi((M^\bullet)^{\otimes\ell}) = \textstyle\sum_n (-1)^n \cdot [(M^{\bullet\otimes\ell})^n] \in \mathrm{K}_0(S_\ell;\, G,\, R),$$

where $\mathrm{K}_0(S_\ell; G, R)$ is the Grothendieck group of finitely generated $R[S_\ell \times G]$-modules that are $R[G]$-projective. As in [Atiyah 1966, Proposition 2.2], we can see that $\chi((M^\bullet)^{\otimes\ell})$ only depends on the class $\chi(M^\bullet) = [M^0] - [M^1]$ in $\mathrm{K}_0(R[G])$ and gives a well-defined map, the "tensor power operation"

$$\tau^\ell : \mathrm{K}_0(R[G]) \to \mathrm{K}_0(S_\ell;\, G,\, R), \quad \tau^\ell([M^0] - [M^1]) = \chi((M^\bullet)^{\otimes\ell}).$$

(This statement also follows from [Grayson 1989]; see for example [Köck 2000, §1].) In general, if $M^\bullet$ is a perfect complex of $R[G]$-modules, then as in [Atiyah 1966],

$$\tau^\ell\big(\textstyle\sum_i (-1)^i [M^i]\big) = \textstyle\sum_n (-1)^n [(M^{\bullet\otimes\ell})^n].$$

**3.2.** ***The formula.*** Suppose that $g : Y \to \mathrm{Spec}(\mathbb{Z})$ is projective and flat and that $\mathscr{E}$ is a coherent $\mathbb{O}_Y[G]$-module that is $G$-c.t. and $\mathbb{O}_Y$-locally free. Consider the exterior tensor product $\mathscr{E}^{\boxtimes\ell} = \bigotimes_{i=1}^{\ell} p_i^* \mathscr{E}$ of $\mathscr{E}$ on $Y^\ell$ with $p_i : Y^\ell \to Y$ the $i$-th projection. Then $\mathscr{E}^{\boxtimes\ell}$ is an $S_\ell \times G$-equivariant coherent $\mathbb{O}_{Y^\ell}[G]$-module that is $\mathbb{O}_{Y^\ell}$-locally free and by Lemma 3.1.2(b) $G$-cohomologically trivial. Denote by $g^\ell : Y^\ell \to \mathrm{Spec}(\mathbb{Z})$ the structure morphism.

**Theorem 3.2.1** (Künneth formula). *We have*

$$\tau^\ell(g_*^{\mathrm{ct}}(\mathscr{E})) = (g^\ell)_*^{\mathrm{ct}}(\mathscr{E}^{\boxtimes\ell}) \tag{3.2.2}$$

*in* $\mathrm{K}_0(S_\ell; G, \mathbb{Z}) = \mathrm{G}_0^{\mathrm{ct}}(S_\ell; G, \mathbb{Z})$.

*Proof.* This follows [Kempf 1980, proof of Theorem 14]. Note that [Köck 2000, Theorem A] gives a similar result for $G = \{1\}$. Recall the construction of a bounded complex $(M^\bullet, d^\bullet)$ of finitely generated projective $\mathbb{Z}[G]$-modules that is quasi-isomorphic to $\mathrm{R}\Gamma(Y, \mathscr{E})$ described in Section 2.2.1 that uses the Čech complex $C^\bullet(\mathcal{U}, \mathscr{E})$. In this case, all the terms of $C^\bullet(\mathcal{U}, \mathscr{E})$ are $\mathbb{Z}$-flat. The complex $C^\bullet(\mathcal{U}, \mathscr{E})$ is given as the global sections of a corresponding complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$ of quasicoherent $\mathbb{O}_Y[G]$-modules whose terms are direct sums of sheaves of the form $j_*\mathscr{E}$, where $j : U_{j_1\cdots j_m} \hookrightarrow Y$ is the open immersion. Since all the intersections $U_{j_1\cdots j_m}$ are affine, the complex $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$ gives an acyclic resolution

$$0 \to \mathscr{E} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$$

of the $\mathbb{O}_Y[G]$-module $\mathscr{E}$. Consider the (exterior) tensor product

$$\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})^{\boxtimes\ell} = \bigotimes_{i=1}^{\ell} p_i^* \mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$$

with $S_\ell$-action defined following the rule of signs as before. Notice that all the terms of $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})$ have $\mathbb{Z}$-flat stalks. We can also see that $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})^{\boxtimes \ell}$ is acyclic; thus, it gives an acyclic resolution

$$0 \to \mathscr{E}^{\boxtimes \ell} \to \mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})^{\boxtimes \ell}$$

that respects the $S_\ell$-action. It follows from the definition that the global sections of $\mathfrak{C}^\bullet(\mathcal{U}, \mathscr{E})^{\boxtimes \ell}$ are $C^\bullet(\mathcal{U}, \mathscr{E})^{\otimes \ell}$. Hence, we obtain an isomorphism in the derived category of complexes of $\mathbb{Z}[S_\ell \times G]$-modules

$$\mathrm{R}\Gamma(Y^\ell, \mathscr{E}^{\boxtimes \ell}) \xrightarrow{\sim} C^\bullet(\mathcal{U}, \mathscr{E})^{\otimes \ell}.$$

Using $\phi : M^\bullet \to C^\bullet(\mathcal{U}, \mathscr{E})$, we also obtain a $\mathbb{Z}[S_\ell \times G]$-morphism of complexes

$$\phi^{\otimes \ell} : (M^\bullet)^{\otimes \ell} \to C^\bullet(\mathcal{U}, \mathscr{E})^{\otimes \ell}.$$

Since $\phi$ is a $\mathbb{Z}[G]$-quasi-isomorphism and the terms of $M^\bullet$ and $C^\bullet(\mathcal{U}, \mathscr{E})$ are $\mathbb{Z}$-flat, $\phi^{\otimes \ell}$ is a quasi-isomorphism. Combining these, we get an isomorphism in the derived category of complexes of $\mathbb{Z}[S_\ell \times G]$-modules

$$(M^\bullet)^{\otimes \ell} \xrightarrow{\sim} \mathrm{R}\Gamma(Y^\ell, \mathscr{E}^{\boxtimes \ell}), \tag{3.2.3}$$

and by Lemma 3.1.2(a), $(M^\bullet)^{\otimes \ell}$ is perfect as a complex of $\mathbb{Z}[G]$-modules. By taking the Euler characteristics of both sides, we obtain the result. □

## 4. Adams operations

### 4.1. *Cyclic powers.*

**4.1.1.** Again $\ell$ is a prime and $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$. By [Köck 1997], there is an "Adams operator" homomorphism

$$\psi^\ell : \mathrm{K}_0(\mathbb{Z}'[G]) \to \mathrm{K}_0(\mathbb{Z}'[G])$$

defined using "cyclic power operations" (following constructions of Kervaire and of Atiyah) as follows. Set $\Delta = \mathrm{Aut}(C_\ell) = (\mathbb{Z}/\ell\mathbb{Z})^*$, and consider the semidirect product $C_\ell \rtimes \Delta$ given by the tautological $\Delta$-action on $C_\ell$. We view $C_\ell \rtimes \Delta$ as a subgroup of $S_\ell = \mathrm{Perm}(C_\ell)$ in the natural way. Denote by $\sigma$ the generator 1 of $\mathbb{Z}/\ell\mathbb{Z} = C_\ell$. If $P$ is a finitely generated projective $\mathbb{Z}'[G]$-module, the tensor product $P^{\otimes \ell}$ is naturally an $S_\ell \times G$-module that is $\mathbb{Z}'[G]$-projective by Lemma 3.1.2. Let $S = \mathbb{Z}'[X]/(X^{\ell-1} + X^{\ell-2} + \cdots + 1)$, and set $z$ for the image of $X$ in $S$. We have $\mathbb{Z}'[C_\ell] = \mathbb{Z}' \times S$ by $\sigma \mapsto z$. Then $z^\ell = 1$. The group $\Delta$ acts on $S$ via automorphisms given by $\delta(z) = z^\delta$ for $\delta \in \Delta = (\mathbb{Z}/\ell\mathbb{Z})^*$. For $a \in \mathbb{Z}/\ell\mathbb{Z}$, we set

$$F_a(P^{\otimes \ell}) := ((S \otimes_{\mathbb{Z}'} P^{\otimes \ell})_a)^\Delta,$$

where by definition

$$(S\otimes_{\mathbb{Z}'}P^{\otimes\ell})_a=\{x\in S\otimes_{\mathbb{Z}'}P^{\otimes\ell}\mid\sigma(x)=z^a\cdot x\}.$$

(Notice that $\Delta$ acts on $(S\otimes_{\mathbb{Z}'}P^{\otimes\ell})_a\subset S\otimes_{\mathbb{Z}'}P^{\otimes\ell}$ by the diagonal action.) By [Köck 1997, Corollary 1.4], $F_a(P^{\otimes\ell})$ are projective $\mathbb{Z}'[G]$-modules, and we have by definition

$$\psi^\ell([P]):=[F_0(P^{\otimes\ell})]-[F_1(P^{\otimes\ell})]$$

in $\mathrm{K}_0(\mathbb{Z}'[G])$.

**4.1.2.** Consider now the Grothendieck rings $\mathrm{G}_0(\mathbb{Z}'[C_\ell])$, $\mathrm{G}_0(\mathbb{Z}'[C_\ell\times G])$, etc., where $\ell$ is a prime that does not divide the order $\#G$. Inflation gives homomorphisms $\mathrm{G}_0(\mathbb{Z}'[C_\ell])\to\mathrm{G}_0(\mathbb{Z}'[C_\ell\times G])$ and $\mathrm{G}_0(\mathbb{Z}'[G])\to\mathrm{G}_0(\mathbb{Z}'[C_\ell\times G])$ that we will suppress in the notation. Set $v=[\mathbb{Z}'[C_\ell]]$ for the class of the free module in $\mathrm{G}_0(\mathbb{Z}'[C_\ell])$, and let $\alpha:=v-1$ be the class of the augmentation ideal.

**4.1.3.** Write $\mathbb{Z}'[C_\ell\times G]=\mathbb{Z}'[G]\times S[G]$. Suppose that $R$ is a $\mathbb{Z}'$-algebra. If $N$ is an $R[C_\ell\times G]$-module, then the $C_\ell$-invariants $N^{C_\ell}$ give an $R[G]$-module that is a direct summand of $N$. Hence, if $N$ is projective as an $R[G]$-module or is $G$-c.t., then $N^{C_\ell}$ is projective as an $R[G]$-module or is $G$-c.t., respectively. The functor $N\mapsto N^{C_\ell}$ from $R[C_\ell\times G]$-modules to $R[G]$-modules is exact. Let us consider the homomorphism [Köck 2000, Lemma 4.3, Corollary 4.4]

$$\zeta:\mathrm{K}_0(R[C_\ell\times G])\to\mathrm{K}_0(R[G]),\quad \zeta([N])=\ell\cdot[N^{C_\ell}]-[N],\tag{4.1.4}$$

where we subtract the class of $N$ as an $R[G]$-module by forgetting the $C_\ell$-action. This is a $\mathrm{G}_0(\mathbb{Z}'[G])$-module homomorphism. We can see that $\zeta$ vanishes on the subgroup $(v)\cdot\mathrm{K}_0(R[C_\ell\times G])$. Indeed, suppose that $Q$ is a finitely generated projective $R[C_\ell\times G]$-module, and let us consider the $R[C_\ell\times G]$-module $R[C_\ell]\otimes_R Q\simeq\mathbb{Z}'[C_\ell]\otimes_{\mathbb{Z}'}Q$. Then there is an isomorphism (Frobenius reciprocity)

$$R[C_\ell]\otimes_R Q\simeq R[C_\ell]\otimes_R P=\mathrm{Ind}_G^{C_\ell\times G}(P),\tag{4.1.5}$$

where $P=\mathrm{Res}_G^{C_\ell\times G}(Q)$ is a projective $G$-module with trivial $C_\ell$-action. We have

$$(R[C_\ell]\otimes_R Q)^{C_\ell}\simeq(R[C_\ell]\otimes_R P)^{C_\ell}\simeq P$$

and so $\zeta(R[C_\ell]\otimes_R Q)=\ell\cdot[P]-[P^{\oplus\ell}]=0$.

**4.1.6.** Here $R$ is still a $\mathbb{Z}'$-algebra. Consider also the map

$$\xi:\mathrm{K}_0(R[G])\to\mathrm{K}_0(R[C_\ell\times G])$$

obtained by inflation (i.e., by considering a $G$-module as a $C_\ell\times G$-module with $C_\ell$ acting trivially). This is a $\mathrm{G}_0(\mathbb{Z}'[G])$-module homomorphism, and we can see from the definitions that

$$\zeta \circ \xi = (\ell - 1) \cdot \mathrm{id} \tag{4.1.7}$$

as maps $\mathrm{K}_0(R[G]) \to \mathrm{K}_0(R[G])$.

### 4.2. *Cyclic and tensor powers.*

**4.2.1.** Recall the tensor power operation

$$\tau^\ell : \mathrm{K}_0(\mathbb{Z}'[G]) = \mathrm{G}_0^{\mathrm{pr}}(G, \mathbb{Z}') \to \mathrm{G}_0^{\mathrm{ct}}(C_\ell; G, \mathbb{Z}') = \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])$$

given by the construction in Section 3.1.3 applied to $R = \mathbb{Z}'$ followed by restriction from $S_\ell \times G$ to $C_\ell \times G$.

**Proposition 4.2.2.** *For each $x$ in $\mathrm{K}_0(\mathbb{Z}'[G])$, we have*

$$\tau^\ell(x) = \psi^\ell(x) \quad \textit{in } \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])/(v) \cdot \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G]).$$

*Proof.* This follows the lines of the proof of [Köck 2000, Proposition 1.13]. First observe that the argument in [loc. cit.] gives that the map

$$\bar{\tau}^\ell : \mathrm{K}_0(\mathbb{Z}'[G]) \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])/(v) \cdot \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])$$

obtained from $\tau^\ell$ is a homomorphism. It is then enough to show the identity for $x = [P]$, the class of a finitely generated projective $\mathbb{Z}'[G]$-module $P$. As above, consider $P^{\otimes\ell}$. Let $e = \ell^{-1} \cdot \left(\sum_{i=0}^{\ell-1} \sigma^i\right)$ be the idempotent in $\mathbb{Z}'[C_\ell]$ so that $e \cdot P^{\otimes\ell} = (P^{\otimes\ell})^{C_\ell}$. Write $P^{\otimes\ell} = (P^{\otimes\ell})^{C_\ell} \oplus Q_1$ with $Q_1 := (1-e) \cdot P^{\otimes\ell}$ an $S[G]$-module (via $\mathbb{Z}'[G] \to S[G]$). As in [Köck 1997, Examples 1.5], we see that

$$F_0(P^{\otimes\ell}) = e \cdot P^{\otimes\ell} = (P^{\otimes\ell})^{C_\ell}, \qquad F_1(P^{\otimes\ell}) = Q_1^\Delta.$$

There is a short exact sequence

$$0 \to Q_1 \to Q_1^\Delta \otimes_{\mathbb{Z}'} \mathbb{Z}'[C_\ell] \to Q_1^\Delta \to 0$$

of $C_\ell \times G$-modules where the first map is given by

$$q \mapsto \sum_{i=0}^{\ell-1} \left( \sum_{a \in \Delta} a\sigma^{-i} \cdot q \right) \otimes \sigma^i$$

and the second map is obtained by tensoring the augmentation map $\mathbb{Z}[C_\ell] \to \mathbb{Z}$; here $Q_1^\Delta$ is viewed as having trivial $C_\ell$-action. This gives $[Q_1] = \alpha \cdot [Q_1^\Delta]$ in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])$. This now implies $[P^{\otimes\ell}] = [F_0(P^{\otimes\ell})] + \alpha \cdot [F_1(P^{\otimes\ell})]$ in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])$, where we regard $F_0(P^{\otimes\ell})$ and $F_1(P^{\otimes\ell})$ as having trivial $C_\ell$-action. Since $\alpha = v - 1$,

$$[P^{\otimes l}] = [F_0(P^{\otimes\ell})] - [F_1(P^{\otimes\ell})] \quad \text{in } \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])/(v) \cdot \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G]),$$

and the result follows. □

**Proposition 4.2.3.** *Let* $x_0 = [\mathbb{Z}'[G]]$ *be the class of the free module in* $\mathrm{K}_0(\mathbb{Z}'[G])$. *We have* $\psi^\ell(x_0) = x_0$ *in* $\mathrm{K}_0(\mathbb{Z}'[G])$.

*Proof.* Note that [Köck 1999, Theorem 1.6(e)] gives a corresponding result for the (a priori different) Adams operators defined via exterior powers. Set $\Gamma = C_\ell \rtimes (\mathbb{Z}/\ell\mathbb{Z})^* \subset S_\ell = \mathrm{Perm}(\mathbb{F}_\ell)$; the element $\gamma = (\sigma^a, b)$ is then the (affine) map $\mathbb{F}_\ell \to \mathbb{F}_\ell$ given by $\gamma(x) = bx + a$. Consider the set $G^\ell = \mathrm{Maps}(\mathbb{F}_\ell, G)$ with $G \times \Gamma$-action given by $(g, \gamma) \cdot (g_x)_{x\in\mathbb{F}_\ell} = (gg_{\gamma^{-1}(x)})_{x\in\mathbb{F}_\ell}$. Suppose $(g, \gamma)$ stabilizes $(g_x)_{x\in\mathbb{F}_\ell}$ so that $gg_x = g_{\gamma(x)}$ for all $x \in \mathbb{F}_\ell$. If $b \neq 1$, there is $y \in \mathbb{F}_\ell$ such that $\gamma(y) = y$. Then $gg_y = g_y$ and so $g = 1$. If $b = 1$, we have $gg_x = g_{x+a}$ for all $x \in \mathbb{F}_\ell$; this gives $g^\ell = 1$ and so again $g = 1$ since $\ell$ is prime to $\#G$. We conclude that the stabilizer in $G \times \Gamma$ of any $\underline{g} = (g_x)_{x\in\mathbb{F}_\ell} \in G^\ell$ lies in $1 \times \Gamma$. Therefore, $G^\ell$ is in $G \times \Gamma$-equivariant bijection with a disjoint union of sets of the form $G \times (\Gamma/\Gamma_{\underline{g}})$ with $\Gamma_{\underline{g}}$ the stabilizer subgroup in $\Gamma$. Hence, the $\mathbb{Z}'[G \times \Gamma]$-module $\mathbb{Z}'[G]^{\otimes\ell} = \mathbb{Z}'[G^\ell]$ is isomorphic to a direct sum of modules of the form $\mathbb{Z}'[G] \otimes_{\mathbb{Z}'} \mathbb{Z}'[\Gamma/\Gamma_{\underline{g}}]$. By the definition of $F_a$ (see [Köck 1997, §1] or Section 4.1.1), we see that $F_a(\mathbb{Z}'[G] \otimes_{\mathbb{Z}'} \mathbb{Z}'[\Gamma/\Gamma_{\underline{g}}]) \simeq \mathbb{Z}'[G] \otimes_{\mathbb{Z}'} F_a(\mathbb{Z}'[\Gamma/\Gamma_{\underline{g}}])$. We conclude that $F_a(\mathbb{Z}'[G]^{\otimes\ell})$ are free $\mathbb{Z}'[G]$-modules. Therefore, $\psi^\ell(x_0) = m \cdot x_0$ in $\mathrm{K}_0(\mathbb{Z}'[G])$ for some $m \in \mathbb{Z}$, and by comparing $\mathbb{Z}'$-ranks (for this we may assume $G = \{1\}$), we can easily see that $m = 1$. ☐

**4.2.4.** If $Z$ is a projective flat $G$-scheme over $\mathrm{Spec}(\mathbb{Z}')$, there is an Adams operation $\psi^\ell : \mathrm{K}_0(G, Z) \to \mathrm{K}_0(G, Z)$ [Köck 1998]. By an argument as above, or by using the equivariant splitting principle as in [Köck 2000], we can see that

$$\psi^\ell(\mathscr{F}) = [\mathscr{F}^{\otimes\ell}] \tag{4.2.5}$$

in the quotient $\mathrm{K}_0(C_\ell \times G, Z)/(v) \cdot \mathrm{K}_0(C_\ell \times G, Z)$.

### 4.3. *The Cassou-Noguès–Taylor Adams operations.*

**4.3.1.** We continue to assume that $\ell$ is prime to the order $\#G$. Then by [Swan 1960], finitely generated projective $\mathbb{Z}'[G]$-modules are locally free.

If $(n, \#G) = 1$, we denote by $\psi_n^{\mathrm{CNT}} : \mathrm{Cl}(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}[G])$ the Adams operator homomorphism defined by Cassou-Noguès and Taylor [1985; Taylor 1984]. (Roughly speaking, this is given, via the Fröhlich description, as the dual of the Adams operation $\psi^n(\chi)(g) := \chi(g^n)$ on the character group.) The operators $\psi_n^{\mathrm{CNT}}$ also restrict to operators on $\mathrm{Cl}(\mathbb{Z}'[G])$; we will denote these by the same symbol. Note that we can identify $\mathrm{Cl}(\mathbb{Z}'[G])$ with the subgroup of $\mathrm{K}_0(\mathbb{Z}'[G])$ of elements of rank 0. Denote by $r : \mathrm{K}_0(\mathbb{Z}'[G]) \to \mathbb{Z}$ the rank homomorphism.

Köck has shown that the Cassou-Noguès–Taylor Adams operators can be described in terms of (arguably more natural) Adams operators $\psi_{\mathrm{ext}}^\ell$ defined via exterior powers and the Newton polynomial [Köck 1999, Theorem 3.7]. Here, we use his arguments to obtain a similar relation with the Adams operators $\psi^\ell$ of [Köck 1997] defined via cyclic powers, which are better suited to our application.

**Proposition 4.3.2.** *Let* $\ell'$ *be a prime with* $\ell\ell' \equiv 1 \bmod \exp(G)$, *and set* $x_0 = [\mathbb{Z}'[G]]$ *for the class of the free module of rank* 1. *Then we have*

$$\psi^\ell(x - r(x)\cdot x_0) = \ell \cdot \psi^{\mathrm{CNT}}_{\ell'}(x - r(x)\cdot x_0) \tag{4.3.3}$$

*for all* $x \in \mathrm{K}_0(\mathbb{Z}'[G])$.

*Proof.* Let us explain how we can deduce this by combining results and arguments from [Köck 1996; 1997; 1999]. Since $\psi^\ell : \mathrm{K}_0(\mathbb{Z}'[G]) \to \mathrm{K}_0(\mathbb{Z}'[G])$ is a group homomorphism [Köck 1997, Proposition 2.5] that preserves the rank, $\psi^\ell$ restricts to an (additive) operation on the subgroup $\mathrm{Cl}(\mathbb{Z}'[G])$ of rank-0 elements. The group $\mathrm{K}_0(\mathbb{Z}'[G])$ is generated by the classes of locally free left ideals in $\mathbb{Z}'[G]$; hence, we can assume $x = [P]$ where $P$ is such a ideal. We may assume that $P \otimes_{\mathbb{Z}'} \mathbb{Z}_v = \mathbb{Z}_v[G]\cdot\lambda_v$, with $\lambda_v \in \mathbb{Q}_v[G]^* \cap \mathbb{Z}_v[G]$, so that $P = \bigcap_{v\neq l}(\mathbb{Z}_v[G]\cdot\lambda_v \cap \mathbb{Q}[G])$. Then a Fröhlich representative of $x - x_0$ is given via the classes (“reduced norms”) $[\lambda_v] \in \mathrm{K}_1(\mathbb{Q}_v[G])$ of $\lambda_v \in \mathbb{Q}_v[G]^*$. Note that we have [Taylor 1984]

$$\mathrm{K}_1(\mathbb{Q}_v[G]) \simeq \mathrm{Hom}_{\mathrm{Gal}(\bar{\mathbb{Q}}_v/\mathbb{Q}_v)}(\mathrm{K}_0(\bar{\mathbb{Q}}_v[G]), \bar{\mathbb{Q}}_v^*). \tag{4.3.4}$$

By [Köck 1997], cyclic power Adams operators $\psi^\ell$ can also be defined on the higher K-groups $\mathrm{K}_i(R[G])$, $i \geq 1$, for every commutative $\mathbb{Z}'$-algebra. In particular, we have $\psi^\ell$ on $\mathrm{K}_1(\mathbb{Z}_v[G])$ and $\mathrm{K}_1(\mathbb{Q}_v[G])$, for $v \neq (\ell)$, and on $\mathrm{K}_1(\mathbb{Q}[G])$. Using [Köck 1997, Corollary 1.4(c)], we see that the base change homomorphism

$$\mathrm{K}_1(\mathbb{Z}_v[G]) \to \mathrm{K}_1(\mathbb{Q}_v[G])$$

commutes with $\psi^\ell$. In fact, by [Köck 1997, §3], the operators $\psi^\ell$ are defined via the cyclic operations $[a]_\ell$ of [loc. cit.] as $\psi^\ell = [0]_\ell - [1]_\ell$. Moreover, the operations $[a]_\ell$ are given via continuous maps on the level of spaces that give K-theory in the style of Gillet and Grayson [loc. cit.]. Using this, we can see that the topological argument of the proof of [Köck 1999, Proposition 3.1] also applies to the operators $[a]_\ell$ and $\psi^\ell$, and so we obtain the commutative diagram of [Köck 1999, Proposition 3.1] for $K = \mathbb{Q}$, $\mathfrak{p} = (v)$, and $\gamma = [a]_\ell$ or $\psi^\ell$, i.e., that $[a]_\ell$ and $\psi^\ell$ commute with the connecting homomorphism $\Phi : \mathrm{K}_1(\mathbb{Q}_v[G]) \to \mathrm{K}_0(\mathbb{Z}'[G])$. As in the proof of [Köck 1999, Corollary 3.5], this, together with the localization sequence, implies that the element $\psi^\ell(x - x_0) = \psi^\ell(x) - \psi^\ell(x_0)$ has Fröhlich representative given by $(\psi^\ell([\lambda_v]))_v$ with $\psi^\ell : \mathrm{K}_1(\mathbb{Q}_v[G]) \to \mathrm{K}_1(\mathbb{Q}_v[G])$ as above (see [Köck 1999] for more details).

Now by [Köck 1996, Corollary (c) of Proposition 1], the operator $\psi^\ell$ on $\mathrm{K}_1(\mathbb{Q}_v[G])$ and $\mathrm{K}_1(\mathbb{Q}[G])$ agrees with the (more standard) Adams operator $\psi^\ell_{\mathrm{ext}}$ defined using exterior powers and the Newton polynomial (as for example in [Köck

1999]). In fact, then by [Köck 1996, Corollary 1 of Theorem 1] (or the proof of [Köck 1999, Theorem 3.7]), $\psi^\ell([\lambda_v])$ is given via (4.3.4) by $\chi \mapsto ([\lambda_v](\psi^{\ell'}(\chi)))^\ell$. The result now follows from the Fröhlich description of $\mathrm{Cl}(\mathbb{Z}'[G])$ [Taylor 1984] and the definition of the Cassou-Noguès–Taylor Adams operator. □

**Remark 4.3.5.** Since $\psi^\ell$ is only defined for $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$-algebras, the above proposition does not give an expression for the Cassou-Noguès–Taylor Adams operators on $\mathrm{Cl}(\mathbb{Z}[G])$. Still, this weaker result is enough for our purposes.

## 5. Localization and Adams–Riemann–Roch identities

### 5.1. *Localization for $C_\ell$- and $C_\ell \times G$-modules.*

**5.1.1.** For simplicity, we set $\mathscr{R}(C_\ell) = \mathrm{G}_0(\mathbb{Z}'[C_\ell])' = \mathrm{G}_0(\mathbb{Z}'[C_\ell])[\ell^{-1}]$, $\mathscr{R}(C_\ell \times G) = \mathrm{G}_0(\mathbb{Z}'[C_\ell \times G])'$, etc., where $\ell$ is a prime that does not divide the order $\#G$. Inflation gives homomorphisms $\mathscr{R}(C_\ell) \to \mathscr{R}(C_\ell \times G)$ and $\mathscr{R}(G) \to \mathscr{R}(C_\ell \times G)$ that we will suppress in the notation. Denote by $\alpha$ the class in $\mathscr{R}(C_\ell)$ of the augmentation ideal of $\mathbb{Z}'[C_\ell]$, and set $v = 1 + \alpha = [\mathbb{Z}'[C_\ell]]$. The ring structure in $\mathscr{R}(C_\ell)$ is such that $v^2 = \ell \cdot v$. Also denote by $I_G$ the ideal of $\mathscr{R}(G)$ given as the kernel of the rank homomorphism. Set

$$\mathscr{R}(C_\ell \times G)^\wedge := \varprojlim_n \mathscr{R}(C_\ell \times G)/(I_G^n \mathscr{R}(C_\ell \times G) + v\mathscr{R}(C_\ell \times G)).$$

In general, if $M$ is an $\mathscr{R}(C_\ell \times G)$-module, we set $M^\wedge := \varprojlim_n M/(I_G^n M + vM)$, which is an $\mathscr{R}(C_\ell \times G)^\wedge$-module. The maximal ideals of $\mathscr{R}(C_\ell \times G)^\wedge$ correspond to maximal ideals of $\mathscr{R}(C_\ell \times G)$ that contain $I_G\mathscr{R}(C_\ell \times G) + v\mathscr{R}(C_\ell \times G)$; we can see (see Section 2.1.2) that these are the maximal ideals of the form $\rho_{((\sigma,1),(q))}$ with $\ell \neq q$ and $\sigma$ is a generator of $C_\ell$. (The ideal $\rho_{((\sigma,1),(q))}$ is independent of the choice of the generator $\sigma$; there is exactly one ideal for each prime $q \neq \ell$.) If $\phi : M \to N$ is an $\mathscr{R}(C_\ell \times G)$-module homomorphism such that the localization $\phi_\rho : M_\rho \to N_\rho$ is an isomorphism for every $\rho$ of the form $\rho_{((\sigma,1),(q))}$ with $q \neq \ell$ as above, then the induced $\phi^\wedge : M^\wedge \to N^\wedge$ is an isomorphism of $\mathscr{R}(C_\ell \times G)^\wedge$-modules.

### 5.2. *Cyclic localization on products.*

**5.2.1.** Fix a prime $\ell$ that does not divide $\#G$, and as before, set $\mathbb{Z}' = \mathbb{Z}[\ell^{-1}]$. Suppose that $Z \to \mathrm{Spec}(\mathbb{Z}')$ is a quasiprojective scheme equipped with an action of $G$. We will consider "localization" on the fixed points for the action of the cyclic group $C_\ell$ on the $\ell$-fold fiber product $Z^\ell$ over $\mathrm{Spec}(\mathbb{Z}')$. In fact, the product $C_\ell \times G$ acts on $Z^\ell$; $C_\ell$ acts by permutation of the factors and $G$ acts diagonally.

**5.2.2.** Consider a maximal ideal $\rho$ of $\mathscr{R}(C_\ell \times G)$ of the form $\rho_{((\sigma,1),(q))}$ with $q \neq \ell$ and $\sigma$ a generator of $C_\ell$ as before. The corresponding fixed-point subscheme $Z^\rho$ of $Z^\ell$ is by definition the reduced union of the translates of the fixed subscheme $Z^{(\sigma,1)}$

of the element $(\sigma, 1)$. In our case, this is the diagonal:

$$(Z^\ell)^\rho = (Z^\ell)^{(\sigma,1)} \cdot (C_\ell \times G) = \Delta(Z) = Z \hookrightarrow Z^\ell. \tag{5.2.3}$$

Consider the homomorphism obtained by push-forward of coherent sheaves

$$\Delta_* : \mathrm{G}_0(C_\ell \times G, Z) \to \mathrm{G}_0(C_\ell \times G, Z^\ell).$$

The "concentration" theorem [Chinburg et al. 1997b, Theorem 6.1] implies that, after localizing at any $\rho$ as above, we obtain an $\mathscr{R}(C_\ell \times G)_\rho$-module isomorphism

$$(\Delta_*)_\rho : \mathrm{G}_0(C_\ell \times G, Z)_\rho \xrightarrow{\sim} \mathrm{G}_0(C_\ell \times G, Z^\ell)_\rho.$$

It now follows as above that $\Delta_*$ gives an $\mathscr{R}(C_\ell \times G)^\wedge$-module isomorphism

$$\Delta_*^\wedge : \mathrm{G}_0(C_\ell \times G, Z)^\wedge \xrightarrow{\sim} \mathrm{G}_0(C_\ell \times G, Z^\ell)^\wedge. \tag{5.2.4}$$

Here the completions are given as above for the $\mathscr{R}(C_\ell \times G)$-modules $\mathrm{G}_0(C_\ell \times G, Z)'$ and $\mathrm{G}_0(C_\ell \times G, Z^\ell)'$.

**Definition 5.2.5.** Let $L_\bullet : \mathrm{G}_0(C_\ell \times G, Z^\ell) \to \mathrm{G}_0(C_\ell \times G, Z)^\wedge$ be the $\mathscr{R}(C_\ell \times G)$-homomorphism defined as the composition of $\mathrm{G}_0(C_\ell \times G, Z^\ell) \to \mathrm{G}_0(C_\ell \times G, Z^\ell)^\wedge$ with the inverse of $\Delta_*^\wedge$. Set $\vartheta^\ell := L_\bullet(1) \in \mathrm{G}_0(C_\ell \times G, Z)^\wedge$. Then

$$\Delta_*^\wedge(\vartheta^\ell) = 1 = [\mathscr{O}_{Z^\ell}]. \tag{5.2.6}$$

**5.2.7.** If $\mathscr{F}$ is a $G$-equivariant locally free coherent $\mathscr{O}_Z$-module on $Z$, then $\mathscr{F}^{\boxtimes \ell}$ is a $C_\ell \times G$-equivariant locally free coherent $\mathscr{O}_{Z^\ell}$-module on $Z^\ell$ and $\mathscr{F}^{\otimes \ell} \simeq \Delta^*(\mathscr{F}^{\boxtimes \ell})$ as $C_\ell \times G$-sheaves on $Z$. Using (5.2.6) and the projection formula, we obtain

$$\mathscr{F}^{\boxtimes \ell} = \Delta_*^\wedge(\vartheta^\ell) \otimes \mathscr{F}^{\boxtimes \ell} = \Delta_*^\wedge(\vartheta^\ell \otimes \Delta^*(\mathscr{F}^{\boxtimes \ell})) = \Delta_*^\wedge(\vartheta^\ell \otimes \mathscr{F}^{\otimes \ell}).$$

Therefore,

$$\mathscr{F}^{\boxtimes \ell} = \Delta_*^\wedge(\vartheta^\ell \otimes \mathscr{F}^{\otimes \ell}) \tag{5.2.8}$$

in $\mathrm{G}_0(C_\ell \times G, Z^\ell)^\wedge$.

**5.2.9.** Suppose that $Z$ is smooth over $\mathrm{Spec}(\mathbb{Z}')$; then $Z^\ell$ is also smooth. Denote by $\mathscr{N}_{Z|Z^\ell}$ the locally free conormal bundle $\mathscr{I}_Z/\mathscr{I}_Z^2$ of $Z \subset Z^\ell$ that gives a class in $\mathrm{K}_0(C_\ell \times G, Z)$. (Here $\mathscr{I}_Z$ is the ideal sheaf of $Z \subset Z^\ell$.) As in the proof of the Lefschetz–Riemann–Roch theorem [Thomason 1992; Chinburg et al. 1997b], we can see using the self-intersection formula that the homomorphism $L_\bullet$ is given as the composition of the restriction

$$\mathrm{G}_0(C_\ell \times G, Z^\ell) \cong \mathrm{K}_0(C_\ell \times G, Z^\ell) \xrightarrow{\Delta^*} \mathrm{K}_0(C_\ell \times G, Z) \cong \mathrm{G}_0(C_\ell \times G, Z)$$

followed by multiplication by $\lambda_{-1}(\mathscr{N}_{Z|Z^\ell})^{-1} \in \mathrm{K}_0(C_\ell \times G, Z)^\wedge$, and so $\vartheta^\ell$ is the image of $\lambda_{-1}(\mathscr{N}_{Z|Z^\ell})^{-1}$ in $\mathrm{G}_0(C_\ell \times G, Z)^\wedge$. By [Chinburg et al. 1997b],

$$\lambda_{-1}(\mathscr{N}_{Z|Z^\ell}) := \sum_{i=0}^{\mathrm{top}} (-1)^i [\wedge^i \mathscr{N}_{Z|Z^\ell}]$$

is invertible in all the localizations $\mathrm{K}_0(C_\ell \times G, Z)_\rho$, with $\rho$ as above, so it is invertible in $\mathrm{K}_0(C_\ell \times G, Z)^\wedge$.

**5.2.10.** If $Z$ is projective but not necessarily smooth, we can describe $L_\bullet$ by following [Baum et al. 1979; Quart 1979]. Embed $Z$ as a closed $G$-subscheme of a smooth projective bundle $P = \mathbb{P}(\mathscr{G}) \to \mathrm{Spec}(\mathbb{Z}')$ with $G$-action [Köck 1998]. Then $P^\ell$ is also smooth over $\mathrm{Spec}(\mathbb{Z}')$. Let us write $i : Z \hookrightarrow P$, $i^\ell : Z^\ell \hookrightarrow P^\ell$, for the corresponding closed immersions. Starting with $x \in \mathrm{G}_0(C_\ell \times G, Z^\ell)$, we first "resolve $x$ on $P^\ell$", i.e., represent the push-forward $(i^\ell)_*(x)$ by a bounded complex $\mathscr{E}^\bullet(x)$ of $C_\ell \times G$-equivariant locally free coherent sheaves on $P^\ell$ that is exact off $Z^\ell$ (so that the homology of $\mathscr{E}^\bullet(x)$ gives back $x$). Next, we restrict the complex $\mathscr{E}^\bullet(x)$ to $P$ to obtain $\mathscr{E}^\bullet(x)_{|P}$, a complex exact off $Z = P \cap Z^\ell$. Finally, we take the class $[h(\mathscr{E}^\bullet(x)_{|P})]$ of the homology of $\mathscr{E}^\bullet(x)_{|P}$ to obtain an element of $\mathrm{G}_0(C_\ell \times G, Z)$ [Soulé et al. 1992]. For simplicity, write $T_\ell = C_\ell \times G$. Then $x \mapsto [h(\mathscr{E}^\bullet(x)_{|P})]$ is the composition

$$\mathrm{G}_0(T_\ell, Z^\ell) \xrightarrow[\sim]{h^{-1}} \mathrm{K}_0^{Z^\ell}(T_\ell, P^\ell) \xrightarrow{|P} \mathrm{K}_0^{P\cap Z^\ell}(T_\ell, P) \xrightarrow{h} \mathrm{G}_0(T_\ell, Z), \tag{5.2.11}$$

where in the middle we have the relative K-groups of complexes of $T_\ell$-equivariant locally free sheaves exact off $Z^\ell$ and $Z^\ell \cap P = Z$, respectively (see [Soulé et al. 1992, §3; Baum et al. 1979, Definition 2.1] for more details). Finally, we multiply $[h(\mathscr{E}^\bullet(x)_{|P})]$ by the restriction $\lambda_{-1}(\mathscr{N}_{P|P^\ell})^{-1}_{|Z}$ in $\mathrm{K}_0(C_\ell \times G, Z)^\wedge$ of the inverse $\lambda_{-1}(\mathscr{N}_{P|P^\ell})^{-1} \in \mathrm{K}_0(C_\ell \times G, P)^\wedge$. We claim that

$$L_\bullet(x) = [h(\mathscr{E}^\bullet(x)_{|P})] \cdot \lambda_{-1}(\mathscr{N}_{P|P^\ell})^{-1}_{|Z}. \tag{5.2.12}$$

To verify (5.2.12), we can follow the argument in the proof of Lemma 1 in [Quart 1979], which applies in this situation. The reader is referred there for more details.

### 5.3. *Adams–Riemann–Roch identities.*

**5.3.1.** Let $f : X \to \mathrm{Spec}(\mathbb{Z})$ be projective and flat with a tame action of $G$. Then the quotient scheme $\pi : X \to Y = X/G$ exists and $\pi$ is finite. Choose a prime $\ell$ with $(\ell, \#G) = 1$. We will apply the setup of the previous section to $Z = X' = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}')$.

We have the projective equivariant Euler characteristics

$$f_* : \mathrm{G}_0(C_\ell \times G, X')' \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])'$$

and similarly $f^{\ell}_* : \mathrm{G}_0(C_\ell \times G, (X')^{\ell})' \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])'$. (Here, we omit the superscript ct from $f^{\mathrm{ct}}_*$ and $(f^{\ell}_*)^{\mathrm{ct}}$.) These are both $\mathscr{R}(C_\ell \times G) = \mathrm{G}_0(\mathbb{Z}'[C_\ell \times G])'$-module homomorphisms and induce $\mathscr{R}(C_\ell \times G)^\wedge$-homomorphisms

$$f^\wedge_* : \mathrm{G}_0(C_\ell \times G, X')^\wedge \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge,$$
$$(f^{\ell}_*)^\wedge : \mathrm{G}_0(C_\ell \times G, (X')^{\ell})^\wedge \to \mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge.$$

Now suppose $\mathscr{F}$ is a $G$-equivariant coherent locally free $\mathscr{O}_X$-module. We apply $(f^{\ell}_*)^\wedge$ on both sides of (5.2.8). Using $f^{\ell}_* \cdot \Delta_* = f_*$, we obtain

$$(f^{\ell})^\wedge_*(\mathscr{F}^{\boxtimes \ell}) = f^\wedge_*(\vartheta^\ell \otimes \mathscr{F}^{\otimes \ell}) \tag{5.3.2}$$

in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge$.

**5.3.3.** Assume in addition that $\pi$ is flat; then so is $\pi^\ell : X^\ell \to Y^\ell$. Set $\mathscr{E} = \pi_* \mathscr{F}$. This is a $G$-c.t. coherent $\mathscr{O}_Y[G]$-module that is $\mathscr{O}_Y$-locally free. We have $\pi^{\ell}_*(\mathscr{F}^{\boxtimes \ell}) = \mathscr{E}^{\boxtimes \ell}$ on $Y^\ell$. The Künneth formula (3.2.2) now gives

$$f^{\ell}_*(\mathscr{F}^{\boxtimes \ell}) = g^{\ell}_*(\pi^{\ell}_*(\mathscr{F}^{\boxtimes \ell})) = g^{\ell}_*(\mathscr{E}^{\boxtimes \ell}) = \tau^\ell(g^{\mathrm{ct}}_*(\mathscr{E})) = \tau^\ell(f^{\mathrm{ct}}_*(\mathscr{F})).$$

Combining this with (5.3.2) gives

$$\tau^\ell(f^{\mathrm{ct}}_*(\mathscr{F})) = f^\wedge_*(\vartheta^\ell \otimes \mathscr{F}^{\otimes \ell})$$

in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge$. Using Proposition 4.2.2, we obtain

$$\psi^\ell(f^{\mathrm{ct}}_*(\mathscr{F})) = f^\wedge_*(\vartheta^\ell \otimes \mathscr{F}^{\otimes \ell}). \tag{5.3.4}$$

Finally, using (4.2.5), this gives the Adams–Riemann–Roch identity

$$\psi^\ell(f^{\mathrm{ct}}_*(\mathscr{F})) = f^\wedge_*(\vartheta^\ell \otimes \psi^\ell(\mathscr{F})) \tag{5.3.5}$$

in $\mathrm{K}_0(\mathbb{Z}'[C_\ell \times G])^\wedge$.

**5.3.6.** Now consider $\zeta : \mathrm{G}_0(C_\ell \times G, Z)' \to \mathrm{G}_0(G, Z)'$ given by $\mathscr{F} \mapsto \ell \cdot [\mathscr{F}^{C_\ell}] - [\mathscr{F}]$ (see Section 4.1.3). This is an $\mathscr{R}(G) = \mathrm{G}_0(\mathbb{Z}'[G])'$-module homomorphism. As in Section 4.1.3, we can see that $\zeta$ vanishes on $(v) \cdot \mathrm{G}_0(C_\ell \times G, Z)'$ using Frobenius reciprocity. Therefore, it also gives

$$\zeta^\wedge : \mathrm{G}_0(C_\ell \times G, Z)^\wedge \to \mathrm{G}_0(G, Z)^\wedge = \varprojlim_n \mathrm{G}_0(G, Z)'/I^n_G \cdot \mathrm{G}_0(G, Z)'.$$

**Theorem 5.3.7.** *Under the above assumptions, we have*

$$(\ell - 1)\psi^\ell(f^{\mathrm{ct}}_*(\mathscr{F})) = f^\wedge_*(\zeta^\wedge(\vartheta^\ell) \otimes \psi^\ell(\mathscr{F})) \tag{5.3.8}$$

*in* $\mathrm{K}_0(\mathbb{Z}'[G])^\wedge = \varprojlim_n \mathrm{K}_0(\mathbb{Z}'[G])'/I^n_G \cdot \mathrm{K}_0(\mathbb{Z}'[G])'$.

*Proof.* Apply to the identity (5.3.5) the natural map induced on the completions by the map $\zeta$ of Section 4.1.3. The result then follows by using (4.1.7). □

Under some additional assumptions, we will see in the next section that $\vartheta^\ell = L_\bullet(1)$ is given as the inverse of a Bott element. This justifies calling (5.3.5) and (5.3.8) Adams–Riemann–Roch identities.

## 5.4. *Localization and Bott classes.*

**5.4.1.** We return to the more general setup of Section 5.2. Suppose in addition that $f : Z \to \mathrm{Spec}(\mathbb{Z}')$ is a local complete intersection. Then, we can find a $\mathbb{Z}'[G]$-lattice $E$ such that $f$ factors $G$-equivariantly $Z \hookrightarrow \mathbb{P}(E) \to \mathrm{Spec}(\mathbb{Z}')$, where $P = \mathbb{P}(E)$ is the projective space with linear $G$-action determined by $E$. In this, the first morphism $i : Z \to \mathbb{P}(E)$ is a closed immersion, and the conormal sheaf $\mathscr{N}_{Z|P} := \mathscr{I}_Z/\mathscr{I}_Z^2$ is a $G$-equivariant sheaf locally free over $Z$ of rank equal to the codimension $c$ of $Z$ in $P$ [Köck 1998, §3]. By definition, the cotangent element of $Z$ is $T_Z^\vee := [i^*\Omega_P^1] - [\mathscr{N}_{Z|P}]$ in $\mathrm{K}_0(G, Z)$; it is independent of the choice of such an embedding. The following result appears in [Köck 2000] if $Z$ is smooth and $G$ acts trivially on $Z$; it is inspired by an observation of Nori [2000]. (The case that $Z$ is smooth and $G$ acts trivially is enough for the proof of our main result in the unramified case, Theorem 1.0.2.)

**Theorem 5.4.2.** *Suppose that $Z \to \mathrm{Spec}(\mathbb{Z}')$ is a projective scheme with $G$-action that is a local complete intersection. Then the element $\vartheta^\ell = L_\bullet(1) \in \mathrm{G}_0(C_\ell \times G, Z)^\wedge$ is the image of the inverse $\theta^\ell(T_Z^\vee)^{-1} \in \mathrm{K}_0(G, Z)^\wedge$ of the Bott class under the natural homomorphism $\mathrm{K}_0(G, Z)^\wedge \to \mathrm{G}_0(C_\ell \times G, Z)^\wedge$.*

Here the inverse of the Bott class $\theta^\ell(T_Z^\vee)^{-1} = \theta^\ell(i^*\Omega_P^1)^{-1} \cdot \theta^\ell(\mathscr{N}_{Z|P})$ is defined in the $I_G$-adic completion $\mathrm{K}_0(G, Z)^\wedge$ as in [Köck 1998, p. 432]. As is remarked in [loc. cit.], if $G$ acts trivially on $Z$, then the completion is not needed: the Bott class $\theta^\ell(T_Z^\vee)$ is then defined and is invertible in $\mathrm{K}_0(Z)' = \mathrm{K}_0(Z)[\ell^{-1}]$.

*Proof.* Starting with $i : Z \hookrightarrow P$ as above, we obtain a similar embedding of the fibered product $i^\ell : Z^\ell \hookrightarrow P^\ell$ that also makes $Z^\ell$ a local complete intersection in the smooth $P^\ell$. We now use Section 5.2.10 to calculate $\vartheta^\ell = L_\bullet(1)$. Since $P \to \mathrm{Spec}(\mathbb{Z}')$ is smooth and projective, we can find a $G$-equivariant resolution $\mathscr{E}^\bullet \to i_*\mathscr{O}_Z$ of $i_*\mathscr{O}_Z$ by a bounded complex of $G$-equivariant locally free coherent $\mathscr{O}_P$-sheaves on $P$. Then, the total exterior tensor product $(\mathscr{E}^\bullet)^{\boxtimes \ell} = \bigotimes_{j=1}^{\ell} p_j^*\mathscr{E}^\bullet$ with its natural $C_\ell$-action (with the rule of signs as in Section 3.1.1) provides a $C_\ell \times G$-equivariant locally free resolution of the $C_\ell \times G$-coherent sheaf $(i^\ell)_*(\mathscr{O}_{Z^\ell}) \simeq (i_*\mathscr{O}_Z)^{\boxtimes \ell} = \bigotimes_{j=1}^{\ell} p_j^*(i_*\mathscr{O}_Z)$ on $P^\ell$. Let us restrict to $P$, i.e., pull back the complex $(\mathscr{E}^\bullet)^{\boxtimes \ell}$ via the diagonal $\Delta_P : P \hookrightarrow P^\ell$. We obtain the total $\ell$-th tensor product

$$(\mathscr{E}^\bullet)^{\otimes \ell} = (\mathscr{E}^\bullet)^{\boxtimes \ell}|_P = \Delta_P^*((\mathscr{E}^\bullet)^{\boxtimes \ell}) \tag{5.4.3}$$

with its natural $C_\ell \times G$-action. Since $\mathscr{E}^\bullet$ is exact off $Z$, so is $(\mathscr{E}^\bullet)^{\otimes \ell}$; consider the element $[h((\mathscr{E}^\bullet)^{\otimes \ell})]$ of $\mathrm{G}_0(C_\ell \times G, Z)$ obtained from its total homology:

$$[h((\mathscr{E}^\bullet)^{\otimes \ell})] := \textstyle\sum_j (-1)^j [\mathrm{H}^j((\mathscr{E}^\bullet)^{\otimes \ell})] \tag{5.4.4}$$

as in [Soulé et al. 1992]. By Section 5.2.10, we have

$$\vartheta^\ell = L_\bullet(1) = [h((\mathscr{E}^\bullet)^{\otimes \ell})] \cdot \lambda_{-1}(\mathscr{N}_{P|P^\ell})^{-1}_{|Z}. \tag{5.4.5}$$

We now use two results of Köck. By [Köck 2001, Theorem 5.1], we have canonical $C_\ell \times G$-isomorphisms $\mathrm{H}^j((\mathscr{E}^\bullet)^{\otimes \ell}) \simeq \wedge^j(\mathscr{N}_{Z|P} \otimes \alpha)$, where $\alpha$ again is the augmentation ideal. Therefore,

$$[h((\mathscr{E}^\bullet)^{\otimes \ell})] = \textstyle\sum_j (-1)^j [\wedge^j(\mathscr{N}_{Z|P} \cdot \alpha)] = \lambda_{-1}(\mathscr{N}_{Z|P} \cdot \alpha). \tag{5.4.6}$$

On the other hand, [Köck 2000, Lemma 3.5], gives a $C_\ell$-isomorphism

$$\Omega^1_P \otimes \alpha \xrightarrow{\sim} \mathscr{N}_{P|P^\ell},$$

which, as we can easily see, is also $G$-equivariant. Therefore,

$$\lambda_{-1}(\mathscr{N}_{P|P^\ell})_{|Z} = \lambda_{-1}(i^*\Omega^1_P \cdot \alpha)$$

in $\mathrm{K}_0(C_\ell \times G, Z)$.

Now, as in [Köck 2000, Proposition 3.2], we see using the $G$-equivariant splitting principle (e.g., [Köck 1998]) that, if $\mathscr{F}$ is a $G$-equivariant locally free coherent sheaf over $Z$, then

$$\theta^\ell(\mathscr{F}) = \lambda_{-1}(\mathscr{F} \cdot \alpha)$$

in $\mathrm{K}_0(C_\ell \times G, Z)'/(v) \cdot \mathrm{K}_0(C_\ell \times G, Z)'$.

Combining (5.4.5), (5.4.6), and the above, we obtain

$$\begin{aligned}\vartheta^\ell = L_\bullet(1) = [h((\mathscr{E}^\bullet)^{\otimes \ell})] \cdot \lambda_{-1}(\mathscr{N}_{P|P^\ell})^{-1}_{|Z} \\ = \lambda_{-1}(\mathscr{N}_{Z|P} \cdot \alpha) \cdot \lambda_{-1}(i^*\Omega^1_P \cdot \alpha)^{-1} = \theta^\ell(\mathscr{N}_{Z|P}) \cdot \theta^\ell(i^*\Omega^1_P)^{-1}\end{aligned}$$

and therefore

$$\vartheta^\ell = L_\bullet(1) = \theta^\ell(T^\vee_Z)^{-1}$$

in $\mathrm{G}_0(C_\ell \times G, Z)^\wedge$. This completes the proof. □

### 5.5. *A general Adams–Riemann–Roch formula.*

**5.5.1.** In this section, we assume that $G$ acts tamely on $f : X \to \mathrm{Spec}(\mathbb{Z})$, which is always projective and flat of relative dimension $d$. We also assume that $f$ is a local complete intersection and that $\pi : X \to Y = X/G$ is flat.

**5.5.2.** We now see that our results imply:

**Theorem 5.5.3.** *Let $\mathscr{F}$ be a $G$-equivariant coherent locally free $\mathcal{O}_X$-module. Under the above assumptions on $X$, if $\ell$ is a prime with $(\ell, \#G) = 1$ and $\ell'$ is another prime with $\ell\ell' \equiv 1 \bmod \exp(G)$, we have*

$$\begin{aligned}(\ell-1)\cdot\psi^{\ell}(\bar{\chi}(X,\mathscr{F}))&\\ =\ell(\ell-1)\cdot\psi^{\mathrm{CNT}}_{\ell'}(\bar{\chi}(X,\mathscr{F}))&=(\ell-1)\cdot f_*^{\wedge}(\theta^{\ell}(T_X^{\vee})^{-1}\otimes\psi^{\ell}(\mathscr{F}))\end{aligned}\tag{5.5.4}$$

*in* $\mathrm{Cl}(\mathbb{Z}'[G])^{\wedge}$. *Here* $\theta^{\ell}(T_X^{\vee})^{-1}$ *belongs to* $\mathrm{K}_0(G, X)^{\wedge}$.

*Proof.* In this, we also denote by $\psi^{\ell}$ the action of this operator on the completion $\mathrm{Cl}(\mathbb{Z}'[G])^{\wedge}$ of the quotient $\mathrm{Cl}(\mathbb{Z}'[G])' = \mathrm{K}_0(\mathbb{Z}'[G])'/\langle[\mathbb{Z}'[G]]\rangle$; this action is well-defined by Proposition 4.2.3 and [Köck 1997, Proposition 2.10]. The statement then follows from the Adams–Riemann–Roch identity (5.3.8) and Theorem 5.4.2 by using also $\zeta^{\wedge}(\vartheta^{\ell}) = \zeta^{\wedge}(\theta^{\ell}(T_X^{\vee})^{-1}) = (\ell-1)\cdot\theta^{\ell}(T_X^{\vee})^{-1}$ as in (4.1.7) and Proposition 4.3.2. ☐

**5.5.5.** The result in this paragraph will be used in Section 7. Suppose in addition that $Y \to \mathrm{Spec}(\mathbb{Z})$ is regular (then the condition that $\pi$ is flat is implied by the rest of our assumptions; see, e.g., [Matsumura 1980, (18.H)]). Recall we have classes $T_X^{\vee}$, $\pi^*T_Y^{\vee}$, and $T_{X/Y}^{\vee} := T_X^{\vee} - \pi^*T_Y^{\vee}$ in the Grothendieck group $\mathrm{K}_0(G, X)$. Notice that $\theta^{\ell}(T_Y^{\vee})$ and $\theta^{\ell}(T_Y^{\vee})^{-1}$ are defined in $\mathrm{K}_0(Y)'$ and $\theta^{\ell}(T_X^{\vee})^{-1}$ in $\mathrm{K}_0(G, X)^{\wedge}$ and we have

$$\theta^{\ell}(T_{X/Y}^{\vee})^{-1} = \theta^{\ell}(T_X^{\vee})^{-1}\pi^*(\theta^{\ell}(T_Y^{\vee}))$$

in $\mathrm{K}_0(G, X)^{\wedge}$. Then also

$$\theta^{\ell}(T_X^{\vee})^{-1} = \theta^{\ell}(T_{X/Y}^{\vee})^{-1}\pi^*(\theta^{\ell}(T_Y^{\vee}))^{-1}\tag{5.5.6}$$

in $\mathrm{K}_0(G, X)^{\wedge}$. We can now write

$$\theta^{\ell}(T_Y^{\vee})^{-1} = \ell^{-d} + c_{\ell}, \qquad \theta^{\ell}(T_{X/Y}^{\vee})^{-1} = 1 + r^{\ell}_{X/Y}$$

with $c_{\ell} \in \mathrm{K}_0(Y)' = \mathrm{G}_0(Y)'$ supported on a proper closed subset of $Y$ and $r^{\ell}_{X/Y} \in \mathrm{K}_0(G, X)^{\wedge}$. (Here $1 = [\mathcal{O}_X]$.) From Theorem 5.5.3 and (5.5.6), we obtain

$$(\ell-1)\cdot\psi^{\ell}(\bar{\chi}(X,\mathcal{O}_X)) = (\ell-1)\cdot f_*^{\wedge}((1+r^{\ell}_{X/Y})\cdot(\ell^{-d}+\pi^*c_{\ell})).\tag{5.5.7}$$

This gives the identity

$$\begin{aligned}(\ell-1)(\psi^{\ell}-\ell^{-d})\cdot(\bar{\chi}(X,\mathcal{O}_X))&\\ =\ell^{-d}(\ell-1)f_*^{\wedge}(r^{\ell}_{X/Y})&+(\ell-1)f_*^{\wedge}(\pi^*(c_{\ell})\cdot\theta^{\ell}(T_{X/Y}^{\vee})^{-1})\end{aligned}\tag{5.5.8}$$

in $\mathrm{Cl}(\mathbb{Z}'[G])^{\wedge}$. In this situation, since $X$ and $Y$ are both local complete intersections over $\mathbb{Z}$, the cotangent complexes $L_{X/\mathbb{Z}}$, $L_{Y/\mathbb{Z}}$, and hence $\pi^*L_{Y/\mathbb{Z}}$ are all perfect of $\mathbb{O}_X$-tor amplitude in $[-1, 0]$. (Here $L_{X/\mathbb{Z}}$ and $\pi^* L_{Y/\mathbb{Z}}$ are complexes of $G$-equivariant $\mathbb{O}_X$-modules; see [Illusie 1971] for the definition and properties of the cotangent complex.) If $i : Y \hookrightarrow P$ is an embedding in a smooth scheme as before, then $i$ is a regular immersion and there is a quasi-isomorphism $L_{Y/\mathbb{Z}} \simeq [\mathcal{N}_{Y|P} \to i^*\Omega^1_{P/\mathbb{Z}}]$ and similarly for $L_{X/\mathbb{Z}}$ after choosing a $G$-equivariant embedding. There is a canonical distinguished triangle

$$\pi^* L_{Y/\mathbb{Z}} \to L_{X/\mathbb{Z}} \to L_{X/Y} \to \pi^* L_{Y/\mathbb{Z}}[1].$$

The cotangent complex $L_{X/Y}$ is then also perfect and gives the class $T^\vee_{X/Y}$ in $\mathrm{K}_0(G, X)$. The morphism $\pi^* L_{Y/\mathbb{Z}} \to L_{X/\mathbb{Z}}$ is an isomorphism over the largest open subscheme $U$ of $X$ such that $\pi : U \to V = U/G$ is étale.

## 6. Unramified covers

### 6.1. *The main identity.*

**6.1.1.** Here we suppose in addition that $\pi : X \to Y$ is unramified, i.e., that the cover $\pi$ is étale. In this case, by étale descent, pull-back by $\pi$ gives isomorphisms $\pi^* : \mathrm{G}_0(Y) \xrightarrow{\sim} \mathrm{G}_0(G, X)$ and $\pi^* : \mathrm{G}_0(C_\ell, Y) \xrightarrow{\sim} \mathrm{G}_0(C_\ell \times G, X)$. We can see that, by using such isomorphisms, the map

$$\Delta_* : \mathrm{G}_0(C_\ell, Y') \to \mathrm{G}_0(C_\ell, (Y')^\ell)$$

can be identified with the map $\Delta_*$ of Section 5.2.2. In this case, we can show more directly, using localization for the $C_\ell$-action on $(Y')^\ell$, that $\Delta_*$ above gives an isomorphism after localizing at each maximal ideal $\rho_{(\sigma,(q))}$, for $q \neq \ell$, of $\mathscr{R}(C_\ell) = \mathrm{G}_0(\mathbb{Z}'[C_\ell])'$ that contains the element $v$. Set $\mathscr{R}(C_\ell)^\flat = \mathscr{R}(C_\ell)/(v)$, and use $^\flat$ to denote base change via $\mathscr{R}(C_\ell) \to \mathscr{R}(C_\ell)^\flat$. We see that $\Delta_*^\flat$ is an isomorphism and there is $\eta^\ell \in \mathrm{G}_0(C_\ell, Y')^\flat$ whose pull-back by $\pi$ maps to $\vartheta^\ell \in \mathrm{G}_0(C_\ell \times G, X')^\wedge$. In particular, $\zeta(\eta^\ell)$ makes sense as an element in $\mathrm{G}_0(Y')$ and pulls back to an element of $\mathrm{G}_0(G, X')$ that is equal to $\zeta^\wedge(\vartheta^\ell)$ in $\mathrm{G}_0(G, X')^\wedge$. Given the above, the proof of the identity (5.3.8) now goes through without having to take completions and we obtain

$$(\ell - 1) \cdot \psi^\ell(f^{\mathrm{ct}}_*(\mathbb{O}_X)) = f^{\mathrm{ct}}_*(\pi^* \zeta(\eta^\ell)) \tag{6.1.2}$$

in $\mathrm{K}_0(\mathbb{Z}'[G])'$.

**6.1.3.** We will now show, by Noetherian induction, the following result:

**Theorem 6.1.4.** *Suppose $\pi : X \to Y$ is a $G$-torsor with $Y \to \mathrm{Spec}(\mathbb{Z})$ projective and flat of relative dimension $d$, and let $\mathscr{F}$ be a $G$-equivariant coherent $\mathbb{O}_X$-module.*

*Let* $\ell$ *be a prime such that* $(\ell, \#G) = 1$. *Then*

$$(\ell-1)^{d+1} \cdot \prod_{i=0}^{d} (\psi^{\ell} - \ell^{-i}) \cdot f_*^{\mathrm{ct}}(\mathscr{F}) = 0 \tag{6.1.5}$$

*in* $\mathrm{Cl}(\mathbb{Z}'[G])' = \mathrm{Cl}(\mathbb{Z}'[G])[\ell^{-1}]$.

*Proof.* Set $\Psi(\ell, d) = (\ell-1)^{d+1} \cdot \prod_{i=0}^{d}(\psi^{\ell} - \ell^{-i})$ for the endomorphism of $\mathrm{Cl}(\mathbb{Z}'[G])' = \mathrm{K}_0(\mathbb{Z}'[G])'/\langle[\mathbb{Z}'[G]]\rangle$; this is well-defined by Proposition 4.2.3. We start by recalling that since $\pi : X \to Y$ is a $G$-torsor, by descent, all $G$-equivariant coherent $\mathbb{O}_X$-modules $\mathscr{F}$ are obtained by pulling back along $\pi$, i.e., are of the form $\mathscr{F} \simeq \pi^*\mathscr{G}$ for $\mathscr{G}$ a coherent $\mathbb{O}_Y$-module. By [Chinburg et al. 1997b, Theorem 6.1], the image of the class $f_*^{\mathrm{ct}}(\pi^*\mathscr{G})$ in $\mathrm{Cl}(\mathbb{Z}'[G])_{\rho}$ is trivial for any prime $\rho$ of $\mathscr{R}(G)$ that does not contain the ideal $I_G$ (see the proof of Proposition 4.5 in [Pappas 1998]). Therefore, it is enough to consider (6.1.5) for the image of $f_*^{\mathrm{ct}}(\pi^*\mathscr{G})$ in $\mathrm{Cl}(\mathbb{Z}'[G])^{\wedge}$. We will argue by induction on $d$. The map $\mathscr{G} \mapsto f_*^{\mathrm{ct}}(\pi^*\mathscr{G})$ induces a group homomorphism

$$\chi : \mathrm{G}_0(Y) \to \mathrm{Cl}(\mathbb{Z}'[G])'.$$

Now note that $\mathrm{G}_0(Y)$ is generated by classes of the form $i_*(\mathbb{O}_T)$ where $i : T \hookrightarrow Y$ is an integral closed subscheme of $Y$. Let us consider the $G$-torsor $\pi_{|T} : \pi^{-1}(T) = X \times_Y T \to T$ obtained by restriction, and denote by $h : \pi^{-1}(T) \to \mathrm{Spec}(\mathbb{Z})$ the structure morphism. Observe that, by the definitions, we have $f_*^{\mathrm{ct}} \cdot \pi^* \cdot i_* = h_*^{\mathrm{ct}} \cdot \pi_{|T}^*$. If $T$ is fibral, i.e., if $T \to \mathrm{Spec}(\mathbb{Z})$ factors through $\mathrm{Spec}(\mathbb{F}_p)$ for some prime $p$, then $\chi(i_*(\mathbb{O}_T)) = 0$ by a result of Nakajima [1984]; see [Chinburg et al. 1997a, Theorem 1.3.2]. It remains to deal with $T$ that are integral and flat over $\mathrm{Spec}(\mathbb{Z})$; the above allows us to reduce to the case when $Y$ is integral and $\mathscr{G} = \mathbb{O}_Y$. We first show that, for $Y$ integral, projective, and flat over $\mathrm{Spec}(\mathbb{Z})$ of dimension $d+1$, there is a proper reduced closed subscheme $i : W \hookrightarrow Y$ (therefore of smaller dimension) and a class $c' \in \mathrm{G}_0(C_{\ell}, W')^{\flat}$ such that

$$\eta^{\ell} - \ell^{-d} = i_*(c'). \tag{6.1.6}$$

To see this, take $W$ to be given by the complement $Y - U$ of an open subset $U$ of $Y$ where $(\Omega^1_{Y/\mathbb{Z}})_{|U} \simeq \mathbb{O}_U^d$. Indeed, then since $U$ is smooth, by the easy case of Theorem 5.4.2 [Köck 2000], the element $\eta^{\ell}_{U'}$ for $U'$ is $\theta^{\ell}(\mathbb{O}^d_{U'})^{-1} = \ell^{-d}$. Since $\Delta_*$ and therefore $L_\bullet$ commutes with restriction to open subschemes, we obtain that the restriction of $\eta^{\ell}$ to $U$ is $\ell^{-d}$ and so there is $c'$ as above. Applying $\zeta$ to (6.1.6), we obtain

$$\zeta(\eta^{\ell}) - (\ell-1)\ell^{-d} = i_*(\zeta(c')) \tag{6.1.7}$$

with $\zeta(c') \in \mathrm{G}_0(W')'$. The identity (6.1.2) now implies

$$(\ell-1)\cdot(\psi^\ell-\ell^{-d})(f_*^{\mathrm{ct}}(\mathbb{O}_X)) = f_*^{\mathrm{ct}}(\pi^* i_*(\zeta(c'))) \tag{6.1.8}$$

in $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$. Consider the $G$-torsor $\pi_{|W}:\pi^{-1}(W)=X\times_Y W\to W$ obtained by restriction, and denote by $h:\pi^{-1}(W)\to\mathrm{Spec}(\mathbb{Z})$ the structure morphism. Again, we have $f_*^{\mathrm{ct}}\cdot\pi^*\cdot i_* = h_*^{\mathrm{ct}}\cdot\pi_{|W}^*$. We can extend $\zeta(c')\in\mathrm{G}_0(W')'$ to a class $z\in\mathrm{G}_0(W)'$. Then

$$f_*^{\mathrm{ct}}(\pi^* i_*(\zeta(c'))) = h_*^{\mathrm{ct}}(\pi_{|W}^* z).$$

This identity allows us to reduce to considering the $G$-torsor $\pi_{|W}$. If $d=0$, then $W$ is fibral and $h_*^{\mathrm{ct}}(\pi_{|W}^* z)=0$ by the result of Nakajima (in this case, the normal basis theorem and dévissage is enough). For $d>0$, we can use the induction hypothesis on $h$ and the $G$-torsor $\pi^{-1}(W)\to W$. This implies that $\Psi(\ell,d-1)$ annihilates $h_*^{\mathrm{ct}}(\pi_{|W}^* z)$, and the result follows from (6.1.8) and the above. ☐

**Remark 6.1.9.** Taylor's [1981] proof of the Fröhlich conjecture easily implies that $2\cdot f_*^{\mathrm{ct}}(\mathscr{F})=0$ in $\mathrm{Cl}(\mathbb{Z}[G])$ if $d=0$ [Pappas 1998, Theorem 4.1]. By using this input at the step $d=0$ of the inductive proof above, we can improve Theorem 6.1.4 and obtain the following: if $d\geq 1$ and $\ell$ is odd, then $f_*^{\mathrm{ct}}(\mathscr{F})$ is actually annihilated by $(\ell-1)^{d+1}\cdot\prod_{i=1}^d(\psi^\ell-\ell^{-i})$ in $\mathrm{Cl}(\mathbb{Z}'[G])'$, i.e., we can omit the factor $\psi^\ell-1$ that corresponds to $i=0$.

**6.2.** ***Class groups of p-groups and class field theory.*** In this section, $G$ is a $p$-group of exponent $p^N$ with $p$ an odd prime.

**6.2.1.** By [Roquette 1958], we can write the group algebra $\mathbb{Q}[G]$ as a direct product of matrix rings $\mathbb{Q}[G]=\prod_i \mathrm{Mat}_{n_i\times n_i}(\mathbb{Q}(\zeta_{p^{s_i}}))$ with center $Z=\prod_i K_i$. Here $K_i=\mathbb{Q}(\zeta_{p^{s_i}})$ is the cyclotomic field, $s_i\leq N$. Denote by $\mathbb{O}_i$ the ring of integers of $K_i$.

**6.2.2.** Denote by $\mathrm{Cl}(\mathbb{Z}[G])_p$ and $\mathrm{Cl}(\mathbb{Z}'[G])_p$ the $p$-power torsion parts of $\mathrm{Cl}(\mathbb{Z}[G])$ and $\mathrm{Cl}(\mathbb{Z}'[G])$. We now show:

**Proposition 6.2.3.** *There exists an infinite set of primes $\ell\neq p$ with the following properties*:

(a) $\ell$ mod $p$ *generates* $(\mathbb{Z}/p\mathbb{Z})^*$,

(b) $\ell^{p-1}\equiv 1$ mod $p^N$, *and*

(c) *for* $\mathbb{Z}'=\mathbb{Z}[\ell^{-1}]$ *the restriction* $\mathrm{Cl}(\mathbb{Z}[G])_p\to\mathrm{Cl}(\mathbb{Z}'[G])_p$ *is an isomorphism.*

*Proof.* In what follows, we will write $\mathbb{A}(K)$ and $\mathbb{A}(K)^*$ for the adeles and ideles, respectively, of the number field $K$ and $\mathbb{A}(\mathbb{O}_K)^*=\prod_v \mathbb{O}_{K,v}^*$ for the integral ideles. Using the Fröhlich description of the class group [Taylor 1984], we can write

$$\mathrm{Cl}(\mathbb{Z}[G]) = (\mathbb{A}(Z)^*/Z^*)/\mathcal{U} = \big(\textstyle\prod_i \mathbb{A}(K_i)^*/K_i^*\big)/\mathcal{U} \tag{6.2.4}$$

where $\mathcal{U} = \prod_v \mathrm{Det}(\mathbb{Z}_v[G]) \subset \mathbb{A}(\mathbb{O}_Z)^* = \prod_i \mathbb{A}(\mathbb{O}_i)^*$. The subgroup $\mathcal{U}$ is open of finite index in $\mathbb{A}(\mathbb{O}_Z)^*$. We can find finite index subgroups $\mathcal{U}_i = \prod_v \mathcal{U}_{i,v} \subset \mathbb{A}(\mathbb{O}_i)^*$, with $\mathcal{U}_{i,v} = (\mathbb{O}_i)^*_v$ if $v$ does not divide $p$, such that

$$\textstyle\prod_i \mathcal{U}_i \subset \mathcal{U} \subset \mathbb{A}(\mathbb{O}_Z)^*. \tag{6.2.5}$$

We can also assume $\mathcal{U}_{i,p}$ is stable for the action of the Galois group $\mathrm{Gal}(K_i/\mathbb{Q}) = \mathrm{Gal}(K_{i,p}/\mathbb{Q}_p)$. It follows now from (6.2.4) and (6.2.5) that we can write $\mathrm{Cl}(\mathbb{Z}[G])$ as a quotient

$$\textstyle\prod_i \mathbb{A}(K_i)^*/(K_i^* \cdot \mathcal{U}_i) \twoheadrightarrow \mathrm{Cl}(\mathbb{Z}[G]). \tag{6.2.6}$$

By class field theory, the source can be identified with the product of the Galois groups of ray class field extensions of $K_i$ that are at most ramified at the unique prime over $p$. In fact, this also implies that the $p$-Sylow $\mathrm{Cl}(\mathbb{Z}[G])_p$ can also be written as a quotient

$$\textstyle\prod_i \mathrm{Gal}(L_i/K_i) \twoheadrightarrow \mathrm{Cl}(\mathbb{Z}[G])_p$$

where $L_i/K_i$ is a $p$-power ray class field of $K_i$ that is ramified at most at the unique prime of $K_i$ over $p$ such that $\mathrm{Gal}(L_i/K_i)$ is identified with the $p$-power quotient of $\mathbb{A}(K_i)^*/(K_i^* \cdot \mathcal{U}_i)$. Let us consider $\mathrm{Cl}(\mathbb{Z}'[G])$ for $\ell \neq p$. The discussion above applies again, and as before, we can write

$$\textstyle\prod_i \mathbb{A}^\ell(K_i)^*/(K_i^* \cdot \mathcal{U}_i^\ell) \twoheadrightarrow \mathrm{Cl}(\mathbb{Z}'[G])$$

where the superscript $\ell$, as in $\mathbb{A}^\ell$, means adeles away from $\ell$. (Also, as usual, $\mathbb{A}_\ell$ will denote adeles over $\ell$.) We can easily see that $\mathrm{Cl}(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathbb{Z}'[G])$ and hence $\mathrm{Cl}(\mathbb{Z}[G])_p \to \mathrm{Cl}(\mathbb{Z}'[G])_p$ is surjective.

Now choose $n$ such that $n \geq N \geq \max_i\{s_i\}$ and such that $\mathbb{Q}(\zeta_{p^n})$ contains all the (cyclotomic) $p$-power ray class fields $L_i$, for $i$ with $s_i = 0$, of $K_i = \mathbb{Q}$. Also set $M_n$ to be a $p$-ray class field of $\mathbb{Q}(\zeta_{p^n})$ ramified only above $p$ that contains all the fields $L_i$, for $s_i \geq 1$, and corresponds to a $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$-stable subgroup of $\mathbb{A}(\mathbb{Q}(\zeta_{p^n}))^*$. The extension $M_n/\mathbb{Q}$ is Galois; set $G_n = \mathrm{Gal}(M_n/\mathbb{Q})$. This is an extension

$$1 \to \mathrm{Gal}(M_n/\mathbb{Q}(\zeta_{p^n})) \to G_n \to \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) = \Delta \times \Gamma_n \to 1 \tag{6.2.7}$$

with $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^*$ and $\Gamma_n = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\zeta_p)) \simeq (\mathbb{Z}/p^{n-1}\mathbb{Z})$. We can find an element $\tau \in G_n$ of order $p-1$ as follows: lift the generator of $\Delta$ to an element $\tau_0$; then a suitable power $\tau = \tau_0^{p^m}$ has order $p-1$. By the Chebotarev density theorem, there is an infinite set of primes $\ell$ so that $\mathrm{Frob}_\ell$ lies in the conjugacy class $\langle\tau\rangle$ of $\tau$ in $G_n$. Then $\mathrm{Frob}_\ell$ generates the group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Hence, the ideal $(\ell)$ remains prime in $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and $\ell$ generates $(\mathbb{Z}/p\mathbb{Z})^*$. Write $\mathfrak{L} = (\ell)$ in $\mathbb{Q}(\zeta_p)$, and consider $\mathrm{Frob}_\mathfrak{L}$ in $\mathrm{Gal}(M_n/\mathbb{Q}(\zeta_p))$. We have $\mathrm{Frob}_\mathfrak{L} = \mathrm{Frob}_\ell^{p-1} = \langle\tau^{p-1}\rangle = \langle 1\rangle$, and so $\mathfrak{L}$ splits completely in $M_n$.

By class field theory and the assumption $\mathcal{U}_{i,v} = \mathcal{O}^*_{i,v}$ for $v \neq p$, we see that the above implies that there is an infinite set of primes $\ell \neq p$ that generate $(\mathbb{Z}/p\mathbb{Z})^*$ and satisfy $\ell^{p-1} \equiv 1 \bmod p^N$ and additionally such that, for all $i$, the image of the subgroup

$$\mathbb{A}_\ell(K_i)^* = \prod_{\mathfrak{L}|\ell} K^*_{i,\mathfrak{L}} \subset \mathbb{A}(K_i)^*$$

in $\mathbb{A}(K_i)^*/K_i^* \cdot \mathcal{U}_i$ has order prime to $p$. Let us denote by $Q_i = Q_i(\ell)$ this order and set $Q = \prod_i Q_i$. For such an $\ell$, suppose $c$ is an element in the kernel of $\mathrm{Cl}(\mathbb{Z}[G])_p \to \mathrm{Cl}(\mathbb{Z}'[G])_p$ that is given by an idele $(a_i) \in \prod_i \mathbb{A}(K_i)^*$. Then

$$(a_i^\ell)_i = (\gamma_i)_i \cdot u^\ell$$

with $\gamma_i \in K_i^*$ (diagonally embedded in the prime to $\ell$-ideles) and $u^\ell \in \mathcal{U}^\ell$. Here $(a_i)_i = (a_i^\ell)_i \cdot (a_{i,\ell})_i$ with $(a_{i,\ell})$ the $\ell$-component of $(a_i)$ (considered as an idele with 1 at all places away from $\ell$). The product idele $(b_i)_i = (\gamma_i^{-1} \cdot a_i)_i$ also produces the class $c$ in $\mathrm{Cl}(\mathbb{Z}[G])_p$. We can write $(b_i)_i = (b_i^\ell)_i \cdot (b_{i,\ell})_i$. The component $(b_{i,v})_i$ at a place $v$ away from $\ell$ is equal to the corresponding component of $u^\ell$, and so it is in $\mathcal{U}_v$. Using our assumption on $\ell$, we can write

$$(b^Q_{i,\ell}) = (\delta_i \cdot u_i)$$

with $u_i \in \mathcal{U}_i$ and $\delta_i \in K_i^* \subset \mathbb{A}(K_i)^*$ (embedded diagonally). Combining these gives

$$(b_i)_i^Q = (\delta_i \cdot u_i)_i \cdot (b_i^\ell)_i^Q,$$

which is in $\left(\prod_i K_i^*\right) \cdot \mathcal{U}$. Therefore, $Q \cdot c$ is trivial in $\mathrm{Cl}(\mathbb{Z}[G])$; hence, $c$ is trivial. □

### 6.3. *Adams eigenspaces and the proof of the main result.*

**6.3.1.** Recall that $G$ is a $p$-group of exponent $p^N$ for an odd prime $p$. The group $(\mathbb{Z}/p^N\mathbb{Z})^* = (\mathbb{Z}/p\mathbb{Z})^* \times \mathbb{Z}/p^{N-1}\mathbb{Z}$ acts on the $p$-power torsion $\mathrm{Cl}(\mathbb{Z}[G])_p$ via the Cassou-Noguès–Taylor Adams operations: indeed, these operators are periodic, and this is essential for our argument. The element $a \in (\mathbb{Z}/p^N\mathbb{Z})^*$ acts via $\psi_a^{\mathrm{CNT}}$; we will simply denote this by $\psi_a$ in what follows. We will restrict this action to the subgroup $(\mathbb{Z}/p\mathbb{Z})^*$. This gives a decomposition into eigenspaces

$$\mathrm{Cl}(\mathbb{Z}[G])_p = \bigoplus_{\chi:(\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{Z}_p^*} \mathrm{Cl}(\mathbb{Z}[G])_p^\chi = \bigoplus_{i=0}^{p-2} \mathrm{Cl}(\mathbb{Z}[G])_p^{(i)}$$

where $\mathrm{Cl}(\mathbb{Z}[G])_p^{(i)} = \{c \in \mathrm{Cl}(\mathbb{Z}[G])_p \mid \psi_a(c) = \omega(a)^i \cdot c \text{ for all } a \in (\mathbb{Z}/p\mathbb{Z})^*\}$. Here $\omega : (\mathbb{Z}/p\mathbb{Z})^* \to \mathbb{Z}_p^*$ is the Teichmüller character. There is a similar result for $\mathrm{Cl}(\mathbb{Z}'[G])_p$ after we choose a prime $\ell \neq p$.

**Theorem 6.3.2.** *Suppose $\pi : X \to Y$ is a $G$-torsor with $Y \to \operatorname{Spec}(\mathbb{Z})$ projective and flat of relative dimension $d$, and assume that $G$ is a $p$-group for an odd prime $p$. Let $\mathscr{F}$ be a $G$-equivariant coherent $\mathbb{O}_X$-module. Then the class $\bar{\chi}(X, \mathscr{F}) \in \mathrm{Cl}(\mathbb{Z}[G])$ is $p$-power torsion. If $p > d$, the class lies in $\bigoplus_{i=2}^{d+1} \mathrm{Cl}(\mathbb{Z}[G])_p^{(i)}$. In particular, if $d = 1$, then $\bar{\chi}(X, \mathscr{F})$ lies in the eigenspace $\mathrm{Cl}(\mathbb{Z}[G])_p^{(2)}$.*

*Proof.* The fact that $\bar{\chi}(X, \mathscr{F})$ is $p$-power torsion follows from the localization theorem of [Chinburg et al. 1997b] as in [Pappas 1998, Proposition 4.5]. Choose an odd prime $\ell$ as in Proposition 6.2.3; in particular, $\ell - 1$ is prime to $p$. Propositions 4.2.3 and 4.3.2 together imply that, for $x \in \mathrm{Cl}(\mathbb{Z}'[G])_p$, we have $\psi^\ell(x) = \ell \cdot \psi_{\ell'}(x)$ for $\ell'\ell \equiv 1 \bmod p^N$. From Theorem 6.1.4 and Remark 6.1.9, we then obtain that $f_*^{\mathrm{ct}}(\mathscr{F})$ lies in $\bigoplus_{i=2}^{d+1} \mathrm{Cl}(\mathbb{Z}'[G])_p^{(i)}$. However, for our choice of $\ell$, the restriction $\mathrm{Cl}(\mathbb{Z}[G])_p \to \mathrm{Cl}(\mathbb{Z}'[G])_p$ is an isomorphism, and this gives the result. □

**6.3.3.** We continue to assume that $G$ is a $p$-group, $p$ an odd prime. Let $\#G = p^n$. We show:

**Proposition 6.3.4.** *If $p \geq 5$, then $\mathrm{Cl}(\mathbb{Z}[G])_p^{(2)} = (0)$. If $p = 3$, then $\mathrm{Cl}(\mathbb{Z}[G])_p^{(2)} = \mathrm{Cl}(\mathbb{Z}[G])_p^{(0)}$ is annihilated by the Artin exponent $A(G) = p^{n-1}$ of $G$.*

*Proof.* As above, we can write

$$\mathrm{Cl}(\mathbb{Z}[G]) = \prod_i \mathbb{A}(K_i)^* / \left(\prod_i K_i^*\right) \cdot \mathcal{U},$$

where $K_i = \mathbb{Q}(\zeta_{p^{s_i}})$ and $\mathcal{U}$ is an open subgroup of the product $\prod_i \mathbb{A}(\mathbb{O}_i)^*$, which is maximal at all $v \neq p$. For $b \in (\mathbb{Z}/p\mathbb{Z})^*$, denote by $\sigma_b$ the Galois automorphism of $\mathbb{Q}(\zeta_{p^\infty}) = \bigcup_m \mathbb{Q}(\zeta_{p^m})$ given by $\sigma(\zeta) = \zeta^{\omega(b)}$. We can see [Cassou-Noguès and Taylor 1985] that the operator $\psi_a$, for $a \in (\mathbb{Z}/p\mathbb{Z})^* \subset (\mathbb{Z}/p^N\mathbb{Z})^*$, is induced by the action of $\sigma_a$ on the product $\prod_i \mathbb{A}(K_i)^*$. Let $\mathcal{M}_G \simeq \bigoplus_i \mathrm{Mat}_{n_i \times n_i}(\mathbb{Z}[\zeta_{p^{s_i}}])$ be a maximal $\mathbb{Z}[G]$-order in $\mathbb{Q}[G]$. Denote by $D(\mathbb{Z}[G])$ the kernel of the natural group homomorphism $\mathrm{Cl}(\mathbb{Z}[G]) \to \mathrm{Cl}(\mathcal{M}_G) = \prod_i \mathrm{Cl}(\mathbb{Q}(\zeta_{p^{s_i}}))$. The kernel group $D(\mathbb{Z}[G])$ has $p$-power order [Taylor 1984, p. 37]. Since $\mathcal{U}$ is maximal at $v \neq p$, we can write

$$D(\mathbb{Z}[G]) = \frac{\prod_i \mathbb{Z}_p[\zeta_{p^{s_i}}]^*}{\left(\prod_i \mathbb{Z}[\zeta_{p^{s_i}}]^*\right) \cdot \mathcal{U}_p}.$$

For $x \in \mathrm{Cl}(\mathbb{Z}[G])_p^{(2)}$, let $(x_i)_i$ be the image of $x$ in the class group $\mathrm{Cl}(\mathcal{M}_G)$. Then $x_i$ is a $p$-power torsion element in $\mathrm{Cl}(\mathbb{Q}(\zeta_{p^{s_i}}))$ that satisfies $\sigma_a(x_i) = \omega(a)^2 \cdot x_i$ for all $a \in (\mathbb{Z}/p\mathbb{Z})^*$. The second eigenspace of the $p$-part of the class group of $\mathbb{Q}(\zeta_{p^m})$ is trivial. (Combine $B_2 = \frac{1}{6}$ with Herbrand's theorem and the "reflection theorems"; see [Washington 1997, Theorems 6.17 and 10.9] to see this for $m = 1$; the result then follows.) It follows that $(x_i)_i = 0$, and so $x$ is in $D(\mathbb{Z}[G])$. Such an $x$ is then represented by $(u_i)_i$ with $u_i \in (\mathbb{Z}_p[\zeta_{p^{s_i}}]^*)^{(2)}$. For $m \geq 0$, consider the pro-$p$-Sylow subgroup $(\mathbb{Z}_p[\zeta_{p^m}]^*)_p$ of $\mathbb{Z}_p[\zeta_{p^m}]^*$. Denote by $(\widehat{\mathbb{Z}[\zeta_{p^m}]^*})_p$ the intersection $\widehat{\mathbb{Z}[\zeta_{p^m}]}^* \cap (\mathbb{Z}_p[\zeta_{p^m}]^*)_p$ of the $p$-adic closure of the global units $\mathbb{Z}[\zeta_{p^m}]^*$ in $\mathbb{Z}_p[\zeta_{p^m}]^*$

with $(\mathbb{Z}_p[\zeta_{p^m}]^*)_p$. If $p \geq 5$, then since the second Bernoulli number $B_2 = \frac{1}{6}$ is not divisible by $p$, we have

$$(\mathbb{Z}_p[\zeta_{p^m}]^*)_p^{(2)} = (\widehat{\mathbb{Z}[\zeta_{p^m}]^*})_p^{(2)}$$

by a classical result of Iwasawa (see for example [Washington 1997, Theorem 13.56; Oliver 1983, p. 296]). This shows that $x$ is trivial in $D(\mathbb{Z}[G])$. If $p = 3$, then since 3 is regular, we have as above $\mathrm{Cl}(\mathbb{Z}[G])_p^{(2)} = \mathrm{Cl}(\mathbb{Z}[G])_p^{(0)} = D(\mathbb{Z}[G])_p^{(0)}$. This group is annihilated by $A(G)$ by [Oliver 1983, Theorem 9]; in this case, $A(G) = p^{n-1} = 3^{n-1}$ by [Lam 1968]. □

**6.3.5.** We can now show Theorems 1.0.2 and 1.0.3 of the introduction. For this, we allow $G$ to stand for an arbitrary finite group.

*Proof of Theorem 1.0.2.* Using Noetherian induction and the 0-dimensional result of Taylor exactly as in [Pappas 1998, Proposition 4.4], we see that

$$\gcd(2, \#G) \cdot \#G \cdot f_*^{\mathrm{ct}}(\mathscr{F}) = 0. \tag{6.3.6}$$

Using localization as in [Pappas 1998, Proposition 4.5], we see that the $p$-power torsion part of the class $f_*^{\mathrm{ct}}(\mathscr{F})$ is annihilated by any power of $p$ that annihilates its restriction $\mathrm{Res}^G_{G_p}(f_*^{\mathrm{ct}}(\pi^*\mathscr{G}))$ in the class group $\mathrm{Cl}(\mathbb{Z}[G_p])$ of a $p$-Sylow $G_p$. By definition, this restriction is the Euler characteristic class for the $G_p$-cover $X \to X/G_p$. If $G$ is a $p$-group of order $p \geq 5$, we have $f_*^{\mathrm{ct}}(\mathscr{F}) = 0$ in $\mathrm{Cl}(\mathbb{Z}[G])$ by Theorem 6.3.2 and Proposition 6.3.4. We can apply this to a $p$-Sylow $G_p$ of $G$ and the $G_p$-torsor $X \to X/G_p$. By the above, we obtain that the prime to 6 part of $f_*^{\mathrm{ct}}(\mathscr{F})$ is trivial. By (6.3.6), the 2-part is always annihilated by $\gcd(2, \#G)^{v_2(\#G)+1}$. When the 2-Sylow $G_2$ of $G$ is abelian, [Pappas 1998, Theorem 1.1], applied to the cover $X \to X/G_2$, shows that the restriction of $f_*^{\mathrm{ct}}(\mathscr{F})$ to $\mathrm{Cl}(\mathbb{Z}[G_2])$ is 2-torsion. In general, by [Pappas 1998, Theorem 1.1], the restriction of $f_*^{\mathrm{ct}}(\mathscr{F})$ to $\mathrm{Cl}(\mathbb{Z}[G_2])$ lies in the kernel subgroup $D(\mathbb{Z}[G_2])$. The kernel subgroup $D(\mathbb{Z}[G_2])$ is trivial when the 2-group $G_2$ has order $\leq 4$, is cyclic of order 8, or is dihedral; it has order 2 when $G_2$ is generalized quaternion or semidihedral [Curtis and Reiner 1987, p. 129, line 30; Taylor 1984, Theorem 2.1, p. 79]. Also, by Theorem 6.3.2 and Proposition 6.3.4, when $p = 3$, the restriction of $f_*^{\mathrm{ct}}(\mathscr{F})$ to $\mathrm{Cl}(\mathbb{Z}[G_3])$ is annihilated by the Artin exponent $A(G_3)$. If $G_3$ is abelian, this restriction is trivial, again by [Pappas 1998, Theorem 1.1]. Theorem 1.0.2 now follows. □

*Proof of Theorem 1.0.3.* Let us note that, in this situation, we have $\mathrm{H}^0(X_{\mathbb{Q}}, \mathbb{O}_{X_{\mathbb{Q}}}) = \mathrm{H}^0(Y_{\mathbb{Q}}, \mathbb{O}_{Y_{\mathbb{Q}}})$, and since the $G$-cover $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is unramified, the Hurwitz formula gives $g_X - h = \#G \cdot (g_Y - h)$. The result then follows from Theorem 1.0.2 exactly as in the proof of [Pappas 1998, Theorem 5.5] provided we show that $\mathrm{H}^1(X, \omega_X) \simeq \mathbb{Z}^{\oplus h}$. Since $G$ acts trivially on $\mathrm{H}^0(X_{\mathbb{Q}}, \mathbb{O}_{X_{\mathbb{Q}}})$, we have $\mathrm{H}^0(X, \mathbb{O}_X) \simeq \mathbb{Z}^{\oplus h}$ with trivial $G$-action. Under the rest of our assumptions, duality implies that there is a $G$-equivariant isomorphism $\mathrm{H}^1(X, \omega_X) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}^0(X, \mathbb{O}_X), \mathbb{Z})$, and the result then follows. ☐

## 7. Tamely ramified covers of curves

**7.1. *Curves over* $\mathbb{Z}$.** We assume that $G$ acts tamely on $f : X \to \mathrm{Spec}(\mathbb{Z})$, which is projective, flat, and a local complete intersection of relative dimension 1. We also assume that $Y = X/G$ is irreducible and regular; then $\pi : X \to Y$ is finite and flat. Let $U$ be the largest open subscheme of $X$ such that $\pi : U \to V = U/G$ is étale. The complements $R(X/Y) = X - U$ and $B(X/Y) = Y - V$ are respectively the ramification and branch loci of $\pi$. The ramification locus is the closed subset of $X$ defined by the annihilator $\mathrm{Ann}(\Omega^1_{X/Y})$. Our assumption of tameness implies that both the ramification and branch loci are fibral, i.e., are subsets of the union of fibers of $X \to \mathrm{Spec}(\mathbb{Z})$ and $Y \to \mathrm{Spec}(\mathbb{Z})$, respectively, over a finite set $S$ of primes $(p)$ [Chinburg et al. 1997a, §1.2]. Fix a prime $\ell \neq 2$ that does not divide $\#G$.

**7.1.1.** Denote by $F_i\mathrm{G}_0(Y)$ the subgroup of elements of $\mathrm{K}_0(Y) = \mathrm{G}_0(Y)$ represented as linear combinations of coherent sheaves supported on subschemes of $Y$ of dimension $\leq i$. Consider the homomorphisms

$$\begin{aligned} \bar{\chi} : F_1\mathrm{G}_0(Y) \to \mathrm{Cl}(\mathbb{Z}[G]), \qquad & \bar{\chi}(c) \;= f_*^{\mathrm{ct}}(\pi^*(c)), \\ \mathrm{cl}_{X/Y} : F_1\mathrm{G}_0(Y) \to \mathrm{Cl}(\mathbb{Z}'[G])^\wedge, \quad & \mathrm{cl}_{X/Y}(c) := f_*^\wedge(\pi^*(c) \cdot \theta^\ell(T^\vee_{X/Y})^{-1}), \end{aligned}$$

where $\theta^\ell(T^\vee_{X/Y})^{-1}$ is as in Section 5.5.5. Note that a value of $\mathrm{cl}_{X/Y}$ appears in the right-hand side of (5.5.8).

**Proposition 7.1.2.** *Under the above assumptions,*

(1) *the image of* $\bar{\chi}$ *is* $\gcd(2, \#G)$*-torsion and*

(2) *the image of* $\mathrm{cl}_{X/Y}$ *is* $(\ell - 1)$*-torsion.*

*Proof.* We note that, under our assumptions, we have isomorphisms $\mathrm{Pic}(Y) = \mathrm{CH}_1(Y) \xrightarrow{\sim} F_1\mathrm{G}_0(Y)/F_0\mathrm{G}_0(Y)$ and $\mathrm{CH}_0(Y) \xrightarrow{\sim} F_0\mathrm{G}_0(Y)$. (This follows from "Riemann–Roch without denominators" as in [Soulé 1985]; see also [Fulton 1998, Example 15.3.6]). Here $\mathrm{CH}_i(Y)$ is the Chow group of dimension-$i$ cycles modulo rational equivalence on $Y$, and both maps are given by sending the class $[V]$ of a dimension-1 or -0, respectively, integral subscheme $V$ of $Y$ to $i_*([\mathbb{O}_V])$ where $i : V \to Y$ is the corresponding morphism.

Now observe:

(a) Both $\mathrm{cl}_{X/Y}$ and $\bar{\chi}$ are trivial on $F_0\mathrm{G}_0(Y)$. By 2-dimensional class field theory [Kato and Saito 1983, Theorem 2], $\mathrm{CH}_0(Y)$ is a finite abelian group and there is a reciprocity isomorphism $\mathrm{CH}_0(Y) \xrightarrow{\sim} \tilde{\pi}_1^{\mathrm{ab}}(Y)$, where $\tilde{\pi}_1^{\mathrm{ab}}(Y)$ classifies unramified abelian covers of $Y$ that split completely over all real-valued points of $Y$. Suppose $Y' \to Y$ is an irreducible unramified abelian Galois cover. A standard argument using the classical description of unramified abelian covers of curves via isogenies of their Jacobians (or, alternatively, smooth base change for étale cohomology) shows that there is an infinite set of primes $q$ such that the base change $Y'_{\mathbb{F}_q} \to Y_{\mathbb{F}_q}$ is nonsplit, i.e., such that $Y'_{\mathbb{F}_q}$ is irreducible. By applying this to the universal cover $Y^{\mathrm{uni}} \to Y$ with Galois group $\mathrm{CH}_0(Y)$, we obtain that there is a prime $q$ not in $\{\ell\} \cup S$ such that $Y_{\mathbb{F}_q}$ is smooth and with the property that $Y^{\mathrm{uni}}_{\mathbb{F}_q}$ is irreducible. This implies that the Frobenius elements of the closed points of the smooth projective curve $Y_{\mathbb{F}_q}$ generate the Galois group or, in other words, that the group $\mathrm{CH}_0(Y)$ is generated by the classes of points that are supported on $Y_{\mathbb{F}_q} \subset Y$. Now $\theta^\ell(T^\vee_{X/Y})^{-1}_{|U} = 1$, and since $X_{\mathbb{F}_q} \subset U$, if $c$ corresponds to a point on $Y_{\mathbb{F}_q}$, we obtain $\mathrm{cl}_{X/Y}(c) = f^{\mathrm{ct}}_*(\pi^*(c)) = 0$ by the normal basis theorem for the $G$-Galois algebra that corresponds to $\pi^{-1}(c)$. (See also [Chinburg et al. 1997a, Theorem 1.3.2]).

(b) By (a) above, $\mathrm{cl}_{X/Y}$ and $\overline{\chi}$ both factor through $F_1\mathrm{G}_0(Y)/F_0\mathrm{G}_0(Y) \simeq \mathrm{Pic}(Y)$. Suppose now that $\delta \in \mathrm{Pic}(Y)$. By [Chinburg et al. 1997a, Proposition 9.1.3] (the assumption that the special fibers are divisors with normal crossings is not needed for this), there is a "harmless" base extension given by a number field $N/\mathbb{Q}$, unramified at all primes over $S$, of degree $[N:\mathbb{Q}]$ a power of a prime number $\neq \ell$, and $[N:\mathbb{Q}] \equiv 1 \bmod \#\mathrm{Cl}(\mathbb{Z}[G])$ such that the following is true: we can write the base change $\delta_{\mathbb{O}_N} \in \mathrm{Pic}(Y_{\mathbb{O}_N})$ as a sum $\delta_{\mathbb{O}_N} = \sum_i m_i[D_i]$ with $m_i = \pm 1$, where $D_i$ are horizontal divisors in $Y_{\mathbb{O}_N}$, which at most intersect each irreducible component of $(Y_{\mathbb{O}_N})_p$, $p \in S$, transversely at closed points that are away from the singular locus of the reduced special fiber $(Y_{\mathbb{O}_N})^{\mathrm{red}}_p$. Denote by $\iota_i : D_i \hookrightarrow Y_{\mathbb{O}_N}$ the closed immersion and by $\mathfrak{D}_i$ the normalization of $D_i$. Then, we can see as in [loc. cit.] that, for each $i$, the morphism $\widetilde{\mathfrak{D}}_i = \pi^{-1}(\mathfrak{D}_i) = \mathfrak{D}_i \times_Y X \to \mathfrak{D}_i$ is a tame $G$-cover of regular affine schemes of dimension 1 flat over $\mathbb{Z}$, which is unramified away from $S$. The normalization morphism $q_i : \mathfrak{D}_i \to D_i$ is an isomorphism over an open subset of $D_i$ that contains all primes over $S$. As in [loc. cit.], we can now see that our conditions on the field $N$ together with $\delta_{\mathbb{O}_N} = \sum_i m_i[D_i]$ imply that

$$\overline{\chi}(\delta) = \textstyle\sum_i m_i \cdot \overline{\chi}(X_{\mathbb{O}_N}, (\iota_i)_*\mathbb{O}_{D_i}) = \sum_i m_i \cdot [\Gamma(\widetilde{\mathfrak{D}}_i, \mathbb{O}_{\widetilde{\mathfrak{D}}_i})]$$

in $\mathrm{Cl}(\mathbb{Z}[G])$. Observe that the classes $[\Gamma(\widetilde{\mathfrak{D}}_i, \mathbb{O}_{\widetilde{\mathfrak{D}}_i})]$ are $\gcd(2, \#G)$-torsion by Taylor's theorem and part (1) follows.

The proof of part (2) is similar. For simplicity, write $D = D_i$, $\mathfrak{D} = \mathfrak{D}_i$, $\widetilde{\mathfrak{D}} = \widetilde{\mathfrak{D}}_i$, and $\iota : D \hookrightarrow Y_{\mathbb{O}_N}$ and denote by $h : \widetilde{\mathfrak{D}} \to \mathrm{Spec}(\mathbb{Z})$ the structure morphism. By Taylor's theorem, $\gcd(2, \#G) \cdot h^{\mathrm{ct}}_*(\mathbb{O}_{\widetilde{\mathfrak{D}}}) = 0$ in $\mathrm{Cl}(\mathbb{Z}'[G])$. Therefore, since $\ell - 1$ is even, by applying (5.5.8) (for $d = 0$) to the cover $X = \widetilde{\mathfrak{D}} \to Y = \mathfrak{D}$, we obtain that

$$(\ell - 1) \cdot \left[h^\wedge_*(\theta^\ell(T^\vee_{\widetilde{\mathfrak{D}}/\mathfrak{D}})^{-1}) + h^\wedge_*(\pi^*(c') \cdot \theta^\ell(T^\vee_{\widetilde{\mathfrak{D}}/\mathfrak{D}})^{-1})\right] = 0$$

in $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ where $c' \in \mathrm{K}_0(\mathfrak{D})' = \mathrm{G}_0(\mathfrak{D})'$ is supported on a proper closed subset of $\mathfrak{D}$. We can assume that $c'$ is supported away from primes in $S$. Then, as in (a) above, we obtain that the second term in the above sum vanishes. Therefore,

$$(\ell - 1) \cdot (h_*^\wedge(\theta^\ell(T^\vee_{\widetilde{\mathfrak{D}}/\mathfrak{D}})^{-1})) = 0$$

also. Observe that $q : \mathfrak{D} \to D$ is an isomorphism over an open subset of $\mathfrak{D}$ whose complement has image in $X$ disjoint from the support of $T^\vee_{X/Y}$. Also the formation of cotangent complexes of flat morphisms commutes with base change [Illusie 1971]: we can see that the base change of $\theta^\ell(T^\vee_{X/Y})^{-1} \in \mathrm{K}_0(G, X)^\wedge$ to $\widetilde{\mathfrak{D}}$ is equal to $\theta^\ell(T^\vee_{\widetilde{\mathfrak{D}}/\mathfrak{D}})^{-1} \in \mathrm{K}_0(G, \widetilde{\mathfrak{D}})^\wedge$. Using these two facts, and the projection formula, we now obtain

$$\begin{aligned}(\ell - 1) \cdot \mathrm{cl}_{X_{\mathbb{O}_N}/Y_{\mathbb{O}_N}}(\iota_*[\mathbb{O}_D]) &= (\ell - 1) \cdot f_*^\wedge(\pi^*(\iota_*[\mathbb{O}_D]) \cdot \theta^\ell(T^\vee_{X_{\mathbb{O}_N}/Y_{\mathbb{O}_N}})^{-1})\\ &= (\ell - 1) \cdot f_*^\wedge(\pi^*(\iota_*[q_*\mathbb{O}_{\mathfrak{D}}]) \cdot \theta^\ell(T^\vee_{X_{\mathbb{O}_N}/Y_{\mathbb{O}_N}})^{-1})\\ &= (\ell - 1) \cdot h_*^\wedge(\theta^\ell(T^\vee_{\widetilde{\mathfrak{D}}/\mathfrak{D}})^{-1}) = 0.\end{aligned}$$

Here, for simplicity, we also write $\pi$ for the cover $X_{\mathbb{O}_N} \to Y_{\mathbb{O}_N}$ and denote by $f$ the structure morphism $X_{\mathbb{O}_N} \to \mathrm{Spec}(\mathbb{Z})$. As above, we can now see that we have

$$\mathrm{cl}_{X/Y}(\delta) = \textstyle\sum_i m_i \cdot \mathrm{cl}_{X_{\mathbb{O}_N}/Y_{\mathbb{O}_N}}((\iota_i)_*[\mathbb{O}_{D_i}]),$$

and this, together with the above, concludes the proof of part (2). □

**Corollary 7.1.3.** *Under the above assumptions, if $\mathscr{G}$ is a locally free coherent $\mathbb{O}_Y$-module of rank $r$, then*

$$\gcd(2, \#G) \cdot (\bar{\chi}(X, \pi^*\mathscr{G}) - r \cdot \bar{\chi}(X, \mathbb{O}_X)) = 0$$

*in* $\mathrm{Cl}(\mathbb{Z}[G])$.

*Proof.* This follows from Proposition 7.1.2(1) since $[\mathscr{G}] - r \cdot [\mathbb{O}_Y] \in F_1\mathrm{G}_0(Y)$. □

**7.2.** ***The input localization theorem.*** Here we let $S$ be the smallest finite set of rational primes that contains the support of the branch locus of $\pi : X \to Y$. For simplicity, let us set $X_S = \bigcup_{p \in S} X_{\mathbb{F}_p}$ and $\widehat{X}_S = \bigcup_{p \in S} X_{\mathbb{Z}_p}$.

**Theorem 7.2.1.** *Let $\pi : X \to Y$ be a tamely ramified $G$-cover of schemes that are projective and flat over* $\mathrm{Spec}(\mathbb{Z})$ *of relative dimension* 1. *Suppose that $Y$ is regular and that $X$ is a local complete intersection. Let $\mathscr{F}$ be a $G$-equivariant coherent $\mathbb{O}_X$-module. Then*

$$\gcd(2, \#G)^{v_2(\#G)+2} \gcd(3, \#G)^{v_3(\#G)-1} \cdot \bar{\chi}(X, \mathscr{F})$$

*in* $\mathrm{Cl}(\mathbb{Z}[G])$ *depends only on the pair $(\widehat{X}_S, \mathscr{F}|_{\widehat{X}_S})$ where $\mathscr{F}|_{\widehat{X}_S}$ denotes the pull-back of $\mathscr{F}$ from $X$ to $\widehat{X}_S$.*

*Proof.* We consider the projections $\overline{\chi}(X, \mathscr{F})_\rho$ on the localizations $\mathrm{Cl}(\mathbb{Z}[G])_\rho$ of the finite $\mathrm{G}_0(\mathbb{Z}[G])$-module $\mathrm{Cl}(\mathbb{Z}[G])$ at the maximal ideals $\rho \subset \mathrm{G}_0(\mathbb{Z}[G])$. Recall

$$\mathrm{Cl}(\mathbb{Z}[G]) = \bigoplus\nolimits_\rho \mathrm{Cl}(\mathbb{Z}[G])_\rho.$$

Consider $\rho$ that do not contain the kernel $I_G$ of the rank map. The projection $\overline{\chi}(X, \mathscr{F})_\rho$ depends only on the inverse image $(\iota_*)_\rho^{-1}([\mathscr{F}])$ of the class of $\mathscr{F}$ under the isomorphism [Chinburg et al. 1997b, Theorem 6.1]

$$(\iota_*)_\rho : \mathrm{G}_0(G, X^\rho)_\rho \xrightarrow{\sim} \mathrm{G}_0(G, X)_\rho \tag{7.2.2}$$

where $\iota : X^\rho \subset X$. For such $\rho$, the fixed-point subscheme $X^\rho$ is contained in the ramification locus $R = R(X/Y) \subset X_S$ and there is a similar isomorphism

$$(\hat{\iota}_*)_\rho : \mathrm{G}_0(G, X^\rho)_\rho \xrightarrow{\sim} \mathrm{G}_0(G, \widehat{X}_S)_\rho \tag{7.2.3}$$

with the property that $(\iota_*)_\rho$ and $(\hat{\iota}_*)_\rho$ commute with the base change homomorphism $\mathrm{G}_0(G, X) \to \mathrm{G}_0(G, \widehat{X}_S)$, $\mathscr{F} \mapsto \mathscr{F}|_{\widehat{X}_S}$. This shows that $(\iota_*)_\rho^{-1}([\mathscr{F}])$ and therefore also $\overline{\chi}(X, \mathscr{F})_\rho$, for $I_G \not\subset \rho$, only depend on $(\widehat{X}_S, \mathscr{F}|_{\widehat{X}_S})$. It remains to deal with maximal $\rho$ such that $I_G \subset \rho$. These are of the form $\rho = \rho_{(1,p)}$ for some prime $p$ that divides $\#G$. The argument in the proof of [Pappas 1998, Proposition 4.5] shows that, for such $\rho$, the component $\overline{\chi}(X, \mathscr{F})_\rho$ depends only on the $p$-power part $\overline{\chi}(X, \mathscr{F})_p$ of the restriction of $\overline{\chi}(X, \mathscr{F})$ to the $p$-Sylow $G_p$. To avoid a conflict in the notation, we will use in this proof the symbol $q$ to denote a prime in the set $S$.

Note that, under our assumptions, $\pi : X \to Y$ is finite and flat. If $\mathscr{G} = (\pi_*(\mathscr{F}))^G$, there is a canonical short exact sequence of $G$-equivariant coherent $\mathbb{O}_X$-modules

$$0 \to \pi^*\mathscr{G} \to \mathscr{F} \to \mathscr{Y} \to 0$$

with $\mathscr{Y}$ supported on the ramification locus $R(X/Y)$. In fact, $\mathscr{Y}$ is canonically isomorphic to the cokernel of $\pi^*\mathscr{G}|_{\widehat{X}_S} \to \mathscr{F}|_{\widehat{X}_S}$ and $\mathscr{G}|_{\widehat{X}_S} = (\pi_*(\mathscr{F}|_{\widehat{X}_S}))^G$. This shows that $\mathscr{Y}$ is determined from $\mathscr{F}|_{\widehat{X}_S}$. Therefore, it is enough to show the statement for sheaves of the form $\mathscr{F} = \pi^*\mathscr{G}$. In view of Corollary 7.1.3, we first consider the case $\mathscr{F} = \mathbb{O}_X$. Notice that $\pi_*\mathbb{O}_X$ is $\mathbb{O}_Y$-locally free of rank $\#G$ on $Y$ and hence, again by Corollary 7.1.3, the difference

$$\overline{\chi}(X, \pi^*(\pi_*(\mathbb{O}_X))) - \#G \cdot \overline{\chi}(X, \mathbb{O}_X)$$

is $\gcd(2, \#G)$-torsion. On the other hand, the $G$-action morphism, $m : X \times G \to X \times_Y X$, $(x, g) \mapsto (x, x \cdot g)$, restricts to an isomorphism over $U$. This gives a $G$-equivariant homomorphism

$$\pi^*(\pi_*(\mathbb{O}_X)) = \pi_*(\mathbb{O}_X) \otimes_{\mathbb{O}_Y} \mathbb{O}_X \xrightarrow{m^*} \bigoplus_{g \in G} \mathbb{O}_X = \mathrm{Maps}(G, \mathbb{O}_X)$$

of $G$-equivariant coherent $\mathbb{O}_X$-modules that is injective. The cokernel of $m^*$ is supported on $R(X/Y)$ and, as above, is determined from $\widehat{X}_S$. Since we have $\overline{\chi}(X, \mathrm{Maps}(G, \mathbb{O}_X)) = 0$, this shows that $\gcd(2, \#G) \cdot (\#G) \cdot \overline{\chi}(X, \mathbb{O}_X)$ depends only on $\widehat{X}_S$. Therefore, $\gcd(2, \#G)^{v_2(\#G)+1} \cdot \overline{\chi}(X, \mathbb{O}_X)_2$ depends only on $\widehat{X}_S$. By Corollary 7.1.3 and the above, we see that $\gcd(2, \#G)^{v_2(\#G)+2} \cdot \overline{\chi}(X, \mathscr{F})_2$ depends only on $(\widehat{X}_S, \mathscr{F}|_{\widehat{X}_S})$.

We now deal with the $p$-power part $\overline{\chi}(X, \mathbb{O}_X)_p$ for $p$ odd. By the above, it is enough to consider the case that $G$ is a $p$-group. We claim that, in this case, the $I_G$-adic completion $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ is the $p$-power part $\mathrm{Cl}(\mathbb{Z}'[G])_p$. Indeed, we observe that the class $[\mathbb{Z}'[G]] \in \mathrm{G}_0(\mathbb{Z}[G])$ annihilates $\mathrm{Cl}(\mathbb{Z}'[G])$, but since it has rank $\#G$, it is invertible in the localizations of $\mathrm{G}_0(\mathbb{Z}[G])$ at $\rho = I_G + (q)$ for all $q \neq p$. Hence, the completion $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ is supported at $p$ and the claim follows since, by Section 2.1.2, the only prime ideal of $\mathrm{G}_0(\mathbb{Z}[G])$ supported over $p$ is $I_G + (p)$ (see the proof of [Pappas 1998, Proposition 4.5]). Combining Proposition 7.1.2(2) and (5.5.8), we obtain

$$(\ell - 1)(\psi^\ell - \ell^{-1}) \cdot \overline{\chi}(X, \mathbb{O}_X) = \ell^{-1}(\ell - 1) \cdot f_*^\wedge(r^\ell_{X/Y}) \tag{7.2.4}$$

in $\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$. Now apply (7.2.4) to a prime $\ell$ as in Proposition 6.2.3. We see that Proposition 6.3.4 implies that the multiple $\gcd(3, \#G)^{v_3(\#G)-1} \cdot \overline{\chi}(X, \mathbb{O}_X)_p$ is determined by $f_*^\wedge(r^\ell_{X/Y}) \in \mathrm{Cl}(\mathbb{Z}'[G])^\wedge$.

We will show that $f_*^\wedge(r^\ell_{X/Y})$ depends only on the $G$-cover $\widehat{X}_S \to \widehat{Y}_S$. Set $U_S = X - X_S$. Recall

$$r^\ell_{X/Y} = \theta^\ell(T^\vee_X)^{-1} \cdot \theta^\ell(\pi^* T^\vee_Y) - 1$$

is in $\mathrm{K}_0(G, X)^\wedge$ with trivial image in $\mathrm{K}_0(G, U_S)^\wedge$ under restriction. Observe that $f_*^\wedge(r^\ell_{X/Y})$ only depends on the image of $r^\ell_{X/Y}$ in $\mathrm{G}_0(G, X)^\wedge$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{G}_1(G, U_S)^\wedge \arrow[r] \arrow[d] & \mathrm{G}_0(G, X_S)^\wedge \arrow[r] \arrow[d] & \mathrm{G}_0(G, X)^\wedge \arrow[r] \arrow[d, "\beta"'] & \mathrm{G}_0(G, U_S)^\wedge \arrow[r] \arrow[d] & 0 \\
\mathrm{G}_1(G, \widehat{U}_S)^\wedge \arrow[r] & \mathrm{G}_0(G, X_S)^\wedge \arrow[r] & \mathrm{G}_0(G, \widehat{X}_S)^\wedge \arrow[r] & \mathrm{G}_0(G, \widehat{U}_S)^\wedge \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and are obtained by completing the standard equivariant localization sequences [Thomason 1987]. Here $\widehat{U}_S = \bigcup_{q \in S} X_{\mathbb{Q}_q}$ and the vertical maps are given by base change; the second vertical map is the identity. Since the action of $G$ on $\widehat{U}_S$ is free, we have by étale descent $\mathrm{G}_1(G, \widehat{U}_S) = \mathrm{G}_1(\widehat{U}_S/G) = \bigoplus_{q \in S} \mathrm{G}_1(Y_{\mathbb{Q}_q})$. (In particular, this also implies that $I_G^m \cdot \mathrm{G}_1(G, \widehat{U}_S) = (0)$ for $m > 1$ and so $\mathrm{G}_1(G, \widehat{U}_S)^\wedge = \mathrm{G}_1(G, \widehat{U}_S)$.) As in [Chinburg 1994; Chinburg et al. 1997a], we can see that $f_*^\wedge \cdot (i_S)_*^\wedge : \mathrm{G}_0(G, X_S)^\wedge \to \mathrm{G}_0(G, X)^\wedge \to \mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ can be written as a composition $(f_S)_*:\mathrm{G}_0(G,X_S)^\wedge\to\bigoplus_{q\in S}\mathrm{K}_0(\mathbb{F}_q[G])^\wedge\to\mathrm{Cl}(\mathbb{Z}'[G])^\wedge$ where the first arrow is given by the sum of the projective equivariant Euler characteristics for the $G$-schemes $f_q: X_{\mathbb{F}_q}\to\mathrm{Spec}(\mathbb{F}_q)$.

**Proposition 7.2.5.** *The homomorphism* $(f_S)_*$ *vanishes on the image of the map* $\mathrm{G}_1(G,\widehat{U}_S)^\wedge\to\mathrm{G}_0(G,X_S)^\wedge$.

*Proof.* For this, we need:

**Lemma 7.2.6.** *The maps above give a commutative diagram*

$$\begin{tikzcd}
\mathrm{G}_1(Y_{\mathbb{Q}_q}) \arrow[r, "\sim"] \arrow[d, "(g_{\mathbb{Q}_q})_*"'] & \mathrm{G}_1(G,X_{\mathbb{Q}_q}) \arrow[r] & \mathrm{G}_0(G,X_{\mathbb{F}_q}) \arrow[d, "(f_q)_*"] \\
\mathbb{Q}_q^* \arrow[r] & \mathrm{G}_1(\mathbb{Q}_q[G]) \arrow[r] & \mathrm{G}_0(\mathbb{F}_q[G])
\end{tikzcd}$$

*where the bottom left horizontal arrow*

$$\mathbb{Q}_q^*\to\mathrm{G}_1(\mathbb{Q}_q[G])=\mathrm{Hom}_{\mathrm{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)}(\mathscr{R}_{\bar{\mathbb{Q}}_q}(G),\bar{\mathbb{Q}}_q^*)$$

*sends* $\mu\in\mathbb{Q}_q^*$ *to the character function* $\chi\mapsto\mu^{\deg(\chi)}$.

*Proof.* Using the Quillen–Gersten spectral sequence, we can see that $\mathrm{G}_1(Y_{\mathbb{Q}_q})$ is generated by the following two types of elements: constants $c\in\mathbb{Q}_q^*$ considered as giving automorphisms of the structure sheaf of $Y_{\mathbb{Q}_q}$ (i.e., elements in the image of $(f_{\mathbb{Q}_q})^*:\mathbb{Q}_q^*=\mathrm{G}_1(\mathrm{Spec}(\mathbb{Q}_q))^*\to\mathrm{G}_1(Y_{\mathbb{Q}_q})$) and elements in the image of $k(y)^*=\mathrm{G}_1(\mathrm{Spec}(k(y)))\to\mathrm{G}_1(Y_{\mathbb{Q}_q})$ where $k(y)$ is the residue field of some closed point $y$ of $Y_{\mathbb{Q}_q}$ [Gillet 1981, Example 4.6]. It is enough to check the commutativity on these two types of elements. The statement for the first type follows easily from the fact (a consequence of Lefschetz–Riemann–Roch or of the main result of [Nakajima 1984]) that the $G$-character $[\mathrm{H}^0(X_{\mathbb{Q}_q},\mathcal{O}_{X_{\mathbb{Q}_q}})]-[\mathrm{H}^1(X_{\mathbb{Q}_q},\mathcal{O}_{X_{\mathbb{Q}_q}})]$ is a multiple of the regular character. The statement for the second type follows from an explicit calculation by using the normal basis theorem. □

The lemma implies that the values of $(f_S)_*$ on the image of $\mathrm{G}_1(G,\widehat{U}_S)^\wedge\to\mathrm{G}_0(G,X_S)^\wedge$ are in the subgroup generated by the sums of images of the character functions $\chi\mapsto\mu^{\deg(\chi)}$. These values are all classes of virtually free $\mathbb{F}_q[G]$-modules in $\mathrm{K}_0(\mathbb{F}_q[G])\subset\mathrm{G}_0(\mathbb{F}_q[G])$, and this shows the desired vanishing. □

Proposition 7.2.5, combined with a chase on the commutative diagram on page 1510, now implies that $f_*^\wedge(r_{X/Y}^\ell)$ only depends on the image of $r_{X/Y}^\ell$ under the base change $\beta:\mathrm{G}_0(G,X)^\wedge\to\mathrm{G}_0(G,\widehat{X}_S)^\wedge$. By flat base change for the cotangent complex [Illusie 1971], this image is the class of

$$r_{\widehat{X}_S/\widehat{Y}_S}^\ell:=\theta^\ell(T_{\widehat{X}_S}^\vee)^{-1}\cdot\theta^\ell(\pi^*T_{\widehat{Y}_S}^\vee)-1.$$

Therefore, $f_*^\wedge(r^\ell_{X/Y})$ only depends on $\widehat{X}_S \to \widehat{Y}_S$, which in turn, since $\widehat{Y}_S = \widehat{X}_S/G$, only depends on the $G$-scheme $\widehat{X}_S$.

Combining all these, we now obtain the statement of Theorem 7.2.1. ☐

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