Algebra & Number Theory

Volume 10

2016

No. 1

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ALGEBRA AND NUMBER THEORY 10:1 (2016)
dx.doi.org/10.2140/ant.2016.10.155

On the image of the Galois representation associated to a non-CM Hida family

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Fix a prime $p > 2$. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{I})$ be the Galois representation coming from a non-CM irreducible component $\mathbb{I}$ of Hida's p-ordinary Hecke algebra. Assume the residual representation $\bar{\rho}$ is absolutely irreducible. Under a minor technical condition we identify a subring $\mathbb{I}_0$ of $\mathbb{I}$ containing $\mathbb{Z}_p[[T]]$ such that the image of ρ is large with respect to $\mathbb{I}_0$. That is, $\mathrm{Im}\,\rho$ contains $\ker\big(\mathrm{SL}_2(\mathbb{I}_0) \to \mathrm{SL}_2(\mathbb{I}_0/\mathfrak{a})\big)$ for some nonzero $\mathbb{I}_0$-ideal $\mathfrak{a}$. This paper builds on recent work of Hida who showed that the image of such a Galois representation is large with respect to $\mathbb{Z}_p[[T]]$. Our result is an $\mathbb{I}$-adic analogue of the description of the image of the Galois representation attached to a non-CM classical modular form obtained by Ribet and Momose in the 1980s.

1. Introduction

A Hida family F that is an eigenform and has coefficients in a domain $\mathbb{I}$ has an associated Galois representation $\rho_F : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(Q(\mathbb{I}))$, where $Q(\mathbb{I})$ is the field of fractions of $\mathbb{I}$. A fundamental problem is to understand the image of such a representation. One expects the image to be "large" in an appropriate sense, so long as F does not have any extra symmetries; that is, as long as F does not have CM. (In the CM case there is a nontrivial character η such that $\rho_F \cong \rho_F \otimes \eta$. This forces the image of ρ_F to be "small".) This notion of "largeness" can be defined relative to any subring $\mathbb{I}_0$ of $\mathbb{I}$, and one can then ask whether $\mathrm{Im}\,\rho_F$ is large with respect to $\mathbb{I}_0$. Even when F does not have CM it might happen that there is an automorphism σ of $\mathbb{I}$ and a nontrivial character η such that $\rho_F^\sigma \cong \rho_F \otimes \eta$. Such automorphisms, called conjugate self-twists of F, can be thought of as weak symmetries of F. In this paper we explain how conjugate self-twists constrict the image of ρ_F. In particular, let $\mathbb{I}_0$ be the subring of $\mathbb{I}$ fixed by all conjugate self-twists of F. Our main result is that $\mathrm{Im}\,\rho_F$ is "large" with respect to $\mathbb{I}_0$.

The study of the image of the Galois representation attached to a modular form,

This work was supported by National Science Foundation grant DGE-1144087.
MSC2010: primary 11F80; secondary 11F85, 11F11.
Keywords: Galois representation, Galois deformation, Hida family.

and showing that it is large in the absence of CM, was first carried out by Serre [1973] and Swinnerton-Dyer [1973]. They studied the Galois representation attached to a modular form of level 1 with integral coefficients. Ribet [1980; 1985] and Momose [1981] generalized the work of Serre and Swinnerton-Dyer to cover all Galois representations coming from classical modular forms. Ribet's work dealt with the weight two case, and Momose proved the general case. The main theorem in this paper is an analogue of their results in the $\mathbb{I}$-adic setting. In fact, their work is a key input for our proof.

Shortly after Hida constructed the representations ρ_F, Mazur and Wiles [1986] showed that if $\mathbb{I} = \mathbb{Z}_p[\![T]\!]$ and the image of the residual representation $\bar{\rho}_F$ contains $\mathrm{SL}_2(\mathbb{F}_p)$ then $\mathrm{Im}\,\rho_F$ contains $\mathrm{SL}_2(\mathbb{Z}_p[\![T]\!])$. Under the assumptions that $\mathbb{I}$ is a power series ring in one variable and the image of the residual representation $\bar{\rho}_F$ contains $\mathrm{SL}_2(\mathbb{F}_p)$, our main result was proved by Fischman [2002]. Fischman's work is the only previous work that considers the effect of conjugate self-twists on $\mathrm{Im}\,\rho_F$. Hida [2015] has shown under some technical hypotheses that if F does not have CM then $\mathrm{Im}\,\rho_F$ is large with respect to the ring $\mathbb{Z}_p[\![T]\!]$, even when $\mathbb{I} \supsetneq \mathbb{Z}_p[\![T]\!]$. The methods he developed play an important role in this paper. The local behavior of ρ_F at p was studied by Ghate and Vatsal [2004] and later by Hida [2013]. They showed, under some assumptions later removed by Zhao [2014], that $\rho_F|_{D_p}$ is indecomposable, where D_p denotes the decomposition group at p in $G_{\mathbb{Q}}$. We will make use of this result later. Finally, Hida and Tilouine [2015] showed that certain GSp_4-representations associated to Siegel modular forms have large image.

Our result is the first to describe the effect of conjugate self-twists on the image of ρ_F without any assumptions on $\mathbb{I}$ and without assuming that the image of $\bar{\rho}_F$ contains $\mathrm{SL}_2(\mathbb{F}_p)$. We do need an assumption on $\bar{\rho}_F$, namely that $\bar{\rho}_F$ is absolutely irreducible and another small technical condition, but this is much weaker than assuming $\mathrm{Im}\,\bar{\rho}_F \supseteq \mathrm{SL}_2(\mathbb{F}_p)$.

2. Main theorems and structure of paper

We begin by fixing notation that will be in place throughout the paper. Let $p > 2$ be prime. Fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ as well as an embedding $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. Let $\mathbb{Z}^+$ denote the set of positive integers. Fix $N_0 \in \mathbb{Z}^+$ prime to p; it will serve as our tame level. Let $N = N_0 p^r$ for some fixed $r \in \mathbb{Z}^+$. Fix a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ which will serve as our nebentypus. Let χ_1 be the product of $\chi|_{(\mathbb{Z}/N_0\mathbb{Z})^\times}$ with the tame p-part of χ, and write $c(\chi)$ for the conductor of χ. During the proof of the main theorem we will assume that the order of χ is a power of 2 and that $2c(\chi)|N$. The fact that we can assume these restrictions on χ for the purpose of demonstrating $\mathbb{I}_0$-fullness is shown in Proposition 3.9.

For a valuation ring W over $\mathbb{Z}_p$, let $\Lambda_W = W[\![T]\!]$. Let $\mathbb{Z}_p[\chi]$ be the extension of $\mathbb{Z}_p$ generated by the values of χ. When $W = \mathbb{Z}_p[\chi]$ we write Λ_χ for Λ_W. When $W = \mathbb{Z}_p$ then we let $\Lambda = \Lambda_{\mathbb{Z}_p}$. For any valuation ring W over $\mathbb{Z}_p$, an *arithmetic prime* of Λ_W is a prime ideal of the form

$$P_{k,\varepsilon} := (1 + T - \varepsilon(1 + p)(1 + p)^k)$$

for an integer $k \geq 2$ and character $\varepsilon : 1 + p\mathbb{Z}_p \to W^\times$ of p-power order. We shall write $r(\varepsilon)$ for the nonnegative integer such that $p^{r(\varepsilon)}$ is the order of ε. If R is a finite extension of Λ_W, then we say a prime of R is *arithmetic* if it lies over an arithmetic prime of Λ_W.

For a Dirichlet character $\psi : (\mathbb{Z}/M\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$, let $S_k(\Gamma_0(M), \psi)$ be the space of classical cusp forms of weight k, level $\Gamma_0(M)$, and nebentypus ψ. Let $h_k(\Gamma_0(M), \psi)$ be the Hecke algebra of $S_k(\Gamma_0(M), \psi)$, and let $h_k^{\mathrm{ord}}(\Gamma_0(M), \psi)$ denote the p-ordinary Hecke algebra. Let ω be the p-adic Teichmüller character. We can describe Hida's big p-ordinary Hecke algebra $\boldsymbol{h}^{\mathrm{ord}}(N, \chi; \Lambda_\chi)$ as follows [Hida 2015]. It is the unique Λ_χ-algebra that is

(1) free of finite rank over Λ_χ,

(2) equipped with Hecke operators $T(n)$ for all $n \in \mathbb{Z}^+$, and

(3) satisfies the following specialization property: for every arithmetic prime $P_{k,\varepsilon}$ of Λ_χ there is an isomorphism

$$\boldsymbol{h}^{\mathrm{ord}}(N, \chi; \Lambda_\chi)/P_{k,\varepsilon}\boldsymbol{h}^{\mathrm{ord}}(N, \chi; \Lambda_\chi) \cong h_k^{\mathrm{ord}}(\Gamma_0(Np^{r(\varepsilon)}), \chi_1\varepsilon\omega^{-k})$$

that sends $T(n)$ to $T(n)$ for all $n \in \mathbb{Z}^+$.

For a commutative ring R, we use $Q(R)$ to denote the total ring of fractions of R. Hida [1986a] has shown that there is a Galois representation

$$\rho_{N_0,\chi} : G_{\mathbb{Q}} \to \mathrm{GL}_2\big(Q(\boldsymbol{h}^{\mathrm{ord}}(N_0, \chi; \Lambda_\chi))\big)$$

that is unramified outside N and satisfies $\operatorname{tr} \rho_{N_0,\chi}(\mathrm{Frob}_\ell) = T(\ell)$ for all primes ℓ not dividing N. Let $\operatorname{Spec} \mathbb{I}$ be an irreducible component of $\operatorname{Spec} \boldsymbol{h}^{\mathrm{ord}}(N_0, \chi; \Lambda_\chi)$. Assume further that $\mathbb{I}$ is primitive in the sense of [Hida 1986b, Section 3]. Let $\lambda_F : \boldsymbol{h}^{\mathrm{ord}}(N_0, \chi; \Lambda_\chi) \to \mathbb{I}$ be the natural Λ_χ-algebra homomorphism coming from the inclusion of spectra. By viewing

$$Q(\boldsymbol{h}^{\mathrm{ord}}(N_0, \chi; \Lambda_\chi)) = \boldsymbol{h}^{\mathrm{ord}}(N_0, \chi; \Lambda_\chi) \otimes_{\Lambda_\chi} Q(\Lambda_\chi)$$

and composing $\rho_{N_0,\chi}$ with $\lambda_F \otimes 1$ we obtain a Galois representation

$$\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(Q(\mathbb{I}))$$

that is unramified outside N and satisfies

$$\operatorname{tr}\rho_F(\mathrm{Frob}_\ell) = \lambda_F(T(\ell))$$

for all primes ℓ not dividing N.

Henceforth for any $n \in \mathbb{Z}^+$ we shall let $a(n, F)$ denote $\lambda_F(T(n))$. Let F be the formal power series in q given by

$$F = \sum_{n=1}^{\infty} a(n, F)q^n.$$

Let $\mathbb{I}' = \Lambda_\chi[\{a(\ell, F) : \ell \nmid N\}]$ which is an order in $Q(\mathbb{I})$ since F is primitive. We shall consider the Hida family F and the associated ring $\mathbb{I}'$ to be fixed throughout the paper. For a local ring R we will use $\mathfrak{m}_R$ to denote the unique maximal ideal of R. Let $\mathbb{F} := \mathbb{I}'/\mathfrak{m}_{\mathbb{I}'}$, the residue field of $\mathbb{I}'$. We exclusively use the letter $\mathfrak{P}$ to denote a prime of $\mathbb{I}$, and $\mathfrak{P}'$ shall always denote $\mathfrak{P} \cap \mathbb{I}'$. Conversely, we exclusively use $\mathfrak{P}'$ to denote a prime of $\mathbb{I}'$ in which case we are implicitly fixing a prime $\mathfrak{P}$ of $\mathbb{I}$ lying over $\mathfrak{P}'$.

If $\mathfrak{P}$ is a height one prime of $\mathbb{I}$ then we write $f_\mathfrak{P}$ for the p-adic modular form obtained by reducing the coefficients of F modulo $\mathfrak{P}$. In particular, if $\mathfrak{P}$ is an arithmetic prime lying over $P_{k,\varepsilon}$ then $f_\mathfrak{P} \in S_k(\Gamma_0(Np^{r(\varepsilon)}), \varepsilon\chi_1\omega^{-k})$.

Recall that Hida [1986a] has shown that there is a well defined residual representation $\bar{\rho}_F : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}/\mathfrak{m}_\mathbb{I})$ of ρ_F. Throughout this paper we impose the following assumption.

$$\text{Assume that } \bar{\rho}_F \text{ is absolutely irreducible.} \tag{abs}$$

By the Chebotarev density theorem, we see that $\operatorname{tr}\bar{\rho}_F$ is valued in $\mathbb{F}$. Under (abs) we may use pseudorepresentations to find a $\mathrm{GL}_2(\mathbb{I}')$-valued representation that is isomorphic to ρ_F over $Q(\mathbb{I})$. Thus we may (and do) assume that ρ_F takes values in $\mathrm{GL}_2(\mathbb{I}')$.

Definition 2.1. Let $g = \sum_{n=1}^{\infty} a(n, g)q^n$ be either a classical Hecke eigenform or a Hida family of such forms. Let K be the field generated by $\{a(n, g) : n \in \mathbb{Z}^+\}$ over either $\mathbb{Q}$ in the classical case or $Q(\Lambda_\chi)$ in the Λ_χ-adic case. We say a pair (σ, η_σ) is a *conjugate self-twist* of g if η_σ is a Dirichlet character, σ is an automorphism of K, and

$$\sigma(a(\ell, g)) = \eta_\sigma(\ell)a(\ell, g)$$

for all but finitely many primes ℓ. If there is a nontrivial character η such that $(1, \eta)$ is a conjugate self-twist of g, then we say that g has *complex multiplication* or *CM*. Otherwise, g does not have CM.

If a modular form does not have CM then a conjugate self-twist is uniquely determined by the automorphism.

We shall always assume that our fixed Hida family F does not have CM. Let

$$\Gamma = \{\sigma \in \mathrm{Aut}(Q(\mathbb{I})) : \sigma \text{ is a conjugate self-twist of } F\}.$$

Under the assumption (abs) it follows from a lemma of Carayol and Serre [Hida 2000b, Proposition 2.13] that if $\sigma \in \Gamma$ then $\rho_F^\sigma \cong \rho_F \otimes \eta_\sigma$ over $\mathbb{I}'$. As ρ_F is unramified outside N we see that in fact $\sigma(a(\ell, F)) = \eta_\sigma(\ell)a(\ell, F)$ for all primes ℓ not dividing N. Therefore σ restricts to an automorphism of $\mathbb{I}'$. Let $\mathbb{I}_0 = (\mathbb{I}')^\Gamma$. Define

$$H_0 := \bigcap_{\sigma \in \Gamma} \ker \eta_\sigma$$

and

$$H := H_0 \cap \ker(\det(\bar{\rho}_F)).$$

These open normal subgroups of $G_{\mathbb{Q}}$ play an important role in our proof.

For a commutative ring B and ideal $\mathfrak{b}$ of B, write

$$\Gamma_B(\mathfrak{b}) := \ker(\mathrm{SL}_2(B) \to \mathrm{SL}_2(B/\mathfrak{b})).$$

We call $\Gamma_B(\mathfrak{b})$ a *congruence subgroup* of $\mathrm{GL}_2(B)$ if $\mathfrak{b} \neq 0$. We can now define what we mean when we say a representation is "large" with respect to a ring.

Definition 2.2. Let G be a group, A a commutative ring, and $r : G \to \mathrm{GL}_2(A)$ a representation. For a subring B of A, we say that r is B*-full* if there is some $\gamma \in \mathrm{GL}_2(A)$ such that $\gamma(\mathrm{Im}\, r)\gamma^{-1}$ contains a congruence subgroup of $\mathrm{GL}_2(B)$.

Let D_p be the decomposition group at p in $G_{\mathbb{Q}}$. That is, D_p is the image of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ under the embedding $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$ induced by ι_p. Recall that over $Q(\mathbb{I})$ the local representation $\rho_F|_{D_p}$ is isomorphic to $\left(\begin{smallmatrix} \varepsilon & u \\ 0 & \delta \end{smallmatrix}\right)$ [Hida 2000a, Theorem 4.3.2]. Let $\bar{\varepsilon}$ and $\bar{\delta}$ denote the residual characters of ε and δ, respectively.

Definition 2.3. For any open subgroup $G_0 \le G_{\mathbb{Q}}$ we say that ρ_F is G_0*-regular* if $\bar{\varepsilon}|_{D_p \cap G_0} \neq \bar{\delta}|_{D_p \cap G_0}$.

The main result of this paper is the following.

Theorem 2.4. *Assume $p > 2$ and let F be a primitive non-CM p-adic Hida family. Assume $|\mathbb{F}| \neq 3$ and that the residual representation $\bar{\rho}_F$ is absolutely irreducible and H_0-regular. Then ρ_F is $\mathbb{I}_0$-full.*

The strategy of the proof is to exploit the results of Ribet [1980; 1985] and Momose [1981]. Since an arithmetic specialization of a non-CM Hida family cannot be CM, their work implies that if $\mathfrak{P}'$ is an arithmetic prime of $\mathbb{I}'$ then there is a certain subring $\mathcal{O} \subseteq \mathbb{I}'/\mathfrak{P}'$ for which $\rho_F \bmod \mathfrak{P}'$ is $\mathcal{O}$-full. To connect their ring $\mathcal{O}$ with $\mathbb{I}_0$, in Section 6 we show that $Q(\mathcal{O}) = Q(\mathbb{I}_0/\mathcal{Q})$, where $\mathcal{Q} = \mathbb{I}_0 \cap \mathfrak{P}'$. The proof that $Q(\mathcal{O}) = Q(\mathbb{I}_0/\mathcal{Q})$ relies on establishing a relationship between conjugate self-twists of F and conjugate self-twists of the arithmetic specializations of F. As this may be of independent interest we state the result here.

Theorem 2.5. *Let $\mathfrak{P}$ be an arithmetic prime of $\mathbb{I}$ and σ be a conjugate self-twist of $f_{\mathfrak{P}}$ that is also an automorphism of the local field $\mathbb{Q}_p(\{a(n, f_{\mathfrak{P}}) : n \in \mathbb{Z}^+\})$. Then σ can be lifted to $\tilde{\sigma} \in \Gamma$ such that $\tilde{\sigma}(\mathfrak{P}') = \mathfrak{P}'$, where $\mathfrak{P}' = \mathfrak{P} \cap \mathbb{I}'$.*

The proof, in Section 3, uses a combination of abstract deformation theory and automorphic techniques. Deformation theory is used to lift σ to an automorphism of the universal deformation ring of $\bar{\rho}_F$. Then we use automorphic methods to show that this lift preserves the irreducible component $\operatorname{Spec} \mathbb{I}$. The key technical input is that $\boldsymbol{h}^{\mathrm{ord}}(N, \chi; \Lambda_\chi)$ is étale over arithmetic points of Λ.

The remainder of the paper consists of a series of reduction steps that allow us to deduce our theorem from the aforementioned results of Ribet and Momose. Our methods make it convenient to modify ρ_F to a related representation $\rho : H \to \mathrm{SL}_2(\mathbb{I}_0)$ and show that ρ is $\mathbb{I}_0$-full. We axiomatize the properties of ρ at the beginning of Section 4 and use ρ in the next three sections to prove Theorem 2.4. Then in Section 7 we explain how to show the existence of ρ with the desired properties.

The task of showing that ρ is $\mathbb{I}_0$-full is done in three steps. In Section 4 we consider the projection of $\operatorname{Im} \rho$ to $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$, where P is an arithmetic prime of Λ and $\mathcal{Q}$ runs over all primes of $\mathbb{I}_0$ lying over P. We show that if the image of $\operatorname{Im} \rho$ in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open then ρ is $\mathbb{I}_0$-full. This uses Pink's theory of Lie algebras for p-profinite subgroups of SL_2 over p-profinite semilocal rings [Pink 1993] and the related techniques developed by Hida [2015].

In Section 5 we show that if the image of $\operatorname{Im} \rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is $\mathbb{I}_0/\mathcal{Q}$-full for all primes $\mathcal{Q}$ of $\mathbb{I}_0$ lying over P, then the image of $\operatorname{Im} \rho$ is indeed open in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$. The argument is by contradiction and uses Goursat's lemma. It was inspired by an argument of Ribet [1975]. This is the only section where we make use of the assumption that $|\mathbb{F}| \neq 3$.

The final step showing that the image of $\operatorname{Im} \rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is $\mathbb{I}_0/\mathcal{Q}$-full for every $\mathcal{Q}$ lying over P is done in Section 6. The key input is Theorem 2.5 from Section 3 together with the work of Ribet and Momose on the image of the Galois representation associated to a non-CM classical modular form. We give a brief exposition of their work and a precise statement of their result at the beginning of Section 6. We reiterate the structure of the proof of Theorem 2.4 at the end of Section 6.

3. Lifting twists

Let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be (not necessarily distinct) arithmetic primes of $\mathbb{I}$, and let $\mathfrak{P}'_i = \mathfrak{P}_i \cap \mathbb{I}'$. We shall often view $\mathfrak{P}_i$ as a geometric point in $\operatorname{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$. Suppose there is an isomorphism $\sigma : \mathbb{I}/\mathfrak{P}_1 \cong \mathbb{I}/\mathfrak{P}_2$ and a Dirichlet character $\eta : G_{\mathbb{Q}} \to Q(\mathbb{I}/\mathfrak{P}_2)^\times$ such that

$$\sigma(a(\ell, f_{\mathfrak{P}_1})) = \eta(\ell)a(\ell, f_{\mathfrak{P}_2})$$

for all primes ℓ not dividing N. We may (and do) assume without loss of generality that η is primitive since the above relation holds even when η is replaced by its primitive character. In this section we show that σ can be lifted to a conjugate self-twist of F.

Theorem 3.1. *Assume that η takes values in $\mathbb{Z}_p[\chi]$ and that the order of χ is a power of 2. If η is ramified at 2, assume further that $2c(\chi)|N$. Then there is an automorphism $\tilde{\sigma} : \mathbb{I}' \to \mathbb{I}'$ such that*

$$\tilde{\sigma}(a(\ell, F)) = \eta(\ell)a(\ell, F)$$

for all but finitely many primes ℓ and $\sigma \circ \mathfrak{P}'_1 = \mathfrak{P}'_2 \circ \tilde{\sigma}$. In particular, $\mathfrak{P}'_1$ and $\mathfrak{P}'_2$ necessarily lie over the same prime of $\mathbb{I}_0$.

Remark. The condition that the order of χ be a power of 2 looks restrictive. However in Proposition 3.9 we show that for the purpose of proving $\mathbb{I}_0$-fullness we may replace F with a family whose nebentypus has order a power of 2. The same proposition shows that the condition that $2c(\chi)|N$ is not restrictive when proving $\mathbb{I}_0$-fullness.

There are two steps in the proof of Theorem 3.1. First we use abstract deformation theory to construct a lift Σ of σ to the universal deformation ring of $\bar{\rho}_F$ (or some base change of that ring). This allows us to show that η is necessarily quadratic. Then we show that the induced map on spectra Σ^* sends the irreducible component $\operatorname{Spec} \mathbb{I}'$ to another *modular* component of the universal deformation ring. Since σ is an isomorphism between $\mathbb{I}/\mathfrak{P}_1$ and $\mathbb{I}/\mathfrak{P}_2$ it follows that the arithmetic point $\mathfrak{P}'_1$ lies on both $\operatorname{Spec} \mathbb{I}'$ and $\Sigma^*(\operatorname{Spec} \mathbb{I}')$. Since the Hecke algebra is étale over arithmetic points of Λ, it follows that $\Sigma^*(\operatorname{Spec} \mathbb{I}') = \operatorname{Spec} \mathbb{I}'$ and hence Σ descends to the desired automorphism of $\mathbb{I}'$.

Lifting σ to the universal deformation ring. Let W be the ring of Witt vectors of $\mathbb{F}$. Let $\mathbb{Q}^N$ be the maximal subfield of $\overline{\mathbb{Q}}$ unramified outside N and infinity, and let $G_{\mathbb{Q}}^N := \operatorname{Gal}(\mathbb{Q}^N/\mathbb{Q})$. Note that ρ_F factors through $G_{\mathbb{Q}}^N$. For the remainder of this section we shall consider $G_{\mathbb{Q}}^N$ to be the domain of ρ_F and $\bar{\rho}_F$.

We set up the notation for deformation theory. For our purposes universal deformation rings of pseudorepresentations are sufficient. However, since we are assuming that $\bar{\rho}_F$ is absolutely irreducible, we use universal deformation rings of representations to avoid introducing extra notation for pseudorepresentations.

Let $\mathcal{C}$ denote the category of complete local p-profinite W-algebras with residue field $\mathbb{F}$. Let $\bar{\pi} : G_{\mathbb{Q}}^N \to \mathrm{GL}_n(\mathbb{F})$ be an absolutely irreducible representation. We say an object $R_{\bar{\pi}} \in \mathcal{C}$ and representation $\bar{\pi}^{\mathrm{univ}} : G_{\mathbb{Q}}^N \to \mathrm{GL}_n(R_{\bar{\pi}})$ is a *universal couple* for $\bar{\pi}$ if: $\bar{\pi}^{\mathrm{univ}} \bmod \mathfrak{m}_{R_{\bar{\pi}}} \cong \bar{\pi}$ and for every $A \in \mathcal{C}$ and representation $r : G_{\mathbb{Q}}^N \to \mathrm{GL}_n(A)$ such that $r \bmod \mathfrak{m}_A \cong \bar{\pi}$, there exists a unique W-algebra homomorphism $\alpha(r) : R_{\bar{\pi}} \to A$ such that $r \cong \alpha(r) \circ \bar{\pi}^{\mathrm{univ}}$. Mazur [1989] proved that a universal couple always exists (and is unique) when $\bar{\pi}$ is absolutely irreducible.

Since η takes values in $\mathbb{Z}_p[\chi]$ which may not be contained in W, we need to extend scalars. Let $\mathcal{O} = W[\eta]$. We recommend the reader assume $\mathcal{O} = W$ on the first read. In fact, in Proposition 3.4 we will use deformation theory to conclude that η is quadratic, but we cannot assume that from the start. For a commutative W-algebra A, let ${}^{\mathcal{O}}\!A := \mathcal{O} \otimes_W A$. It will be important that we are tensoring on the left by $\mathcal{O}$ as we will sometimes want to view ${}^{\mathcal{O}}\!A$ as a right W-algebra.

Let $\bar{\sigma}$ denote the automorphism of $\mathbb{F}$ induced by σ and $\bar{\eta}$ the projection of η to $\mathbb{F}$. The automorphism $\bar{\sigma}$ of $\mathbb{F}$ induces an automorphism $W(\bar{\sigma})$ on W. For any W-algebra A, let $A^{\bar{\sigma}} := A \otimes_{W(\bar{\sigma})} W$, where W is considered as a W-algebra via $W(\bar{\sigma})$. Note that $A^{\bar{\sigma}}$ is a W-bimodule with different left and right actions. Namely there is the left action given by $w(a \otimes w') = aw \otimes w'$, which may be different from the right action given by $(a \otimes w')w = a \otimes ww'$. In particular, ${}^{\mathcal{O}}\!A^{\bar{\sigma}} = \mathcal{O} \otimes_W A \otimes_{W(\bar{\sigma})} W$. Let $\iota(\bar{\sigma}, A) : A \to A^{\bar{\sigma}}$ be the usual map given by $\iota(\bar{\sigma}, A)(a) = a \otimes 1$. It is an isomorphism of rings with inverse given by $\iota(\bar{\sigma}^{-1}, A)$. Furthermore, $\iota(\bar{\sigma}, A)$ is a *left* W-algebra homomorphism.

The next lemma describes the relationship between the deformation rings arising from the universal couples $(R_{\bar{\rho}_F}, \bar{\rho}_F^{\mathrm{univ}})$, $(R_{\bar{\rho}_F^{\bar{\sigma}}}, (\bar{\rho}_F^{\bar{\sigma}})^{\mathrm{univ}})$, and $(R_{\bar{\eta}\otimes\bar{\rho}_F}, (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}})$.

Lemma 3.2. (1) *If $\bar{\rho}_F^{\bar{\sigma}} \cong \bar{\eta} \otimes \bar{\rho}_F$ then the universal couples $(R_{\bar{\rho}_F^{\bar{\sigma}}}, (\bar{\rho}_F^{\bar{\sigma}})^{\mathrm{univ}})$ and $(R_{\bar{\eta}\otimes\bar{\rho}_F}, (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}})$ are canonically isomorphic.*

(2) *There is a canonical isomorphism $\varphi : R_{\bar{\rho}_F}^{\bar{\sigma}} \to R_{\bar{\rho}_F^{\bar{\sigma}}}$ of right W-algebras such that*

$$(\bar{\rho}_F^{\bar{\sigma}})^{\mathrm{univ}} \cong \varphi \circ \iota(\bar{\sigma}, R_{\bar{\rho}_F}) \circ \bar{\rho}_F^{\mathrm{univ}}.$$

(3) *Viewing $(\bar{\eta}\otimes\bar{\rho}_F)^{\mathrm{univ}}$ as a representation valued in $\mathrm{GL}_2({}^{\mathcal{O}}R_{\bar{\eta}\otimes\bar{\rho}_F})$ via the natural map $R_{\bar{\eta}\otimes\bar{\rho}_F} \to {}^{\mathcal{O}}R_{\bar{\eta}\otimes\bar{\rho}_F}$, there is a natural W-algebra homomorphism $\psi : R_{\bar{\eta}\otimes\bar{\rho}_F} \to {}^{\mathcal{O}}R_{\bar{\rho}_F}$ such that*

$$\eta \otimes \bar{\rho}_F^{\mathrm{univ}} \cong (1 \otimes \psi) \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}}.$$

Proof. The first statement follows directly from the definition of universal couples.

For (2), we show that the right W-algebra $R_{\bar{\rho}_F}^{\bar{\sigma}}$ satisfies the universal property for $\bar{\rho}_F^{\bar{\sigma}}$. Let $A \in \mathcal{C}$ and $r : G_{\mathbb{Q}}^N \to \mathrm{GL}_2(A)$ be a deformation of $\bar{\rho}_F^{\bar{\sigma}}$. Then $\iota(\bar{\sigma}^{-1}, A) \circ r$

is a deformation of $\bar{\rho}_F$, viewing $A^{\bar{\sigma}^{-1}}$ as a right W-algebra. By universality there is a unique *right* W-algebra homomorphism $\alpha(\iota(\bar{\sigma}^{-1}, A) \circ r) : R_{\bar{\rho}_F} \to A^{\bar{\sigma}^{-1}}$ such that $\iota(\bar{\sigma}^{-1}, A) \circ r \cong \alpha(\iota(\bar{\sigma}^{-1}, A) \circ r) \circ \bar{\rho}_F^{\mathrm{univ}}$. Tensoring $\alpha(\iota(\bar{\sigma}^{-1}, A) \circ r)$ with W over $W(\bar{\sigma})$ gives a homomorphism of right W-algebras $\alpha(\iota(\bar{\sigma}^{-1}, A) \circ r) \otimes_{W(\bar{\sigma})} 1 : R_{\bar{\rho}_F}^{\bar{\sigma}} \to A$ such that $r \cong (\alpha(\iota(\bar{\sigma}^{-1}, A) \circ r) \otimes_{W(\bar{\sigma})} 1) \circ \iota(\bar{\sigma}, R_{\bar{\rho}_F}) \circ \bar{\rho}_F^{\mathrm{univ}}$. This shows that the right W-algebra $R_{\bar{\rho}_F}^{\bar{\sigma}}$ satisfies the universal property for $\bar{\rho}_F^{\bar{\sigma}}$. With notation as above, when $r = (\bar{\rho}_F^{\bar{\sigma}})^{\mathrm{univ}}$ we set $\varphi = \alpha(\iota(\bar{\sigma}^{-1}, R_{\bar{\rho}_F^{\bar{\sigma}}}) \circ (\bar{\rho}_F^{\bar{\sigma}})^{\mathrm{univ}}) \otimes_{W(\bar{\sigma})} 1$, so

$$(\bar{\rho}_F^{\bar{\sigma}})^{\mathrm{univ}} \cong \varphi \circ \iota(\bar{\sigma}, R_{\bar{\rho}_F}) \circ \bar{\rho}_F^{\mathrm{univ}}. \tag{1}$$

In particular, φ is a right W-algebra homomorphism.

Finally, let $i : R_{\bar{\eta} \otimes \bar{\rho}_F} \to {}^{\mathcal{O}}R_{\bar{\eta} \otimes \bar{\rho}_F}$ be the map given by $x \mapsto 1 \otimes x$. If A is a W-algebra and $r : G_{\mathbb{Q}}^N \to \mathrm{GL}_2(A)$ is a deformation of $\bar{\rho}_F$ then $\eta \otimes r : G_{\mathbb{Q}}^N \to \mathrm{GL}_2({}^{\mathcal{O}}A)$ is a deformation of $\bar{\eta} \otimes \bar{\rho}_F$. Hence there is a unique W-algebra homomorphism $\alpha(\eta \otimes r) : R_{\bar{\eta} \otimes \bar{\rho}_F} \to {}^{\mathcal{O}}A$ such that $\eta \otimes r \cong \alpha(\eta \otimes r) \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}}$. We can extend $\alpha(\eta \otimes r)$ to an $\mathcal{O}$-algebra homomorphism $1 \otimes \alpha(\eta \otimes r) : {}^{\mathcal{O}}R_{\bar{\eta} \otimes \bar{\rho}_F} \to {}^{\mathcal{O}}A$ by sending $x \otimes y$ to $(x \otimes 1)\alpha(\eta \otimes r)(y)$. In particular, $\eta \otimes r \cong (1 \otimes \alpha(\eta \otimes r)) \circ i \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}}$. When $r = \bar{\rho}_F^{\mathrm{univ}}$, let ψ denote $\alpha(\eta \otimes \bar{\rho}_F^{\mathrm{univ}})$, so

$$\eta \otimes \bar{\rho}_F^{\mathrm{univ}} \cong (1 \otimes \psi) \circ i \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}}. \qquad \square$$

Let A be a W-algebra. We would like to define a ring homomorphism $m(\bar{\sigma}, A) : A^{\bar{\sigma}} \to A$ such that $m(\bar{\sigma}, A) \circ \iota(\bar{\sigma}, A)$ is a lift of $\bar{\sigma}$. When $A = \mathbb{F}$ we can do this by defining $m(\bar{\sigma}, \mathbb{F})(x \otimes y) = \bar{\sigma}(x)y$. Similarly, when $A = W$ we can define $m(\bar{\sigma}, W)(x \otimes y) = W(\bar{\sigma})(x)y$. If $A = W[T]$ or $W[\![T]\!]$ then $A^{\bar{\sigma}} = W^{\bar{\sigma}}[T]$ or $W^{\bar{\sigma}}[\![T]\!]$, and we can define $m(\bar{\sigma}, A)$ by simply applying $m(\bar{\sigma}, W)$ to the coefficients of the polynomials or power series. However, for a general W-algebra A it is not necessarily possible to define $m(\bar{\sigma}, A)$ or to lift $\bar{\sigma}$. (If A happens to be smooth over W then it is always possible to lift $\bar{\sigma}$ to A.) Note that by Nakayama's lemma, if $m(\bar{\sigma}, A)$ exists then $m(\bar{\sigma}, A) \circ \iota(\bar{\sigma}, A)$ is a ring automorphism of A.

Fortunately, we do not need $m(\bar{\sigma}, A)$ to exist for all W-algebras; just for $\mathbb{I}'$. Our strategy is to prove that if $\bar{\rho}_F^{\bar{\sigma}} \cong \bar{\eta} \otimes \bar{\rho}_F$, then the ring homomorphism $m(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F})$ exists.

Lemma 3.3. *If $\bar{\rho}_F$ is absolutely irreducible and $\bar{\rho}_F^{\bar{\sigma}} \cong \bar{\eta} \otimes \bar{\rho}_F$ then there is a ring homomorphism $m(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F}) : {}^{\mathcal{O}}R_{\bar{\rho}_F}^{\bar{\sigma}} \to {}^{\mathcal{O}}R_{\bar{\rho}_F}$ that is a lift of $m(\bar{\sigma}, \mathbb{F})$. In particular, $m(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F}) \circ \iota(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F})$ is a lift of $\bar{\sigma}$.*

Proof. With notation as in Lemma 3.2 define $m(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F}) = (1 \otimes \psi) \circ (1 \otimes \varphi)$. We will show that $1 \otimes \varphi$ induces $m(\bar{\sigma}, \mathbb{F})$ and $1 \otimes \psi$ induces the identity on $\mathbb{F}$. Note that $\mathbb{F}$ is the residue field of $\mathcal{O}$ since $\bar{\chi}$, and hence $\bar{\eta}$, takes values in $\mathbb{F}$. Therefore all of the tensor products with $\mathcal{O}$ residually disappear. Hence it suffices to show that φ induces $m(\bar{\sigma}, \mathbb{F})$ and ψ acts trivially on $\mathbb{F}$.

By definition $\mathbb{F}$ is generated by $\{\overline{a(\ell, F)} : \ell \nmid N\}$. Therefore it suffices to check that ψ acts trivially on $\overline{a(\ell, F)}$ for any prime ℓ not dividing N. But $\psi \circ (\bar{\eta} \otimes \bar{\rho}_F)^{\mathrm{univ}} \cong \eta \otimes \bar{\rho}_F^{\mathrm{univ}}$. Evaluating at Frob_ℓ, taking traces, and reducing to the residue field shows that ψ induces the identity on $\mathbb{F}$.

Let $\bar{\varphi} : \mathbb{F} \otimes_{\bar{\sigma}} \mathbb{F} \to \mathbb{F}$ be the residual map induced by φ. By reducing (1) to the residue field we find that $\bar{\sigma} \circ \bar{\rho}_F \cong \bar{\varphi} \circ \iota(\bar{\sigma}, \mathbb{F}) \circ \bar{\rho}_F$. By universality we conclude that $\bar{\sigma} = \bar{\varphi} \circ \iota(\bar{\sigma}, \mathbb{F})$. But $\bar{\sigma} = m(\bar{\sigma}, \mathbb{F}) \circ \iota(\bar{\sigma}, \mathbb{F})$ and hence $\bar{\varphi} = m(\bar{\sigma}, \mathbb{F})$, as desired. $\square$

Define $\Sigma = (1 \otimes \psi) \circ (1 \otimes \varphi) \circ (1 \otimes \iota(\bar{\sigma}, R_{\bar{\rho}_F}))$. By the proof of Lemma 3.3 we see that Σ is a lift of $\bar{\sigma}$ to ${}^{\mathcal{O}}R_{\bar{\rho}_F}$. In the next subsection we use automorphic techniques to descend Σ to $\mathbb{I}'$. In order to do so we need the following properties of Σ.

Proposition 3.4. (1) *For all $w \in W$ we have $\Sigma(1 \otimes w) = 1 \otimes W(\bar{\sigma})(w)$.*

(2) *For all $x \in \mathcal{O}$ we have $\Sigma(x \otimes 1) = x \otimes 1$.*

(3) *The automorphism $\bar{\sigma}$ of $\mathbb{F}$ is necessarily trivial and hence, under the assumption that the order of χ is a power of 2 and $p \neq 2$, it follows that η is a quadratic character.*

(4) *The automorphism Σ of $R_{\bar{\rho}_F}$ is a lift of σ.*

Proof. The first point is the most subtle. The key point is that φ is a *right* W-algebra homomorphism. Let $w \in W$. Then

$$(1 \otimes \iota(\bar{\sigma}, {}^{\mathcal{O}}R_{\bar{\rho}_F}))(1 \otimes w) = 1 \otimes w \otimes 1 = 1 \otimes 1 \otimes W(\bar{\sigma})(w).$$

Since φ is a right W-algebra homomorphism and ψ is a W-algebra homomorphism we see that $\Sigma(1 \otimes w) = 1 \otimes W(\bar{\sigma})(w)$, as claimed.

The fact that $\Sigma(x \otimes 1) = x \otimes 1$ for all $x \in \mathcal{O}$ follows directly from the definition of Σ.

The first two facts imply that $W(\bar{\sigma})$ is trivial. Indeed, for any $w \in W$ we have $w \otimes 1 = 1 \otimes w \in {}^{\mathcal{O}}R_{\bar{\rho}_F}$. Therefore by the first two facts, in ${}^{\mathcal{O}}R_{\bar{\rho}_F}$ we have

$$w \otimes 1 = \Sigma(w \otimes 1) = \Sigma(1 \otimes w) = 1 \otimes W(\bar{\sigma})(w) = W(\bar{\sigma})(w) \otimes 1.$$

The ring homomorphism $\mathcal{O} \to {}^{\mathcal{O}}R_{\rho_F}$ is injective since $R_{\bar{\rho}_F}$ covers $\mathbb{I}'$ and $\mathbb{I}' \supset \mathcal{O}$. Therefore $W(\bar{\sigma})$ and hence $\bar{\sigma}$ must be trivial.

Therefore $\bar{\rho}_F \cong \bar{\eta} \otimes \bar{\rho}_F$. Taking determinants we find that $\det \bar{\rho}_F = \bar{\eta}^2 \det \bar{\rho}_F$ and hence $\bar{\eta}$ is quadratic. Therefore the values of η are of the form $\pm\zeta$, where ζ is a p-power root of unity. But by assumption η takes values in $\mathbb{Z}_p[\chi]$ and χ has 2-power order. Since $p \neq 2$ it follows that η must be quadratic.

In view of the previous parts of the current proposition we see that $\mathcal{O} = W$ and hence ${}^{\mathcal{O}}R_{\bar{\rho}_F} = R_{\bar{\rho}_F}$. Furthermore, the first two maps in the definition of Σ become trivial and hence $\Sigma = \psi$. By definition of ψ we have $\psi \circ \bar{\rho}_F^{\mathrm{univ}} \cong \eta \otimes \bar{\rho}_F^{\mathrm{univ}}$.

Let $\alpha = \alpha(\rho_F) : R_{\bar{\rho}_F} \to \mathbb{I}'$ and regard $\mathfrak{P}'_i : \mathbb{I}' \to \overline{\mathbb{Q}}_p$ as an algebra homomorphism. Since $\rho_1^\sigma \cong \eta \otimes \rho_2$ it follows from the definitions of all maps involved that

$$\sigma \circ \mathfrak{P}'_1 \circ \alpha \circ \bar{\rho}_F^{\mathrm{univ}} \cong \mathfrak{P}'_2 \circ \alpha \circ \Sigma \circ \bar{\rho}_F^{\mathrm{univ}}.$$

By universality $\sigma \circ \mathfrak{P}'_1 \circ \alpha = \mathfrak{P}'_2 \circ \alpha \circ \Sigma$ and thus Σ is a lift of σ. □

Descending Σ to $\mathbb{I}'$ via automorphic methods. To prove Theorem 3.1 it now remains to show that Σ descends to an automorphism of $\mathbb{I}'$. Let us describe the strategy of proof before proceeding. We begin by showing that the character η is unramified at p. Once we know this, it is fairly straightforward to check that the irreducible component $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ is modular in the sense that it is an irreducible component of an ordinary Hecke algebra of *some* tame level and nebentypus. We then verify that the tame level and nebentypus of $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ match those for $\operatorname{Spec} \mathbb{I}'$, so we have two irreducible components of the same Hecke algebra. Finally, $\mathfrak{P}'_1$ is an arithmetic point on both $\operatorname{Spec} \mathbb{I}'$ and $\Sigma^*(\operatorname{Spec} \mathbb{I}')$. As the ordinary Hecke algebra is étale over Λ at arithmetic points [Hida 2006, Proposition 3.78], the two irreducible components $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ and $\operatorname{Spec} \mathbb{I}'$ must coincide. In other words, Σ descends to $\mathbb{I}'$ as desired. There is a technical point that $\operatorname{Spec} \mathbb{I}'$ and $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ are only irreducible components of the algebra generated by Hecke operators away from N, so in order to use étaleness we must associate to $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ a primitive irreducible component $\operatorname{Spec} \mathbb{J}$ of the full Hecke algebra. See the discussion after Corollary 3.7.

Lemma 3.5. *Let $\rho_1, \rho_2 : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be ordinary representations such that the inertia group acts by an infinite order character on the kernel of the unique p-unramified quotient of each ρ_i. Assume there is an automorphism $\sigma \in G_{\mathbb{Q}_p}$ and a finite order character η such that $\rho_1^\sigma \cong \eta \otimes \rho_2$. Then η is unramified at p.*

Proof. Since ρ_i is p-ordinary, by choosing bases appropriately we may assume $\rho_i = \left(\begin{smallmatrix} \varepsilon_i & * \\ 0 & \delta_i \end{smallmatrix}\right)$ with δ_i unramified. By assumption $\varepsilon_i|_{I_p}$ has infinite order. As $\rho_1^\sigma \cong \eta \otimes \rho_2$, it follows that for some $\boldsymbol{x} \in \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ we have $\rho_1^\sigma = \boldsymbol{x}(\eta \otimes \rho_2)\boldsymbol{x}^{-1}$. Write $\boldsymbol{x} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\eta \otimes \rho_2 = \left(\begin{smallmatrix} \eta\varepsilon_2 & u \\ 0 & \eta\delta_2 \end{smallmatrix}\right)$. A straightforward matrix computation shows that on I_p we have

$$\begin{pmatrix} \varepsilon_1^\sigma & * \\ 0 & 1 \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} (ad\varepsilon_2 - bc)\eta - acu & * \\ c(d\eta(\varepsilon_2 - 1) - cu) & (ad - bc\varepsilon_2)\eta + acu \end{pmatrix}.$$

Hence either $c = 0$ or $cu = d\eta(\varepsilon_2 - 1)$.

If $c = 0$ then on I_p we have

$$\begin{pmatrix} \varepsilon_1^\sigma & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \eta\varepsilon_2 & * \\ 0 & \eta \end{pmatrix},$$

and so $\eta|_{I_p} = 1$, as desired. If $cu = d\eta(\varepsilon_2 - 1)$ then on I_p we have

$$\begin{pmatrix} \varepsilon_1^\sigma & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \eta & * \\ 0 & \eta\varepsilon_2 \end{pmatrix}.$$

Therefore we have $\varepsilon_1^\sigma|_{I_p} = \eta|_{I_p} = \varepsilon_2^{-1}|_{I_p}$. But this is impossible since $\varepsilon_i|_{I_p}$ has infinite order by assumption while η has finite order. Therefore η must be unramified. $\square$

In what follows we use Wiles's interpretation of Hida families [Wiles 1988]. Namely for a finite extension $\mathbb{J}$ of Λ_χ, a formal power series $G = \sum_{n=1}^{\infty} a(n, G)q^n$ is a $\mathbb{J}$-*adic cusp form* of level $\Gamma_0(N)$ and character χ if for almost all arithmetic primes $\mathfrak{P}$ of $\mathbb{J}$, the specialization of G at $\mathfrak{P}$ gives the q-expansion of an element $g_{\mathfrak{P}}$ of $S_k(\Gamma_0(Np^{r(\varepsilon)}), \varepsilon\chi\omega^{-k})$, where $\mathfrak{P}$ lies over $P_{k,\varepsilon}$. One defines the Hecke operators by the usual formulae on coefficients of q-expansions. We say G is *ordinary* if it is an eigenform for the Hecke operators whose eigenvalue under $U(p)$ is in $\mathbb{J}^\times$. Let $\mathbb{S}(N, \chi; \mathbb{J})$ be the $\mathbb{J}$-submodule of $\mathbb{J}[[q]]$ spanned by all $\mathbb{J}$-adic cusp forms of level $\Gamma_0(N)$ and character χ that are also Hecke eigenforms. Let $\mathbb{S}^{\mathrm{ord}}(N, \chi; \mathbb{J})$ denote the $\mathbb{J}$-subspace of $\mathbb{S}(N, \chi; \mathbb{J})$ spanned by all ordinary $\mathbb{J}$-adic cusp forms.

For each Dirichlet character ψ, we shall write $c(\psi) \in \mathbb{Z}^+$ for the conductor of ψ. Let $\psi : (\mathbb{Z}/L\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be a Dirichlet character. Let η be a primitive Dirichlet character with values in $\mathbb{Z}[\chi]$. (Every twist character of F has this property by Lemma 3.11.) Denote by $M(\psi, \eta)$ the least common multiple of L, $c(\eta)^2$, and $c(\psi)c(\eta)$. By [Shimura 1971, Proposition 3.64], there is a linear map

$$\begin{aligned} R_{\psi,\eta} : S_k(\Gamma_0(M(\psi, \eta)), \psi) &\to S_k(\Gamma_0(M(\psi, \eta)), \eta^2\psi) \\ f = \sum_{n=1}^{\infty} a(n, f)q^n &\mapsto \eta f = \sum_{n=1}^{\infty} \eta(n)a(n, f)q^n. \end{aligned}$$

We now show that there is an analogous map in the $\mathbb{J}$-adic setting.

Lemma 3.6. *There is a well defined $\mathbb{J}$-linear map*

$$\begin{aligned} \mathbb{R}_{\chi,\eta} : \mathbb{S}(M(\chi, \eta), \chi; \mathbb{J}) &\to \mathbb{S}(M(\chi, \eta), \eta^2\chi; \mathbb{J}) \\ G = \sum_{n=1}^{\infty} a(n, G)q^n &\mapsto \eta G = \sum_{n=1}^{\infty} \eta(n)a(n, G)q^n. \end{aligned}$$

If $p \nmid c(\eta)$ then $\mathbb{R}_{\chi,\eta}$ sends $\mathbb{S}^{\mathrm{ord}}(M(\chi, \eta), \chi; \mathbb{J})$ to $\mathbb{S}^{\mathrm{ord}}(M(\chi, \eta), \eta^2\chi; \mathbb{J})$.

Proof. Let $\mathfrak{P}$ be an arithmetic prime of $\mathbb{J}$, and let $P_{k,\varepsilon}$ be the arithmetic prime of Λ lying under $\mathfrak{P}$. If $G \in \mathbb{S}^{\mathrm{ord}}(M(\chi, \eta), \chi; \mathbb{J})$ then

$$g_{\mathfrak{P}} \in S_k(\Gamma_0(M(\chi, \eta)p^{r(\varepsilon)}), \varepsilon\chi\omega^{-k}).$$

Let $\psi = \varepsilon\chi\omega^{-k}$. It follows easily from the definitions that $M(\psi, \eta) = M(\chi, \eta)p^{r(\varepsilon)}$. Therefore

$$\eta g_{\mathfrak{P}} = R_{\psi,\eta}(g_{\mathfrak{P}}) \in S_k(\Gamma_0(M(\psi, \eta)), \eta^2\psi) = S_k(\Gamma_0(M(\chi, \eta)p^{r(\varepsilon)}), \eta^2\varepsilon\chi\omega^{-k}),$$

so $\eta G \in \mathbb{S}(M(\chi,\eta), \eta^2\chi; \mathbb{J})$.

For the statement about ordinarity, we may assume G is a normalized eigenform, so $a(p, G)$ is the eigenvalue of G under the $U(p)$ operator. If G is ordinary then $a(p, G) \in \mathbb{J}^\times$. Hence $\eta(p)a(p, G) = a(p, \eta G) \in \mathbb{J}^\times$ if and only if $\eta(p) \neq 0$. □

Corollary 3.7. *The representation associated to* $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ *is modular of level* $M(\chi,\eta)$ *and nebentypus* χ.

Proof. The representation associated to $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ is isomorphic to $\eta \otimes \rho_F$. Consider the formal q-expansion $\eta F := \sum_{n=1}^\infty \eta(n)a(n, F)q^n \in \mathbb{I}[[q]]$. By Lemma 3.6 and Lemma 3.5 we see that ηF is a Hida family of level $\Gamma_0(M(\chi,\eta))$ and nebentypus $\eta^2\chi$. Clearly the Galois representation of ηF is isomorphic to $\eta \otimes \rho_F$ since their traces on Frobenius elements agree on all but finitely many primes. Since $\eta \otimes \rho_F \cong \alpha \circ \Sigma \circ \bar{\rho}_F^{\text{univ}}$, it follows that $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ is modular of level $M(\chi,\eta)$ and nebentypus $\eta^2\chi$. By Proposition 3.4 we know that η is quadratic and hence $\eta^2\chi = \chi$. □

For any integer multiple M of N, let $\boldsymbol{h}^{\text{ord}}(M, \chi; \Lambda_\chi)'$ be the Λ_χ-subalgebra of $\boldsymbol{h}^{\text{ord}}(M, \chi; \Lambda_\chi)$ generated by $\{T(n) : (n, N) = 1\}$. Corollary 3.7 shows that $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ is an irreducible component of $\operatorname{Spec} \boldsymbol{h}^{\text{ord}}(M(\chi,\eta), \chi; \Lambda_\chi)'$. There is a natural map $\beta : \operatorname{Spec} \boldsymbol{h}^{\text{ord}}(M, \chi; \Lambda_\chi) \to \operatorname{Spec} \boldsymbol{h}^{\text{ord}}(M, \chi; \Lambda_\chi)'$ coming from the natural inclusion of algebras. An irreducible component $\operatorname{Spec} \mathbb{J}'$ of $\boldsymbol{h}^{\text{ord}}(M, \chi; \Lambda_\chi)'$ essentially corresponds to the data of the Fourier coefficients away from N. The preimage $\beta^{-1}(\operatorname{Spec} \mathbb{J}')$ is a union of irreducible components whose Fourier coefficients agree with those of $\mathbb{J}'$ away from N. By the theory of newforms we know that there is a unique primitive irreducible component $\operatorname{Spec} \mathbb{J}$ of $\boldsymbol{h}^{\text{ord}}(M, \chi; \Lambda_\chi)$ that projects to $\operatorname{Spec} \mathbb{J}'$ under β. Let $\operatorname{Spec} \mathbb{J}$ be the primitive component of $\boldsymbol{h}^{\text{ord}}(M(\chi,\eta), \chi; \Lambda_\chi)$ that projects to $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ under β. By the proof of Corollary 3.7, $\mathbb{J}$ is the primitive form associated to ηF and so $\rho_{\mathbb{J}} \cong \eta \otimes \rho_F$.

Since $N | M(\chi,\eta)$ there is a natural inclusion

$$\operatorname{Spec} \boldsymbol{h}^{\text{ord}}(N, \chi; \Lambda_\chi) \hookrightarrow \operatorname{Spec} \boldsymbol{h}^{\text{ord}}(M(\chi,\eta), \chi; \Lambda_\chi).$$

We wish to show that $\operatorname{Spec} \mathbb{J}$ is an irreducible component of $\operatorname{Spec} \boldsymbol{h}^{\text{ord}}(N, \chi; \Lambda_\chi)$. We do this locally by computing the level of $\operatorname{Spec} \mathbb{J}$ at each prime ℓ. Let v_ℓ denote the usual ℓ-adic valuation on the integers, normalized such that $v_\ell(\ell) = 1$.

Proposition 3.8. *The primitive component* $\operatorname{Spec} \mathbb{J}$ *is an irreducible component of* $\operatorname{Spec} \boldsymbol{h}^{\text{ord}}(N, \chi; \Lambda_\chi)$.

Proof. First note that if $\ell \nmid c(\eta)$ then $v_\ell(M(\chi,\eta)) = v_\ell(N)$ since $c(\chi)|N$. In particular, by Lemma 3.5 we have $v_p(M(\chi,\eta)) = v_p(N)$.

Fix a prime $\ell \neq p$ at which η is ramified. For a pro-p ring A and representation $\pi : G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(A)$, let $C_\ell(\pi)$ denote the ℓ-conductor of π. See [Hida 2015, p. 659] for the precise definition. When π is the representation associated to a classical form f, the ℓ-conductor of π is related to the level of f by the proof of the local Langlands conjecture for GL_2. Indeed, when f is a classical newform of level N we have $C_\ell(\rho_f) = \ell^{v_\ell(N)}$. If f is new away from p and $\ell \neq p$ then we still have $C_\ell(\rho_f) = \ell^{v_\ell(N)}$.

First suppose that $\rho_F|_{I_\ell}$ is not reducible indecomposable. Then $(\eta \otimes \rho_F)|_{I_\ell}$ is not reducible indecomposable either. Therefore $C_\ell(\rho_F) = C_\ell(\rho_{f_{\mathfrak{P}_1}})$ and $C_\ell(\eta \otimes \rho_F) = C_\ell(\eta \otimes \rho_{f_{\mathfrak{P}_2}})$ [Hida 2015, Lemma 10.2(2)]. Since Galois action does not change conductors we have

$$C_\ell(\rho_F) = C_\ell(\rho_{f_{\mathfrak{P}_1}}) = C_\ell(\rho^\sigma_{f_{\mathfrak{P}_1}}) = C_\ell(\eta \otimes \rho_{f_{\mathfrak{P}_2}}) = C_\ell(\eta \otimes \rho_F).$$

Since F is a primitive form we have that $f_{\mathfrak{P}_1}$ is new away from p and hence $C_\ell(\rho_{f_{\mathfrak{P}_1}}) = \ell^{v_\ell(N)}$. On the other hand since $\mathbb{J}$ is primitive we have $C_\ell(\rho_{\mathbb{J}}) = C_\ell(\eta \otimes \rho_F)$ is equal to the ℓ-part of the level of $\mathbb{J}$, which gives the desired result at ℓ.

Now assume that $\rho_F|_{I_\ell}$ is reducible indecomposable. By Lemma 10.1(4) of [Hida 2015] we have a character $\psi : G_{\mathbb{Q}_\ell} \to \mathbb{I}^\times$ such that $\rho_F|_{G_{\mathbb{Q}_\ell}} \cong \left(\begin{smallmatrix} \mathcal{N}\psi & * \\ 0 & \psi \end{smallmatrix}\right)$, where $\mathcal{N}$ is the unramified cyclotomic character acting on p-power roots of unity and $\psi|_{I_\ell}$ has finite order. Note that since η is a quadratic character, $c(\eta)$ is squarefree away from 2. Similarly, since χ has 2-power order it follows that $c(\chi)$ is a power of 2 times a product of distinct odd primes. Therefore, for odd primes ℓ it is enough to show that $c_\ell(\eta)^2 | N$. We use the description of the conductor of a locally reducible indecomposable representation given on page 660 of [Hida 2015]. Let $\psi_1 = \psi \mod \mathfrak{P}_1$. Then $\rho_{f_{\mathfrak{P}_2}}|_{I_\ell} \cong \left(\begin{smallmatrix} \eta^{-1}\psi_1^\sigma & * \\ 0 & \eta^{-1}\psi_1^\sigma \end{smallmatrix}\right)$. If ψ_1 is unramified then $\eta^{-1}\psi_1^\sigma$ is ramified and hence

$$c_\ell(\eta)^2 = c_\ell(\eta^{-1})^2 = c_\ell(\eta^{-1}\psi_1^\sigma)^2 = C_\ell(\rho_{f_{\mathfrak{P}_2}}).$$

Since $\rho_{f_{\mathfrak{P}_2}}$ is a specialization of ρ_F we have $C_\ell(\rho_{f_{\mathfrak{P}_2}}) | N$ giving the desired result. Now suppose that ψ_1 is ramified. Then $c_\ell(\eta) = \ell | c_\ell(\psi_1)$ and $C_\ell(\rho_{f_{\mathfrak{P}_1}}) = c_\ell(\psi_1)^2$. Again, since $\rho_{f_{\mathfrak{P}_1}}$ is a specialization of ρ_F we see that $c_\ell(\eta)^2 | N$.

Finally the case $\ell = 2$ follows from the assumption that $2c(\chi) | N$. We are able to make this hypothesis by Proposition 3.9.

Therefore $\operatorname{Spec} \mathbb{J}$ is an irreducible component of $\boldsymbol{h}^{\mathrm{ord}}(N, \chi; \Lambda_\chi)$, as desired. □

Proof of Theorem 3.1. We first lift σ to an automorphism Σ of ${}^{\mathcal{O}}R_{\bar{\rho}_F}$ by Lemma 3.3. We are able to use the definition of Σ to show that ${}^{\mathcal{O}}R_{\bar{\rho}_F} = R_{\bar{\rho}_F}$ and that η is quadratic in Proposition 3.4. By Proposition 3.8 we see that $\Sigma^*(\operatorname{Spec} \mathbb{I}')$ is a component of $\operatorname{Spec} \boldsymbol{h}^{\mathrm{ord}}(N, \chi; \Lambda_\chi)'$. Since $\rho^\sigma_{f_{\mathfrak{P}_1}} \cong \eta \otimes \rho_{f_{\mathfrak{P}_2}}$ it follows that the arithmetic point $\mathfrak{P}'_1$ is a point on both $\operatorname{Spec} \mathbb{I}'$ and $\Sigma^*(\operatorname{Spec} \mathbb{I}')$. We claim that in fact $\mathfrak{P}_1 \in \operatorname{Spec} \mathbb{I} \cap \operatorname{Spec} \mathbb{J}$.

Note that $\mathbb{J}$ is the primitive family passing through $f^\sigma_{\mathfrak{P}_1}$. (We know $f^\sigma_{\mathfrak{P}_1}$ is primitive since $f_{\mathfrak{P}_1}$ is an arithmetic specialization of the primitive family F, and

Galois conjugation does not change the level.) Indeed, $\mathbb{J}$ is the primitive form of ηF. Let $\mathfrak{P} \in \operatorname{Spec} \mathbb{J}$ such that $\mathbb{J} \mod \mathfrak{P} = f^{\sigma}_{\mathfrak{P}_1}$. On the other hand the kernel of the specialization map giving rise to $f^{\sigma}_{\mathfrak{P}_1}$ is $\mathfrak{P}_1$, since $f^{\sigma}_{\mathfrak{P}_1} = \sigma(F \mod \mathfrak{P}_1)$. Therefore $\mathfrak{P} = \mathfrak{P}_1 \in \operatorname{Spec} \mathbb{I} \cap \operatorname{Spec} \mathbb{J}$.

Since $\boldsymbol{h}^{\mathrm{ord}}(N, \chi; \Lambda_\chi)$ is étale over arithmetic points of Λ by [Hida 2006, Proposition 3.78] it follows that the irreducible components $\operatorname{Spec} \mathbb{I}$ and $\operatorname{Spec} \mathbb{J}$ must coincide and hence $\Sigma^*(\operatorname{Spec} \mathbb{I}') = \operatorname{Spec} \mathbb{I}'$. Therefore Σ descends to the desired automorphism $\tilde{\sigma}$ of $\mathbb{I}'$. The fact that $\tilde{\sigma}(a(\ell, F)) = \eta(\ell)a(\ell, F)$ for almost all primes ℓ follows from specializing $\Sigma \circ \bar{\rho}_F^{\mathrm{univ}} \cong \eta \otimes \bar{\rho}_F^{\mathrm{univ}}$ to $\mathbb{I}'$ and taking traces. Finally, $\sigma \circ \mathfrak{P}_1' = \mathfrak{P}_2' \circ \tilde{\sigma}$ since Σ is a lift of σ by Proposition 3.4. □

Nebentypus and twist characters. We end this section with some information about twist characters. In particular Proposition 3.9 shows that we may assume from the beginning that χ has 2-power order with $2c(\chi)|N$.

Note that the ring $\mathbb{I}_0$ depends on F. However, if ψ is a character then ψF has the same group of conjugate self-twists as that of F, and thus the same fixed ring $\mathbb{I}_0$. Indeed, if σ is a conjugate self-twist of F with character η, then a straightforward calculation shows that $\psi^\sigma \eta \psi^{-1}$ is the twist character of σ on ψF.

Proposition 3.9. *There is a Dirichlet character ψ such that the nebentypus $\psi^2\chi$ of ψF has order a power of 2 and $2c(\psi^2\chi)|M(\chi, \psi)$. Furthermore, ρ_F is $\mathbb{I}_0$-full if and only if $\rho_{\psi F}$ is $\mathbb{I}_0$-full.*

Proof. It is well known that the nebentypus of ψF is $\psi^2\chi$ [Shimura 1971, Proposition 3.64]. Write $\chi = \chi_2\xi$, where χ_2 is a character whose order is a power of 2 and ξ is an odd order character. Let $2n-1$ denote the order of ξ. Then $\xi^{2n} = \xi$, so taking $\psi_{\mathrm{odd}} = \xi^{-n}$ we see that $\psi_{\mathrm{odd}}^2\chi = \chi_2\xi^{-2n}\xi = \chi_2$ is a character whose order is a power of 2.

Let 2^{t-1} be the order of $\psi_{\mathrm{odd}}^2\chi$, and let $\psi_2 : (\mathbb{Z}/2^t\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$ be the associated primitive character. Let $\psi = \psi_2\psi_{\mathrm{odd}}$. Then $2^{2t}|M(\chi, \psi)$ whereas $c_2(\psi^2\chi)|2^{t-1}$. Since $t \geq 1$ we see that

$$2c_2(\psi^2\chi)|2^t|M(\chi, \psi),$$

as desired.

Suppose that $\rho_{\psi F}$ is $\mathbb{I}_0$-full. Since ψ is a finite order character, $\ker\psi$ is an open subgroup of $G_{\mathbb{Q}}$. Thus $\rho_{\psi F}|_{\ker\psi}$ is also $\mathbb{I}_0$-full. Note that $\rho_{\psi F}|_{\ker\psi} = \rho_F|_{\ker\psi}$. Thus ρ_F is $\mathbb{I}_0$-full. □

We finish this section by recalling a lemma of Momose that shows that twist characters are valued in $\mathbb{Z}_p[\chi]$. Thus Theorem 3.1 says that whenever a conjugate self-twist of a classical specialization $f_{\mathfrak{P}}$ of F induces an automorphism of $\mathbb{Q}_p(f_{\mathfrak{P}})$, that conjugate self-twist can be lifted to a conjugate self-twist of the whole family F.

Lemma 3.10 [Momose 1981, Lemma 1.5]. *If σ is a conjugate self-twist of $f \in S_k(\Gamma_0(N), \chi)$, then η_σ is the product of a quadratic character with some power of χ. In particular, η_σ takes values in $\mathbb{Z}[\chi]$.*

The proof of Lemma 3.10 is not difficult and goes through without change in the $\mathbb{I}$-adic setting. For completeness, we give the proof in that setting.

Lemma 3.11. *If σ is a conjugate self-twist of F then η_σ is the product of a quadratic character with some power of χ. In particular, η_σ has values in $\mathbb{Z}[\chi]$.*

Proof. As $\bar{\rho}_F$ is absolutely irreducible, $\rho_F^\sigma \cong \eta_\sigma \otimes \rho_F$. Thus $\sigma(\det \rho_F) = \eta_\sigma^2 \det \rho_F$. Define $\kappa : 1 + p\mathbb{Z}_p \to \Lambda^\times$ by $\kappa((1+p)^s) = (1+T)^s$ for $s \in \mathbb{Z}_p$. Recall that for all primes ℓ not dividing N we have

$$\det \rho_F(\mathrm{Frob}_\ell) = \chi(\ell)\kappa(\langle \ell \rangle)\ell^{-1}.$$

Substituting this expression for $\det \rho_F$ into $\sigma(\det \rho_F) = \eta_\sigma^2 \det \rho_F$ yields $\eta_\sigma^2 = \chi^\sigma \chi^{-1}$.

Recall that $\chi^\sigma = \chi^\alpha$ for some integer $\alpha > 0$. To prove the result it suffices to show that there is some $i \in \mathbb{Z}$ such that $\eta_\sigma^2 = \chi^{2i}$. If χ has odd order then there is a positive integer j for which $\chi = \chi^{2j}$. Thus $\eta_\sigma^2 = \chi^{\sigma-1} = \chi^{2j(\alpha-1)}$. If χ has even order then χ^σ also has even order since σ is an automorphism. Thus α must be odd. Then $\alpha - 1$ is even and $\eta_\sigma^2 = \chi^\sigma \chi^{-1} = \chi^{\alpha-1}$, as desired. □

4. Sufficiency of open image in product

Recall that $H_0 = \bigcap_{\sigma\in\Gamma} \ker(\eta_\sigma)$ and $H = H_0 \cap \ker(\det \bar{\rho}_F)$. For a variety of reasons, our methods work best for representations valued in $\mathrm{SL}_2(\mathbb{I}_0)$ rather than $\mathrm{GL}_2(\mathbb{I}')$. Therefore, for the next three sections we assume the following theorem, the proof of which is given in Section 7.

Theorem 4.1. *Assume that $\bar{\rho}_F$ is absolutely irreducible and H_0-regular. If $V = \mathbb{I}'^2$ is the module on which $G_\mathbb{Q}$ acts via ρ_F, then there is a basis for V such that all of the following happen simultaneously*:

1. *ρ_F is valued in $\mathrm{GL}_2(\mathbb{I}')$.*
2. *$\rho_F|_{D_p}$ is upper triangular.*
3. *$\rho_F|_{H_0}$ is valued in $\mathrm{GL}_2(\mathbb{I}_0)$.*
4. *There is a matrix $\boldsymbol{j} = \left(\begin{smallmatrix}\zeta & 0\\ 0 & \zeta'\end{smallmatrix}\right)$, where ζ and ζ' are roots of unity, such that $\boldsymbol{j}$ normalizes the image of ρ_F and $\zeta \not\equiv \zeta' \bmod p$.*

Let $H' = \ker(\det \bar{\rho}_F)$. For any $h \in H'$ we have $\det \rho_F(h) \in 1 + \mathfrak{m}_{\mathbb{I}'}$. Since $p \neq 2$ and $\mathbb{I}'$ is p-adically complete, we have

$$\sqrt{\det \rho_F(h)} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (\det \rho_F(h) - 1)^n \in \mathbb{I}'^\times.$$

Since ρ_F is a 2-dimensional representation $\rho_F|_{H'} \otimes \left(\sqrt{\det \rho_F|_{H'}}\right)^{-1}$ takes values in $\mathrm{SL}_2(\mathbb{I}')$. Restricting further it follows from Theorem 4.1 that

$$\rho := \rho_F|_H \otimes \left(\sqrt{\det \rho_F|_H}\right)^{-1}$$

takes values in $\mathrm{SL}_2(\mathbb{I}_0)$. Note that the image of ρ is still normalized by the matrix $\boldsymbol{j}$ of Theorem 4.1 since we only modified ρ_F by scalars, which commute with $\boldsymbol{j}$. In Proposition 4.10 we show that ρ_F is $\mathbb{I}_0$-full if and only if ρ is $\mathbb{I}_0$-full. The proof of Proposition 4.10 is postponed until the end of the current section since it uses the theory of Pink–Lie algebras developed below. In the next three sections we prove that ρ is $\mathbb{I}_0$-full.

The purpose of the current section is to make the following reduction step in the proof of Theorem 2.4.

Proposition 4.2. *Assume there is an arithmetic prime P of Λ such that the image of* $\operatorname{Im}\rho$ *in* $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ *is open in the product topology. Then ρ (and hence ρ_F) is $\mathbb{I}_0$-full.*

In the proof we use a result of Pink [1993] that classifies p-profinite subgroups of $\mathrm{SL}_2(A)$ for a complete semilocal p-profinite ring A. (Our assumption that $p > 2$ is necessary for Pink's theory.) We give a brief exposition of the relevant parts of his work for the sake of establishing notation. Define

$$\Theta : \mathrm{SL}_2(A) \to \mathfrak{sl}_2(A), \quad \boldsymbol{x} \mapsto \boldsymbol{x} - \tfrac{1}{2}\operatorname{tr}(\boldsymbol{x}),$$

where we consider $\frac{1}{2}\operatorname{tr}(\boldsymbol{x})$ as a scalar matrix. Let $\mathcal{G}$ be a p-profinite subgroup of $\mathrm{SL}_2(A)$. Define $L_1(\mathcal{G})$ to be the closed subgroup of $\mathfrak{sl}_2(A)$ that is topologically generated by $\Theta(\mathcal{G})$. Let $L_1 \cdot L_1$ be the closed (additive) subgroup of $M_2(A)$ topologically generated by $\{\boldsymbol{x}\boldsymbol{y} : \boldsymbol{x}, \boldsymbol{y} \in \mathcal{G}\}$. Let C denote $\operatorname{tr}(L_1 \cdot L_1)$. Sometimes we will view $C \subset M_2(A)$ as a set of scalar matrices. For $n \geq 2$ define $L_n(\mathcal{G})$ to be the closed (additive) subgroup of $\mathfrak{sl}_2(A)$ generated by

$$[L_1(\mathcal{G}), L_{n-1}(\mathcal{G})] := \{\boldsymbol{x}\boldsymbol{y} - \boldsymbol{y}\boldsymbol{x} : \boldsymbol{x} \in L_1(\mathcal{G}), \boldsymbol{y} \in L_{n-1}(\mathcal{G})\}.$$

Definition 4.3. The *Pink–Lie algebra* of a p-profinite group $\mathcal{G}$ is $L_2(\mathcal{G})$. Whenever we write $L(\mathcal{G})$ without a subscript we shall always mean $L_2(\mathcal{G})$.

As an example one can compute that for an ideal $\mathfrak{a}$ of A, the p-profinite subgroup $\mathcal{G} = \Gamma_A(\mathfrak{a})$ has Pink–Lie algebra $L_2(\mathcal{G}) = \mathfrak{a}^2\mathfrak{sl}_2(A)$. This example plays an important role in what follows.

For $n \geq 1$, define

$$\mathcal{M}_n(\mathcal{G}) = C \oplus L_n(\mathcal{G}) \subset M_2(A)$$
$$\mathcal{H}_n(\mathcal{G}) = \{\boldsymbol{x} \in \mathrm{SL}_2(A) : \Theta(\boldsymbol{x}) \in L_n(\mathcal{G}) \text{ and } \operatorname{tr}(\boldsymbol{x}) - 2 \in C\}.$$

Pink proves that $\mathcal{M}_n(\mathcal{G})$ is a closed $\mathbb{Z}_p$-Lie algebra of $M_2(A)$ and that $\mathcal{H}_n = \mathrm{SL}_2(A) \cap (1 + \mathcal{M}_n)$ for all $n \geq 1$. Furthermore, write

$$\mathcal{G}_1 = \mathcal{G}, \quad \mathcal{G}_{n+1} = (\mathcal{G}, \mathcal{G}_n),$$

where $(\mathcal{G}, \mathcal{G}_n)$ is the closed subgroup of $\mathcal{G}$ topologically generated by the commutators $\{g g_n g^{-1} g_n^{-1} : g \in \mathcal{G}, g_n \in \mathcal{G}_n\}$.

Theorem 4.4 [Pink 1993]. *With notation as above, $\mathcal{G}$ is a closed normal subgroup of $\mathcal{H}_1(\mathcal{G})$. Furthermore, $\mathcal{H}_n(\mathcal{G}) = (\mathcal{G}, \mathcal{G}_n)$ for $n \geq 2$.*

There are two important functoriality properties of the correspondence $\mathcal{G} \mapsto L(\mathcal{G})$ that we will use. First, since Θ is constant on conjugacy classes of $\mathcal{G}$ it follows that $L_n(\mathcal{G})$ is stable under the adjoint action of the normalizer $N_{\mathrm{SL}_2(A)}(\mathcal{G})$ of $\mathcal{G}$ in $\mathrm{SL}_2(A)$. That is, for $\boldsymbol{g} \in N_{\mathrm{SL}_2(A)}(\mathcal{G})$, $\boldsymbol{x} \in L_n(\mathcal{G})$ we have $\boldsymbol{g}\boldsymbol{x}\boldsymbol{g}^{-1} \in L_n(\mathcal{G})$. If $\mathfrak{a}$ is an ideal of A such that $A/\mathfrak{a}$ is p-profinite, then we write $\overline{\mathcal{G}}_{\mathfrak{a}}$ for the p-profinite group $\mathcal{G} \cdot \Gamma_A(\mathfrak{a})/\Gamma_A(\mathfrak{a}) \subseteq \mathrm{SL}_2(A/\mathfrak{a})$. The second functoriality property is that the canonical linear map $L(\mathcal{G}) \to L(\overline{\mathcal{G}}_{\mathfrak{a}})$ induced by $\boldsymbol{x} \mapsto \boldsymbol{x} \bmod \mathfrak{a}$ is surjective.

Let $\mathfrak{m}_0$ be the maximal ideal of $\mathbb{I}_0$, and let $\mathbb{G}$ denote the p-profinite group $\operatorname{Im}\rho \cap \Gamma_{\mathbb{I}_0}(\mathfrak{m}_0)$. The proof of Proposition 4.2 consists of showing that if $\overline{\mathbb{G}}_{P\mathbb{I}_0}$ is open in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ then $\mathbb{G}$ contains $\Gamma_{\mathbb{I}_0}(\mathfrak{a}_0)$ for some nonzero $\mathbb{I}_0$-ideal $\mathfrak{a}_0$. Let $L = L(\mathbb{G})$ be the Pink–Lie algebra of $\mathbb{G}$. Since $\overline{\mathbb{G}}_{P\mathbb{I}_0}$ is open, for every prime $\mathcal{Q}$ of $\mathbb{I}_0$ lying over P there is a nonzero $\mathbb{I}_0/\mathcal{Q}$-ideal $\bar{\mathfrak{a}}_{\mathcal{Q}}$ such that

$$\overline{\mathbb{G}}_{P\mathbb{I}_0} \supseteq \prod_{\mathcal{Q}|P} \Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}}).$$

Thus $L(\overline{\mathbb{G}}_{P\mathbb{I}_0}) \supseteq \oplus_{\mathcal{Q}|P} \bar{\mathfrak{a}}_{\mathcal{Q}}^2 \mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q})$.

Recall from Theorem 4.1 that we have roots of unity ζ and ζ' such that $\zeta \not\equiv \zeta' \bmod p$ and the matrix $\boldsymbol{j} := \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta' \end{smallmatrix}\right)$ normalizes $\mathbb{G}$. Let $\alpha = \zeta\zeta'^{-1}$. A straightforward calculation shows that the eigenvalues of $\mathrm{Ad}(\boldsymbol{j})$ acting on $\mathfrak{sl}_2(\mathbb{I}_0)$ are α, 1, α^{-1}. Note that since $\zeta \neq \zeta'$ either all of α, 1, α^{-1} are distinct or else $\alpha = -1$. For $\lambda \in \{\alpha, 1, \alpha^{-1}\}$ let $L[\lambda]$ be the λ-eigenspace of $\mathrm{Ad}(\boldsymbol{j})$ acting on L. One computes that $L[1]$ is the set of diagonal matrices in L. If $\alpha = -1$ then $L[-1]$ is the set of antidiagonal matrices in L. If $\alpha \neq -1$ then $L[\alpha]$ is the set of upper nilpotent matrices in L, and $L[\alpha^{-1}]$ is the set of lower nilpotent matrices in L. Regardless of the value of α, let $\mathfrak{u}$ denote the set of upper nilpotent matrices in L and $\mathfrak{u}^t$ denote the set of lower nilpotent matrices in L. Let $\mathcal{L}$ be the $\mathbb{Z}_p$-Lie algebra generated by $\mathfrak{u}$ and $\mathfrak{u}^t$ in $\mathfrak{sl}_2(\mathbb{I}_0)$.

Lemma 4.5. *The matrix $\boldsymbol{J} := \left(\begin{smallmatrix} 1+T & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes* $\operatorname{Im}\rho$, *and $\mathcal{L}$ is a Λ-submodule of $\mathfrak{sl}_2(\mathbb{I}_0)$.*

Proof. First we show that $\mathcal{L}$ is a Λ-module assuming that $\boldsymbol{J}$ normalizes $\operatorname{Im}\rho$. Since $\mathcal{L}$ is a $\mathbb{Z}_p$-Lie algebra and $\Lambda = \mathbb{Z}_p[[T]]$, it suffices to show that $\boldsymbol{x} \in \mathcal{L}$ implies $T\boldsymbol{x} \in \mathcal{L}$.

If $\boldsymbol{x} \in \mathfrak{u}$ then a simple computation shows that $\boldsymbol{J}\boldsymbol{x}\boldsymbol{J}^{-1} = (1+T)\boldsymbol{x}$. As $\mathcal{L}$ is an abelian group it follows that $T\boldsymbol{x} = (1+T)\boldsymbol{x} - \boldsymbol{x} \in \mathfrak{u}$. Similarly, for $\boldsymbol{y} \in \mathfrak{u}^t$ we have $T\boldsymbol{y} \in \mathfrak{u}^t$. It follows that $T[\boldsymbol{x}, \boldsymbol{y}] = [T\boldsymbol{x}, \boldsymbol{y}] \in \mathcal{L}$. Any element in $\mathcal{L}$ can be written as a sum of elements in $\mathfrak{u}$, $\mathfrak{u}^t$, and $[\mathfrak{u}, \mathfrak{u}^t]$. Therefore $\mathcal{L}$ is a Λ-submodule of $\mathfrak{sl}_2(\mathbb{I}_0)$.

Now we show that $\boldsymbol{J}$ normalizes $\operatorname{Im}\rho$. The proof is nearly identical to the proof of [Hida 2015, Lemma 1.4] except we do not require $\zeta, \zeta' \in \mathbb{Z}_p$. As in the proof of Proposition 4.10, we know there is an element $\boldsymbol{\tau} = \left(\begin{smallmatrix} 1+T & u \\ 0 & 1 \end{smallmatrix}\right) \in \operatorname{Im}\rho_F$. A straightforward matrix calculation shows that $\boldsymbol{\tau} \in \operatorname{Im}\rho_F|_H$. Writing $t = (1+T)^{1/2}$ and $u' = t^{-1}u$ we see that $\boldsymbol{\tau}' = \left(\begin{smallmatrix} t & u' \\ 0 & t^{-1} \end{smallmatrix}\right) \in \operatorname{Im}\rho$. Since $\rho_F|_H$ and ρ differ only by a character, their images have the same normalizer. In particular, the matrix $\boldsymbol{j}$ from Theorem 4.1 normalizes $\operatorname{Im}\rho$. Hence the commutator $(\boldsymbol{\tau}', \boldsymbol{j}) \in \operatorname{Im}\rho$ and we can compute

$$(\boldsymbol{\tau}', \boldsymbol{j}) = \begin{pmatrix} 1 & u't(1-\alpha) \\ 0 & 1 \end{pmatrix}.$$

Let $\mathfrak{v} = \left\{x \in \mathbb{I}_0 : \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{u}\right\}$. Then $\mathfrak{v}$ is a $\mathbb{Z}_p[\alpha]$-module. Indeed, it is a $\mathbb{Z}_p$-module since we can raise unipotent matrices to $\mathbb{Z}_p$-powers, so it suffices to show that $\mathfrak{v}$ is closed under multiplication by α. This follows by conjugating unipotent elements by $\boldsymbol{j}$. Since $\alpha \not\equiv 1 \mod p$ we have that $1-\alpha$ is a unit in $\mathbb{Z}_p[\alpha]$. Therefore $u't \in \mathfrak{v}$. Let $\boldsymbol{\beta} = \boldsymbol{\tau}'^{-1}\left(\begin{smallmatrix} 1 & u't \\ 0 & 1 \end{smallmatrix}\right)\boldsymbol{\tau}' \in \operatorname{Im}\rho$. Then $t^{-1}\boldsymbol{J} = \boldsymbol{\tau}'\boldsymbol{\beta}^{-1}$ (and hence $\boldsymbol{J}$) normalizes $\operatorname{Im}\rho$. □

The proof of Proposition 4.2 is easier when $\alpha \neq -1$, so we start with that case.

Proof of Proposition 4.2 when $\alpha \neq -1$. We will show that the finitely generated Λ-module

$$X := \mathfrak{sl}_2(\mathbb{I}_0)/\mathcal{L}$$

is a torsion Λ-module. From this it follows that there is a nonzero Λ-ideal $\mathfrak{a}$ such that $\mathfrak{a}\mathfrak{sl}_2(\mathbb{I}_0) \subseteq \mathcal{L}$. Thus

$$(\mathfrak{a}\mathbb{I}_0)^2\mathfrak{sl}_2(\mathbb{I}_0) \subseteq \mathcal{L} \subseteq L$$

since $\mathbb{I}_0\mathfrak{sl}_2(\mathbb{I}_0) = \mathfrak{sl}_2(\mathbb{I}_0)$. But $(\mathfrak{a}\mathbb{I}_0)^2\mathfrak{sl}_2(\mathbb{I}_0)$ is the Pink–Lie algebra of $\Gamma_{\mathbb{I}_0}(\mathfrak{a}\mathbb{I}_0)$ and so $\Gamma_{\mathbb{I}_0}(\mathfrak{a}\mathbb{I}_0) \subseteq \mathbb{G}_2 \subseteq \mathbb{G}$, as desired.

To show that X is a finitely generated Λ-module, recall that the arithmetic prime P in the statement of Proposition 4.2 is a height one prime of Λ. By Nakayama's lemma it suffices to show that X/PX is Λ/P-torsion. The natural epimorphism $\mathfrak{sl}_2(\mathbb{I}_0)/P\mathfrak{sl}_2(\mathbb{I}_0) \twoheadrightarrow X/PX$ has kernel $\mathcal{L} \cdot P\mathfrak{sl}_2(\mathbb{I}_0)/P\mathfrak{sl}_2(\mathbb{I}_0)$, so

$$X/PX \cong \mathfrak{sl}_2(\mathbb{I}_0/P\mathbb{I}_0)/(\mathcal{L} \cdot P\mathfrak{sl}_2(\mathbb{I}_0)/P\mathfrak{sl}_2(\mathbb{I}_0)).$$

We use the following notation:

$$\bar{L} = L(\overline{\mathbb{G}}_{P\mathbb{I}_0}) = \text{ the Pink–Lie algebra of } \overline{\mathbb{G}}_{P\mathbb{I}_0},$$

$$\bar{L}[\lambda] = \text{the } \lambda\text{-eigenspace of } \operatorname{Ad}(\boldsymbol{j}) \text{ on } \bar{L}, \text{ for } \lambda \in \{\alpha, 1, \alpha^{-1}\},$$
$$\bar{\mathcal{L}} = \text{the } \mathbb{Z}_p\text{-algebra generated by } \bar{L}[\alpha] \text{ and } \bar{L}[\alpha^{-1}].$$

By the functoriality of Pink's construction, the canonical surjection $\mathbb{I}_0 \twoheadrightarrow \mathbb{I}_0/P\mathbb{I}_0$ induces surjections

$$L[\lambda] \twoheadrightarrow \bar{L}[\lambda]$$

for all $\lambda \in \{\alpha, 1, \alpha^{-1}\}$. Therefore the canonical linear map $\mathcal{L} \to \bar{\mathcal{L}}$ is also a surjection. That is, $\mathcal{L} \cdot P\mathfrak{sl}_2(\mathbb{I}_0)/P\mathfrak{sl}_2(\mathbb{I}_0) = \bar{\mathcal{L}}$ and so $X/PX \cong \mathfrak{sl}_2(\mathbb{I}_0/P\mathbb{I}_0)/\bar{\mathcal{L}}$. Since $\bar{\mathbb{G}}_{P\mathbb{I}_0} \supseteq \prod_{\mathcal{Q}|P} \Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}})$, it follows that

$$\bar{L}[\alpha] \supseteq \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \oplus_{\mathcal{Q}|P} \bar{\mathfrak{a}}_{\mathcal{Q}}^2 \right\},$$
$$\bar{L}[\alpha^{-1}] \supseteq \left\{ \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} : x \in \oplus_{\mathcal{Q}|P} \bar{\mathfrak{a}}_{\mathcal{Q}}^2 \right\}.$$

Since $\alpha \neq -1$ we have $\mathfrak{u} = \bar{L}[\alpha]$ and $\mathfrak{u}^t = \bar{L}[\alpha^{-1}]$. Therefore

$$\bar{\mathcal{L}} \supseteq \oplus_{\mathcal{Q}|P} \bar{\mathfrak{a}}_{\mathcal{Q}}^4 \mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q}).$$

Since each $\bar{\mathfrak{a}}_{\mathcal{Q}}$ is a nonzero $\mathbb{I}_0/\mathcal{Q}$-ideal, it follows that $\oplus_{\mathcal{Q}|P}\mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q})/\bar{\mathfrak{a}}_{\mathcal{Q}}^4\mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q})$ is Λ/P-torsion. Finally, the inclusions

$$\oplus_{\mathcal{Q}|P} \bar{\mathfrak{a}}_{\mathcal{Q}}^4 \mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q}) \subseteq \bar{\mathcal{L}} \subseteq \mathfrak{sl}_2(\mathbb{I}_0/P\mathbb{I}_0) \subseteq \oplus_{\mathcal{Q}|P} \mathfrak{sl}_2(\mathbb{I}_0/\mathcal{Q}\mathbb{I})$$

show that $\mathfrak{sl}_2(\mathbb{I}_0/P\mathbb{I}_0)/\bar{\mathcal{L}} \cong X/PX$ is Λ/P-torsion. □

Let

$$\mathfrak{v} = \left\{ v \in \mathbb{I}_0 : \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{u} \right\} \text{ and } \mathfrak{v}^t = \left\{ v \in \mathbb{I}_0 : \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \in \mathfrak{u}^t \right\}.$$

Definition 4.6. A Λ-*lattice* in $Q(\mathbb{I}_0)$ is a finitely generated Λ-submodule M of $Q(\mathbb{I}_0)$ such that the $Q(\Lambda)$-span of M is equal to $Q(\mathbb{I}_0)$. If in addition M is a subring of $\mathbb{I}_0$ then we say M is a Λ-*order*.

Proof of Proposition 4.2 when $\alpha = -1$. We show in Lemmas 4.7 and 4.8 that $\mathfrak{v}$ and $\mathfrak{v}^t$ are Λ-lattices in $Q(\mathbb{I}_0)$. To do this we use the fact that the local Galois representation $\rho_F|_{D_p}$ is indecomposable [Ghate and Vatsal 2004; Zhao 2014].

We then show in Proposition 4.9 that any Λ-lattice in $Q(\mathbb{I}_0)$ contains a nonzero $\mathbb{I}_0$-ideal. Let $\mathfrak{b}$ and $\mathfrak{b}^t$ be nonzero $\mathbb{I}_0$-ideals such that $\mathfrak{b} \subseteq \mathfrak{v}$ and $\mathfrak{b}^t \subseteq \mathfrak{v}^t$. Let $\mathfrak{a}_0 = \mathfrak{b}\mathfrak{b}^t$. Then from the definitions of $\mathfrak{v}$, $\mathfrak{v}^t$, and $\mathcal{L}$, we find that

$$\mathcal{L} \supseteq \mathfrak{a}_0^2 \mathfrak{sl}_2(\mathbb{I}_0).$$

By Pink's theory it follows that $\mathbb{G} \supseteq \Gamma_{\mathbb{I}_0}(\mathfrak{a}_0)$. □

Finally, we prove the three key facts used in the proof of Proposition 4.2 when $\alpha = -1$.

Lemma 4.7. *With notation as above,* $\mathfrak{v}$ *is a* Λ*-lattice in* $Q(\mathbb{I}_0)$.

Proof. Let $\bar{L} = L(\bar{\mathbb{G}}_{P\mathbb{I}_0})$. Recall that $L[1]$ surjects onto $\bar{L}[1]$. Now $\bar{L}[1]$ contains

$$\left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in \oplus_{Q|P}\bar{\mathfrak{a}}_Q^2\right\},$$

and $\oplus_{Q|P}\bar{\mathfrak{a}}_Q^2$ is a Λ/P-lattice in $Q(\mathbb{I}_0/P\mathbb{I}_0)$. It follows from Nakayama's lemma that the set of entries in the matrices of $L[1]$ contains a Λ-lattice $\mathfrak{a}$ for $Q(\mathbb{I}_0)$.

By a theorem of Ghate and Vatsal [2004] (later generalized by Hida [2013] and Zhao [2014]) we know that $\rho_F|_{D_p}$ is indecomposable. Hence there is a matrix in the image of ρ whose upper right entry is nonzero. This produces a nonzero nilpotent matrix in L_1. Taking the Lie bracket of this matrix with a nonzero element of $L[1]$ produces a nonzero nilpotent matrix in L which we will call $\left(\begin{smallmatrix} 0 & v \\ 0 & 0 \end{smallmatrix}\right)$. Note that for any $a \in \mathfrak{a}$ we have

$$\begin{pmatrix} 0 & 2av \\ 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix}\right] \in L.$$

Thus the lattice $\mathfrak{a}v$ is contained in $\mathfrak{v}$, so $Q(\Lambda)\mathfrak{v} = Q(\mathbb{I}_0)$. The fact that $\mathfrak{v}$ is finitely generated follows from the fact that Λ is noetherian and $\mathfrak{v}$ is contained in the finitely generated Λ-module $\mathbb{I}_0$. □

Lemma 4.8. *With notation as above,* $\mathfrak{v}^t$ *is a* Λ*-lattice in* $Q(\mathbb{I}_0)$.

Proof. Let $\bar{c} \in \oplus_{Q|P}\bar{\mathfrak{a}}_Q^2$. Since $L[-1]$ surjects to $\bar{L}[-1]$ there is some $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in L$ such that $b \in P\mathbb{I}_0$ and $c \bmod P\mathbb{I}_0 = \bar{c}$. Since $\mathfrak{v}$ is a Λ-lattice in $Q(\mathbb{I}_0)$ by Lemma 4.7, it follows that there is some nonzero $\alpha \in \Lambda$ such that $\alpha b \in \mathfrak{v}$.

We claim that there is some nonzero $\beta \in \Lambda$ for which $\left(\begin{smallmatrix} 0 & \alpha b \\ \beta c & 0 \end{smallmatrix}\right) \in L$. Assuming the existence of β, since $\alpha b \in \mathfrak{v}$ it follows that $\beta c \in \mathfrak{v}^t$. That is, $c \in Q(\Lambda)\mathfrak{v}^t$. Since $\bar{c}$ runs over $\oplus_{Q|P}\bar{\mathfrak{a}}_Q^2$, it follows from Nakayama's lemma that $\mathfrak{v}^t$ is a Λ-lattice in $Q(\mathbb{I}_0)$.

To see that β exists, recall that L is normalized by the matrix $\boldsymbol{J} = \left(\begin{smallmatrix} 1+T & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by Lemma 4.5. Thus

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} 0 & Tb \\ ((1+T)^{-1}-1)c & 0 \end{pmatrix} = \begin{pmatrix} 1+T & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} (1+T)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in L.$$

Write $\alpha = f(T)$ as a power series in T. Since $(1+T)^{-1} - 1$ is divisible by T, we can evaluate f at $(1+T)^{-1} - 1$ to get another element of $\mathbb{Z}_p[\![T]\!]$. Taking $\beta = f((1+T)^{-1} - 1)$, the calculation above shows the desired inclusion:

$$\begin{pmatrix} 0 & \alpha b \\ \beta c & 0 \end{pmatrix} \in L, \qquad \square$$

Proposition 4.9. *Every Λ-lattice in $Q(\mathbb{I}_0)$ contains a nonzero $\mathbb{I}_0$-ideal.*

Proof. Let M be a Λ-lattice in $Q(\mathbb{I}_0)$. Define

$$R = \{x \in \mathbb{I}_0 : xM \subseteq M\}.$$

Then R is a subring of $\mathbb{I}_0$ that is also a Λ-lattice for $Q(\mathbb{I}_0)$. Thus R is a Λ-order in $\mathbb{I}_0$, and M is an R-module. Therefore

$$\mathfrak{c} := \{x \in \mathbb{I}_0 : x\mathbb{I}_0 \subseteq R\}$$

is a nonzero $\mathbb{I}_0$-ideal. Note that $Q(R) = Q(\mathbb{I}_0) = Q(\Lambda)M$. Since M is a finitely generated Λ-module there is some nonzero $r \in \mathbb{I}_0$ such that $rM \subseteq R$. As rM is still a Λ-lattice for $Q(\mathbb{I}_0)$, by replacing M with rM we may assume that M is an R-ideal.

Now consider $\mathfrak{a} = \mathfrak{c} \cdot (M\mathbb{I}_0)$, where $M\mathbb{I}_0$ is the ideal generated by M in $\mathbb{I}_0$. Note that $\mathfrak{a}$ is a nonzero $\mathbb{I}_0$-ideal since both $\mathfrak{c}$ and $M\mathbb{I}_0$ are nonzero $\mathbb{I}_0$-ideals. To see that $\mathfrak{a} \subseteq M$, let $x \in \mathbb{I}_0$ and $c \in \mathfrak{c}$. Then $xc \in R$ by definition of $\mathfrak{c}$. If $a \in M$ then $xca \in M$ since M is an R-ideal. Thus $xca \in M$, so $\mathfrak{a} \subseteq M$. □

Remark. Note that the only property of $\mathbb{I}_0$ that is used in the proof of Proposition 4.9 is that $\mathbb{I}_0$ is a Λ-order in $Q(\mathbb{I}_0)$. Thus, once we have shown that ρ (or ρ_F) is $\mathbb{I}_0$-full, it follows that the representation is R-full for *any* Λ-order R in $Q(\mathbb{I}_0)$. In particular, if $\tilde{\mathbb{I}}_0$ is the maximal Λ-order in $Q(\mathbb{I}_0)$ then ρ_F is $\tilde{\mathbb{I}}_0$-full.

Finally, we show that for the purposes of proving $\mathbb{I}_0$-fullness it suffices to work with ρ instead of ρ_F.

Proposition 4.10. *The representation ρ_F is $\mathbb{I}_0$-full if and only if ρ is $\mathbb{I}_0$-full.*

Proof. Note that $\operatorname{Im}\rho_F|_{H_0} \cap \mathrm{SL}_2(\mathbb{I}_0) \subseteq \operatorname{Im}\rho$ by definition. Thus if ρ_F is $\mathbb{I}_0$-full then so is ρ.

Now assume that ρ is $\mathbb{I}_0$-full. As in the proof of [Hida 2015, Theorem 8.2], let $\Gamma = \{(1+T)^s : s \in \mathbb{Z}_p\}$ and

$$\mathbb{K} = \{x \in \rho_F(H_0) : \det x \in \Gamma\}.$$

Note that $\mathbb{K}$ is a finite index subgroup of $\operatorname{Im}\rho_F$. Since F is ordinary and non-CM we can find an element of the form $\boldsymbol{\tau} = \left(\begin{smallmatrix} 1+T & u \\ 0 & 1 \end{smallmatrix}\right) \in \operatorname{Im}\rho_F$ [Hida 2000a, Theorem 4.3.2]. Let $n = [G_{\mathbb{Q}} : H_0]$. By replacing Γ with $\{(1+T)^{ns} : s \in \mathbb{Z}_p\}$ and $\boldsymbol{\tau}$ with $\boldsymbol{\tau}^n$, we may assume that $\boldsymbol{\tau} \in \mathbb{K}$.

Let $\mathbb{S} = \mathbb{K} \cap \mathrm{SL}_2(\mathbb{I}_0)$ and $\mathcal{T} = \{\boldsymbol{\tau}^s : s \in \mathbb{Z}_p\}$. We can write $\mathbb{K}$ as a semidirect product

$$\mathbb{K} = \mathcal{T} \ltimes \mathbb{S}.$$

Indeed, given $\boldsymbol{x} \in \mathbb{K}$ there is a unique $s \in \mathbb{Z}_p$ such that $\det \boldsymbol{x} = (1+T)^s$. Thus we identify $\boldsymbol{x}$ with $(\boldsymbol{\tau}^s, \boldsymbol{\tau}^{-s}\boldsymbol{x}) \in \mathcal{T} \ltimes \mathbb{S}$.

Let $\mathbb{K}'$ be the image of $\mathbb{K}$ under the natural map

$$\Phi : \mathbb{K} \to \operatorname{Im}\rho, \quad \boldsymbol{x} \mapsto \boldsymbol{x}(\det \boldsymbol{x})^{-1/2}.$$

Then $\mathbb{K}'$ is a finite index subgroup of $\operatorname{Im}\rho$ and therefore contains $\Gamma_{\mathbb{I}_0}(\mathfrak{a})$ for some nonzero $\mathbb{I}_0$-ideal $\mathfrak{a}$ since ρ is $\mathbb{I}_0$-full. Note that $\ker\Phi$ is precisely the set of scalar matrices in $\mathbb{K}$. Therefore, for some $0 \le r \le \infty$,

$$\ker\Phi \cong \{(1+T)^s : s \in p^r\mathbb{Z}_p\},$$

where $r = \infty$ means $\ker\Phi = \{1\}$. If $r \neq \infty$ then by passing to finite index subgroups of $\mathbb{K}$, $\mathbb{K}'$, and Γ we may assume that $\ker\Phi = \Gamma$. Thus, given any $\boldsymbol{y} \in \Gamma_{\mathbb{I}_0}(\mathfrak{a})$ we can find $\boldsymbol{x} \in \mathbb{K}$ such that $\Phi(\boldsymbol{x}) = \boldsymbol{y}$. Let $s \in \mathbb{Z}_p$ such that $\det \boldsymbol{x} = (1+T)^{s/2}$. Then the scalar matrix $(1+T)^{-s/2}$ is in Γ hence in $\mathbb{K}$. Hence $\boldsymbol{x}(1+T)^{-s/2} \in \mathbb{S}$ and $\Phi(\boldsymbol{x}(1+T)^{s/2}) = \boldsymbol{y}$. But Φ is the identity on $\mathbb{S}$, so $\boldsymbol{y} = \boldsymbol{x}(1+T)^{-s/2} \in \mathbb{S}$. Therefore $\Gamma_{\mathbb{I}_0}(\mathfrak{a}) \subseteq \mathbb{S}$ and ρ_F is $\mathbb{I}_0$-full.

It remains to deal with the case when $\ker\Phi = \{1\}$. In this case Φ is an isomorphism onto $\mathbb{K}'$ and we can use Φ^{-1} to get a continuous group homomorphism from $\mathbb{K}'$ onto $\mathbb{Z}_p$:

$$s : \mathbb{K}' \cong \mathbb{K} \cong \mathcal{T} \ltimes \mathbb{S} \twoheadrightarrow \mathcal{T} \cong \mathbb{Z}_p.$$

Note that $\ker s = \mathbb{S}$, so we want to show that $\ker s$ is $\mathbb{I}_0$-full. By assumption there is a nonzero $\mathbb{I}_0$-ideal $\mathfrak{a}_0$ such that $\Gamma_{\mathbb{I}_0}(\mathfrak{a}_0) \subseteq \mathbb{K}'$. Let $\mathfrak{v} = \{b \in \mathfrak{a}_0 : \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \ker s\}$ and $\mathfrak{v}^t = \{c \in \mathfrak{a}_0 : \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) \in \ker s\}$. Both $\mathfrak{v}$ and $\mathfrak{v}^t$ are Λ-lattices in $Q(\mathbb{I}_0)$. We shall prove this for $\mathfrak{v}$; the proof for $\mathfrak{v}^t$ is similar. Note that $\mathfrak{v}$ is a $\mathbb{Z}_p$-module: if $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathbb{S}$ then $\left(\begin{smallmatrix} 1 & sb \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)^s \in \mathbb{S}$ since $\mathbb{S}$ is closed (as it is the determinant 1 image of a Galois representation). To see that $\mathfrak{v}$ is a Λ-module, recall that $\mathbb{S}$ is normalized by $\boldsymbol{J} = \left(\begin{smallmatrix} 1+T & 0 \\ 0 & 1 \end{smallmatrix}\right)$ by the proof of Lemma 4.5. Therefore conjugation by $\boldsymbol{J}$ gives an action of T on $\mathfrak{v}$ as in the proof of Lemma 4.5. Now we consider the Λ-module $\mathfrak{a}_0/\mathfrak{v}$ which, as a group, is isomorphic to a closed subgroup of $\mathbb{Z}_p$. Therefore $\mathfrak{a}_0/\mathfrak{v}$ is a torsion Λ-module. Since $\mathfrak{a}_0$ is a Λ-lattice in $Q(\mathbb{I}_0)$ it follows that $\mathfrak{v}$ must also be a Λ-lattice in $Q(\mathbb{I}_0)$, as claimed.

We have shown that there are nonzero $\mathbb{I}_0$-ideals $\mathfrak{b} \subseteq \mathfrak{v}$ and $\mathfrak{b}^t \subseteq \mathfrak{v}^t$ such that the Pink–Lie algebra $L(\mathbb{S})$ contains

$$\left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathfrak{b}\right\} \cup \left\{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c \in \mathfrak{b}^t\right\}.$$

By letting $\mathfrak{c} = \mathfrak{b}\mathfrak{b}^t$ and taking Lie brackets of the upper and lower nilpotent matrices above we find that $L(\mathbb{S}) \supseteq \mathfrak{c}^2\mathfrak{sl}_2(\mathbb{I}_0)$. Therefore $\mathbb{S}$ is $\mathbb{I}_0$-full, as desired. □

5. Open image in product

The purpose of this section is to prove the following reduction step in the proof of Theorem 2.4.

Proposition 5.1. *Assume that $|\mathbb{F}| \neq 3$. Fix an arithmetic prime P of Λ. Assume that for every prime $\mathcal{Q}$ of $\mathbb{I}_0$ lying over P, the image of* $\operatorname{Im}\rho$ *in* $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ *is open. Then the image of* $\operatorname{Im}\rho$ *in* $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ *is open in the product topology.*

Thus if we can show that there is some arithmetic prime P of Λ satisfying the hypothesis of Proposition 5.1, then combining the above result with Proposition 4.2 yields Theorem 2.4.

Fix an arithmetic prime P of Λ satisfying the hypothesis of Proposition 5.1. Note that $\mathbb{Z}_p$ does not contain any p-power roots of unity since $p > 2$. Therefore $P = P_{k,1}$ for some $k \geq 2$. Recall that $\mathbb{G} = \operatorname{Im}\rho \cap \Gamma_{\mathbb{I}_0}(\mathfrak{m}_0)$, and write $\overline{\mathbb{G}}$ for the image of $\mathbb{G}$ in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$. We begin our proof of Proposition 5.1 with the following lemma of Ribet which allows us to reduce to considering products of only two copies of SL_2.

Lemma 5.2 [Ribet 1975, Lemma 3.4]. *Let $S_1, \dots, S_t (t > 1)$ be profinite groups. Assume for each i that the following condition is satisfied: for each open subgroup U of S_i, the closure of the commutator subgroup of U is open in S_i. Let $\mathcal{G}$ be a closed subgroup of $S = S_1 \times \cdots \times S_t$ that maps to an open subgroup of each group $S_i \times S_j (i \neq j)$. Then $\mathcal{G}$ is open in S.*

Apply this lemma to our situation with $\{S_1, \dots, S_t\} = \{\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}) : \mathcal{Q}|P\}$ and $\mathcal{G} = \overline{\mathbb{G}}$. The lemma implies that it is enough to prove that for all primes $\mathcal{Q}_1 \neq \mathcal{Q}_2$ of $\mathbb{I}_0$ lying over P, the image G of $\overline{\mathbb{G}}$ under the projection to $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ is open. We shall now consider what happens when this is not the case. Indeed, the reader should be warned that the rest of this section is a proof by contradiction.

Proposition 5.3. *Let P be an arithmetic prime of Λ satisfying the hypotheses of Proposition 5.1, and assume $|\mathbb{F}| \neq 3$. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be distinct primes of $\mathbb{I}_0$ lying over P. Let $\mathfrak{P}_i$ be a prime of $\mathbb{I}$ lying over $\mathcal{Q}_i$. If G is not open in* $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ *then there is an isomorphism $\sigma : \mathbb{I}_0/\mathcal{Q}_1 \cong \mathbb{I}_0/\mathcal{Q}_2$ and a character $\varphi : H \to Q(\mathbb{I}_0/\mathcal{Q}_2)^\times$ such that*

$$\sigma(a(\ell, f_{\mathfrak{P}_1})) = \varphi(\ell) a(\ell, f_{\mathfrak{P}_2})$$

for all primes ℓ for which $\mathrm{Frob}_\ell \in H$.

Proof. Our strategy is to mimic the proof of [Ribet 1975, Theorem 3.5]. Let G_i be the projection of G to $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_i)$, so $G \subseteq G_1 \times G_2$. By hypothesis G_i is open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_i)$. Let $\pi_i : G \to G_i$ be the projection maps and set $N_1 = \ker \pi_2$ and $N_2 = \ker \pi_1$. Though a slight abuse of notation, we view N_i as a subset of G_i.

Goursat's lemma implies that the image of G in $G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism

$$\alpha : G_1/N_1 \cong G_2/N_2.$$

Since G is not open in $G_1 \times G_2$ by hypothesis, either N_1 is not open in G_1 or N_2 is not open in G_2. (Otherwise $N_1 \times N_2$ is open and hence G is open.) Without loss of generality we may assume that N_1 is not open in G_1. From the classification of subnormal subgroups of $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1)$ in [Tazhetdinov 1983] it follows that $N_1 \subseteq \{\pm 1\}$ since N_1 is not open. If N_2 is open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ then α gives an isomorphism from either G_1 or $\mathrm{PSL}_2(\mathbb{I}_0/\mathcal{Q}_1)$ to the finite group G_2/N_2. Clearly this is impossible, so N_2 is not open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$. Again by [Tazhetdinov 1983] we have $N_2 \subseteq \{\pm 1\}$. Recall that G_i comes from $\mathbb{G} = \operatorname{Im} \rho \cap \Gamma_{\mathbb{I}_0}(\mathfrak{m}_0)$ by reduction. In particular, $-1 \notin G_i$ since all elements of $\mathbb{G}$ reduce to the identity in $\mathrm{SL}_2(\mathbb{F})$. Thus we must have $N_i = \{1\}$. Hence α gives an isomorphism $G_1 \cong G_2$. We note that the theorem in [loc. cit.] requires $|\mathbb{F}| \neq 3$. This invocation of [loc. cit.] is the only reason we assume $|\mathbb{F}| \neq 3$.

The isomorphism theory of open subgroups of SL_2 over a local ring was studied by Merzljakov [1973]. (There is a unique theorem in his paper, and that is the result to which we refer. His theorem applies to more general groups and rings, but it is relevant in particular to our situation.) Although his result is stated only for automorphisms of open subgroups, his proof goes through without change for isomorphisms. His result implies that α must be of the form

$$\alpha(\boldsymbol{x}) = \eta(\boldsymbol{x})\boldsymbol{y}^{-1}\sigma(\boldsymbol{x})\boldsymbol{y}, \tag{2}$$

where $\eta \in \operatorname{Hom}(G_1, Q(\mathbb{I}_0/\mathcal{Q}_2)^\times)$, $\boldsymbol{y} \in \mathrm{GL}_2(Q(\mathbb{I}_0/\mathcal{Q}_2))$ and $\sigma : \mathbb{I}_0/\mathcal{Q}_1 \cong \mathbb{I}_0/\mathcal{Q}_2$ is a ring isomorphism. By $\sigma(\boldsymbol{x})$ we mean that we apply σ to each entry of the matrix $\boldsymbol{x}$.

For any $\boldsymbol{g} \in G$ we can write $\boldsymbol{g} = (\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{x} \in G_1$, $\boldsymbol{y} \in G_2$. Since G is the graph of α we have $\alpha(\boldsymbol{x}) = \boldsymbol{y}$. By definition of G there is some $h \in H$ such that $\boldsymbol{x} = \mathfrak{P}_1(\rho(h))$ and $\boldsymbol{y} = \mathfrak{P}_2(\rho(h))$. Recall that for almost all primes ℓ for which $\mathrm{Frob}_\ell \in H$ we have $\operatorname{tr}(\rho(\mathrm{Frob}_\ell)) = \left(\sqrt{\det \rho_F(\mathrm{Frob}_\ell)}\right)^{-1} a(\ell, F)$. Furthermore $\det \rho_F(\mathrm{Frob}_\ell) \bmod P = \chi(\ell)\ell^{k-1}$ since $P = P_{k,1}$. Using these facts together with Equation (2) we see that for almost any $\mathrm{Frob}_\ell \in H$ we have

$$\sigma(a(\ell, f_{\mathfrak{P}_1})) = \varphi(\ell) a(\ell, f_{\mathfrak{P}_2}),$$

where

$$\varphi(\ell) := \eta^{-1}\left(\mathfrak{P}_1(\rho(\mathrm{Frob}_\ell))\right) \frac{\sigma\left(\sqrt{\chi(\ell)\ell^{k-1}}\right)}{\sqrt{\chi(\ell)\ell^{k-1}}},$$

as claimed. □

To finish the proof of Proposition 5.1 we need to remove the condition that $\mathrm{Frob}_\ell \in H$ from the conclusion of Proposition 5.3. That is, we would like to show that there is an isomorphism $\tilde{\sigma} : \mathbb{I}'/\mathfrak{P}'_1 \cong \mathbb{I}'/\mathfrak{P}'_2$ extending σ and a character $\tilde{\varphi} : G_{\mathbb{Q}} \to Q(\mathbb{I}'/\mathfrak{P}'_2)^\times$ extending φ such that

$$\tilde{\sigma}(a(\ell, f_{\mathfrak{P}_1})) = \tilde{\varphi}(\ell)a(\ell, f_{\mathfrak{P}_2})$$

for almost all primes ℓ. If we can do this, then applying Theorem 3.1 allows us to lift $\tilde{\sigma}$ to an element of Γ that sends $\mathfrak{P}'_1$ to $\mathfrak{P}'_2$. (We also need to verify that $\tilde{\varphi}$ takes values in $\mathbb{Z}_p[\chi]$ in order to apply Theorem 3.1.) But this is a contradiction since $\mathfrak{P}'_1$ and $\mathfrak{P}'_2$ lie over different primes of $\mathbb{I}_0$. Hence it follows from Proposition 5.3 that G must be open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ and Lemma 5.2 implies Proposition 5.1.

We show the existence of $\tilde{\sigma}$ and $\tilde{\varphi}$ using obstruction theory as developed in [Hida 2000b, §4.3.5]. For the sake of notation, we briefly recall the theory here. For the proofs we refer the reader to [Hida 2000b]. Let K be a finite extension of $\mathbb{Q}_p$, $n \in \mathbb{Z}^+$, and $r : H \to \mathrm{GL}_n(K)$ be an absolutely irreducible representation. For all $g \in G_{\mathbb{Q}}$ define a twisted representation on H by $r^g(h) := r(ghg^{-1})$. Assume the following condition:

$$r \cong r^g \text{ over } K \text{ for all } g \in G_{\mathbb{Q}}. \tag{3}$$

Under Hypothesis (3) it can be shown that there is a function $c : G_{\mathbb{Q}} \to \mathrm{GL}_n(K)$ with the following properties:

(1) $r = c(g)^{-1} r^g c(g)$ for all $g \in G_{\mathbb{Q}}$;

(2) $c(hg) = r(h)c(g)$ for all $h \in H$, $g \in G_{\mathbb{Q}}$;

(3) $c(1) = 1$.

As r is absolutely irreducible, it follows that $b(g, g') := c(g)c(g')c(gg')^{-1}$ is a 2-cocycle with values in $K^\times$. In fact b factors through $\Delta := G_{\mathbb{Q}}/H$ and hence represents a class in $H^2(\Delta, K^\times)$. We call this class $\mathrm{Ob}(r)$. It is independent of the function c satisfying the above three properties. The class $\mathrm{Ob}(r)$ measures the obstruction to lifting r to a representation of $G_{\mathbb{Q}}$. We say a continuous representation $\tilde{r} : G_{\mathbb{Q}} \to \mathrm{GL}_n(K)$ is an *extension* of r to $G_{\mathbb{Q}}$ if $\tilde{r}|_H = r$.

Proposition 5.4. (1) *There is an extension $\tilde{r}$ of r to $G_{\mathbb{Q}}$ if and only if* $\mathrm{Ob}(r) = 0 \in H^2(\Delta, K^\times)$.

(2) *If* $\mathrm{Ob}(r) = 0$ *and $\tilde{r}$ is an extension of r to $G_{\mathbb{Q}}$, then all other extensions of r to $G_{\mathbb{Q}}$ are of the form $\tilde{r} \otimes \psi$ for some character $\psi : \Delta \to K^\times$.*

For ease of notation we shall write $K_i = Q(\mathbb{I}/\mathfrak{P}_i)$ and $E_i = Q(\mathbb{I}_0/\mathcal{Q}_i)$. Write $\rho_i : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_i)$ for $\rho_{f_{\mathfrak{P}_i}}$. By Theorem 4.1 we see that $\rho_i|_H$ takes values in $\mathrm{GL}_2(E_i)$. Proposition 5.3 gives an isomorphism $\sigma : E_1 \cong E_2$ and a character $\varphi : H \to E_2^\times$ such that

$$\mathrm{tr}(\rho_1|_H^\sigma) = \mathrm{tr}(\rho_2|_H \otimes \varphi).$$

In order to use obstruction theory to show the existence of $\tilde{\sigma}$ and $\tilde{\varphi}$ we must show that all of the representations in question satisfy Hypothesis (3).

Lemma 5.5. *Let L_i be a finite extension of K_i. View ρ_1 as a representation over L_1 and $\rho_2|_H, \rho_1|_H^\sigma, \rho_2|_H \otimes \varphi$, and φ as representations over L_2. Then $\rho_i|_H, \rho_1|_H^\sigma, \rho_2|_H \otimes \varphi$, and φ all satisfy Hypothesis (3). Furthermore we have $\mathrm{Ob}(\rho_i|_H) = 0$, $\mathrm{Ob}(\rho_1|_H^\sigma) = \mathrm{Ob}(\rho_2|_H \otimes \varphi)$, and*

$$\mathrm{Ob}(\rho_2|_H \otimes \varphi) = \mathrm{Ob}(\rho_2|_H) + \mathrm{Ob}(\varphi) \in H^2(\Delta, (L_2)^\times).$$

Proof. Recall that a continuous representation of a compact group over a field of characteristic 0 is determined up to isomorphism by its trace. Therefore to verify (3) it suffices to show that if r is any of the representations listed in the statement of the lemma, then

$$\operatorname{tr} r = \operatorname{tr} r^g$$

for all $g \in G_{\mathbb{Q}}$. This is obvious when r is $\rho_1|_H$ or $\rho_2|_H$ since both extend to representations of $G_{\mathbb{Q}}$ and hence

$$\operatorname{tr} \rho_i^g(h) = \operatorname{tr} \rho_i(g)\rho_i(h)\rho_i(g)^{-1} = \operatorname{tr} \rho_i(h).$$

Since ρ_i is an extension of $\rho_i|_H$ and $L_i \supseteq K_i$ we have $\mathrm{Ob}(\rho_i|_H) = 0$.

When $r = \rho_1|_H^\sigma$, let $\tau : K_1 \hookrightarrow \overline{\mathbb{Q}}_p$ be an extension of σ. Then ρ_1^τ is an extension of $\rho_1|_H^\sigma$ and hence we can use the same argument as above to conclude that $\operatorname{tr} \rho_1|_H^\sigma = \operatorname{tr}(\rho_1|_H^\sigma)^g$. (Note that for this particular purpose, we do not care about the field in which τ takes values.)

When $r = \rho_2|_H \otimes \varphi$, recall that $\operatorname{tr} \rho_1|_H^\sigma = \varphi \operatorname{tr} \rho_2|_H$. Since both $\rho_1|_H^\sigma$ and $\rho_2|_H$ satisfy Hypothesis (3) so does $\rho_2|_H \otimes \varphi$. Furthermore, $\operatorname{tr} \rho_1|_H^\sigma = \operatorname{tr}(\rho_2|_H \otimes \varphi)$ implies that $\rho_1|_H^\sigma \cong \rho_2|_H \otimes \varphi$ and hence $\mathrm{Ob}(\rho_1|_H^\sigma) = \mathrm{Ob}(\rho_2|_H \otimes \varphi)$.

Since $(\rho_1|_H^\sigma)^g \cong \rho_2|_H^g \otimes \varphi^g$ for any $g \in G_{\mathbb{Q}}$ and since both $\rho_i|_H$ satisfy (3) we see that

$$\varphi^g \operatorname{tr} \rho_2|_H = \varphi \operatorname{tr} \rho_2|_H. \tag{4}$$

Thus if we know $\operatorname{tr} \rho_2|_H$ is nonzero sufficiently often then we can deduce that φ satisfies (3) . More precisely, let $m \in \mathbb{Z}^+$ be the conductor for φ, so $\varphi : (\mathbb{Z}/m\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times$. Then we have a surjection $H \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$ with kernel κ. Choose a set S of coset representatives of κ in H, so $H = \sqcup_{s \in S} s\kappa$. If we can show that $\operatorname{tr} \rho_2(s\kappa) \neq \{0\}$ for all $s \in S$, then it follows from Equation (4) that $\varphi^g = \varphi$ for all $g \in G_{\mathbb{Q}}$. Recall that ρ_2 is a Galois representation attached to a classical modular form, and so by Ribet [1980; 1985] and Momose's [1981] result we know that its image is open. (See Theorem 6.1 for a precise statement of their result.) Then the restriction of ρ_2 to any open subset of $G_{\mathbb{Q}}$ also has open image and hence $\operatorname{tr} \rho_2$ is not identically zero. Each $s\kappa$ is open in $G_{\mathbb{Q}}$, so $\varphi^g = \varphi$.

Finally, note that if $c : G_{\mathbb{Q}} \to \mathrm{GL}_2(L_2)$ is a function satisfying conditions 1–3 above for $r = \rho_2|_H$ and $\eta : G_{\mathbb{Q}} \to L_2^\times$ is a function satisfying conditions 1–3 above for φ, then ηc is a function satisfying conditions 1–3 for $\rho_2|_H \otimes \varphi$. From this it follows that $\mathrm{Ob}(\rho_2|_H \otimes \varphi) = \mathrm{Ob}(\rho_2|_H) + \mathrm{Ob}(\varphi)$. □

With L_i as in the previous lemma, suppose there is an extension $\tilde{\sigma} : L_1 \cong L_2$ of σ and an extension $\tilde{\varphi} : G_{\mathbb{Q}} \to L_2^\times$ of φ. We now show that this gives us the desired relation among traces.

Lemma 5.6. *If there exists extensions $\tilde{\sigma}$ of σ and $\tilde{\varphi}$ of φ, then there exists a character $\eta : G_{\mathbb{Q}} \to L_2^\times$ that is also a lift of φ such that $\rho_1^{\tilde{\sigma}} \cong \rho_2 \otimes \eta$.*

Proof. Note that since F does not have CM, $\rho_1|_H$ and $\rho_2|_H$ are absolutely irreducible by results of Ribet [1977]. For any absolutely irreducible representation $\pi : G_{\mathbb{Q}} \to \mathrm{GL}_2(L_2)$ Frobenius reciprocity gives

$$\langle \pi, \mathrm{Ind}_H^{G_{\mathbb{Q}}}(\rho_1|_H^\sigma)\rangle_{G_{\mathbb{Q}}} = \langle \pi|_H, \rho_1|_H^\sigma\rangle_H = \langle \pi|_H, \rho_2|_H \otimes \varphi\rangle_H. \tag{5}$$

Thus if π is a 2-dimensional irreducible constituent of $\mathrm{Ind}(\rho_1|_H^\sigma)$ then $\rho_1|_H^\sigma$ is a constituent of $\pi|_H$. As both are 2-dimensional, it follows that $\rho_1|_H^\sigma \cong \pi|_H$ and thus π is an extension of $\rho_1|_H^\sigma$. Since $\tilde{\sigma}$ exists by hypothesis, we know that $\rho_1^{\tilde{\sigma}}$ is also an extension of $\rho_1|_H^\sigma$.

Since $\tilde{\varphi}$ exists by hypothesis, we can take $\pi = \rho_2 \otimes \tilde{\varphi}$. Then (5) implies that π is an irreducible constituent of $\mathrm{Ind}_H^G(\rho_1|_H^\sigma)$. By Proposition 5.4 there is a character $\psi : \Delta \to L_2^\times$ such that $\rho_2 \otimes \tilde{\varphi} \cong \rho_1^{\tilde{\sigma}} \otimes \psi$. That is,

$$\rho_1^{\tilde{\sigma}} \cong \rho_2 \otimes (\tilde{\varphi}\psi^{-1}).$$

Setting $\eta = \tilde{\varphi}\psi^{-1}$ gives the desired conclusion. □

Finally, we turn to showing the existence of $\tilde{\sigma}$ and $\tilde{\varphi}$. With notation as in Lemma 5.5, suppose there exists $\tilde{\sigma}^{-1} : L_2 \cong L_1$ that lifts σ^{-1}. Then $\tilde{\sigma}^{-1}$ induces an isomorphism $H^2(\Delta, L_2^\times) \cong H^2(\Delta, L_1^\times)$ that sends $\mathrm{Ob}(\rho_1|_H^\sigma)$ to $\mathrm{Ob}(\rho_1|_H)$. It follows from Lemma 5.5 that $\mathrm{Ob}(\rho_1|_H^\sigma) = 0$ and hence $\mathrm{Ob}(\rho_2|_H \otimes \varphi) = 0$. But $0 = \mathrm{Ob}(\rho_2|_H \otimes \varphi) = \mathrm{Ob}(\rho_2|_H) + \mathrm{Ob}(\varphi) = \mathrm{Ob}(\varphi)$, and thus we can extend φ to $\tilde{\varphi} : G_{\mathbb{Q}} \to L_2^\times$.

The above argument requires that we find $L_i \supseteq K_i$ such that L_1 is isomorphic to L_2 via a lift of σ. We can achieve this as follows. Let $\tau : K_1 \hookrightarrow \overline{\mathbb{Q}}_p$ be an extension of σ. Let $L_2 = K_2\tau(K_1)$. Let $\tilde{\sigma}^{-1} : L_2 \hookrightarrow \overline{\mathbb{Q}}_p$ be an extension of τ^{-1} and set $L_1 = \tilde{\sigma}^{-1}(L_2)$. This construction satisfies the desired properties. Applying Lemma 5.6 we see that there is a character $\eta : G_{\mathbb{Q}} \to L_2^\times$ such that

$$\mathrm{tr}\, \rho_1^{\tilde{\sigma}} = \mathrm{tr}\, \rho_2 \otimes \eta. \tag{6}$$

This is almost what we want. Note that by (6) it follows that $\tilde{\sigma}$ restricts to an isomorphism from $(\mathbb{I}'/\mathfrak{P}_1')[\eta]$ to $(\mathbb{I}'/\mathfrak{P}_2')[\eta]$. The only problem is that $\tilde{\sigma}$ may not send $\mathbb{I}'/\mathfrak{P}'_1$ to $\mathbb{I}'/\mathfrak{P}'_2$ and η may have values in L_2 that are not in $(\mathbb{I}'/\mathfrak{P}'_2)^\times$. We shall show that this cannot be the case.

Recall that χ is the nebentypus of F, and $\mathfrak{P}_1$ and $\mathfrak{P}_2$ lie over the arithmetic prime $P_{k,1}$ of Λ. Thus for almost all primes ℓ we have $\det \rho_i(\mathrm{Frob}_\ell) = \chi(\ell)\ell^{k-1}$. Applying this to Equation (6) we find that

$$\chi^{\tilde{\sigma}}(\ell)\ell^{k-1} = \eta^2(\ell)\chi(\ell)\ell^{k-1}.$$

Recall that $\chi(\ell)$ is a root of unity and hence $\chi^{\tilde{\sigma}}(\ell)$ is just a power of $\chi(\ell)$. Thus $\eta^2(\ell) \in \mathbb{Z}_p[\chi] \subseteq \mathbb{I}'/\mathfrak{P}'_i$ and hence $[(\mathbb{I}'/\mathfrak{P}'_i)[\eta] : \mathbb{I}'/\mathfrak{P}'_i] \leq 2$. Thus we may assume that $L_2 = K_2[\eta]$, which is at most a quadratic extension of K_2.

Note that since η^2 takes values in $\mathbb{I}'/\mathfrak{P}'_i$ we can obtain $(\mathbb{I}'/\mathfrak{P}'_i)[\eta]$ from $\mathbb{I}'/\mathfrak{P}'_i$ by adjoining a 2-power root of unity. (Write η as the product of a 2-power order character and an odd order character and note that any odd order root of unity is automatically a square in any ring in which it is an element.)

Lemma 5.7. *We have $(\mathbb{I}'/\mathfrak{P}'_i)[\eta] = \mathbb{I}'/\mathfrak{P}'_i$ for $i = 1, 2$. Therefore $\tilde{\sigma} : \mathbb{I}'/\mathfrak{P}'_1 \cong \mathbb{I}'/\mathfrak{P}'_2$ and η takes values in $\mathbb{Z}_p[\chi]$.*

Proof. Suppose first that $\mathbb{I}'/\mathfrak{P}'_2 = (\mathbb{I}'/\mathfrak{P}'_2)[\eta]$ but $[(\mathbb{I}'/\mathfrak{P}'_1)[\eta] : \mathbb{I}'/\mathfrak{P}'_1] = 2$. Then we have that $\tilde{\sigma} : (\mathbb{I}'/\mathfrak{P}'_1)[\eta] \cong \mathbb{I}'/\mathfrak{P}'_2$. Note that $(\mathbb{I}'/\mathfrak{P}'_1)[\eta]$ is unramified over $\mathbb{I}'/\mathfrak{P}'_1$ since it is obtained by adjoining a prime-to-p root of unity (namely a 2-power root of unity). Thus the residue field of $(\mathbb{I}'/\mathfrak{P}'_1)[\eta]$ must be a quadratic extension of the residue field $\mathbb{F}$ of $\mathbb{I}'/\mathfrak{P}'_1$. But $\mathbb{F}$ is also the residue field of $\mathbb{I}'/\mathfrak{P}'_2$ and since $(\mathbb{I}'/\mathfrak{P}'_1)[\eta] \cong \mathbb{I}'/\mathfrak{P}'_2$ they must have the same residue field, a contradiction. Therefore we must have $(\mathbb{I}'/\mathfrak{P}'_1)[\eta] = \mathbb{I}'/\mathfrak{P}'_1$.

It remains to deal with the case when $[(\mathbb{I}'/\mathfrak{P}'_1)[\eta] : \mathbb{I}'/\mathfrak{P}'_1] = [(\mathbb{I}'/\mathfrak{P}'_2)[\eta] : \mathbb{I}'/\mathfrak{P}'_2] = 2$. As noted above, these extensions must be unramified and hence the residue field of $(\mathbb{I}'/\mathfrak{P}'_i)[\eta]$ must be the unique quadratic extension $\mathbb{E} = \mathbb{F}[\bar{\eta}]$ of $\mathbb{F}$. Note that $\tilde{\sigma}$ induces an automorphism $\hat{\sigma}$ of $\mathbb{E}$ that necessarily restricts to an automorphism of $\mathbb{F}$. From $\chi^{\tilde{\sigma}} = \eta^2\chi$ we find that

$$\bar{\chi}^{\hat{\sigma}} = \bar{\eta}^2\bar{\chi}.$$

On the other hand $\hat{\sigma}$ is an automorphism of $\mathbb{F}$ and hence is equal to some power of Frobenius. So we see that for some $s \in \mathbb{Z}$ we have $\bar{\eta}^2 = \bar{\chi}^{p^s-1}$. Since p is odd, $p^s - 1$ is even and hence $\bar{\eta}^2$ takes values in $\mathbb{F}_p[\bar{\chi}^2]$. Thus $\bar{\eta}$ takes values in $\mathbb{F}_p[\bar{\chi}] \subseteq \mathbb{F}$, a contradiction to the assumption that $[\mathbb{F}[\bar{\eta}] : \mathbb{F}] = 2$.

Since η^2 takes values in $\mathbb{Z}_p[\chi]$ and $\mathbb{F}_p[\bar{\eta}] \subseteq \mathbb{F}_p[\bar{\chi}]$, it follows that in fact η must take values in $\mathbb{Z}_p[\chi]$. Hence we may take $L_i = K_i$ and $\tilde{\sigma} : \mathbb{I}'/\mathfrak{P}'_1 \cong \mathbb{I}'/\mathfrak{P}'_2$. □

Proof of Proposition 5.1. By Lemma 5.2 it suffices to show that, for any two primes $\mathcal{Q}_1 \neq \mathcal{Q}_2$ of $\mathbb{I}_0$ lying over $P_{k,1}$, the image of $\operatorname{Im}\rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ is open. Proposition 5.3 says that if that is not the case, then there is an isomorphism

$\sigma : \mathbb{I}_0/\mathcal{Q}_1 \cong \mathbb{I}_0/\mathcal{Q}_2$ and a character $\varphi : H \to Q(\mathbb{I}_0/\mathcal{Q}_2)^\times$ such that $\operatorname{tr} \rho_{f_{\mathfrak{P}_1}}|_H^\sigma = \operatorname{tr} \rho_{f_{\mathfrak{P}_2}}|_H \otimes \varphi$. The obstruction theory arguments allow us to lift σ and φ to $\tilde{\sigma} : \mathbb{I}'/\mathfrak{P}_1' \cong \mathbb{I}'/\mathfrak{P}_2'$ and $\tilde{\varphi} : G_\mathbb{Q} \to Q(\mathbb{I}/\mathfrak{P}_2)^\times$ such that $\operatorname{tr} \rho_{f_{\mathfrak{P}_1}}^{\tilde{\sigma}} = \operatorname{tr} \rho_{f_{\mathfrak{P}_2}} \otimes \tilde{\varphi}$. Theorem 3.1 allows us to lift $\tilde{\sigma}$ to an element of Γ that sends $\mathfrak{P}_1'$ to $\mathfrak{P}_2'$. But $\mathfrak{P}_1'$ and $\mathfrak{P}_2'$ lie over different primes of $\mathbb{I}_0$ and Γ fixes $\mathbb{I}_0$, so we reach a contradiction. Therefore the image of $\operatorname{Im} \rho$ in the product $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_1) \times \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q}_2)$ is open. □

6. Proof of main theorem

In this section we use the compatibility between the conjugate self-twists of F and those of its classical specializations established in Section 3 to relate $\mathbb{I}_0/\mathcal{Q}$ to the ring appearing in the work of Ribet [1980; 1985] and Momose [1981]. This allows us to use their results to finish the proof of Theorem 2.4.

We begin by recalling the work of Ribet and Momose. We follow [Ribet 1985] closely. Let $f = \sum_{n=1}^\infty a(n, f)q^n$ be a classical eigenform of weight k. Let $K = \mathbb{Q}(\{a(n, f) : n \in \mathbb{Z}^+\})$ with ring of integers $\mathcal{O}$. Denote by Γ_f the group of conjugate self-twists of f. Let $E = K^{\Gamma_f}$ and $H_f = \bigcap_{\sigma\in\Gamma_f} \ker \eta_\sigma$. For any character ψ, let $G(\psi)$ denote the Gauss sum of the primitive character of ψ. For $\sigma, \tau \in \Gamma_f$ Ribet defined

$$c(\sigma, \tau) := \frac{G(\eta_\sigma^{-1})G(\eta_\tau^{-\sigma})}{G(\eta_{\sigma\tau}^{-1})}.$$

One shows that c is a 2-cocycle on Γ_f with values in $K^\times$.

Let $\mathfrak{X}$ be the central simple E-algebra associated to c. Then K is the maximal commutative semisimple subalgebra of $\mathfrak{X}$. It can be shown that $\mathfrak{X}$ has order two in the Brauer group of E, and hence there is a 4-dimensional E-algebra D that represents the same element as $\mathfrak{X}$ in the Brauer group of E. Namely, if $\mathfrak{X}$ has order one then $D = M_2(E)$ and otherwise D is a quaternion division algebra over E.

For a prime p, recall that we have a Galois representation

$$\rho_{f,p} : G_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{O}_K \otimes_\mathbb{Z} \mathbb{Z}_p)$$

associated to f. The following theorem is due to Ribet [1980] in the case when f has weight 2.

Theorem 6.1 [Momose 1981]. *We may view $\rho_{f,p}|_{H_f}$ as a representation valued in $(D \otimes_\mathbb{Q} \mathbb{Q}_p)^\times$. Furthermore, letting $\mathfrak{n}$ denote the reduced norm map on D, the image of $\rho_{f,p}|_{H_f}$ is open in*

$$\{x \in (D \otimes_\mathbb{Q} \mathbb{Q}_p)^\times : \mathfrak{n}x \in \mathbb{Q}_p^\times\}.$$

In particular, when $D \otimes_\mathbb{Q} \mathbb{Q}_p$ is a matrix algebra, the above theorem tell us that $\operatorname{Im} \rho_{f,p}|_{H_f}$ is open in

$$\{\boldsymbol{x} \in \mathrm{GL}_2(\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}_p) : \det \boldsymbol{x} \in (\mathbb{Z}_p^\times)^{k-1}\}.$$

Let $\mathfrak{p}$ be a prime of $\mathcal{O}_E$ lying over p, and let $\rho_{f,\mathfrak{p}}$ be the representation obtained by projecting $\rho_{f,p}|_{H_f}$ to the $\mathcal{O}_{E_\mathfrak{p}}$-component. Under the assumption that $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a matrix algebra Theorem 6.1 implies that $\rho_{f,\mathfrak{p}}$ is $\mathcal{O}_{E_\mathfrak{p}}$-full. Finally, Brown and Ghate [2003, Theorem 3.3.1] proved that if f is ordinary at p, then $D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a matrix algebra.

Thus, the Galois representation associated to each classical specialization of our $\mathbb{I}$-adic form F is $\mathcal{O}_{E_\mathfrak{p}}$-full with respect to the appropriate ring $\mathcal{O}_{E_\mathfrak{p}}$. We must show that $E_\mathfrak{p}$ is equal to $Q(\mathbb{I}_0/\mathcal{Q})$, where $\mathcal{Q}$ corresponds to $\mathfrak{p}$ in a way we will make precise below.

Recall that we have a fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\mathfrak{P} \in \operatorname{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ be an arithmetic prime of $\mathbb{I}$, and let $\mathcal{Q}$ be the prime of $\mathbb{I}_0$ lying under $\mathfrak{P}$. As usual, let $\mathfrak{P}' = \mathfrak{P} \cap \mathbb{I}'$. Let $D(\mathfrak{P}'|\mathcal{Q}) \subseteq \Gamma$ be the decomposition group of $\mathfrak{P}'$ over $\mathcal{Q}$. Let

$$K_\mathfrak{P} = \mathbb{Q}(\{\iota_p^{-1}(a(n, f_\mathfrak{P})) : n \in \mathbb{Z}^+\}) \subset \overline{\mathbb{Q}},$$

and let $\Gamma_\mathfrak{P}$ be the group of all conjugate self-twists of the classical modular form $f_\mathfrak{P}$. Set $E_\mathfrak{P} = K_\mathfrak{P}^{\Gamma_\mathfrak{P}}$. Let $\mathfrak{q}_\mathfrak{P}$ be the prime of $K_\mathfrak{P}$ corresponding to the embedding $\iota_p|_{K_\mathfrak{P}}$, and set $\mathfrak{p}_\mathfrak{P} = \mathfrak{q}_\mathfrak{P} \cap E_\mathfrak{P}$. Let $D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P}) \subseteq \Gamma_\mathfrak{P}$ be the decomposition group of $\mathfrak{q}_\mathfrak{P}$ over $\mathfrak{p}_\mathfrak{P}$. Thus we have that the completion $K_{\mathfrak{P},\mathfrak{q}_\mathfrak{P}}$ of $K_\mathfrak{P}$ at $\mathfrak{q}_\mathfrak{P}$ is equal to $Q(\mathbb{I}/\mathfrak{P})$ and $\operatorname{Gal}(K_{\mathfrak{P},\mathfrak{q}_\mathfrak{P}}/E_{\mathfrak{P},\mathfrak{p}_\mathfrak{P}}) = D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P})$. Thus we may view $D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P})$ as the set of all automorphisms of $K_{\mathfrak{P},\mathfrak{q}_\mathfrak{P}}$ that are conjugate self-twists of $f_\mathfrak{P}$.

With this in mind, we see that there is a natural group homomorphism

$$\Phi : D(\mathfrak{P}'|\mathcal{Q}) \to D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P})$$

since any element of $D(\mathfrak{P}'|\mathcal{Q})$ stabilizes $\mathfrak{P}'$ and hence induces an automorphism of $Q(\mathbb{I}'/\mathfrak{P}') = Q(\mathbb{I}/\mathfrak{P}) = K_{\mathfrak{P},\mathfrak{q}_\mathfrak{P}}$. The induced automorphism will necessarily be a conjugate self-twist of $f_\mathfrak{P}$ since we started with a conjugate self-twist of F. Thus we get an element of $D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P})$. The main compatibility result is that Φ is an isomorphism.

Proposition 6.2. *The natural group homomorphism Φ is an isomorphism. Hence $Q(\mathbb{I}_0/\mathcal{Q}) = E_{\mathfrak{P},\mathfrak{p}_\mathfrak{P}}$.*

Proof. The fact that Φ is injective is easy. Namely, if $\sigma \in D(\mathfrak{P}'|\mathcal{Q})$ acts trivially on $K_{\mathfrak{P},\mathfrak{q}_\mathfrak{P}}$ then for almost all ℓ we have

$$a(\ell, f_\mathfrak{P}) = a(\ell, f_\mathfrak{P})^\sigma = \eta_\sigma(\ell) a(\ell, f_\mathfrak{P}).$$

Since F (and hence its arithmetic specialization $f_\mathfrak{P}$) does not have CM it follows that $\eta_\sigma = 1$. Hence $\sigma = 1$ and Φ is injective.

To see that Φ is surjective, let $\sigma \in D(\mathfrak{q}_\mathfrak{P}|\mathfrak{p}_\mathfrak{P})$. By Theorem 3.1 we see that there is $\tilde{\sigma} \in \operatorname{Aut} \mathbb{I}'$ that is a conjugate self-twist of F and $\sigma \circ \mathfrak{P} = \mathfrak{P} \circ \tilde{\sigma}$. That is,

$\tilde{\sigma} \in D(\mathfrak{P}'|\mathcal{Q})$ and $\Phi(\tilde{\sigma}) = \sigma$. We have

$$E_{\mathfrak{P},\mathfrak{p}_{\mathfrak{P}}} = K_{\mathfrak{P},\mathfrak{q}_{\mathfrak{P}}}^{D(\mathfrak{q}_{\mathfrak{P}}|\mathfrak{p}_{\mathfrak{P}})} = Q(\mathbb{I}'/\mathfrak{P}')^{D(\mathfrak{P}|\mathcal{Q})}.$$

A general fact from commutative algebra [Bourbaki 1972, Theorem V.2.2.2] tells us that $Q(\mathbb{I}'/\mathfrak{P}')^{D(\mathfrak{P}|\mathcal{Q})} = Q(\mathbb{I}_0/\mathcal{Q})$, as desired. □

Corollary 6.3. *Let $\mathcal{Q}$ be a prime of $\mathbb{I}_0$ lying over an arithmetic prime of Λ. There is a nonzero $\mathbb{I}_0/\mathcal{Q}$-ideal $\bar{\mathfrak{a}}_{\mathcal{Q}}$ such that*

$$\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}}) \subseteq \mathrm{Im}(\rho_F \bmod \mathcal{Q}\mathbb{I}') \subseteq \prod_{\mathfrak{P}'|\mathcal{Q}} \mathrm{GL}_2(\mathbb{I}'/\mathfrak{P}'),$$

where the inclusion of $\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}})$ in the product is via the diagonal embedding $\mathrm{GL}_2(\mathbb{I}_0/\mathcal{Q}) \hookrightarrow \prod_{\mathfrak{P}'|\mathcal{Q}} \mathrm{GL}_2(\mathbb{I}'/\mathfrak{P}')$. Hence the image of $\mathrm{Im}\,\rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open.

Proof. For a prime $\mathfrak{P}$ of $\mathbb{I}$, write $\mathcal{O}_{\mathfrak{P}}$ for the ring of integers of $E_{\mathfrak{P},\mathfrak{p}_{\mathfrak{P}}}$. By Theorem 6.1 and the remarks following it, for each prime $\mathfrak{P}$ of $\mathbb{I}$ lying over $\mathcal{Q}$ we have that $\mathrm{Im}\,\rho_{f_{\mathfrak{P}}}$ contains $\Gamma_{\mathcal{O}_{\mathfrak{P}}}(\bar{\mathfrak{a}}_{\mathfrak{P}})$ for some nonzero $\mathcal{O}_{\mathfrak{P}}$-ideal $\bar{\mathfrak{a}}_{\mathfrak{P}}$. While $\mathbb{I}_0/\mathcal{Q}$ need not be integrally closed, by Proposition 6.2 we see that $\bar{\mathfrak{a}}_{\mathfrak{P}} \cap (\mathbb{I}_0/\mathcal{Q})$ is a nonzero $\mathbb{I}_0/\mathcal{Q}$-ideal.

Thus we have

$$\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathfrak{P}} \cap \mathbb{I}_0/\mathcal{Q}) \subseteq \Gamma_{\mathcal{O}_{\mathfrak{P}}}(\bar{\mathfrak{a}}_{\mathfrak{P}}) \subseteq \mathrm{Im}\,\rho_{f_{\mathfrak{P}}} = \mathrm{Im}\,\rho_F \bmod \mathfrak{P} \subseteq \mathrm{GL}_2(\mathbb{I}'/\mathfrak{P}').$$

Let $\bar{\mathfrak{a}}_{\mathcal{Q}} = \bigcap_{\mathfrak{P}|\mathcal{Q}} \bar{\mathfrak{a}}_{\mathfrak{P}} \cap \mathbb{I}_0/\mathcal{Q}$. This is a finite intersection of nonzero $\mathbb{I}_0/\mathcal{Q}$-ideals and hence is nonzero. The first statement follows from the above inclusions.

For the statement about ρ, recall that $\rho_F|_{H_0}$ is valued in $\mathrm{GL}_2(\mathbb{I}_0)$ and consequently $\mathrm{Im}\,\rho_F|_{H_0} \bmod \mathcal{Q}$ lies in the diagonally embedded copy of $\mathrm{GL}_2(\mathbb{I}_0/\mathcal{Q})$ in $\prod_{\mathfrak{P}'|\mathcal{Q}} \mathrm{GL}_2(\mathbb{I}'/\mathfrak{P}')$. Since H is open in $G_{\mathbb{Q}}$, by replacing $\bar{\mathfrak{a}}_{\mathcal{Q}}$ with a smaller $\mathbb{I}_0/\mathcal{Q}$-ideal if necessary, we may assume that $\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}})$ is contained in the image of $\rho_F|_H$ in $\mathrm{GL}_2(\mathbb{I}_0/\mathcal{Q})$. Since ρ and ρ_F are equal on elements of determinant 1 and $\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}}) \subseteq \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$, it follows that $\Gamma_{\mathbb{I}_0/\mathcal{Q}}(\bar{\mathfrak{a}}_{\mathcal{Q}})$ is contained in the image of $\mathrm{Im}\,\rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$. That is, the image of $\mathrm{Im}\,\rho$ in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open. □

Summary of Proof of Theorem 2.4. Theorem 4.1, which will be proved in the next section, allows us to create a representation $\rho : H \to \mathrm{SL}_2(\mathbb{I}_0)$ with the property that if ρ is $\mathbb{I}_0$-full then so is ρ_F. This is important for the use of Pink's theory in Section 4 as well as for the techniques of Section 5. Proposition 4.2 shows that it is sufficient to prove that the image of $\mathrm{Im}\,\rho$ in $\prod_{\mathcal{Q}|P} \mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ is open for some arithmetic prime P of Λ. Proposition 5.1 further reduces the problem to showing that the image of ρ modulo $\mathcal{Q}$ is open in $\mathrm{SL}_2(\mathbb{I}_0/\mathcal{Q})$ for all primes $\mathcal{Q}$ of $\mathbb{I}_0$ lying over a fixed arithmetic prime P of Λ.

This reduces the problem to studying the image of a Galois representation attached to one of the classical specializations of F (twisted by the inverse square root of the determinant). Hence we can apply the work of Ribet and Momose, but only after we show that $Q(\mathbb{I}_0/\mathcal{Q})$ is the same field that occurs in their work. This is done in Proposition 6.2, though the main input is Theorem 3.1. □

7. Obtaining an $\mathrm{SL}_2(\mathbb{I}_0)$-valued representation

In this section we prove:

Theorem 4.1. *Assume that $\bar{\rho}_F$ is absolutely irreducible and H_0-regular. If $V = \mathbb{I}'^2$ is the module on which $G_{\mathbb{Q}}$ acts via ρ_F, then there is a basis for V such that all of the following happen simultaneously*:

(1) ρ_F *is valued in* $\mathrm{GL}_2(\mathbb{I}')$;

(2) $\rho_F|_{D_p}$ *is upper triangular*;

(3) $\rho_F|_{H_0}$ *is valued in* $\mathrm{GL}_2(\mathbb{I}_0)$;

(4) *There is a matrix $\boldsymbol{j} = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta' \end{smallmatrix}\right)$, where ζ and ζ' are roots of unity, such that $\boldsymbol{j}$ normalizes the image of ρ_F and $\zeta \not\equiv \zeta'$* mod *p.*

It is well known that so long as $\bar{\rho}_F$ is absolutely irreducible we may assume that ρ_F has values in $\mathrm{GL}_2(\mathbb{I}')$ and the local representation $\rho_F|_{D_p}$ is upper triangular [Hida 2000a, Theorem 4.3.2]. To show that $\rho_F|_{H_0}$ has values in $\mathrm{GL}_2(\mathbb{I}_0)$ we begin by investigating the structure of Γ.

Proposition 7.1. *The group Γ is a finite abelian 2-group.*

Proof. Let S be the set of primes ℓ for which $a(\ell, F)^\sigma = \eta_\sigma(\ell)a(\ell, F)$ for all $\sigma \in \Gamma$, so S excludes only finitely many primes. For $\ell \in S$, let

$$b_\ell := \frac{a(\ell, F)^2}{\det \rho_F(\mathrm{Frob}_\ell)}.$$

It turns out that $b_\ell \in \mathbb{I}_0$. To see this, note that since $\bar{\rho}_F$ is absolutely irreducible, for any $\sigma \in \Gamma$ we have $\rho_F^\sigma \cong \eta_\sigma \otimes \rho_F$ over $\mathbb{I}'$. Taking determinants we find that $\det \rho_F^{\sigma-1} = \eta_\sigma^2$. Thus we have

$$(a(\ell, F)^\sigma)^2 = \eta_\sigma(\ell)^2 a(\ell, F)^2 = \det \rho_F(\mathrm{Frob}_\ell)^{\sigma-1} a(\ell, F)^2,$$

from which it follows that $b_\ell^\sigma = b_\ell$. Solving for $a(\ell, F)$ in the definition of b_ℓ we find that

$$Q(\mathbb{I}') = Q(\mathbb{I}_0)\left[\sqrt{b_\ell \det \rho_F(\mathrm{Frob}_\ell)} : \ell \in S\right].$$

Recall that for $\ell \in S$ we have $\det \rho_F(\mathrm{Frob}_\ell) = \chi(\ell)\kappa(\langle\ell\rangle)\ell^{-1}$, where $\kappa(\langle\ell\rangle) \in 1 + \mathfrak{m}_\Lambda$. (Currently all that matters is that κ is valued in $1 + \mathfrak{m}_\Lambda$. For a precise definition of κ, see the proof of Lemma 3.11.) In particular, $\sqrt{\kappa(\langle\ell\rangle)} \in \Lambda$. Similarly, we can write $\ell = \langle\ell\rangle\omega(\ell)$ with $\langle\ell\rangle \in 1 + p\mathbb{Z}_p$ and $\omega(\ell) \in \mu_{p-1}$. So $\sqrt{\langle\ell\rangle} \in \Lambda$ as well.

Let

$$\mathcal{K} = Q(\mathbb{I}_0)\big[\sqrt{b_\ell}, \sqrt{\det \rho_F(\mathrm{Frob}_\ell)} : \ell \in S\big],$$

which is an abelian extension of $Q(\mathbb{I}_0)$ since it is obtained by adjoining square roots. The above argument shows that in fact $\mathcal{K}$ is obtained from $Q(\mathbb{I}_0)[\sqrt{b_\ell} : \ell \in S]$ by adjoining finitely many roots of unity, namely the square roots of the values of χ and the square roots of μ_{p-1}. As odd order roots of unity are automatically squares, we can write $\mathcal{K} = Q(\mathbb{I}_0)[\sqrt{b_\ell} : \ell \in S][\mu_{2^s}]$ for some $s \in \mathbb{Z}^+$. Thus we have

$$\mathrm{Gal}(\mathcal{K}/Q(\mathbb{I}_0)) \cong \mathrm{Gal}\big(Q(\mathbb{I}_0)[\sqrt{b_\ell} : \ell \in S]/Q(\mathbb{I}_0)\big) \times \mathrm{Gal}\big(Q(\mathbb{I}_0)[\mu_{2^s}]/Q(\mathbb{I}_0)\big).$$

By Kummer theory the first group is an elementary abelian 2-group. The second group is isomorphic to $(\mathbb{Z}/2^s\mathbb{Z})^\times$ and hence is a 2-group. As Γ is a quotient of $\mathrm{Gal}(\mathcal{K}/Q(\mathbb{I}_0))$ it follows that Γ is a finite abelian 2-group, as claimed. □

For ease of notation let $\pi = \bar{\rho}_F|_{H_0} : H_0 \to \mathrm{GL}_2(\mathbb{F})$. Let D be a nonsquare in $\mathbb{F}$, and let $\mathbb{E} = \mathbb{F}[\sqrt{D}]$ be the unique quadratic extension of $\mathbb{F}$.

Lemma 7.2. *Let K be a field and $\mathcal{S} \subset \mathrm{GL}_n(K)$ a set of nonconstant semisimple operators that can be simultaneously diagonalized over $\overline{K}$. If $\boldsymbol{y} \in \mathrm{GL}_n(\overline{K})$ such that $\boldsymbol{y}\mathcal{S}\boldsymbol{y}^{-1} \subset \mathrm{GL}_n(K)$, then there is a matrix $\boldsymbol{z} \in \mathrm{GL}_n(K)$ such that $\boldsymbol{z}\mathcal{S}\boldsymbol{z}^{-1} = \boldsymbol{y}\mathcal{S}\boldsymbol{y}^{-1}$. In particular, if π is irreducible over $\mathbb{F}$ but not absolutely irreducible, then $\mathbb{E}$ is the splitting field for π.*

Proof. Let $\sigma \in G_K := \mathrm{Gal}(\overline{K}/K)$. Then for any $\boldsymbol{x} \in \mathcal{S}$ we have $\boldsymbol{y}^\sigma \boldsymbol{x}\boldsymbol{y}^{-\sigma} = (\boldsymbol{y}\boldsymbol{x}\boldsymbol{y}^{-1})^\sigma = \boldsymbol{y}\boldsymbol{x}\boldsymbol{y}^{-1}$, so $\boldsymbol{y}^{-1}\boldsymbol{y}^\sigma$ centralizes $\boldsymbol{x}$. As elements in $\mathcal{S}$ are simultaneously diagonalizable, they have the same centralizer in $\mathrm{GL}_n(\overline{K})$. Since elements of $\mathcal{S}$ are semisimple, their centralizer is a torus and hence isomorphic to $(\overline{K})^{\oplus n}$. It's not hard to show that $a : G_K \to (\overline{K}^\times)^{\oplus n}$ given by $\sigma \mapsto \boldsymbol{y}^{-1}\boldsymbol{y}^\sigma$ is a 1-cocycle. (Here we view $(\overline{K}^\times)^{\oplus n}$ as a G_K-module by letting elements of G_K act componentwise.) By Hilbert's theorem 90 we have $H^1(G_K, (\overline{K}^\times)^{\oplus n}) = H^1(G_K, \overline{K}^\times)^{\oplus n} = 0$. Hence a is a coboundary. That is, there is some $\alpha \in (\overline{K}^\times)^{\oplus n}$ such that

$$a_\sigma = \boldsymbol{y}^{-1}\boldsymbol{y}^\sigma = \alpha^{-1}\alpha^\sigma$$

for all $\sigma \in G_K$. Thus $(\boldsymbol{y}\alpha^{-1})^\sigma = \boldsymbol{y}\alpha^{-1}$ for all $\sigma \in G_K$, so $\boldsymbol{z} := \boldsymbol{y}\alpha^{-1} \in \mathrm{GL}_n(K)$. But α commutes with $\mathcal{S}$ and so $\boldsymbol{z}\mathcal{S}\boldsymbol{z}^{-1} = \boldsymbol{y}\mathcal{S}\boldsymbol{y}^{-1}$, as claimed.

To deduce the claim about π, let $\mathcal{S} = \mathrm{Im}\,\pi$. The fact that $\mathcal{S}$ is semisimple follows from Clifford's theorem since $\bar{\rho}_F$ is absolutely irreducible [Isaacs 1976, Theorem 6.5, Corollary 6.6]. If π is not absolutely irreducible then there is a matrix $\boldsymbol{y} \in \mathrm{GL}_2(\overline{\mathbb{F}})$ that simultaneously diagonalizes $\mathcal{S}$. Note that every matrix in $\mathrm{Im}\,\pi$ has eigenvalues in $\mathbb{E}$. Indeed every matrix has a quadratic characteristic polynomial and $\mathbb{E}$ is the unique quadratic extension of $\mathbb{F}$. Thus, taking $K = \mathbb{E}$ we see that $\boldsymbol{y}\mathcal{S}\boldsymbol{y}^{-1} \subset \mathrm{GL}_2(K)$. The first statement of the lemma tells us that $\mathrm{Im}\,\pi$ is diagonalizable over $\mathbb{E}$. Since π is irreducible over $\mathbb{F}$ and $[\mathbb{E}:\mathbb{F}]=2$, it follows that $\mathbb{E}$ is the smallest extension of $\mathbb{F}$ over which $\operatorname{Im}\pi$ is diagonalizable. $\square$

Let Z be the centralizer of $\operatorname{Im}\pi$ in $M_2(\mathbb{F})$. Since $\bar{\rho}_F$ is H_0-regular, exactly one of the three cases must occur:

1. The representation π is absolutely irreducible. In this case Z consists of scalar matrices over $\mathbb{F}$.
2. The representation π is not absolutely irreducible, but π is irreducible over $\mathbb{F}$. In this case we may assume
$$Z=\left\{\begin{pmatrix}\alpha & \beta D\\ \beta & \alpha\end{pmatrix} : \alpha,\beta\in\mathbb{F}\right\}\cong\mathbb{E}.$$
3. The representation π is reducible over $\mathbb{F}$. In this case we may assume that Z consists of diagonal matrices over $\mathbb{F}$.

Recall that since $\bar{\rho}_F$ is absolutely irreducible, for any $\sigma\in\Gamma$ we have $\rho_F^\sigma\cong\eta_\sigma\otimes\rho_F$. That is, there is some $\boldsymbol{t}_\sigma\in\mathrm{GL}_2(\mathbb{I}')$ such that
$$\rho_F(g)^\sigma=\eta_\sigma(g)\boldsymbol{t}_\sigma\rho_F(g)\boldsymbol{t}_\sigma^{-1}$$
for all $g\in G_{\mathbb{Q}}$. Then for all $\sigma,\tau\in\Gamma$, $g\in G_{\mathbb{Q}}$ we have
$$\eta_{\sigma\tau}(g)\boldsymbol{t}_{\sigma\tau}\rho_F(g)\boldsymbol{t}_{\sigma\tau}^{-1}=\rho(g)^{\sigma\tau}=\eta_\sigma^\tau(g)\eta_\tau(g)\boldsymbol{t}_\sigma^\tau\boldsymbol{t}_\tau\rho_F(g)\boldsymbol{t}_\tau^{-1}\boldsymbol{t}_\sigma^{-\tau}.$$
Using the fact that $\eta_{\sigma\tau}=\eta_\sigma^\tau\eta_\tau$ we see that $c(\sigma,\tau):=\boldsymbol{t}_{\sigma\tau}^{-1}\boldsymbol{t}_\sigma^\tau\boldsymbol{t}_\tau$ commutes with the image of ρ_F. As ρ_F is absolutely irreducible, $c(\sigma,\tau)$ must be a scalar. Hence c represents a 2-cocycle of Γ with values in $\mathbb{I}'^\times$.

We will need to treat case 2 (π is irreducible over $\mathbb{F}$ but not absolutely irreducible) a bit differently, so we establish notation that will unify the proofs that follow. For a finite extension M of $\mathbb{Q}_p$, let $\mathcal{O}_M$ denote the ring of integers of M. Let K be the largest finite extension of $\mathbb{Q}_p$ for which $\mathcal{O}_K[[T]]$ is contained in $\mathbb{I}'$. So K has residue field $\mathbb{F}$. Let L be the unique unramified quadratic extension of K. Write $\mathbb{J}=\Lambda_{\mathcal{O}_L}[\{a(\ell,F):\ell\nmid N\}]$. Note that the residue field of $\mathbb{J}$ is the unique quadratic extension of $\mathbb{F}$. Let
$$A=\begin{cases}\mathbb{J} & \text{in case 2,}\\ \mathbb{I}' & \text{else.}\end{cases}$$
Let κ be the residue field of A, so $\kappa=\mathbb{E}$ in case 2 and $\kappa=\mathbb{F}$ otherwise.

Since L is obtained from K by adjoining some prime-to-p root of unity, in case 2 it follows that $Q(A)$ is Galois over $Q(\mathbb{I}_0)$ with Galois group isomorphic to $\Gamma\times\mathbb{Z}/2\mathbb{Z}$. In particular, we have an action of Γ on A in all cases. Let $B=A^\Gamma$. In case 2, A is a quadratic extension of B and $B\cap\mathbb{I}'=\mathbb{I}_0$. Otherwise $B=\mathbb{I}_0$. We may consider the 2-cocycle c in $H^2(\Gamma,A^\times)$.

Lemma 7.3. *With notation as above,* $[c]=0\in H^2(\Gamma, A^\times)$. *Thus there is a function* $\zeta:\Gamma\to A^\times$ *such that* $c(\sigma,\tau)=\zeta(\sigma\tau)^{-1}\zeta(\sigma)^\tau\zeta(\tau)$ *for all* $\sigma,\tau\in\Gamma$.

Proof. Consider the exact sequence $1\to 1+\mathfrak{m}_A\to A^\times\to\kappa^\times\to 1$. Note that for $j>0$ we have $H^j(\Gamma, 1+\mathfrak{m}_A)=0$ since $1+\mathfrak{m}_A$ is a p-profinite group for $p>2$ and Γ is a 2-group by Proposition 7.1. Thus the long exact sequence in cohomology gives isomorphisms

$$H^j(\Gamma, A^\times)\cong H^j(\Gamma,\kappa^\times)$$

for all $j>0$. Hence it suffices to prove that $[\bar{c}]=0\in H^2(\Gamma,\kappa^\times)$.

Let $\sigma\in\Gamma$ and $h\in H_0$. Recall that Γ acts trivially on $\mathbb{F}$ by Proposition 3.4. Since $\rho_F^\sigma(h)=\eta_\sigma(h)\boldsymbol{t}_\sigma\rho_F(h)\boldsymbol{t}_\sigma^{-1}$ and $\eta_\sigma(h)=1$ it follows that $\bar{\boldsymbol{t}}_\sigma\in Z$.

We now split into the three cases depending on the irreducibility of π. Suppose we are in case 1, so π is absolutely irreducible and $\kappa=\mathbb{F}$. Then $\bar{\boldsymbol{t}}_\sigma$ must be a scalar in $\mathbb{F}^\times$. Call it $\bar{\zeta}(\sigma)$. Then $\bar{c}(\sigma,\tau)=\bar{\zeta}(\sigma\tau)^{-1}\bar{\zeta}(\sigma)^\tau\bar{\zeta}(\tau)$, and so $[\bar{c}]=0\in H^2(\Gamma,\mathbb{F}^\times)$.

In case 2, using the description of Z above we see that $\bar{\boldsymbol{t}}_\sigma=\left(\begin{smallmatrix}\alpha_\sigma & \beta_\sigma D\\ \beta_\sigma & \alpha_\sigma\end{smallmatrix}\right)$ for some $\alpha_\sigma,\beta_\sigma\in\mathbb{F}$. This becomes a scalar, say $\bar{\zeta}(\sigma)=\alpha_\sigma+\beta_\sigma\sqrt{D}$, over $\mathbb{E}=\kappa$. Thus $\bar{\boldsymbol{t}}_\sigma=\bar{\zeta}(\sigma)$. As above $\bar{c}(\sigma,\tau)=\bar{\zeta}(\sigma\tau)^{-1}\bar{\zeta}(\sigma)^\tau\bar{\zeta}(\tau)$, and thus $[\bar{c}]=0\in H^2(\Gamma,\kappa^\times)$.

Finally, in case 3 we have that $\bar{\boldsymbol{t}}_\sigma$ is a diagonal matrix. The diagonal map $\mathbb{F}\hookrightarrow\mathbb{F}\oplus\mathbb{F}$ induces an injection $H^2(\Gamma,\mathbb{F}^\times)\hookrightarrow H^2(\Gamma,\mathbb{F}^\times\oplus\mathbb{F}^\times)$. The fact that $\bar{\boldsymbol{t}}_\sigma$ is a diagonal matrix allows us to calculate that the image of $[\bar{c}]$ in $H^2(\Gamma,\mathbb{F}^\times\oplus\mathbb{F}^\times)$ is 0. Since the map is an injection, it follows that $[\bar{c}]=0\in H^2(\Gamma,\mathbb{F}^\times)$, as desired. □

Replace $\boldsymbol{t}_\sigma\in\mathrm{GL}_2(\mathbb{I}')$ by $\boldsymbol{t}_\sigma\zeta(\sigma)^{-1}\in\mathrm{GL}_2(A)$. Then we still have $\rho_F^\sigma=\eta_\sigma\boldsymbol{t}_\sigma\rho_F\boldsymbol{t}_\sigma^{-1}$, and now $\boldsymbol{t}_{\sigma\tau}=\boldsymbol{t}_\sigma^\tau\boldsymbol{t}_\tau$. That is, $\sigma\mapsto\boldsymbol{t}_\sigma$ is a nonabelian 1-cocycle with values in $\mathrm{GL}_2(A)$. Since F is primitive we have $Q(\mathbb{I})=Q(\mathbb{I}')$. Thus by [Hida 2000a, Theorem 4.3.2] we see that $\rho_F|_{D_p}$ is isomorphic to an upper triangular representation over $Q(\mathbb{I}')$. Under the assumptions that $\bar{\rho}_F$ is absolutely irreducible and H_0-regular, the proof of [Hida 2000a, Theorem 4.3.2] goes through with $\mathbb{I}'$ in place of $\mathbb{I}$. That is, $\rho_F|_{D_p}$ is isomorphic to an upper triangular representation over $\mathbb{I}'$. Let $V=\mathbb{I}'^2$ be the representation space for ρ_F with basis chosen such that

$$\rho_F|_{D_p}=\begin{pmatrix}\varepsilon & u\\ 0 & \delta\end{pmatrix},$$

and assume $\bar{\varepsilon}\neq\bar{\delta}$. Let $V[\varepsilon]\subset V$ be the free direct summand of V on which D_p acts by ε and $V[\delta]$ be the quotient of V on which D_p acts by δ. Let $V_A=V\otimes_{\mathbb{I}'}A$. Similarly for $\lambda\in\{\varepsilon,\delta\}$ let $V_A[\lambda]:=V[\lambda]\otimes_{\mathbb{I}'}A$. For $\boldsymbol{v}\in V_A$, define

$$\boldsymbol{v}^{[\sigma]}:=\boldsymbol{t}_\sigma^{-1}\boldsymbol{v}^\sigma, \tag{7}$$

where σ acts on $\boldsymbol{v}$ componentwise. Note that in case 2 we are using the action of Γ on A described prior to Lemma 7.3.

Lemma 7.4. *For all $\sigma, \tau \in \Gamma$ we have $(\boldsymbol{v}^{[\sigma]})^{[\tau]} = \boldsymbol{v}^{[\sigma\tau]}$, so this defines an action of Γ on V_A. Furthermore, this action stabilizes $V_A[\varepsilon]$ and $V_A[\delta]$.*

Proof. The formula (7) defines an action since $\sigma \mapsto \boldsymbol{t}_\sigma$ is a nonabelian 1-cocycle. Let λ be either δ or ε. Let $\boldsymbol{v} \in V_A[\lambda]$ and $\sigma \in \Gamma$. We must show that $\boldsymbol{v}^{[\sigma]} \in V_A[\lambda]$. Let $d \in D_p$. Using the fact that $\boldsymbol{v} \in V_A[\lambda]$ and $\rho_F^\sigma = \eta_\sigma \boldsymbol{t}_\sigma \rho_F \boldsymbol{t}_\sigma^{-1}$ we find that

$$\rho_F(d)\boldsymbol{v}^{[\sigma]} = \eta_\sigma^{-1}(d)\lambda^\sigma(d)\boldsymbol{v}^{[\sigma]}.$$

Note that for all $d \in D_p$

$$\begin{pmatrix} \varepsilon^\sigma(d) & u^\sigma(d) \\ 0 & \delta^\sigma(d) \end{pmatrix} = \rho_F^\sigma(d) = \eta_\sigma(d)\boldsymbol{t}_\sigma \rho_F(d)\boldsymbol{t}_\sigma^{-1} = \eta_\sigma(d)\boldsymbol{t}_\sigma \begin{pmatrix} \varepsilon(d) & u(d) \\ 0 & \delta(d) \end{pmatrix} \boldsymbol{t}_\sigma^{-1}. \tag{8}$$

Using the fact that $\varepsilon \neq \delta$ and that $\rho_F|_{D_p}$ is indecomposable [Ghate and Vatsal 2004; Zhao 2014] we see that $u/(\varepsilon - \delta)$ cannot be a constant. (If $u/(\varepsilon - \delta) = \alpha$ is a constant, then conjugating by $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ makes $\rho_F|_{D_p}$ diagonal.) Hence $\boldsymbol{t}_\sigma$ must be upper triangular. Therefore (8) implies that $\lambda^\sigma(d) = \eta_\sigma(d)\lambda(d)$, and thus

$$\rho_F(d)\boldsymbol{v}^{[\sigma]} = \eta_\sigma^{-1}(d)\lambda^\sigma(d)\boldsymbol{v}^{[\sigma]} = \lambda(d)\boldsymbol{v}^{[\sigma]}. \qquad \square$$

We are now ready to show that $\rho_F|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0)$.

Theorem 7.5. *Let $\rho_F : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{I}')$ such that $\rho_F|_{D_p}$ is upper triangular. Assume that $\bar{\rho}_F$ is absolutely irreducible and H_0-regular. Then $\rho_F|_{H_0}$ takes values in $\mathrm{GL}_2(\mathbb{I}_0)$.*

Proof. We have an exact sequence of $A[D_p]$-modules

$$0 \to V_A[\varepsilon] \to V_A \to V_A[\delta] \to 0 \tag{9}$$

that is stable under the new action of Γ defined in Lemma 7.4. Tensoring with κ over A we get an exact sequence of κ-vector spaces

$$V_\kappa[\bar{\varepsilon}] \to V_\kappa \to V_\kappa[\bar{\delta}] \to 0. \tag{10}$$

Since $V_A[\varepsilon]$ is a direct summand of V_A, the first arrow is injective. Since $V_A[\varepsilon]$ and V_A are free A-modules, it follows that $\dim_\kappa V_\kappa[\bar{\varepsilon}] = 1$ and $\dim_\kappa V_\kappa = 2$. Counting dimensions in (10) now tells us that $\dim_\kappa V_\kappa[\bar{\delta}] = 1$.

Going back to the exact sequence (9) we can take Γ-invariants since all of the modules are stable under the new action of Γ. This gives an exact sequence of $B[D_p \cap H_0]$-modules

$$0 \to V_A[\varepsilon]^\Gamma \to V_A^\Gamma \to V_A[\delta]^\Gamma \to H^1(\Gamma, V_A[\varepsilon]).$$

Since Γ is a 2-group by Proposition 7.1 and $V_A[\varepsilon] \cong A$ is p-profinite, we find that $H^1(\Gamma, V_A[\varepsilon]) = 0$. Tensoring with κ^Γ over B we get an exact sequence

$$V_A[\varepsilon]^\Gamma \otimes_B \kappa^\Gamma \to V_A^\Gamma \otimes_B \kappa^\Gamma \to V_A[\delta]^\Gamma \otimes_B \kappa^\Gamma \to 0.$$

If $\dim_{\kappa^\Gamma} V_A[\lambda]^\Gamma \otimes_B \kappa^\Gamma = 1$ for $\lambda \in \{\varepsilon, \delta\}$, then it follows from Nakayama's lemma that $V_A[\lambda]^\Gamma$ is a free B-module of rank 1. Hence V_A^Γ is a free B-module of rank 2. In all the cases except case 2, this completes the proof. In case 2 the above argument tells us that if we view ρ_F as a $\mathrm{GL}_2(A)$-valued representation, then $\rho_F|_{H_0}$ takes values in $\mathrm{GL}_2(B)$. We know that ρ_F actually has values in $\mathrm{GL}_2(\mathbb{I}')$ and hence $\rho_F|_{H_0}$ has values in $\mathrm{GL}_2(B \cap \mathbb{I}') = \mathrm{GL}_2(\mathbb{I}_0)$.

Thus we must show that for $\lambda \in \{\varepsilon, \delta\}$ we have $\dim_{\kappa^\Gamma} V_A[\lambda]^\Gamma \otimes_B \kappa^\Gamma = 1$. Note that $V_A[\lambda]^\Gamma \otimes_B \kappa^\Gamma = V_\kappa[\bar{\lambda}]^\Gamma$. When we are not in case 2, Γ acts trivially on κ and hence

$$\dim_{\mathbb{F}} V_{\mathbb{F}}[\bar{\lambda}]^\Gamma = \dim_{\mathbb{F}} V_{\mathbb{F}}[\bar{\lambda}] = 1.$$

Now assume we are in case 2, so $\kappa = \mathbb{E}$. Write $\overline{\Gamma}$ for the quotient of Γ that acts on $\mathbb{E}$. That is, $\overline{\Gamma} = \mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Let $\sigma \in \overline{\Gamma}$ be a generator. Since $\dim_{\mathbb{E}} V_{\mathbb{E}}[\bar{\lambda}] = 1$ we can choose some nonzero $\boldsymbol{v} \in V_{\mathbb{E}}[\bar{\lambda}]$. We would like to show that

$$\boldsymbol{v} + \boldsymbol{v}^{[\sigma]} \neq 0$$

since the right hand side is $\overline{\Gamma}$-invariant.

Since $V_{\mathbb{E}}[\bar{\lambda}]$ is 1-dimensional, there is some $\alpha \in \mathbb{E}^\times$ such that $\boldsymbol{v}^{[\sigma]} = \alpha \boldsymbol{v}$. Thus $\boldsymbol{v} + \boldsymbol{v}^{[\sigma]} = (1+\alpha)\boldsymbol{v}$. If $\alpha \neq -1$ then we are done. Otherwise we can change $\boldsymbol{v}$ to $a\boldsymbol{v}$ for any $a \in \mathbb{E}^\times$. It is easy to see that $(a\boldsymbol{v})^{[\sigma]} = a^\sigma \alpha a^{-1}(a\boldsymbol{v})$ and thus changing $\boldsymbol{v}$ to $a\boldsymbol{v}$ changes α to $a^\sigma a^{-1}\alpha$. So we need to show that there is some $a \in \mathbb{E}^\times$ such that $a^\sigma a^{-1} \neq 1$. But clearly this holds for any $a \in \mathbb{E} \setminus \mathbb{F}$. Therefore $\dim_{\mathbb{F}} V_{\mathbb{E}}[\bar{\lambda}]^\Gamma \geq 1$.

To get equality, let $0 \neq \boldsymbol{w} \in V_{\mathbb{E}}[\bar{\lambda}]^\Gamma$. Since $V_{\mathbb{E}}[\bar{\lambda}]^\Gamma \subseteq V_{\mathbb{E}}[\bar{\lambda}]$ and $\dim_{\mathbb{E}} V_{\mathbb{E}}[\bar{\lambda}] = 1$, any element of $V_{\mathbb{E}}[\bar{\lambda}]^\Gamma$ is an $\mathbb{E}$-multiple of $\boldsymbol{w}$. If $\beta \in \mathbb{E} \setminus \mathbb{F}$ then σ does not fix β. Thus

$$(\beta \boldsymbol{w})^{[\sigma]} = \beta^\sigma \boldsymbol{w}^{[\sigma]} = \beta^\sigma \boldsymbol{w} \neq \beta \boldsymbol{w}.$$

Hence $V_{\mathbb{E}}[\bar{\lambda}]^\Gamma = \mathbb{F}\boldsymbol{w}$ and $\dim_{\mathbb{F}} V_{\mathbb{E}}[\bar{\lambda}]^\Gamma = 1$, as desired. □

Finally, we modify ρ_F to obtain the normalizing matrix $\boldsymbol{j}$ in the last part of Theorem 4.1.

Lemma 7.6. *Suppose $\rho_F : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{I}')$ such that $\rho_F|_{D_p}$ is upper triangular and $\rho_F|_{H_0}$ is valued in $\mathrm{GL}_2(\mathbb{I}_0)$. Assume $\bar{\rho}_F$ is absolutely irreducible and H_0-regular. Then there is an upper triangular matrix $\boldsymbol{x} \in \mathrm{GL}_2(\mathbb{I}_0)$ and roots of unity ζ and ζ' such that $\boldsymbol{j} := \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta' \end{smallmatrix}\right)$ normalizes the image of $\boldsymbol{x}\rho_F\boldsymbol{x}^{-1}$ and $\zeta \not\equiv \zeta' \bmod p$.*

Proof. This argument is due to Hida [2000a, Lemma 4.3.20]. As $\bar{\rho}_F$ is H_0-regular there is an $h \in H_0$ such that $\bar{\varepsilon}(h) \neq \bar{\delta}(h)$. Let ζ and ζ' be the roots of unity in $\mathbb{I}_0$ satisfying $\zeta \equiv \varepsilon(h) \bmod \mathfrak{m}_0$ and $\zeta' \equiv \delta(h) \bmod \mathfrak{m}_0$. By our choice of h we have $\zeta \not\equiv \zeta' \bmod p$.

Let $q = |\mathbb{F}|$. Then for some $u \in \mathbb{I}_0$

$$\lim_{n\to\infty} \rho_F(h)^{q^n} = \begin{pmatrix} \zeta & u \\ 0 & \zeta' \end{pmatrix}.$$

Conjugating ρ_F by $\left(\begin{smallmatrix} 1 & u/(\zeta-\zeta') \\ 0 & 1 \end{smallmatrix}\right)$ preserves all three of the desired properties, and the image of the resulting representation is normalized by $\boldsymbol{j} = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta' \end{smallmatrix}\right)$. □

Acknowledgements

I am grateful to my advisor, Haruzo Hida, for suggesting this problem to me and for his endless patience and insights as I worked on it. I would like to thank Ashay Burungale for many helpful and encouraging conversations about this project. Richard Pink and Jacques Tilouine provided me with insights into the larger context of this problem. The referee provided many useful comments that prompted me to make major corrections to earlier versions of this paper. Tomomi Ozawa pointed out many typos. Finally, I am grateful for the financial support provided by UCLA and National Science Foundation that allowed me to complete this work.

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Communicated by Barry Mazur
Received 2015-01-07 Revised 2015-10-06 Accepted 2015-11-27

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

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Algebra & Number Theory

Volume 10 No. 1 2016