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Constructible isocrystals

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We introduce a new category of coefficients for p-adic cohomology called constructible isocrystals. Conjecturally, the category of constructible isocrystals endowed with a Frobenius structure is equivalent to the category of perverse holonomic arithmetic $\mathcal{D}$-modules. We prove here that a constructible isocrystal is completely determined by any of its geometric realizations.

Introduction

The relation between topological invariants and differential invariants of a manifold is always fascinating. We may first recall de Rham's theorem, which implies the existence of an isomorphism

$$\mathrm{H}^i_{\mathrm{dR}}(V) \simeq \mathrm{Hom}(\mathrm{H}_i(V), \mathbb{C})$$

on any complex analytic manifold V. The nonabelian version is an equivalence of categories

$$\mathrm{MIC}(V) \simeq \mathrm{Rep}_{\mathbb{C}}(\pi_1(V, x))$$

between coherent modules endowed with an integrable connection and finite-dimensional representations of the fundamental group. The analogous result holds on a smooth complex algebraic variety X if we stick to regular connections (see

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[Deligne 1970] or Bernard Malgrange's lecture in [Borel et al. 1987]). It has been generalized by Masaki Kashiwara [1984] to an equivalence

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{reg,hol}}(X) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{cons}}(X^{\mathrm{an}})$$

between the categories of bounded complexes of $\mathcal{D}_X$-modules with regular holonomic cohomology and bounded complexes of $\mathbb{C}_{X^{\mathrm{an}}}$-modules with constructible cohomology.

Both categories come with a so-called t-structure but these t-structures do not correspond under this equivalence. Actually, they define a new t-structure on the other side that may be called *perverse*. The notion of a perverse sheaf on X^{an} has been studied for some time now (see [Borel et al. 1987], for example). On the $\mathcal{D}$-module side, however, this notion only appeared in the recent article [Kashiwara 2004], even if he does not give it a name (we call it perverse but it might as well be called constructible; see [Abe 2013]). In any case, he shows that the perverse t-structure on $\mathrm{D}^{\mathrm{b}}_{\mathrm{reg,hol}}(X)$ is given by

$$\begin{cases} \mathrm{D}^{\leq 0} : \operatorname{codim} \operatorname{supp} \mathcal{H}^n(\mathcal{F}^\bullet) \geq n & \text{for } n \geq 0, \\ \mathrm{D}^{\geq 0} : \mathcal{H}^n_Z(\mathcal{F}^\bullet) = 0 & \text{for } n < \operatorname{codim} Z. \end{cases}$$

In particular, if we call *perverse* a complex of $\mathcal{D}_X$-modules satisfying both conditions, there exists an equivalence of categories

$$\mathrm{D}^{\mathrm{perv}}_{\mathrm{reg,hol}}(X) \simeq \mathrm{Cons}(X^{\mathrm{an}})$$

between the categories of perverse (complexes of) $\mathcal{D}_X$-modules with regular holonomic cohomology and constructible $\mathbb{C}_{X^{\mathrm{an}}}$-modules.

In a handwritten note called "Cristaux discontinus", Pierre Deligne gave an algebraic interpretation of the right-hand side of this equivalence. More precisely, he introduces the notion of a constructible procoherent crystal and proves an equivalence

$$\mathrm{Cons}_{\mathrm{reg,procoh}}(X/\mathbb{C}) \simeq \mathrm{Cons}(X^{\mathrm{an}})$$

between the categories of regular constructible procoherent crystals and constructible $\mathbb{C}_{X^{\mathrm{an}}}$-modules.

By composition, we obtain what may be called the *Deligne–Kashiwara correspondence*:

$$\mathrm{Cons}_{\mathrm{reg,procoh}}(X/\mathbb{C}) \simeq \mathrm{D}^{\mathrm{perv}}_{\mathrm{reg,hol}}(X).$$

It would be quite interesting to give an algebraic construction of this equivalence but this is not our purpose here. Actually, we would like to describe an arithmetic analog.

Let K be a p-adic field with discrete valuation ring $\mathcal{V}$ and perfect residue field k. Let $X \hookrightarrow P$ be a locally closed embedding of an algebraic k-variety into a formal $\mathcal{V}$-scheme. Assume for the moment that P is smooth and quasicompact, and that

the locus of X at infinity inside P has the form $D \cap \overline{X}$, where D is a divisor in P. We may consider the category $\mathrm{D}^{\mathrm{b}}(X \subset P/K)$ of bounded complexes of $\mathcal{D}^{\dagger}_{P}({}^{\dagger}D)_{\mathbb{Q}}$-modules on P with support on $\overline{X}$ (see [Berthelot 2002], for example). On the other hand, we may also consider the category of overconvergent isocrystals on $(X \subset P/K)$. Daniel Caro proved [2009] that there exists a fully faithful functor

$$\mathrm{sp}_{+} : \mathrm{Isoc}^{\dagger}_{\mathrm{coh}}(X \subset P/K) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X \subset P/K)$$

(the index coh simply means overconvergent isocrystals in Berthelot's sense — see below). This is the first step towards an overconvergent Deligne–Kashiwara correspondence. Note that this construction is extended to a slightly more general situation by Tomoyuki Abe and Caro [2013] and was already known to Pierre Berthelot [1996b, Proposition 4.4.3] in the case $\overline{X} = P_k$.

In [Le Stum 2014], we defined a category, which we may denote $\mathrm{MIC}^{\dagger}_{\mathrm{cons}}(P/K)$, of convergent constructible ∇-modules on P_K when P is a geometrically connected smooth proper curve over $\mathcal{V}$, as well as a category $\mathrm{D}^{\mathrm{perv}}(P/K)$ of perverse (complexes of) $\mathcal{D}^{\dagger}_{P\mathbb{Q}}$-modules on P, and we built a functor

$$\mathrm{R}\widetilde{\mathrm{sp}}_{*} : \mathrm{MIC}^{\dagger}_{\mathrm{cons}}(P/K) \to \mathrm{D}^{\mathrm{perv}}_{\mathrm{coh}}(P/K).$$

Actually, we proved the overconvergent Deligne–Kashiwara correspondence in this situation: this functor induces an equivalence of categories

$$\mathrm{R}\widetilde{\mathrm{sp}}_{*} : F\text{-}\mathrm{MIC}^{\dagger}_{\mathrm{cons}}(P/K) \simeq F\text{-}\mathrm{D}^{\mathrm{perv}}_{\mathrm{hol}}(P/K)$$

between (convergent) constructible F-∇-modules on P_K and perverse holonomic F-$\mathcal{D}^{\dagger}_{P\mathbb{Q}}$-modules on P. Note that this is compatible with Caro's sp_{+} functor.

In order to extend this theorem to a higher dimension, it is necessary to develop a general theory of *constructible* (*overconvergent*) *isocrystals*. One could try to mimic Berthelot's original definition and let $\mathrm{Isoc}^{\dagger}_{\mathrm{cons}}(X \subset Y \subset P/K)$ be the category of $j^{\dagger}_{X}\mathcal{O}_{]Y[}$-modules $\mathcal{F}$ endowed with an overconvergent connection which are only "constructible" and not necessarily coherent (here X is open in Y and Y is closed in P). It means that there exists a locally finite covering of X by locally closed subvarieties Z such that $j^{\dagger}_{Z}\mathcal{F}$ is a coherent $j^{\dagger}_{Z}\mathcal{O}_{]Y[}$-module. It would then be necessary to show that the definition is essentially independent of P as long as P is smooth and Y proper, and that they glue when there does not exist any global geometric realization.

We choose here an equivalent but different approach with built-in functoriality. I introduced in [Le Stum 2011] the overconvergent site of the algebraic variety X and showed that we can identify the category of locally finitely presented modules on this site with the category of overconvergent isocrystals in the sense of Berthelot. Actually, we can define a broader category of overconvergent isocrystals (without any finiteness condition) and call an overconvergent isocrystal E *constructible* when

there exists a locally finite covering of X by locally closed subvarieties Y such that $E_{|Y}$ is locally finitely presented. Note that K may be any nontrivial complete ultrametric field and that there exists a relative theory (over some base O). We denote by $\mathrm{Isoc}^{\dagger}_{\mathrm{cons}}(X/O)$ the category of constructible overconvergent isocrystals on X/O. We expect a "Grothendieck's six operations formalism" for overconvergent F-isocrystals and, more generally, that all usual properties of constructible coefficients will hold in our context.

As usual, when we are given a crystalline solution to a coefficient problem, it is necessary to be able to give an interpretation in terms of modules with an integrable connection. Here, one may define a category $\mathrm{MIC}^{\dagger}_{\mathrm{cons}}(X, V/O)$ of constructible modules endowed with an overconvergent connection on any "geometric realization" V of X/O, as in Berthelot's approach. We will prove (Theorem 4.12 below) that, when $\mathrm{Char}(K) = 0$, there exists an equivalence of categories

$$\mathrm{Isoc}^{\dagger}_{\mathrm{cons}}(X/O) \simeq \mathrm{MIC}^{\dagger}_{\mathrm{cons}}(X, V/O).$$

As a corollary, we obtain that the later category is essentially independent of the choice of the geometric realization (and that they glue when there does not exist such a geometric realization). Note that this applies in particular to the case of the curve P above which "is" a geometric realization of P_k so that

$$\mathrm{Isoc}^{\dagger}_{\mathrm{cons}}(P_k/K) = \mathrm{MIC}^{\dagger}_{\mathrm{cons}}(P/K).$$

In Section 1, we briefly present the overconvergent site and review some material that will be needed afterwards. In Section 2, we study some functors between overconvergent sites that are associated to locally closed embeddings. We do a little more that what is necessary for the study of constructible isocrystals, hoping that this will be useful in the future. In Section 3, we introduce overconvergent isocrystals and explain how one can construct and deconstruct them. In Section 4, we show that constructible isocrystals may be interpreted in terms of modules with integrable connections.

Notation and conventions

Throughout this article, K denotes a nontrivial complete ultrametric field with valuation ring $\mathcal{V}$ and residue field k.

An *algebraic variety* over k is a scheme over k that admits a locally finite covering by schemes of finite type over k. A *formal scheme* over $\mathcal{V}$ always admits a locally finite covering by π-adic formal schemes of finite presentation over $\mathcal{V}$. An *analytic variety* over K is a strictly analytic K-space in the sense of [Berkovich 1993], for example. We will use the letters $X, Y, Z, U, C, D, \ldots$ to denote algebraic varieties over k, P, Q, S for formal schemes over $\mathcal{V}$ and V, W, O for analytic varieties over K.

An analytic variety over K is said to be *good* if it is locally affinoid. This is the case, for example, if V is affinoid, proper or algebraic, or more generally if V is an open subset of such a variety. Note that in Berkovich's original definition [1990] all analytic varieties were good.

As usual, we will write $\mathbb{A}^1$ and $\mathbb{P}^1$ for the affine and projective lines. We will also use $\mathbb{D}(0, 1^{\pm})$ for the open or closed disc of radius 1.

1. The overconvergent site

We briefly recall the definition of the overconvergent site from [Le Stum 2011]. An object is made of

(1) a locally closed embedding $X \hookrightarrow P$ of an algebraic variety (over k) into a formal scheme (over $\mathcal{V}$) and

(2) a morphism $\lambda : V \to P_K$ of analytic varieties (over K).

We denote this object by $X \subset P \xleftarrow{\mathrm{sp}} P_K \leftarrow V$ and call it an *overconvergent variety*. Here, sp denotes the *specialization* map and we also introduce the notion of a *tube* of X in V:

$$]X[_V := \lambda^{-1}(\mathrm{sp}^{-1}(X)).$$

We call the overconvergent variety *good* if any point of $]X[_V$ has an affinoid neighborhood in V. It makes it simpler to assume from the beginning that all overconvergent varieties are good since the important theorems can only hold for those (and bad overconvergent varieties play no role in the theory). But, on the other hand, most constructions can be carried out without this assumption.

We define a *formal morphism* between overconvergent varieties as a triple of compatible morphisms:

$$\begin{array}{ccccccc} X' & \hookrightarrow & P' & \leftarrow & P'_K & \leftarrow & V' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v_K} & & \downarrow{\scriptstyle u} \\ X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V \end{array}$$

Such a formal morphism induces a continuous map

$$]f[_u :]X'[_{V'} \to]X[_V$$

between the tubes.

Actually, the notion of a formal morphism is too rigid to reflect the true nature of the algebraic variety X and it is necessary to make invertible what we call a *strict neighborhood*, which we define now: it is a formal morphism as above such that f is an isomorphism $X' \simeq X$ and u is an open immersion that induces an isomorphism between the tubes $]X'[_{V'} \simeq]X[_V$. Formal morphisms admit a calculus

of right fractions with respect to strict neighborhoods, and the quotient category is the *overconvergent site* $\mathrm{An}^{\dagger}_{/\mathcal{V}}$. Roughly speaking, we allow the replacement of V by any neighborhood of $]X[_V$ in V and we make the role of P secondary (only existence is required).

Since we call our category a site, we must endow it with a topology which is actually defined by the pretopology of families of formal morphisms

$$\begin{array}{ccccccc} X & \hookrightarrow & P_i & \leftarrow & P_{iK} & \leftarrow & V_i \\ \| & & \downarrow{\scriptstyle v_i} & & \downarrow{\scriptstyle v_{iK}} & & \downarrow \\ X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V \end{array}$$

in which V_i is open in V and $]X[_V \subset \bigcup V_i$ (this is a *standard* site).

Since the formal scheme plays a very loose role in the theory, we usually denote by (X, V) an overconvergent variety and write (f, u) for a morphism.

We use the general formalism of *restricted* category (also called *localized* or *comma* or *slice* category) to define relative overconvergent sites. First of all, we define an *overconvergent presheaf* as a presheaf (of sets) T on $\mathrm{An}^{\dagger}_{/\mathcal{V}}$. If we are given an overconvergent presheaf T, we may consider the restricted site $\mathrm{An}^{\dagger}_{/T}$. An object is a section s of T on some overconvergent variety (X, V) but we like to see s as a morphism from (the presheaf represented by) (X, V) to T. We will then say that (X, V) is a (*overconvergent*) *variety over* T. A morphism between varieties over T is just a morphism of overconvergent varieties which is compatible with the given sections. The above pretopology is still a pretopology on $\mathrm{An}^{\dagger}_{/T}$ and we denote by $T_{\mathrm{An}^{\dagger}}$ the corresponding topos. As explained by David Zureick-Brown [2010; 2014], one may as well replace $\mathrm{An}^{\dagger}_{/T}$ by any fibered category over $\mathrm{An}^{\dagger}_{/\mathcal{V}}$. This is necessary if one wishes to work with algebraic stacks instead of algebraic varieties.

As a first example, we can apply our construction to the case of a representable sheaf $T := (X, V)$. Another very important case is the following: we are given an overconvergent variety (C, O) and an algebraic variety X over C. Then, we define the overconvergent sheaf X/O as follows: a section of X/O is a variety (X', V') over (C, O) with a given factorization $X' \to X \to C$ (this definition extends immediately to algebraic spaces — or even algebraic stacks if one is ready to work with fibered categories). Alternatively, if we are actually given a variety (X, V) over (C, O), we may also consider the overconvergent presheaf X_V/O: a section is a variety (X', V') over (C, O) with a given factorization $X' \to X \to C$ which extends to *some* factorization $(X', V') \to (X, V) \to (C, O)$. Note that we only require the *existence* of the second factorization. In other words, X_V/O is the image presheaf of the natural map $(X, V) \to X/O$. An important theorem (more precisely Corollary 2.5.12 in [Le Stum 2011]) states that, if we work only with *good* overconvergent varieties, then there exists an isomorphism of topos

$(X_V/O)_{\mathrm{An}^\dagger} \simeq (X/O)_{\mathrm{An}^\dagger}$ when we start from a *geometric* situation

$$\begin{array}{ccccccc} X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v_K} & & \downarrow{\scriptstyle u} \\ C & \hookrightarrow & S & \leftarrow & S_K & \leftarrow & O \end{array} \tag{1}$$

with P proper and smooth around X over S and V a neighborhood of the tube of X in $P_K \times_{S_K} O$ (and (C, O) is good).

If we are given a morphism of overconvergent presheaves $v : T' \to T$, we will also say that T' is a (*overconvergent*) *presheaf over* T. It will induce a morphism of topos $v_{\mathrm{An}^\dagger} : T'_{\mathrm{An}^\dagger} \to T_{\mathrm{An}^\dagger}$. We will often drop the index $\mathrm{An}^\dagger$ and keep writing v instead of $v_{\mathrm{An}^\dagger}$. Also, we will usually write the inverse image of a sheaf $\mathcal{F}$ as $\mathcal{F}_{|T'}$ when there is no ambiguity about v. Note that there will exist a triple of adjoint functors $v_!$, v^{-1}, v_* with $v_!$ exact.

For example, any morphism $(f, u) : (Y, W) \to (X, V)$ of overconvergent varieties will give rise to a morphism of topos

$$(f, u)_{\mathrm{An}^\dagger} : (Y, W)_{\mathrm{An}^\dagger} \to (X, V)_{\mathrm{An}^\dagger}.$$

It will also induce a morphism of overconvergent presheaves $f_u : Y_W/O \to X_V/O$ giving rise to a morphism of topos

$$f_{u\mathrm{An}^\dagger} : (Y_W/O)_{\mathrm{An}^\dagger} \to (X_V/O)_{\mathrm{An}^\dagger}.$$

Finally, if (C, O) is an overconvergent variety, then any morphism $f : Y \to X$ of algebraic varieties over C induces a morphism of overconvergent presheaves $f : Y/O \to X/O$ giving rise to a morphism of topos

$$f_{\mathrm{An}^\dagger} : (Y/O)_{\mathrm{An}^\dagger} \to (X/O)_{\mathrm{An}^\dagger}.$$

If we are given an overconvergent variety (X, V), there exists a *realization* map (morphism of topos)

$$(X, V)_{\mathrm{An}^\dagger} \xrightarrow{\varphi}]X[_{V\mathrm{an}}, \quad (X, V') \leftarrow\!| \;]X[_{V'},$$

where $]X[_{V\mathrm{an}}$ denotes the category of sheaves (of sets) on the analytic variety $]X[_V$ (which has a section ψ). Now, if T is any overconvergent presheaf and (X, V) is a variety over T, then there exists a canonical morphism $(X, V) \to T$. Therefore, if $\mathcal{F}$ is a sheaf on T, we may consider its restriction $\mathcal{F}_{|(X,V)}$, which is a sheaf on (X, V). We define the *realization* of $\mathcal{F}$ on (X, V) as

$$\mathcal{F}_{X,V} := \varphi_{V*}(\mathcal{F}_{|(X,V)})$$

(we shall simply write $\mathcal{F}_V$ in practice unless we want to emphasize the role of X). As one might expect, the sheaf $\mathcal{F}$ is completely determined by its realizations $\mathcal{F}_V$

and the transition morphisms $]f[_u^{-1}\mathcal{F}_V \to \mathcal{F}_{V'}$ obtained by functoriality whenever $(f,u):(X',V')\to(X,V)$ is a morphism over T.

We will need below the following result:

Proposition 1.1. *If we are given a* ***cartesian*** *diagram of overconvergent presheaves* (*with a representable upper map*)

$$\begin{array}{ccc} (X',V') & \xrightarrow{(f,u)} & (X,V) \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\ T' & \xrightarrow{v} & T \end{array}$$

and $\mathcal{F}'$ is a sheaf on T', then

$$(v_*\mathcal{F}')_V =]f[_{u*}\mathcal{F}'_{V'}.$$

Proof. Since the diagram is cartesian, we have (this is formal)

$$s^{-1}v_*\mathcal{F}' = (f,u)_*(s')^{-1}\mathcal{F}'.$$

It follows that

$$\begin{aligned}(v_*\mathcal{F}')_V &= \varphi_{V*}s^{-1}v_*\mathcal{F}' = \varphi_{V*}(f,u)_*(s')^{-1}\mathcal{F}' \\ &=]f[_{u*}\varphi_{V'*}(s')^{-1}\mathcal{F}' =]f[_{u*}\mathcal{F}'_{V'}.\end{aligned}$$ □

If (X,V) is an overconvergent variety, we will denote by $i_X :]X[_V \hookrightarrow V$ the inclusion map. Then, if T is an overconvergent presheaf, we define the structural sheaf of $\mathrm{An}^\dagger_{/T}$ as the sheaf $\mathcal{O}_T^\dagger$ whose realization on any (X,V) is $i_X^{-1}\mathcal{O}_V$. An $\mathcal{O}_T^\dagger$-module E will also be called a (*overconvergent*) *module* on T. As it was the case for sheaves of sets, the module E is completely determined by its realizations E_V and the transition morphisms

$$]f[_u^\dagger E_V := i_{X'}^{-1}u^*i_{X*}E_V \to E_{V'} \tag{2}$$

obtained by functoriality whenever $(f,u):(X',V')\to(X,V)$ is a morphism over T. A module on T is called an (*overconvergent*) *isocrystal* if all the transition maps (2) are actually isomorphisms (used to be called a crystal in [Le Stum 2011]). We will denote by

$$\mathrm{Isoc}^\dagger(T) \subset \mathcal{O}_T^\dagger\text{-Mod}$$

the full subcategory made of all isocrystals on T (used to be denoted by $\mathrm{Cris}^\dagger(T)$ in [Le Stum 2011]). Be careful that inclusion is only right exact in general.

If we are given a morphism of overconvergent presheaves $v : T' \to T$ then the functors $v_!$, v^{-1}, v_* preserve modules (we use the same notation $v_!$ for sheaves of sets and abelian groups; this should not create any confusion) and v^{-1} preserves isocrystals.

One can show that a module on T is locally finitely presented if and only if it is an isocrystal with coherent realizations. We will denote their category by $\mathrm{Isoc}^{\dagger}_{\mathrm{coh}}(T)$ (be careful that it only means that the realizations are coherent: $\mathcal{O}^{\dagger}_{T}$ is not a coherent ring in general). In the case $T = X/S_K$ and $\mathrm{Char}(K) = 0$, this is equivalent to Berthelot's original definition [1996a, Definition 2.3.6] of an overconvergent isocrystal.

Back to our examples, it is not difficult to see that, when (X, V) is an overconvergent variety, the realization functor induces an equivalence of categories

$$\mathrm{Isoc}^{\dagger}(X, V) \simeq i_X^{-1}\mathcal{O}_V\text{-Mod}$$

between isocrystals on (X, V) and $i_X^{-1}\mathcal{O}_V$-modules. Now, if (X, V) is a variety over an overconvergent variety (C, O) and

$$p_1, p_2 : (X, V \times_O V) \to (X, V)$$

denote the projections, we define an *overconvergent stratification* on an $i_X^{-1}\mathcal{O}_V$-module $\mathcal{F}$ as an isomorphism

$$\epsilon :]p_2[^{\dagger}\mathcal{F} \simeq]p_1[^{\dagger}\mathcal{F}$$

that satisfies the cocycle condition on triple products and the normalization condition along the diagonal. They form an additive category $\mathrm{Strat}^{\dagger}(X, V/O)$ with cokernels and tensor products. It is even an abelian category when V is universally flat over O in a neighborhood of $]X[_V$. In any case, the realization functor will induce an equivalence

$$\mathrm{Isoc}^{\dagger}(X_V/O) \simeq \mathrm{Strat}^{\dagger}(X, V/O).$$

We may also consider, for $n \in \mathbb{N}$, the n-th infinitesimal neighborhood $V^{(n)}$ of V in $V \times_O V$. Then, a (*usual*) *stratification* on an $i_X^{-1}\mathcal{O}_V$-module $\mathcal{F}$ is a compatible family of isomorphisms

$$\epsilon^{(n)} : i_X^{-1}\mathcal{O}_{V^{(n)}} \otimes_{i_X^{-1}\mathcal{O}_V} \mathcal{F} \simeq \mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\mathcal{O}_{V^{(n)}}$$

that satisfy the cocycle condition on triple products and the normalization condition along the diagonal. Again, they form an additive category $\mathrm{Strat}(X, V/O)$ with cokernels and tensor products, and even an abelian category when V is smooth over O in a neighborhood of $]X[_V$. There exists an obvious faithful functor

$$\mathrm{Strat}^{\dagger}(X, V/O) \to \mathrm{Strat}(X, V/O). \tag{3}$$

Note that, *a priori*, different overconvergent stratifications might give rise to the same usual stratification (and of course many usual stratifications will *not* extend at all to an overconvergent one). Finally, a *connection* on an $i_X^{-1}\mathcal{O}_V$-module $\mathcal{F}$ is an

$\mathcal{O}_O$-linear map

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^1_V$$

that satisfies the Leibniz rule. Integrability is defined as usual. They form an additive category $\mathrm{MIC}(X, V/O)$ and there exists again a faithful functor

$$\mathrm{Strat}(X, V/O) \to \mathrm{MIC}(X, V/O) \tag{4}$$

(∇ is induced by $\epsilon^{(1)} - \sigma$, where σ switches the factors in $V \times_O V$). When V is smooth over O in a neighborhood of $]X[_V$ and $\mathrm{Char}(K) = 0$, then the functor (4) is an equivalence. Actually, both categories are then equivalent to the category of $i_X^{-1}\mathcal{D}_{V/O}$-modules. In general, we will denote by $\mathrm{MIC}^\dagger(X, V/O)$ the image of the composition of the functors (3) and (4) and then call the connection *overconvergent* (and add an index coh when we consider only coherent modules). Thus, there exists a realization functor

$$\mathrm{Isoc}^\dagger(X_V/O) \to \mathrm{MIC}^\dagger(X, V/O) \tag{5}$$

which is faithful and essentially surjective (but not an equivalence in general). In practice, we are interested in isocrystals on X/O, where (C, O) is an overconvergent variety and X is an algebraic variety over C. We can localize in order to find a geometric realization V for X over O such as (1) and work directly on (X, V): there exists an equivalence of categories

$$\mathrm{Isoc}^\dagger(X/O) \simeq \mathrm{Isoc}^\dagger(X_V/O)$$

that may be composed with (5) in order to get the realization functor

$$\mathrm{Isoc}^\dagger(X/O) \to \mathrm{MIC}^\dagger(X, V/O).$$

In [Le Stum 2011], we proved that, when $\mathrm{Char}(K) = 0$, it induces an equivalence

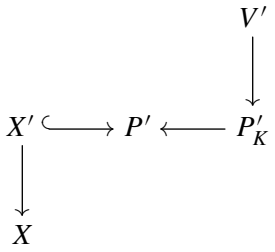
$$\mathrm{Isoc}^\dagger_{\mathrm{coh}}(X/O) \simeq \mathrm{MIC}^\dagger_{\mathrm{coh}}(X, V/O)$$

(showing in particular that the right-hand side is independent of the choice of the geometric realization and that they glue). We will extend this below to what we call *constructible isocrystals*.

2. Locally closed embeddings

In this section, we fix an algebraic variety X over k. Recall that a (overconvergent) variety over $X/\mathcal{M}(K)$ (we will simply write X/K in the future) is a pair made of an overconvergent variety (X', V') and a morphism $X' \to X$. In other words,

it is a diagram



in which P' is a formal scheme.

We also fix a presheaf T over X/K. For example, T could be (the presheaf represented by) an overconvergent variety (X', V') over X/K. Also, if (C, O) is an overconvergent variety and X is an algebraic variety over C, then we may consider the sheaf $T := X/O$ (see Section 1). Finally, if we are given a morphism of overconvergent varieties $(X, V) \to (C, O)$, then we could set $T := X_V/O$ (see Section 1 again).

Finally, we also fix an open immersion $\alpha : U \hookrightarrow X$ and denote by $\beta : Z \hookrightarrow X$ the embedding of a closed complement. Actually, in the beginning, we consider more generally a locally closed embedding $\gamma : Y \hookrightarrow X$.

Definition 2.1. The *restriction* of T to Y is the inverse image

$$T_Y := (Y/K) \times_{(X/K)} T$$

of T over Y/K. We will still denote by $\gamma : T_Y \hookrightarrow T$ the corresponding map. When $\mathcal{F}$ is a sheaf on T, the *restriction* of T to Y is $\mathcal{F}_{|Y} := \gamma^{-1}\mathcal{F}$.

For example, if $T = (X', V')$ is a variety over X/K, then $T_Y = (Y', V')$, where Y' is the inverse image of Y in X'. Also, if (C, O) is an overconvergent variety, X is an algebraic variety over C and $T = X/O$, then $T_Y = Y/O$. Finally, if we are given a morphism of overconvergent varieties $(X, V) \to (C, O)$ and $T = X_V/O$, then we will have $T_Y = Y_V/O$.

If (X, V) is an overconvergent variety, we may consider the morphism of overconvergent varieties $(\gamma, \mathrm{Id}_V) : (Y, V) \hookrightarrow (X, V)$. We will then denote by $]\gamma[_V :]Y[_V \hookrightarrow]X[_V$, or simply $]\gamma[$ if there is no ambiguity, the corresponding map on the tubes. Recall that $]\gamma[$ is the inclusion of an analytic domain. This is an open immersion when γ is a closed embedding and vice versa (we use Berkovich topology).

The next result generalizes Proposition 3.1.10 of [Le Stum 2011].

Proposition 2.2. *Let (X', V') be an overconvergent variety over T and $\gamma' : Y' \hookrightarrow X'$ be the inclusion of the inverse image of Y inside X'. If $\mathcal{F}$ is a sheaf on T_Y, then*

$$(\gamma_*\mathcal{F})_{X',V'} =]\gamma'[_*\mathcal{F}_{Y',V'}.$$

Proof. Using [Le Stum 2011, Corollary 2.4.15], this follows from Proposition 1.1. □

Since we will use it in some of our examples, we should also mention that $\mathrm{R}^i\gamma_* E = 0$ for $i > 0$ when E is an isocrystal with coherent realizations. This follows from the fact that, with the notation of the proposition, $]\gamma'[$ is a quasi-Stein map.

We can work out very simple examples right now. We will do our computations on the overconvergent variety

$$\mathbb{P}^1_{k/K} := \mathbb{P}^1_k \hookrightarrow \widehat{\mathbb{P}}^1_{\mathcal{V}} \leftarrow \mathbb{P}^{1,\mathrm{an}}_K.$$

We consider first the open immersion $\alpha : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ and the structural sheaf $\mathcal{O}^\dagger_{\mathbb{A}^1_k/K}$. If we let $i : \mathbb{D}(0, 1^+) \hookrightarrow \mathbb{P}^{1,\mathrm{an}}_K$ denote the inclusion map, we have

$$\mathrm{R}\Gamma\big(\mathbb{P}^1_{k/K}, \alpha_*\mathcal{O}^\dagger_{\mathbb{A}^1_k/K}\big) = \mathrm{R}\Gamma\big(\mathbb{P}^{1,\mathrm{an}}_K, i_*i^{-1}\mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_K}\big) = K[t]^\dagger := \bigcup_{\lambda>1} K\{t/\lambda\}$$

(functions with radius of convergence (strictly) bigger than one at the origin).

On the other hand, if we start from $\beta : \infty \hookrightarrow \mathbb{P}^1_k$ and let $j : \mathbb{D}(\infty, 1^-) \hookrightarrow \mathbb{P}^{1,\mathrm{an}}_K$ denote the inclusion map, we have

$$\mathrm{R}\Gamma\big(\mathbb{P}^1_{k/K}, \beta_*\mathcal{O}^\dagger_{\infty/K}\big) = \mathrm{R}\Gamma\big(\mathbb{P}^{1,\mathrm{an}}_K, j_*j^{-1}\mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_K}\big) = K[1/t]^{\mathrm{an}} := \bigcap_{\lambda>1} K\{\lambda/t\}$$

(functions with radius of convergence at least one at infinity).

The following is immediate from Proposition 2.2:

Corollary 2.3. (1) $\gamma^{-1}_{\mathrm{An}^\dagger} \circ \gamma_{\mathrm{An}^\dagger *} = \mathrm{Id}$, *and*

(2) *if* $\gamma' : Y' \hookrightarrow X$ *is another locally closed embedding with* $Y \cap Y' = \varnothing$, *then*

$$\gamma^{-1}_{\mathrm{An}^\dagger} \circ \gamma'_{\mathrm{An}^\dagger *} = 0.$$

Alternatively, one may say that if $\mathcal{F}$ is a sheaf on T_Y, we have

$$(\gamma_*\mathcal{F})_{|Y} = \mathcal{F} \quad \text{and} \quad (\gamma_*\mathcal{F})_{|Y'} = 0.$$

The first assertion of the corollary means that $\gamma_{\mathrm{An}^\dagger}$ is an embedding of topos (direct image is fully faithful). Actually, from the fact that Y is a subobject of X in the category of varieties, one easily deduces that T_Y is a subobject of T in the overconvergent topos and $\gamma_{\mathrm{An}^\dagger}$ is therefore an *open* immersion of topos. Note also that the second assertion applies in particular to open and closed complements (both ways): in particular, these functors *cannot* be used to glue along open and closed complements. We will need some refinement.

We focus now on the case of an *open* immersion $\alpha : U \hookrightarrow X$ which gives rise to a closed embedding on the tubes.

Proposition 2.4. *The functor* $\alpha_{\mathrm{An}^\dagger *} : T_{U,\mathrm{An}^\dagger} \to T_{\mathrm{An}^\dagger}$ *is exact and preserves isocrystals.*

Proof. This is not trivial but can be proved exactly as in Corollary 3.1.12 and Proposition 3.3.15 of [Le Stum 2011] (which is the case $T = X/O$). □

The following definition is related to rigid cohomology with compact support (recall that $\beta : Z \hookrightarrow X$ denotes the embedding of a closed complement of U):

Definition 2.5. If $\mathcal{F}$ is a sheaf of abelian groups on T, then

$$\underline{\Gamma}_U \mathcal{F} = \ker(\mathcal{F} \to \beta_* \mathcal{F}_{|Z})$$

is the subsheaf of *sections of $\mathcal{F}$ with support in U.*

If we denote by $\mathcal{U}$ the *closed* subtopos of $T_{\mathrm{An}^\dagger}$ which is the complement of the open topos $T_{Z,\mathrm{An}^\dagger}$, then $\underline{\Gamma}_U$ is the same thing as the functor $\mathcal{H}^0_{\mathcal{U}}$ of sections with support in $\mathcal{U}$. With this in mind, the first two assertions of the next proposition below are completely formal. One may also show that the functor $\mathcal{F} \mapsto \mathcal{F}/\beta_! \beta^{-1} \mathcal{F}$ is an exact left adjoint to $\underline{\Gamma}_U$; it follows that $\underline{\Gamma}_U$ preserves injectives.

Actually, we shall use the open/closed formalism only in the classical situation. Recall (see [Iversen 1986, Section II.6], for example, for these kinds of things) that if $i : W \hookrightarrow V$ is a closed embedding of topological spaces, then i_* has a right adjoint $i^!$ (and one usually sets $\underline{\Gamma}_W := i_* i^!$) which commutes with direct images. If (X, V) is an overconvergent variety, we know that $]\alpha[:]U[\hookrightarrow]X[$ is a closed embedding and we may therefore consider the functors $]\alpha[^!$ and $\underline{\Gamma}_{]U[}$.

Proposition 2.6. (1) *The functor $\underline{\Gamma}_U$ is left exact and preserves modules.*

(2) *If $\mathcal{F}$ is a sheaf of abelian groups on T, then there exists a distinguished triangle*

$$\mathrm{R}\underline{\Gamma}_U \mathcal{F} \to \mathcal{F} \to \mathrm{R}\beta_* \mathcal{F}_{|Z} \to .$$

(3) *If (X', V') is a variety over T and $\alpha' : U' \hookrightarrow X'$ denotes the immersion of the inverse image of U into X', we have*

$$(\mathrm{R}\underline{\Gamma}_U E)_{V'} = \mathrm{R}\underline{\Gamma}_{]U'[_{V'}} E_{V'}$$

*for any **isocrystal** E on T.*

Proof. The first assertion follows immediately from the fact that all the functors involved (β^{-1}, β_* and ker) do have these properties. The second assertion results from the fact that the map $\mathcal{F} \to \beta_* \mathcal{F}_{|Z}$ is surjective when $\mathcal{F}$ is an injective sheaf (this is formal). In order to prove the last assertion, it is sufficient to remember (this is a standard fact) that there exists a distinguished triangle

$$\mathrm{R}\underline{\Gamma}_{]U'[_{V'}} E_{V'} \to E_{V'} \to \mathrm{R}]\beta'[_*]\beta[^{-1} E_{V'} \to,$$

where $\beta' : Z' \hookrightarrow X'$ denotes the inverse image of the inclusion of a closed complement of U. Since E is an isocrystal, we have $(E_{|Z})_{Z',V'} =]\beta'[^{-1} E_{X',V'}$. □

Note that the second assertion means that there exists an exact sequence

$$0 \to \underline{\Gamma}_U \mathcal{F} \to \mathcal{F} \to \beta_* \mathcal{F}_{|Z} \to \mathrm{R}^1 \underline{\Gamma}_U \mathcal{F} \to 0$$

and that $\mathrm{R}^i \beta_* \mathcal{F}_{|Z} = \mathrm{R}^{i+1} \underline{\Gamma}_U \mathcal{F}$ for $i > 0$. We can do the exercise with $\alpha : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ and $\beta : \infty \hookrightarrow \mathbb{P}^1_k$ as above. We obtain

$$\mathrm{R}\Gamma\big(\mathbb{P}^1_{k/K}, \mathrm{R}\underline{\Gamma}_{\mathbb{A}^1_k} \mathcal{O}^\dagger_{\mathbb{P}^1_k/K}\big) = [K \to K[1/t]^{\mathrm{an}}] = (K[1/t]^{\mathrm{an}}/K)[-1].$$

Since realization does not commute with the inverse image in general, we need to introduce a new functor. Recall that in order to define a sheaf on T, it is sufficient (and even equivalent) to give a compatible family of sheaves on the tubes $]X'[_{V'}$ for all (X', V') over T.

Lemma 2.7. *If $\mathcal{F}$ is a sheaf on T, then the assignment*

$$(X', V') \mapsto \big(j_U^\dagger \mathcal{F}\big)_{V'} :=]\alpha'[_*]\alpha'[^{-1}\mathcal{F}_{V'},$$

where $\alpha' : U' \hookrightarrow X'$ denotes the immersion of the inverse image of U into X', defines a sheaf on T.

Proof. We give ourselves a morphism $(f, u) : (X'', V'') \to (X', V')$ over T, we denote by $g : U'' \to U'$ the map induced by f on the inverse images of U into X' and X'', respectively, and by $\alpha'' : U'' \hookrightarrow X''$ the inclusion map. We consider the cartesian diagram (forgetful functor to algebraic varieties is left exact)

$$\begin{array}{ccc} (U'', V'') & \hookrightarrow & (X'', V'') \\ \downarrow{\scriptstyle (g,u)} & & \downarrow{\scriptstyle (f,u)} \\ (U', V') & \hookrightarrow & (X', V') \end{array}$$

which gives rise to a cartesian diagram (tube is left exact)

$$\begin{array}{ccc}]U''[_{V''} & \hookrightarrow &]X''[_{V''} \\ \downarrow{\scriptstyle]g[_u} & & \downarrow{\scriptstyle]f[_u} \\ ]U'[_{V'} & \hookrightarrow &]X'[_{V'} \end{array}$$

Since $]\alpha'[$ is a closed embedding, we have $]f[_u^{-1} \circ]\alpha'[_* =]\alpha''[_* \circ]g[_u^{-1}$ and there exists a canonical map

$$]f[_u^{-1}]\alpha'[_*]\alpha'[^{-1}\mathcal{F}_{V'} =]\alpha''[_*]g[_u^{-1}]\alpha'[^{-1}\mathcal{F}_{V'} =]\alpha''[_*]\alpha''[^{-1}]f[_u^{-1}\mathcal{F}_{V''} \to]\alpha''[_*]\alpha''[^{-1}\mathcal{F}_{V''}. \quad \square$$

Definition 2.8. If $\mathcal{F}$ is a sheaf on T, then $j_U^\dagger \mathcal{F}$ is the sheaf of *overconvergent sections of $\mathcal{F}$ around U.*

Proposition 2.9. (1) *The functor $j_U^\dagger$ is exact and preserves isocrystals.*

(2) *If E is an isocrystal on T, we have $j_U^\dagger E = \alpha_* \alpha^{-1} E$.*

Proof. Exactness can be checked on realizations. But, if (X', V') is a variety over T and $\alpha' : U' \hookrightarrow X'$ denotes the immersion of the inverse image of U in X', then we know the exactness of $]\alpha'[_*$ (because $]\alpha'[$ is a closed embedding) and $]\alpha'[^{-1}$. The second part of the first assertion is a consequence of the second assertion which follows from the fact that $(\alpha^{-1}E)_{V'} =]\alpha'[^{-1} E_{V'}$ when E is an isocrystal. □

Note that the canonical map $j_U^\dagger \mathcal{F} \to \alpha_* \alpha^{-1} \mathcal{F}$ is still bijective when $\mathcal{F}$ is a sheaf of *Zariski type* (see Definition 4.6.1[1] of [Le Stum 2011]) but there are important concrete situations where equality fails, as we shall see right now.

In order to exhibit a counterexample, we let again $\alpha : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ and $\beta : \infty \hookrightarrow \mathbb{P}^1_k$ denote the inclusion maps and consider the sheaf $\mathcal{F} := \beta_* \mathcal{O}^\dagger_{\infty/K}$, which is *not* an isocrystal (and not even of Zariski type). Since $\alpha^{-1} \circ \beta_* = 0$, we have $\alpha_* \alpha^{-1} \mathcal{F} = 0$. Now, let us denote by $i_\xi : \xi \hookrightarrow \mathbb{P}^{1,\mathrm{an}}_K$ the inclusion of the generic point of the unit disc (corresponding to the Gauss norm) and let $i : \mathbb{D}(0, 1^+) \hookrightarrow \mathbb{P}^{1,\mathrm{an}}_K$ and $j : \mathbb{D}(\infty, 1^-) \hookrightarrow \mathbb{P}^{1,\mathrm{an}}_K$ be the inclusion maps as above. Let

$$\mathcal{R} := \left\{ \sum_{n \in \mathbb{Z}} a_n t^n : \begin{cases} \exists \lambda > 1, \lambda^n a_n \to 0 \text{ for } n \to +\infty \\ \forall \lambda > 1, \lambda^n a_n \to 0 \text{ for } n \to -\infty \end{cases} \right\}$$

be the *Robba ring* (functions that converge on some open annulus of outer radius one at infinity). Then, one easily sees that

$$\left(j^\dagger_{\mathbb{A}^1_k} \beta_* \mathcal{O}^\dagger_{\infty/K}\right)_{\mathbb{P}^1_{k/K}} = i_* i^{-1} j_* \mathcal{O}_{\mathbb{D}(0,1^-)} = i_{\xi *} \mathcal{R}$$

so that $j^\dagger_{\mathbb{A}^1_k} \mathcal{F} \neq 0$. This computation also shows that

$$\mathrm{R}\Gamma\left(\mathbb{P}^1_{k/K}, j^\dagger_{\mathbb{A}^1_k} \beta_* \mathcal{O}^\dagger_{\infty/K}\right) = \mathcal{R}.$$

We now turn to the study of the *closed* embedding $\beta : Z \hookrightarrow X$, which requires some care (as we just experienced, the direct image of an isocrystal need not be an isocrystal).

The following definition has to do with cohomology with support in a closed subset.

Definition 2.10. For any sheaf of abelian groups $\mathcal{F}$ on T,

$$\underline{\Gamma}^\dagger_Z \mathcal{F} := \ker(\mathcal{F} \to \alpha_* \mathcal{F}_{|U})$$

is the subsheaf of *overconvergent sections of $\mathcal{F}$ with support in Z*.

[1] The comment following Definition 4.6.1 in [Le Stum 2011] is not correct and Lemma 4.6.2 is only valid for an *open* immersion.

We will do some examples below when we have more material at our disposal.

As above, if we denote by $\mathcal{Z}$ the closed subtopos of $T_{\mathrm{An}^\dagger}$ which is the complement of the open topos $T_{U,\mathrm{An}^\dagger}$, then $\underline{\Gamma}^\dagger_Z$ is the same thing as the functor $\mathcal{H}^0_{\mathcal{Z}}$ of sections with support in $\mathcal{Z}$. This is the approach taken by David Zureick-Brown [2010; 2014] in order to define cohomology with support in Z on the overconvergent site. The next proposition is completely formal if one uses Zureick-Brown's approach. Also, as above, one may prove that $\underline{\Gamma}^\dagger_Z$ preserves injectives because the functor $\mathcal{F} \mapsto \mathcal{F}/\alpha_!\alpha^{-1}\mathcal{F}$ is an exact left adjoint.

Proposition 2.11. (1) *The functor $\underline{\Gamma}^\dagger_Z$ is left exact and preserves modules.*

(2) *If $\mathcal{F}$ is an abelian sheaf on T, then there exists a distinguished triangle*

$$0 \to \mathrm{R}\underline{\Gamma}^\dagger_Z\mathcal{F} \to \mathcal{F} \to \alpha_*\mathcal{F}_{|U} \to .$$

We will also show below that $\underline{\Gamma}^\dagger_Z$ preserves isocrystals.

Proof. As in the proof of Proposition 2.6, the first assertion follows from the fact that all the functors involved (and the kernel as well) are left exact and preserve overconvergent modules. Similarly the second one is a formal consequence of the definition because α_* and α^{-1} both preserve injectives (they both have an exact left adjoint) and the map $\mathcal{F} \to \alpha_*\mathcal{F}_{|U}$ is an epimorphism when $\mathcal{F}$ is injective (standard). □

Note that the last assertion of the proposition means that there exists an exact sequence

$$0 \to \underline{\Gamma}^\dagger_Z\mathcal{F} \to \mathcal{F} \to \alpha_*\mathcal{F}_{|U} \to \mathrm{R}^1\underline{\Gamma}^\dagger_Z\mathcal{F} \to 0$$

and that $\mathrm{R}^i\underline{\Gamma}^\dagger_Z\mathcal{F} = 0$ for $i > 1$.

Before going any further, we want to stress the fact that β^{-1} has an adjoint $\beta_!$ on the left in the category of all modules (or abelian groups or even sets with a light modification) but $\beta_!$ does not preserve isocrystals in general. Actually, we always have $(\beta_!\mathcal{F})_{X',V'} = 0$ unless the morphism $X' \to X$ factors through Z (recall that we use the coarse topology on the algebraic side). Again, the workaround consists in working directly with the realizations. If $j : W \hookrightarrow V$ is an open immersion of topological spaces, then j^{-1} has an adjoint $j_!$ on the left also (on sheaves of abelian groups or sheaves of sets with a light modification). This is an exact functor that commutes with inverse images (see [Iversen 1986, Section II.6] again). Now, if (X, V) is an overconvergent variety, then $]\beta[:]Z[\hookrightarrow]X[$ is an *open* immersion and we may consider the functor $]\beta[_!$.

In the next lemma again, we use realizations and transition maps in order to define a sheaf.

Lemma 2.12. *If $\mathcal{F}$ is a sheaf (of sets or abelian groups) on T_Z, then the assignment*

$$(X', V') \mapsto (\beta_\dagger\mathcal{F})_{X',V'} :=]\beta'[_!\mathcal{F}_{Z',V'},$$

where $\beta' : Z' \hookrightarrow X'$ denotes the embedding of the inverse image of Z into X', defines a sheaf on T. Moreover, if E is an isocrystal on T_Z, then $\beta_\dagger E$ is an isocrystal on T.

Proof. As above, we consider a morphism $(f, u) : (X'', V'') \to (X', V')$ over T. We denote by $h : Z'' \to Z'$ the map induced by f on the inverse images of Z into X' and X'', respectively, and by $\beta'' : Z'' \hookrightarrow X''$ the inclusion map. We have a cartesian diagram

$$\begin{array}{ccc} (Z'', V'') & \hookrightarrow & (X'', V'') \\ \downarrow{\scriptstyle (h,u)} & & \downarrow{\scriptstyle (f,u)} \\ (Z', V') & \hookrightarrow & (X', V') \end{array}$$

giving rise to a cartesian diagram

$$\begin{array}{ccc}]Z''[_{V''} & \hookrightarrow &]X''[_{V''} \\ \downarrow{\scriptstyle]h[_u} & & \downarrow{\scriptstyle]f[_u} \\ ]Z'[_{V'} & \hookrightarrow &]X'[_{V'} \end{array}$$

It follows that there exists a canonical map

$$]f[_u^{-1}]\beta'[_!\mathcal{F}_{V'} =]\beta''[_!]h[_u^{-1}\mathcal{F}_{V'} \to]\beta''[_!\mathcal{F}_{V''}$$

as asserted. We consider now an isocrystal E and we want to show that

$$]f[_u^{\dagger}]\beta'[_!E_{V'} \simeq]\beta''[_!E_{V''}.$$

This immediately follows from the equality (which is formal)

$$i_{X''}^{-1}\mathcal{O}_{V''} \otimes_{i_{X''}^{-1}u^{-1}\mathcal{O}_{V'}}]\beta''[_!]h[_u^{-1}E_{V'} =]\beta''[_!\big(i_{Z''}^{-1}\mathcal{O}_{V''} \otimes_{i_{Z''}^{-1}u^{-1}\mathcal{O}_{V'}}]h[_u^{-1}E_{V'}\big). \quad \square$$

Definition 2.13. The sheaf $\beta_\dagger\mathcal{F}$ is the *overconvergent direct image* of $\mathcal{F}$.

Note that there exist two flavors of $\beta_\dagger$: for sheaves of sets and for sheaves of abelian groups. Whichever we consider should be clear from the context.

Proposition 2.14. (1) *If $\mathcal{F}$ is a sheaf on T_Z, then*:

(a) $(\beta_\dagger\mathcal{F})_{|Z} = \mathcal{F}$.

(b) $(\beta_\dagger\mathcal{F})_{|U} = 0$.

(c) *If E is an isocrystal on T, then*

$$\mathcal{H}om(\beta_\dagger\mathcal{F}, E) = \beta_*\,\mathcal{H}om(\mathcal{F}, \beta^{-1}E). \tag{6}$$

(d) *There exists a short exact sequence*

$$0 \to \beta_\dagger\mathcal{F} \to \beta_*\mathcal{F} \to j_U^\dagger\beta_*\mathcal{F} \to 0. \tag{7}$$

(2) *The functor $\beta_\dagger$ is fully faithful, exact, and preserves isocrystals, and the induced functor*

$$\beta_\dagger : \mathrm{Isoc}^\dagger(T_Z) \to \mathrm{Isoc}^\dagger(T)$$

is left adjoint to

$$\beta^{-1} : \mathrm{Isoc}^\dagger(T) \to \mathrm{Isoc}^\dagger(T_Z).$$

Proof. As usual, if (X', V') is a variety over T, then we denote by $\alpha' : U' \hookrightarrow X'$ and $\beta' : Z' \hookrightarrow X'$ the inclusions of the inverse images of U and Z, respectively.

When (X', V') is an overconvergent variety over T_Z, then we will have $]\beta'[= \mathrm{Id}$, and when (X', V') is an overconvergent variety over T_U then $]\beta'[= \varnothing$. We obtain the first two assertions. When E is an isocrystal on T, we have an isomorphism (this is standard)

$$\mathcal{H}om(]\beta'[_!\mathcal{F}_{V'}, E_{V'}) =]\beta'[_*\mathcal{H}om(\mathcal{F}_{V'},]\beta'[^{-1}E_{V'}),$$

from which the third assertion follows. Also, there exists a short exact sequence

$$0 \to]\beta'[_!\mathcal{F}_{Z',V'} \to]\beta'[_*\mathcal{F}_{Z',V'} \to]\alpha'[_*]\alpha'[^{-1}]\beta'[_*\mathcal{F}_{Z',V'} \to 0$$

which provides the fourth assertion.

Full faithfulness and exactness of $\beta_\dagger$ follow from the full faithfulness and exactness of $]\beta'[_!$ for all (X', V'). The fact that $\beta_\dagger$ preserves isocrystals was proved in Lemma 2.12. The last assertion may be obtained by taking global sections of the equality (6). □

We can also mention that there exists a distinguished triangle

$$\beta_\dagger\mathcal{F} \to \mathrm{R}\beta_*\mathcal{F} \to j_U^\dagger\mathrm{R}\beta_*\mathcal{F} \to .$$

Now, we prove that the exact sequence (7) is universal:

Proposition 2.15. *If $\mathcal{F}'$ and $\mathcal{F}''$ are modules on T_Z and T_U, respectively, then any extension*

$$0 \to \beta_\dagger\mathcal{F}' \to \mathcal{F} \to \alpha_*\mathcal{F}'' \to 0$$

is a pull-back of the fundamental extension (7) *through a unique morphism*

$$\alpha_*\mathcal{F}'' \to j_U^\dagger\beta_*\mathcal{F}'.$$

Proof. We know that $\beta^{-1}\alpha_*\mathcal{F}'' = 0$ and it follows that

$$\mathrm{Hom}(\alpha_*\mathcal{F}'', \beta_*\mathcal{F}') = \mathrm{Hom}(\beta^{-1}\alpha_*\mathcal{F}'', \mathcal{F}') = 0.$$

This being true for any sheaves, we see that, actually, $\mathrm{R}\,\mathrm{Hom}(\alpha_*\mathcal{F}'', \mathrm{R}\beta_*\mathcal{F}') = 0$. It formally follows that $\mathrm{R}^i\,\mathrm{Hom}(\alpha_*\mathcal{F}'', \beta_*\mathcal{F}') = 0$ for $i \leq 1$. As a consequence, we

obtain a canonical isomorphism

$$\mathrm{Hom}(\alpha_*\mathcal{F}'', j_U^\dagger\beta_*\mathcal{F}') \simeq \mathrm{Ext}(\alpha_*\mathcal{F}'', \beta_\dagger\mathcal{F}').$$

This is exactly the content of our assertion. □

We should observe that we always have $\mathrm{Hom}(\alpha_*\mathcal{F}'', \beta_\dagger\mathcal{F}') = 0$. However, it is *not* true that $\mathrm{Ext}(\alpha_*\mathcal{F}'', \beta_\dagger\mathcal{F}') = 0$ in general. This can happen because $\beta_\dagger$ does not preserve injectives (although it is exact).

The overconvergent direct image is related to overconvergent support as follows:

Proposition 2.16. *If E is an isocrystal on T, then*

$$\underline{\Gamma}_Z^\dagger E = \beta_\dagger E_{|Z}$$

and, for all $i > 0$, $\mathrm{R}^i\underline{\Gamma}_Z^\dagger E = 0$.

Proof. Recall from Proposition 2.11 that there exists an exact sequence

$$0 \to \underline{\Gamma}_Z^\dagger E \to E \to \alpha_* E_{|U} \to \mathrm{R}^1\underline{\Gamma}_Z^\dagger E \to 0$$

and that $\mathrm{R}^i\underline{\Gamma}_Z^\dagger E = 0$ for $i > 1$. Now, let (X', V') be a variety over T. Denote by $\beta' : Z' \hookrightarrow X'$, $\alpha' : U' \hookrightarrow X'$ the embeddings of the inverse images of Z and U into X'. There exists a short exact sequence (standard again)

$$0 \to]\beta'[_!]\beta'[^{-1}E_{V'} \to E_{V'} \to]\alpha'[_*]\alpha'[^{-1}E_{V'} \to 0.$$

Since E is an isocrystal, we have $(\alpha_* E_{|U})_{V'} =]\alpha'[_*]\alpha'[^{-1}E_{V'}$. It follows that $(\mathrm{R}^1\underline{\Gamma}_Z^\dagger E)_{V'} = 0$ and we also see that

$$(\underline{\Gamma}_Z^\dagger E)_{V'} =]\beta'[_!]\beta'[^{-1}E_{V'} = (\beta_\dagger E_{|Z})_{V'}.$$ □

Note that the proposition is still valid for sheaves of Zariski type and not merely for isocrystals. Be careful however that $\beta_\dagger E \neq \underline{\Gamma}_Z^\dagger\beta_* E$ in general, even when E is an isocrystal on T_Z. With our favorite example in mind, we have

$$\underline{\Gamma}_Z^\dagger\beta_*\mathcal{O}_{\infty/K}^\dagger = \beta_*\mathcal{O}_{\infty/K}^\dagger \neq \beta_\dagger\mathcal{O}_{\infty/K}^\dagger,$$

as our computations below will show.

Corollary 2.17. *The functor $\underline{\Gamma}_Z^\dagger$ preserves isocrystals, and the induced functor*

$$\underline{\Gamma}_Z^\dagger : \mathrm{Isoc}^\dagger(T) \to \mathrm{Isoc}^\dagger(T)$$

is exact. Moreover, if E is an isocrystal on T, then there exists a short exact sequence

$$0 \to \underline{\Gamma}_Z^\dagger E \to E \to j_U^\dagger E \to 0.$$

We might as well write this last short exact sequence as

$$0 \to \beta_\dagger E_{|Z} \to E \to \alpha_* E_{|U} \to 0.$$

As promised above, we can do an example and consider the closed embedding $\beta : \infty \hookrightarrow \mathbb{P}^1_k$ again. We compute

$$\beta_\dagger \mathcal{O}^\dagger_{\infty/K} = \underline{\Gamma}^\dagger_\infty \mathcal{O}^\dagger_{\mathbb{P}^1_k/K}.$$

We have

$$\mathrm{R}\Gamma\big(\mathbb{P}^1_{k/K}, \beta_\dagger \mathcal{O}^\dagger_{\infty/K}\big) = [K \to K[t]^\dagger] = (K[t]^\dagger/K)[-1].$$

We can also remark that the (long) exact sequence obtained by applying $\mathrm{R}\Gamma(\mathbb{P}^1_{k/K}, -)$ to the fundamental short exact sequence

$$0 \to \beta_\dagger \mathcal{O}^\dagger_{\infty/K} \to \beta_* \mathcal{O}^\dagger_{\infty/K} \to j^\dagger_U \beta_* \mathcal{O}^\dagger_{\infty/K} \to 0$$

reads

$$0 \to K[1/t]^{\mathrm{an}} \to \mathcal{R} \to K[t]^\dagger/K \to 0. \tag{8}$$

Corollary 2.18. (1) *The functors α_* and α^{-1} induce an equivalence between isocrystals on T_U and isocrystals on T such that $\underline{\Gamma}^\dagger_Z E = 0$ (or $j^\dagger_U E = E$).*

(2) *The functors $\beta_\dagger$ and β^{-1} induce an equivalence between isocrystals on T_Z and isocrystals on T such that $\underline{\Gamma}^\dagger_Z E = E$ (or $j^\dagger_U E = 0$).*

Proof. If E'' is an isocrystal on T_U, then $\alpha_* E''$ is an isocrystal on T and therefore

$$\underline{\Gamma}^\dagger_Z \alpha_* E'' = \beta_\dagger \beta^{-1} \alpha_* E'' = 0.$$

Conversely, if E is an isocrystal on T such that $\underline{\Gamma}^\dagger_Z E = 0$, then $E = j^\dagger_U E = \alpha_* \alpha^{-1} E$. This shows the first part.

Now, if E' is an isocrystal on T_Z, then $\beta_\dagger E'$ is an isocrystal on T and therefore

$$\underline{\Gamma}^\dagger_Z \beta_\dagger E' = \beta_\dagger \beta^{-1} \beta_\dagger E' = \beta_\dagger E'.$$

Conversely, if E is an isocrystal on T such that $\underline{\Gamma}^\dagger_Z E = E$, then $E = \beta_\dagger \beta^{-1} E$. □

We can also make the functor of sections with support in an open subset come back into the picture:

Corollary 2.19. *If E is an isocrystal on T, then there exists a distinguished triangle*

$$\mathrm{R}\underline{\Gamma}_U E \to j^\dagger_U E \to j^\dagger_U \mathrm{R}\beta_* E_{|Z} \to .$$

Proof. There exists actually a commutative diagram of distinguished triangles:

$$\begin{array}{ccccccc}
 & & \underline{\Gamma}^{\dagger}_{Z}E & = & \underline{\Gamma}^{\dagger}_{Z}E & & \\
 & & \downarrow & & \downarrow & & \\
\mathrm{R}\underline{\Gamma}_{U}E & \longrightarrow & E & \longrightarrow & \mathrm{R}\beta_{*}E_{|Z} & \longrightarrow & \\
\| & & \downarrow & & \downarrow & & \\
\mathrm{R}\underline{\Gamma}_{U}E & \longrightarrow & j_{U}^{\dagger}E & \longrightarrow & j_{U}^{\dagger}\mathrm{R}\beta_{*}E_{|Z} & \longrightarrow & \\
 & & \downarrow & & \downarrow & &
\end{array}$$

More precisely, we know that the vertical triangles as well as the middle horizontal one are all distinguished. The bottom one must be distinguished too. □

Back to our running example, we see that the long exact sequence obtained by applying $\mathrm{R}\Gamma(\mathbb{P}^1_{k/K}, -)$ to the distinguished triangle

$$\mathrm{R}\underline{\Gamma}_{\mathbb{A}^1_k}\mathcal{O}^{\dagger}_{\mathbb{P}^1_k/K} \to j^{\dagger}_{\mathbb{A}^1_k}\mathcal{O}^{\dagger}_{\mathbb{P}^1_k/K} \to j^{\dagger}_{\mathbb{A}^1_k}\mathrm{R}\beta_{*}\mathcal{O}^{\dagger}_{\infty/K} \to$$

reads

$$0 \to K[t]^{\dagger} \to \mathcal{R} \to K[1/t]^{\mathrm{an}}/K \to 0.$$

We can summarize the situation as follows:

(1) There exist two triples of adjoint functors (up means left):

$$\mathcal{O}^{\dagger}_{T_U}\text{-Mod} \underset{\alpha_*}{\overset{\alpha_!}{\underset{\alpha^{-1}}{\rightleftarrows}}} \mathcal{O}^{\dagger}_{T}\text{-Mod} \underset{\beta*}{\overset{\beta!}{\underset{\beta^{-1}}{\rightleftarrows}}} \mathcal{O}^{\dagger}_{T_Z}\text{-Mod}.$$

Moreover, α_* is *exact* and *preserves isocrystals* (and so do α^{-1} and β^{-1}).

(2) There exist two functors with support (that preserve injectives):

$$\underline{\Gamma}_{U} \circlearrowright \mathcal{O}^{\dagger}_{T}\text{-Mod} \circlearrowleft \underline{\Gamma}^{\dagger}_{Z}.$$

Moreover, $\Gamma^{\dagger}_{Z}$ *preserves isocrystals* and is *exact on isocrystals*.

(3) There exist two other functors:

$$j^{\dagger}_{U} \circlearrowright \mathcal{O}^{\dagger}_{T}\text{-Mod} \xleftarrow{\beta_{\dagger}} \mathcal{O}^{\dagger}_{T_Z}\text{-Mod}.$$

They are both *exact* and *preserve isocrystals* (but not injectives). If E is an isocrystal on T, we have

$$j^{\dagger}_{U}E = \alpha_{*}E_{|U} \quad \text{and} \quad \underline{\Gamma}^{\dagger}_{Z}E = \beta_{\dagger}E_{|Z}.$$

3. Constructibility

Recall that K denotes a complete ultrametric field with ring of integers $\mathcal{V}$ and residue field k. We let X be an algebraic variety over k and T a (overconvergent) presheaf over X/K. Roughly speaking, T is some family of varieties X' over X which embed into a formal $\mathcal{V}$-scheme P', together with a morphism of analytic K-varieties $V' \to P'_K$. A (*overconvergent*) *module* $\mathcal{F}$ on T is then a compatible family of $i_{X'}^{-1}\mathcal{O}_{V'}$-modules $\mathcal{F}_{V'}$, where $i_{X'} :]X'[_{V'} \hookrightarrow V'$ denotes the inclusion of the tube (the reader is redirected to Section 1 for the details).

Definition 3.1. A module $\mathcal{F}$ on T is said to be *constructible* (with respect to X) if there exists a locally finite covering of X by locally closed subvarieties Y such that $\mathcal{F}_{|Y}$ is locally finitely presented.

Recall that a locally finitely presented module is the same thing as an isocrystal with coherent realizations. It is important to notice however that a constructible module is *not* necessarily an isocrystal (the transition maps might not be bijective). We'll give an example later.

Proposition 3.2. (1) *Constructible modules on T form an additive category which is stable under cokernel, extension, tensor product and internal Hom.*

(2) *Constructible isocrystals on T form an additive category* $\mathrm{Isoc}^{\dagger}_{\mathrm{cons}}(T)$ *which is stable under cokernel, extension and tensor product.*

Proof. The analog to the first assertion for locally finitely presented modules is completely formal besides the internal Hom question that was proved in Proposition 3.3.12 of [Le Stum 2011]. The analog to the second assertion for all isocrystals was proved in Corollary 3.3.9 of [Le Stum 2011]. Since the restriction maps $\mathcal{F} \mapsto \mathcal{F}_{|Y}$ are exact and commute with tensor product and internal Hom, everything follows. □

Note however that $\mathcal{H}om(E_1, E_2)$ need *not* be an isocrystal (see example below) when E_1 and E_2 are two constructible isocrystals.

Proposition 3.3. *Let $\mathcal{F}$ be a module on T.*

(1) *The module $\mathcal{F}$ is constructible if and only if there exists a locally finite covering by locally closed subvarieties Y of X such that $\mathcal{F}_{|Y}$ is constructible.*

(2) *If $T' \to T$ is any morphism of overconvergent presheaves and $\mathcal{F}$ is constructible, then $\mathcal{F}_{|T'}$ is constructible. The converse also is true if $T' \to T$ is a covering.*

(3) *Assume that T is actually a presheaf on X'/K for some $f : X' \to X$. If $\mathcal{F}$ is constructible with respect to X, then it is also constructible with respect to X'.*

Proof. The first assertion is an immediate consequence of the transitivity of locally finite coverings by locally closed subsets: if $X = \bigcup X_i$ and $X_i = \bigcup X_{ij}$ are such coverings, so is the covering $X = \bigcup X_{ij}$.

In order to prove the second assertion, note first that it is formally satisfied by locally finitely presented modules. Moreover, if Y is a locally closed subvariety of X, we have $(\mathcal{F}_{|T'})_{|Y} = (\mathcal{F}_{|Y})_{|T'_Y}$. The result follows.

Finally, for the third assertion, if $X = \bigcup X_i$ is a locally finite covering by locally closed subvarieties, so is $X' = \bigcup f^{-1}(X_i)$. Moreover, by definition $\mathcal{F}_{|f^{-1}(X_i)} = \mathcal{F}_{|X_i}$ and there is nothing to do. □

Together with Corollary 2.18 above, the next proposition will allow us to move freely along a closed or open embedding when we consider constructible isocrystals (note that this is obviously wrong for overconvergent isocrystals with coherent realizations):

Proposition 3.4. (1) *If $\alpha : U \hookrightarrow X$ is an open immersion of algebraic varieties, then a module $\mathcal{F}''$ on T_U is constructible if and only if $\alpha_*\mathcal{F}''$ is constructible.*

(2) *If $\beta : Z \hookrightarrow X$ is a closed embedding of algebraic varieties, then a module $\mathcal{F}'$ on T_Z is constructible if and only if $\beta_\dagger\mathcal{F}'$ is constructible.*

Proof. We may assume that U and Z are open and closed complements. We saw in Corollary 2.3 that $(\alpha_*\mathcal{F}'')_{|Z} = \mathcal{F}''$ and $(\alpha_*\mathcal{F}')_{|U} = 0$. We also saw in Proposition 2.14 that $(\beta_\dagger\mathcal{F}')_{|Z} = \mathcal{F}'$ and $(\beta_\dagger\mathcal{F}')_{|U} = 0$. □

It is easy to see that the *usual* dual to a constructible isocrystal is *not* an isocrystal in general: if $\beta : Z \hookrightarrow X$ is a closed embedding of algebraic varieties and E is an overconvergent isocrystal on Z with coherent realizations, it follows from Proposition 2.14 that

$$(\beta_\dagger E)^\vee := \mathcal{H}om(\beta_\dagger E, \mathcal{O}_T^\dagger) = \beta_* \mathcal{H}om(E, \mathcal{O}_{T_Z}^\dagger) = \beta_* E^\vee,$$

which is constructible but is not an isocrystal in general (as we saw in Section 2).

The next property is also very important because it allows the use of noetherian induction to reduce some assertions about constructible isocrystals to analogous assertions about overconvergent isocrystals with coherent realizations.

Lemma 3.5. *A module $\mathcal{F}$ on T is constructible if and only if there exists a closed subvariety Z of X such that, if $U := X \setminus Z$, then both $\mathcal{F}_{|Z}$ and $\mathcal{F}_{|U}$ are constructible. We may even assume that U is dense in X and $\mathcal{F}_{|U}$ is locally finitely presented.*

Proof. The condition is sufficient thanks to assertion (1) of Proposition 3.3. Conversely, if ξ is a generic point of X, then there exists a locally closed subset Y of X such that $\xi \in Y$ and $\mathcal{F}_{|Y}$ is locally finitely presented. The subset Y contains necessarily an open neighborhood U_ξ of ξ in X. We may choose $U := \bigcup U_\xi$. □

Proposition 3.6. *An isocrystal E on T is constructible if and only if there exists an exact sequence*

$$0 \to \beta_\dagger E' \to E \to \alpha_* E'' \to 0, \tag{9}$$

where E'' (resp. E') is a constructible isocrystal on a closed subvariety Z of X (resp. on $U := X \setminus Z$) and $\beta : Z \hookrightarrow X$ (resp. $\alpha : U \hookrightarrow X$) denotes the inclusion map. We may assume that U is dense in X and that E'' has coherent realizations.

Proof. If we are given such an exact sequence, we may pull back along α and β in order to obtain $E' \simeq E_{|Z}$ and $E'' \simeq E_{|U}$. Conversely, we may set $E' := E_{|Z}$ and $E'' := E_{|U}$ in order to get such an exact sequence by Proposition 2.16. □

Note that this property is specific to constructible *isocrystals* and that the analog for constructible modules is wrong.

It follows from Proposition 2.15 that any extension such as (9) comes from a unique morphism $\alpha_* E'' \to j_U^\dagger \beta_* E'$. This is a classical gluing method and the correspondence is given by the morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \beta_\dagger E' & \longrightarrow & \beta_* E' & \longrightarrow & j_U^\dagger \beta_* E' & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \beta_\dagger E' & \longrightarrow & E & \longrightarrow & \alpha_* E'' & \longrightarrow & 0 \end{array}$$

We can do the computations in the very special case of $\alpha : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ and $\beta : \infty \hookrightarrow \mathbb{P}^1_k$. We have $E' = \mathcal{O}^\dagger_{\infty/K} \otimes_K H$ for some finite-dimensional vector space H, and E'' is given by a finite free $K[t]^\dagger$-module M of finite rank endowed with a (overconvergent) connection. One can show that there exists a canonical isomorphism

$$\begin{aligned} \mathrm{Ext}(\alpha_* E'', \beta_\dagger E') &= \mathrm{Hom}(\alpha_* E'', j_U^\dagger \beta_* E') \\ &= \mathrm{Hom}_\nabla(M, \mathcal{R} \otimes_K H) \\ &= H^0_{\mathrm{dR}}(M^\vee \otimes_{K[t]^\dagger} \mathcal{R}) \otimes_K H \end{aligned}$$

(the second identity is not trivial). A slight generalization will give a classification of constructible isocrystals on smooth projective curves as in Theorem 6.15 of [Le Stum 2014].

4. Integrable connections and constructibility

In this section, we will give a more concrete description of constructible isocrystals in the case when T is representable by some overconvergent variety (X, V), in the case $T = X_V/O$, where (X, V) is a variety over some overconvergent variety (C, O), and finally when $T = X/O$, where X is a variety over C (see Section 1).

Definition 4.1. Let (X, V) be an overconvergent variety. An $i_X^{-1}\mathcal{O}_V$-module $\mathcal{F}$ is *constructible* if there exists a locally finite covering of X by locally closed subvarieties Y such that $i_Y^{-1} i_{X*} \mathcal{F}$ is a coherent $i_Y^{-1}\mathcal{O}_V$-module.

Of course, we have $i_Y^{-1} i_{X*}\mathcal{F} = i_{Y\subset X}^{-1}\mathcal{F}$ if we denote by $i_{Y\subset X} :]Y[_V \hookrightarrow]X[_V$ the inclusion of the tubes.

Proposition 4.2. *Let (X, V) be an overconvergent variety. Then*:

(1) $\mathrm{Isoc}^\dagger_{\mathrm{cons}}(X, V)$ *is an abelian subcategory of* $\mathrm{Isoc}^\dagger(X, V)$.

(2) *The realization functor induces an equivalence between* $\mathrm{Isoc}^\dagger_{\mathrm{cons}}(X, V)$ *and the category of all constructible $i_X^{-1}\mathcal{O}_V$-modules.*

Proof. It was shown in Proposition 3.3.8 of [Le Stum 2011] that the realization functor induces an equivalence between $\mathrm{Isoc}^\dagger(X, V)$ and the category of all $i_X^{-1}\mathcal{O}_V$-modules. Overconvergent isocrystals correspond to coherent modules. The second assertion is an immediate consequence of these observations. The first assertion then follows immediately from the analogous result about coherent modules. □

Recall that an $i_X^{-1}\mathcal{O}_V$-module may be endowed with an overconvergent stratification. Then, we have:

Proposition 4.3. *Let (X, V) be an overconvergent variety over another overconvergent variety (C, O).*

(1) *If V is universally flat over O in a neighborhood of* $]X[$, *then* $\mathrm{Isoc}^\dagger_{\mathrm{cons}}(X_V/O)$ *is an abelian subcategory of* $\mathrm{Isoc}^\dagger(X_V/O)$.

(2) *The realization functor induces an equivalence between* $\mathrm{Isoc}^\dagger_{\mathrm{cons}}(X_V/O)$ *and the category of constructible $i_X^{-1}\mathcal{O}_V$-modules $\mathcal{F}$ endowed with an overconvergent stratification.*

Proof. According to Proposition 3.5.3 of [Le Stum 2011], its corollary and Proposition 3.5.5 of the same paper, the proof goes exactly as in Proposition 4.2. □

The next corollary is valid if we work with *good* overconvergent varieties (which we may have assumed from the beginning).

Corollary 4.4. *If (C, O) is an* (*good*) *overconvergent variety and X is an algebraic variety over C, then* $\mathrm{Isoc}^\dagger_{\mathrm{cons}}(X/O)$ *is an abelian subcategory of* $\mathrm{Isoc}^\dagger(X/O)$.

Proof. Using Proposition 4.6.3 of [Le Stum 2011], we may assume that X has a geometric realization over (C, O) and use the second part of Proposition 3.5.8 in the same paper. □

We could have included a description of constructible isocrystal as modules endowed with an overconvergent stratification on some geometric realization of X/O but we are heading towards a finer description (this is what the rest of this section is all about).

Recall that any overconvergent stratification will induce, by pull-back at each level, a usual stratification. This is a faithful construction and we want to show that

it is actually *fully* faithful when we work with constructible modules (in suitable geometric situations). Thus, we have the sequence of injective maps

$$\mathrm{Hom}_{\mathrm{Strat}^\dagger}(\mathcal{F},\mathcal{G}) \hookrightarrow \mathrm{Hom}_{\mathrm{Strat}}(\mathcal{F},\mathcal{G}) \hookrightarrow \mathrm{Hom}(\mathcal{F},\mathcal{G})$$

and we wonder whether the first one is actually bijective. In order to do so, we will also have to study the injectivity of the maps in the sequence

$$\mathrm{Ext}_{\mathrm{Strat}^\dagger}(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}_{\mathrm{Strat}}(\mathcal{F},\mathcal{G}) \to \mathrm{Ext}(\mathcal{F},\mathcal{G}).$$

We start with the following observation:

Proposition 4.5. *Let (X,V) be a variety over an overconvergent variety (C,O), $\alpha : U \hookrightarrow X$ the inclusion of an open subvariety of X and $\beta : Z \hookrightarrow X$ the inclusion of a closed complement. Let $\mathcal{F}'$ be an $i_Z^{-1}\mathcal{O}_V$-module and $\mathcal{F}''$ an $i_U^{-1}\mathcal{O}_V$-module. Then a usual (resp. an overconvergent) stratification on the direct sum $]\beta[_!\mathcal{F}' \oplus]\alpha[_*\mathcal{F}''$ is uniquely determined by its restrictions to $\mathcal{F}'$ and $\mathcal{F}''$.*

Proof. Let us denote by

$$\epsilon^{(n)} = \begin{pmatrix}]\beta[_!\epsilon'^{(n)} & \varphi_n \\ \psi_n &]\alpha[_*\epsilon''^{(n)} \end{pmatrix}$$

the stratification of $]\beta[_!\mathcal{F}' \oplus]\alpha[_*\mathcal{F}''$ (recall that the maps $]\beta[_!$ and $]\alpha[_*$ are fully faithful). Then the maps

$$\varphi_n : i_X^{-1}\mathcal{O}_{V^{(n)}} \otimes_{i_X^{-1}\mathcal{O}_V}]\alpha[_*\mathcal{F}'' \to]\beta[_!\mathcal{F}' \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\mathcal{O}_{V^{(n)}}$$

and

$$\psi_n : i_X^{-1}\mathcal{O}_{V^{(n)}} \otimes_{i_X^{-1}\mathcal{O}_V}]\beta[_!\mathcal{F}' \to]\alpha[_*\mathcal{F}'' \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\mathcal{O}_{V^{(n)}}$$

are necessarily zero, as one may see by considering the fibers.

On the other hand, denote by

$$\epsilon = \begin{pmatrix}]\beta[_!\epsilon' & \varphi \\ \psi &]\alpha[_*\epsilon'' \end{pmatrix}$$

the overconvergent stratification of $]\beta[_!\mathcal{F}' \oplus]\alpha[_*\mathcal{F}''$. Then the maps

$$\varphi :]p_2[^\dagger]\alpha[_*\mathcal{F}'' \simeq]p_1[^\dagger]\beta[_!\mathcal{F}' \quad \text{and} \quad \psi :]p_2[^\dagger]\beta[_!\mathcal{F}' \simeq]p_1[^\dagger]\alpha[_*\mathcal{F}''$$

are necessarily zero, as one may see by considering the fibers and using the fact that $p_i^\dagger$ commutes with $]\alpha[_*$ and $]\beta[_!$. □

We keep the assumptions and the notation of the proposition for a while and assume that $\mathcal{F}'$ and $\mathcal{F}''$ are both endowed with a usual (resp. an overconvergent) stratification. From the general fact that

$$\mathrm{Hom}(]\beta[_!\mathcal{F}',]\alpha[_*\mathcal{F}'') = 0 \quad \text{and} \quad \mathrm{Hom}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{F}') = 0,$$

we can deduce that

$$\mathrm{Hom}_{\mathrm{Strat}}(]\beta[_!\mathcal{F}',]\alpha[_*\mathcal{F}'') = 0 \quad \big(\text{resp. } \mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\beta[_!\mathcal{F}',]\alpha[_*\mathcal{F}'') = 0\big),$$
$$\mathrm{Hom}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{F}') = 0 \quad \big(\text{resp. } \mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{F}') = 0\big).$$

Since we also know that

$$\mathrm{Ext}(]\beta[_!\mathcal{F}',]\alpha[_*\mathcal{F}'') = 0,$$

we can deduce the following result from the proposition:

Corollary 4.6. *If $\mathcal{F}'$ and $\mathcal{F}''$ are both endowed with a usual* (*resp. an overconvergent*) *stratification*, *then we have*

$$\mathrm{Ext}_{\mathrm{Strat}}(]\beta[_!\mathcal{F}',]\alpha[_*\mathcal{F}'') = 0 \quad \big(\text{resp. } \mathrm{Ext}_{\mathrm{Strat}^\dagger}(]\beta[_!\mathcal{F}',]\alpha[_*\mathcal{F}'') = 0\big).$$

Alternatively, it means that any short exact sequence of $i_Z^{-1}\mathcal{O}_V$-modules (resp. with a usual stratification, resp. with an overconvergent stratification)

$$0 \to]\alpha[_*\mathcal{F}'' \to \mathcal{F} \to]\beta[_!\mathcal{F}' \to 0$$

splits (and the splitting is compatible with the extra structure).

From the proposition, we may also deduce the following:

Corollary 4.7. *If $\mathcal{F}'$ and $\mathcal{F}''$ are both endowed with a usual stratification*, *then the map*

$$\mathrm{Ext}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{F}') \hookrightarrow \mathrm{Ext}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{F}')$$

is injective. If $\mathcal{F}'$ and $\mathcal{F}''$ are both endowed with an overconvergent stratification, *then the maps*

$$\mathrm{Ext}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{F}') \hookrightarrow \mathrm{Ext}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{F}') \hookrightarrow \mathrm{Ext}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{F}')$$

are injective.

Alternatively, it means that if $\mathcal{F}$ is an $i_X^{-1}\mathcal{O}_V$-module with a usual (resp. an overconvergent) stratification, *and* if the exact sequence of $i_X^{-1}\mathcal{O}_V$-modules

$$0 \to]\beta[_!\mathcal{F}_{]Z[} \to \mathcal{F} \to]\alpha[_*\mathcal{F}_{]U[} \to 0$$

splits, then the splitting is always compatible with the (resp. the overconvergent) stratifications.

We are now ready to prove our main result:

Proposition 4.8. *Let*

$$\begin{array}{ccccccc}
X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v_K} & & \downarrow{\scriptstyle u} \\
C & \hookrightarrow & S & \leftarrow & S_K & \leftarrow & O
\end{array}$$

be a formal morphism of overconvergent varieties with f quasicompact, v smooth at X, O locally separated and V a good neighborhood of X in $P_K \times_{S_K} O$. If $\mathcal{F}$ and $\mathcal{G}$ are two constructible $i_X^{-1}\mathcal{O}_V$-modules endowed with an overconvergent stratification, then

$$\mathrm{Hom}_{\mathrm{Strat}^\dagger}(\mathcal{F},\mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Strat}}(\mathcal{F},\mathcal{G}).$$

Proof. Since we know that the map is injective, we may rephrase the assertion as follows: we are given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of constructible $i_X^{-1}\mathcal{O}_V$-modules which is compatible with the usual stratifications and we have to show that φ is actually compatible with the overconvergent stratifications. This question is clearly local on O, which is locally compact. We may therefore assume that the image of O in S_K is contained in some S'_K with S' quasicompact. We may then pull back the diagram along $S' \to S$ and assume that X is finite-dimensional (use assertion (3) of Proposition 3.3). This will allow us to use noetherian induction.

We know (use, for example, Propositions 3.5 and 4.3) that there exists a dense open subset U of X such that the restrictions $\mathcal{F}''$ and $\mathcal{G}''$ to U of $\mathcal{F}$ and $\mathcal{G}$ are coherent. Moreover, it was shown in Corollary 3.4.10 of [Le Stum 2011] that the proposition is valid for $\mathcal{F}''$ and $\mathcal{G}''$ on U. Let us denote as usual by $\alpha : U \hookrightarrow X$ the inclusion map. Since $]\alpha[_*$ is fully faithful, we see that the proposition is valid for $]\alpha[_*\mathcal{F}''$ and $]\alpha[_*\mathcal{G}''$. In other words, we have a bijection

$$\mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'',]\alpha[_*\mathcal{G}'') \simeq \mathrm{Hom}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'',]\alpha[_*\mathcal{G}''). \tag{10}$$

We denote now by $\beta : Z \hookrightarrow X$ the inclusion of a closed complement of U and let $\mathcal{F}'$ and $\mathcal{G}'$ be the restrictions of $\mathcal{F}$ and $\mathcal{G}$ to Z. We observe the following commutative diagram:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'', \mathcal{G}) & \hookrightarrow & \mathrm{Hom}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'', \mathcal{G}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'',]\alpha[_*\mathcal{G}'') & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'',]\alpha[_*\mathcal{G}'') \\
\downarrow & & \downarrow \\
\mathrm{Ext}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{G}') & \hookrightarrow & \mathrm{Ext}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{G}')
\end{array}$$

The columns are exact because $\mathrm{Hom}(]\alpha[_*\mathcal{F}'',]\beta[_!\mathcal{G}') = 0$, the bottom map is injective thanks to Corollary 4.7 and the middle map is the isomorphism (10). It follows from the five lemma (or an easy diagram chase) that the upper map is

necessarily bijective: we have

$$\mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'', \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'', \mathcal{G}''). \tag{11}$$

We turn now to the other side: by induction, the proposition is valid for $\mathcal{F}'$ and $\mathcal{G}'$ on Z, and since $]\beta[_!$ is fully faithful, it also holds for $]\beta[_!\mathcal{F}'$ and $]\beta[_!\mathcal{G}'$. Hence, we have

$$\mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\beta[_!\mathcal{F}',]\beta[_!\mathcal{G}') \simeq \mathrm{Hom}_{\mathrm{Strat}}(]\beta[_!\mathcal{F}',]\beta[_!\mathcal{G}'). \tag{12}$$

Now, we consider the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\beta[_!\mathcal{F}',]\beta[_!\mathcal{G}') & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{Strat}}(]\beta[_!\mathcal{F}',]\beta[_!\mathcal{G}') \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\beta[_!\mathcal{F}', \mathcal{G}) & \hookrightarrow & \mathrm{Hom}_{\mathrm{Strat}}(]\beta[_!\mathcal{F}', \mathcal{G}) \end{array}$$

The vertical maps are bijective because $\mathrm{Hom}(]\beta[_!\mathcal{F}',]\alpha[_*\mathcal{G}'') = 0$ and the upper map is simply the isomorphism (12). It follows that we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\beta[_!\mathcal{F}', \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Strat}}(]\beta[_!\mathcal{F}', \mathcal{G}). \tag{13}$$

In order to end the proof, we will need to kill another obstruction. Since the proposition holds for $]\alpha[_*\mathcal{F}''$ and *any* constructible $\mathcal{G}$, the following canonical map is necessarily injective:

$$\mathrm{Ext}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'', \mathcal{G}) \hookrightarrow \mathrm{Ext}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'', \mathcal{G}). \tag{14}$$

We consider now the commutative diagram with exact columns

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'', \mathcal{G}) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'', \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Strat}^\dagger}(\mathcal{F}, \mathcal{G}) & \hookrightarrow & \mathrm{Hom}_{\mathrm{Strat}}(\mathcal{F}, \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{Strat}^\dagger}(]\beta[_\dagger\mathcal{F}', \mathcal{G}) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{Strat}}(]\beta[_\dagger\mathcal{F}', \mathcal{G}) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{\mathrm{Strat}^\dagger}(]\alpha[_*\mathcal{F}'', \mathcal{G}) & \hookrightarrow & \mathrm{Ext}_{\mathrm{Strat}}(]\alpha[_*\mathcal{F}'', \mathcal{G}) \end{array}$$

The horizontal isomorphisms are just (11) and (13) and the bottom injection is (14). It is then sufficient to apply the five lemma again. □

We may reformulate the statement of the proposition as follows:

Corollary 4.9. *The forgetful functor from constructible $i_X^{-1}\mathcal{O}_V$-modules endowed with an overconvergent stratification to $i_X^{-1}\mathcal{O}_V$-modules endowed with a usual stratification is fully faithful.*

It is also worth mentioning the following immediate consequence:

Corollary 4.10. *If $\mathcal{F}$ and $\mathcal{G}$ are two constructible $i_X^{-1}\mathcal{O}_V$-modules endowed with an overconvergent stratification, then we have an injective map*

$$\mathrm{Ext}_{\mathrm{Strat}^\dagger}(\mathcal{F},\mathcal{G}) \hookrightarrow \mathrm{Ext}_{\mathrm{Strat}}(\mathcal{F},\mathcal{G}). \tag{15}$$

It means that if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{16}$$

is a short exact sequence of constructible $i_X^{-1}\mathcal{O}_V$-modules endowed with an overconvergent stratification, then any splitting for the usual stratifications will be compatible with the overconvergent stratifications. I strongly suspect that much more is actually true: if we are given an exact sequence (16) of constructible $i_X^{-1}\mathcal{O}_V$-modules endowed with usual stratifications and if the stratifications of $\mathcal{F}'$ and $\mathcal{F}''$ are overconvergent, then the stratification of $\mathcal{F}$ should also be overconvergent. In other words, the injective map (15) would be an isomorphism.

If (X, V) is a variety over an overconvergent variety (C, O), we will denote by

$$\mathrm{MIC}^\dagger_{\mathrm{cons}}(X, V/O)$$

the category of constructible $i_X^{-1}\mathcal{O}_V$-modules $\mathcal{F}$ endowed with an overconvergent connection (recall that it means that the connection extends to some overconvergent stratification). Then, we can also state the following corollary:

Corollary 4.11. *If* $\mathrm{Char}(K) = 0$, *then the realization functor induces an equivalence of categories*

$$\mathrm{Isoc}^\dagger_{\mathrm{cons}}(X_V/O) \simeq \mathrm{MIC}^\dagger_{\mathrm{cons}}(X, V/O).$$

As a consequence, we observe that we will have, for a constructible isocrystal E on X_V/O,

$$\Gamma(X_V/O, E) \simeq H^0_{\mathrm{dR}}(E_V),$$

and we expect the same to hold for higher cohomology spaces; we only know at this point that

$$H^1(X_V/O, E) \subset H^1_{\mathrm{dR}}(E_V).$$

Again, we need to work with good overconvergent varieties for the theorem to hold:

Theorem 4.12. *Assume that* $\mathrm{Char}(K) = 0$ *and that we are given a commutative diagram*

$$\begin{array}{ccccccc} X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v_K} & & \downarrow{\scriptstyle u} \\ C & \hookrightarrow & S & \leftarrow & S_K & \leftarrow & O \end{array} \tag{17}$$

in which P *is a formal scheme over* S *which is proper and smooth around* X, *and* V *is a neighborhood of the tube of* X *in* $P_K \times_{S_K} O$ (*and* O *is good in the neighborhood of* $]C[$). *Then the realization functor induces an equivalence of categories*

$$\mathrm{Isoc}^{\dagger}_{\mathrm{cons}}(X/O) \simeq \mathrm{MIC}^{\dagger}_{\mathrm{cons}}(X, V/O)$$

between constructible overconvergent isocrystals on X/O *and constructible* $i_X^{-1}\mathcal{O}_V$*-modules endowed with an overconvergent connection.*

Proof. Using the second assertion of Proposition 3.5.8 in [Le Stum 2011], this follows immediately from Corollary 4.11. □

As a consequence of the theorem, we see that the notion of a constructible module endowed with an overconvergent connection only depends on X and *not* on the choice of the geometric realization (17). It is likely that this could have been proven directly using Berthelot's technique of diagonal embedding. However, we believe that our method is much more natural because functoriality is built in.

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