Algebra & Number Theory

Volume 10

2016

No. 10

The Voronoi formula and double Dirichlet series

Eren Mehmet Kıral and Fan Zhou

msp

ALGEBRA AND NUMBER THEORY 10:10 (2016)
dx.doi.org/10.2140/ant.2016.10.2267

The Voronoi formula and double Dirichlet series

Eren Mehmet Kıral and Fan Zhou

We prove a Voronoi formula for coefficients of a large class of L-functions including Maass cusp forms, Rankin–Selberg convolutions, and certain noncuspidal forms. Our proof is based on the functional equations of L-functions twisted by Dirichlet characters and does not directly depend on automorphy. Hence it has wider application than previous proofs. The key ingredient is the construction of a double Dirichlet series.

1. Introduction

A Voronoi formula is an identity involving Fourier coefficients of automorphic forms, with the coefficients twisted by additive characters on either side. A history of the Voronoi formula can be found in [Miller and Schmid 2004]. Since its introduction in [loc. cit.], the Voronoi formula on GL(3) of Miller and Schmid has become a standard tool in the study of L-functions arising from GL(3), and has found important applications such as those in [Blomer 2012; Blomer et al. 2013; Khan 2012; Li 2009; 2011; Li and Young 2012; Miller 2006; Munshi 2013; 2015]. As of yet the general GL(N) formula has had fewer applications, a notable one being found in [Kowalski and Ricotta 2014].

The first proof of a Voronoi formula on GL(3) was found by Miller and Schmid [2006] using the theory of automorphic distributions. Later, a Voronoi formula was established for GL(N) with $N \geq 4$ in [Goldfeld and Li 2006; 2008; Miller and Schmid 2011], with [Miller and Schmid 2011] being more general and earlier than [Goldfeld and Li 2008] (see the addendum there). Goldfeld and Li's proof [2008] is more akin to the classical proof in GL(2) [Good 1981], obtaining the associated Dirichlet series through a shifted "vertical" period integral and making use of automorphy. An adelic version was established by Ichino and Templier [2013], allowing ramifications and applications to number fields. Another direction

MSC2010: primary 11F30; secondary 11F68, 11L05.
Keywords: Voronoi formula, automorphic form, Maass form, multiple Dirichlet series, Gauss sum, Kloosterman sum, Rankin–Selberg L-function.

of generalization with more complicated additive twists on either side has been considered in an unpublished work of Li and Miller and in [Zhou 2016].

In this article, we prove a Voronoi formula for a large class of automorphic objects or L-functions, including cusp forms for $\mathrm{SL}(N, \mathbb{Z})$, Rankin–Selberg convolutions, and certain noncuspidal forms. Previous works [Miller and Schmid 2011; Goldfeld and Li 2008; Ichino and Templier 2013] do not offer a Voronoi formula for Rankin–Selberg convolutions or noncuspidal forms. Even for Maass cusp forms, our new proof is shorter than any previous one, and uses a completely different set of techniques.

Let us briefly summarize our method of proof. We first reduce the statement of a Voronoi formula to a formula involving Gauss sums of Dirichlet characters. We construct a complex function of two variables and write it as double Dirichlet series in two different ways by applying a functional equation. Using the uniqueness theorem of Dirichlet series, we get an identity between coefficients of these two double Dirichlet series. This leads us to the Voronoi formula with Gauss sums.

One of our key steps in obtaining the Voronoi formula is the use of functional equations of L-functions twisted by Dirichlet characters. The relationship between the Dirichlet twists and the additive twists was expected, but not fully understood, such as in [Duke and Iwaniec 1990; Goldfeld and Li 2006, Section 4; Buttcane and Khan 2015; Zhou 2016]. In these works, only prime modulus is dealt with, which is a significant restriction. Miller and Schmid [2006, Section 6] derived the functional equation of L-functions twisted by a Dirichlet character of prime conductor from the Voronoi formula. However there is a combinatorial difficulty in reversing this process, i.e., obtaining additive twists of general nonprime conductors from multiplicative ones, which was acknowledged in both [Miller and Schmid 2006, p. 430] and [Ichino and Templier 2013, p. 68]. The method presented here is able to overcome this difficulty by discovering an interlocking structure among a family of Voronoi formulas with different conductors.

Our proof of the Voronoi formula is complete for additive twists of all conductors, prime or not, and unlike [Ichino and Templier 2013], [Miller and Schmid 2006], or [Miller and Schmid 2011], does not depend directly on automorphy of the cusp forms. This fact allows us to apply our theorem to many conjectural Langlands functorial transfers. For example, the Rankin–Selberg convolutions (also called functorial products) for $\mathrm{GL}(m) \times \mathrm{GL}(n)$ are not yet known to be automorphic on $\mathrm{GL}(m \times n)$ in general. Yet we know the functional equations of $\mathrm{GL}(m) \times \mathrm{GL}(n)$ L-functions twisted by Dirichlet characters. Thus, our proof provides a Voronoi formula for the Rankin–Selberg convolutions on $\mathrm{GL}(m) \times \mathrm{GL}(n)$ (see Example 1.7). Voronoi formulas for these functorial cases are unavailable from [Goldfeld and Li 2008], [Miller and Schmid 2011] or [Ichino and Templier 2013]. In Theorem 1.3 we reformulate our Voronoi formula like the classical converse theorem of Weil, i.e., assuming every L-function twisted by a Dirichlet character is entire, has an Euler product (or satisfies Hecke relations), and satisfies the precise functional equations, then the Voronoi formula as in Theorem 1.1 is valid. We do not have to assume it is a standard L-function coming from a cusp form.

Furthermore, by Theorem 1.3, we obtain a Voronoi formula for certain noncuspidal forms, such as isobaric sums (see Example 1.8). This is not readily available from any previous work but it is believed (see [Miller and Schmid 2011, p. 176]) that one may derive a formula by using formulas on smaller groups through a possibly complicated procedure. Such complication does not occur in our method because we work directly with L-functions.

We first state the main results for Maass cusp forms. Denote

$$e(x) := \exp(2\pi i x)$$

for $x \in \mathbb{R}$. Let $N \geq 3$ be an integer. Let $a, n \in \mathbb{Z}$, $c \in \mathbb{N}$ and let

$$\boldsymbol{q} = (q_1, q_2, \dots, q_{N-2}) \quad \text{and} \quad \boldsymbol{d} = (d_1, d_2, \dots, d_{N-2})$$

be tuples of positive integers satisfying the divisibility conditions

$$d_1 | q_1 c, \quad d_2 \,\Big|\, \frac{q_1 q_2 c}{d_1}, \quad \dots, \quad d_{N-2} \,\Big|\, \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-3}}. \tag{1}$$

In this case, to simplify notation we set

$$\xi_i := \frac{q_1 \cdots q_i c}{d_1 \cdots d_i}.$$

Define the hyper-Kloosterman sum as

$$\mathrm{Kl}(a, n, c; \boldsymbol{q}, \boldsymbol{d}) = \sideset{}{^*}\sum_{x_1 \bmod \xi_1} \sideset{}{^*}\sum_{x_2 \bmod \xi_2} \cdots \sideset{}{^*}\sum_{x_{N-2} \bmod \xi_{N-2}} e\biggl(\frac{d_1 x_1 a}{c} + \frac{d_2 x_2 \overline{x_1}}{\xi_1} + \cdots + \frac{d_{N-2} x_{N-2} \overline{x_{N-3}}}{\xi_{N-3}} + \frac{n \overline{x_{N-2}}}{\xi_{N-2}}\biggr),$$

where $\sum^*$ indicates that the summation is over reduced residue classes, and $\overline{x_i}$ denotes the multiplicative inverse of x_i modulo ξ_i. When $N = 3$, $\mathrm{Kl}(a, n, c; q_1, d_1)$ becomes the classical Kloosterman sum $S(aq_1, n; \xi_1)$. For the degenerate case $N = 2$, we define $\mathrm{Kl}(a, n, c;\ ,\ ) := e(an/c)$.

Let F be a Hecke–Maass cusp form for $\mathrm{SL}(N, \mathbb{Z})$ with the spectral parameters $(\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$. Let $A(m_1, \dots, m_{N-1})$, with $(m_1, \dots, m_{N-1}) \in \mathbb{N}^{N-1}$, be the Fourier–Whittaker coefficients of F normalized as $A(1, \dots, 1) = 1$. We refer to [Goldfeld 2006] for the definitions and the basic results of Maass forms for $\mathrm{SL}(N, \mathbb{Z})$.

The Fourier coefficients satisfy the Hecke relations

$$A(m_1m_1',\dots,m_{N-1}m_{N-1}') = A(m_1,\dots,m_{N-1})A(m_1',\dots,m_{N-1}') \tag{2}$$

if $(m_1\cdots m_{N-1}, m_1'\cdots m_{N-1}') = 1$ is satisfied,

$$A(1,\dots,1,n)A(m_{N-1},\dots,m_1) = \sum_{\substack{d_0\cdots d_{N-1}=n\\ d_1|m_1,\dots,d_{N-1}|m_{N-1}}} A\left(\frac{m_{N-1}d_{N-2}}{d_{N-1}},\dots,\frac{m_2d_1}{d_2},\frac{m_1d_0}{d_1}\right), \tag{3}$$

and

$$A(n,1,\dots,1)A(m_1,\dots,m_{N-1}) = \sum_{\substack{d_0\cdots d_{N-1}=n\\ d_1|m_1,\dots,d_{N-1}|m_{N-1}}} A\left(\frac{m_1d_0}{d_1},\frac{m_2d_1}{d_2},\dots,\frac{m_{N-1}d_{N-2}}{d_{N-1}}\right). \tag{4}$$

The dual Maass form of F is denoted by $\widetilde{F}$. Let $B(*,\dots,*)$ be the Fourier–Whittaker coefficients of $\widetilde{F}$. These coefficients satisfy

$$B(m_1,\dots,m_{N-1}) = A(m_{N-1},\dots,m_1). \tag{5}$$

Define the ratio of Gamma factors

$$G_\pm(s) := i^{-N\delta}\pi^{-N(1/2-s)}\prod_{j=1}^{N}\Gamma\left(\frac{\delta+1-s-\overline{\lambda_j}}{2}\right)\Gamma\left(\frac{\delta+s-\lambda_j}{2}\right)^{-1}, \tag{6}$$

where for even Maass forms, we define $\delta = 0$ in G_+ and $\delta = 1$ in G_-, and for odd Maass forms, we define $\delta = 1$ in G_+ and $\delta = 0$ in G_-. We refer to [Goldfeld 2006, Section 9.2] for the definition of even and odd Maass forms.

Theorem 1.1 (Voronoi formula on GL(N) of Miller and Schmid [2011]). *Let F be a Hecke–Maass cusp form with coefficients $A(*,\dots,*)$, and $G_\pm$ a ratio of Gamma factors as in* (6). *Let $c > 0$ be an integer and let a be any integer with $(a, c) = 1$. Denote by $\bar{a}$ the multiplicative inverse of a modulo c. Let the additively twisted Dirichlet series be given as*

$$L_{\boldsymbol{q}}\left(s, F, \frac{a}{c}\right) = \sum_{n=1}^{\infty}\frac{A(q_{N-2},\dots,q_1,n)}{n^s}e\left(\frac{\bar{a}n}{c}\right) \tag{7}$$

for $\Re(s) > 1$. This Dirichlet series has an analytic continuation to all $s \in \mathbb{C}$ and satisfies the functional equation

$$
\begin{aligned}
&L_{\boldsymbol{q}}(s, F, a/c)\\
&\quad= \frac{G_+(s) - G_-(s)}{2} \sum_{d_1 \mid q_1 c} \sum_{d_2 \mid \frac{q_1 q_2 c}{d_1}} \cdots \sum_{d_{N-2} \mid \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-3}}}\\
&\qquad\quad \sum_{n=1}^{\infty} \frac{A(n, d_{N-2}, \ldots, d_2, d_1)\, \mathrm{Kl}(a, n, c; \boldsymbol{q}, \boldsymbol{d})}{n^{1-s} c^{Ns-1} d_1 d_2 \cdots d_{N-2}} \frac{d_1^{(N-1)s} d_2^{(N-2)s} \cdots d_{N-2}^{2s}}{q_1^{(N-2)s} q_2^{(N-3)s} \cdots q_{N-2}^{s}}\\
&\qquad + \frac{G_+(s) + G_-(s)}{2} \sum_{d_1 \mid q_1 c} \sum_{d_2 \mid \frac{q_1 q_2 c}{d_1}} \cdots \sum_{d_{N-2} \mid \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-3}}}\\
&\qquad\quad \sum_{n=1}^{\infty} \frac{A(n, d_{N-2}, \ldots, d_2, d_1)\, \mathrm{Kl}(a, -n, c; \boldsymbol{q}, \boldsymbol{d})}{n^{1-s} c^{Ns-1} d_1 d_2 \cdots d_{N-2}} \frac{d_1^{(N-1)s} d_2^{(N-2)s} \cdots d_{N-2}^{2s}}{q_1^{(N-2)s} q_2^{(N-3)s} \cdots q_{N-2}^{s}}, \qquad (8)
\end{aligned}
$$

in the region of convergence of the expression on the right-hand side ($\Re(s) < 0$).

The traditional Voronoi formula, involving weight functions instead of Dirichlet series, is obtained after taking an inverse Mellin transform against a suitable test function.

Choose a Dirichlet character χ modulo c, which is not necessarily primitive, multiply both sides of (8) by $\chi(a)$, and sum this equality over the reduced residue system modulo c. We obtain the following Voronoi formula with Gauss sums. In Section 3B we show through elementary finite arithmetic that the formulas (8) and (11) are equivalent.

Theorem 1.2 (Voronoi formula with Gauss sums). *Let χ be a Dirichlet character modulo c, induced from the primitive character χ^* modulo c^* with $c^* \mid c$. Define for $\boldsymbol{q} = (q_1, \ldots, q_{N-2})$ a tuple of positive integers*

$$
H(\boldsymbol{q}; c, \chi^*, s) = \sum_{n=1}^{\infty} \frac{A(q_{N-2}, \ldots, q_1, n)\, g(\overline{\chi^*}, c, n)}{n^s (c/c^*)^{1-2s}} \qquad (9)
$$

for $\Re(s) > 1$, *and*

$$
\begin{aligned}
G(\boldsymbol{q}; c, \chi^*, s) &= \frac{G(s)\chi^*(-1)}{c^{Ns-1}(c/c^*)^{1-2s}} \sum_{d_1 c^* \mid q_1 c} \sum_{d_2 c^* \mid \frac{q_1 q_2 c}{d_1}} \cdots \sum_{d_{N-2} c^* \mid \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-3}}}\\
&\quad \sum_{n=1}^{\infty} \frac{A(n, d_{N-2}, \ldots, d_1)}{n^{1-s} d_1 d_2 \cdots d_{N-2}} \frac{d_1^{(N-1)s} d_2^{(N-2)s} \cdots d_{N-2}^{2s}}{q_1^{(N-2)s} q_2^{(N-3)s} \cdots q_{N-2}^{s}}\\
&\qquad \times g(\chi^*, c, d_1) g(\chi^*, \xi_1, d_2) \cdots g(\chi^*, \xi_{N-3}, d_{N-2}) g(\chi^*, \xi_{N-2}, n) \qquad (10)
\end{aligned}
$$

for $\Re(s) < 0$, *where G equals G_+ or G_- depending on whether $\chi^*(-1)$ is* 1 *or* -1, *and $g(\chi^*, \ell c^*, *)$ is the Gauss sum of the induced character modulo ℓc^* from χ^*,* *which is defined in Definition 2.1. Both functions have analytic continuation to all* $s \in \mathbb{C}$, *and the equality*

$$H(\boldsymbol{q}; c, \chi^*, s) = G(\boldsymbol{q}; c, \chi^*, s) \tag{11}$$

is satisfied.

In proving (11), we define

$$Z(s, w) = \frac{L_{\boldsymbol{q}}(2w - s, F)L(s, F \times \chi^*)}{L(2w - 2s + 1, \overline{\chi^*})}, \tag{12}$$

where $\boldsymbol{q} = (q_1, \ldots, q_{N-2})$ is a tuple of positive integers, and the function $L_{\boldsymbol{q}}(s, F)$ is given as the Dirichlet series

$$L_{\boldsymbol{q}}(s, F) = \sum_{n=1}^{\infty} \frac{A(q_{N-2}, \ldots, q_1, n)}{n^s}$$

for $\Re(s) \gg 1$. We express $Z(s, w)$ as a double Dirichlet series in two different ways. In one region of convergence we express the L-functions as Dirichlet series and obtain

$$Z(s, w) = \sum_{n=1}^{\infty} \frac{a_n(s)}{n^{2w}}.$$

On the other hand, we apply the functional equation of $L(s, F \times \chi^*)$, replacing s with $1 - s$, and write $Z(s, w)$ as the Dirichlet series

$$Z(s, w) = \sum_{n} \frac{b_n(s)}{n^{2w}}.$$

By the uniqueness of Dirichlet series, we must have $a_n(s) = b_n(s)$. This equality leads us to the Voronoi formula with Gauss sums.

Our proof only uses the Hecke relations about the Fourier coefficients of F and the exact form of the functional equations. The expression of Gamma factors, or the automorphy of F, plays no role. Hence we can formulate our theorem in a style similar to the classical converse theorem of Weil. First, let us list the properties of Fourier coefficients that we use in order to state the following theorem.

The Fourier coefficients of F grow moderately, i.e.,

$$A(m_1, \ldots, m_{N-1}) \ll (m_1 \cdots m_{N-1})^{\sigma} \tag{13}$$

for some $\sigma > 0$. Given a primitive Dirichlet character χ^* modulo c^*, define the twisted L-function

$$L(s, F \times \chi^*) = \sum_{n=1}^{\infty} \frac{A(1, \ldots, 1, n)\chi^*(n)}{n^s} \tag{14}$$

for $\Re(s) > \sigma + 1$. It has analytic continuation to the whole complex plane, and satisfies the functional equation

$$L(s, F \times \chi^*) = \tau(\chi^*)^N c^{*-Ns} G(s) L(1-s, \widetilde{F} \times \overline{\chi^*}), \tag{15}$$

where $G(s) = G_+(s)$ or $G_-(s)$ depending on whether $\chi^*(-1) = 1$ or -1.

Theorem 1.3. *Let F be a symbol and assume that with F come numbers*

$$A(m_1, \dots, m_{N-1}) \in \mathbb{C}$$

attached to every $(N-1)$-tuple $(m_1, \dots, m_{N-1})$ of natural numbers. Assume $A(1, \dots, 1) = 1$.

Assume that these "coefficients" $A(*, \dots, *)$ satisfy the aforementioned Hecke relations (2), (3) *and* (4). *Further assume that they grow moderately as in* (13).

Let $\widetilde{F}$ be another symbol whose associated coefficients $B(*, \dots, *) \in \mathbb{C}$ are given as in (5) *and assume that they also satisfy the same properties. Further, assume that there are two meromorphic functions $G_+(s)$ and $G_-(s)$ associated to the pair $(F, \widetilde{F})$, so that for a given primitive character χ^*, the function $L(s, F \times \chi^*)$ as defined in* (14) *satisfies the functional equation* (15).

Under all these assumptions, $L_{\boldsymbol{q}}(s, F, a/c)$, defined as in (7) *for $\Re(s) > 1 + \sigma$, has analytic continuation to all $s \in \mathbb{C}$, and satisfies the Voronoi formula* (8). (*The Dirichlet series on the right side of* (8) *is absolutely convergent for $\Re(s) < -\sigma$.*)

Equivalently the functions $H(\boldsymbol{q}; c, \chi^*, s)$ and $G(\boldsymbol{q}; c, \chi^*, s)$ as defined by the formulas (9) *and* (10) *have analytic continuations to all s and equal each other as in* (11).

Remark 1.4 (the structure of this article). Theorem 1.3 is our main result. For the most part our focus is on the case $N \geq 3$, and we deal with the case $N = 2$ in Remark 3.2. The Voronoi formula (8) is proved to be equivalent to a formula (11) involving Gauss sums. The equivalence is shown in Proposition 3.5. A convolved version of (11) is obtained in Theorem 3.1 by comparing Dirichlet coefficients of two different expressions of a double Dirichlet series. We later show in Proposition 3.3 that this convolved version yields (11).

Remark 1.5. If we start with an L-series $L(s, F)$ with an Euler product

$$L(s, F) = \sum_{n=1}^{\infty} \frac{A(1, \dots, 1, n)}{n^s} = \prod_p \prod_{i=1}^{N} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}$$

and with $\prod_i \alpha_i(p) = 1$ for any p, then we can define $A(p^{k_1}, \dots, p^{k_{N-1}})$ by the Casselman–Shalika formula [Zhou 2014, Proposition 5.1] and they are compatible with the Hecke relations. More explicitly, for a prime number p, we define $A(p^{k_1}, \dots, p^{k_{N-1}}) = S_{k_1, \dots, k_{N-1}}(\alpha_1(p), \dots, \alpha_N(p))$ by the work of Shintani, where

$S_{k_1,\ldots,k_{N-1}}(x_1, \ldots, x_N)$ is the Schur polynomial, which can be found in [Goldfeld 2006, p. 233].

We extend the definition to all $A(*, \ldots, *)$ multiplicatively by (2). One can prove that $A(*, \ldots, *)$ satisfies the Hecke relations (2)–(4). In summary, the "coefficients" $A(*, \ldots, *)$ along with the Hecke relations can be generated by an L-function with an Euler product.

The following examples satisfy the conditions in Theorem 1.3, and hence we have a Voronoi formula for each of them.

Example 1.6 (automorphic form for $\mathrm{SL}(N, \mathbb{Z})$). Any cuspidal automorphic form for $\mathrm{SL}(N, \mathbb{Z})$ satisfies the conditions in Theorem 1.3. It can have an unramified or ramified component at the archimedean place, because only the exact form of the $G_\pm$ function would change; see [Godement and Jacquet 1972]. The Hecke–Maass cusp forms considered in Theorem 1.1 are included in this category, and therefore, we prove Theorem 1.3 instead of Theorem 1.1.

Example 1.7 (Rankin–Selberg convolution). Let F_1 and F_2 be even Hecke–Maass cusp forms for $\mathrm{SL}(N_1, \mathbb{Z})$ and $\mathrm{SL}(N_2, \mathbb{Z})$ with the spectral parameters

$$(\lambda_1, \ldots, \lambda_{N_1}) \in \mathbb{C}^{N_1} \quad \text{and} \quad (\mu_1, \ldots, \mu_{N_2}) \in \mathbb{C}^{N_2},$$

respectively. Assume $F_1 \neq \widetilde{F}_2$ if $N_1 = N_2$. The automorphic forms F_1 and F_2 have the standard L-functions

$$L(s, F_1) = \prod_p \prod_{i=1}^{N_1} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1} \quad \text{and} \quad L(s, F_2) = \prod_p \prod_{i=1}^{N_2} \left(1 - \frac{\beta_i(p)}{p^s}\right)^{-1}.$$

Let $L(s, F_1 \times F_2)$ be the Rankin–Selberg L-function of F_1 and F_2 defined by

$$L(s, F_1 \times F_2) = \prod_p \prod_{i_1=1}^{N_1} \prod_{i_2=1}^{N_2} \left(1 - \frac{\alpha_{i_1}(p)\beta_{i_2}(p)}{p^s}\right)^{-1}.$$

The L-function is of degree $N := N_1 N_2$. The work of Jacquet, Piatetskii-Shapiro, and Shalika [Jacquet et al. 1983] shows that $L(s, F \times \chi^*) = L(s, (F_1 \times \chi^*) \times F_2)$ is holomorphic and satisfies the functional equation (15) for $F := F_1 \times F_2$.

Define $A(p^{k_1}, \ldots, p^{k_{N-1}})$ by the Schur polynomials as in Remark 1.5:

$$A(p^{k_1}, \ldots, p^{k_{N-1}}) := S_{k_1,\ldots,k_{N-1}}\big(\alpha_1(p)\beta_1(p), \ldots, \alpha_{i_1}(p)\beta_{i_2}(p), \ldots, \alpha_{N_1}(p)\beta_{N_2}(p)\big).$$

Extend the definition to all $A(*, \dots, *)$ multiplicatively by (2). Define

$$G_{\pm}(s) := \\ i^{-N\delta}\pi^{-N(1/2-s)} \prod_{i_1=1}^{N_1}\prod_{i_2=1}^{N_2} \Gamma\Big(\frac{\delta+1-s-\overline{\lambda_{i_1}}-\overline{\mu_{i_2}}}{2}\Big)\Gamma\Big(\frac{\delta+s-\lambda_{i_1}-\mu_{i_2}}{2}\Big)^{-1},$$

where one takes $\delta=0$ and $\delta=1$ for G_+ and G_-, respectively. Theorem 1.3 gives us a Voronoi formula for the Rankin–Selberg convolution $F=F_1\times F_2$ with the $A(*, \dots, *)$ and $G_{\pm}$ defined above.

Example 1.8 (isobaric sum, Eisenstein series). For $i=1,\dots,k$, let F_i be a Hecke–Maass cusp form for $\mathrm{SL}(N_i,\mathbb{Z})$. Let s_i be complex numbers with $\sum_i N_i s_i=0$. Define the isobaric sum $F=\big(F_1\times|\cdot|_{\mathbb{A}}^{s_1}\big)\boxplus\big(F_2\times|\cdot|_{\mathbb{A}}^{s_2}\big)\boxplus\cdots\boxplus\big(F_k\times|\cdot|_{\mathbb{A}}^{s_k}\big)$, whose L-function is $L(s,F)=\prod_i L(s+s_i,F_i)$. This isobaric sum F is associated with a noncuspidal automorphic form on $\mathrm{GL}(N)$, an Eisenstein series twisted by Maass forms, where $N=\sum_i N_i$; see [Goldfeld 2006, Section 10.5]. The L-function twisted by a character is simply given by $L(s,F\times\chi^*)=\prod_i L(s+s_i,F_i\times\chi^*)$, which satisfies the conditions of Theorem 1.3.

Example 1.9 (symmetric powers on GL(2)). Let f be a modular form of weight k for $\mathrm{SL}(2,\mathbb{Z})$, and define $F:=\mathrm{Sym}^2 f$. The symmetric square F satisfies the conditions in Theorem 1.3 by the work of Shimura [1975]. Here we do not need to involve automorphy using Gelbart–Jacquet lifting. One may have similar results for higher symmetric powers depending on the recent progress in the theory of Galois representations.

As a last remark, let us explain the construction of the double Dirichlet series $Z(s,w)$ given by (12). This construction originates from the Rankin–Selberg convolution of a cusp form F and an Eisenstein series on GL(2). The Fourier coefficients of the Eisenstein series $E(z,s,\chi^*)$ can be written in terms of the divisor function $\sigma_{2s-1}(n,\chi^*)$ defined in Definition 2.1:

$$\frac{1}{n^{2s-1}}\frac{\sigma_{2s-1}(n,\chi^*)}{L(2s,\overline{\chi^*})} \quad\text{or}\quad \sum_{\ell=1}^{\infty}\frac{g(\overline{\chi^*},\ell c^*,n)}{(\ell c^*)^{2s}}.$$

Therefore, in the case of F on GL(2), the Rankin–Selberg integral of F and $E\big(*,w-s+\frac{1}{2},\chi^*\big)$ produces the double Dirichlet series

$$\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{A(n)g(\overline{\chi^*},\ell c^*,n)}{n^s(\ell c^*)^{2w+1-2s}}.$$

A similar expression appears on the left-hand side of the Voronoi formula with Gauss sums (9). The Rankin–Selberg convolution of the cusp form F and an

Eisenstein series can be written as a product of two copies of a standard L-function of F, namely

$$\frac{L(2w-s,F)L(s,F\times\chi^*)}{L(2w-2s+1,\overline{\chi^*})}.$$

Applying the functional equation to only $L(s,F\times\chi^*)$ gives us another expression, which is similar to the right-hand side (10) of the Voronoi formula with Gauss sums. Since $L(2w-s,F)$ was not used in this process, we have the freedom to replace $L(2w-s,F)$ by $L_{\boldsymbol{q}}(2w-s,F)$ in the case of $\mathrm{GL}(N)$, and it gives us enough generality to prove the Voronoi formula (11) with Gauss sums. In the case of GL(3), this construction is similar to Bump's double Dirichlet series; see [Goldfeld 2006, Chapter 6.6] or [Bump 1984, Chapter X].

2. Background on Gauss sums

Here we collect information about the Gauss sums of Dirichlet characters which are not necessarily primitive.

Definition 2.1. Let χ be a Dirichlet character modulo c induced from a primitive Dirichlet character χ^* modulo c^*. Define the divisor function

$$\sigma_s(m,\chi)=\sum_{d\mid m}\chi(d)d^s.$$

Define the Gauss sum of χ to be

$$g(\chi^*,c,m)=\sum_{\substack{(u,c)=1\\ u \bmod c}}\chi(u)e\Big(\frac{mu}{c}\Big).$$

The standard Gauss sum for χ^* is given as $\tau(\chi^*)=g(\chi^*,c^*,1)$.

The Gauss sum $g(\chi^*,c,m)$ is the same as the Gauss sum $\tau_m(\chi)$ in other literature. However we prefer our notation because we come upon numerous Gauss sums of characters χ induced from a single primitive character χ^*.

Lemma 2.2 (Gauss sum of nonprimitive characters [Miyake 1989, Lemma 3.1.3(2)]). *Let χ be a character modulo c induced from a primitive character χ^* modulo c^*. Then the Gauss sum of χ is given by*

$$g(\chi^*,c,a)=\tau(\chi^*)\sum_{d\mid(a,c/c^*)}d\chi^*\Big(\frac{c}{c^*d}\Big)\overline{\chi^*}\Big(\frac{a}{d}\Big)\mu\Big(\frac{c}{c^*d}\Big).$$

Lemma 2.3 [Montgomery and Vaughan 2007, Theorem 9.12]. *Let χ^* be a primitive character modulo c^* and assume $c^*\mid c$. Then we have*

$$g(\chi^*,c,a)=\tau(\chi^*)\frac{\phi(c)}{\phi(c/(c,a))}\mu\Big(\frac{c}{c^*(c,a)}\Big)\chi^*\Big(\frac{c}{c^*(c,a)}\Big)\overline{\chi^*}\Big(\frac{a}{(c,a)}\Big)$$

if $c^*\mid c/(a,c)$. Otherwise, $g(\chi^*,c,a)$ is zero.

The next lemma is a generalization of a famous formula of Ramanujan:

$$\frac{\sigma_{s-1}(n)}{n^{s-1}} = \zeta(s) \sum_{\ell=1}^{\infty} \frac{c_\ell(n)}{\ell^s},$$

where $c_\ell(n)$ is the Ramanujan sum.

Lemma 2.4. *Let* $\Re(s) > 1$. *Define a Dirichlet series*

$$I(s, \chi^*, c^*, m) = \sum_{\ell=1}^{\infty} \frac{g(\chi^*, \ell c^*, m)}{\ell^s}$$

as a generating function for the nonprimitive Gauss sums induced from χ^*. *It satisfies the identity*

$$\tau(\chi^*)\sigma_{s-1}(m, \overline{\chi^*}) = m^{s-1} I(s, \chi^*, c^*, m) L(s, \chi^*).$$

Proof. We prove the equivalent formula

$$\tau(\chi^*) m^{1-s} \sigma_{s-1}(m, \overline{\chi^*}) L(s, \chi^*)^{-1} = I(s, \chi^*, c^*, m).$$

For $\Re(s) > 1$, the function $\tau(\chi^*) m^{1-s} \sigma_{s-1}(m, \overline{\chi^*}) L(s, \chi^*)^{-1}$ can be written as a Dirichlet series

$$\tau(\chi^*) \sum_{d|m} \frac{d\overline{\chi^*}(m/d)}{d^s} \sum_{n=1}^{\infty} \frac{\chi^*(n)\mu(n)}{n^s} = \tau(\chi^*) \sum_{\ell=1}^{\infty} \frac{\sum_{d|(m,\ell)} d\overline{\chi^*}(m/d)\mu(\ell/d)\chi^*(\ell/d)}{\ell^s},$$

and this equals $I(s, \chi^*, c^*, m)$ by Lemma 2.2. □

Lemma 2.5. *For any two positive integers n and m, and a primitive Dirichlet character χ^* modulo c^*, we have*

$$\sum_{\ell d = n} \chi^*(d) g(\chi^*, \ell c^*, m) = \begin{cases} \tau(\chi^*)\overline{\chi^*(m/n)}n & \text{if } n \mid m, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We start with the formula,

$$\frac{\tau(\chi^*)\sigma_{s-1}(m, \overline{\chi^*})}{m^{s-1}} = I(s, \chi^*, c^*, m) L(s, \chi^*).$$

Both sides are Dirichlet series and we equate coefficients. The left-hand side is given as

$$\tau(\chi^*) \sum_{e|m} \frac{\overline{\chi^*(m/e)}e}{e^s},$$

whereas the right-hand side is

$$\sum_{\ell=1}^{\infty}\frac{g(\chi^*,\ell c^*,m)}{\ell^s}\sum_{d=1}^{\infty}\frac{\chi^*(d)}{d^s}=\sum_{n=1}^{\infty}\frac{\sum_{d\ell=n}\chi^*(d)g(\chi^*,\ell c^*,m)}{n^s}. \qquad \square$$

3. The Voronoi formula

3A. ***Double Dirichlet series.*** We begin by proving a convolved version of (11).

Theorem 3.1. *For $N\geq 3$, $\boldsymbol{q}=(q_1,\dots,q_{N-2})\in\mathbb{N}^{N-2}$, and $n\in\mathbb{N}$, define*

$$\mathcal{H}(\boldsymbol{q};n,s):=\sum_{d_1|q_1,\dots,d_{N-2}|q_{N-2}}\frac{\chi^*(d_1\cdots d_{N-2})}{(d_1\cdots d_{N-2})^s}\sum_{d\ell=n}\chi^*(d)H(\boldsymbol{q}';\ell c^*,\chi^*,s)$$

for $\Re(s)\gg 1$, and

$$\mathcal{G}(\boldsymbol{q};n,s):=\sum_{d_1|q_1,\dots,d_{N-2}|q_{N-2}}\frac{\chi^*(d_1\cdots d_{N-2})}{(d_1\cdots d_{N-2})^s}\sum_{d\ell=n}\chi^*(d)G(\boldsymbol{q}';\ell c^*,\chi^*,s)$$

for $\Re(1-s)\gg 1$, where we abbreviate

$$\boldsymbol{q}'=\left(\frac{q_1 d}{d_1},\frac{q_2 d_1}{d_2},\dots,\frac{q_{N-2}d_{N-3}}{d_{N-2}}\right). \tag{16}$$

The functions $\mathcal{H}(\boldsymbol{q};n,s)$ and $\mathcal{G}(\boldsymbol{q};n,s)$ have analytic continuation to all $s\in\mathbb{C}$ and these analytic continuations satisfy

$$\mathcal{H}(\boldsymbol{q};n,s)=\mathcal{G}(\boldsymbol{q};n,s). \tag{17}$$

Proof. The region of absolute convergence for $\mathcal{H}(\boldsymbol{q};n,s)$ is a right half plane $\Re(s)\gg 1$, and the region of absolute convergence of $\mathcal{G}(\boldsymbol{q};n;s)$ is a left half plane $\Re(1-s)\gg 1$. Let $Z(s,w)$ be defined as in (12). For any $s\in\mathbb{C}$ and w with $\Re(w)$ large enough so that $\Re(2w-s)\gg 1$ and $\Re(w-s)>0$, writing $L_{\boldsymbol{q}}(2w-s,F)$ and $L(2w-2s+1,\overline{\chi^*})^{-1}$ as Dirichlet series, we derive

$$Z(s,w)=L(s,F\times\chi^*)\sum_{n=1}^{\infty}\frac{\sum_{d|n}A(q_{N-2},\dots,q_1,d)d^s\,\overline{\chi^*}(n/d)\mu(n/d)(n/d)^{2s-1}}{n^{2w}}.$$

Hence, we have

$$Z(s,w)=\sum_{n=1}^{\infty}\frac{a_n(s)}{n^{2w}},$$

where

$$a_n(s)=L(s,F\times\chi^*)\sum_{d|n}A(q_{N-2},\dots,q_1,d)d^s\,\overline{\chi^*}(n/d)\mu(n/d)(n/d)^{2s-1}.$$

Here $a_n(s)$ is an analytic function of $s\in\mathbb{C}$, because $L(s,F\times\chi^*)$ is entire. The computation below shows that $a_n(s)$ equals either side of (17) in their respective regions of absolute convergence, up to scaling by a constant $\tau(\overline{\chi^*})$. This proves the analytic continuation of $\mathscr{H}$ and $\mathscr{G}$ as well as their equality.

For $\Re(s) \gg 1$, $\Re(w-s) > 0$, we expand the two L-functions in the numerator of $Z(s, w)$ as Dirichlet series, obtaining

$$\begin{aligned} Z(s,w) &= \frac{1}{L(2w-2s+1, \overline{\chi^*})} \sum_{n,m=1}^{\infty} \frac{A(q_{N-2}, \dots, q_1, n) A(1, \dots, 1, m) \chi^*(m)}{n^{2w-s} m^s} \\ &= \frac{1}{L(2w-2s+1, \overline{\chi^*})} \sum_{n,m=1}^{\infty} \biggl(\frac{\chi^*(m)}{n^{2w-s} m^s} \\ &\qquad \times \sum_{\substack{d_0 d_1 \cdots d_{N-1} = m \\ d_0 | n, d_1 | q_1, \dots, d_{N-2} | q_{N-2}}} A\Bigl(\frac{q_{N-2} d_{N-3}}{d_{N-2}}, \dots, \frac{q_1 d_0}{d_1}, \frac{n d_{N-1}}{d_0} \Bigr) \biggr), \end{aligned}$$

where we have used the Hecke relation (3). We change the variable $n/d_0 \to n$ and combine $h = n d_{N-1}$, giving

$$\begin{aligned} Z(s,w) &= \frac{1}{L(2w-2s+1, \overline{\chi^*})} \sum_{n, d_0, d_{N-1}=1}^{\infty} \sum_{\substack{d_i | q_i \\ i=1,\dots,N-2}} \frac{\chi^*(d_0 \cdots d_{N-1})}{n^{2w-s} d_0^{2w-s} (d_0 \cdots d_{N-1})^s} \\ &\qquad \times A\Bigl(\frac{q_{N-2} d_{N-3}}{d_{N-2}}, \dots, \frac{q_1 d_0}{d_1}, n d_{N-1} \Bigr) \\ &= \frac{1}{L(2w-2s+1, \overline{\chi^*})} \sum_{d_0, h=1}^{\infty} \sum_{\substack{d_i | q_i \\ i=1,\dots,N-2}} \frac{\chi^*(d_0 \cdots d_{N-2})}{d_0^{2w-s} (d_0 \cdots d_{N-2})^s} \\ &\qquad \times A\Bigl(\frac{q_{N-2} d_{N-3}}{d_{N-2}}, \dots, \frac{q_1 d_0}{d_1}, h \Bigr) \frac{\sigma_{2w-2s}(h, \chi^*)}{h^{2w-s}}. \end{aligned}$$

Applying Lemma 2.4, we get

$$\begin{aligned} Z(s,w) &= \tau(\overline{\chi^*})^{-1} \sum_{d_0=1}^{\infty} \sum_{\substack{d_i | q_i \\ i=1,\dots,N-2}} \biggl(\frac{\chi^*(d_0 \cdots d_{N-2})}{d_0^{2w} (d_1 \cdots d_{N-2})^s} \\ &\qquad \times \sum_{h=1}^{\infty} \frac{1}{h^s} A\Bigl(\frac{q_{N-2} d_{N-3}}{d_{N-2}}, \dots, \frac{q_1 d_0}{d_1}, h \Bigr) \sum_{\ell=1}^{\infty} \frac{g(\overline{\chi^*}, \ell c^*, h)}{\ell^{2w-2s+1}} \biggr). \end{aligned}$$

Therefore, defining $\boldsymbol{q}'$ as in (16), we reach

$$\begin{aligned} Z(s,w) &= \tau(\overline{\chi^*})^{-1} \sum_{n=1}^{\infty} \frac{1}{n^{2w}} \sum_{d_1 | q_1, \dots, d_{N-2} | q_{N-2}} \biggl(\frac{\chi^*(d_1 \cdots d_{N-2})}{(d_1 \cdots d_{N-2})^s} \\ &\qquad \times \sum_{d\ell = n} \chi^*(d) H(\boldsymbol{q}'; \ell c^*, \chi^*, s) \biggr). \end{aligned} \tag{18}$$

On the other hand, let us apply the functional (15) to $L(s, F\times\chi^*)$ in $Z(s,w)$, giving

$$Z(s,w)=\frac{G(s)\tau(\chi^*)^N}{c^{*Ns}}\,\frac{L_{\boldsymbol{q}}(2w-s,F)L(1-s,\widetilde{F}\times\overline{\chi^*})}{L(2w-2s+1,\overline{\chi^*})}.$$

Given $\Re(1-s)\gg 1$ and $\Re(2w-s)\gg 1$, we open the expression as a Dirichlet series:

$$\begin{aligned}
&Z(s,w)\\
&=\frac{G(s)\tau(\chi^*)^N c^{*-Ns}}{L(2w-2s+1,\overline{\chi^*})}\sum_{n,m=1}^{\infty}\frac{A(q_{N-2},\dots,q_1,n)A(m,1,\dots,1)\overline{\chi^*}(m)}{n^{2w-s}m^{1-s}}\\
&=\frac{G(s)\tau(\chi^*)^N c^{*-Ns}}{L(2w-2s+1,\overline{\chi^*})}\\
&\quad\times\sum_{n,m=1}^{\infty}\frac{\overline{\chi^*}(m)}{n^{2w-s}m^{1-s}}\sum_{\substack{d_0d_1\cdots d_{N-1}=m\\ d_0|n,d_1|q_1,\dots,d_{N-2}|q_{N-2}}}A\Big(\frac{q_{N-2}d_{N-1}}{d_{N-2}},\dots,\frac{q_1d_2}{d_1},\frac{nd_1}{d_0}\Big)\\
&=\frac{G(s)\tau(\chi^*)^N c^{*-Ns}}{L(2w-2s+1,\overline{\chi^*})}\\
&\quad\times\sum_{n,m=1}^{\infty}\sum_{\substack{d_0d_1\cdots d_{N-1}=m\\ d_0|n,d_1|q_1,\dots,d_{N-2}|q_{N-2}}}\frac{\overline{\chi^*}(d_0d_1\cdots d_{N-1})A\Big(\frac{q_{N-2}d_{N-1}}{d_{N-2}},\dots,\frac{q_1d_2}{d_1},\frac{nd_1}{d_0}\Big)}{(n/d_0)^{2w-s}d_0^{1+2w-2s}(d_1\cdots d_{N-1})^{1-s}},
\end{aligned}$$

where we have combined the Fourier coefficients by the Hecke relation (4). We change the variable $n/d_0\to n$. Then the sum over d_0 cancels with $L(2w-2s+1,\overline{\chi^*})$ in the denominator, giving

$$\begin{aligned}
&Z(s,w)\\
&=\frac{G(s)\tau(\chi^*)^N c^{*-Ns}}{L(2w-2s+1,\overline{\chi^*})}\sum_{n,d_0,d_{N-1}=1}^{\infty}\sum_{\substack{d_i|q_i\\ i=1,\dots,N-2}}A\Big(\frac{q_{N-2}d_{N-1}}{d_{N-2}},\dots,\frac{q_1d_2}{d_1},d_1n\Big)\\
&\qquad\qquad\times\frac{\overline{\chi^*}(d_0d_1\cdots d_{N-1})}{n^{2w-s}d_0^{1+2w-2s}(d_1\cdots d_{N-1})^{1-s}}\\
&=\frac{G(s)\tau(\chi^*)^N}{c^{*Ns}}\sum_{n,d_{N-1}=1}^{\infty}\sum_{\substack{d_i|q_i\\ i=1,\dots,N-2}}\frac{\overline{\chi^*}(d_1\cdots d_{N-1})}{n^{2w-s}(d_1\cdots d_{N-1})^{1-s}}.
\end{aligned}\tag{19}$$

If we denote the right-hand side of (17) by $\tau(\overline{\chi^*})b_n(s)$, our goal is to transform (19) into $R:=\sum_{n=1}^{\infty}b_n(s)n^{-2w}$. But at this point it is easier to start from R. More explicitly, we have

$$R=\tau(\overline{\chi^*})^{-1}\sum_{h=1}^{\infty}\frac{1}{h^{2w}}\sum_{d_1|q_1,\dots,d_{N-2}|q_{N-2}}\biggl(\frac{\chi^*(d_1\cdots d_{N-2})}{(d_1\cdots d_{N-2})^s}\times\sum_{d\ell=h}\chi^*(d)G(\boldsymbol{q}';\ell c^*,\chi^*,s)\biggr). \tag{20}$$

Here $\boldsymbol{q}'$ has been defined in (16). We plug in the definition of $G(\boldsymbol{q}';\ell c^*,\chi^*,s)$ from (10) for $\boldsymbol{q}'$, giving

$$\begin{aligned}G(\boldsymbol{q}';\ell c^*,\chi^*,s)&\\=\frac{G(s)\chi^*(-1)}{c^{*Ns-1}\ell^{(N-2)s}}&\sum_{f_1|\frac{q_1d\ell}{d_1}}\sum_{f_2|\frac{q_1q_2d\ell}{f_1d_2}}\cdots\sum_{f_{N-2}|\frac{q_1\cdots q_{N-2}d\ell}{f_1\cdots f_{N-3}d_{N-2}}}\\&\sum_{n=1}^{\infty}\frac{A(n,f_{N-2},\dots,f_1)}{n^{1-s}f_1f_2\cdots f_{N-2}}\frac{f_1^{(N-1)s}f_2^{(N-2)s}\cdots f_{N-2}^{2s}}{q_1^{(N-2)s}q_2^{(N-3)s}\cdots q_{N-2}^{s}}\frac{(d_1\cdots d_{N-2})^s}{d^{(N-2)s}}\\&\times g(\chi^*,\ell c^*,f_1)g\Bigl(\chi^*,\frac{q_1d\ell c^*}{f_1d_1},f_2\Bigr)\\&\cdots\times g\Bigl(\chi^*,\frac{q_1\cdots q_{N-3}d\ell c^*}{f_1\cdots f_{N-3}d_{N-3}},f_{N-2}\Bigr)g\Bigl(\chi^*,\frac{q_1\cdots q_{N-2}d\ell c^*}{f_1\cdots f_{N-2}d_{N-2}},n\Bigr).\end{aligned}$$

We substitute $G(\boldsymbol{q}';\ell c^*,\chi^*,s)$ with this expression in (20) and change the orders of summation between f_i and d_i. The summations over d and d_i collapse with the repeated use of Lemma 2.5, giving

$$\begin{aligned}R=\tau(\overline{\chi^*})^{-1}\frac{G(s)\chi^*(-1)}{c^{*Ns-1}}\sum_{h=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\substack{h|f_1\\f_1|q_1h}}\sum_{\substack{\frac{q_1h}{f_1}|f_2\\f_2|\frac{q_1q_2h}{f_1}}}\cdots\sum_{\substack{\frac{q_1\cdots q_{N-3}h}{f_1\cdots f_{N-3}}|f_{N-2}\\f_{N-2}|\frac{q_1\cdots q_{N-2}h}{f_1\cdots f_{N-3}}}}\sum_{\frac{q_1\cdots q_{N-2}h}{f_1\cdots f_{N-2}}|n}\frac{\tau(\chi^*)^{N-1}}{h^{2w}}\\\times\overline{\chi^*}\Bigl(\frac{f_1}{h}\Bigr)\overline{\chi^*}\Bigl(\frac{f_1f_2}{hq_1}\Bigr)\cdots\overline{\chi^*}\Bigl(\frac{f_1f_2\cdots f_{N-2}}{hq_1\cdots q_{N-3}}\Bigr)\overline{\chi^*}\Bigl(\frac{f_1f_2\cdots f_{N-2}n}{hq_1\cdots q_{N-2}}\Bigr)\\\times\Bigl(\frac{q_1}{f_1}\Bigr)^{N-2}\Bigl(\frac{q_2}{f_2}\Bigr)^{N-3}\cdots\Bigl(\frac{q_{N-2}}{f_{N-2}}\Bigr)h^{N-1-Ns+2s}\\\times\frac{A(n,f_{N-2},\dots,f_1)}{n^{1-s}f_1\cdots f_{N-2}}\frac{f_1^{(N-1)s}\cdots f_{N-2}^{2s}}{q_1^{(N-2)s}\cdots q_{N-2}^{s}}.\end{aligned}$$

Define $e_1=f_1/h$ and $e_i=(f_1\cdots f_i)/(q_1\cdots q_{i-1}h)$ for $i=2,\dots,N-2$, so that the double conditions under the sums simplify to $e_i|q_i$. Extend this to all positive integers by setting $e_{N-1}=(f_1\cdots f_{N-2}n)/(hq_1\cdots q_{N-2})$. Finally, noting $\tau(\overline{\chi^*})^{-1}=\chi^*(-1)\tau(\chi^*)/c^*$, we get

$$R = \frac{G(s)\tau(\chi^*)^N}{c^{*Ns}} \sum_{h, e_{N-1}=1}^{\infty} \frac{1}{h^{2w-s}} \sum_{\substack{e_i | q_i \\ i=1,\ldots,N-2}} \frac{\overline{\chi^*}(e_1 \cdots e_{N-2} e_{N-1})}{(e_1 \cdots e_{N-1})^{1-s}} \times A\left(\frac{e_{N-1} q_{N-2}}{e_{N-2}}, \ldots, \frac{e_2 q_1}{e_1}, e_1 h\right),$$

which in turn, by (19), equals $Z(s, w)$ as well as (18). We complete the proof by applying the uniqueness theorem for Dirichlet series [Apostol 1976, Theorem 11.3] to the equality between (18) and (20). □

Remark 3.2. The above proof works for $N \geq 3$ but not for $N = 2$. We can prove the Voronoi formula for $\mathrm{SL}(2, \mathbb{Z})$ similarly and easily by considering

$$Z(s, w) = \frac{L(2w - s, F) L(s, F \times \chi^*)}{L(2w - 2s + 1, \overline{\chi^*}) L(2w, \chi^*)}.$$

We have, from the Hecke relations on GL(2),

$$Z(s, w) = \tau(\overline{\chi^*})^{-1} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{A(n)}{n^s} \frac{g(\overline{\chi^*}, \ell c^*, n)}{\ell^{1+2w-2s}},$$

and applying the functional equation for $L(s, F \times \chi^*)$ we have

$$Z(s, w) = \tau(\chi^*) c^{*-2s} G(s) \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{A(n)}{n^{1-s}} \frac{g(\chi^*, \ell c^*, n)}{\ell^{2w}}.$$

Applying the uniqueness theorem for Dirichlet series to the variable w, we get the Voronoi formula with Gauss sums on GL(2).

Proposition 3.3. *Equation* (11) *is equivalent to Theorem 3.1.*

Proof. Construct the following summation:

$$\begin{aligned} T := &\sum_{e_0 | n} \sum_{e_1 | q_1 e_0} \cdots \sum_{e_{N-2} | q_{N-2} e_{N-3}} \frac{\mu(e_0 \cdots e_{N-2}) \chi^*(e_0 \cdots e_{N-2})}{(e_1 \cdots e_{N-2})^s} \times \mathcal{H}\left(\frac{q_1 e_0}{e_1}, \ldots, \frac{q_{N-2} e_{N-3}}{e_{N-2}}; \frac{n}{e_0}, s\right) \\ = &\sum_{e_0 | n} \sum_{e_1 | q_1 e_0} \cdots \sum_{e_{N-2} | q_{N-2} e_{N-3}} \Biggl(\frac{\mu(e_0 \cdots e_{N-2}) \chi^*(e_0 \cdots e_{N-2})}{(e_1 \cdots e_{N-2})^s} \\ &\times \sum_{\substack{d_i | q_i e_{i-1}/e_i \\ i=1,\ldots,N-2}} \frac{\chi^*(d_1 \cdots d_{N-2})}{(d_1 \cdots d_{N-2})^s} \sum_{d_0 | n/e_0} \chi^*(d_0) \\ &\times H\left(\frac{q_1 e_0 d_0}{e_1 d_1}, \ldots, \frac{q_{N-2} e_{N-3} d_{N-3}}{e_{N-2} d_{N-2}}; \frac{n}{e_0 d_0} c^*, \chi^*, s\right)\Biggr). \end{aligned}$$

Change variables $e_i d_i \to a_i$ for $i = 0, \dots, N-2$, and change orders of summation, getting

$$T = \sum_{a_0|n} \sum_{e_0|a_0} \sum_{a_1|q_1 e_0} \sum_{e_1|a_1} \cdots \sum_{a_{N-2}|q_{N-2}e_{N-3}} \sum_{e_{N-2}|a_{N-2}} \frac{\chi^*(a_0 \cdots a_{N-2})}{(a_1 \cdots a_{N-2})^s} \\ \times H\left(\frac{q_1 a_0}{a_1}, \frac{q_2 a_1}{a_2}, \dots, \frac{q_{N-2} a_{N-3}}{a_{N-2}}; \frac{nc^*}{a_0}, \chi^*, s\right) \mu(e_0) \cdots \mu(e_{N-2}).$$

One by one, the Möbius summation over e_i forces $a_i = 1$, and thus we obtain $T = H(\boldsymbol{q}; nc^*, \chi^*, s)$. By Theorem 3.1, we have $\mathscr{H} = \mathscr{G}$, and the same calculations yield $T = G(\boldsymbol{q}; nc^*, \chi^*, s)$. This proves the theorem. $\square$

3B. ***Equivalence between equations* (8) *and* (11).** First we prove a lemma showing that the hyper-Kloosterman sum on the right-hand side of (8) becomes a product of $(N-2)$ Gauss sums after averaging against a Dirichlet character.

Lemma 3.4. *Let χ be a Dirichlet character modulo c which is induced from the primitive character χ^* modulo c^*. Let $\boldsymbol{q} = (q_1, \dots, q_{N-2})$ and $\boldsymbol{d} = (d_1, \dots, d_{N-2})$ be two tuples of positive integers, and assume that all the divisibility conditions in* (1) *are met. Consider the summation*

$$S := \sum_{\substack{a \bmod c \\ (a,c)=1}} \chi(a) \operatorname{Kl}(a, n, c; \boldsymbol{q}, \boldsymbol{d}).$$

The quantity S is zero unless the divisibility conditions

$$d_1 c^* | q_1 c, \quad d_2 c^* \,\Big|\, \frac{q_1 q_2 c}{d_1}, \quad d_3 c^* \,\Big|\, \frac{q_1 q_2 q_3 c}{d_1 d_2}, \quad \dots, \quad d_{N-2} c^* \,\Big|\, \frac{q_1 \cdots q_{N-2} c}{d_1 \cdots d_{N-3}} \tag{21}$$

are satisfied. Under such divisibility conditions, setting $\xi_i := (q_1 \cdots q_i c)/(d_1 \cdots d_i)$, S can be written as a product of Gauss sums:

$$S = g(\chi^*, c, d_1) g(\chi^*, \xi_1, d_2) \cdots g(\chi^*, \xi_{N-3}, d_{N-2}) g(\chi^*, \xi_{N-2}, n).$$

Proof. The divisibility conditions (1) imply

$$d_1 \mid q_1(c, d_1), \quad d_2 \mid q_2(\xi_1, d_2), \quad \dots, \quad d_{N-2} \mid q_{N-2}(\xi_{N-3}, d_{N-2}). \tag{22}$$

We open up the hyper-Kloosterman sum in S. The forthcoming computation is an iterative process. The summation over a yields a Gauss sum, which in turn produces the term $\overline{\chi^*}(x_1)$. Then the summation over x_1 yields another Gauss sum, which produces the term $\overline{\chi^*}(x_2)$, and so on.

First, we sum over a modulo c:

$$\begin{aligned} S &= \sum_{a \bmod c} \chi(a) \sideset{}{^*}\sum_{x_1 \bmod \xi_1} e\Big(\frac{d_1 x_1 a}{c}\Big)\Big(\sideset{}{^*}\sum_{x_2 \bmod \xi_2} e\Big(\frac{d_2 x_2 \overline{x_1}}{\xi_1}\Big) \cdots \Big) \\ &= \sideset{}{^*}\sum_{x_1 \bmod \xi_1} g(\chi^*, c, x_1 d_1)\Big(\sideset{}{^*}\sum_{x_2 \bmod \xi_2} e\Big(\frac{d_2 x_2 \overline{x_1}}{\xi_1}\Big) \cdots \Big). \end{aligned}$$

Now, because $(c, x_1 d_1) = ((c, q_1 c), x_1 d_1) = (c, (q_1 c, x_1 d_1)) = (c, d_1)$, we deduce from Lemma 2.3 that

$$g(\chi^*, c, x_1 d_1) = \overline{\chi^*(x_1)} g(\chi^*, c, d_1).$$

By Lemma 2.3, this Gauss sum is zero unless $c^* \mid c/(c, d_1)$, which implies the first divisibility condition of (21) because, by (22),

$$c^* \,\Big|\, \frac{c}{(c, d_1)} = \frac{d_1}{(c, d_1)} \frac{c}{d_1} \,\Big|\, \frac{q_1 c}{d_1}.$$

Next we sum over x_1. Notice that $\overline{x_1}$ is its multiplicative inverse modulo $q_1 c/d_1$, and hence modulo c^*. This means that $\chi^*(\overline{x_1}) = \overline{\chi^*(x_1)}$. We change variables in the x_1 summation $x_1 \to \overline{x_1}$, and change orders of summation to obtain

$$\begin{aligned} S &= g(\chi^*, c, d_1) \sideset{}{^*}\sum_{x_1 \bmod \xi_1} \overline{\chi^*(x_1)}\Big(\sideset{}{^*}\sum_{x_2 \bmod \xi_2} e\Big(\frac{d_2 x_2 \overline{x_1}}{\xi_1}\Big) \cdots \Big) \\ &= g(\chi^*, c, d_1) \sideset{}{^*}\sum_{x_2 \bmod \xi_2} \sideset{}{^*}\sum_{x_1 \bmod \xi_1} \chi^*(x_1) e\Big(\frac{d_2 x_2 x_1}{\xi_1}\Big)\Big(\sideset{}{^*}\sum_{x_3 \bmod \xi_3} e\Big(\frac{d_3 x_3 \overline{x_2}}{\xi_2}\Big) \cdots \Big) \\ &= g(\chi^*, c, d_1) \sideset{}{^*}\sum_{x_2 \bmod \xi_2} g(\chi^*, \xi_1, d_2 x_2)\Big(\sideset{}{^*}\sum_{x_3 \bmod \xi_3} e\Big(\frac{d_3 x_3 \overline{x_2}}{\xi_2}\Big) \cdots \Big). \end{aligned}$$

Once again, the equalities $(\xi_1, d_2 x_2) = ((\xi_1, d_2 \xi_2), d_2 x_2) = (\xi_1, (d_2 \xi_2, d_2 x_2)) = (\xi_1, d_2)$ imply that we can pull out $\overline{\chi^*}(x_2)$ from the Gauss sum. Then we have

$$S = g(\chi^*, c, d_1) g(\chi^*, \xi_1, d_2) \sideset{}{^*}\sum_{x_2 \bmod \xi_2} \overline{\chi^*(x_2)}\Big(\sideset{}{^*}\sum_{x_3 \bmod \xi_3} e\Big(\frac{d_3 x_3 \overline{x_2}}{\xi_2}\Big) \cdots \Big).$$

The second Gauss sum $g(\chi^*, \xi_1, d_2)$ vanishes unless $c^* \mid \xi_1/(\xi_1, d_2)$ by Lemma 2.3. This in turn implies $c^* \mid \xi_1/(\xi_1, d_2) \mid \xi_2$ by (22), which is the second divisibility condition of (21). We complete the proof after repeating this process $(N-2)$ times. □

Proposition 3.5. *The equations* (8) *and* (11) *are equivalent.*

Proof. Let χ be a Dirichlet character modulo c induced from the primitive Dirichlet character χ^* modulo c^*. Multiply both sides of (8) by $\chi(a)$ and sum over reduced residue classes modulo c. On the left-hand side of (8), one gets

$$\sum_{\substack{a \bmod c \\ (a,c)=1}} \chi(a) L_{\boldsymbol{q}}(s, F, a/c) = (c/c^*)^{1-2s} H(\boldsymbol{q}; c, \chi^*, s),$$

whereas on the right-hand side of (8), one obtains $(c/c^*)^{1-2s} G(\boldsymbol{q}; c, \chi^*, s)$ by making use of Lemma 3.4 and the fact that

$$g(\chi^*, \xi_{N-2}, -n) = \pm g(\chi^*, \xi_{N-2}, n),$$

depending on whether $\chi(-1)$ is 1 or -1. This shows that (8) implies (11).

Conversely, if we multiply both sides of (11) by $\overline{\chi(a)}/\phi(c)$ and sum over all Dirichlet characters (both primitive and nonprimitive) modulo c, we obtain (8) by using the orthogonality relation for Dirichlet characters. Since both of the aforementioned summations that shuttle between (8) and (11) are finite, the properties of absolute convergence and analytic continuation are preserved. $\square$

Acknowledgements

The authors thank Dorian Goldfeld and Wenzhi Luo for helpful suggestions, Matthew Young for his extensive help in organization of the manuscript and his encouragement, and Jeffrey Hoffstein, in whose class the seed for this work originated.

References

[Apostol 1976] T. M. Apostol, *Introduction to analytic number theory*, Springer, New York, 1976. MR Zbl

[Blomer 2012] V. Blomer, "Subconvexity for twisted L-functions on GL(3)", *Amer. J. Math.* **134**:5 (2012), 1385–1421. MR Zbl

[Blomer et al. 2013] V. Blomer, R. Khan, and M. Young, "Distribution of mass of holomorphic cusp forms", *Duke Math. J.* **162**:14 (2013), 2609–2644. MR Zbl

[Bump 1984] D. Bump, *Automorphic forms on* $\mathrm{GL}(3, \mathbb{R})$, Lecture Notes in Mathematics **1083**, Springer, Berlin, 1984. MR Zbl

[Buttcane and Khan 2015] J. Buttcane and R. Khan, "L^4-norms of Hecke newforms of large level", *Math. Ann.* **362**:3-4 (2015), 699–715. MR Zbl

[Duke and Iwaniec 1990] W. Duke and H. Iwaniec, "Estimates for coefficients of L-functions, I", pp. 43–47 in *Automorphic forms and analytic number theory* (Montreal, 1989), edited by R. Murty, Univ. Montréal, 1990. MR Zbl

[Godement and Jacquet 1972] R. Godement and H. Jacquet, *Zeta functions of simple algebras*, Lecture Notes in Mathematics **260**, Springer, Berlin, 1972. MR Zbl

[Goldfeld 2006] D. Goldfeld, *Automorphic forms and L-functions for the group* $\mathrm{GL}(n, \mathbb{R})$, Cambridge Studies in Advanced Mathematics **99**, Cambridge University Press, 2006. MR Zbl

[Goldfeld and Li 2006] D. Goldfeld and X. Li, "Voronoi formulas on $\mathrm{GL}(n)$", *Int. Math. Res. Not.* **2006** (2006), art. id 86295. MR Zbl

[Goldfeld and Li 2008] D. Goldfeld and X. Li, "The Voronoi formula for $\mathrm{GL}(n, \mathbb{R})$", *Int. Math. Res. Not.* **2008**:2 (2008), art. id rnm144. MR Zbl

[Good 1981] A. Good, “Cusp forms and eigenfunctions of the Laplacian”, *Math. Ann.* **255**:4 (1981), 523–548. MR Zbl

[Ichino and Templier 2013] A. Ichino and N. Templier, “On the Voronoĭ formula for $\mathrm{GL}(n)$”, *Amer. J. Math.* **135**:1 (2013), 65–101. MR Zbl

[Jacquet et al. 1983] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, “Rankin–Selberg convolutions”, *Amer. J. Math.* **105**:2 (1983), 367–464. MR Zbl

[Khan 2012] R. Khan, “Simultaneous non-vanishing of $\mathrm{GL}(3) \times \mathrm{GL}(2)$ and $\mathrm{GL}(2)$ L-functions”, *Math. Proc. Cambridge Philos. Soc.* **152**:3 (2012), 535–553. MR Zbl

[Kowalski and Ricotta 2014] E. Kowalski and G. Ricotta, “Fourier coefficients of $\mathrm{GL}(N)$ automorphic forms in arithmetic progressions”, *Geom. Funct. Anal.* **24**:4 (2014), 1229–1297. MR Zbl

[Li 2009] X. Li, “The central value of the Rankin–Selberg L-functions”, *Geom. Funct. Anal.* **18**:5 (2009), 1660–1695. MR Zbl

[Li 2011] X. Li, “Bounds for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ L-functions and $\mathrm{GL}(3)$ L-functions”, *Ann. of Math.* (2) **173**:1 (2011), 301–336. MR Zbl

[Li and Young 2012] X. Li and M. P. Young, “The L^2 restriction norm of a GL_3 Maass form”, *Compos. Math.* **148**:3 (2012), 675–717. MR Zbl

[Miller 2006] S. D. Miller, “Cancellation in additively twisted sums on $\mathrm{GL}(n)$”, *Amer. J. Math.* **128**:3 (2006), 699–729. MR Zbl

[Miller and Schmid 2004] S. D. Miller and W. Schmid, “Summation formulas, from Poisson and Voronoi to the present”, pp. 419–440 in *Noncommutative harmonic analysis*, edited by P. Delorme and M. Vergne, Progress in Mathematics **220**, Birkhäuser, Boston, 2004. MR Zbl

[Miller and Schmid 2006] S. D. Miller and W. Schmid, “Automorphic distributions, L-functions, and Voronoi summation for $\mathrm{GL}(3)$”, *Ann. of Math.* (2) **164**:2 (2006), 423–488. MR Zbl

[Miller and Schmid 2011] S. D. Miller and W. Schmid, “A general Voronoi summation formula for $\mathrm{GL}(n, \mathbb{Z})$”, pp. 173–224 in *Geometry and analysis, No. 2*, edited by L. Ji, Advanced Lectures in Mathematics **18**, Int. Press, Somerville, MA, 2011. MR Zbl

[Miyake 1989] T. Miyake, *Modular forms*, Springer, Berlin, 1989. MR Zbl

[Montgomery and Vaughan 2007] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory, I: Classical theory*, Cambridge Studies in Advanced Mathematics **97**, Cambridge University Press, 2007. MR Zbl

[Munshi 2013] R. Munshi, “Shifted convolution sums for $\mathrm{GL}(3) \times \mathrm{GL}(2)$”, *Duke Math. J.* **162**:13 (2013), 2345–2362. MR Zbl

[Munshi 2015] R. Munshi, “The circle method and bounds for L-functions, IV: Subconvexity for twists of $\mathrm{GL}(3)$ L-functions”, *Ann. of Math.* (2) **182**:2 (2015), 617–672. MR Zbl

[Shimura 1975] G. Shimura, “On the holomorphy of certain Dirichlet series”, *Proc. London Math. Soc.* (3) **31**:1 (1975), 79–98. MR Zbl

[Zhou 2014] F. Zhou, “Weighted Sato–Tate vertical distribution of the Satake parameter of Maass forms on $\mathrm{PGL}(N)$”, *Ramanujan J.* **35**:3 (2014), 405–425. MR Zbl

[Zhou 2016] F. Zhou, “Voronoi summation formulae on $\mathrm{GL}(n)$”, *J. Number Theory* **162** (2016), 483–495. MR Zbl

Communicated by Philippe Michel
Received 2016-05-08 Revised 2016-07-19 Accepted 2016-09-23

ekiral@math.tamu.edu *Department of Mathematics, Texas A&M University, College Station, TX 77843, United States*

zhou.1406@math.osu.edu *Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210, United States*

mathematical sciences publishers msp

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2016 is US $290/year for the electronic version, and $485/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 10 No. 10 2016
