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Presentation of affine Kac–Moody groups over rings

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Tits has defined Steinberg groups and Kac–Moody groups for any root system and any commutative ring R. We establish a Curtis–Tits-style presentation for the Steinberg group $\mathfrak{St}$ of any irreducible affine root system with rank ≥ 3, for any R. Namely, $\mathfrak{St}$ is the direct limit of the Steinberg groups coming from the 1- and 2-node subdiagrams of the Dynkin diagram. In fact, we give a completely explicit presentation. Using this we show that $\mathfrak{St}$ is finitely presented if the rank is ≥ 4 and R is finitely generated as a ring, or if the rank is 3 and R is finitely generated as a module over a subring generated by finitely many units. Similar results hold for the corresponding Kac–Moody groups when R is a Dedekind domain of arithmetic type.

1. Introduction

Suppose R is a commutative ring and A is one of the *ABCDEFG* Dynkin diagrams, or equivalently its Cartan matrix. Steinberg [1968] defined what is now called the Steinberg group $\mathfrak{St}_A(R)$, by generators and relations. It plays a central role in K-theory and some aspects of Lie theory.

Kac–Moody algebras are infinite-dimensional generalizations of the semisimple Lie algebras. When $R = \mathbb{R}$ and A is an affine Dynkin diagram, the corresponding Kac–Moody group is a central extension of the loop group of a finite-dimensional Lie group. For a general ring R and any generalized Cartan matrix A, the definition of a Kac–Moody group is due to Tits [1987]. A difficulty in tracing the story is that Tits began by defining a "Steinberg group" which unfortunately differs from Steinberg's original group when A has an A_1 component. This was resolved by Morita and Rehmann [1990] by adding extra relations to Tits' definition. So there are two definitions of the Steinberg group. Increasing the chance of confusion, they agree for most A of interest, including the irreducible affine diagrams of rank ≥ 3. We follow Morita and Rehmann, so the Steinberg group $\mathfrak{St}_A(R)$ reduces to

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Steinberg's original group when this is defined. See Section 3 for further background on $\mathfrak{St}$.

Tits then defined another functor $R \mapsto \tilde{\mathfrak{G}}_A(R)$ as a quotient of his version of the Steinberg group. In this paper we will omit the tilde and refer to $\mathfrak{G}_A(R)$ as the Kac–Moody group of type A over R. The relations added by Morita and Rehmann to the definition of $\mathfrak{St}_A(R)$ are among the relations that Tits imposed in his definition of $\mathfrak{G}_A(R)$. Therefore, we may regard $\mathfrak{G}_A(R)$ as a quotient of $\mathfrak{St}_A(R)$, just as Tits did, even though our $\mathfrak{St}_A(R)$ is not quite the same as his. (Tits actually defined $\tilde{\mathfrak{G}}_D(R)$ where D is a root datum; by $\mathfrak{G}_A(R)$ we refer to the root datum whose generalized Cartan matrix is A and which is "simply connected in the strong sense" [Tits 1987, p. 551]. The general case differs from this one by enlarging or shrinking the center of $\tilde{\mathfrak{G}}_D(R)$.) See Section 3 for further background on $\mathfrak{G}$.

The meaning of "Kac–Moody group" is far from standardized. Tits [1987] wrote down axioms (KMG1)–(KMG9) that one could demand of a functor from rings to groups before calling it a Kac–Moody functor. He showed in [loc. cit., Theorem $1'$] that any such functor admits a natural homomorphism from $\mathfrak{G}_A$, which is an isomorphism at every field. So Kac–Moody groups over fields are well-defined, and over general rings $\mathfrak{G}_A$ approximates the yet unknown ultimate definition. This is why we refer to $\mathfrak{G}_A$ as the Kac–Moody group. But $\mathfrak{G}_A$ does not quite satisfy Tits' axioms, so ultimately some other language may be better. See Section 6 for more on this.

The purpose of this paper is to simplify Tits' presentations of $\mathfrak{St}_A(R)$ and $\mathfrak{G}_A(R)$ when A is an affine Dynkin diagram of rank (number of nodes) ≥ 3. We will always take affine diagrams to be irreducible. We will show that $\mathfrak{St}_A(R)$ and $\mathfrak{G}_A(R)$ are finitely presented under quite weak hypotheses on R. This is surprising because there is no obvious reason for an infinite-dimensional group over (say) $\mathbb{Z}$ to be finitely presented, and Tits' presentations are "very" infinite. His generators are indexed by all pairs (root, ring element), and his relations specify the commutators of many pairs of these generators. Subtle implicitly defined coefficients appear throughout his relations.

The main step in proving our finite presentation results is to first establish smaller, and more explicit, presentations for $\mathfrak{St}_A(R)$ and $\mathfrak{G}_A(R)$. These presentations are not necessarily finite, but they do apply to all R. In [Allcock 2015] we wrote down a presentation for a group functor we called the pre-Steinberg group $\mathfrak{PSt}_A$. We have reproduced it in Section 2, for any generalized Cartan matrix A. The generators are S_i and $X_i(t)$ with i varying over the nodes of the Dynkin diagram and t varying over R. The relations are (2-1)–(2-28), but (2-27)–(2-28) may be omitted when A is 2-spherical (it has no edges labeled "∞") and has no A_1 components. This case includes all affine diagrams of rank ≥ 3. The only way the presentation fails to be finite is that the $X_i(t)$ are parameterized by elements of R, and each Chevalley relation is parameterized by pairs of elements of R.

The name "pre-Steinberg group" reflects the fact that there is a natural map from $\mathfrak{PSt}_A(R)$ to the Steinberg group $\mathfrak{St}_A(R)$. In Section 3 we will describe this in a conceptual manner. But in terms of presentations it suffices to say that our $X_i(t)$ and S_i map to the group elements $x_{\alpha_i}(t)$ and $\hat{w}_{\alpha_i}(1)$ in Morita and Rehmann's definition of $\mathfrak{St}_A(R)$ [1990, §2]. Our general philosophy is that $\mathfrak{PSt}_A(R)$ is interesting only as a means of approaching $\mathfrak{St}_A(R)$, as in the following theorem, which is our main result.

Theorem 1.1 (presentation of affine Steinberg and Kac–Moody groups). *Suppose A is an affine Dynkin diagram of rank ≥ 3 and R is a commutative ring. Then the natural map from the pre-Steinberg group $\mathfrak{PSt}_A(R)$ to the Steinberg group $\mathfrak{St}_A(R)$ is an isomorphism. In particular, $\mathfrak{St}_A(R)$ has a presentation with generators S_i and $X_i(t)$, with i varying over the simple roots and t over R, and relations* (2-1)–(2-26).

One obtains Tits' Kac–Moody group $\mathfrak{G}_A(R)$ by adjoining the relations

$$\tilde{h}_i(u)\tilde{h}_i(v) = \tilde{h}_i(uv) \tag{1-1}$$

for all simple roots i and all units u, v of R, where

$$\tilde{h}_i(u) := \tilde{s}_i(u)\tilde{s}_i(-1), \quad \tilde{s}_i(u) := X_i(u)S_iX_i(1/u)S_i^{-1}X_i(u).$$

We remark that if A is a spherical diagram (that is, its Weyl group is finite) then it follows immediately from an alternate description of $\mathfrak{PSt}_A$ that $\mathfrak{PSt}_A \to \mathfrak{St}_A$ is an isomorphism; see Section 3 or [Allcock 2015, §7]. So Theorem 1.1 extends the isomorphism $\mathfrak{PSt}_A \cong \mathfrak{St}_A$ from the spherical case to the affine case, except for the two affine diagrams of rank 2. See [Allcock and Carbone 2016] for a further extension, to the simply laced hyperbolic case, and [Allcock 2015] for generalizations beyond the hyperbolic case.

For a moment we return to the case where A is an arbitrary generalized Cartan matrix. If $B_1 \subseteq B_2$ are two subdiagrams of A then there is a natural homomorphism $\mathfrak{PSt}_{B_1}(R) \to \mathfrak{PSt}_{B_2}(R)$. This is because the generators and relations of $\mathfrak{PSt}_{B_1}(R)$ are among those of $\mathfrak{PSt}_{B_2}(R)$, by the fact that our presentations of these groups are defined in terms of the nodes and edges of these subdiagrams of A. Using these maps, we consider the directed system of groups $\mathfrak{PSt}_B(R)$, where B varies over the subdiagrams of A of rank ≤ 2. It is a formality that the direct limit is $\mathfrak{PSt}_A(R)$; this is just an abstract way of saying that each generator or relation of $\mathfrak{PSt}_A(R)$ already appears in the presentation of some $\mathfrak{PSt}_B(R)$ with B of rank ≤ 2.

When A is affine of rank ≥ 3, $\mathfrak{PSt}_A(R) \to \mathfrak{St}_A(R)$ is an isomorphism by Theorem 1.1. And $\mathfrak{PSt}_B(R) \to \mathfrak{St}_B(R)$ is an isomorphism for every proper subdiagram B of A, since such subdiagrams are spherical. It follows that we may replace $\mathfrak{PSt}$ by $\mathfrak{St}$ throughout the preceding paragraph, proving the following result. The point is that affine Steinberg groups of rank ≥ 3 are built up from the classical Steinberg groups of types A_1, A_1^2, A_2, B_2 and G_2.

Corollary 1.2 (Curtis–Tits presentation). *Suppose A is an affine Dynkin diagram of rank ≥ 3 and R is a commutative ring. Then $\mathfrak{St}_A(R)$ is the direct limit of the groups $\mathfrak{St}_B(R)$, where B varies over the subdiagrams of A of rank ≤ 2, and the maps between these groups are as specified above. The same result also holds with $\mathfrak{St}$ replaced by $\mathfrak{G}$ throughout.* □

An informal way to restate Corollary 1.2 is that a presentation for $\mathfrak{St}_A(R)$ can be got by amalgamating one's favorite presentations for the $\mathfrak{St}_B(R)$. Splitthoff [1986] discovered quite weak sufficient conditions for the latter groups to be finitely presented. When these hold, one would therefore expect $\mathfrak{St}_A(R)$ also to be finitely presented. The next theorem expresses this idea precisely. Claim (ii) is part of [Allcock 2015, Theorem 1.4]. See Section 6 for the proof of claim (i).

Theorem 1.3 (finite presentability). *Suppose A is an affine Dynkin diagram and R is any commutative ring. Then the Steinberg group $\mathfrak{St}_A(R)$ is finitely presented as a group if either*

(i) $\operatorname{rk} A > 3$ *and R is finitely generated as a ring*, *or*

(ii) $\operatorname{rk} A = 3$ *and R is finitely generated as a module over a subring generated by finitely many units.*

In either case, *if the unit group of R is finitely generated as an abelian group*, *then Tits' Kac–Moody group $\mathfrak{G}_A(R)$ is finitely presented as a group.*

One of the main motivations for Splitthoff's work was to understand when the Chevalley–Demazure groups, over Dedekind domains of interest in number theory, are finitely presented. This was finally settled by Behr [1967; 1998], capping a long series of works by many authors. The following analogue of these results follows immediately from Theorem 1.3. How close the analogy is depends on how well $\mathfrak{G}_A$ approximates whatever plays the role of the Chevalley–Demazure group scheme in the setting of Kac–Moody theory.

Corollary 1.4 (finite presentation in arithmetic contexts). *Suppose K is a global field*, *meaning a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$. Suppose S is a nonempty finite set of places of K*, *including all infinite places in the number field case. Let R be the ring of S-integers in K.*

Suppose A is an affine Dynkin diagram. Then Tits' Kac–Moody group $\mathfrak{G}_A(R)$ is finitely presented if $\operatorname{rk} A \geq 3$, *unless K is a function field and $|S| = 1$*, *when* $\operatorname{rk} A > 3$ *suffices.* □

We remark that if R is a field then the $\mathfrak{G}_A$ case of Corollary 1.2 is due to Abramenko and Mühlherr [1997] (see also [Devillers and Mühlherr 2007]). Namely, suppose A is any generalized Cartan matrix which is 2-spherical, and that R is a field (but not $\mathbb{F}_2$ if A has a double bond, and neither $\mathbb{F}_2$ nor $\mathbb{F}_3$ if A has a triple bond). Then $\mathfrak{G}_A(R)$ is the direct limit of the groups $\mathfrak{G}_B(R)$. Abramenko and Mühlherr

[1997, p. 702] state that if A is affine then one can remove the restrictions $R \neq \mathbb{F}_2, \mathbb{F}_3$.

One of our goals is to bring Kac–Moody groups into the world of geometric and combinatorial group theory, which mostly addresses finitely presented groups. For example, which Kac–Moody groups admit classifying spaces with finitely many cells below some chosen dimension? What other finiteness properties do they have? Do they have Kazhdan's property T? What isoperimetric inequalities do they satisfy in various dimensions? Are there (nonsplit) Kac–Moody groups over local fields whose uniform lattices (suitably defined) are word hyperbolic? Are some Kac–Moody groups (or classes of them) quasi-isometrically rigid? We find the last question very attractive, since the corresponding answer for lattices in Lie groups is deep (see [Eskin and Farb 1997; Farb and Schwartz 1996; Kleiner and Leeb 1997; Schwartz 1995]).

Regarding property T we would like to mention work of Hartnick and Köhl [2015] who showed that many Kac–Moody groups over local fields have property T when equipped with the Kac–Peterson topology. Also, Shalom [1999] and Neuhauser [2003] respectively showed that the loop groups of (i.e., the spaces of continuous maps from S^1 to) $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ have property T.

2. Presentation of the pre-Steinberg group $\mathfrak{PSt}_A(R)$

Suppose R is any commutative ring and A is any generalized Cartan matrix. Write I for the set of A's nodes, and for $i, j \in I$ write m_{ij} for the order of the product of the corresponding generators of the Weyl group. Following [Allcock 2015, §7], the pre-Steinberg group $\mathfrak{PSt}_A(R)$ is defined by the following presentation. The generators are S_i and $X_i(t)$ with $t \in R$. The relations are (2-1)–(2-28) below, in which i, j vary over I and t, u vary over R. We use the notation $Y \rightleftarrows Z$ to say that Y and Z commute.

If A has no A_1 components and is 2-spherical (all m_{ij} are finite), then the last two relations (2-27)–(2-28) follow from the others and may be omitted [Allcock 2015, Remark 7.13]. If A is affine of rank ≥ 3 then it satisfies this condition, and our main result (Theorem 1.1) is that the presentation equally well defines the Steinberg group $\mathfrak{St}_A(R)$.

For every $i \in I$ we impose the relations

$$X_i(t)X_i(u) = X_i(t+u), \tag{2-1}$$

$$S_i = X_i(1)S_iX_i(1)S_i^{-1}X_i(1). \tag{2-2}$$

For all i, j we impose the relations

$$S_i^2 S_j S_i^{-2} = S_j^{\varepsilon}, \tag{2-3}$$

$$S_i^2 X_j(t) S_i^{-2} = X_j(\varepsilon t), \tag{2-4}$$

where $\varepsilon = (-1)^{A_{ij}}$.

Whenever $m_{ij} = 2$ we impose the relations

$$S_i S_j = S_j S_i, \tag{2-5}$$
$$S_i \rightleftarrows X_j(t), \tag{2-6}$$
$$X_i(t) \rightleftarrows X_j(u). \tag{2-7}$$

Whenever $m_{ij} = 3$ we impose the relations

$$S_i S_j S_i = S_j S_i S_j, \tag{2-8}$$
$$S_j S_i X_j(t) = X_i(t) S_j S_i, \tag{2-9}$$
$$X_i(t) \rightleftarrows S_i X_j(u) S_i^{-1}, \tag{2-10}$$
$$[X_i(t), X_j(u)] = S_i X_j(tu) S_i^{-1}. \tag{2-11}$$

Whenever $m_{ij} = 4$ we impose the following relations; in (2-14)–(2-17), s (resp. l) refers to whichever of i and j is the shorter (resp. longer) root:

$$S_i S_j S_i S_j = S_j S_i S_j S_i, \tag{2-12}$$
$$S_i S_j S_i \rightleftarrows X_j(t), \tag{2-13}$$
$$S_s X_l(t) S_s^{-1} \rightleftarrows S_l X_s(u) S_l^{-1}, \tag{2-14}$$
$$X_l(t) \rightleftarrows S_s X_l(u) S_s^{-1}, \tag{2-15}$$
$$[X_s(t), S_l X_s(u) S_l^{-1}] = S_s X_l(-2tu) S_s^{-1}, \tag{2-16}$$
$$[X_s(t), X_l(u)] = S_l X_s(-tu) S_l^{-1} \cdot S_s X_l(t^2 u) S_s^{-1}. \tag{2-17}$$

Whenever $m_{ij} = 6$ we impose the following relations; s and l have the same meaning they had in the previous paragraph:

$$S_i S_j S_i S_j S_i S_j = S_j S_i S_j S_i S_j S_i, \tag{2-18}$$
$$S_i S_j S_i S_j S_i \rightleftarrows X_j(t), \tag{2-19}$$
$$X_l(t) \rightleftarrows S_l S_s X_l(u) S_s^{-1} S_l^{-1}, \tag{2-20}$$
$$S_s S_l X_s(t) S_l^{-1} S_s^{-1} \rightleftarrows S_l S_s X_l(u) S_s^{-1} S_l^{-1}, \tag{2-21}$$
$$S_s X_l(t) S_s^{-1} \rightleftarrows S_l X_s(u) S_l^{-1}, \tag{2-22}$$
$$[X_l(t), S_s X_l(u) S_s^{-1}] = S_l S_s X_l(tu) S_s^{-1} S_l^{-1}, \tag{2-23}$$
$$[X_s(t), S_s S_l X_s(u) S_l^{-1} S_s^{-1}] = S_s X_l(3tu) S_s^{-1}, \tag{2-24}$$
$$\begin{aligned}[X_s(t), S_l X_s(u) S_l^{-1}] = S_s S_l X_s(-2tu) S_l^{-1} S_s^{-1} \cdot S_s X_l(-3t^2 u) S_s^{-1} \\ \cdot S_l S_s X_l(-3tu^2) S_s^{-1} S_l^{-1},\end{aligned} \tag{2-25}$$
$$\begin{aligned}[X_s(t), X_l(u)] = S_s S_l X_s(t^2 u) S_l^{-1} S_s^{-1} \cdot S_l X_s(-tu) S_l^{-1} \\ \cdot S_s X_l(t^3 u) S_s^{-1} \cdot S_l S_s X_l(-t^3 u^2) S_s^{-1} S_l^{-1}.\end{aligned} \tag{2-26}$$

This paper	[Allcock 2015]
(2-1)	(7.4)
(2-2)	(7.26)
(2-3)	(7.2)–(7.3)
(2-4)	(7.5)
(2-5) ∪ (2-8) ∪ (2-12) ∪ (2-18)	(7.1)
(2-6)	(7.6)
(2-7)	(7.10), the A_1^2 Chevalley relation
(2-9)	(7.7)
(2-10)–(2-11)	(7.11)–(7.12), the A_2 Chevalley relations
(2-13)	(7.8)
(2-14)–(2-17)	(7.13)–(7.16), the B_2 Chevalley relations
(2-19)	(7.9)
(2-20)–(2-26)	(7.17)–(7.23), the G_2 Chevalley relations
(2-27)	(7.24)
(2-28)	(7.25)

Table 1. Correspondence between our relations and those of [Allcock 2015].

Officially, the next two relations are part of the presentation of $\mathfrak{PSt}_A(R)$. But as mentioned above, they may be omitted if A is 2-spherical without A_1 components. We let r vary over the units of R and impose the relations

$$\tilde{h}_i(r) X_j(t) \tilde{h}_i(r)^{-1} = X_j(r^{A_{ij}} t), \tag{2-27}$$

$$\tilde{h}_i(r)\, S_j X_j(t) S_j^{-1}\, \tilde{h}_i(r)^{-1} = S_j X_j(r^{-A_{ij}} t) S_j^{-1}, \tag{2-28}$$

where $\tilde{h}_i(r)$ is defined in Theorem 1.1.

Because we have organized the relations differently than we did in [Allcock 2015], we will state the correspondence explicitly. See Table 1.

3. Steinberg and pre-Steinberg groups

Our goal in this section is to describe the Steinberg group and to give a second description of the pre-Steinberg group. This description makes visible the latter group's natural map to the Steinberg group, and is the form we will use for our calculations in Section 5.

We work in the setting of [Tits 1987] and [Allcock 2015], so R is a commutative ring and A is a generalized Cartan matrix. This matrix determines a complex Lie algebra $\mathfrak{g}$ called the Kac–Moody algebra, and we write Φ for the set of real roots of $\mathfrak{g}$. For each real root α, its root space $\mathfrak{g}_\alpha$ comes with a distinguished pair of (complex vector space) generators, each the negative of the other. We write $\mathfrak{g}_{\alpha,\mathbb{Z}}$ for their integral span, and we define the root group $\mathfrak{U}_\alpha$ as $\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes R \cong R$. Tits' definition of the Steinberg group begins with the free product $*_{\alpha\in\Phi}\, \mathfrak{U}_\alpha$.

We emphasize that there is no natural way to choose an isomorphism $R \to \mathfrak{U}_\alpha$. If $\pm e$ are the two distinguished generators for $\mathfrak{g}_\alpha$, then there are two natural choices for the parameterization of $\mathfrak{U}_\alpha$, namely $t \mapsto (\pm e) \otimes t$. Often we will choose one of these and call it X_α; we speak of this as a "sign choice". Making such a choice makes computations more concrete, but breaks the symmetry.

In Tits' definition of $\mathfrak{St}_A(R)$, the relations have the following form. He calls a pair $\alpha, \beta \in \Phi$ *prenilpotent* if some element of the Weyl group W sends both α, β to positive roots, and some other element of W sends both to negative roots. A consequence of this condition is that every root in $\mathbb{N}\alpha + \mathbb{N}\beta$ is real, which enables Tits to write down Chevalley-style relators for α, β. That is, for every prenilpotent pair α, β he imposes relations of the form

$$[\text{element of } \mathfrak{U}_\alpha, \text{element of } \mathfrak{U}_\beta] = \prod_{\gamma \in \theta(\alpha,\beta) - \{\alpha,\beta\}} (\text{element of } \mathfrak{U}_\gamma), \tag{3-1}$$

where $\theta(\alpha, \beta) := (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Phi$ and $\mathbb{N} = \{0, 1, 2, \dots\}$. The exact relations are given in a rather implicit form in [Tits 1987, §3.6]. Writing them down explicitly requires choosing parameterizations of $\mathfrak{U}_\alpha$, $\mathfrak{U}_\beta$ and each $\mathfrak{U}_\gamma$. We suppose this has been done as above, with the parameterizations being X_α, X_β and the various X_γ. Then the relations take the form

$$[X_\alpha(t), X_\beta(u)] = \prod_{\substack{\text{roots } \gamma = m\alpha + n\beta \\ \text{with } m,n \geq 1}} X_\gamma\big(N_{\alpha\beta\gamma} t^m u^n\big), \tag{3-2}$$

where the $N_{\alpha\beta\gamma}$ are integers determined by the structure constants of $\mathfrak{g}$, the sign choices made in parameterizing the root groups, and the ordering of the terms on the right side. See [Tits 1987, §§3.4–3.6] for details, or Section 5 for the cases we will need. Morita [1987; 1988] showed that the right side has at most 1 term except when $(\mathbb{Q}\alpha \oplus \mathbb{Q}\beta) \cap \Phi$ has type B_2 or G_2, and found simple formulas for the constants (up to sign).

For Tits, this is the end of the definition of the Steinberg group. We called this group $\mathfrak{St}_A^{\mathrm{Tits}}(R)$ in [Allcock 2015] to avoid confusion with $\mathfrak{St}_A(R)$ itself, which we take to also satisfy the Morita–Rehmann relations. These extra relations play the role of making the "maximal torus" and "Weyl group" in $\mathfrak{St}_A^{\mathrm{Tits}}(R)$ act in the expected way on root spaces. These relations follow from the Chevalley relations when A is 2-spherical without A_1 components, so the reader could skip down to the definition of the Kac–Moody group $\mathfrak{G}_A(R)$.

Here is a terse description of the Morita–Rehmann relations; see [Morita and Rehmann 1990, Relations (B′)] or [Allcock 2015, §6] for details. For each simple root $\alpha \in \Phi$ and each of the two choices e for a generator of $\mathfrak{g}_{\alpha,\mathbb{Z}}$, we impose relations as follows. By a standard construction, the choice of e distinguishes a generator f for $\mathfrak{g}_{-\alpha,\mathbb{Z}}$. Using e and f as above, we obtain parameterizations of $\mathfrak{U}_\alpha$ and $\mathfrak{U}_{-\alpha}$

which we will call X_e and X_f. For $r \in R^*$ we define $\tilde{s}_e(r) = X_e(r)X_f(1/r)X_e(r)$ and $\tilde{h}_e(r) = \tilde{s}_e(r)\tilde{s}_e(-1)$. Morita and Rehmann impose relations that describe the actions of $\tilde{s}_e(1)$ and $\tilde{h}_e(r)$ on every $\mathfrak{U}_\beta$, where β varies over Φ. First, conjugation by $\tilde{s}_e(1)$ sends $\mathfrak{U}_\beta$ to $\mathfrak{U}_{s_\alpha(\beta)}$ in the same way that $s_e^* := (\exp \operatorname{ad}_e)(\exp \operatorname{ad}_f)(\exp \operatorname{ad}_e) \in \operatorname{Aut} \mathfrak{g}$ does. (Here s_α is the reflection in α, and for the relation to make sense one must check that s_e^* sends $\mathfrak{g}_{\beta,\mathbb{Z}}$ to $\mathfrak{g}_{s_\alpha(\beta),\mathbb{Z}}$.) Second, every $\tilde{h}_e(r)$ acts on $\mathfrak{U}_\beta \cong R$ by scaling by $r^{\langle \alpha^\vee, \beta \rangle}$, where $\alpha^\vee$ is the coroot associated to α.

The quotient of $\mathfrak{St}_A^{\text{Tits}}(R)$ by all these relations is the definition of the Steinberg group $\mathfrak{St}_A(R)$ and agrees with Steinberg's original group when A is spherical. We remark that we let e vary over both possible choices of generator for $\mathfrak{g}_{\alpha,\mathbb{Z}}$ just to avoid choosing one. But one could choose one without harm, because it turns out that the relations imposed for e are the same as those imposed for $-e$. Also, Morita and Rehmann write $\hat{w}_\alpha$ rather than $\tilde{s}_e$, and their definition of it uses $X_f(-1/r)$ rather than $X_f(1/r)$. This sign just reflects the fact that they use a different sign on f than Tits does, in the "standard" basis e, f, h for $\mathfrak{sl}_2$.

The Kac–Moody group $\mathfrak{G}_A(R)$ is defined as the quotient of $\mathfrak{St}_A(R)$ by the relations (1-1).

In Section 2 we defined the pre-Steinberg group $\mathfrak{PSt}_A(R)$ in terms of generators and relations. It also has an "intrinsic" definition: the same as $\mathfrak{St}_A(R)$, except that Tits' Chevalley relations are imposed only for *classically nilpotent* pairs α, β. This means $(\mathbb{Q}\alpha+\mathbb{Q}\beta)\cap\Phi$ is finite and $\alpha+\beta$ is nonzero, which is equivalent to α, β satisfying $\alpha+\beta \neq 0$ and lying in some A_1, A_1^2, A_2, B_2 or G_2 root system. As the name suggests, such a pair is prenilpotent. So $\mathfrak{PSt}_A(R)$ is defined the same way as $\mathfrak{St}_A(R)$, just omitting the Chevalley relations for prenilpotent pairs that are not classically prenilpotent. In particular, $\mathfrak{St}_A(R)$ is a quotient of $\mathfrak{PSt}_A(R)$, hence the prefix "pre-".

In [Allcock 2015] we defined $\mathfrak{PSt}_A(R)$ this way and then showed that it has the presentation in Section 2. In this paper, for ease of exposition we defined $\mathfrak{PSt}_A(R)$ by this presentation. But we will use the above "intrinsic" description in the proof of Theorem 1.1. So equality between the two versions of $\mathfrak{PSt}_A(R)$ is essential for our work. We proved it in [loc. cit., Theorem 1.2] and restate it now:

Theorem 3.1 (the two models of $\mathfrak{PSt}_A(R)$). *Let A be a generalized Cartan matrix and R a commutative ring. For each simple root α_i, choose one of the two distinguished parameterizations $X_{e_i} : R \to \mathfrak{U}_{\alpha_i}$. Then the pre-Steinberg group as defined in Section 2 is isomorphic to the pre-Steinberg group as defined above*, *by $S_i \mapsto \tilde{s}_{e_i}(1)$ and $X_i(t) \mapsto X_{e_i}(t)$.* □

4. Nomenclature for affine root systems

Our proof of Theorem 1.1, appearing in the next section, refers to the root system as a whole, with the simple roots playing no special role. It is natural in this setting to

	[Moody and Pianzola 1995]	[Kac 1990]	condition
$\widetilde{A}_n$	$A_n^{(1)}$	$A_n^{(1)}$	$n \geq 1$
$\widetilde{B}_n$	$B_n^{(1)}$	$B_n^{(1)}$	$n \geq 2$
$\widetilde{C}_n$	$C_n^{(1)}$	$C_n^{(1)}$	$n \geq 2$
$\widetilde{D}_n$	$D_n^{(1)}$	$D_n^{(1)}$	$n \geq 3$
$\widetilde{E}_n$	$E_n^{(1)}$	$E_n^{(1)}$	$n = 6, 7, 8$
$\widetilde{F}_4$	$F_4^{(1)}$	$F_4^{(1)}$	
$\widetilde{G}_2$	$G_2^{(1)}$	$G_2^{(1)}$	
$\widetilde{B}_n^{\text{even}}$	$B_n^{(2)}$	$D_{n+1}^{(2)}$	$n \geq 2$
$\widetilde{C}_n^{\text{even}}$	$C_n^{(2)}$	$A_{2n-1}^{(2)}$	$n \geq 2$
$\widetilde{BC}_n^{\text{odd}}$	$BC_n^{(2)}$	$A_{2n}^{(2)}$	$n \geq 1$
$\widetilde{F}_4^{\text{even}}$	$F_4^{(2)}$	$E_6^{(2)}$	
$\widetilde{G}_2^{0 \bmod 3}$	$G_2^{(3)}$	$D_4^{(3)}$	

Table 2. Our and others' names for affine root systems; see Section 4.

use nomenclature for the affine root systems that emphasizes this global perspective. Our notation $\widetilde{X}_n^{\cdots}$ in Table 2 is close to that in [Moody and Pianzola 1995, §3.5]. The differences are that our superscripts describe the construction of the root systems, and that we use a tilde to indicate affineness. For the affine root systems obtained by "folding", Kac's nomenclature [1990, pp. 54–55] emphasizes not the affine root system itself but rather the one being folded.

It is very easy to describe the set Φ of real roots in the root system $\widetilde{X}_n^{\cdots}$. Let $\bar{\Phi}$ be a root system of type X_n, let $\bar{\Lambda}$ be its root lattice, and let Λ be $\bar{\Lambda} \oplus \mathbb{Z}$. Then $\Phi \subseteq \Lambda$ is the set of pairs (root of $X_n, m \in \mathbb{Z}$) satisfying the condition that if the root is long then m has the property "$\cdots$" indicated in the superscript, if any.

A set of simple roots can be described as follows. We begin with a set of simple roots for the root system $\Phi_0 \subseteq \Phi$ consisting of roots of the form $(\bar{\alpha}, 0)$. This is an X_n root system except for $\widetilde{BC}_n^{\text{odd}}$, when it has type B_n. The affinizing simple root is $(\bar{\alpha}, 1)$, where $\bar{\alpha}$ is the lowest root of Φ_0 in the absence of a superscript, or twice the lowest short root for $\widetilde{BC}_n^{\text{odd}}$, or the lowest short root in all other cases. This can be used to verify the correspondences between our nomenclature and those of [Kac 1990] and [Moody and Pianzola 1995].

The condition on n in Table 2 is the weakest condition for which the definition of $\widetilde{X}_n^{\cdots}$ makes sense. If one wishes to avoid duplication, so that each isomorphism class of affine root system appears exactly once, then one should omit one of $\widetilde{A}_3 \cong \widetilde{D}_3$, one of $\widetilde{B}_2 \cong \widetilde{C}_2$ and one of $\widetilde{B}_2^{\text{even}} \cong \widetilde{C}_2^{\text{even}}$. Both [Kac 1990] and [Moody and Pianzola 1995] omit $\widetilde{D}_3$, $\widetilde{B}_2$ and $\widetilde{C}_2^{\text{even}}$. Also, [Moody and Pianzola 1995] gives $A_1^{(2)}$ as an alternate name for $BC_1^{(2)}$.

5. The isomorphism $\mathfrak{PSt}_A(R) \to \mathfrak{St}_A(R)$

This section is devoted to proving Theorem 1.1, whose hypotheses we assume throughout. In light of Theorem 3.1, our goal is to show that the Chevalley relations for the classically prenilpotent pairs imply those of the remaining prenilpotent pairs. We will begin by saying which pairs of real roots are prenilpotent and which are classically prenilpotent. Then we will analyze the pairs that are prenilpotent but not classically prenilpotent.

We fix the affine Dynkin diagram A, write Φ, $\bar{\Phi}$, Λ, $\bar{\Lambda}$ as in Section 4, and use an overbar to indicate projections of roots from Φ to $\bar{\Phi}$. It is easy to see that $\alpha, \beta \in \Phi$ are classically prenilpotent if and only if $\alpha = \beta$ or $\bar{\alpha}, \bar{\beta} \in \bar{\Phi}$ are linearly independent. The following lemma describes which pairs of roots are prenilpotent but not classically prenilpotent, and what their Chevalley relations are (except for one special case discussed later).

Lemma 5.1. *The following are equivalent*:

(i) α, β *are prenilpotent but not classically prenilpotent.*

(ii) $\alpha \neq \beta$ *are not equal and* $\bar{\alpha}, \bar{\beta}$ *differ by a positive scalar factor.*

(iii) $\alpha \neq \beta$, *and either* $\bar{\alpha} = \bar{\beta}$ *are equal or else one is twice the other and* $\Phi = \widetilde{BC}_n^{\text{odd}}$.

When these equivalent conditions hold, *the Chevalley relations between* $\mathfrak{U}_\alpha, \mathfrak{U}_\beta$ *are* $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$, *unless* $\Phi = \widetilde{BC}_n^{\text{odd}}$, $\bar{\alpha}$ *and* $\bar{\beta}$ *are the same short root of* $\bar{\Phi} = BC_n$, *and* $\alpha + \beta \in \Phi$.

Proof. We think of the Weyl group W acting on affine space in the usual way, with each root corresponding to an open halfspace. A root is positive if its halfspace contains the fundamental chamber, or negative if not. Recall that two roots $\alpha, \beta \in \Phi$ form a prenilpotent pair if some element w_+ of W sends both to positive roots, and some $w_- \in W$ sends both to negative roots. The existence of both $w_\pm$ is equivalent to saying that some chamber lies in the halfspaces of both α and β, and some other chamber lies in neither of them. (Proof: apply $w_\pm$ to the fundamental chamber rather than to $\{\alpha, \beta\}$.) By Euclidean geometry, this happens only if either their bounding hyperplanes are nonparallel or their bounding hyperplanes are parallel and one halfspace contains the other. In the first case, $\bar{\alpha}$ and $\bar{\beta}$ are linearly independent, so α, β are classically prenilpotent. In the second case, $\bar{\alpha}$ and $\bar{\beta}$ differ by a positive scalar. If α and β are equal then they form a classically prenilpotent pair. Otherwise they do not, because $(\mathbb{Q}\alpha \oplus \mathbb{Q}\beta) \cap \Phi$ is infinite. This proves the equivalence of (i) and (ii).

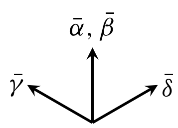
To see the equivalence of (ii) and (iii) we refer to the fact that $\bar{\Phi}$ is a reduced root system (i.e, the only positive multiple of a root that can be a root is that root itself) except in the case $\Phi = \widetilde{BC}_n^{\text{odd}}$. In this last case, the only way one root of $\bar{\Phi} = BC_n$ can be a positive multiple of a different root is that the long roots are got by doubling the short roots.

The proof of the final claim is similar. Except in the excluded case, we have $\bar{\Phi}\cap(\mathbb{N}\bar{\alpha}+\mathbb{N}\bar{\beta})=\{\bar{\alpha},\bar{\beta}\}$. The corresponding claim for Φ follows, so $\theta(\alpha,\beta)-\{\alpha,\beta\}$ is empty and the right-hand side of (3-2) is the identity. That is, the Chevalley relations for α,β read $[\mathfrak{U}_\alpha,\mathfrak{U}_\beta]=1$. (In the excluded case, $\Phi\cap(\mathbb{N}\alpha+\mathbb{N}\beta)=\{\alpha,\beta,\alpha+\beta\}$. So the Chevalley relations set the commutators of elements of $\mathfrak{U}_\alpha$ with elements of $\mathfrak{U}_\beta$ equal to certain elements of $\mathfrak{U}_{\alpha+\beta}$. See Case 6 below.) $\square$

Recall from Theorem 3.1 that $\mathfrak{St}_A(R)$ may be got from $\mathfrak{PSt}_A(R)$ by adjoining the Chevalley relations for every prenilpotent pair α, β that is not classically prenilpotent. So to prove Theorem 1.1 it suffices to show that these relations already hold in $\mathfrak{PSt}:=\mathfrak{PSt}_A(R)$. In light of Lemma 5.1, the proof falls into seven cases, according to Φ and the relative position of $\bar{\alpha}$ and $\bar{\beta}$. Conceptually, they are organized as follows; see below for their exact hypotheses. Case 1 applies if $\bar{\alpha}=\bar{\beta}$ is a long root of some A_2 root system in $\bar{\Phi}$. Case 2 (resp. 3) applies if $\bar{\alpha}=\bar{\beta}$ is a long (resp. short) root of some B_2 root system in $\bar{\Phi}$. Case 4 applies if $\bar{\alpha}=\bar{\beta}$ is a short root of $\bar{\Phi}=G_2$. The rest of the cases are specific to $\Phi=\widetilde{BC}_n^{\text{odd}}$. Case 5 applies if $\bar{\beta}=2\bar{\alpha}$. Case 6 or 7 applies if $\bar{\alpha}=\bar{\beta}$ is a short root of BC_n. There are two cases because $\alpha+\beta$ may or may not be a root.

In every case but one we must establish $[\mathfrak{U}_\alpha,\mathfrak{U}_\beta]=1$. Each case begins by choosing two roots in Φ, of which β is a specified linear combination, and whose projections to $\bar{\Phi}$ are specified. Given the global description of Φ from Section 4, this is always easy. Then we use the Chevalley relations for various classically prenilpotent pairs to deduce the Chevalley relations for α, β.

Case 1 of Theorem 1.1. Assume $\bar{\alpha}=\bar{\beta}$ is a root of $\bar{\Phi}=A_{n\geq 2}$, $\bar{\Phi}=D_n$ or $\bar{\Phi}=E_n$, or a long root of $\bar{\Phi}=G_2$. Choose $\bar{\gamma},\bar{\delta}\in\bar{\Phi}$ as shown, and choose lifts $\gamma,\delta\in\Phi$ summing to β. (Choose any $\gamma\in\Phi$ lying over $\bar{\gamma}$, define $\delta=\beta-\gamma$, and use the global description of Φ to check that $\delta\in\Phi$. This is trivial except in the case $\Phi=\widetilde{G}_2^{0\bmod 3}$, when it is easy.)



Because $\bar{\alpha}+\bar{\gamma},\bar{\alpha}+\bar{\delta}\notin\bar{\Phi}$, it follows that $\alpha+\gamma,\alpha+\delta\notin\Phi$. So the Chevalley relations $[\mathfrak{U}_\alpha,\mathfrak{U}_\gamma]=[\mathfrak{U}_\alpha,\mathfrak{U}_\delta]=1$ hold. The Chevalley relations for γ, δ imply $[\mathfrak{U}_\gamma,\mathfrak{U}_\delta]=\mathfrak{U}_{\gamma+\delta}=\mathfrak{U}_\beta$. (These relations are (2-23) in the G_2 case and (2-11) in the others. One can write them as $[X_\gamma(t),X_\delta(u)]=X_{\gamma+\delta}(tu)$ in the notation of the next paragraph.) Since $\mathfrak{U}_\alpha$ commutes with $\mathfrak{U}_\gamma$ and $\mathfrak{U}_\delta$, it commutes with the group they generate, hence $\mathfrak{U}_\beta$. $\square$

The other cases use the same strategy: express an element of $\mathfrak{U}_\beta$ in terms of other root groups, and then evaluate its commutator with an element of $\mathfrak{U}_\alpha$. But the calculations are more delicate. We will work with explicit elements $X_\gamma(t) \in \mathfrak{U}_\gamma$ for various roots $\gamma \in \Phi$. Here t varies over R, and the definition of $X_\gamma(t)$ depends on choosing a basis vector e_γ for $\mathfrak{g}_{\gamma,\mathbb{Z}} \subseteq \mathfrak{g}$, as explained in Section 3. For each γ there are two possibilities for e_γ. The point of making these sign choices is to write down the relations explicitly.

For example, if $s, l \in I$ are the short and long roots of a B_2 subdiagram of A, then we copy their relations from (2-17): for all $t, u \in R$,

$$[X_s(t), X_l(u)] = S_l X_s(-tu) S_l^{-1} \cdot S_s X_l(t^2 u) S_s^{-1}. \tag{5-1}$$

The reason for writing the right side this way is to avoid making choices: to write down the relation, one only needs to specify generators e_s and e_l for $\mathfrak{g}_{s,\mathbb{Z}}$ and $\mathfrak{g}_{l,\mathbb{Z}}$, not the other root spaces involved. But for explicit computation one must choose generators for these other root spaces. Because S_s and S_l permute the root spaces in the same way the reflections in s and l do, the terms on the right in (5-1) lie in $\mathfrak{U}_{l+s}$ and $\mathfrak{U}_{l+2s}$. Therefore, after choosing suitable generators e_{l+s} and e_{l+2s} for $\mathfrak{g}_{l+s,\mathbb{Z}}$ and $\mathfrak{g}_{l+2s,\mathbb{Z}}$, we may rewrite (5-1) as

$$[X_s(t), X_l(u)] = X_{l+s}(-tu) \cdot X_{l+2s}(t^2 u). \tag{5-2}$$

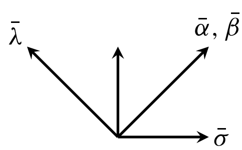
Now, if σ and λ are short and long simple roots for *any* copy of B_2 in Φ, then some element w of the Weyl group sends some pair of simple roots to them. Taking s and l to be this pair, and defining X_σ, X_λ, $X_{\lambda+\sigma}$ and $X_{\lambda+2\sigma}$ as the w-conjugates of X_s, X_l, X_{l+s} and X_{l+2s}, we can write the Chevalley relation for σ and λ by applying the substitution $s \mapsto \sigma$ and $l \mapsto \lambda$ to (5-2):

$$[X_\sigma(t), X_\lambda(u)] = X_{\lambda+\sigma}(-tu) \cdot X_{\lambda+2\sigma}(t^2 u). \tag{5-3}$$

In this way we can obtain the Chevalley relations we will need, for any classically prenilpotent pair, from the ones listed explicitly in Section 2. One could also refer to other standard references, for example, [Carter 1972, §5.2].

The root system $\widetilde{BC}_{n\geq 2}^{\text{odd}}$ appears as a possibility in several cases, including the next one. We will use "short", "middling" and "long" to refer to its three root lengths.

Case 2 of Theorem 1.1. Assume $\bar\alpha = \bar\beta$ is a long root of $\bar\Phi = B_{n\geq 2}$, $\bar\Phi = C_{n\geq 2}$, $\bar\Phi = BC_{n\geq 2}$ or $\bar\Phi = F_4$. Our first step is to choose roots $\bar\lambda, \bar\sigma \in \bar\Phi$ as pictured:



This is easily done using any standard description of $\bar\Phi$. (Note: although $\bar\lambda$ stands for "long" and $\bar\sigma$ for "short", $\bar\sigma$ is actually a middling root in the case $\bar\Phi = BC_n$.)

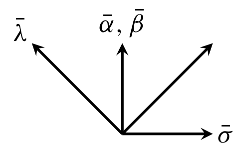
Our second step is to choose lifts $\lambda, \sigma \in \Phi$ with $\beta = \lambda + 2\sigma$. If Φ equals $\widetilde{B}_n$, $\widetilde{C}_n$ or $\widetilde{F}_4$ then one chooses any lift σ of $\bar{\sigma}$ and defines λ as $\beta - 2\sigma$. This works since every element of Λ lying over a root of $\bar{\Phi}$ is a root of Φ. If Φ equals $\widetilde{B}_n^{\text{even}}$, $\widetilde{C}_n^{\text{even}}$, $\widetilde{F}_4^{\text{even}}$ or $\widetilde{BC}_n^{\text{odd}}$ then this argument might fail since Φ is "missing" some long roots. Instead, one chooses any $\lambda \in \Phi$ lying over $\bar{\lambda}$ and defines σ as $(\beta - \lambda)/2$. Now, $\beta - \lambda = (\bar{\beta} - \bar{\lambda}, m)$ with m being even by the meaning of the superscript "even" or "odd". Also, $\bar{\beta} - \bar{\lambda}$ is divisible by 2 in $\bar{\Lambda}$ by the figure above. It follows that $\sigma \in \Lambda$. Then, as an element of Λ lying over a short (or middling) root of $\bar{\Phi}$, σ lies in Φ.

Because σ, λ are simple roots for a B_2 root system inside Φ, their Chevalley relation (5-3) holds in $\mathfrak{PSt}$. This shows that any element of $\mathfrak{U}_\beta = \mathfrak{U}_{\lambda+2\sigma}$ can be written in the form

$$(\text{some } x_{\lambda+\sigma} \in \mathfrak{U}_{\lambda+\sigma}) \cdot \big[(\text{some } x_\sigma \in \mathfrak{U}_\sigma), (\text{some } x_\lambda \in \mathfrak{U}_\lambda)\big]. \tag{5-4}$$

Referring to the picture of $\bar{\Phi}$ shows that $\alpha + \lambda + \sigma \notin \Phi$. Therefore, the Chevalley relations in $\mathfrak{PSt}$ include $[\mathfrak{U}_\alpha, \mathfrak{U}_{\lambda+\sigma}] = 1$. In particular, $\mathfrak{U}_\alpha$ commutes with the first term of (5-4). The same argument shows that $\mathfrak{U}_\alpha$ also commutes with the other terms, hence with any element of $\mathfrak{U}_\beta$. This shows that the Chevalley relations present in $\mathfrak{PSt}$ imply $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$, as desired. $\square$

Case 3 of Theorem 1.1. Assume $\bar{\alpha} = \bar{\beta}$ is a short root of $\bar{\Phi} = B_{n\geq 2}$, $\bar{\Phi} = C_{n\geq 2}$ or $\bar{\Phi} = F_4$, or a middling root of $\bar{\Phi} = BC_{n\geq 2}$. We may choose $\lambda, \sigma \in \Phi$ with sum β and the following projections to $\bar{\Phi}$ (by a simpler argument than in the previous case):



The Chevalley relations for σ, λ are (5-3), showing that any element of $\mathfrak{U}_\beta = \mathfrak{U}_{\sigma+\lambda}$ can be written in the form

$$\big[(\text{some } x_\sigma \in \mathfrak{U}_\sigma), (\text{some } x_\lambda \in \mathfrak{U}_\lambda)\big] \cdot (\text{some } x_{\lambda+2\sigma} \in \mathfrak{U}_{\lambda+2\sigma}). \tag{5-5}$$

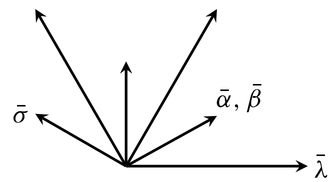
As in the previous case, we will conjugate this by an arbitrary element of $\mathfrak{U}_\alpha$. This requires the following Chevalley relations. We have $[\mathfrak{U}_\alpha, \mathfrak{U}_\lambda] = 1$ and $[\mathfrak{U}_\alpha, \mathfrak{U}_{\lambda+2\sigma}] = 1$ by the same argument as before. What is new is that the Chevalley relations for α, σ depend on whether $\alpha + \sigma$ is a root. If it is not, then $\mathfrak{U}_\alpha$ commutes with $\mathfrak{U}_\sigma$ and therefore with (5-5). That is, $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$ as desired. If $\alpha + \sigma$ is a root then $[\mathfrak{U}_\alpha, \mathfrak{U}_\sigma] \subseteq \mathfrak{U}_{\alpha+\sigma}$. Then conjugating (5-5) by an element of $\mathfrak{U}_\alpha$ yields

$$\big[x_\sigma \cdot (\text{some } x_{\alpha+\sigma} \in \mathfrak{U}_{\alpha+\sigma}), x_\lambda\big] \cdot x_{\lambda+2\sigma},$$

which we can simplify by further use of Chevalley relations. Namely, neither $\lambda + \alpha + \sigma$ nor $\alpha + 2\sigma$ is a root, so $\mathfrak{U}_{\alpha+\sigma}$ centralizes $\mathfrak{U}_\lambda$ and $\mathfrak{U}_\sigma$. Therefore, $x_{\alpha+\sigma}$

centralizes the other terms in the commutator, and hence drops out, leaving (5-5). This shows that conjugation by any element of $\mathfrak{U}_\alpha$ leaves invariant every element of $\mathfrak{U}_\beta$. That is, $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$. $\square$

Case 4 of Theorem 1.1. Assume $\bar{\alpha} = \bar{\beta}$ is a short root of $\bar{\Phi} = G_2$. This is the hardest case by far. Begin by choosing roots $\bar{\sigma}, \bar{\lambda} \in \bar{\Phi}$ as shown, with lifts $\sigma, \lambda \in \Phi$ summing to β:



Many different root groups appear in the argument, so we choose a generator e_γ of γ's root space, for each $\gamma \in \Phi$ that is a nonnegative linear combination of α, σ, λ.

Next we write down the G_2 Chevalley relations in $\mathfrak{PSt}$ that we will need, derived from (2-20)–(2-26). We will record them in the $\Phi = \widetilde{G}_2$ case and then comment on the simplifications that occur if $\Phi = \widetilde{G}_2^{0 \bmod 3}$. After negating some of the e_γ, for γ involving σ and λ but not α, we may suppose that the Chevalley relations (2-26) for σ, λ read

$$[X_\sigma(t), X_\lambda(u)] = X_{2\sigma+\lambda}(t^2u) X_{\sigma+\lambda}(-tu) X_{3\sigma+\lambda}(t^3u) X_{3\sigma+2\lambda}(-t^3u^2). \tag{5-6}$$

Then we may negate $e_{\alpha+2\sigma+\lambda}$, if necessary, to suppose the Chevalley relations (2-24) for α, $2\sigma + \lambda$ read

$$[X_\alpha(t), X_{2\sigma+\lambda}(u)] = X_{\alpha+2\sigma+\lambda}(3tu). \tag{5-7}$$

After negating some of the e_γ, for γ involving α and σ but not λ, we may suppose that the Chevalley relations (2-25) for σ and α read

$$[X_\sigma(t), X_\alpha(u)] = X_{\alpha+\sigma}(-2tu) X_{\alpha+2\sigma}(-3t^2u) X_{2\alpha+\sigma}(-3tu^2). \tag{5-8}$$

We know the Chevalley relations (2-24) for σ and $\alpha + \sigma$ have the form

$$[X_\sigma(t), X_{\alpha+\sigma}(u)] = X_{\alpha+2\sigma}(3\varepsilon tu), \tag{5-9}$$

where $\varepsilon = \pm 1$. We cannot choose the sign because we've already used our freedom to negate $e_{\alpha+2\sigma}$ in order to get (5-8). Similarly, we know that the Chevalley relations (2-23) for λ and $\alpha + 2\sigma$ are

$$[X_\lambda(t), X_{\alpha+2\sigma}(u)] = X_{\alpha+2\sigma+\lambda}(\varepsilon' tu) \tag{5-10}$$

for some $\varepsilon' = \pm 1$. (We will see at the very end that $\varepsilon = \varepsilon' = 1$.)

We were able to write down these relations because we could work out the roots in the positive span of any two given roots. This used the assumption $\Phi = \widetilde{G}_2$, but now suppose $\Phi = \widetilde{G}_2^{0 \bmod 3}$. It may happen that some of the vectors appearing in the previous paragraph, projecting to long roots of $\bar{\Phi} = G_2$, are not roots of Φ. One can check that if $\alpha - \beta$ is divisible by 3 in Λ then there is no change. On the other hand, if $\alpha - \beta \not\equiv 0 \pmod 3$ then $\alpha + 2\sigma + \lambda$, $\alpha + 2\sigma$ and $2\alpha + \sigma$ are not roots. Because $(\mathbb{Q}\alpha \oplus \mathbb{Q}(2\sigma + \lambda)) \cap \Phi$ now has type A_2 rather than G_2, (5-7) is replaced by $[\mathfrak{U}_\alpha, \mathfrak{U}_{2\sigma+\lambda}] = 1$, from (2-10). And $(\mathbb{Q}\alpha \oplus \mathbb{Q}\sigma) \cap \Phi$ also has type A_2 now, so (5-8) is replaced by $[X_\sigma(t), X_\alpha(t)] = X_{\alpha+\sigma}(tu)$, obtained from (2-11), and (5-9) is replaced by $[\mathfrak{U}_\sigma, \mathfrak{U}_{\alpha+\sigma}] = 1$, from (2-10). Finally, there is no relation (5-10) because there is no longer a root group $\mathfrak{U}_{\alpha+2\sigma}$. The calculations below use the relations (5-6)–(5-10). To complete the proof, one must also carry out a similar calculation using (5-6) and the altered versions of (5-7)–(5-9). This calculation is so much easier that we omit it.

The long roots $3\sigma + 2\lambda$, $\alpha + 2\sigma + \lambda$ and $2\alpha + \sigma$ all lie over $3\bar{\sigma} + 2\bar{\lambda}$. These root groups commute with all others that will appear, by the Chevalley relations in $\mathfrak{PSt}$, and they commute with each other by Case 1 above. We will use this without specific mention.

Since $\beta = \sigma + \lambda$, we may take (5-6) with $t = 1$ and rearrange, to express any element of $\mathfrak{U}_\beta$ as

$$X_\beta(u) = X_{3\sigma+\lambda}(u) X_{3\sigma+2\lambda}(-u^2) \big[X_\lambda(u), X_\sigma(1)\big] X_{2\sigma+\lambda}(u). \tag{5-11}$$

We use this to express the commutators generating $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta]$:

$$\begin{aligned}
[X_\alpha(t), X_\beta(u)] = {} & X_\alpha(t) X_{3\sigma+\lambda}(u) X_\alpha(t)^{-1} \cdot X_\alpha(t) X_{3\sigma+2\lambda}(-u^2) X_\alpha(t)^{-1} \\
& \cdot \big[X_\alpha(t) X_\lambda(u) X_\alpha(t)^{-1}, X_\alpha(t) X_\sigma(1) X_\alpha(t)^{-1}\big] \\
& \cdot X_\alpha(t) X_{2\sigma+\lambda}(u) X_\alpha(t)^{-1} \\
& \cdot X_{2\sigma+\lambda}(-u) \big[X_\sigma(1), X_\lambda(u)\big] X_{3\sigma+2\lambda}(u^2) X_{3\sigma+\lambda}(-u).
\end{aligned} \tag{5-12}$$

Because $\mathfrak{U}_\alpha$ centralizes $\mathfrak{U}_{3\sigma+\lambda}$, $\mathfrak{U}_{3\sigma+2\lambda}$ and $\mathfrak{U}_\lambda$, we may cancel all the $X_\alpha(t)$ in the first two terms, and in the first term of the first commutator. Because $\mathfrak{U}_{3\sigma+2\lambda}$ centralizes all terms present, we may cancel the terms $X_{3\sigma+2\lambda}(\pm u^2)$. The terms between the commutators assemble themselves into $[X_\alpha(t), X_{2\sigma+\lambda}(u)]$, which equals $X_{\alpha+2\sigma+\lambda}(3tu)$ by (5-7). Because $\mathfrak{U}_{\alpha+2\sigma+\lambda}$ centralizes all terms present, we may move this term to the very beginning. Finally, from (5-8) one can rewrite the second term of the first commutator as

$$X_\alpha(t) X_\sigma(1) X_\alpha(t)^{-1} = X_{2\alpha+\sigma}(3t^2) X_{\alpha+2\sigma}(3t) X_{\alpha+\sigma}(2t) X_\sigma(1).$$

After all these simplifications, (5-12) reduces to

$$\begin{aligned}
[X_\alpha(t), X_\beta(u)] = {} & X_{\alpha+2\sigma+\lambda}(3tu) X_{3\sigma+\lambda}(u) \\
& \cdot [X_\lambda(u), X_{2\alpha+\sigma}(3t^2) X_{\alpha+2\sigma}(3t) X_{\alpha+\sigma}(2t) X_\sigma(1)][X_\sigma(1), X_\lambda(u)] X_{3\sigma+\lambda}(-u).
\end{aligned} \tag{5-13}$$

Now we focus on the first commutator $[\cdots,\ \cdots]$ on the right side. All its terms commute with $\mathfrak{U}_{2\alpha+\sigma}$, so we may drop the $X_{2\alpha+\sigma}(3t^2)$ term. Writing out what remains gives

$$[\cdots,\ \cdots] = X_\lambda(u)X_{\alpha+2\sigma}(3t)X_{\alpha+\sigma}(2t)X_\sigma(1) \\ \cdot X_\lambda(-u)X_\sigma(-1)X_{\alpha+\sigma}(-2t)X_{\alpha+2\sigma}(-3t).$$

By repeatedly using (5-9)–(5-10) and the commutativity of various pairs of root groups, we move all the X_λ and X_σ terms to the far right. A page-long computation yields

$$[\cdots,\ \cdots] = X_{\alpha+2\sigma+\lambda}(3\varepsilon' tu - 6\varepsilon\varepsilon' tu)\big[X_\lambda(u),\, X_\sigma(1)\big].$$

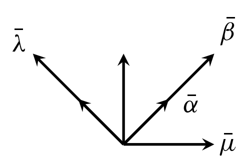
Plugging this into (5-13), and canceling the commutators and the $X_{3\sigma+\lambda}(\pm u)$ terms, yields

$$[X_\alpha(t),\, X_\beta(u)] = X_{\alpha+2\sigma+\lambda}(3tu + 3\varepsilon' tu - 6\varepsilon\varepsilon' tu) = X_{\alpha+2\sigma+\lambda}(Ctu),$$

where C equals 0, ± 6 or 12 depending on $\varepsilon, \varepsilon' \in \{\pm 1\}$.

If $C = 0$ (i.e., $\varepsilon = \varepsilon' = 1$) then we have established the desired Chevalley relation $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$ and the proof is complete. Otherwise we pass to the quotient $\mathfrak{St}$ of $\mathfrak{PSt}$. Here $\mathfrak{U}_\alpha$ and $\mathfrak{U}_\beta$ commute, so we derive the relation $X_{\alpha+2\sigma+\lambda}(Ct) = 1$ in $\mathfrak{St}$. Since this identity holds universally, it holds for $R = \mathbb{C}$, so the image of $\mathfrak{U}_{\alpha+2\sigma+\lambda}(\mathbb{C})$ in $\mathfrak{St}(\mathbb{C})$ is the trivial group. This is a contradiction, since $\mathfrak{St}(\mathbb{C})$ acts on the Kac–Moody algebra $\mathfrak{g}$, with $X_{\alpha+2\sigma+\lambda}(t)$ acting (nontrivially for $t \neq 0$) by $\exp \operatorname{ad}(te_{\alpha+2\sigma+\lambda})$. Since $C \neq 0$ leads to a contradiction, we must have $C = 0$ and so the Chevalley relation $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$ holds in $\mathfrak{PSt}$. □

Case 5 of Theorem 1.1. Assume $\bar\beta = 2\bar\alpha$ in $\bar\Phi = BC_{n\geq 2}$. Choose $\bar\mu, \bar\lambda \in \bar\Phi$ as shown, and lift them to $\mu, \lambda \in \Phi$ with $2\mu + \lambda = \beta$. (Mnemonic: μ is middling and λ is long.)

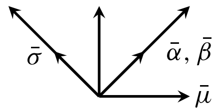


As in Case 2 (when $\bar\alpha$ and $\bar\beta$ were the same long root of $\bar\Phi = B_n$), we can express any element of $\mathfrak{U}_\beta$ in the form

$$(\text{some } x_{\mu+\lambda} \in \mathfrak{U}_{\mu+\lambda}) \cdot \big[(\text{some } x_\lambda \in \mathfrak{U}_\lambda),\, (\text{some } x_\mu \in \mathfrak{U}_\mu)\big].$$

The Chevalley relations in $\mathfrak{PSt}$ include the commutativity of $\mathfrak{U}_\alpha$ with $\mathfrak{U}_\lambda$, $\mathfrak{U}_\mu$ and $\mathfrak{U}_{\mu+\lambda}$. So $\mathfrak{U}_\alpha$ also centralizes $\mathfrak{U}_\beta$. □

Case 6 of Theorem 1.1. Assume $\bar\alpha = \bar\beta$ is a short root of $\bar\Phi = BC_{n\geq 2}$ and $\alpha + \beta$ is a root. This is the exceptional case of Lemma 5.1, and the Chevalley relation we must establish is not $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$. We will determine the correct relation during the proof. We begin by choosing $\bar{\mu}, \bar{\sigma} \in \bar{\Phi}$ as shown and lifting them to $\mu, \sigma \in \Phi$ with $\mu + \sigma = \beta$, so that σ, μ generate a B_2 root system:



We choose a generator e_γ for the root space of each nonnegative linear combination $\gamma \in \Phi$ of α, σ, μ. By changing the signs of $e_{\sigma+\mu}$ and $e_{2\sigma+\mu}$ if necessary, we may suppose that the Chevalley relations (2-17) for σ, μ are

$$[X_\sigma(t), X_\mu(u)] = X_{\sigma+\mu}(-tu) X_{2\sigma+\mu}(t^2 u). \tag{5-14}$$

Since $\sigma + \mu = \beta$ we may take $t = 1$ in (5-14) to express any element of $\mathfrak{U}_\beta$:

$$X_\beta(u) = X_{2\sigma+\mu}(u)[X_\mu(u), X_\sigma(1)]. \tag{5-15}$$

Using this one can express any generator for $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta]$:

$$\begin{aligned}[X_\alpha(t), X_\beta(u)] = X_\alpha(t) X_{2\sigma+\mu}(u) X_\alpha(t)^{-1} & \\ \cdot [X_\alpha(t) X_\mu(u) X_\alpha(t)^{-1}, X_\alpha(t) X_\sigma(1) X_\alpha(t)^{-1}] & \\ \cdot [X_\sigma(1), X_\mu(u)] \cdot X_{2\sigma+\mu}(-u). & \end{aligned} \tag{5-16}$$

By the Chevalley relations $[\mathfrak{U}_\alpha, \mathfrak{U}_{2\sigma+\mu}] = [\mathfrak{U}_\alpha, \mathfrak{U}_\mu] = 1$, the $X_\alpha(t)^{\pm 1}$ cancel in the first term on the right side and in the first term of the first commutator.

Now we consider the Chevalley relations of α and σ. Since $\bar{\alpha} + \bar{\sigma}$ is a middling root of $\bar{\Phi}$, and Φ contains every element of Λ lying over every such root, we see that $\alpha + \sigma$ is a root of Φ. In particular, $(\mathbb{Q}\alpha \oplus \mathbb{Q}\sigma) \cap \Phi$ is a B_2 root system, in which α and σ are orthogonal short roots. The Chevalley relations (2-16) for α, σ are therefore

$$[X_\alpha(t), X_\sigma(u)] = X_{\alpha+\sigma}(-2tu), \tag{5-17}$$

after changing the sign of $e_{\alpha+\sigma}$ if necessary.

Next, $\mu + \sigma + \alpha = \alpha + \beta$ is a root by hypothesis. We choose $e_{\mu+\sigma+\alpha}$ so that the Chevalley relations (2-16) for μ, $\alpha + \sigma$ are

$$[X_\mu(t), X_{\alpha+\sigma}(u)] = X_{\mu+\alpha+\sigma}(-2tu). \tag{5-18}$$

Now we rewrite (5-16), applying the cancellations mentioned above and rewriting the second term in the first commutator using (5-17):

$$\begin{aligned}&[X_\alpha(t), X_\beta(u)] \\ &\quad = X_{2\sigma+\mu}(u) \cdot \big[X_\mu(u), X_{\alpha+\sigma}(-2t) X_\sigma(1)\big] \cdot \big[X_\sigma(1), X_\mu(u)\big] \cdot X_{2\sigma+\mu}(-u).\end{aligned} \tag{5-19}$$

Now we restrict attention to the first commutator on the right side and use the Chevalley relations $[\mathfrak{U}_{\alpha+\sigma}, \mathfrak{U}_\sigma] = 1$ and (5-18) to obtain

$$\begin{aligned}[X_\mu(u), X_{\alpha+\sigma}(-2t)X_\sigma(1)] &= X_\mu(u)X_{\alpha+\sigma}(-2t)\cdot X_\sigma(1)\cdot X_\mu(-u)X_\sigma(-1)X_{\alpha+\sigma}(2t)\\ &= X_{\mu+\alpha+\sigma}(4tu)X_{\alpha+\sigma}(-2t)X_\mu(u)\cdot X_\sigma(1)\\ &\qquad\cdot X_{\mu+\alpha+\sigma}(4tu)X_{\alpha+\sigma}(2t)X_\mu(-u)X_\sigma(-1).\end{aligned}$$

The projections to $\bar{\Phi}$ of any two roots occurring as subscripts are linearly independent. Therefore, any two of them are classically prenilpotent, so their Chevalley relations are present in $\mathfrak{PSt}$. In particular, $\mathfrak{U}_{\mu+\alpha+\sigma}$ centralizes all the other terms; we gather the $X_{\mu+\alpha+\sigma}(4tu)$ terms at the beginning. Next, $[\mathfrak{U}_\sigma, \mathfrak{U}_{\alpha+\sigma}] = 1$, so we may move $X_\sigma(1)$ to the right across $X_{\alpha+\sigma}(2t)$. Then we can use (5-18) again to move $X_\mu(u)$ rightward across $X_{\alpha+\sigma}(2t)$. The result is

$$[X_\mu(u),\, X_{\alpha+\sigma}(-2t)X_\sigma(1)] = X_{\mu+\alpha+\sigma}(4tu)[X_\mu(u),\, X_\sigma(1)].$$

Plugging this into (5-19) and canceling the commutators gives

$$[X_\alpha(t),\, X_\beta(u)] = X_{2\sigma+\mu}(u)X_{\mu+\alpha+\sigma}(4tu)X_{2\sigma+\mu}(-u) = X_{\alpha+\beta}(4tu).$$

Tits' Chevalley relation in his definition of $\mathfrak{St}$ has the same form, with the factor 4 replaced by some integer C. (Although we don't need it, we remark that $C = \pm 4$ by the second displayed equation in [Tits 1987, §3.5], or from [Morita 1988, Theorem 2(2)]. This is related to the fact that $(\mathbb{Q}\alpha \oplus \mathbb{Q}\beta) \cap \Phi$ is a rank 1 affine root system, of type $\widetilde{BC}_1^{\text{odd}}$.) If $C \neq 4$ then in $\mathfrak{St}$ we deduce $X_{\alpha+\beta}((C-4)tu) = 1$ for all $t, u \in R$ and all rings R, leading to the same contradiction we found in Case 4. Therefore, $C = 4$ and we have established that Tits' relation already holds in $\mathfrak{PSt}$. $\square$

Case 7 of Theorem 1.1. Assume $\bar{\alpha} = \bar{\beta}$ is a short root of $\bar{\Phi} = BC_{n\geq 2}$ and $\alpha+\beta$ is not a root. This is similar to the previous case but much easier. We choose μ, σ and the e_γ in the same way, except that $\mu+\sigma+\alpha$ is no longer a root, so the Chevalley relation (5-18) is replaced by $[\mathfrak{U}_\mu, \mathfrak{U}_{\alpha+\sigma}] = 1$. We expand $X_\beta(u)$ as in (5-15) and obtain (5-19) as before. But this time the $X_{\alpha+\sigma}(-2t)$ term centralizes both $\mathfrak{U}_\mu$ and $\mathfrak{U}_\sigma$, so it vanishes from the commutator. The right side of (5-19) then collapses to 1 and we have proven $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$ in $\mathfrak{PSt}$. $\square$

6. Finite presentations

In this section we prove Theorem 1.3, that various Steinberg and Kac–Moody groups are finitely presented. At the end we make several remarks about possible variations on the definition of Kac–Moody groups.

Proof of Theorem 1.3. We must show that $\mathfrak{St}_A(R)$ is finitely presented under either of the two stated hypotheses. By Theorem 1.1 it suffices to prove this with $\mathfrak{PSt}$ in place of $\mathfrak{St}$.

(ii) We are assuming that $\operatorname{rk} A = 3$ and that R is finitely generated as a module over a subring generated by finitely many units. Theorem 1.4(ii) of [Allcock 2015] shows that if R satisfies this hypothesis and A is 2-spherical, then $\mathfrak{PSt}_A(R)$ is finitely presented. This proves (ii).

(i) Now we are assuming that $\operatorname{rk} A > 3$ and that R is finitely generated as a ring. Theorem 1.4(iii) of [Allcock 2015] gives the finite presentability of $\mathfrak{PSt}_A(R)$ if every pair of nodes of the Dynkin diagram lies in some irreducible spherical diagram of rank ≥ 3. (This use of a covering of A by spherical diagrams was also used by Capdeboscq [2013].) By inspecting the list of affine Dynkin diagrams of rank > 3, one checks that this treats all cases of (i) except

$A =$ 

(with some orientations of the double edges). In this case, no irreducible spherical diagram contains α and δ. Note that $\beta \neq \gamma$ since $\operatorname{rk} A > 3$.

For this case we use a variation on the proof of Theorem 1.4(iii) of [Allcock 2015]. Consider the direct limit G of the groups $\mathfrak{St}_B(R)$ as B varies over all irreducible spherical diagrams of rank ≥ 2. If $\operatorname{rk} B \geq 3$ then $\mathfrak{St}_B(R)$ is finitely presented by Theorem I of [Splitthoff 1986]. If $\operatorname{rk} B = 2$ then $\mathfrak{St}_B(R)$ is finitely generated by [Allcock 2015, Lemma 12.2]. Since every irreducible rank 2 diagram lies in one of rank > 2, it follows that G is finitely presented. Now, G satisfies all the relations of $\mathfrak{St}_A(R)$ except for the commutativity of $\mathfrak{St}_{\{\alpha\}}$ with $\mathfrak{St}_{\{\delta\}}$. Because these groups may not be finitely generated, we might need infinitely many additional relations to impose commutativity in the obvious way.

So we proceed indirectly. Let Y_α be a finite subset of $\mathfrak{St}_{\{\alpha\}}$ which together with $\mathfrak{St}_{\{\beta\}}$ generates $\mathfrak{St}_{\{\alpha,\beta\}}$. This is possible since $\mathfrak{St}_{\{\alpha,\beta\}}$ is finitely generated. We define Y_δ similarly, with γ in place of β. We define H as the quotient of G by the finitely many relations $[Y_\alpha, Y_\delta] = 1$, and we claim that the images in H of $\mathfrak{St}_{\{\alpha\}}$ and $\mathfrak{St}_{\{\delta\}}$ commute.

A computation in H establishes this: First, every element of Y_δ centralizes $\mathfrak{St}_{\{\beta\}}$ by the definition of G, and every element of Y_α by that of H. Therefore, it centralizes $\mathfrak{St}_{\{\alpha,\beta\}}$, hence $\mathfrak{St}_{\{\alpha\}}$. We've shown that $\mathfrak{St}_{\{\alpha\}}$ centralizes Y_δ, and it centralizes $\mathfrak{St}_{\{\gamma\}}$ by the definition of G. Therefore, it centralizes $\mathfrak{St}_{\{\gamma,\delta\}}$, hence $\mathfrak{St}_{\{\delta\}}$.

H has the same generators as $\mathfrak{PSt}_A(R)$, and its defining relations are among those defining $\mathfrak{PSt}_A(R)$. On the other hand, we have shown that the generators of H satisfy all the relations in $\mathfrak{PSt}_A(R)$. So $H \cong \mathfrak{PSt}_A(R)$. In particular, $\mathfrak{PSt}_A(R)$ is finitely presented.

It remains to prove the finite presentability of $\mathfrak{G}_A(R)$ under the extra hypothesis that the unit group of R is finitely generated as an abelian group. This follows from [Allcock 2015, Lemma 12.4], which says that the quotient of $\mathfrak{PSt}_A(R)$ by all the relations (1-1) is equally well-defined by finitely many of them. Choosing finitely many such relations, and imposing them on the quotient $\mathfrak{St}_A(R)$ of $\mathfrak{PSt}_A(R)$, gives all the relations (1-1). The quotient of $\mathfrak{St}_A(R)$ by these is the definition of $\mathfrak{G}_A(R)$, proving its finite presentation. □

Remark 6.1 (completions). We have worked with the "minimal" or "algebraic" forms of Kac–Moody groups. One can consider various completions, such as those surveyed in [Tits 1985]. None of these completions can possibly be finitely presented, so no analogue of Theorem 1.3 exists. But it is reasonable to hope for an analogue of Corollary 1.2.

Remark 6.2 (Chevalley–Demazure group schemes). If A is spherical then we write $\mathfrak{CD}_A$ for the simply connected version of the associated Chevalley–Demazure group scheme. This is the unique most natural (in a certain technical sense) algebraic group over $\mathbb{Z}$ of type A. If R is a Dedekind domain of arithmetic type, then the question of whether $\mathfrak{CD}_A(R)$ is finitely presented was settled by Behr [1967; 1998]. We emphasize that our Theorem 1.3 does not give a new proof of his results, because $\mathfrak{CD}_A(R)$ may be a proper quotient of $\mathfrak{G}_A(R)$. The kernel of $\mathfrak{St}_A(R) \to \mathfrak{CD}_A(R)$ is called $K_2(A; R)$ and contains the relators (1-1). It can be extremely complicated.

For a nonspherical Dynkin diagram A, the functor $\mathfrak{CD}_A$ is not defined. The question of whether there is a good definition, and what it would be, seems to be completely open. Only when R is a field is there known to be a unique "best" definition of a Kac–Moody group [Tits 1987, Theorem 1′, p. 553]. The main problem is what extra relations to impose on $\mathfrak{G}_A(R)$. The remarks below discuss the possible forms of some additional relations.

Remark 6.3 (Kac–Moody groups over integral domains). If R is an integral domain with fraction field k, then it is open whether $\mathfrak{G}_A(R) \to \mathfrak{G}_A(k)$ is injective. If $\mathfrak{G}_A$ satisfies Tits' axioms then this would follow from (KMG4), but Tits does not assert that $\mathfrak{G}_A$ satisfies his axioms. If $\mathfrak{G}_A(R) \to \mathfrak{G}_A(k)$ is not injective, then the image seems a better candidate than $\mathfrak{G}_A(R)$ itself for the role of "the" Kac–Moody group.

Remark 6.4 (Kac–Moody groups via representations). Fix a root datum D and a commutative ring R. By using Kostant's $\mathbb{Z}$-form of the universal enveloping algebra of $\mathfrak{g}$, one can construct a $\mathbb{Z}$-form $V^\lambda_{\mathbb{Z}}$ of any integrable highest-weight module V^λ of $\mathfrak{g}$. Then one defines V^λ_R as $V^\lambda_{\mathbb{Z}} \otimes R$. For each real root α, one can exponentiate $\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes R \cong R$ to get an action of $\mathfrak{U}_\alpha \cong R$ on V^λ_R. One can define the action of the torus $(R^*)^n$ directly. Then one can take the group $\mathfrak{G}^\lambda_D(R)$ generated by these transformations and call it a Kac–Moody group. This approach is extremely natural and not yet fully worked out. The first such work for Kac–Moody groups over rings is Garland's landmark paper [1980] treating affine groups; see also Tits' survey [1985, §5], its references, and the recent articles [Bao and Carbone 2015] and [Carbone and Garland 2012].

Tits [1987, p. 554] asserts that this construction allows one to build a Kac–Moody functor satisfying all his axioms (KMG1)–(KMG9). We imagine that he reasoned as follows. First, show that each $\mathfrak{G}_D^\lambda$ is a Kac–Moody functor and therefore by Tits' theorem admits a canonical functorial homomorphism from $\mathfrak{G}_A$, where A is the generalized Cartan matrix of D. One cannot directly apply Tits' theorem, because $\mathfrak{G}_D^\lambda(R)$ only comes equipped with the homomorphisms $\mathrm{SL}_2(R) \to \mathfrak{G}_D^\lambda(R)$ required by Tits when $\mathrm{SL}_2(R)$ is generated by its subgroups $\left(\begin{smallmatrix}1 & * \\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0 \\ * & 1\end{smallmatrix}\right)$. Presumably this difficulty can be overcome. Second, define I as the intersection of the kernels of all the homomorphisms $\mathfrak{G}_A \to \mathfrak{G}_D^\lambda$, and then define the desired Kac–Moody functor as $\mathfrak{G}_A/I$.

Remark 6.5 (Kac–Moody groups as amalgams of Chevalley–Demazure groups). The difficulty in the previous remark, that $\mathrm{SL}_2(R)$ is not always generated by unipotent elements, might be resolved as follows. One can consider the spherical subdiagrams B of A, construct the corresponding Chevalley–Demazure groups $\mathfrak{CD}_B(R)$, and amalgamate these as in Corollary 1.2, rather than amalgamating Steinberg groups. Our results here and in [Allcock 2015] show that this amalgam satisfies the Chevalley relations of all of the prenilpotent pairs that are not classically prenilpotent. (For nonaffine diagrams this requires A to be 3-spherical; 2-sphericity will do if A is simply laced or R has no tiny quotients.) And it is the smallest extension of Tits' construction that recovers $\mathfrak{CD}_A(R)$ when A is spherical. We propose this amalgam, possibly with extra relations, as a reasonable candidate for the definition of Kac–Moody groups.

Remark 6.6 (loop groups). Suppose X is one of the *ABCDEFG* diagrams, $\widetilde{X}$ is its affine extension as in Section 4, and R is a commutative ring. The well-known description of affine Kac–Moody algebras and loop groups makes it natural to expect that $\mathfrak{G}_{\widetilde{X}}(R)$ is a central extension of $\mathfrak{G}_X(R[t^{\pm 1}])$ by R^*. The most general results along these lines that I know of are Theorems 10.1 and B.1 in [Garland 1980], although they concern slightly different groups. Instead, one might simply *define* the loop group $G_{\widetilde{X}}(R)$ as a central extension of $\mathfrak{CD}_X(R[t^{\pm 1}])$ by R^*, where the 2-cocycle defining the extension would have to be made explicit. Then one could try to show that $G_{\widetilde{X}}$ satisfies Tits' axioms.

It is natural to ask whether such a group $G_{\widetilde{X}}(R)$ would be finitely presented if R is finitely generated. If R^* is finitely generated then this is equivalent to the finite presentation of the quotient $\mathfrak{CD}_X\big(R[t^{\pm 1}]\big)$. If $\operatorname{rk} X \geq 3$ then $\mathfrak{St}_X\big(R[t^{\pm 1}]\big)$ is finitely presented by Theorem I of [Splitthoff 1986]. Then, as explained in Section 7 of [loc. cit.], the finite presentability of $\mathfrak{CD}_X\big(R[t^{\pm 1}]\big)$ boils down to properties of $K_1\big(X, R[t^{\pm 1}]\big)$ and $K_2\big(X, R[t^{\pm 1}]\big)$.

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