The Prym map of degree-7 cyclic coverings

Herbert Lange and Angela Ortega

msp

msp

The Prym map of degree-7 cyclic coverings

Herbert Lange and Angela Ortega

We study the Prym map for degree-7 étale cyclic coverings over a curve of genus 2. We extend this map to a proper map on a partial compactification of the moduli space and prove that the Prym map is generically finite onto its image of degree 10.

1. Introduction

Consider an étale finite covering $f : Y \to X$ of degree p of a smooth complex projective curve X of genus $g \geq 2$. Let $\mathrm{Nm}_f : JY \to JX$ denote the norm map of the corresponding Jacobians. One can associate to the covering f its Prym variety

$$P(f) := (\mathrm{Ker}\,\mathrm{Nm}_f)^0,$$

the connected component containing 0 of the kernel of the norm map, which is an abelian variety of dimension

$$\dim P(f) = g(Y) - g(X) = (p-1)(g-1).$$

The variety $P(f)$ carries a natural polarization, namely, the restriction of the principal polarization Θ_Y of JY to $P(f)$. Let D denote the type of this polarization. If, moreover, $f : Y \to X$ is a cyclic covering of degree p, then the group action induces an action on the Prym variety. Let $\mathcal{B}_D$ denote the moduli space of abelian varieties of dimension $(p-1)(g-1)$ with a polarization of type D and an automorphism of order p compatible with the polarization. If $\mathcal{R}_{g,p}$ denotes the moduli space of étale cyclic coverings of degree p of curves of genus g, we get a map

$$\mathrm{Pr}_{g,p} : \mathcal{R}_{g,p} \to \mathcal{B}_D$$

associating to every covering in $\mathcal{R}_{g,p}$ its Prym variety, called the *Prym map*.

Particularly interesting are the cases where $\dim \mathcal{R}_{g,p} = \dim \mathcal{B}_D$. For instance, for $p = 2$ this occurs only if $g = 6$. In this case the Prym map $\mathrm{Pr}_{6,2} : \mathcal{R}_6 \to \mathcal{A}_5$ is generically finite of degree 27 (see [Donagi and Smith 1981]) and the fibers carry

the structure of the 27 lines on a smooth cubic surface. For $(g, p) = (4, 3)$, it is also known that $\text{Pr}_{4,3}$ is generically finite of degree 16 onto its 9-dimensional image $\mathcal{B}_D$ (see [Faber 1988]).

In this paper we investigate the case $(g, p) = (2, 7)$, where $\dim \mathcal{R}_{g,p} = \dim \mathcal{B}_D$. The main result of the paper is the following theorem. Let G be the cyclic group of order 7.

Theorem 1.1. *For any cyclic étale G-cover $f : \widetilde{C} \to C$ of a curve C of genus 2, the Prym variety $\text{Pr}(f)$ is an abelian variety of dimension 6 with a polarization of type $D = (1, 1, 1, 1, 1, 7)$ and a G-action. The Prym map*

$$\text{Pr}_{2,7} : \mathcal{R}_{2,7} \to \mathcal{B}_D$$

is generically finite of degree 10.

The paper is organized as follows. First we compute in Section 2 the dimension of the moduli space $\mathcal{B}_D$ when $(g, p) = (2, 7)$. In Sections 3–5, we extend the Prym map to a partial compactification of admissible coverings $\widetilde{\mathcal{R}}_{2,7}$ such that $\text{Pr}_{2,7} : \widetilde{\mathcal{R}}_{2,7} \to \mathcal{B}_D$ is a proper map. We prove the generic finiteness of the Prym map in Section 6 by specializing to a curve in the boundary. In order to compute the degree of the Prym map, we describe in Section 7 a complete fiber over a special abelian sixfold with polarization type $(1, 1, 1, 1, 1, 7)$, and in Section 8 we give a basis for the Prym differentials for the different types of admissible coverings appearing in the special fiber. Finally, in Section 9 we determine the degree of the Prym map by computing the local degrees along the special fiber.

2. Dimension of the moduli space $\mathcal{B}_D$

As in the introduction, let $\mathcal{R}_{2,7}$ denote the moduli space of nontrivial cyclic étale coverings $f : \widetilde{C} \to C$ of degree 7 of curves of genus 2. The Hurwitz formula gives $g(\widetilde{C}) = 8$. Hence the Prym variety $P = P(f)$ is of dimension 6 and the canonical polarization of the Jacobian $J\widetilde{C}$ induces a polarization of type $(1, 1, 1, 1, 1, 7)$ on P (see [Lange and Ortega 2011, p. 397]). Let σ denote an automorphism of $J\widetilde{C}$ generating the group of automorphisms of $\widetilde{C}/C$. It induces an automorphism of P, also of order 7, which is compatible with the polarization. The Prym map $\text{Pr}_{2,7} : \mathcal{R}_{2,7} \to \mathcal{B}_D$ is the morphism defined by $f \mapsto P(f)$. Here $\mathcal{B}_D$ is the moduli space of abelian varieties of dimension 6 with a polarization of type $(1, 1, 1, 1, 1, 7)$ and an automorphism of order 7 compatible with the polarization. The main result of this section is the following proposition.

Proposition 2.1. $\dim \mathcal{B}_D = \dim \mathcal{R}_{2,7} = 3$.

Proof. Clearly $\dim \mathcal{R}_{2,7} = \dim \mathcal{M}_2 = 3$. So we have to show that also $\dim \mathcal{B}_D = 3$. For this we use Shimura's theory of abelian varieties with endomorphism structure (see [Shimura 1963] or [Birkenhake and Lange 2004, Chapter 9]).

Let $K = \mathbb{Q}(\rho_7)$ denote the cyclotomic field generated by a primitive 7-th root of unity ρ_7. Clearly $\mathcal{B}_D$ coincides with one of Shimura's moduli spaces of polarized abelian varieties with endomorphism structure in K. The field K is a totally complex quadratic extension of a totally real number field of degree $e_0 = 3$. Define

$$m := \frac{\dim P}{e_0} = 2.$$

The polarization of P depends on the lattice of P and a matrix $T \in M_m(\mathbb{Q}(\rho_7))$. The signature of T (see [Birkenhake and Lange 2004, p. 264]) is an e_0-tuple of nonnegative integers $((r_1, s_1), \ldots, (r_{e_0}, s_{e_0}))$ satisfying

$$r_\nu + s_\nu = m = 2$$

for all ν, where e_0 is the number of real embeddings of the totally real subfield of $\mathbb{Q}(\rho_7)$. Recall that for each embedding $\mathbb{Q}(\rho_7) \hookrightarrow \mathbb{C}$, the matrix T is skew-hermitian, and the (r_ν, s_ν) are the signatures of the corresponding skew-hermitian matrices. Then, according to [Shimura 1963, p. 162] or [Birkenhake and Lange 2004, p. 266, lines 6–8], we have

$$\dim \mathcal{B}_D = \sum_{\nu=1}^{e_0} r_\nu s_\nu \leq 3, \tag{2-1}$$

with equality if and only if $r_\nu = s_\nu = 1$ for all ν.

On the other hand, in Section 6 we will see that the map $\mathrm{Pr}_{2,7}$ is generically injective. This implies that

$$\dim \mathcal{B}_D \geq \dim \mathcal{R}_{2,7} = 3,$$

which completes the proof of the proposition.   □

Remark 2.2. According to [Ortega 2003], we know that P is isogenous to the product of a Jacobian of dimension 3 with itself. Then $\mathrm{End}_{\mathbb{Q}}(P)$ is not a simple algebra. Hence, if one knows that $\mathrm{Pr}_{2,7}$ is dominant onto the component $\mathcal{B}_D$, then [Birkenhake and Lange 2004, Proposition 9.9.1] implies that $r_\nu = s_\nu = 1$ for $\nu = 1, 2, 3$, which also gives $\dim \mathcal{B}_D = 3$.

Remark 2.3. It is claimed in [Faber 1988] that the Prym map $\mathrm{Pr}_{2,6} : \mathcal{R}_{2,6} \to \mathcal{B}_D$ satisfies $\dim \mathcal{B}_D = \dim \mathcal{R}_{2,6} = 3$. In a subsequent paper [Lange and Ortega $\geq$ 2016], we show that this does not occur. Moreover, we prove that there are no further examples with this property in the case of étale cyclic coverings of degree $2p$ for a prime p, but there are 3 more cases for cyclic ramified coverings of these degrees.

3. The condition (∗)

In this section we study the Prym map for coverings of degree 7 between stable curves. Let $G = \mathbb{Z}/7\mathbb{Z}$ be the cyclic group of order 7 with generator σ and let

$f : \widetilde{C} \to C$ be a G-cover of a connected stable curve C of arithmetic genus g. We fix in the sequel a primitive 7-th root of unity ρ. We assume the following condition for the covering f:

($*$) The fixed points of σ are exactly the nodes of $\widetilde{C}$ and at each node one local parameter is multiplied by ρ^δ and the other by $\rho^{-\delta}$ for some δ, $1 \le \delta \le 3$.

As in [Beauville 1977], we have $f^*\omega_C \simeq \omega_{\widetilde{C}}$, which implies

$$p_a(\widetilde{C}) = 7g - 6.$$

Let $\widetilde{N}$ and N be the normalizations of $\widetilde{C}$ and C, respectively, and let $\widetilde{f} : \widetilde{N} \to N$ be the induced map. At each node s of $\widetilde{C}$ we make the usual identification

$$\mathcal{K}_s^*/\mathcal{O}_s^* \simeq \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z}.$$

Then the action of σ on $\mathcal{K}_s^*/\mathcal{O}_s^*$ is

$$\sigma^*((z, m, n)_s) = (\rho^{\delta(m-n)}z, m, n)_s$$

for some δ, $1 \le \delta \le 3$. Here we label the branches at the node s such that a local parameter at the first branch (corresponding to m) is multiplied by ρ^δ with $1 \le \delta \le 3$. Then we have

$$f_*((z, m, n)_s) = (z^7, m, n)_{f(s)}$$

since $f_*((z, m, n)_s) = \sum_{k=0}^6 (\sigma^*)^k(z, m, n)_s$, viewed as a divisor at $f(s)$, and

$$
\begin{aligned}
\sum_{k=0}^6 (\sigma^*)^k(z, m, n)_s &= \left(\prod_{k=0}^6 (\rho^{\delta(m-n)})^k z, 7m, 7n \right)_s \\
&= \left(\rho^{\sum_{k=0}^6 \delta(m-n)k} z^7, 7m, 7n \right)_s \\
&= \left(\rho^{\delta(m-n)\frac{6\cdot 7}{2}} z^7, 7m, 7n \right)_s \\
&= (z^7, 7m, 7n)_s \\
&= f^*(z^7, m, n)_{f(s)}.
\end{aligned}
$$

We define the multidegree of a line bundle L on $\widetilde{C}$ by

$$\deg L = (d_1, \ldots, d_v),$$

where v is the number of components of $\widetilde{C}$ and d_i is the degree of L on the i-th component of $\widetilde{C}$.

Lemma 3.1. *Let $L \in \operatorname{Pic}\widetilde{C}$ with $\operatorname{Nm}L \simeq \mathcal{O}_{\widetilde{C}}$. Then*

$$L \simeq M \otimes \sigma^* M^{-1}$$

for some $M \in \operatorname{Pic} \tilde{C}$. Moreover, M can be chosen of multidegree $(k, 0, \ldots, 0)$ with $0 \le k \le 6$.

Proof. As in [Mumford 1971, Lemma 1], using Tsen's theorem, there is a divisor D such that $L \simeq \mathcal{O}_{\tilde{C}}(D)$ and $f_* D = 0$. We write

$$D = \underbrace{\sum_{x \in \tilde{C}_{\mathrm{reg}}} x}_{D_{\mathrm{reg}}} + \underbrace{\sum_{s \in \tilde{C}_{\mathrm{sing}}} (z_s, m_s, n_s)_s}_{D_{\mathrm{sing}}}. \tag{3-1}$$

If $x \in \tilde{C}_{\mathrm{reg}}$ is in the support of D_{reg}, then $\sigma^k(x)$ is in the support of D_{reg} for some $1 \le k \le 6$. Since

$$x - \sigma^k(x) = (x + \sigma(x) + \cdots + \sigma^{k-1}(x)) - \sigma(x + \sigma(x) + \cdots + \sigma^{k-1}(x)),$$

there is a divisor E_{reg} such that $D_{\mathrm{reg}} = E_{\mathrm{reg}} - \sigma^* E_{\mathrm{reg}}$. From (3-1) one sees that the divisor D_{sing} is the sum of divisors of the form $(\rho^{a_s}, 0, 0)_s$ for some $1 \le a_s \le 6$. Choosing an integer i_s such that $-i_s \delta_s \equiv a_s \bmod 7$, we have

$$(1, i_s, 0) - \sigma^*(1, i_s, 0) = (\rho^{-\delta_s i_s}, 0, 0) = (\rho^{a_s}, 0, 0)_s.$$

Then there is a divisor E_{sing} such that $D_{\mathrm{sing}} = E_{\mathrm{sing}} - \sigma^* E_{\mathrm{sing}}$. Thus $D = E - \sigma^* E$ with $E = E_{\mathrm{reg}} + E_{\mathrm{sing}}$. Set $M = \mathcal{O}_{\tilde{C}}(E)$. By replacing M with $M \otimes f^* N$, where N is a line bundle on C, we may assume that the multidegree of M is $(\epsilon_1, \ldots, \epsilon_\nu)$ with $0 \le \epsilon_i \le 6$. Using the fact that $\tilde{C}$ is connected, the multidegree can be accumulated on one of the components by further tensoring M with line bundles associated to divisors of the form $(1, 1, -1)_s$, since $(1, 1, -1)_s - \sigma^*(1, 1, -1)_s = (1, 0, 0)_s = 0$. $\square$

Let P denote the Prym variety of $f : \tilde{C} \to C$, i.e., the connected component containing 0 of the kernel of norm map $\operatorname{Nm} : J\tilde{C} \to JC$. By definition, it is a connected commutative algebraic group. Lemma 3.1 implies that P is the variety of line bundles in $\ker \operatorname{Nm}$ of the form $M \otimes \sigma^* M^{-1}$ with M of multidegree $(0, \ldots, 0)$. Let $\nu : N \to C$ and $\tilde{\nu} : \tilde{N} \to \tilde{C}$ denote the normalizations of C and $\tilde{C}$, respectively. Hence there is a map $\tilde{f}$ making the following diagram commutative:

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{\nu}} & \tilde{C} \\ \tilde{f} \downarrow & & \downarrow f \\ N & \xrightarrow{\nu} & C \end{array}$$

Proposition 3.2. *Suppose $p_a(C) = g$. Then P is an abelian variety of dimension $6g - 6$.*

Proof. (As in [Beauville 1977] and [Faber 1988].) Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{T} & \longrightarrow & J\widetilde{C} & \xrightarrow{\tilde{\nu}^*} & J\widetilde{N} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \\
0 & \longrightarrow & T & \longrightarrow & JC & \xrightarrow{\nu^*} & JN & \longrightarrow & 0
\end{array}
\tag{3-2}
$$

of commutative algebraic groups, where the vertical arrows are the norm maps and T and $\widetilde{T}$ are the groups of classes of divisors of multidegree $(0, \ldots, 0)$ with singular support. Since f^* restricted to T is an isomorphism and $\mathrm{Nm} \circ f^* = 7$, the norm on $\widetilde{T}$ is surjective and

$$
\ker \mathrm{Nm}_{|\widetilde{T}} \simeq \widetilde{T}[7] = \{\text{points of order 7 in } \widetilde{T}\}.
$$

On the other hand, Lemma 3.1 implies that $\ker \mathrm{Nm}$ consists either of 7 components or is connected. Let R be the kernel of $\mathrm{Nm} : J\widetilde{N} \to JN$. Then one obtains an exact sequence

$$
0 \to \widetilde{T}[7] \to \widetilde{R} \to R \to 0
\tag{3-3}
$$

with $\widetilde{R} = P$ if $\ker \mathrm{Nm}$ is connected, and $\widetilde{R} = P \times \mathbb{Z}/7\mathbb{Z}$ if $\ker \mathrm{Nm}$ is not connected. We will see that in both cases P is an abelian variety.

Suppose first that C and hence $\widetilde{C}$ are nonsingular. Then $\widetilde{C} = \widetilde{N}$ and hence $\widetilde{T} = 0$. Since $\ker(J\widetilde{N} \to JN)$ has 7 components, P is an abelian variety.

Suppose that C and thus also $\widetilde{C}$ have $s > 0$ singular points. Then $\dim \widetilde{T} = \dim T = s$. Then R is an abelian variety, since $\tilde{f}$ is ramified. We get a surjective homomorphism $P \to R$ with kernel consisting of 7^s elements when $\mathrm{Ker}\, \mathrm{Nm}$ is connected and 7^{s-1} when it is not. Hence also P is an abelian variety. Moreover,

$$
\dim P = \dim R = \dim J\widetilde{C} - \dim JC.
$$

Now, if C has s nodes and N has t connected components, then also $\widetilde{C}$ has s nodes and $\widetilde{N}$ has t connected components. This implies

$$
\dim J\widetilde{C} - \dim JC = p_a(\widetilde{C}) - p_a(C) = 6g - 6. \qquad \square
$$

Let $\widetilde{\Theta}$ denote the canonical polarization of the generalized Jacobian $J\widetilde{C}$ (see [Beauville 1977]). It restricts to a polarization Σ on the abelian subvariety P. We denote the isogeny $P \to \widehat{P}$ associated to Σ by the same letter.

Proposition 3.3. *The polarization Σ on P is of type $D := (1, \ldots, 1, 7, \ldots, 7)$, where 7 occurs*

$$
\begin{cases}
g \text{ times} & \text{if } \ker \mathrm{Nm} \text{ is connected}, \\
g - 1 \text{ times} & \text{if } \ker \mathrm{Nm} \text{ is not connected}.
\end{cases}
$$

Proof. (Similar to [Faber 1988, Proposition 2.4]; we use also [Beauville 1977, Corollary 2.3, p. 156].) If C is smooth, this is well known. In this case $\ker f$ consists of 7 components and 7 occurs $g - 1$ times in the polarization D (see [Birkenhake and Lange 2004, §12.3]). So suppose C is not smooth and f is a covering of type $(*)$. In this case the maps f^* and $\tilde{f}^*$ are injective.

Consider the isogeny

$$h : P \times JN \to J\widetilde{N}, \quad (L, M) \mapsto \tilde{v}^*(L) \otimes \tilde{f}^*M.$$

We first claim that $\ker h \subset P[7] \times JN[7]$. To see this, let $(L, M) \in \ker h$, i.e., $L \in P$ and $M \in JN$ and $\tilde{v}^*(L) \otimes \tilde{f}^*M \simeq \mathcal{O}_{\widetilde{N}}$. Choose $M' \in JC$ with

$$M \simeq v^*M'. \tag{3-4}$$

Then

$$\tilde{v}^*(L \otimes f^*M') \simeq \tilde{v}^*L \otimes \tilde{f}^*v^*M' \simeq \tilde{v}^*L \otimes \tilde{f}^*M \simeq \mathcal{O}_{\widetilde{N}}, \tag{3-5}$$

so $L \otimes f^*M' \in \ker \tilde{v}^* \simeq \widetilde{T}$. Since $f^* : T \to \widetilde{T}$ is an isomorphism, there is a unique $N \in T$ such that

$$L \simeq f^*(N \otimes M'^{-1}). \tag{3-6}$$

This implies

$$(N \otimes M'^{-1})^{\otimes 7} \simeq \mathrm{Nm} \circ f^*(N \otimes M'^{-1}) \simeq \mathrm{Nm}\, L \simeq \mathcal{O}_C$$

and hence $L \in P[7]$ and using (3-5) we get

$$M^{\otimes 7} \simeq \mathrm{Nm}\, \tilde{f}^*M \simeq \mathrm{Nm}\, \tilde{f}^*v^*M' \simeq \mathrm{Nm}\, \tilde{v}^*L^{-1} \simeq v^*\, \mathrm{Nm}\, L^{-1} \simeq v^*\mathcal{O}_N \simeq \mathcal{O}_C.$$

This completes the proof of the claim.

Now, since the map $v^* : JC[7] \to JN[7]$ is surjective, we can choose M' in (3-4) even as an element of $JC[7]$. Moreover, from (3-6) we also get $N \in T[7]$, since f^* is injective. Now consider the following extension of the map h to the whole of the kernel of the norm map of f:

$$\tilde{h} : \ker \mathrm{Nm} \times JN \to J\widetilde{N}, \quad (L, M) \mapsto \tilde{v}^*(L) \otimes \tilde{f}^*M.$$

We claim that

$$\ker \tilde{h} = \{(f^*(N \otimes M'^{-1}), v^*M') \mid M' \in JC[7],\ N \in T[7]\}. \tag{3-7}$$

So $\ker h$ consists of those elements of the right-hand side of (3-7) for which $f^*(N \otimes M'^{-1})$ is contained in the connected component containing 0 of $\ker \mathrm{Nm}_f$.

Proof of (3-7). The inclusion of $\ker \tilde{h}$ in the left-hand side of the equation follows from (3-4) and (3-6), which are valid for the extension $\tilde{h}$.

For the converse inclusion, suppose that $M' \in JC[7]$ and $N \in T[7]$. First note that $f^*(N \otimes M'^{-1}) \subset \ker \mathrm{Nm}$, since $\mathrm{Nm}\, f^*(N \otimes M'^{-1}) \simeq (N \otimes M'^{-1})^{\otimes 7} \simeq \mathcal{O}_C$. Moreover,

$$\tilde{\nu}^* f^*(N \otimes M'^{-1}) \otimes \tilde{f}^* \nu^* M' \simeq \tilde{\nu}^* f^*(N \otimes M'^{-1}) \otimes \tilde{\nu}^* f^* M' \simeq \tilde{\nu}^* f^* N \simeq \mathcal{O}_{\tilde{N}}.$$

This completes the proof of (3-7).   □

Since for any $\alpha \in T[7]$ we have $(N \otimes \alpha) \otimes (\alpha^{-1} \otimes M'^{-1}) \simeq N \otimes M'^{-1}$ and $\nu^*(M' \otimes \alpha) \simeq \nu^* M'$, we conclude that

$$\dim_{\mathbb{F}_7} \ker \tilde{h} = \dim_{\mathbb{F}_7} JC[7] = 2g - t,$$

with $t = \dim T = \dim \tilde{T}$. This implies

$$\dim_{\mathbb{F}_7} \ker h = \begin{cases} 2g - t & \text{if } \ker \text{ is connected,} \\ 2g - t - 1 & \text{if } \ker \text{ is not connected,} \end{cases}$$

since in the not-connected case $\ker \mathrm{Nm}$ consists of 7 components.

Now $\ker h$ is a maximal isotropic subgroup of the kernel of the polarization of $P \times JN$, since this polarization is the pullback under h of the principal polarization of $J\tilde{N}$. This implies $\dim_{\mathbb{F}_7}(\ker \Sigma \times JN[7]) = 2(\dim_{\mathbb{F}_7} \ker h)$. Since $\dim_{\mathbb{F}_7}(JN[7]) = 2g - 2t$, it follows that

$$\dim_{\mathbb{F}_7}(\ker \Sigma) = \begin{cases} 2g & \text{if } \ker \mathrm{Nm} \text{ is connected,} \\ 2g - 2 & \text{if } \ker \mathrm{Nm} \text{ is not connected.} \end{cases}$$

Since $\ker \Sigma \subset P[7]$, this gives the assertion.   □

Corollary 3.4. $\ker \mathrm{Nm}$ *consists of 7 components.*

Proof. Consider $\tilde{C}_t \to C_t$, a family of coverings, where the central fiber $\tilde{C}_0 \to C_0$ satisfies condition $(*)$ and all the other fibers are coverings of smooth curves. The fibers in the associated family of abelian varieties $\ker \mathrm{Nm}_t$ have 7 components for $t \neq 0$ and, according to Proposition 3.3, they have a polarization of type $(1, \dots, 1, 7, \dots, 7)$, where 7 appears $g - 1$ times. The type of a polarization in a family of abelian varieties is constant since it is given by integers; therefore, $\ker \mathrm{Nm}_0$ has the same polarization type as the nearby fibers and, again by Proposition 3.3, $\ker \mathrm{Nm}_0$ is nonconnected, so it consists of 7 components.   □

4. The condition $(**)$

As in the last section, let $f : \tilde{C} \to C$ be a G-covering of stable curves. Recall that a node $z \in \tilde{C}$ is either

- of index 1, i.e., $|\mathrm{Stab}\, z| = 1$, in which case $f^{-1}(f(z))$ consists of 7 nodes which are cyclically permuted under σ, or

- of index 7, i.e., $|\mathrm{Stab}\, z| = 7$, in which case z is the only preimage of the node $f(z)$ and f is totally ramified at both branches of z. Since σ is of order 7, the two branches of z are not exchanged.

We also say a node of C is *of index i* if a preimage (and hence every preimage) under f is a node of index i. We assume the following conditions for the G-covering $f : \widetilde{C} \to C$ of connected stable curves:

$$(**) \begin{cases} p_a(C) = g \text{ and } p_a(\widetilde{C}) = 7g - 6; \\ \sigma \text{ is not the identity on any irreducible component of } \widetilde{C}; \\ \text{if at a fixed node of } \sigma \text{ one local parameter is multiplied by } \rho^i, \text{ the other} \\ \quad \text{is multiplied by } \rho^{-i}, \text{ where } \rho \text{ denotes a fixed 7-th root of unity}; \\ P := \mathrm{Pr}(f) \text{ is an abelian variety.} \end{cases}$$

Under these assumptions the nodes of $\widetilde{C}$ are exactly the preimages of the nodes of C. We define for $i = 1$ and 7:

- $n_i :=$ the number of nodes of C of index i, i.e., nodes whose preimage consists of $7/i$ nodes of $\widetilde{C}$,

- $c_i :=$ the number irreducible components of C whose preimage consists of $7/i$ irreducible components of $\widetilde{C}$,

- $r :=$ the number of fixed nonsingular points under σ.

Lemma 4.1. *The covering satisfies* $(**)$ *if and only if $r = 0$ and $c_1 = n_1$.*

In particular, any covering satisfying $(**)$ is an admissible G-cover (for the definition, see Section 5), and coverings satisfying condition $(*)$ also verify $(**)$.

Proof. (As in [Beauville 1977] and [Faber 1988].) Let $\widetilde{N}$ and N be the normalizations of $\widetilde{C}$ and C, respectively. The covering $\tilde{f} : \widetilde{N} \to N$ is ramified exactly at the points lying over the fixed points of $\sigma : \widetilde{C} \to \widetilde{C}$. Hence the Hurwitz formula says

$$p_a(\widetilde{N}) - 1 = 7(p_a(N) - 1) + 3r + 6n_7.$$

So

$$p_a(\widetilde{C}) - 1 = p_a(\widetilde{N}) - 1 + 7n_1 + n_7 = 7(p_a(N) - 1) + 3r + 7n_1 + 7n_7.$$

Moreover,

$$p_a(C) - 1 = p_a(N) - 1 + n_1 + n_7,$$

which altogether gives

$$p_a(\widetilde{C}) - 1 = 7(p_a(C) - 1) + 3r.$$

Hence the first condition in $(**)$ is equivalent to $r = 0$.

Now we discuss the condition that P is an abelian variety. For this consider again the diagram (3-2). From the surjectivity of the norm maps it follows that P is an abelian variety if and only if $\dim \widetilde{T} = \dim T$. Now

$$\dim J\widetilde{N} = p_a(\widetilde{N}) - n_7 - 7n_1 + c_7 + 7c_1 - 1$$

and thus

$$\dim \widetilde{T} = (n_7 - c_7) + 7(n_1 - c_1) + 1 \quad \text{and} \quad \dim T = (n_7 - c_7) + (n_1 - c_1) + 1.$$

Hence $\dim \widetilde{T} = \dim T$ if and only if $c_1 = n_1$. $\qquad\qquad\square$

Let $f : \widetilde{C} \to C$ be a G-covering satisfying condition $(**)$ with generating automorphism σ. We denote by B the union of the components of $\widetilde{C}$ fixed under σ and write

$$\widetilde{C} = A_1 \cup \cdots \cup A_7 \cup B$$

with $\sigma(A_i) = A_{i+1}$, where $A_8 = A_1$. Observe that the covering $B \to B/\sigma$ satisfies condition $(*)$.

Proposition 4.2. (i) *If $B = \varnothing$, then $\widetilde{C} = A_1 \cup \cdots \cup A_7$, where A_1 can be chosen connected and tree-like, and $\#A_i \cap A_{i+1} = 1$ for $i = 1, \ldots, 7$.*

(ii) *If $B \neq \varnothing$, then $A_i \cap A_{i+1} = \varnothing$ for $i = 1, \ldots, 7$. Each connected component of A_1 is tree-like and meets B at only one point. Also B is connected.*

For the proof we need the following elementary lemma (the analogue of [Beauville 1977, Lemma 5.3] and [Faber 1988, Lemma 2.6]), which will be applied to the dual graph of $\widetilde{C}$.

Lemma 4.3. *Let Γ be a connected graph with a fixed-point free automorphism σ of order 7. Then there exists a connected subgraph S of γ such that $\sigma^i(S) \cap \sigma^{i+1}(S)$ is empty for $i = 0, \ldots, 6$ and $\bigcup_{i=0}^{6} \sigma^i(S)$ contains every vertex of Γ.*

Proof of Proposition 4.2. (As in [Beauville 1977] and [Faber 1988].) Let Γ denote the dual graph of $\widetilde{C}$. If $B = \varnothing$, let A_1 correspond to the subgraph S of Lemma 4.3. Let v be the number of vertices of S, e the number of edges of S and s the number of nodes of A_1 which belong to only one component. The equality $c_1 = n_1$ implies

$$v = e + s - \#A_1 \cap A_2.$$

Since $1 - v + e \geq 0$ and $\#A_1 \cap A_2 \geq 1$ give $s = 0$, we have $\#A_1 \cap A_2 = 1$ and $1 - v + e = 0$. So A_1 is tree-like. This proves (i).

Assume $B \neq \varnothing$ and define

- $t := \#A_1 \cap A_2$,
- $m := \#A_i \cap B$ for $i = 1, \ldots, 7$,
- $i_{A_1} := \#$ irreducible components of A_1,

- $c_{A_1} := \#$ of connected components of A_1,

- $n_{A_1} := \#$ nodes of A_1.

Recall that assumption $(**)$ implies that B does not contain any node which moves under σ. Then

$$c_1 = i_{A_1} \quad \text{and} \quad n_1 = n_{A_1} + r + m.$$

For any curve we have $n_{A_1} - i_{A_1} + c_{A_1} \geq 0$ (see [Beauville 1977, Proof of Lemma 5.3]). Thus, if $c_1 = n_1$,

$$0 = n_{A_1} + t + m - i_{A_1} \geq t + m - c_{A_1}.$$

Since $\widetilde{C}$ is connected, any connected component of $\bigcup_{i=1}^{7} A_i$ meets B. But then any connected component of A_1 meets B, which implies $m \geq c_{A_1}$. Hence

$$0 \geq t + m - c_{A_1} \geq t \geq 0.$$

Hence $t = 0$, $m = c_{A_1}$ and $n_{A_1} - i_{A_1} + c_{A_1} = 0$. So $A_i \cap A_{i+1} = \varnothing$ and B is connected. $\qquad\square$

In the next section we will extend the Prym map to the G-covers satisfying $(**)$, thereby obtaining a proper map. Theorem 4.4 describes the associated Pryms by reducing to the easier situation of coverings verifying $(*)$.

Theorem 4.4. *Suppose that $f : \widetilde{C} \to C$ satisfies condition $(**)$. Then there exist the following isomorphisms of polarized abelian varieties:*

- *In case* (i) *of Proposition* 4.2, $(P, \Sigma) \simeq \ker((JA_1)^7 \to JA_1)$ *with the polarization induced by the principal polarization on* $(JA_1)^7$.

- *In case* (ii) *of Proposition* 4.2, $(P, \Sigma) \simeq \ker((JA_1)^7 \to JA_1) \times Q$, *where* Q *is the generalized Prym variety associated to the covering* $B \to B/\sigma$.

Proof. (As in [Beauville 1977, Theorem 5.4].) In case (i), $\widetilde{C}$ is obtained from the disjoint union of 7 copies of A_1 by fixing 2 smooth points p and q of A_1 and identifying q in the i-th copy with p in the $(i + 1)$-st copy of A_1 cyclically. The curve $C = \widetilde{C}/G$ is obtained from A_1 by identifying p and q, and $f : \widetilde{C} \to C$ is an étale covering. Note that JA_1 is an abelian variety, since A_1 is tree-like.

Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & J\widetilde{C} & \longrightarrow & (JA_1)^7 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle m} & & \\
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & JC & \longrightarrow & JA_1 & \longrightarrow & 0
\end{array}
$$

in which m is the addition map. One checks immediately that $\mathrm{Nm} : \mathbb{C}^* \to \mathbb{C}^*$ is an isomorphism. This implies the assertion. Notice that the polarization of P is of type

$$(1, \ldots, 1, \underbrace{7, \ldots, 7}_{g-1}).$$

In case (ii), we have

$$J\widetilde{C} \simeq (JA_1)^7 \times JB \quad \text{and} \quad P = (\ker \mathrm{Nm})^0 \simeq \ker((JA_1)^7 \to JA_1) \times Q,$$

which immediately implies the assertion. $\square$

5. The extension of the Prym map to a proper map

Let $\mathcal{R}_{g,7}$ denote the moduli space of nontrivial étale G-covers $f : \widetilde{C} \to C$ of smooth curves C of genus g, and let $\mathcal{B}_D$ denote the moduli space of polarized abelian varieties of dimension $6g - 6$ with polarization of type D, with D as in Proposition 3.3 and compatible with the G-action. As in the introduction we denote by

$$\mathrm{Pr}_{g,7} : \mathcal{R}_{g,7} \to \mathcal{B}_D$$

the corresponding Prym map associating to the covering f the Prym variety $\mathrm{Pr}(f)$. In order to extend this map to a proper map, we consider the compactification $\overline{\mathcal{R}}_{g,7}$ of $\mathcal{R}_{g,7}$ consisting of admissible G-coverings of stable curves of genus g introduced in [Abramovich et al. 2003].

Let $\mathcal{X} \to S$ be a family of stable curves of arithmetic genus g. A *family of admissible G-covers* of $\mathcal{X}$ over S is a finite morphism $\mathcal{Z} \to \mathcal{X}$ such that

(1) the composition $\mathcal{Z} \to \mathcal{X} \to S$ is a family of stable curves;

(2) every node of a fiber of $\mathcal{Z} \to S$ maps to a node of the corresponding fiber of $\mathcal{X} \to S$;

(3) $\mathcal{Z} \to \mathcal{X}$ is a principal G-bundle away from the nodes;

(4) if z is a node of index 7 in a fiber of $\mathcal{Z} \to S$ and ξ and η are local coordinates of the two branches near z, any element of the stabilizer $\mathrm{Stab}_G(z)$ acts as

$$(\xi, \eta) \mapsto (\rho\xi, \rho^{-1}\eta),$$

where ρ is a primitive 7-th root of unity.

In the case of $S = \mathrm{Spec}\,\mathbb{C}$ we just speak of an admissible G-cover. In this case the ramification index at any node z over x equals the order of the stabilizer of z and depends only on x. It is called the *index* of the G-cover $\mathcal{Z} \to \mathcal{X}$ at x. Since 7 is a prime, the index of a node is either 1 or 7. Note that, for any admissible G-cover $Z \to X$, the curve Z is stable if and only if X is stable.

As shown in [Abramovich et al. 2003] or [Arbarello et al. 2011, Chapter 16], the moduli space $\overline{\mathcal{R}}_{g,7}$ of admissible G-covers of stable curves of genus g is a natural compactification of $\mathcal{R}_{g,7}$. Clearly the coverings satisfying condition (∗∗) are admissible and form an open subspace $\widetilde{\mathcal{R}}_{g,7}$ of $\overline{\mathcal{R}}_{g,7}$.

Theorem 5.1. *The map* $\mathrm{Pr}_{g,7} : \mathcal{R}_{g,7} \to \mathcal{B}_D$ *extends to a proper map* $\widetilde{\mathrm{Pr}}_{g,7} : \widetilde{\mathcal{R}}_{g,7} \to \mathcal{B}_D$.

Proof. The proof is the same as the proof of [Faber 1988, Theorem 2.8] just replacing 3-fold covers by 7-fold covers. So we will omit it. □

6. Generic finiteness of $\mathrm{Pr}_{2,7}$

From now on we consider only the case $g = 2$, i.e., of G-covers of curves of genus 2. So $\dim \mathcal{R}_{2,7} = \dim \mathcal{M}_2 = 3$ and $\mathcal{B}_D$ is the moduli space of polarized abelian varieties of type $(1, 1, 1, 1, 1, 7)$ with G-action which is also of dimension 3. Let $[f : \widetilde{C} \to C] \in \mathcal{R}_{2,7}$ be a general point and let the covering f be given by the 7-division point $\eta \in JC$. The next lemma is a particular case of the results in [Lange and Ortega 2011, p. 397–398] that we include here for the sake of completeness.

Lemma 6.1. (i) *The cotangent space of $\mathcal{B}_D$ at the point* $\mathrm{Pr}_{2,7}([f : \widetilde{C} \to C]) \in \mathcal{B}_D$ *is identified with the vector space* $\bigoplus_{i=1}^{3} \left(H^0(\omega_C \otimes \eta^i) \otimes H^0(\omega_C \otimes \eta^{7-i}) \right)$.

(ii) *The codifferential of the map* $\mathrm{Pr}_{2,7} : \mathcal{R}_{2,7} \to \mathcal{B}_D$ *at the point* (f, η) *is given by the sum of the multiplication maps*

$$\bigoplus_{i=1}^{3} \left(H^0(\omega_C \otimes \eta^i) \otimes H^0(\omega_C \otimes \eta^{7-i}) \right) \longrightarrow H^0(\omega_C^2).$$

Proof. (i) Consider the composed map $\mathcal{R}_{g,7} \xrightarrow{\mathrm{Pr}_{2,7}} \mathcal{B}_D \xrightarrow{\pi} \mathcal{A}_D$, where $\mathcal{A}_D$ denotes the moduli space of abelian varieties with polarization of type D. The cotangent space of the image of $[f : \widetilde{C} \to C]$ in $\mathcal{A}_D$ is by definition the cotangent at the Prym variety P of f. It is well known that the cotangent space $T_{P,0}^*$ at 0 is

$$T_{P,0}^* = H^0(\widetilde{C}, \omega_{\widetilde{C}})^- = \bigoplus_{i=1}^{6} H^0(C, \omega_C \otimes \eta^i). \tag{6-1}$$

According to [Welters 1983] the cotangent space of $\mathcal{A}_D$ at the point P can be identified with the second symmetric product of $H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$. This gives

$$T_{\mathcal{A}_D, P}^* = \bigoplus_{i=1}^{6} S^2 H^0(\omega_C \otimes \eta^i) \oplus \bigoplus_{i=1}^{3} \left(H^0(\omega_C \otimes \eta^i) \otimes H^0(\omega_C \otimes \eta^{7-i}) \right). \tag{6-2}$$

Since the map $\pi : \mathcal{B}_D \to \mathcal{A}_D$ is finite onto its image and the group G acts on the cotangent space of $\mathcal{B}_D$ at the point, we conclude that this space can be identified with a 3-dimensional G-subspace of the G-space $T_{\mathcal{A}_D, P}^*$, which is defined over

the rationals. But there is only one such subspace, namely,

$$\bigoplus_{i=1}^{3}\left(H^0(\omega_C \otimes \eta^i) \otimes H^0(\omega_C \otimes \eta^{7-i})\right).$$

This gives (i).

(ii) It is well known that the cotangent space of $\mathcal{R}_{2,7}$ at a point (C, η) without automorphism is given by $H^0(\omega_C^2)$ and the codifferential of $\mathrm{Pr}_{2,7} : \mathcal{R}_{2,7} \to \mathcal{A}_D$ at (C, η) by the natural map $S^2(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-) \longrightarrow H^0(\omega_C^2)$. The assertion follows immediately from Lemma 6.1(i) and equations (6-1) and (6-2). $\qquad\square$

Theorem 6.2. *The map* $\widetilde{\mathrm{Pr}}_{2,7} : \widetilde{\mathcal{R}}_{2,7} \to \mathcal{B}_D$ *is surjective and hence of finite degree.*

Proof. Since the extension $\widetilde{\mathrm{Pr}}_{2,7}$ is proper according to Theorem 5.1, it suffices to show that the map $\mathrm{Pr}_{2,7}$ is generically finite. Now $\mathrm{Pr}_{2,7}$ is generically finite as soon as its differential at the generic point $[f : \widetilde{C} \to C] \in \mathcal{R}_{2,7}$ is injective. Let f be given by the 7-division point η. According to Lemma 6.1, the codifferential of $\mathrm{Pr}_{2,7}$ at $[f : \widetilde{C} \to C]$ is given by (the sum of) the multiplication of sections

$$\mu_{C,\eta} : \bigoplus_{j=1}^{3} H^0(C, \omega_C \otimes \eta^j) \otimes H^0(C, \omega_C \otimes \eta^{7-j}) \longrightarrow H^0(C, \omega_C^2).$$

Note that the surjectivity of $\mu_{C,\eta}$ is an open condition in $\bar{\mathcal{R}}_{2,7}$. Since $\bar{\mathcal{R}}_{2,7}$ is irreducible and $\widetilde{\mathcal{R}}_{2,7}$ is open and dense in $\bar{\mathcal{R}}_{2,7}$, it suffices to show that the map $\mu_{X,\eta}$ is surjective at a point (X, η) in the compactification $\bar{\mathcal{R}}_{2,7}$, even if $\mathrm{Pr}_{2,7}$ is not defined at (X, η). So if $\mu_{X,\eta}$ is surjective at this point, it will be surjective at a general point of $\mathcal{R}_{2,7}$. Moreover, it suffices to show that $\mu_{X,\eta}$ is injective, since both sides of the map are of dimension 3.

Consider the curve

$$X = Y \cup Z,$$

the union of two rational curves intersecting in 3 points q_1, q_2, q_3 which we can assume to be $[1, 0]$, $[0, 1]$, $[1, 1]$, respectively. The line bundle $\eta_X = (\eta_Y, \eta_Z)$ on X of degree 0 is uniquely determined by the gluing of the fiber over the nodes

$$\mathcal{O}_{Y|q_i} \xrightarrow{\cdot c_i} \mathcal{O}_{Z|q_i},$$

given by the multiplication by a nonzero constant c_i. We may assume $c_3 = 1$ and since $\eta_X^7 \simeq \mathcal{O}_X$ we have $c_1^7 = c_2^7 = 1$. Notice that $\omega_{X|Y} = \mathcal{O}_Y(1)$ and $\omega_{X|Z} = \mathcal{O}_Z(1)$ and the restrictions η_Y, η_Z are trivial line bundles.

From the exact sequence

$$0 \to \mathcal{O}_Z(-2) \to \omega_X \otimes \eta_X^i \xrightarrow{\beta_i} \mathcal{O}_Y(1) \to 0$$

we have $h^0(X, \omega_X \otimes \eta_X^i) = 1$ for $i = 1, \ldots, 6$. Moreover, since $H^0(Z, \mathcal{O}_Z(-2)) = 0$, the map β_i induces an inclusion $H^0(X, \omega_X \otimes \eta_X^i) \hookrightarrow H^0(Y, \mathcal{O}_Y(1))$ for $i = 1, \ldots, 6$.

Therefore, to study the injectivity of $\mu_{X,\eta}$, it is enough to check whether the projection of $\bigoplus_{i=1}^3 H^0(X, \omega_X \otimes \eta_X^i) \otimes H^0(X, \omega_X \otimes \eta_X^{7-i})$ to $H^0(Y, \mathcal{O}_Y(1)) \otimes H^0(Y, \mathcal{O}_Y(1))$ is contained in the kernel of the multiplication map

$$H^0(Y, \mathcal{O}_Y(1)) \otimes H^0(Y, \mathcal{O}_Y(1)) \longrightarrow H^0(Y, \mathcal{O}_Y(2)).$$

We claim that the line bundle $\omega_X = (\omega_{|Y}, \omega_{|Z})$ is uniquely determined and one can choose the gluing c_i at the nodes q_i to be the multiplication by the same constant. To see this, first notice that, since (X, ω_X) is a limit linear series of canonical line bundles, the nodes of X are necessary Weierstrass points of X. Let $s_3 \in H^0(X, \omega_X(-2q_3))$ be a section giving a trivialization of ω_Y and ω_Z away from q_3. For $i = 1, 2$, we have

$$\mathcal{O}_Y(1)_{|q_i} \xrightarrow{s_3^{-1}} \mathcal{O}_{X|q_i} \xrightarrow{s_3} \mathcal{O}_Z(1)_{|q_i},$$

which implies that $c_1 = c_2$. Similarly, by using a section in $H^0(X, \omega_X(-2q_2))$ one shows that $c_1 = c_3$.

A section of $\omega_{X_{|Y}} \otimes \eta_Y^i \simeq \mathcal{O}_Y(1)$ for $i = 1, 2, 3$ is of the form $f_i(x, y) = a_i x + b_i y$, with a_i, b_i constants. Suppose that the sections f_i are in the image of the inclusion

$$H^0(X, \omega_X \otimes \eta_X^i) \hookrightarrow H^0(Y, \mathcal{O}_Y(1)).$$

By evaluating the section at the points q_i and using the gluing conditions, one gets $a_i = c_1^i - 1$ and $b_i = c_2^i - 1$. One obtains a similar condition for the image of the sections of $H^0(X, \omega_X \otimes \eta_X^{7-i})$ in $H^0(Y, \mathcal{O}_Y(1))$. Set $j = 7 - i$. By multiplying the corresponding sections of $\omega_X \otimes \eta_X^i$ and $\omega_X \otimes \eta_X^j$ we have that an element in the image of $\mu_{X,\eta}$ is of the form

$$\left(2 - c_1^i - c_1^j\right)x^2 + \left(2 - c_2^i - c_2^j\right)y^2 - \left(2 - c_1^j - c_2^i + c_1^j c_2^i - c_1^i - c_2^j + c_1^i c_2^j\right)xy.$$

Hence, after taking the sum of such sections for $i = 1, 2, 3$ we conclude that there is a nontrivial element in the kernel of $\mu_{X,\eta}$ if and only if there is a nontrivial solution for the linear system $A\boldsymbol{x} = 0$ with

$$A = \begin{pmatrix} 2 - c_1 - c_1^6 & 2 - c_1 c_2^6 - c_1^6 c_2 & 2 - c_2 - c_2^6 \\ 2 - c_1^2 - c_1^5 & 2 - c_1^2 c_2^5 - c_1^5 c_2^2 & 2 - c_2^2 - c_2^5 \\ 2 - c_1^3 - c_1^4 & 2 - c_1^3 c_2^4 - c_1^4 c_2^3 & 2 - c_2^3 - c_2^4 \end{pmatrix}.$$

Clearly, if $c_i = 1$ for some i or $c_1 = c_2$, the determinant of A vanishes. We compute

$$\tfrac{1}{7}\det A = c_1^6(c_2^3 - c_2^5) + c_1^5(c_2^6 - c_2^3) + c_1^4(c_2^2 - c_2) + c_1^3(c_2^5 - c_2^6) + c_1^2(c_2 - c_2^4) + c_1(c_2^4 - c_2^2).$$

Suppose that $c_i \neq 1$ and $c_2 = c_1^k$ for some $2 \leq k \leq 6$. Then a straightforward computation shows that $\det A \neq 0$ if and only if $k = 3$ or $k = 5$.

In conclusion, we can find a limit linear series (X, η_X) with $\eta_X^7 \simeq \mathcal{O}_X$, for suitable values of the c_i, such that the composition map in the commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{i=1}^{3} H^0(X, \omega_X \otimes \eta^i) \otimes H^0(X, \omega_X \otimes \eta^{7-i}) & \longrightarrow & H^0(X, \omega_X^2) \\
\downarrow & & \downarrow{\scriptstyle\simeq} \\
H^0(Y, \mathcal{O}_Y(1)) \otimes H^0(Y, \mathcal{O}_Y(1)) & \longrightarrow & H^0(Y, \mathcal{O}_Y(2))
\end{array}
$$

is an isomorphism. $\square$

7. A complete fiber of $\widetilde{\mathrm{Pr}}_{2,7}$

For a special point of $\mathcal{B}_D$, consider a smooth curve E of genus 1. Then the kernel of the addition map

$$X = X(E) := \ker(m : E^7 \to E) \quad \text{with} \quad m(x_1, \ldots, x_7) = x_1 + \cdots + x_7$$

is an abelian variety of dimension 6, isomorphic to E^6. The kernel of the induced polarization of the canonical principal polarization of E^7 is $\{(x, \ldots, x) : x \in E[7]\}$, which consists of 7^2 elements. So the polarization on X induced by the canonical polarization of E^7 is of type $D = (1, 1, 1, 1, 1, 7)$. Since the symmetric group $\mathcal{S}_7$ acts on E^7 in the obvious way, X admits an automorphism of order 7. Hence X with the induced polarization is an element of $\mathcal{B}_D$. To be more precise, the group $\mathcal{S}_7$ admits exactly 120 subgroups of order 7. Hence to every elliptic curve there exist exactly 120 abelian varieties X as above with G-action. All of them are isomorphic to each other, since the corresponding subgroups are conjugate to each other according the Sylow theorems. We want to determine the complete preimage $\widetilde{\mathrm{Pr}}_{2,7}^{-1}(X)$ of X. We need some lemmas. For simplicity we denote by $\mathrm{Pr}(f)$ the Prym variety of a covering f in $\widetilde{\mathcal{R}}_{2,7}$.

Lemma 7.1. *Let $f : \widetilde{C} \to C$ be a covering satisfying* (∗∗) *with $g = 2$ such that $\mathrm{Pr}(f) \simeq X$. Then C contains a node of index 1.*

Proof. Suppose that either C is smooth or all the nodes of C are of index 7. Then the exact sequence (3-3) gives an isogeny $j : \mathrm{Pr}(f) \to \mathrm{Pr}(\tilde{f})$ onto the Prym variety of the normalization $\tilde{f}$ of f. Actually, in the smooth case, j is an isomorphism and, if there is a node of index 1, the kernel $\widetilde{T}[7]$ is positive dimensional. The isomorphism $\mathrm{Pr}(f) = X$ implies that the kernel of j is of the form $\{(x, \ldots, x)\}$ with $x \in X[7]$. Hence the action of the symmetric group S_7 on X descends to a nontrivial action on $\mathrm{Pr}(\tilde{f})$.

We can extend this action to $J\widetilde{N}$ by combining it with the identity on JN. Namely, $J\widetilde{N} \simeq (JN \times \mathrm{Pr}(\tilde{f}))/H$, where H is constructed as follows. Let $\langle \eta \rangle \subset JN[7]$

be the subgroup defining the covering $\tilde{f}$ and let $H_1 \subset JN[7]$ be its orthogonal complement with respect to the Weil pairing. Then $H = \{(\alpha, -f^*\alpha) : \alpha \in H_1\}$. Since $f^*H_1 = \{(x, \ldots, x) : x \in E[7]\} \subset \mathrm{Pr}(f)$, we get an S_7-action on $J\tilde{N}$ which is clearly nontrivial.

If C is smooth, then $\tilde{N} \simeq \tilde{C}$. On the other hand, if all the nodes of C are of index 7, then $\tilde{N}$ consists of at least two components. In any case, for each component $\tilde{N}_i$ of $\tilde{N}$ we have

$$\# \mathrm{Aut}(J\tilde{N}_i) \geq \tfrac{1}{2}\# S_7 = 2520.$$

Moreover, according to a classical theorem of Weil, $\mathrm{Aut}\,\tilde{N}_i$ embeds into $\mathrm{Aut}\,J\tilde{N}_i$ with quotient of order ≤ 2. So

$$\# \mathrm{Aut}\,\tilde{N}_i \geq \tfrac{1}{2}\# \mathrm{Aut}(J\tilde{N}_i).$$

On the other hand, $\tilde{N}_i$ is a smooth curve of genus ≤ 8. So Hurwitz's theorem implies

$$\# \mathrm{Aut}(\tilde{N}_i) \leq 84 \cdot (8-1) = 588.$$

Together, this gives a contradiction.   □

Lemma 7.2. *Let $f : \tilde{C} \to C$ be a covering satisfying* (∗∗) *such that C has a component containing nodes of index 1 and 7. Then any node of index 1 is the intersection with another component of C.*

Proof. Suppose x and y are nodes of C in a component C_i of index 1 and 7, respectively. Then the preimage $f^{-1}(C_i)$ is a component, since over y the map f is totally ramified. Since $f^{-1}(x)$ consists of 7 nodes, the equality $n_1 = c_1$ implies that x is the intersection of 2 components.   □

Theorem 7.3. *Let $X = \ker(m : E^7 \to E)$ be a polarized abelian variety as above. The fiber $\tilde{\mathrm{Pr}}_{2,7}^{-1}(X)$ consists of the following 4 types of elements of $\tilde{\mathcal{R}}_{2,7}$ (see Figure 1).*

(i) *$C = E/p \sim q$ and $\tilde{C} = \bigsqcup_{i=1}^{7} E_i/p_i \sim q_{i+1}$ with $E_i \simeq E$ for all i and $q_8 = q_1$ and we can enumerate in such a way that the preimages of p and q are $p_i, q_i \in E_i$.*

(ii) *$C = E_1 \cup_p E_2$ consists of 2 elliptic curves intersecting in one point p. Then up to exchanging E_1 and E_2 we have: $\tilde{C}$ consists of an elliptic curve F_1, which is a 7-fold cover of E_1 and 7 copies of $E_2 \simeq E$ not intersecting each other and intersecting F_1 each in one point.*

(iii) *$C = E_1 \cup_p E_2$ with E_2 elliptic and E_1 rational with a node at q. Then $E_2 \simeq E$ and $f^{-1}(E_2)$ consists of 7 disjoint curves all isomorphic to E and $f^{-1}(E_1)$ is a rational curve with one node lying 7:1 over E_1 and intersecting each component of $f^{-1}(E_2)$ in a point over p.*

(iv) *$C = E_1 \cup_p E_2$ as in (iii) and $\tilde{C}$ is an étale G-cover over C.*

We call the coverings of the theorem *of type* (i), (ii), (iii) and (iv), respectively. Theorem 7.3 will be used in Proposition 7.5 to describe the complete fibers of the Prym map over $X(E)$.

Proof. There are 7 types of stable curves of genus 2. We determine the coverings $f : \widetilde{C} \to C$ in $\widetilde{\mathrm{Pr}}_{2,7}^{-1}(X)$ in each case separately.

(1) There is no étale G-cover $f : \widetilde{C} \to C$ of a smooth curve C of genus 2 such that $P = \mathrm{Pr}(f) \simeq X$. This is a direct consequence of Lemma 7.1.

(2) If $C = E/p \sim q$ then the singular point of C is of index 1 and $f : \widetilde{C} \to C$ is a G-covering satisfying ($\ast\ast$) such that $P = \mathrm{Pr}(f) \simeq X$. Then

$$\widetilde{C} = \bigsqcup_{i=1}^{7} E_i / (p_i \sim q_{i+1})$$

with $E_i \simeq E$ and we enumerate in such a way that the preimages of p and q are p_i and q_i with $q_8 = q_1$. In this case $\mathrm{Pr}(f) \simeq X$.

Proof: According to Lemma 7.1 the node is necessarily of index 1 and thus the map $\tilde{f} : \widetilde{C} \to C$ is étale. The exact sequence (3-3) together with Corollary 3.4 gives an isomorphism $P \simeq R$ with R the Prym variety of the map $\tilde{f}$. Clearly we can enumerate the components of $\widetilde{N}$ in such a way that $\widetilde{C}$ is as above and R is the kernel of the map $m : \bigtimes_{i=1}^{7} E \to E$, i.e., $R \simeq X$. We are in case (i) of the theorem.

(3) There is no rational curve C with 2 nodes admitting a G-cover $f : \widetilde{C} \to C$ satisfying ($\ast\ast$) such that $P = \mathrm{Pr}(f) \simeq X$.

Proof: Suppose there is such a covering. By Lemmas 7.1 and 7.2 both nodes are of index 1 and hence the map $\widetilde{C} \to C$ is étale. Then all components of $\widetilde{C}$ are rational. This implies that $P \simeq \mathbb{C}^{\ast 6}$ is not an abelian variety, a contradiction.

(4) Let $C = E_1 \cup_p E_2$ consist of 2 elliptic curves intersecting in one point and let $f : \widetilde{C} \to C$ be a covering satisfying ($\ast\ast$) such that $P = \mathrm{Pr}(f) \simeq X$. Then, up to exchanging E_1 and E_2, we have that $\widetilde{C}$ consists of an elliptic curve F_1, which is a 7-fold cover of E_1 and 7 copies of $E_2 \simeq E$ not intersecting each other and intersecting F_1 each in one point. So $X = \ker(m : E_2^7 \to E_2)$.

Proof: By Lemma 7.1 the node is of index 1 and the map $f : \widetilde{C} \to C$ is étale. Since there is no connected graph with 14 vertices and 7 edges, we are necessarily in case (ii) of the theorem.

(5) Suppose $C = E_1 \cup_p E_2$ with components E_2 elliptic and E_1 rational with a node q and $f : \widetilde{C} \to C$ a G-covering satisfying ($\ast\ast$) such that $P = \mathrm{Pr}(f) \simeq X$. Then $E_2 \simeq E$ and either $f : \widetilde{C} \to C$ is étale and connected or $\widetilde{C}$ consists of 7 components all isomorphic to E and a rational component F_2 over E_2 totally ramified exactly over q and intersecting each E_i exactly in one point lying over p. So $\mathrm{Pr}(f) \simeq X$.

Proof: According to Lemma 7.1 at least one node of C is of index 1. Suppose first that both nodes are of index 1. Then clearly f is étale and we are in case (iv) of the theorem. If only one node is of index 1, then according to Lemma 7.2 q is of index 7 and p of index 1. This gives case (iii) of the theorem.

(6) There is no curve C consisting of 2 rational components intersecting in one point p admitting a G-cover $f : \widetilde{C} \to C$ satisfying ($**$) such that $P = \Pr(f) \simeq X$.

Proof: According to Lemmas 7.1 and 7.2 the node p is of index 1 and the nodes q_1 and q_2 of the rational components of C are of index 1 or 7. By the Hurwitz formula all components of $\widetilde{C}$ are rational. This implies $\Pr(f) \simeq \mathbb{C}^{*6}$, contradicting ($**$).

(7) If C is the union of 2 rational curves intersecting in 3 points, there is no cover $f : \widetilde{C} \to C$ satisfying ($**$) such that $P = \Pr(f) \simeq X$.

Proof: By Lemma 7.1 at least one of the 3 nodes of C is of index 1. So $\widetilde{C}$ consists of at least 8 components. But then the other nodes also are of index 1, because if one node is of index 7, the curve $\widetilde{C}$ consists of 2 components only. Hence all 3 nodes are of index 1. But then all components of $\widetilde{C}$ are rational. So $P = \Pr(f)$ cannot be an abelian variety, contradicting ($**$).

Together, steps (1)–(7) prove the theorem. $\qquad\square$

Corollary 7.4. *Let E be a general elliptic curve. Then the fiber $\widetilde{\Pr}_{2,7}^{-1}(X(E))$ consists of two irreducible components S_1 and S_2. The component S_1 is a covering over E whose points correspond to coverings of type* (i), *except for one point, which corresponds to a covering of type* (iv). *The component S_2 is a finite covering of the moduli space of elliptic curves, and every point corresponds to a covering of type* (ii), *except for two points, one of which corresponds to a covering of type* (iii) *and the other to a covering of type* (iv). *The components S_1 and S_2 intersect at a point corresponding to a covering of type* (iv) *(see Figure 1).*

Proof. It is known that for 2 elliptic curves $E_1 \neq E_2$ we can have $X(E_1) \simeq X(E_2)$ as abelian varieties, but not necessarily as polarized abelian varieties. Hence $X(E)$ determines E (which can be seen also from Theorem 7.3).

We claim that the coverings of type (iv) are contained in S_1 and S_2, whereas the coverings of type (iii) are contained in S_2 only: it is known that a curve $\widetilde{C}$ degenerates to a curve $\widetilde{C}'$ of some other type if and only if the dual graph of $\widetilde{C}'$ can be contracted to the dual graph of $\widetilde{C}$. On the other hand, the locus of curves covering some curve of genus ≥ 2 of some fixed degree is closed in the moduli space of curves. Now considering the dual graphs of the curves $\widetilde{C}$ of the coverings of the different types gives the assertion.

In the case of coverings of type (i) we have $C = E/p_1 \sim p_2$. We can use the translations of E to fix p_1, and then p_2 is free, which gives the assertion, since there are only finitely many étale coverings of C.
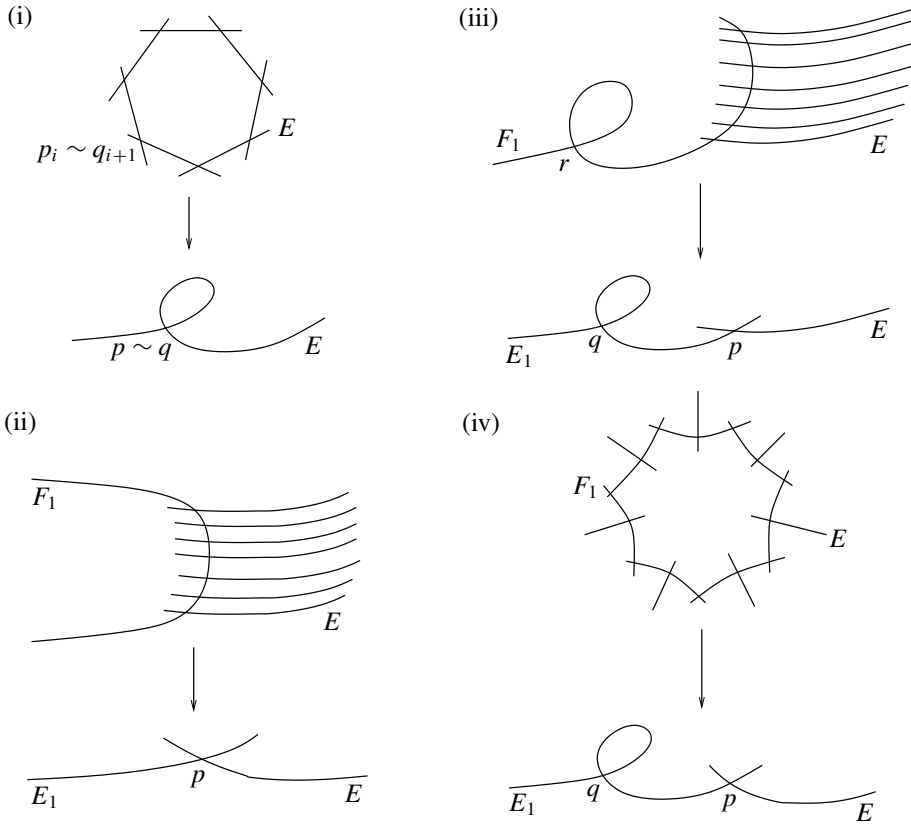
Figure 1. Admissible coverings on the fiber of $X(E)$.

In case (ii) we have $C = E_1 \cup_p E_2$, where E_1 is an arbitrary elliptic curve and $E_2 \simeq E$. Since p may be fixed with an isomorphism of E and E_2, this gives the isomorphism of $\widetilde{\mathrm{Pr}}_{2,7}^{-1}(E)$ with a finite covering of the moduli space of elliptic curves, again since there are only finitely many coverings $\widetilde{C}$ of type (ii) of C.

Finally, in cases (iii) and (iv), the 3 points of the normalization of E_1 given by p and the 2 preimages of the node, which we can assume to be 1, 0, and ∞, respectively, determine the curve C uniquely. For type (iii) the induced map on the normalization of F_1 is a 7:1 map $h : \mathbb{P}^1 \to \mathbb{P}^1$ totally ramified at 2 points, which we assume to be ∞ and 0. So h can be expressed as a polynomial in one variable of degree 7, with vanishing order 7 at 0 and such that $h(1) = 1$, that is, $h(x) = x^7$. Then the map h, and hence the covering, is uniquely determined. For a covering of type (iv) over C we consider 7 copies of $\mathbb{P}^1$, where the point 1 on every rational component is identified to the point ∞ of another rational component, and we attach elliptic curves isomorphic to E_2 at each point 0. The number of étale coverings is

the number of subgroups of order 7 in $JE_1[7] \simeq \mathbb{Z}/7\mathbb{Z}$ (the 7-torsion points in the nodal curve E_1 are determined by a 7-th root of unity). So there is only one such covering up to isomorphism. $\qquad\square$

Varying the elliptic curve E, we obtain a one-dimensional locus $\mathcal{E} \subset \mathcal{B}_D$ consisting of the polarized abelian varieties $X(E)$ with G-action as above. Let $\mathcal{S}$ denote the preimage of $\mathcal{E}$ under the extended Prym map $\widetilde{\mathrm{Pr}}_{2,7} : \widetilde{\mathcal{R}}_{2,7} \to \mathcal{B}_D$. The next proposition is a direct consequence of Corollary 7.4.

Proposition 7.5. *The scheme $\mathcal{S}$ has dimension 2 and is the union of 2 closed subschemes*

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2,$$

where $\mathcal{S}_1$ parametrizes coverings of type (i) *and* (iv), *and $\mathcal{S}_2$ parametrizes coverings of type* (ii), (iii) *and* (iv). *In particular, they intersect exactly in the points parametrizing coverings of type* (iv).

8. The codifferential on the boundary divisors

In this section we will give bases of the Prym differentials and an explicit description of the codifferential of the Prym map.

Let $f : \widetilde{C} \to C$ be a covering corresponding to a point of $\mathcal{S}$. We want to compute the rank of the codifferential of the Prym map $\mathrm{Pr}_{2,7} : \widetilde{\mathcal{R}}_{2,7} \to \mathcal{B}_D$ at the point $[f : \widetilde{C} \to C] \in \widetilde{\mathcal{R}}_{2,7}$. According to [Donagi and Smith 1981] this codifferential is the map

$$\mathcal{P}^* : S^2 \big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^- \big)^G \to H^0(C, \Omega_C \otimes \omega_C),$$

where Ω_C is the sheaf of Kähler differentials on C and $S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)^G$ is the cotangent space to $\mathcal{B}_D$ at the Prym variety of the covering $f : \widetilde{C} \to C$. Letting $j : \Omega_C \to \omega_C$ denote the canonical map, we first compute the rank of the composed map

$$S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)^G \xrightarrow{\mathcal{P}^*} H^0(C, \Omega_C \otimes \omega_C) \xrightarrow{j} H^0(C, \omega^2).$$

Suppose first that f is of type (i), (ii) or (iv). In these cases the covering f is étale and hence given by a 7-division point η of JC. According to the analogue of [Lange and Ortega 2011, Equation (3.4)] we have

$$H^0(\widetilde{C}, \omega_C)^- = \bigoplus_{i=1}^{6} H^0(C, \omega_C \otimes \eta^i) \tag{8-1}$$

and hence

$$S^2\big(H^0(\widetilde{C}, \omega_C)^-\big)^G = \bigoplus_{i=1}^{3} \big(H^0(C, \omega_C \otimes \eta^i) \otimes H^0(C, \omega_C \otimes \eta^{7-i})\big). \tag{8-2}$$

Using this, the above composed map is just the sum of the cup product map

$$\phi : \bigoplus_{i=1}^{3} \left(H^0(C, \omega_C \otimes \eta^i) \otimes H^0(C, \omega_C \otimes \eta^{7-i}) \right) \longrightarrow H^0(C, \omega_C^2), \qquad (8\text{-}3)$$

whose rank we want to compute first.

We shall give a suitable basis for the space of Prym differentials. First we consider a covering of type (i) constructed as follows. Let E be a smooth curve of genus 1 and let $q \neq q'$ be two fixed points of E. Then

$$C := E / q \sim q'$$

is a stable curve of genus 2 with normalization $n : E \to C$ and node $p := n(q) = n(q')$. Let $f : \widetilde{C} \to C$ be a cyclic étale covering with Galois group $G = \langle \sigma \rangle \simeq \mathbb{Z}/7\mathbb{Z}$. The normalization $\tilde{n} : \widetilde{N} \to \widetilde{C}$ consists of 7 components $N_i \simeq E$ with $\sigma(N_i) = N_{i+1}$ for $i = 1, \ldots, 7$ and $N_8 = N_1$. Let q_i and q_i' be the elements of N_i corresponding to q and q'. Then $\tilde{n}(q_i) = \tilde{n}(q_{i+1}') =: p_i$ for $i = 1, \ldots, 7$ with $q_8' = q_1'$. Clearly $\sigma(p_i) = p_{i+1}$ for all i.

Recall that $\omega_{\widetilde{C}}$ is the subsheaf of $\tilde{n}_*\left(\mathcal{O}_{\widetilde{N}} \sum (q_i + q_i')\right)$ consisting of (local) sections φ which considered as sections of $\mathcal{O}_{\widetilde{N}} \sum (q_i + q_i')$ satisfy the condition

$$\mathrm{Res}_{q_i}(\varphi) + \mathrm{Res}_{q_{i+1}'}(\varphi) = 0$$

for $i = 1, \ldots, 7$. Here we use the fact that $\omega_{\widetilde{N}} = \mathcal{O}_{\widetilde{N}}$. Consider the following elements of $H^0\left(\widetilde{C}, \tilde{n}_*\left(\mathcal{O}_{\widetilde{N}} \sum (q_i + q_i')\right)\right)$, regarded as sections on $\widetilde{N}$:

$$\omega_1 := \begin{cases} \text{nonzero section of } \mathcal{O}_{N_1}(q_1 + q_1') \text{ vanishing at } q_1 \text{ and } q_1', \\ 0 \text{ elsewhere}, \end{cases}$$

$$\omega_i := (\sigma^{-i})^*(\omega_1) \quad \text{for } i = 2, \ldots, 7.$$

Note that ω_i is nonzero on N_i vanishing at q_i and q_i' and zero elsewhere.

Now we construct similar differentials for coverings of type (ii). Let

$$C = E_1 \cup_p E_2$$

consist of 2 elliptic curves E_1 and E_2 intersecting transversally in one point p, and let $f : \widetilde{C} \to C$ be a covering of type (ii). So $\widetilde{C}$ consists of an elliptic curve F_1, which is an étale cyclic cover of E_1 of degree 7 and 7 disjoint curves $E_2^1, \ldots, E_2^7$ all isomorphic to E_2. The curve E_i intersects F_1 transversally in a point p_i, such that the group G permutes the curves E_i and the points p_i cyclically, i.e., $\sigma(E_2^i) = E_2^{i+1}$ and $\sigma(p_i) = p_{i+1}$ with $E_2^8 = E_2^1$ and $p_8 = p_1$.

Let $\tilde{n} : \widetilde{N} \to \widetilde{C}$ denote the normalization map. Then $\widetilde{N}$ is the disjoint union of the 8 elliptic curves $F_1, E_2^1, \ldots, E_2^7$. We denote the point p_i by the same letter

when considered as a point of F_1 and E_2^i. Consider the line bundle

$$L = \mathcal{O}_{F_1}(p_1 + \cdots + p_7) \sqcup \mathcal{O}_{E_2^1}(p_1) \sqcup \cdots \sqcup \mathcal{O}_{E_2^7}(p_7).$$

Then $\omega_{\widetilde{C}}$ is the subsheaf of $\tilde{n}_*(L)$ consisting of (local) sections φ which considered as sections of L satisfy the condition

$$(\mathrm{Res}_{p_i})_{|F_1}(\varphi) + (\mathrm{Res}_{p_i})_{|E_2^i}(\varphi) = 0 \tag{8-4}$$

for $i = 1, \ldots, 7$. Consider the following sections of $\tilde{n}_*(L)$, regarded as sections on $\widetilde{N}$:

$$\omega_1 := \begin{cases} \text{nonzero sections of } \mathcal{O}_{F_1}(p_1) \text{ and } \mathcal{O}_{E_2^1}(p_1) \text{ satisfying (8-4) at } p_1, \\ 0 \text{ elsewhere,} \end{cases}$$

$$\omega_i := (\sigma^{-i})^*(\omega_1) \quad \text{for } i = 2, \ldots, 7.$$

Thus ω_i is nonzero on $F_1(p_i) \sqcup E_2^i(p_i)$ vanishing at p_i and zero elsewhere.

We construct the analogous differentials for the covering f of type (iv), which is uniquely determined according to Proposition 7.5. So let

$$C = E_1 \cup_p E_2$$

with E_2 elliptic and E_1 a rational curve with one node q and let $f : \widetilde{C} \to C$ be the covering of type (iv). Then $\widetilde{C}$ consists of 14 components $F_1, \ldots, F_7$ isomorphic to $\mathbb{P}^1$ with $f_{|F_i} : F_i \to E_1$ the normalization and $E_2^1, \ldots, E_2^7$ all isomorphic to E_2 with $f_{|E_2^i} : E_2^i \to E_2$ the isomorphism. Then E_2^i intersects F_i in the point p_i lying over p and no other component of $\widetilde{C}$. If q_i and q_i' are the points of F_i lying over q, the F_i and F_{i+1} intersect transversally in the points q_i and q_{i+1}' for $i = 1, \ldots, 7$, where $q_8' = q_1$. The group G permutes the components and points cyclically, i.e., $\sigma(F_1) = F_{i+1}$ and similarly for E_2^i, p_i, q_i and q_i'.

The normalization $\tilde{n} : \widetilde{N} \to \widetilde{C}$ of the curve $\widetilde{C}$ is the disjoint union of the components F_i and E_2^i. We denote also the point p_i by the same letter when considered as a point of F_i and E_2^i. Consider the following line bundle on $\widetilde{N}$:

$$L = \bigsqcup_{i=1}^{7} \mathcal{O}_{F_i}(q_i + q_i' + p_i) \sqcup \bigsqcup_{i=1}^{7} \mathcal{O}_{E_2^i}(p_i).$$

Then $\omega_{\widetilde{C}}$ is the subsheaf of $\tilde{n}_*(L)$ consisting of (local) sections φ which viewed as sections of L satisfy the conditions

$$(\mathrm{Res}_{p_i})_{|F_i}(\varphi) + (\mathrm{Res}_{p_i})_{|E_2^i}(\varphi) = 0, \quad (\mathrm{Res}_{q_i})_{|F_i}(\varphi) + (\mathrm{Res}_{q_{i+1}'})_{|F_{i+1}}(\varphi) = 0 \tag{8-5}$$

for $i = 1, \ldots, 7$. Consider the following sections of $\tilde{n}_*(L)$, regarded as sections on $\tilde{N}$:

$$\omega_1 := \begin{cases} \text{nonzero sections of } \omega_{F_1}(q_1 + q_1' + p_1) \text{ and } \mathcal{O}_{E_2^1}(p_1) \text{ satisfying (8-5)} \\ \quad \text{at } p_1 \text{ and vanishing at } q_1 \text{ and } q_1', \\ 0 \text{ elsewhere,} \end{cases}$$

$$\omega_i := (\sigma^{-i})^*(\omega_1) \quad \text{for } i = 2, \ldots, 7.$$

Note that up to a multiplicative constant there is exactly one such section ω_1, since $h^0(\omega_{F_1}(q_1 + q_1' + p_1)) = 2$ and $h^0(\mathcal{O}_{E_2^1}(p_1)) = 1$.

Finally, we consider coverings of type (iii). Let $C = E_1 \cup_p E_2$, as for the covering of type (iv) above, and let $f : \tilde{C} \to C$ be a covering of type (iii). So $\tilde{C}$ consists of a rational curve F_1 with a node r lying over the node q of C and 7 components $E_2^1, \ldots, E_2^7$ all isomorphic to E_2. Then E_2^i intersects F_1 in the point p_i lying over p and intersects no other component of $\tilde{C}$. The group G acts on F_1 with only a fixed point r and permutes the E_i^2 and p_i cyclically as above. We use the following partial normalization $\tilde{n} : \tilde{N} \to \tilde{C}$ of $\tilde{C}$:

$$\tilde{N} := F_1 \sqcup \bigsqcup_{i=1}^{7} E_2^i.$$

Consider the following line bundle on $\tilde{N}$:

$$L = \mathcal{O}_{F_1}(p_1 + \cdots + p_7) \sqcup \bigsqcup_{i=1}^{7} \mathcal{O}_{E_2^i}(p_i).$$

Since the canonical bundles of F_1 and E_2^i are trivial, it is clear that $\omega_{\tilde{C}}$ is the subsheaf of $\tilde{n}_*(L)$ consisting of (local) sections φ which regarded as sections of L satisfy the relations

$$(\mathrm{Res}_{p_i})_{|F_1}(\varphi) + (\mathrm{Res}_{p_i})_{|E_2^i}(\varphi) = 0 \tag{8-6}$$

for $i = 1, \ldots, 7$. As before, define a section

$$\omega_1 := \begin{cases} \text{nonzero sections of } \mathcal{O}_{F_1}(p_1 + \cdots + p_7) \text{ and } \mathcal{O}_{E_2^i}(p_1) \\ \quad \text{vanishing at } p_1, \ldots, p_7, \\ 0 \text{ elsewhere,} \end{cases}$$

of $\tilde{n}_*(L)$, considered as a section of $\tilde{N}$, and define the sections ω_i, for $i = 2, \ldots, 7$, as in the previous cases. Note that up to a multiplicative constant there is exactly one such section ω_1.

From now on, $f : \tilde{C} \to C$ will be a covering of type (i)–(iv) as above. We fix a primitive 7-th root of unity, for example, $\rho := e^{2\pi i/7}$, and define for $i = 0, \ldots, 6$

the section

$$\Omega_i := \sum_{j=1}^{7} \rho^{ij} \omega_j.$$

Clearly Ω_i is a global section of L that defines a section of $\omega_{\widetilde{C}}$, which we denote with the same symbol.

Lemma 8.1. $\sigma^*(\Omega_i) = \rho^i \Omega_i$ *for* $i = 0, \dots, 6$. *In particular,* $\Omega_0 \in H^0(\widetilde{C}, \omega_{\widetilde{C}})^+$ *and* $\{\Omega_1, \dots, \Omega_6\}$ *is a basis of* $H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$.

Proof. The first assertion follows from a simple calculation using the definition of ω_i. So clearly $\Omega_0 \in H^0(\widetilde{C}, \omega_{\widetilde{C}})^+$ and $\Omega_i \in H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$ for $i = 1, \dots, 6$. Since $\Omega_1, \dots, \Omega_6$ are in different eigenspaces of σ, they are linearly independent and since $H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$ is of dimension 6, they form a basis. □

Remark 8.2. In cases (i), (ii) and (iv), $H^0(C, \omega_C \otimes \eta^{7-i})$ is the eigenspace of σ^i and Ω_i is a generator for $i = 1, \dots, 6$.

Proposition 8.3. *The map*

$$\phi : S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)^G \longrightarrow H^0(C, \omega_C^2)$$

is of rank 1.

Proof. We have to show that the kernel of ϕ is 2-dimensional. A basis of $S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)^G$ is given by $\{\Omega_1 \otimes \Omega_6, \Omega_2 \otimes \Omega_5, \Omega_3 \otimes \Omega_4\}$. Let a, b, c be complex numbers with

$$\phi(a\Omega_1 \otimes \Omega_6 + b\Omega_2 \otimes \Omega_5 + c\Omega_3 \otimes \Omega_4) = 0.$$

Define, for $i = 1, \dots, 7$,

$$\psi_i := \sum_{i=1}^{7} \omega_j \otimes \omega_{j+i-1}.$$

An easy but tedious computation gives

$$\Omega_1 \otimes \Omega_6 = \psi_1 + \rho\psi_7 + \rho^2\psi_6 + \rho^3\psi_5 + \rho^4\psi_4 + \rho^5\psi_3 + \rho^6\psi_2,$$
$$\Omega_2 \otimes \Omega_5 = \psi_1 + \rho\psi_4 + \rho^2\psi_7 + \rho^3\psi_3 + \rho^4\psi_6 + \rho^5\psi_2 + \rho^6\psi_5,$$
$$\Omega_3 \otimes \Omega_4 = \psi_1 + \rho\psi_3 + \rho^2\psi_5 + \rho^3\psi_7 + \rho^4\psi_2 + \rho^5\psi_4 + \rho^6\psi_6.$$

So we get

$$
\begin{aligned}
0 = {}& \phi\big((a+b+c)\psi_1 + (a\rho^6 + b\rho^5 + c\rho^4)\psi_2 + (a\rho^5 + b\rho^3 + c\rho)\psi_3 \\
& + (a\rho^4 + b\rho + c\rho^5)\psi_4 + (a\rho^3 + b\rho^6 + c\rho^2)\psi_5 \\
& + (a\rho^2 + b\rho^4 + c\rho^6)\psi_6 + (a\rho + b\rho^2 + c\rho^3)\psi_7\big) \\
= {}& (a+b+c)(\omega_1^2 + \cdots + \omega_7^2) \\
& + \big(a(\rho + \rho^6) + b(\rho^2 + \rho^5) + c(\rho^3 + \rho^4)\big) \sum_{j=1}^{7} \omega_j \omega_{j+1} \\
& + \big(a(\rho^2 + \rho^5) + b(\rho^3 + \rho^4) + c(\rho + \rho^6)\big) \sum_{j=1}^{7} \omega_j \omega_{j+2} \\
& + \big(a(\rho^3 + \rho^4) + b(\rho + \rho^6) + c(\rho^2 + \rho^5)\big) \sum_{j=1}^{7} \omega_j \omega_{j+3}.
\end{aligned}
$$

This section is zero if and only if its restriction to any component is zero. Now the restriction to N_i for all i gives

$$
0 = a + b + c + (6a + 6b + 6c)\sum_{j=1}^{6} \rho^j = -5(a+b+c).
$$

So $\phi(a\Omega_1 \otimes \Omega_6 + b\Omega_2 \otimes \Omega_5 + c\Omega_3 \otimes \Omega_4) = 0$ if and only if $a + b + c = 0$. Hence the kernel of ϕ is of dimension 2, which proves the proposition. $\square$

   Proposition 8.3 shows that the codifferential map along the divisors $\mathcal{S}_1$ and $\mathcal{S}_2$ in Proposition 7.5 is not surjective. In fact, as we will see later, the kernel of ϕ coincides with the conormal bundle of the image of these divisors in $\mathcal{B}_D$. In order to compute the degree we will perform a blow-up along these divisors.

   Let $\mathcal{E} \subset \mathcal{B}_D$ denote the one-dimensional locus consisting of the abelian varieties which are of the form $X = \ker(m : E^7 \to E)$ with $m(x_1, \ldots, x_7) = x_1 + \cdots + x_7$ for a given elliptic curve E. As we saw in Section 7, the induced polarization is of type D. Note that $\mathcal{E}$ is a closed subset of $\mathcal{B}_D$. The aim is to compute the degree of $\mathrm{Pr}_{7,2}$ above a point $X \in \mathcal{E}$. We denote by $\mathcal{S} \subset \widetilde{\mathcal{R}}_{2,7}$ the inverse image of $\mathcal{E}$ under $\widetilde{\mathrm{Pr}}_{2,7}$. According to Proposition 7.5, $\mathcal{S}$ is a divisor consisting of 2 irreducible components in the boundary $\widetilde{\mathcal{R}}_{2,7} \setminus \mathcal{R}_{2,7}$. We have $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where a general point of $\mathcal{S}_1$ corresponds to the G-covers with base an irreducible nodal curve of genus 1, and a point of $\mathcal{S}_2$ corresponds to a product of elliptic curves intersecting in a point. Moreover, for any fixed elliptic curve E, $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect in the unique point given by the covering of type (iv).

   As in [Donagi and Smith 1981], let $\widetilde{\mathcal{B}}_D$ be the blow-up of $\mathcal{B}_D$ along $\mathcal{E}$ and $\widetilde{\mathcal{R}} \simeq \widetilde{\mathcal{R}}_{2,7}$ the blow-up of $\widetilde{\mathcal{R}}_{2,7}$ along the divisor $\mathcal{S} = \widetilde{\mathrm{Pr}}_{2,7}^{-1}(\mathcal{E})$. We then obtain the

commutative diagrams

$$
\begin{array}{ccc}
\widetilde{\mathcal{R}} & \xrightarrow{\widetilde{\mathcal{P}}} & \widetilde{\mathcal{B}}_D \\
\simeq \downarrow & & \downarrow \\
\widetilde{\mathcal{R}}_{2,7} & \xrightarrow{\widetilde{\mathrm{Pr}}_{2,7}} & \mathcal{B}_D
\end{array}
\qquad
\begin{array}{ccc}
\widetilde{\mathcal{S}} & \xrightarrow{\widetilde{\mathcal{P}}} & \widetilde{\mathcal{E}} \\
\simeq \downarrow & & \downarrow {\scriptstyle \mathbb{P}^1\text{-bundle}} \\
\mathcal{S} & \xrightarrow{\widetilde{\mathrm{Pr}}_{2,7}} & \mathcal{E}
\end{array}
$$

in which $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{E}}$ are the exceptional loci.

Lemma I.3.2 of [Donagi and Smith 1981] guarantees that the local degree of $\widetilde{\mathrm{Pr}}_{2,7}$ along a component of $\mathcal{S}$ equals the degree of the induced map on the exceptional divisors $\widetilde{\mathcal{P}}_{|\mathcal{S}_i} : \mathcal{S}_i \to \widetilde{\mathcal{E}}$ if the codifferential map $\mathcal{P}^*$ is surjective on the respective conormal bundles. Recall that the fibers of the conormal bundles at the point X are given by

$$
\mathcal{N}^*_{X,\mathcal{E}/\mathcal{B}_D} = \ker(T^*_X \mathcal{B}_D \to T^*_X \mathcal{E}),
$$

$$
\mathcal{N}^*_{(C,\eta),\mathcal{S}_i/\widetilde{\mathcal{R}}_{2,7}} = \ker(T^*_{(C,\eta)}\widetilde{\mathcal{R}}_{2,7} \to T^*_{(C,\eta)}\mathcal{S}_i)
$$

for $i = 1, 2$. As in [Donagi and Smith 1981], by taking level structures on the moduli spaces we can assume we are working on fine moduli spaces, which allows us to identify the tangent space to $\mathcal{S}_1$ at the G-admissible cover $[\widetilde{C} \to C]$, where $C = E/(p \sim q)$, with the tangent space to $\overline{\mathcal{M}}_2$ at C. Thus the conormal bundle $\mathcal{N}^*_{(C,\eta),\mathcal{S}_1/\widetilde{\mathcal{R}}_{2,7}}$ can be identified with the conormal bundle $\mathcal{N}^*_{C,\Delta_0/\overline{\mathcal{M}}_2} \subset H^0(\Omega_C \otimes \omega_C)$, where Δ_0 is the divisor of irreducible nodal curves in $\overline{\mathcal{M}}_2$. Similarly, we can identify the tangent space to $\mathcal{S}_2$ at $[\widetilde{C} \to C]$, where $C = E_1 \cup_p E_2$, with the tangent space to $\overline{\mathcal{M}}_2$ at C. Thus $\mathcal{N}^*_{(C,\eta),\mathcal{S}_2/\widetilde{\mathcal{R}}_{2,7}}$ can be identified with $\mathcal{N}^*_{C,\Delta_1/\overline{\mathcal{M}}_2} \subset H^0(\Omega_C \otimes \omega_C)$, where Δ_1 is the divisor of reducible nodal curves in $\overline{\mathcal{M}}_2$.

Using the fact that $X = \mathrm{Pr}(\widetilde{C}, C)$ we can identify $(T_X \mathcal{A}_D)^* \simeq \bigoplus_{i=1}^6 \Omega_i \mathbb{C}$ and $S^2(H^0(\widetilde{C}, \omega_C)^-)^G$ as in (8-2). Then for a covering (C, η) the conormal bundles fit in the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N}^*_{X,\mathcal{E}/\mathcal{B}_D} & \longrightarrow & S^2(H^0(\widetilde{C}, \omega_C)^-)^G & \longrightarrow & T^*_X \mathcal{E} & \longrightarrow & 0 \\
& & \downarrow {\scriptstyle n^*} & & \downarrow {\scriptstyle \mathcal{P}^*} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}^*_{(C,\eta),\mathcal{S}_i/\widetilde{\mathcal{R}}_{2,7}} & \longrightarrow & H^0(C, \Omega_C \otimes \omega_C) & \longrightarrow & T^*_{(C,\eta)}\mathcal{S}_i & \longrightarrow & 0 \\
& & & & \downarrow {\scriptstyle H^0(j)} & & & & \\
& & & & H^0(C, \omega_C^2) & & & &
\end{array}
\qquad (8\text{-}7)
$$

in which n^* is the conormal map, $i = 1, 2$ and j is induced by the canonical map $\Omega_C \to \omega_C$ (see [Donagi and Smith 1981, IV, 2.3.3]).

Lemma 8.4. *The kernel of $H^0(j)$ is one-dimensional.*

Proof. As a map of sheaves, the canonical map $j : \Omega_C \otimes \omega_C \to \omega_C^2$ has one-dimensional kernel, namely, the one-dimensional torsion sheaf with support the node of C (see [Donagi and Smith 1981, IV, 2.3.3]). On the other hand, the map $H^0(j)$ is the composition of the pullback to the normalization with the push forward to C. This implies that the kernel of $H^0(j)$ consists exactly of the sections of the skyscraper sheaf supported at the node, and hence is one-dimensional. $\qquad\square$

Proposition 8.5. *For coverings of type* (i)–(iv) *the restricted codifferential map* $n^* : \mathcal{N}_{X,\mathcal{E}/\mathcal{B}_D}^* \to \mathcal{N}_{(C,\eta),\mathcal{S}_i/\widetilde{\mathcal{R}}_{2,7}}^*$ *is surjective, for $i = 1, 2$.*

Proof. First notice that from the "local-global" exact sequence (see [Bardelli 1989]) we have

$$\operatorname{Ker} H^0(j) = \mathcal{N}_{C,\Delta_0/\overline{\mathcal{M}}_2}^* \subset H^0(\Omega_C \otimes \omega_C).$$

Therefore, $\operatorname{Ker} H^0(j) \subset \operatorname{Im} \mathcal{P}^*$. Since $\dim \operatorname{Ker} H^0(j) = 1$ and, by Proposition 8.3, $\dim \operatorname{Ker}(H^0(j) \circ \mathcal{P}^*) = 2$, we have $\dim \operatorname{Ker}(\mathcal{P}^*) = 1$. By diagram (8-7) this implies that the kernel of n^* is of dimension ≤ 1. Since $\mathcal{N}_{X,\mathcal{E}/\mathcal{B}_D}^*$ is a vector space of dimension 2 and $\mathcal{N}_{(C,\eta),\mathcal{S}_i/\widetilde{\mathcal{R}}_{2,7}}^*$ a vector space of dimension 1, it follows that n^* is surjective. $\qquad\square$

9. Local degree of $\mathrm{Pr}_{2,7}$ over the boundary divisors

First we compute the local degree of the Prym map $\widetilde{\mathrm{Pr}}_{2,7}$ along the divisor $\mathcal{S}_1$. Since the conormal map of $\mathrm{Pr}_{2,7}$ along $\mathcal{S}_1$ is surjective according to Proposition 8.5, [Donagi and Smith 1981, I, Lemma 3.2] implies that the local degree along $\mathcal{S}_1$ is given by the degree of the induced map $\widetilde{\mathcal{P}} : \widetilde{\mathcal{S}}_1 \to \widetilde{\mathcal{E}}$ on the exceptional divisor $\widetilde{\mathcal{S}}_1$. Now the polarized abelian variety $X(E)$ is uniquely determined by the elliptic curve E, according to its definition. Hence the curve $\mathcal{E}$ can be identified with the moduli space of elliptic curves, i.e., with the affine line. The exceptional divisor $\widetilde{\mathcal{E}}$ is then a $\mathbb{P}^1$-bundle over $\mathcal{E}$. On the other hand, $\mathcal{S}_1$ is a divisor in $\widetilde{\mathcal{R}}_{2,7}$, so $\widetilde{\mathcal{S}}_1$ is isomorphic to $\mathcal{S}_1$. Clearly $\widetilde{\mathcal{P}}$ maps the fibers $\widetilde{\mathrm{Pr}}^{-1}(X(E)) \cap \widetilde{\mathcal{S}}_1$ onto the fibers $\mathbb{P}^1$ over the elliptic curves E.

Now $\widetilde{\mathrm{Pr}}^{-1}(X(E)) \cap \widetilde{\mathcal{S}}_1$ consists of coverings of type (i) and one covering of type (iv), which we denote by $C_E^{(iv)}$. The coverings of type (i) have as base a nodal curve of the form $C = E/p \sim q$ and we can assume that $p = 0$, thus $\widetilde{\mathrm{Pr}}^{-1}(X(E)) \cap \widetilde{\mathcal{S}}_1$ is parametrized by E itself (the point $q = 0$ corresponds to the covering of type (iv)). Hence the induced conormal map on the exceptional divisors $\widetilde{\mathcal{P}} : \widetilde{\mathcal{S}}_1 \to \widetilde{\mathcal{E}}$ restricted to the fiber over $X(E)$ is a map $\phi : E \to \mathbb{P}^1$. Combining everything, we conclude that the local degree of the Prym map along $\mathcal{S}_1$ coincides with the degree of the induced map $\phi : E \to \mathbb{P}^1$.

Proposition 9.1. *The local degree of the Prym map $\widetilde{\mathrm{Pr}}_{2,7}$ along $\mathcal{S}_1$ is 2.*

Proof. According to what we have written above, it is sufficient to show that the map $\phi : E \to \mathbb{P}^1$ induced by $\widetilde{\mathcal{P}}$ is a double covering. We use again the identification (8-2) (and its analogue for coverings of type (iii)). As in [Donagi and Smith 1981], let x, y be local coordinates at 0 and q, and let dx, dy be the corresponding differentials. If $(a, b, c) \in S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)$ are coordinates in the basis of Lemma 8.1, then $\mathbb{P}(\mathrm{Ker}(H^0(j) \circ \mathcal{P}^*)) \simeq \mathbb{P}^1$ has coordinates $[a, b]$ and its dual is identified with $\mathbb{P}(\mathrm{Im}\,\widetilde{\mathcal{P}}_{|\mathrm{fiber}})$. In order to describe the kernel of $\mathcal{P}^*$, we look at the multiplication on the stalk over the node $p = (q \sim 0)$. Around p the line bundles η^i are trivial; therefore, the element $(a, b, c) \in \bigoplus_{i=1}^3 (\omega_{C,p} \otimes \omega_{C,p})$ (in coordinates $a, b, c \in \mathcal{O}_p$ for a fixed basis of $\omega_{C,p} \otimes \omega_{C,p}$) is sent to $a + b + c \in (\Omega_C \otimes \omega_C)_p$ under $\widetilde{\mathcal{P}}$. Thus the germ $a + b + c \in \mathcal{O}_p$ is zero if it is in the kernel of $\mathcal{P}^*$. In particular, the coefficient of $dx\,dy$ must vanish. Set $\alpha = a_0\,dx$, $\beta = b_0\,dx$, $\gamma = c_0\,dx$, and let $\alpha = a_q\,dy$, $\beta = b_q\,dy$, $\gamma = c_q\,dy$ be the local description of the differentials. Then the coefficient of $dx\,dy$ must satisfy

$$a_0 a_q + b_0 b_q + c_0 c_q = 0. \tag{9-1}$$

Now, by looking at the dual picture, we consider $\mathbb{P}^1 = \mathbb{P}(\mathrm{Ker}\,\mathcal{P}^*)^* \subset \mathbb{P}^{2*}$. Let E be embedded in $\mathbb{P}^{2*}$ by the linear system $|3 \cdot 0|$. The coordinate functions $[a, b, c] \in \mathbb{P}^{2*}$ satisfy condition (9-1) for all $q \in E$. Then the points on the fiber over $[a_q, b_q, c_q]$ are points in E over the line passing through the origin $0 \in E \subset \mathbb{P}^{2*}$ and q. Hence the map $E \to \mathbb{P}^1$ corresponds to the restriction to E of the projection $\mathbb{P}^{2*} \to \mathbb{P}^1$ from the origin, which is the double covering $E \to \mathbb{P}^1$ determined by the divisor $0 + q$ of E and thus of degree two. $\qquad\square$

We now turn our attention to the Prym map on $\mathcal{S}_2$. By the surjectivity of the conormal map of $\mathrm{Pr}_{2,7}$ on $\mathcal{S}_2$ (Proposition 8.5), the local degree along $\mathcal{S}_2$ is computed by the degree of the map $\widetilde{\mathcal{P}} : \widetilde{\mathcal{S}}_2 \to \widetilde{\mathcal{E}}$ on the divisor $\widetilde{\mathcal{S}}_2$, which is a $\mathbb{P}^1$-bundle over $\mathcal{E}$. Given an elliptic curve E, the fiber of $\widetilde{\mathrm{Pr}}^{-1}(X(E))$ intersected with the divisor $\mathcal{S}_2$ consists of coverings of type (ii), one covering of type (iii), denoted by $\mathcal{C}_E^{(\mathrm{iii})}$, and one covering of type (iv), $\mathcal{C}_E^{(\mathrm{iv})}$, which lies in the intersection with the divisor $\mathcal{S}_1$.

Recall that the type (ii) coverings have base curve $C = E_1 \cup E$ intersecting at one point, which we can assume to be 0, and with E_1 an arbitrary elliptic curve. The covering over C is the union of a degree-7 étale cyclic covering F_1 over E_1 and 7 elliptic curves E_i attached to F_1 mapping each one of them isomorphically to E. So the type (ii) coverings on the fiber over E are parametrized by pairs $(E_1, \langle \eta \rangle)$, where E_1 is an elliptic curve and $\langle \eta \rangle \subset E_1$ is a subgroup of order 7.

It is known that the parametrization space of the pairs $(E_1, \langle \eta \rangle)$ is the modular curve $Y_0(7) := \Gamma_0(7) \setminus \mathbb{H}$. The natural projection $(E_1, \eta) \mapsto E_1$ defines a map $\pi_0 : Y_0(7) \to \mathbb{C}$. Moreover, the curve $Y_0(7)$ admits a compactification $X_0(7) := \overline{Y_0(7)}$ such that the map π_0 extends to a map $\pi : X_0(7) \to \mathbb{P}^1$ (see [Silverman 1986]). The genus of $X_0(7)$ can be computed by the Hurwitz formula using the fact that π is of

degree 8, and it is ramified over the points corresponding to elliptic curves with j-invariant 0 and 12^3 (with ramification degree 4 on each fiber) and over ∞, where the inverse image consists of two cusps, one étale and the other of ramification index 7. The two cusps over ∞ represent the coverings $\mathcal{C}_E^{(\mathrm{iii})}$ and $\mathcal{C}_E^{(\mathrm{iv})}$ above $X(E)$ (see Remark 9.4). This gives that $X_0(7)$ is of genus zero. Thus, we can identify $\widetilde{\mathcal{S}}_2 \cap \widetilde{\mathrm{Pr}}^{-1}(X(E))$ with $X_0(7) \simeq \mathbb{P}^1$.

Then, since $\widetilde{\mathcal{S}}_2 \simeq \mathcal{S}_2$, the restriction of the conormal map $\widetilde{\mathcal{P}}$ to a fiber over the point $[E] \in \mathcal{E}$ is a map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$.

Proposition 9.2. *The map π coincides with the map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ of the fibers of $\widetilde{\mathcal{S}}_2 \to \widetilde{\mathcal{E}}$ over a point $[E] \in \mathcal{E}$.*

Proof. Let o and o' be the zero elements of E and E_1, respectively, with local coordinates x and y, respectively. Set $\alpha = a_o\,dx$, $\beta = b_o\,dx$, $\gamma = c_o\,dx$, and let $\alpha = a_{o'}\,dy$, $\beta = b_{o'}\,dy$, $\gamma = c_{o'}\,dy$ be the local description of elements of $(\omega_{C,p} \otimes \omega_{C,p})$ around the node $p = (o \sim o')$. As in the proof of Proposition 9.1, for an element (a, b, c) in the kernel of $\mathcal{P}^*$, the coefficient of $dx\,dy$ must vanish, i.e., it satisfies

$$a_o a_{o'} + b_o b_{o'} + c_o c_{o'} = 0 \tag{9-2}$$

in a neighborhood of the node. Considering the dual map, one sees that the fiber of $\widetilde{\mathcal{P}}$ over a point $[a_o, b_o, c_o] \in \mathbb{P}(\mathrm{Ker}\,\mathcal{P}^*)^* \subset \mathbb{P}^2$ with $c_o = -a_o - b_o$ corresponds to the pairs $(E_1, \langle \eta \rangle)$ such that the local functions $a, b, c \in \mathcal{O}_p$ take the values $[a_{o'}, b_{o'}, c_{o'}]$ around the $o' \in E_1$ with $c_{o'} = -a_{o'} - b_{o'}$ and such that they verify (9-2). This determines completely the triple $[a_{o'}, b_{o'}, c_{o'}]$, which depends only on the values at the node $o' \in E_1$ of the base curve. Note that a, b, c are elements of the local ring $\mathcal{O}_p$, which determines the curve E_1 uniquely. In fact, its quotient ring is the direct product of the function fields of E_1 and E, which in turn determines the curves. Therefore, the map ψ can be identified with the projection $(E_1, \langle \eta \rangle) \mapsto E_1$. $\square$

As an immediate consequence we have:

Corollary 9.3. *The local degree of the Prym map $\widetilde{\mathrm{Pr}}_{2,7}$ along the divisor $\mathcal{S}_2$ is 8.*

Using Proposition 9.1 and Corollary 9.3 we conclude that the degree of the Prym map $\widetilde{\mathrm{Pr}}_{2,7}$ is 10, which finishes the proof of Theorem 1.1

Remark 9.4. The moduli interpretation of $X_0(7) \setminus Y_0(7)$ is given by the *Néron polygons*: one of the cusps represents a 1-*gon*, that is, a nodal cubic curve, corresponding to the covering $\mathcal{C}_E^{(\mathrm{iii})}$, and the other represents a 7-*gon*, that is, 7 copies of $\mathbb{P}^1$ with the point 0 of one attached to the point ∞ of the other in a closed chain, which corresponds to the covering $\mathcal{C}_E^{(\mathrm{iv})}$ (see [Silverman 1994, IV, §8]).

Acknowledgements

We thank E. Esteves for his useful suggestions for the proof of Theorem 6.2 and the referee for pointing out an error and suggesting some improvements in the presentation. Ortega is thankful to G. Farkas for stimulating discussions.

References

[Abramovich et al. 2003] D. Abramovich, A. Corti, and A. Vistoli, "Twisted bundles and admissible covers", *Comm. Algebra* **31**:8 (2003), 3547–3618. MR Zbl

[Arbarello et al. 2011] E. Arbarello, M. Cornalba, and P. A. Griffiths, *Geometry of algebraic curves*, vol. II, Grundlehren der Mathematischen Wissenschaften **268**, Springer, Heidelberg, 2011. MR Zbl

[Bardelli 1989] F. Bardelli, "Lectures on stable curves", pp. 648–704 in *Lectures on Riemann surfaces* (Trieste, 1987), edited by M. Cornalba et al., World Scientific, Teaneck, NJ, 1989. MR Zbl

[Beauville 1977] A. Beauville, "Prym varieties and the Schottky problem", *Invent. Math.* **41**:2 (1977), 149–196. MR Zbl

[Birkenhake and Lange 2004] C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **302**, Springer, Berlin, 2004. MR Zbl

[Donagi and Smith 1981] R. Donagi and R. C. Smith, "The structure of the Prym map", *Acta Math.* **146**:1 (1981), 25–102. MR Zbl

[Faber 1988] C. Faber, "Prym varieties of triple cyclic covers", *Math. Z.* **199**:1 (1988), 61–79. MR Zbl

[Lange and Ortega 2011] H. Lange and A. Ortega, "Prym varieties of cyclic coverings", *Geom. Dedicata* **150** (2011), 391–403. MR Zbl

[Lange and Ortega ≥ 2016] H. Lange and A. Ortega, "On the Prym map of cyclic coverings", in preparation.

[Mumford 1971] D. Mumford, "Theta characteristics of an algebraic curve", *Ann. Sci. École Norm. Sup.* (4) **4**:2 (1971), 181–192. MR Zbl

[Ortega 2003] A. Ortega, "Variétés de Prym associées aux revêtements n-cycliques d'une courbe hyperelliptique", *Math. Z.* **245**:1 (2003), 97–103. MR Zbl

[Shimura 1963] G. Shimura, "On analytic families of polarized abelian varieties and automorphic functions", *Ann. of Math.* (2) **78** (1963), 149–192. MR Zbl

[Silverman 1986] J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics **106**, Springer, New York, NY, 1986. MR Zbl

[Silverman 1994] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics **151**, Springer, New York, NY, 1994. MR Zbl

[Welters 1983] G. E. Welters, "Polarized abelian varieties and the heat equations", *Compositio Math.* **49**:2 (1983), 173–194. MR Zbl

lange@math.fau.de   *Department Mathematik, Universität Erlangen, Cauerstr. 11, D-91058 Erlangen, Germany*

ortega@math.hu-berlin.de   *Institut für Mathematik, Humboldt Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany*

Algebra & Number Theory

msp.org/ant

Algebra & Number Theory