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The Prym map of degree-7 cyclic coverings

Herbert Lange and Angela Ortega

We study the Prym map for degree-7 étale cyclic coverings over a curve of genus 2. We extend this map to a proper map on a partial compactification of the moduli space and prove that the Prym map is generically finite onto its image of degree 10.

1. Introduction

Consider an étale finite covering $f : Y \to X$ of degree p of a smooth complex projective curve X of genus $g \geq 2$. Let $\mathrm{Nm}_f : JY \to JX$ denote the norm map of the corresponding Jacobians. One can associate to the covering f its Prym variety

$$P(f) := (\mathrm{Ker}\,\mathrm{Nm}_f)^0,$$

the connected component containing 0 of the kernel of the norm map, which is an abelian variety of dimension

$$\dim P(f) = g(Y) - g(X) = (p-1)(g-1).$$

The variety $P(f)$ carries a natural polarization, namely, the restriction of the principal polarization Θ_Y of JY to $P(f)$. Let D denote the type of this polarization. If, moreover, $f : Y \to X$ is a cyclic covering of degree p, then the group action induces an action on the Prym variety. Let $\mathcal{B}_D$ denote the moduli space of abelian varieties of dimension $(p-1)(g-1)$ with a polarization of type D and an automorphism of order p compatible with the polarization. If $\mathcal{R}_{g,p}$ denotes the moduli space of étale cyclic coverings of degree p of curves of genus g, we get a map

$$\mathrm{Pr}_{g,p} : \mathcal{R}_{g,p} \to \mathcal{B}_D$$

associating to every covering in $\mathcal{R}_{g,p}$ its Prym variety, called the *Prym map*.

Particularly interesting are the cases where $\dim \mathcal{R}_{g,p} = \dim \mathcal{B}_D$. For instance, for $p = 2$ this occurs only if $g = 6$. In this case the Prym map $\mathrm{Pr}_{6,2} : \mathcal{R}_6 \to \mathcal{A}_5$ is generically finite of degree 27 (see [Donagi and Smith 1981]) and the fibers carry

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the structure of the 27 lines on a smooth cubic surface. For $(g, p) = (4, 3)$, it is also known that $\mathrm{Pr}_{4,3}$ is generically finite of degree 16 onto its 9-dimensional image $\mathcal{B}_D$ (see [Faber 1988]).

In this paper we investigate the case $(g, p) = (2, 7)$, where $\dim \mathcal{R}_{g,p} = \dim \mathcal{B}_D$. The main result of the paper is the following theorem. Let G be the cyclic group of order 7.

Theorem 1.1. *For any cyclic étale G-cover $f : \widetilde{C} \to C$ of a curve C of genus* 2, *the Prym variety* $\mathrm{Pr}(f)$ *is an abelian variety of dimension* 6 *with a polarization of type* $D = (1, 1, 1, 1, 1, 7)$ *and a G-action. The Prym map*

$$\mathrm{Pr}_{2,7} : \mathcal{R}_{2,7} \to \mathcal{B}_D$$

is generically finite of degree 10.

The paper is organized as follows. First we compute in Section 2 the dimension of the moduli space $\mathcal{B}_D$ when $(g, p) = (2, 7)$. In Sections 3–5, we extend the Prym map to a partial compactification of admissible coverings $\widetilde{\mathcal{R}}_{2,7}$ such that $\mathrm{Pr}_{2,7} : \widetilde{\mathcal{R}}_{2,7} \to \mathcal{B}_D$ is a proper map. We prove the generic finiteness of the Prym map in Section 6 by specializing to a curve in the boundary. In order to compute the degree of the Prym map, we describe in Section 7 a complete fiber over a special abelian sixfold with polarization type $(1, 1, 1, 1, 1, 7)$, and in Section 8 we give a basis for the Prym differentials for the different types of admissible coverings appearing in the special fiber. Finally, in Section 9 we determine the degree of the Prym map by computing the local degrees along the special fiber.

2. Dimension of the moduli space $\mathcal{B}_D$

As in the introduction, let $\mathcal{R}_{2,7}$ denote the moduli space of nontrivial cyclic étale coverings $f : \widetilde{C} \to C$ of degree 7 of curves of genus 2. The Hurwitz formula gives $g(\widetilde{C}) = 8$. Hence the Prym variety $P = P(f)$ is of dimension 6 and the canonical polarization of the Jacobian $J\widetilde{C}$ induces a polarization of type $(1, 1, 1, 1, 1, 7)$ on P (see [Lange and Ortega 2011, p. 397]). Let σ denote an automorphism of $J\widetilde{C}$ generating the group of automorphisms of $\widetilde{C}/C$. It induces an automorphism of P, also of order 7, which is compatible with the polarization. The Prym map $\mathrm{Pr}_{2,7} : \mathcal{R}_{2,7} \to \mathcal{B}_D$ is the morphism defined by $f \mapsto P(f)$. Here $\mathcal{B}_D$ is the moduli space of abelian varieties of dimension 6 with a polarization of type $(1, 1, 1, 1, 1, 7)$ and an automorphism of order 7 compatible with the polarization. The main result of this section is the following proposition.

Proposition 2.1. $$\dim \mathcal{B}_D = \dim \mathcal{R}_{2,7} = 3.$$

Proof. Clearly $\dim \mathcal{R}_{2,7} = \dim \mathcal{M}_2 = 3$. So we have to show that also $\dim \mathcal{B}_D = 3$. For this we use Shimura's theory of abelian varieties with endomorphism structure (see [Shimura 1963] or [Birkenhake and Lange 2004, Chapter 9]).

Let $K = \mathbb{Q}(\rho_7)$ denote the cyclotomic field generated by a primitive 7-th root of unity ρ_7. Clearly $\mathcal{B}_D$ coincides with one of Shimura's moduli spaces of polarized abelian varieties with endomorphism structure in K. The field K is a totally complex quadratic extension of a totally real number field of degree $e_0 = 3$. Define

$$m := \frac{\dim P}{e_0} = 2.$$

The polarization of P depends on the lattice of P and a matrix $T \in M_m(\mathbb{Q}(\rho_7))$. The signature of T (see [Birkenhake and Lange 2004, p. 264]) is an e_0-tuple of nonnegative integers $((r_1, s_1), \dots, (r_{e_0}, s_{e_0}))$ satisfying

$$r_\nu + s_\nu = m = 2$$

for all ν, where e_0 is the number of real embeddings of the totally real subfield of $\mathbb{Q}(\rho_7)$. Recall that for each embedding $\mathbb{Q}(\rho_7) \hookrightarrow \mathbb{C}$, the matrix T is skew-hermitian, and the (r_ν, s_ν) are the signatures of the corresponding skew-hermitian matrices. Then, according to [Shimura 1963, p. 162] or [Birkenhake and Lange 2004, p. 266, lines 6–8], we have

$$\dim \mathcal{B}_D = \sum_{\nu=1}^{e_0} r_\nu s_\nu \le 3, \tag{2-1}$$

with equality if and only if $r_\nu = s_\nu = 1$ for all ν.

On the other hand, in Section 6 we will see that the map $\mathrm{Pr}_{2,7}$ is generically injective. This implies that

$$\dim \mathcal{B}_D \ge \dim \mathcal{R}_{2,7} = 3,$$

which completes the proof of the proposition. □

Remark 2.2. According to [Ortega 2003], we know that P is isogenous to the product of a Jacobian of dimension 3 with itself. Then $\mathrm{End}_{\mathbb{Q}}(P)$ is not a simple algebra. Hence, if one knows that $\mathrm{Pr}_{2,7}$ is dominant onto the component $\mathcal{B}_D$, then [Birkenhake and Lange 2004, Proposition 9.9.1] implies that $r_\nu = s_\nu = 1$ for $\nu = 1, 2, 3$, which also gives $\dim \mathcal{B}_D = 3$.

Remark 2.3. It is claimed in [Faber 1988] that the Prym map $\mathrm{Pr}_{2,6} : \mathcal{R}_{2,6} \to \mathcal{B}_D$ satisfies $\dim \mathcal{B}_D = \dim \mathcal{R}_{2,6} = 3$. In a subsequent paper [Lange and Ortega ≥ 2016], we show that this does not occur. Moreover, we prove that there are no further examples with this property in the case of étale cyclic coverings of degree $2p$ for a prime p, but there are 3 more cases for cyclic ramified coverings of these degrees.

3. The condition (∗)

In this section we study the Prym map for coverings of degree 7 between stable curves. Let $G = \mathbb{Z}/7\mathbb{Z}$ be the cyclic group of order 7 with generator σ and let

$f : \widetilde{C} \to C$ be a G-cover of a connected stable curve C of arithmetic genus g. We fix in the sequel a primitive 7-th root of unity ρ. We assume the following condition for the covering f:

(∗) The fixed points of σ are exactly the nodes of $\widetilde{C}$ and at each node one local parameter is multiplied by ρ^{δ} and the other by $\rho^{-\delta}$ for some $\delta,\ 1 \le \delta \le 3$.

As in [Beauville 1977], we have $f^*\omega_C \simeq \omega_{\widetilde{C}}$, which implies

$$p_a(\widetilde{C}) = 7g - 6.$$

Let $\widetilde{N}$ and N be the normalizations of $\widetilde{C}$ and C, respectively, and let $\tilde{f} : \widetilde{N} \to N$ be the induced map. At each node s of $\widetilde{C}$ we make the usual identification

$$\mathcal{K}_s^*/\mathcal{O}_s^* \simeq \mathbb{C}^* \times \mathbb{Z} \times \mathbb{Z}.$$

Then the action of σ on $\mathcal{K}_s^*/\mathcal{O}_s^*$ is

$$\sigma^*((z, m, n)_s) = (\rho^{\delta(m-n)} z, m, n)_s$$

for some $\delta,\ 1 \le \delta \le 3$. Here we label the branches at the node s such that a local parameter at the first branch (corresponding to m) is multiplied by ρ^{δ} with $1 \le \delta \le 3$. Then we have

$$f_*((z, m, n)_s) = (z^7, m, n)_{f(s)}$$

since $f_*((z, m, n)_s) = \sum_{k=0}^{6} (\sigma^*)^k (z, m, n)_s$, viewed as a divisor at $f(s)$, and

$$\begin{aligned}
\sum_{k=0}^{6} (\sigma^*)^k (z, m, n)_s &= \left(\prod_{k=0}^{6} (\rho^{\delta(m-n)})^k z, 7m, 7n \right)_s \\
&= \left(\rho^{\sum_{k=0}^{6} \delta(m-n)k} z^7, 7m, 7n \right)_s \\
&= \left(\rho^{\delta(m-n)\frac{6\cdot 7}{2}} z^7, 7m, 7n \right)_s \\
&= (z^7, 7m, 7n)_s \\
&= f^*(z^7, m, n)_{f(s)}.
\end{aligned}$$

We define the multidegree of a line bundle L on $\widetilde{C}$ by

$$\deg L = (d_1, \dots, d_v),$$

where v is the number of components of $\widetilde{C}$ and d_i is the degree of L on the i-th component of $\widetilde{C}$.

Lemma 3.1. *Let $L \in \operatorname{Pic} \widetilde{C}$ with $\operatorname{Nm} L \simeq \mathcal{O}_{\widetilde{C}}$. Then*

$$L \simeq M \otimes \sigma^* M^{-1}$$

for some $M \in \operatorname{Pic} \widetilde{C}$. *Moreover*, M *can be chosen of multidegree* $(k, 0, \dots, 0)$ *with* $0 \le k \le 6$.

Proof. As in [Mumford 1971, Lemma 1], using Tsen's theorem, there is a divisor D such that $L \simeq \mathcal{O}_{\widetilde{C}}(D)$ and $f_* D = 0$. We write

$$D = \underbrace{\sum_{x \in \widetilde{C}_{\mathrm{reg}}} x}_{D_{\mathrm{reg}}} + \underbrace{\sum_{s \in \widetilde{C}_{\mathrm{sing}}} (z_s, m_s, n_s)_s}_{D_{\mathrm{sing}}} . \tag{3-1}$$

If $x \in \widetilde{C}_{\mathrm{reg}}$ is in the support of D_{reg}, then $\sigma^k(x)$ is in the support of D_{reg} for some $1 \le k \le 6$. Since

$$x - \sigma^k(x) = (x + \sigma(x) + \cdots + \sigma^{k-1}(x)) - \sigma(x + \sigma(x) + \cdots + \sigma^{k-1}(x)),$$

there is a divisor E_{reg} such that $D_{\mathrm{reg}} = E_{\mathrm{reg}} - \sigma^* E_{\mathrm{reg}}$. From (3-1) one sees that the divisor D_{sing} is the sum of divisors of the form $(\rho^{a_s}, 0, 0)_s$ for some $1 \le a_s \le 6$. Choosing an integer i_s such that $-i_s \delta_s \equiv a_s \mod 7$, we have

$$(1, i_s, 0) - \sigma^*(1, i_s, 0) = (\rho^{-\delta_s i_s}, 0, 0) = (\rho^{a_s}, 0, 0)_s.$$

Then there is a divisor E_{sing} such that $D_{\mathrm{sing}} = E_{\mathrm{sing}} - \sigma^* E_{\mathrm{sing}}$. Thus $D = E - \sigma^* E$ with $E = E_{\mathrm{reg}} + E_{\mathrm{sing}}$. Set $M = \mathcal{O}_{\widetilde{C}}(E)$. By replacing M with $M \otimes f^* N$, where N is a line bundle on C, we may assume that the multidegree of M is $(\epsilon_1, \dots, \epsilon_\nu)$ with $0 \le \epsilon_i \le 6$. Using the fact that $\widetilde{C}$ is connected, the multidegree can be accumulated on one of the components by further tensoring M with line bundles associated to divisors of the form $(1, 1, -1)_s$, since $(1, 1, -1)_s - \sigma^*(1, 1, -1)_s = (1, 0, 0)_s = 0$. □

Let P denote the Prym variety of $f : \widetilde{C} \to C$, i.e., the connected component containing 0 of the kernel of norm map $\mathrm{Nm} : J\widetilde{C} \to JC$. By definition, it is a connected commutative algebraic group. Lemma 3.1 implies that P is the variety of line bundles in $\ker \mathrm{Nm}$ of the form $M \otimes \sigma^* M^{-1}$ with M of multidegree $(0, \dots, 0)$. Let $\nu : N \to C$ and $\tilde{\nu} : \widetilde{N} \to \widetilde{C}$ denote the normalizations of C and $\widetilde{C}$, respectively. Hence there is a map $\tilde{f}$ making the following diagram commutative:

$$\begin{tikzcd} \widetilde{N} \arrow[r, "\tilde{\nu}"] \arrow[d, "\tilde{f}"'] & \widetilde{C} \arrow[d, "f"] \\ N \arrow[r, "\nu"] & C \end{tikzcd}$$

Proposition 3.2. *Suppose* $p_a(C) = g$. *Then* P *is an abelian variety of dimension* $6g - 6$.

Proof. (As in [Beauville 1977] and [Faber 1988].) Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{T} \arrow[r] \arrow[d, "\mathrm{Nm}"] & J\widetilde{C} \arrow[r, "\tilde{\nu}^*"] \arrow[d, "\mathrm{Nm}"] & J\widetilde{N} \arrow[r] \arrow[d, "\mathrm{Nm}"] & 0 \\ 0 \arrow[r] & T \arrow[r] & JC \arrow[r, "\nu^*"] & JN \arrow[r] & 0 \end{tikzcd} \tag{3-2}$$

of commutative algebraic groups, where the vertical arrows are the norm maps and T and $\widetilde{T}$ are the groups of classes of divisors of multidegree $(0, \dots, 0)$ with singular support. Since f^* restricted to T is an isomorphism and $\mathrm{Nm} \circ f^* = 7$, the norm on $\widetilde{T}$ is surjective and

$$\ker \mathrm{Nm}_{|\widetilde{T}} \simeq \widetilde{T}[7] = \{\text{points of order 7 in } \widetilde{T}\}.$$

On the other hand, Lemma 3.1 implies that $\ker \mathrm{Nm}$ consists either of 7 components or is connected. Let R be the kernel of $\mathrm{Nm} : J\widetilde{N} \to JN$. Then one obtains an exact sequence

$$0 \to \widetilde{T}[7] \to \widetilde{R} \to R \to 0 \tag{3-3}$$

with $\widetilde{R} = P$ if $\ker \mathrm{Nm}$ is connected, and $\widetilde{R} = P \times \mathbb{Z}/7\mathbb{Z}$ if $\ker \mathrm{Nm}$ is not connected. We will see that in both cases P is an abelian variety.

Suppose first that C and hence $\widetilde{C}$ are nonsingular. Then $\widetilde{C} = \widetilde{N}$ and hence $\widetilde{T} = 0$. Since $\ker(J\widetilde{N} \to JN)$ has 7 components, P is an abelian variety.

Suppose that C and thus also $\widetilde{C}$ have $s > 0$ singular points. Then $\dim \widetilde{T} = \dim T = s$. Then R is an abelian variety, since $\tilde{f}$ is ramified. We get a surjective homomorphism $P \to R$ with kernel consisting of 7^s elements when $\mathrm{Ker}\,\mathrm{Nm}$ is connected and 7^{s-1} when it is not. Hence also P is an abelian variety. Moreover,

$$\dim P = \dim R = \dim J\widetilde{C} - \dim JC.$$

Now, if C has s nodes and N has t connected components, then also $\widetilde{C}$ has s nodes and $\widetilde{N}$ has t connected components. This implies

$$\dim J\widetilde{C} - \dim JC = p_a(\widetilde{C}) - p_a(C) = 6g - 6. \qquad \square$$

Let $\widetilde{\Theta}$ denote the canonical polarization of the generalized Jacobian $J\widetilde{C}$ (see [Beauville 1977]). It restricts to a polarization Σ on the abelian subvariety P. We denote the isogeny $P \to \widehat{P}$ associated to Σ by the same letter.

Proposition 3.3. *The polarization Σ on P is of type $D := (1, \dots, 1, 7, \dots, 7)$, where 7 occurs*

$$\begin{cases} g \text{ times} & \text{if } \ker \mathrm{Nm} \text{ is connected}, \\ g-1 \text{ times} & \text{if } \ker \mathrm{Nm} \text{ is not connected}. \end{cases}$$

Proof. (Similar to [Faber 1988, Proposition 2.4]; we use also [Beauville 1977, Corollary 2.3, p. 156].) If C is smooth, this is well known. In this case $\ker f$ consists of 7 components and 7 occurs $g-1$ times in the polarization D (see [Birkenhake and Lange 2004, §12.3]). So suppose C is not smooth and f is a covering of type $(*)$. In this case the maps f^* and $\tilde{f}^*$ are injective.

Consider the isogeny

$$h : P \times JN \to J\widetilde{N}, \quad (L, M) \mapsto \tilde{\nu}^*(L) \otimes \tilde{f}^* M.$$

We first claim that $\ker h \subset P[7] \times JN[7]$. To see this, let $(L, M) \in \ker h$, i.e., $L \in P$ and $M \in JN$ and $\tilde{\nu}^*(L) \otimes \tilde{f}^* M \simeq \mathcal{O}_{\widetilde{N}}$. Choose $M' \in JC$ with

$$M \simeq \nu^* M'. \tag{3-4}$$

Then

$$\tilde{\nu}^*(L \otimes f^* M') \simeq \tilde{\nu}^* L \otimes \tilde{f}^* \nu^* M' \simeq \tilde{\nu}^* L \otimes \tilde{f}^* M \simeq \mathcal{O}_{\widetilde{N}}, \tag{3-5}$$

so $L \otimes f^* M' \in \ker \tilde{\nu}^* \simeq \widetilde{T}$. Since $f^* : T \to \widetilde{T}$ is an isomorphism, there is a unique $N \in T$ such that

$$L \simeq f^*(N \otimes M'^{-1}). \tag{3-6}$$

This implies

$$(N \otimes M'^{-1})^{\otimes 7} \simeq \operatorname{Nm} \circ f^*(N \otimes M'^{-1}) \simeq \operatorname{Nm} L \simeq \mathcal{O}_C$$

and hence $L \in P[7]$ and using (3-5) we get

$$M^{\otimes 7} \simeq \operatorname{Nm} \tilde{f}^* M \simeq \operatorname{Nm} \tilde{f}^* \nu^* M' \simeq \operatorname{Nm} \tilde{\nu}^* L^{-1} \simeq \nu^* \operatorname{Nm} L^{-1} \simeq \nu^* \mathcal{O}_N \simeq \mathcal{O}_C.$$

This completes the proof of the claim.

Now, since the map $\nu^* : JC[7] \to JN[7]$ is surjective, we can choose M' in (3-4) even as an element of $JC[7]$. Moreover, from (3-6) we also get $N \in T[7]$, since f^* is injective. Now consider the following extension of the map h to the whole of the kernel of the norm map of f:

$$\tilde{h} : \ker \operatorname{Nm} \times JN \to J\widetilde{N}, \quad (L, M) \mapsto \tilde{\nu}^*(L) \otimes \tilde{f}^* M.$$

We claim that

$$\ker \tilde{h} = \{(f^*(N \otimes M'^{-1}), \nu^* M') \mid M' \in JC[7],\ N \in T[7]\}. \tag{3-7}$$

So $\ker h$ consists of those elements of the right-hand side of (3-7) for which $f^*(N \otimes M'^{-1})$ is contained in the connected component containing 0 of $\ker \operatorname{Nm}_f$.

Proof of (3-7). The inclusion of $\ker \tilde{h}$ in the left-hand side of the equation follows from (3-4) and (3-6), which are valid for the extension $\tilde{h}$.

For the converse inclusion, suppose that $M' \in JC[7]$ and $N \in T[7]$. First note that $f^*(N \otimes M'^{-1}) \subset \ker \mathrm{Nm}$, since $\mathrm{Nm}\, f^*(N \otimes M'^{-1}) \simeq (N \otimes M'^{-1})^{\otimes 7} \simeq \mathcal{O}_C$. Moreover,

$$\tilde{\nu}^* f^*(N \otimes M'^{-1}) \otimes \tilde{f}^* \nu^* M' \simeq \tilde{\nu}^* f^*(N \otimes M'^{-1}) \otimes \tilde{\nu}^* f^* M' \simeq \tilde{\nu}^* f^* N \simeq \mathcal{O}_{\widetilde{N}}.$$

This completes the proof of (3-7). □

Since for any $\alpha \in T[7]$ we have $(N \otimes \alpha) \otimes (\alpha^{-1} \otimes M'^{-1}) \simeq N \otimes M'^{-1}$ and $\nu^*(M' \otimes \alpha) \simeq \nu^* M'$, we conclude that

$$\dim_{\mathbb{F}_7} \ker \tilde{h} = \dim_{\mathbb{F}_7} JC[7] = 2g - t,$$

with $t = \dim T = \dim \widetilde{T}$. This implies

$$\dim_{\mathbb{F}_7} \ker h = \begin{cases} 2g - t & \text{if } \ker \text{ is connected,} \\ 2g - t - 1 & \text{if } \ker \text{ is not connected,} \end{cases}$$

since in the not-connected case $\ker \mathrm{Nm}$ consists of 7 components.

Now $\ker h$ is a maximal isotropic subgroup of the kernel of the polarization of $P \times JN$, since this polarization is the pullback under h of the principal polarization of $J\widetilde{N}$. This implies $\dim_{\mathbb{F}_7}(\ker \Sigma \times JN[7]) = 2(\dim_{\mathbb{F}_7} \ker h)$. Since $\dim_{\mathbb{F}_7}(JN[7]) = 2g - 2t$, it follows that

$$\dim_{\mathbb{F}_7}(\ker \Sigma) = \begin{cases} 2g & \text{if } \ker \mathrm{Nm} \text{ is connected,} \\ 2g - 2 & \text{if } \ker \mathrm{Nm} \text{ is not connected.} \end{cases}$$

Since $\ker \Sigma \subset P[7]$, this gives the assertion. □

Corollary 3.4. $\ker \mathrm{Nm}$ *consists of* 7 *components.*

Proof. Consider $\widetilde{C}_t \to C_t$, a family of coverings, where the central fiber $\widetilde{C}_0 \to C_0$ satisfies condition $(*)$ and all the other fibers are coverings of smooth curves. The fibers in the associated family of abelian varieties $\ker \mathrm{Nm}_t$ have 7 components for $t \neq 0$ and, according to Proposition 3.3, they have a polarization of type $(1, \dots, 1, 7, \dots, 7)$, where 7 appears $g - 1$ times. The type of a polarization in a family of abelian varieties is constant since it is given by integers; therefore, $\ker \mathrm{Nm}_0$ has the same polarization type as the nearby fibers and, again by Proposition 3.3, $\ker \mathrm{Nm}_0$ is nonconnected, so it consists of 7 components. □

4. The condition $(**)$

As in the last section, let $f : \widetilde{C} \to C$ be a G-covering of stable curves. Recall that a node $z \in \widetilde{C}$ is either

- of index 1, i.e., $|\mathrm{Stab}\, z| = 1$, in which case $f^{-1}(f(z))$ consists of 7 nodes which are cyclically permuted under σ, or

- of index 7, i.e., $|\mathrm{Stab}\, z| = 7$, in which case z is the only preimage of the node $f(z)$ and f is totally ramified at both branches of z. Since σ is of order 7, the two branches of z are not exchanged.

We also say a node of C is *of index i* if a preimage (and hence every preimage) under f is a node of index i. We assume the following conditions for the G-covering $f : \widetilde{C} \to C$ of connected stable curves:

$$(**)\ \begin{cases} p_a(C) = g \text{ and } p_a(\widetilde{C}) = 7g-6; \\ \sigma \text{ is not the identity on any irreducible component of } \widetilde{C}; \\ \text{if at a fixed node of } \sigma \text{ one local parameter is multiplied by } \rho^i, \text{ the other} \\ \quad \text{is multiplied by } \rho^{-i}, \text{ where } \rho \text{ denotes a fixed 7-th root of unity;} \\ P := \mathrm{Pr}(f) \text{ is an abelian variety.} \end{cases}$$

Under these assumptions the nodes of $\widetilde{C}$ are exactly the preimages of the nodes of C. We define for $i = 1$ and 7:

- $n_i :=$ the number of nodes of C of index i, i.e., nodes whose preimage consists of $7/i$ nodes of $\widetilde{C}$,
- $c_i :=$ the number irreducible components of C whose preimage consists of $7/i$ irreducible components of $\widetilde{C}$,
- $r :=$ the number of fixed nonsingular points under σ.

Lemma 4.1. *The covering satisfies* $(**)$ *if and only if* $r = 0$ *and* $c_1 = n_1$.

In particular, any covering satisfying $(**)$ is an admissible G-cover (for the definition, see Section 5), and coverings satisfying condition $(*)$ also verify $(**)$.

Proof. (As in [Beauville 1977] and [Faber 1988].) Let $\widetilde{N}$ and N be the normalizations of $\widetilde{C}$ and C, respectively. The covering $\tilde{f} : \widetilde{N} \to N$ is ramified exactly at the points lying over the fixed points of $\sigma : \widetilde{C} \to \widetilde{C}$. Hence the Hurwitz formula says

$$p_a(\widetilde{N}) - 1 = 7(p_a(N) - 1) + 3r + 6n_7.$$

So

$$p_a(\widetilde{C}) - 1 = p_a(\widetilde{N}) - 1 + 7n_1 + n_7 = 7(p_a(N) - 1) + 3r + 7n_1 + 7n_7.$$

Moreover,

$$p_a(C) - 1 = p_a(N) - 1 + n_1 + n_7,$$

which altogether gives

$$p_a(\widetilde{C}) - 1 = 7(p_a(C) - 1) + 3r.$$

Hence the first condition in $(**)$ is equivalent to $r = 0$.

Now we discuss the condition that P is an abelian variety. For this consider again the diagram (3-2). From the surjectivity of the norm maps it follows that P is an abelian variety if and only if $\dim \widetilde{T} = \dim T$. Now

$$\dim J\widetilde{N} = p_a(\widetilde{N}) - n_7 - 7n_1 + c_7 + 7c_1 - 1$$

and thus

$$\dim \widetilde{T} = (n_7 - c_7) + 7(n_1 - c_1) + 1 \quad \text{and} \quad \dim T = (n_7 - c_7) + (n_1 - c_1) + 1.$$

Hence $\dim \widetilde{T} = \dim T$ if and only if $c_1 = n_1$. □

Let $f : \widetilde{C} \to C$ be a G-covering satisfying condition $(**)$ with generating automorphism σ. We denote by B the union of the components of $\widetilde{C}$ fixed under σ and write

$$\widetilde{C} = A_1 \cup \cdots \cup A_7 \cup B$$

with $\sigma(A_i) = A_{i+1}$, where $A_8 = A_1$. Observe that the covering $B \to B/\sigma$ satisfies condition $(*)$.

Proposition 4.2. (i) *If $B = \varnothing$, then $\widetilde{C} = A_1 \cup \cdots \cup A_7$, where A_1 can be chosen connected and tree-like, and $\#A_i \cap A_{i+1} = 1$ for $i = 1, \ldots, 7$.*

(ii) *If $B \neq \varnothing$, then $A_i \cap A_{i+1} = \varnothing$ for $i = 1, \ldots, 7$. Each connected component of A_1 is tree-like and meets B at only one point. Also B is connected.*

For the proof we need the following elementary lemma (the analogue of [Beauville 1977, Lemma 5.3] and [Faber 1988, Lemma 2.6]), which will be applied to the dual graph of $\widetilde{C}$.

Lemma 4.3. *Let Γ be a connected graph with a fixed-point free automorphism σ of order 7. Then there exists a connected subgraph S of γ such that $\sigma^i(S) \cap \sigma^{i+1}(S)$ is empty for $i = 0, \ldots, 6$ and $\bigcup_{i=0}^{6} \sigma^i(S)$ contains every vertex of Γ.*

Proof of Proposition 4.2. (As in [Beauville 1977] and [Faber 1988].) Let Γ denote the dual graph of $\widetilde{C}$. If $B = \varnothing$, let A_1 correspond to the subgraph S of Lemma 4.3. Let v be the number of vertices of S, e the number of edges of S and s the number of nodes of A_1 which belong to only one component. The equality $c_1 = n_1$ implies

$$v = e + s - \#A_1 \cap A_2.$$

Since $1 - v + e \geq 0$ and $\#A_1 \cap A_2 \geq 1$ give $s = 0$, we have $\#A_1 \cap A_2 = 1$ and $1 - v + e = 0$. So A_1 is tree-like. This proves (i).

Assume $B \neq \varnothing$ and define

- $t := \#A_1 \cap A_2$,
- $m := \#A_i \cap B$ for $i = 1, \ldots, 7$,
- $i_{A_1} :=$ # irreducible components of A_1,

- $c_{A_1} := \#$ of connected components of A_1,
- $n_{A_1} := \#$ nodes of A_1.

Recall that assumption $(**)$ implies that B does not contain any node which moves under σ. Then

$$c_1 = i_{A_1} \quad \text{and} \quad n_1 = n_{A_1} + r + m.$$

For any curve we have $n_{A_1} - i_{A_1} + c_{A_1} \geq 0$ (see [Beauville 1977, Proof of Lemma 5.3]). Thus, if $c_1 = n_1$,

$$0 = n_{A_1} + t + m - i_{A_1} \geq t + m - c_{A_1}.$$

Since $\widetilde{C}$ is connected, any connected component of $\bigcup_{i=1}^{7} A_i$ meets B. But then any connected component of A_1 meets B, which implies $m \geq c_{A_1}$. Hence

$$0 \geq t + m - c_{A_1} \geq t \geq 0.$$

Hence $t = 0$, $m = c_{A_1}$ and $n_{A_1} - i_{A_1} + c_{A_1} = 0$. So $A_i \cap A_{i+1} = \varnothing$ and B is connected. $\square$

In the next section we will extend the Prym map to the G-covers satisfying $(**)$, thereby obtaining a proper map. Theorem 4.4 describes the associated Pryms by reducing to the easier situation of coverings verifying $(*)$.

Theorem 4.4. *Suppose that $f : \widetilde{C} \to C$ satisfies condition $(**)$. Then there exist the following isomorphisms of polarized abelian varieties*:

- *In case* (i) *of Proposition* 4.2, $(P, \Sigma) \simeq \ker((JA_1)^7 \to JA_1)$ *with the polarization induced by the principal polarization on* $(JA_1)^7$.
- *In case* (ii) *of Proposition* 4.2, $(P, \Sigma) \simeq \ker((JA_1)^7 \to JA_1) \times Q$, *where Q is the generalized Prym variety associated to the covering $B \to B/\sigma$.*

Proof. (As in [Beauville 1977, Theorem 5.4].) In case (i), $\widetilde{C}$ is obtained from the disjoint union of 7 copies of A_1 by fixing 2 smooth points p and q of A_1 and identifying q in the i-th copy with p in the $(i+1)$-st copy of A_1 cyclically. The curve $C = \widetilde{C}/G$ is obtained from A_1 by identifying p and q, and $f : \widetilde{C} \to C$ is an étale covering. Note that JA_1 is an abelian variety, since A_1 is tree-like.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C}^* \arrow[r] \arrow[d, "\mathrm{Nm}"] & J\widetilde{C} \arrow[r] \arrow[d, "\mathrm{Nm}"] & (JA_1)^7 \arrow[r] \arrow[d, "m"] & 0 \\
0 \arrow[r] & \mathbb{C}^* \arrow[r] & JC \arrow[r] & JA_1 \arrow[r] & 0
\end{tikzcd}$$

in which m is the addition map. One checks immediately that $\mathrm{Nm}:\mathbb{C}^*\to\mathbb{C}^*$ is an isomorphism. This implies the assertion. Notice that the polarization of P is of type

$$(1,\dots,1,\underbrace{7,\dots,7}_{g-1}).$$

In case (ii), we have

$$J\widetilde{C}\simeq (JA_1)^7\times JB \quad \text{and} \quad P=(\ker\mathrm{Nm})^0\simeq \ker((JA_1)^7\to JA_1)\times Q,$$

which immediately implies the assertion. □

5. The extension of the Prym map to a proper map

Let $\mathcal{R}_{g,7}$ denote the moduli space of nontrivial étale G-covers $f:\widetilde{C}\to C$ of smooth curves C of genus g, and let $\mathcal{B}_D$ denote the moduli space of polarized abelian varieties of dimension $6g-6$ with polarization of type D, with D as in Proposition 3.3 and compatible with the G-action. As in the introduction we denote by

$$\mathrm{Pr}_{g,7}:\mathcal{R}_{g,7}\to\mathcal{B}_D$$

the corresponding Prym map associating to the covering f the Prym variety $\mathrm{Pr}(f)$. In order to extend this map to a proper map, we consider the compactification $\overline{\mathcal{R}}_{g,7}$ of $\mathcal{R}_{g,7}$ consisting of admissible G-coverings of stable curves of genus g introduced in [Abramovich et al. 2003].

Let $\mathcal{X}\to S$ be a family of stable curves of arithmetic genus g. A *family of admissible G-covers* of $\mathcal{X}$ over S is a finite morphism $\mathcal{Z}\to\mathcal{X}$ such that

(1) the composition $\mathcal{Z}\to\mathcal{X}\to S$ is a family of stable curves;

(2) every node of a fiber of $\mathcal{Z}\to S$ maps to a node of the corresponding fiber of $\mathcal{X}\to S$;

(3) $\mathcal{Z}\to\mathcal{X}$ is a principal G-bundle away from the nodes;

(4) if z is a node of index 7 in a fiber of $\mathcal{Z}\to S$ and ξ and η are local coordinates of the two branches near z, any element of the stabilizer $\mathrm{Stab}_G(z)$ acts as

$$(\xi,\eta)\mapsto(\rho\xi,\rho^{-1}\eta),$$

where ρ is a primitive 7-th root of unity.

In the case of $S=\operatorname{Spec}\mathbb{C}$ we just speak of an admissible G-cover. In this case the ramification index at any node z over x equals the order of the stabilizer of z and depends only on x. It is called the *index* of the G-cover $\mathcal{Z}\to\mathcal{X}$ at x. Since 7 is a prime, the index of a node is either 1 or 7. Note that, for any admissible G-cover $Z\to X$, the curve Z is stable if and only if X is stable.

As shown in [Abramovich et al. 2003] or [Arbarello et al. 2011, Chapter 16], the moduli space $\overline{\mathcal{R}}_{g,7}$ of admissible G-covers of stable curves of genus g is a natural compactification of $\mathcal{R}_{g,7}$. Clearly the coverings satisfying condition $(**)$ are admissible and form an open subspace $\widetilde{\mathcal{R}}_{g,7}$ of $\overline{\mathcal{R}}_{g,7}$.

Theorem 5.1. *The map* $\mathrm{Pr}_{g,7} : \mathcal{R}_{g,7} \to \mathcal{B}_D$ *extends to a proper map* $\widetilde{\mathrm{Pr}}_{g,7} : \widetilde{\mathcal{R}}_{g,7} \to \mathcal{B}_D$.

Proof. The proof is the same as the proof of [Faber 1988, Theorem 2.8] just replacing 3-fold covers by 7-fold covers. So we will omit it. □

6. Generic finiteness of $\mathrm{Pr}_{2,7}$

From now on we consider only the case $g = 2$, i.e., of G-covers of curves of genus 2. So $\dim \mathcal{R}_{2,7} = \dim \mathcal{M}_2 = 3$ and $\mathcal{B}_D$ is the moduli space of polarized abelian varieties of type $(1, 1, 1, 1, 1, 7)$ with G-action which is also of dimension 3. Let $[f : \widetilde{C} \to C] \in \mathcal{R}_{2,7}$ be a general point and let the covering f be given by the 7-division point $\eta \in JC$. The next lemma is a particular case of the results in [Lange and Ortega 2011, p. 397–398] that we include here for the sake of completeness.

Lemma 6.1. (i) *The cotangent space of* $\mathcal{B}_D$ *at the point* $\mathrm{Pr}_{2,7}([f : \widetilde{C} \to C]) \in \mathcal{B}_D$ *is identified with the vector space* $\bigoplus_{i=1}^{3}\big(H^0(\omega_C \otimes \eta^i) \otimes H^0(\omega_C \otimes \eta^{7-i})\big)$.

(ii) *The codifferential of the map* $\mathrm{Pr}_{2,7} : \mathcal{R}_{2,7} \to \mathcal{B}_D$ *at the point* (f, η) *is given by the sum of the multiplication maps*

$$\bigoplus_{i=1}^{3}\big(H^0(\omega_C \otimes \eta^i) \otimes H^0(\omega_C \otimes \eta^{7-i})\big) \longrightarrow H^0(\omega_C^2).$$

Proof. (i) Consider the composed map $\mathcal{R}_{g,7} \xrightarrow{\mathrm{Pr}_{2,7}} \mathcal{B}_D \xrightarrow{\pi} \mathcal{A}_D$, where $\mathcal{A}_D$ denotes the moduli space of abelian varieties with polarization of type D. The cotangent space of the image of $[f : \widetilde{C} \to C]$ in $\mathcal{A}_D$ is by definition the cotangent at the Prym variety P of f. It is well known that the cotangent space $T^*_{P,0}$ at 0 is

$$T^*_{P,0} = H^0(\widetilde{C}, \omega_{\widetilde{C}})^- = \bigoplus_{i=1}^{6} H^0(C, \omega_C \otimes \eta^i). \tag{6-1}$$

According to [Welters 1983] the cotangent space of $\mathcal{A}_D$ at the point P can be identified with the second symmetric product of $H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$. This gives

$$T^*_{\mathcal{A}_D,P} = \bigoplus_{i=1}^{6} S^2 H^0(\omega_C \otimes \eta^i) \oplus \bigoplus_{i=1}^{3}\big(H^0(\omega_C \otimes \eta^i) \otimes H^0(\omega_C \otimes \eta^{7-i})\big). \tag{6-2}$$

Since the map $\pi : \mathcal{B}_D \to \mathcal{A}_D$ is finite onto its image and the group G acts on the cotangent space of $\mathcal{B}_D$ at the point, we conclude that this space can be identified with a 3-dimensional G-subspace of the G-space $T^*_{\mathcal{A}_D,P}$, which is defined over the rationals. But there is only one such subspace, namely,

$$\bigoplus_{i=1}^{3}\big(H^0(\omega_C \otimes \eta^i) \otimes H^0(\omega_C \otimes \eta^{7-i})\big).$$

This gives (i).

(ii) It is well known that the cotangent space of $\mathcal{R}_{2,7}$ at a point (C, η) without automorphism is given by $H^0(\omega_C^2)$ and the codifferential of $\mathrm{Pr}_{2,7} : \mathcal{R}_{2,7} \to \mathcal{A}_D$ at (C, η) by the natural map $S^2(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-) \longrightarrow H^0(\omega_C^2)$. The assertion follows immediately from Lemma 6.1(i) and equations (6-1) and (6-2). $\square$

Theorem 6.2. *The map* $\widetilde{\mathrm{Pr}}_{2,7} : \widetilde{\mathcal{R}}_{2,7} \to \mathcal{B}_D$ *is surjective and hence of finite degree.*

Proof. Since the extension $\widetilde{\mathrm{Pr}}_{2,7}$ is proper according to Theorem 5.1, it suffices to show that the map $\mathrm{Pr}_{2,7}$ is generically finite. Now $\mathrm{Pr}_{2,7}$ is generically finite as soon as its differential at the generic point $[f : \widetilde{C} \to C] \in \mathcal{R}_{2,7}$ is injective. Let f be given by the 7-division point η. According to Lemma 6.1, the codifferential of $\mathrm{Pr}_{2,7}$ at $[f : \widetilde{C} \to C]$ is given by (the sum of) the multiplication of sections

$$\mu_{C,\eta} : \bigoplus_{j=1}^{3} H^0(C, \omega_C \otimes \eta^j) \otimes H^0(C, \omega_C \otimes \eta^{7-j}) \longrightarrow H^0(C, \omega_C^2).$$

Note that the surjectivity of $\mu_{C,\eta}$ is an open condition in $\overline{\mathcal{R}}_{2,7}$. Since $\overline{\mathcal{R}}_{2,7}$ is irreducible and $\widetilde{\mathcal{R}}_{2,7}$ is open and dense in $\overline{\mathcal{R}}_{2,7}$, it suffices to show that the map $\mu_{X,\eta}$ is surjective at a point (X, η) in the compactification $\overline{\mathcal{R}}_{2,7}$, even if $\mathrm{Pr}_{2,7}$ is not defined at (X, η). So if $\mu_{X,\eta}$ is surjective at this point, it will be surjective at a general point of $\mathcal{R}_{2,7}$. Moreover, it suffices to show that $\mu_{X,\eta}$ is injective, since both sides of the map are of dimension 3.

Consider the curve

$$X = Y \cup Z,$$

the union of two rational curves intersecting in 3 points q_1, q_2, q_3 which we can assume to be $[1, 0]$, $[0, 1]$, $[1, 1]$, respectively. The line bundle $\eta_X = (\eta_Y, \eta_Z)$ on X of degree 0 is uniquely determined by the gluing of the fiber over the nodes

$$\mathcal{O}_{Y_{|q_i}} \xrightarrow{\cdot c_i} \mathcal{O}_{Z_{|q_i}},$$

given by the multiplication by a nonzero constant c_i. We may assume $c_3 = 1$ and since $\eta_X^7 \simeq \mathcal{O}_X$ we have $c_1^7 = c_2^7 = 1$. Notice that $\omega_{X_{|Y}} = \mathcal{O}_Y(1)$ and $\omega_{X_{|Z}} = \mathcal{O}_Z(1)$ and the restrictions η_Y, η_Z are trivial line bundles.

From the exact sequence

$$0 \to \mathcal{O}_Z(-2) \to \omega_X \otimes \eta_X^i \xrightarrow{\beta_i} \mathcal{O}_Y(1) \to 0$$

we have $h^0(X,\omega_X\otimes\eta_X^i)=1$ for $i=1,\dots,6$. Moreover, since $H^0(Z,\mathcal{O}_Z(-2))=0$, the map β_i induces an inclusion $H^0(X,\omega_X\otimes\eta_X^i)\hookrightarrow H^0(Y,\mathcal{O}_Y(1))$ for $i=1,\dots,6$.

Therefore, to study the injectivity of $\mu_{X,\eta}$, it is enough to check whether the projection of $\bigoplus_{i=1}^3 H^0(X,\omega_X\otimes\eta_X^i)\otimes H^0(X,\omega_X\otimes\eta_X^{7-i})$ to $H^0(Y,\mathcal{O}_Y(1))\otimes H^0(Y,\mathcal{O}_Y(1))$ is contained in the kernel of the multiplication map

$$H^0(Y,\mathcal{O}_Y(1))\otimes H^0(Y,\mathcal{O}_Y(1))\longrightarrow H^0(Y,\mathcal{O}_Y(2)).$$

We claim that the line bundle $\omega_X=(\omega_{|Y},\omega_{|Z})$ is uniquely determined and one can choose the gluing c_i at the nodes q_i to be the multiplication by the same constant. To see this, first notice that, since (X,ω_X) is a limit linear series of canonical line bundles, the nodes of X are necessary Weierstrass points of X. Let $s_3\in H^0(X,\omega_X(-2q_3))$ be a section giving a trivialization of ω_Y and ω_Z away from q_3. For $i=1,2$, we have

$$\mathcal{O}_Y(1)_{|q_i}\xrightarrow{s_3^{-1}}\mathcal{O}_{X\,|q_i}\xrightarrow{s_3}\mathcal{O}_Z(1)_{|q_i},$$

which implies that $c_1=c_2$. Similarly, by using a section in $H^0(X,\omega_X(-2q_2))$ one shows that $c_1=c_3$.

A section of $\omega_{X_{|Y}}\otimes\eta_Y^i\simeq\mathcal{O}_Y(1)$ for $i=1,2,3$ is of the form $f_i(x,y)=a_ix+b_iy$, with a_i,b_i constants. Suppose that the sections f_i are in the image of the inclusion

$$H^0(X,\omega_X\otimes\eta_X^i)\hookrightarrow H^0(Y,\mathcal{O}_Y(1)).$$

By evaluating the section at the points q_i and using the gluing conditions, one gets $a_i=c_1^i-1$ and $b_i=c_2^i-1$. One obtains a similar condition for the image of the sections of $H^0(X,\omega_X\otimes\eta_X^{7-i})$ in $H^0(Y,\mathcal{O}_Y(1))$. Set $j=7-i$. By multiplying the corresponding sections of $\omega_X\otimes\eta_X^i$ and $\omega_X\otimes\eta_X^j$ we have that an element in the image of $\mu_{X,\eta}$ is of the form

$$\big(2-c_1^i-c_1^j\big)x^2+\big(2-c_2^i-c_2^j\big)y^2-\big(2-c_1^j-c_2^i+c_1^jc_2^i-c_1^i-c_2^j+c_1^ic_2^j\big)xy.$$

Hence, after taking the sum of such sections for $i=1,2,3$ we conclude that there is a nontrivial element in the kernel of $\mu_{X,\eta}$ if and only if there is a nontrivial solution for the linear system $A\boldsymbol{x}=0$ with

$$A=\begin{pmatrix}2-c_1-c_1^6 & 2-c_1c_2^6-c_1^6c_2 & 2-c_2-c_2^6\\ 2-c_1^2-c_1^5 & 2-c_1^2c_2^5-c_1^5c_2^2 & 2-c_2^2-c_2^5\\ 2-c_1^3-c_1^4 & 2-c_1^3c_2^4-c_1^4c_2^3 & 2-c_2^3-c_2^4\end{pmatrix}.$$

Clearly, if $c_i=1$ for some i or $c_1=c_2$, the determinant of A vanishes. We compute

$$\tfrac{1}{7}\det A=c_1^6(c_2^3-c_2^5)+c_1^5(c_2^6-c_2^3)+c_1^4(c_2^2-c_2)+c_1^3(c_2^5-c_2^6)+c_1^2(c_2-c_2^4)+c_1(c_2^4-c_2^2).$$

Suppose that $c_i \neq 1$ and $c_2 = c_1^k$ for some $2 \leq k \leq 6$. Then a straightforward computation shows that $\det A \neq 0$ if and only if $k = 3$ or $k = 5$.

In conclusion, we can find a limit linear series (X, η_X) with $\eta_X^7 \simeq \mathcal{O}_X$, for suitable values of the c_i, such that the composition map in the commutative diagram

$$\begin{tikzcd} \bigoplus_{i=1}^3 H^0(X, \omega_X \otimes \eta^i) \otimes H^0(X, \omega_X \otimes \eta^{7-i}) \arrow[r] \arrow[d, hook] & H^0(X, \omega_X^2) \arrow[d, "\simeq"] \\ H^0(Y, \mathcal{O}_Y(1)) \otimes H^0(Y, \mathcal{O}_Y(1)) \arrow[r] & H^0(Y, \mathcal{O}_Y(2)) \end{tikzcd}$$

is an isomorphism. $\square$

7. A complete fiber of $\widetilde{\mathrm{Pr}}_{2,7}$

For a special point of $\mathcal{B}_D$, consider a smooth curve E of genus 1. Then the kernel of the addition map

$$X = X(E) := \ker(m : E^7 \to E) \quad \text{with} \quad m(x_1, \dots, x_7) = x_1 + \dots + x_7$$

is an abelian variety of dimension 6, isomorphic to E^6. The kernel of the induced polarization of the canonical principal polarization of E^7 is $\{(x, \dots, x) : x \in E[7]\}$, which consists of 7^2 elements. So the polarization on X induced by the canonical polarization of E^7 is of type $D = (1, 1, 1, 1, 1, 7)$. Since the symmetric group $\mathcal{S}_7$ acts on E^7 in the obvious way, X admits an automorphism of order 7. Hence X with the induced polarization is an element of $\mathcal{B}_D$. To be more precise, the group $\mathcal{S}_7$ admits exactly 120 subgroups of order 7. Hence to every elliptic curve there exist exactly 120 abelian varieties X as above with G-action. All of them are isomorphic to each other, since the corresponding subgroups are conjugate to each other according the Sylow theorems. We want to determine the complete preimage $\widetilde{\mathrm{Pr}}_{2,7}^{-1}(X)$ of X. We need some lemmas. For simplicity we denote by $\mathrm{Pr}(f)$ the Prym variety of a covering f in $\widetilde{\mathcal{R}}_{2,7}$.

Lemma 7.1. *Let $f : \widetilde{C} \to C$ be a covering satisfying $(**)$ with $g = 2$ such that $\mathrm{Pr}(f) \simeq X$. Then C contains a node of index* 1.

Proof. Suppose that either C is smooth or all the nodes of C are of index 7. Then the exact sequence (3-3) gives an isogeny $j : \mathrm{Pr}(f) \to \mathrm{Pr}(\tilde{f})$ onto the Prym variety of the normalization $\tilde{f}$ of f. Actually, in the smooth case, j is an isomorphism and, if there is a node of index 1, the kernel $\widetilde{T}[7]$ is positive dimensional. The isomorphism $\mathrm{Pr}(f) = X$ implies that the kernel of j is of the form $\{(x, \dots, x)\}$ with $x \in X[7]$. Hence the action of the symmetric group S_7 on X descends to a nontrivial action on $\mathrm{Pr}(\tilde{f})$.

We can extend this action to $J\widetilde{N}$ by combining it with the identity on JN. Namely, $J\widetilde{N} \simeq (JN \times \mathrm{Pr}(\tilde{f}))/H$, where H is constructed as follows. Let $\langle \eta \rangle \subset JN[7]$

be the subgroup defining the covering $\tilde{f}$ and let $H_1 \subset JN[7]$ be its orthogonal complement with respect to the Weil pairing. Then $H = \{(\alpha, -f^*\alpha) : \alpha \in H_1\}$. Since $f^*H_1 = \{(x, \dots, x) : x \in E[7]\} \subset \mathrm{Pr}(f)$, we get an S_7-action on $J\widetilde{N}$ which is clearly nontrivial.

If C is smooth, then $\widetilde{N} \simeq \widetilde{C}$. On the other hand, if all the nodes of C are of index 7, then $\widetilde{N}$ consists of at least two components. In any case, for each component $\widetilde{N}_i$ of $\widetilde{N}$ we have

$$\#\,\mathrm{Aut}(J\widetilde{N}_i) \geq \tfrac{1}{2}\#S_7 = 2520.$$

Moreover, according to a classical theorem of Weil, $\mathrm{Aut}\,\widetilde{N}_i$ embeds into $\mathrm{Aut}\,J\widetilde{N}_i$ with quotient of order ≤ 2. So

$$\#\,\mathrm{Aut}\,\widetilde{N}_i \geq \tfrac{1}{2}\#\,\mathrm{Aut}(J\widetilde{N}_i).$$

On the other hand, $\widetilde{N}_i$ is a smooth curve of genus ≤ 8. So Hurwitz's theorem implies

$$\#\,\mathrm{Aut}(\widetilde{N}_i) \leq 84 \cdot (8-1) = 588.$$

Together, this gives a contradiction. □

Lemma 7.2. *Let $f : \widetilde{C} \to C$ be a covering satisfying $(**)$ such that C has a component containing nodes of index 1 and 7. Then any node of index 1 is the intersection with another component of C.*

Proof. Suppose x and y are nodes of C in a component C_i of index 1 and 7, respectively. Then the preimage $f^{-1}(C_i)$ is a component, since over y the map f is totally ramified. Since $f^{-1}(x)$ consists of 7 nodes, the equality $n_1 = c_1$ implies that x is the intersection of 2 components. □

Theorem 7.3. *Let $X = \ker(m : E^7 \to E)$ be a polarized abelian variety as above. The fiber $\widetilde{\mathrm{Pr}}_{2,7}^{-1}(X)$ consists of the following 4 types of elements of $\widetilde{\mathcal{R}}_{2,7}$ (see Figure 1).*

(i) *$C = E/p \sim q$ and $\widetilde{C} = \bigsqcup_{i=1}^{7} E_i/p_i \sim q_{i+1}$ with $E_i \simeq E$ for all i and $q_8 = q_1$ and we can enumerate in such a way that the preimages of p and q are $p_i, q_i \in E_i$.*

(ii) *$C = E_1 \cup_p E_2$ consists of 2 elliptic curves intersecting in one point p. Then up to exchanging E_1 and E_2 we have: $\widetilde{C}$ consists of an elliptic curve F_1, which is a 7-fold cover of E_1 and 7 copies of $E_2 \simeq E$ not intersecting each other and intersecting F_1 each in one point.*

(iii) *$C = E_1 \cup_p E_2$ with E_2 elliptic and E_1 rational with a node at q. Then $E_2 \simeq E$ and $f^{-1}(E_2)$ consists of 7 disjoint curves all isomorphic to E and $f^{-1}(E_1)$ is a rational curve with one node lying 7:1 over E_1 and intersecting each component of $f^{-1}(E_2)$ in a point over p.*

(iv) *$C = E_1 \cup_p E_2$ as in* (iii) *and $\widetilde{C}$ is an étale G-cover over C.*

We call the coverings of the theorem *of type* (i), (ii), (iii) and (iv), respectively. Theorem 7.3 will be used in Proposition 7.5 to describe the complete fibers of the Prym map over $X(E)$.

Proof. There are 7 types of stable curves of genus 2. We determine the coverings $f:\widetilde{C}\to C$ in $\widetilde{\mathrm{Pr}}_{2,7}^{-1}(X)$ in each case separately.

(1) There is no étale G-cover $f:\widetilde{C}\to C$ of a smooth curve C of genus 2 such that $P=\mathrm{Pr}(f)\simeq X$. This is a direct consequence of Lemma 7.1.

(2) If $C=E/p\sim q$ then the singular point of C is of index 1 and $f:\widetilde{C}\to C$ is a G-covering satisfying $(**)$ such that $P=\mathrm{Pr}(f)\simeq X$. Then

$$\widetilde{C}=\bigsqcup_{i=1}^{7}E_i/(p_i\sim q_{i+1})$$

with $E_i\simeq E$ and we enumerate in such a way that the preimages of p and q are p_i and q_i with $q_8=q_1$. In this case $\mathrm{Pr}(f)\simeq X$.

Proof: According to Lemma 7.1 the node is necessarily of index 1 and thus the map $\tilde{f}:\widetilde{C}\to C$ is étale. The exact sequence (3-3) together with Corollary 3.4 gives an isomorphism $P\simeq R$ with R the Prym variety of the map $\tilde{f}$. Clearly we can enumerate the components of $\widetilde{N}$ in such a way that $\widetilde{C}$ is as above and R is the kernel of the map $m:\times_{i=1}^{7}E\to E$, i.e., $R\simeq X$. We are in case (i) of the theorem.

(3) There is no rational curve C with 2 nodes admitting a G-cover $f:\widetilde{C}\to C$ satisfying $(**)$ such that $P=\mathrm{Pr}(f)\simeq X$.

Proof: Suppose there is such a covering. By Lemmas 7.1 and 7.2 both nodes are of index 1 and hence the map $\widetilde{C}\to C$ is étale. Then all components of $\widetilde{C}$ are rational. This implies that $P\simeq\mathbb{C}^{*6}$ is not an abelian variety, a contradiction.

(4) Let $C=E_1\cup_p E_2$ consist of 2 elliptic curves intersecting in one point and let $f:\widetilde{C}\to C$ be a covering satisfying $(**)$ such that $P=\mathrm{Pr}(f)\simeq X$. Then, up to exchanging E_1 and E_2, we have that $\widetilde{C}$ consists of an elliptic curve F_1, which is a 7-fold cover of E_1 and 7 copies of $E_2\simeq E$ not intersecting each other and intersecting F_1 each in one point. So $X=\ker(m:E_2^7\to E_2)$.

Proof: By Lemma 7.1 the node is of index 1 and the map $f:\widetilde{C}\to C$ is étale. Since there is no connected graph with 14 vertices and 7 edges, we are necessarily in case (ii) of the theorem.

(5) Suppose $C=E_1\cup_p E_2$ with components E_2 elliptic and E_1 rational with a node q and $f:\widetilde{C}\to C$ a G-covering satisfying $(**)$ such that $P=\mathrm{Pr}(f)\simeq X$. Then $E_2\simeq E$ and either $f:\widetilde{C}\to C$ is étale and connected or $\widetilde{C}$ consists of 7 components all isomorphic to E and a rational component F_2 over E_2 totally ramified exactly over q and intersecting each E_i exactly in one point lying over p. So $\mathrm{Pr}(f)\simeq X$.

Proof: According to Lemma 7.1 at least one node of C is of index 1. Suppose first that both nodes are of index 1. Then clearly f is étale and we are in case (iv) of the theorem. If only one node is of index 1, then according to Lemma 7.2 q is of index 7 and p of index 1. This gives case (iii) of the theorem.

(6) There is no curve C consisting of 2 rational components intersecting in one point p admitting a G-cover $f : \widetilde{C} \to C$ satisfying $(**)$ such that $P = \mathrm{Pr}(f) \simeq X$.

Proof: According to Lemmas 7.1 and 7.2 the node p is of index 1 and the nodes q_1 and q_2 of the rational components of C are of index 1 or 7. By the Hurwitz formula all components of $\widetilde{C}$ are rational. This implies $\mathrm{Pr}(f) \simeq \mathbb{C}^{*6}$, contradicting $(**)$.

(7) If C is the union of 2 rational curves intersecting in 3 points, there is no cover $f : \widetilde{C} \to C$ satisfying $(**)$ such that $P = \mathrm{Pr}(f) \simeq X$.

Proof: By Lemma 7.1 at least one of the 3 nodes of C is of index 1. So $\widetilde{C}$ consists of at least 8 components. But then the other nodes also are of index 1, because if one node is of index 7, the curve $\widetilde{C}$ consists of 2 components only. Hence all 3 nodes are of index 1. But then all components of $\widetilde{C}$ are rational. So $P = \mathrm{Pr}(f)$ cannot be an abelian variety, contradicting $(**)$.

Together, steps (1)–(7) prove the theorem. □

Corollary 7.4. *Let E be a general elliptic curve. Then the fiber $\widetilde{\mathrm{Pr}}_{2,7}^{-1}(X(E))$ consists of two irreducible components S_1 and S_2. The component S_1 is a covering over E whose points correspond to coverings of type* (i), *except for one point, which corresponds to a covering of type* (iv). *The component S_2 is a finite covering of the moduli space of elliptic curves, and every point corresponds to a covering of type* (ii), *except for two points, one of which corresponds to a covering of type* (iii) *and the other to a covering of type* (iv). *The components S_1 and S_2 intersect at a point corresponding to a covering of type* (iv) (see Figure 1).

Proof. It is known that for 2 elliptic curves $E_1 \neq E_2$ we can have $X(E_1) \simeq X(E_2)$ as abelian varieties, but not necessarily as polarized abelian varieties. Hence $X(E)$ determines E (which can be seen also from Theorem 7.3).

We claim that the coverings of type (iv) are contained in S_1 and S_2, whereas the coverings of type (iii) are contained in S_2 only: it is known that a curve $\widetilde{C}$ degenerates to a curve $\widetilde{C}'$ of some other type if and only if the dual graph of $\widetilde{C}'$ can be contracted to the dual graph of $\widetilde{C}$. On the other hand, the locus of curves covering some curve of genus ≥ 2 of some fixed degree is closed in the moduli space of curves. Now considering the dual graphs of the curves $\widetilde{C}$ of the coverings of the different types gives the assertion.

In the case of coverings of type (i) we have $C = E/p_1 \sim p_2$. We can use the translations of E to fix p_1, and then p_2 is free, which gives the assertion, since there are only finitely many étale coverings of C.

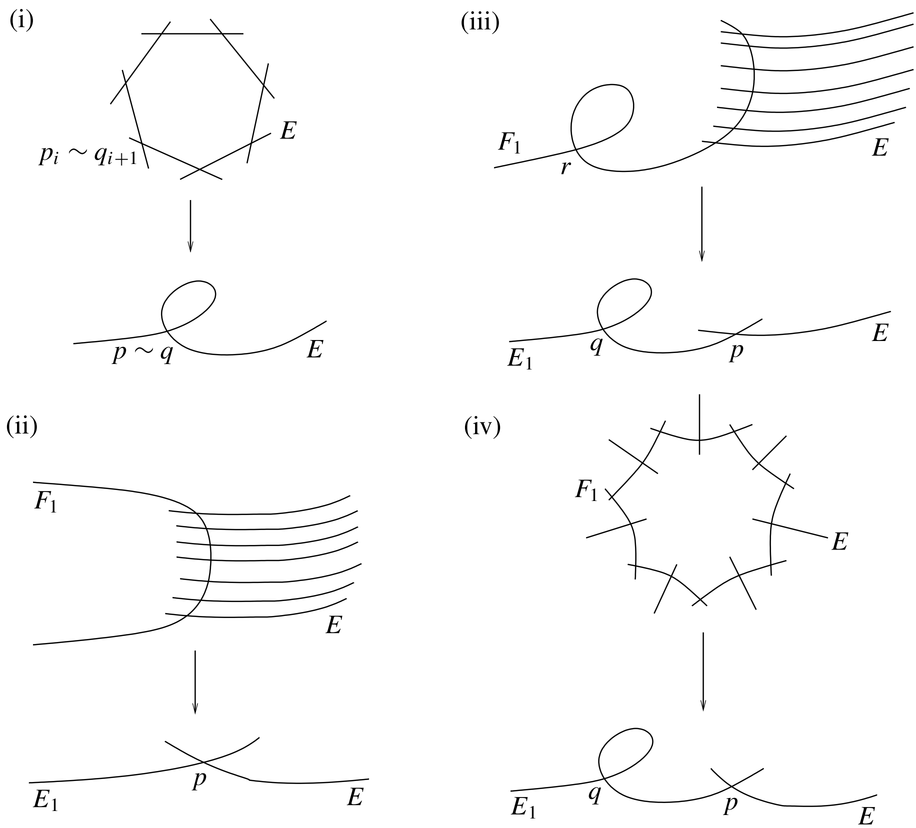

Figure 1. Admissible coverings on the fiber of $X(E)$.

In case (ii) we have $C = E_1 \cup_p E_2$, where E_1 is an arbitrary elliptic curve and $E_2 \simeq E$. Since p may be fixed with an isomorphism of E and E_2, this gives the isomorphism of $\widetilde{\mathrm{Pr}}_{2,7}^{-1}(E)$ with a finite covering of the moduli space of elliptic curves, again since there are only finitely many coverings $\widetilde{C}$ of type (ii) of C.

Finally, in cases (iii) and (iv), the 3 points of the normalization of E_1 given by p and the 2 preimages of the node, which we can assume to be 1, 0, and ∞, respectively, determine the curve C uniquely. For type (iii) the induced map on the normalization of F_1 is a 7:1 map $h : \mathbb{P}^1 \to \mathbb{P}^1$ totally ramified at 2 points, which we assume to be ∞ and 0. So h can be expressed as a polynomial in one variable of degree 7, with vanishing order 7 at 0 and such that $h(1) = 1$, that is, $h(x) = x^7$. Then the map h, and hence the covering, is uniquely determined. For a covering of type (iv) over C we consider 7 copies of $\mathbb{P}^1$, where the point 1 on every rational component is identified to the point ∞ of another rational component, and we attach elliptic curves isomorphic to E_2 at each point 0. The number of étale coverings is the number of subgroups of order 7 in $JE_1[7] \simeq \mathbb{Z}/7\mathbb{Z}$ (the 7-torsion points in the nodal curve E_1 are determined by a 7-th root of unity). So there is only one such covering up to isomorphism. $\square$

Varying the elliptic curve E, we obtain a one-dimensional locus $\mathcal{E} \subset \mathcal{B}_D$ consisting of the polarized abelian varieties $X(E)$ with G-action as above. Let $\mathcal{S}$ denote the preimage of $\mathcal{E}$ under the extended Prym map $\widetilde{\mathrm{Pr}}_{2,7} : \widetilde{\mathcal{R}}_{2,7} \to \mathcal{B}_D$. The next proposition is a direct consequence of Corollary 7.4.

Proposition 7.5. *The scheme $\mathcal{S}$ has dimension* 2 *and is the union of* 2 *closed subschemes*

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2,$$

where $\mathcal{S}_1$ parametrizes coverings of type (i) *and* (iv), *and $\mathcal{S}_2$ parametrizes coverings of type* (ii), (iii) *and* (iv). *In particular, they intersect exactly in the points parametrizing coverings of type* (iv).

8. The codifferential on the boundary divisors

In this section we will give bases of the Prym differentials and an explicit description of the codifferential of the Prym map.

Let $f : \widetilde{C} \to C$ be a covering corresponding to a point of $\mathcal{S}$. We want to compute the rank of the codifferential of the Prym map $\mathrm{Pr}_{2,7} : \widetilde{\mathcal{R}}_{2,7} \to \mathcal{B}_D$ at the point $[f : \widetilde{C} \to C] \in \widetilde{\mathcal{R}}_{2,7}$. According to [Donagi and Smith 1981] this codifferential is the map

$$\mathcal{P}^* : S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)^G \to H^0(C, \Omega_C \otimes \omega_C),$$

where Ω_C is the sheaf of Kähler differentials on C and $S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)^G$ is the cotangent space to $\mathcal{B}_D$ at the Prym variety of the covering $f : \widetilde{C} \to C$. Letting $j : \Omega_C \to \omega_C$ denote the canonical map, we first compute the rank of the composed map

$$S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)^G \xrightarrow{\mathcal{P}^*} H^0(C, \Omega_C \otimes \omega_C) \xrightarrow{j} H^0(C, \omega^2).$$

Suppose first that f is of type (i), (ii) or (iv). In these cases the covering f is étale and hence given by a 7-division point η of JC. According to the analogue of [Lange and Ortega 2011, Equation (3.4)] we have

$$H^0(\widetilde{C}, \omega_C)^- = \bigoplus_{i=1}^{6} H^0(C, \omega_C \otimes \eta^i) \tag{8-1}$$

and hence

$$S^2\big(H^0(\widetilde{C}, \omega_C)^-\big)^G = \bigoplus_{i=1}^{3} \big(H^0(C, \omega_C \otimes \eta^i) \otimes H^0(C, \omega_C \otimes \eta^{7-i})\big). \tag{8-2}$$

Using this, the above composed map is just the sum of the cup product map

$$\phi : \bigoplus_{i=1}^{3} \big(H^0(C, \omega_C \otimes \eta^i) \otimes H^0(C, \omega_C \otimes \eta^{7-i})\big) \longrightarrow H^0(C, \omega_C^2), \tag{8-3}$$

whose rank we want to compute first.

We shall give a suitable basis for the space of Prym differentials. First we consider a covering of type (i) constructed as follows. Let E be a smooth curve of genus 1 and let $q \neq q'$ be two fixed points of E. Then

$$C := E/q \sim q'$$

is a stable curve of genus 2 with normalization $n : E \to C$ and node $p := n(q) = n(q')$. Let $f : \widetilde{C} \to C$ be a cyclic étale covering with Galois group $G = \langle \sigma \rangle \simeq \mathbb{Z}/7\mathbb{Z}$. The normalization $\tilde{n} : \widetilde{N} \to \widetilde{C}$ consists of 7 components $N_i \simeq E$ with $\sigma(N_i) = N_{i+1}$ for $i = 1, \dots, 7$ and $N_8 = N_1$. Let q_i and q_i' be the elements of N_i corresponding to q and q'. Then $\tilde{n}(q_i) = \tilde{n}(q_{i+1}') =: p_i$ for $i = 1, \dots, 7$ with $q_8' = q_1'$. Clearly $\sigma(p_i) = p_{i+1}$ for all i.

Recall that $\omega_{\widetilde{C}}$ is the subsheaf of $\tilde{n}_*\big(\mathcal{O}_{\widetilde{N}} \sum (q_i + q_i')\big)$ consisting of (local) sections φ which considered as sections of $\mathcal{O}_{\widetilde{N}} \sum (q_i + q_i')$ satisfy the condition

$$\mathrm{Res}_{q_i}(\varphi) + \mathrm{Res}_{q_{i+1}'}(\varphi) = 0$$

for $i = 1, \dots, 7$. Here we use the fact that $\omega_{\widetilde{N}} = \mathcal{O}_{\widetilde{N}}$. Consider the following elements of $H^0\big(\widetilde{C}, \tilde{n}_*\big(\mathcal{O}_{\widetilde{N}} \sum (q_i + q_i')\big)\big)$, regarded as sections on $\widetilde{N}$:

$$\omega_1 := \begin{cases} \text{nonzero section of } \mathcal{O}_{N_1}(q_1 + q_1') \text{ vanishing at } q_1 \text{ and } q_1', \\ 0 \text{ elsewhere}, \end{cases}$$
$$\omega_i := (\sigma^{-i})^*(\omega_1) \quad \text{for } i = 2, \dots, 7.$$

Note that ω_i is nonzero on N_i vanishing at q_i and q_i' and zero elsewhere.

Now we construct similar differentials for coverings of type (ii). Let

$$C = E_1 \cup_p E_2$$

consist of 2 elliptic curves E_1 and E_2 intersecting transversally in one point p, and let $f : \widetilde{C} \to C$ be a covering of type (ii). So $\widetilde{C}$ consists of an elliptic curve F_1, which is an étale cyclic cover of E_1 of degree 7 and 7 disjoint curves $E_2^1, \dots, E_2^7$ all isomorphic to E_2. The curve E_i intersects F_1 transversally in a point p_i, such that the group G permutes the curves E_i and the points p_i cyclically, i.e., $\sigma(E_2^i) = E_2^{i+1}$ and $\sigma(p_i) = p_{i+1}$ with $E_2^8 = E_2^1$ and $p_8 = p_1$.

Let $\tilde{n} : \widetilde{N} \to \widetilde{C}$ denote the normalization map. Then $\widetilde{N}$ is the disjoint union of the 8 elliptic curves $F_1, E_2^1, \dots, E_2^7$. We denote the point p_i by the same letter when considered as a point of F_1 and E_2^i. Consider the line bundle

$$L = \mathcal{O}_{F_1}(p_1 + \cdots + p_7) \sqcup \mathcal{O}_{E_2^1}(p_1) \sqcup \cdots \sqcup \mathcal{O}_{E_2^7}(p_7).$$

Then $\omega_{\widetilde{C}}$ is the subsheaf of $\tilde{n}_*(L)$ consisting of (local) sections φ which considered as sections of L satisfy the condition

$$(\mathrm{Res}_{p_i})_{|F_1}(\varphi) + (\mathrm{Res}_{p_i})_{|E_2^i}(\varphi) = 0 \tag{8-4}$$

for $i = 1, \dots, 7$. Consider the following sections of $\tilde{n}_*(L)$, regarded as sections on $\widetilde{N}$:

$$\omega_1 := \begin{cases} \text{nonzero sections of } \mathcal{O}_{F_1}(p_1) \text{ and } \mathcal{O}_{E_2^1}(p_1) \text{ satisfying (8-4) at } p_1, \\ 0 \text{ elsewhere,} \end{cases}$$
$$\omega_i := (\sigma^{-i})^*(\omega_1) \quad \text{for } i = 2, \dots, 7.$$

Thus ω_i is nonzero on $F_1(p_i) \sqcup E_2^i(p_i)$ vanishing at p_i and zero elsewhere.

We construct the analogous differentials for the covering f of type (iv), which is uniquely determined according to Proposition 7.5. So let

$$C = E_1 \cup_p E_2$$

with E_2 elliptic and E_1 a rational curve with one node q and let $f : \widetilde{C} \to C$ be the covering of type (iv). Then $\widetilde{C}$ consists of 14 components $F_1, \dots, F_7$ isomorphic to $\mathbb{P}^1$ with $f_{|F_i} : F_i \to E_1$ the normalization and $E_2^1, \dots, E_2^7$ all isomorphic to E_2 with $f_{|E_2^i} : E_2^i \to E_2$ the isomorphism. Then E_2^i intersects F_i in the point p_i lying over p and no other component of $\widetilde{C}$. If q_i and q_i' are the points of F_i lying over q, the F_i and F_{i+1} intersect transversally in the points q_i and q_{i+1}' for $i = 1, \dots, 7$, where $q_8' = q_1$. The group G permutes the components and points cyclically, i.e., $\sigma(F_1) = F_{i+1}$ and similarly for E_2^i, p_i, q_i and q_i'.

The normalization $\tilde{n} : \widetilde{N} \to \widetilde{C}$ of the curve $\widetilde{C}$ is the disjoint union of the components F_i and E_2^i. We denote also the point p_i by the same letter when considered as a point of F_i and E_2^i. Consider the following line bundle on $\widetilde{N}$:

$$L = \bigsqcup_{i=1}^{7} \mathcal{O}_{F_i}(q_i + q_i' + p_i) \sqcup \bigsqcup_{i=1}^{7} \mathcal{O}_{E_2^i}(p_i).$$

Then $\omega_{\widetilde{C}}$ is the subsheaf of $\tilde{n}_*(L)$ consisting of (local) sections φ which viewed as sections of L satisfy the conditions

$$(\mathrm{Res}_{p_i})_{|F_i}(\varphi) + (\mathrm{Res}_{p_i})_{|E_2^i}(\varphi) = 0, \quad (\mathrm{Res}_{q_i})_{|F_i}(\varphi) + (\mathrm{Res}_{q_{i+1}'})_{|F_{i+1}}(\varphi) = 0 \tag{8-5}$$

for $i = 1, \dots, 7$. Consider the following sections of $\tilde{n}_*(L)$, regarded as sections on $\widetilde{N}$:

$$\omega_1 := \begin{cases} \text{nonzero sections of } \omega_{F_1}(q_1 + q_1' + p_1) \text{ and } \mathcal{O}_{E_2^1}(p_1) \text{ satisfying (8-5)} \\ \quad \text{at } p_1 \text{ and vanishing at } q_1 \text{ and } q_1', \\ 0 \text{ elsewhere,} \end{cases}$$

$$\omega_i := (\sigma^{-i})^*(\omega_1) \quad \text{for } i = 2, \dots, 7.$$

Note that up to a multiplicative constant there is exactly one such section ω_1, since $h^0(\omega_{F_1}(q_1 + q_1' + p_1)) = 2$ and $h^0(\mathcal{O}_{E_2^1}(p_1)) = 1$.

Finally, we consider coverings of type (iii). Let $C = E_1 \cup_p E_2$, as for the covering of type (iv) above, and let $f : \widetilde{C} \to C$ be a covering of type (iii). So $\widetilde{C}$ consists of a rational curve F_1 with a node r lying over the node q of C and 7 components $E_2^1, \dots, E_2^7$ all isomorphic to E_2. Then E_2^i intersects F_1 in the point p_i lying over p and intersects no other component of $\widetilde{C}$. The group G acts on F_1 with only a fixed point r and permutes the E_i^2 and p_i cyclically as above. We use the following partial normalization $\tilde{n} : \widetilde{N} \to \widetilde{C}$ of $\widetilde{C}$:

$$\widetilde{N} := F_1 \sqcup \bigsqcup_{i=1}^{7} E_2^i.$$

Consider the following line bundle on $\widetilde{N}$:

$$L = \mathcal{O}_{F_1}(p_1 + \cdots + p_7) \sqcup \bigsqcup_{i=1}^{7} \mathcal{O}_{E_2^i}(p_i).$$

Since the canonical bundles of F_1 and E_2^i are trivial, it is clear that $\omega_{\widetilde{C}}$ is the subsheaf of $\tilde{n}_*(L)$ consisting of (local) sections φ which regarded as sections of L satisfy the relations

$$(\mathrm{Res}_{p_i})_{|F_1}(\varphi) + (\mathrm{Res}_{p_i})_{|E_2^i}(\varphi) = 0 \tag{8-6}$$

for $i = 1, \dots, 7$. As before, define a section

$$\omega_1 := \begin{cases} \text{nonzero sections of } \mathcal{O}_{F_1}(p_1 + \cdots + p_7) \text{ and } \mathcal{O}_{E_2^i}(p_1) \\ \quad \text{vanishing at } p_1, \dots, p_7, \\ 0 \text{ elsewhere,} \end{cases}$$

of $\tilde{n}_*(L)$, considered as a section of $\widetilde{N}$, and define the sections ω_i, for $i = 2, \dots, 7$, as in the previous cases. Note that up to a multiplicative constant there is exactly one such section ω_1.

From now on, $f : \widetilde{C} \to C$ will be a covering of type (i)–(iv) as above. We fix a primitive 7-th root of unity, for example, $\rho := e^{2\pi i/7}$, and define for $i = 0, \dots, 6$

the section

$$\Omega_i := \sum_{j=1}^{7} \rho^{ij} \omega_j.$$

Clearly Ω_i is a global section of L that defines a section of $\omega_{\widetilde{C}}$, which we denote with the same symbol.

Lemma 8.1. $\sigma^*(\Omega_i) = \rho^i \Omega_i$ *for* $i = 0, \dots, 6$. *In particular,* $\Omega_0 \in H^0(\widetilde{C}, \omega_{\widetilde{C}})^+$ *and* $\{\Omega_1, \dots, \Omega_6\}$ *is a basis of* $H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$.

Proof. The first assertion follows from a simple calculation using the definition of ω_i. So clearly $\Omega_0 \in H^0(\widetilde{C}, \omega_{\widetilde{C}})^+$ and $\Omega_i \in H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$ for $i = 1, \dots, 6$. Since $\Omega_1, \dots, \Omega_6$ are in different eigenspaces of σ, they are linearly independent and since $H^0(\widetilde{C}, \omega_{\widetilde{C}})^-$ is of dimension 6, they form a basis. □

Remark 8.2. In cases (i), (ii) and (iv), $H^0(C, \omega_C \otimes \eta^{7-i})$ is the eigenspace of σ^i and Ω_i is a generator for $i = 1, \dots, 6$.

Proposition 8.3. *The map*

$$\phi : S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)^G \longrightarrow H^0(C, \omega_C^2)$$

is of rank 1.

Proof. We have to show that the kernel of ϕ is 2-dimensional. A basis of $S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)^G$ is given by $\{\Omega_1 \otimes \Omega_6,\ \Omega_2 \otimes \Omega_5,\ \Omega_3 \otimes \Omega_4\}$. Let a, b, c be complex numbers with

$$\phi(a\Omega_1 \otimes \Omega_6 + b\Omega_2 \otimes \Omega_5 + c\Omega_3 \otimes \Omega_4) = 0.$$

Define, for $i = 1, \dots, 7$,

$$\psi_i := \sum_{i=1}^{7} \omega_j \otimes \omega_{j+i-1}.$$

An easy but tedious computation gives

$$\begin{aligned}
\Omega_1 \otimes \Omega_6 &= \psi_1 + \rho\psi_7 + \rho^2\psi_6 + \rho^3\psi_5 + \rho^4\psi_4 + \rho^5\psi_3 + \rho^6\psi_2,\\
\Omega_2 \otimes \Omega_5 &= \psi_1 + \rho\psi_4 + \rho^2\psi_7 + \rho^3\psi_3 + \rho^4\psi_6 + \rho^5\psi_2 + \rho^6\psi_5,\\
\Omega_3 \otimes \Omega_4 &= \psi_1 + \rho\psi_3 + \rho^2\psi_5 + \rho^3\psi_7 + \rho^4\psi_2 + \rho^5\psi_4 + \rho^6\psi_6.
\end{aligned}$$

So we get

$$\begin{aligned}
0 &= \phi\big((a+b+c)\psi_1 + (a\rho^6+b\rho^5+c\rho^4)\psi_2 + (a\rho^5+b\rho^3+c\rho)\psi_3 \\
&\qquad + (a\rho^4+b\rho+c\rho^5)\psi_4 + (a\rho^3+b\rho^6+c\rho^2)\psi_5 \\
&\qquad + (a\rho^2+b\rho^4+c\rho^6)\psi_6 + (a\rho+b\rho^2+c\rho^3)\psi_7\big) \\
&= (a+b+c)(\omega_1^2+\cdots+\omega_7^2) \\
&\qquad + \big(a(\rho+\rho^6)+b(\rho^2+\rho^5)+c(\rho^3+\rho^4)\big)\sum_{j=1}^{7}\omega_j\omega_{j+1} \\
&\qquad + \big(a(\rho^2+\rho^5)+b(\rho^3+\rho^4)+c(\rho+\rho^6)\big)\sum_{j=1}^{7}\omega_j\omega_{j+2} \\
&\qquad + \big(a(\rho^3+\rho^4)+b(\rho+\rho^6)+c(\rho^2+\rho^5)\big)\sum_{j=1}^{7}\omega_j\omega_{j+3}.
\end{aligned}$$

This section is zero if and only if its restriction to any component is zero. Now the restriction to N_i for all i gives

$$0 = a+b+c+(6a+6b+6c)\sum_{j=1}^{6}\rho^j = -5(a+b+c).$$

So $\phi(a\Omega_1\otimes\Omega_6 + b\Omega_2\otimes\Omega_5 + c\Omega_3\otimes\Omega_4) = 0$ if and only if $a+b+c=0$. Hence the kernel of ϕ is of dimension 2, which proves the proposition. $\square$

Proposition 8.3 shows that the codifferential map along the divisors $\mathcal{S}_1$ and $\mathcal{S}_2$ in Proposition 7.5 is not surjective. In fact, as we will see later, the kernel of ϕ coincides with the conormal bundle of the image of these divisors in $\mathcal{B}_D$. In order to compute the degree we will perform a blow-up along these divisors.

Let $\mathcal{E} \subset \mathcal{B}_D$ denote the one-dimensional locus consisting of the abelian varieties which are of the form $X = \ker(m : E^7 \to E)$ with $m(x_1,\dots,x_7) = x_1+\cdots+x_7$ for a given elliptic curve E. As we saw in Section 7, the induced polarization is of type D. Note that $\mathcal{E}$ is a closed subset of $\mathcal{B}_D$. The aim is to compute the degree of $\mathrm{Pr}_{7,2}$ above a point $X \in \mathcal{E}$. We denote by $\mathcal{S} \subset \widetilde{\mathcal{R}}_{2,7}$ the inverse image of $\mathcal{E}$ under $\widetilde{\mathrm{Pr}}_{2,7}$. According to Proposition 7.5, $\mathcal{S}$ is a divisor consisting of 2 irreducible components in the boundary $\widetilde{\mathcal{R}}_{2,7} \setminus \mathcal{R}_{2,7}$. We have $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where a general point of $\mathcal{S}_1$ corresponds to the G-covers with base an irreducible nodal curve of genus 1, and a point of $\mathcal{S}_2$ corresponds to a product of elliptic curves intersecting in a point. Moreover, for any fixed elliptic curve E, $\mathcal{S}_1$ and $\mathcal{S}_2$ intersect in the unique point given by the covering of type (iv).

As in [Donagi and Smith 1981], let $\widetilde{\mathcal{B}}_D$ be the blow-up of $\mathcal{B}_D$ along $\mathcal{E}$ and $\widetilde{\mathcal{R}} \simeq \widetilde{\mathcal{R}}_{2,7}$ the blow-up of $\widetilde{\mathcal{R}}_{2,7}$ along the divisor $\mathcal{S} = \widetilde{\mathrm{Pr}}_{2,7}^{-1}(\mathcal{E})$. We then obtain the commutative diagrams

$$\begin{tikzcd} \widetilde{\mathcal{R}} \arrow[r, "\widetilde{\mathcal{P}}"] \arrow[d, "\simeq"'] & \widetilde{\mathcal{B}}_D \arrow[d] \\ \widetilde{\mathcal{R}}_{2,7} \arrow[r, "\widetilde{\Pr}_{2,7}"] & \mathcal{B}_D \end{tikzcd} \qquad \begin{tikzcd} \widetilde{\mathcal{S}} \arrow[r, "\widetilde{\mathcal{P}}"] \arrow[d, "\simeq"'] & \widetilde{\mathcal{E}} \arrow[d, "\mathbb{P}^1\text{-bundle}"] \\ \mathcal{S} \arrow[r, "\widetilde{\Pr}_{2,7}"] & \mathcal{E} \end{tikzcd}$$

in which $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{E}}$ are the exceptional loci.

Lemma I.3.2 of [Donagi and Smith 1981] guarantees that the local degree of $\widetilde{\Pr}_{2,7}$ along a component of $\mathcal{S}$ equals the degree of the induced map on the exceptional divisors $\widetilde{\mathcal{P}}_{|\mathcal{S}_i} : \mathcal{S}_i \to \widetilde{\mathcal{E}}$ if the codifferential map $\mathcal{P}^*$ is surjective on the respective conormal bundles. Recall that the fibers of the conormal bundles at the point X are given by

$$\mathcal{N}^*_{X,\mathcal{E}/\mathcal{B}_D} = \ker(T^*_X\mathcal{B}_D \to T^*_X\mathcal{E}),$$

$$\mathcal{N}^*_{(C,\eta),\mathcal{S}_i/\widetilde{\mathcal{R}}_{2,7}} = \ker(T^*_{(C,\eta)}\widetilde{\mathcal{R}}_{2,7} \to T^*_{(C,\eta)}\mathcal{S}_i)$$

for $i = 1, 2$. As in [Donagi and Smith 1981], by taking level structures on the moduli spaces we can assume we are working on fine moduli spaces, which allows us to identify the tangent space to $\mathcal{S}_1$ at the G-admissible cover $[\widetilde{C} \to C]$, where $C = E/(p \sim q)$, with the tangent space to $\overline{\mathcal{M}}_2$ at C. Thus the conormal bundle $\mathcal{N}^*_{(C,\eta),\mathcal{S}_1/\widetilde{\mathcal{R}}_{2,7}}$ can be identified with the conormal bundle $\mathcal{N}^*_{C,\Delta_0/\overline{\mathcal{M}}_2} \subset H^0(\Omega_C \otimes \omega_C)$, where Δ_0 is the divisor of irreducible nodal curves in $\overline{\mathcal{M}}_2$. Similarly, we can identify the tangent space to $\mathcal{S}_2$ at $[\widetilde{C} \to C]$, where $C = E_1 \cup_p E_2$, with the tangent space to $\overline{\mathcal{M}}_2$ at C. Thus $\mathcal{N}^*_{(C,\eta),\mathcal{S}_2/\widetilde{\mathcal{R}}_{2,7}}$ can be identified with $\mathcal{N}^*_{C,\Delta_1/\overline{\mathcal{M}}_2} \subset H^0(\Omega_C \otimes \omega_C)$, where Δ_1 is the divisor of reducible nodal curves in $\overline{\mathcal{M}}_2$.

Using the fact that $X = \Pr(\widetilde{C}, C)$ we can identify $(T_X\mathcal{A}_D)^* \simeq \bigoplus_{i=1}^6 \Omega_i\mathbb{C}$ and $S^2\big(H^0(\widetilde{C}, \omega_C)^-\big)^G$ as in (8-2). Then for a covering (C, η) the conormal bundles fit in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{N}^*_{X,\mathcal{E}/\mathcal{B}_D} \arrow[r] \arrow[d, "n^*"] & S^2\big(H^0(\widetilde{C}, \omega_C)^-\big)^G \arrow[r] \arrow[d, "\mathcal{P}^*"] & T^*_X\mathcal{E} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{N}^*_{(C,\eta),\mathcal{S}_i/\widetilde{\mathcal{R}}_{2,7}} \arrow[r] & H^0(C, \Omega_C \otimes \omega_C) \arrow[r] \arrow[d, "H^0(j)"] & T^*_{(C,\eta)}\mathcal{S}_i \arrow[r] & 0 \\ & & H^0(C, \omega_C^2) & & \end{tikzcd} \tag{8-7}$$

in which n^* is the conormal map, $i = 1, 2$ and j is induced by the canonical map $\Omega_C \to \omega_C$ (see [Donagi and Smith 1981, IV, 2.3.3]).

Lemma 8.4. *The kernel of $H^0(j)$ is one-dimensional.*

Proof. As a map of sheaves, the canonical map $j : \Omega_C \otimes \omega_C \to \omega_C^2$ has one-dimensional kernel, namely, the one-dimensional torsion sheaf with support the node of C (see [Donagi and Smith 1981, IV, 2.3.3]). On the other hand, the map $H^0(j)$ is the composition of the pullback to the normalization with the push forward to C. This implies that the kernel of $H^0(j)$ consists exactly of the sections of the skyscraper sheaf supported at the node, and hence is one-dimensional. □

Proposition 8.5. *For coverings of type* (i)–(iv) *the restricted codifferential map* $n^* : \mathcal{N}^*_{X,\mathcal{E}/\mathcal{B}_D} \to \mathcal{N}^*_{(C,\eta),\mathcal{S}_i/\widetilde{\mathcal{R}}_{2,7}}$ *is surjective, for* $i = 1, 2$.

Proof. First notice that from the "local-global" exact sequence (see [Bardelli 1989]) we have

$$\operatorname{Ker} H^0(j) = \mathcal{N}^*_{C,\Delta_0/\overline{\mathcal{M}}_2} \subset H^0(\Omega_C \otimes \omega_C).$$

Therefore, $\operatorname{Ker} H^0(j) \subset \operatorname{Im} \mathcal{P}^*$. Since $\dim \operatorname{Ker} H^0(j) = 1$ and, by Proposition 8.3, $\dim \operatorname{Ker}(H^0(j) \circ \mathcal{P}^*) = 2$, we have $\dim \operatorname{Ker}(\mathcal{P}^*) = 1$. By diagram (8-7) this implies that the kernel of n^* is of dimension ≤ 1. Since $\mathcal{N}^*_{X,\mathcal{E}/\mathcal{B}_D}$ is a vector space of dimension 2 and $\mathcal{N}^*_{(C,\eta),\mathcal{S}_i/\widetilde{\mathcal{R}}_{2,7}}$ a vector space of dimension 1, it follows that n^* is surjective. □

9. Local degree of $\mathrm{Pr}_{2,7}$ over the boundary divisors

First we compute the local degree of the Prym map $\widetilde{\mathrm{Pr}}_{2,7}$ along the divisor $\mathcal{S}_1$. Since the conormal map of $\mathrm{Pr}_{2,7}$ along $\mathcal{S}_1$ is surjective according to Proposition 8.5, [Donagi and Smith 1981, I, Lemma 3.2] implies that the local degree along $\mathcal{S}_1$ is given by the degree of the induced map $\widetilde{\mathcal{P}} : \widetilde{\mathcal{S}}_1 \to \widetilde{\mathcal{E}}$ on the exceptional divisor $\widetilde{\mathcal{S}}_1$. Now the polarized abelian variety $X(E)$ is uniquely determined by the elliptic curve E, according to its definition. Hence the curve $\mathcal{E}$ can be identified with the moduli space of elliptic curves, i.e., with the affine line. The exceptional divisor $\widetilde{\mathcal{E}}$ is then a $\mathbb{P}^1$-bundle over $\mathcal{E}$. On the other hand, $\mathcal{S}_1$ is a divisor in $\widetilde{\mathcal{R}}_{2,7}$, so $\widetilde{\mathcal{S}}_1$ is isomorphic to $\mathcal{S}_1$. Clearly $\widetilde{\mathcal{P}}$ maps the fibers $\widetilde{\mathrm{Pr}}^{-1}(X(E)) \cap \widetilde{\mathcal{S}}_1$ onto the fibers $\mathbb{P}^1$ over the elliptic curves E.

Now $\widetilde{\mathrm{Pr}}^{-1}(X(E)) \cap \widetilde{\mathcal{S}}_1$ consists of coverings of type (i) and one covering of type (iv), which we denote by $\mathcal{C}_E^{(\mathrm{iv})}$. The coverings of type (i) have as base a nodal curve of the form $C = E/p \sim q$ and we can assume that $p = 0$, thus $\widetilde{\mathrm{Pr}}^{-1}(X(E)) \cap \widetilde{\mathcal{S}}_1$ is parametrized by E itself (the point $q = 0$ corresponds to the covering of type (iv)). Hence the induced conormal map on the exceptional divisors $\widetilde{\mathcal{P}} : \widetilde{\mathcal{S}}_1 \to \widetilde{\mathcal{E}}$ restricted to the fiber over $X(E)$ is a map $\phi : E \to \mathbb{P}^1$. Combining everything, we conclude that the local degree of the Prym map along $\mathcal{S}_1$ coincides with the degree of the induced map $\phi : E \to \mathbb{P}^1$.

Proposition 9.1. *The local degree of the Prym map* $\widetilde{\mathrm{Pr}}_{2,7}$ *along* $\mathcal{S}_1$ *is* 2.

Proof. According to what we have written above, it is sufficient to show that the map $\phi : E \to \mathbb{P}^1$ induced by $\widetilde{\mathcal{P}}$ is a double covering. We use again the identification (8-2) (and its analogue for coverings of type (iii)). As in [Donagi and Smith 1981], let x, y be local coordinates at 0 and q, and let dx, dy be the corresponding differentials. If $(a, b, c) \in S^2\big(H^0(\widetilde{C}, \omega_{\widetilde{C}})^-\big)$ are coordinates in the basis of Lemma 8.1, then $\mathbb{P}(\operatorname{Ker}(H^0(j) \circ \mathcal{P}^*)) \simeq \mathbb{P}^1$ has coordinates $[a, b]$ and its dual is identified with $\mathbb{P}(\operatorname{Im} \widetilde{\mathcal{P}}_{|\text{fiber}})$. In order to describe the kernel of $\mathcal{P}^*$, we look at the multiplication on the stalk over the node $p = (q \sim 0)$. Around p the line bundles η^i are trivial; therefore, the element $(a, b, c) \in \bigoplus_{i=1}^3 (\omega_{C,p} \otimes \omega_{C,p})$ (in coordinates $a, b, c \in \mathcal{O}_p$ for a fixed basis of $\omega_{C,p} \otimes \omega_{C,p}$) is sent to $a + b + c \in (\Omega_C \otimes \omega_C)_p$ under $\widetilde{\mathcal{P}}$. Thus the germ $a + b + c \in \mathcal{O}_p$ is zero if it is in the kernel of $\mathcal{P}^*$. In particular, the coefficient of $dx\,dy$ must vanish. Set $\alpha = a_0\,dx$, $\beta = b_0\,dx$, $\gamma = c_0\,dx$, and let $\alpha = a_q\,dy$, $\beta = b_q\,dy$, $\gamma = c_q\,dy$ be the local description of the differentials. Then the coefficient of $dx\,dy$ must satisfy

$$a_0 a_q + b_0 b_q + c_0 c_q = 0. \tag{9-1}$$

Now, by looking at the dual picture, we consider $\mathbb{P}^1 = \mathbb{P}(\operatorname{Ker} \mathcal{P}^*)^* \subset \mathbb{P}^{2*}$. Let E be embedded in $\mathbb{P}^{2*}$ by the linear system $|3\cdot 0|$. The coordinate functions $[a, b, c] \in \mathbb{P}^{2*}$ satisfy condition (9-1) for all $q \in E$. Then the points on the fiber over $[a_q, b_q, c_q]$ are points in E over the line passing through the origin $0 \in E \subset \mathbb{P}^{2*}$ and q. Hence the map $E \to \mathbb{P}^1$ corresponds to the restriction to E of the projection $\mathbb{P}^{2*} \to \mathbb{P}^1$ from the origin, which is the double covering $E \to \mathbb{P}^1$ determined by the divisor $0 + q$ of E and thus of degree two. □

We now turn our attention to the Prym map on $\mathcal{S}_2$. By the surjectivity of the conormal map of $\mathrm{Pr}_{2,7}$ on $\mathcal{S}_2$ (Proposition 8.5), the local degree along $\mathcal{S}_2$ is computed by the degree of the map $\widetilde{\mathcal{P}} : \widetilde{S}_2 \to \widetilde{\mathcal{E}}$ on the divisor $\widetilde{\mathcal{S}}_2$, which is a $\mathbb{P}^1$-bundle over $\mathcal{E}$. Given an elliptic curve E, the fiber of $\widetilde{\mathrm{Pr}}^{-1}(X(E))$ intersected with the divisor $\mathcal{S}_2$ consists of coverings of type (ii), one covering of type (iii), denoted by $\mathcal{C}_E^{(\text{iii})}$, and one covering of type (iv), $\mathcal{C}_E^{(\text{iv})}$, which lies in the intersection with the divisor $\mathcal{S}_1$.

Recall that the type (ii) coverings have base curve $C = E_1 \cup E$ intersecting at one point, which we can assume to be 0, and with E_1 an arbitrary elliptic curve. The covering over C is the union of a degree-7 étale cyclic covering F_1 over E_1 and 7 elliptic curves E_i attached to F_1 mapping each one of them isomorphically to E. So the type (ii) coverings on the fiber over E are parametrized by pairs $(E_1, \langle\eta\rangle)$, where E_1 is an elliptic curve and $\langle\eta\rangle \subset E_1$ is a subgroup of order 7.

It is known that the parametrization space of the pairs $(E_1, \langle\eta\rangle)$ is the modular curve $Y_0(7) := \Gamma_0(7) \setminus \mathbb{H}$. The natural projection $(E_1, \eta) \mapsto E_1$ defines a map $\pi_0 : Y_0(7) \to \mathbb{C}$. Moreover, the curve $Y_0(7)$ admits a compactification $X_0(7) := \overline{Y_0(7)}$ such that the map π_0 extends to a map $\pi : X_0(7) \to \mathbb{P}^1$ (see [Silverman 1986]). The genus of $X_0(7)$ can be computed by the Hurwitz formula using the fact that π is of degree 8, and it is ramified over the points corresponding to elliptic curves with j-invariant 0 and 12^3 (with ramification degree 4 on each fiber) and over ∞, where the inverse image consists of two cusps, one étale and the other of ramification index 7. The two cusps over ∞ represent the coverings $\mathcal{C}_E^{(\mathrm{iii})}$ and $\mathcal{C}_E^{(\mathrm{iv})}$ above $X(E)$ (see Remark 9.4). This gives that $X_0(7)$ is of genus zero. Thus, we can identify $\widetilde{\mathcal{S}}_2 \cap \widetilde{\mathrm{Pr}}^{-1}(X(E))$ with $X_0(7) \simeq \mathbb{P}^1$.

Then, since $\widetilde{\mathcal{S}}_2 \simeq \mathcal{S}_2$, the restriction of the conormal map $\widetilde{\mathcal{P}}$ to a fiber over the point $[E] \in \mathcal{E}$ is a map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$.

Proposition 9.2. *The map π coincides with the map $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ of the fibers of $\widetilde{\mathcal{S}}_2 \to \widetilde{\mathcal{E}}$ over a point $[E] \in \mathcal{E}$.*

Proof. Let o and o' be the zero elements of E and E_1, respectively, with local coordinates x and y, respectively. Set $\alpha = a_o dx$, $\beta = b_o dx$, $\gamma = c_o dx$, and let $\alpha = a_{o'} dy$, $\beta = b_{o'} dy$, $\gamma = c_{o'} dy$ be the local description of elements of $(\omega_{C,p} \otimes \omega_{C,p})$ around the node $p = (o \sim o')$. As in the proof of Proposition 9.1, for an element (a, b, c) in the kernel of $\mathcal{P}^*$, the coefficient of $dx\,dy$ must vanish, i.e., it satisfies

$$a_o a_{o'} + b_o b_{o'} + c_o c_{o'} = 0 \tag{9-2}$$

in a neighborhood of the node. Considering the dual map, one sees that the fiber of $\widetilde{\mathcal{P}}$ over a point $[a_o, b_o, c_o] \in \mathbb{P}(\mathrm{Ker}\,\mathcal{P}^*)^* \subset \mathbb{P}^2$ with $c_o = -a_o - b_o$ corresponds to the pairs $(E_1, \langle\eta\rangle)$ such that the local functions $a, b, c \in \mathcal{O}_p$ take the values $[a_{o'}, b_{o'}, c_{o'}]$ around the $o' \in E_1$ with $c_{o'} = -a_{o'} - b_{o'}$ and such that they verify (9-2). This determines completely the triple $[a_{o'}, b_{o'}, c_{o'}]$, which depends only on the values at the node $o' \in E_1$ of the base curve. Note that a, b, c are elements of the local ring $\mathcal{O}_p$, which determines the curve E_1 uniquely. In fact, its quotient ring is the direct product of the function fields of E_1 and E, which in turn determines the curves. Therefore, the map ψ can be identified with the projection $(E_1, \langle\eta\rangle) \mapsto E_1$. $\square$

As an immediate consequence we have:

Corollary 9.3. *The local degree of the Prym map $\widetilde{\mathrm{Pr}}_{2,7}$ along the divisor $\mathcal{S}_2$ is* 8.

Using Proposition 9.1 and Corollary 9.3 we conclude that the degree of the Prym map $\widetilde{\mathrm{Pr}}_{2,7}$ is 10, which finishes the proof of Theorem 1.1

Remark 9.4. The moduli interpretation of $X_0(7) \setminus Y_0(7)$ is given by the *Néron polygons*: one of the cusps represents a 1-*gon*, that is, a nodal cubic curve, corresponding to the covering $\mathcal{C}_E^{(\mathrm{iii})}$, and the other represents a 7-*gon*, that is, 7 copies of $\mathbb{P}^1$ with the point 0 of one attached to the point ∞ of the other in a closed chain, which corresponds to the covering $\mathcal{C}_E^{(\mathrm{iv})}$ (see [Silverman 1994, IV, §8]).

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References

[Abramovich et al. 2003] D. Abramovich, A. Corti, and A. Vistoli, "Twisted bundles and admissible covers", *Comm. Algebra* **31**:8 (2003), 3547–3618. MR Zbl

[Arbarello et al. 2011] E. Arbarello, M. Cornalba, and P. A. Griffiths, *Geometry of algebraic curves*, vol. II, Grundlehren der Mathematischen Wissenschaften **268**, Springer, Heidelberg, 2011. MR Zbl

[Bardelli 1989] F. Bardelli, "Lectures on stable curves", pp. 648–704 in *Lectures on Riemann surfaces* (Trieste, 1987), edited by M. Cornalba et al., World Scientific, Teaneck, NJ, 1989. MR Zbl

[Beauville 1977] A. Beauville, "Prym varieties and the Schottky problem", *Invent. Math.* **41**:2 (1977), 149–196. MR Zbl

[Birkenhake and Lange 2004] C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **302**, Springer, Berlin, 2004. MR Zbl

[Donagi and Smith 1981] R. Donagi and R. C. Smith, "The structure of the Prym map", *Acta Math.* **146**:1 (1981), 25–102. MR Zbl

[Faber 1988] C. Faber, "Prym varieties of triple cyclic covers", *Math. Z.* **199**:1 (1988), 61–79. MR Zbl

[Lange and Ortega 2011] H. Lange and A. Ortega, "Prym varieties of cyclic coverings", *Geom. Dedicata* **150** (2011), 391–403. MR Zbl

[Lange and Ortega $\geq$ 2016] H. Lange and A. Ortega, "On the Prym map of cyclic coverings", in preparation.

[Mumford 1971] D. Mumford, "Theta characteristics of an algebraic curve", *Ann. Sci. École Norm. Sup.* (4) **4**:2 (1971), 181–192. MR Zbl

[Ortega 2003] A. Ortega, "Variétés de Prym associées aux revêtements n-cycliques d'une courbe hyperelliptique", *Math. Z.* **245**:1 (2003), 97–103. MR Zbl

[Shimura 1963] G. Shimura, "On analytic families of polarized abelian varieties and automorphic functions", *Ann. of Math.* (2) **78** (1963), 149–192. MR Zbl

[Silverman 1986] J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics **106**, Springer, New York, NY, 1986. MR Zbl

[Silverman 1994] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics **151**, Springer, New York, NY, 1994. MR Zbl

[Welters 1983] G. E. Welters, "Polarized abelian varieties and the heat equations", *Compositio Math.* **49**:2 (1983), 173–194. MR Zbl

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