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Local bounds for L^p norms of Maass forms in the level aspect

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We apply techniques from harmonic analysis to study the L^p norms of Maass forms of varying level on a quaternion division algebra. Our first result gives a candidate for the local bound for the sup norm in terms of the level, which is new when the level is not squarefree. The second result is a bound for L^p norms in the level aspect that is analogous to Sogge's theorem on L^p norms of Laplace eigenfunctions.

1. Introduction

Let ϕ be a cuspidal newform of level $\Gamma_0(N)$ on $GL_2/\mathbb{Q}$ or a quaternion division algebra over $\mathbb{Q}$, which we shall assume is L^2-normalised with respect to the measure that gives $\Gamma_0(N)\backslash\mathbb{H}^2$ mass 1. There has recently been interest in bounding the sup norm $\|\phi\|_\infty$ in terms of N and the infinite component of ϕ, see [Blomer and Holowinsky 2010; Harcos and Templier 2013; 2012; Saha 2015b; Templier 2010; 2014; 2015]. The "trivial" bound in the level aspect (with the infinite component remaining bounded) is generally considered to be $\|\phi\|_\infty \ll_\epsilon N^{1/2+\epsilon}$, provided N is squarefree; see [Abbes and Ullmo 1995] or any of the previously cited papers. Our first result is a candidate for the generalisation of this to arbitrary N.

Theorem 1. *Let $D/\mathbb{Q}$ be a quaternion division algebra that is split at infinity. Let ϕ be an L^2-normalised newform of level $K_0(N)$ on* $\mathrm{PGL}_1(D)$, *where N is odd and coprime to the primes that ramify in D. Assume that ϕ is spherical at infinity with spectral parameter t, which is related to the Laplace eigenvalue by the equation $(\Delta+1/4+t^2)\phi=0$. Let $N_0\geq 1$ be the smallest number with $N|N_0^2$. We have*

$$\|\phi\|_\infty \ll (1+|t|)^{1/2} N_0^{1/2} \prod_{p|N} (1+1/p)^{1/2}. \tag{1}$$

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Notation is standard, and specified below. When t is bounded, Theorem 1 gives a bound of $N^{1/2+\epsilon}$ for N squarefree, but roughly $N^{1/4+\epsilon}$ for powerful N. While the theorem is restricted to compact quotients, Section 3.1 gives a weaker result in the case of $\mathrm{PGL}_2/\mathbb{Q}$. This has been strengthened by Saha [2015a], who proves a bound on $\mathrm{PGL}_2/\mathbb{Q}$ that combines (1) in the level aspect with the $t^{5/12+\epsilon}$ bound of Iwaniec and Sarnak [1995] in the eigenvalue aspect. He does this by combining the methods of Iwaniec and Sarnak with bounds for Whittaker functions and an amplification argument.

Our second result is the analogue in the level aspect of a classical theorem of Sogge [1988], which we now recall. Let M be a compact Riemannian surface with Laplacian Δ, and let ψ be a function on M satisfying $(\Delta+\lambda^2)\psi=0$ and $\|\psi\|_2=1$. Define $\delta:[2,\infty]\to\mathbb{R}$ by

$$\delta(p)=\begin{cases}\frac{1}{2}-\frac{2}{p}, & 0\le\frac{1}{p}\le\frac{1}{6},\\ \frac{1}{4}-\frac{1}{2p}, & \frac{1}{6}\le\frac{1}{p}\le\frac{1}{2}.\end{cases}\tag{2}$$

Sogge's theorem states that

$$\|\psi\|_p\ll\lambda^{\delta(p)}\quad\text{for}\quad 2\le p\le\infty.\tag{3}$$

In particular, this is stronger than the bound obtained by interpolating between bounds for the L^2 and L^∞ norms. Our next theorem demonstrates that something similar is possible in the level aspect.

Theorem 2. *Let $D/\mathbb{Q}$ be a quaternion division algebra that is split at infinity. Let ϕ be an L^2-normalised newform of level $K_0(q^2)$ on $\mathrm{PGL}_1(D)$, where q is an odd prime that does not ramify in D. Assume that ϕ is principal series at q, that ϕ is spherical at infinity with spectral parameter t, and that $|t|\le A$ for some $A>0$. We have*

$$\|\phi\|_p\ll_A q^{\delta(p)}.$$

It should be possible to give some extension of Theorem 2 to general ϕ, although in some cases the method may not give any improvement over the bound given by interpolating between L^2 and L^∞ norms. In particular, this seems to occur when ϕ is special at q. We have chosen to work in the simplest case where the method gives a nontrivial result.

Theorems 1 and 2 are an attempt to prove the correct local bounds for L^p norms of eigenfunctions in the level aspect, in the same way that (3) is the local bound in the eigenvalue aspect. The term "local bound" means the best bound that may be proved by only considering the behaviour of ϕ in one small open set at a time, without taking the global structure of the space into account.

The bound (3) is sharp on the round sphere. In the same way, one may obtain limited evidence that Theorem 1, and Theorem 2 for $p\ge 6$, are the sharp local bounds by comparing them with what may be proved on the "compact form" of the arithmetic quotient being considered. In the case of Theorem 2, this means taking an L^2-normalised function ψ on $\mathrm{PGL}(2, \mathbb{Z}_q)$ of the same type as ϕ' defined below — in other words, invariant under the group $K(q, q)$ defined in Section 2.1, and generating an irreducible representation of the same type as ϕ' under right translation — and proving bounds for $\|\psi\|_p$. We may prove that $\|\psi\|_p \ll q^{\delta(p)}$ in the same way as Theorem 2, after which Equation (4) and Lemma 11 imply that this is sharp for $p \geq 6$. An analogous statement may be proved for Theorem 1 when N is a growing power of a fixed prime. However, we do not yet know if Theorem 2 is sharp in this sense for $2 \leq p \leq 6$. We expect the bound of Theorem 1 to have a natural expression as the square root of the Plancherel density around the representation of ϕ.

Because the proofs do not make use of the global structure of the arithmetic quotient, it should be possible to improve the exponents by using arithmetic amplification.

2. Notation

2.1. ***Adelic groups.*** Let $\mathbb{A}$ and $\mathbb{A}_f$ be the adeles and finite adeles of $\mathbb{Q}$. Let $D/\mathbb{Q}$ be a quaternion division algebra that is split at infinity. Let S be the set containing 2 and all primes that ramify in D, and let $S_\infty = S \cup \{\infty\}$. Let $G = \mathrm{PGL}_1(D)$. If v is a place of $\mathbb{Q}$, let $G_v = G(\mathbb{Q}_v)$. Let $X = G(\mathbb{Q})\backslash G(\mathbb{A})$. Let $\mathcal{O} \subset D$ be a maximal order. Let $K = \bigotimes_p K_p \subset G(\mathbb{A}_f)$ be a compact subgroup with the properties that K_p is open in G_p for $p \in S$, and K_p is isomorphic to the image of $\mathcal{O}_p^\times$ in G_p when $p \notin S$. This allows us to choose isomorphisms $K_p \simeq \mathrm{PGL}(2, \mathbb{Z}_p)$ when $p \notin S$. When $M, N \geq 1$ are prime to S, we shall use these isomorphisms to define the upper triangular congruence subgroup $K_0(N)$, principal congruence subgroup $K(N)$, and

$$K(M, N) = \left\{ k \in K \;:\; k \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}(M),\; k \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}(N) \right\}$$

in the natural way. We choose a maximal compact subgroup $K_\infty \subset G_\infty$.

We fix a Haar measure on $G(\mathbb{A})$ by taking the product of the measures on G_p assigning mass 1 to K_p, and any Haar measure on G_∞. We use this measure to define convolution of functions on $G(\mathbb{A})$, which we denote by $*$, and if $f \in C_0^\infty(G(\mathbb{A}))$ we use it to define the operator $R(f)$ by which f acts on $L^2(X)$. If H is a group and f is a function on H, we define the function $f^\vee$ by $f^\vee(h) = \overline{f}(h^{-1})$. If $f \in C_0^\infty(G(\mathbb{A}))$, the operators $R(f)$ and $R(f^\vee)$ are adjoints.

2.2. ***Newforms.*** Let $N \geq 1$ be prime to S. We shall say that $\phi \in L^2(X)$ is a newform of level $K_0(N)$ if ϕ lies in an automorphic representation $\pi = \bigotimes_v \pi_v$ of G, ϕ is invariant under $K_0(N)$, and we have a factorisation $\phi = \bigotimes_v \phi_v$ where ϕ_v is a local newvector of level N for $v \notin S_\infty$. We shall say that ϕ is spherical with spectral parameter $t \in \mathbb{C}$ if π_∞ satisfies these conditions, and ϕ is invariant under K_∞. Note that our normalisation of t is such that the tempered principal series corresponds to $t \in \mathbb{R}$.

2.3. ***The Harish-Chandra transform.*** Given $k \in C_0^\infty(G_\infty)$, we define its Harish-Chandra transform by

$$\hat{k}(t) = \int_{G_\infty} k(g)\varphi_t(g)\, dg$$

for $t \in \mathbb{C}$, where φ_t is the standard spherical function with spectral parameter t. We will use the following standard result on the existence of a K_∞-biinvariant function with concentrated spectral support.

Lemma 3. *There is a compact set $B \subset G_\infty$ such that for any $t \in \mathbb{R} \cup [0, i/2]$, there is a K_∞-biinvariant function $k \in C_0^\infty(G_\infty)$ with the following properties*:

(a) *The function k is supported in B, and $\|k\|_\infty \ll 1 + |t|$.*

(b) *The Harish-Chandra transform $\hat{k}$ is nonnegative on $\mathbb{R} \cup [0, i/2]$, and satisfies $\hat{k}(t) \geq 1$.*

Proof. When $t \in \mathbb{R}$ and $|t| \geq 1$, this is, e.g., Lemma 2.1 of [Templier 2015]. When $|t| \leq 1$, one may fix a K_∞-biinvariant real bump function k_0 supported near the identity and define $k = k_0 * k_0$. □

Note that condition (b) implies that $k = k^\vee$.

2.4. ***Inner products of matrix coefficients.*** The following lemma is known as Schur orthogonality, see Theorem 2.4 and Proposition 2.11 of [Bump 2013] for the proof.

Lemma 4. *Let H be a finite group, and let dh be the Haar measure of mass 1 on H. Let (ρ, V) be an irreducible representation of H, and let $\langle \cdot, \cdot \rangle$ be a positive definite H-invariant Hermitian form on V. If $v_i \in V$ for $1 \leq i \leq 4$, we have*

$$\int_H \langle \rho(h)v_1, v_2\rangle \overline{\langle \rho(h)v_3, v_4\rangle}\, dh = \frac{\langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}}{\dim V}. \tag{4}$$

3. Proof of Theorem 1

Choose $h \in G(\mathbb{A})$ by setting $h_v = 1$ for $v \in S_\infty$, and

$$h_v = \begin{pmatrix} N & \\ & N_0 \end{pmatrix}$$

when $v \notin S_\infty$. We define $\phi' = R(h)\phi$, so that ϕ' is invariant under $K(N_0, N/N_0)$. Let $V = \bigotimes V_v \subset \pi$ be the space generated by ϕ' under the action of K.

Lemma 5. *V is an irreducible representation of K, and*

$$\dim V \leq N_0 \prod_{p|N} (1 + 1/p).$$

Proof. It suffices to prove the analogous statement for the tensor factors V_p, and we may assume that $p \notin S$. If we could write $V_p = V^1 + V^2$, where V^i are nontrivial K_p-invariant subspaces, then the projection of ϕ'_p to each subspace would be invariant under $K_p(N_0, N/N_0)$. However, this contradicts the uniqueness of the newvector.

As V_p is irreducible and factors through $K_p/K_p(N_0)$, the lemma now follows from the results of Silberger [1970, §3.4], in particular the remarks on pages 96–97. Note that we use our assumption that $2 \in S$ at this point. □

We define $k_f \in C_0^\infty(G(\mathbb{A}_f))$ to be $\overline{\langle R(g)\phi', \phi' \rangle}$ for $g \in K$ and 0 otherwise. Choose a function $k_\infty \in C_0^\infty(G_\infty)$ as in Lemma 3, and define $k = k_\infty k_f$. It may be seen that $k = k^\vee$, which implies that $R(k)$ is self-adjoint. Lemma 5 and Equation (4) imply that $k_f = \dim V k_f * k_f$, and combining this with Lemma 3(b) gives that $R(k)$ is nonnegative. Lemmas 3 and 5 and Equation (4) imply that $R(k)\phi' = \lambda\phi'$, where $\lambda > 0$ and

$$\lambda^{-1} \leq \dim V \leq N_0 \prod_{p|N} (1 + 1/p). \tag{5}$$

Extend ϕ' to an orthonormal basis $\{\phi_i\}$ of eigenfunctions for $R(k)$ with eigenvalues $\lambda_i \geq 0$. The pretrace formula associated to k is

$$\sum_i \lambda_i |\phi_i(x)|^2 = \sum_{\gamma \in G(\mathbb{Q})} k(x^{-1}\gamma x)$$

and dropping all terms from the left hand side but ϕ' gives

$$\lambda |\phi'(x)|^2 \leq \sum_{\gamma \in G(\mathbb{Q})} k(x^{-1}\gamma x). \tag{6}$$

The compactness of X and uniformly bounded support of k implies that the number of nonzero terms on the right hand side is bounded independently of x, and combining (5) and Lemma 3(a) completes the proof.

3.1. ***A result in the noncompact case.*** If we set $G = \mathrm{PGL}_2/\mathbb{Q}$, it may be seen that we have the following analogue of Theorem 1.

Proposition 6. *Let $\Omega \subset G(\mathbb{Q})\backslash G(\mathbb{A})$ be compact. Let ϕ be an L^2-normalised newform of level $K_0(N)$ on G, where N is odd. Assume that ϕ is spherical at infinity with spectral parameter t. Let $N_0 \geq 1$ be the smallest number with $N | N_0^2$. If ϕ' is related to ϕ as above, we have $\|\phi'|_\Omega\|_\infty \ll (1 + |t|)^{1/2} N_0^{1/2} \prod_{p|N} (1 + 1/p)^{1/2}$.*

See [Saha 2015a] for a strengthening of this result.

4. Proof of Theorem 2

We maintain the notation ϕ', V, and k_f from Section 3. Our assumption $2 \in S$ implies that $q \geq 3$.

Lemma 7. *We have* $\dim V = q$ *or* $q + 1$.

Proof. We have assumed that π_q is isomorphic to an irreducible principal series representation $\mathcal{I}(\chi, \chi^{-1})$, for some character χ of $\mathbb{Q}_q^\times$ with conductor q. By considering the compact model of $\mathcal{I}(\chi, \chi^{-1})$, and applying the fact that V factors through $K/K(q) \simeq \mathrm{PGL}(2, \mathbb{F}_q)$, we see that V must be a subrepresentation of a principal series representation of $\mathrm{PGL}(2, \mathbb{F}_q)$. (Here $\mathbb{F}_q$ denotes the field with q elements.) It follows that $\dim V$ must be either 1, q, or $q + 1$. The possibility $\dim V = 1$ is ruled out because any one-dimensional representation of K that is trivial on $K(q, q)$ must be trivial, and this contradicts our assumption that ϕ is new at $K_0(q^2)$. □

Let $k_\infty^0 \in C_0^\infty(G_\infty)$ be a real-valued K_∞-biinvariant function, so that $k_\infty^0 = (k_\infty^0)^\vee$. If we choose k_∞^0 to be a nonnegative bump function with sufficiently small support, we may assume that its Harish-Chandra transform satisfies $\hat{k}_\infty^0(t) \geq 1$ for $t \in [0, A] \cup [0, i/2]$. We define $k_\infty = k_\infty^0 * k_\infty^0$. Let $k_0 = k_\infty^0 k_f$, and $k = k_\infty k_f$. Let $T_0 = R(k_0)$ and $T = R(k)$. We see that T_0 is self-adjoint, and Equation (4) implies that $T = \dim V T_0^2$. Let $W \subset K_q$ be the subgroup $\{1, w\}$, where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $k_{1,f} \in C_0^\infty(G(\mathbb{A}_f))$ be k_f times the characteristic function of $WK(q, q)$, and let $k_{2,f} = k_f - k_{1,f}$. Let $k_i = k_\infty k_{i,f}$, and $T_i = R(k_i)$. The proof of Theorem 2 works by combining the decomposition $T = T_1 + T_2$ with interpolation between the following bounds.

Lemma 8. *We have*

$$\|T_1 f\|_\infty \ll \|f\|_1, \quad \|T_2 f\|_\infty \ll q^{-1/2} \|f\|_1,$$

for any $f \in C^\infty(X)$.

Proof. The integral kernels of T_i are given by

$$\sum_{\gamma \in G(\mathbb{Q})} k_i(x^{-1} \gamma y).$$

The result now follows from the compactness of $G(\mathbb{Q}) \backslash G(\mathbb{A})$, the bound $\|k_1\|_\infty \ll 1$, and the bound $\|k_2\|_\infty \ll q^{-1/2}$ which follows from Lemma 11 below. □

Lemma 9. *We have*

$$\|T_1 f\|_2 \ll q^{-2}\|f\|_2, \quad \|T_2 f\|_2 \ll q^{-1}\|f\|_2,$$

for any $f \in C^\infty(X)$.

Proof. The choice of k_∞ and the identity $k_f = \dim V k_f * k_f$ imply that the $L^2 \to L^2$ norm of T is $\ll (\dim V)^{-1} \le q^{-1}$. Lemma 11 implies that $k_{1,f} = k_{1,f}^\vee$ and

$$k_{1,f} = [K : WK(q,q)] k_{1,f} * k_{1,f} = (q(q+1)/2) k_{1,f} * k_{1,f},$$

and this implies that the $L^2 \to L^2$ norm of T_1 is $\ll q^{-2}$. The bound for T_2 follows from the triangle inequality. □

Interpolating between these bounds gives the following.

Lemma 10. *We have* $\|Tf\|_p \ll q^{2\delta(p)-1}\|f\|_{p'}$ *for any* $2 \le p \le \infty$, *where* p' *is the dual exponent to* p.

Proof. Applying the Riesz–Thorin interpolation theorem [Folland 1999, Theorem 6.27] to the bounds

$$\|T_1 f\|_\infty \ll \|f\|_1, \quad \|T_1 f\|_2 \ll q^{-2}\|f\|_2$$

gives $\|T_1 f\|_p \ll q^{-4/p}\|f\|_{p'}$ for $2 \le p \le \infty$, and applying it to

$$\|T_2 f\|_\infty \ll q^{-1/2}\|f\|_1, \quad \|T_2 f\|_2 \ll q^{-1}\|f\|_2$$

gives $\|T_2 f\|_p \ll q^{-1/2-1/p}\|f\|_{p'}$ for $2 \le p \le \infty$. Applying Minkowski's inequality then gives

$$\|Tf\|_p \le \|T_1 f\|_p + \|T_2 f\|_p \ll (q^{-4/p} + q^{-1/2-1/p})\|f\|_{p'},$$

and the observation $2\delta(p) - 1 = \max(-4/p, -1/2 - 1/p)$ completes the proof. □

We now combine Lemma 10 with the usual adjoint-square argument: we have

$$\begin{aligned}
\langle \dim V T_0^2 f, f\rangle &= \langle Tf, f\rangle \\
&\ll q^{2\delta(p)-1}\|f\|_{p'}^2 \\
\langle T_0 f, T_0 f\rangle &\ll q^{2\delta(p)-2}\|f\|_{p'}^2 \\
\|T_0 f\|_2 &\ll q^{\delta(p)-1}\|f\|_{p'}.
\end{aligned}$$

Taking adjoints gives $\|T_0 f\|_p \ll q^{\delta(p)-1}\|f\|_2$. Applying this with $f = \phi'$ and estimating the eigenvalue of T_0 on ϕ' as in Section 3 completes the proof.

Lemma 11. *Let π_q be isomorphic to an irreducible principal series representation $\mathcal{I}(\chi, \chi^{-1})$, for some character χ of $\mathbb{Q}_q^\times$ with conductor q. When $g \in K_q$, the matrix coefficient $\langle \pi_q(g)\phi'_q, \phi'_q \rangle$ satisfies*

$$\langle \pi_q(g)\phi'_q, \phi'_q \rangle = 1, \qquad g \in K_q(q, q), \tag{7}$$

$$\langle \pi_q(g)\phi'_q, \phi'_q \rangle = \chi(-1), \quad g \in wK_q(q, q), \tag{8}$$

$$\langle \pi_q(g)\phi'_q, \phi'_q \rangle \ll q^{-1/2}, \quad g \notin WK_q(q, q), \tag{9}$$

where the implied constant is absolute.

Proof. We may reduce the problem to one for the group $\mathrm{PGL}(2, \mathbb{F}_q)$ as in Lemma 7. We let T and B be the usual diagonal and upper triangular subgroups of $\mathrm{PGL}(2, \mathbb{F}_q)$. We now think of χ as a nontrivial character of $\mathbb{F}_q^\times$, and let (ρ, H) denote the corresponding induced representation of $\mathrm{PGL}(2, \mathbb{F}_q)$. We realise H as the space of functions $f : \mathrm{PGL}(2, \mathbb{F}_q) \to \mathbb{C}$ satisfying

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi(a/d) f(g)$$

with the invariant Hermitian form

$$\langle f_1, f_2 \rangle = \sum_{g \in B\backslash \mathrm{PGL}(2,\mathbb{F}_q)} f_1(g) \overline{f_2}(g). \tag{10}$$

It may be seen that there is a unique function $f_0 \in H$ that is invariant under T, up to scaling, and we may choose it to be

$$f_0 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{cases} \chi(\det/cd)/\sqrt{q-1}, & cd \neq 0, \\ 0, & cd = 0. \end{cases} \tag{11}$$

It follows that $\|f_0\| = 1$, and so $\langle \pi_q(g)\phi'_q, \phi'_q \rangle = \langle \rho(g) f_0, f_0 \rangle$. Equation (7) is immediate, and (8) follows from $\rho(w) f_0 = \chi(-1) f_0$. To prove (9), we assume that

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin WT.$$

We may write $\langle \rho(g) f_0, f_0 \rangle$ as a sum over $\mathbb{F}_q$ as follows.

Lemma 12. *We have*

$$\langle \rho(g) f_0, f_0 \rangle = (q-1)^{-1} \chi(\det(g)) \sum_n \chi^{-1}((c+an)(d+bn)) \chi(n). \tag{12}$$

Proof. We choose a set of coset representatives for $B\backslash \mathrm{PGL}(2, \mathbb{F}_q)$ consisting of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, \quad n \in \mathbb{F}_q.$$

Applying (11) for these representatives gives

$$f_0\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = 0 \quad \text{and} \quad f_0\left(\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}\right) = \chi^{-1}(n)/\sqrt{q-1}.$$

The first coset representative therefore makes no contribution to $\langle \rho(g) f_0, f_0 \rangle$. For the others, we calculate

$$\begin{aligned} [\rho(g) f_0]\left(\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}\right) &= f_0\left(\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\ &= f_0\left(\begin{pmatrix} a & b \\ c+an & d+bn \end{pmatrix}\right) \\ &= \chi(\det(g))\chi^{-1}((c+an)(d+bn))/\sqrt{q-1}. \end{aligned}$$

Substituting these into (10) completes the proof. □

We bound the sum (12) by rewriting it as

$$\sum_n \chi^{-1}\big((c+an)(d+bn)n^{q-2}\big)$$

and applying [Schmidt 2004, Chapter 2, Theorem 2.4] (see also [Iwaniec and Kowalski 2004, Theorem 11.23]). We must first check that $(c+an)(d+bn)n^{q-2}$ is not a proper power. The assumption $g \notin WT$ implies that one or both of $a+cn$ and $b+dn$ have a root distinct from 0. If they both have the same root distinct from 0, this contradicts the invertability of g. Therefore $(c+an)(d+bn)n^{q-2}$ must have at least one root of multiplicity 1, so it cannot be a power. As $(c+an)(d+bn)n^{q-2}$ has at most 3 distinct roots, [Schmidt 2004] or [Iwaniec and Kowalski 2004] therefore give

$$\left|\sum_n \chi\big((c+an)(d+bn)n^{q-2}\big)\right| \le 2\sqrt{q},$$

which completes the proof of (9). □

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