Analytic continuation on Shimura varieties with
μ-ordinary locus

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We study the geometry of unitary Shimura varieties without assuming the existence of an ordinary locus. We prove, by a simple argument, the existence of canonical subgroups on a strict neighborhood of the μ-ordinary locus (with an explicit bound). We then define the overconvergent modular forms (of classical weight) as well as the relevant Hecke operators. Finally, we show how an analytic continuation argument can be adapted to this case to prove a classicality theorem, namely that an overconvergent modular form which is an eigenform for the Hecke operators is classical under certain assumptions.

Introduction

A modular form is defined as a global section of a certain sheaf on the modular curve. To study congruences between modular forms, one is led to introduce new objects, namely p-adic and overconvergent modular forms. These are sections of a sheaf on the ordinary locus of the modular curve and on a strict neighborhood of the ordinary locus, respectively. A lot of work has been done using these objects; one can for example construct families of overconvergent modular forms.

It is possible to generalize the definition of p-adic and overconvergent modular forms to other varieties, for example the Hilbert modular variety or the Siegel variety. The natural definition of an overconvergent modular form is a section of a certain sheaf on a strict neighborhood of the ordinary locus. In greater generality, one can consider Shimura varieties with a nonempty ordinary locus.

But for Shimura varieties without ordinary locus, this definition fails. Recall that, for Shimura varieties of PEL type, the criterion for the existence of an ordinary locus at p is that p splits completely in its reflex field. For Shimura varieties of PEL type (C) (associated to symplectic groups), the reflex field is $\mathbb{Q}$, so there is always an ordinary locus. But, if one considers Shimura varieties of type (A) (associated to unitary groups), then the reflex field may not be $\mathbb{Q}$ and the ordinary locus may be empty.

Let us look at an example. Let F be a CM-field with totally real subfield F_0, and consider the special fiber of the Shimura variety associated to a unitary group with signature (a_σ, b_σ) at each real place σ of F_0. Suppose for simplicity that p is a prime number inert in F_0 that splits as $\pi^+\pi^-$ in F. The choice of π^+ gives an order for the elements of the pair (a_σ, b_σ) for each real place σ of F_0. The existence of the ordinary locus at p is then equivalent to the fact that there exist integers a and b such that $a_\sigma = a$ and $b_\sigma = b$ for each real place σ. The structure of the p-torsion of the abelian variety is then well known on the ordinary locus. In our setting, the p-divisible group $A[p^\infty]$ splits as $A[(\pi^+)^\infty] \oplus A[(\pi^-)^\infty]$, and $A[(\pi^-)^\infty]$ is the dual of $A[(\pi^+)^\infty]$. On the ordinary locus, the p-divisible group $A[(\pi^+)^\infty]$ is an extension of a multiplicative part of height da and an étale part of height db (d is the degree of F_0 over $\mathbb{Q}$). In particular, there exists a multiplicative subgroup $H_a \subset A[\pi^+]$ of height da. Actually, this property characterizes the ordinary locus: if there exists a multiplicative subgroup H_a of height ha, then the abelian variety is ordinary at p. On the rigid space associated to the Shimura variety (i.e., the generic fiber of its formal completion along its special fiber), one can define the ordinary locus. There is still a multiplicative subgroup of $A[\pi^+]$ of height da on this locus, and work of Fargues [2011] shows that this subgroup extends to a canonical subgroup on a strict neighborhood of the ordinary locus.

If the (a_σ) are not equal to a certain integer a, then the ordinary locus is empty. There is always a special locus in the special fiber of the Shimura variety, called the μ-ordinary locus, but the situation is more involved. Suppose we are in the same setting as before, and let us order the elements $a_1 \leq a_2 \leq \cdots \leq a_d$. Then the μ-ordinary locus is characterized by the fact that the p-divisible group $A[(\pi^+)^\infty]$ has a filtration $0 \subset X_1 \subset X_2 \subset \cdots \subset X_{d+1} = A[(\pi^+)^\infty]$ such that X_{i+1}/X_i is a p-divisible group of height $d(a_{i+1} - a_i)$ with its structure explicitly described (by convention we set $a_0 = 0$ and $a_{d+1} = a_d + b_d$). The fact that A is μ-ordinary is then also equivalent to the existence of subgroups $0 \subset H_{a_1} \subset \cdots \subset H_{a_d} \subset A[\pi^+]$, with H_{a_i} of height da_i and the structure of $H_{a_{i+1}}/H_{a_i}$ explicitly described.

If one looks at the rigid space of the Shimura variety, then one can define the μ-ordinary locus. There are several canonical subgroups in $A[\pi^+]$ on this locus. We lack a good theory for these canonical subgroups, which should be analogous to the one given by Fargues [2011]. However, by simple arguments, one can prove the following fact:

Theorem. *On a strict neighborhood of the μ-ordinary locus, there exist canonical subgroups $H_{a_1} \subset \cdots \subset H_{a_d}$ in $A[\pi^+]$. These subgroups are characterized by the fact that their degree* (in the sense of [Fargues 2010]) *is maximal among the subgroups of the same height of $A[\pi^+]$.*

The proof is actually very simple: Let us consider X_{Iw}, the variety with Iwahori level at p, and let $f : X_{\mathrm{Iw}} \to X$ be the projection. The μ-ordinary locus is the image under f of the locus where the subgroups H_{a_i} are of maximal degree. Let $0 < \varepsilon < \frac{1}{2}$, and consider the locus in X_{Iw} where the degree of H_{a_i} is bigger than the maximal degree minus ε. The image under f of this locus is a strict neighborhood of the μ-ordinary locus. The existence of canonical subgroups follows from the definition. Their uniqueness is a simple computation using the properties of the degree function (see Proposition 1.24).

If we consider the space X_{Iw}, then one can call the locus where the degree of each H_{a_i} is maximal the μ-ordinary-multiplicative locus. It then makes sense to define an overconvergent modular form as the section of a certain sheaf on a strict neighborhood of the μ-ordinary-multiplicative locus.

The Hecke algebra at p acts both on the rigid space and on the space of modular forms. In the case of the existence of the ordinary locus, there is one relevant Hecke operator, parametrizing complements of the canonical subgroup. In the general case, there will be as many relevant Hecke operators as the number of canonical subgroups. We will denote by U_{p,a_i} these Hecke operators. One can show that these operators increase the degrees of all the subgroups of $A[\pi^+]$, then act on the space of overconvergent modular forms.

We can now state the main result of the paper, namely that an overconvergent modular form, which is an eigenform for the Hecke operators U_{p,a_i}, can be analytically continued to the whole variety under a certain assumption, and thus is classical. Let κ be a weight; explicitly, it is a collection of integers $(\kappa_{i,1} \geq \cdots \geq \kappa_{i,a_i}, \lambda_{i,a} \geq \cdots \geq \lambda_{i,b_i})_{1 \leq i \leq d}$. Let $S = \{a_1, \ldots, a_d\} \cap [1, a_d + b_d - 1]$. The cardinality of S is exactly the number of canonical subgroups. Let us write $\Sigma_i = \{j : a_j = a_i\}$ for all i.

Theorem 3.17. *Let f be an overconvergent modular form of weight κ. Suppose that f is an eigenform for the Hecke operators U_{p,a_i}, with eigenvalue α_i for $a_i \in S$, and that we have the relations*

$$n_i + v(\alpha_i) < \inf_{j \in \Sigma_i} (\kappa_{j,a_j} + \lambda_{j,b_j}).$$

Then f is classical.

Here n_i is a constant depending on the variety. It actually comes from the normalization factor of the Hecke operator U_{p,a_i}. This theorem is a classicality result, analogous to the one proved by Coleman [1996] in the case of the modular

curve. Actually, this result has also been proven by Buzzard [2003] and Kassaei [2006] using an analytic continuation method, from which we take inspiration here. Note that there has been extensive work on the classicality problem in the case of the existence of the ordinary locus. We can cite the work of Sasaki [2010], Johansson [2013], Tian and Xiao [2013] and Pilloni and Stroh [2011] in the case of Hilbert varieties, and the work of the author with Pilloni and Stroh [Bijakowski et al. 2016] for more general PEL Shimura varieties (see also [Bijakowski 2014] for Shimura varieties with ramification).

In this introduction, we have assumed that p is inert in F_0 and splits in F. The group associated to the Shimura variety is then a linear group at p. If p is inert in F, then the group is a unitary group at p. Everything we have said adapts to that context: the description of the μ-ordinary locus, the existence of the canonical subgroups, and the analytic continuation theorem. Note that the geometry is more involved in that case; for example, to define Hecke operators, one has to deal with subgroups of $A[p^2]$.

Of course, the assumption that p is inert in F_0 is for simplicity, so what we have said can be formulated for any prime p unramified in F. In the writing of the paper, we have tried to formulate propositions valid in both the linear and unitary cases as much as possible, but of course we often had to treat the proofs separately.

Let us now talk briefly about the text. In Section 1, we introduce the varieties we are dealing with, define the μ-ordinary locus and study the canonical subgroups. In Section 2 we define the classical and overconvergent modular, as well as the Hecke operators. In Section 3 we prove the analytic continuation result. For simplicity, we suppose that the prime p is inert in Sections 2 and 3, and Section 4 briefly shows how to handle the general case.

1. Shimura varieties of type (A)

1A. *The moduli space.*

1A1. *Shimura datum.* We will introduce the objects needed to define the Shimura variety of unitary type we will work with. We refer to [Kottwitz 1992, Section 5] for more details.

Let F_0 be a totally real field of degree d, and F a CM-extension of F_0. Let $(U_{\mathbb{Q}}, \langle\,,\,\rangle)$ be a nondegenerate hermitian F-module and G its automorphism group. For any $\mathbb{Q}$-algebra R, we have

$$G(R) = \big\{ (g, c) \in \mathrm{GL}_F(U_{\mathbb{Q}} \otimes_{\mathbb{Q}} R) \times R^* : \langle gx, gy \rangle = c \langle x, y \rangle \text{ for all } x, y \in U_{\mathbb{Q}} \otimes_{\mathbb{Q}} R \big\}.$$

Let $\tau_1, \ldots, \tau_d$ be the embeddings of F_0 into $\mathbb{R}$, and let σ_i and $\bar{\sigma}_i$ be the two embeddings of F into $\mathbb{C}$ extending τ_i. The choice of σ_i gives an isomorphism $F \otimes_{F_0, \tau_i} \mathbb{R} \simeq \mathbb{C}$. Let $U_i = U_{\mathbb{Q}} \otimes_{F_0, \tau_i} \mathbb{R}$. We write (a_i, b_i) for the signature of the

antihermitian structure on U_i. Then $G_{\mathbb{R}}$ is isomorphic to

$$G\left(\prod_{i=1}^{d} \mathrm{U}(a_i, b_i)\right),$$

where $a_i + b_i$ is independent of i, and is equal to $(1/2d) \dim_{\mathbb{Q}} U_{\mathbb{Q}}$. We'll call this quantity $a + b$.

We also give ourselves a morphism of $\mathbb{R}$-algebras $h : \mathbb{C} \to \mathrm{End}_F U_{\mathbb{R}}$ such that $\langle h(z)v, w \rangle = \langle v, h(\bar{z})w \rangle$ and $(v, w) \mapsto \langle v, h(i)w \rangle$ is positive definite. This morphism gives a complex structure on $U_{\mathbb{R}}$; let $U_{\mathbb{C}}^{1,0}$ be the subspace of $U_{\mathbb{C}}$ on which $h(z)$ acts by multiplication by z.

We then have $U_{\mathbb{C}}^{1,0} \simeq \prod_{i=1}^{d} \mathbb{C}^{a_i} \oplus \overline{\mathbb{C}}^{b_i}$ as $F \otimes_{\mathbb{Q}} \mathbb{R}$-modules and $F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \bigoplus_{i=1}^{d} \mathbb{C}$ (where the action of $\mathbb{C}$ on $\mathbb{C}^{a_i} \oplus \overline{\mathbb{C}}^{b_i}$ is the standard action on the first factor and the conjugated action on the second).

Let U be an O_F-stable lattice of $U_{\mathbb{Q}}$, and assume that the pairing $\langle\,,\,\rangle$ induces a pairing $U \times U \to \mathbb{Z}$ which is perfect at p.

The ring O_F is a free $\mathbb{Z}$-module. Let $\alpha_1, \ldots, \alpha_t$ be a basis of this module and

$$\mathrm{det}_{U^{1,0}} = f(X_1, \ldots, X_t) = \det\left(X_1\alpha_1 + \cdots + X_t\alpha_t; U_{\mathbb{C}}^{1,0} \otimes_{\mathbb{C}} \mathbb{C}[X_1, \ldots, X_t]\right).$$

We can show that f is polynomial with algebraic coefficients. The number field E generated by its coefficients is called the reflex field.

Remark 1.1. We chose for simplicity to work with a central algebra. One can easily adapt the arguments here, replacing F by a simple algebra B with center F.

1A2. *The Shimura variety.* Let us now define the PEL Shimura variety of type (A) associated to G. Let K be an extension of $\mathbb{Q}_p$ containing the images of all the embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$. Assume that p is unramified in F. We fix an embedding $E \hookrightarrow K$, so that the coefficients of the polynomial $\mathrm{det}_{U^{1,0}}$ can be seen as elements of O_K. We also fix an integer $N \geq 3$ prime to p.

Definition 1.2. Let X be the moduli space over O_K whose S-points are the isomorphism classes of $(A, \lambda, \iota, \eta)$, where

- $A \to S$ is an abelian scheme;

- $\lambda : A \to A^t$ is a prime-to-p polarization;

- $\iota : O_F \to \mathrm{End}\,A$ is compatible with complex conjugation and the Rosati involution, and the polynomials $\mathrm{det}_{U^{1,0}}$ and $\mathrm{det}_{\mathrm{Lie}(A)}$ are equal;

- $\eta : A[N] \to U/NU$ is an O_F-linear symplectic similitude, which lifts locally for the étale topology to an O_F-linear symplectic similitude

$$H_1(A, \mathbb{A}_f^p) \to U \otimes_{\mathbb{Z}} \mathbb{A}_f^p.$$

The moduli space X is representable by a quasiprojective scheme over O_K. We will make a slight assumption on the variety we consider.

Hypothesis 1.3. We suppose that we are not in the case $d = 1$ and $(a, b) = (1, 1)$.

This condition is technical, and will ensure that we can neglect the cusps in the definition of the modular forms. The case we exclude corresponds essentially to the modular curve, which is well known.

1A3. *Iwahori level.* Let π be a prime of F_0 above p; we will call f the residual degree and write $q := p^f$. Since p is unramified in F, we have two possibilities for the behavior of p in F:

- π splits as $\pi^+ \pi^-$ in F. We say that π is in case (L).
- π is inert in F. We say that π is in case (U).

The terminology (L) and (U) comes from the fact that the group G at π is a linear or a unitary group, respectively. To define the Iwahori structure, we will break into the two cases.

Definition 1.4. Let $X_{\mathrm{Iw},\pi}$ be the moduli space of isomorphism classes of tuples $(A, \lambda, \iota, \eta, H_\bullet)$, where

- $(A, \lambda, \iota, \eta)$ is a point of X;
- $0 \subset H_1 \subset \cdots \subset H_{a+b} = A[\pi^+]$, where each H_i is a subgroup of $A[\pi^+]$ which is locally isomorphic for the étale topology to $(O_F/\pi^+)^i$ in the case (L);
- $0 \subset H_1 \subset \cdots \subset H_{a+b} = A[\pi]$, where each H_i is a subgroup of $A[\pi]$ which is locally isomorphic for the étale topology to $(O_F/\pi)^i$ and such that $H_{a+b-i} = H_i^\perp$ in the case (U).

The moduli spaces $X_{\mathrm{Iw},\pi}$ are representable by quasiprojective schemes over O_K. We also define the full Iwahori space by $X_{\mathrm{Iw}} = X_{\mathrm{Iw},\pi_1} \times_X X_{\mathrm{Iw},\pi_2} \times_X \cdots \times_X X_{\mathrm{Iw},\pi_g}$, where $\pi_1, \ldots, \pi_g$ are the primes of F_0 above p and the maps $X_{\mathrm{Iw},\pi_i} \to X$ are the natural morphisms corresponding to forgetting the (H_i).

Remark 1.5. In the case (L), the subgroups $A[\pi^+]$ and $A[\pi^-]$ are Cartier duals. This comes from the compatibility between the complex conjugation and the Rosati involution. Therefore, each of these groups is totally isotropic. A flag $(H_\bullet)$ of $A[\pi^+]$ gives naturally a flag $(H_\bullet^\perp)$ of $A[\pi^-]$ with $H_i^\perp = (A[\pi^+]/H_i)^D \subset A[\pi^+]^D = A[\pi^-]$. Choosing the prime π^- instead of π^+ would have given the same definition.

Now we will explicitly describe the determinant condition for the abelian scheme A. We are still working with a prime π of F_0 above p, and assume it is of type (L). Let Σ_π be the decomposition group at π, i.e., the elements $\sigma \in \mathrm{Hom}(F_0, \overline{\mathbb{Q}}_p)$ such that σ sends π into the maximal ideal of $\overline{\mathbb{Q}}_p$. For every $\sigma \in \Sigma_\pi$, there are two embeddings σ^+ and σ^- of F into $\overline{\mathbb{Q}}_p$ above σ; the embedding

σ^+ sends π^+ into the maximal ideal of $\overline{\mathbb{Q}}_p$, and similarly for π^-. To σ we have a pair of integers (a_σ, b_σ) and the choice of the embedding σ^+ gives an order for the two elements of the pair. Let $A \to R$ be an abelian scheme over an O_K-algebra R. If we let $\omega_\pi := e^*\Omega^1_{A[\pi^\infty]}$, then we have $\omega_\pi = \omega_\pi^+ \oplus \omega_\pi^-$, where $\omega_\pi^+ := e^*\Omega^1_{A[(\pi^+)^\infty]}$. The determinant condition for A then implies that

$$\omega_\pi^+ = \bigoplus_{\sigma \in \Sigma_\pi} R^{a_\sigma}$$

with O_F acting on R^{a_σ} by σ^+. Similarly, we have

$$\omega_\pi^- = \bigoplus_{\sigma \in \Sigma_\pi} R^{b_\sigma}$$

with O_F acting on R^{b_σ} by σ^-.

Now suppose that π is of type (U). We still denote by Σ_π the decomposition group at π of F_0. For each $\sigma \in \Sigma_\pi$, there are two embeddings σ_1 and σ_2 of F above σ; the choice of σ_1 gives an order for the elements of the pair (a_σ, b_σ). Let $A \to R$ be an abelian scheme over an O_K-algebra R. If we let $\omega_\pi := e^*\Omega^1_{A[\pi^\infty]}$, then the determinant condition for A implies

$$\omega_\pi = \bigoplus_{\sigma \in \Sigma_\pi} R^{a_\sigma} \oplus R^{b_\sigma},$$

where O_F acts by σ_1 on R^{a_σ} and by σ_2 on R^{b_σ}.

1B. *The μ-ordinary locus.* We will describe in this section the μ-ordinary locus of the Shimura variety. Let us first introduce some notation.

Let L be an unramified extension of $\mathbb{Q}_p$ of degree f_0 and k a field of characteristic p containing the residue field of L. Let D be the Galois group of L over $\mathbb{Q}_p$; there is an isomorphism $D \simeq \mathbb{Z}/f_0\mathbb{Z}$, where 1 is identified with the Frobenius σ of L. We will write $W(k)$ for the ring of Witt vectors of k. Let $\underline{\varepsilon} = (\varepsilon_\tau)_{\tau \in D}$ be a sequence of integers equal to 0 or 1. We define a Dieudonné module $M_{\underline{\varepsilon}}$ in the following way: it is a free $W(k)$-module of rank f_0 and, if $(e_\tau)_{\tau \in D}$ is a basis of this module, then the Frobenius and Verschiebung are defined by

$$F e_{\sigma^{-1}\tau} = p^{\varepsilon_\tau} e_\tau \quad \text{and} \quad V e_\tau = p^{1-\varepsilon_\tau} e_{\sigma^{-1}\tau}.$$

The module $M_{\underline{\varepsilon}}$ is given an action of the ring of integers of L, namely O_L acts on $W(k) \cdot e_\tau$ by τ.

We'll call $\mathrm{BT}_{\underline{\varepsilon}}$ the p-divisible group over k corresponding to the Dieudonné module $M_{\underline{\varepsilon}}$, and $H_{\underline{\varepsilon}}$ the p-torsion of this p-divisible group.

1B1. *Linear case.* Now we come back to our Shimura variety. Consider first the case (L); we are still considering a place π of F_0 above π which splits as $\pi = \pi^+\pi^-$

in F. If k is a field containing the residue field of O_K and $x = (A, \lambda, \iota, \eta)$ is a k-point of X, then whether the abelian variety A is μ-ordinary at π will depend on the p-divisible group $A[(\pi^+)^\infty]$. Recall that this p-divisible group has an action of O_{F,π^+}, the completion of O_F at π^+; this is an unramified extension of $\mathbb{Q}_p$ of degree f. If Σ_π denotes, as before, the decomposition group of π in F_0, then there is a bijection between Σ_π and the Galois group of O_{F,π^+} and, for each $\sigma \in \Sigma_\pi$, we have a pair of integers (a_σ, b_σ). We put the elements (a_σ) in increasing order; we then have $a_1 \leq a_2 \leq \cdots \leq a_f$. For each integer $0 \leq i \leq f$ we define the sequence $\underline{\varepsilon}_i = (\varepsilon_{i,j})_{1 \leq j \leq f}$ by $\varepsilon_{i,j} = 1$ if $j \geq i+1$ and $\varepsilon_{i,j} = 0$ otherwise. We also set by convention $a_0 = 0$ and $a_{f+1} = a + b$.

Definition 1.6. Let k be an algebraically closed field of characteristic p, and let $x = (A, \lambda, \iota, \eta)$ be a k-point of X. Then x is μ-ordinary at π if there is an isomorphism

$$A[(\pi^+)^\infty] \simeq \prod_{i=0}^{f} \mathrm{BT}_{\underline{\varepsilon}_i}^{a_{i+1}-a_i}$$

of p-divisible groups with O_{F,π^+} action.

Note that the term on the right-hand side is explicitly

$$\mathrm{BT}_{(1,\ldots,1)}^{a_1} \times \mathrm{BT}_{(0,1,\ldots,1)}^{a_2-a_1} \times \cdots \times \mathrm{BT}_{(0,\ldots,0,1)}^{a_f-a_{f-1}} \times \mathrm{BT}_{(0,\ldots,0)}^{b_f} .$$

Let X_0 denote the special fiber of X, and $X_0^{\mu\text{-}\pi\text{-ord}}$ the μ-ordinary locus at the place π. We have the following proposition, due to Wedhorn [1999, Theorem 1.6.2]:

Proposition 1.7. *The μ-ordinary locus $X_0^{\mu\text{-}\pi\text{-ord}}$ is open and dense in X_0.*

We also have the following characterization of the μ-ordinary locus:

Proposition 1.8 [Moonen 2004, Theorem 1.3.7]. *Let k be an algebraically closed field of characteristic p, and let $x = (A, \lambda, \iota, \eta)$ be a k-point of X. Then x is μ-ordinary at π if and only if there is an isomorphism*

$$A[\pi^+] \simeq \prod_{i=0}^{f} \mathrm{BT}_{\underline{\varepsilon}_i}^{a_{i+1}-a_i}[\pi^+]$$

of finite flat group schemes with O_{F,π^+} action.

Since $\mathrm{BT}_{\underline{\varepsilon}_i}$ is multiplicative for $i = 0$, étale for $i = f$ and bi-infinitesimal otherwise, we have the following criterion for the existence of the ordinary locus at π:

Proposition 1.9. *The μ-ordinary locus equals the ordinary locus (at the place π) if and only if there exists an integer a such that $a_\sigma = a$ for all $\sigma \in \Sigma_\pi$.*

This last condition is also equivalent to the fact that the local reflex field at π is equal to $\mathbb{Q}_p$ (see [Wedhorn 1999, Section 1.6] for more details).

1B2. *Unitary case.* Let us now consider the case (U). Let k be a field containing the residue field of O_K, and let $x = (A, \lambda, \iota, \eta)$ be a k-point of X. Whether the abelian variety A is μ-ordinary at π will depend on the p-divisible group $A[\pi^\infty]$. Recall that this p-divisible group has an action of $O_{F,\pi}$, the completion of O_F at π; this is an unramified extension of $\mathbb{Q}_p$ of degree $2f$. Recall that Σ_π is the decomposition group at π of F_0 and it is of cardinality f. If $\sigma \in \Sigma_\pi$, there are two embeddings σ_1 and σ_2 of F into $\overline{\mathbb{Q}}_p$, and the choice of one of the two gives elements a_σ and b_σ. We suppose that the choice is made so that $a_\sigma \leq b_\sigma$; we also order the elements in Σ_π so that the sequence (a_σ) is increasing. We then have

$$a_1 \leq a_2 \leq \cdots \leq a_f \leq b_f \leq b_{f-1} \leq \cdots \leq b_1.$$

This gives an order on the embeddings of F into $\overline{\mathbb{Q}}_p$. For each integer $0 \leq i \leq 2f$ we define the sequence $\underline{\varepsilon}_i = (\varepsilon_{i,j})_{1 \leq j \leq 2f}$ by $\varepsilon_{i,j} = 1$ if $j \geq i+1$ and $\varepsilon_{i,j} = 0$ otherwise. We define a sequence $(\alpha_i)_{0 \leq i \leq 2f+1}$ by $\alpha_0 = 0$, $\alpha_i = a_i$ for $1 \leq i \leq f$, $\alpha_i = b_{2f+1-i}$ for $f+1 \leq i \leq 2f$ and $\alpha_{2f+1} = a+b$.

Definition 1.10. Let k be an algebraically closed field of characteristic p, and let $x = (A, \lambda, \iota, \eta)$ be a k-point of X. Then x is μ-ordinary at π if there is an isomorphism

$$A[\pi^\infty] \simeq \prod_{i=0}^{2f} \mathrm{BT}_{\underline{\varepsilon}_i}^{\alpha_{i+1}-\alpha_i}$$

of p-divisible groups with $O_{F,\pi}$ action.

Note that the term in the right-hand side is explicitly

$$\mathrm{BT}_{(1,\ldots,1)}^{a_1} \times \mathrm{BT}_{(0,1,\ldots,1)}^{a_2-a_1} \times \cdots \times \mathrm{BT}_{(0,\ldots,0,1,1\ldots,1)}^{a_f-a_{f-1}}$$
$$\times \mathrm{BT}_{(0,\ldots,0,1\ldots,1)}^{b_f-a_f} \times \mathrm{BT}_{(0,\ldots,0,0,1\ldots,1)}^{a_f-a_{f-1}} \times \cdots \times \mathrm{BT}_{(0,\ldots,0)}^{a_1}.$$

(We have used the fact that $b_i - b_{i-1} = a_{i-1} - a_i$, since the quantity $a_j + b_j$ is independent of j.)

Let X_0 denote the special fiber of X, and $X_0^{\mu\text{-}\pi\text{-ord}}$ the μ-ordinary locus. We have the following proposition, due to Wedhorn [1999, Theorem 1.6.2]:

Proposition 1.11. *The μ-ordinary locus $X_0^{\mu\text{-}\pi\text{-ord}}$ is open and dense in X_0.*

We also have the following characterization of the μ-ordinary locus:

Proposition 1.12 [Moonen 2004, Theorem 1.3.7]. *Let k be an algebraically closed field of characteristic p, and let $x = (A, \lambda, \iota, \eta)$ be a k-point of X. Then x is μ-ordinary at π if and only if there is an isomorphism*

$$A[\pi] \simeq \prod_{i=0}^{2f} \mathrm{BT}_{\underline{\varepsilon}_i}^{\alpha_{i+1}-\alpha_i}[\pi]$$

of finite flat group schemes with $O_{F,\pi}$ action.

Since $\mathrm{BT}_{\mathcal{E}_i}$ is multiplicative for $i = 0$, étale for $i = 2f$ and bi-infinitesimal otherwise, we have the following criterion for the existence of the ordinary locus at π:

Proposition 1.13. *The μ-ordinary locus equals the ordinary locus (at the place π) if and only if $a_\sigma = b_\sigma = \frac{1}{2}(a + b)$ for all $\sigma \in \Sigma_\pi$.*

Again, this last condition is equivalent to the fact that the local reflex field at π is equal to $\mathbb{Q}_p$.

We'll need later to work with the rigid space associated to X. Let us call this rigid space X_{rig}; it is the generic fiber of the formal completion of X along its special fiber. We refer to [Berthelot 1996] for more details on rigid spaces. We have a specialization map $\mathrm{sp} : X_{\mathrm{rig}} \to X_0$, and we'll write $X_{\mathrm{rig}}^{\mu\text{-}\pi\text{-ord}}$ for the inverse image of the μ-ordinary locus under the specialization map. We'll also write $X_{\mathrm{Iw,rig}}$ for the rigid space associated to X_{Iw}.

1C. *Canonical subgroups.*

1C1. *Degrees and partial degrees.* Before introducing the canonical subgroups on the μ-ordinary locus, we'll define the degree for a finite flat group scheme defined over a finite extension of $\mathbb{Q}_p$, and the partial degrees for these endowed with an action of the ring of integers of an unramified extension of $\mathbb{Q}_p$.

Definition 1.14. Let L be a finite extension of $\mathbb{Q}_p$ and let G be a finite flat group scheme of p-power order over O_L. Let ω_G be the conormal module along the unit section. Then the degree of G is, by definition,

$$\deg G := v(\mathrm{Fitt}_0 \, \omega_G),$$

where Fitt_0 denotes the Fitting ideal, and the valuation of an ideal is defined by $v(x\,O_L) = v(x)$, normalized by $v(p) = 1$.

We now state some properties of this function. We refer to [Fargues 2010, Section 3] for more details.

Proposition 1.15. *The degree function has the following properties*:

- *Let G be as before. Then, if G^D denotes the Cartier dual of G, we have*

$$\deg G^D = \mathrm{ht} \, G - \deg G.$$

  In particular, $\deg G \in [0, \mathrm{ht} \, G]$.

- *The degree function is additive*: *if we have an exact sequence*

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

  with G_i finite flat, then $\deg G_2 = \deg G_1 + \deg G_3$.

- *Let G and G' be two finite flat group schemes, and suppose that there exists a morphism $f : G \to G'$ which is an isomorphism in generic fiber. Then $\deg G \leq \deg G'$, and we have equality if and only if f is an isomorphism.*

We deduce from the last property the following corollary:

Corollary 1.16. *Let G be a finite flat group scheme of p-power order defined over a finite extension of $\mathbb{Q}_p$. Suppose that H_1 and H_2 are two finite flat subgroups of G. Then*

$$\deg H_1 + \deg H_2 \leq \deg(H_1 + H_2) + \deg(H_1 \cap H_2).$$

Proof. By dividing everything by $H_1 \cap H_2$, we are reduced to the case $H_1 \cap H_2 = \{0\}$. The morphism $H_1 \times H_2 \to H_1 + H_2$ is an isomorphism in generic fiber, thus by the previous proposition we get

$$\deg(H_1 \times H_2) \leq \deg(H_1 + H_2).$$

But, since the degree function is additive, $\deg(H_1 \times H_2) = \deg H_1 + \deg H_2$.   □

Let G be as in the previous definition and suppose now that G has an action of O_M, where M is a finite unramified extension of $\mathbb{Q}_p$. Let Σ be the set of embeddings of M into $\overline{\mathbb{Q}}_p$. Then the module ω_G has an action of O_M and has the decomposition

$$\omega_G = \bigoplus_{\sigma \in \Sigma} \omega_{G,\sigma},$$

where O_M acts on $\omega_{G,\sigma}$ by σ.

Definition 1.17. The partial degree of G is defined for all $\sigma \in \Sigma$ as

$$\deg_\sigma G = v(\mathrm{Fitt}_0 \, \omega_{G,\sigma}).$$

Proposition 1.18. *The partial degree functions have the following properties*:

- *We have $\deg G = \sum_{\sigma \in \Sigma} \deg_\sigma G$.*
- *Suppose that G has height $[M : \mathbb{Q}_p]h$. If G^D denotes the Cartier dual of G, we have, for all $\sigma \in \Sigma$,*

$$\deg_\sigma G^D = h - \deg_\sigma G.$$

  In particular, $\deg_\sigma G \in [0, h]$.
- *The partial degree functions are additive*: *if we have an exact sequence*

$$0 \to G_1 \to G_2 \to G_3 \to 0$$

  with G_i finite flat with an action of O_M, then $\deg_\sigma G_2 = \deg_\sigma G_1 + \deg_\sigma G_3$ for all $\sigma \in \Sigma$.

We refer to [Bijakowski 2014, Section 1.1.5] for more details.

Remark 1.19. If we are in the situation of the third point of Proposition 1.15, i.e., if we have a morphism of finite flat group schemes $f : G \to G'$ that is an isomorphism in generic fiber and if G and G' have an action of O_M, then it is not true that the partial degrees increase. Indeed, the functions that increase are linear combinations of the partial degrees. For example, if $[M : \mathbb{Q}_p] = 2$, there are two partial degrees $\deg_1$ and $\deg_2$, and the functions that increase are $\deg_1 + p \deg_2$ and $p \deg_1 + \deg_2$. See [Bijakowski 2014, Proposition 1.1.33] for more details.

Example 1.20. Let us apply what we have said to our Shimura variety. Let $x = (A, \lambda, \iota, \eta)$ be an O_L-point of X (where L is a finite extension of $\mathbb{Q}_p$) and suppose that π is a place of F_0 above p in case (L). Then $A[\pi^+]$ has an action of $O_{F_0,\pi}$. Moreover, we have

$$\deg_\sigma A[\pi^+] = a_\sigma$$

for all $\sigma \in \Sigma_\pi$. If H is an O_F-stable subgroup of $A[\pi^+]$ of height fh, then the orthogonal $H^\perp$ is a subgroup of $A[\pi^-]$ of height $f(a + b - h)$. We have $H^\perp \simeq (A[\pi^+]/H)^D$, and thus

$$\deg_\sigma H^\perp = (a + b - h) - (a_\sigma - \deg_\sigma H) = b_\sigma - h + \deg_\sigma H$$

for all $\sigma \in \Sigma_\pi$. We see that one has the inequalities

$$\deg_\sigma H \geq h - b_\sigma \quad \text{and} \quad \deg_\sigma H^\perp \geq b_\sigma - h$$

for all $\sigma \in \Sigma_\pi$.

Example 1.21. Suppose now that π is in case (U). Then the group scheme $A[\pi]$ has an action of $O_{F,\pi}$. Recall that if $\sigma \in \Sigma_\pi$ is an embedding of F_0 into $\overline{\mathbb{Q}}_p$ above π, then there are two embeddings σ_1 and σ_2 of F extending σ. With our previous conventions, we have

$$\deg_{\sigma_1} A[\pi] = a_\sigma \quad \text{and} \quad \deg_{\sigma_2} A[\pi] = b_\sigma.$$

If H is an O_F-stable subgroup of $A[\pi]$ of height $2fh$, then the orthogonal $H^\perp$ is a subgroup of $A[\pi]$ of height $2f(a + b - h)$. We have $H^\perp \simeq (A[\pi]/H)^{D,c}$, where the superscript c means that the action of O_F on $(A[\pi]/H)^{D,c}$ is the conjugate of the natural one. This comes from the compatibility between the Rosati involution and the complex conjugation. Thus,

$$\deg_{\sigma_1} H^\perp = a_\sigma - h + \deg_{\sigma_2} H \quad \text{and} \quad \deg_{\sigma_2} H^\perp = b_\sigma - h + \deg_{\sigma_1} H$$

for all $\sigma \in \Sigma_\pi$. We see that one has the inequalities

$$\deg_{\sigma_1} H \geq h - b_\sigma \quad \text{and} \quad \deg_{\sigma_2} H \geq h - a_\sigma$$

for all $\sigma \in \Sigma_\pi$.

1C2. *Siegel variety.* Let us now recall some facts for the canonical subgroup of the Siegel variety.

Let $g \geq 1$ be an integer and $\mathcal{A}_g$ the Siegel variety. There is a universal abelian scheme A on $\mathcal{A}_g$. There is also a Hasse invariant Ha on $\mathcal{A}_g$. We quote the main result obtained by Fargues [2011] on the canonical subgroup.

Proposition 1.22. *Let A be an abelian scheme of dimension g defined over O_L (L is a finite extension of $\mathbb{Q}_p$). Suppose that the valuation w of the Hasse invariant is strictly less than $\frac{1}{2}$. Then there is a canonical subgroup $H \subset A[p]$, of height g, totally isotropic, with*

$$\deg H = g - w.$$

Let $\mathcal{A}_{g,\mathrm{rig}}$ be the rigid space associated to $\mathcal{A}_g$. Then the ordinary locus of $\mathcal{A}_{g,\mathrm{rig}}$ is defined as the locus where the associated abelian scheme is ordinary; it is also the locus where the Hasse invariant is invertible. The proposition says that, on a strict neighborhood of the ordinary locus, there exists a canonical subgroup of high degree in $A[p]$. We propose a simple reformulation of this property. We have the following observation:

Proposition 1.23. *Let A be an abelian scheme of dimension g defined over O_L (L is a finite extension of $\mathbb{Q}_p$). There exists at most one subgroup H of height g of $A[p]$ with*

$$\deg H > g - \tfrac{1}{2}.$$

Proof. Suppose not, and let H_1 and H_2 be two subgroups with $\deg H_i > g - \frac{1}{2}$. Then we have

$$2g - 1 < \deg H_1 + \deg H_2 \leq \deg(H_1 + H_2) + \deg(H_1 \cap H_2).$$

But $\deg(H_1 + H_2) \leq \deg A[p] = g$ and, since $H_1 \cap H_2$ is of height $h \leq g - 1$, we have $\deg(H_1 \cap H_2) \leq g - 1$. We get a contradiction. $\square$

This can be used to prove the existence of the canonical subgroup in the following way. Let $\mathcal{A}'_g$ be the Siegel variety parametrizing a g-dimensional abelian scheme with polarization and a subgroup H totally isotropic of height g. We have a map $f : \mathcal{A}'_g \to \mathcal{A}_g$ corresponding to forgetting H. If we let $\mathcal{A}'_{g,\mathrm{rig}}$ be the rigid space associated to $\mathcal{A}'_g$, we still have a morphism $f : \mathcal{A}'_{g,\mathrm{rig}} \to \mathcal{A}_{g,\mathrm{rig}}$. Define $X_r = \{x \in \mathcal{A}'_{g,\mathrm{rig}} : \deg H(x) \geq g - r\}$ for any rational r; it is an admissible open subset of $\mathcal{A}'_{g,\mathrm{rig}}$. Then the ordinary locus of $\mathcal{A}_{g,\mathrm{rig}}$ is $f(X_0)$, and it follows from [Bijakowski et al. 2016, Proposition 4.1.7] that $(f(X_r))_{r>0}$ forms a basis of strict neighborhoods of the ordinary locus (the map f is finite étale, and the $(X_r)_{r>0}$ are strict neighborhoods of X_0). The previous proposition shows that on $f(X_r)$ there is exactly one subgroup of height g of degree greater than or equal to $g - r$ for $r < \frac{1}{2}$; this is the canonical subgroup.

1C3. *Linear case.* Now let's get back to our Shimura varieties. Suppose we are in case (L). Then we have the following proposition:

Proposition 1.24. *Let L be a finite extension of $\mathbb{Q}_p$ and $x = (A, \lambda, \iota, \eta)$ an O_L-point of X. Let $1 \leq i \leq f$ be an integer. Then there exists at most one subgroup $H \subset A[\pi^+]$ that is stable by O_F of height $f a_i$ such that*

$$\deg H > \sum_{j=1}^{f} \min(a_j, a_i) - \tfrac{1}{2}.$$

Proof. Suppose not, and let H_1 and H_2 be two such subgroups. Let us denote by $f h$ the height of $H_1 \cap H_2$; the height of $H_1 + H_2$ is then $f(2a_i - h)$. We have

$$\deg H_1 + \deg H_2 \leq \deg(H_1 + H_2) + \deg(H_1 \cap H_2).$$

But

$$\deg(H_1 + H_2) \leq \sum_{j=1}^{f} \min(a_j, 2a_i - h) \leq \sum_{j=1}^{i} a_j + \sum_{j=i+1}^{f} (2a_i - h)$$

and

$$\deg(H_1 \cap H_2) \leq \sum_{j=1}^{f} \min(a_j, h) \leq \sum_{j=1}^{i-1} a_j + \sum_{j=i}^{f} h.$$

We finally get

$$\deg H_1 + \deg H_2 \leq 2\sum_{j=1}^{i-1} a_j + 2\sum_{j=i+1}^{f} a_i + a_i + h \leq 2\sum_{j=1}^{f} \min(a_j, a_i) - 1$$

since $h \leq a_i - 1$, a contradiction. $\qquad\square$

The proposition shows that there exists at most one subgroup of height $f a_i$ and of big degree. The next proposition shows that if two such subgroups exist (with different heights), then we automatically have an inclusion.

Proposition 1.25. *Let $i < j$ be two integers between 1 and f. Let $x = (A, \lambda, \iota, \eta)$ be an O_L-point of X, and suppose there exists, for $l \in \{i, j\}$, a subgroup $H_l \subset A[\pi^+]$ that is stable by O_F of height $f a_l$ such that*

$$\deg H_l > \sum_{k=1}^{f} \min(a_k, a_l) - \tfrac{1}{2}.$$

Then we have $H_i \subset H_j$.

Proof. Let fh denote the height of $H_i \cap H_j$. We have the following inequalities:

$$\deg(H_i + H_j) \leq \sum_{k=1}^{f} \min(a_k, a_i + a_j - h) \leq \sum_{k=1}^{j} a_k + \sum_{k=j+1}^{f} (a_i + a_j - h),$$

$$\deg(H_i \cap H_j) \leq \sum_{k=1}^{f} \min(a_k, h) \leq \sum_{k=1}^{i-1} a_k + \sum_{k=i}^{f} h.$$

We then get

$$\deg H_i + \deg H_j \leq \deg(H_i + H_j) + \deg(H_i \cap H_j)$$

$$\leq 2 \sum_{k=1}^{i-1} a_k + \sum_{k=i}^{j} (a_k + h) + \sum_{k=j+1}^{f} (a_i + a_j).$$

If we do not have the inclusion $H_i \subset H_j$, then $h \leq a_i - 1$. We then get

$$\deg H_i + \deg H_j \leq \sum_{k=1}^{f} (\min(a_k, a_i) + \min(a_k, a_j)) - (j - i + 1).$$

We get a contradiction with the hypothesis that H_i and H_j have big degrees. $\qquad\square$

As a consequence, we directly get the existence of canonical subgroups for μ-ordinary abelian schemes. Let s be the cardinality of $\{a_1, \ldots, a_f\} \cap [1, a+b-1]$, and denote by $A_1 < \cdots < A_s$ the different elements of this set.

Corollary 1.26. *Let L be a finite extension of $\mathbb{Q}_p$, and let $x = (A, \lambda, \iota, \eta)$ be an O_L-point of X. Assume that x is μ-ordinary (i.e., the special fiber of A is μ-ordinary). Then, for any integer $1 \leq k \leq s$, there exists a unique subgroup $H_k \subset A[\pi^+]$ of height fA_k, with*

$$\deg_\sigma H_k = \min(a_\sigma, A_k)$$

for all $\sigma \in \Sigma_\pi$.

Proof. We can work over $\overline{\mathbb{Q}}_p$. Since A is μ-ordinary we have, by [Moonen 2004, Proposition 2.1.9], a filtration

$$X_0 = 0 \subset X_1 \subset \cdots \subset X_{f+1} = A[(\pi^+)^\infty]$$

on $A[(\pi^+)^\infty]$ with X_i p-divisible groups such that $(X_{i+1}/X_i) \times \overline{\mathbb{F}}_p \simeq \mathrm{BT}_{\underline{\varepsilon}i}^{a_{i+1}-a_i}$ for $0 \leq i \leq f$. Let $Y_i = X_{i+1}/X_i$ for $0 \leq i \leq f$. Then Y_i is a p-divisible group over the ring of integers of $\overline{\mathbb{Q}}_p$. The module ω_{Y_i} decomposes into $\bigoplus_{\sigma \in \Sigma_\pi} \omega_{Y_i, \sigma}$, and each $\omega_{Y_i, \sigma}$ is free over the ring of integers of $\overline{\mathbb{Q}}_p$. Recall that we have chosen an ordering for the set $\Sigma_\pi = \{\sigma_1, \ldots, \sigma_f\}$. From the description of the special fiber of Y_i, one sees that ω_{Y_i, σ_j} is 0 if $j \leq i$, and is free of rank $a_{i+1} - a_i$ over the ring of integers of $\overline{\mathbb{Q}}_p$ if $j \geq i+1$. Thus, $\deg_{\sigma_j} Y_i[\pi^+]$ is 0 if $j \leq i$ and $a_{i+1} - a_i$ otherwise.

Let $H_i = X_i[\pi^+]$ for $0 \le i \le f$; it is a finite flat subgroup of $A[\pi^+]$ of height $f a_i$. Moreover, we have

$$\deg_{\sigma_j} H_i = \sum_{k=1}^{i} \deg_{\sigma_j} H_k / H_{k-1} = \sum_{k=1}^{\min(i,j)} (a_k - a_{k-1}) = a_{\min(i,j)} = \min(a_i, a_j)$$

for all i and j between 1 and f. This gives the existence of the desired subgroups. The uniqueness follows from the previous proposition. $\qquad\square$

Remark 1.27. We also have the following description of the canonical subgroups in the special fiber. Let L be a finite extension of $\mathbb{Q}_p$, and let $x = (A, \lambda, \iota, \eta)$ be a μ-ordinary point of X defined over O_L. Let A_s denote the special fiber of A; then the Frobenius F acts on $A_s[(\pi^+)^\infty]$, and we can form the subgroups

$$C_i := (\pi^+)^{f-i} A_s[F^f, (\pi^+)^{f-i+1}]$$

for $1 \le i \le f$. Then we have $0 \subset C_1 \subset \cdots \subset C_f \subset A_s[\pi^+]$, and the special fiber of each of the canonical subgroups is equal to one of the C_i. More precisely, we have $H_j \times k(L) = C_{r(j)}$ for $1 \le j \le s$, with $k(L)$ the residue field of L and $r(j) = \min\{l : a_l = A_j\}$.

We can then define the relevant degree functions on $X_{\mathrm{Iw},\pi}$. For each integer j, define

$$d_j = \sum_{\sigma \in \Sigma_\pi} \min(a_\sigma, a_j).$$

Let $X_{\mathrm{Iw},\pi,\mathrm{rig}}$ be the associated rigid space. We define the degree function Deg : $X_{\mathrm{Iw},\pi,\mathrm{rig}} \to \prod_{j=1}^{s}[0, d_j]$ on this space by

$$\mathrm{Deg}(A, \lambda, \iota, \eta, H_\bullet) := (\deg H_{a_j})_{1 \le j \le s}.$$

We also define the j-th degree function by $\mathrm{Deg}_j(A, \lambda, \iota, \eta, H_\bullet) := \deg H_{a_j}$ for $1 \le j \le s$.

Remark 1.28. The integers s and d_j as well as the functions Deg and Deg_j depend on the place π. If the context is clear, we choose not to write the dependence on π to lighten the notation.

We then have the following description of the μ-ordinary locus.

Proposition 1.29. *The space* $X_{\mathrm{rig}}^{\mu\text{-}\pi\text{-ord}} \subset X_{\mathrm{rig}}$ *is exactly the image of*

$$\mathrm{Deg}^{-1}(\{d_1\} \times \cdots \times \{d_s\})$$

under the map $X_{\mathrm{Iw},\pi,\mathrm{rig}} \to X_{\mathrm{rig}}$.

Proof. If x is a μ-ordinary point, it then follows from the previous corollary that there exist subgroups H_j of height fa_j with $\deg H_j = d_j$ for all $1 \leq j \leq s$. Conversely, suppose that $(A, \lambda, \iota, \eta)$ is a point of X_{rig}, with A defined over the ring of integers of an extension L of $\mathbb{Q}_p$, and that there exist subgroups H_j of height fa_j with $\deg H_j = d_j$ for all $1 \leq j \leq s$. We want to show that A is μ-ordinary; it suffices to show that $A[\pi^+]$ has a nice description. We have $\deg_\sigma H_j \leq \min(a_\sigma, a_j)$ for all $\sigma \in \Sigma_\pi$ and, since $d_j = \sum_{\sigma \in \Sigma_\pi} \min(a_\sigma, a_j)$, this forces the last inequality to be an equality. Define $H_0 = 0$, $H_{s+1} = A[\pi^+]$, and let H_j' be a complement of H_{j-1} in H_j for all $1 \leq j \leq s+1$. This is possible if the field L is large enough. We claim that, for $1 \leq j \leq s+1$,

$$H_j \simeq H_j' \times H_{j-1}.$$

Indeed we have a morphism $H_{j-1} \to H_j/H_j'$ that is an isomorphism in generic fiber. The degree of the image of H_{j-1} in H_j/H_j' thus increases, but since the degree of H_{j-1} is maximal, it must be an equality. We deduce that $\deg H_j = \deg H_j' + \deg H_{j-1}$ and that the morphism $H_j' \times H_{j-1} \to H_j$ is an isomorphism.

We finally get

$$A[\pi^+] \simeq H_1' \times H_2' \times \cdots \times H_{s+1}'.$$

But we can explicitly describe the groups H_j'. Indeed, for all $1 \leq j \leq s+1$ and $\sigma \in \Sigma_\pi$, we have (setting $A_0 = 0$ and $A_{s+1} = a + b$)

$$\deg_\sigma H_j' = \deg_\sigma H_j - \deg_\sigma H_{j-1} = \min(a_\sigma, a_j) - \min(a_\sigma, A_{j-1}).$$

This quantity is 0 if $a_\sigma \leq A_{j-1}$ and $a_j - A_{j-1}$ if $a_\sigma \geq a_j$. Since the height of H_j' is $f(a_j - A_{j-1})$, one can see that the special fiber of H_j' is isomorphic to $BT_{\varepsilon_{s(j)}}[\pi^+]^{A_j - A_{j-1}}$, where $s(j)$ is the number of $\sigma \in D_\pi$ with $a_\sigma \leq A_{j-1}$ (one can see this by looking at the Dieudonné module associated to the special fiber of H_j', for example). We then conclude that A is μ-ordinary by Proposition 1.8. □

1C4. *Unitary case.* Suppose now we are in case (U). Then we have the following proposition:

Proposition 1.30. *Let L be a finite extension of $\mathbb{Q}_p$ and $x = (A, \lambda, \iota, \eta)$ an O_L-point of X. Let $1 \leq i \leq f$ be an integer. Then there exists at most one subgroup $H \subset A[\pi]$ that is stable by O_F of height $2fa_i$ such that*

$$\deg H > \sum_{j=1}^{2f} \min(\alpha_j, a_i) - \tfrac{1}{2}.$$

Moreover, if such a subgroup exists, it is totally isotropic.

Proof. The proof of the first part of the proposition is exactly the same as in the linear case (see Proposition 1.24). To prove that the subgroup is totally isotropic,

we will use the same argument as in the proof of Proposition 1.25. We only need to get a bound for the degree of $H^\perp$. But we have $H^\perp \simeq (A[\pi]/H)^D$, so

$$\deg H^\perp = 2fb_i - \deg(A[\pi]/H)$$

$$= \deg H + \sum_{j=1}^{2f}(b_i - \alpha_j) > \sum_{j=1}^{2f}(b_i - \alpha_j + \min(\alpha_j, a_i)) - \tfrac{1}{2}.$$

But $b_i - \alpha_j + \min(\alpha_j, a_i) = \min(b_i, a_i + b_i - \alpha_j)$ and, since $a_i + b_i$ is constant, we have $a_i + b_i - \alpha_j = \alpha_{2f+1-i}$. In conclusion, we get

$$\deg H^\perp > \sum_{j=1}^{2f} \min(b_i, \alpha_j) - \tfrac{1}{2}.$$

We conclude that $H \subset H^\perp$ by applying the proof of Proposition 1.25 directly (note that $H^\perp$ is of height $2fb_i$). □

The proposition shows that there exists at most one subgroup of height $2fa_i$ and of big degree. The next proposition shows that if two such subgroups exists (with different heights), then we automatically have an inclusion.

Proposition 1.31. *Let $i < j$ be two integers between 1 and f. Let $x = (A, \lambda, \iota, \eta)$ be an O_L-point of X, and suppose there exists, for $l \in \{i, j\}$, a subgroup $H_l \subset A[\pi]$ that is stable by O_F of height $2fa_l$ such that*

$$\deg H_l > \sum_{k=1}^{2f} \min(\alpha_k, a_l) - \tfrac{1}{2}.$$

Then $H_i \subset H_j$.

Proof. The proof is the same as in the linear case (see Proposition 1.25). □

As a consequence, we directly get the existence of canonical subgroups for μ-ordinary abelian schemes. Let s be the cardinality of $\{a_1, \ldots, a_f\} \cap [1, \tfrac{1}{2}(a+b)]$, and denote by $A_1 < \cdots < A_s$ the different elements of this set.

Corollary 1.32. *Let L be a finite extension of $\mathbb{Q}_p$ and let $x = (A, \lambda, \iota, \eta)$ be an O_L-point of X. Assume that x is μ-ordinary (i.e., the special fiber of A is μ-ordinary). Then, for any integer $1 \le k \le s$, there exists a unique totally isotropic subgroup $H_k \subset A[\pi]$ of height $2fA_k$, with*

$$\deg_\sigma H_k = \min(\alpha_\sigma, A_k)$$

for all $\sigma \in \Sigma_\pi$.

Proof. The proof is similar to the linear case (see Corollary 1.26). □

Remark 1.33. We also have the following description of the canonical subgroups in the special fiber. Let L be a finite extension of $\mathbb{Q}_p$, and let $x = (A, \lambda, \iota, \eta)$ be a μ-ordinary point of X defined over O_L. Let A_s denote the special fiber of A; then the Frobenius F acts on $A_s[\pi^\infty]$, and we can form the subgroups

$$C_i := \pi^{2f-i} A_s[F^{2f}, \pi^{2f-i+1}]$$

for $1 \le i \le 2f$. Then we have $0 \subset C_1 \subset \cdots \subset C_{2f} \subset A_s[\pi]$, and the special fiber of each of the canonical subgroups is equal to one of the C_i. More precisely, we have $H_j \times k(L) = C_{r(j)}$ for $1 \le j \le s$, with $k(L)$ the residue field of L and $r(j) = \min\{l : a_l = A_j\}$. Note also that $C_i^\perp = C_{2f+1-i}$.

We can then define the relevant degree functions on $X_{\mathrm{Iw},\pi}$. For each integer k, define

$$d_k = \sum_{j=1}^{2f} \min(\alpha_j, A_k).$$

Let $X_{\mathrm{Iw},\pi,\mathrm{rig}}$ be the associated rigid space. We define the degree function $\mathrm{Deg} : X_{\mathrm{Iw},\pi,\mathrm{rig}} \to \prod_{k=1}^s [0, d_k]$ on this space by

$$\mathrm{Deg}(A, \lambda, \iota, \eta, H_\bullet) := (\deg H_{A_k})_{1 \le k \le s}.$$

We also define the k-th degree function by $\mathrm{Deg}_k(A, \lambda, \iota, \eta, H_\bullet) := \deg H_{A_k}$ for $1 \le k \le s$. We then have the following description of the μ-ordinary locus:

Proposition 1.34. *The space $X_{\mathrm{rig}}^{\mu\text{-ord}} \subset X_{\mathrm{rig}}$ is exactly the image of*

$$\mathrm{Deg}^{-1}(\{d_1\} \times \cdots \times \{d_s\})$$

under the map $X_{\mathrm{Iw},\pi,\mathrm{rig}} \to X_{\mathrm{rig}}$.

Proof. If x is a μ-ordinary point, it follows from the previous corollary that there exist subgroups H_k of height $2fA_k$ with $\deg H_k = d_k$ for all $1 \le k \le s$. Conversely, suppose that $(A, \lambda, \iota, \eta)$ is a point of X_{rig}, with A defined over the ring of integers of an extension L of $\mathbb{Q}_p$, and that there exist subgroups H_k of height $2fA_k$ with $\deg H_k = d_k$ for all $1 \le k \le s$. This implies that we have $\deg_\sigma H_k = \min(\alpha_\sigma, A_k)$ for all $\sigma \in D_\pi$ and $1 \le k \le s$. We want to show that A is μ-ordinary; it suffices to show that $A[\pi]$ has a nice description. Define $H_0 = 0$, $H_i = H_{2s+1-i}^\perp$ for $s+1 \le i \le 2s+1$, and let H_j' be a complement of H_{j-1} in H_j for all $1 \le j \le 2s+1$. This is possible if the field L is big enough. We claim that

$$H_j \simeq H_j' \times H_{j-1}.$$

Indeed we have a morphism $H_{j-1} \to H_j/H_j'$ that is an isomorphism in generic fiber. The degree of the image of H_{j-1} in H_j/H_j' thus increases, but since the degree of

H_{j-1} is maximal, it must be an equality. We deduce that $\deg H_j = \deg H'_j + \deg H_{j-1}$ and that the morphism $H'_j \times H_{j-1} \to H_j$ is an isomorphism.

We finally get

$$A[\pi] \simeq H'_1 \times H'_2 \times \cdots \times H'_{2s+1}.$$

But we can explicitly describe the groups H'_j using the same proof as Proposition 1.29. We then conclude that A is μ-ordinary by Proposition 1.12. $\qquad\square$

2. Modular forms and Hecke operators

2A. *Modular forms.* Let's now define the modular forms for the Shimura variety X. Let π be a place of F_0 above p, and suppose it is in case (L). Then we define the $O_F \otimes O_K$-module St_π by

$$\mathrm{St}_\pi := O_K^{a_\sigma} \oplus O_K^{b_\sigma},$$

where O_F acts on $O_K^{a_\sigma}$ by σ^+ and on $O_K^{b_\sigma}$ by σ^-. If π is in case (U), we define the $O_F \otimes O_K$-module St_π by

$$\mathrm{St}_\pi := O_K^{a_\sigma} \oplus O_K^{b_\sigma},$$

where O_F acts on $O_K^{a_\sigma}$ by σ_1 and on $O_K^{b_\sigma}$ by σ_2. Finally, we define the $O_F \otimes O_K$-module St by

$$\mathrm{St} = \bigoplus_\pi \mathrm{St}_\pi,$$

where π runs over the places of F_0 above p. If R is an O_K-algebra and $(A, \lambda, \iota, \eta)$ is an R-point of X, then the $R \otimes O_F$-module $e^* \Omega^1_{A/R}$ is isomorphic to $\mathrm{St} \otimes_{O_K} R$. The sheaf $\omega_A := e^* \Omega^1_{A/X}$ is then locally isomorphic to $\mathrm{St} \otimes_{O_K} \mathcal{O}_X$ (it is a locally free sheaf on $\mathcal{O}_X$).

Define

$$\mathcal{T} = \mathrm{Isom}_{O_F \otimes \mathcal{O}_X}(\mathrm{St} \otimes \mathcal{O}_X, \omega_A).$$

This is a torsor on X under the group, defined over O_K,

$$M = \prod_{\pi \in \mathcal{P}} \prod_{\sigma \in \Sigma_\pi} \mathrm{GL}_{a_\sigma} \times \mathrm{GL}_{b_\sigma},$$

where $\mathcal{P}$ is the set of primes of F_0 above π. Let B_M be the upper Borel subgroup of M, U_M its unipotent radical, and T_M its maximal torus. Let $X(T_M)$ be the character group of T_M and $X(T_M)^+$ the cone of dominant weights for B_M. If $\kappa \in X(T_M)^+$, we let $\kappa' = -w_0 \kappa \in X(T_M)^+$, where w_0 is the element of highest length in the Weyl group of M relative to T_M.

Let $\phi : \mathcal{T} \to X$ be the projection morphism.

Definition 2.1. Let $\kappa \in X(T_M)^+$. The sheaf of modular forms of weight κ is $\omega^\kappa = \phi_* O_{\mathcal{T}}[\kappa']$, where $\phi_* O_{\mathcal{T}}[\kappa']$ is the subsheaf of $\phi_* O_{\mathcal{T}}$, on which $B_M = T_M U_M$ acts by κ' on T_M and trivially on U_M.

A modular form of weight κ on X with coefficients in an O_L-algebra R is thus a global section of ω^κ, so an element of $H^0(X \times_{O_K} R, \omega^\kappa)$. Using the projection $X_{\mathrm{Iw}} \to X$, we similarly define the sheaf ω^κ on X_{Iw}, as well as the modular forms on X_{Iw}.

2B. *Overconvergent modular forms.* For simplicity, we will now assume that there is only one place π of F_0 above p, that is to say that p is inert in F_0. The case with several places does not add any difficulty, and will be treated in Section 4.

We can then define the space of overconvergent modular forms. These will be sections of the sheaf of modular forms defined over a strict neighborhood of the μ-ordinary locus. Recall that we have defined in both cases a degree function

$$\mathrm{Deg} : X_{\mathrm{Iw,rig}} \to \prod_{k=1}^{s}[0, d_k].$$

Since there is only one place above p in F_0, we have $X_{\mathrm{Iw}} = X_{\mathrm{Iw},\pi}$.

We define the μ-ordinary-multiplicative locus as $\mathrm{Deg}^{-1}(\{d_1\} \times \cdots \times \{d_s\})$. By Proposition 1.29 or Proposition 1.34, this locus lies in the μ-ordinary locus.

Definition 2.2. The space of overconvergent modular forms of weight κ is defined as

$$M^\dagger := \mathrm{colim}_{\mathcal{V}} H^0(\mathcal{V}, \omega^\kappa),$$

where $\mathcal{V}$ runs over the strict neighborhoods of the μ-ordinary-multiplicative locus in $X_{\mathrm{Iw,rig}}$.

An overconvergent modular form is then defined over a space of the form

$$\mathrm{Deg}^{-1}([d_1 - \varepsilon, d_1] \times \cdots \times [d_s - \varepsilon, d_s])$$

for some $\varepsilon > 0$.

2C. *Hecke operators.* We now define the Hecke operators. These operators will act both on the rigid space and on the space of modular forms. We will fix the weight κ. Explicitly, κ is a collection of integers

$$\left((\kappa_{\sigma,1} \geq \cdots \geq \kappa_{\sigma,a_\sigma}), \ (\lambda_{\sigma,1} \geq \cdots \geq \lambda_{\sigma,b_\sigma}) \right)_{\sigma \in \Sigma_\pi}.$$

We recall that we still assume that π is the only place of F_0 above p. To simplify the notation, we define $\kappa_\sigma := \kappa_{\sigma,a_\sigma}$ and $\lambda_\sigma := \lambda_{\sigma,b_\sigma}$.

2C1. *Linear case.* Assume that π is in case (L). Let $1 \le i \le a+b-1$ be an integer, and let C_i be the moduli space defined over K parameterizing $(A, \lambda, \iota, \eta, H_\bullet, L)$, with $(A, \lambda, \iota, \eta, H_\bullet)$ a point of X_{Iw} and $L = L_0 \oplus L_0^\perp$ a subgroup of $A[\pi]$, where L_0 is an O_F-stable subgroup of $A[\pi^+]$ with $A[\pi^+] = H_i \oplus L_0$. We have two morphisms $p_1, p_2 : C_i \to X_{\mathrm{Iw}} \times_{O_K} K$. The morphism p_1 corresponds to forgetting L, and the morphism p_2 is defined as $p_2(A, \lambda, \iota, \eta, H_\bullet, L) = (A/L, \lambda', \iota', \eta', H'_\bullet)$, with

- $H'_j = (H_j + L_0)/L_0$ if $j \le i$;
- $H'_j = ((\pi^+)^{-1}(H_j \cap L_0))/L_0$ if $j > i$.

We take the polarization λ' to be equal to $p \cdot \lambda$, which is a prime-to-p polarization. Let C_i^{an} be the analytic space associated to C_i, and define $C_{i,\mathrm{rig}} := p_1^{-1}(X_{\mathrm{Iw},\mathrm{rig}})$. The morphisms p_1 and p_2 give morphisms $C_{i,\mathrm{rig}} \to X_{\mathrm{Iw},\mathrm{rig}}$.

Definition 2.3. The i-th Hecke operator acting on the subsets of $X_{\mathrm{Iw},\mathrm{rig}}$ is defined by

$$U_{\pi,i}(S) = p_2(p_1^{-1}(S)).$$

This operator preserves the admissible open subsets and quasicompact admissible open subsets.

Let us denote by $p : A \to A/L$ the universal isogeny over C_i. This induces an isomorphism $p^* : \omega_{(A/L)/X} \to \omega_{A/X}$, and thus a morphism $p^*(\kappa) : p_2^* \omega^\kappa \to p_1^* \omega^\kappa$. For every admissible open $\mathcal{U}$ of $X_{\mathrm{Iw},\mathrm{rig}}$, we form the composed morphism

$$\widetilde{U}_{\pi,i} : H^0(U_{\pi,i}(\mathcal{U}), \omega^\kappa) \to H^0(p_1^{-1}(\mathcal{U}), p_2^* \omega^\kappa)$$

$$\xrightarrow{p^*(\kappa)} H^0(p_1^{-1}(\mathcal{U}), p_1^* \omega^\kappa) \xrightarrow{\mathrm{Tr}_{p_1}} H^0(\mathcal{U}, \omega^\kappa).$$

Definition 2.4. The Hecke operator acting on modular forms is defined by $U_{\pi,i} = (1/p^{N_i})\widetilde{U}_{\pi,i}$ with

$$N_i = \sum_{\sigma \in \Sigma_\pi} \big(\min(i, a_\sigma)\min(a+b-i, b_\sigma) + \max(a_\sigma - i, 0)\kappa_\sigma + \max(i - a_\sigma, 0)\lambda_\sigma\big).$$

We will also write

$$n_i = \sum_{\sigma \in \Sigma_\pi} \min(i, a_\sigma)\min(a+b-i, b_\sigma)$$

for the constant term of N_i, which is independent of the weight.

Let us explain briefly the meaning of the normalization factor N_i. The term $\min(i, a_\sigma)\min(a+b-i, b_\sigma)$ comes from the inseparability degree of the projection p_1. The term $\max(a_\sigma - i, 0)\kappa_\sigma + \max(i - a_\sigma, 0)\lambda_\sigma$ comes from the morphism $p^*(\kappa)$. Indeed, we have the following proposition:

Proposition 2.5. *Let M be a finite extension of $\mathbb{Q}_p$, let $(A, \lambda, \iota, \eta, H_\bullet)$ be an O_M-point of X_{Iw} and let $L = L_0 \oplus L_0^\perp$ be a subgroup of $A[\pi]$, where L_0 is an O_F-stable subgroup of $A[\pi^+]$ with $A[\pi^+] = H_i \oplus L_0$ in generic fiber. Then we have, for all $\sigma \in \Sigma_\pi$,*

$$\deg_\sigma L_0 \geq a_\sigma - i \quad \text{and} \quad \deg_\sigma L_0^\perp \geq i - a_\sigma.$$

Proof. The group $A[\pi^+]/L_0$ is of height fi and has partial degree $(a_\sigma - \deg_\sigma L_0)_\sigma$. Hence, $a_\sigma - \deg_\sigma L_0 \leq i$ and $\deg_\sigma L_0 \geq a_\sigma - i$. We get the other equality by duality (note that $b_\sigma - (a + b - i) = i - a_\sigma$). $\qquad\square$

We have the following proposition concerning the behavior of the Hecke operator regarding the degree function:

Proposition 2.6. *Let $x = (A, \lambda, \iota, \eta, H_\bullet)$ be a point of $X_{\text{Iw,rig}}$, and $y \in U_{\pi,i}(x)$ corresponding to a subgroup $L \in A[\pi]$ as before. Write $y = (A/L, \lambda, \iota, \eta, H'_\bullet)$; then we have*

$$\deg H'_j \geq \deg H_j$$

for all $1 \leq j \leq a + b - 1$. Moreover, we have

$$\deg H'_i = \deg A[\pi^+] - \deg L_0.$$

If $\deg H'_i = \deg H_i$, then $\deg H_i \in \mathbb{Z}$.

Proof. For all $1 \leq j \leq i$, the morphism $H_j \to H'_j$ is an isomorphism in the generic fiber. Thus, $\deg H'_j \geq \deg H_j$. If $i < j \leq a + b - 1$, we have

$$\deg H'_j = \deg((\pi^+)^{-1}(H_j \cap L_0)) - \deg L_0.$$

Since $\deg((\pi^+)^{-1}H) = \deg A[\pi^+] + \deg H$ for every subgroup H of $A[\pi^+]$, we get

$$\deg H'_j = \deg A[\pi^+] + \deg(H_j \cap L_0) - \deg L_0$$
$$= \deg(H_j + L_0) + \deg(H_j \cap L_0) - \deg L_0 \geq \deg H_j$$

from the properties of the degree function.

We have $H'_i = (A[\pi^+]/L_0)$, hence the formula for the degree of H'_i. If $\deg H'_i = \deg H_i$, then we have $\deg H_i + \deg L_0 = \deg A[\pi^+]$, and $A[\pi^+] = H_i \times L_0$. Since $A[\pi^+]$ is a BT_1, so is H_i, and its degree is an integer. $\qquad\square$

2C2. *Unitary case.* Assume now that π is in case (U). Let $1 \leq i \leq \frac{1}{2}(a + b)$ be an integer, and let C_i be the moduli space defined over K parameterizing $(A, \lambda, \iota, \eta, H_\bullet, L)$, with $(A, \lambda, \iota, \eta, H_\bullet)$ a point of X_{Iw}, where

- L is an O_F-stable, totally isotropic subgroup of $A[\pi^2]$ such that $A[\pi] = H_i \oplus L[\pi] = H_i^\perp \oplus \pi L$ if $i < \frac{1}{2}(a + b)$;

- L is an O_F-stable, totally isotropic subgroup of $A[\pi]$ such that $A[\pi] = H_i \oplus L$ if $i = \frac{1}{2}(a + b)$.

We have two morphisms p_1, $p_2 : C_i \to X_{\mathrm{Iw}} \times_{O_K} K$. The morphism p_1 corresponds to forgetting L, and the morphism p_2 is defined as $p_2(A, \lambda, \iota, \eta, H_\bullet, L) = (A/L, \lambda', \iota', \eta', H'_\bullet)$, with

- $H'_j = (H_j + L)/L$ if $j \leq i$;
- $H'_j = (\pi^{-1}(H_j \cap L) + L)/L$ if $i < j \leq \frac{1}{2}(a + b)$.

We take the polarization λ' to be equal to $p \cdot \lambda$, which is a prime-to-p polarization. Let C_i^{an} be the analytic space associated to C_i, and define $C_{i,\mathrm{rig}} := p_1^{-1}(X_{\mathrm{Iw},\mathrm{rig}})$. The morphisms p_1 and p_2 give morphisms $C_{i,\mathrm{rig}} \to X_{\mathrm{Iw},\mathrm{rig}}$.

Definition 2.7. The i-th Hecke operator acting on the subsets of $X_{\mathrm{Iw},\mathrm{rig}}$ is defined by

$$U_{\pi,i}(S) = p_2(p_1^{-1}(S)).$$

This operator preserves the admissible open subsets and quasicompact admissible open subsets.

Remark 2.8. The condition $A[\pi] = H_i^\perp \oplus \pi L$ is actually redundant with the condition $A[\pi] = H_i \oplus L[\pi]$. Indeed, since L is totally isotropic, we have $\pi L \subset L[\pi]^\perp$ (we denote by $L[\pi]^\perp$ the orthogonal in $A[\pi]$ of $L[\pi]$). Comparing the heights, we see that we have the equality $\pi L = L[\pi]^\perp$.

Let us denote by $p : A \to A/L$ the universal isogeny over C_i. This induces an isomorphism $p^* : \omega_{(A/L)/X} \to \omega_{A/X}$, and thus a morphism $p^*(\kappa) : p_2^* \omega^\kappa \to p_1^* \omega^\kappa$. For every admissible open subset $\mathcal{U}$ of $X_{\mathrm{Iw},\mathrm{rig}}$, we form the composed morphism

$$\widetilde{U}_{\pi,i} : H^0(U_{\pi,i}(\mathcal{U}), \omega^\kappa) \to H^0(p_1^{-1}(\mathcal{U}), p_2^* \omega^\kappa)$$

$$\xrightarrow{p^*(\kappa)} H^0(p_1^{-1}(\mathcal{U}), p_1^* \omega^\kappa) \xrightarrow{\mathrm{Tr}_{p_1}} H^0(\mathcal{U}, \omega^\kappa).$$

Definition 2.9. The Hecke operator acting on modular forms is defined by $U_{\pi,i} = (1/p^{N_i})\widetilde{U}_{\pi,i}$ with

$$N_i = \sum_{\sigma \in \Sigma_\pi} \left((a + b) \min(i, a_\sigma) + \max(a_\sigma - i, 0)\kappa_\sigma + \max(b_\sigma - a_\sigma, b_\sigma - i)\lambda_\sigma \right)$$

if $i < \frac{1}{2}(a + b)$, and

$$N_{(a+b)/2} = \sum_{\sigma \in \Sigma_\pi} \left(\tfrac{1}{2}(a + b)a_\sigma + \tfrac{1}{2}(b_\sigma - a_\sigma)\lambda_\sigma \right).$$

We will also write

$$n_i = \sum_{\sigma \in \Sigma_\pi} (a + b) \min(i, a_\sigma)$$

if $i < \frac{1}{2}(a + b)$, and

$$n_{(a+b)/2} = \sum_{\sigma \in \Sigma_\pi}{}' \tfrac{1}{2}(a + b)a_\sigma$$

for the constant term of N_i, which is independent of the weight.

Again, the reason for the normalization factor n_i comes in two parts. The term $(a + b)\min(i, a_\sigma)$ comes from the inseparability degree of the projection p_1. The term $\max(a_\sigma - i, 0)\kappa_\sigma + \max(b_\sigma - a_\sigma, b_\sigma - i)\lambda_\sigma$ comes from the morphism $p^*(\kappa)$. Indeed, we have the following proposition:

Proposition 2.10. *Let M be a finite extension of $\mathbb{Q}_p$, let $(A, \lambda, \iota, \eta, H_\bullet)$ be an O_M-point of X_{Iw} and let L be a subgroup of $A[\pi^2]$ as before. If $i < \frac{1}{2}(a + b)$, we have*

$$\deg_{\sigma_1} L \geq a_\sigma - i \quad \text{and} \quad \deg_{\sigma_2} L \geq \max(b_\sigma - i, b_\sigma - a_\sigma).$$

If $i = \frac{1}{2}(a + b)$, we have

$$\deg_{\sigma_2} L \geq \tfrac{1}{2}(b_\sigma - a_\sigma).$$

Proof. Suppose first that $i < \frac{1}{2}(a + b)$. The subgroup L being totally isotropic, we get

$$\deg_{\sigma_2} L = \deg_{\sigma_1} L + b_\sigma - a_\sigma.$$

The group $A[\pi]/L[\pi]$ is of height $2fi$, hence we have $\deg_{\sigma_1} A[\pi]/L[\pi] \leq i$. We get

$$\deg_{\sigma_1} L \geq \deg_{\sigma_1} L[\pi] \geq a_\sigma - i.$$

We deduce that $\deg_{\sigma_2} L = \deg_{\sigma_1} L + b_\sigma - a_\sigma \geq \max(b_\sigma - i, b_\sigma - a_\sigma)$.

If $i = \frac{1}{2}(a + b)$, then L is a maximal totally isotropic subgroup of $A[\pi]$. Thus, we have

$$\deg_{\sigma_2} L = \tfrac{1}{2}(b_\sigma - a_\sigma) + \deg_{\sigma_1} L \geq \tfrac{1}{2}(b_\sigma - a_\sigma). \qquad \square$$

We have the following proposition concerning the behavior of the Hecke operator regarding the degree function:

Proposition 2.11. *Let $x = (A, \lambda, \iota, \eta, H_\bullet)$ be a point of $X_{\mathrm{Iw,rig}}$, and $y \in U_{\pi,i}(x)$ corresponding to a subgroup $L \in A[\pi^2]$ as before. Write $y = (A/L, \lambda, \iota, \eta, H'_\bullet)$; then we have*

$$\deg H'_j \geq \deg H_j$$

for all $1 \leq j \leq \frac{1}{2}(a + b)$. Moreover, if $i < \frac{1}{2}(a + b)$, we have

$$\deg H'_i = 2fi - \deg(L/L[\pi]).$$

If $\deg H'_i = \deg H_i$, then $\deg H_i \in \mathbb{Z}$.

If $i = \frac{1}{2}(a+b)$, then

$$\deg H'_{(a+b)/2} = f(a+b) - \deg(L).$$

If $\deg H'_{(a+b)/2} = \deg H_{(a+b)/2}$, then $d_{(a+b)/2} - \deg H_{(a+b)/2} \in 2\mathbb{Z}$.

Proof. Suppose first that $i < \frac{1}{2}(a+b)$. For each $1 \leq j \leq i$, the morphism $H_j \to H'_j$ is an isomorphism in the generic fiber. Thus, $\deg H'_j \geq \deg H_j$. Suppose $i < j \leq \frac{1}{2}(a+b)$. Then, observing that $\pi^{-1}(H_j \cap L) \cap L = L[\pi]$, we get

$$\deg H'_j = \deg(\pi^{-1}(H_j \cap L) + L) - \deg L$$

$$\geq \deg(\pi^{-1}(H_j \cap L)) - \deg L[\pi]$$

$$\geq \deg A[\pi] + \deg(H_j \cap L) - \deg L[\pi]$$

$$= \deg(H_j + L[\pi]) + \deg(H_j \cap L[\pi]) - \deg L[\pi]$$

$$\geq \deg H_j$$

from the properties of the degree function.

Let us calculate the degree of H'_i. We have

$$\deg H'_i = \deg(A[\pi] + L)/L$$

$$= \deg(\pi^{-1} L[\pi]^{\perp})/L = \deg A[\pi] + \deg L[\pi]^{\perp} - \deg L$$

$$= 2fi + \deg L[\pi] - \deg L = 2fi - \deg(L/L[\pi]).$$

If $\deg H'_i = \deg H_i$, then we have $\deg H_i + \deg L[\pi] = \deg A[\pi]$ and $A[\pi] = H_i \times L[\pi]$. Since $A[\pi]$ is a BT_1, so is H_i, and its degree is an integer.

Suppose now that $i = \frac{1}{2}(a+b)$. The same argument as before shows that we still have $\deg H'_j \geq \deg H_j$ for all $1 \leq j \leq \frac{1}{2}(a+b)$. Since $H'_{(a+b)/2} = A[\pi]/L$, we have

$$\deg H'_{(a+b)/2} = \deg A[\pi] - \deg L = f(a+b) - \deg L.$$

If we have the equality $\deg H'_{(a+b)/2} = \deg H_{(a+b)/2}$, then $H_{(a+b)/2}$ is a BT_1, and all its partial degrees are integers. Since $H_{(a+b)/2}$ is totally isotropic, we have the relations

$$\deg_{\sigma_2} H_{(a+b)/2} = \deg_{\sigma_1} H_{(a+b)/2} + \tfrac{1}{2}(b_\sigma - a_\sigma)$$

for all $\sigma \in \Sigma_\pi$. If we write $h_\sigma = \deg_{\sigma_1} H_{(a+b)/2}$, which is an integer under the previous assumption, we have

$$d_{(a+b)/2} - \deg H_{(a+b)/2} = \sum_{\sigma \in \Sigma_\pi} \left(a_\sigma + \tfrac{1}{2}(a+b) - 2h_\sigma - \tfrac{1}{2}(b_\sigma - a_\sigma) \right)$$

$$= 2 \sum_{\sigma \in \Sigma_\pi} (a_\sigma - h_\sigma) \in 2\mathbb{Z}. \qquad \square$$

We will also need the following useful lemma:

Lemma 2.12. *Let $x = (A, \lambda, \iota, \eta, H_\bullet)$ be a point of $X_{\mathrm{Iw,rig}}$, and let $L \in A[\pi^2]$ be a totally isotropic subgroup as before, corresponding to a point of $U_{\pi,i}(x)$, with $i < \frac{1}{2}(a+b)$. We have the inequality*

$$\deg(L/L[\pi]) \leq 2fi - \deg A[\pi] + \deg L[\pi].$$

Proof. The multiplication by π gives a morphism $L/L[\pi] \to \pi L$. But, L being totally isotropic, we have the relation $\pi L = L[\pi]^\perp$. We thus get a morphism $L/L[\pi] \to L[\pi]^\perp$, which is an isomorphism in generic fiber. Thus,

$$\deg(L/L[\pi]) \leq \deg L[\pi]^\perp = 2fi - \deg A[\pi] + \deg L[\pi]. \qquad \square$$

3. A classicality result

We will now prove a control theorem, that is to say that an overconvergent modular form is indeed classical under a certain assumption.

3A. *Decomposition of the Hecke operators.* Fix a rational $\varepsilon > 0$. We will fix rationals $\varepsilon_k \in \{\varepsilon, d_k\}$ for $1 \leq k \leq s$. Also fix an integer i between 1 and s such that $\varepsilon_i = \varepsilon$. Since we have assumed that there is only one place π of F_0 above p, we will simply let $\Sigma = \Sigma_\pi$. We define a partition of this set. First, we let, for $1 \leq k \leq s$,

$$\Sigma_k := \{\sigma \in \Sigma : a_\sigma = A_k\}.$$

We will also let $\Sigma_0 := \{\sigma \in \Sigma : a_\sigma = 0\}$ and $\Sigma_{s+1} := \{\sigma \in \Sigma : a_\sigma = a+b\}$ (of course Σ_{s+1} is always empty in case (U)). The sets $(\Sigma_k)_{0 \leq k \leq s+1}$ form a partition of Σ.

From the collection $(\varepsilon_k)_{1 \leq k \leq s}$, we define another partition of Σ. The set S_1 is defined to be

$$S_1 = \Sigma_0 \cup \bigcup_{\substack{k \neq i \\ \varepsilon_k = \varepsilon}} \Sigma_k \cup \Sigma_{s+1}.$$

The complement is the set

$$S_2 = \Sigma_i \cup \bigcup_{k, \varepsilon_k = d_k} \Sigma_k.$$

Define

$$\mathcal{U}_0 := \mathrm{Deg}^{-1}([d_1 - \varepsilon_1, d_1] \times \cdots \times [d_s - \varepsilon_s, d_s]) = \bigcap_{k=1}^{s} \mathrm{Deg}_k^{-1}[d_k - \varepsilon_k, d_k],$$

$$\mathcal{U}_1 := \bigcap_{k \neq i} \mathrm{Deg}_k^{-1}[d_k - \varepsilon_k, d_k].$$

We will define a decomposition of the Hecke operator U_{π, A_i} on subsets of $\mathcal{U}_1$. Fix a rational $\alpha > 0$, and define the integer t to be 1, except in the unitary case

and if $i = \frac{1}{2}(a+b)$, where we set $t = 2$. For simplicity, we will write U_i for U_{π, A_i}. Define

$$\mathcal{U} := \mathrm{Deg}_i^{-1}([0, d_i - t(1-\alpha)]) \cap \mathcal{U}_1.$$

Theorem 3.1. *Let $N \geq 1$ and let $\beta < \varepsilon$ be a positive rational. There exists a finite ordered set M_N and a decreasing sequence of admissible open subsets $(\mathcal{U}_k(N))_{k \in M_N}$ of $\mathcal{U}$ such that, for all $k \geq 0$, we have a decomposition of the operator U_i^N on $\mathcal{U}_k(N) \backslash \mathcal{U}_{k+1}(N)$ of the form*

$$U_i^N = \left(\coprod_{j=0}^{N-1} U_i^{N-1-j} \circ T_j \right) \amalg T_N,$$

with $T_0 = U_{i,k,N}^{\mathrm{good}}$,

$$T_j = \coprod_{k_1 \in M_{N-1}, \ldots, k_j \in M_{N-j}} U_{i,k_j,N}^{\mathrm{good}} U_{i,k_{j-1},k_j,N}^{\mathrm{bad}} \cdots U_{i,k,k_1,N}^{\mathrm{bad}} \quad \text{for } 0 < j < N,$$

and

$$T_N = \coprod_{k_1 \in M_{N-1}, \ldots, k_{N-1} \in M_1} U_{i,k_{N-1},N}^{\mathrm{bad}} U_{i,k_{N-2},k_{N-1},N}^{\mathrm{bad}} \cdots U_{i,k,k_1,N}^{\mathrm{bad}}$$

such that

- *the images of $U_{i,j,N}^{\mathrm{good}}$ for $j \in M_k$ are in $\mathrm{Deg}_i^{-1}(]d_i - t(1-\beta), d_i])$;*
- *the images of $U_{i,l,l',N}^{\mathrm{bad}}$ for $l \in M_k$ and $l' \in M_{k-1}$, and $U_{i,l,N}^{\mathrm{bad}}$ for $l \in M_1$ are in $\mathrm{Deg}_i^{-1}([0, d_i - t(1-\beta)])$.*

The idea is that the points in $]d_i - t(1-\beta), d_i]$ are "good" (because the overconvergent modular form will be defined at these points), and the points in $\mathrm{Deg}_i^{-1}([0, d_i - t(1-\beta)])$ are "bad". The decomposition in the theorem is then made to ensure that all the operators have their image either in good points or in bad points.

Let us describe this decomposition by looking at a point $x \in \mathcal{U}$. The set $U_i(\{x\})$ is finite and has, say, N_1 points in $\mathrm{Deg}_i^{-1}([0, d_i - t(1-\beta)])$ and N_2 in its complement $\mathrm{Deg}_i^{-1}(]d_i - t(1-\beta), d_i])$. Thus, one can decompose the operator U_i over x as

$$U_i = U_{i,x}^{\mathrm{good}} \amalg U_{i,x}^{\mathrm{bad}}$$

with $U_{i,x}^{\mathrm{bad}}$ corresponding to the N_1 points in $\mathrm{Deg}_i^{-1}([0, d_i - t(1-\beta)])$ and $U_{i,x}^{\mathrm{good}}$ to the N_2 other points. This is a decomposition of operators, meaning that the set $U_i(\{x\})$ is the disjoint union of the sets $U_{i,x}^{\mathrm{good}}(\{x\})$ and $U_{i,x}^{\mathrm{bad}}(\{x\})$; moreover, the operators $U_{i,x}^{\mathrm{good}}$ and $U_{i,x}^{\mathrm{bad}}$ induce morphisms $H^0(U_i(\{x\}), \omega^\kappa) \to H^0(\{x\}, \omega^\kappa)$ for each weight κ, such that

$$U_i f = U_{i,x}^{\mathrm{good}} f + U_{i,x}^{\mathrm{bad}} f$$

for all $f \in H^0\big(U_i(\{x\}), \omega^\kappa\big)$. These morphisms are defined in the same way as the morphism U_i, with the same normalization factor. This gives the decomposition of the theorem for $N = 1$ at the point x.

To get the decomposition for $N = 2$, one needs to study the operator $U_{i,x}^{\text{bad}}$. Let $x_1, \ldots, x_{N_1}$ be the elements of the set $U_{i,x}^{\text{bad}}(\{x\})$. We can thus decompose the operator $U_{i,x}^{\text{bad}}$ into

$$U_{i,x}^{\text{bad}} = \coprod_{j=1}^{N_1} U_{i,x,j}^{\text{bad}},$$

where $U_{i,x,j}^{\text{bad}}$ corresponds to the point x_j. Then, for each $1 \le j \le N_1$, we have the decomposition of the operator U_i at the point x_j as $U_{i,x_j}^{\text{good}} \sqcup U_{i,x_j}^{\text{bad}}$. One then gets the decomposition of U_i^2 at the point x,

$$U_i^2 = U_i \circ U_{i,x}^{\text{good}} \sqcup \coprod_{j=1}^{N_1} U_{i,x_j}^{\text{good}} \circ U_{i,x,j}^{\text{bad}} \sqcup \coprod_{j=1}^{N_1} U_{i,x_j}^{\text{bad}} \circ U_{i,x,j}^{\text{bad}}.$$

Repeating this argument, one gets the desired decomposition of the operator U_i^N at the point x. Of course, this decomposition does not have a meaning on the whole open subset $\mathcal{U}$ because the morphisms used to define the different operators will not be finite (the integer N_1 depends on the point x). But, on an adequate subset, this will be the case. That is why one has to construct the subsets $(\mathcal{U}_k(N))_{k \in M_N}$. This construction and the proof of its properties have been done in [Bijakowski et al. 2016, Theorem 4.4.1].

3B. *Analytic continuation.* Let f be a section of the sheaf ω^κ on $\mathcal{U}_0$. We will show that f can be extended to $\mathcal{U}_1$ under a certain condition. More precisely, suppose in this section that f is an eigenform for the Hecke operator U_i with eigenvalue α_i and that

$$v(\alpha_i) + n_{A_i} < (1 - 2f\varepsilon) \inf_{\sigma \in S_2} (\kappa_\sigma + \lambda_\sigma).$$

The first step is to extend f to $\text{Deg}_i^{-1}(\,]d_i - t, d_i])\cap\mathcal{U}_1$. We will use the following proposition:

Proposition 3.2. *Let $0 < \gamma < 1$ be a rational. Then there exists an integer N such that*

$$U_i^N\big(\text{Deg}_i^{-1}([d_i - t\gamma, d_i])\cap\mathcal{U}_1\big) \subset \text{Deg}_i^{-1}([d_i - \varepsilon, d_i])\cap\mathcal{U}_1.$$

Proof. The key point is that on $\text{Deg}_i^{-1}([d_i - t\gamma, d_i])\cap\mathcal{U}_1$ the operator U_i strictly increases the degree function Deg_i. Since $\text{Deg}_i^{-1}([d_i - t\gamma, d_i])\cap\mathcal{U}_1$ is a quasi-compact open subset of $X_{\text{Iw,rig}}$, one can then apply the argument in [Pilloni 2011, Proposition 2.5]. □

Corollary 3.3. *The overconvergent form f extends to $\text{Deg}_i^{-1}(\,]d_i - t, d_i])\cap\mathcal{U}_1$.*

Proof. Let $0 < \gamma < 1$ be a rational. By the previous proposition, there exists an integer N such that

$$U_i^N \big(\mathrm{Deg}_i^{-1}([d_i - t\gamma, d_i]) \cap \mathcal{U}_1\big) \subset \mathrm{Deg}_i^{-1}([d_i - \varepsilon, d_i]) \cap \mathcal{U}_1.$$

The quantity $\alpha_i^{-N} U_i^N f$ is thus defined on $\mathrm{Deg}_i^{-1}([d_i - t\gamma, d_i]) \cap \mathcal{U}_1$. This formula allows us to extend f to $\mathrm{Deg}_i^{-1}(]d_i - t, d_i]) \cap \mathcal{U}_1$. $\qquad\square$

The second step is to define some series on $\mathcal{U} := \mathrm{Deg}_i^{-1}\big([0, d_i - t(1-\alpha)]\big) \cap \mathcal{U}_1$ for some sufficiently small rational α. We will use the decomposition of the Hecke operator U_i. First, we recall the definition of the norm of an operator. If $\mathcal{U}$ is an open subset of $X_{\mathrm{Iw,rig}}$, and $T : H^0(T(\mathcal{U}), \omega^\kappa) \to H^0(\mathcal{U}, \omega^\kappa)$ is an operator, then we define the norm of T as

$$\|T\| := \inf\big\{r > 0 : |Tf|_{\mathcal{U}} \le r|f|_{T(\mathcal{U})} \text{ for all } f \in H^0(T(\mathcal{U}), \omega^\kappa)\big\}.$$

Theorem 3.4. *Suppose that all the operators U_i^{bad} introduced in Theorem 3.1 satisfy the relation*

$$\|\alpha_i^{-1} U_i^{\mathrm{bad}}\| < 1.$$

Then it is possible to construct sections $f_N \in H^0(\mathcal{U}, \omega^\kappa / p^{A_N})$ such that $A_N \to \infty$. Moreover, the functions f_N can be glued together with the initial form f to give an element of $H^0(\mathcal{U}_1, \omega^\kappa)$.

The construction of the series f_N and the proof of the gluing process have been done in [Bijakowski et al. 2016, Section 4.5]. Let us briefly describe the method. We take an element $\beta > 0$ and consider the open subsets $(\mathcal{U}_k(N))_{k \in M_N}$ constructed in Theorem 3.1 for this element. Let K_N be the largest element of M_N; neglecting the bad points for the operator U_i^N gives a function $g_{K_N} \in H^0(\mathcal{U}_{K_N}(N), \omega^\kappa)$ (we refer to [Bijakowski et al. 2016, Definition 4.5.2] for the precise definition of the function g_{K_N}). Now take $\beta' < \beta$, and again apply Theorem 3.1 for this other element: we get open subsets $(\mathcal{U}_k(N)')_{k \in M_N}$. We consider the subset $\mathcal{U}_{K_N-1}(N)' \backslash \mathcal{U}_{K_N}(N)'$ and the decomposition of U_i^N on it. Neglecting the bad points gives a function $g_{K_N-1} \in H^0(\mathcal{U}_{K_N-1}(N)' \backslash \mathcal{U}_{K_N}(N)', \omega^\kappa)$. One can then show that g_{K_N} and g_{K_N-1} can be glued together to give a function modulo p^{A_N}, where A_N is an explicit constant. One thus gets a function $f_N \in H^0(\mathcal{U}_{K_N-1}(N)', \omega^\kappa / p^{A_N})$ (see [Bijakowski et al. 2016, Proposition 4.5.6] for more details). Repeating this argument, one thus extends the function f_N to the whole $\mathcal{U}$. The hypothesis on the operators implies that the sequence $(A_N)_N$ tends to infinity. A gluing lemma ([Bijakowski et al. 2016, Proposition 4.5.7], following [Kassaei 2006]) then ensures the existence of a function $f \in H^0(\mathcal{U}, \omega^\kappa)$, which can be glued with the initial function f, thus getting a section on $\mathcal{U}_1$.

We will now prove that, under the assumption made at the beginning of Section 3B, the condition in the theorem is fulfilled, that is to say that the norm of the operators

$\alpha_i^{-1} U_i^{\mathrm{bad}}$ is strictly less than 1. We will split the discussion between the linear and unitary cases.

3B1. *Linear case.* We recall that the integer t is equal to 1 in this case. Let M be a finite extension of $\mathbb{Q}_p$, let $x = (A, \lambda, \iota, \eta, H_\bullet)$ be a point of $\mathcal{U}$ defined over O_M, and let $L = L_0 \oplus L_0^\perp$ be a subgroup of $A[\pi]$, where L_0 is an O_F-stable subgroup of $A[\pi^+]$ with $A[\pi^+] = H_{A_i} \oplus L_0$ in generic fiber. Let us write $A' = A/L$, and let H'_{A_i} be the image of H_{A_i} in A/L. Thus,

$$H'_{A_i} = A[\pi^+]/L_0.$$

We suppose that the subgroup L corresponds to a bad point, that is to say that $\deg H'_{A_i} \leq d_i - 1 + \alpha$ for a certain rational $\alpha > 0$. We write

$$\deg_\sigma L_0 = \max(a_\sigma - A_i, 0) + l_\sigma$$

for all $\sigma \in \Sigma$. As is shown by Proposition 2.5, l_σ is a positive rational for all σ. We then have, for all σ,

$$\deg_\sigma H'_{A_i} = a_\sigma - \deg_\sigma L_0 = \min(A_i, a_\sigma) - l_\sigma.$$

We deduce that $\deg H'_{A_i} = d_i - \sum_{\sigma \in \Sigma} l_\sigma$. The condition of being a bad point gives

$$\sum_{\sigma \in \Sigma} l_\sigma \geq 1 - \alpha.$$

Actually, we can control some of the l_σ. First, we prove the following technical lemma:

Lemma 3.5. *Let $x = (A, \lambda, \iota, \eta, H_\bullet)$ be a point as before and let $H \subset A[\pi^+]$ be an O_F-stable subgroup. If H is of height fA_k and $\deg H \geq d_k - \varepsilon$, then $\deg_\sigma H \geq \min(a_\sigma, A_k) - \varepsilon$ for all $\sigma \in \Sigma$.*

If H is of height $f(a+b - A_k)$ and $\deg H \leq \deg A[\pi^+] - d_k + \varepsilon$, then $\deg_\sigma H \leq \max(a_\sigma - A_k, 0) + \varepsilon$ for all $\sigma \in \Sigma$.

Proof. Suppose that H is of height fA_k and $\deg H \geq d_k - \varepsilon$. If $\deg_\sigma H < \min(a_\sigma, A_k) - \varepsilon$ for some $\sigma \in \Sigma$, then

$$\deg H = \sum_{\sigma' \in \Sigma} \deg_{\sigma'} H < \min(a_\sigma, A_k) - \varepsilon + \sum_{\sigma' \neq \sigma} \min(a_{\sigma'}, A_k) = d_k - \varepsilon$$

and we get a contradiction.

If H is of height $f(a+b-A_k)$ and $\deg H \leq \deg A[\pi^+] - d_k + \varepsilon$, then we can apply the previous argument to $A[\pi^+]/H$. We get $\deg_\sigma(A[\pi^+]/H) \geq \min(a_\sigma, A_k) - \varepsilon$ for all $\sigma \in \Sigma$, and therefore $\deg_\sigma H \leq \max(a_\sigma - A_k, 0) + \varepsilon$. $\square$

Now we prove a bound for some of the l_σ.

Lemma 3.6. *If $\sigma \in S_1$, we have $l_\sigma \leq \varepsilon$.*

Proof. If $\sigma \in \Sigma_0 \cup \Sigma_{s+1}$, then $l_\sigma = 0$. If not, $\sigma \in \Sigma_k$, with $1 \leq k \leq s$, $k \neq i$, and $\varepsilon_k = \varepsilon$. Since x is a point of $\mathcal{U}$, we have $\deg H_{A_k} \geq d_k - \varepsilon$. Suppose first that $k < i$. Since H_{A_k} and L_0 are disjoint, the morphism $H_{A_k} \to H'_{A_k} := (H_{A_k} + L_0)/L_0$ is an isomorphism in the generic fiber. Thus, by the properties of the degree function, $\deg H'_{A_k} \geq \deg H_{A_k} \geq d_k - \varepsilon$. By Lemma 3.5, we get $\deg_\sigma H'_{A_k} \geq A_k - \varepsilon$. But we also have $\deg_\sigma H'_{A_k} \leq \deg_\sigma H'_{A_i} = \min(A_i, a_\sigma) - l_\sigma = A_k - l_\sigma$. In conclusion, we get $l_\sigma \leq \varepsilon$.

The case $k > i$ can be treated by duality, considering the group $A[\pi^-]$. We can also give a direct argument. Let us denote the group $L_0/(L_0 \cap H_{A_k})$ by L'_k. Since H_{A_k} and L_0 generate $A[\pi^+]$, the morphism $L'_k \to A[\pi^+]/H_{A_k}$ is an isomorphism in generic fiber, so we get $\deg L'_k \leq \deg(A[\pi^+]/H_{A_k})$. But we have

$$\deg(A[\pi^+]/H_{A_k}) = \deg A[\pi^+] - \deg H_{A_k} \leq \deg A[\pi^+] - d_k + \varepsilon.$$

From Lemma 3.5, we get $\deg_\sigma L'_k \leq \varepsilon$. Since $L_0 \cap H_{A_k}$ is of height $f(A_k - A_i)$, we have

$$l_\sigma = \deg_\sigma L_0 - (A_k - A_i) = \deg_\sigma L'_k + \deg_\sigma(L_0 \cap H_{A_k}) - (A_k - A_i) \leq \varepsilon. \qquad \square$$

Remark 3.7. Actually, we can have more control on the l_σ. Indeed, suppose that there exists $\sigma \in \Sigma_k \cap S_1$. If $k < i$, then we have $l_\sigma \leq \varepsilon$ for all $\sigma \in \Sigma_j$ with $j \leq k$. If $k > i$, then we have $l_\sigma \leq \varepsilon$ for all $\sigma \in \Sigma_j$ with $j \geq k$.

Putting together all the calculations made, we get the following result:

Proposition 3.8. *We have*

$$\sum_{\sigma \in S_2} l_\sigma \geq 1 - \alpha - f\varepsilon.$$

Proof. If $\sigma \in S_1$ we have $l_\sigma \leq \varepsilon$, and we also have $\sum_{\sigma \in \Sigma} l_\sigma \geq 1 - \alpha$. Thus,

$$\sum_{\sigma \in S_2} l_\sigma \geq 1 - \alpha - \sum_{\sigma \in S_1} l_\sigma \geq 1 - \alpha - f\varepsilon. \qquad \square$$

We can now prove the bound for the norm of the operator U_i^{bad}.

Proposition 3.9. *We have*

$$\|\alpha_i^{-1} U_i^{\mathrm{bad}}\| \leq p^{v(\alpha_i) + n_{A_i} - (1 - \alpha - 2f\varepsilon)\inf_{\sigma \in S_2}(\kappa_\sigma + \lambda_\sigma)}.$$

Proof. The term $p^{v(\alpha_i)}$ is the norm of the element α_i^{-1}. Recall also the normalization factor for the Hecke operator,

$$N_{A_i} = n_{A_i} + \sum_{\sigma \in \Sigma_\pi} \left(\max(a_\sigma - A_i, 0)\kappa_\sigma + \max(A_i - a_\sigma, 0)\lambda_\sigma\right).$$

The first term will come in the bound and the second term will be canceled out by the bounds for the partial degrees of L_0 and $L_0^\perp$.

Indeed, let us calculate the norm of the morphism $\omega_A^\kappa \to \omega_{A/L}^\kappa$. Let κ_1 be the weight defined by $((\kappa_\sigma, \ldots, \kappa_\sigma), (\lambda_\sigma, \ldots, \lambda_\sigma))_{\sigma \in \Sigma}$ and $\kappa_2 = \kappa - \kappa_1$. Thus, $\omega^\kappa = \omega^{\kappa_1} \otimes \omega^{\kappa_2}$ and ω^{κ_1} is a line bundle. Since κ_2 has nonnegative coefficients, the morphism $\omega_A^{\kappa_2} \to \omega_{A/L}^{\kappa_2}$ has norm less than 1. It then suffices to study the morphism $\omega_A^{\kappa_1} \to \omega_{A/L}^{\kappa_1}$. But we have

$$\omega_A^{\kappa_1} = \bigotimes_{\sigma \in \Sigma} (\det \omega_{A,\sigma}^+)^{\kappa_\sigma} \otimes (\det \omega_{A,\sigma}^-)^{\lambda_\sigma},$$

where $\omega_{A,\sigma}^+$ is the submodule of ω_A where O_F acts on σ^+, and similarly for $\omega_{A,\sigma}^-$. We recall that $L = L_0 \oplus L_0^\perp$. The norm of the map $\omega_A^{\kappa_1} \to \omega_{A/L}^{\kappa_1}$ is exactly p^A, with

$$A = -\sum_{\sigma \in \Sigma} (\kappa_\sigma \deg_\sigma L_0 + \lambda_\sigma \deg_\sigma L_0^\perp).$$

But recall that $\deg_\sigma L_0 = \max(a_\sigma - A_i, 0) + l_\sigma$ and that

$$\deg_\sigma L_0^\perp = A_i - a_\sigma + \deg_\sigma L_0 = \max(A_i - a_\sigma, 0) + l_\sigma.$$

Thus,

$$A = -\sum_{\sigma \in \Sigma} \left(\max(a_\sigma - A_i, 0)\kappa_\sigma + \max(A_i - a_\sigma, 0)\lambda_\sigma + l_\sigma(\kappa_\sigma + \lambda_\sigma) \right)$$

$$= -(N_{A_i} - n_{A_i}) + B$$

with

$$B = -\sum_{\sigma \in \Sigma} (l_\sigma(\kappa_\sigma + \lambda_\sigma)) \leq -\sum_{\sigma \in S_2} (l_\sigma(\kappa_\sigma + \lambda_\sigma))$$

$$\leq -\inf_{\sigma \in S_2}(\kappa_\sigma + \lambda_\sigma) \sum_{\sigma \in S_2} l_\sigma \leq -(1 - \alpha - f\varepsilon) \inf_{\sigma \in S_2}(\kappa_\sigma + \lambda_\sigma).$$

Hence the bound for the operator $\alpha_i^{-1} U_i^{\text{bad}}$ follows, noting that $1 - \alpha - f\varepsilon \geq 1 - \alpha - 2f\varepsilon$. $\square$

Since we have made the assumption

$$v(\alpha_i) + n_{A_i} < (1 - 2f\varepsilon) \inf_{\sigma \in S_2}(\kappa_\sigma + \lambda_\sigma),$$

we see that, if α is small enough, the norm of the operator $\alpha_i^{-1} U_i^{\text{bad}}$ is strictly less than 1.

Remark 3.10. Actually, we only needed the weaker hypothesis

$$v(\alpha_i) + n_{A_i} < (1 - f\varepsilon) \inf_{\sigma \in S_2}(\kappa_\sigma + \lambda_\sigma),$$

but we used ours to have the same condition in both the linear and unitary cases.

3B2. *Unitary case.* We now turn to the case (U). We first assume that $i < \frac{1}{2}(a+b)$, so that the integer t is equal to 1. Let M be a finite extension of $\mathbb{Q}_p$, let $x = (A, \lambda, \iota, \eta, H_\bullet)$ be a point of $\mathcal{U}$ defined over O_M, and let L be a totally isotropic O_F-stable subgroup of $A[\pi^2]$ such that $A[\pi] = H_i \oplus L[\pi]$. Let $A' = A/L$, and let H'_{A_i} be the image of H_{A_i} in A/L. Thus,

$$H'_{A_i} = (A[\pi] + L)/L.$$

We suppose that the subgroup L corresponds to a bad point, that is to say that $\deg H'_{A_i} \leq d_i - 1 + \alpha$ for a certain rational $\alpha > 0$. We write

$$\deg_{\sigma_1} L = \max(a_\sigma - A_i, 0) + l_\sigma,$$

and

$$\deg_{\sigma_2} L = b_\sigma - a_\sigma + \deg_{\sigma_1} L = \max(b_\sigma - A_i, b_\sigma - a_\sigma) + l_\sigma$$

for all $\sigma \in \Sigma$. As it is shown by Proposition 2.10, l_σ is a positive rational for all σ. We then have $\deg H'_{A_i} = 2fA_i - \deg(L/L[\pi])$; therefore,

$$\deg(L/L[\pi]) \geq 2fA_i - d_i + 1 - \alpha.$$

We also have, by Lemma 2.12, $\deg(L/L[\pi]) \leq 2fA_i - \deg A[\pi] + \deg L[\pi]$. Thus,

$$\deg L[\pi] \geq \deg A[\pi] - d_i + 1 - \alpha.$$

We conclude that

$$\deg L \geq 2fA_i - 2d_i + \deg A[\pi] + 2(1 - \alpha).$$

By an explicit computation, one checks the equality

$$2fA_i - 2d_i + \deg A[\pi] = \sum_{\sigma \in \Sigma}(\max(a_\sigma - A_i, 0) + \max(b_\sigma - A_i, b_\sigma - a_\sigma)).$$

We conclude that the condition of being a bad point gives

$$\sum_{\sigma \in \Sigma} l_\sigma \geq 1 - \alpha.$$

Actually, we can control some of the l_σ. First, we prove the following technical lemma:

Lemma 3.11. *Let $x = (A, \lambda, \iota, \eta, H_\bullet)$ be a point as before, and let $H \subset A[\pi]$ be an O_F-stable subgroup. If H is of height $2fA_k$ and $\deg H \geq d_k - \varepsilon$, then $\deg_{\sigma_1} H \geq \min(a_\sigma, A_k) - \varepsilon$ for all $\sigma \in \Sigma$.*

If H is of height $2f(a+b-A_k)$ and $\deg H \leq \deg A[\pi] - d_k + \varepsilon$, then $\deg_{\sigma_1} H \leq \max(a_\sigma - A_k, 0) + \varepsilon$ for all $\sigma \in \Sigma$.

Proof. Suppose that H is of height $2fA_k$ and $\deg H \geq d_k - \varepsilon$. If $\deg_{\sigma_1} H < \min(a_\sigma, A_k) - \varepsilon$ for some $\sigma \in \Sigma$, then

$$\deg H = \sum_{\sigma' \in \Sigma} (\deg_{\sigma_1'} H + \deg_{\sigma_2'} H)$$

$$< \min(a_\sigma, A_k) - \varepsilon + A_k + \sum_{\sigma' \neq \sigma} (\min(a_{\sigma'}, A_k) + A_k) = d_k - \varepsilon,$$

and we get a contradiction.

If H is of height $2f(a + b - A_k)$ and $\deg H \leq \deg A[\pi] - d_k + \varepsilon$, then we can apply the previous argument to $A[\pi]/H$. We get $\deg_{\sigma_1}(A[\pi]/H) \geq \min(a_\sigma, A_k) - \varepsilon$ for all $\sigma \in \Sigma$, and therefore $\deg_{\sigma_1} H \leq \max(a_\sigma - A_k, 0) + \varepsilon$.   □

Now we prove a bound for some of the l_σ.

Lemma 3.12. *If* $\sigma \in S_1$*, we have* $l_\sigma \leq 2\varepsilon$.

Proof. If $\sigma \in \Sigma_0$ then $l_\sigma = 0$. If not, $\sigma \in \Sigma_k$ with $1 \leq k \leq s$, $k \neq i$, and $\varepsilon_k = \varepsilon$. Since x is a point of $\mathcal{U}$, we have $\deg H_{A_k} \geq d_k - \varepsilon$. Suppose first that $k < i$. Since H_{A_k} and $L[\pi]$ are disjoint, the morphism $H_{A_k} \to H'_{A_k} := (H_{A_k} + L[\pi])/L[\pi]$ is an isomorphism in the generic fiber. Thus, by the properties of the degree function, $\deg H'_{A_k} \geq \deg H_{A_k} \geq d_k - \varepsilon$. By Lemma 3.11, we have $\deg_{\sigma_1} H'_{A_k} \geq A_k - \varepsilon$. But we also have $\deg_{\sigma_1} H'_{A_k} \leq \deg_{\sigma_1} A[\pi]/L[\pi] = a_\sigma - \deg_{\sigma_1} L[\pi]$, and therefore $\deg_{\sigma_1} L[\pi] \leq \varepsilon$. Next, we study $H''_{A_k} := (H'_{A_k} + L/L[\pi])/(L/L[\pi])$. Using Lemma 3.11, we get

$$A_k - \varepsilon \leq \deg_{\sigma_1} H_{A_k''} \leq A_k - \deg_{\sigma_1} L/L[\pi].$$

Therefore, $\deg_{\sigma_1}(L/L[\pi]) \leq \varepsilon$. In conclusion, we get $l_\sigma \leq 2\varepsilon$.

The case $k > i$ cannot be treated by duality in this case. Let us denote the group $L[\pi]/(L[\pi] \cap H_{A_k})$ by L'_k. Since H_{A_k} and $L[\pi]$ generate $A[\pi]$, the morphism $L'_k \to A[\pi]/H_{A_k}$ is an isomorphism in generic fiber, so we get $\deg L'_k \leq \deg(A[\pi]/H_{A_k})$. But we have

$$\deg(A[\pi]/H_{A_k}) = \deg A[\pi] - \deg H_{A_k} \leq \deg A[\pi] - d_k + \varepsilon.$$

Lemma 3.11 gives $\deg_{\sigma_1} L'_k \leq \varepsilon$. Since $L[\pi] \cap H_{A_k}$ is of height $2f(A_k - A_i)$, we have

$$\deg_{\sigma_1} L[\pi] = \deg_{\sigma_1} L'_k + \deg_{\sigma_1} L[\pi] \cap H_{A_k} \leq \varepsilon + A_k - A_i.$$

Now we study $A' := A/L[\pi]$ and set $H'_{A_k} := (\pi^{-1}(H_{A_k} \cap L[\pi]))/L[\pi]$. If H''_{A_k} is the image of the morphism $H'_{A_k} \to A'[\pi]/(L/L[\pi])$, then we get $\deg H''_{A_k} \geq d_k - \varepsilon$. Lemma 3.11 gives

$$A_k - \varepsilon \leq \deg_{\sigma_1} H''_{A_k} \leq A_k - \deg_{\sigma_1}(L/L[\pi]).$$

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We deduce that $\deg_{\sigma_1} L/L[\pi] \leq \varepsilon$. Finally, we have

$$l_\sigma = \deg_{\sigma_1} L - (A_k - A_i) \leq 2\varepsilon. \qquad \square$$

Remark 3.13. Actually, we can have more control on the l_σ. Indeed, suppose that there exists $\sigma \in \Sigma_k \cap S_1$. If $k < i$, then we have $l_\sigma \leq \varepsilon$ for all $\sigma \in \Sigma_j$ with $j \leq k$. If $k > i$, then we have $l_\sigma \leq \varepsilon$ for all $\sigma \in \Sigma_j$ with $j \geq k$.

Putting together all the calculations made, we get the following result:

Proposition 3.14. *We have*

$$\sum_{\sigma \in S_2} l_\sigma \geq 1 - \alpha - 2f\varepsilon.$$

Proof. If $\sigma \in S_1$, we have $l_\sigma \leq 2\varepsilon$, and we also have $\sum_{\sigma \in \Sigma} l_\sigma \geq 1 - \alpha$. Thus,

$$\sum_{\sigma \in S_2} l_\sigma \geq 1 - \alpha - \sum_{\sigma \in S_1} l_\sigma \geq 1 - \alpha - 2f\varepsilon. \qquad \square$$

We can now prove the bound for the norm of the operator U_i^{bad}.

Proposition 3.15. *We have*

$$\|\alpha_i^{-1} U_i^{\mathrm{bad}}\| \leq p^{v(\alpha_i) + n_{A_i} - (1 - \alpha - 2f\varepsilon) \inf_{\sigma \in S_2} (\kappa_\sigma + \lambda_\sigma)}.$$

Proof. The proof is exactly the same as in the linear case. $\qquad \square$

Since we have made the assumption

$$v(\alpha_i) + n_{A_i} < (1 - 2f\varepsilon) \inf_{\sigma \in S_2} (\kappa_\sigma + \lambda_\sigma),$$

we see that, if α is small enough, the norm of the operator $\alpha_i^{-1} U_i^{\mathrm{bad}}$ is strictly less than 1.

We now deal with the case $i = \frac{1}{2}(a + b)$; the integer t is now equal to 2. The subgroup L is now a maximal totally isotropic subgroup of $A[\pi]$. We write $\deg_{\sigma_1} L = l_\sigma$, and we have $\deg_{\sigma_2} L = \frac{1}{2}(b_\sigma - a_\sigma) + l_\sigma$ for all $\sigma \in \Sigma$. Recall that

$$\deg H'_{(a+b)/2} = f(a+b) - \deg L = \sum_{\sigma \in \Sigma} \left(a_\sigma + \tfrac{1}{2}(a+b) - 2l_\sigma\right),$$

so that

$$d_{(a+b)/2} - \deg H'_{(a+b)/2} = 2 \sum_{\sigma \in \Sigma} l_\sigma.$$

The condition of being a bad point gives

$$\sum_{\sigma \in \Sigma} l_\sigma \geq 1 - \alpha$$

for a certain $\alpha > 0$. The calculations are now exactly the same as previously, and the bound obtained in Proposition 3.15 is still valid in the case $i = \frac{1}{2}(a + b)$.

3B3. *Conclusion.* We now recall the analytic continuation result we got, putting together all the results of the past sections.

Recall that we have defined subsets

$$\mathcal{U}_0 := \mathrm{Deg}^{-1}([d_1 - \varepsilon_1, d_1] \times \cdots \times [d_s - \varepsilon_s, d_s]) = \bigcap_{k=1}^{s} \mathrm{Deg}_k^{-1}[d_k - \varepsilon_k, d_k],$$

$$\mathcal{U}_1 := \bigcap_{k \neq i} \mathrm{Deg}_k^{-1}[d_k - \varepsilon_k, d_k].$$

We also have defined a partition $\Sigma = S_1 \amalg S_2$. The result of this section is thus contained in the following theorem:

Theorem 3.16. *Let κ be a weight and f a section of ω^κ on $\mathcal{U}_0$. Suppose that f is an eigenform for the Hecke operators U_i, with eigenvalues α_i, for all $1 \leq i \leq s$, and that we have the relations*

$$n_{A_i} + v(\alpha_i) < (1 - 2f\varepsilon) \inf_{\sigma \in S_2} (\kappa_\sigma + \lambda_\sigma).$$

Then f can be extended to a section of ω^κ on $\mathcal{U}_1$.

3C. *The classicality theorem.* We recall that we have assumed that p is inert in F_0. We have defined integers $(A_i)_{1 \leq i \leq s}$, and to each of these integers corresponds a canonical subgroup and a relevant Hecke operator. We also have a partition

$$\Sigma = \Sigma_0 \amalg \coprod_{j=1}^{s} \Sigma_j \amalg \Sigma_{s+1}.$$

We can now state the classicality result.

Theorem 3.17. *Let f be an overconvergent modular form of weight κ. Suppose that f is an eigenform for the Hecke operators U_{π, A_i}, with eigenvalues α_i, and that*

$$n_{A_i} + v(\alpha_i) < \inf_{\sigma \in \Sigma_i} (\kappa_\sigma + \lambda_\sigma)$$

for $1 \leq i \leq s$. Then f is classical.

Before giving the proof of the theorem, let us make the conditions explicit. In the case (L), the conditions become

$$v(\alpha_i) + \sum_{j=1}^{f} \min(a_j, A_i) \min(b_j, B_i) < \inf_{\sigma \in \Sigma_i} (\kappa_\sigma + \lambda_\sigma).$$

In the special case where all the (a_i) are distinct and different from 0 and $a + b$, we have $s = d$ conditions, which can be written

$$v(\alpha_\sigma) + \sum_{\sigma' \in \Sigma} \min(a_\sigma, a_{\sigma'}) \min(b_\sigma, b_{\sigma'}) < \kappa_\sigma + \lambda_\sigma,$$

where α_σ is the eigenvalue of U_{π, a_σ}.

In the case (U), they become

$$v(\alpha_i) + \sum_{j=1}^{f} (a+b)\min(a_j, A_i) < \inf_{\sigma \in \Sigma_i}(\kappa_\sigma + \lambda_\sigma)$$

if $A_i < \frac{1}{2}(a+b)$, and

$$v(\alpha_i) + \sum_{j=1}^{f} \tfrac{1}{2}(a+b)a_j < \inf_{\sigma \in \Sigma_i}(\kappa_\sigma + \lambda_\sigma)$$

if $A_i = \frac{1}{2}(a+b)$.

In the special case where all the (a_i) are distinct and different from 0 and $\frac{1}{2}(a+b)$, we have $s = d$ conditions, which can be written

$$v(\alpha_\sigma) + \sum_{\sigma' \in \Sigma} (a+b)\min(a_\sigma, a_{\sigma'}) < \kappa_\sigma + \lambda_\sigma,$$

where α_σ is the eigenvalue of U_{π, a_σ}.

Of course, in the case where the ordinary locus is nonempty, we find the same conditions as in [Bijakowski et al. 2016]. In the case (L), the condition of ordinariness is $a_\sigma = a$, $b_\sigma = b$ for some pair (a, b) and for all $\sigma \in \Sigma$. There is one relevant Hecke operator, $U_{\pi, a}$, and the classicality condition is

$$fab + v(\alpha) < \inf_{\sigma \in \Sigma}(\kappa_\sigma + \lambda_\sigma),$$

where α is the eigenvalue of $U_{\pi, a}$.

In the case (U), the condition of ordinariness is

$$a_\sigma = b_\sigma = \tfrac{1}{2}(a+b)$$

for all $\sigma \in \Sigma$. There is one relevant Hecke operator, $U_{\pi, (a+b)/2}$, and the condition is

$$\tfrac{1}{4}f(a+b)^2 + v(\alpha) < \inf_{\sigma \in \Sigma}(\kappa_\sigma + \lambda_\sigma),$$

where α is the eigenvalue of $U_{\pi, (a+b)/2}$.

Remark 3.18. Since we need to use all the Hecke operators U_{π, A_i} for the classicality result, maybe the relevant operator is a product of these ones. For example, in the linear case the operator $\prod_\sigma U_{\pi, a_\sigma}$ parametrizes complements of (a lifting of) the kernel of the f-th power of the Frobenius on the μ-ordinary locus.

3C1. *Proof of the theorem.* The overconvergent form f is a section of the sheaf ω^κ defined over

$$\mathcal{V}_0 := \bigcap_{i=1}^{s} \mathrm{Deg}_i^{-1}([d_i - \varepsilon, d_i])$$

for some $\varepsilon > 0$. Of course, ε can be taken as small as we want. Let us write $K_i = \inf_{\sigma \in \Sigma_i}(\kappa_\sigma + \lambda_\sigma)$ for all $1 \leq i \leq s$. We put the elements $(K_i)_{1 \leq i \leq s}$ in decreasing order:

$$K_{i_1} \geq K_{i_2} \geq \cdots \geq K_{i_s}.$$

We will use the analytic continuation theorem successively for the operators $U_{i_1}, \ldots, U_{i_s}$, in that order.

We first consider the operator U_{i_1} (we recall that we let $U_i := U_{\pi, A_i}$). We take all the rationals ε_k to be equal to ε. In that case, $S_2 = \Sigma_{i_1}$. We can apply the analytic continuation theorem if the condition

$$n_{A_{i_1}} + v(\alpha_{i_1}) < (1 - 2f\varepsilon) \inf_{\sigma \in S_2}(\kappa_\sigma + \lambda_\sigma)$$

is fulfilled. But $\inf_{\sigma \in S_2}(\kappa_\sigma + \lambda_\sigma) = K_{i_1}$ and we have, by hypothesis,

$$n_{A_{i_1}} + v(\alpha_{i_1}) < K_{i_1}.$$

If ε is small enough, then we can apply the theorem. We can thus extend f to

$$\mathcal{V}_1 := \bigcap_{i \neq i_1} \mathrm{Deg}_i^{-1}([d_i - \varepsilon, d_i]).$$

We then use the operator U_{i_2}. In this case, we take all the rationals ε_k to be equal to ε, except $\varepsilon_{i_1} = d_{i_1}$. In that case, $S_2 = \Sigma_{i_1} \cup \Sigma_{i_2}$. We can apply the analytic continuation theorem if the condition

$$n_{A_{i_2}} + v(\alpha_{i_2}) < (1 - 2f\varepsilon) \inf_{\sigma \in S_2}(\kappa_\sigma + \lambda_\sigma)$$

is fulfilled. But $\inf_{\sigma \in S_2}(\kappa_\sigma + \lambda_\sigma) = \inf(K_{i_1}, K_{i_2}) = K_{i_2}$ and we have, by hypothesis,

$$n_{A_{i_2}} + v(\alpha_{i_2}) < K_{i_2}.$$

If ε is small enough, then we can apply the theorem. We extend f to

$$\mathcal{V}_2 := \bigcap_{i \notin \{i_1, i_2\}} \mathrm{Deg}_i^{-1}([d_i - \varepsilon, d_i]).$$

Repeating this argument, we can extend the overconvergent form f to the whole rigid variety $X_{\mathrm{Iw,rig}}$. We conclude by applying a Koecher principle and a GAGA theorem, which proves that the space

$$H^0(X_{\mathrm{Iw,rig}}, \omega^\kappa)$$

consists of classical modular forms. This will be done in the next section.

3C2. *Compactifications and Koecher's principle.* To complete the proof of the theorem, we need to prove a Koecher principle and to introduce compactifications of the Shimura variety.

Proposition 3.19. *There exists a toroidal compactification $\overline{X}_{\mathrm{Iw}}$ of X_{Iw} defined over O_K. It is a proper scheme, and the sheaf ω^κ extends to $\overline{X}_{\mathrm{Iw}}$.*

The construction of the compactification of X has been done in [Lan 2008], and the one for X_{Iw} follows from [Lan 2015, Section 3] (see also [Bijakowski et al. 2016, Section 5.1]). One can also construct the minimal compactification X_{Iw}^* of X_{Iw}. The Koecher principle states that, under a certain condition, the sections on X_{Iw} automatically extend to the toroidal compactification. The next proposition follows from [Lan 2015, Theorem 8.7] (see also [Bijakowski et al. 2016, Section 5.2]).

Proposition 3.20. *Suppose that the codimension of the boundary of X_{Iw}^* is greater than 2. Then for any O_K-algebra R the restriction map*

$$H^0(\overline{X}_{\mathrm{Iw}} \times R, \omega^\kappa) \to H^0(X_{\mathrm{Iw}} \times R, \omega^\kappa)$$

is an isomorphism.

From this, we deduce a rigid Koecher principle (under the same dimension assumption):

$$H^0(\overline{X}_{\mathrm{Iw,rig}}, \omega^\kappa) \simeq H^0(X_{\mathrm{Iw,rig}}, \omega^\kappa),$$

where $\overline{X}_{\mathrm{Iw,rig}}$ is the rigid space associated to $\overline{X}_{\mathrm{Iw}}$. Finally, since $\overline{X}_{\mathrm{Iw}}$ is proper, we have a GAGA theorem.

Proposition 3.21. *The analytification morphism*

$$H^0(\overline{X}_{\mathrm{Iw}} \times K, \omega^\kappa) \to H^0(\overline{X}_{\mathrm{Iw,rig}}, \omega^\kappa)$$

is an isomorphism.

To conclude, we need to make explicit the dimension condition. We have the following cases:

• If there exists σ such that $a_\sigma b_\sigma = 0$, then the varieties X and X_{Iw} are compact [Lan 2008, Remark 5.3.3.2].

• If this is not the case, the codimension of the boundary of X_{Iw}^* is equal to $d(a + b - 1)$. Indeed, the dimension of the variety is equal to $\sum_\sigma a_\sigma b_\sigma$. From [Lan 2008, Theorem 7.2.4.1], there is a stratification of X_{Iw}^*, and any top-dimensional strata of the boundary is isomorphic to a Shimura variety with signatures $(a_\sigma - 1, b_\sigma - 1)$, hence has dimension $\sum_\sigma (a_\sigma - 1)(b_\sigma - 1)$.

If $d > 1$ or $a + b \geq 3$, then the condition for the Koecher principle is fulfilled.

• If $d = a = b = 1$, we cannot apply the Koecher principle.

Since we have excluded the third case by Hypothesis 1.3, it follows from what has been said before that we have an isomorphism

$$H^0(\overline{X}_{\mathrm{Iw}} \times K, \omega^\kappa) \simeq H^0(X_{\mathrm{Iw,rig}}, \omega^\kappa),$$

that is to say that the space $H^0(X_{\mathrm{Iw,rig}}, \omega^\kappa)$ consists of classical modular forms. If the variety is compact, this is only the GAGA theorem, and in the noncompact case it is a combination of the rigid Koecher principle and the GAGA theorem. This concludes the proof of the theorem.

In the exceptional remaining case, the variety is essentially a modular curve. To prove the classicality theorem, one has to take the cusps into account in the series constructed. We refer to [Kassaei 2006] for more details.

4. The case with several primes above p

In this section, we no longer assume that there is only one prime of F_0 above p. We will define the degree functions and the overconvergent modular forms, and prove the classicality result.

Let $\mathcal{P}$ be the set of primes of F_0 above p. We write $\mathcal{P} = \{\pi_1, \ldots, \pi_g\}$. Recall that we have defined the Shimura variety of Iwahori level at p

$$X_{\mathrm{Iw}} = X_{\mathrm{Iw},\pi_1} \times_X X_{\mathrm{Iw},\pi_2} \times_X \cdots \times_X X_{\mathrm{Iw},\pi_g}.$$

We will denote by $X_{\mathrm{Iw,rig}}$ the rigid space associated to the scheme X_{Iw}. Let π be an element of $\mathcal{P}$. In the previous sections, we have defined an integer s_π, integers $A_{\pi,1} < \cdots < A_{\pi,s_\pi}$, other integers $d_{\pi,1}, \ldots, d_{\pi,s_\pi}$, together with a degree function

$$\mathrm{Deg}_\pi : X_{\mathrm{Iw},\pi,\mathrm{rig}} \to \prod_{i=1}^{s_\pi} [0, d_{\pi,i}].$$

We will define the degree function on $X_{\mathrm{Iw,rig}}$ by

$$\mathrm{Deg} : X_{\mathrm{Iw,rig}} \to \prod_{\pi \in \mathcal{P}} \prod_{i=1}^{s_\pi} [0, d_{\pi,i}],$$
$$x \mapsto \left(\mathrm{Deg}_\pi(p_\pi(x)) \right)_{\pi \in \mathcal{P}}.$$

Here p_π is the projection $X_{\mathrm{Iw,rig}} \to X_{\mathrm{Iw},\pi,\mathrm{rig}}$.

Let $X_{\mathrm{Iw}}^{\mu\text{-ord-mult}}$ be the space $\mathrm{Deg}^{-1}(\{d_{\pi,i}\})$. Let us fix a weight κ.

Definition 4.1. The space of overconvergent modular forms of weight κ is defined as

$$M^\dagger := \mathrm{colim}_{\mathcal{V}} H^0(\mathcal{V}, \omega^\kappa),$$

where $\mathcal{V}$ runs over the strict neighborhoods of $X_{\mathrm{Iw}}^{\mu\text{-ord-mult}}$ in $X_{\mathrm{Iw,rig}}$.

We have defined Hecke operators $U_{\pi, A_{\pi,1}}, \ldots, U_{\pi, A_{\pi, s_\pi}}$. There is a slight ambiguity for the polarization λ'. If π is the only place above p, one can take $p \cdot \lambda$, which is a prime-to-p polarization. In general, λ' is not well defined. Let x be a totally positive element of O_F such that $v_{\pi'}(x)$ equals 1 if $\pi' = \pi$, and 0 if π' is a place of F_0 above p different from π. Then one takes for λ' the polarization $x \cdot \lambda$.

These operators act both on classical and overconvergent modular forms. We can finally state and prove the classicality theorem.

Theorem 4.2. *Let f be an overconvergent modular form. Suppose that f is an eigenform for the Hecke operators $U_{\pi, A_{\pi,i}}$, with eigenvalues $\alpha_{\pi,i}$. If we have the relations*

$$n_{\pi, A_{\pi,i}} + v(\alpha_{\pi,i}) < \inf_{\sigma \in \Sigma_{\pi,i}} (\kappa_\sigma + \lambda_\sigma)$$

for all $\pi \in \mathcal{P}$, $1 \leq i \leq s_\pi$, then f is classical.

We recall that $n_{\pi,i}$ is the constant term of the normalization factor for the Hecke operator $U_{\pi,i}$ and that $\Sigma_{\pi,i} = \{\sigma \in \Sigma_\pi : a_\sigma = A_{\pi,i}\}$.

Proof. We will use for each place π the analytic continuation theorem we got in the previous section to extend the modular form to the whole rigid variety, and deduce its classicality. Luckily, the order of the places π is not important here.

We start with an overconvergent form f. It is a section of ω^κ defined on a set of the form

$$\mathrm{Deg}^{-1}\left(\prod_{\pi \in \mathcal{P}} \prod_{i=1}^{s_\pi} [d_{\pi,i} - \varepsilon, d_{\pi,i}] \right).$$

First, we consider the place π_1. Using the Hecke operators and the relations satisfied by the eigenvalues, by the previous section one can extend f to

$$\mathrm{Deg}^{-1}\left(\prod_{i=1}^{s_{\pi_1}} [0, d_{\pi_1,i}] \times \prod_{\pi \neq \pi_1} \prod_{i=1}^{s_\pi} [d_{\pi,i} - \varepsilon, d_{\pi,i}] \right).$$

Using this argument successively with the different places, one can extend f to the whole rigid variety $X_{\mathrm{Iw,rig}}$. Using a Koecher principle (if the variety is not compact) and a GAGA theorem as in Section 3C2, one can prove that f is a classical modular form. $\square$

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s.bijakowski@imperial.ac.uk   *Department of Mathematics, Imperial College, 180 Queen's Gate, London, SW7 2AZ, United Kingdom*

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