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Bifurcations, intersections, and heights

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We prove the equivalence of dynamical stability, preperiodicity, and canonical height 0, for algebraic families of rational maps $f_t : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, parameterized by t in a quasiprojective complex variety. We use this to prove one implication in the if-and-only-if statement of a certain conjecture on unlikely intersections in the moduli space of rational maps (see "Special curves and postcritically finite polynomials", *Forum Math. Pi* **1** (2013), e3). We present the conjecture here in a more general form.

1. Introduction

Let $f : V \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be an *algebraic family* of rational maps of degree $d \geq 2$. That is, V is an irreducible quasiprojective complex variety, and f is a morphism such that $f_t := f(t, \cdot) : \mathbb{P}^1 \to \mathbb{P}^1$ has degree d for all $t \in V$. Fix a morphism $a : V \to \mathbb{P}^1$, which we view as a *marked point* on $\mathbb{P}^1$. When V is a curve, we will alternatively view f as a rational function defined over the function field $k = \mathbb{C}(V)$, with $a \in \mathbb{P}^1(k)$. In this article, we study the relation between dynamical stability of the pair (f, a), preperiodicity of the point a, and the canonical height of a (defined over the field k). In the final section, we present the general form of a conjecture on density of "special points" in this setting of dynamics on $\mathbb{P}^1$ (which includes as a special case some known statements about points on elliptic curves) — see Conjecture 6.1; compare [Baker and DeMarco 2013, Conjecture 1.10]. We finish the article with the proof of one part of the conjecture, as an application of this study of stability in algebraic families.

Stability. The pair (f, a) is said to be *stable* if the sequence of iterates

$$\{t \mapsto f_t^n(a(t))\}_{n \geq 1}$$

forms a normal family on V. (Recall that a family of holomorphic maps is normal if it is precompact in the topology of uniform convergence on compact subsets; i.e., any sequence contains a locally uniformly convergent subsequence.) The pair (f, a) is *preperiodic* if there exist integers $m > n \geq 0$ such that $f_t^m(a(t)) = f_t^n(a(t))$ for

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all $t \in V$. The pair (f, a) is *isotrivial* if there exists a branched cover $W \to V$ and an algebraic family of Möbius transformations $M : W \times \mathbb{P}^1 \to \mathbb{P}^1$ such that $M_t \circ f_t \circ M_t^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$ and $M_t(a(t)) \in \mathbb{P}^1$ are independent of t.

It is immediate from the definitions that either preperiodicity or isotriviality will imply stability. In this article, we prove the converse:

Theorem 1.1. *Let f be an algebraic family of rational maps of degree $d \geq 2$, and let a be a marked point. Suppose (f, a) is stable. Then either (f, a) is isotrivial or it is preperiodic.*

This is a generalization of [Dujardin and Favre 2008, Theorem 2.5], which itself extends [McMullen 1987, Theorem 2.2], treating the case where a is a critical point of f. In the study of complex dynamics, it is well known that a holomorphic family $f : X \times \mathbb{P}^1 \to \mathbb{P}^1$, for X any complex manifold, is dynamically stable if and only if the pair (f, c) is stable for all critical points c of f [Lyubich 1983; Mañé et al. 1983; McMullen 1994, Chapter 4]. For nonisotrivial algebraic families $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$, McMullen [1987, Lemma 2.1] proved that dynamical stability on all of V implies that all critical points are preperiodic. Combining this with Thurston's rigidity theorem, he concluded that a nonisotrivial stable family must be a family of flexible Lattès maps (i.e., covered by an endomorphism of a nonisotrivial family of elliptic curves).

Canonical height. One step in the proof of Theorem 1.1 provides an elementary geometric proof of Baker's theorem [2009, Theorem 1.6] on the finiteness of rational points with small height, for the canonical height $\hat{h}_f$ associated to the function field $\mathbb{C}(V)$, when the variety V has dimension 1. In fact, we obtain his statement under the weaker hypothesis that f is not isotrivial over $k = \mathbb{C}(V)$ (rather than assuming f is nonisotrivial over any extension of k); see [Baker 2009, Remark 1.7(i)]. The map f is *isotrivial over $k = \mathbb{C}(V)$* if there exists an algebraic family of Möbius transformations $M : V \times \mathbb{P}^1 \to \mathbb{P}^1$ such that $M_t \circ f_t \circ M_t^{-1}$ is independent of t.

Theorem 1.2. *Suppose f is a rational function defined over the function field $k = \mathbb{C}(V)$, of degree ≥ 2, and assume that V has dimension 1. Let $\hat{h}_f$ be the canonical height of f. If f is not isotrivial over k, then there exists a $b > 0$ such that the set $\{a \in \mathbb{P}^1(k) : \hat{h}_f(a) < b\}$ is finite.*

Remark 1.3. Theorem 1.2 contains as a special case the corresponding result about rational points on an elliptic curve E over k, equipped with the Néron–Tate height, generally attributed to Lang and Néron [1959]; see [Silverman 1994, Theorem III.5.4] for a proof. It was a step in proving the Mordell–Weil theorem for function fields. (To treat this case, we project E to $\mathbb{P}^1$ and let f be the rational function induced by multiplication-by-2 on E.) In this setting, one can say more: the set of points in $\mathbb{P}^1(k)$ with height $< b$ that lift to rational points in $E(k)$ is finite for *all* $b > 0$; see [Baker 2009, Theorem B.9], where this is deduced from the conclusion of Theorem 1.2.

The canonical height $\hat{h}_f$ was introduced in [Call and Silverman 1993], and it satisfies $\hat{h}_f(f(a)) = d\hat{h}_f(a)$ when f has degree d. So Theorem 1.2 implies that rational points of height 0 are preperiodic unless f is isotrivial over k. This was proved for polynomials f in [Benedetto 2005]. When k is a number field, this was observed in [Call and Silverman 1993], and the conclusion of Theorem 1.2 holds for all $b > 0$. In the function field setting, the conclusion of Theorem 1.2 *cannot* hold for all $b > 0$, since, for example, the union of all constant points $a \in \mathbb{P}^1(\mathbb{C})$ will form an infinite set of bounded canonical height for any f. On page 1040, we provide explicit examples of functions f for which we compute the sharp bound b and the total number of rational preperiodic points. Also, note that examples do exist of rational functions $f \in k(z)$ that are isotrivial but *not* isotrivial over $k = \mathbb{C}(V)$. A necessary condition is a nontrivial automorphism group of f; see Example 2.2.

Combining Theorem 1.1 with Theorem 1.2, we have:

Theorem 1.4. *Suppose* $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ *is a nonisotrivial algebraic family of rational maps, where* V *has dimension* 1. *Let* $\hat{h}_f : \mathbb{P}^1(\bar{k}) \to \mathbb{R}$ *be a canonical height of* f, *defined over the function field* $k = \mathbb{C}(V)$. *For each* $a \in \mathbb{P}^1(\bar{k})$, *the following are equivalent*:

(1) *The pair* (f, a) *is stable.*

(2) $\hat{h}_f(a) = 0$.

(3) (f, a) *is preperiodic.*

Moreover, the set $\{a \in \mathbb{P}^1(k) : (f, a)$ *is stable*$\}$ *is finite.*

Application to intersection theory. Combining Theorem 1.1 with Montel's theorem on normal families, we obtain another argument for the "easy" implication of the Masser–Zannier theorems [2010; 2012] on anomalous torsion for elliptic curves.

Proposition 1.5. *Suppose that* E *is any nonisotrivial elliptic curve defined over a function field* $k = \mathbb{C}(V)$, *where* V *has dimension* 1, *and let* P *be a point of* $E(k)$. *Then the set of* $t \in V$ *for which the point* P_t *is torsion on* E_t *is infinite.*

The harder part of the Masser–Zannier theorems is the following statement: if two points P and Q in $E(k)$ are independent on E and neither is torsion, meaning that they do not satisfy a relation of the form $mP + nQ = 0$ with integers m and n not both zero, then the set of $t \in V$ for which P_t and Q_t are both torsion on E_t is finite. (In [DeMarco et al. 2016], we gave a dynamical proof of this harder implication for the Legendre family E_t.)

The theorems of [Masser and Zannier 2012], followed by a series of analogous results in the dynamical setting (e.g., [Baker and DeMarco 2011; Ghioca et al.

2013; 2015; DeMarco et al. 2015]), led to the development of a general conjecture about rational maps and marked points — addressing a question first posed by Umberto Zannier, but also encompassing a case of intrinsic dynamical interest, where the marked points are the critical points of the map. A precise statement of this conjecture appears as Conjecture 6.1 in Section 6. In [Baker and DeMarco 2013], we formulated the conjecture in the setting of marked critical points; an error in one of our definitions is corrected here (Remark 6.3). In this article, we use Theorem 1.1 to give a proof of one implication of the more general statement of Conjecture 6.1. This implication reduces to proving the following statement.

Every algebraic family $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ induces a (regular) projection $V \to \mathrm{M}_d$ from the parameter space to the moduli space M_d of conformal conjugacy classes of maps. We say the family f has *dimension N in moduli* if the image of V under this projection has dimension N in M_d. Since V is irreducible, f has dimension 0 in moduli if and only if f is isotrivial.

Theorem 1.6. *Let $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$, of dimension $N > 0$ in moduli. Let $a_1, \dots, a_k$, with $k \leq N$, be any marked points. Then the set*

$$S(a_1, \dots, a_k) = \bigcap_{i=1}^{k} \{t \in V : a_i(t) \textit{ is preperiodic for } f_t\}$$

is Zariski-dense in V.

Remark 1.7. Conjecture 6.1 asserts that the set $S(a_1, \dots, a_k)$ with $k > N$ will be Zariski-dense if and only if at most N of the points $a_1, \dots, a_k$ are dynamically "independent" on V.

Idea of the proof of Theorem 1.1. The proofs of [McMullen 1987, Theorem 2.2] and [Dujardin and Favre 2008, Theorem 2.5] use crucially that the point is critical. The first ingredient of our proof is similar to their proofs, building upon the fact that there are only finitely many nonconstant morphisms from a quasiprojective algebraic curve V to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. This leads to the proof of Theorem 1.2. As a special case of Theorem 1.2, we have:

Proposition 1.8. *Let $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$, and assume that V has complex dimension 1. Fix a marked point $a : V \to \mathbb{P}^1$, and define $g_n : V \to \mathbb{P}^1$ by $g_n(t) = f_t^n(a(t))$. If f is not isotrivial, and if the degrees of $\{g_n\}$ are bounded, then there exist integers $m > n \geq 0$ such that*

$$f_t^m(a(t)) = f_t^n(a(t)) \quad \textit{for all } t \in V.$$

Remark 1.9. If f is isotrivial, then the degrees of $g_n(t) = f_t^n(a(t))$ are bounded if and only if the pair (f, a) is isotrivial. (See Proposition 2.3.)

For Theorem 1.1, it suffices to treat the case where V has complex dimension 1 and so is a finitely punctured Riemann surface. Then each $g_n(t) = f_t^n(a(t))$ extends to a holomorphic map on the compactification of V. In light of Proposition 1.8, it remains to prove that stability on V implies the degrees of $\{g_n\}$ are bounded.

The second ingredient of the proof is a study of normality and escape rates near the punctures of V. The arguments given in Section 3 are inspired by the methods of [DeMarco et al. 2015] and [DeMarco et al. 2016].

From a geometric point of view, the idea to show that the degrees of $\{g_n\}$ are bounded is as follows. Let X denote (the normalization of) a compactification of V. Under iteration, one would typically expect that $\deg g_n \approx d \deg g_{n-1}$, where $d = \deg f$. Viewing g_n as a curve in $X \times \mathbb{P}^1$ (by identifying the function with its graph), the expected degree growth fails when the graph of g_n passes through the indeterminacy points of $(t, z) \mapsto (t, f_t(z))$ and thus its image contains "vertical components" over the punctures of V. Lemma 3.3 shows that the multiplicity of the vertical component in the image of any curve is uniformly bounded by some integer q. On the other hand, normality on V implies that the graphs of g_n are converging over compact subsets of V. Therefore, if $\deg g_n \to \infty$, the graph of g_n must be fluctuating wildly near the punctures of V when n is large (Lemma 4.1). But this fluctuation is controlled by Proposition 3.1.

2. Isotriviality and Theorem 1.2

Throughout this section, we assume that V is an irreducible quasiprojective complex variety of dimension 1; i.e., V is obtained from a compact Riemann surface by removing finitely many points. Let $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$. We prove Theorem 1.2. We also prove Proposition 2.3, to provide a characterization of isotriviality which is used in the proof of Theorem 1.1; it is not needed for the proof of Theorem 1.2.

Isotrivial maps. By definition, f is isotrivial if there exists a family $\{M_t\}$ of Möbius transformations, regular over a branched cover $p : W \to V$, such that $M_t \circ f_{p(t)} \circ M_t^{-1}$ is constant in t. For a marked point $a : V \to \mathbb{P}^1$, the pair (f, a) is isotrivial if, in addition, the function $M_t(a(p(t)))$ is constant on W. (Note that this is well defined, even if the family M_t is not uniquely determined.) The map f (or the pair (f, a)) is isotrivial over $k = \mathbb{C}(V)$ if M can be chosen to be an algebraic family that is regular on V.

Lemma 2.1. *Let V have dimension* 1. *If (f, a) is isotrivial, and if $\{a, f(a), f^2(a)\}$ is a set of three distinct functions on V, then (f, a) is isotrivial over k.*

Proof. Suppose (f, a) is isotrivial. Let $M : W \times \mathbb{P}^1 \to \mathbb{P}^1$ be an algebraic family of Möbius transformations, with branched cover $p : W \to V$, such that $R =$

$M_w \circ f_{p(w)} \circ M_w^{-1} : \mathbb{P}^1 \to \mathbb{P}^1$ and $b = M_w(a(p(w)))$ are independent of $w \in W$. By removing finitely many points from V, we may assume that $p : W \to V$ is a covering map.

Fix a basepoint $t_0 \in V$ and choose a point $w_0 \in p^{-1}(t_0)$. The choices of basepoint determine a representation

$$\rho_f : \pi_1(V, t_0) \to \mathrm{Aut}(f_{t_0}) \subset \mathrm{PSL}_2(\mathbb{C})$$

that is trivial if and only if (f, a) is isotrivial over k. Indeed, choose any $\gamma \in \pi_1(V, t_0)$ and let $\eta : [0, 1] \to W$ be a lift of γ with $\eta(0) = w_0$. Write w_t for $\eta(t)$. Then the equality $f_{p(w_0)} = f_{p(w_1)} = f_{t_0}$ and isotriviality imply that $M_{w_0} f_{t_0} M_{w_0}^{-1} = M_{w_1} f_{t_0} M_{w_1}^{-1}$, so $\rho_f(\gamma) := M_{w_1}^{-1} M_{w_0}$ is an automorphism of f_{t_0}. The triviality of ρ_f is equivalent to the statement that $M_{w_0} = M_{w_1}$ for all such paths, so that M descends to a regular map $M : V \times \mathbb{P}^1 \to \mathbb{P}^1$.

Now choose the basepoint t_0 so that $\{a(t_0), f_{t_0}(a(t_0)), f_{t_0}^2(a(t_0))\}$ are three distinct points in $\mathbb{P}^1$. Fix any $\gamma \in \pi_1(V, t_0)$. For each $n \geq 0$, we have

$$\begin{aligned}\rho_f(\gamma)(f_{t_0}^n(a(t_0))) &= f_{t_0}^n(\rho_f(\gamma)(a(t_0))) = f_{t_0}^n M_{w_1}^{-1} M_{w_0}(a(p(w_0))) \\ &= f_{t_0}^n M_{w_1}^{-1}(b) = f_{t_0}^n(a(p(w_1)) = f_{t_0}^n(a(t_0)),\end{aligned}$$

so the full orbit of $a(t_0)$ under f_{t_0} lies in the fixed set of $\rho_f(\gamma)$. By the assumption on a we deduce that $\rho_f(\gamma)$ is the identity. Therefore, (f, a) is isotrivial over k. □

Example 2.2. Consider the rational function $f_1(z) = z + 1/z$ or the cubic polynomial $P_1(z) = z^3 - 3z$. Both of these functions have $z \mapsto -z$ as an automorphism. Conjugating by $M_t(z) = tz$ and setting $s = t^2$, we see that the families

$$f_s(z) = z + \frac{1}{sz} \quad \text{and} \quad P_s(z) = sz^3 - 3z$$

are isotrivial over a degree-2 extension of $k = \mathbb{C}(s)$. On the other hand, neither f nor P is isotrivial over k. This can be seen by computing the critical points ($= \pm\sqrt{1/s}$ in both examples), and observing that the critical points are interchanged by a nontrivial loop in $V = \mathbb{C} \setminus \{0\}$.

Proposition 2.3. *Suppose V has dimension 1 and f is isotrivial. Let $a : V \to \mathbb{P}^1$ be any marked point. The following are equivalent*:

(1) *The pair (f, a) is isotrivial.*

(2) *The pair (f, a) is stable.*

(3) *The degrees of $g_n(t) = f_t^n(a(t))$ are bounded.*

Proof. Since f is isotrivial, there exist a finite branched cover $p : W \to V$, an algebraic family of Möbius transformations $M : W \times \mathbb{P}^1 \to \mathbb{P}^1$, and a map $R : \mathbb{P}^1 \to \mathbb{P}^1$ such that $M_t \circ f_{p(t)} \circ M_t^{-1} = R$ for all $t \in W$. Set $s = p(t)$.

If the pair (f, a) is isotrivial, then $b = M_t(a(s))$ is also independent of t, so the degrees of $g_n(t) = f_s^n(a(s)) = M_t^{-1}(R^n(b))$ are clearly bounded. Thus (1) implies (3). In addition, note that any sequence in the set $\{R^n(b)\}_{n\geq 1} \subset \mathbb{P}^1$ has a convergent subsequence. This implies the normality of $\{M_t^{-1}(R^n(b))\}_{n\geq 1}$ on W which in turn implies the normality of $\{f_t^n(a(t))\}_n$ on V. Thus (1) implies (2).

Now suppose that (f, a) is not isotrivial, so that $b(t) = M_t(a(s))$ is nonconstant on W. We observe first that the degrees of $\{g_n\}$ must be unbounded, showing that (3) implies (1). Indeed, since b is nonconstant, it extends to a surjective map of finite degree from a compactification $\overline{W}$ to $\mathbb{P}^1$. Choose any point $z_0 \in \mathbb{P}^1$ which is nonexceptional for R, that is, such that the set of preimages $R^{-n}(z_0)$ is growing in cardinality as $n \to \infty$. For any $D > 0$, choose n so that the size of the set $R^{-n}(z_0)$ is larger than D. Then there is a set $P \subset \overline{W}$ of cardinality $|P| \geq D$ such that $b(t)$ is in $R^{-n}(z_0)$ for all $t \in P$. Then $R^n(b(t)) = z_0$ for all $t \in P$. Taking D as large as desired, this shows that the degrees of $\{t \mapsto R^n(b(t))\}_n$ are unbounded. This in turn implies that the degrees of $g_n(t) = M_t^{-1}(R^n(b(t)))$ are unbounded.

Continuing to assume that (f, a) is not isotrivial, we also see that $b(t) = M_t(a(s))$ has only finitely many critical points in W and the image $b(W)$ omits at most finitely many points in $\mathbb{P}^1$. The map R has infinitely many repelling cycles in its Julia set, so there must exist a $t_0 \in W$ such that $b'(t_0) \neq 0$ and $b(t_0)$ is a repelling periodic point of R for all $t \in P$. It follows that the sequence of derivatives $\frac{\partial}{\partial t} R^n(b(t))|_{t=t_0}$ is unbounded; so the sequence $\{R^n(b(t)) = M_t(f_s^n(a(s)))\}_n$ cannot be a normal family on all of W. Therefore, $\{g_n\}_n$ also fails to be normal, and so we have proved that (2) implies (1). □

Finiteness of nonconstant maps. As in the proofs of Theorem 2.2 (and specifically Proposition 4.3) in [McMullen 1987] and of Theorem 2.5 in [Dujardin and Favre 2008], we will need the following statement for our proof of Theorem 1.2.

Lemma 2.4. *Let Λ be any quasiprojective, complex algebraic curve. There are only finitely many nonconstant holomorphic maps from Λ to the triply punctured sphere $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The bound depends only on the Euler characteristic $\chi(\Lambda)$.*

Proof. Any holomorphic map $h : \Lambda \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ extends to a meromorphic function on a (smooth) compactification X of Λ. From Riemann–Hurwitz, the degree of h is bounded by the Euler characteristic $-\chi(\Lambda)$. Any meromorphic function on X is determined by its zeros, poles, and ones; indeed, the ratio of two functions h_1 and h_2 with the same zeros and poles must be constant on X, and if $h_1(x) = 1 = h_2(x)$ for some x, then $h_1 \equiv h_2$. Thus, there are only finitely many combinatorial possibilities for h. □

Proof of Theorem 1.2. Let $d = \deg f \geq 2$. The canonical height $\hat{h}_f(a)$ computes the growth rate of the degrees of $t \mapsto f_t^n(a(t))$ as $n \to \infty$. Precisely, each $a \in \mathbb{P}^1(\bar{k})$

determines a meromorphic function $W \to \mathbb{P}^1$ on a branched cover $p_a : W \to V$, where the topological degree of p_a coincides with the algebraic degree of the field extension $k(a)$ over k. The canonical height is computed as

$$\hat{h}_f(a) = \frac{1}{\deg p_a} \lim_{n\to\infty} \frac{1}{d^n} \deg g_n$$

for the maps $g_n : W \to \mathbb{P}^1$ defined by $g_n(s) = f^n_{p_a(s)}(a(s))$. This height function $\hat{h}_f$ is characterized by two conditions [Call and Silverman 1993, Theorem 1.1]:

(1) The difference $|\hat{h}_f(a) - \deg a/\deg p_a|$ is uniformly bounded on $\mathbb{P}^1(\bar{k})$.

(2) $\hat{h}_f(f(a)) = d\hat{h}_f(a)$ for all $a \in \mathbb{P}^1(\bar{k})$.

In particular, the degrees of $\{g_n\}$ are growing to infinity if and only if the canonical height of a is positive.

Suppose there is a sequence of rational points $a_m \in \mathbb{P}^1(k)$, $m \geq 1$, such that

$$1 > \hat{h}_f(a_m) \to 0$$

as $m \to \infty$. For each point $a \in \mathbb{P}^1(k)$, the *length* of the orbit of a is defined to be the cardinality of the set $\{f^n(a) : n \geq 0\}$ in $\mathbb{P}^1(k)$. Suppose further that only finitely many of the a_m have infinite orbit, and that the finite orbit lengths are uniformly bounded. In this case, all but finitely many of the a_m satisfy a finite number of equations of the form $f^n(a) = f^\ell(a)$ with $n \neq \ell$; thus the set $\{a_m\}$ will be finite. If this holds for any such sequence, then the theorem is proved.

We can assume, therefore, that the orbit lengths of the a_m are tending to infinity with m or are equal to infinity for all m.

From property (1) of the height function, there exists a degree D such that $\deg a \geq D$ with $a \in \mathbb{P}^1(k)$ implies $\hat{h}_f(a) \geq 1$. For each m, choose an integer $N_m \geq 0$ so that the orbit of a_m has length greater than N_m and so that

$$\deg f^i(a_m) \leq D$$

for all $i \leq N_m$. Property (2) of the height function and the condition $\hat{h}_f(a_m) \to 0$ imply that we may take $N_m \to \infty$ as $m \to \infty$. We will deduce that f must be isotrivial over k.

Suppose now that f_t has at least three distinct fixed points for general $t \in V$. Remove the finitely many parameters in V where these three fixed points have collisions. Then there exists a branched cover $p : W \to V$ and an algebraic family of Möbius transformations $M : W \times \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$R_t = M_t \circ f_{p(t)} \circ M_t^{-1}$$

has its fixed points at 0, 1, ∞ for all $t \in W$.

Set $b_m(t) = M_t(a_m(p(t)))$ for each m, so that $R^i(b_m) = M(f^i(a_m))$ for all iterates. The uniform bound of D on the degrees of $\bigcup_m \{f^i(a_m) : i \leq N_m\}$ implies that there is a uniform bound of D' on the degrees of $\bigcup_m \{R^i(b_m) : i \leq N_m\}$. For each point b_m, we define

$$S_{0,m}(n) = \{t \in W : R_t^n(b_m(t)) = 0\},$$
$$S_{1,m}(n) = \{t \in W : R_t^n(b_m(t)) = 1\},$$
$$S_{\infty,m}(n) = \{t \in W : R_t^n(b_m(t)) = \infty\}.$$

Since $\{0, 1, \infty\}$ are fixed points of R_t for all t, we have

$$S_{0,m}(n) \subset S_{0,m}(n+1)$$

for all m and all n; similarly for $S_{1,m}(n)$ and $S_{\infty,m}(n)$. Let

$$S_m = S_{0,m}(N_m) \cup S_{1,m}(N_m) \cup S_{\infty,m}(N_m).$$

For each m and each $n \leq N_m$, the iterate $R^n(b_m)$ determines a holomorphic map

$$W \setminus S_m \longrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}.$$

By construction, the degree of $R^{N_m}(b_m)$ is bounded by D', so we have $|S_m| \leq 3D'$ for all m. Therefore, there is a uniform bound B (independent of m) on the number of nonconstant maps from $W \setminus S_m$ to the triply punctured sphere $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (Lemma 2.4). In other words, for each m, at most B of the first N_m iterates of b_m are nonconstant. Therefore, since $N_m \to \infty$, there exists an m_0 such that b_{m_0} has at least $2d + 2$ consecutive iterates that are constant and distinct.

Lemma 2.5. *Suppose A and B are rational functions of degree d such that we have $A(x_i) = B(x_i)$ for a sequence of $2d + 1$ distinct points $x_1, \ldots, x_{2d+1}$. Then $A = B$.*

Proof. By postcomposing A and B with a Möbius transformation, we may assume that the values $A(x_i)$ and $B(x_i)$ are finite for each i. Consider the difference $F = A - B$. Then F is a rational function of degree $\leq 2d$. But F vanishes in at least $2d + 1$ distinct points, so $F \equiv 0$. □

Set $x_i = R^i(b_{m_0})$ for the consecutive indices i for which x_i is constant in t. Then $R_t(x_i) = x_{i+1}$ for all t along a set of $2d + 1$ distinct points $x_i \in \mathbb{P}^1(\mathbb{C})$. Applying Lemma 2.5, we conclude that the rational function R_t of degree d is independent of t. In other words, f is isotrivial.

It remains to show that f is in fact isotrivial over k, but this follows from Lemma 2.1. Indeed, for the point b_{m_0} in the preceding paragraph, we had $x_i = R^i(b_{m_0})$ independent of t. In other words, the pair $(f, f^i(a_{m_0}))$ is isotrivial. In addition, the orbit length of $f^i(a_{m_0})$ is at least $2d + 1 \geq 3$. Lemma 2.1 states that the pair $(f, f^i(a_{m_0}))$ must be isotrivial over k, so f itself is isotrivial over k.

Finally, suppose that f_t has only 1 or 2 fixed points, generally in V. For a general parameter t_0, choose a forward-invariant set of fixed points and preimages, consisting of at least three distinct points. Pass to a branched cover $W \to V$ on which these points can be marked holomorphically, excluding the finitely many points where collisions occur. For each of these three points, we define the sets $S_{i,m}(n)$ as above. If the i-th point is mapped to the j-th point by f_t, then $S_i(n) \subset S_j(n+1)$ for all n. The rest of the proof goes through exactly the same. This completes the proof of Theorem 1.2.

Two examples: computing canonical height and the number of rational preperiodic points. Consider $Q_t(z) = z^2 + t$, the family of quadratic polynomials. This family defines a nonisotrivial rational function Q over the function field $k = \mathbb{C}(t)$. There is a unique point in $\mathbb{P}^1(k)$ with finite orbit for Q, namely the point $a = \infty$. Indeed, writing $a(t) = a_1(t)/a_2(t)$ for $a \in \mathbb{P}^1(k) \setminus \{\infty\}$, we can compute explicitly that

$$\deg(Q(a)) = \begin{cases} 2 \deg a & \text{if } \deg a_1 > \deg a_2, \\ 2 \deg a + 1 & \text{if } \deg a_1 \le \deg a_2. \end{cases}$$

In both cases, the image $Q(a)$ will satisfy the hypothesis of the first case. Inductively then, we have $\deg Q^n(a) = 2^{n-1} \deg Q(a)$. Consequently, the largest possible b in the statement of Theorem 1.2 is $b = \frac{1}{2}$, since the set $\left\{a \in \mathbb{P}^1(k) : \hat{h}_Q(a) = \frac{1}{2}\right\}$ is precisely the constant points $a \in \mathbb{C}$, while

$$\left|\left\{a \in \mathbb{P}^1(k) : \hat{h}_Q(a) < \tfrac{1}{2}\right\}\right| = 1.$$

As a second example, consider the family of flexible Lattès maps,

$$L_t(z) = \frac{(z^2 - t)^2}{4z(z-1)(z-t)},$$

defining a nonisotrivial L over the field $k = \mathbb{C}(t)$. This family is the quotient of the endomorphism $P \mapsto P + P$ on the Legendre family of elliptic curves $E_t = \{y^2 = x(x-1)(x-t)\}$. (See also the beginning of Section 5, where this is discussed further.) For this example, Proposition 1.4 of [DeMarco et al. 2016] shows there are exactly 4 rational preperiodic points, namely $\{0, 1, t, \infty\}$. We also explicitly computed the height of any starting point $a \in \mathbb{P}^1(k)$ in Proposition 3.1 of the same work. The constant points $a \in \mathbb{C} \setminus \{0, 1\}$ form an infinite set of points of canonical height $\frac{1}{2}$, while

$$\left|\left\{a \in \mathbb{P}^1(k) : \hat{h}_L(a) < \tfrac{1}{2}\right\}\right| = 4.$$

Again, $b = \frac{1}{2}$ is the largest possible constant in the statement of Theorem 1.2.

3. Escape rate at a degenerate parameter

In this section, we construct a "good" escape-rate function associated to a pair of holomorphic maps $f : \mathbb{D}^* \times \mathbb{P}^1 \to \mathbb{P}^1$ and $a : \mathbb{D}^* \to \mathbb{P}^1$, on the punctured unit disk $\mathbb{D}^* = \{t \in \mathbb{C} : 0 < |t| < 1\}$. The construction follows that of [DeMarco et al. 2015; 2016]. I am indebted to Hexi Ye for his assistance in the proof of Proposition 3.1.

The setting. Throughout this section, we work in homogeneous coordinates on $\mathbb{P}^1$. We assume we are given a family of homogeneous polynomial maps

$$F_t : \mathbb{C}^2 \to \mathbb{C}^2,$$

of degree $d \geq 2$, parameterized by $t \in \mathbb{D} = \{t \in \mathbb{C} : |t| < 1\}$, such that the coefficients of F_t are holomorphic in t. Each F_t is given by a pair of homogeneous polynomials (P_t, Q_t), and we define $\mathrm{Res}(F_t)$ to be the homogeneous resultant of the polynomials P_t and Q_t. Recall that the resultant is a polynomial function of the coefficients of F_t, vanishing if and only if P_t and Q_t share a root in $\mathbb{P}^1$. See [Silverman 2007, §2.4] for more information. We assume further that $\mathrm{Res}(F_t) = 0$ if and only if $t = 0$, and also that at least one coefficient of F_0 is nonzero.

We use the norm

$$\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\} \quad \text{on } \mathbb{C}^2.$$

The escape-rate function. Let $A : \mathbb{D} \to \mathbb{C}^2 \setminus \{(0, 0)\}$ be any holomorphic map. Write A_t for $A(t)$.

For each $n \geq 0$, the iterate $F_t^n(A_t)$ is a pair of holomorphic functions in t; we define

$$a_n = \mathrm{ord}_{t=0}\, F_t^n(A_t)$$

to be the minimum of the order of vanishing of the two coordinate functions at $t = 0$; so $a_0 = 0$ and a_n is a nonnegative integer for all $n \geq 1$. Set

$$F_n(t) = t^{-a_n} F_t^n(A_t)$$

so that F_n is a holomorphic map from $\mathbb{D}$ to $\mathbb{C}^2 \setminus \{(0, 0)\}$ for each n. Our main goal in this section is to prove the following statement.

Proposition 3.1. *The functions*

$$G_n(t) = \frac{1}{d^n} \log \|F_n(t)\|$$

converge locally uniformly on the punctured disk $\mathbb{D}^*$ to a continuous function G satisfying

$$G(t) = o(\log|t|) \quad \text{as } t \to 0.$$

Remark 3.2. In [DeMarco et al. 2015; 2016], we used explicit expressions for F_t to deduce that the function G was *continuous* at $t = 0$ for our examples. It remains an interesting open question to determine necessary and sufficient conditions for the functions G_n to converge uniformly to a continuous function G on a neighborhood of $t = 0$.

Order of vanishing. Let F_t and $\mathrm{Res}(F_t)$ be defined as in page 1041.

Lemma 3.3. *Let* $q = \mathrm{ord}_{t=0} \mathrm{Res}(F_t)$. *There are constants* $0 < \alpha < 1 < \beta$ *and* $\delta > 0$ *such that*

$$\alpha |t|^q \leq \frac{\|F_t(z_1, z_2)\|}{\|(z_1, z_2)\|^d} \leq \beta,$$

for all $(z_1, z_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ *and* $0 < |t| < \delta$.

Proof. The statement of Lemma 3.3 is essentially the content of [Baker and Rumely 2010, Lemma 10.1], letting k be the field of Laurent series in t, equipped with the nonarchimedean valuation measuring the order of vanishing at $t = 0$. But to obtain our estimate with the Euclidean norm, we work directly with their proof.

By the homogeneity of F_t, it suffices to prove the estimate assuming $\|(z_1, z_2)\| = 1$ with either $z_1 = 1$ or $z_2 = 1$. The upper bound is immediate from the presentation of F, with bounded coefficients on compact subsets of $\mathbb{D}$.

Write $F = (F_1(x, y), F_2(x, y))$. The resultant $\mathrm{Res}(F)$ is a nonzero element of the valuation ring $\mathcal{O}_k = \{z \in k : \mathrm{ord}_{t=0} z \geq 0\}$. From basic properties of the resultant (e.g., [Silverman 2007, Proposition 2.13]), there exist polynomials $g_1, g_2, h_1, h_2 \in \mathcal{O}_k[x, y]$ such that

$$g_1(x, y)F_1(x, y) + g_2(x, y)F_2(x, y) = \mathrm{Res}(F)x^{2d-1} \tag{3-1}$$

and

$$h_1(x, y)F_1(x, y) + h_2(x, y)F_2(x, y) = \mathrm{Res}(F)y^{2d-1}. \tag{3-2}$$

Setting $x = 1$, equation (3-1) shows that

$$\min\{\mathrm{ord}\, F_1(1, z_2), \mathrm{ord}\, F_2(1, z_2)\} \leq q$$

for any choice of $z_2 \in \mathcal{O}_k$. In fact, taking

$$M = 2 \max\Big\{\sup\Big\{|g_1(1, y)| : |t| \leq \tfrac{1}{2}, |y| \leq 1\Big\}, \sup\Big\{|g_2(1, y)| : |t| \leq \tfrac{1}{2}, |y| \leq 1\Big\}\Big\}$$

we may find $\alpha_1 > 0$ and $0 < \delta_1 < \frac{1}{2}$ such that

$$\min\{\inf\{|F_1(1, z_2)| : |z_2| \leq 1\}, \inf\{|F_2(1, y)| : |z_2| \leq 1\}\} \geq |\mathrm{Res}(F)|/M \geq \alpha_1 |t|^q$$

for all $|t| < \delta_1$.

Similarly, setting $y = 1$ in equation (3-2), we may define the analogous α_2 and δ_2 to estimate $F_1(z_1, 1)$ and $F_2(z_1, 1)$ for any $|z_1| \leq 1$; the conclusion follows by setting $\alpha = \min\{\alpha_1, \alpha_2\}$ and $\delta = \min\{\delta_1, \delta_2\}$. □

Proof of Proposition 3.1. It is a standard convergence argument in complex dynamics that the functions $d^{-n}\log\|F_t^n(A_t)\|$ converge locally uniformly in the region where $\mathrm{Res}(F_t)\neq 0$, exactly as in [Hubbard and Papadopol 1994], [Fornæss and Sibony 1994], or [Branner and Hubbard 1988]. To see that the functions G_n converge locally uniformly on $\mathbb{D}^*$, we must look at the growth of the orders $\{a_n\}$ as $n\to\infty$. From the definition of a_n, we have $a_0=0$ and

$$a_{n+1}=da_n+\mathrm{ord}_{t=0}\,F_t(F_n(t)) \tag{3-3}$$

for all n. Hence by Lemma 3.3, noting that $z=F_n(t)$ has norm bounded away from both 0 and ∞ as $t\to 0$, we find that

$$0\le k_{n+1}:=a_{n+1}-d\cdot a_n\le q.$$

Consequently, the sequence $a_n/d^n=\sum_{i=1}^n k_i/d^i$ has a finite limit. In particular, we may conclude that the sequence

$$G_n(t)=\frac{1}{d^n}\log\|F_n(t)\|=\frac{1}{d^n}\log\|F_t^n(A_t)\|-\frac{a_n}{d^n}\log|t|$$

converges locally uniformly (to $G(t)$) in the punctured unit disk.

To show that $G(t)=o(\log|t|)$ it suffices to show that, for any $\varepsilon>0$, there is a constant C and a $\delta>0$ such that

$$|G(t)|\le\varepsilon|\log|t||+C$$

for all t in the disk of radius δ.

Fix a positive integer N, and define

$$b_n:=\mathrm{ord}_{t=0}\,F_t^{n-N}(F_N(t))$$

for $n\ge N$, so that $b_N=0$ and

$$0\le \ell_{n+1}:=b_{n+1}-d\cdot b_n\le q$$

by Lemma 3.3. In particular, we have

$$\frac{b_n}{d^n}=\sum_{i=N+1}^{n}\frac{\ell_i}{d^i}\le\sum_{i=N+1}^{\infty}\frac{q}{d^i}$$

for all $n>N$. By increasing N if necessary, we can assume that

$$\sum_{i=N+1}^{\infty}\frac{q}{d^i}<\varepsilon.$$

Therefore (recalling the constants $0 < \alpha < 1$ and $\delta > 0$ from Lemma 3.3),

$$\begin{aligned}
&\frac{1}{d^n}\log\|F_n(t)\| + \frac{b_n}{d^n}\log|t| \\
&\quad = \frac{1}{d^n}\log\|F_t^{n-N}(F_N(t))\| \\
&\quad = \sum_{i=1}^{n-N}\left(\frac{1}{d^{i+N}}\log\|F_t^i(F_N(t))\| - \frac{1}{d^{i+N-1}}\log\|F_t^{i-1}(F_N(t))\|\right) + \frac{1}{d^N}\log\|F_N(t))\| \\
&\quad = \sum_{i=1}^{n-N}\frac{1}{d^{i+N}}\log\frac{\|F_t^i(F_N(t))\|}{\|F_t^{i-1}(F_N(t))\|^d} + \frac{1}{d^N}\log\|F_N(t))\| \\
&\quad \geq \sum_{i=1}^{n-N}\frac{1}{d^{i+N}}(\log|t|^q + \log\alpha) + \frac{1}{d^N}\log\|F_N(t))\| \\
&\quad \geq \varepsilon\log|t| + \frac{1}{d^N}\log\|F_N(t))\| + \sum_{i=N}^{\infty}\frac{1}{d^i}\log\alpha.
\end{aligned}$$

Let $C = \sup\left|\log\|F_N(t))\|/d^N\right|$ for t in the disk of radius $\frac{1}{2}$. Then

$$\frac{1}{d^n}\log\|F_n(t)\| \geq \varepsilon\log|t| - C + \sum_{i=N}^{\infty}\frac{1}{d^i}\log\alpha - \frac{b_n}{d^n}\log|t| \geq \varepsilon\log|t| - C + \sum_{i=N}^{\infty}\frac{1}{d^i}\log\alpha,$$

for all $|t| < \delta$ and all $n \geq N$.

For the reverse estimate, we have

$$\begin{aligned}
\frac{1}{d^n}\log\|F_n(t)\| + \frac{b_n}{d^n}\log|t| &= \sum_{i=1}^{n-N}\frac{1}{d^{i+N}}\log\frac{\|F_t^i(F_N(t))\|}{\|F_t^{i-1}(F_N(t))\|^d} + \frac{1}{d^N}\log\|F_N(t))\| \\
&\leq \sum_{i=N}^{\infty}\frac{1}{d^i}\log\beta + \frac{1}{d^N}\log\|F_N(t))\|,
\end{aligned}$$

where $\beta > 1$ is the constant from Lemma 3.3. With the same C as above, we conclude that

$$\frac{1}{d^n}\log\|F_n(t)\| \leq -\frac{b_n}{d^n}\log|t| + \sum_{i=N}^{\infty}\frac{1}{d^i}\log\beta + C \leq -\varepsilon\log|t| + \sum_{i=N}^{\infty}\frac{1}{d^i}\log\beta + C$$

for all $|t| < \delta$ and all $n \geq N$. Passing to the limit as $n \to \infty$, we conclude that $G(t) = o(\log|t|)$ for t near 0. This concludes the proof of Proposition 3.1.

Stability. We now gather some consequences of Proposition 3.1 that will be used in the proof of Theorem 1.1. Let F_t be given as on page 1041, so it induces a family of rational maps f of degree d, parameterized by the punctured disk $\mathbb{D}^*$. Let $a : \mathbb{D} \to \mathbb{P}^1$ be a holomorphic map with a holomorphic lift $A : \mathbb{D} \to \mathbb{C}^2 \setminus \{(0, 0)\}$.

We define the functions F_n and G_n as on page 1041, with

$$G(t) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F_n(t)\|.$$

Recall that the pair (f, a) is stable on $\mathbb{D}^*$ if the sequence of holomorphic functions $\{g_n(t) := f_t^n(a(t))\}$ forms a normal family on $\mathbb{D}^*$.

Corollary 3.4. *Suppose the pair (f, a) is stable on the punctured disk $\mathbb{D}^*$. Then there exists a choice of holomorphic lift $A : \mathbb{D} \to \mathbb{C}^2 \setminus \{(0, 0)\}$ of a such that $G \equiv 0$.*

Proof. Stability of (f, a) on $\mathbb{D}^*$ implies that G is harmonic where $t \neq 0$ for any choice of holomorphic lift A of a [DeMarco 2003, Theorem 9.1]. Indeed, take a subsequence of $\{g_n\}$ that converges uniformly on a small neighborhood U in $\mathbb{D}^*$ to a holomorphic map h into $\mathbb{P}^1$. Shrinking U if necessary, we may select the norm on $\mathbb{C}^2$ so that $\log \|s(\cdot)\|$ is harmonic on a region containing the image $h(U)$ in $\mathbb{P}^1$, where s is any holomorphic section of $\mathbb{C}^2 \setminus \{(0, 0)\} \to \mathbb{P}^1$. Then the corresponding subsequence of the harmonic functions G_n are converging uniformly to a harmonic limit.

Fix a choice of $A : \mathbb{D} \to \mathbb{C}^2 \setminus \{(0, 0)\}$, and construct the escape-rate function G. The bound on G from Proposition 3.1 implies that G extends to a harmonic function on the entire disk. Indeed, by a standard argument in complex analysis, we fix a small disk of radius r and let h be the unique harmonic function on this disk with $h = G$ on the boundary circle. For each $\varepsilon > 0$, consider

$$u_\varepsilon(t) = G(t) - h(t) + \varepsilon \log |t|$$

for t in the punctured disk. The function u_ε extends to an upper-semicontinuous function, setting $u_\varepsilon(0) = -\infty$, and so u_ε is subharmonic on the disk because it satisfies the sub-mean-value property. Thus, $u_\varepsilon \le \varepsilon \log r$ on the disk by the maximum principle. Letting $\varepsilon \to 0$, we deduce that $G \le h$ on the punctured disk. Applying the same reasoning to

$$v_\varepsilon(t) = h(t) - G(t) + \varepsilon \log |t|$$

we obtain the reverse inequality, that $h \le G$, and therefore, $h = G$.

The harmonic function G can now be expressed locally as $\operatorname{Re} \eta$ for a holomorphic function η on $\mathbb{D}$. Now replace A_t with $\tilde{A}_t = e^{-\eta(t)} A_t$. Then

$$F_t^n(\tilde{A}_t) = e^{-d^n \eta(t)} F_t^n(A_t),$$

so the order of vanishing at $t = 0$ is unchanged. We obtain a new escape-rate function

$$\begin{aligned}\tilde{G}(t) &= \lim_{n\to\infty} \frac{1}{d^n} \log \|t^{-a_n} F_t^n(\tilde{A}_t)\| = \lim_{n\to\infty} \frac{1}{d^n} \log \|t^{-a_n} e^{-d^n \eta(t)} F_t^n(A_t)\| \\ &= \lim_{n\to\infty} \frac{1}{d^n} \log \|e^{-d^n \eta(t)} F_n(t)\| = G(t) + \log |e^{-\eta}| = G(t) - G(t) \equiv 0,\end{aligned}$$

completing the proof of the corollary. □

Lemma 3.5. *Suppose that $G \equiv 0$. The functions $\{F_n(t) = t^{-a_n} F_t^n(A_t)\}$ are uniformly bounded in $\mathbb{C}^2 \setminus \{(0,0)\}$ on compact subsets of $\mathbb{D}^*$.*

Proof. Recall that the "usual" escape rate of F_t is defined by

$$G_{F_t}(z) = \lim_{n\to\infty} \frac{1}{d^n} \log \|F_t^n(z)\|$$

for z in $\mathbb{C}^2$. The local uniform convergence of the limit (on $\mathbb{D}^* \times \mathbb{C}^2 \setminus \{(0,0)\}$) implies that $G_{F_t}(z)$ is continuous as a function of (t, z); it is proper in $z \in \mathbb{C}^2 \setminus \{(0,0)\}$, since the function satisfies

$$G_{F_t}(\alpha z) = G_{F_t}(z) + \log|\alpha|$$

for all $\alpha \in \mathbb{C}^*$ and all (t, z). So our desired result follows if we can show that, for each compact subset K of $\mathbb{D}^*$, there exist constants $-\infty < c < C < \infty$ such that

$$c \le G_{F_t}(F_n(t)) \le C$$

for all n and all $t \in K$.

Indeed, note that $G(t) = 0$ implies that $G_{F_t}(A_t) = \eta \log|t|$ for

$$\eta = \lim \frac{a_n}{d^n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{k_i}{d^i}$$

as in the proof of Proposition 3.1, with $0 \le k_i \le q$ for all i. Therefore,

$$\begin{aligned} G_{F_t}(F_n(t)) &= d^n G_{F_t}(A_t) - a_n \log|t| = (d^n\eta - a_n)\log|t| \\ &= \left(\sum_{i=n+1}^{\infty} \frac{k_i}{d^{i-n}}\right) \log|t| \ge \left(\sum_{i=0}^{\infty} \frac{q}{d^i}\right) \log|t|. \end{aligned}$$

On the other hand, the sequence a_n/d^n increases to η, so $(d^n\eta - a_n)\log|t| \le 0$ for all n and all $t \in \mathbb{D}^*$; therefore, $G_{F_t}(F_n(t)) \le 0$ for all t and all n. □

4. Proof of Theorem 1.1

Let f be an algebraic family of rational maps of degree $d \ge 2$, parameterized by the irreducible quasiprojective complex variety V. Let $a : V \to \mathbb{P}^1$ be a marked point. In this section, we prove Theorem 1.1.

It suffices to prove the theorem when V is one-dimensional. For, if t_0 is any parameter in V at which $a(t_0)$ is not preperiodic, taking any one-dimensional slice through t_0 on which (f, a) is not isotrivial, we conclude that (f, a) will not be stable on this slice. Therefore, the family of iterates cannot be normal on all of V.

If f is isotrivial, then the result follows immediately from Proposition 2.3. If f is not isotrivial and if the degrees of $g_n(t) := f_t^n(a(t))$ are bounded, then the conclusion follows immediately from Proposition 1.8.

For the rest of the proof, assume that the degrees of $\{g_n\}$ are unbounded. Suppose also that (f, a) is stable and f is not isotrivial. We will derive a contradiction.

Let C denote the normalization of a compactification of V, so that we may view f as a family defined over the punctured Riemann surface $C \setminus \{x_1, \dots, x_n\}$. The stability of (f, a) implies that $\{g_n\}$ is normal on V. As such, there exists a subsequence $\{g_{n_k}\}$ with unbounded degree that converges locally uniformly on V to a holomorphic function $h : V \to \mathbb{P}^1$. Note that h might have finite degree or it may have essential singularities at the punctures x_i of V. In either case, we find:

Lemma 4.1. *There exists a puncture of V such that, for any neighborhood U of this puncture, and for any point $b \in \mathbb{P}^1$ (with at most one exception), the cardinality of $g_{n_k}^{-1}(b) \cap U$ (counted with multiplicities) is unbounded as $n_k \to \infty$.*

Proof. We apply the argument principle. Fix any $b \in \mathbb{P}^1$ such that $h \not\equiv b$. Choose coordinates on $\mathbb{P}^1$ such that $b = 0$ and $h \not\equiv \infty$. Choose a small loop γ_j around each puncture x_j of V on which h has no zeros or poles.

Consider the integral

$$N(\gamma_j) = \frac{1}{2\pi i} \int_{\gamma_j} \frac{h'}{h} \in \mathbb{Z}$$

computing the winding number of the loop $h \circ \gamma_j$ around the origin. By uniform convergence of $g_{n_k} \to h$ on γ_j, and since g_{n_k} is meromorphic, the number $N(\gamma_j)$ is equal to the difference between the number of zeros and number of poles of g_{n_k} inside the circle for all n_k sufficiently large. But since $\deg g_{n_k} \to \infty$ and the functions converge uniformly to h outside these small loops, the actual count of zeros and poles must be growing to infinity inside one of these circles. □

Fix a puncture x_i of V satisfying the condition of Lemma 4.1. Choose local coordinate t on C on a small disk D around the puncture x_i of V. Choose coordinates on $\mathbb{P}^1$ such that the conclusion of Lemma 4.1 holds for $b = 0$ and $b = \infty$. Let B be a small annulus in the disk D of the form

$$B = \{r_0 < |t| < r_1\}$$

with $0 < 2r_0 < r_1 - r_0 < 1$. Passing to a further subsequence if necessary, let

$$\kappa(n_k) = \min\left\{|g_{n_k}^{-1}(0) \cap D_{r_0}|, |g_{n_k}^{-1}(\infty) \cap D_{r_0}|\right\}$$

so that $\kappa(n_k) \to \infty$ with n_k.

As in Section 3, choose a homogeneous polynomial lift F_t of f_t to $\mathbb{C}^2$, normalized so that the coefficients of F_t are holomorphic in t and not all 0 at $t = 0$. By Proposition 3.1 and Corollary 3.4, we may choose a holomorphic lift A of a with values in $\mathbb{C}^2 \setminus \{(0, 0)\}$ such that the escape-rate function G satisfies $G(t) \equiv 0$. From Lemma 3.5, we deduce that the sequence $\{F_n(t)\}$ is uniformly bounded — away from (0, 0) and ∞ — on the closed annulus $\bar{B} = \{r_0 \leq |t| \leq r_1\}$. In other words, there exist constants $0 < c \leq C < \infty$ such that

$$c \leq \|F_n(t)\| \leq C$$

for all n and all $t \in B$.

Write

$$F_{n_k}(t) = (P_{n_k}(t) R_{n_k}(t),\ Q_{n_k}(t) S_{n_k}(t))$$

for holomorphic P_{n_k}, R_{n_k}, Q_{n_k}, S_{n_k} where $P_{n_k}, Q_{n_k} \approx t^{\kappa(n_k)}$ on the disk D. More precisely, there exist factors

$$P_{n_k}(t) = \prod_{i=1}^{\kappa(n_k)} (t - t_i) \quad \text{and} \quad Q_{n_k}(t) = \prod_{i=1}^{\kappa(n_k)} (t - s_i)$$

for two disjoint sets of roots $\{t_i\}$ and $\{s_j\}$ contained in the small disk D_{r_0}. Note that

$$|P_{n_k}(t)|, |Q_{n_k}(t)| \leq (2r_0)^{\kappa(n_k)}$$

for $|t| = r_0$, and

$$|P_{n_k}(t)|, |Q_{n_k}(t)| \geq (r_1 - r_0)^{\kappa(n_k)}$$

for $|t| = r_1$. The uniform bounds on $F_n(t)$ imply that

$$|R_{n_k}(t)|, |S_{n_k}(t)| \leq \frac{C}{(r_1 - r_0)^{\kappa(n_k)}}$$

on the circle $|t| = r_1$ and

$$\max\{|R_{n_k}(t)|, |S_{n_k}(t)|\} \geq \frac{c}{(2r_0)^{\kappa(n_k)}}$$

for each t on the circle $|t| = r_0$ and all n_k. But $\kappa(n_k) \to \infty$ with n_k and $2r_0 < r_1 - r_0$, so for large n_k these estimates will violate the maximum principle applied to the holomorphic function P'_{n_k} or Q'_{n_k}.

The contradiction obtained shows that if (f, a) is stable on V with f not isotrivial, then the degrees of $\{g_n\}$ must be bounded, returning us to the setting treated by Proposition 1.8. This completes the proof of Theorem 1.1.

5. Density of intersections

In this section, we prove Proposition 1.5 and Theorem 1.6.

Elliptic curves. We begin by explaining the connection between Proposition 1.5 and the theme of this article. Let E_t be a family of smooth elliptic curves, parameterized by a quasiprojective algebraic curve V. The equivalence relation $x \sim -x$ on E_t induces a projection to $\mathbb{P}^1$. Via this projection, the multiplication-by-2 map on E_t descends to a rational function f_t on $\mathbb{P}^1$ of degree 4, called a *Lattès map*. (A formula for the resulting f_t is shown for the Legendre family E_t at the bottom of page 1040, defined there as L_t.) The family f_t is nonisotrivial if and only if the family E_t is nonisotrivial. A point $P_t \in E_t$ projects to a preperiodic point for f_t if and only if P_t is torsion.

In Theorem 1.2, we also refer to the canonical height function: by its definition, the Néron–Tate height on an elliptic curve is equal to $\frac{1}{2}$ times the canonical height for the associated multiplication-by-2 Lattès map. Therefore, height 0 on the elliptic curve coincides with height 0 for the rational function.

Proof of Proposition 1.5. Let E be a nonisotrivial elliptic curve defined over a function field $k = \mathbb{C}(X)$ for an irreducible complex algebraic curve X. We view E as a family E_t of smooth elliptic curves, for all but finitely many $t \in X$; alternatively, we view E as a complex surface, equipped with an elliptic fibration $E \to X$. Fix $P \in E(k)$. Then P determines a section $P : X \to E$. Composing this section P with the degree-two quotient from each $E_t,\ t \in X,$ to $\mathbb{P}^1$, we obtain a marked point $a_P : X \to \mathbb{P}^1$ and a nonisotrivial algebraic family of Lattès maps $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ on a Zariski-open subset $V \subset X$.

From Theorem 1.1, we know that the pair (f, a_P) is stable if and only if the pair (f, a_P) is preperiodic. So either $a_P(t)$ is preperiodic for f_t for all $t \in V$ (in which case P is torsion on E/k), or the pair (f, a_P) is not stable.

In this way, the proposition is a consequence of the following statement, which is a direct application of Montel's theorem on normal families; see, e.g., [Milnor 2006] for background on Montel's theorem.

Given a pair (f, a), the *stable set* $\Omega(f, a) \subset V$ is the largest open set on which $\{t \mapsto f_t^n(a(t))\}$ forms a normal family. The *bifurcation locus* $B(f, a) \subset V$ is the complement of $\Omega(f, a)$ in V.

Proposition 5.1. *Suppose $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ is an algebraic family of rational maps of degree $d \geq 2$. Let $a : V \to \mathbb{P}^1$ be a marked point, and suppose that the bifurcation locus $B(f, a)$ is nonempty. Then for each open $U \subset V$ intersecting $B(f, a)$, there are infinitely many $t \in U$ where $a(t)$ is preperiodic to a repelling cycle of f_t.*

Proof. Fix an open set U having nonempty intersection with $B(f, a)$. Choose a point t_0 in $B(f, a) \cap U$. Choose three distinct repelling periodic points $z_1(t_0)$, $z_2(t_0)$, $z_3(t_0)$ of f_{t_0} that are not in the forward orbit of $a(t_0)$. Shrinking U if necessary, the implicit function theorem implies that these periodic points can be holomorphically parameterized by $t \in U$. By Montel's theorem, the failure of normality of $\{t \mapsto f_t^n(a(t))\}$ on U implies that there exists a parameter $t_1 \in U$ and an integer $n_1 > 0$ such that $f_{t_1}^{n_1}(a(t_1))$ is an element of the set $\{z_1(t_1), z_2(t_1), z_3(t_1)\}$. In particular, $a(t_1)$ is preperiodic for f_{t_1}. Shrinking the neighborhood U, we may find infinitely many such parameters. □

Remark 5.2. In [DeMarco et al. 2016], we studied the *distribution* of the parameters $t \in X$ for which a marked point $P_t \in E_t$ is torsion, with E_t the Legendre family of elliptic curves. The set of such parameters is dense in the parameter space (in the usual analytic topology).

Proof of Theorem 1.6. By hypothesis, the family $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ has dimension N in moduli; this means that the image of the induced projection $V \to \mathrm{M}_d$ to the moduli space of rational maps has dimension N.

To prove Zariski density, we need to show that, for any algebraic subvariety $Y \subset V$ (possibly reducible), the complement $\Lambda = V \setminus Y$ contains a parameter t at which all points $a_1(t), \ldots, a_k(t)$ are preperiodic. Note that Λ is itself an irreducible, quasiprojective complex algebraic variety, so that $f : \Lambda \times \mathbb{P}^1 \to \mathbb{P}^1$ is again an algebraic family of rational maps, of dimension N in moduli.

Consider the marked point a_1. Since f projects to an N-dimensional family in the moduli space, with $N > 0$, it follows that (f, a_1) is not isotrivial on Λ. By Theorem 1.1, the pair (f, a_1) is either preperiodic or it fails to be stable on Λ. If (f, a_1) is preperiodic, we set $\Lambda_1 = \Lambda$. If (f, a_1) is unstable, then Proposition 5.1 shows that there exists a parameter $t_1 \in \Lambda$ where $a_1(t_1)$ is preperiodic to a repelling cycle of f_{t_1}. If a_1 satisfies the equation $f^{n_1}(a_1) = f^{m_1}(a_1)$ at the parameter t_1, we define $\Lambda_1 \subset \Lambda$ to be an irreducible component of the subvariety defined by the equation $f_t^{n_1}(a_1(t)) = f_t^{m_1}(a_1(t))$ that contains t_1. Then Λ_1 is a nonempty quasiprojective variety, of codimension 1 in Λ. Furthermore, since the cycle persists under perturbation, the condition defining Λ_1 will also cut out a codimension-1 subvariety in the moduli space. In other words, Λ_1 must project to a family of dimension $N-1$ in the moduli space. By construction, (f, a_1) is preperiodic on Λ_1.

We continue inductively. Fix $1 \le i < k$. Suppose Λ_i is a quasiprojective subvariety of dimension $\ge N-i$ in moduli on which $(f, a_1), \ldots, (f, a_i)$ are preperiodic. Since $N-i > N-k \ge 0$, the pair (f, a_{i+1}) is not isotrivial on Λ_i. As above, we combine Theorem 1.1 with Proposition 5.1 to find a parameter $t_{i+1} \in \Lambda_i$ where a_{i+1} is preperiodic. We define $\Lambda_{i+1} \subset \Lambda_i$ so that (f, a_{i+1}) is preperiodic on Λ_{i+1}, and the family $f : \Lambda_{i+1} \times \mathbb{P}^1 \to \mathbb{P}^1$ has dimension at least $N - i - 1$ in moduli. In conclusion, all of the points $(f, a_1), \ldots, (f, a_k)$ are preperiodic on Λ_k, and Λ_k has dimension at least $N - k \ge 0$ in moduli. In particular, Λ_k is nonempty, and the theorem is proved.

6. A conjecture on intersections and dynamical relations

We conclude this article with a revised statement of the conjecture from [Baker and DeMarco 2013] on "unlikely intersections" and density of "special points" — and we provide the proof of one implication, as an application of Theorem 1.1. Specifically, we look at algebraic families $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ of dimension $N > 0$

in moduli. We prove that if an $(N+1)$-tuple of marked points is dynamically related, then the set of parameters $t \in V$ where they are simultaneously preperiodic is Zariski-dense in V. We conclude the article with an explanation of how this implies one implication of [Baker and DeMarco 2013, Conjecture 1.10].

Density of special points. In [Baker and DeMarco 2013, Conjecture 1.10], we formulated a conjecture about the arrangement of postcritically finite maps ("special points") in the moduli space of rational maps of degree $d \geq 2$. It was presented as a dynamical analog of the André–Oort conjecture in arithmetic geometry, with the aim of characterizing the "special subvarieties" of the moduli space, meaning the algebraic families $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ with a Zariski-dense subset of postcritically finite maps. Roughly speaking, the special subvarieties should be those that are defined by (a general notion of) critical orbit relations. We proved special cases of the conjecture, for certain families of polynomial maps, and we sketched the proof of one implication in the general case.

If we formulate the conjecture to handle arbitrary marked points, not only critical points, then the statement encompasses recent results about elliptic curves, as in the work of Masser and Zannier (and therefore has overlap with the Pink and Zilber conjectures); see [Masser and Zannier 2012] and the references therein. Evidence towards the more general result is given by [Baker and DeMarco 2013, Theorem 1.3] and the results of [Ghioca et al. 2013; 2015]. Conjecture 6.1 presented here is, therefore, more than just an analogy with statements in arithmetic geometry.

Let V be an irreducible, quasiprojective complex algebraic variety, and let $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$. For a collection of n marked points $a_1, \ldots, a_n : V \to \mathbb{P}^1$, we define

$$S(a_1, \ldots, a_n) = \bigcap_{i=1}^{n} \{t \in V : a_i(t) \text{ is preperiodic for } f_t\}.$$

We say the marked points $a_1, \ldots, a_n$ are *coincident* along V if there exists a marked point a_i and a Zariski-open subset $V' \subset V$ such that

$$S(a_1, \ldots, a_n) \cap V' = S(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \cap V'.$$

In other words, if $\{a_1(t), \ldots, a_{i-1}(t), a_{i+1}(t), \ldots, a_n(t)\}$ are all preperiodic for f_t at a parameter $t \in V'$, then the remaining point $a_i(t)$ must also be preperiodic for f_t. For example, if a pair (f, a) is preperiodic on V, then any collection of points $\{a_1, \ldots, a_n\}$ containing a will be coincident.

A stronger notion than coincidence is that of the dynamical relation, requiring an f-invariant algebraic relation between the points $\{a_1, \ldots, a_n\}$. A formal definition is given below.

Conjecture 6.1. *Let $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ be an algebraic family of rational maps of degree $d \geq 2$, of dimension $N > 0$ in moduli. Let $a_0, \dots, a_N$ be any collection of $N+1$ marked points. The following are equivalent*:

(1) *The set $S(a_0, \dots, a_N)$ is Zariski-dense in V.*

(2) *The points $a_0, \dots, a_N$ are coincident along V.*

(3) *The points $a_0, \dots, a_N$ are dynamically related along V.*

Theorem 6.2. *We have* (3) $\Longrightarrow$ (2) *and* (2) $\Longrightarrow$ (1) *in Conjecture 6.1.*

The implication (3) $\Longrightarrow$ (2) will be a formal consequence of the definitions, while (2) $\Longrightarrow$ (1) is presented below as an application of Theorem 1.1. The remaining challenge is to show that (1) implies (3). We expect that (1) $\Longrightarrow$ (2) should be a consequence of "arithmetic equidistribution" as in the proofs of [Baker and DeMarco 2011; 2013; Ghioca et al. 2013; 2015; DeMarco et al. 2016] when V is a curve.

Dynamical relations. The basic example of a dynamical relation between two marked points $a, b : V \to \mathbb{P}^1$ is an orbit relation: the existence of integers n, m such that

$$f_t^n(a(t)) = f_t^m(b(t))$$

for all $t \in V$. To allow for complicated symmetries, we will say that N marked points $a_1, \dots, a_N$ are *dynamically related* along V if there exists a (possibly reducible) algebraic subvariety

$$X \subset (\mathbb{P}^1)^N$$

defined over the function field $k = \mathbb{C}(V)$, such that three conditions are satisfied:

(R1) $(a_1, \dots, a_N) \in X$.

(R2) (invariance) $F(X) \subset X$, where $F = (f, f, \dots, f) : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N$.

(R3) (nondegeneracy) There exists an $i \in \{1, \dots, N\}$ and a Zariski-open subset $V' \subset V$ such that the projection from the specialization X_t to the i-th coordinate hyperplane in $(\mathbb{P}^1_{\mathbb{C}})^N$ is a finite map for all $t \in V'$.

Remark 6.3. In [Baker and DeMarco 2013], after stating Conjecture 1.10, we offhandedly remarked that one implication of the conjecture "follows easily from an argument mimicking the proof of Proposition 2.6 and the following observation". The proof of Theorem 1.6 in this article is the argument we had in mind, mimicking [Baker and DeMarco 2013, Proposition 2.6], but the stated "observation" was not formulated correctly. The inclusion of condition (R3) and the argument in the proof of Theorem 6.2 on the next page are an attempt to correct that error.

To illustrate the dynamical relation, observe that any N points $a_1, \dots, a_N$ are dynamically related if one of the pairs (f, a_i) is preperiodic. Indeed, if a_i satisfies

$f_t^n(a_i(t)) = f_t^m(a_i(t))$ for all $t \in V$, for some pair of integers $n \neq m \geq 0$, then we could take X to be the hypersurface

$$\{x \in (\mathbb{P}^1)^N : f^n(x_i) = f^m(x_i)\}$$

defined over the field $k = \mathbb{C}(V)$. As a nontrivial example, we look at the relation arising in the Masser–Zannier theorems [2012]. Let $f_{[k]}$ be the Lattès map induced from multiplication by $k \in \mathbb{N}$ on a nonisotrivial elliptic curve E over $k = \mathbb{C}(V)$. Let a_p, a_q be the projections to $\mathbb{P}^1$ of two points p and q in $E(k)$. The linear relation $n \cdot p = m \cdot q$ between points p and q on E, for integers n and m, translates into a dynamical relation in $(\mathbb{P}^1)^2$ defined by

$$f_{[n]}(x_1) = f_{[m]}(x_2).$$

This relation satisfies condition (R2) for $F = (f_{[k]}, f_{[k]})$ because all Lattès maps descended from the same elliptic curve must commute.

Since the writing of [Baker and DeMarco 2013], we have learned about the results in [Medvedev 2007] which significantly simplify the form of possible dynamical relations. In particular, Medvedev has shown that the varieties X satisfying condition (R2) should depend nontrivially on only two input variables. In other words, the rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ will be *disintegrated* in the sense of [Medvedev and Scanlon 2014, Definition 2.20]; see the first theorem in the introduction of the same paper, treating the case where f is a polynomial. An affirmative answer to the following question would provide a further refinement — and simplification — to the notion of dynamical relation, extending the results of [Medvedev and Scanlon 2014] beyond the polynomial setting. (The work of Medvedev and Scanlon relied on Ritt's decomposition theory [1922] for polynomials; the analogous decomposition theory for rational functions is not completely understood.)

Question 6.4. Assume that f is not isotrivial, and suppose that points $a_1, \ldots, a_N$ are dynamically related. Does there always exist a pair of indices i, j (allowing possibly $i = j$) such that the point $(a_1, \ldots, a_N)$ satisfies a relation of the form

$$A(x_i) = B(x_j), \tag{6-1}$$

where $A, B \in k(z)$ are nonconstant rational functions that commute with an iterate of f?

Hypersurfaces in $(\mathbb{P}^1)^N$ defined by relations of the form (6-1) satisfy condition (R2) in the definition of the dynamical relation because f commutes with A and B; they satisfy condition (R3) taking either coordinate i or j.

Proof of Theorem 6.2. We begin by proving (3) $\Longrightarrow$ (2); namely, that a dynamical relation among the points $a_0, \ldots, a_N$ implies that the points are coincident. This follows from the definition of dynamical relation, and it does not depend on the number of points.

Lemma 6.5. *Let $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ be any algebraic family of rational maps. Suppose marked points $a_0, a_1, \ldots, a_n$ are dynamically related along V. Then the points $a_0, \ldots, a_n$ are coincident along V.*

Proof. Let X denote the $(f, \ldots, f)$-invariant subvariety in $(\mathbb{P}^1)^{n+1}$ for the point $(a_1, \ldots, a_n)$, given in the definition of the dynamical relation. Suppose the points are labeled so that x_0 is the coordinate satisfying condition (R3). Then the projection from X_t to $(\mathbb{P}^1_{\mathbb{C}})^n$, forgetting the 0-th coordinate, is finite, for all t in the Zariski-open subset $V' \subset V$.

Now let t_0 be any parameter in V' at which $a_1(t_0), \ldots, a_n(t_0)$ are preperiodic for f_{t_0}. The point $a_0(t_0)$ must lie in the fiber of $X_{t_0} \to (\mathbb{P}^1)^N$ over $(a_1(t_0), \ldots, a_n(t_0))$. Invariance of X implies the invariance of X_{t_0}, so that $f^m_{t_0}(a_0(t_0))$ lies in the fiber over $(f^m(a_1(t_0)), \ldots, f^m(a_N(t_0)))$ for all $m \geq 1$. The preperiodicity of the points guarantees that there are only finitely many points in the base in the orbit of $(a_1(t_0), \ldots, a_n(t_0))$, so the orbit of $a_0(t_0)$ must be contained in a finite set. In other words, $a_0(t_0)$ is preperiodic. This completes the proof. □

Now assume (2), that the given points $a_0, \ldots, a_N$ are coincident. Assume the points are labeled so that a_0 is the dependent point, in the sense that

$$S(a_0, \ldots, a_N) \cap V' = S(a_1, \ldots, a_N) \cap V'$$

for some Zariski-open subset $V' \subset V$. Since V has dimension N in moduli, Theorem 1.6 tells us that the set $S(a_1, \ldots, a_N)$ is Zariski-dense in V. Therefore, so is $S(a_0, \ldots, a_N)$, and the implication (2) $\Longrightarrow$ (1) is proved.

***Proof of one implication of* [Baker and DeMarco 2013, Conjecture 1.10].** Suppose that $f : V \times \mathbb{P}^1 \to \mathbb{P}^1$ is an algebraic family of rational maps of degree $d \geq 2$ and dimension $N > 0$ in moduli. We assume that all $2d - 2$ critical points of f are marked. Conjecture 1.10 of [Baker and DeMarco 2013] states: *the map f_t is postcritically finite for a Zariski-dense set of $t \in V$ if and only if there are at most N dynamically independent critical points.*

Let $c_1, \ldots, c_{2d-2}$ denote the marked critical points. Assume that f has at most N dynamically independent critical points; in other words, given any $n > N$ marked critical points $c_{i_1}, \ldots, c_{i_n}$, there is a dynamical relation among them.

Note that $S(c_1, \ldots, c_{2d-2})$ is precisely the set of parameters t for which f_t is postcritically finite. Applying Lemma 6.5 repeatedly, and reordering the points as needed, there exists a Zariski-open subset $V' \subset V$ such that

$$S(c_1, \ldots, c_{2d-2}) \cap V' = S(c_1, \ldots, c_{2d-3}) \cap V' = \cdots = S(c_1, \ldots, c_N) \cap V'.$$

From Theorem 1.6, we know that $S(c_1, \ldots, c_N)$ is Zariski-dense in V. This proves that the postcritically finite maps form a Zariski-dense subset of V.

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