Algebra & Number Theory

Volume 10

2016

No. 5

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ALGEBRA AND NUMBER THEORY 10:5 (2016)
dx.doi.org/10.2140/ant.2016.10.1057

Frobenius and valuation rings

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The behavior of the Frobenius map is investigated for valuation rings of prime characteristic. We show that valuation rings are always F-pure. We introduce a generalization of the notion of strong F-regularity, which we call F-pure regularity, and show that a valuation ring is F-pure regular if and only if it is Noetherian. For valuations on function fields, we show that the Frobenius map is finite if and only if the valuation is Abhyankar; in this case the valuation ring is Frobenius split. For Noetherian valuation rings in function fields, we show that the valuation ring is Frobenius split if and only if Frobenius is finite, or equivalently, if and only if the valuation ring is excellent.

1. Introduction

Classes of singularities defined using Frobenius — F-purity, Frobenius splitting, and the various variants of F-regularity — have played a central role in commutative algebra and algebraic geometry over the past forty years. The goal of this paper is a systematic study of these F-singularities in the novel, but increasingly important *non-Noetherian* setting of *valuation rings*.

Datta was partially supported by the Juha Heinonen Memorial Graduate Fellowship at the University of Michigan. The second author was partially supported by NSF grants DMS-1001764 and DMS-1501625.

MSC2010: primary 13A35; secondary 13F30, 14B05.

Keywords: valuation rings, Abhyankar valuations, characteristic p commutative algebra, F-pure, F-regular, Frobenius split.

Let R be a commutative ring of prime characteristic p. The Frobenius map is the ring homomorphism $R \xrightarrow{F} R$ sending each element to its p-th power. While simple enough, the Frobenius map reveals deep structural properties of a Noetherian ring of prime characteristic, and is a powerful tool for proving theorems for rings containing an arbitrary field (or varieties, say, over $\mathbb{C}$) by standard reduction to characteristic p techniques. Theories such as Frobenius splitting [Mehta and Ramanathan 1985] and tight closure [Hochster and Huneke 1990] are well-developed in the Noetherian setting, often under the additional assumption that the Frobenius map is finite. Since classically most motivating problems were inspired by algebraic geometry and representation theory, these assumptions seemed natural and not very restrictive. Now, however, good reasons are emerging to study F-singularities in certain non-Noetherian settings as well.

One such setting is cluster algebras [Fomin and Zelevinsky 2002]. An upper cluster algebra over $\mathbb{F}_p$ need not be Noetherian, but recently it was shown that it is always Frobenius split, and indeed, admits a "cluster canonical" Frobenius splitting [Benito et al. 2015]. Likewise valuation rings are enjoying a resurgence of popularity despite rarely being Noetherian, with renewed interest in non-Archimedean geometry [Conrad 2008], the development of tropical geometry [Gubler et al. 2016], and the valuative tree [Favre and Jonsson 2004], to name just a few examples, as well as fresh uses in higher dimensional birational geometry (e.g., [Cutkosky 2004; Fernández de Bobadilla and Pereira 2012; Boucksom 2014]).

For a Noetherian ring R, the Frobenius map is flat if and only if R is regular, by a famous theorem of Kunz [1969]. As we observe in Theorem 3.1, the Frobenius map is always flat for a valuation ring. So in some sense, a valuation ring of characteristic p might be interpreted as a "non-Noetherian regular ring."

On the other hand, some valuation rings are decidedly more like the local rings of smooth points on varieties than others. For example, for a variety X (over, say, an algebraically closed field of characteristic p), the Frobenius map is always finite. For valuation rings of the function field of X, however, we show that the Frobenius is finite if and only the valuation is Abhyankar; see Theorem 5.1. In particular, for discrete valuations, finiteness of Frobenius is equivalent to the valuation being *divisorial* — that is, given by the order of vanishing along a prime divisor on some birational model. Abhyankar valuations might be considered the geometrically most interesting ones (see [Ein et al. 2003]), so it is fitting that their valuation rings behave the most like the rings of smooth points on a variety. Indeed, recently, the local uniformization problem for Abhyankar valuations was settled in positive characteristic [Knaf and Kuhlmann 2005].

One can weaken the demand that Frobenius is flat and instead require only that the Frobenius map is *pure* (see Section 2.5). Hochster and Roberts observed that this condition, which they dubbed *F-purity*, is often sufficient for controlling singularities of a Noetherian local ring, an observation at the heart of their famous theorem on the Cohen–Macaulayness of invariant rings [Hochster and Roberts 1976; 1974]. We show in Corollary 3.3 that any valuation ring of characteristic p is F-pure. Purity of a map is equivalent to its splitting under suitable finiteness hypotheses, but at least for valuation rings (which rarely satisfy said hypotheses), the purity of Frobenius seems to be better behaved and more straightforward than its splitting. Example 4.5.1 shows that not all valuation rings are Frobenius split, even in the Noetherian case.

Frobenius splitting has well known deep local and global consequences for algebraic varieties. In the local case, Frobenius splitting has been said to be a "characteristic p analog" of log canonical singularities for complex varieties, whereas related properties correspond to other singularities in the minimal model program [Hara and Watanabe 2002; Schwede 2009b; Smith 1997; Takagi 2008]. For projective varieties, Frobenius splitting is related to positivity of the anticanonical bundle; see [Brion and Kumar 2005; Mehta and Ramanathan 1985; Smith 2000; Schwede and Smith 2010]. Although valuation rings are always F-pure, the question of their Frobenius splitting is subtle. Abhyankar valuations in function fields are Frobenius split (Theorem 5.1), but a discrete valuation ring is Frobenius split if and only if it is *excellent* in the sense of Grothendieck (Corollary 4.2.2). Along the way, we prove a simple characterization of the finiteness of Frobenius for a Noetherian domain in terms of excellence, which gives a large class of Noetherian domains in which Frobenius splitting implies excellence; see Section 2.6 for details.

Closely related to F-purity and Frobenius splitting are the various variants of F-regularity. Strong F-regularity was introduced by Hochster and Huneke [1989] as a proxy for weak F-regularity — the property that all ideals are tightly closed — because it is easily shown to pass to localizations. Whether or not a weakly F-regular ring remains so after localization is a long standing open question in tight closure theory, as is the equivalence of weak F-regularity and strong F-regularity. Strong F-regularity has found many applications beyond tight closure, and is closely related to Ramanathan's notion of "Frobenius split along a divisor" [Ramanathan 1991; Smith 2000]. A smattering of applications might include [Aberbach and Leuschke 2003; Benito et al. 2015; Blickle 2008; Brion and Kumar 2005; Gongyo et al. 2015; Hacon and Xu 2015; Patakfalvi 2014; Schwede and Tucker 2012; Schwede 2009a; Schwede and Smith 2010; Smith and Van den Bergh 1997; Smith and Zhang 2015; Smith 2000].

Traditionally, strong F-regularity has been defined only for Noetherian rings in which Frobenius is finite. To clarify the situation for valuation rings, we introduce a new definition which we call *F-pure regularity* (see Definition 6.1.1) requiring *purity* rather than *splitting* of certain maps. We show that F-pure regularity is better suited for arbitrary rings, but equivalent to strong F-regularity under the standard finiteness hypotheses; it also agrees with another generalization of strong F-regularity proposed by Hochster [2007] (using tight closure) in the local Noetherian case. Likewise, we show that F-pure regularity is a natural and straightforward generalization of strong F-regularity, satisfying many expected properties — for example, regular rings are F-pure regular. Returning to valuation rings, in Theorem 6.5.1 we characterize F-pure regular valuation rings as precisely those that are Noetherian.

Finally, in Section 6.6, we compare our generalization of strong F-regularity with the obvious competing generalization, in which the standard definition in terms of splitting certain maps is naively extended without assuming any finiteness conditions. To avoid confusion,[1] we call this *split F-regularity*. We characterize split F-regular valuation rings (at least in a certain large class of fields) as precisely those that are Frobenius split, or equivalently *excellent;* see Corollary 6.6.3. But we also point out that there are regular local rings that fail to be split F-regular, so perhaps split F-regularity is not a reasonable notion of "singularity."

2. Preliminaries

Throughout this paper, all rings are assumed to be commutative, and of prime characteristic p unless explicitly stated otherwise. By a local ring, we mean a ring with a unique maximal ideal, *not necessarily Noetherian.*

2.1. *Valuation rings.* We recall some basic facts and definitions about valuation rings (of arbitrary characteristic), while fixing notation. See [Bourbaki 1989, Chapter VI] or [Matsumura 1989, Chapter 4] for proofs and details.

The symbol Γ denotes an ordered abelian group. Recall that such an abelian group is torsion free. The rational rank of Γ, denoted $\operatorname{rat.rank}\Gamma$, is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$.

Let K be a field. A *valuation on K* is a homomorphism

$$v : K^{\times} \to \Gamma$$

from the group of units of K, satisfying

$$v(x+y) \geq \min\{v(x), v(y)\}$$

for all $x, y \in K^{\times}$. We say that v is *defined over a subfield k of K*, or that v is a *valuation on K/k*, if v takes the value 0 on elements of k.

There is no loss of generality in assuming that v is surjective, in which case we say Γ (or Γ_v) is the *value group* of v. Two valuations v_1 and v_2 on K are said to

[1] An earlier version of this paper used the terms *pure F-regularity* and *split F-regularity* for the two generalizations of classical strong F-regularity, depending upon whether maps were required to be pure or split. The names were changed at Karl Schwede's urging to avoid confusion with terminology for pairs in [Takagi 2004].

be *equivalent* if there is an order preserving isomorphism of their value groups identifying $v_1(x)$ and $v_2(x)$ for all $x \in K^\times$. Throughout this paper, we identify equivalent valuations.

The *valuation ring of* v is the subring $R_v \subseteq K$ consisting of all elements $x \in K^\times$ such that $v(x) \geq 0$ (together with the zero element of K). Two valuations on a field K are equivalent if and only if they determine the same valuation ring. Hence a valuation ring of K is essentially the same thing as an equivalence class of valuations on K.

The valuation ring of v is local, with *maximal ideal* m_v consisting of elements of strictly positive values (and zero). The *residue field* R_v/m_v is denoted $\kappa(v)$. If v is a valuation over k, then both R_v and $\kappa(v)$ are k-algebras.

A valuation ring V of K can be characterized directly, without reference to a valuation, as a subring with the property that for every $x \in K$, either $x \in V$ or $x^{-1} \in V$. The valuation ring V uniquely determines a valuation v on K (up to equivalence), whose valuation ring in turn recovers V. Indeed, it is easy to see that the set of ideals of a valuation ring is totally ordered by inclusion, so the set of principal ideals Γ^+ forms a monoid under multiplication, ordered by $(f) \leq (g)$ whenever f *divides* g. Thus, Γ can be taken to be the ordered abelian group generated by the principal ideals, and the valuation $v : K^\times \to \Gamma$ is induced by the monoid map sending each nonzero $x \in V$ to the ideal generated by x. Clearly, the valuation ring of v is V. See [Matsumura 1989, Chapter 4].

2.2. ***Extension of valuations.*** Consider an extension of fields $K \subseteq L$. By definition, a valuation w on L is an *extension* of a valuation v on K if the restriction of w to the subfield K is v. Equivalently, w extends v if R_w *dominates* R_v, meaning that $R_v = R_w \cap K$ with $m_w \cap R_v = m_v$. In this case, there is an induced map of residue fields

$$\kappa(v) \hookrightarrow \kappa(w).$$

The *residue degree of* w *over* v, denoted by $f(w/v)$, is the degree of the residue field extension $\kappa(v) \hookrightarrow \kappa(w)$.

If w extends v, there is a natural injection of ordered groups $\Gamma_v \hookrightarrow \Gamma_w$, since Γ_v is the image of w restricted to the subset K. The *ramification index of* w *over* v, denoted by $e(w/v)$, is the index of Γ_v in Γ_w.

If $K \hookrightarrow L$ is a finite extension, then both the ramification index $e(w/v)$ and the residue degree $f(w/v)$ are *finite*. Indeed, if $K \subseteq L$ is a degree n extension, then

$$e(w/v) f(w/v) \leq n. \tag{2.2.0.1}$$

More precisely:

Proposition 2.2.1 [Bourbaki 1989, VI.8]. *Let $K \subseteq L$ be an extension of fields of finite degree n. For a valuation v on K, consider the set $\mathcal{S}$ of all extensions* (*up to* *equivalence*) w *of* v *to* L. *Then*

$$\sum_{w_i \in \mathcal{S}} e(w_i/v) f(w_i/v) \leq n.$$

In particular, *the set* $\mathcal{S}$ *is finite. Furthermore*, *equality holds if and only if the integral closure of* R_v *in* L *is a finitely generated* R_v*-module.*

2.3. ***Abhyankar valuations.*** Fix a field K finitely generated over a fixed ground field k, and let v be a valuation on K/k. By definition, the *transcendence degree* of v is the transcendence degree of the field extension

$$k \hookrightarrow \kappa(v).$$

The main result about the transcendence degree of valuations is due to Abhyankar [1956]. See also [Bourbaki 1989, VI.10.3, Corollary 1].

Theorem 2.3.1 (Abhyankar's inequality). *Let* K *be a finitely generated field extension of* k, *and let* v *be a valuation on* K/k. *Then*

$$\text{trans. deg}\, v + \text{rat. rank}\, \Gamma_v \leq \text{trans. deg}\, K/k. \tag{2.3.1.1}$$

Moreover if equality holds, *then* Γ_v *is a finitely generated abelian group*, *and* $\kappa(v)$ *is a finitely generated extension of* k.

We say v is an *Abhyankar valuation* if equality holds in Abhyankar's inequality (2.3.1.1). Note that an Abyhankar valuation has a finitely generated value group, and its residue field is finitely generated over the ground field k.

Example 2.3.2. Let K/k be the function field of a normal algebraic variety X of dimension n over a ground field k. For a prime divisor Y of X, consider the local ring $\mathcal{O}_{X,Y}$ of rational functions on X regular at Y. The ring $\mathcal{O}_{X,Y}$ is a discrete valuation ring, corresponding to a valuation v (the order of vanishing along Y) on K/k; this valuation is of rational rank one and transcendence degree $n-1$ over k, hence Abhyankar. Such a valuation is called a *divisorial valuation*. Conversely, every rational rank one Abhyankar valuation is divisorial: for such a v, there exists some normal model X of K/k and a divisor Y such that v is the order of vanishing along Y [Zariski and Samuel 1960, VI, §14, Theorem 31].

Proposition 2.3.3. *Let* $K \subseteq L$ *be a finite extension of finitely generated field extensions of* k, *and suppose that* w *is valuation on* L/k *extending a valuation* v *on* K/k. *Then* w *is Abhyankar if and only if* v *is Abhyankar.*

Proof. Since L/K is finite, L and K have the same transcendence degree over k. On the other hand, the extension $\kappa(v) \subseteq \kappa(w)$ is also finite by (2.2.0.1), and so

$\kappa(v)$ and $\kappa(w)$ also have the same transcendence degree over k. Again by (2.2.0.1), since Γ_w/Γ_v is a finite abelian group, $\mathbb{Q}\otimes_{\mathbb{Z}}\Gamma_w/\Gamma_v = 0$. By exactness of

$$0 \to \mathbb{Q}\otimes_{\mathbb{Z}}\Gamma_v \to \mathbb{Q}\otimes_{\mathbb{Z}}\Gamma_w \to \mathbb{Q}\otimes_{\mathbb{Z}}\Gamma_w/\Gamma_z \to 0$$

we conclude that Γ_w and Γ_v have the same rational rank. The result is now clear from the definition of an Abhyankar valuation. □

2.4. ***Frobenius.*** Let R be a ring of prime characteristic p. The Frobenius map $R \xrightarrow{F} R$ is defined by $F(x) = x^p$. We can denote the target copy of R by F_*R and view it as an R-module via restriction of scalars by F; thus F_*R is both a ring (indeed, it is precisely R) and an R-module in which the action of $r \in R$ on $x \in F_*R$ produces $r^p x$. With this notation, the Frobenius map $F : R \to F_*R$ and its iterates $F^e : R \to F^e_*R$ are ring maps, as well as R-module maps. See [Smith and Zhang 2015, p. 294] for a further discussion of this notation.

We note that F^e_* gives us an exact covariant functor from the category of R-modules to itself. This is nothing but the usual restriction of scalars functor associated to the ring homomorphism $F^e : R \to R$.

For an ideal $I \subset R$, the notation $I^{[p^e]}$ denotes the ideal generated by the p^e-th powers of the elements of I. Equivalently, $I^{[p^e]}$ is the expansion of I under the Frobenius map, that is, $I^{[p^e]} = IF^e_*R$ as subsets of R.

The image of F^e is the subring $R^{p^e} \subset R$ of p^e-th powers. If R is reduced (which is equivalent to the injectivity of Frobenius), statements about the R-module F^e_*R are equivalent to statements about the R^{p^e}-module R.

Definition 2.4.1. A ring R of characteristic p is *F-finite* if $F : R \to F_*R$ is a finite map of rings, or equivalently, if R is a finitely generated R^p-module. Note that $F : R \to F_*R$ is a finite map if and only if $F^e : R \to F^e_*R$ is a finite map for all $e > 0$.

F-finite rings are ubiquitous. For example, every perfect field is F-finite, and a finitely generated algebra over an F-finite ring is F-finite. Furthermore, F-finiteness is preserved under homomorphic images, localization and completion. This means that nearly every ring classically arising in algebraic geometry is F-finite. However, valuation rings even of F-finite fields are often not F-finite.

2.5. ***F-purity and Frobenius splitting.*** We first review purity and splitting for maps of modules over an arbitrary commutative ring A, not necessarily Noetherian or of prime characteristic. A map of A-modules $M \xrightarrow{\varphi} N$ is *pure* if for any A-module Q, the induced map

$$M \otimes_A Q \to N \otimes_A Q$$

is injective. The map $M \xrightarrow{\varphi} N$ is *split* if φ has a left inverse in the category of A-modules. Clearly, a split map is always pure. Although it is not obvious, the converse holds under a weak hypothesis:

Lemma 2.5.1 [Hochster and Roberts 1976, Corollary 5.2]. *Let $M \xrightarrow{\varphi} N$ be a pure map of A-modules where A is a commutative ring. Then φ is split if the cokernel $N/\varphi(M)$ is finitely presented.*

Definition 2.5.2. Let R be an arbitrary commutative ring of prime characteristic p.

(a) The ring R is *Frobenius split* if the map $F : R \to F_*R$ splits as a map of R-modules, that is, there exists an R-module map $F_*R \to R$ such that the composition

$$R \xrightarrow{F} F_*R \to R$$

is the identity map.

(b) The ring R is *F-pure* if $F : R \to F_*R$ is a pure map of R-modules.

A Frobenius split ring is always F-pure. The converse is also true under modest hypothesis:

Corollary 2.5.3. *A Noetherian F-finite ring of characteristic p is Frobenius split if and only if it is F-pure.*

Proof. The F-finiteness hypothesis implies that F_*R is a finitely generated R-module. So a quotient of F_*R is also finitely generated. Since a finitely generated module over a Noetherian ring is finitely presented, the result follows from Lemma 2.5.1. □

2.6. *F-finiteness and excellence.* Although we are mainly concerned with non-Noetherian rings in this paper, it is worth pointing out the following curiosity for readers familiar with Grothendieck's concept of an *excellent ring,* a particular kind of Noetherian ring expected to be the most general setting for many algebro-geometric statements [EGA IV$_2$ 1965, définition 7.8.2].

Proposition 2.6.1. *A Noetherian domain is F-finite if and only if it is excellent and its fraction field is F-finite.*

Proof. If R is F-finite with fraction field K, then also $R \otimes_{R^p} K^p \cong K$ is finite over K^p, so the fraction field of R is F-finite. Furthermore, Kunz [1976, Theorem 2.5] showed that F-finite Noetherian rings are excellent.

We need to show that an excellent Noetherian domain with F-finite fraction field is F-finite. We make use of the following well-known property[2] of an excellent domain A: the integral closure of A in any finite extension of its fraction field is *finite* as an A-module [EGA IV$_2$ 1965, IV, 7.8.3(vi)]. The ring R^p is excellent

[2]sometimes called the *Japanese* or *N2* property.

because it is isomorphic to R, and its fraction field is K^p. Since $K^p \hookrightarrow K$ is finite, the integral closure S of R^p in K is a finite R^p-module. But clearly $R \subset S$, so R is also a finitely generated R^p module, since submodules of a Noetherian module over a Noetherian ring are Noetherian. That is, R is F-finite. □

Using this observation, we can clarify the relationship between F-purity and Frobenius splitting in an important class of rings.

Corollary 2.6.2. *For an excellent Noetherian domain whose fraction field is F-finite, Frobenius splitting is equivalent to F-purity.*

Proof. Our hypothesis implies F-finiteness, so splitting and purity are equivalent by Lemma 2.5.1. □

3. Flatness and purity of Frobenius in valuation rings

Kunz [1969, Theorem 2.1] showed that for a Noetherian ring of characteristic p, the Frobenius map is flat if and only if the ring is regular. In this section, we show how standard results on valuations yield the following result.

Theorem 3.1. *Let V be a valuation ring of characteristic p. Then the Frobenius map $F : V \to F_*V$ is faithfully flat.*

This suggests that we can imagine a valuation ring to be "regular" in some sense. Of course, a Noetherian valuation ring is either a field or a one dimensional regular local ring, but because valuation rings are rarely Noetherian, Theorem 3.1 is not a consequence of Kunz's theorem.

Theorem 3.1 follows from the following general result, whose proof we include for the sake of completeness.

Lemma 3.2 [Bourbaki 1989, VI.3.6, Lemma 1]. *A finitely generated, torsion-free module over a valuation ring is free. In particular, a torsion free module over a valuation ring is flat.*

Proof. Let $M \neq 0$ be a finitely generated, torsion-free V-module. Choose a minimal set of generators $\{m_1, \dots, m_n\}$. If there is a nontrivial relation among these generators, then there exists $v_1, \dots, v_n \in V$ (not all zero) such that $v_1 m_1 + \cdots + v_n m_n = 0$. Re-ordering if necessary, we may assume that v_1 is minimal among (nonzero) coefficients, that is, $(v_i) \subset (v_1)$ for all $i \in \{1, \dots, n\}$. Then for each $i > 1$, there exists $a_i \in V$ such that $v_i = a_i v_1$. This implies that

$$v_1(m_1 + a_2 m_2 + \cdots + a_n m_n) = 0.$$

Since $v_1 \neq 0$ and M is torsion free, we get

$$m_1 + a_2 m_2 + \cdots + a_n m_n = 0.$$

Then $m_1 = -(a_2m_2 + \cdots + a_nm_n)$. So M can be generated by the smaller set $\{m_2, \ldots, m_n\}$ which contradicts the minimality of n. Hence $\{m_1, \ldots, m_n\}$ must be a free generating set.

The second statement follows by considering a torsion-free module as a directed union of its finitely generated submodules, since a directed union of flat modules is flat [Bourbaki 1989, I.2.7 Proposition 9] □

Proof of Theorem 3.1. Observe that F_*V is a torsion free V-module. So by Lemma 3.2, the module F_*V is flat, which means the Frobenius map is flat. To see that Frobenius is faithfully flat, we need only check that $mF_*V \neq F_*V$ for m the maximal ideal of V [Bourbaki 1989, I.3.5 Proposition 9(e)]. But this is clear: the element $1 \in F_*V$ is not in mF_*V, since $1 \in V$ is not in the ideal $m^{[p]}$. □

Corollary 3.3. *Every valuation ring of characteristic p is F-pure.*

Proof. Fix a valuation ring V of characteristic p. We have already seen that the Frobenius map $V \to F_*V$ is faithfully flat (Theorem 3.1). But any faithfully flat map of rings $A \to B$ is pure as a map of A-modules [Bourbaki 1989, I.3.5 Proposition 9(c)]. □

4. F-finite valuation rings

In this section, we investigate F-finiteness in valuation rings. We first prove Theorem 4.1.1 characterizing F-finite valuation rings as those V for which F_*V is a *free* V-module. We then prove a numerical characterization of F-finiteness in terms of ramification index and residue degree for extensions of valuations under Frobenius in Theorem 4.3.1. This characterization is useful for constructing interesting examples, and later for showing that F-finite valuations are Abhyankar.

4.1. *Finiteness and freeness of Frobenius.* For any domain R of characteristic p, we have already observed (see the proof of Proposition 2.6.1) that a necessary condition for F-finiteness is the F-finiteness of its fraction field. For this reason, we investigate F-finiteness of valuation rings only in F-finite ambient fields.

Theorem 4.1.1. *Let K be an F-finite field. A valuation ring V of K is F-finite if and only if F_*V is a free V-module.*

Proof. First assume F_*V is free over V. Since $K \otimes_R F_*V \cong F_*K$ as K-vector spaces, the rank of F_*V over V must be the same as the rank of F_*K over K, namely the degree $[F_*K : K] = [K : K^p]$. Since K is F-finite, this degree is finite, and so F_*V is a free V-module of finite rank. In particular, V is F-finite.

Conversely, suppose that V is F-finite. Then F_*V is a finitely generated, torsion-free V-module. So it is free by Lemma 3.2. □

Corollary 4.1.2. *An F-finite valuation ring is Frobenius split.*

Proof. One of the rank one free summands of F_*V is the copy of V under F, so this copy of V splits off F_*V. Alternatively, since $V \to F_*V$ is pure, we can use Lemma 2.5.1: the cokernel of $V \to F_*V$ is finitely presented because it is finitely generated (being a quotient of the finitely generated V-module F_*V) and the module of relations is finitely generated (by $1 \in F_*V$). □

Remark 4.1.3. The same argument shows that any module finite extension $V \hookrightarrow S$ splits — in other words, every valuation ring is a *splinter* in the sense of [Ma 1988]; see also [Ma 1988, Lemma 1.2].

4.2. *Frobenius splitting in the Noetherian case.* We can say more for Noetherian valuation rings. First we make a general observation about F-finiteness in Noetherian rings.

Theorem 4.2.1. *For a Noetherian domain whose fraction field is F-finite, Frobenius splitting implies F-finiteness (and hence excellence).*

Before embarking on the proof, we point out a consequence for valuation rings.

Corollary 4.2.2. *For a discrete valuation ring V whose fraction field is F-finite, the following are equivalent:*

(i) *V is Frobenius split;*

(ii) *V is F-finite;*

(iii) *V is excellent.*

Proof. A DVR is Noetherian, so equivalence of (i) and (ii) follows from combining Theorem 4.2.1 and Corollary 4.1.2. The equivalence with excellence follows from Proposition 2.6.1. □

Remark 4.2.3. We have proved that all valuation rings are F-pure. However, not all valuation rings, even discrete ones on $\mathbb{F}_p(x, y)$, are Frobenius split, as Example 4.5.1 below shows.

The proof of Theorem 4.2.1 relies on the following lemma.

Lemma 4.2.4. *A Noetherian domain with F-finite fraction field is F-finite if and only if there exists $\phi \in \operatorname{Hom}_{R^p}(R, R^p)$ such that $\phi(1) \neq 0$.*

Proof. Assuming such ϕ exists, we first observe that the canonical map $R \to R^{\vee\vee}$ is injective, where $R^{\vee\vee} := \operatorname{Hom}_{R^p}(\operatorname{Hom}_{R^p}(R, R^p), R^p)$ is the double dual. Indeed, let $x \in R$ be a nonzero element. It suffices to show that there exists $f \in R^\vee := \operatorname{Hom}_{R^p}(R, R^p)$ such that $f(x) \neq 0$. Let $f = \phi \circ x^{p-1}$, where x^{p-1} is the R^p-linear map $R \to R$ given by multiplication by x^{p-1}. Then $f(x) = \phi(x^p) = x^p\phi(1) \neq 0$. This shows that the double dual map is injective.

Now, to show that R is a finitely generated R^p-module, it suffices to show that the larger module $R^{\vee\vee}$ is finitely generated. For this it suffices to show that $R^\vee$ is a finitely generated R^p-module, since the dual of a finitely generated module is finitely generated.

We now show that $R^\vee$ is finitely generated. Let M be a maximal free R^p-submodule of R. Note that M has finite rank (equal to $[K : K^p]$, where K is the fraction field of R) and that R/M is a torsion R^p-module. Since the dual of a torsion module is zero, dualizing the exact sequence $0 \to M \to R \to R/M \to 0$ induces an injection

$$R^\vee := \mathrm{Hom}_{R^p}(R, R^p) \hookrightarrow \mathrm{Hom}_{R^p}(M, R) = M^\vee.$$

Since M is a finitely generated R^p-module, also $M^\vee$, and hence its submodule $R^\vee$ is finitely generated (R is Noetherian). This completes the proof that R is F-finite.

For the converse, fix any K^p-linear splitting $\psi : K \to K^p$. Restricting to R produces an R^p-linear map to K^p. Since R is finitely generated over R^p, we can multiply by some nonzero element c of R^p to produce a nonzero map $\phi : R \to R^p$ such that $\phi(1) = c \neq 0$, completing the proof. □

Proof of Theorem 4.2.1. Let R be a domain with F-finite fraction field. A Frobenius splitting is a map $\phi \in \mathrm{Hom}_{R^p}(R, R^p)$ such that $\phi(1) = 1$. Theorem 4.2.1 then follows immediately from Lemma 4.2.4. □

4.3. ***A numerical criterion for F-finiteness.*** Consider the extension

$$K^p \subseteq K$$

where K any field of characteristic p. For any valuation v on K, let v^p denote the restriction to K^p. We next characterize F-finite valuations in terms of the ramification index and residue degree of v over v^p.

Theorem 4.3.1. *A valuation ring V of an F-finite field K of prime characteristic p is F-finite if and only if*

$$e(v/v^p) f(v/v^p) = [K : K^p],$$

where v is the corresponding valuation on K and v^p is its restriction to K^p.

Proof. First note that v is the only valuation of K extending v^p. Indeed, v is uniquely determined by its values on elements of K^p, since $v(x^p) = pv(x)$ and the value group of v is torsion-free. Furthermore, the valuation ring of v^p is easily checked to be V^p.

Observe that V is the integral closure of V^p in K. Indeed, since V is a valuation ring, it is integrally closed in K, but it is also obviously integral over V^p. We now apply Proposition 2.2.1. Since there is only one valuation extending v^p, the inequality

$$e(v/v^p) f(v/v^p) \leq [K : K^p]$$

will be an equality if and only if the integral closure of V^p in K, namely V, is finite over V^p. □

The following simple consequence has useful applications to the construction of interesting examples of F-finite and non-F-finite valuations.

Corollary 4.3.2. *Let V be a valuation ring of an F-finite field K of characteristic p. If $e(v/v^p) = [K : K^p]$ or $f(v/v^p) = [K : K^p]$, then V is F-finite.*

Remark 4.3.3. Theorem 4.3.1 and its corollary are easy to apply, because the ramification index and residue degree for the extension $V^p \hookrightarrow V$ can be computed in practice. Indeed, since Γ_{v^p} is clearly the subgroup $p\Gamma_v$ of Γ_v, we see that

$$e(v/v^p) = [\Gamma_v : p\Gamma_v]. \tag{4.3.3.1}$$

Also, the local map $V^p \hookrightarrow V$ induces the residue field extension $\kappa(v^p) \hookrightarrow \kappa(v)$, which identifies the field $\kappa(v^p)$ with the subfield $(\kappa(v))^p$. This means that

$$f(v/v^p) = [\kappa(v) : \kappa(v)^p]. \tag{4.3.3.2}$$

4.4. ***Examples of Frobenius split valuations.*** We can use our characterization of F-finite valuations to easily give examples of valuations on $\mathbb{F}_p(x, y)$ that are nondiscrete but Frobenius split.

Example 4.4.1. Consider the rational function field $K = k(x, y)$ over a perfect field k of characteristic p. For an irrational number $\alpha \in \mathbb{R}$, let Γ be the ordered additive subgroup of $\mathbb{R}$ generated by 1 and α. Consider the unique valuation $v : K^\times \to \Gamma$ determined by

$$v(x^i y^j) = i + j\alpha,$$

and let V be the corresponding valuation ring. Since $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$ via the map which sends $a + b\alpha \mapsto (a, b)$, we see that the value group of v^p is $p\Gamma \cong p(\mathbb{Z} \oplus \mathbb{Z})$. Hence

$$e(v/v^p) = [\Gamma : p\Gamma] = p^2 = [K : K^p].$$

So V is F-finite by Corollary 4.3.2. Thus V is also Frobenius split by Corollary 4.1.2.

Example 4.4.2. Consider the lex valuation on the rational function field $K = k(x_1, x_2, \ldots, x_n)$ over a perfect field k of characteristic p. This is the valuation $v : K^\times \to \mathbb{Z}^n$ on K/k defined by sending a monomial $x_1^{a_1} \cdots x_n^{a_n}$ to $(a_1, \ldots, a_n) \in \mathbb{Z}^{\oplus n}$, where $\Gamma = \mathbb{Z}^{\oplus n}$ is ordered lexicographically. Let V be the corresponding valuation ring. The value group of V^p is $p\Gamma$, so $e(v/v^p) = [\Gamma : p\Gamma] = p^n = [K : K^p]$. As in the previous example, Corollary 4.3.2 implies that V is F-finite, and so again F-split.

4.5. ***Example of a non-Frobenius split valuation.*** Our next example shows that discrete valuation rings are not always F-finite, even in the rational function field $\mathbb{F}_p(x, y)$. This is adapted from [Zariski and Samuel 1960, Example, p. 62], where it is credited to F. K. Schmidt.

Example 4.5.1. Let $\mathbb{F}_p((t))$ be the fraction field of the discrete valuation ring $\mathbb{F}_p[[t]]$ of power series in one variable. Since the field of rational functions $\mathbb{F}_p(t)$ is countable, the uncountable field $\mathbb{F}_p((t))$ cannot be algebraic over $\mathbb{F}_p(t)$. So we can find some power series

$$f(t) = \sum_{n=1}^{\infty} a_n t^n$$

in $\mathbb{F}_p[[t]]$ transcendental over $\mathbb{F}_p(t)$.

Since t and $f(t)$ are algebraically independent, there is an *injective* ring map

$$\mathbb{F}_p[x, y] \hookrightarrow \mathbb{F}_p[[t]] \quad \text{such that} \quad x \mapsto t \text{ and } y \mapsto f(t)$$

which induces an extension of fields

$$\mathbb{F}_p(x, y) \hookrightarrow \mathbb{F}_p((t)).$$

Restricting the t-adic valuation on $\mathbb{F}_p((t))$ to the subfield $\mathbb{F}_p(x, y)$ produces a discrete valuation v of $\mathbb{F}_p(x, y)$. Let V denote its valuation ring.

We claim that V is not F-finite, a statement we can verify with Theorem 4.3.1. Note that $L = \mathbb{F}_p(x, y)$ is F-finite, with $[L : L^p] = p^2$. Since the value group Γ_v is $\mathbb{Z}$, we see that

$$e(v/v^p) = [\Gamma_v : p\Gamma_v] = p.$$

On the other hand, to compute the residue degree $f(v/v^p)$, we must understand the field extension $\kappa(v)^p \hookrightarrow \kappa(v)$. Observe that for an element $u \in \mathbb{F}_p(x, y)$ to be in V, its image in $\mathbb{F}_p((t))$ must be a power series of the form

$$\sum_{n=0}^{\infty} b_n t^n,$$

where $b_n \in \mathbb{F}_p$. Clearly

$$v(u - b_0) > 0,$$

which means that the class of $u = \sum_{n=0}^{\infty} b_n t^n$ in $\kappa(v)$ is equal to the class of b_0 in $\kappa(v)$. This implies that $\kappa(v) \cong \mathbb{F}_p$, so that $[\kappa(v) : \kappa(v)^p] = 1$. That is, $f(v/v^p) = 1$.

Finally, we then have that

$$e(v/v^p) f(v/v^p) = p \neq p^2 = [\mathbb{F}_p(x, y) : (\mathbb{F}_p(x, y))^p].$$

So V *cannot* be F-finite by Theorem 4.3.1. Thus this Noetherian ring is neither Frobenius split nor excellent by Corollary 4.2.2.

4.6. *Finite extensions.* Frobenius properties of valuations are largely preserved under finite extension. First note that if $K \hookrightarrow L$ is a finite extension of F-finite fields, then $[L : L^p] = [K : K^p]$; this follows immediately from the commutative diagram of fields

$$\begin{array}{ccc} L & \hookleftarrow & K \\ \uparrow & & \uparrow \\ L^p & \hookleftarrow & K^p \end{array}$$

To wit, $[L : K^p] = [L : K][K : K^p] = [L : L^p][L^p : K^p]$ and $[L : K] = [L^p : K^p]$, so that $[L : L^p] = [K : K^p]$. Moreover, we have:

Proposition 4.6.1. *Let $K \hookrightarrow L$ be a finite extension of F-finite fields of characteristic p. Let v be a valuation on K and w an extension of v to L. Then*:

(i) *The ramification indices $e(v/v^p)$ and $e(w/w^p)$ are equal.*

(ii) *The residue degrees $f(v/v^p)$ and $f(w/w^p)$ are equal.*

(iii) *The valuation ring for v is F-finite if and only if the valuation ring for w is F-finite.*

Proof. By (2.2.0.1), we have

$$[\Gamma_w : \Gamma_v][\kappa(w) : \kappa(v)] \leq [L : K],$$

so both $[\Gamma_w : \Gamma_v]$ and $[\kappa(w) : \kappa(v)]$ are finite. Of course, we also know that the ramification indices $e(w/w^p) = [\Gamma_w : p\Gamma_w]$ and $e(v/v^p) = [\Gamma_v : p\Gamma_v]$ are finite, as are the residue degrees $f(w/w^p) = [\kappa(w) : \kappa(w)^p]$ and $f(v/v^p) = [\kappa(v) : \kappa(v)^p]$.

(i) In light of (4.3.3.1), we need to show that $[\Gamma_w : p\Gamma_w] = [\Gamma_v : p\Gamma_v]$. Since Γ_w is torsion-free, multiplication by p induces an isomorphism $\Gamma_w \cong p\Gamma_w$, under which the subgroup Γ_v corresponds to $p\Gamma_v$. Thus $[p\Gamma_w : p\Gamma_v] = [\Gamma_w : \Gamma_v]$. Using the commutative diagram of finite index abelian subgroups

$$\begin{array}{ccc} \Gamma_w & \hookleftarrow & \Gamma_v \\ \uparrow & & \uparrow \\ p\Gamma_w & \hookleftarrow & p\Gamma_v \end{array}$$

we see that $[\Gamma_w : p\Gamma_w][p\Gamma_w : p\Gamma_v] = [\Gamma_w : \Gamma_v][\Gamma_v : p\Gamma_v]$. Whence $[\Gamma_w : p\Gamma_w] = [\Gamma_v : p\Gamma_v]$.

(ii) In light of (4.3.3.2), we need to show that $[\kappa(w) : \kappa(w)^p] = [\kappa(v) : \kappa(v)^p]$. We have $[\kappa(w)^p : \kappa(v)^p] = [\kappa(w) : \kappa(v)]$, so the result follows from computing the extension degrees in the commutative diagram of finite field extensions:

$$\begin{array}{ccc} \kappa(w) & \hookleftarrow & \kappa(v) \\ \uparrow & & \uparrow \\ \kappa(w)^p & \hookleftarrow & \kappa(v)^p \end{array}$$

(iii) By (i) and (ii) we get $e(w/w^p) = e(v/v^p)$ and $f(w/w^p) = f(v/v^p)$. Therefore

$$e(w/w^p)f(w/w^p) = e(v/v^p)f(v/v^p).$$

Since also $[L : L^p] = [K : K^p]$, we see using Theorem 4.3.1 that w is F-finite if and only if v is F-finite. □

5. F-finiteness in function fields

An important class of fields are *function fields over a ground field k*. By definition, a field K is a function field over k if it is a finitely generated field extension of k. These are the fields that arise as function fields of varieties over a (typically algebraically closed) ground field k. What more can be said about valuation rings in this important class of fields?

We saw in Example 4.5.1 that not every valuation of an F-finite function field is F-finite. However, the following theorem gives a nice characterization of those that are.

Theorem 5.1. *Let K be a finitely generated field extension of an F-finite ground field k. The following are equivalent for a valuation v on K/k:*

(i) *The valuation v is Abhyankar.*

(ii) *The valuation ring R_v is F-finite.*

(iii) *The valuation ring R_v is a free R_v^p-module.*

Furthermore, when these equivalent conditions hold, it is also true that R_v is Frobenius split.

Since Abhyankar valuations have finitely generated value groups and residue fields, the following corollary holds.

Corollary 5.2. *An F-finite valuation of a function field over an F-finite field k has a finitely generated value group and its residue field is a finitely generated field extension of k.*

For example, valuations whose value groups are $\mathbb{Q}$ can never be F-finite.

Remark 5.3. In light of Proposition 2.6.1, we could add a fourth item to the list of equivalent conditions in Theorem 5.1 in the *Noetherian* case: the valuation R_v is excellent. The theorem says that the only discrete valuation rings (of function fields) that are F-finite are the divisorial valuation rings or equivalently, the excellent DVRs.

To prove Theorem 5.1, first recall that the equivalence of (ii) and (iii) was already established in Theorem 4.1.1. The point is to connect these conditions with the Abyhankar property. Our strategy is to use Theorem 4.3.1, which tells us that a valuation v on K is F-finite if and only if

$$e(v/v^p)f(v/v^p) = [K : K^p].$$

We do this by proving two propositions, one comparing the rational rank of v to the ramification index $e(v/v^p)$, and the other comparing the transcendence degree of v to the residue degree $f(v/v^p)$.

Proposition 5.4. *Let v be a valuation of rational rank s on an F-finite field K. Then*

$$e(v/v^p) \le p^s,$$

with equality when the value group Γ_v is finitely generated.

Proof. To see that equality holds when Γ_v is finitely generated, note that in this case, $\Gamma_v \cong \mathbb{Z}^{\oplus s}$. So $\Gamma_v/p\Gamma_v \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus s}$, which has cardinality p^s. That is, $e(v/v^p) = p^s$.

It remains to consider the case where Γ may not be finitely generated. Nonetheless, since $e(v/v^p)$ is finite (see (2.2.0.1)), we do know that $[\Gamma_v : p\Gamma_v] = e(v/v^p)$ is finite. So the proof of Proposition 5.4 comes down to the following simple lemma about abelian groups.

Lemma 5.5. *Let Γ be a torsion free abelian group of rational rank s. Then*

$$[\Gamma : p\Gamma] \le p^s.$$

It suffices to show that $\Gamma/p\Gamma$ is a vector space of dimension $\le s$ over $\mathbb{Z}/p\mathbb{Z}$. So let $t_1, \dots, t_n$ be elements of Γ whose classes modulo $p\Gamma$ are linearly independent over $\mathbb{Z}/p\mathbb{Z}$. Then we claim that the t_i are $\mathbb{Z}$-independent elements of Γ. Assume to the contrary that there is some nontrivial relation $a_1t_1 + \cdots + a_nt_n = 0$, for some integers a_i. Since Γ is torsion-free, we can assume without loss of generality, that at least one a_j is not divisible by p. But now modulo $p\Gamma$, this relation produces a nontrivial relation on classes of the t_i in $\Gamma/p\Gamma$, contrary to the fact that these are linearly independent. This shows that any $\mathbb{Z}/p\mathbb{Z}$-linearly independent subset of $\Gamma/p\Gamma$ must have cardinality at most s. Thus the lemma, and hence Proposition 5.4, is proved. □

Proposition 5.6. *Let K be a finitely generated field extension of an F-finite ground field k. Let v be a valuation of transcendence degree t on K over k. Then*

$$f(v/v^p) \leq p^t[k : k^p],$$

with equality when $\kappa(v)$ is finitely generated over k.

Proof. The second statement follows immediately from the following well-known fact, whose proof is an easy computation.

Lemma 5.7. *A finitely generated field L of characteristic p and transcendence degree n over k satisfies $[L : L^p] = [k : k^p]p^n$.*

It remains to consider the case where $\kappa(v)$ may not be finitely generated. Because K/k is a function field, Abhyankar's inequality (2.3.1.1) guarantees that the transcendence degree of $\kappa(v)$ over k is finite. Let $x_1, \dots, x_t$ be a transcendence basis. There is a factorization

$$k \hookrightarrow k(x_1, \dots, x_t) \hookrightarrow \kappa(v),$$

where the second inclusion is algebraic. The proposition follows immediately from the next lemma.

Lemma 5.8. *If $L' \subseteq L$ is an algebraic extension of F-finite fields, then $[L : L^p] \leq [L' : L'^p]$.*

To prove this lemma, recall that Proposition 4.6.1 ensures that $[L : L^p] = [L' : L'^p]$ when $L' \subseteq L$ is *finite.* So suppose L is algebraic but not necessarily finite over L'. Fix a basis $\{\alpha_1, \dots, \alpha_n\}$ for L over L^p, and consider the intermediate field

$$L' \hookrightarrow L'(\alpha_1, \dots, \alpha_n) \hookrightarrow L.$$

Since each α_i is algebraic over L', it follows that $\tilde{L} := L'(\alpha_1, \dots, \alpha_n)$ is finite over L', so again $[\tilde{L} : \tilde{L}^p] = [L' : L'^p]$ by Proposition 4.6.1. Now observe that $\tilde{L}^p \subset L^p$, and so the L^p-linearly independent set $\{\alpha_1, \dots, \alpha_n\}$ is also linearly independent over $\tilde{L}^p$. This means that $[L : L^p] \leq [\tilde{L} : \tilde{L}^p]$ and hence $[L : L^p] \leq [L' : L'^p]$. This proves Lemma 5.8.

Finally, Proposition 5.6 is proved by applying Lemma 5.8 to the inclusion

$$L' = k(x_1, \dots, x_t) \hookrightarrow L = \kappa(v).$$

(Note that $\kappa(v)$ is F-finite, because $[\kappa(v) : (\kappa(v))^p] = f(v/v^p) \leq [K : K^p]$ from the general inequality (2.2.0.1).) So we get

$$f(v/v^p) = [\kappa(v) : (\kappa(v))^p] \leq [k(x_1, \dots, x_t) : (k(x_1, \dots, x_t))^p] = p^t[k : k^p]. \quad \square$$

Proof of Theorem 5.1. It only remains to prove the equivalence of (i) and (ii). First assume v is Abhyankar. Then its value group Γ_v is finitely generated and its residue field $\kappa(v)$ is finitely generated over k. According to Proposition 5.4, we have $e(v/v^p) = p^s$, where s is the rational rank of v. According to Proposition 5.6, we have $f(v/v^p) = p^t[k : k^p]$, where t is the transcendence degree of v. By definition of Abhyankar, $s + t = n$, where n is the transcendence degree of K/k. But then

$$e(v/v^p)f(v/v^p) = (p^s)(p^t)[k : k^p] = p^n[k : k^p] = [K : K^p].$$

By Theorem 4.3.1, we can conclude that v is F-finite.

Conversely, we want to prove that a valuation v with F-finite valuation ring R_v is Abhyankar. Let s denote the rational rank and t denote the transcendence degree of v. From Theorem 4.3.1, the F-finiteness of v gives

$$e(v/v^p)f(v/v^p) = [K : K^p] = p^n[k : k^p].$$

Using the bounds $p^s \geq e(v/v^p)$ and $p^t[k : k^p] \geq f(v/v^p)$ provided by Propositions 5.4 and 5.6, respectively, we substitute to get

$$p^s p^t[k : k^p] \geq e(v/v^p)f(v/v^p) = p^n[k : k^p].$$

It follows that $s + t \geq n$. Then $s + t = n$ by (2.3.1.1), and v is Abhyankar. $\square$

6. F-regularity

An important class of F-pure rings are the strongly F-regular rings. Originally, strongly F-regular rings were defined only in the Noetherian F-finite case. By definition, a Noetherian F-finite reduced ring R of prime characteristic p is strongly F-regular if for every non-zerodivisor c, there exists e such that the map

$$R \to F^e_* R, \quad 1 \mapsto c$$

splits in the category of R-modules [Hochster and Huneke 1989]. In this section, we show that by replacing the word "splits" with the words "is pure" in the above definition, we obtain a well-behaved notion of F-regularity in a broader setting. Hochster and Huneke [1994, Remark 5.3] themselves suggested, but never pursued, this possibility.

Strong F-regularity first arose as a technical tool in the theory of *tight closure*: Hochster and Huneke [1994] made use of it in their deep proof of the existence of test elements. Indeed, the original motivation for (and the name of) strong F-regularity was born from a desire to better understand *weak F-regularity*, the property of a Noetherian ring characterized by all ideals being tightly closed. In many contexts, strong and weak F-regularity are known to be equivalent (see, e.g., [Lyubeznik and Smith 1999] for the graded case, [Hochster and Huneke 1989] for the Gorenstein case), but it is becoming clear that at least for many applications, strong F-regularity is the more useful and flexible notion. Applications beyond tight closure include commutative algebra more generally [Aberbach and Leuschke 2003; Blickle 2008; Schwede and Tucker 2012; Schwede 2009a; Smith and Zhang 2015], algebraic geometry [Gongyo et al. 2015; Hacon and Xu 2015; Patakfalvi 2014; Schwede and Smith 2010; Smith 2000], representation theory [Brion and Kumar 2005; Mehta and Ramanathan 1985; Ramanathan 1991; Smith and Van den Bergh 1997] and combinatorics [Benito et al. 2015].

6.1. *Basic properties of F-pure regularity.* We propose the following definition, intended to be a generalization of strong F-regularity to arbitrary commutative rings of characteristic p, not necessarily F-finite or Noetherian.

Definition 6.1.1. Let c be an element in a ring R of prime characteristic p. Then R is said to be *F-pure along* c if there exists $e > 0$ such that the R-linear map

$$\lambda_c^e : R \to F_*^e R, \quad 1 \mapsto c$$

is a pure map of R-modules. We say R is *F-pure regular* if it is F-pure along every non-zerodivisor.

A ring R is F-pure if and only if it is F-pure along the element 1. Thus F-pure regularity is a substantial strengthening of F-purity, requiring F-purity along *all* non-zerodivisors instead of just along the unit.

Remark 6.1.2. (i) If R is Noetherian and F-finite, then the map $\lambda_c^e : R \to F_*^e R$ is pure if and only if it splits (by Lemma 2.5.1). So F-pure regularity for a Noetherian F-finite ring is the same as strong F-regularity.

(ii) If c is a zerodivisor, then the map λ_c^e is never injective for any $e \geq 1$. In particular, a ring is never F-pure along a zerodivisor.

(iii) The terminology "F-pure along c" is chosen to honor Ramanathan's [1991] closely related notion of "Frobenius splitting along a divisor". See [Smith 2000].

The following proposition gathers up some basic properties of F-pure regularity for arbitrary commutative rings.

Proposition 6.1.3. *Let R be a commutative ring of characteristic p, not necessarily Noetherian or F-finite.*

(a) *If R is F-pure along some element, then R is F-pure. More generally, if R is F-pure along a product cd, then R is F-pure along the factors c and d.*

(b) *If R is F-pure along some element, then R is reduced.*

(c) *If R is an F-pure regular ring with finitely many minimal primes, and $S \subset R$ is a multiplicative set, then $S^{-1}R$ is F-pure regular. In particular, F-pure regularity is preserved under localization in Noetherian rings, as well as in domains.*

(d) *Let $\varphi : R \to T$ be a pure ring map which maps non-zerodivisors of R to non-zerodivisors of T. If T is F-pure regular, then R is F-pure regular. In particular, if $\varphi : R \to T$ is faithfully flat and T is F-pure regular, then R is F-pure regular.*

(e) *Let $R_1, \ldots, R_n$ be rings of characteristic p. If $R_1 \times \cdots \times R_n$ is F-pure regular, then each R_i is F-pure regular.*

The proof of Proposition 6.1.3 consists mostly of applying general facts about purity to the special case of the maps λ_c^e. For the convenience of the reader, we gather these basic facts together in one lemma.

Lemma 6.1.4. *Let A be an arbitrary commutative ring A, not necessarily Noetherian nor of characteristic p.*

(a) *If $M \to N$ and $N \to Q$ are pure maps of A-modules, then the composition $M \to N \to Q$ is also pure.*

(b) *If a composition $M \to N \to Q$ of A-modules is pure, then $M \to N$ is pure.*

(c) *If B is an A-algebra and $M \to N$ is pure map of A-modules, then $B \otimes_A M \to B \otimes_A N$ is a pure map of B-modules.*

(d) *Let B be an A-algebra. If $M \to N$ is a pure map of B-modules, then it is also pure as a map of A-modules.*

(e) *An A-module map $M \to N$ is pure if and only if for all prime ideals $\mathcal{P} \subset A$, $M_{\mathcal{P}} \to N_{\mathcal{P}}$ is pure.*

(f) *A faithfully flat map of rings is pure.*

(g) *If $(\Lambda, \leq)$ is a directed set with a least element λ_0, and $\{N_\lambda\}_{\lambda\in\Lambda}$ is a direct limit system of A-modules indexed by Λ and $M \to N_{\lambda_0}$ is an A-linear map, then $M \to \varinjlim_\lambda N_\lambda$ is pure if and only if $M \to N_\lambda$ is pure for all λ.*

(h) *A map of modules $A \to N$ over a Noetherian local ring (A, m) is pure if and only if $E \otimes_A A \to E \otimes_A N$ is injective, where E is the injective hull of the residue field of R.*

Proof. Properties (a)–(d) follow easily from the definition of purity and elementary properties of tensor products. As an example, let us prove (d). If P is an A-module, we want to show that $P \otimes_A M \to P \otimes_A N$ is injective. The map of B-modules

$$(P \otimes_A B) \otimes_B M \to (P \otimes_A B) \otimes_B N$$

is injective by purity of $M \to N$ as a map of B-modules. Using the natural A-module isomorphisms $(P \otimes_A B) \otimes_B M \cong P \otimes_A M$ and $(P \otimes_A B) \otimes_B N \cong P \otimes_A N$, we conclude that $P \otimes_A M \to P \otimes_A N$ is injective in the category of A-modules.

Property (e) follows from (c) by tensoring with A_p and the fact that injectivity of a map of modules is a local property. Property (f) follows from [Bourbaki 1989, I.3.5, Proposition 9(c)]. Properties (g) and (h) are proved in [Hochster and Huneke 1995, Lemma 2.1]. □

Proof of Proposition 6.1.3. (a) Multiplication by d is an R-linear map, so by restriction of scalars also

$$F^e_* R \xrightarrow{\times d} F^e_* R$$

is R-linear. Precomposing with λ^e_c we have

$$R \xrightarrow{\lambda^e_c} F^e_* R \xrightarrow{\times d} F^e_* R, \quad 1 \mapsto cd,$$

which is λ^e_{cd}. Our hypothesis that R is F-pure along cd means that there is some e for which this composition is pure. So by Lemma 6.1.4(b), it follows also that λ^e_c is pure. That is, R is F-pure along c (and since R is commutative, along d). The second statement follows since F-purity along the product $c \times 1$ implies R is F-pure along 1. So some iterate of Frobenius is a pure map, and so F-purity follows from Lemma 6.1.4(b).

(b) By (a) we see that R is F-pure. In particular, the Frobenius map is pure and hence injective, so R is reduced.

(c) Note that by (b), R is reduced. Let $\alpha \in S^{-1}R$ be a non-zerodivisor. Because R has finitely many minimal primes, a standard prime avoidance argument shows that there exists a non-zerodivisor $c \in R$ and $s \in S$ such that $\alpha = c/s$ (a minor modification of [Hochster 2007, Proposition, p. 57]). By hypothesis, R is F-pure along c. Hence there exists $e > 0$ such that the map $\lambda^e_c : R \to F^e_* R$ is pure. Then the map

$$\lambda^e_{c/1} : S^{-1}R \longrightarrow F^e_*(S^{-1}R), \quad 1 \mapsto c/1$$

is pure by 6.1.4(e) and the fact that $S^{-1}(F^e_* R) \cong F^e_*(S^{-1}R)$ as $S^{-1}R$-modules (the isomorphism $S^{-1}(F^e_* R) \cong F^e_*(S^{-1}R)$ is given by $r/s \mapsto r/s^{p^e}$). Now the $S^{-1}R$-linear map

$$\ell_{1/s} : S^{-1}R \to S^{-1}R, \quad 1 \mapsto 1/s$$

is an isomorphism. Applying F^e_*, we see that

$$F^e_*(\ell_{1/s}) : F^e_*(S^{-1}R) \to F^e_*(S^{-1}R), \quad 1 \mapsto 1/s$$

is also an isomorphism of $S^{-1}R$-modules. In particular, $F^e_*(\ell_{1/s})$ is a pure map of $S^{-1}R$-modules. So purity of

$$F^e_*(\ell_{1/s}) \circ \lambda^e_{c/1}$$

follows by 6.1.4(a). But $F^e_*(\ell_{1/s}) \circ \lambda^e_{c/1}$ is precisely the map

$$\lambda^e_{c/s} : S^{-1}R \to F^e_*(S^{-1}R), \quad 1 \mapsto c/s.$$

(d) Let $c \in R$ be a non-zerodivisor. Then $\varphi(c)$ is a non-zerodivisor in T by hypothesis. Pick $e > 0$ such that the map $\lambda^e_{\varphi(c)} : T \to F^e_* T$ is a pure map of T-modules. By 6.1.4(f) and 6.1.4(a),

$$R \xrightarrow{\varphi} T \xrightarrow{\lambda^e_{\varphi(c)}} F^e_* T$$

is a pure map of R-modules. We have commutative diagram of R-linear maps

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & T \\ \downarrow{\lambda^e_c} & & \downarrow{\lambda^e_{\varphi(c)}} \\ F^e_* R & \xrightarrow{F^e_*(\varphi)} & F^e_* T \end{array}$$

The purity of λ^e_c follows by 6.1.4(b). Note that if φ is faithfully flat, then it is pure by 6.1.4(f) and maps non-zerodivisors to non-zerodivisors.

(e) Let $R := R_1 \times \cdots \times R_n$. Consider the multiplicative set

$$S := R_1 \times \cdots \times R_{i-1} \times \{1\} \times R_{i+1} \times \cdots \times R_n.$$

Since $S^{-1}R \cong R_i$, it suffices to show that $S^{-1}R$ is F-pure regular. So let $\alpha \in S^{-1}R$ be a non-zerodivisor. Note that we can select $u \in R$ and $s \in S$ such that u is a non-zerodivisor and $\alpha = u/s$. So we can now repeat the proof of (c) verbatim to see that $S^{-1}R$ must be pure along α. □

Remark 6.1.5. It is worth observing that in Definition 6.1.1, if the map λ^e_c is a pure map, then λ^f_c is also a pure map for all $f \geq e$. Indeed, to see this note that it suffices to show that λ^{e+1}_c is pure. We know R is F-pure by 6.1.3(a). So Frobenius

$$F : R \to F_* R$$

is a pure map of R-modules. By hypothesis,

$$\lambda^e_c : R \to F^e_* R$$

is pure. Hence 6.1.4(d) tell us that

$$F_*(\lambda^e_c) : F_* R \to F_*(F^e_* R)$$

is a pure map of R-modules. Hence the composition

$$R \xrightarrow{F} F_*R \xrightarrow{F_*(\lambda_c^e)} F_*(F_*^e R), \quad 1 \mapsto c$$

is a pure map of R-modules by 6.1.4(a). But $F_*(F_*^e R)$ as an R-module is precisely $F_*^{e+1} R$. So

$$\lambda_c^{e+1} : R \to F_*^{e+1} R.$$

is pure.

Example 6.1.6. The polynomial ring over $\mathbb{F}_p$ in infinitely many variables (localized at the obvious maximal ideal) is an example of a F-pure regular ring which is not Noetherian.

6.2. ***Relationship of F-pure regularity to other singularities.*** We show that our generalization of strong F-regularity continues to enjoy many important properties of the more restricted version.

Theorem 6.2.1 [Hochster and Huneke 1989, Theorem 3.1(c)]. *A regular local ring, not necessarily F-finite, is F-pure regular.*

Proof. Let (R, m) be a regular local ring. By Krull's intersection theorem we know that

$$\bigcap_{e>0} m^{[p^e]} = 0.$$

Since R is a domain, the non-zerodivisors are precisely the nonzero elements of R. So let $c \in R$ be a nonzero element. Choose e such that $c \notin m^{[p^e]}$. We show that the map

$$\lambda_c^e : R \to F_*^e R, \quad 1 \mapsto c$$

is pure.

By Lemma 6.1.4, it suffices to check that for the injective hull E of the residue field of R, the induced map

$$\lambda_c^e \otimes \mathrm{id}_E : R \otimes_R E \to F_*^e R \otimes_R E$$

is injective, and for this, in turn, we need only check that the socle generator is not in the kernel.

Recall that E is the direct limit of the injective maps

$$R/(x_1, \ldots, x_n) \xrightarrow{x} R/(x_1^2, \ldots, x_n^2) \xrightarrow{x} R/(x_1^3, \ldots, x_n^3) \xrightarrow{x} \cdots$$

where $x_1, \ldots, x_n$ is a minimal set of generators for m, and the maps are given by multiplication by $x = \Pi_{i=1}^d x_i$. So the module $F_*^e R \otimes_R E$ is the direct limit of the maps

$$R/(x_1^{p^e},\dots,x_n^{p^e}) \xrightarrow{x^{p^e}} R/(x_1^{2p^e},\dots,x_n^{2p^e}) \xrightarrow{x^{p^e}} R/(x_1^{3p^e},\dots,x_n^{3p^e}) \xrightarrow{x^{p^e}} \cdots$$

which remains injective by the faithful flatness of $F_*^e R$. The induced map $\lambda_c^e \otimes \mathrm{id}_E : E \to F_*^e R \otimes E$ sends the socle (namely the image of 1 in R/m) to the class of c in $R/m^{[p^e]}$, so it is nonzero provided $c \notin m^{[p^e]}$. Thus for every nonzero c in a regular local (Noetherian) ring, we have found an e, such that the map λ_c^e is pure. So regular local rings are F-pure regular. □

Proposition 6.2.2. *An F-pure regular ring is normal*, *that is*, *it is integrally closed in its total quotient ring.*

Proof. Take a fraction r/s in the total quotient ring integral over R. Then clearing denominators in an equation of integral dependence, we have $r \in \overline{(s)}$, the integral closure of the ideal (s). This implies that there exists an h such that $(r, s)^{n+h} = (s)^n (r, s)^h$ for all n [Matsumura 1989, p. 64]. Setting $c = s^h$, this implies $cr^n \in (s)^n$ for all large n. In particular, taking $n = p^e$, we see that the class of r modulo (s) is in the kernel of the map induced by tensoring the map

$$R \to F_*^e R, \quad 1 \mapsto c \tag{6.2.2.1}$$

with the quotient module $R/(s)$. By purity of the map (6.2.2.1), it follows that $r \in (s)$. We conclude that r/s is in R and that R is normal. □

6.3. ***Connections with tight closure.*** In his lecture notes on tight closure, Hochster [2007] suggests another way to generalize strong F-regularity to non-F-finite (but Noetherian) rings using tight closure. We show here that his generalized strong F-regularity is the same as F-pure regularity for local Noetherian rings.

Although Hochster and Huneke introduced tight closure only in Noetherian rings, we can make the same definition in general for an arbitrary ring of prime characteristic p. Let $N \hookrightarrow M$ be an inclusion of R-modules. The *tight closure* of N in M is an R-module N_M^* containing N. By definition, an element $x \in M$ is in N_M^* if there exists $c \in R$, not in any minimal prime, such that for all sufficiently large e, the element $c \otimes x \in F_*^e R \otimes_R M$ belongs to the image of the module $F_*^e R \otimes_R N$ under the natural map $F_*^e R \otimes_R N \to F_*^e R \otimes_R M$ induced by tensoring the inclusion $N \hookrightarrow M$ with the R-module F_R^e. We say that N is tightly closed in M if $N_M^* = N$.

Definition 6.3.1. Let R be a Noetherian ring of prime characteristic p. We say that R is *strongly F-regular* in the sense of Hochster if, for any inclusion of R modules $N \hookrightarrow M$, $N_M^* = N$.

The next result compares F-pure regularity with strong F-regularity in the sense of Hochster.

Proposition 6.3.2. *Let R be an arbitrary commutative ring of prime characteristic. If R is F-pure regular, then N is tightly closed in M for any pair of R modules with $N \subset M$. The converse also holds if R is Noetherian and local.*

Proof. Suppose $x \in N^*_M$. Equivalently the class $\bar{x}$ of x in M/N is in $0^*_{M/N}$. So there exists c not in any minimal prime such that $c \otimes \bar{x} = 0$ in $F^e_* R \otimes_R M/N$ for all large e. But this means that the map

$$R \to F^e_* R, \quad 1 \mapsto c$$

is not pure for any e, since the naturally induced map

$$R \otimes M/N \to F^e_* R \otimes M/N$$

has $1 \otimes \bar{x}$ in its kernel.

For the converse, let $c \in R$ be not in any minimal prime. We need to show that there exists some e such that the map $R \to F^e_* R$ sending 1 to c is pure. Let E be the injective hull of the residue field of R. According to Lemma 6.1.4(i), it suffices to show that there exists an e such that after tensoring E, the induced map

$$R \otimes E \to F^e_* R \otimes E$$

is injective. But if not, then a generator η for the socle of E is in the kernel for every e, that is, for all e, $c \otimes \eta = 0$ in $F^e_* R \otimes E$. In this case, $\eta \in 0^*_E$, contrary to our hypothesis that all modules are tightly closed. □

Remark 6.3.3. We do not know whether Proposition 6.3.2 holds in the nonlocal case. Indeed, we do not know if F-pure regularity is a local property: if R_m is F-pure regular for all maximal ideals m of R, does it follow that R is F-pure regular? If this were the case, then our argument above extends to arbitrary Noetherian rings.

Remark 6.3.4. A Noetherian ring of characteristic p is *weakly F-regular* if N is tightly closed in M for any pair of *Noetherian* R modules with $N \subset M$. Clearly F-pure regular implies weakly F-regular. The converse is a long standing open question in the F-finite Noetherian case. For valuation rings, however, our arguments show that weak and pure F-regularity are equivalent (and both are equivalent to the valuation ring being Noetherian); See Corollary 6.5.4.

6.4. ***Elements along which F-purity fails.*** We now observe an analog of the splitting prime of Aberbach and Enescu [2005]; See also [Tucker 2012, Lemma 4.7].

Proposition 6.4.1. *Let R be a ring of characteristic p. The set*

$$\mathcal{I} := \{c \in R \, : \, R \text{ is not F-pure along } c\}.$$

is closed under multiplication by R, and $R \setminus \mathcal{I}$ is multiplicatively closed. Thus, if $\mathcal{I}$ is closed under addition, then $\mathcal{I}$ is a prime ideal (or the whole ring R).

Proof. We first note that $\mathcal{I}$ is closed under multiplication by elements of R. Indeed, suppose that $c \in \mathcal{I}$ and $r \in R$. Then if $rc \notin \mathcal{I}$, we have that R is F-pure along rc, but this implies R is F-pure along c by Proposition 6.1.3(a), contrary to $c \in \mathcal{I}$.

We next show that the complement $R \setminus \mathcal{I}$ is a multiplicatively closed set (if nonempty). To wit, take $c, d \notin \mathcal{I}$. Because R is F-pure along both c and d, we have that there exist e and f such that the maps

$$R \xrightarrow{\lambda_c^e} F_*^e R, \quad 1 \mapsto c, \quad \text{and} \quad R \xrightarrow{\lambda_d^f} F_*^f R, \quad 1 \mapsto d$$

are both pure. Since purity is preserved by restriction of scalars (Lemma 6.1.4(d)), we also have that

$$F_*^e R \xrightarrow{F_*^e(\lambda_d^f)} F_*^e F_*^f R = F_*^{e+f} R$$

is pure. Hence the composition

$$R \xrightarrow{\lambda_c^e} F_*^e R \xrightarrow{\lambda_d^f} F_*^e F_*^f R, \quad 1 \mapsto c^{p^e} d$$

is pure as well (Lemma 6.1.4(a)). This means that $c^{p^e} d$ is not in $\mathcal{I}$, and since $\mathcal{I}$ is closed under multiplication, neither is cd. Note also that if $R \setminus \mathcal{I}$ is nonempty, then $1 \in R \setminus \mathcal{I}$ by Proposition 6.1.3(a). Thus $R \setminus \mathcal{I}$ is a multiplicative set.

Finally, if $\mathcal{I}$ is closed under addition (and $\mathcal{I} \neq R$), we conclude that $\mathcal{I}$ is a prime ideal since it is an ideal whose complement is a multiplicative set. □

Remark 6.4.2. If R is a Noetherian local domain, then the set $\mathcal{I}$ of Proposition 6.4.1 can be checked to be closed under addition (see, for example, [Tucker 2012, Lemma 4.7] for the F-finite case). Likewise, for valuation rings, the set $\mathcal{I}$ is also an ideal: we construct it explicitly in the next section. However, for an arbitrary ring, $\mathcal{I}$ can fail to be an ideal. For example, under suitable hypothesis, the set $\mathcal{I}$ is also the union of the centers of F-purity in the sense of Schwede, hence in this case, $\mathcal{I}$ is a finite union of ideals but not necessarily an ideal in the nonlocal case; see [Schwede 2010].

6.5. ***F-pure regularity and valuation rings.*** In this subsection we characterize valuation rings that are F-pure regular, as summarized by the following main result.

Theorem 6.5.1. *A valuation ring is F-pure regular if and only if it is Noetherian. Equivalently, a valuation ring is F-pure regular if and only if it is a field or a DVR.*

A key ingredient in the proof is the following theorem about the set of elements along which V fails to be F-pure (see Definition 6.1.1).

Theorem 6.5.2. *The set of elements c along which a valuation ring (V, m) fails to be F-pure is the prime ideal*

$$\mathcal{Q} := \bigcap_{e>0} m^{[p^e]}.$$

Proof. First, take any $c \in \mathcal{Q}$. We need to show that V is not F-pure along c, that is, that the map

$$\lambda_c^e : V \to F_*^e V, \quad 1 \mapsto c$$

is not pure for any e. Because $c \in m^{[p^e]}$, we see that tensoring with $\kappa := V/m$ produces the zero map. So λ_c^e is not pure for any e, which means V is not F-pure along c.

For the other inclusion, let $c \notin m^{[p^e]}$ for some $e > 0$. We claim that $\lambda_c^e : V \to F_*^e V$ is pure. Apply Lemma 6.1.4(g) to the set Σ of finitely generated submodules of $F_*^e V$ which contain c. Note that Σ is a directed set under inclusion with a least element, namely the V-submodule of $F_*^e V$ generated by c, and $F_*^e V$ is the direct limit of the elements of Σ. It suffices to show that if $T \in \Sigma$, then

$$\lambda_T : V \to T, \quad 1 \mapsto c$$

is pure. But T is free since it is a finitely generated, torsion-free module over a valuation ring (Lemma 3.2). Since $c \notin m^{[p^e]}$, by the V module structure on T, we get $c \notin mT$. By Nakayama's lemma, we know c is part of a free basis for T. So λ_T splits, and is pure in particular.

Now that we know that the set of elements along which R is not F-pure is an ideal, it follows that it is a prime ideal from Proposition 6.4.1. □

Corollary 6.5.3. *For a valuation ring* (V, m) *of characteristic* p, *define*

$$\mathcal{Q} := \bigcap_{e>0} m^{[p^e]}.$$

Then the quotient $V/\mathcal{Q}$ *is an F-pure regular valuation ring. Furthermore,* V *is F-pure regular if and only if* $\mathcal{Q}$ *is zero.*

Proof. The second statement follows immediately from Theorem 6.5.2. For the first, observe that $V/\mathcal{Q}$ is a domain since $\mathcal{Q}$ is prime. So ideals of $V/\mathcal{Q}$ inherit the total ordering under inclusion from V, and $V/\mathcal{Q}$ is a valuation ring whose maximal ideal $\overline{m}$ satisfies $\bigcap_{e>0} \overline{m}^{[p^e]} = 0$. So $V/\mathcal{Q}$ is F-pure regular. □

Corollary 6.5.4. *For a valuation ring, F-pure regularity is equivalent to all ideals* (*equivalently, the maximal ideal*) *being tightly closed.*

Proof. Proposition 6.3.2 ensures that F-pure regularity implies all ideals are tightly closed. For the converse, note that if there is some nonzero c in $\bigcap_{e>0} m^{[p^e]}$, then $1 \in m^*$. So for any proper ideal m, the condition that $m^* = m$ implies that $\bigcap_{e>0} m^{[p^e]} = 0$. In particular, if the maximal ideal of a valuation ring V is tightly closed, then Corollary 6.5.3 implies that V is F-pure regular. □

Proof of Theorem 6.5.1. First observe that if V is a field or DVR, then it is F-pure regular. Indeed, every map of modules over a field is pure (since all vector space maps split). And a DVR is a one dimensional regular local ring, so it is F-pure regular by Theorem 6.2.1.

Conversely, we show that if (V, m) is F-pure regular, its dimension is at most one. Suppose (V, m) admits a nonzero prime ideal $P \neq m$. Choose $x \in m \setminus P$, and a nonzero element $c \in P$. The element c *cannot* divide x^n in V, since in that case we would have $x^n \subset (c) \subset P$, but P is a prime ideal not containing x. It then follows from the definition of a valuation ring that x^n divides c for all n. This means in particular that $c \in (x)^{[p^e]} \subset m^{[p^e]}$ for all e. So $c \in \mathcal{Q}$. According to Theorem 6.5.2, R is not F-pure regular.

It remains to show that an F-pure regular valuation ring V of dimension one is discrete. Recall that the value group Γ of V is (order isomorphic to) an additive subgroup of $\mathbb{R}$ [Matsumura 1989, Theorem 10.7].

We claim that Γ has a least positive element. To see this, let η be the greatest lower bound of all positive elements in Γ. First observe that η is strictly positive. Indeed, for fixed $c \in m$, the sequence $v(c)/p^e$ consists of positive real numbers approaching zero as e gets large. If Γ contains elements of arbitrarily small positive values, then we could find $x \in V$ such that

$$0 < v(x) < \frac{v(c)}{p^e}.$$

But then $0 < v(x^{p^e}) < v(c)$, which says that $c \in (x)^{[p^e]} \subset m^{[p^e]}$ for all e. This contradicts our assumption that V is F-pure along c (again, using Theorem 6.5.2).

Now that we know the greatest lower bound η of Γ is positive, it remains to show that $\eta \in \Gamma$. Choose ϵ such that $0 < \epsilon < \eta$. If $\eta \notin \Gamma$, we know $\eta < v(y)$ for all $y \in m$. Since η is the greatest lower bound, we can find y such that

$$\eta < v(y) < \eta + \epsilon,$$

as well as x such that

$$\eta < v(x) < v(y) < \eta + \epsilon.$$

Then

$$0 < v(y/x) < \epsilon < \eta,$$

contradicting the fact that η is a lower bound for Γ. We conclude that $\eta \in \Gamma$, and that Γ has a least positive element.

It is now easy to see, using the Archimedean axiom for real numbers, that the ordered subgroup Γ of $\mathbb{R}$ is generated by its least positive element η. In particular, Γ is order isomorphic to $\mathbb{Z}$. We conclude that V is a DVR. □

Remark 6.5.5. For a valuation ring (V, m) of dimension $n \geq 1$, our results show that in general $\mathcal{Q} = \bigcap_{e \in \mathbb{N}} m^{[p^e]}$ is a prime ideal of height at least $n - 1$. It is easy to see that the situation where $V/\mathcal{Q}$ is a DVR arises if and only if m is principal, which in turn is equivalent to the value group Γ having a least positive element. For example, this is the case for the lex valuation in Example 4.4.2. It is not hard to check that $\mathcal{Q}$ is a uniformly F-compatible ideal in the sense of Schwede [2010] (see also [Smith and Zhang 2015, §3A] for further discussion of uniformly F-compatible ideals), generalizing of course to the non-Noetherian and non-F-finite setting. A general investigation of uniformly F-compatible ideals appears to be fruitful, and is being undertaken by the first author.

6.6. ***Split F-regularity.*** Of course, there is another obvious way[3] to adapt Hochster and Huneke's definition of strongly F-regular to arbitrary rings of prime characteristic p:

Definition 6.6.1. A ring R is *split F-regular* if for all nonzero divisors c, there exists e such that the map $R \to F^e_* R$ sending 1 to c splits as a map of R-modules.

Since split maps are pure, a split F-regular ring is F-pure regular. Split F-regular rings are also clearly Frobenius split. On the other hand, Example 4.5.1 shows that a discrete valuation ring need not be Frobenius split, so split F-regularity is strictly stronger than F-pure regularity. In particular, not every regular local ring is split F-regular, so split F-regularity should not really be considered a class of "singularities" even for Noetherian rings.

Remark 6.6.2. In Noetherian rings, split F-regularity is very close to F-pure regularity. For example, if R is an F-pure regular Noetherian domain whose fraction field is F-finite, then the only obstruction to split F-regularity is the splitting of Frobenius. This is a consequence of Lemma 4.2.4, which tells us R is F-finite if it is Frobenius split, and Lemma 2.5.1, which tells us F-split and F-pure are the same in F-finite Noetherian rings.

Corollary 6.6.3. *For a discrete valuation ring V whose fraction field is F-finite, the following are equivalent*:

(i) *V is split F-regular.*

(ii) *V is Frobenius split.*

(iii) *V is F-finite.*

(iv) *V is free over V^p.*

(v) *V is excellent.*

Moreover, if K is a function field over an F-finite ground field k, and V is a valuation of K/k, then (i)–(v) *are equivalent to V being a divisorial valuation ring.*

[3]This generalization is used for cluster algebras in [Benito et al. 2015] for example.

Proof. All this has been proved already. Recall that a DVR is a regular local ring, so it is always F-pure regular and hence split F-regular if it is F-finite. Also, the final statement follows from Theorem 5.1 because an Abyhankar valuation of rational rank one is necessarily divisorial, and a divisorial valuation of a functional field over an F-finite field is necessarily F-finite. □

To summarize: a valuation ring is F-pure regular if and only if it is Noetherian, and split F-regular (under the additional assumption that its fraction field is F-finite) if and only if it is excellent.

7. Concluding remarks

We have argued that for valuation rings, F-purity and F-pure regularity (a version of strong F-regularity defined using pure maps instead of split maps) are natural and robust properties. We have also seen that the conditions of Frobenius splitting and split F-regularity are more subtle, and that even regular rings can fail to satisfy these.

For Noetherian valuation rings in F-finite fields, we have seen that the Frobenius splitting property is equivalent to F-finiteness and also to excellence, but we do not know what happens in the non-Noetherian case: does there exist an example of a (necessarily non-Noetherian) Frobenius split valuation ring of an F-finite field that is *not* F-finite? By Corollary 5.2, a possible strategy could be to construct a Frobenius split valuation ring in a function field whose value group is infinitely generated. For example, can one construct an F-split valuation in $\mathbb{F}_p(x, y)$ with value group $\mathbb{Q}$? On the other hand, perhaps Frobenius splitting is equivalent to F-finiteness (just as in the Noetherian case). One might then ask whether a generalized version of Theorem 4.1.1 holds for arbitrary fields: is a valuation ring Frobenius split if and only if Frobenius is free?

We propose that F-pure regularity is a more natural generalization of strong F-regularity to the non-F-finite case than a suggested generalization of strong F-regularity using tight closure due to Hochster. We have seen that F-pure regularity implies Hochster's notion, and that they are equivalent for local Noetherian rings. However, we do not know whether F-pure regularity is a local notion: if R_m is F-pure regular for all maximal ideals, does it follow that R is F-pure regular? We expect this to be true, but the standard arguments are insufficient to prove it. (In the Noetherian F-finite case, this is well known; see [Hochster and Huneke 1994, Theorem 5.5(a)]. Furthermore, the answer is affirmative for excellent rings with F-finite total quotient rings, by Proposition 2.6.1.) If true, then F-pure regularity would be equivalent to all modules being tightly closed in the Noetherian case. More generally, might F-pure regularity be equivalent to the property that all modules are tightly closed even in the non-Noetherian case? Or even that all ideals are tightly closed? An affirmative answer to this last question would imply that strong and weak F-regularity are equivalent.

Acknowledgements

The authors gratefully acknowledge the conversation with Karl Schwede and Zsolt Patakfalvi that inspired our investigation of Frobenius in valuation rings, and especially Karl, with whom we later had many fruitful discussions. The authors also acknowledge helpful discussions with Mel Hochster, Daniel Hernandez, Mattias Jonsson, Eric Canton, Linquan Ma, and Juan Felipe Pérez, some of which occurred at the American Mathematical Society's 2015 Mathematics Research Community in commutative algebra, attended by the first author. In addition, Karl and Linquan made detailed comments on an earlier version, which greatly improved the paper. In particular, Linquan suggested Corollary 6.5.4. We also thank the referee for helpful suggestions.

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Communicated by János Kollár
Received 2015-07-30 Accepted 2016-01-18

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

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Algebra & Number Theory

Volume 10 No. 5 2016

1937-0652(2016)10:5;1-K