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Tropical independence II: The maximal rank conjecture for quadrics

David Jensen and Sam Payne

Building on our earlier results on tropical independence and shapes of divisors in tropical linear series, we give a tropical proof of the maximal rank conjecture for quadrics. We also prove a tropical analogue of Max Noether's theorem on quadrics containing a canonically embedded curve, and state a combinatorial conjecture about tropical independence on chains of loops that implies the maximal rank conjecture for algebraic curves.

1. Introduction

Let $X \subset \mathbb{P}^r$ be a smooth curve of genus g, and recall that a linear map between finite dimensional vector spaces has *maximal rank* if it is either injective or surjective. The kernel of the restriction map

$$\rho_m : H^0(\mathbb{P}^r, \mathcal{O}(m)) \to H^0(X, \mathcal{O}(m)|_X)$$

is the space of homogeneous polynomials of degree m that vanish on X. The conjecture that ρ_m should have maximal rank for sufficiently general embeddings of sufficiently general curves, attributed to Noether in [Arbarello and Ciliberto 1983, p. 4][1], was studied classically by Severi [1915, §10], and popularized by Harris [1982, p. 79].

Maximal rank conjecture. *Suppose X is general and $V \subset \mathcal{L}(D_X)$ is a general linear series of given degree and rank. Then the multiplication maps*

$$\mu_m : \mathrm{Sym}^m V \to \mathcal{L}(mD_X)$$

have maximal rank for all m.

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[1] Noether considered the case of space curves in [Noether 1882, §8]. See also [Castelnuovo et al. 1925, pp. 172–173] for hints toward Noether's understanding of the general problem.

Recall that the general curve of genus g has a linear series of degree d and rank r if and only if the Brill–Noether number $\rho(g,r,d) = g-(r+1)(g-d+r)$ is nonnegative, and in this case there is an open dense subset of $\mathcal{M}_g$ over which the universal space parametrizing curves with a linear series of degree d and rank r is irreducible. Therefore, it makes sense to talk about a general linear series of degree d and rank r on a general curve of genus g when $\rho(g,r,d)$ is nonnegative.

Our main result gives a combinatorial condition on the skeleton of a curve over a nonarchimedean field to ensure the existence of such a linear series for which μ_2 has maximal rank. Let Γ be a chain of loops connected by bridges with admissible edge lengths, as defined in Section 4. See Figure 1 for a schematic illustration, and note that our conditions on the edge lengths are more restrictive than those in [Cools et al. 2012; Jensen and Payne 2014].

Theorem 1.1. *Let X be a smooth projective curve of genus g over a nonarchimedean field such that the minimal skeleton of the Berkovich analytic space X^{an} is isometric to Γ. Suppose $r \geq 3$, $\rho(g,r,d) \geq 0$, and $d < g+r$. Then there is a very ample complete linear series $\mathcal{L}(D_X)$ of degree d and rank r on X such that the multiplication map $\mu_2 : \operatorname{Sym}^2 \mathcal{L}(D_X) \to \mathcal{L}(2D_X)$ has maximal rank.*

Such curves do exist, over fields of arbitrary characteristic, and the condition that X^{an} has skeleton Γ ensures that X is Brill–Noether–Petri general [Jensen and Payne 2014]. As explained in Section 2, to prove the maximal rank conjecture for fixed g, r, d, and m over an algebraically closed field of given characteristic, it is enough to produce a single linear series $V \subset \mathcal{L}(D_X)$ on a single Brill–Noether–Petri general curve over a field of the same characteristic for which μ_m has maximal rank. In particular, the maximal rank conjecture for $m = 2$, and arbitrary g, r, and d, follows from Theorem 1.1, so we recover the main result of [Ballico 2012a] and extend this from characteristic zero to arbitrary characteristic; whenever the general curve of genus g admits a nondegenerate embedding of degree d in $\mathbb{P}^r$ then the image of a general nondegenerate embedding is contained in the expected number of independent quadrics.

Surjectivity of μ_m for small values of m can often be used, together with uniform position arguments, to prove surjectivity for larger values of m. See, for instance, [Arbarello et al. 1985, pp. 140–141]. When Theorem 1.1 gives surjectivity of μ_2, we apply such uniform position arguments, together with some analysis of a few special cases where uniform position is not known to hold in positive characteristic, to deduce surjectivity of μ_m for all m.

Theorem 1.2. *Let X and D_X be as in Theorem 1.1, and suppose μ_2 is surjective. Then μ_m is surjective for all $m \geq 2$.*

This proves the maximal rank conjecture for all m in the range where μ_2 is surjective, recovering the main result of [Ballico and Fontanari 2010], which determines when the general embedding of the general curve in characteristic zero is projectively normal, and extending this result to arbitrary characteristic.

Remark 1.3. The tropical methods presented here give a manifestly characteristic free approach to the maximal rank conjecture (see Conjecture 4.6). This is also the first approach to the maximal rank conjecture based on the intrinsic geometry of curves; all prior work depends in one way or another on degenerations of embedded curves in projective space. Most of these papers are written with a characteristic zero hypothesis, which is used, for instance, in uniform position arguments, but in most cases this seems to be a matter of convenience rather than necessity. Our proof of Theorem 1.2 circumvents the cases where uniform position is not known in positive characteristic, and with some care it should be possible to use similar arguments to remove the characteristic zero hypotheses from results such as those in [Ballico and Fontanari 2010; Ballico 2012a] more directly, without this tropical approach.

Remark 1.4. The maximal rank conjecture is known, for all m, when $r = 3$ [Ballico and Ellia 1987a], and in the nonspecial case $d \geq r + g$ [Ballico and Ellia 1987b]. There is a rich history of partial results on the maximal rank conjecture for $m = 2$, including some with significant applications, prior to the work of Ballico and Fontanari mentioned above. Voisin [1992, §4] proved the case of adjoint bundles of gonality pencils and deduced the surjectivity of the Wahl map for generic curves. Teixidor [2003] proved that μ_2 is injective for all linear series on the general curve when $d < g + 2$, over fields of characteristic not equal to two. Farkas proved the case where $\rho(g, r, d)$ is zero and $\dim \operatorname{Sym}^2 \mathcal{L}(D_X) = \dim \mathcal{L}(2D_X)$, and used this to deduce an infinite sequence of counterexamples to the slope conjecture [Farkas 2009, Theorem 1.5]. Another special case is Noether's theorem on canonically embedded curves, discussed below. Furthermore, Larson [2012] has proved an analogue of the maximal rank conjecture for hyperplane sections of curves.

This is only a small sampling of prior work related to the maximal rank conjecture. Other notable results include the asymptotic theorem from [Ballico and Ellia 1989]. The difficulties involved in applying the same classical degeneration method to the remaining cases of the conjecture are discussed in [Ballico and Ellia 1989, §11], and the evidence for the conjecture and best known results as of a few years ago are surveyed in [Harris 2009; Ballico 2012b].

Two key tools in the proof of Theorem 1.1 are the lifting theorem from [Cartwright et al. 2015] and the notion of *tropical independence* developed in [Jensen and Payne 2014]. The lifting theorem allows us to realize any divisor D of rank r on Γ as the tropicalization of a divisor D_X of rank r on X. Our understanding of tropical linear series on Γ, together with the nonarchimedean Poincaré–Lelong formula, produces rational functions $\{f_0, \ldots, f_r\}$ in the linear series $\mathcal{L}(D_X)$ whose tropicalizations

$\{\psi_0, \ldots, \psi_r\}$ are a specific well-understood collection of piecewise linear functions on Γ. We then show the tropical independence of a large subset of the piecewise linear functions $\{\psi_i + \psi_j\}_{0 \le i \le j \le r}$. Since $\psi_i + \psi_j$ is the tropicalization of $f_i \cdot f_j$, the size of this subset is a lower bound for the rank of μ_2, and this is the bound we use to prove Theorem 1.1.

There is no obvious obstruction to proving the maximal rank conjecture in full generality using this approach, although the combinatorics become more challenging as the parameters increase. We state a precise combinatorial conjecture in Section 4, which, for any given g, r, d, and m, implies the maximal rank conjecture for the same g, r, d, and m. We prove this conjecture not only for $m = 2$, but also for $md < 2g + 4$. (See Theorem 5.3.)

We also present advances in understanding multiplication maps by tropical methods on skeletons other than a chain of loops. Recall that Noether's theorem on canonically embedded curves says that $\mu_2 : \operatorname{Sym}^2 \mathcal{L}(K_X) \to \mathcal{L}(2K_X)$ is surjective whenever X is not hyperelliptic. This may be viewed as a strong form of the maximal rank conjecture for quadrics in the case where $r = g - 1$ and $d = 2g - 2$.

On the purely tropical side, we prove an analogue of Noether's theorem for trivalent, 3-edge-connected graphs.

Theorem 1.5. *Let Γ be a trivalent, 3-edge-connected metric graph. Then there is a tropically independent set of $3g - 3$ functions in $2R(K_\Gamma)$.*

Furthermore, we prove the appropriate lifting statements to leverage this tropical result into a maximal rank statement for canonical embeddings of curves with trivalent and 3-connected skeletons.

Theorem 1.6. *Let X be a smooth projective curve of genus g over a nonarchimedean field such that the minimal skeleton Γ of X^{an} is trivalent and 3-edge-connected with first Betti number g. Then there are $3g - 3$ rational functions in the image of $\mu_2 : \operatorname{Sym}^2 \mathcal{L}(K_X) \to \mathcal{L}(2K_X)$ whose tropicalizations are tropically independent. In particular, μ_2 is surjective.*

The last statement, on surjectivity of μ_2, also follows from Noether's theorem, because trivalent, 3-edge connected graphs are never hyperelliptic [Baker and Norine 2009, Lemma 5.3].

Remark 1.7. The present article is a sequel to [Jensen and Payne 2014], further developing the method of tropical independence. This is just one aspect of the tropical approach to linear series, an array of techniques for handling degenerations of linear series over a one parameter family of curves where the special fiber is not of compact type, combining discrete methods with computations on skeletons of Berkovich analytifications. Seminal works in the development of this theory include [Baker and Norine 2007; Baker 2008; Amini and Baker 2015]. Combined with techniques from p-adic integration, this method also leads to uniform bounds on rational points for curves of fixed genus with small Mordell–Weil rank [Katz et al. 2015].

This tropical approach is in some ways analogous to the theory of limit linear series, developed by Eisenbud and Harris in the 1980s, which systematically studies the degeneration of linear series to singular curves of compact type [Eisenbud and Harris 1986]. This theory led to simplified proofs of the Brill–Noether and Gieseker–Petri theorems [Eisenbud and Harris 1983], along with many new results about the geometry of curves, linear series, and moduli [Eisenbud and Harris 1987a; 1987b; 1987c; 1989]. Tropical methods have also led to new proofs of the Brill–Noether and Gieseker–Petri theorems [Cools et al. 2012; Jensen and Payne 2014]. Some progress has been made toward building frameworks that include both classical limit linear series and also generalizations of limit linear series for curves not of compact type [Amini and Baker 2015; Osserman 2014; 2016], which are helpful for explaining connections between the tropical and limit linear series proofs of the Brill–Noether theorem. These relations are also addressed in [Jensen and Payne 2014, Remark 1.4] and [Castorena et al. 2014]. The nature of the relations between the tropical approach and more classical approaches for results involving multiplication maps, such as the Gieseker–Petri theorem and other maximal rank results, remain unclear, as do the relations between such basic and essential facts as the Riemann–Roch theorems for algebraic and tropical curves.

Note that several families of curves appearing in proofs of the Brill–Noether and Gieseker–Petri theorems are not contained in the open subset of $\overline{\mathcal{M}}_g$ for which the maximal rank condition holds. For example, the sections of $K3$ surfaces used by Lazarsfeld [1986] in his proof of the Brill–Noether and Gieseker–Petri theorems without degenerations do not satisfy the maximal rank conjecture for $m = 2$ [Voisin 1992, Theorem 0.3 and Proposition 3.2]. Furthermore, the stabilizations of the flag curves used by Eisenbud and Harris are limits of such curves [Farkas and Popa 2005, Proposition 7.2].

2. Preliminaries

Recall that a general curve X of genus g has a linear series of rank r and degree d if and only if the Brill–Noether number

$$\rho(g, r, d) = g - (r+1)(g - d + r)$$

is nonnegative, and the scheme $\mathcal{G}^r_d(X)$ parametrizing its linear series of degree d and rank r is smooth of pure dimension $\rho(g, r, d)$. This scheme is irreducible when $\rho(g, r, d)$ is positive, and monodromy acts transitively when $\rho(g, r, d) = 0$. Therefore, if $U \subset \mathcal{M}_g$ is the dense open set parametrizing such *Brill–Noether–Petri* *general* curves, then $\mathcal{G}^r_d(U)$, the universal linear series of rank r and degree d over U, is smooth and irreducible of relative dimension $\rho(g,r,d)$. The *general linear series of degree d and rank r on a general curve of genus g* appearing in the statement of the maximal rank conjecture refers simply to a general point in the irreducible space $\mathcal{G}^r_d(U)$.

When X is Brill–Noether–Petri general and D_X is a basepoint-free divisor of rank at least 1, the basepoint-free pencil trick shows that its multiples mD_X are nonspecial for $m \geq 2$ (see Remark 2.1). Therefore, by standard upper semicontinuity arguments from algebraic geometry and the fact that $\mathcal{G}^r_d$ is defined over Spec $\mathbb{Z}$, to prove the maximal rank conjecture for fixed g, r, d, and m, over an arbitrary algebraically closed field of given characteristic, it suffices to produce a single Brill–Noether–Petri general curve X of genus g over a field of the same characteristic with a linear series $V \subset \mathcal{L}(D_X)$ of degree d and rank r such that μ_m has maximal rank. As mentioned in the introduction, the maximal rank conjecture is known when the linear series is nonspecial. In the remaining cases, the general linear series is complete, so we can and do assume that $V = \mathcal{L}(D_X)$.

Remark 2.1. Suppose D_X is a basepoint-free special divisor of rank $r \geq 1$ on a Brill–Noether–Petri general curve X. The fact that mD_X is nonspecial for $m \geq 2$ is an application of the basepoint-free pencil trick, as follows. Choose a basepoint-free pencil $V \subset \mathcal{L}(D_X)$. Then the trick identifies $\mathcal{L}(K_X - 2D_X)$ with the kernel of the multiplication map

$$\mu : V \otimes \mathcal{L}(K_X - D_X) \to \mathcal{L}(K_X).$$

The Petri condition says that this multiplication map is injective, even after replacing V by $\mathcal{L}(D_X)$. Therefore, there are no sections of K_X that vanish on $2D_X$ and hence no sections that vanish on mD_X for $m \geq 2$, which means that mD_X is nonspecial.

Remark 2.2. When $r \geq 3$ and $\rho(g,r,d) \geq 0$, the general linear series of degree d on a general curve of genus g defines an embedding in $\mathbb{P}^r$, and hence the conjecture can be rephrased in terms of a general point of the corresponding component of the appropriate Hilbert scheme. One can also consider analogues of the maximal rank conjecture for curves that are general in a given irreducible component of a given Hilbert scheme, rather than general in moduli. However, the maximal rank condition can fail when the Hilbert scheme in question does not dominate $\mathcal{M}_g$. Suppose, for example, that X is a curve of genus 8 and degree 8 in $\mathbb{P}^3$. Then $h^0(\mathcal{O}_X(2)) = 9$, and hence X is contained in a quadric surface. It follows that the kernel of μ_3 has dimension at least 4, and therefore μ_3 is not surjective. This does not contradict the maximal rank conjecture, since the general curve of genus 8 has no linear series of rank 3 and degree 8.

Proof of Theorem 1.2. We assume μ_2 is surjective. Suppose $r \geq 4$. We begin by showing that μ_3 is surjective. Note the polynomial identity

$$\binom{r+2}{2} - (2d - g + 1) = \binom{d-g}{2} - \binom{g-d+r}{2} - \rho(g, r, d).$$

(This identity reappears as Lemma 8.2, in the special case $\rho(g, r, d) = 0$.) By assumption, the left-hand side is nonnegative, as are $\rho(g, r, d)$ and $g - d + r$. It follows that $d \geq g$. By [Arbarello et al. 1985, Exercise B-6, p. 138][2], it follows that the dimension of the linear series spanned by sums of divisors in $|D_X|$ and $|2D_X|$ is at least

$$\min\{4d - 2g, 3d - g\} = 3d - g.$$

Therefore, if μ_2 is surjective then μ_3 is also surjective.

We now show, by induction on m, that μ_m is surjective for all $m > 3$. Let $V \subset \mathcal{L}(D_X)$ be a basepoint-free pencil. By the basepoint-free pencil trick, we have an exact sequence

$$0 \to \mathcal{L}((m-1)D_X) \to V \otimes \mathcal{L}(mD_X) \to \mathcal{L}((m+1)D_X).$$

Since $(m-1)D_X$ and mD_X are both nonspecial, the image of the right hand map has dimension

$$2(md - g + 1) - ((m-1)d - g + 1) = (m+1)d - g + 1,$$

hence it is surjective.

It remains to consider the cases where $r = 3$. By assumption, the divisor D_X is special, so $d < g + 3$. Furthermore, μ_2 is surjective, so $2d - g + 1 \leq 10$, and $\rho(g, r, d) \geq 0$, so $3g \leq 4d - 12$. This leaves exactly two possibilities for (g, d), namely $(4, 6)$ and $(5, 7)$. In each of these cases, $h^1(\mathcal{O}(D_X)) = 1$ and, since X is Brill–Noether general, $\mathrm{Cliff}(X) = \lfloor (g-1)/2 \rfloor$. Then $d = 2g + 1 - h^1(\mathcal{O}(D_X)) - \mathrm{Cliff}(X)$ and hence $\mathcal{O}(D_X)$ gives a projectively normal embedding, by [Green and Lazarsfeld 1986, Theorem 1]. $\square$

Since we are trying to produce a single sufficiently general curve of each genus over a field of each characteristic, we may, for simplicity, assume that we are working over an algebraically closed field that is spherically complete with respect to a valuation that surjects onto the real numbers. Any metric graph Γ of first Betti number g appears as the skeleton of a smooth projective genus g curve X over such a field (see, for instance, [Abramovich et al. 2015]).

Recall that the skeleton is a subset of the set of valuations on the function field of X, and evaluation of these valuations, also called tropicalization, takes each

[2]The statement of the exercise is missing a necessary hypothesis, that $\mathcal{D}$ has rank at least 3. The solution following the hint requires the uniform position lemma, which is known for $r \geq 3$ in characteristic zero [Harris 1980] and, over arbitrary fields, when $r \geq 4$ [Rathmann 1987].

rational function f on X to a piecewise linear function with integer slopes on Γ, denoted $\operatorname{trop}(f)$.

Our primary tool for using the skeleton of a curve and tropicalizations of rational functions to make statements about ranks of multiplication maps is the notion of tropical independence developed in [Jensen and Payne 2014].

Definition 2.3. A set of piecewise linear functions $\{\psi_0, \dots, \psi_r\}$ on a metric graph Γ is *tropically dependent* if there are real numbers $b_0, \dots, b_r$ such that for every point v in Γ the minimum

$$\min\{\psi_0(v) + b_0, \dots, \psi_r(v) + b_r\}$$

occurs at least twice. If there are no such real numbers then $\{\psi_0, \dots, \psi_r\}$ is *tropically independent*.

One key basic property of this notion is that if $\{\operatorname{trop}(f_i)\}_i$ is tropically independent on Γ, then the corresponding set of rational functions $\{f_i\}_i$ is linearly independent in the function field of X [Jensen and Payne 2014, §3.1]. Note also that if f and g are rational functions, then $\operatorname{trop}(f \cdot g) = \operatorname{trop}(f) + \operatorname{trop}(g)$.

Remark 2.4. Adding a constant to each piecewise linear function does not affect the tropical independence of a given collection. When $\{\psi_0, \dots, \psi_r\}$ is tropically dependent, we often replace each ψ_i with $\psi_i + b_i$ and assume that the minimum of the set $\{\psi_0(v), \dots, \psi_r(v)\}$ occurs at least twice at every point $v \in \Gamma$.

Lemma 2.5. *Let D_X be a divisor on X, and let $\{f_0, \dots, f_r\}$ be rational functions in $\mathcal{L}(D_X)$. If there exist k multisets $I_1, \dots, I_k \subset \{0, \dots, r\}$, each of size m, such that $\left\{\sum_{i \in I_j} \operatorname{trop}(f_i)\right\}_j$ is tropically independent, then the multiplication map*

$$\mu_m : \operatorname{Sym}^m \mathcal{L}(D_X) \to \mathcal{L}(mD_X)$$

has rank at least k.

Proof. The tropicalization of $\prod_{i \in I_j} f_i$ is the corresponding sum $\sum_{i \in I_j} \operatorname{trop}(f_i)$. If these sums $\left\{\sum_{i \in I_j} \operatorname{trop}(f_i)\right\}_j$ are tropically independent then the rational functions $\left\{\prod_{i \in I_j} f_i\right\}_j$ are linearly independent. These k rational functions are in the image of μ_m, and the lemma follows. □

Remark 2.6. If $f_0, \dots, f_r$ are rational functions in a linear series $\mathcal{L}(D_X)$, and $b_0, \dots, b_r$ are real numbers, then the pointwise minimum

$$\theta = \min\{\operatorname{trop}(f_0) + b_0, \dots, \operatorname{trop}(f_r) + b_r\}$$

is the tropicalization of a rational function in $\mathcal{L}(D_X)$. The rational function may be chosen of the form $a_0 f_0 + \cdots + a_r f_r$ where a_i is a sufficiently general element of the ground field such that $\operatorname{val}(a_i) = b_i$.

We will also repeatedly use the following basic fact about the shapes of divisors associated to a pointwise minimum of functions in a tropical linear series.

Shape lemma for minima [Jensen and Payne 2014, Lemma 3.4]. *Let D be a divisor on a metric graph Γ, with $\psi_0, \dots, \psi_r$ piecewise linear functions in $R(D)$, and let*

$$\theta = \min\{\psi_0, \dots, \psi_r\}.$$

Let $\Gamma_j \subset \Gamma$ be the closed set where θ is equal to ψ_j. Then $\operatorname{div}(\theta) + D$ contains a point $v \in \Gamma_j$ if and only if v is in either

(1) *the divisor $\operatorname{div}(\psi_j) + D$, or*

(2) *the boundary of Γ_j.*

In [Jensen and Payne 2014], this shape lemma for minima is combined with another lemma about shapes of canonical divisors to reach the contradiction that proves the Gieseker–Petri theorem.

3. Max Noether's theorem

Here we examine functions in the canonical and 2-canonical linear series using trivalent and 3-edge-connected graphs. This section is not logically necessary for the proof of Theorem 1.1, and can be safely skipped by a reader who is interested only in the proof of the maximal rank conjecture for quadrics. Nevertheless, the two are not unrelated and we include this section because, as explained in the introduction, Noether's theorem is a strong form of one case of the maximal rank conjecture for quadrics. Also, the arguments presented here illustrate the potential for applying our methods to the study of linear series and multiplication maps using skeletons other than a chain of loops, which may be important for future work.

Our arguments in this section depend on a careful analysis of the loci where piecewise linear functions attain their minima. Recall that, for a divisor D on a metric graph Γ, the tropical linear series $R(D)$ is the set of piecewise linear functions with integer slope ψ on Γ such that $\operatorname{div}(\psi) + D$ is effective. The tropical linear series $R(D)$ is a tropical module, which means that it is closed under scalar addition and pointwise minimum [Haase et al. 2012, Lemma 4]. For $v \in \Gamma$, we write $\deg_v(D)$ for the coefficient of v in the divisor D, and for a piecewise linear function ψ, we write

$$\Gamma_\psi = \{v \in \Gamma \mid \psi(v) = \min_{w \in \Gamma} \psi(w)\}$$

for the subgraph on which ψ attains its global minimum.

Lemma 3.1. *Let D be a divisor on Γ with $\psi \in R(D)$. Then, for any point $v \in \Gamma_\psi$,*

$$\operatorname{outdeg}_{\Gamma_\psi}(v) \leq \deg_v(D),$$

where $\mathrm{outdeg}_{\Gamma_\psi}(v)$ *denotes the number of tangent vectors based at* v *that are not contained in* Γ_ψ.

Proof. Since ψ obtains its minimum value at v, all of the outgoing slopes of ψ at v are nonnegative, and those along edges that are not in Γ_ψ are strictly positive. Since all of these slopes are integers and $\mathrm{div}(\psi) + D$ is effective, it follows that $\mathrm{outdeg}_{\Gamma_\psi}(v)$ is at most $\deg_v(D)$. □

Recall that the canonical divisor K_Γ is given by

$$\deg_v(K_\Gamma) = \mathrm{val}(v) - 2,$$

where $\mathrm{val}(v)$ is the valence (or number of outgoing edges) of v in Γ. The following lemma restricts the loci on which functions in $R(K_\Gamma)$ attain their minimum.

Lemma 3.2. *Let* ψ *be a piecewise linear function in* $R(K_\Gamma)$. *Then the subgraph* Γ_ψ *on which* ψ *attains its minimum is a union of edges in* Γ *and has no leaves.*

Proof. By Lemma 3.1, the outdegree $\mathrm{outdeg}_v(\Gamma_\psi)$ is at most $\deg(v) - 2$ at each point $v \in \Gamma_\psi$. It follows that any edge which contains a point of Γ_ψ in its interior is entirely contained in ψ, and the number of edges in Γ_ψ containing any vertex v is at least two, so Γ_ψ has no leaves. □

As a first application, we show that every loop in Γ is the locus where some function in $R(K_\Gamma)$ attains its minimum, and that this function lifts to a canonical section on any totally degenerate curve whose skeleton is Γ. Here, a *loop* is an embedded circle in Γ or, equivalently, a connected subgraph in which every point has valence 2.

Proposition 3.3. *Let* Γ *be a metric graph and let* $\Gamma' \subset \Gamma$ *be a loop. Then there is a function* $\psi \in R(K_\Gamma)$ *such that the subgraph* Γ_ψ *on which* ψ *attains its minimum is exactly* Γ'.

Furthermore, if X *is a smooth projective curve over a nonarchimedean field such that the minimal skeleton of the Berkovich analytic space* X^{an} *is isometric to* Γ, *and* K_X *is a canonical divisor that tropicalizes to* K_Γ, *then* ψ *can be chosen to be* $\mathrm{trop}(f)$ *for some* $f \in \mathcal{L}(K_X)$.

Proof. Let g be the first Betti number of Γ. Choose points $v_1, \ldots, v_{g-1}$ of valence 2 in $\Gamma \smallsetminus \Gamma'$ such that $\Gamma \smallsetminus \{v_1, \ldots, v_{g-1}\}$ is connected. Since K_Γ has rank $g - 1$, there is a divisor $D \sim K_\Gamma$ such that $D - v_1 - \cdots - v_{g-1}$ is effective. Let ψ be a piecewise linear function such that $K_\Gamma + \mathrm{div}(\psi) = D$.

By Lemma 3.1, the subgraph $\Gamma_\psi \subset \Gamma$ where ψ attains its minimum is a union of edges of Γ and has no leaves. Since $\mathrm{ord}_{v_i}(\psi)$ is positive for $1 \leq i \leq g - 1$, it follows that Γ_ψ does not contain any of the points $v_1, \ldots, v_{g-1}$. Being a subgraph of $\Gamma \smallsetminus \{v_1, \ldots, v_{g-1}\}$, the first Betti number of Γ_ψ is at most 1. On the other hand, every point has valence at least two in Γ_ψ. It follows that Γ_ψ is a loop, and hence must be the unique loop Γ' contained in $\Gamma \smallsetminus \{v_1, \dots, v_{g-1}\}$.

We now prove the last part of the proposition. Let $p_1, \dots, p_{g-1}$ be points in X specializing to $v_1, \dots, v_{g-1}$, respectively. Since K_X has rank $g-1$, there is a rational function $f \in \mathcal{L}(K_X)$ such that $\operatorname{div}(f) + K_\Gamma - p_1 - \cdots - p_{g-1}$ is effective. From this we see that $\operatorname{div}(\operatorname{trop}(f)) + K_\Gamma - v_1 - \cdots - v_{g-1}$ is effective, and the proposition follows. □

Our next lemma controls the locus where a piecewise linear function in $R(2K_\Gamma)$ attains its minimum, when Γ is trivalent.

Lemma 3.4. *Suppose Γ is trivalent and let ψ be a piecewise linear function in $R(2K_\Gamma)$. Then Γ_ψ is a union of edges in Γ.*

Proof. If $v \in \Gamma_\psi$ lies in the interior of an edge of Γ then, by Lemma 3.1, we have $\operatorname{outdeg}_{\Gamma_\psi}(v) = 0$, so Γ_ψ contains the entire edge. On the other hand, if $v \in \Gamma_\psi$ is a trivalent vertex of Γ then Lemma 3.1 says that $\operatorname{outdeg}_{\Gamma_\psi}(v) \le 2$. It follows that Γ_ψ contains at least one of the three edges adjacent to v. □

We conclude this section by applying this lemma and the preceding proposition together with Menger's theorem to prove Theorem 1.5, the analogue of Noether's theorem for trivalent 3-connected graphs.

Remark 3.5. A similar application of Menger's theorem is used to prove an analogue of Noether's theorem for graph curves in [Bayer and Eisenbud 1991, §4].

Proof of Theorems 1.5 and 1.6. Assume Γ is trivalent and 3-edge-connected. Let $e \subset \Gamma$ be an edge with endpoints v and w. Since Γ is 3-edge-connected, Menger's theorem says that there are two distinct paths from v to w that do not share an edge and do not pass through e. Equivalently, there are two loops Γ_1 and Γ_2 in Γ such that $\Gamma_1 \cap \Gamma_2 = e$. By Proposition 3.3 there are functions ψ_1 and ψ_2 in $R(K_\Gamma)$ such that $\Gamma_{\psi_i} = \Gamma_i$. We write $\psi_e = \psi_1 + \psi_2$, which is a piecewise linear function in $R(2K_\Gamma)$. Note that $\Gamma_{\psi_e} = e$. Furthermore, again by Proposition 3.3, if X is a curve with skeleton Γ and K_X is a canonical divisor tropicalizing to K_Γ, then we can choose f_1 and f_2 in $\mathcal{L}(K_X)$ such that $\psi_i = \operatorname{trop}(f_i)$, and hence $\psi_e = \operatorname{trop}(f_1 \cdot f_2)$ is the tropicalization of a function in the image of $\mu_2 : \operatorname{Sym}^2(\mathcal{L}(K_X)) \to \mathcal{L}(2K_X)$.

We claim that the set of $3g-3$ functions $\{\psi_e\}_e$ is tropically independent. Suppose not. Then there are constants b_e such that $\min_e\{\psi_e + b_e\}$ occurs twice at every point of Γ. Let

$$\theta = \min_e\{\psi_e + b_e\},$$

which is a piecewise linear function in $R(2K_\Gamma)$. By Lemma 3.4, the function θ achieves its minimum along an edge, and hence there must be two functions in the set $\{\psi_e + b_e\}_e$ that achieve their minima along this edge. However, by construction, the functions $\psi_e + b_e$ achieve their minima along distinct edges, which is a contradiction. We conclude that $\{\psi_e\}_e$ is tropically independent, as claimed. □

4. Special divisors on a chain of loops

For the remainder of the paper, we focus our attention on a chain of loops with bridges Γ, as pictured in Figure 1. Here, we briefly recall the classification of special divisors on Γ from [Cools et al. 2012], along with the characterization of vertex avoiding classes and their basic properties.

The graph Γ has $2g+2$ vertices, one on the left-hand side of each bridge, which we label $w_0, \dots, w_g$, and one on the right-hand side of each bridge, which we label $v_1, \dots, v_{g+1}$. There are two edges connecting the vertices v_k and w_k, the top and bottom edges of the k-th loop, whose lengths are denoted ℓ_k and m_k, respectively, as shown schematically in Figure 1. For $1 \leq k \leq g+1$ there is a bridge connecting w_k and v_{k+1}, which we refer to as the k-th bridge β_k, of length n_k. Throughout, we assume that Γ has admissible edge lengths in the following sense, which is stronger than the genericity conditions in [Cools et al. 2012; Jensen and Payne 2014].

Definition 4.1. The graph Γ has *admissible edge lengths* if

$$4gm_k < \ell_k \ll \min\{n_{k-1}, n_k\} \text{ for all } k,$$

and there are no nontrivial linear relations $c_1 m_1 + \cdots + c_g m_g = 0$ with integer coefficients of absolute value at most $g+1$.

Remark 4.2. The inequality $4gm_k < \ell_k$ is required to ensure that the shapes of the functions ψ_i and ψ_{ij} are as described in Sections 6 and 7. Both inequalities are used in the proof of Lemma 6.2, and the required upper bound on ℓ_k depends on the size of the multisets. For multisets of size m, we assume $2m\ell_k < \min\{n_{k-1}, n_k\}$. In particular, for Theorem 1.1, the inequality $4\ell_k < \min\{n_{k-1}, n_k\}$ would suffice. The condition on integer linear relations is used in the proof of Proposition 7.6.

The special divisor classes on a chain of loops, i.e., the classes of effective divisors D such that $r(D) > \deg(D) - g$, are explicitly classified in [Cools et al. 2012]. Every effective divisor on Γ is equivalent to an effective w_0-reduced divisor, which has d_0 chips at the vertex w_0, together with at most one chip on every loop.

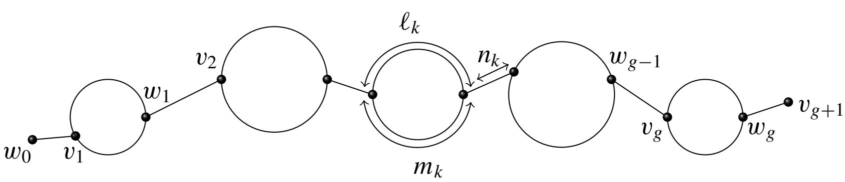

Figure 1. The graph Γ.

We may therefore associate to each equivalence class the data $(d_0, x_1, x_2, \dots x_g)$, where $x_i \in \mathbb{R}/(\ell_i + m_i)\mathbb{Z}$ is the distance from v_i to the chip on the i-th loop in the counterclockwise direction, if such a chip exists, and $x_i = 0$ otherwise. The associated lingering lattice path in $\mathbb{Z}^r$, whose coordinates we number from 0 to $r-1$, is a sequence of points $p_0, \dots, p_g$ starting at

$$p_0 = (d_0, d_0 - 1, \dots, d_0 - r + 1).$$

We write $p_i(j)$ for the j-th coordinate of p_i. With this notation, the i-th step in the lingering lattice path is given by

$$p_i - p_{i-1} = \begin{cases} (-1, -1, \dots, -1) & \text{if } x_i = 0, \\ e_j & \text{if } x_i = (p_{i-1}(j) + 1)m_i \mod \ell_i + m_i \\ & \text{and both } p_{i-1} \text{ and } p_{i-1} + e_j \text{ are in } \mathcal{C}, \\ 0 & \text{otherwise,} \end{cases}$$

where $e_0, \dots e_{r-1}$ are the basis vectors in $\mathbb{Z}^r$ and $\mathcal{C}$ is the set of lattice points in the open Weyl chamber

$$\mathcal{C} = \{y \in \mathbb{Z}^r \mid y_0 > \cdots > y_{r-1} > 0\}.$$

By [Cools et al. 2012, Theorem 4.6], a divisor D on Γ has rank at least r if and only if the associated lingering lattice path lies entirely in the open Weyl chamber $\mathcal{C}$.

Remark 4.3. Although the lingering lattice path associated to D, as defined above and in [Cools et al. 2012], is a sequence of points in $\mathbb{Z}^r$ with coordinates labeled from 0 to $r-1$, we find it convenient to consider this $\mathbb{Z}^r$ as being embedded in $\mathbb{Z}^{r+1}$, with coordinates labeled from 0 to r, as the sublattice in which the last coordinate is zero. In other words, we set $p_j(r) = 0$ for all j.

The steps in the direction 0 are referred to as *lingering steps*, and the number of lingering steps cannot exceed the Brill–Noether number $\rho(g, r, d)$. In the case where $\rho(g, r, d) = 0$, such lattice paths are in bijection with rectangular tableaux of size $(r+1) \times (g - d + r)$. This bijection is given as follows. We label the columns of the tableau from 0 to r and place i in the j-th column when the i-th step is in the direction e_j, and we place i in the last column when the i-th step is in the direction $(-1, \dots, -1)$.

An open dense subset of the special divisor classes of degree d and rank r on Γ are *vertex avoiding*, in the sense of [Cartwright et al. 2015, Definition 2.3], which means that

- the associated lingering lattice path has exactly $\rho(g, r, d)$ lingering steps,
- for any i, $x_i \neq m_i \mod (\ell_i + m_i)$, and
- for any i and j, $x_i \neq (p_{i-1}(j))m_i \mod (\ell_i + m_i)$.

Vertex avoiding classes come with a useful collection of canonical representatives. If D is a divisor of rank r on Γ whose class is vertex avoiding, then there is a unique effective divisor $D_i \sim D$ such that $\deg_{w_0}(D_i) = i$ and $\deg_{v_{g+1}}(D_i) = r - i$. Equivalently, D_i is the unique divisor equivalent to D such that $D_i - i w_0 - (r-i) v_{g+1}$ is effective. Furthermore,

- the divisor D_i has no points on any of the bridges,
- for $i < r$, the divisor D_i fails to have a point on the j-th loop if and only if the j-th step of the associated lingering lattice path is in the direction e_i,
- the divisor D_r fails to have a point on the j-th loop if and only if the j-th step of the associated lingering lattice path is in the direction $(-1, \dots, -1)$.

Notation 4.4. Throughout, we let X be a smooth projective curve of genus g whose analytification has skeleton Γ. For the remainder of the paper, we let D be a w_0-reduced divisor on Γ of degree d and rank r whose class is vertex avoiding, D_X a lift of D to X, and ψ_i a piecewise linear function on Γ such that $D + \operatorname{div}(\psi_i) = D_i$. By a *lift* of D to X, we mean that D_X is a divisor of degree d and rank r on X whose tropicalization is D.

Note that ψ_i is uniquely determined up to an additive constant, and for $i < r$ the slope of ψ_i along the bridge β_j is $p_j(i)$. In this context, being w_0-reduced means that $D = D_r$, so the function ψ_r is constant. In particular, the functions $\psi_0, \dots, \psi_r$ have distinct slopes along bridges, so $\{\psi_0, \dots, \psi_r\}$ is tropically independent. Recall that, for convenience, we set $p_j(r) = 0$ for all j.

Proposition 4.5. *There is a rational function* $f_i \in \mathcal{L}(D_X)$ *such that* $\operatorname{trop}(f_i) = \psi_i$.

Proof. The proof is identical to the proof of [Jensen and Payne 2014, Proposition 6.5], which is the special case where $\rho(g, r, d) = 0$. □

When $\rho(g, r, d) = 0$, all divisor classes of degree d and rank r are vertex avoiding. Note that, since $\{\psi_0, \dots, \psi_r\}$ is tropically independent of size $r + 1$, the set of rational functions $\{f_0, \dots, f_r\}$ is a basis for $\mathcal{L}(D_X)$.

For a multiset $I \subset \{0, \dots, r\}$ of size m, let $D_I = \sum_{i \in I} D_i$ and let ψ_I be a piecewise linear function such that $mD + \operatorname{div} \psi_I = D_I$. By construction, the function ψ_I is in $R(mD)$ and agrees with $\sum_{i \in I} \psi_i$ up to an additive constant.

Conjecture 4.6. *Suppose* $r \geq 3$, $\rho(g, r, d) \geq 0$, *and* $d < g + r$. *Then there is a divisor* D *of rank* r *and degree* d *whose class is vertex avoiding on a chain of loops* Γ *with generic edge lengths, and a tropically independent subset* $\mathcal{A} \subset \{\psi_I \mid \#I = m\}$ *of size*

$$\#\mathcal{A} = \min\left\{ \binom{r+m}{m},\ md - g + 1 \right\}.$$

The conjecture is trivial for $r = 0$ and easy for $r = 1$, since the functions $k\psi_0$ have distinct nonzero slopes on every bridge and hence $\{0, \psi_0, \dots, m\psi_0\}$ is tropically independent. Yet another easy case is $m = 1$, since $\{\psi_0, \dots \psi_r\}$ is tropically independent. In the remainder of the paper we prove the conjecture for $m = 2$ and for $md < 2g + 4$.

Proposition 4.7. *For any fixed g, r, d, and m, the maximal rank conjecture follows from Conjecture 4.6.*

Proof. Choose a smooth projective curve X over a nonarchimedean field whose skeleton is Γ. Then X is Brill–Noether–Petri general [Jensen and Payne 2014] and D lifts to a divisor D_X of degree d and rank r on X [Cartwright et al. 2015]. We may assume $r \geq 1$, and it follows that mD_X is nonspecial for $m \geq 2$ by Remark 2.1. By Lemma 2.5, the rank of μ_m is at least as large as any set $\mathcal{A}$ such that $\{\psi_I \mid I \in \mathcal{A}\}$ is tropically independent. Therefore, Conjecture 4.6 implies that $\mu_m : \mathrm{Sym}^m \mathcal{L}(D_X) \to \mathcal{L}(mD_X)$ has maximal rank and, as discussed in Section 2, the maximal rank conjecture for g, r, d, and m follows. □

5. Two points on each loop

Let D be a w_0-reduced vertex avoiding divisor on Γ. We continue to use the notation established in the previous section and recall, in particular, that since D is vertex avoiding there are piecewise linear functions ψ_i, unique up to an additive constant, such that $D + \operatorname{div} \psi_i$ is the unique effective divisor equivalent to D such that $\deg_{w_0}(D_i) = i$ and $\deg_{v_{g+1}}(D_i) = r - i$. Furthermore, since D is w_0-reduced, $D = D_r$ and ψ_r is a constant function.

We now show that any nontrivial tropical dependence among the piecewise linear functions $\psi_I = \sum_{i \in I} \psi_i$, for multisets I of size m, gives rise to a divisor equivalent to mD with degree at least 2 at w_0, degree at least 2 at v_{g+1}, and degree at least 2 on each loop. As a consequence, we deduce Theorem 5.3, which confirms Conjecture 4.6 and the maximal rank conjecture for $md < 2g + 4$.

Lemma 5.1. *Let I and J be distinct multisets of size m. Then, for each loop γ° in Γ, the restrictions $D_I|_{\gamma^\circ}$ and $D_J|_{\gamma^\circ}$ are distinct.*

Proof. Suppose γ° is the j-th loop. Let q_i be the point on γ° whose distance from v_j in the counterclockwise direction is $x_j - p_{j-1}(i)m_j$. Then the degree of q_i in D_I is equal to the multiplicity of i in the multiset I, unless the j-th step of the lingering lattice path is in the direction e_i, in which case the degree of q_i in D_I is zero. It follows that the multiset I can be recovered from the restriction $D_I|_{\gamma^\circ}$. □

Let θ be the piecewise linear function

$$\theta = \min_I\{\psi_I\},$$

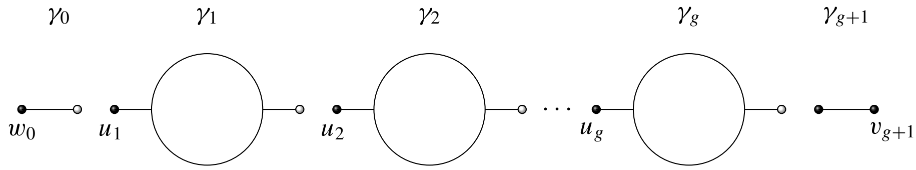

Figure 2. Decomposition of the graph Γ into locally closed pieces $\{\gamma_k\}$.

which is in $R(mD)$, and let Δ be the corresponding effective divisor

$$\Delta = mD + \operatorname{div}\theta.$$

By Lemma 5.1, no two functions ψ_I can agree on an entire loop, so if the minimum occurs everywhere at least twice on a loop, then there are at least three functions ψ_I that achieve the minimum at some point of the loop. We will study θ and Δ by systematically using observations like this one, examining behavior on each piece of Γ and controlling which functions ψ_I can achieve the minimum at some point in each loop.

Recall that, for $0 \le k \le g$, the k-th bridge β_k connects w_k to v_{k+1}. Let u_k be the midpoint of β_{k-1}. We then decompose Γ into $g+2$ locally closed subgraphs $\gamma_0, \dots, \gamma_{g+1}$, as follows. The subgraph γ_0 is the half-open interval $[w_0, u_1)$. For $1 \le i \le g$, the subgraph γ_i, which includes the i-th loop of Γ, is the union of the two half-open intervals $[u_i, u_{i+1})$, which contain the top and bottom edges of the i-th loop, respectively. Finally, the subgraph γ_{g+1} is the closed interval $[u_{g+1}, v_{g+1}]$. We further write γ_i° for the i-th embedded loop in Γ, which is a closed subset of γ_i, for $1 \le i \le g$. The decomposition

$$\Gamma = \gamma_0 \sqcup \cdots \sqcup \gamma_{g+1}$$

is illustrated by Figure 2.

Proposition 5.2. *Suppose the minimum of $\{\psi_I(v)\}_I$ occurs at least twice at every point v in Γ. Then $\deg_{w_0}(\Delta)$, $\deg_{v_{g+1}}(\Delta)$, and $\deg(\Delta|_{\gamma_i^\circ})$ are all at least 2.*

Proof. Note that exactly one function ψ_I has slope mr on the first bridge; this is the function corresponding to the multiset $I = \{0, \dots, 0\}$. Similarly, the only multiset that gives slope $mr - 1$ is $\{1, 0, \dots, 0\}$. Therefore, if the minimum occurs twice along the first bridge, then the outgoing slope of θ at w_0 is at most $mr - 2$, and hence $\deg_{w_0}(\Delta) \ge 2$, as required. Similarly, we have $\deg_{v_{g+1}}(\Delta) \ge 2$.

It remains to show that $\deg(\Delta|_{\gamma_i^\circ}) \ge 2$ for $1 \le i \le g$. Choose a point $v \in \gamma_i^\circ$. By assumption, there are at least two distinct multisets I and I' such that both ψ_I and $\psi_{I'}$ obtain the minimum on some closed interval containing v. By Lemma 5.1, the functions ψ_I and $\psi_{I'}$ do not agree on all of γ_i°, so there is another point $v' \in \gamma_i^\circ$ where at least one of these two functions does not obtain the minimum. Without loss of generality, assume that ψ_I does not obtain the minimum at v'. Then ψ_I obtains the minimum on a proper closed subset of γ_i°, and since γ_i° is a loop, this set has outdegree at least two. By the shape lemma for minima (see Section 2), it follows that $(\mathrm{div}(\theta) + mD)|_{\gamma_i^\circ}$ has degree at least two. □

As an immediate application of this proposition, we prove Conjecture 4.6 for $md < 2g + 4$.

Theorem 5.3. *If $md < 2g + 4$ then $\{\psi_I \mid \#I = m\}$ is tropically independent.*

Proof. Suppose that $\{\psi_I\}_I$ is tropically dependent. After adding a constant to each ψ_I, we may assume that the minimum $\theta(v) = \min_I \psi_I(v)$ occurs at least twice at every point v in Γ. By Proposition 5.2, the restriction of $\Delta = mD + \mathrm{div}(\theta)$ to each of the $g+2$ locally closed subgraphs $\gamma_k \subset \Gamma$ has degree at least two. Therefore the degree of Δ is at least $2g + 4$, and the theorem follows. □

In particular, the maximal rank conjecture holds for $md < 2g + 4$. This partially generalizes the case where $m = 2$ and $d < g + 2$, proved by Teixidor i Bigas [2003]. Note, however, that [Teixidor 2003] proves that the maximal rank condition holds for *all* divisors of degree less than $g+2$, whereas Theorem 5.3 implies this statement only for a general divisor.

6. Permissible functions

In the preceding section, we introduced a decomposition of Γ as the disjoint union of locally closed subgraphs $\gamma_0, \ldots, \gamma_{g+1}$ and proved that if $\theta(v) = \min_I \psi_I(v)$ occurs at least twice at every point v in γ_i then the degree of $\Delta = mD + \mathrm{div}(\theta)$ restricted to γ_i is at least 2. These degrees of restrictions $\Delta|_{\gamma_i}$ appear repeatedly throughout the rest of the paper, so we fix

$$\delta_i = \deg(\Delta|_{\gamma_i}).$$

By Proposition 5.2, we have $\delta_i \geq 2$ for all i.

We now discuss how the nonnegative integer vector $\delta = (\delta_0, \ldots, \delta_{g+1})$ restricts the multisets I such that ψ_I can achieve the minimum on the k-th loop of Γ.

For $a \leq b$, let $\Gamma_{[a,b]}$ be the locally closed, connected subgraph

$$\Gamma_{[a,b]} = \gamma_a \sqcup \cdots \sqcup \gamma_b.$$

Note that the degrees of divisors in a tropical linear series restricted to such subgraphs are governed by the slopes of the associated piecewise linear functions, as follows.

Suppose $\Gamma' \subset \Gamma$ is a closed connected subgraph and ψ is a piecewise linear function with integer slopes on Γ. Then $\mathrm{div}(\psi|_{\Gamma'})$ has degree zero and the multiplicity of each boundary point $v \in \partial\Gamma'$ is the sum of the incoming slopes at v, along the edges in Γ'. Now $\mathrm{div}(\psi)|_{\Gamma'}$ agrees with $\mathrm{div}(\psi|_{\Gamma'})$ except at the boundary points and a simple computation at the boundary points of the locally closed subgraph γ_k, for $1 \leq k \leq g$ shows that

$$\deg(\operatorname{div}(\psi)|_{\gamma_k}) = \sigma_k(\psi) - \sigma_{k+1}(\psi),$$

where $\sigma_k(\psi)$ is the incoming slope of ψ from the left at u_k. Similarly,

$$\deg(\operatorname{div}(\psi)|_{\Gamma_{[0,k]}}) = -\sigma_{k+1}(\psi).$$

Our indexing conventions for lingering lattice paths are chosen for consistency with [Cools et al. 2012], and with this notation we have

$$\sigma_k(\psi_i) = p_{k-1}(i).$$

These slopes, and the conditions on the edge lengths on Γ, lead to restrictions on the multisets I such that ψ_I achieves the minimum at some point in the k-th loop γ_k°.

Definition 6.1. Let $I \subset \{0, \dots, r\}$ be a multiset of size m. We say that ψ_I is *δ-permissible* on γ_k° if

$$\deg(D_I|_{\Gamma_{\leq k-1}}) \geq \delta_0 + \cdots + \delta_{k-1}$$

and

$$\deg(D_I|_{\Gamma_{\leq k}}) \leq \delta_0 + \cdots + \delta_k.$$

We say that ψ_I is *δ-permissible* on $\Gamma_{[a,b]}$ if there is some $k \in [a, b]$ such that ψ_I is δ-permissible on γ_k°.

Lemma 6.2. *If $\psi_I(v) = \theta(v)$ for some $v \in \gamma_k^\circ$ then ψ_I is δ-permissible on γ_k°.*

Proof. Recall that the edge lengths of Γ are assumed to be admissible, in the sense of Definition 4.1.

Suppose $\psi_I(v) = \theta(v)$ for some point v in γ_k°. We claim that the slope of ψ_I along the bridge β_{k-1} to the left of the loop is at most the incoming slope of θ from the left at u_{k-1}. Indeed, if the slope of ψ_I is strictly greater than that of θ then, since $\psi_I(u_{k-1}) \geq \theta(u_{k-1})$ and the slope of θ can only decrease when going from u_{k-1} to v_k, the difference $\psi_I(v_k) - \theta(v_k)$ will be at least the distance from u_{k-1} to v_k, which is $n_{k-1}/2$.

The slopes of ψ_I and θ along the bottom edge are between 0 and mg, and the slopes along the top edge are between 0 and m. Since $\ell_k > 4gm_k$ by assumption, it follows that $|\psi_I - \theta|$ changes by at most $m\ell_k$ between v_k and any other point in γ_k°. Assuming $2m\ell_k < n_{k-1}$, this proves the claim.

Note that the incoming slopes of ψ_I and θ from the left at u_k are

$$\deg(mD|_{\Gamma_{[0,k-1]}}) - \deg(D_I|_{\Gamma_{[0,k-1]}}), \quad \text{and} \quad \deg(mD|_{\Gamma_{[0,k-1]}}) - \delta_0 - \cdots - \delta_{k-1},$$

respectively. Therefore, the claim implies that $\deg(D_I|_{\Gamma_{[0,k-1]}}) \geq \delta_0 + \cdots + \delta_{k-1}$.

A similar argument using slopes along the bridge β_k to the right of γ_k° and assuming $2m\ell_k < n_k$ shows that $\deg(D_I|_{\Gamma_{\leq k}}) \leq \delta_0 + \cdots + \delta_k$, and the lemma follows. $\square$

Our general strategy for proving Conjecture 4.6 in the case $m = 2$ is to choose the set $\mathcal{A}$ carefully, assume that the minimum occurs everywhere at least twice, and then bound $\delta_0 + \ldots + \delta_i$ inductively, moving from left to right across the graph. By induction, we assume a lower bound on $\delta_0 + \cdots + \delta_i$. Then, for a carefully chosen $j > i$, we consider $\delta_0 + \cdots + \delta_j$. If this is too small, then Lemma 6.2 severely restricts which functions ψ_I can achieve the minimum on loops in $\Gamma_{[i+1,j]}$, making it impossible for the minimum to occur everywhere at least twice unless the bottom edge lengths $m_{i+1}, \ldots, m_j$ satisfy a nontrivial linear relation with small integer coefficients. We deduce a lower bound on $\delta_0 + \cdots + \delta_j$ and continue until we can show $\delta_0 + \cdots + \delta_{g+1} > 2d$, a contradiction. We give a first taste of this type of argument in Lemma 6.4 and Example 6.6. Example 6.7 illustrates how similar techniques may be applied to understand the kernel of μ_m when it is not injective. A more general (and more technical) version of the key step in this argument, using the assumption that a small number of functions ψ_I achieve the minimum everywhere at least twice on $\Gamma_{[i+1,j]}$ to produce a nontrivial linear relation with small integer coefficients, appears in the proof of Proposition 7.6.

Notation 6.3. For the remainder, we fix $m = 2$, and I and I_j will always denote multisets of size 2 in $\{0, \ldots, r\}$, which we identify with pairs (i, j) with $0 \leq i \leq j \leq r$. We write ψ_{ij} for the piecewise linear function $\psi_i + \psi_j$ corresponding to the multiset $I = \{i, j\}$.

Lemma 6.4. *Suppose that* $\delta_k = 2$ *and* $\theta(v) = \min\{\psi_{I_1}(v), \psi_{I_2}(v), \psi_{I_3}(v)\}$ *occurs at least twice at every point in* γ_k°. *Then,* $\theta|_{\gamma_k^\circ} = \psi_{I_j}|_{\gamma_k^\circ}$, *for some* $1 \leq j \leq 3$.

Proof. By Lemma 5.1, no two of the functions may obtain the minimum on all of γ_k°. After renumbering, we may assume that ψ_{I_3} obtains the minimum on some but not all of the loop. Let v be a boundary point of the locus where ψ_{I_3} obtains the minimum. Since there are only three functions that obtain the minimum, one must obtain the minimum in a neighborhood of v. After renumbering we may assume that this is ψ_{I_1}. We claim that θ is equal to ψ_{I_1} on the whole loop. If not, then by the shape lemma for minima, $D + \operatorname{div}\theta$ would contain the two points in the boundary of the locus where ψ_{I_1} obtains the minimum, in addition to v, contradicting the assumption that δ_k, the degree of $D + \operatorname{div}\theta$ on γ_k, is 2. $\square$

Remark 6.5. It follows from Lemma 6.4 that, under the given hypotheses, the tropical dependence on the k-th loop is essentially unique, in the sense that if b_1, b_2, and b_3 are real numbers such that

$$\theta(v) = \min\{\psi_{I_1}(v) + b_1, \psi_{I_2}(v) + b_2, \psi_{I_3}(v) + b_3\}$$

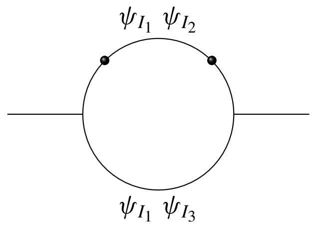

Figure 3. An illustration of the regions where different functions obtain the minimum in the situation of Lemma 6.4.

occurs at least twice at every point on the k-th loop, then $b_1 = b_2 = b_3$. Furthermore, since each ψ_{I_j} has constant slope along the bottom edge of γ_k and no two agree on the entire top edge, there must be one pair that agrees on the full bottom edge and part of the top edge and another pair that agrees on part of the top edge, as shown in Figure 3. Note that the divisor $D + \operatorname{div}\theta$ consists of two points on the top edge and one (but not both) of these points may lie at one of the end points, v_k or w_k.

Before we turn to the proof of the main theorem, we illustrate the techniques involved with a pair of examples.

Example 6.6. Suppose $g = 10$, and let D be the divisor of rank 4 and degree 12 corresponding to the tableau pictured in Figure 4. We note that this special case of the maximal rank conjecture for $m = 2$ is used to produce a counterexample to the slope conjecture in [Farkas and Popa 2005].

Assume that the minimum $\theta = \min\{\psi_I\}$ occurs at least twice at every point of Γ. By Proposition 5.2, the divisor $\Delta = \operatorname{div}(\theta) + 2D$ has degree at least two on each of the 12 locally closed subgraphs γ_k. Since $\deg(2D) = 24$, the degree of Δ on each of these subgraphs must be exactly 2. In other words, $\delta = (2, \dots, 2)$.

In the lingering lattice path for D, we have

$$p_4 = (6, 5, 2, 1, 0), \quad p_5 = (6, 5, 3, 1, 0), \quad p_6 = (6, 5, 4, 1, 0).$$

The δ-permissible functions ψ_{ij} on $\Gamma_{[5,6]}$ are those such that either

$$p_4(i) + p_4(j) \le 6 \quad \text{and} \quad p_5(i) + p_5(j) \ge 6, \quad \text{or}$$
$$p_5(i) + p_5(j) \le 6 \quad \text{and} \quad p_6(i) + p_6(j) \ge 6.$$

1	3	5	7	9
2	4	6	8	10

Figure 4. The tableau corresponding to the divisor D in Example 6.6.

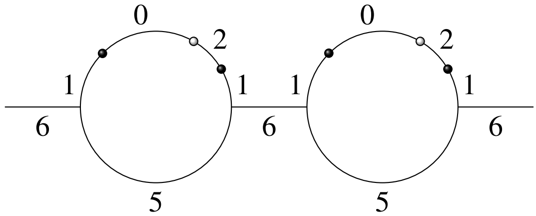

Figure 5. The shape of ψ_{04} and ψ_{13} on $\Gamma_{[5,6]}$.

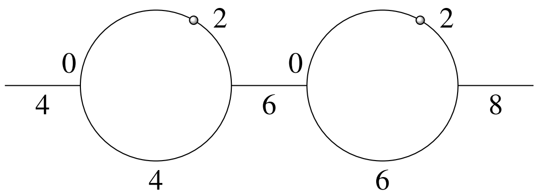

Figure 6. The shape of ψ_{22} on $\Gamma_{[5,6]}$.

There are only 3 such pairs: $(0, 4)$, $(1, 3)$, and $(2, 2)$. These functions are illustrated in Figures 5 and 6. Each domain of linearity is labeled with the slope of ψ_{ij} from left to right. The point of D is marked with a white circle. There are no points on $\Gamma_{[5,6]}$ where the function ψ_{22} has positive order of vanishing. The points where ψ_{04} has positive order of vanishing lie on either side of the white circle, and similarly for ψ_{13}. These points are marked with black circles. For the function ψ_{04}, these black circles occur closer to the white circle than they do for ψ_{13}. The region on which these two functions disagree on γ_5 is the disjoint union of two line segments, each of length m_5. Similarly, the region on which these two functions disagree on γ_6 is again the disjoint union of two line segments, each of length m_6.

By Lemma 6.4, in order for the minimum to occur at least twice at every point of $\Gamma_{[5,6]}$, on each of the two loops there must be a single function ψ_{ij} that obtains the minimum at every point. Because the slope of ψ_{22} along the bottom edge differs from that of ψ_{13} and ψ_{04}, we see that, on either loop, the function that obtains the minimum at every point cannot be ψ_{22}. Similarly, because on each loop the function ψ_{04} has slope 1 on a region where both of the other functions have slope 2, we see that the function that obtains the minimum at every point cannot be ψ_{04}. We therefore see that ψ_{13} obtains the minimum at every point of $\Gamma_{[5,6]}$, and ψ_{04} must achieve the minimum on both bottom edges. Let q_5 and q_6 be the points of D on γ_5 and γ_6, respectively, as shown in Figure 7.

The regions of the graph are labeled by the pairs of functions $\psi_{ij}, \psi_{i'j'}$ that obtain the minimum on that region. For each i, the change in value $\psi_i(q_6) - \psi_i(q_5)$ may be expressed as a function of the entries in the lattice path and the lengths of the edges in Γ. Specifically, as we travel from q_5 to q_6, the slopes of ψ_{22} and ψ_{13}

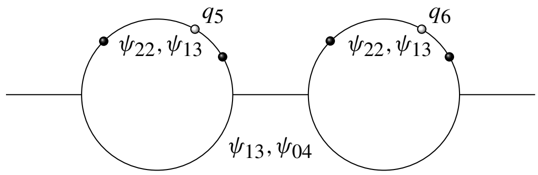

Figure 7. Regions of $\Gamma_{[5,6]}$ on which the functions obtain the minimum.

differ by 1 on an interval of length m_5 along the top edge of γ_5, and again on an interval of length m_6 along the top edge of γ_6. This computation shows that

$$(\psi_{22}(q_5) - \psi_{13}(q_5)) - (\psi_{22}(q_6) - \psi_{13}(q_6)) = m_5 - m_6.$$

Since ψ_{13} and ψ_{22} agree at q_5 and q_6, it follows that m_5 must equal m_6, contradicting the hypothesis that Γ has admissible edge lengths in the sense of Definition 4.1.

We conclude that the minimum cannot occur everywhere at least twice, so $\{\psi_I\}_I$ is tropically independent. Therefore, for any curve X with skeleton Γ and any lift of D to a divisor D_X of rank 4, the map

$$\mu_2 : \mathrm{Sym}^2 \mathcal{L}(D_X) \to \mathcal{L}(2D_X)$$

is injective.

We now consider an example illustrating our approach via tropical independence when μ_2 is not injective. Recall that the canonical divisor on a nonhyperelliptic curve of genus 4 gives an embedding in $\mathbb{P}^3$ whose image is contained in a unique quadric. This is the special case of the maximal rank conjecture where g, r, d, and m are 4, 3, 6, and 2, respectively.

Example 6.7. Suppose $g = 4$ and $m = 2$. Note that the class of the canonical divisor $D = K_\Gamma$ is vertex avoiding of rank 3. Since Γ is the skeleton of a curve whose canonical embedding lies on a quadric, the functions ψ_I are tropically dependent, and we may assume $\min_I \psi_I(v)$ occurs at least twice at every point $v \in \Gamma$.

Let $\theta(v) = \min_I \psi_I(v)$, and let $\Delta = 2K_\Gamma + \operatorname{div}\theta$. By Proposition 5.2, the degree δ_k of Δ on γ_k is at least 2 for $k = 0, \ldots, 5$. Since $\deg(\Delta) = 12$, it follows that $\delta = (2, \ldots, 2)$.

The lingering lattice path associated to K_Γ is given by

$$p_0 = (3, 2, 1, 0), \quad p_1 = (4, 2, 1, 0), \quad p_2 = (4, 3, 1, 0),$$
$$p_3 = (4, 3, 2, 0), \quad p_4 = (3, 2, 1, 0).$$

Since $\delta_0 = \delta_1 = 2$, the δ-permissible functions ψ_{ij} on γ_1 are those such that

$$p_0(i) + p_0(j) \le 4 \quad \text{and} \quad p_1(i) + p_1(j) \ge 4.$$

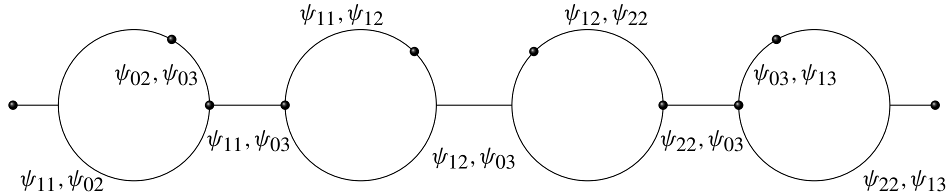

Figure 8. The unique tropical dependence for the canonical linear system when $g = 4$ and $m = 2$.

There are only three such pairs: $(0, 2)$, $(1, 1)$, and $(0, 3)$. In a similar way, we see that there are precisely three δ-permissible functions on each loop γ_k. By Lemma 6.4 and Remark 6.5, the tropical dependence among the three functions that achieve the minimum on each loop is essentially unique. Figure 8 illustrates the combinatorial structure of this dependence.

Since this dependence among the functions that realize the minimum at some point in Γ is essentially unique, omitting any one of the six functions that appear leaves a tropically independent set of size 9. Therefore, the map

$$\mu_2 : \mathrm{Sym}^2 \mathcal{L}(D_X) \to \mathcal{L}(2D_X)$$

has rank at least 9. Since $\mathcal{L}(2D_X)$ has dimension 9, it follows that μ_2 is surjective.

7. Shapes of functions, excess degree, and linear relations among edge lengths

In this section, and in Section 8, below, we assume that $\rho(g, r, d) = 0$. All of the essential difficulties appear already in this special case. The case $\rho(g, r, d) > 0$ is treated in Section 9 through a minor variation on these arguments.

We now proceed with the more delicate and precise combinatorial arguments required to prove Theorem 1.1. With g, r, and d fixed, and assuming $d - g \leq r$, we must produce a divisor D of degree d and rank r on Γ, together with a set

$$\mathcal{A} \subset \{(i, j) \mid 0 \leq i \leq j \leq r\}$$

of size

$$\#\mathcal{A} = \min\left\{\binom{r+2}{2}, 2d - g + 1\right\},$$

such that the corresponding collection of rational functions

$$\{\psi_{ij} \mid (i, j) \in \mathcal{A}\}$$

is tropically independent.

Notation 7.1. The quantity $g - d + r$ appears repeatedly throughout, so we set

$$s = g - d + r,$$

which simplifies various formulas. The condition that $\rho(g, r, d) = 0$ means that $g = (r+1)s$.

We now specify the divisor D that we will use to prove Conjecture 4.6 for $m = 2$ when $\rho(g, r, d) = 0$. The set $\mathcal{A}$ is described in Section 8.

Notation 7.2. For the remainder of this section and Section 8, let D be the divisor of degree d and rank r on Γ corresponding to the standard tableau with $r+1$ columns and s rows in which the numbers $1, \ldots, s$ appear in the leftmost column; $s+1, \ldots, 2s$ appear in the next column, and so on. We number the columns from zero to r, so the ℓ-th column contains the numbers $\ell s+1, \ldots, (\ell+1)s$. The specific case $g = 10$, $r = 4$, $d = 12$ is illustrated in Figure 4 from Example 6.6.

Remark 7.3. Our choice of divisor is particularly convenient for the inductive step in the proof of Theorem 1.1, in which we divide the graph Γ into the $r+1$ regions $\Gamma_{[\ell s+1,(\ell+1)s]}$, for $0 \leq \ell \leq r$, and move from left to right across the graph, one region at a time, studying the consequences of the existence of a tropical dependence. Since the numbers $\ell s+1, \ldots, (\ell+1)s$ all appear in the ℓ-th column, the slopes of the functions ψ_i, for $i \neq \ell$, are the same along all bridges and bottom edges, respectively, in the subgraph $\Gamma_{[\ell s+1,(\ell+1)s]}$. Only the slopes of ψ_ℓ are changing in this region.

Here we describe the *shape* of the function ψ_i, by which we mean the combinatorial configuration of regions on the loops and bridges on which ψ_i has constant slope, as well as the slopes from left to right on each region. These data determine (and are determined by) the combinatorial configurations of the points in $D_i = D + \operatorname{div}(\psi_i)$.

Fix $0 \leq \ell \leq r$. Suppose $\ell s+1 \leq k \leq (\ell+1)s$, so γ_k° is a loop in the subgraph $\Gamma_{[\ell s+1,(\ell+1)s]}$. Recall from Section 4 and Section 6 that if $\ell \neq r$ then D contains one point on the top edge of γ_k°, at distance

$$p_{k-1}(\ell) = \sigma_k(\ell)$$

in the counterclockwise direction from w_k, where $\sigma_k(\ell)$ is the slope of ψ_ℓ along the bridge β_k.

Case 1: The shape of ψ_i, for $i < \ell$. If $i < \ell$ then $D_i = D + \operatorname{div}\psi_i$ contains one point on the top edge of γ_k°, at distance $(r+s-i-1-\sigma_k(\ell)) \cdot m_k$ from v_k, the left endpoint of γ_k°. This is illustrated schematically in Figure 9. The point of D_i on the top edge of γ_k° is marked with a black circle, and the point of D is marked with a white circle. (In the case where $\ell = r$, the white circle is located at the right-hand vertex w_k.) Each region of constant slope is labeled with the slope of ψ_i from left to right. The slope of ψ_i from left to right along each bridge adjacent to γ_k° is $r+s-i$, and the slope along the bottom edge is $r+s-i-1$.

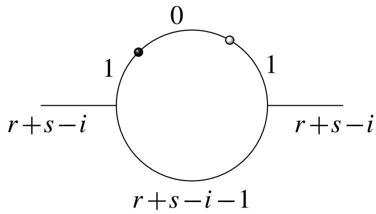

Figure 9. The shape of ψ_i on γ_k, for $i < \ell$.

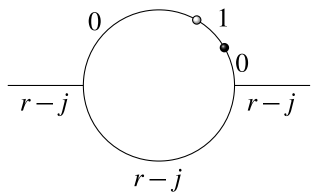

Figure 10. The shape of ψ_j on γ_k, for $j > \ell$.

Case 2: The shape of ψ_j, for $j > \ell$. If $j > \ell$ then $D_j = D + \operatorname{div} \psi_j$ contains one point on the top edge of γ_k°, at distance $(\sigma_k(\ell) - r + j)$ from w_k, as shown in Figure 10. The slope of ψ_j along the bottom edge and both adjacent bridges is $r - j$.

Case 3: The shape of ψ_ℓ. The divisor D_ℓ has no points on γ_k°, as shown in Figure 11. Note that this is the only case in which the slope is not the same along the two bridges adjacent to γ_k°.

We use the shapes of the functions ψ_i to control the set of pairs (i, j) such that ψ_{ij} is δ-permissible on certain loops, as follows. Suppose $\{\psi_{ij}\}$ is tropically dependent, so there are constants b_{ij} such that $\min\{\psi_{ij}(v) + b_{ij}\}$ occurs at least twice at every point $v \in \Gamma$. Replacing ψ_{ij} with $\psi_{ij} + b_{ij}$, we may assume $\min\{\psi_{ij}(v)\}$ occurs at least twice at every point. Let

$$\theta = \min_{ij}\{\psi_{ij}\}, \quad \Delta = 2D + \operatorname{div}(\theta), \quad \text{and} \quad \delta_i = \deg(\Delta|_{\gamma_i}).$$

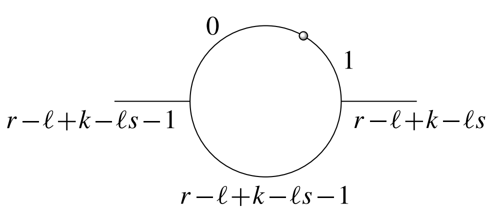

Figure 11. The shape of ψ_ℓ on γ_k.

By Proposition 5.2, each δ_i is at least 2, and some may be strictly greater. We keep track of the *excess degree function*

$$e(k) = \delta_0 + \cdots + \delta_k - 2k.$$

It contains exactly the same information as δ, but in a form that is somewhat more convenient for our inductive arguments in Section 8. Note that $e(k)$ is positive and nondecreasing as a function of k.

In the induction step, we study the δ-permissible functions ψ_{ij} on subgraphs

$$\Gamma_{[a(\ell),b(\ell)]} \subseteq \Gamma_{[\ell s+1,(\ell+1)s]},$$

where $a(\ell)$ and $b(\ell)$ are given by

$$a(\ell) = \begin{cases} \ell s + 1 & \text{for } \ell \le \lceil r/2 \rceil, \\ \ell(s+1) - \lceil r/2 \rceil + 1 & \text{for } \ell > \lceil r/2 \rceil, \end{cases}$$

and

$$b(\ell) = \begin{cases} \ell(s+1) - \lceil r/2 \rceil + s & \text{for } \ell \le \lceil r/2 \rceil, \\ (\ell+1)s & \text{for } \ell > \lceil r/2 \rceil. \end{cases}$$

Note that the subgraph $\Gamma_{[a(\ell),b(\ell)]}$ is only well-defined if $a(\ell) \le b(\ell)$. This is the case when ℓ is in the range

$$\max\left\{0, \left\lceil \frac{r}{2} \right\rceil - s\right\} \le \ell < \min\left\{r, \left\lceil \frac{r}{2} \right\rceil + s\right\}.$$

We focus on the situation where $e(\ell s)$ and $e((\ell+1)s)$ are both equal to $\ell - s + \lceil r/2 \rceil$, which is the critical case for our argument.

Lemma 7.4. *Suppose*

$$e(\ell s) = e((\ell+1)s) = \ell - s + \left\lceil \frac{r}{2} \right\rceil,$$

for some $0 \le \ell \le r$. *If* ψ_{ij} *is* δ*-permissible on* $\Gamma_{[a(\ell),b(\ell)]}$, *then either*

(1) $i < \ell < j$, *and* $i + j = \ell + \lceil r/2 \rceil$, *or*

(2) $i = j = \ell$.

Proof. Note that, by our choice of D,

$$\deg(D_i|_{\Gamma_{[0,k]}}) = \begin{cases} i + k & \text{for } i > \ell, \\ i + \ell s & \text{for } i = \ell, \\ i + k - s & \text{for } i < \ell. \end{cases}$$

Also, since $e(k)$ is nondecreasing,

$$e(k) = \ell - s + \left\lceil \frac{r}{2} \right\rceil$$

for all k in $[\ell s, (\ell+1)s]$, and in particular for k in $[a(\ell), b(\ell)]$.

We now prove the lemma in the case where $\ell \le \lceil r/2 \rceil$. The proof in the case where $\ell > \lceil r/2 \rceil$ is similar. Suppose $i \ge \ell$, $j > \ell$, and $k \in [a(\ell), b(\ell)]$. Then

$$\begin{aligned}\deg(D_{ij}|_{\Gamma_{[0,k]}}) &\ge i + j + k + \ell s \\ &> 2\ell + k + \ell s.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\deg(\Delta|_{\Gamma_{[0,k]}}) &= 2k + \ell - s + \left\lceil \frac{r}{2} \right\rceil \\ &\le 2\ell + k + \ell s,\end{aligned}$$

where the inequality is given by using $k \le b(\ell)$ and $b(\ell) = \ell(s+1) - \lceil r/2 \rceil + s$. Combining the two displayed inequalities shows that $\deg(D_{ij}|_{\Gamma_{[0,k]}})$ is greater than $\deg(\Delta|_{\Gamma_{[0,k]}})$, and hence ψ_{ij} is not δ-permissible on γ_k°.

A similar argument shows that, if $i < \ell$, $j \le \ell$, and $k \in [a(\ell), b(\ell)]$, then

$$\begin{aligned}\deg(D_{ij}|_{\Gamma_{[0,k-1]}}) &\le i + j + \ell s + k - 1 - s \\ &< 2\ell + \ell s + k - 1 - s.\end{aligned}$$

On the other hand, since $\ell \le \lceil r/2 \rceil$ by hypothesis, and $k \ge \ell s + 1$, we have

$$\begin{aligned}\deg(\Delta|_{\Gamma_{[0,k-1]}}) &= 2k - 2 + \ell - s + \left\lceil \frac{r}{2} \right\rceil \\ &\ge 2k - 2 + 2\ell - s \\ &\ge 2\ell + \ell s + k - 1 - s.\end{aligned}$$

In this case, we conclude that $\deg(D_{ij}|_{\Gamma_{[0,k-1]}})$ is less than $\deg(\Delta|_{\Gamma_{[0,k-1]}})$, and hence ψ_{ij} is not δ-permissible on γ_k°.

We have shown that, if ψ_{ij} is δ-permissible on $\Gamma_{[a(\ell),b(\ell)]}$, then either $i = j = \ell$ or $i < \ell < j$. It remains to show that if $i < \ell < j$ then $i + j = \ell + \lceil r/2 \rceil$. Suppose $i < \ell < j$. Then

$$\deg(D_{ij}|_{\Gamma_{[0,k]}}) = i + j + 2k - s.$$

If ψ_{ij} is δ-permissible on γ_k°, then this is less than or equal to $\deg(\Delta|_{\Gamma_{[0,k]}})$, which is $2k+\ell-s+\lceil r/2 \rceil$. It follows that $i+j \le \ell+\lceil r/2 \rceil$. Similarly, if ψ_{ij} is δ-permissible on γ_k° then $\deg(D_{ij}|_{\Gamma_{[0,k-1]}}) \ge \deg(\Delta|_{\Gamma_{[0,k-1]}})$, and it follows that $i + j \ge \ell + \lceil r/2 \rceil$. Therefore, $i + j = \ell + \lceil r/2 \rceil$, as required. □

We continue with the notation from Lemma 7.4, with ℓ a fixed integer between 0 and r, and $[a(\ell), b(\ell)]$ the corresponding subinterval of $[\ell s+1, (\ell+1)s]$, when this is nonempty. We also fix a subset $\mathcal{A} \subset \{(i, j) \in \mathbb{Z}^2 \mid 0 \le i \le j \le r\}$ and suppose that $\theta(v) = \min\{\psi_{ij}(v) \mid (i, j) \in \mathcal{A}\}$ occurs at least twice at every point. Equivalently, in the set up of Lemma 7.4, we assume that $b_{ij} \gg 0$ for (i, j) not in $\mathcal{A}$.

Remark 7.5. The following proposition is the key technical step in our inductive argument, and may be seen as a generalization of the following two simple facts. In order for the minimum to be achieved everywhere at least twice, on a chain of zero loops (i.e., a single edge), at least two functions are required, and on a chain of one loop, at least three functions are required (Lemma 5.1).

Proposition 7.6. *Suppose $e(a(\ell)) = e(b(\ell)) = \ell - s + \lceil r/2 \rceil$. Then there are at least $b(\ell) - a(\ell) + 3$ functions ψ_{ij}, with $(i, j) \in \mathcal{A}$, that are δ-permissible on $\Gamma_{[a(\ell),b(\ell)]}$.*

Proof. Let $a = a(\ell)$ and $b = b(\ell)$. Assume that there are at most $b - a + 2$ functions that are δ-permissible on $\Gamma_{[a,b]}$. We will show that the bottom edge lengths m_k for $k \in [a, b]$ satisfy a linear relation with small integer coefficients, contradicting the admissibility of the edge lengths of Γ (Definition 4.1).

Since $e(k)$ is nondecreasing, the assumption that $e(a) = e(b)$ implies that Δ contains exactly two points on each loop in $\Gamma_{[a,b]}$, and no points in the interiors of the bridges. It follows that θ has constant slope on each of these bridges. As discussed in Section 6, the slope at the midpoint of β_k is determined by the degree of $\operatorname{div}\theta$ on $\Gamma_{[0,k]}$, and one computes that this slope is $2r - e(k)$. Therefore, the slope of θ is constant on every bridge in $\Gamma_{[a,b]}$, and equal to

$$\sigma := 2r - \ell + s - \left\lceil \frac{r}{2} \right\rceil.$$

We begin by describing the shapes of the δ-permissible functions ψ_{ij} on $\Gamma_{[a,b]}$. By Lemma 7.4, the δ-permissible functions ψ_{ij} satisfy either $i = j = \ell$ or $i < \ell < j$ and $i + j = \ell + \lceil r/2 \rceil$. Suppose $i < \ell < j$. In this case, the shape of ψ_{ij} on the subgraph γ_k is as pictured in Figure 12.

Note that the shape of ψ_{ij} is determined by the shapes of ψ_i and ψ_j, as shown in Figures 9 and 10, respectively. The point q_k of D on γ_k° is marked with a white circle. The fact that the slopes of ψ_{ij} along the bridges are equal to σ is due to the condition $i + j = \ell + \lceil r/2 \rceil$.

We now describe the shape of the function $\psi_{\ell\ell}$. Note that the slope of $\psi_{\ell\ell}$ along the bridge $\beta_{\ell s}$ is $2r - 2\ell$, and the slope increases by two along each successive bridge

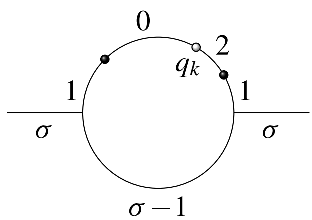

Figure 12. The shape of ψ_{ij} on γ_k, for $i < \ell < j$.

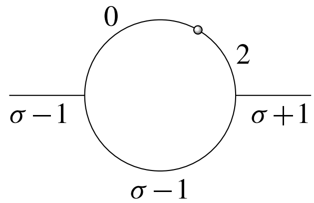

Figure 13. The shape of $\psi_{\ell\ell}$ on γ_h, when σ is odd.

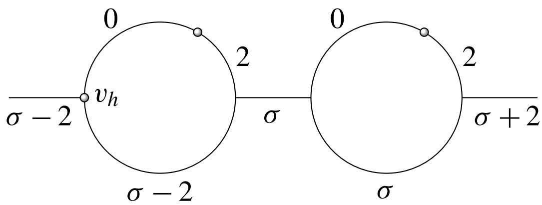

Figure 14. The shape of $\psi_{\ell\ell}$ on $\Gamma_{[h,h+1]}$, when σ is even.

β_k, for $k \in [\ell s+1, (\ell+1)s]$. It follows that if σ is odd then $\psi_{\ell\ell}$ is δ-permissible on only one loop γ_h°, as shown in Figure 13.

If σ is even, then $\psi_{\ell\ell}$ is δ-permissible on two consecutive loops γ_h and γ_{h+1}, as shown in Figure 14. We choose h so that γ_h° is the leftmost loop on which $\psi_{\ell\ell}$ is δ-permissible. We will use v_h as a point of reference for the remaining calculations in the proof of the proposition. (The values of ψ_{ij} and $\psi_{\ell\ell}$ at every point in $\Gamma_{[a,b]}$ are determined by the shape computations above and the values at v_h.)

For the permissible functions ψ_{ij} with $i < \ell < j$, the slopes along the bridges and bottom edges are independent of (i, j). One then computes directly that

$$\psi_{ij}(q_k) - \psi_{i'j'}(q_k) = \psi_{ij}(v_h) - \psi_{i'j'}(v_h) + (i'-i)m_k. \tag{1}$$

Similarly, one computes

$$\psi_{ij}(q_h) - \psi_{\ell\ell}(q_h) = \psi_{ij}(v_h) - \psi_{\ell\ell}(v_h) + (r+s-i-1-\sigma_h(\ell))m_h, \tag{2}$$

and, when σ is even,

$$\psi_{ij}(q_{h+1}) - \psi_{\ell\ell}(q_{h+1}) = \psi_{ij}(v_h) - \psi_{\ell\ell}(v_h) + (r+s-i-1-\sigma_{h+1}(\ell))m_{h+1} + m_h. \tag{3}$$

We use these expressions, together with the tropical dependence hypothesis (our standing assumption that $\min\{\psi_{ij}(v) \mid (i, j) \in \mathcal{A}\}$ occurs at least twice at every point) to produce a linear relation with small integer coefficients among the bottom edge lengths $m_a, \ldots, m_b$, as follows.

Let $\mathcal{A}' \subset \mathcal{A}$ be the set of pairs (i, j) such that ψ_{ij} is δ-permissible on $\Gamma_{[a,b]}$. We now build a graph whose vertices are the pairs $(i, j) \in \mathcal{A}'$, and whose edges record the pairs that achieve the minimum at one of the points q_k or at the point v_h. Say

$\psi_{i_0 j_0}$ and $\psi_{i'_0 j'_0}$ achieve the minimum at v_h. Then we add an edge e_0 from (i_0, j_0) to (i'_0, j'_0) in the graph. Associated to this edge, we have the equation

$$\psi_{i_0 j_0}(v_h) - \psi_{i'_0 j'_0}(v_h) = 0. \tag{E_0}$$

Next, for $a \le k \le b$, say $\psi_{i_k j_k}$ and $\psi_{i'_k j'_k}$ achieve the minimum at q_k. Then we add an edge e_k from (i_k, j_k) to (i'_k, j'_k) and, associated to this edge, we have the equation

$$\psi_{i_k j_k}(v_h) - \psi_{i'_k j'_k}(v_h) = \alpha_k m_k + \lambda_k m_{k-1}, \tag{E_k}$$

where α_k and λ_k are small positive integers determined by formula (1), (2), or (3), according to whether one of the pairs is equal to (ℓ, ℓ) and, if so, whether k is equal to h or $h+1$. Note that, in every case, α_k is nonzero.

The graph now has $b-a+2$ edges and, by hypothesis, it has at most $b-a+2$ vertices. Therefore, it must contain a loop. If the edges $e_{k_1}, \dots e_{k_t}$ form a loop then we can take a linear combination of the equations $E_{k_1}, \dots, E_{k_t}$, each with coefficient ± 1, so that the left-hand sides add up to zero. This gives a linear relation among the bottom edge lengths $m_{k_1}, \dots, m_{k_t}$, with small integer coefficients. Furthermore, if $k_t > k_j$ for all $j \neq t$, then m_{k_t} appears with nonzero coefficient in E_{k_t}, and does not appear in E_{k_j} for $j < t$, so this linear relation is nontrivial. Finally, note that $|\alpha_k| \le r+s \le g$ for all k, and λ_k is either 0 or 1, so the coefficient of each edge length m_k is an integer of absolute value less than or equal to $g+1$. This contradicts the hypothesis that Γ has admissible edge lengths, and proves the proposition. □

8. Proof of Theorem 1.1 for $\rho(g, r, d) = 0$

In this section, we continue with the assumption from Section 7 that $\rho(g, r, d) = 0$ and prove Conjecture 4.6 for $m = 2$, applying an inductive argument that relies on Lemma 7.4 and Proposition 7.6 in the inductive step. The case $\rho(g, r, d) > 0$ is handled by a minor variation on these arguments in Section 9.

Remark 8.1. Wang [2015] has recently shown that the maximal rank conjecture for $m = 2$ follows from the special case where $\rho(g, r, d) = 0$. Our proof of Theorem 1.1 does not rely on this reduction. We prove Conjecture 4.6 for $m = 2$ and arbitrary $\rho(g, r, d)$.

We separate the argument into two cases, according to whether or not μ_2 is injective. The following identity is used to characterize the range of cases in which μ_2 is injective and to count the set $\mathcal{A}$ that we define in the remaining cases.

Lemma 8.2. *Suppose $s \le r$. Then*

$$\binom{r+2}{2} - \binom{r-s}{2} + \binom{s}{2} = 2d - g + 1.$$

Proof. The lemma follows from a series of algebraic manipulations. Expand the left-hand side as a polynomial in r and s, collect terms, and apply the identities $s = g - d + r$ and $g = (r+1)s$. $\square$

It follows immediately from Lemma 8.2 that

$$\binom{r+2}{2} \leq 2d - g + 1 \quad \text{if and only if} \quad r - s \leq s.$$

In particular, the maximal rank conjecture predicts that μ_2 is injective for a general linear series on a general curve exactly when $r \leq 2s$. We now proceed with the proof that $\{\psi_{ij} \mid 0 \leq i \leq j \leq r\}$ is tropically independent in the injective case.

Proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) = 0$, and $r \leq 2s$. We must show that the set of functions $\{\psi_{ij} \mid 0 \leq i \leq j \leq r\}$ is tropically independent. Suppose not. Then there are constants b_{ij} such that the minimum

$$\theta(v) = \min_{ij}\{\psi_{ij}(v) + b_{ij}\}$$

occurs at least twice at every point $v \in \Gamma$. We continue with the notation from Section 7, setting

$$\Delta = 2D + \operatorname{div}\theta, \quad \delta_i = \deg(\Delta)|_{\gamma_i}, \quad \text{and} \quad e(k) = \delta_0 + \cdots + \delta_k - 2k.$$

As described above, our strategy is to bound the excess degree function $e(k) = \delta_0 + \cdots + \delta_k - 2k$ inductively, moving from left to right across the graph.

More precisely, we claim that

$$e(\ell s) \geq \ell - s + \left\lceil \frac{r}{2} \right\rceil \quad \text{for } \ell \leq r. \tag{4}$$

We prove this claim by induction on ℓ, using Lemma 7.4 and Proposition 7.6. To see that the theorem follows from the claim, note that the claim implies that

$$\deg(\Delta) \geq 2g + r - s + \left\lceil \frac{r}{2} \right\rceil + 2.$$

Since $d = g + r - s$, this gives $\deg(\Delta) \geq 2d + s - \lfloor r/2 \rfloor + 2$, a contradiction, since $r \leq 2s$. It remains to prove the claim (4).

The claim is clear for $\ell = 1$, since $e(k) \geq \delta_0 \geq 2$ for all k and $\lceil r/2 \rceil \leq s$, by assumption. We proceed by induction on ℓ. Assume that $\ell < 2r - s - 1 - \lceil r/2 \rceil$ and

$$e(\ell s) \geq \ell - s + \left\lceil \frac{r}{2} \right\rceil.$$

We must show that $e((\ell+1)s) \geq \ell - s + \lceil r/2 \rceil + 1$. If $e(\ell s) > \ell - s + \lceil r/2 \rceil$ then there is nothing to prove, since e is nondecreasing. It remains to rule out the possibility that $e(\ell s) = e((\ell+1)s) = \ell - s + \lceil r/2 \rceil$.

Suppose that $e(\ell s) = e((\ell+1)s) = \ell - s + \lceil r/2 \rceil$. Fix $a = a(\ell)$ and $b = b(\ell)$ as in Section 7. By Lemma 7.4, if ψ_{ij} is δ-admissible on $\Gamma_{[a,b]}$ then either $i = j = \ell$

or $i < \ell < j$ and $i + j = \ell + \lceil r/2 \rceil$. We consider two cases and use Proposition 7.6 to reach a contradiction in each case.

Case 1: If $1 \leq \ell \leq \lceil r/2 \rceil$ then there are exactly $\ell + 1$ possibilities for i, and j is uniquely determined by i. In this case $b - a = \ell + s - \lceil r/2 \rceil - 1$. Since $r \leq 2s$, this implies that the number of δ-permissible functions is at most $b - a + 2$, which contradicts Proposition 7.6, and the claim follows.

Case 2: If $\lceil r/2 \rceil < \ell < r$ then there are exactly $r - \ell + 1$ possibilities for j, and i is uniquely determined by j. In this case, $b - a = s - \ell + \lceil r/2 \rceil + 1$, which is at least $r - \ell - 1$, since $r \leq 2s$. Therefore, the number of δ-permissible functions on $\Gamma_{[a,b]}$ is at most $b - a + 2$, which contradicts Proposition 7.6, and the claim follows.

This completes the proof of Conjecture 4.6 (and hence Theorem 1.1) in the case where $m = 2$, $\rho(g, r, d) = 0$, and $r \leq 2s$. □

Our proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) = 0$, and $r > 2s$ is similar to the argument above, bounding the excess degree function $e(\ell s)$ by induction on ℓ, with Lemma 7.4 and Proposition 7.6 playing a key role in the inductive step. The one essential new feature is that we must specify the subset $\mathcal{A}$. The description of this set, and the argument that follows, depend in a minor way on the parity of r, so we fix

$$\epsilon(r) = \begin{cases} 0 & \text{if } r \text{ is even,} \\ 1 & \text{if } r \text{ is odd.} \end{cases}$$

Let $\mathcal{A}$ be the subset of the integer points in the triangle $0 \leq i \leq j \leq r$ that are not in any of the following three regions:

(1) the half-open triangle where $j \geq i + 2$ and $i + j < r - 2s + \epsilon(r)$,

(2) the half-open triangle where $j \geq i + 2$ and $i + j > r + 2s$,

(3) the closed chevron where $r - s + \epsilon(r) \ \leq \ i + j \leq r + s$, and either

$$i \leq \tfrac{1}{2}(r - 2s - 2 + \epsilon(r)) \quad \text{or} \quad j \geq \tfrac{1}{2}(r + 2s + 2).$$

Figure 15 illustrates the case $g = 36$, $r = 11$, $d = 44$, and $s = 3$. The points of $\mathcal{A}$ are marked with black dots, the three regions are shaded gray, and the omitted integer points are marked with white circles.

Remark 8.3. There are many possible choices for $\mathcal{A}$, as one can see even in relatively simple cases, such as Example 6.7. We present one particular choice that works uniformly for all g, r, and d. (In the situation of Example 6.7, the two half-open triangles are empty, and the closed chevron contains a single integer point, namely (0, 3).) The essential property for the purposes of our inductive argument is the number of points (i, j) in $\mathcal{A}$, with $i \neq j$, on each diagonal line $i + j = k$, for $0 \leq k \leq 2r$. The argument presented here works essentially verbatim for any other subset of the integer points in the triangle with this property, and can be adapted to work somewhat more generally. We have made no effort to characterize those subsets that are tropically independent, since producing a single such subset is sufficient for the proof of Theorem 1.1.

Remark 8.4. Our choice of $\mathcal{A}$, suitably interpreted, works even in the injective case. When $r - s \leq s$, the shaded regions are empty, since the half space $i \leq \frac{1}{2}(r - 2s - 2 + \epsilon(r))$ lies entirely to the left of the triangle $0 \leq i \leq j \leq r$, and the half space $j \geq \frac{1}{2}(r + 2s + 2)$ lies above it.

We now verify that the set $\mathcal{A}$ described above has the correct size.

Lemma 8.5. *The size of $\mathcal{A}$ is* $\#\mathcal{A} = 2d - g + 1$.

Proof. As shown in Figure 15, moving the lower left triangle vertically and the upper right triangle horizontally by integer translations, we can assemble the shaded regions to form a closed triangle minus a half-open triangle. These translations show that the two half-open triangles plus the convex hull of the chevron shape are scissors congruent to a triangle of side length $r - s - 2$ that contains $\binom{r-s}{2}$ integer points. The difference between the chevron shape and its convex hull is

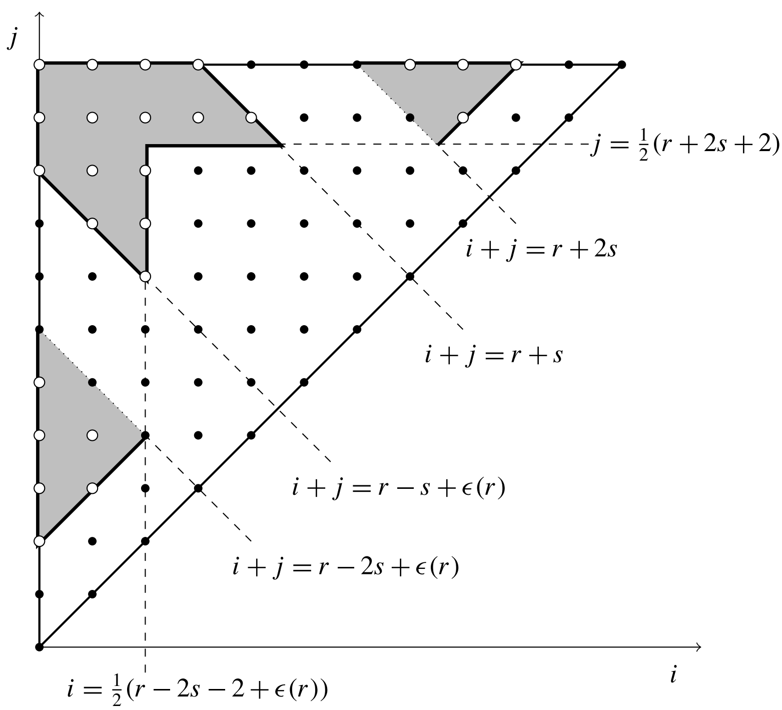

Figure 15. Points in the set $\mathcal{A}$ are marked by black dots. The integer points in the triangle $0 \leq i \leq j \leq r$ that are omitted from $\mathcal{A}$ are marked with white circles.

a half-open triangle that contains $\binom{s}{2}$ integer points. Therefore, the shaded region contains exactly $\binom{r-s}{2} - \binom{s}{2}$ lattice points, and the proposition then follows from the identity in Lemma 8.2. $\square$

Proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) = 0$, and $r > 2s$. We will show that

$$\{\psi_{ij} \mid (i, j) \in \mathcal{A}\}$$

is tropically independent. Suppose not. Then there are constants b_{ij} such that $\theta(v) = \min_{(i,j)\in\mathcal{A}}\{\psi_{ij}(v) + b_{ij}\}$ occurs at least twice at every point v in Γ. Let

$$\Delta = \mathrm{div}(\theta) + 2D, \quad \delta_i = \deg(\Delta)|_{\gamma_i}, \quad \text{and} \quad e(k) = \delta_0 + \cdots + \delta_k - 2k.$$

Note that $\deg_{w_0}(\Delta)$ is $2r - \sigma_0(\theta)$, where $\sigma_0(\theta)$ is the outgoing slope of θ at w_0. Since the minimum is achieved twice at every point, this slope must agree with the slope $\sigma_0(\psi_{ij}) = 2r - i - j$ for at least two pairs $(i, j) \in \mathcal{A}$. The points in the half-open triangle where $j \geq i + 2$ and $i + j > r + 2s$ are omitted from $\mathcal{A}$, so there is only one pair $(i, j) \in \mathcal{A}$ such that $i + j = k$, for $k < r - 2s + \epsilon(r)$. It follows that $\deg_{w_0}(\Delta) \geq r - 2s + \epsilon(r)$. Similarly, $\deg_{v_{g+1}}(\Delta) \geq r - 2s$.

We claim that

$$e(\ell s) \geq \ell - s + \left\lceil \frac{r}{2} \right\rceil \quad \text{for} \quad \ell \leq \left\lfloor \frac{r}{2} \right\rfloor + s + 1. \tag{5}$$

Note that the assumption $r > 2s$ implies that $\lfloor r/2 \rfloor + s + 1 \leq r$. Since e is a nondecreasing function of k, and $\deg_{v_{g+1}}(\Delta) \geq r - 2s$, the claim implies that

$$\deg(\Delta) \geq 2g + \left(\left\lfloor \frac{r}{2} \right\rfloor + s + 1 - s + \left\lceil \frac{r}{2} \right\rceil \right) + r - 2s.$$

Collecting terms gives $\deg(\Delta) \geq 2g + 2r - 2s + 1 = 2d + 1$, a contradiction.

It remains to prove claim (5). Since $\delta_0 \geq r - 2s + \epsilon(r)$, the claim holds for $\ell \leq \lceil r/2 \rceil - s$. We proceed by induction on ℓ. Assume that $e(\ell s) \geq \ell - s + \lceil r/2 \rceil$ and $\ell \leq \lfloor r/2 \rfloor + s$. We must show that $e((\ell + 1)s) \geq \ell - s + \lceil r/2 \rceil + 1$. If $e(\ell s) > \ell - s + \lceil r/2 \rceil$ then there is nothing to prove, since e is nondecreasing. It remains to rule out the possibility that $e(\ell s) = e((\ell + 1)s) = \ell - s + \lceil r/2 \rceil$.

Suppose $e(\ell s) = e((\ell + 1)s) = \ell - s + \lceil r/2 \rceil$. Fix $a = a(\ell)$ and $b = b(\ell)$ as in Section 7. By Lemma 7.4, if ψ_{ij} is δ-admissible on $\Gamma_{[a,b]}$ then either $i = j = \ell$ or $i < \ell < j$ and $i + j = \ell + \lceil r/2 \rceil$. We consider three cases.

Case 1: If $\lceil r/2 \rceil - s \leq \ell \leq \lfloor r/2 \rfloor$ then there are $\lceil r/2 \rceil - s$ pairs (i, j) with $i \neq j$ and $i + j = \ell + \lceil r/2 \rceil$ that are contained in the closed chevron and hence omitted from $\mathcal{A}$. This leaves

$$\ell + 1 - \left\lceil \frac{r}{2} \right\rceil + s = b - a + 2$$

pairs $(i, j) \in \mathcal{A}$ such that ψ_{ij} is δ-permissible on $\Gamma_{[a,b]}$. We can then apply Proposition 7.6, and the claim follows.

Case 2: If $\lfloor r/2 \rfloor < \ell < r/2 + s$ then there are $\lfloor r/2 \rfloor - s$ pairs (i, j) with $i + j = \ell + \lceil r/2 \rceil$ that are in the closed chevron and hence omitted from $\mathcal{A}$. This leaves

$$r - \ell + 1 - \left\lfloor \frac{r}{2} \right\rfloor + s = b - a + 2$$

pairs $(i, j) \in \mathcal{A}$ such that ψ_{ij} is δ-permissible on $\Gamma_{[a,b]}$. We can then apply Proposition 7.6, and the claim follows.

Case 3: If $\ell = r/2 + s$, then there are $r/2 - s$ pairs (i, j) with $i + j = r + s$ that are contained in the closed chevron and hence omitted from $\mathcal{A}$. This leaves one pair $(i, j) \in \mathcal{A}$ such that ψ_{ij} has slope r on the bridge $\beta_{(r/2+s)s}$. It follows that θ cannot have slope r at any point of this bridge. If $e((\ell + 1)s) \leq r$, however, then the inductive hypothesis implies that $e(\ell s) = e(\ell s + 1) = r$, hence θ has constant slope r on this bridge, a contradiction, and the claim follows. $\square$

Remark 8.6. In Case 3 of the above argument, the formulas for $a(\ell)$ and $b(\ell)$ would give $a(\ell) = b(\ell) + 1$, so the subgraph $\Gamma_{[a(\ell),b(\ell)]}$ might be thought of as a chain of $b(\ell) - a(\ell) + 1 = 0$ loops. The inductive step in this case is then an application of a degenerate version of Proposition 7.6 for a chain of zero loops, i.e., for a single edge. See also Remark 7.5.

9. Proof of Theorem 1.1 for $\rho(g, r, d) > 0$

Fix $\rho = \rho(g, r, d)$, $g' = g - \rho$, and $d' = d - \rho$. Let Γ' be a chain of g' loops with admissible edge lengths. Note that $\rho(g', r, d') = 0$. Therefore, the constructions in Sections 7 and 8 produce a divisor D' on Γ' of rank r and degree d' whose class is vertex avoiding, together with a set $\mathcal{A}'$ of integer points (i, j) with $0 \leq i \leq j \leq r$ of size

$$\#\mathcal{A}' = \min\left\{\binom{r+2}{2}, 2d' - g' + 1\right\} = \min\left\{\binom{r+2}{2}, 2d - g + 1 - \rho\right\},$$

such that the collection of piecewise linear functions $\{\psi'_{ij} \in R(D') \mid (i, j) \in \mathcal{A}'\}$ is tropically independent.

We use Γ', D', and $\mathcal{A}'$ as starting points to construct a chain of g loops with admissible edge lengths Γ, a divisor D of degree d and rank r whose class is vertex avoiding, and a set $\mathcal{A}$ with size $\#\mathcal{A} = \min\left\{\binom{r+2}{2}, 2d - g + 1\right\}$ such that $\{\psi_{ij} \in R(D) \mid (i, j) \in \mathcal{A}\}$ is tropically independent. Note that

$$g = g' + \rho, \quad d = d' + \rho, \quad \text{and} \quad \#\mathcal{A} - \#\mathcal{A}' = \min\left\{\rho, \binom{r+2}{2} - \#\mathcal{A}'\right\}.$$

Proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) > 0$, and $\binom{r+2}{2} \geq 2d - g + 1$. We construct Γ, D and $\mathcal{A}$ by adding ρ new loops to Γ', ρ new points to D', and ρ new points to $\mathcal{A}'$. Any collection of ρ points in the complement of $\mathcal{A}'$ will work, but the location of the new loops added to Γ' depends on the set $\mathcal{A} \smallsetminus \mathcal{A}'$.

Recall that the complement of $\mathcal{A}'$ consists of the integer points in the closed chevron, the lower left half-open triangle, and the upper right half-open triangle, as shown in Figure 15. Suppose $\mathcal{A} \smallsetminus \mathcal{A}'$ consists of ν new points in the chevron, ν_1 new points in the lower left half-open triangle, and ν_2 new points in the upper right half-open triangle. Then construct Γ from Γ' by adding ν_1 new loops to the left end of Γ', ν_2 new loops to the right end of Γ', and ν new loops in the middle of the chain, at locations that are specified as follows.

For $\lceil r/2 \rceil - s \leq \ell < \lceil r/2 \rceil + s$, let $a(\ell)$ and $b(\ell)$ be as defined in Section 7. For each new element (i, j) from the chevron, we add a corresponding loop to the end of the subgraph $\Gamma'_{[a(\ell), b(\ell)]}$, where ℓ is the unique integer such that $i + j = \ell + \lceil r/2 \rceil$. In other words, if there are t points (i, j) in $\mathcal{A} \smallsetminus \mathcal{A}'$ such that $i + j = \ell + \lceil r/2 \rceil$, we add t new loops immediately to the right of the $b(\ell)$-th loop in Γ'.

Let $\alpha(k)$ denote the number of new points $(i, j) \in \mathcal{A} \smallsetminus \mathcal{A}'$ such that $i + j \leq k$. We construct our divisor D' so that it has one chip on each of the new loops. The new loops correspond to lingering steps in the associated lattice path, and the location of the points on the new loops are chosen in specific regions on the top edges, as described below, and sufficiently general so that the class of D' is vertex avoiding.

Just as in Sections 7 and 8, we suppose that $\{\psi_{ij} \mid (i, j) \in \mathcal{A}\}$ is tropically dependent, choose constants b_{ij} such that the minimum

$$\theta(v) = \min_{ij}\{\psi_{ij}(v) + b_{ij})\}$$

occurs at least twice at every point v in Γ, and fix

$$\Delta = \operatorname{div}(\theta) + 2D, \quad \delta_i = \deg(\Delta)|_{\gamma_i}, \quad \text{and} \quad e(k) = \delta_0 + \cdots + \delta_k - 2k.$$

We again fix $s = g - d + r$, which is the same as $g' - d' + r$. We claim that

(1) $\delta_0 + \cdots + \delta_{\nu_1} \geq r - 2s + \epsilon(r)$,

(2) $e(\ell s + \alpha(\ell + \lceil r/2 \rceil)) \geq \ell - s + \lceil r/2 \rceil \quad$ for $\lfloor r/2 \rfloor - s + \epsilon(r) \leq \ell \leq \lfloor r/2 \rfloor + s + 1$,

(3) $\delta_{g - \nu_2 + 1} + \cdots + \delta_{g+1} \geq r - 2s$.

Just as in the proof for $\rho = 0$ and $r > 2s$, the claim implies that $\deg(\Delta) \geq 2d + 1$, which is a contradiction. It remains to prove the claim, which we do inductively, moving from left to right across the graph.

To prove (1), we show that $e(\alpha(k)) \geq k$ for $0 \leq k \leq r - 2s + \epsilon(r)$. For $k = 0$, there is nothing to prove, and we proceed by induction on k. Let $a = \alpha(k) + 1$ and $b = \alpha(k + 1)$. As in the $\rho = 0$ case, we must rule out the possibility that $e(a) = e(b) = k$. Just as in Lemma 7.4, if $e(a) = e(b) = k$ then the δ-permissible functions on $\Gamma_{[a,b]}$ are exactly those ψ_{ij} such that $i + j = k$. We choose the location of the new points on the loops in $\Gamma_{[a,b]}$ so that the functions ψ_i for $i \leq k/2$ have the combinatorial shape shown in Figure 9, on each loop in $\Gamma_{[a,b]}$, and those for

$i > k/2$ have the combinatorial shape shown in Figure 10. It follows that each δ-permissible ψ_{ij} has the combinatorial shape shown in Figure 12. By construction, there are exactly $b - a + 2$ pairs $(i, j) \in \mathcal{A}$ such that $i + j = k$. Then, just as in Proposition 7.6, we conclude that $e(b) \geq k + 1$, which proves (1). (The argument in this case is somewhat simpler than in Proposition 7.6, since the combinatorial shapes appearing in Figures 13 and 14 do not occur.) The proof of (3) is similar.

It remains to prove (2). Note that (2) follows from (1) for $\ell = \lfloor r/2 \rfloor - s + \epsilon(r)$. We proceed by induction on ℓ. Let $a = a(\ell) + \alpha(\ell + \lceil r/2 \rceil - 1)$ and let $b = b(\ell) + \alpha(\ell + \lceil r/2 \rceil)$. As in the $\rho = 0$ case, it suffices to rule out the possibility that $e(a) = e(b) = \ell - s + \lceil r/2 \rceil$.

Suppose $e(a) = e(b) = \ell - s + \lceil r/2 \rceil$. Then, just as in Lemma 7.4, if ψ_{ij} is δ-permissible on $\Gamma_{[a,b]}$, then either $i = j = \ell$ or $i < \ell < j$ and $i + j = \ell + \lceil r/2 \rceil$. We choose the location of the points on the new loops in $\Gamma_{[a,b]}$ so that ψ_{ij} has the shape shown in Figure 12 for $i < \ell < j$. Then, just as in Proposition 7.6, it follows that there must be at least $b - a + 3$ functions that are δ-permissible on $\Gamma_{[a,b]}$. However, by construction, there are only $b - a + 2$ functions that are δ-permissible on $\Gamma_{[a,b]}$, a contradiction. We conclude that $e(b) > \ell - s + \lceil r/2 \rceil$, as required. This completes the proof of the claim, and the theorem follows. $\square$

Remark 9.1. The analogue of (1) in the case $\rho(g, r, d) = 0$ and $r > 2s$ is the lower bound $\delta_0 \geq r - 2s + \epsilon(r)$ which comes from having only one pair $(i, j) \in \mathcal{A}$ such that ψ_{ij} has a given slope σ at w_0, for $0 \leq \sigma < r - 2s + \epsilon(r)$. This bound may be seen as coming from $r - 2s + \epsilon(r)$ applications of the degenerate version of Proposition 7.6 for a chain of zero loops, i.e., a single edge. As we add points to $\mathcal{A}$ and add loops to the left of w_0, these chains of zero loops become actual chains of loops, and we then use the usual version of Proposition 7.6. A similar remark applies to (3).

Proof of Conjecture 4.6 for $m = 2$, $\rho(g, r, d) > 0$, and $\binom{r+2}{2} \leq 2d - g + 1$. Again, it suffices to construct a divisor D on Γ of rank r and degree d whose class is vertex avoiding such that all of the functions ψ_{ij} are tropically independent. Let

$$\eta = \min\left\{\rho, 2d - g + 1 - \binom{r+2}{2}\right\}.$$

By the arguments in the preceding case, on the chain of $g - \eta$ loops with bridges, there exists a vertex avoiding divisor D' of rank r and degree $d - \eta$ such that the functions ψ_{ij} are tropically independent. We construct a divisor D on Γ of rank r and degree d by specifying that $D|_{\Gamma_{[0,g-\eta]}} = D'$, and the remaining η steps of the corresponding lattice path are all lingering, with the points on the last η loops chosen sufficiently general so that the class of D is vertex avoiding. Then the restrictions of the functions ψ_{ij} to $\Gamma_{[0,g-\eta]}$ are tropically independent, so the functions themselves are tropically independent as well. $\square$

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dave.h.jensen@gmail.com *Department of Mathematics, 719 Patterson Office Tower, Lexington, KY 40506-0027, United States*

sam.payne@yale.edu *Mathematics Department, Yale University, 10 Hillhouse Ave, New Haven, CT 06511, United States*

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Algebraicity of normal analytic compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity

Pinaki Mondal

We present an effective criterion to determine if a normal analytic compactification of $\mathbb{C}^2$ with one irreducible curve at infinity is algebraic or not. As a byproduct we establish a correspondence between normal algebraic compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity and algebraic curves contained in $\mathbb{C}^2$ with one place at infinity. Using our criterion we construct pairs of homeomorphic normal analytic surfaces with minimally elliptic singularities such that one of the surfaces is algebraic and the other is not. Our main technical tool is the sequence of *key forms* — a "global" variant of the sequence of *key polynomials* introduced by MacLane [1936] to study valuations in the "local" setting — which also extends the notion of *approximate roots* of polynomials considered by Abhyankar and Moh [1973].

1. Introduction

Algebraic compactifications of $\mathbb{C}^2$ (i.e., compact algebraic surfaces containing $\mathbb{C}^2$) are in a sense the simplest compact algebraic surfaces. The simplest among these are the *primitive compactifications*, i.e., those for which the complement of $\mathbb{C}^2$ (a.k.a. the *curve at infinity*) is irreducible. It follows from a famous result of and Van de Ven that up to isomorphism, $\mathbb{P}^2$ is the only *nonsingular* primitive compactification of $\mathbb{C}^2$. In some sense a more natural category than nonsingular algebraic surfaces is the category of *normal* algebraic surfaces[1]. In this article we tackle the problem of understanding the simplest normal algebraic compactifications of $\mathbb{C}^2$:

Question 1.1. What are the normal primitive algebraic compactifications of $\mathbb{C}^2$?

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[1]This is true for example from the perspective of valuation theory: the irreducible components of the curve at infinity of a normal compactification $\bar{X}$ of $\mathbb{C}^2$ correspond precisely to the discrete valuations on $\mathbb{C}[x, y]$ which are *centered at infinity* with positive dimensional center on $\bar{X}$. Therefore $\bar{X}$ is primitive and normal if and only if $\bar{X}$ corresponds to precisely one discrete valuation centered at infinity on $\mathbb{C}[x, y]$.

We give a complete answer to this question; in particular, we characterize both algebraic and *nonalgebraic* primitive compactifications of $\mathbb{C}^2$. Our answer is equivalent to an *explicit* criterion for determining *algebraicity* of (analytic) contractions of a class of curves: indeed, it follows from well-known results of Kodaira, and independently of Morrow, that any normal analytic compactification $\bar{X}$ of $\mathbb{C}^2$ is the result of contraction of a (possibly reducible) curve E from a nonsingular surface constructed from $\mathbb{P}^2$ by a sequence of blow-ups. On the other hand, a well-known result of Grauert completely and *effectively* characterizes all curves on a nonsingular analytic surface which can be *analytically* contracted: namely it is necessary and sufficient that the matrix of intersection numbers of the irreducible components of E is negative definite. It follows that the question of understanding algebraicity of analytic compactifications of $\mathbb{C}^2$ is equivalent to the following question.

Question 1.2. Let $\pi : Y \to \mathbb{P}^2$ be a birational morphism of nonsingular complex algebraic surfaces and $L \subseteq \mathbb{P}^2$ be a line. Assume π restricts to an isomorphism on $\pi^{-1}(\mathbb{P}^2 \setminus L)$. Let E be the *exceptional divisor* of π (i.e., E is the union of curves on Y which map to points in $\mathbb{P}^2$) and $E_1, \dots, E_N$ be irreducible curves contained in E. Let E' be the union of the strict transform L' (on Y) of L and all components of E *excluding* $E_1, \dots, E_N$. Assume E' is analytically contractible; let $\pi' : Y \to Y'$ be the contraction of E'. When is Y' algebraic?

Question 1.1 is equivalent to the $N = 1$ case of Question 1.2. We give a complete solution (Theorem 4.1) to this case of Question 1.2. Our answer is in particular *effective*, i.e., given a description of Y (e.g., if we know a sequence of blow ups which construct Y from $\mathbb{P}^2$, or if we know precisely the discrete valuation ν on $\mathbb{C}(x, y)$ associated to the unique curve on $Y \setminus \mathbb{C}^2$ which does *not* get contracted), our algorithm determines in finite time if the contraction is algebraic. In fact the algorithm is a one-liner: "Compute the *key forms* of ν. Y' is algebraic if and only if the last key form is a polynomial." The only previously known effective criteria for determining the algebraicity of contraction of curves on surfaces was the well-known criteria of Artin [1962] which states that a normal surface is algebraic if all its singularities are *rational*. We refer the reader to [Morrow and Rossi 1975; Brenton 1977; Franco and Lascu 1999; Schröer 2000; Bădescu 2001; Palka 2013] for other criteria — some of these are more general, but none is effective in the above sense. Moreover, as opposed to Artin's criterion, ours is *not* numerical, i.e., it is not determined by numerical invariants of the associated singularities. We give an example (in Section 2) which shows that in fact there is *no topological*, let alone numerical, answer to Question 1.2 even for $N = 1$.

As a corollary of our criterion, we establish a new correspondence between normal primitive algebraic compactifications of $\mathbb{C}^2$ and algebraic curves in $\mathbb{C}^2$ with *one place at infinity*[2] (Theorem 4.3). Curves with one place at infinity have been extensively studied in affine algebraic geometry (see, e.g., [Abhyankar and Moh 1973; 1975; Ganong 1979; Russell 1980; Nakazawa and Oka 1997; Suzuki 1999; Wightwick 2007]), and we believe the connection we found between these and compactifications of $\mathbb{C}^2$ will be useful for the study of both[3].

Our main technical tool is the sequence of *key forms*, which is a direct analogue of the sequence *key polynomials* introduced by MacLane [1936]. The key polynomials were introduced (and have been extensively used — see, e.g., [Moyls 1951; Favre and Jonsson 2004; Vaquié 2007; Herrera Govantes et al. 2007]) to study valuations in a *local* setting. However, our criterion shows how they retain information about the *global geometry* when computed in "global coordinates."

The example in Section 2 shows that algebraicity of Y' from Question 1.2 can not be determined only from the (weighted) *dual graph* (Definition 3.25) of E'. However, at least when $N = 1$, it is possible to completely characterize the weighted dual graphs (more precisely, *augmented and marked* weighted dual graphs — see Definition 3.26) which correspond to *only algebraic* contractions, those which correspond to *only nonalgebraic* contractions, and those which correspond to *both* types of contractions (Theorem 4.4). The characterization involves two sets of *semigroup conditions* (S1-k) and (S2-k). We note that the first set of semigroup conditions (S1-k) are equivalent to the semigroup conditions that appear in the theory of plane curves with one place at infinity developed in [Abhyankar and Moh 1973; Abhyankar 1977; 1978; Sathaye and Stenerson 1994].

Finally we would like to point that Question 1.1 is equivalent to a two dimensional *Cousin-type problem at infinity*: let $O_1, \dots, O_N \in \mathbb{P}^2 \setminus \mathbb{C}^2$ be *points at infinity*. Let (u_j, v_j) be coordinates near O_j, $\psi_j(u_j)$ be a *Puiseux series* (Definition 3.2) in u_j, and r_j be a positive rational number, $1 \le j \le N$.

Question 1.3. Determine if there exists a polynomial $f \in \mathbb{C}[x, y]$ such that for each analytic branch C of the curve $f = 0$ at infinity, there exists j, $1 \le j \le N$, such that

- C intersects L_∞ at O_j,
- C has a Puiseux expansion $v_j = \theta(u_j)$ at O_j such that $\operatorname{ord}_{u_j}(\theta - \psi_j) \ge r_j$.

On our way to understand normal primitive compactifications of $\mathbb{C}^2$, we solve the $N = 1$ case of Question 1.3 (Theorem 4.7).

Remark 1.4. We use *Puiseux series* in an essential way in this article. However, instead of the usual Puiseux series, from Section 3 onward, we almost exclusively

[2]Let $C \subseteq \mathbb{C}^2$ be an algebraic curve, and let $\bar{C}'$ be the closure of C in $\mathbb{P}^2$ and $\sigma : \bar{C} \to \bar{C}'$ be the desingularization of $\bar{C}'$. C has *one place at infinity* if and only if $|\sigma^{-1}(\bar{C}' \setminus C)| = 1$.

[3]For example, we use this connection in [Mondal 2013b] to solve completely the main problem studied in [Campillo et al. 2002].

work with *descending* Puiseux series (a descending Puiseux series in x is simply a meromorphic Puiseux series in x^{-1} — see Definition 3.4). The choice was enforced on us "naturally" from the context — while key polynomials and Puiseux series are natural tools in the study of valuations in the local setting, when we need to study the relation of valuations corresponding to curves at infinity (on a compactification of $\mathbb{C}^2$) to global properties of the surface, key forms and descending Puiseux series are sometimes more convenient.

1A. ***Organization.*** We start with an example in Section 2 to illustrate that the answer to Question 1.2 can not be numerical or topological. The construction also serves as an example of nonalgebraic normal *Moishezon surfaces*[4] with the "simplest possible" singularities (see Remark 2.1). In Section 3 we recall some background material and in Section 4 we state our results. The rest of the article is devoted to the proof of the results of Section 4. In Section 5 we recall some more background material needed for the proof; in particular in Section 5A we state the algorithm to compute key forms of a valuation from the associated descending Puiseux series, and illustrate the algorithm via an elaborate example (we note that this algorithm is essentially the same as the algorithm used in [Makar-Limanov 2015] for a different purpose). In Section 6 we build some tools for dealing with descending Puiseux series and in Section 7 we use these tools to prove the results from Section 4. The appendices contain proof of two lemmas from Section 6 — the proofs were relegated to the appendix essentially because of their length.

2. Algebraic and nonalgebraic compactifications with homeomorphic singularities

Let (u, v) be a system of "affine" coordinates near a point $O \in \mathbb{P}^2$ ("affine" means that both $u = 0$ and $v = 0$ are lines on $\mathbb{P}^2$) and L be the line $\{u = 0\}$. Let C_1 and C_2 be curve-germs at O defined respectively by $f_1 := v^5 - u^3$ and $f_2 := (v - u^2)^5 - u^3$. For each i, let Y_i be the surface constructed by resolving the singularity of C_i at O and then blowing up 8 more times the point of intersection of the (successive) strict transform of C_i with the exceptional divisor. Let E_i^* be the *last* exceptional curve, and $E'^{(i)}$ be the union of the strict transform L'_i (on Y_i) of L and (the strict transforms of) all exceptional curves except E_i^*.

Note that the pairs of germs (C_1, L) and (C_2, L) are *isomorphic* via the map $(u, v) \mapsto (u, v + u^2)$. It follows that "weighted dual graphs" (Definition 3.25) of the $E'^{(i)}$ are *identical*; they are depicted in Figure 1, left (we labeled the vertices according to the order of appearance of the corresponding curves in the sequence of blow-ups). It is straightforward to compute that the matrices of intersection

[4] *Moishezon surfaces* are analytic surfaces for which the fields of meromorphic functions have transcendence degree 2 over $\mathbb{C}$.

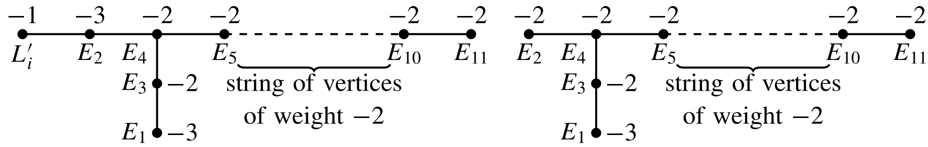

Figure 1. Singularity of Y_i'. Left: weighted dual graph of $E'^{(i)}$. Right: weighted dual graph of the minimal resolution of the singularity of Y_i'.

numbers of the components of the $E'^{(i)}$ are negative definite, so that there is a bimeromorphic analytic map $Y_i \to Y_i'$ contracting $E'^{(i)}$. Note that each Y_i' is a normal analytic surface with one singular point P_i. It follows from the construction that the weighted dual graphs of the minimal resolution of singularities of Y_i' are identical (see Figure 1, right), so that the numerical invariants of the singularities of the Y_i' are also *identical*.

In fact it follows (from, e.g., [Neumann 1981, Section 8]) that the singularities of the Y_i' are also *homeomorphic*. However, Theorem 4.1 and Example 3.19 imply that Y_1' is algebraic, but Y_2' is *not*.

Remark 2.1. It is straightforward to verify that the weighted dual graph presented in Figure 1, right, is precisely the graph labeled $D_{9,*,0}$ in [Laufer 1977]. It then follows from [Laufer 1977] that the singularities at P_i are *Gorenstein hypersurface singularities* of multiplicity 2 and geometric genus 1, which are also *minimally elliptic* (in the sense of [Laufer 1977]). Minimally elliptic Gorenstein singularities have been extensively studied (see, e.g., [Yau 1979; Ohyanagi 1981; Némethi 1999]), and in a sense they form the simplest class of nonrational singularities[5]. Since having only rational singularities implies algebraicity of the surface [Artin 1962], it follows that the surface Y_2' we constructed above is a normal nonalgebraic Moishezon surface with the "simplest possible" singularity.

It follows from [Laufer 1977, Table 2] that the singularity at the origin of $z^2 = x^5 + xy^5$ (Figure 2) is of the same type as the singularity of each Y_i', $1 \le i \le 2$.

3. Background I

Here we compile the background material needed to state the results. In Section 5 we compile further background material that we use for the proof.

Notation 3.1. Throughout the rest of the article we use X to denote $\mathbb{C}^2$ with coordinate ring $\mathbb{C}[x, y]$ and $\bar{X}_{(x,y)}$ to denote the copy of $\mathbb{P}^2$ such that X is embedded

[5]Indeed, every connected proper subvariety of the exceptional divisor of the minimal resolution of a minimally elliptic singularity is the exceptional divisor of the minimal resolution of a rational singularity [Laufer 1977].

Figure 2. The singularity of $z^2 = x^5 + xy^5$ at the origin (*whirling dervish*).

into $\bar{X}_{(x,y)}$ via the map $(x, y) \mapsto [1 : x : y]$. We also denote by L_∞ the *line at infinity* $\bar{X}_{(x,y)} \setminus X$, and by Q_y the point of intersection of L_∞ and (the closure of) the y-axis. Finally, if $\omega_0, \dots, \omega_n$ are positive integers, we denote by $\mathbb{P}^n(\omega_0, \dots, \omega_n)$ the complex n-dimensional weighted projective space corresponding to weights $\omega_0, \dots, \omega_n$.

3A. *Meromorphic and descending Puiseux series.*

Definition 3.2 (meromorphic Puiseux series). A *meromorphic Puiseux series* in a variable u is a fractional power series of the form $\sum_{m \geq M} a_m u^{m/p}$ for some $m, M \in \mathbb{Z}$, $p \geq 1$ and $a_m \in \mathbb{C}$ for all $m \in \mathbb{Z}$. If all exponents of u appearing in a meromorphic Puiseux series are positive, then it is simply called a *Puiseux series* (in u). Given a meromorphic Puiseux series $\phi(u)$ in u, write it in the form

$$\phi(u) = \cdots + a_1 u^{q_1/p_1} + \cdots + a_2 u^{q_2/(p_1 p_2)} + \cdots + a_l u^{q_l/(p_1 p_2 \cdots p_l)} + \cdots,$$

where q_1/p_1 is the smallest noninteger exponent, and for each k, $1 \leq k \leq l$, we have $a_k \neq 0$, $p_k \geq 2$, $\gcd(p_k, q_k) = 1$, and the exponents of all terms with order between $q_k/(p_1 \cdots p_k)$ and $q_k/(p_1 \cdots p_{k+1})$ (or, if $k = l$, all terms of order above $1/(p_1 \cdots p_l)$) belong to $1/(p_1 \cdots p_k)\mathbb{Z}$. Then the pairs $(q_1, p_1), \dots, (q_l, p_l)$, are called the *Puiseux pairs* of ϕ and the exponents $q_k/(p_1 \cdots p_k)$, $1 \leq k \leq l$, are called *characteristic exponents* of ϕ. The *polydromy order* [Casas-Alvero 2000, Chapter 1] of ϕ is $p := p_1 \cdots p_l$, i.e., the polydromy order of ϕ is the smallest p such that $\phi \in \mathbb{C}((u^{1/p}))$. Let ζ be a primitive p-th root of unity. The *conjugates* of ϕ are

$$\phi_j(u) := \cdots + a_1 \zeta^{j q_1 p_2 \cdots p_l} u^{q_1/p_1} + \cdots + a_2 \zeta^{j q_2 p_3 \cdots p_l} u^{q_2/(p_1 p_2)} + \cdots + a_l \zeta^{j q_l} u^{q_l/(p_1 p_2 \cdots p_l)} + \cdots$$

for $1 \le j \le p$ (i.e., ϕ_j is constructed by multiplying the coefficients of terms of ϕ with order n/p by ζ^{jn}).

We recall the standard fact that the field of meromorphic Puiseux series in u is the algebraic closure of the field $\mathbb{C}((u))$ of Laurent polynomials in u:

Theorem 3.3. *Let $f \in \mathbb{C}((u))[v]$ be an irreducible monic polynomial in v of degree d. Then there exists a meromorphic Puiseux series $\phi(u)$ in u of polydromy order d such that*

$$f = \prod_{i=1}^{d} (v - \phi_i(u)),$$

where the ϕ_i are conjugates of ϕ.

Definition 3.4 (descending Puiseux series). A *descending Puiseux series* in x is a meromorphic Puiseux series in x^{-1}. The notions regarding meromorphic Puiseux series defined in Definition 3.2 extend naturally to the setting of descending Puiseux series. In particular, if $\phi(x)$ is a descending Puiseux series and the Puiseux pairs of $\phi(1/x)$ are $(q_1, p_1), \ldots, (q_l, p_l)$, then ϕ has Puiseux pairs $(-q_1, p_1), \ldots, (-q_l, p_l)$, polydromy order $p := p_1 \cdots p_l$, and characteristic exponents $-q_k/(p_1 \cdots p_k)$ for $1 \le k \le l$.

Notation 3.5. We use $\mathbb{C}\langle\langle x \rangle\rangle$ to denote the field of descending Puiseux series in x. For $\phi \in \mathbb{C}\langle\langle x \rangle\rangle$ and $r \in \mathbb{R}$, we denote by $[\phi]_{>r}$ the *descending Puiseux polynomial* (i.e., descending Puiseux series with finitely many terms) consisting of all terms of ϕ of degree $> r$. If ψ is also in $\mathbb{C}\langle\langle x \rangle\rangle$, then we write

$$\phi \equiv_r \psi \iff [\phi]_{>r} = [\psi]_{>r} \iff \deg_x(\phi - \psi) \le r.$$

The following is an immediate Corollary of Theorem 3.3:

Theorem 3.6. *Let $f \in \mathbb{C}[x, x^{-1}, y]$. Then there are (up to conjugacy) unique descending Puiseux series $\phi_1, \ldots, \phi_k$ in x, a unique nonnegative integer m and $c \in \mathbb{C}^*$ such that*

$$f = cx^m \prod_{i=1}^{k} \prod_{\substack{\phi_{ij} \text{ is a} \\ \text{conjugate} \\ \text{of } \phi_i}} (y - \phi_{ij}(x)).$$

3B. ***Divisorial discrete valuation and semidegree.*** Let $\sigma : \tilde{Y} \dashrightarrow Y$ be a birational correspondence of normal complex algebraic surfaces and C be an irreducible analytic curve on $\tilde{Y}$. Then the local ring $\mathcal{O}_{\tilde{Y},C}$ of C on $\tilde{Y}$ is a discrete valuation ring. Let ν be the associated valuation on the field $\mathbb{C}(Y)$ of rational functions on Y; in other words ν is the order of vanishing along C. We say that ν is a *divisorial discrete valuation* on $\mathbb{C}(Y)$; the *center* of ν on Y is $\sigma(C \setminus S)$, where S is the set of points of indeterminacy of σ (the normality of Y ensures that S is a discrete set, so that $C \setminus S \neq \varnothing$). Moreover, if U is an open subset of Y, we say that ν is *centered at infinity* with respect to U if and only if $\sigma(C \setminus S) \subseteq Y \setminus U$.

Definition 3.7 (semidegree). Let U be an affine variety and ν be a divisorial discrete valuation on the ring $\mathbb{C}[U]$ of regular functions on U which is centered at infinity with respect to U. Then we say that $\delta := -\nu$ is a *semidegree* on $\mathbb{C}[U]$.

The following result, which connects semidegrees on $\mathbb{C}[x, y]$ with descending Puiseux series in x, is a reformulation of [Favre and Jonsson 2004, Proposition 4.1].

Theorem 3.8. *Let δ be a semidegree on $\mathbb{C}[x, y]$. Assume that $\delta(x) > 0$. Then there is a* descending Puiseux polynomial (*i.e., a descending Puiseux series with finitely many terms*) $\phi_\delta(x)$ (*unique up to conjugacy*) *in x and a* (*unique*) *rational number $r_\delta < \operatorname{ord}_x(\phi_\delta)$ such that for every $f \in \mathbb{C}[x, y]$,*

$$\delta(f) = \delta(x) \deg_x(f(x, \phi_\delta(x) + \xi x^{r_\delta})), \tag{3-1}$$

where ξ is an indeterminate.

Definition 3.9. In the situation of Theorem 3.8, we say that $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ is the *generic descending Puiseux series* associated to δ. Moreover, if $\bar{X}$ is an analytic compactification of $X = \mathbb{C}^2$ and $Z \subseteq \bar{X} \setminus \mathbb{C}^2$ is a curve at infinity such that δ is the order of pole along Z, then we also say that $\tilde{\phi}_\delta(x, \xi)$ is the *generic descending Puiseux series associated to Z*.

Example 3.10. If δ is a weighted degree in (x, y)-coordinates corresponding to weights p for x and q for y with p, q positive integers, then the generic descending Puiseux series corresponding to δ is $\tilde{\phi}_\delta = \xi x^{q/p}$. Note that if we embed $\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[x, y]$ into the weighted projective space $\mathbb{P}^2(1, p, q)$ via $(x, y) \mapsto [1 : x : y]$, then δ is precisely the order of the pole along the curve at infinity.

Example 3.11. Recall the setup of the example from Section 2. Then the C_i have Puiseux expansions $v = \psi_i(u)$ at O, where

$$\psi_1(u) = u^{3/5}, \quad \psi_2(u) = u^{3/5} + u^2.$$

Now note that $(x, y) := (1/u, v/u)$ are coordinates on $\mathbb{P}^2 \setminus L \cong \mathbb{C}^2$, and with respect to (x, y) coordinates the C_1 has a *descending* Puiseux expansion of the form $y = x\psi_1(1/x) = x^{2/5}$. Similarly, C_2 has a descending Puiseux expansion of the form $y = x\psi_2(1/x) = x^{2/5} + x^{-1}$. Let δ_i be the order of pole along E_i^*, $1 \le i \le 2$. Then the generic descending Puiseux series corresponding to δ_1 and δ_2 are respectively of the form

$$\tilde{\phi}_{\delta_1}(x, \xi_1) = x^{2/5} + \xi_1 x^{-6/5}, \quad \tilde{\phi}_{\delta_2}(x, \xi_2) = x^{2/5} + x^{-1} + \xi_2 x^{-6/5}. \tag{3-2}$$

Definition 3.12 (formal Puiseux pairs of generic descending Puiseux series). Let δ and $\tilde{\phi}_\delta(x,\xi) := \phi_\delta(x) + \xi x^{r_\delta}$ be as in Definition 3.9. Let the Puiseux pairs of ϕ_δ be $(q_1, p_1), \ldots, (q_l, p_l)$. Express r_δ as $q_{l+1}/(p_1 \cdots p_l p_{l+1})$ where $p_{l+1} \geq 1$ and $\gcd(q_{l+1}, p_{l+1}) = 1$. The *formal Puiseux pairs* of $\tilde{\phi}_\delta$ are $(q_1, p_1), \ldots, (q_{l+1}, p_{l+1})$, with (q_{l+1}, p_{l+1}) being the *generic* formal Puiseux pair. Note that

(1) $\delta(x) = p_1 \cdots p_{l+1}$,

(2) it is possible that $p_{l+1} = 1$ (whereas every other p_k is ≥ 2).

3C. *Geometric interpretation of generic descending Puiseux series.* In this subsection we recall from [Mondal 2016] the geometric interpretation of generic descending Puiseux series. We keep the conventions introduced in Notation 3.1.

Definition 3.13. An *irreducible analytic curve germ at infinity* on X is the image γ of an analytic map h from a punctured neighborhood Δ' of the origin in $\mathbb{C}$ to X such that $|h(s)| \to \infty$ as $|s| \to 0$ (in other words, h is analytic on Δ' and has a pole at the origin). If $\bar{X}$ is an analytic compactification of X, then there is a unique point $P \in \bar{X} \setminus X$ such that $|h(s)| \to P$ as $|s| \to 0$. We call P the *center* of γ on $\bar{X}$, and write $P = \lim_{\bar{X}} \gamma$.

Let $\bar{X}$ be a primitive normal analytic compactification of X with an irreducible curve C_∞ at infinity. Let $\sigma : \bar{X}_{(x,y)} \dashrightarrow \bar{X}$ be the natural bimeromorphic map, and let Y be a *resolution of indeterminacies* of σ, i.e., Y is a nonsingular rational surface equipped with analytic maps $\pi : Y \to \bar{X}_{(x,y)}$ and $\pi' : Y \to \bar{X}$ such that $\pi' = \sigma \circ \pi$. Let L'_∞ be the strict transform of $L_\infty \subseteq \bar{X}_{(x,y)}$ on Y and $Q'_y \in L'_\infty$ be (the unique point) such that $\pi(Q'_y) = Q_y$. Let $P_\infty := \pi'(Q'_y) \in C_\infty$.

Proposition 3.14 [Mondal 2016, Proposition 3.5]. *Let δ be the order of pole along C_∞, $\tilde{\phi}_\delta(x,\xi)$ be the generic descending Puiseux series associated to δ and γ be an irreducible analytic curve germ on X. Then $\lim_{\bar{X}} \gamma \in C_\infty \setminus \{P_\infty\}$ if and only if γ has a parametrization of the form*

$$t \mapsto (t, \tilde{\phi}_\delta(t,\xi)|_{\xi=c} + \text{l.d.t.}) \quad \textit{for } |t| \gg 0$$

for some $c \in \mathbb{C}$, where l.d.t. *means* lower degree terms (*in t*).

Remark-Definition 3.15. We call P_∞ *a center of $\mathbb{P}^2$-infinity on $\bar{X}$*. P_∞ is in fact *unique* in the case of "generic" primitive normal compactifications of $\mathbb{C}^2$ (we do not use this uniqueness in this article, so we state it without a proof):

- If $\bar{X} \cong \mathbb{P}^2(1,1,q)$ for some $q > 0$, then *every* point of C_∞ is a center of $\mathbb{P}^2$-infinity on $\bar{X}$.
- If $\bar{X} \cong \mathbb{P}^2(1,p,q)$ for some $p, q > 1$, then $\bar{X}$ has *two* singular points, and these are precisely the centers of $\mathbb{P}^2$-infinity on $\bar{X}$.

- In all other cases, there is a *unique* center of $\mathbb{P}^2$-infinity on $\bar{X}$ — it is precisely the unique point on $\bar{X}$ which has a nonquotient singularity.

3D. *Key forms of a semidegree.* Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Pick $k > 1$ such that $\delta(y/x^k) < 0$. Set $(u, v) := (1/x, y/x^k)$. Then $\nu := -\delta$ is a discrete valuation on $\mathbb{C}[u, v]$ which is *centered at the origin*. It follows that ν can be completely described in terms of a finite sequence of *key polynomials* in (u, v) [MacLane 1936]. The *key forms* of δ that we introduce in this section are precisely the analogue of key polynomials of ν. We refer to [Favre and Jonsson 2004, Chapter 2] for the properties of key polynomials that we used as a model for our definition of key forms.

Definition 3.16 (key forms). Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. A sequence of elements $g_0, g_1, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ is called the sequence of *key forms* for δ if the following properties are satisfied with $\eta_j := \delta(g_j)$, $0 \le j \le n+1$:

(P0) $\eta_{j+1} < \alpha_j \eta_j = \sum_{i=0}^{j-1} \beta_{j,i} \eta_i$ for $1 \le j \le n$, where
 (a) $\alpha_j = \min\{\alpha \in \mathbb{Z}_{>0} : \alpha \eta_j \in \mathbb{Z}\eta_0 + \cdots + \mathbb{Z}\eta_{j-1}\}$ for $1 \le j \le n$,
 (b) the $\beta_{j,i}$ are integers such that $0 \le \beta_{j,i} < \alpha_i$ for $1 \le i < j \le n$ (in particular, only the $\beta_{j,0}$ are allowed to be negative).

(P1) $g_0 = x$, $g_1 = y$.

(P2) For $1 \le j \le n$, there exists $\theta_j \in \mathbb{C}^*$ such that

$$g_{j+1} = g_j^{\alpha_j} - \theta_j g_0^{\beta_{j,0}} \cdots g_{j-1}^{\beta_{j,j-1}}.$$

(P3) Let $z_1, \dots, z_{n+1}$ be indeterminates and η be the *weighted degree* on $B := \mathbb{C}[x, x^{-1}, z_1, \dots, z_{n+1}]$ corresponding to weights η_0 for x and η_j for z_j, where $1 \le j \le n+1$ (i.e., the value of η on a polynomial is the maximum "weight" of its monomials). Then for every polynomial $g \in \mathbb{C}[x, x^{-1}, y]$,

$$\delta(g) = \min\{\eta(G) : G(x, z_1, \dots, z_{n+1}) \in B,\ G(x, g_1, \dots, g_{n+1}) = g\}. \tag{3-3}$$

The properties of key forms of semidegrees compiled in the following theorem are straightforward analogues of corresponding (standard) properties of key polynomials of valuations.

Theorem 3.17. (1) *Every semidegree δ on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$ has a unique and finite sequence of key forms.*

(2) *Conversely, given $g_0, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ and integers $\eta_0, \dots, \eta_{l+1}$ with $\eta_0 > 0$ which satisfy properties* (P0)–(P2), *there is a unique semidegree δ on $\mathbb{C}[x, y]$ such that the g_j are key forms of δ and $\eta_j = \delta(g_j)$, $0 \le j \le n+1$.*

(3) (Recall Notation 3.1.) *Assume $\sigma : \bar{X}^* \to \bar{X}_{(x,y)}$ is a composition of point blow-ups and $E^* \subseteq \bar{X}^*$ is an exceptional curve of σ. Let δ be the order of pole*

along E^*. Assume $\delta(x) > 0$. Then the following data are equivalent: given any one of them, there is an explicit algorithm to construct the others in finite time.

(a) *A minimal sequence of points on successive blow-ups of $\bar{X}_{(x,y)}$ such that σ factors through the composition of these blow-ups and E^* is the strict transform of the exceptional curve of the last blow-up.*

(b) *A generic descending Puiseux series of δ.*

(c) *The sequence of key forms of δ.*

(4) *Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Let*

$$\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$$

be the generic descending Puiseux series and let g_{n+1} be the last key form of δ. Then the descending Puiseux factorization of g_{n+1} is of the form

$$g_{n+1} = \prod_{\substack{\psi_j \text{ is a} \\ \text{conjugate} \\ \text{of } \psi}} (y - \psi_j(x))$$

for some $\psi \in \mathbb{C}\langle\langle x \rangle\rangle$ such that $\psi \equiv_{r_\delta} \phi_\delta$ (see Notation 3.5).

Example 3.18. Let δ be the weighted degree from Example 3.10. The key forms of δ are $g_0 = x$ and $g_1 = y$.

Example 3.19. Let δ_1 and δ_2 be the semidegrees from Example 3.11. Then the key forms of δ_1 are $x, y, y^5 - x^2$. On the other hand the key forms of δ_2 are $x, y, y^5 - x^2, y^5 - x^2 - 5x^{-1}y^4$ (see Algorithm 5.1 for the general algorithm to compute key forms from generic descending Puiseux series).

Definition 3.20 (essential key forms). Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$, and let $g_0, \dots, g_{n+1}$ be the key forms of δ. Pick the subsequence $j_1, j_2, \dots, j_m$ of $1, \dots, n$ consisting of all j_k such that $\alpha_{j_k} > 1$ (where α_{j_k} is as in property (P0) of Definition 3.16). Set

$$f_k := \begin{cases} g_0 = x & \text{if } k = 0, \\ g_{j_k} & \text{if } 1 \le k \le m, \\ g_{n+1} & \text{if } k = m + 1. \end{cases}$$

We say that $f_0, \dots, f_{m+1}$ are the *essential key forms* of δ.

The following properties of essential key forms follow in a straightforward manner from the defining properties of key forms.

Proposition 3.21. (Let the notation be as in Definition 3.20.) *Let $\tilde{\phi}_\delta(x, \xi)$ be the generic descending Puiseux series of δ and $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$ be the formal Puiseux pairs of $\tilde{\phi}_\delta$. Then:*

(1) *$l = m$, i.e., the number of essential key forms of δ is precisely $l + 1$.*

(2) *Set* $\omega_k := \delta(f_k)$, $0 \le k \le l+1$. *Then the sequence* $\omega_0, \dots, \omega_{l+1}$ *depends only on the formal Puiseux pairs of* $\tilde{\phi}_\delta$. *More precisely, with* $p_0 := q_0 := 1$, *we have*

$$\omega_k = \begin{cases} p_1 \cdots p_{l+1} & \text{if } k = 0, \\ p_{k-1}\omega_{k-1} + (q_k - q_{k-1}p_k)p_{k+1} \cdots p_{l+1} & \text{if } 1 \le k \le l+1. \end{cases} \tag{3-4}$$

(3) *Let* $\alpha_1, \dots, \alpha_{n+1}$ *be as in property* (P0) *of key forms. Then*

$$\alpha_j = \begin{cases} p_k & \text{if } j = j_k, \quad 1 \le k \le l+1, \\ 1 & \text{otherwise.} \end{cases}$$

(4) *Pick* j, $0 \le j \le n+1$. *Assume* $j_k < j < j_{k+1}$ *for some* k, $0 \le k \le l$. *Then* $\delta(g_j)$ *is in the group generated by* $\omega_0, \dots, \omega_k$.

Definition 3.22. We call $\omega_0, \dots, \omega_{l+1}$ of Proposition 3.21 the sequence of *essential key values* of δ.

Example 3.23. Let δ_1, δ_2 be as in Examples 3.11 and 3.19. Then all the key forms of δ_1 are essential, and the essential key values are $\omega_0 = \delta_1(x) = 5$, $\omega_1 = \delta_1(y) = 2$, $\omega_2 = \delta_1(y^5 - x^2) = 2$. The key forms of δ_2 are x, y, $y^5 - x^2 - 5x^{-1}y^4$. The sequence of essential key values of δ_2 is the *same* as that of δ_1.

3E. *Resolution of singularities of primitive normal compactifications.* Given two birational algebraic surfaces Y_1, Y_2, we say that Y_1 *dominates* Y_2 if the birational map $Y_1 \dashrightarrow Y_2$ is in fact a morphism. Let $\bar{X}$ be a primitive normal analytic compactification of $X := \mathbb{C}^2$ and $\pi : Y \to \bar{X}$ be a resolution of singularities of $\bar{X}$. We say that π or Y is $\mathbb{P}^2$-*dominating* if Y dominates $\mathbb{P}^2$. The resolution π is a *minimal* $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}$ if up to isomorphism (of algebraic varieties) Y is the only $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}$ which is dominated by Y.

Theorem 3.24. *Every primitive normal analytic compactification of* $\mathbb{C}^2$ *has a unique minimal* $\mathbb{P}^2$*-dominating resolution of singularities.*

We have not found any proof of Theorem 3.24 in the literature. We give a proof in [Mondal 2013a] (using Theorem 4.1 of this article). In this section we recall from [Mondal 2016] a description of the *dual graphs* of minimal $\mathbb{P}^2$-dominating resolutions of singularities of primitive normal analytic compactifications of $\mathbb{C}^2$.

Definition 3.25. Let $E_1, \dots, E_k$ be nonsingular curves on a (nonsingular) surface such that for each $i \ne j$, either $E_i \cap E_j = \varnothing$, or E_i and E_j intersect transversally at a single point. Then $E = E_1 \cup \cdots \cup E_k$ is called a *simple normal crossing curve*. The *(weighted) dual graph* of E is a weighted graph with k vertices $V_1, \dots, V_k$ such that

- there is an edge between V_i and V_j if and only if $E_i \cap E_j \ne \varnothing$,
- the weight of V_i is the self intersection number of E_i.

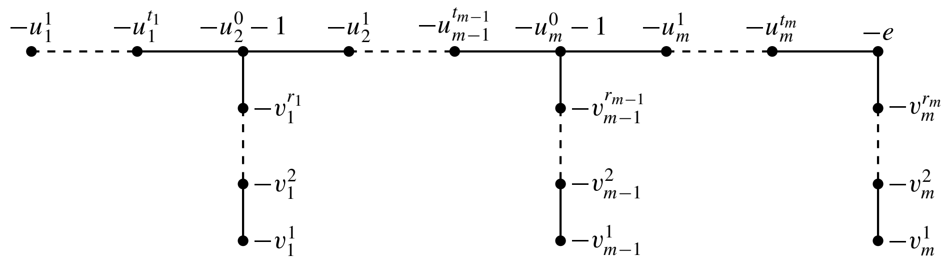

Figure 3. $\tilde{\Gamma}_{\vec{\tilde{q}},\vec{\tilde{p}},m,e}$.

Usually we will abuse the notation, and label V_i also by E_i.

Definition 3.26. Let $\bar{X}$ be a primitive normal analytic compactification of $X := \mathbb{C}^2$ and $\pi : Y \to \bar{X}$ be a resolution of singularities of $\bar{X}$ such that $Y \setminus X$ is a simple normal crossing curve. The *augmented dual graph* of π is the dual graph (Definition 3.25) of $Y \setminus X$. If Y is $\mathbb{P}^2$-dominating, we define the *augmented and marked dual graph* of π to be its augmented dual graph with the strict transforms of the curves at infinity on $\mathbb{P}^2$ and $\bar{X}$ marked (e.g., by different colors or labels).

Given a sequence $(\tilde{q}_1, \tilde{p}_1), \dots, (\tilde{q}_n, \tilde{p}_n)$ of pairs of relatively prime integers, and positive integers m, e such that $1 \le m \le n$, we denote by $\tilde{\Gamma}_{\vec{\tilde{q}},\vec{\tilde{p}},m,e}$ the weighted graph in Figure 3, where the right-most vertex in the top row has weight $-e$, and the other weights satisfy: $u_i^0, v_i^0 \ge 1$ and $u_i^j, v_i^j \ge 2$ for $j > 0$, and are uniquely determined from the continued fractions

$$\frac{\tilde{p}_i}{q'_i} = u_i^0 - \cfrac{1}{u_i^1 - \cfrac{1}{\ddots - \cfrac{1}{u_i^{t_i}}}}, \qquad \frac{q'_i}{\tilde{p}_i} = v_i^0 - \cfrac{1}{v_i^1 - \cfrac{1}{\ddots - \cfrac{1}{v_i^{r_i}}}}, \tag{3-5}$$

where $q'_1 := q_1$ and $q'_i := \tilde{q}_i - \tilde{q}_{i-1}\tilde{p}_i$ if $i \ne 1$.

Remark 3.27. $\tilde{\Gamma}_{\vec{\tilde{q}},\vec{\tilde{p}},m,1}$ is the weighted dual graph of the exceptional divisor of the minimal resolution of an irreducible plane curve singularity with Puiseux pairs $(\tilde{q}_1, \tilde{p}_1), \dots, (\tilde{q}_m, \tilde{p}_m)$ (see, e.g., [Mendris and Némethi 2005, Section 2.2]).

Theorem 3.28 [Mondal 2016, Proposition 4.2, Corollary 6.3]. *Let $\bar{X}$ be a primitive normal compactification of $X := \operatorname{Spec} \mathbb{C}[x, y] \cong \mathbb{C}^2$.*

(1) *If $\bar{X}$ is nonsingular, then $\bar{X} \cong \mathbb{P}^2$.*

(2) *Assume $\bar{X}$ is singular. Let $\tilde{\phi}_\delta(x, \xi)$ be the generic descending Puiseux series* (Definition 3.9) *associated to $E^* := \bar{X} \setminus X$ and $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$*

be the formal Puiseux pairs *of* $\tilde{\phi}(x, \xi)$ (Definition 3.12). *Define* $(\tilde{q}_i, \tilde{p}_i) := (p_1 \cdots p_i - q_i, p_i)$, $1 \le i \le l+1$.

(a) *After a (polynomial) change of coordinates of* $\mathbb{C}^2$ *if necessary, we may assume that* $q_1 < p_1$ *and either* $l = 0$ *or* $q_1 > 1$.

(b) *Assume* (2a) *holds. If* $p_{l+1} > 1$, *then the augmented and marked dual graph of the minimal* $\mathbb{P}^2$*-dominating resolution of singularities of* $\bar{X}$ *is as in Figure 4, left, where the strict transform of the curve at infinity on* $\mathbb{P}^2$ *(resp.* $\bar{X}$*) is marked by* L *(resp.* E^**).*

(c) *Assume* (2a) *holds. If* $p_{l+1} = 1$, *then* ($l \ge 1$, *and) the augmented and marked dual graph of the minimal* $\mathbb{P}^2$*-dominating resolution of singularities of* $\bar{X}$ *is as in Figure 4, right, where the strict transform of the curve at infinity on* $\mathbb{P}^2$ *(resp.* $\bar{X}$*) is marked by* L *(resp.* E^**).*

(3) *Conversely, let* $0 \le l$, *and* $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$ *be pairs of integers such that*

(a) $p_k \ge 2, \ 1 \le k \le l$,

(b) $p_{l+1} \ge 1$,

(c) $\tilde{q}_k := p_1 \cdots p_k - q_k > 0, \ 1 \le k \le l+1$,

(d) $\gcd(p_k, q_k) = 1, \ 1 \le k \le l+1$.

Assume moreover that (2a) *holds, i.e., either* $l = 0$ *or* $q_{l+1} > 1$. *Define* $\omega_0, \dots, \omega_{l+1}$ *as in* (3-4). *Let*

$$\Gamma_{\vec{p},\vec{q}} := \begin{cases} \text{the graph from Figure 4, left} & \text{if } p_{l+1} > 1, \\ \text{the graph from Figure 4, right} & \text{if } p_{l+1} = 1. \end{cases} \tag{3-6}$$

Then $\Gamma_{\vec{p},\vec{q}}$ *is the augmented and marked dual graph of the minimal* $\mathbb{P}^2$*-dominating resolution of singularities of a primitive normal analytic compactification of* $\mathbb{C}^2$ *if and only if* $\omega_{l+1} > 0$.

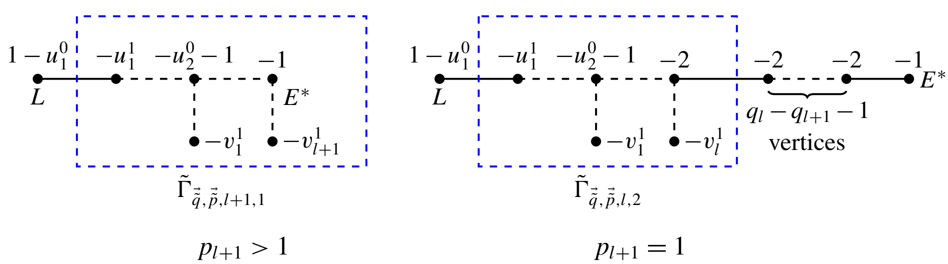

Figure 4. Augmented and marked dual graph for the minimal $\mathbb{P}^2$-dominating resolutions of singularities of primitive normal analytic compactifications of $\mathbb{C}^2$.

Remark 3.29. Let $\bar{X}$ be a primitive normal analytic compactification of $\mathbb{C}^2 = \operatorname{Spec}\mathbb{C}[x, y]$ and Γ be the augmented and marked dual graph for the minimal $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}$. Theorem 3.28 and identity (3-5) imply that Γ determines, and is determined by, the formal Puiseux pairs of the generic descending Puiseux series associated to the curve E^* at infinity on $\bar{X}$. Let δ be the semidegree on $\mathbb{C}[x, y]$ corresponding to E^*. Proposition 3.21 then implies that the δ-value of essential key forms of δ are also uniquely determined by Γ; we call these the *essential key values* of Γ.

4. Main results

Consider the setup of Question 1.2. Assume $N = 1$. Choose coordinates (x, y) on $\mathbb{P}^2 \setminus L$. Let δ be the semidegree on $\mathbb{C}[x, y]$ associated to E_1 (i.e., δ is the order of pole along E_1) and let $g_0, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ be the key forms of δ.

Theorem 4.1 (answering Question 1.2 in the case $N = 1$). *The following are equivalent*:

(1) *Y' is algebraic.*

(2) *g_j is a polynomial*, $0 \le j \le n + 1$.

(3) *g_{n+1} is a polynomial.*

If any of these conditions holds, then Y' is isomorphic to the closure of the image of $\mathbb{C}^2$ in the weighted projective variety $\mathbb{P}^{n+2}(1, \delta(g_0), \dots, \delta(g_{n+1}))$ under the mapping $(x, y) \mapsto [1 : g_0 : \cdots : g_{n+1}]$.

Remark 4.2. To ask Question 1.2 we need to determine if the given curve E' is analytically contractible. We would like to point out that in addition to the direct application of Grauert's criterion, the contractibility of E' can be determined in terms of the semidegrees associated to $E_1, \dots, E_N$ [Mondal 2016, Theorem 1.4]. In particular, in the $N = 1$ case, E' of Question 1.2 is analytically contractible if and only if $\delta(g_{n+1}) > 0$ (where δ and g_{n+1} are as above).

We now state the correspondence between primitive normal algebraic compactifications of $\mathbb{C}^2$ and algebraic curves in $\mathbb{C}^2$ with one place at infinity.

Theorem 4.3. *Let $\bar{X}$ be a primitive normal analytic compactification of $\mathbb{C}^2$. Let $P \in \bar{X} \setminus \mathbb{C}^2$ be a center of a $\mathbb{P}^2$-infinity on $\bar{X}$* (Remark-Definition 3.15). *Then the following are equivalent*:

(1) *$\bar{X}$ is algebraic.*

(2) *There is an algebraic curve C in $\mathbb{C}^2$ with* one place at infinity *such that P is not on the closure of C in $\bar{X}$.*

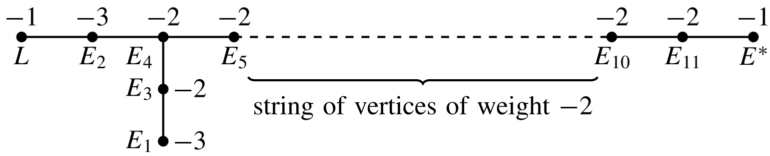

Figure 5. Augmented and marked dual graph of the minimal $\mathbb{P}^2$-dominating resolution of singularities of Y_i' from Section 2.

Let δ be the semidegree on $\mathbb{C}[x, y]$ corresponding to the curve at infinity on $\bar{X}$, and $g_0, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ be the sequence of key forms of δ. If either (1) or (2) is true, then g_{n+1} is a polynomial, and defines a curve C as in (2).

Now we come to the question of characterization of augmented and marked dual graphs of the resolution of singularities of primitive normal analytic compactifications of $\mathbb{C}^2$. For a primitive normal analytic compactification $\bar{X}$ of $\mathbb{C}^2$, let $\Gamma_{\bar{X}}$ be the *augmented and marked* dual graph (from Theorem 3.28) associated to the minimal $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}$. Let $\mathcal{G}$ be the collection of $\Gamma_{\bar{X}}$ as $\bar{X}$ varies over all primitive normal analytic compactifications of $\mathbb{C}^2$; note that assertions (2) and (3) of Theorem 3.28 give a complete description of $\mathcal{G}$. Pick $\Gamma \in \mathcal{G}$. Let $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$ be the formal Puiseux pairs, and $\omega_0, \dots, \omega_{l+1}$ be the sequence of *essential key values* of Γ (Remark 3.29). Fix k, $1 \le k \le l$. The *semigroup conditions* for k are:

$$p_k\omega_k \in \mathbb{Z}_{\ge 0}\langle \omega_0, \dots, \omega_{k-1}\rangle, \tag{S1-k}$$

$$(\omega_{k+1}, p_k\omega_k) \cap \mathbb{Z}\langle \omega_0, \dots, \omega_k\rangle = (\omega_{k+1}, p_k\omega_k) \cap \mathbb{Z}_{\ge 0}\langle \omega_0, \dots, \omega_k\rangle, \tag{S2-k}$$

where $(\omega_{k+1}, p_k\omega_k) := \{a \in \mathbb{R} : \omega_{k+1} < a < p_k\omega_k\}$ and $\mathbb{Z}_{\ge 0}\langle \omega_0, \dots, \omega_k\rangle$ (respectively, $\mathbb{Z}\langle \omega_0, \dots, \omega_k\rangle$) denotes the semigroup (respectively, group) generated by linear combinations of $\omega_0, \dots, \omega_k$ with nonnegative integer (respectively, integer) coefficients.

Theorem 4.4. (1) $\Gamma = \Gamma_{\bar{X}}$ *for some primitive normal* algebraic *compactification $\bar{X}$ of $\mathbb{C}^2$ if and only if the semigroup conditions* (S1-k) *hold for all k, $1 \le k \le l$.*

(2) $\Gamma = \Gamma_{\bar{X}}$ *for some primitive normal* nonalgebraic *compactification $\bar{X}$ of $\mathbb{C}^2$ if and only if either* (S1-k) *or* (S2-k) fails *for some k, $1 \le k \le l$.*

Remark-Example 4.5. Note that if (S1-k) holds for all k, $1 \le k \le l$, but (S2-k) *fails* for some k, $1 \le k \le l$, then Theorem 4.4 implies that there exist primitive normal analytic compactifications $\bar{X}_1$, $\bar{X}_2$ of $\mathbb{C}^2$ such that $\bar{X}_1$ is algebraic, $\bar{X}_2$ is *not* algebraic, and $\Gamma = \Gamma_{\bar{X}_1} = \Gamma_{\bar{X}_2}$. Indeed, that is precisely what happens in the setup of Section 2: let Γ be the augmented and marked dual graph corresponding to the minimal $\mathbb{P}^2$-dominating resolution of singularities of the Y_i' (Figure 5). It follows from (3-2) that the formal Puiseux pairs associated to Γ are $(2, 5)$, $(-6, 1)$;

in particular $l = 1$. Example 3.23 implies that the sequence of essential key values of Γ is $(5, 2, 2)$. It is straightforward to verify that (S1-k) is satisfied for $k = 1$. On the other hand,

$$3 \in (2, 10) \cap \mathbb{Z}\langle 5, 2\rangle \setminus \mathbb{Z}_{\geq 0}\langle 5, 2\rangle$$

so that (S2-k) is violated for $k = 1$. This implies that Γ corresponds to both algebraic and nonalgebraic normal compactifications of $\mathbb{C}^2$, as we have already seen in Section 2.

Remark-Example 4.6. We state some straightforward corollaries of Theorem 4.4 and of the fact — a special case of [Herzog 1970, Proposition 2.1] — that if p, q are relatively prime positive integers, then the greatest integer not belonging to $\mathbb{Z}_{\geq 0}\langle p, q\rangle$ is $pq - p - q$.

(1) Pick relatively prime positive integers p, q such that $p > q$. Then $\Gamma_{p,q}$ (defined as in (3-6)) corresponds to only algebraic compactifications of $\mathbb{C}^2$.

(2) Pick integers p, q, r such that p, q are relatively prime, $p > q > 0$ and $q > r$. Set $l := 1$, $(q_1, p_1) := (q, p)$, $(q_2, p_2) := (r, 1)$. Then (3-4) implies that $\omega_0 = p$, $\omega_1 = q$ and $\omega_2 = (p-1)q + r$. Assertion (3) of Theorem 3.28 therefore implies that $\Gamma_{\vec{p},\vec{q}}$ corresponds to a compactification of $\mathbb{C}^2$ if and only if $(p-1)q + r > 0$. So assume $q > r > -(p-1)q$.

   (a) If $r \geq -p$, then $\Gamma_{\vec{p},\vec{q}}$ corresponds to only algebraic compactifications of $\mathbb{C}^2$.

   (b) If $-p > r > -(p-1)q$, then $\Gamma_{\vec{p},\vec{q}}$ corresponds to both algebraic and nonalgebraic compactifications of $\mathbb{C}^2$.

(3) Let p_1, q_1, p_2 be integers such that $p_1 > q_1 > 1$, $p_2 \geq 2$, p_1 is relatively prime to q_1, and p_2 is relatively prime to $p_1 q_1 - p_1 - q_1$. Set

$$q_2 := p_1 q_1 - p_1 - q_1 - q_1(p_1 - 1)p_2, \quad q_3 := q_2 - 1, \quad p_3 := 1.$$

In this case $\omega_0 = p_1 p_2$, $\omega_1 = q_1 p_2$, $\omega_2 = p_1 q_1 - p_1 - q_1$ and $\omega_3 = p_2 \omega_2 - 1$. It follows that (S1-k) fails for $k = 2$ and therefore $\Gamma_{\vec{p},\vec{q}}$ corresponds to *only nonalgebraic* compactifications of $\mathbb{C}^2$.

Finally we formulate our answer to Question 1.3 in the case $N = 1$. Consider $O \in L_\infty := \mathbb{P}^2 \setminus \mathbb{C}^2$. Let (u, v) be coordinates near O, $\psi(u)$ be a Puiseux series in u, and r be a positive rational number. After a change of coordinates near O if necessary, we may assume that the coordinate of O is $(0, 0)$ with respect to the (u, v)-coordinates, and $(x, y) := (1/u, v/u)$ is a system of coordinates on $\mathbb{P}^2 \setminus L_\infty \cong \mathbb{C}^2$. Let

$$\phi(x) := x\psi(1/x).$$

Note that $\phi(x)$ is a descending Puiseux series in x. Let ξ be an indeterminate, and define, following Notation 3.5,

$$\tilde{\phi}(x,\xi) := [\phi(x)]_{>1-r} + \xi x^{1-r}.$$

Let δ be the semidegree on $\mathbb{C}[x, y]$ with generic descending Puiseux series $\tilde{\phi}$, and let $g_0, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ be the key forms of δ (see Algorithm 5.1 for the algorithm to determine key forms of δ from $\tilde{\phi}$).

Theorem 4.7 (answering Question 1.3 in the case $N = 1$). *The following are equivalent*:

(1) *There exists a polynomial $f \in \mathbb{C}[x, y]$ such that for each analytic branch C of the curve $f = 0$ at infinity*,
   - *C intersects L_∞ at O*,
   - *C has a Puiseux expansion $v = \theta(u)$ at O such that* $\operatorname{ord}_u(\theta - \psi) \geq r$.

(2) *g_j is a polynomial*, $0 \leq j \leq n+1$.

(3) *g_{n+1} is a polynomial.*

If any of these conditions holds, g_{n+1} *satisfies the properties of f from condition* (1).

5. Background II: notions required for the proof

In this section we collect more background material we use in the proof of the results stated in Section 4.

5A. ***Key forms from descending Puiseux series.*** Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Assume the generic descending Puiseux series for δ is

$$\tilde{\phi}_\delta(x,\xi) := \sum_{j=1}^{k'_0} a_{0j} x^{q_{0j}} + a_1 x^{q_1/p_1} + \cdots + a_2 x^{q_2/(p_1 p_2)} + \cdots \\ + a_l x^{q_l/(p_1 p_2 \cdots p_l)} + \xi x^{q_{l+1}/(p_1 p_2 \cdots p_{l+1})},$$

where $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$ are the *formal* Puiseux pairs of $\tilde{\phi}_\delta$ (Definition 3.12), $k'_0 \geq 0$, and $q_{01} > \cdots > q_{0k'_0}$ are integers greater than q_1/p_1. Let

$$g_0 = x, g_1 = y, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$$

be the key forms of δ. Recall from Proposition 3.21 that precisely $l+2$ of the key forms of δ are *essential*. Let $0 = j_0 < \cdots < j_{l+1} = n+1$ be the subsequence of $(0, \dots, n)$ consisting of indices of essential key forms of δ.

Algorithm 5.1 (construction of key forms from descending Puiseux series; cf. the algorithm in [Makar-Limanov 2015]).

1. *Base step.* Set $j_0 := 0,\ g_0 := x,\ g_1 := y$. Also define $p_0 := 1$. Now assume

(i) $g_0, \dots, g_s$ have been calculated, $s \geq 1$,

(ii) $j_0, \dots, j_k$ have been calculated, $k \geq 0$,

(iii) $j_k < s \leq j_{k+1}$.

2. *Inductive step for* (s, k). Let

$$\tilde{\omega}_s := \deg_x(g_s|_{y=\tilde{\phi}_\delta(x,\xi)}), \quad \tilde{c}_s := \text{coefficient of } x^{\tilde{\omega}_s} \text{ in } g_s|_{y=\tilde{\phi}_\delta(x,\xi)}.$$

Case 2.1: If $\tilde{c}_s \in \mathbb{C}[\xi] \setminus \mathbb{C}$, then set $n := s-1$, $j_{k+1} := s$, and stop the process.

Case 2.2: Otherwise if $\tilde{\omega}_s \in 1/(p_0 \cdots p_k)\mathbb{Z}$, then there are unique integers $\beta_0^s, \cdots, \beta_k^s$ and unique $c \in \mathbb{C}^*$ such that

(1) $0 \leq \beta_i^s < p_i$ for $1 \leq i \leq k$,

(2) $\sum_{i=0}^k \beta_i^s \tilde{\omega}_{j_i} = \tilde{\omega}_s$, and

(3) the coefficient of $x^{\tilde{\omega}_s}$ in $cg_{j_0}^{\beta_0^s} \cdots g_{j_k}^{\beta_k^s}|_{y=\tilde{\phi}_\delta(x,\xi)}$ is $\tilde{c}_s$.

Then set $g_{s+1} := g_s - cg_{j_0}^{\beta_0^s} \cdots g_{j_k}^{\beta_k^s}$, and repeat the inductive step for $(s+1, k)$.

Case 2.3: Otherwise

$$\tilde{\omega}_s \in \frac{1}{p_0 \cdots p_{k+1}}\mathbb{Z} \setminus \frac{1}{p_0 \cdots p_k}\mathbb{Z},$$

and there are unique integers $\beta_0^s, \cdots, \beta_k^s$ and unique $c \in \mathbb{C}^*$ such that

(1) $0 \leq \beta_i^s < p_i$ for $1 \leq i \leq k$,

(2) $\sum_{i=0}^k \beta_i^s \tilde{\omega}_{j_i} = p_{k+1}\tilde{\omega}_s$, and

(3) the coefficient of $x^{\tilde{\omega}_s}$ in $cg_{j_0}^{\beta_0^s} \cdots g_{j_k}^{\beta_k^s}|_{y=\tilde{\phi}_\delta(x,\xi)}$ is $(\tilde{c}_s)^{p_{k+1}}$.

Then set $j_{k+1} := s$, $g_{s+1} := g_s^{p_{k+1}} - cg_{j_0}^{\beta_0^s} \cdots g_{j_k}^{\beta_k^s}$, and repeat the inductive step for $(s+1, k+1)$.

Example 5.2. Let $\tilde{\phi}_\delta(x, \xi) := x^3 + x^2 + x^{5/3} + x + x^{-13/6} + x^{-7/3} + \xi x^{-8/3}$. The formal Puiseux pairs of $\tilde{\phi}_\delta$ are $(5, 3), (-13, 2), (-16, 1)$. We compute the key forms of δ following Algorithm 5.1: by definition we have $g_0 = x$, $g_1 = y$, $j_0 = 0$. Since the exponents of x in the first two terms of $\tilde{\psi}_\delta$ are integers, subsequent applications of Case 2.2 of Algorithm 5.1 implies that the next two key forms are $g_2 = y - x^3$ and $g_3 = y - x^3 - x^2$. Note that

$$g_3|_{y=\tilde{\psi}_\delta} = x^{5/3} + x + x^{-13/6} + x^{-7/3} + \xi x^{-8/3}, \tag{5-1}$$

In the notation of Algorithm 5.1, we have $\tilde{\omega}_3 = 5/3 \notin \mathbb{Z}$. It follows that $j_1 = 3$. Since

$$g_3^3|_{y=\tilde{\psi}_\delta} = x^5 + 3x^{13/3} + 3x^{11/3} + x^3 + 3x^{7/6} + 3x + 3\xi x^{2/3} + \text{l.d.t.}, \tag{5-2}$$

(where l.d.t. denotes terms with smaller degree in x), Case 2.3 of Algorithm 5.1 implies that $g_4 = g_3^3 - x^5$. Now note that $13/3 = 1 + 2 \cdot (5/3)$ and $11/3 = 2 + 5/3$, so that (5-1) and (5-2) imply that

$$\begin{aligned} g_4|_{y=\tilde{\psi}_\delta} &= 3x\big(g_3|_{y=\tilde{\psi}_\delta} - x - x^{-13/6} - x^{-7/3} - \xi x^{-8/3}\big)^2 \\ &\qquad + 3x^2\big(g_3|_{y=\tilde{\psi}_\delta} - x - x^{-13/6} - x^{-7/3} - \xi x^{-8/3}\big) \\ &\qquad\qquad + x^3 + 3x^{7/6} + 3x + 3\xi x^{2/3} + \text{l.d.t.} \\ &= 3xg_3^2|_{y=\tilde{\psi}_\delta} - 3x^2 g_3|_{y=\tilde{\psi}_\delta} + x^3 + 3x^{7/6} + 3x + 3\xi x^{2/3} + \text{l.d.t.} \end{aligned}$$

Repeated applications of Case 2.2 of Algorithm 5.1 then imply that

$$g_5 = g_4 - 3xg_3^2, \quad g_6 = g_4 - 3xg_3^2 - 3x^2 g_3, \quad g_7 = g_4 - 3xg_3^2 - 3x^2 g_3 - x^3.$$

Note that

$$g_7|_{y=\tilde{\psi}_\delta} = 3x^{7/6} + 3x + 3\xi x^{2/3} + \text{l.d.t.} \tag{5-3}$$

Since $\tilde{\omega}_7 = \frac{7}{6} \notin \frac{1}{3}\mathbb{Z}$, following Case 2.3 of Algorithm 5.1 we have $j_2 = 7$. Since $p_2 = 2$, we compute

$$g_7^2|_{y=\tilde{\psi}_\delta} = 9x^{7/3} + 18x^{13/6} + 18\xi x^{11/6} + \text{l.d.t.},$$

Since $\frac{7}{3} = -1 + 2 \cdot \frac{5}{3} + 0 \cdot \frac{7}{6}$ and $\frac{13}{6} = 1 + 0 \cdot \frac{5}{3} + \frac{7}{6}$, identities (5-1) and (5-3) imply that

$$\begin{aligned} g_7^2|_{y=\tilde{\psi}_\delta} &= 9x^{-1}\big(g_3|_{y=\tilde{\psi}_\delta} - x - x^{-13/6} - x^{-5/2} - \xi x^{-8/3}\big)^2 \\ &\qquad + 6x\big(g_7|_{y=\tilde{\psi}_\delta} - 3x - 3\xi x^{2/3} - \text{l.d.t.}\big) + 18\xi x^{11/6} + \text{l.d.t.} \\ &= 9x^{-1} g_3^2|_{y=\tilde{\psi}_\delta} + 6xg_7|_{y=\tilde{\psi}_\delta} - 18x^2 + 18\xi x^{11/6} + \text{l.d.t.} \end{aligned}$$

Cases 2.3 and 2.2 of Algorithm 5.1 then imply that the next key forms are

$$g_8 = g_7^2 - 9x^{-1}g_3^2, \quad g_9 = g_7^2 - 9x^{-1}g_3^2 - 6xg_7, \quad g_{10} = g_7^2 - 9x^{-1}g_3^2 - 6xg_7 + 18x^2.$$

Since

$$g_{10}|_{y=\tilde{\psi}_\delta} = 18\xi x^{11/6} + \text{l.d.t.},$$

Case 2.1 of Algorithm 5.1 implies that g_{10} is the last key form of δ, and $n = 9$, $j_3 = 10$. In particular, note that there are precisely 4 essential key forms (namely g_0, g_3, g_7, g_{10}) of δ, as predicted by Proposition 3.21.

The assertions of the following proposition are straightforward implications of Algorithm 5.1.

Proposition 5.3. *Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Let $g_0, \dots, g_{n+1}$ be key forms and $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ be the generic descending Puiseux series of δ,*

(1) *Let $n_* \le n$ and let δ_* be the unique semidegree such that the key forms of δ_* are $g_0, \dots, g_{n_*+1}$ and $\delta^*(g_j) = \delta(g_j)/e$, $0 \le j \le n_* + 1$, where $e := \gcd\big(\delta(g_0), \dots, \delta(g_{n_*+1})\big)$. Then δ_* has a generic descending Puiseux series of the form*

$$\tilde{\phi}_{\delta_*}(x, \xi) = \phi_*(x) + \xi x^{r_*},$$

where

(a) $r_* \ge r_\delta$, *and*

(b) $\phi_*(x) = [\phi_\delta(x)]_{>r_*}$.

(2) *Let α_i, $1 \le i \le n$, be the smallest positive integer such that $\alpha_i \delta(g_i)$ is in the (abelian) group generated by $\delta(g_0), \dots, \delta(g_{i-1})$. Fix m, $0 \le m \le n$. Recall that each g_i is an element in $\mathbb{C}[x, x^{-1}, y]$ which is monic in y. The following are equivalent*:

(a) *g_i is a polynomial*, $0 \le i \le m + 1$.

(b) *For each i, $1 \le i \le m$, the semigroup generated by $\delta(g_0), \dots, \delta(g_{i-1})$ contains $\alpha_i \delta(g_i)$.*

5B. *Degree-like functions and compactifications.* In this subsection we recall from [Mondal 2014b] the basic facts of compactifications of affine varieties via *degree-like functions*. Recall that $X = \mathbb{C}^2$ in our notation; however the results in this subsection remain valid if X is an arbitrary affine variety.

Definition 5.4. A map $\delta : \mathbb{C}[X] \setminus \{0\} \to \mathbb{Z}$ is called a *degree-like function* if

(1) $\delta(f + g) \le \max\{\delta(f), \delta(g)\}$ for all $f, g \in \mathbb{C}[X]$, with $<$ in the preceding inequality implying $\delta(f) = \delta(g)$.

(2) $\delta(fg) \le \delta(f) + \delta(g)$ for all $f, g \in \mathbb{C}[X]$.

Every degree-like function δ on $\mathbb{C}[X]$ defines an *ascending filtration* $\{\mathcal{F}_d^\delta\}_{d\ge 0}$ on $\mathbb{C}[X]$, where $\mathcal{F}_d^\delta := \{f \in \mathbb{C}[X] : \delta(f) \le d\}$. Define

$$\mathbb{C}[X]^\delta := \bigoplus_{d\ge 0} \mathcal{F}_d^\delta, \quad \operatorname{gr} \mathbb{C}[X]^\delta := \bigoplus_{d\ge 0} \mathcal{F}_d^\delta / \mathcal{F}_{d-1}^\delta.$$

Remark 5.5. For every $f \in \mathbb{C}[X]$, there are infinitely many "copies" of f in $\mathbb{C}[X]^\delta$, namely the copy of f in $\mathcal{F}_d^\delta$ for *each* $d \ge \delta(f)$; we denote the copy of f in $\mathcal{F}_d^\delta$ by $(f)_d$. If t is a new indeterminate, then

$$\mathbb{C}[X]^\delta \cong \sum_{d\ge 0} \mathcal{F}_d^\delta t^d,$$

via the isomorphism $(f)_d \mapsto f t^d$. Note that t corresponds to $(1)_1$ under this isomorphism.

We say that δ is *finitely generated* if $\mathbb{C}[X]^\delta$ is a finitely generated algebra over $\mathbb{C}$ and that δ is *projective* if in addition $\mathcal{F}_0^\delta = \mathbb{C}$. The motivation for the terminology comes from the following straightforward result.

Proposition 5.6 [Mondal 2014b, Proposition 2.8]. *If δ is a projective degree-like function on $\mathbb{C}[X]$, then $\bar{X}^\delta := \operatorname{Proj}\mathbb{C}[X]^\delta$ is a projective compactification of X. The* hypersurface at infinity $\bar{X}^\delta_\infty := \bar{X}^\delta \setminus X$ *is the zero set of the $\mathbb{Q}$-Cartier divisor defined by* $(1)_1$ *and is isomorphic to* $\operatorname{Proj}\operatorname{gr}\mathbb{C}[X]^\delta$. *Conversely, if $\bar{X}$ is any projective compactification of X such that $\bar{X} \setminus X$ is the support of an effective ample divisor, then there is a projective degree-like function δ on $\mathbb{C}[X]$ such that $\bar{X}^\delta \cong \bar{X}$.*

Remark 5.7. A *semidegree*, which we already defined in Section 3B, is a degree-like function which always satisfies property (2) of Definition 5.4 with an equality.

The following proposition (which is straightforward to prove) is a special case of [Mondal 2014b, Theorem 4.1].

Proposition 5.8. *Let δ be a projective degree-like function on $\mathbb{C}[X]$, and $\bar{X}^\delta$ be the corresponding projective compactification from Proposition 5.6. Assume δ is a semidegree. Then $\bar{X}^\delta$ is a normal variety and $\bar{X}^\delta_\infty := \bar{X}^\delta \setminus X$ is an irreducible codimension-one subvariety. Moreover, there is a positive integer d_δ such that δ agrees with d_δ times the order of* pole *along $\bar{X}^\delta_\infty$.*

6. Some preparatory results

In this section we develop some preliminary results to be used in Section 7 for the proofs of our main results.

Convention 6.1. Let $y_1, \dots, y_k$ be indeterminates. From now on we write A_k, $\tilde{A}_k$, R, $\tilde{R}$ to denote respectively $\mathbb{C}[x, x^{-1}, y_1, \dots, y_k]$, $\mathbb{C}\langle\langle x\rangle\rangle[y_1, \dots, y_k]$, $\mathbb{C}[x, x^{-1}, y]$, $\mathbb{C}\langle\langle x\rangle\rangle[y]$. Below we frequently deal with maps $A_k \to R$. We always (unless there is a misprint!) use upper-case letters $F, G, \dots$ for elements in A_k and corresponding lower-case letters $f, g, \dots$ for their images in R.

6A. *The "star action" on descending Puiseux series.*

Definition 6.2. Let $\phi = \sum_j a_j x^{q_j/p} \in \mathbb{C}\langle\langle x\rangle\rangle$ be a descending Puiseux series with polydromy order p and r be a multiple of p. Then for all $c \in \mathbb{C}$ we define

$$c \star_r \phi := \sum_j a_j c^{q_j r/p} x^{q_j/p}.$$

For $\Phi = \sum_{\alpha\in\mathbb{Z}^k_{\geq 0}} \phi_\alpha(x) y_1^{\alpha_1} \cdots y_k^{\alpha_k} \in \tilde{A}_k$, the *polydromy order* of Φ is the lowest common multiple of the polydromy orders of all the nonzero ϕ_α. Let r be a multiple of the polydromy order Φ. Then we define

$$c \star_r \Phi := \sum_\alpha (c \star_r \phi_\alpha) y_1^{\alpha_1} \cdots y_k^{\alpha_k}.$$

Remark 6.3. It is straightforward to see that in the case that c is an r-th root of unity (and r is a multiple of the polydromy order of ϕ), $c \star_r \phi$ is a *conjugate* of ϕ (cf. Remark-Notation 6.5).

The following properties of the $\star_r$ operator are straightforward to see:

Lemma 6.4. (1) *Let p be the polydromy order of $\Phi \in \tilde{A}_k$, d and e be positive integers, and $c \in \mathbb{C}$. Then $c \star_{pde} \Phi = c^e \star_{pd} \Phi = c^{de} \star_p \Phi$.*

(2) *Let $\Phi_j = \sum_j \phi_{j,\alpha}(x) y_1^{\alpha_1} \cdots y_k^{\alpha_k} \in \tilde{A}_k$ for $j = 1, 2$, and r be a multiple of the polydromy order of each nonzero $\phi_{j,\alpha}$. Then*

$$c \star_r (\Phi_1 + \Phi_2) = (c \star_r \Phi_1) + (c \star_r \Phi_2),$$
$$c \star_r (\Phi_1 \Phi_2) = (c \star_r \Phi_1)(c \star_r \Phi_2).$$

(3) *Let $\pi : \tilde{A}_k \to \tilde{R}$ be a $\mathbb{C}$-algebra homomorphism defined by $x \mapsto x$ and*

$$y_j \mapsto f_j \in R \quad \textit{for } 1 \le j \le k.$$

Let $\Phi = \sum_\alpha \phi_\alpha(x) y_1^{\alpha_1} \cdots y_k^{\alpha_k} \in \tilde{A}_k$, let r be a multiple of the polydromy order of each nonzero ϕ_α, and μ be a (not necessarily primitive) r-th root of unity. Then $\pi(\mu \star_r \Phi) = \mu \star_r \pi(\Phi)$. □

Remark-Notation 6.5. If ϕ is a descending Puiseux series in x with polydromy order p, then we write

$$f_\phi := \prod_{\substack{\phi_j \text{ is a} \\ \text{conjugate} \\ \text{of } \phi}} \bigl(y - \phi_j(x)\bigr) = \prod_{j=0}^{p-1} \bigl(y - \zeta^j \star_p \phi(x)\bigr) \in \tilde{R}, \tag{6-1}$$

where ζ is a primitive p-th root of unity. If $f \in \mathbb{C}[x, y]$, then its descending Puiseux factorization (Theorem 3.6) can be described as follows: There are unique (up to conjugacy) descending Puiseux series $\phi_1, \ldots, \phi_k$, a unique nonnegative integer m, and $c \in \mathbb{C}^*$ such that

$$f = c x^m \prod_{i=1}^{k} f_{\phi_i}.$$

Let $(q_1, p_1), \ldots, (q_l, p_l)$ be Puiseux pairs of ϕ. Set $p_0 := 1$. For each k, $0 \le k \le l$, we write

$$f_\phi^{(k)} := \prod_{j=0}^{p_0 p_1 \cdots p_k - 1} (y - \zeta^j \star_p \phi(x)), \tag{6-2}$$

where ζ is a primitive $(p_1 \cdots p_l)$-th root of unity. Note that $f_\phi^{(l)} = f_\phi$, and for each $m, n, 0 \le m < n \le l$,

$$\begin{aligned}
f_\phi^{(n)} &= \prod_{j=0}^{p_0 p_1 \cdots p_n - 1} \bigl(y - \zeta^j \star_p \phi(x)\bigr) \\
&= \prod_{i=0}^{p_{m+1} \cdots p_n - 1} \prod_{j=0}^{p_0 p_1 \cdots p_m - 1} \bigl(y - \zeta^{i p_0 p_1 \cdots p_m + j} \star_p \phi(x)\bigr) \\
&= \prod_{i=0}^{p_{m+1} \cdots p_n - 1} \zeta^{i p_0 p_1 \cdots p_m} \star_p \left(\prod_{j=0}^{p_0 p_1 \cdots p_m - 1} \bigl(y - \zeta^j \star_p \phi(x)\bigr) \right) \\
&= \prod_{i=0}^{p_{m+1} \cdots p_n - 1} \zeta^{i p_0 p_1 \cdots p_m} \star_p (f_\phi^{(m)}).
\end{aligned} \tag{6-3}$$

6B. *"Canonical" preimages of polynomials and their comparison.*

Lemma 6.6 ("canonical" preimages of elements in $\mathbb{C}\langle\langle x \rangle\rangle[y]$). *Let $p_0 := 1$, and $p_1, \ldots, p_{k-1}$ be positive integers, and $\pi : A_k \to R$ be a ring homomorphism which sends $x \mapsto x$ and $y_j \mapsto f_j$, where f_j is monic in y of degree $p_0 \cdots p_{j-1}$, $1 \le j \le k$. Then π induces a homomorphism $\tilde{A}_k \to \tilde{R}$ which we also denote by π. If f is a nonzero element in $\tilde{R}$, then there is a unique $F_f^\pi \in \tilde{A}_k$ such that*

(1) *$\pi(F_f^\pi) = f$, and*

(2) *$\deg_{y_j}(F_f^\pi) < p_j$ for all j, $1 \le j \le k-1$.*

Moreover, if f is monic in y of degree $p_1 \cdots p_{k-1} d$ for some integer d, then

(3) *F_f^π is monic in y_k of degree d,*

(4) *if the coefficient of $x^\alpha y_1^{\beta_1} \cdots y_k^{\beta_k}$ in $F_f^\pi - y_k^d$ is nonzero, then*

$$\sum_{i=1}^{j} p_0 \cdots p_{i-1} \beta_i < p_1 \cdots p_j \quad \text{for all } j, \ 1 \le j \le k-1,$$

and

$$\sum_{i=1}^{k} p_0 \cdots p_{i-1} \beta_i < p_1 \cdots p_{k-1} d.$$

Finally,

(5) *if each of $f, f_1, \ldots, f_k$ is in $\mathbb{C}[x, y]$ (resp. R), then F_f^π is in $\mathbb{C}[x, y_1, \ldots, y_k]$ (resp. A_k).*

Proof. This results from an application of Theorem 2.13 of [Abhyankar 1977]. □

Now assume δ is a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Assume the generic descending Puiseux series for δ is

$$\begin{aligned}\tilde{\phi}_\delta(x, \xi) &:= \phi_\delta(x) + \xi x^{r_\delta}\\ &= \cdots + a_1 x^{q_1/p_1} + \cdots + a_2 x^{q_2/(p_1 p_2)} + \cdots + a_l x^{q_l/(p_1 \cdots p_l)} + \xi x^{q_{l+1}/(p_1 \cdots p_{l+1})},\end{aligned}$$

where $(q_1, p_1), \ldots, (q_{l+1}, p_{l+1})$ are the formal Puiseux pairs of $\tilde{\phi}_\delta$. Let $g_0 = x$, $g_1 = y, \ldots, g_{n+1} \in R$ be the sequence of key forms of δ and $g_{j_0}, \ldots, g_{j_{l+1}}$ be the subsequence of *essential* key forms. For $0 \le k \le l+1$, define

$$f_k := g_{j_k}, \qquad \omega_k := \delta(f_k). \tag{6-4}$$

Lemma 6.7. *f_1 has the form $y -$ a polynomial in x. If $1 \le k \le l$, one can write*

$$f_{k+1} = f_k^{p_k} - \sum_{i=0}^{m_k} c_{k,i} f_0^{\beta^i_{k,0}} \cdots f_k^{\beta^i_{k,k}}$$

where

(1) $m_k \ge 0$,

(2) $c_{k,i} \in \mathbb{C}^*$ *for all* i, $0 \le i \le m_k$,

(3) *the* $\beta^i_{k,j}$ *are integers such that* $0 \le \beta^i_{k,j} < p_j$ *for* $1 \le j \le k$ *and* $0 \le i \le m_k$,

(4) $\beta^0_{k,k} = 0$,

(5) $p_k \omega_k = \sum_{j=0}^{k-1} \beta^0_{k,j}\omega_j > \sum_{j=0}^{k} \beta^1_{k,j}\omega_j > \cdots > \sum_{j=0}^{k} \beta^{m_k}_{k,j}\omega_j > \omega_{k+1}$.

Proof. Combine property (P2) of key forms, assertion (3) of Proposition 3.21, and the defining property of essential key forms (Definition 3.20). ☐

Let $\pi : A_{l+1} \to R$ be the $\mathbb{C}$-algebra homomorphism which maps $x \mapsto x$ and $y_k \to f_k$, $1 \le k \le l+1$, and let $\pi_k := \pi|_{A_k} : A_k \to R$, $1 \le k \le l+1$. Let ω be the weighted degree on A_{l+1} corresponding to weights ω_0 for x and ω_k for y_k, $1 \le k \le l+1$. We will often abuse the notation and write π and ω respectively for π_k and $\omega|_{A_k}$ for each k, $1 \le k \le l+1$. Define

$$F_{k+1} := \begin{cases} y_1 & \text{if } k = 0,\\ y_k^{p_k} - \sum_{i=0}^{m_k} c_{k,i} x^{\beta^i_{k,0}} y_1^{\beta^i_{k,1}} \cdots y_k^{\beta^i_{k,k}} & \text{if } 1 \le k \le l, \end{cases} \tag{6-5}$$

where the $c_{k,i}$ and $\beta^i_{k,j}$ are as in Lemma 6.7. Note that $F_1 \in A_1$ and $F_k \in A_{k-1}$ for $2 \le k \le l+1$. Moreover, $\pi(F_k) = f_k$ for each k, $1 \le k \le l+1$.

Lemma 6.8. *Fix* k, $1 \le k \le l+1$.

(1) *Let H_1, H_2 be two monomials in A_k such that* $\deg_{y_j}(H) < p_j$ *for all* j, $1 \le j \le k$. *Then* $\omega(H_1) \ne \omega(H_2)$.

(2) *Suppose* $H \in A_k$ *is such that* $\deg_{y_j}(H) < p_j$ *for all* j, $1 \le j \le k$. *Then* $\delta(\pi(H)) = \omega(H)$.

Proof. Assertion (3) of Proposition 3.21 implies that for each j, $1 \le j \le k$, p_j is the smallest positive integer such that $p_j\omega_j$ is in the group generated by $\omega_0, \dots, \omega_{j-1}$. This immediately implies assertion (1). For assertion (2), write $H = \sum_{i\ge 1} H_i$, where the H_i are monomials in A_k. By assertion (1) we may assume w.l.o.g. that $\omega(H) = \omega(H_1) > \omega(H_2) > \cdots$. Since $\delta(\pi(y_j)) = \delta(f_j) = \omega_j = \omega(y_j)$ for each j, $1 \le j \le k$, it follows that $\delta(\pi(H_i)) = \omega(H_i)$ for all i. It then follows from the definition of degree-like functions (Definition 5.4) that $\delta(\pi(H)) = \omega(H_1) = \omega(H)$. □

Lemma 6.9. *For each k, $1 \le k \le l+1$, define*

$$r_k := \frac{q_k}{p_1 p_2 \cdots p_k}, \tag{6-6}$$

$$\phi_k := [\phi_\delta]_{>r_k}. \tag{6-7}$$

Define $f_{\phi_k} \in \tilde{R}$ as in (6-1). *Also define*

$$F_{\phi_k} := \begin{cases} F^{\pi_1}_{f_{\phi_1}} \in \tilde{A}_1 & \text{for } k = 1, \\ F^{\pi_{k-1}}_{f_{\phi_k}} \in \tilde{A}_{k-1} & \text{for } 2 \le k \le l+1. \end{cases}$$

Then:

(a) $\delta(f_{\phi_k}) = \omega_k$.

(b) $F_1 = F_{\phi_1} = y_1$.

(c) *For $k \ge 1$, F_{k+1} is precisely the sum of all monomial terms T (in $x, y_1, \dots, y_k$) of $F_{\phi_{k+1}}$ such that $\omega(T) > \omega_{k+1}$.*

Proof. We compute $\delta(f_{\phi_k})$ using (3-1). Let $\tilde{p}_k := p_1 \cdots p_{k-1}$. It is straightforward to see that ϕ_k has precisely $\tilde{p}_k$ conjugates $\phi_{k,1}, \dots, \phi_{k,\tilde{p}_k}$, and $\deg_x(\tilde{\phi}_\delta(x,\xi) - \phi_{k,j}(x))$ equals r_1 for $(p_1-1)p_2\cdots p_{k-1}$ of the $\phi_{k,j}$, equals r_2 for $(p_2-1)p_3\cdots p_{k-1}$ of the $\phi_{k,j}$, and so on. Identity (3-1) then implies that

$$\begin{aligned} \delta(f_{\phi_k}) &= \delta(x) \sum_{j=1}^{\tilde{p}_k} \deg_x(\tilde{\phi}_\delta(x,\xi) - \phi_{k,j}(x)) \\ &= p_1 \cdots p_{l+1}\biggl((p_1-1)p_2\cdots p_{k-1}\frac{q_1}{p_1} + (p_2-1)p_3\cdots p_{k-1}\frac{q_2}{p_1p_2} + \cdots \\ &\qquad\qquad + (p_{k-1}-1)\frac{q_{k-1}}{p_1\cdots p_{k-1}} + \frac{q_k}{p_1\cdots p_k}\biggr). \end{aligned}$$

A straightforward induction on k then yields that

$$\delta(f_{\phi_k}) = p_{k-1}\delta(f_{\phi_{k-1}}) + (q_k - q_{k-1}p_k)p_{k+1}\cdots p_{l+1}.$$

Identity (3-4) then implies that $\delta(f_{\phi_k}) = \omega_k$, which proves assertion (a). Assertion (b) follows immediately from the definitions. We now prove assertion (c). Fix k, $1 \le k \le l$. Let $\tilde{F}$ be the sum of all monomial terms T (in $x, y_1, \dots, y_k$) of $F_{\phi_{k+1}}$

such that $\omega(T) > \omega_{k+1}$, i.e., $F_{\phi_{k+1}} = \tilde{F} + \tilde{G}$ for some $\tilde{G} \in \tilde{A}_k$ with $\omega(\tilde{G}) \le \omega_{k+1}$. It follows that

$$\begin{aligned}\delta(\pi(\tilde{F})) = \delta(\pi(F_{\phi_{k+1}}) - \pi(\tilde{G})) &\le \max\{\delta(\pi(F_{\phi_{k+1}})), \delta(\pi(\tilde{G}))\} \\ &\le \max\{\delta(f_{\phi_{k+1}}), \omega(\tilde{G})\} \le \omega_{k+1}.\end{aligned}$$

On the other hand, $\delta(\pi(F_{k+1})) = \delta(f_{k+1}) = \omega_{k+1}$. It follows that

$$\delta(\pi(\tilde{F}-F_{k+1})) = \delta(\pi(\tilde{F})-\pi(F_{k+1})) \le \max\{\delta(\pi(\tilde{F})), \delta(\pi(F_{k+1}))\} = \omega_{k+1}. \tag{6-8}$$

Now, (6-5) and the defining properties of F_{ϕ_k} in Lemma 6.6 imply that $H := \tilde{F} - F_{k+1}$ satisfies the hypothesis of assertion (2) of Lemma 6.8, so that $\delta(\pi(\tilde{F} - F_{k+1})) = \omega(\tilde{F} - F_{k+1})$. Inequality (6-8) then implies that

$$\omega(\tilde{F} - F_{k+1}) \le \omega_{k+1}. \tag{6-9}$$

But the construction of $\tilde{F}$ and assertion (5) of Lemma 6.7 imply that all the monomials that appear in $\tilde{F}$ or F_{k+1} have ω-value *greater than* ω_{k+1}. Therefore (6-9) implies that $\tilde{F} = F_{k+1}$, as required to complete the proof. □

The proof of the next lemma is long, and we put it in Appendix A.

Lemma 6.10. *Fix k, $0 \le k \le l$. Pick $\psi \in \mathbb{C}\langle\langle x \rangle\rangle$ such that $\psi \equiv_{r_{k+1}} \phi_\delta$; in particular, the first k Puiseux pairs of ψ are $(q_1, p_1), \dots, (q_k, p_k)$. As in* (6-2), *define*

$$f_\psi^{(k)} := \prod_{j=0}^{p_0 p_1 \cdots p_k - 1} \big(y - \zeta^j \star_q \psi(x)\big),$$

where q is the polydromy order of ψ and ζ is a primitive q-th root of unity. Define

$$F_\psi^{(k)} := \begin{cases} F^{\pi_1}_{f_\psi^{(0)}} \in \tilde{A}_1 & \textit{for } k = 0, \\ F^{\pi_k}_{f_\psi^{(k)}} \in \tilde{A}_k & \textit{for } 1 \le k \le l. \end{cases} \tag{6-10}$$

Then

$$\omega(F_\psi^{(k)} - F_{k+1}) \le \omega_{k+1}.$$

6C. ***Implications of polynomial key forms.*** We continue with the notation of Section 6B. Let $\xi_1, \dots, \xi_{l+1}$ be new indeterminates, and for each k, $1 \le k \le l+1$, let δ_k be the semidegree on $\mathbb{C}[x, y]$ corresponding to the generic degreewise Puiseux series

$$\tilde{\phi}_k(x, \xi_k) := \phi_k(x) + \xi_k x^{r_k},$$

i.e., $\delta_k(x) = p_1 \cdots p_k$ and for each $f \in \mathbb{C}[x, y] \setminus \{0\}$,

$$\delta_k(f(x, y)) = \delta_k(x) \deg_x\big(f(x, \tilde{\phi}_k(x, \xi_k))\big). \tag{6-11}$$

The following lemma follows from a straightforward examination of Algorithm 5.1.

Lemma 6.11. *For each k, $1 \le k \le l+1$, the following hold*:

(1) *The key forms of δ_k are $g_0, g_1, \dots, g_{j_k}$.*

(2) *The essential key forms of δ_k are $f_0, \dots, f_k$.*

(3) $\delta_k(g_j) = \frac{\delta(g_j)}{p_{k+1}\cdots p_{l+1}}$, $0 \le j \le j_k$. □

Fix k, $1 \le k \le l+1$. In this subsection we assume condition (Polynomial$_k$) below is satisfied, and examine some of its implications.

$$\text{All the key forms of } \delta_k \text{ are polynomials.} \tag{Polynomial$_k$}$$

Lemma 6.12. *Assume* (Polynomial$_k$) *holds. Then $\delta(g_j) \ge 0$ for $0 \le j \le j_k - 1$.*

Proof. This follows from combining assertion (2) of Proposition 5.3 with assertion (3) of Lemma 6.11. □

Let $C_k := \mathbb{C}[x, y_1, \dots, y_k] \subseteq A_k$. Since the g_j are polynomial for $0 \le j \le j_k$, Algorithm 5.1 implies that the F_j (defined in (6-5)) are also polynomial for $0 \le j \le k$; in particular, $F_1 \in C_1$ and $F_{j+1} \in C_j$, $1 \le j \le k-1$. For $1 \le j \le k-1$, let H_{j+1} be the *leading form* of F_{j+1} with respect to ω, i.e.,

$$H_{j+1} := y_j^{p_j} - c_{j,0} x^{\beta^0_{j,0}} y_1^{\beta^0_{j,1}} \cdots y_{j-1}^{\beta^0_{j,j-1}}, \quad 1 \le j \le k-1. \tag{6-12}$$

Let $\prec$ be the reverse lexicographic order on C_k, i.e., $x^{\beta_0} y_1^{\beta_1} \cdots y_k^{\beta_k} \prec x^{\beta'_0} y_1^{\beta'_1} \cdots y_k^{\beta'_k}$ if and only if the right-most nonzero entry of $(\beta_0 - \beta'_0, \dots, \beta_k - \beta'_k)$ is negative.

The following lemma is the main result of this subsection. Its proof is long, and we put it in Appendix B.

Lemma 6.13. *Assume* (Polynomial$_{l+1}$) *holds. Then*

(1) (Recall the notation of Section 5B.) *Define*

$$S^\delta := \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^\delta_d \supseteq \mathbb{C}[x, y]^\delta.$$

Then S^δ is generated as a $\mathbb{C}$-algebra by $(1)_1, (x)_{\omega_0}, (y_1)_{\omega_1}, \dots, (y_{l+1})_{\omega_{l+1}}$.

(2) *Let J_{l+1} be the ideal in C_{l+1} generated by the* leading weighted homogeneous forms (*with respect to ω*) *of polynomials $F \in C_{l+1}$ such that $\delta(\pi(F)) < \omega(F)$. Then $\mathcal{B}_{l+1} := (H_{l+1}, \dots, H_2)$ is a Gröbner basis of J_{l+1} with respect to $\prec$.*

7. Proof of the main results

In this section we give proofs of Theorems 4.1, 4.3, 4.4 and 4.7.

Proof of Theorem 4.7. The implication (2) ⇒ (3) is obvious. We prepare the ground for the rest with an easily seen reformulation:

Lemma 7.1. *Assertion* (1) *of Theorem 4.7 is equivalent to the following assertion*:

(1′) *There exists a polynomial* $f \in \mathbb{C}[x, y]$ *such that for each analytic branch* C *of the curve* $f = 0$ *at infinity*,

- C *intersects* L_∞ *at* O, *and*
- C *has a* ***descending*** *Puiseux expansion* $y = \theta(x)$ *at* O *such that* $\deg_x(\theta - \phi)$ *is at most* $1 - r$. □

Assertion (4) of Theorem 3.17 implies that if g_{n+1} is a polynomial, then g_{n+1} satisfies the properties of f from (1′); in particular (3) ⇒ (1′). To finish the proof of Theorem 4.7 it remains to prove that (1′) ⇒ (2). So assume (1′) holds. We proceed by contradiction, i.e., we also assume that there exists m, $1 \le m \le n$, such that g_{m+1} is not a polynomial, and show that this leads to a contradiction. By assertion (1) of Proposition 5.3, we may (and will) assume that $m = n$.

We adopt the notation of Sections 6B and 6C. In particular, we write $\tilde{\phi}_\delta(x, \xi)$ and $\phi_\delta(x)$, r_δ for $\tilde{\phi}(x, \xi)$ and $[\phi(x)]_{>1-r}$, $1 - r$, respectively, and we denote by $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$ the formal Puiseux pairs of $\tilde{\phi}_\delta$. We also denote by $g_{j_0}, \dots, g_{j_{l+1}}$ the sequence of essential key forms, and set $f_k := g_{j_k}$, $0 \le k \le l+1$.

Let $f \in \mathbb{C}[x, y]$ be as in (1′). By assumption f has a descending Puiseux factorization of the form

$$f = a \prod_{m=1}^{M} f_{\psi_m} \tag{7-1}$$

for some $a \in \mathbb{C}^*$ and $\psi_1, \dots, \psi_m \in \mathbb{C}\langle\langle x \rangle\rangle$ such that

$$\psi_m \equiv_{r_\delta} \phi_\delta, \quad \text{for each } m,\ 1 \le m \le M, \tag{7-2}$$

where the f_{ψ_m} are defined as in (6-1). Without loss of generality we may (and will) assume that $a = 1$.

At first we claim that $l \ge 1$. Indeed, assume to the contrary that $l = 0$. Then

$$\tilde{\phi}_\delta(x, \xi) = h(x) + \xi x^{r_\delta}$$

for some $h \in \mathbb{C}[x, x^{-1}]$. Since g_{n+1} is not a polynomial, assertion (2) of Proposition 5.3 implies that $h(x) = h_1(x) + h_2(x)$, where $h_1 \in \mathbb{C}[x]$, $h_2 \in \mathbb{C}[x^{-1}] \setminus \mathbb{C}$, and $0 > \deg_x(h_2(x)) > r_\delta$. Let $e := -\deg_x(h_2(x)) > 0$ and $y' := y - h_1(x)$. Then (7-1) implies that f is a product of elements in $\mathbb{C}\langle\langle x \rangle\rangle[y']$ of the form $y' - \psi_{m,i}(x)$ for $\psi_{m,i} \in \mathbb{C}\langle\langle x \rangle\rangle$ such that each $\psi_{m,i}(x) = h_2(x) + \text{l.d.t.}$, where l.d.t. denotes terms with degree smaller than $\mathrm{ord}_x(h_2) < -e$. It is then straightforward to see that $f \notin \mathbb{C}[x, y'] = \mathbb{C}[x, y]$, which contradicts our choice of f. It follows that $l \ge 1$, as claimed.

Let F_k, $1 \le k \le l+1$, be as in (6-5). Fix m, $1 \le m \le M$. Then (7-2) and Lemma 6.10 imply that

$$F^{(l)}_{\psi_m} = F_{l+1} + \tilde{F}_m, \tag{7-3}$$

where $\tilde{F}_m \in \tilde{A}_l := \mathbb{C}\langle\langle x\rangle\rangle[y_1, \dots, y_l]$ and $\omega(\tilde{F}_m) \le \omega_{l+1}$. Let s_m denote the polydromy order of ψ_m and μ_m be a primitive s_m-th root of unity. Identity (7-2) implies that $t_m := s_m/(p_1 p_2 \cdots p_l)$ is a positive integer. Therefore (6-3) and assertion (3) of Lemma 6.4 imply that

$$f_{\psi_m} = \prod_{j=0}^{t_m-1} \mu_k^{jp_1\cdots p_l} \star_{s_m} (f^{(l)}_{\psi_m}) = \prod_{j=0}^{t_m-1} \mu_m^{jp_1\cdots p_l} \star_{s_m} (\pi_l(F_{l+1} + \tilde{F}_m)) = \pi_l(G_m), \tag{7-4}$$

where

$$G_m := \prod_{j=0}^{t_m-1} \bigl(F_{l+1} + \mu_m^{jp_1\cdots p_l} \star_{s_m} (\tilde{F}_m)\bigr) \in \tilde{B}_l. \tag{7-5}$$

Recall that $F_{l+1} = y_l^{p_l} - \sum_{i=1}^{m_l} c_{l,i} x^{\beta^i_{l,0}} y_1^{\beta^i_{l,1}} \cdots y_l^{\beta^i_{l,l}}$. Since by our assumption all the key forms but the last one are polynomials, it follows from assertion (2) of Proposition 5.3 that $\beta^i_{l,0} \ge 0$ for all $i < m_l$, but $\beta^i_{m_l,0} < 0$; set

$$\omega'_{l+1} := \omega(x^{\beta^{m_l}_{l,0}} y_1^{\beta^{m_l}_{l,1}} \cdots y_l^{\beta^{m_l}_{l,l}}) = \sum_{i=0}^{l} \beta^{m_l}_{l,i} \omega_i. \tag{7-6}$$

Then $\omega'_{l+1} > \omega_{l+1}$ and therefore we may express G_m as

$$G_m = \prod_{j=0}^{t_m-1} \biggl(y_l^{p_l} - \sum_{i=0}^{m_l} c_{l,i} x^{\beta^i_{l,0}} y_1^{\beta^i_{l,1}} \cdots y_l^{\beta^i_{l,l}} - G_{m,j} \biggr), \tag{7-7}$$

for some $G_{m,j} \in \tilde{B}_l$ with $\omega(G_{m,j}) < \omega'_{l+1}$. Identities (7-1), (7-4) and (7-7) imply that $f = \pi_l(F)$ for some $F \in \tilde{A}_l$ of the form

$$F = \prod_{m'=1}^{M'} \biggl(y_l^{p_l} - \sum_{i=0}^{m_l} c_{l,i} x^{\beta^i_{l,0}} y_1^{\beta^i_{l,1}} \cdots y_l^{\beta^i_{l,l}} - G'_{m'} \biggr), \tag{7-8}$$

where $\omega(G'_{m'}) < \omega'_{l+1}$ for all m', $1 \le m' \le M'$. Let

$$G := \begin{cases} F - y_l^{M' p_l} & \text{if } m_l = 0, \\ F - \bigl(y_l^{p_l} - \sum_{i=0}^{m_l-1} c_{l,i} x^{\beta^i_{l,0}} y_1^{\beta^i_{l,1}} \cdots y_l^{\beta^i_{l,l}}\bigr)^{M'} & \text{otherwise.} \end{cases}$$

Recall from assertion (4) of Lemma 6.7 that $\beta^0_{l,l} = 0$. It follows that the *leading weighted homogeneous form* of G with respect to ω is

$$\mathfrak{L}_\omega(G) = \begin{cases} -c_{l,0}x^{\beta^0_{l,0}}y_1^{\beta^0_{l,1}}\cdots y_{l-1}^{\beta^0_{l,l-1}} & \text{if } m_l=0,\ M'=1,\\ \sum_{i=1}^{M'}\binom{M'}{i}(-c_{l,0})^i y_l^{(M'-i)p_l}x^{i\beta^0_{l,0}}y_1^{i\beta^0_{l,1}}\cdots y_{l-1}^{i\beta^0_{l,l-1}} & \text{if } m_l=0,\ M'>1,\\ M'c_{l,m_l}(y_l^{p_l}-c_{l,0}x^{\beta^0_{l,0}}y_1^{\beta^0_{l,1}}\cdots y_{l-1}^{\beta^0_{l,l-1}})^{M'-1}x^{\beta^{m_l}_{l,0}}y_1^{\beta^{m_l}_{l,1}}\cdots y_l^{\beta^{m_l}_{l,l}} & \text{otherwise.}\end{cases} \tag{7-9}$$

Since $\pi_l(F) = f \in \mathbb{C}[x, y]$, it follows that $g := \pi_l(G)$ is also a polynomial in x and y. Assertion (1) of Lemma 6.13 then implies that there is a polynomial $\tilde{G} \in C_l := \mathbb{C}[x, y_1, \dots, y_l]$ such that $\pi_l(\tilde{G}) = g$ and $\omega(\tilde{G}) = \delta_l(g)$. In particular, $\omega(\tilde{G}) \le \omega(G)$.

Claim 7.2. $\omega(\tilde{G}) = \omega(G)$.

Proof. Let $\prec$ be the reverse lexicographic monomial ordering on C_l from Section 6C and let α be the smallest positive integer such that $x^\alpha \mathfrak{L}_\omega(G)$ is a polynomial. Then (7-9) implies that the *leading term* of $x^\alpha \mathfrak{L}_\omega(G)$ with respect to $\prec$ is

$$\mathrm{LT}_\prec(x^\alpha \mathfrak{L}_\omega(G)) = \begin{cases} -c_{l,0}y_1^{\beta^0_{l,1}}\cdots y_{l-1}^{\beta^0_{l,l-1}} & \text{if } m_l=0,\ M'=1,\\ -c_{l,0}M'y_l^{(M'-1)p_l}x^{(M'-1)|\beta^0_{l,0}|}y_1^{\beta^0_{l,1}}\cdots y_{l-1}^{\beta^0_{l,l-1}} & \text{if } m_l=0,\ M'>1,\\ M'c_{l,m_l}y_l^{(M'-1)p_l+\beta^{m_l}_{l,l}}y_1^{\beta^{m_l}_{l,1}}\cdots y_{l-1}^{\beta^{m_l}_{l,l-1}} & \text{otherwise.}\end{cases} \tag{7-10}$$

Assume contrary to the claim that $\omega(G) > \omega(\tilde{G}) = \delta_l(g)$. Then $x^\alpha \mathfrak{L}_\omega(G) \in J_l$, where J_l is the ideal from assertion (2) of Lemma 6.13. Assertion (2) of Lemma 6.13 then implies that there exists j, $1 \le j \le l-1$, such that $y_j^{p_j} = \mathrm{LT}_\prec(H_{j+1})$ divides $\mathrm{LT}_\prec(x^\alpha \mathfrak{L}_\omega(G))$. But this contradicts the fact that $\beta^{m_l}_{l,j'} < p_{j'}$ for all j', $1 \le j' \le l-1$ (assertion (3) of Lemma 6.7) and completes the proof of the claim. □

Let J_l and α be as in the proof of Claim 7.2. Note that $\mathfrak{L}_\omega(x^\alpha \tilde{G}) \notin J_l$ by our choice of $\tilde{G}$. Therefore, after "dividing out" $\tilde{G}$ by the Gröbner basis $\mathcal{B}_l$ of Lemma 6.13 (which does not change $\omega(\tilde{G})$) if necessary, we may (and will) assume that

$$y_j^{p_j} \text{ does } \textit{not} \text{ divide any of the monomial terms of } \mathfrak{L}_\omega(x^\alpha \tilde{G}) \text{ for any } j,\ 1 \le j \le l-1. \tag{7-11}$$

Since $\pi_l(x^\alpha G - x^\alpha \tilde{G}) = 0$, it follows that $\mathfrak{L}_\omega(x^\alpha G - x^\alpha \tilde{G}) \in J_l$. Since $\omega(G) = \omega(\tilde{G})$ by Claim 7.2, it follows that $H^* := \mathfrak{L}_\omega(x^\alpha G) - \mathfrak{L}_\omega(x^\alpha \tilde{G}) \in J_l$. Let

$$H := \mathrm{LT}_\prec(\mathfrak{L}_\omega(x^\alpha G)) \quad \text{and} \quad \tilde{H} := \mathrm{LT}_\prec(\mathfrak{L}_\omega(x^\alpha \tilde{G})).$$

Since $\tilde{G} \in \mathbb{C}[x, y_1, \dots, y_l]$, it follows that $\deg_x(\tilde{H}) \ge \alpha$. On the other hand, (7-10) implies that $\deg_x(H) = \alpha + \beta^0_{m_l,0} < \alpha$. It follows in particular that $H \ne \tilde{H}$ and $\mathrm{LT}_\prec(H^*) = \max_\prec\{H, -\tilde{H}\}$. Then (7-10) and (7-11) imply that $y_j^{p_j} = \mathrm{LT}_\prec(H_j)$ does *not* divide $\mathrm{LT}_\prec(H^*)$ for any j, $1 \le j \le l-1$. This contradicts assertion (2) of Lemma 6.13 and finishes the proof of the implication $(1') \Rightarrow (2)$, as required to complete the proof of Theorem 4.7. $\square$

Proof of Theorem 4.1. Theorem 4.7 implies that (2) $\Leftrightarrow$ (3). Now assume (2) is true. Note that $\delta(f) > 0$ for each nonconstant $f \in \mathbb{C}[x, y]$ (since such an f must have a pole at the irreducible curve $E'_1 := \pi'(E_1) \subseteq Y'$); so that the ring S^δ defined in Lemma 6.13 is precisely the ring $\mathbb{C}[x, y]^\delta$ from Section 5B. Assertion (1) of Lemma 6.13 and Proposition 5.8 then imply that Y' is isomorphic to the closure of the image of $\mathbb{C}^2$ in the weighted projective variety $\mathbb{P}^{l+2}(1, \delta(f_0), \dots, \delta(f_{l+1}))$ under the mapping $(x, y) \mapsto [1 : f_0 : \cdots : f_{l+1}]$. In particular this shows (2) $\Rightarrow$ (1).

It remains to show that (1) $\Rightarrow$ (2). So assume that Y' is algebraic. Recall the setup of Proposition 3.14. We can identify Y' with $\bar{X}$ and E'_1 with C_∞ (where $\bar{X}$ and C_∞ are as in Proposition 3.14). Let $P_\infty \in C_\infty$ be as in Proposition 3.14. Since Y' is algebraic, there exists a compact algebraic curve C on Y' such that $P_\infty \notin C$. Let $f \in \mathbb{C}[x, y]$ be the polynomial that generates the ideal of C in $\mathbb{C}[x, y]$. Proposition 3.14 then implies that f satisfies the condition of property $(1')$ from Lemma 7.1. Theorem 4.7 and Lemma 7.1 then show that (2) is true, as required. $\square$

Proof of Theorem 4.3. Let δ be the semidegree on $\mathbb{C}[x, y]$ corresponding to the curve at infinity on $\bar{X}$, $\tilde{\phi}_\delta(x, \xi)$ be the associated generic descending Puiseux series, and $g_0, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ be the key forms. If $\bar{X}$ is algebraic, then Theorem 4.1 implies that g_{n+1} is a polynomial. Proposition 3.14 and assertion (4) of Theorem 3.17 then imply that $g_{n+1} = 0$ defines a curve C as in assertion (2) of Theorem 4.3. This proves the implication (1) $\Rightarrow$ (2), and also the last assertion of Theorem 4.3. It remains to prove the implication (2) $\Rightarrow$ (1). So assume there exists $f \in \mathbb{C}[x, y]$ such that $C := \{f = 0\}$ is as in (2). Proposition 3.14 implies that f satisfies the condition of property $(1')$ from Lemma 7.1. Then Lemma 7.1, Theorem 4.7 and Theorem 4.1 imply that $\bar{X}$ is algebraic, as required. $\square$

Proof of Theorem 4.4. The ($\Rightarrow$) direction of assertion (1) follows from Theorem 4.1 and assertion (2) of Proposition 5.3. For the ($\Leftarrow$) implication, note that assertion (3) of Proposition 3.21 and Property (P0) of key forms imply for each k, $1 \le k \le l$, that

$$p_k \omega_k = \sum_{j=0}^{k-1} \beta'_{k,j} \omega_j,$$

where the $\beta'_{k,j}$ are integers such that $0 \le \beta'_{k,j} < p_j$ for $1 \le j < k$. Define g^0_k, $0 \le k \le l + 1$, by

$$g^0_k = \begin{cases} x & \text{if } k = 0, \\ y & \text{if } k = 1, \\ (g^0_{k-1})^{p_{k-1}} - \prod_{j=0}^{k-2} (g^0_j)^{\beta'_{k-1,j}} & \text{if } 2 \le k \le l + 1. \end{cases}$$

Assertion (2) of Theorem 3.17 implies that there is a unique semidegree δ^0 on $\mathbb{C}[x, y]$ such that its key forms are $g_0^0, \ldots, g_{l+1}^0$ and $\delta^0(g_k^0) = \omega_k$, $0 \le k \le l+1$. Since $\omega_{l+1} > 0$ (assertion (3) of Theorem 3.28) it follows that δ^0 defines a primitive normal compactification $\bar{X}^0$ of $\mathbb{C}^2$ (Remark 4.2). It follows from Proposition 3.21 that (q_k, p_k), $1 \le k \le l+1$, are uniquely determined in terms of $\omega_0, \ldots, \omega_{l+1}$. Therefore Γ is precisely the augmented and marked dual graph associated to the minimal $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}^0$. Now, if (S1-k) holds for each k, $1 \le k \le l$, then each $\beta'_{k,0}$ is nonnegative, so that each g_k^0 is a polynomial. Theorem 4.1 then implies that $\bar{X}^0$ is algebraic, which proves the ($\Leftarrow$) implication of assertion (1).

Now we prove assertion (2). For the ($\Rightarrow$) implication, pick a nonalgebraic normal primitive compactification $\bar{X}$ of $\mathbb{C}^2$ such that $\Gamma = \Gamma_{\bar{X}}$. Let δ be the order of pole along the curve at infinity on $\bar{X}$. Theorem 4.1 implies that at least one of the key forms of δ is not a polynomial. Assertions (2) and (4) of Proposition 3.21 and assertion (2) of Proposition 5.3 now imply that either (S1-k) or (S2-k) fails, as required. It remains to prove the ($\Leftarrow$) implication of assertion (2). Let g_k^0, $0 \le k \le l+1$, be as in the preceding paragraph. If (S1-k) fails for some k, $1 \le k \le l$, take the smallest such k. Then by construction g_k^0 is *not* a polynomial, so that $\bar{X}^0$ is not algebraic (Theorem 4.1), as required. Now assume that (S1-k) holds for all k, $1 \le k \le l$, but there exists k, $1 \le k \le l$, such that (S2-k) fails; let k be the smallest such integer. Pick $\tilde{\omega} \in (\omega_{k+1}, p_k\omega_k) \cap \mathbb{Z}\langle\omega_0, \ldots, \omega_k\rangle \setminus \mathbb{Z}_{\ge 0}\langle\omega_0, \ldots, \omega_k\rangle$. Then it is straightforward to see that there exist integers $\tilde{\beta}_0, \ldots, \tilde{\beta}_k$ such that $\tilde{\beta}_0 < 0$, $0 \le \tilde{\beta}_j < p_j$, $1 \le j < k$, and

$$\tilde{\omega} = \sum_{j=0}^{k-1} \tilde{\beta}_j \omega_j.$$

Define g_i^1, where $0 \le i \le l+2$, by

$$g_i^1 = \begin{cases} g_i^0 & \text{if } 0 \le i \le k+1, \\ g_{k+1}^0 - \prod_{j=0}^{k} (g_j^0)^{\tilde{\beta}_j} & \text{if } i = k+2, \\ (g_{i-1}^1)^{p_{i-2}} - \prod_{j=0}^{k} (g_j^1)^{\beta'_{i-2,j}} \prod_{j=k+2}^{i-2} (g_j^1)^{\beta'_{i-2,j-1}} & \text{if } k+3 \le i \le l+2. \end{cases}$$

The same arguments as in the proof of assertion (1) imply that there is a primitive normal compactification $\bar{X}^1$ of $\mathbb{C}^2$ such that:

- $g_0^1, \ldots, g_{l+2}^1$ are the key forms of the semidegree δ^1 corresponding to its curve at infinity, and

$$\delta^1(g_i^1) = \begin{cases} \omega_i & \text{if } 0 \le i \le k, \\ \tilde{\omega} & \text{if } i = k+1, \\ \omega_{i-1} & \text{if } k+2 \le i \le l+2. \end{cases}$$

- Γ is the augmented and marked dual graph associated to the minimal $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}^1$.

Since g^1_{k+2} is not a polynomial, $\bar{X}^1$ is not algebraic (Theorem 4.1), as required to complete the proof of assertion (2). $\square$

Appendix A: Proof of Lemma 6.10

Notation A.1. Fix k, $1 \leq k \leq l+1$. For $F \in \tilde{A}_k$ and $\mu \in \mathbb{R}$, we write $[F]_{>\mu}$ for the sum of all monomial terms H of F such that $\omega(H) > \mu$.

Lemma A.2. *Fix k, $1 \leq k \leq l$. Pick $\psi_1, \psi_2 \in \mathbb{C}\langle\langle x \rangle\rangle$ and $\mu \leq \omega_k \in \mathbb{R}$. Assume*

(1) *the first k Puiseux pairs of each ψ_j are $(q_1, p_1), \dots, (q_k, p_k)$.*

Assumption (1) implies that we can define $F^{(k-1)}_{\psi_j}$, $F^{(k)}_{\psi_j}$, $1 \leq j \leq 2$, as in Lemma 6.10. Assume

(2) $[F^{(k-1)}_{\psi_1}]_{>\mu} = [F^{(k-1)}_{\psi_2}]_{>\mu}$, *and*
(3) $[F^{(k-1)}_{\psi_j}]_{>\omega_k} = [F_k]_{>\omega_k}$ *for each j, $1 \leq j \leq 2$.*

Then $[F^{(k)}_{\psi_1}]_{>(p_k-1)\omega_k+\mu} = [F^{(k)}_{\psi_2}]_{>(p_k-1)\omega_k+\mu}$.

Proof. Let
$$\tilde{A} := \begin{cases} \tilde{A}_1 & \text{if } k = 1, \\ \tilde{A}_{k-1} & \text{otherwise.} \end{cases}$$

Assumptions (2) and (3) imply that there exists $G \in \tilde{A}$ with $\omega(G) \leq \omega_k$ such that for both j, $1 \leq j \leq 2$,
$$F^{(k-1)}_{\psi_j} = F_k + G + G_j$$

for some $G_j \in \tilde{A}$ with $\omega(G_j) \leq \mu$. Fix j, $1 \leq j \leq 2$. Let m_j be the polydromy order of ψ_j and μ_j be a primitive m_j-th root of unity. Then identity (6-3) and assertion (3) of Lemma 6.4 imply that
$$f^{(k)}_{\psi_j} = \prod_{i=0}^{p_k-1} \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} (f^{(k-1)}_{\psi_j}) = \pi_{k-1}(G_j^*),$$
where
$$\begin{aligned} G_j^* &:= \prod_{i=0}^{p_k-1} \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} (F^{(k-1)}_{\psi_j}) = \prod_{i=0}^{p_k-1} \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} (F_k + G + G_j) \\ &= \prod_{i=0}^{p_k-1} \big(F_k + \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} G + \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} G_j\big) \\ &= \prod_{i=0}^{p_k-1} \big(F_k + \mu^{ip_1\cdots p_{k-1}} \star_{m} G + \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} G_j\big), \end{aligned}$$

m is the *polydromy order* of G (Definition 6.2), and μ is a primitive m-th root of unity (the last equality is an implication of assertion (1) of Lemma 6.4). Let

$$G_{j,0} := \prod_{i=0}^{p_k-1} \left(y_k + \mu^{ip_1\cdots p_{k-1}} \star_m G + \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} G_j \right) \in \tilde{A}_k. \tag{A-1}$$

Note that $\pi_k(G_{j,0}) = f_{\psi_j}^{(k)} = \pi_k(F_{\psi_j}^{(k)})$. Now we construct $F_{\psi_j}^{(k)}$ from $G_{j,0}$ via constructing a sequence of elements $G_{j,0}, G_{j,1}, \ldots$ as follows:

- For $\beta := (\beta_1, \ldots, \beta_k) \in \mathbb{Z}_{\geq 0}^k$, define

$$|\beta|_{k-1} := \sum_{j=1}^{k-1} p_0 \cdots p_{j-1} \beta_j.$$

  Consider the well order $\prec_{k-1}^*$ on $\mathbb{Z}_{\geq 0}^k$ defined as follows: $\beta \prec_{k-1}^* \beta'$ if and only if

  (1) $|\beta|_{k-1} < |\beta'|_{k-1}$, or

  (2) $|\beta|_{k-1} = |\beta'|_{k-1}$ and the left-most nonzero entry of $\beta - \beta'$ is negative.

- Assume $G_{j,N}$ has been constructed, $N \geq 0$. Express $G_{j,N}$ as

$$G_{j,N} = \sum_{\beta \in \mathbb{Z}_{\geq 0}^k} g_{j,N,\beta}(x) y_1^{\beta_1} \cdots y_k^{\beta_k}$$

  and define

$$\mathcal{E}_{j,N} := \left\{ \beta \in \mathbb{Z}_{\geq 0}^k : g_{j,N,\beta} \neq 0 \text{ and } \beta_i \geq p_i \text{ for some } i,\ 1 \leq i \leq k-1 \right\}.$$

- If $\mathcal{E}_{j,N} = \varnothing$, then **stop.**
- Otherwise pick the maximal element $\beta^* = (\beta_1^*, \ldots, \beta_k^*)$ of $\mathcal{E}_{j,N}$ with respect to $\prec_{k-1}^*$, and the maximal i^*, $1 \leq i^* \leq k-1$, such that $\beta_{i^*}^* \geq p_{i^*}$, and set

$$G_{j,N+1} = \sum_{\beta \neq \beta^*} g_{j,N,\beta}(x) y_1^{\beta_1} \cdots y_k^{\beta_k} + g_{j,N,\beta^*}(x) \times \prod_{i \neq i^*} (y_i)^{\beta_i^*} (y_i)^{\beta_{i^*}^* - p_{i^*}} (y_{i+1} - (F_{i+1} - y_{i^*}^{p_{i^*}})). \tag{A-2}$$

Assertion (c) of Lemma 6.9 and assertion (4) of Lemma 6.6 imply that all the "new" exponents of $(y_1, \ldots, y_k)$ that appear in $G_{j,N+1}$ are smaller (with respect to $\prec_{k-1}^*$) than β^*, and it follows that the sequence of $G_{j,N}$'s stops at some finite value N^* of N.

Claim A.2.1. $$G_{j,N^*} = F_{\psi_j}^{(k)}.$$

Proof. Indeed, (A-2) implies that $\pi_k(G_{j,N^*}) = \pi_k(G_{j,0}) = f_{\psi_j}^{(k)}$. Since we must have $\mathcal{E}_{j,N^*} = \varnothing$ for G_{j,N^*} to be the last element of the sequence of $G_{j,N}$'s, G_{j,N^*} satisfies the characterizing properties of $F_{\psi_j}^{(k)} = F^{\pi_k}_{f_{\psi_j}^{(k)}}$ from Lemma 6.6. $\square$

Now note that, for each i, $1 \le i \le k-1$, every monomial term in $y_{i+1}-(F_{i+1}-y_i^{p_i})$ has ω-value smaller than or equal to $\omega(y_i^{p_i})$ (assertion (5), Lemma 6.7). It then follows from (A-1) and (A-2) that G_j has no effect on $[G_{j,N}]_{>(p_k-1)\omega_k+\mu}$ for any N, i.e., $[G_{1,N}]_{>(p_k-1)\omega_k+\mu} = [G_{2,N}]_{>(p_k-1)\omega_k+\mu}$ for all N. Claim A.2.1 then implies the lemma. $\square$

Corollary A.3. *Let*

$$\omega_{i,j} := \omega_i + q_j p_{j+1} \cdots p_{l+1} - q_i p_{i+1} \cdots p_{l+1} \quad \text{for } 1 \le i \le j \le l+1.$$

Fix j, $0 \le j \le l$. Let $\psi \in \mathbb{C}\langle\langle x \rangle\rangle$ be such that $\psi \equiv_{r_{j+1}} \phi_{j+1}$ (where $r_1, \dots, r_{l+1}$ and $\phi_1, \dots, \phi_{l+1}$ are as in (6-6) *and* (6-7), *respectively). Then for all i such that $0 \le i \le j$,*

$$[F_\psi^{(i)}]_{>\omega_{i+1,j+1}} = [F_{\phi_{j+1}}^{(i)}]_{>\omega_{i+1,j+1}}.$$

Proof. At first we consider the $i=0$ case. Equation (6-1) implies that $f_\psi^{(0)} = y - \psi(x)$ and $f_{\phi_{k+1}}^{(0)} = y - \phi_{k+1}(x)$. Then (6-10) implies that

$$F_\psi^{(0)} = y_1 + \phi_1(x) - \psi(x), \quad F_{\phi_{k+1}}^{(0)} = y_1 + \phi_1(x) - \phi_{j+1}(x).$$

It follows that

$$\begin{aligned}\omega(F_\psi^{(0)} - F_{\phi_{j+1}}^{(0)}) &= \omega_0 \deg_x(\phi_{j+1}(x) - \psi(x)) \\ &\le p_1 \cdots p_{l+1} r_{j+1} = q_{j+1} p_{j+2} \cdots p_{l+1} = \omega_{1,j+1}.\end{aligned}$$

It follows that the corollary is true for $i=0$ and all j, $0 \le j \le l$.

Now we start the proof of the general case. We proceed by induction on j. It follows from the preceding discussion that the corollary is true for $j=0$. So assume it holds for $0 \le j \le j' \le l-1$. To show that it holds for $j = j'+1$, we proceed by induction on i. By the same reasoning, we may assume that it also holds for $j = j'+1$ and $0 \le i \le i' \le j'$. Pick ψ such that $\psi \equiv_{r_{j'+2}} \phi_{j'+2}$. Then applying the induction hypothesis with $j = j'+1$ and $i = i'$, we have

$$[F_\psi^{(i')}]_{>\omega_{i'+1,j'+2}} = [F_{\phi_{j'+2}}^{(i')}]_{>\omega_{i'+1,j'+2}}. \tag{A-3}$$

On the other hand, since $\psi \equiv_{r_{i'+1}} \phi_{i'+1}$, we can apply the induction hypothesis with $j = i'$ and $i = i'$ to obtain

$$[F_\psi^{(i')}]_{>\omega_{i'+1,i'+1}} = [F_{\phi_{i'+1}}^{(i')}]_{>\omega_{i'+1,i'+1}}.$$

Similarly, since $\phi_{j'+2} \equiv_{r_{i'+1}} \phi_{i'+1}$, we have

$$[F_{\phi_{j'+2}}^{(i')}]_{>\omega_{i'+1,i'+1}} = [F_{\phi_{i'+1}}^{(i')}]_{>\omega_{i'+1,i'+1}}.$$

Since $\omega_{i'+1,i'+1} = \omega_{i'+1}$, it follows that

$$[F_\psi^{(i')}]_{>\omega_{i'+1}} = [F_{\phi_{j'+2}}^{(i')}]_{>\omega_{i'+1}} = [F_{\phi_{i'+1}}^{(i')}]_{>\omega_{i'+1}}. \tag{A-4}$$

Identities (A-3), (A-4) and assertion (c) of Lemma 6.9 imply that ψ and $\phi_{j'+2}$ satisfy the hypotheses of Lemma A.2 with $\mu = \omega_{i'+1,j'+2}$ and $k = i'+1$. Lemma A.2 therefore implies that

$$[F_{\psi}^{(i'+1)}]_{>\mu'} = [F_{\phi_{j'+2}}^{(i'+1)}]_{>\mu'},$$

where $\mu' = (p_{i'+1} - 1)\omega_{i'+1} + \omega_{i'+1,j'+2}$. It is straightforward to check using (3-4) that $\mu' = \omega_{i'+2,j'+2}$, as required to complete the induction. □

Proof of Lemma 6.10. Since $\omega_{k+1} = \omega_{k+1,k+1}$ and $F_{\phi_{k+1}}^{(k)} = F_{\phi_{k+1}}$, Lemma 6.10 is simply a special case of Corollary A.3. □

Appendix B: Proof of Lemma 6.13

We freely use the notation of Section 6C.

Proof of assertion (1) of Lemma 6.13. Since $f_0 = x$ and each f_j, $1 \le j \le l+1$, is monic in y with $\deg_y(f_j) = p_0 \cdots p_{j-1}$ (where $p_0 := 1$), it is straightforward to see that each polynomial $f \in \mathbb{C}[x, y]$ can be represented as a finite sum of the form

$$f = \sum_{\beta \in \mathbb{Z}_{\ge 0}^{l+2}} a_\beta f_0^{\beta_0} \cdots f_{l+1}^{\beta_{l+1}},$$

where for each $\beta = (\beta_0, \dots, \beta_{l+1})$, we have $a_\beta \in \mathbb{C}$ and $\beta_j < p_j$, $1 \le j \le l$. It suffices to show that

$$\delta(f) = \max\{\delta(f_0^{\beta_0} \cdots f_{l+1}^{\beta_{l+1}}) : c_\beta \neq 0\}.$$

We compute $\delta(f)$ via identity (3-1). Assertion (4) of Theorem 3.17 implies that

$$f_j|_{y=\tilde{\phi}_\delta(x,\xi)} = \begin{cases} c_j^* x^{\omega_j/\omega_0} + \text{l.d.t.} & \text{for } 0 \le j \le l, \\ (c_{l+1}^* \xi + c_{l+1}) x^{\omega_{l+1}/\omega_0} + \text{l.d.t.} & \text{for } j = l+1, \end{cases} \tag{B-1}$$

where $c_j^* \in \mathbb{C}^*$, $0 \le j \le l$, $c_{l+1} \in \mathbb{C}$, and l.d.t. denotes terms with lower degree in x. Let $d := \max\{\delta(f_0^{\beta_0} \cdots f_{l+1}^{\beta_{l+1}}) : a_\beta \neq 0\}$ and $\mathcal{B} := \{\beta : a_\beta \neq 0,\ \delta(f_0^{\beta_0} \cdots f_{l+1}^{\beta_{l+1}}) = d\}$. It follows that

$$f|_{y=\tilde{\phi}_\delta(x,\xi)} = c(\xi) x^{d/\omega_0} + \text{l.d.t.}, \tag{B-2}$$

where

$$c(\xi) := \sum_{\beta \in \mathcal{B}} a_\beta (c_{l+1}^* \xi + c_{l+1})^{\beta_{l+1}} \prod_{j=0}^{l} (c_j^*)^{\beta_j}. \tag{B-3}$$

Now, assertion (1) of Lemma 6.8 implies that for two distinct elements β, β' of $\mathcal{B}$, $\beta_{l+1} \neq \beta'_{l+1}$. Identity (B-3) then implies that $c(\xi) \neq 0$, so that (B-2) implies that $\delta(f) = d$, as required to complete the proof of assertion (1). □

For each j, $0 \le j \le l+1$, let $\Omega_j \subseteq \mathbb{Z}$ be the semigroup generated by $\omega_0, \dots, \omega_j$; recall that for $j \ge 1$, condition (Polynomial$_j$) implies that $\Omega_{j-1} \subseteq \mathbb{Z}_{\ge 0}$ (Lemma 6.12).

Lemma B.1. *Assume* (Polynomial$_{l+1}$) *holds. Fix j, $1 \le j \le l$. Let $\bar{J}_{j+1}$ be the ideal in C_j generated by $H_2, \dots, H_{j+1}$. Let t be an indeterminate. Then*

$$C_j/\bar{J}_{j+1} \cong \mathbb{C}[\Omega_j] \cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_j}],$$

via the mapping $x \mapsto t^{\omega_0}$ and $y_i \mapsto b_i t^{\omega_i}$, $1 \le i \le j$, for some $b_1, \dots, b_j \in \mathbb{C}^*$.

Proof. We proceed by induction on j. For $j = 1$, identity (6-12) and assertions (4) and (5) of Lemma 6.7 imply that

$$C_1/\bar{J}_2 = \mathbb{C}[x, y_1]/\langle y_1^{p_1} - c_{1,0}x^{q_1}\rangle \cong \mathbb{C}[t^{p_1}, t^{q_1}],$$

where t is an indeterminate and the isomorphism maps $x \mapsto t^{p_1}$ and $y_1 \mapsto c_{1,0}^{1/p_1} t^{q_1}$, where $c_{1,0}^{1/p_1}$ is a p_1-th root of $c_{1,0} \in \mathbb{C}^*$. Since $\omega_0 = p_1 p_2 \cdots p_l$ and $\omega_1 = q_1 p_2 \cdots p_l$, this proves the lemma for $j = 1$. Now assume that the lemma is true for $j - 1$, $2 \le j \le l$, i.e., there exists an isomorphism

$$C_{j-1}/\bar{J}_j \cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_{j-1}}]$$

which maps $x \mapsto t^{\omega_0}$ and $y_i \mapsto b_i t^{\omega_i}$, $1 \le i \le j-1$ for some $b_1, \cdots, b_{j-1} \in \mathbb{C}^*$. It follows that

$$\begin{aligned} C_j/\bar{J}_{j+1} &= C_{j-1}[y_j]/\langle \bar{J}_j, y_j^{p_j} - c_{j,0}x^{\beta^0_{j,0}} y_1^{\beta^0_{j,1}} \cdots y_{j-1}^{\beta^0_{j,j-1}}\rangle \\ &\cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_{j-1}}, y_j]/\langle y_j^{p_j} - \tilde{c}t^{p_j\omega_j}\rangle \end{aligned}$$

for some $\tilde{c} \in \mathbb{C}^*$ (the last isomorphism uses assertion (5) of Lemma 6.7). Since $p_j = \min\{\alpha \in \mathbb{Z}_{>0}; \alpha\omega_j \in \mathbb{Z}\omega_0 + \cdots + \mathbb{Z}\omega_{j-1}\}$ (assertion (3) of Proposition 3.21), it follows that

$$\mathbb{C}[t^{\omega_0}, \dots, t^{\omega_{j-1}}, y_j]/\langle y_j^{p_j} - \tilde{c}t^{p_j\omega_j}\rangle \cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_j}]$$

via a map which sends $y_j \mapsto (\tilde{c})^{1/p_j} t^{\omega_j}$ (where $(\tilde{c})^{1/p_j}$ is a p_j-th root of $\tilde{c}$), which completes the induction. ☐

Let z be an indeterminate and $\hat{C}_{l+1} := C_{l+1}[z] = \mathbb{C}[z, x, y_1, \dots, y_{l+1}]$. Let $\hat{\omega}$ be the weighted degree on $\hat{C}_{l+1}$ such that $\hat{\omega}(z) = 1$ and $\hat{\omega}|_{C_{l+1}} = \omega$. Equip $\hat{C}^{l+1}$ with the grading determined by $\hat{\omega}$. Let S^δ be as in assertion (1) of Lemma 6.13 and $\hat{\pi} : \hat{C}_{l+1} \to S^\delta$ be the map which sends $z \mapsto (1)_1$, $x \mapsto (x)_{\omega_0}$, and $y_j \mapsto (f_j)_{\omega_j}$, $1 \le j \le l+1$. Assertion (1) implies that $\hat{\pi}$ is a surjective homomorphism of graded rings. Let I be the ideal generated by $(1)_1$ in S^δ and $\hat{J}_{l+1} := \hat{\pi}^{-1}(I) \subseteq \hat{C}_{l+1}$.

Claim B.2. *$\hat{J}_{l+1}$ is generated by $\hat{\mathcal{B}}_{l+1} := (H_{l+1}, \dots, H_2, z)$.*

Proof. Let $\bar{J}_{l+1}$ be the ideal of C_l as defined in Lemma B.1, and $\hat{J}'_{l+1}$ be the ideal of $\hat{C}_{l+1}$ generated by $\bar{J}_{l+1}$ and z. It is straightforward to see that $\hat{J}'_{l+1} \subseteq \hat{J}_{l+1}$. Lemma B.1 implies that

$$\hat{C}_{l+1}/\hat{J}'_{l+1} \cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_l}, y_{l+1}].$$

Let $R := \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_l}, y_{l+1}]$. Then $S^\delta/I \cong \hat{C}_{l+1}/\hat{J}_{l+1} \cong R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R. Now, it follows from the construction of S^δ that $\dim(S^\delta) = 3$. Since I is the principal ideal generated by a nonzero divisor in S^δ, it follows that $\dim(R/\mathfrak{p}) = \dim(S^\delta/I) = 2$. Since R is an integral domain of dimension 2, we must have $\mathfrak{p} = 0$, which implies the claim. □

Proof of assertion (2) of Lemma 6.13. Since $J_{l+1} = \hat{J}_{l+1} \cap C_{l+1}$, Claim B.2 shows that $\mathcal{B}_{l+1}$ generates J_{l+1}. Therefore, to show that $\mathcal{B}_k$ is a Gröbner basis of J_k with respect to $\prec_k$, it suffices to show that running a step of *Buchberger's algorithm* with $\mathcal{B}_{l+1}$ as input leaves $\mathcal{B}_{l+1}$ unchanged. We follow Buchberger's algorithm as described in [Cox et al. 1997, Section 2.7], which consists of performing the following steps for each pair of $H_i, H_j \in \mathcal{B}_{l+1}$, $2 \le i < j \le l+1$:

Step 1: Compute the *S-polynomial* $S(H_i, H_j)$ of H_i and H_j. The leading terms of H_i and H_j with respect to $\prec$ are respectively $\mathrm{LT}_\prec(H_i) = y_{i-1}^{p_{i-1}}$ and $\mathrm{LT}_\prec(H_j) = y_{j-1}^{p_{j-1}}$, so that the *S-polynomial* of H_i and H_j is

$$\begin{aligned} S(H_i, H_j) &:= y_{j-1}^{p_{j-1}} H_i - y_{i-1}^{p_{i-1}} H_j \\ &= -\,(c_{i-1,0} x^{\beta^0_{i-1,0}} y_1^{\beta^0_{i-1,1}} \cdots y_{i-2}^{\beta^0_{i-1,i-2}}) y_{j-1}^{p_{j-1}} \\ &\qquad\qquad + (c_{j-1,0} x^{\beta^0_{j-1,0}} y_1^{\beta^0_{j-1,1}} \cdots y_{j-2}^{\beta^0_{j-1,j-2}}) y_{i-1}^{p_{i-1}}. \end{aligned}$$

Step 2: Divide $S(H_i, H_j)$ by $\mathcal{B}_k$ and if the remainder is nonzero, then adjoin it to $\mathcal{B}_{l+1}$. Since $i < j$, the leading term of $S(H_i, H_j)$ is

$$\mathrm{LT}_\prec(S(H_i, H_j)) = -(c_{i-1,0} x^{\beta^0_{i-1,0}} y_1^{\beta^0_{i-1,1}} \cdots y_{i-2}^{\beta^0_{i-1,i-2}}) y_{j-1}^{p_{j-1}}.$$

Since $\beta^0_{i-1,j'} < p_{j'}$ for all j', $1 \le j' \le i-1$ (assertion (3) of Lemma 6.7), it follows that H_j is the only element of $\mathcal{B}_{l+1}$ such that $\mathrm{LT}_\prec(H_j)$ divides $\mathrm{LT}_\prec(S(H_i, H_j))$. The remainder of the division of $S(H_i, H_j)$ by H_j is

$$\begin{aligned} S_1 &:= S(H_i, H_j) + (c_{i-1,0} x^{\beta^0_{i-1,0}} y_1^{\beta^0_{i-1,1}} \cdots y_{i-2}^{\beta^0_{i-1,i-2}}) H_j \\ &= (c_{j-1,0} x^{\beta^0_{j-1,0}} y_1^{\beta^0_{j-1,1}} \cdots y_{j-2}^{\beta^0_{j-1,j-2}}) H_i, \end{aligned}$$

so that the leading term of S_1 is

$$\mathrm{LT}_\prec(S_1) = (c_{j-1,0} x^{\beta^0_{j-1,0}} y_1^{\beta^0_{j-1,1}} \cdots y_{j-2}^{\beta^0_{j-1,j-2}}) y_{i-1}^{p_{i-1}}.$$

It follows as in the case of $S(H_i, H_j)$ that H_i is the only element of $\mathcal{B}_{l+1}$ whose leading term divides $\mathrm{LT}_\prec(S_1)$. Since H_i divides S_1, the remainder of the division of S_1 by H_i is zero, and it follows that the remainder of the division of $S(H_i, H_k)$ by $\mathcal{B}_k$ is zero. Consequently *Step 2 concludes without changing $\mathcal{B}_{l+1}$*.

It follows from the preceding paragraphs that running one step of Buchberger's algorithm keeps $\mathcal{B}_{l+1}$ unchanged, and consequently $\mathcal{B}_{l+1}$ is a Gröbner basis of J_{l+1} with respect to $\prec$ [Cox et al. 1997, Theorem 2.7.2]. This completes the proof of assertion (2) of Lemma 6.13. □

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pinakio@gmail.com *College of The Bahamas, Thompson Boulevard, Nassau, Bahamas*

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The local lifting problem for A_4

Andrew Obus

We solve the local lifting problem for the alternating group A_4, thus showing that it is a *local Oort group*. Specifically, if k is an algebraically closed field of characteristic 2, we prove that every A_4-extension of $k[[s]]$ lifts to characteristic zero.

1. Introduction

This paper concerns lifting Galois extensions of power series rings from characteristic p to characteristic zero, the so-called *local lifting problem*:

Problem 1.1. (The local lifting problem) Let k be an algebraically closed field of characteristic p and G a finite group. Let $k[[z]]/k[[s]]$ be a G-Galois extension (that is, G acts on $k[[z]]$ by k-automorphisms with fixed ring $k[[s]]$). Does this extension lift to characteristic zero? That is, does there exist a DVR R of characteristic zero with residue field k and a G-Galois extension $R[[Z]]/R[[S]]$, that reduces to $k[[z]]/k[[s]]$?

We will refer to a G-Galois extension $k[[z]]/k[[s]]$ as a *local G-extension*. Basic ramification theory shows that any group G that occurs as the Galois group of a local extension is of the form $P \rtimes \mathbb{Z}/m$, with P a p-group and $p \nmid m$. Chinburg et al. [2011] ask, given a prime p, for which groups G (of the form $P \rtimes \mathbb{Z}/m$) is it true that all local G-actions (over all algebraically closed fields of characteristic p) lift to characteristic zero? Such a group is called a *local Oort group* (for p). Due to various obstructions (the *Bertin obstruction* of [Bertin 1998], the *KGB obstruction* of [Chinburg et al. 2011], and the *Hurwitz tree obstruction* of [Brewis and Wewers 2009]), the list of possible local Oort groups is quite limited. In particular, due to [Chinburg et al. 2011, Theorem 1.2; Brewis and Wewers 2009], if a group G is a local Oort group for p, then G is either cyclic, dihedral of order $2p^n$, or the alternating group A_4 ($p = 2$). Cyclic groups are known to be local Oort — this is

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the so-called *Oort conjecture*, proven by Obus–Wewers [2014] and Pop [2014]. Dihedral groups of order $2p$ are known to be local Oort for odd p due to Bouw and Wewers [2006] and for $p = 2$ due to Pagot [2002]. The group D_9 is local Oort by [Obus 2015]. Our main theorem is:

Theorem 1.2. *If k is an algebraically closed field of characteristic* 2, *then every A_4-extension of $k[[s]]$ lifts to characteristic zero. That is*, *the group A_4 is a local Oort group for $p = 2$.*

This result was announced by Bouw (see the beginning of [Bouw and Wewers 2006]), but the proof has not been written down. Our proof uses a simple idea that avoids the "Hurwitz tree" machinery of [Bouw and Wewers 2006]. Namely, one first classifies local A_4-extensions by what we call their "break" (this is a jump in the higher ramification filtration). One then uses the following strategy of Pop [2014], sometimes known as the "Mumford method": First, make an equicharacteristic deformation of a local A_4-extension such that, generically, the break of the extension goes down. If one can lift the local extensions arising from the generic fiber of this deformation, Pop's work shows that one can lift the original extension. On the other hand, we show explicitly that local A_4-extensions with small break lift. An induction finishes the proof.

We remark that Florian Pop has his own similar proof of Theorem 1.2, which was communicated to the author after the first draft of this paper was written (see Remark 5.3).

The main motivation for the local lifting problem is the following *global lifting problem*, about deformation of curves with an action of a finite group (or equivalently, deformation of Galois branched covers of curves).

Problem 1.3. (The global lifting problem) Let X/k be a smooth, connected, projective curve over an algebraically closed field of characteristic p. Suppose a finite group Γ acts on X. Does (X, Γ) lift to characteristic zero? That is, does there exist a DVR R of characteristic zero with residue field k and a relative projective curve X_R/R with Γ-action, such that X_R, along with its Γ-action, reduces to X?

It is a major result of Grothendieck [SGA 1 1971, XIII, Corollaire 2.12] that the global lifting problem can be solved whenever Γ acts with *tame* (prime-to-p) inertia groups, and R can be taken to be the Witt ring $W(k)$. More generally, the *local-global principle* states that (X, Γ) lifts to characteristic zero over R, a complete DVR, if and only if the local lifting problem holds (over R) for each point of X with nontrivial stabilizer in Γ. Specifically, if x is such a point, then its complete local ring is isomorphic to $k[[z]]$. The stabilizer $I_x \subseteq \Gamma$ acts on $k[[z]]$ by k-automorphisms, and we check the local lifting problem for the local I_x-extension $k[[z]]/k[[z]]^{I_x}$. Thus, the global lifting problem is reduced to the local lifting problem. Proofs of the local-global principle have been given by Bertin and Mézard [2000], Green and Matignon [1998], and Garuti [1996].

One consequence of the local-global principle and Theorem 1.2 is the following:

Corollary 1.4. *The groups A_4 and A_5 are so-called* Oort groups *for every prime. That is, if $\Gamma \in \{A_4, A_5\}$ acts on a smooth projective curve X over an algebraically closed field of positive characteristic p, then (X, Γ) lifts to characteristic zero.*

Proof. By the local-global principle (see also [Chinburg et al. 2008, Theorem 2.4]), it suffices to show that every cyclic-by-p subgroup of A_4 or A_5 is a local Oort group for p. The only subgroups of A_4 of this form for any p are isomorphic to the trivial group, $\mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/3$, or A_4. The subgroups of A_5 of this form are isomorphic to the trivial group, $\mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/5$, D_3, A_4, and D_5. All of these are local Oort groups for the relevant primes, as has been noted above. $\square$

Conventions and notation. Throughout, k is an algebraically closed field of characteristic 2. The ring R is a large enough complete discrete valuation ring of characteristic zero with residue field k, maximal ideal $\mathfrak{m}$, and uniformizer π. We normalize the valuation v on R so that $v(2) = 1$. In any polynomial or power series ring with coefficients in R, the expression $o(x)$ for $x \in R$ means a polynomial or power series with coefficients in $x\mathfrak{m}$.

The ring $k[[t]]$ is always a $\mathbb{Z}/3$-extension of $k[[s]]$ with $t^3 = s$. Likewise, $R[[T]]$ is always a $\mathbb{Z}/3$-extension of $R[[S]]$ with $T^3 = S$. If $k[[z]]/k[[s]]$ is an extension, it is always assumed to contain $k[[t]]$. Our convention for variables is that lowercase letters represent the reduction of capital letters from characteristic 0 to characteristic 2 (e.g., t is the reduction of T).

We write ζ_3 for a primitive third root of unity in any ring.

2. A_4-extensions

We start with the basic structure theory of A_4-extensions.

A_4-extensions in characteristic 2.

Lemma 2.1. *Let $K \subseteq L \subseteq M$ be a tower of field extensions of characteristic 2 such that L/K is $\mathbb{Z}/3$-Galois and M/L is $\mathbb{Z}/2$-Galois. Let σ be a generator of $\mathrm{Gal}(L/K)$. For $\ell \in L$, let $\bar{\ell}$ denote the image of ℓ in $L/(F-1)L$, where F is Frobenius. Suppose $M \cong L[x]/(x^2 - x - a)$, and let d be the dimension of the $\mathbb{F}_2$-vector space generated by $\bar{a}$, $\overline{\sigma(a)}$, and $\overline{\sigma^2(a)}$. If N is the Galois closure of M over L, then $\mathrm{Gal}(N/K)$ can be expressed as a semidirect product*

$$\mathrm{Gal}(N/K) \cong (\mathbb{Z}/2)^d \rtimes \mathbb{Z}/3.$$

Proof. By the Schur–Zassenhaus theorem, it suffices to prove $\mathrm{Gal}(N/L) \cong (\mathbb{Z}/2)^d$. But N/L is clearly generated by Artin–Schreier roots of a, $\sigma(a)$, and $\sigma^2(a)$. Thus the result follows from Artin–Schreier theory. $\square$

Corollary 2.2. *If $d = 2$ in Lemma 2.1, then* $\mathrm{Gal}(N/K) \cong A_4$.

Proof. The group $\mathrm{Gal}(N/K)$ must be a semidirect product $(\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3$ that is nonabelian (as there exists a non-Galois subextension). The only such group is A_4. $\square$

If $K = k((s))$ in Lemma 2.1 above, then after a change of variable, we may assume that $L = k((t))$ with $t^3 = s$. Then, it is easy to see that an Artin–Schreier representative a of M/L may be chosen uniquely such that $a \in t^{-1}k[t^{-1}]$ and a has only odd-degree terms. We say that such an a is in *standard form*. In this case, a standard exercise shows that the break in the higher ramification filtration of M/L (i.e., the largest i such that the higher ramification group G_i is nontrivial) occurs at $\deg(a)$, thought of as a polynomial in t^{-1}.

Corollary 2.3. *Suppose $K = k((s))$ and $L = k((t))$. Suppose $a \in t^{-1}k[t^{-1}] \subseteq L$ is in standard form. Using the notation of Lemma 2.1, we have* $\mathrm{Gal}(N/K) \cong A_4$ *if and only if a has no nonzero terms of degree divisible by* 3.

Proof. Since linear combinations of elements of L in standard form are also in standard form, Lemma 2.1 and Corollary 2.2 imply that $\mathrm{Gal}(N/K) = A_4$ if and only if the $\mathbb{F}_2$-subspace V of L generated by a, $\sigma(a)$, and $\sigma^2(a)$ has dimension 2. If a has no nonzero terms of degree divisible by 3, then $a + \sigma(a) + \sigma^2(a) = 0$ is the only $\mathbb{F}_2$-linear relation that holds among the conjugates of a, so $\dim V = 2$ (note that $a \neq 0$ since it is an Artin–Schreier representative of M/L). Conversely, if a has a nonzero term of degree divisible by 3, then either no $\mathbb{F}_2$-linear relation holds, or $a \in k((s))$ (in which case $a = \sigma(a) = \sigma^2(a)$). In either case, $\dim V \neq 2$. $\square$

If $d = 2$ in the context of Lemma 2.1, then we say that $a \in L$ *gives rise to* the A_4-extension N/K. By abuse of notation, if $K \cong k((s))$, we say that the break of N/K is the ramification break of M/L. This is the same as the unique ramification break of N/L in either the upper or lower numbering. Furthermore, if $K = k((s))$ and $N = k((z))$, we also say that a gives rise to the extension $k[[z]]/k[[s]]$.

Proposition 2.4. *If $K = k((s))$ and N/K is an A_4-extension with* break $= \nu$, *then* $\nu \equiv 1$ *or* $5 \pmod 6$.

Proof. If a gives rise to N/K and is in standard form, we know that ν is the degree of a in t^{-1}. This must be odd, and by Corollary 2.3, it cannot be divisible by 3. $\square$

A_4-extensions in characteristic zero. The story in characteristic zero (or odd characteristic) is completely analogous. We state the result for reference and omit the proof, which is the same as in Lemma 2.1 with Kummer theory substituted for Artin–Schreier theory.

Proposition 2.5. *Let $K \subseteq L \subseteq M$ be a tower of separable field extensions of characteristic $\neq 2$ such that L/K is $\mathbb{Z}/3$-Galois and M/L is $\mathbb{Z}/2$-Galois. Let σ be a generator of* $\mathrm{Gal}(L/K)$. *For $\ell \in L^\times$, let $\bar{\ell}$ denote the image of ℓ in $L^\times/(L^\times)^2$. Suppose $M \cong L[x]/(x^2 - a)$, and let d be the dimension of the $\mathbb{F}_2$-subspace of $L^\times/(L^\times)^2$ generated by $\bar{a}$, $\overline{\sigma(a)}$, and $\overline{\sigma^2(a)}$. If N is the Galois closure of M over L, then* $\mathrm{Gal}(N/K)$ *can be expressed as a semidirect product*: $\mathrm{Gal}(N/K) \cong (\mathbb{Z}/2)^d \rtimes \mathbb{Z}/3$. *In particular, if $d = 2$, then* $\mathrm{Gal}(N/K) \cong A_4$.

In the context of Proposition 2.5, we again say that $a \in L$ *gives rise* to N/K.

3. Characteristic 2 deformations

For this section, let K, L, M, N be as in Lemma 2.1, with $K = k((s))$, $L = k((t))$, and $N = k((z))$. Suppose $\mathrm{Gal}(N/K) \cong A_4$, and N/K is given rise to by $a \in t^{-1}k[t^{-1}]$ in standard form. Let ν be the break of N/K. Our goal is to prove the following proposition.

Proposition 3.1. *Suppose that $\nu > 6$ and all A_4-extensions N'/K with* break $\leq \nu - 6$ *lift to characteristic zero. Then N/K lifts to characteristic zero.*

Our proof follows an idea of Pop [2014]. As in [Pop 2014; Obus 2015], we make a deformation in characteristic 2 so that the generic fiber has "milder" ramification than the special fiber.

Proposition 3.2. *Let $\mathcal{A} = k[\![\varpi, s]\!] \supseteq k[\![s]\!]$, and let $\mathcal{K} =$* $\mathrm{Frac}(\mathcal{A})$. *There exists an A_4-extension $\mathcal{N}/\mathcal{K}$, with $\mathcal{N} \supseteq N$, having the following properties*:

(1) *The unique $\mathbb{Z}/3$-subextension $\mathcal{L}/\mathcal{K}$ of $\mathcal{N}/\mathcal{K}$ is given by $\mathcal{L} = \mathcal{K}[t] \subseteq \mathcal{N}$.*

(2) *If $\mathcal{C}$ is the integral closure of $\mathcal{A}$ in $\mathcal{N}$, we have $\mathcal{C} \cong k[\![\varpi, z]\!]$. In particular, $(\mathcal{C}/(\varpi))/(\mathcal{A}/(\varpi))$ is A_4-isomorphic to the original extension $k[\![z]\!]/k[\![s]\!]$.*

(3) *Let $\mathcal{B} = \mathcal{A}[t] \subseteq \mathcal{L}$, let $\mathcal{R} = \mathcal{A}[\varpi^{-1}]$, let $\mathcal{S} = \mathcal{B}[\varpi^{-1}]$, and let $\mathcal{T} = \mathcal{C}[\varpi^{-1}]$. Then $\mathcal{T}/\mathcal{R}$ is an A_4-extension of Dedekind rings, branched at* 2 *maximal ideals. Above the ideal (s), the inertia group is A_4, and the* break *is $\nu - 6$. The other branched ideal has inertia group $\mathbb{Z}/2 \times \mathbb{Z}/2$, unique* ramification break $= 1$, *and can be chosen to be of the form $(s - \mu^3)$, where $\mu \in \varpi^2 k[\![\varpi^2]\!] \backslash \{0\}$ is arbitrary.*

Proof. Define $\mathcal{L}$ by adjoining t to $\mathcal{K}$. We proceed by deforming a to an element of $\mathcal{L}$. Let $\mu \in \varpi^2 k[\![\varpi^2]\!] \backslash \{0\}$. Let $a' = a/t^{-6} = a/s^{-2}$, and deform a to the element $\tilde{a} := a'(s - \mu^3)^{-2} = a' \prod_{\alpha=1}^{3} (\zeta_3^\alpha t - \mu)^{-2} \in \mathcal{B}_{(\varpi)} \subseteq \mathcal{L}$. Note that $\tilde{a}$ reduces to

$a \pmod{\varpi}$. Observe also that $\mathrm{Gal}(\mathscr{L}/\mathscr{K}) \cong \mathbb{Z}/3$, and the $\mathbb{F}_2$-vector space generated by the images of a' (and thus $\tilde{a}$) under this Galois action has dimension 2. By Corollary 2.2, $\tilde{a} \in \mathscr{L}$ gives rise to an A_4-extension $\mathscr{N}/\mathscr{K}$. We claim that this is the extension we seek.

Property (1) is obvious. To show property (3), first note that $\mathscr{S}/\mathscr{R}$ is branched exactly above the ideal (s). The $\mathbb{Z}/2$-subextensions of $\mathscr{N}/\mathscr{L}$ are the Artin–Schreier extensions corresponding to $\tilde{a}$, $\sigma(\tilde{a})$, and $\sigma^2(\tilde{a})$, where σ generates $\mathrm{Gal}(\mathscr{L}/\mathscr{K})$. Each of these is ramified at most above the ideals (t) and $(\zeta_3^{\alpha} t - \mu)$, for $\alpha \in \{1, 2, 3\}$. We will see in the next paragraph that all of these ideals ramify in each $\mathbb{Z}/2$-subextension. Thus the ramification groups of $\mathscr{T}/\mathscr{S}$ above these ideals are all $\mathbb{Z}/2 \times \mathbb{Z}/2$. Since the three $\mathbb{Z}/2$-subextensions are Galois conjugate over $\mathscr{K}$, there can only be one higher ramification jump for each ideal, and it is determined, say, by the Artin–Schreier subextension corresponding to $\tilde{a}$.

To determine the ramification, we consider the Artin–Schreier extension of the complete discrete valuation field $k((\varpi))((t))$ (resp. $k((\varpi))((\zeta_3^{\alpha} t - \mu))$ for $\alpha \in \{1, 2, 3\}$) given by $\tilde{a}$. Since t is a unit in $k((\varpi))[[\zeta_3^{\alpha} t - \mu]]$ for any α and $\zeta_3^{\alpha} t - \mu$ is a unit in $k((\varpi))[[t]]$ and in $k((\varpi))[[\zeta_3^{\alpha'} t - \mu]]$ for any $\alpha' \neq \alpha$ in $\{1, 2, 3\}$, the degree of the pole of $\tilde{a}$ with respect to t (resp. $\zeta_3^{\alpha} t - \mu$) is $\nu - 6$ (resp. 2). Since $\nu - 6$ is odd, we have that the Artin–Schreier extension of $k((\varpi))((t))$ given by $\tilde{a}$ ramifies and has ramification break $= \nu - 6$. To calculate the ramification break for the corresponding extension of $k((\varpi))((\zeta_3^{\alpha} t - \mu))$, we assume $\alpha = 3$ for simplicity and we write $\tilde{a}$ as a Laurent series in $(t - \mu)$. Note that $\tilde{a} = t^{-1}(t - \mu)^{-2} x^2$ for some $x \in k((\varpi))[[t - \mu]]^{\times}$, and that

$$t^{-1} = \mu^{-1} + \mu^{-2}(t - \mu) + \text{higher order terms in } (t - \mu).$$

So

$$\tilde{a} = c\mu^{-1}(t - \mu)^{-2} + c\mu^{-2}(t - \mu)^{-1} + \theta,$$

where $\theta \in k((\varpi))[[t - \mu]]$ and $c \in k((\varpi))$ is the "constant" term of x^2 (in fact, it is easy to see that $c \in k((\mu^2)) = k((\varpi^4)))$. Let $b = \sqrt{c\mu^{-1}}(t - \mu)^{-1}$. After replacing $\tilde{a}$ with $\tilde{a} + b^2 - b$, which does not change the Artin–Schreier extension, we see that $\tilde{a}$ has a simple pole (since $c \neq \mu^3$, the principal part does not vanish). So this extension ramifies with ramification break $= 1$. This shows property (3).

For property (2), it suffices by [Green and Matignon 1998, I, Theorem 3.4] to show that the total degree of the different of $\mathscr{T}/\mathscr{R}$ is equal to the degree of the different of N/K. Clearly, we may replace $\mathscr{R}$ by $\mathscr{S}$ and K by L. Call these degrees $\delta_{\mathscr{T}/\mathscr{S}}$ and $\delta_{N/L}$, respectively.

Since the ramification break of M/L is ν, and N/L is the compositum of Galois conjugates of M/L, we have that N/L has ν as its single ramification break in the upper numbering, and all nontrivial higher ramification groups of N/L have order 4. Using Serre's different formula [Serre 1968, IV, Proposition 4], we obtain $\delta_{N/L} = 3(\nu+1)$.

For $\delta_{\mathcal{T}/\mathcal{S}}$, we add up the contributions from the different branched ideals separately. For the ideal (t), arguing as in the previous paragraph, we have a $\mathbb{Z}/2\times\mathbb{Z}/2$-extension with single ramification break $= \nu-6$. This gives a contribution of $3(\nu-5)$ to $\delta_{\mathcal{T}/\mathcal{S}}$. For each of the branched ideals $(\zeta^\alpha t-\mu)$ $(\alpha\in\{1,2,3\})$, we have ramification group $\mathbb{Z}/2\times\mathbb{Z}/2$ with ramification break $= 1$. Using Serre's different formula again, we get a contribution of $3\cdot 3\cdot 2 = 18$ to $\delta_{\mathcal{T}/\mathcal{S}}$. Thus $\delta_{\mathcal{T}/\mathcal{S}} = 3(\nu-5)+18 = \delta_{N/L}$, and we are done. □

We omit the proof of the next proposition, which follows from Proposition 3.2 exactly as Theorem 3.6 follows from Key Lemma 3.2 in [Pop 2014].

Proposition 3.3. *Let $Y \to W$ be a branched A_4-cover of projective smooth k-curves. Suppose that the local inertia at each totally ramified point is an extension $k[[z]]/k[[s]]$ having* break $\leq \nu$ *and given rise to by an Artin–Schreier generator in standard form divisible by t^{-6} in $k[t^{-1}]$. Set $\mathcal{W} = W\times_k k[[\varpi]]$. Then there is an A_4-cover of projective smooth $k[[\varpi]]$-curves $\mathcal{Y}\to\mathcal{W}$ with special fiber $Y\to W$ such that the totally ramified points on the generic fiber $\mathcal{Y}_\eta \to \mathcal{W}_\eta$ have* break $\leq \nu-6$.

Before we prove Proposition 3.1, we recall *Harbater–Katz–Gabber* covers (or HKG-*covers*) from [Katz 1986]. Let $G\cong P\rtimes\mathbb{Z}/m$, with P a p-group and $p\nmid m$. If $k[[z]]/k[[s]]$ is a local G-extension, then the associated HKG-cover is the unique branched G-cover $X\to\mathbb{P}^1_k$ tamely ramified of index m above $s=\infty$ and totally ramified above $s=0$ (where s is a coordinate on $\mathbb{P}^1_k$), such that the formal completion of $X\to\mathbb{P}^1_k$ above 0 yields $k[[z]]/k[[s]]$.

Proof of Proposition 3.1. The proof is essentially the same as the proof of [Obus 2015, Proposition 1.11], which itself is adapted from [Pop 2014]. We include it here for completeness.

Let $Y\to W=\mathbb{P}^1$ be the Harbater–Katz–Gabber cover associated to $k[[z]]/k[[s]]$, let $\mathcal{Y}\to\mathcal{W}$ be the A_4-cover over $k[[\varpi]]$ guaranteed by Proposition 3.3, let $\overline{\mathcal{Y}}\to\overline{\mathcal{W}}$ be its base change to the integral closure of $k[[\varpi]]$ in $\overline{k((\varpi))}$, and let $\overline{\mathcal{Y}}_\eta\to\overline{\mathcal{W}}_\eta$ be the generic fiber of $\mathcal{Y}\to\mathcal{W}$. Recall that we assume that *every* local A_4-extension with break $\leq\nu-6$ lifts to characteristic zero. Furthermore, by [Pagot 2002] and the theory of tame ramification, every abelian extension of $k[[s]]$ (and thus of $\overline{k((\varpi))}((s))$) with Galois group a proper subgroup of A_4 lifts to characteristic zero. So the local-global principle tells us that $\overline{\mathcal{Y}}_\eta\to\overline{\mathcal{W}}_\eta$ lifts to a cover $\mathcal{Y}_{\mathbb{O}_1}\to\mathcal{W}_{\mathbb{O}_1}$ over some characteristic zero complete discrete valuation ring $\mathbb{O}_1$ with residue field $\overline{k((\varpi))}$. Then, [Pop 2014, Lemma 4.3] shows that we can "glue" the covers $\overline{\mathcal{Y}}\to\overline{\mathcal{W}}$ and $\mathcal{Y}_{\mathbb{O}_1}\to\mathcal{W}_{\mathbb{O}_1}$ along the generic fiber of the former and the special fiber of the latter, in order to get a cover defined over a *rank two* characteristic zero valuation ring $\mathbb{O}$ with residue field k lifting $Y\to W$ (compare [Pop 2014, p. 319, second paragraph]). Note that this process works starting with any A_4-extension of $k[\![s]\!]$ with break $= \nu$, and that such extensions can be parametrized by some affine space $\mathbb{A}^N$ (with one coordinate corresponding to each possible coefficient in an entry of an Artin–Schreier generator in standard form).

To conclude, we remark that [Pop 2014, Proposition 4.7] and its setup carry through exactly in our situation, with our $\mathbb{A}^N$ playing the role of $\mathbb{A}^{|\iota|}$ in [Pop 2014]. Indeed, we have that the analog of Σ_ι in that proposition contains all closed points, by the paragraph above. Thus we can in fact lift $Y \to W$ over a *discrete* characteristic zero valuation ring. Applying the easy direction of the local-global principle, we obtain a lift of $k[\![z]\!]/k[\![s]\!]$. This concludes the proof of Proposition 3.1. □

4. The form of a lift

We start by reviewing lifts of $\mathbb{Z}/2$-extensions of $k[\![t]\!]$. The following lemma is well-known, but difficult to cite directly from the literature. We provide a proof.

Lemma 4.1. *Let $k((u))/k((t))$ be a $\mathbb{Z}/2$-extension with Artin–Schreier generator $a \in t^{-1}k[t^{-1}]$ in standard form and* ramification break $= \nu$. *Let A be a lift of a to $T^{-1}R[T^{-1}]$ of degree ν. If $\Phi \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$ has degree ν or $\nu + 1$ and there exists $H \in 1 + T^{-1}\mathfrak{m}[T^{-1}]$ such that*

$$\Phi = H^2 + 4A + o(4),$$

then the normalization of $R[\![T]\!]$ in $M := \mathrm{Frac}(R[\![T]\!])[\sqrt{\Phi}]$ is a lift of $k[\![u]\!]/k[\![t]\!]$ to characteristic zero. Furthermore, Φ has simple roots.

Proof. The extension $k((u))/k((t))$ is given by adjoining an element y such that $y^2 - y = a$. Making a substitution $\sqrt{\Phi} = H + 2Y$, the expression for Φ given in the lemma yields

$$H^2 + 4HY + 4Y^2 = H^2 + 4A + o(4),$$

or $Y^2 - Y = A + o(1)$. Thus we see that the normalization of $R[\![T]\!]_{(\pi)}$ in M gives $k((u))/k((t))$ upon reduction modulo π. By Serre's different formula [Serre 1968, IV, Proposition 4], the degree of the different of $k[\![u]\!]/k[\![t]\!]$ is $\nu + 1$. On the other hand, the normalization of $R[\![T]\!] \otimes_R \mathrm{Frac}(R)$ in M is branched at at most $\nu + 1$ maximal ideals, corresponding to the roots of Φ and also 0 if Φ has degree ν. Since this is a tamely ramified $\mathbb{Z}/2$-extension, the degree of its different is at most $\nu + 1$. By [Green and Matignon 1998, I, 3.4], the degree of the different is exactly $\nu + 1$ and the normalization of $R[\![T]\!]$ in M is a lift of $k[\![u]\!]/k[\![t]\!]$. This also shows that the roots of Φ are all simple. □

For Proposition 4.2 below, recall that $s = t^3$ and $S = T^3$.

Proposition 4.2. *Let $k[[z]]/k[[s]]$ be a local A_4-extension with* break $= \nu$ *given rise to by $a \in t^{-1}k[t^{-1}]$ in standard form. If $F(T^{-1})$ and $H(T^{-1})$ are in $1+T^{-1}\mathfrak{m}[T^{-1}]$ such that F has degree $(\nu+1)/2$ and*

$$F(\zeta_3 T^{-1})F(\zeta_3^2 T^{-1}) = H^2 + 4A + o(4),$$

where A is a lift of a to $T^{-1}R[T^{-1}]$ of degree ν, then the normalization of $R[[S]]$ in the A_4-extension of $\mathrm{Frac}(R[[S]])$ *given rise to by $F(\zeta_3 T^{-1})F(\zeta_3^2 T^{-1})$ is a lift of $k[[z]]/k[[s]]$ to characteristic zero.*

Proof. Let the local $\mathbb{Z}/2$-extension $k[[u]]/k[[t]]$ be given by normalizing $k[[t]]$ in the Artin–Schreier $\mathbb{Z}/2$-extension of $k((t))$ given by a. Let $L = \mathrm{Frac}(R[[T]])$. By Lemma 4.1, normalizing $R[[T]]$ in the degree 2 Kummer extension M/L given by some polynomial $\Phi \in 1+T^{-1}\mathfrak{m}[T^{-1}]$ of degree $\nu+1$ in T^{-1} such that $\Phi = H^2+4A+o(4)$ with A as in the proposition gives a lift of $k[[u]]/k[[t]]$ to characteristic zero, and such a Φ has simple roots.

Let σ generate $\mathrm{Gal}(L/\mathrm{Frac}(R[[S]]))$ and, by abuse of notation, $\mathrm{Gal}(k((t))/k((s)))$. Write $\Phi = F(\zeta_3 T^{-1})F(\zeta_3^2 T^{-1})$ for some polynomial $F \in 1+T^{-1}\mathfrak{m}[T^{-1}]$ of degree $(\nu+1)/2$ as in the proposition. Then Φ has simple roots, and thus $F(T^{-1})$, $F(\zeta_3 T^{-1})$, and $F(\zeta_3^2 T^{-1})$ have pairwise disjoint simple roots. Consequently, the $\mathbb{F}_2$-subspace of $L^\times/(L^\times)^2$ generated by Φ, $\sigma(\Phi)$, and $\sigma^2(\Phi)$ has dimension 2. By Proposition 2.5, this is equivalent to the Galois closure N (over $\mathrm{Frac}(R[[S]])$) of M having Galois group A_4.

Let $k((u'))/k((t))$ be the Artin–Schreier extension given by $\sigma(a)$. Clearly, the normalization of $R[[T]]$ in $\mathrm{Frac}(R[[T]])[\sqrt{\sigma(\Phi)}]$ is a lift of $k[[u']]/k[[t]]$. Note that $k[[z]]$ is the normalization of $k[[t]]$ in the compositum of $k((u))$ and $k((u'))$. Analogously, $N := \mathrm{Frac}(R[[T]])(\sqrt{\Phi}, \sqrt{\sigma(\Phi)})$ is the A_4-extension given rise to by Φ. Now, Φ and $\sigma(\Phi)$ have exactly $(\nu+1)/2$ zeroes in common. Thus [Green and Matignon 1998, I, Theorem 5.1] shows that the normalization of $R[[T]]$ in N is a lift of the Klein four extension $k[[z]]/k[[t]]$ (and is isomorphic to $R[[Z]]/R[[T]]$ for Z reducing to z). We conclude that $R[[Z]]/R[[S]]$ is a lift of $k[[z]]/k[[s]]$. □

5. Proof of Theorem 1.2

In this section, let $k[[z]]/k[[s]]$ be a local A_4-extension given rise to by $a \in t^{-1}k[t^{-1}]$ in standard form. Recall that $\deg(a) = \nu$, where ν is the break in $k[[z]]/k[[s]]$. We will prove that $k[[z]]/k[[s]]$ lifts to characteristic zero by strong induction on ν.

Proposition 5.1. *If $\nu = 1$, then $k[[z]]/k[[s]]$ lifts to characteristic zero.*

Proof. Since $\nu = 1$, we have $a = \bar{c}_1 t^{-1}$, with $\bar{c}_1 \in k$. By Proposition 4.2, it suffices to find $F(T^{-1})$ and $H(T^{-1})$ in $1+T^{-1}R[T^{-1}]$ such that F has degree 1 and

$$F(\zeta_3 T^{-1})F(\zeta_3^2 T^{-1}) = H^2 + 4c_1 T^{-1} + o(4),$$

where c_1 is a lift of $\bar{c}_1$ to R. This is done by taking $H=1$ and $F=1-4c_1T^{-1}$. □

Proposition 5.2. *If $\nu=5$, then $k[\![z]\!]/k[\![s]\!]$ lifts to characteristic zero.*

Proof. Since $\nu=5$, we have $a=\bar{c}_1t^{-1}+\bar{c}_5t^{-5}$, with $\bar{c}_1, \bar{c}_5 \in k$. By Proposition 4.2, it suffices to find $F(T^{-1})$ and $H(T^{-1})$ in $1+T^{-1}R[T^{-1}]$ such that F has degree 3 and

$$F(\zeta_3T^{-1})F(\zeta_3^2T^{-1}) = H^2+4c_1T^{-1}+4c_5T^{-5}+o(4),$$

where each c_i is a lift of $\bar{c}_i$ to R.

Let $b\in R$ be any element such that $v(b)=\frac{2}{5}$. Write

$$F(T^{-1}) = 1+a_1T^{-1}+a_2T^{-2}+a_3T^{-3},$$

where

$$a_1=-2b-4c_1, \quad a_2=b^2, \quad a_3=-4c_5/b^2.$$

Note that $v(a_1)=\frac{7}{5}$, $v(a_2)=\frac{4}{5}$, and $v(a_3)=\frac{6}{5}$. Then

$$\begin{aligned} F(\zeta_3T^{-1})F(\zeta_3^2T^{-1}) &= 1-a_1T^{-1}-a_2T^{-2}+a_2^2T^{-4}-a_2a_3T^{-5}+o(4) \\ &= 1+(4c_1+2b)T^{-1}-b^2T^{-2}+b^4T^{-4}+4c_5T^{-5}+o(4) \\ &= (1+bT^{-1}+b^2T^{-2})^2+4c_1T^{-1}+4c_5T^{-5}+o(4). \end{aligned}$$

We conclude by taking $H=1+bT^{-1}+b^2T^{-2}$. □

Proof of Theorem 1.2. We use strong induction on the break $=\nu$ of $k[\![z]\!]/k[\![s]\!]$, which only takes values congruent to 1 or 5 modulo 6 (Proposition 2.4). The base cases $\nu=1$ and $\nu=5$ are Propositions 5.1 and 5.2, respectively. The induction step is Proposition 3.1. □

Remark 5.3. Florian Pop has informed the author of his own proof, which uses much the same method. In place of the deformation in Proposition 3.2, he uses one for which it is slightly more difficult to verify that it yields an A_4-extension, but which immediately reduces Theorem 1.2 to the case $\nu=1$ (eliminating the need for Proposition 5.2).

Question 5.4. Given k, does there exist a particular DVR R in characteristic zero such that all local A_4-extensions over k lift over R? This is known for local G-extensions in characteristic p where G is cyclic with $v_p(|G|)\le 2$ (see [Green and Matignon 1998], where it is shown that $W(k)[\zeta_{p^2}]$ works). Since our proof is rather inexplicit, this question remains open for A_4.

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andrewobus@gmail.com *Department of Mathematics, University of Virginia, 141 Cabell Drive, Charlottesville, VA 22904, United States*

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Syntomic cohomology and p-adic regulators for varieties over p-adic fields

Jan Nekovář and Wiesława Nizioł

With appendices by Laurent Berger and Frédéric Déglise

We show that the logarithmic version of the syntomic cohomology of Fontaine and Messing for semistable varieties over p-adic rings extends uniquely to a cohomology theory for varieties over p-adic fields that satisfies h-descent. This new cohomology — syntomic cohomology — is a Bloch–Ogus cohomology theory, admits a period map to étale cohomology, and has a syntomic descent spectral sequence (from an algebraic closure of the given field to the field itself) that is compatible with the Hochschild–Serre spectral sequence on the étale side and is related to the Bloch–Kato exponential map. In relative dimension zero we recover the potentially semistable Selmer groups and, as an application, we prove that Soulé's étale regulators land in the potentially semistable Selmer groups.

Our construction of syntomic cohomology is based on new ideas and techniques developed by Beilinson and Bhatt in their recent work on p-adic comparison theorems.

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1. Introduction

In this article we define syntomic cohomology for varieties over p-adic fields, relate it to the Bloch–Kato exponential map, and use it to study the images of Soulé's étale regulators. Contrary to all the previous constructions of syntomic cohomology (see below for a brief review), we do not restrict ourselves to varieties coming with a nice model over the integers. Hence our syntomic regulators make no integrality assumptions on the K-theory classes in the domain.

1A. ***Statement of the main result.*** Recall that, for varieties proper and smooth over a p-adic ring of mixed characteristic, syntomic cohomology (or its nonproper variant: syntomic-étale cohomology) was introduced by Fontaine and Messing [1987] in their proof of the crystalline comparison theorem as a natural bridge between crystalline cohomology and étale cohomology. It was generalized to log-syntomic cohomology for semistable varieties by Kato [1994]. For a log-smooth scheme $\mathscr{X}$ over a complete discrete valuation ring V of mixed characteristic $(0, p)$ and a perfect residue field, and for any $r \geq 0$, rational log-syntomic cohomology of $\mathscr{X}$ can be defined as the "filtered Frobenius eigenspace" in log-crystalline cohomology, i.e., as the mapping fiber

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, r) := \mathrm{Cone}\big(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}, \mathscr{J}^{[r]}) \xrightarrow{1-\varphi_r} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\big)[-1], \tag{1}$$

where $\mathrm{R}\Gamma_{\mathrm{cr}}(\,\cdot\,, \mathscr{J}^{[r]})$ denotes the absolute rational log-crystalline cohomology (i.e., over $\mathbb{Z}_p$) of the r-th Hodge filtration sheaf $\mathscr{J}^{[r]}$ and φ_r is the crystalline Frobenius divided by p^r. This definition suggested that the log-syntomic cohomology could be the sought-for p-adic analog of Deligne–Beilinson cohomology. Recall that, for a complex manifold X, the latter can be defined as the cohomology $\mathrm{R}\Gamma(X, \mathbb{Z}(r)_{\mathscr{D}})$ of Deligne complex $\mathbb{Z}(r)_{\mathscr{D}}$:

$$0 \to \mathbb{Z}(r) \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{r-1}_X \to 0.$$

And, indeed, since its introduction, log-syntomic cohomology has been used with some success in the study of special values of p-adic L-functions and in formulating p-adic Beilinson conjectures (see [Besser et al. 2009] for a review).

The syntomic cohomology theory with $\mathbb{Q}_p$-coefficients $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ $(r \geq 0)$ for arbitrary varieties — more generally, for arbitrary essentially finite diagrams of varieties — over the p-adic field K (the fraction field of V) that we construct in this article is a generalization of Fontaine–Messing(–Kato) log-syntomic cohomology. That is, for a semistable scheme[1] $\mathscr{X}$ over V, we have $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, r) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$, where X is the largest subvariety of $\mathscr{X}_K$ with trivial log-structure. An analogous theory $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, r)$ $(r \geq 0)$ exists for (diagrams of) varieties over $\overline{K}$, where $\overline{K}$ is an algebraic closure of K.

[1] Throughout the Introduction, the divisors at infinity of semistable schemes have no multiplicities.

Our main result can be stated as follows.

Theorem A. *For any variety X over K, there is a canonical graded commutative dg $\mathbb{Q}_p$-algebra $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,*)$ such that*:

(1) *It is the unique extension of log-syntomic cohomology to varieties over K that satisfies h-descent*; *i.e., for any hypercovering $\pi : Y_\bullet \to X$ in the h-topology, we have a quasi-isomorphism*

$$\pi^* : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h,*) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{syn}}(Y_{\bullet,h},*).$$

(2) *It is a Bloch–Ogus cohomology theory* [1974].

(3) *For $X = \mathrm{Spec}(K)$, we have*

$$H^*_{\mathrm{syn}}(X_h,r) \simeq H^*_{\mathrm{st}}(G_K,\mathbb{Q}_p(r)),$$

where $H^i_{\mathrm{st}}(G_K,-)$ denotes the Ext-*group* $\mathrm{Ext}^i(\mathbb{Q}_p,-)$ *in the category of* (*potentially*) *semistable representations of $G_K = \mathrm{Gal}(\overline{K}/K)$.*

(4) *There are functorial syntomic period morphisms*

$$\rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r) \to \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p(r)),$$
$$\rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h},r) \to \mathrm{R}\Gamma(X_{\overline{K},\text{ét}},\mathbb{Q}_p(r))$$

compatible with products which induce quasi-isomorphisms

$$\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p(r)),$$
$$\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h},r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\overline{K},\text{ét}},\mathbb{Q}_p(r)).$$

(5) *The Hochschild–Serre spectral sequence for étale cohomology*

$${}^{\text{ét}}E_2^{i,j} = H^i(G_K, H^j(X_{\overline{K},\text{ét}},\mathbb{Q}_p(r))) \Rightarrow H^{i+j}(X_{\text{ét}},\mathbb{Q}_p(r))$$

has a syntomic analog

$${}^{\mathrm{syn}}E_2^{i,j} = H^i_{\mathrm{st}}(G_K, H^j(X_{\overline{K},\text{ét}},\mathbb{Q}_p(r))) \Rightarrow H^{i+j}_{\mathrm{syn}}(X_h,r).$$

(6) *There is a canonical morphism of spectral sequences ${}^{\mathrm{syn}}E_t \to {}^{\text{ét}}E_t$ compatible with the syntomic period map.*

(7) *There are syntomic Chern classes*

$$c^{\mathrm{syn}}_{i,j} : K_j(X) \to H^{2i-j}_{\mathrm{syn}}(X_h,i)$$

compatible with étale Chern classes via the syntomic period map.

As is shown in [Déglise and Nizioł 2015], syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,*)$ can be interpreted as an absolute p-adic Hodge cohomology. That is, it is a derived Hom in the category of admissible (φ, N, G_K)-modules between the trivial module and a complex of such modules canonically associated to a variety. Alternatively, it is a derived Hom in the category of potentially semistable representations between the trivial representation and a complex of such representations canonically associated to a variety. A particularly simple construction of such a complex, using Beilinson's basic lemma, was proposed by Beilinson (and is presented in [Déglise and Nizioł 2015]). The category of modules over the syntomic cohomology algebra $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, *)$ (taken in a motivic sense) yields a category of p-adic Galois representations that better approximates the category of geometric representations than the category of potentially semistable representations [Déglise and Nizioł 2015]. For further applications of the syntomic cohomology algebra, we refer the interested reader to [loc. cit.].

Similarly, as is shown in [Nizioł 2016a], geometric syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, *)$ is a derived Hom in the category of effective φ-gauges (with one paw) [Fargues 2015] between the trivial gauge and a complex of such gauges canonically associated to a variety. In particular, geometric syntomic cohomology group is a finite-dimensional Banach–Colmez space [Colmez 2002], and hence has a very rigid structure.

The syntomic descent spectral sequence and its compatibility with the Hochschild–Serre spectral sequence in étale cohomology imply the following proposition.

Proposition 1.1. *Let $i \geq 0$. The composition*

$$H^{i-1}_{\mathrm{dR}}(X)/F^r \xrightarrow{\partial} H^i_{\mathrm{syn}}(X_h, r) \xrightarrow{\rho_{\mathrm{syn}}} H^i_{\text{ét}}(X, \mathbb{Q}_p(r)) \longrightarrow H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))$$

is the zero map. The induced (from the syntomic descent spectral sequence) map

$$H^{i-1}_{\mathrm{dR}}(X)/F^r \to H^1(G_K, H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)))$$

is equal to the Bloch–Kato exponential associated with the Galois representation $H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))$.

This yields a comparison between p-adic étale regulators, syntomic regulators, and the Bloch–Kato exponential (which was proved in the good reduction case in [Nekovář 1998] and [Nizioł 2001, Theorem 5.2][2]) that is of fundamental importance for the theory of special values of L-functions, both complex valued and p-adic. The point is that syntomic regulators can be thought of as an abstract p-adic integration theory. The comparison results stated above then relate certain p-adic integrals to the values of the p-adic étale regulator via the Bloch–Kato exponential map. A modification of syntomic cohomology developed in [Besser 2000] in the good reduction case (resp. in [Besser et al. 2016] — using the techniques of the present article — in the case of arbitrary reduction) can be used to perform explicit computations. For example, the formulas from [Besser et al. 2016, §3] were applied to a calculation of certain p-adic regulators in [Bertolini et al. 2015; Darmon and Rotger 2016].

[2] The Bloch–Kato exponential is called l there.

1B. ***Construction of syntomic cohomology.*** We will now sketch the proof of Theorem A. Recall first that a little bit after log-syntomic cohomology had appeared on the scene, Selmer groups of Galois representations — describing extensions in certain categories of Galois representations — were introduced by Bloch and Kato [1990] and linked to special values of L-functions. And a syntomic cohomology (in the good reduction case), a priori different than that of Fontaine and Messing, was defined in [Nizioł 2001] and by Besser [2000] as a higher-dimensional analog of the complexes computing these groups. The guiding idea here was that just as Selmer groups classify extensions in certain categories of "geometric" Galois representations, their higher-dimensional analogs — syntomic cohomology groups — should classify extensions in a category of "p-adic motivic sheaves". This was shown to be the case for H^1 by Bannai [2002], who has also shown that Besser's (rigid) syntomic cohomology is a p-adic analog of Beilinson's absolute Hodge cohomology [1986].

Complexes computing the semistable and potentially semistable Selmer groups were introduced in [Nekovář 1993; Fontaine and Perrin-Riou 1994]. For a semistable scheme $\mathscr{X}$ over V, their higher-dimensional analog can be written as the homotopy limit[3]

$$\mathrm{R}\Gamma'_{\mathrm{syn}}(\mathscr{X},r) := \left[\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) & \xrightarrow{(1-\varphi_r,\iota_{\mathrm{dR}})} & \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)\oplus \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)/F^r \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \end{array}\right], \qquad (2)$$

where $\mathscr{X}_0$ is the special fiber of $\mathscr{X}$, $\mathrm{R}\Gamma_{\mathrm{HK}}(\,\cdot\,)$ is the Hyodo–Kato cohomology, N denotes the Hyodo–Kato monodromy, and $\mathrm{R}\Gamma_{\mathrm{dR}}(\,\cdot\,)$ is the logarithmic de Rham cohomology. The map ι_{dR} is the Hyodo–Kato morphism that induces a quasi-isomorphism $\iota_{\mathrm{dR}}:\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)\otimes_{K_0}K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)$ for K_0 — the fraction field of Witt vectors of the residue field of V.

Using Dwork's trick, we prove (see Proposition 3.8) that the two definitions of log-syntomic cohomology are the same, i.e., that there is a quasi-isomorphism

$$\alpha_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},r) \xrightarrow{\sim} \mathrm{R}\Gamma'_{\mathrm{syn}}(\mathscr{X},r).$$

It follows that log-syntomic cohomology groups vanish in degrees strictly higher than $2\dim X_K+2$ and that, if $\mathscr{X}=\mathrm{Spec}(V)$, then $H^i\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},r)\simeq H^i_{\mathrm{st}}(G_K,\mathbb{Q}_p(r))$.

The syntomic cohomology for varieties over p-adic fields that we introduce in this article is a generalization of the log-syntomic cohomology of Fontaine and Messing. Observe that it is clear how one can try to use log-syntomic cohomology

[3] See Section 1E for an explanation of the notation we use for certain homotopy limits.

to define syntomic cohomology for varieties over fields that satisfies h-descent. Namely, for a variety X over K, consider the h-topology of X and recall that (using alterations) one can show that it has a basis consisting of semistable models over finite extensions of V [Beilinson 2012]. By h-sheafifying the complexes $Y \mapsto \mathrm{R}\Gamma_{\mathrm{syn}}(Y, r)$ (for a semistable model Y) we get syntomic complexes $\mathscr{S}(r)$. We define the (*arithmetic*) syntomic cohomology as

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) := \mathrm{R}\Gamma(X_h, \mathscr{S}(r)).$$

A priori it is not clear that the so-defined syntomic cohomology behaves well: the finite ramified field extensions introduced by alterations are in general a problem for log-crystalline cohomology. For example, the related complexes $\mathrm{R}\Gamma_{\mathrm{cr}}(X_h, \mathscr{J}^{[r]})$ are huge. However, taking Frobenius eigenspaces cuts off the "noise" and the resulting syntomic complexes do indeed behave well. To get an idea why this is the case, h-sheafify the complexes $Y \mapsto \mathrm{R}\Gamma'_{\mathrm{syn}}(Y, r)$ and imagine that you can sheafify the maps α_{syn} as well. We get sheaves $\mathscr{S}'(r)$ and quasi-isomorphisms $\alpha_{\mathrm{syn}} : \mathscr{S}(r) \xrightarrow{\sim} \mathscr{S}'(r)$. Setting $\mathrm{R}\Gamma'_{\mathrm{syn}}(X_h, r) := \mathrm{R}\Gamma(X_h, \mathscr{S}'(r))$, we obtain the quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \simeq \mathrm{R}\Gamma'_{\mathrm{syn}}(X_h, r) \simeq \left[\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) & \xrightarrow{(1-\varphi_r, \iota_{\mathrm{dR}})} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) \oplus \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) \end{array}\right], \tag{3}$$

where $\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)$ denotes the Hyodo–Kato cohomology (defined as h-cohomology of the presheaf: $Y \mapsto \mathrm{R}\Gamma_{\mathrm{HK}}(Y_0)$) and $\mathrm{R}\Gamma_{\mathrm{dR}}(\,\cdot\,)$ is Deligne's de Rham cohomology [1974]. The Hyodo–Kato map ι_{dR} is the h-sheafification of the logarithmic Hyodo–Kato map. It is well-known that Deligne's de Rham cohomology groups are finite-rank K-vector spaces; it turns out that the Hyodo–Kato cohomology groups are finite-rank K_0-vector spaces: we have a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{HK}}(X_h) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\bar{K},h})^{G_K}$, and the geometric Hyodo–Kato groups $H^*\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\bar{K},h})$ are finite-rank K_0^{nr}-vector spaces, where K_0^{nr} is the maximal unramified extension of K_0 (see (4) below).

It follows that syntomic cohomology groups vanish in degrees higher than $2\dim X_K + 2$ and that syntomic cohomology is, in fact, a generalization of the classical log-syntomic cohomology; i.e., for a semistable scheme $\mathscr{X}$ over V, we have $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, r) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$, where X is the largest subvariety of $\mathscr{X}_K$ with trivial log-structure. This follows from the quasi-isomorphism α_{syn}: logarithmic Hyodo–Kato and de Rham cohomologies (over a fixed base) satisfy proper descent and the finite field extensions that appear as the "noise" in alterations do not destroy anything since logarithmic Hyodo–Kato and de Rham cohomologies satisfy finite Galois descent.

Alas, we were not able to sheafify the map α_{syn}. The reason for that is that the construction of α_{syn} uses a twist by a high power of Frobenius — a power depending on the field K. And alterations are going to introduce a finite extension of K — hence a need for higher and higher powers of Frobenius. So instead we construct directly the map

$$\alpha_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \to \mathrm{R}\Gamma'_{\mathrm{syn}}(X_h, r).$$

To do that, we show first that the syntomic cohomological dimension of X is finite. Then we take a semistable h-hypercovering of X, truncate it at an appropriate level, extend the base field K to K', and base-change everything to K'. There we can work with one field and use the map α_{syn} defined earlier. Finally, we show that we can descend.

1C. ***Syntomic period maps.*** We pass now to the construction of the period maps from syntomic to étale cohomology that appear in Theorem A. They are easier to define over $\overline{K}$, i.e., from the *geometric* syntomic cohomology. In this setting, things go smoother with h-sheafification since going all the way up to $\overline{K}$ before completing kills a lot of "noise" in log-crystalline cohomology. More precisely, for a semistable scheme $\mathscr{X}$ over V, we have the canonical quasi-isomorphisms [Beilinson 2013]

$$\iota_{\mathrm{cr}} : \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\overline{V}})^{\tau}_{B^+_{\mathrm{cr}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\overline{V}}), \quad \iota_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\overline{V}})^{\tau}_{\overline{K}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_{\overline{K}}), \tag{4}$$

where $\overline{V}$ is the integral closure of V in $\overline{K}$, B^+_{cr} is the crystalline period ring, and τ denotes certain twist. These quasi-isomorphisms h-sheafify well: for a variety X over K, they induce the quasi-isomorphisms [Beilinson 2013]

$$\iota_{\mathrm{cr}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K},h})^{\tau}_{B^+_{\mathrm{cr}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_{\overline{K},h}), \quad \iota_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K},h})^{\tau}_{\overline{K}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}}), \tag{5}$$

where the terms have obvious meaning. Since Deligne's de Rham cohomology has proper descent (by definition), it follows that h-crystalline cohomology behaves well. That is, if we define crystalline sheaves $\mathscr{J}^{[r]}_{\mathrm{cr}}$ and $\mathscr{A}_{\mathrm{cr}}$ on $X_{\overline{K},h}$ by h-sheafifying the complexes $Y \mapsto \mathrm{R}\Gamma_{\mathrm{cr}}(Y, \mathscr{J}^{[r]})$ and $Y \mapsto \mathrm{R}\Gamma_{\mathrm{cr}}(Y)$, respectively, for Y which are a base change to $\overline{V}$ of a semistable scheme over a finite extension of V (such schemes Y form a basis of $X_{\overline{K},h}$) then the complexes $\mathrm{R}\Gamma(X_{\overline{K},h}, \mathscr{J}^{[r]})$ and $\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\overline{V},h}) := \mathrm{R}\Gamma(X_{\overline{K},h}, \mathscr{A}_{\mathrm{cr}})$ generalize log-crystalline cohomology (in the sense described above) and the latter one is a perfect complex of B^+_{cr}-modules.

We obtain syntomic complexes $\mathscr{S}(r)$ on $X_{\overline{K},h}$ by h-sheafifying the complexes $Y \mapsto \mathrm{R}\Gamma_{\mathrm{syn}}(Y, r)$ and (geometric) syntomic cohomology by setting $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, r) :=$

$\mathrm{R}\Gamma(X_{\bar{K},h}, \mathscr{S}(r))$. They fit into an analog of the exact sequence (1) and, by the above, generalize log-syntomic cohomology.

To construct the syntomic period maps

$$\begin{aligned} \rho_{\mathrm{syn}} &: \mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h}, r) \to \mathrm{R}\Gamma(X_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)), \\ \rho_{\mathrm{syn}} &: \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \to \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)), \end{aligned} \tag{6}$$

consider the syntomic complexes $\mathscr{S}_n(r)$: the mod-p^n version of the syntomic complexes $\mathscr{S}(r)$ on $X_{\bar{K},h}$. We have the distinguished triangle

$$\mathscr{S}_n(r) \to \mathscr{J}^{[r]}_{\mathrm{cr},n} \xrightarrow{p^r-\varphi} \mathscr{A}_{\mathrm{cr},n}.$$

Recall that the filtered Poincaré lemma of Beilinson [2013] and Bhatt [2012] yields a quasi-isomorphism $\rho_{\mathrm{cr}} : J^{[r]}_{\mathrm{cr},n} \xrightarrow{\sim} \mathscr{J}^{[r]}_{\mathrm{cr},n}$, where $J^{[r]}_{\mathrm{cr}} \subset A_{\mathrm{cr}}$ is the r-th filtration level of the period ring A_{cr}. Using the fundamental sequence of p-adic Hodge theory,

$$0 \to \mathbb{Z}/p^n(r)' \to J^{\langle r\rangle}_{\mathrm{cr},n} \xrightarrow{1-\varphi_r} A_{\mathrm{cr},n} \to 0,$$

where $\mathbb{Z}/p^n(r)' := (1/(p^a a!)\mathbb{Z}_p(r)) \otimes \mathbb{Z}/p^n$ and a denotes the largest integer $\le r/(p-1)$, we obtain the syntomic period map $\rho_{\mathrm{syn}} : \mathscr{S}_n(r) \to \mathbb{Z}/p^n(r)'$. It is a quasi-isomorphism modulo a universal constant. It induces the geometric syntomic period map in (6), and, by Galois descent, its arithmetic analog.

To study the descent spectral sequences from Theorem A, we need to consider the other version of syntomic cohomology, i.e., the complexes

$$\mathrm{R}\Gamma'_{\mathrm{syn}}(X_{\bar{K},h}, r) := \left[\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\bar{K},h}) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} & \xrightarrow{(1-\varphi_r,\iota_{\mathrm{dR}})} & \begin{array}{c}\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\bar{K},h}) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} \\ \oplus(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}}) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r\end{array} \\ \downarrow N & & \downarrow (N,0) \\ \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\bar{K},h}) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\bar{K},h}) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} \end{array}\right], \tag{7}$$

where B^+_{st} and B^+_{dR} are the semistable and de Rham p-adic period rings, respectively. We deduce a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h}, r) \xrightarrow{\sim} \mathrm{R}\Gamma'_{\mathrm{syn}}(X_{\bar{K},h}, r)$.

Remark 1.2. This quasi-isomorphism yields, for a semistable scheme $\mathscr{X}$ over V, the exact sequence

$$\begin{aligned} \cdots \to H^i_{\mathrm{syn}}(\mathscr{X}_{\bar{K}}, r) &\to (H^i_{\mathrm{HK}}(\mathscr{X})_{\mathbb{Q}} \otimes_{K_0} B^+_{\mathrm{st}})^{\varphi=p^r, N=0} \\ &\to (H^i_{\mathrm{dR}}(\mathscr{X}_K) \otimes_K B^+_{\mathrm{dR}})/F^r \to H^{i+1}_{\mathrm{syn}}(\mathscr{X}_{\bar{K}}, r) \to \cdots. \end{aligned}$$

It is a sequence of finite-dimensional Banach–Colmez Spaces [Colmez 2002] and as such is a key in the proof of the semistable comparison theorem for formal schemes in [Colmez and Niziol 2015].

We also have a syntomic period map

$$\rho'_{\mathrm{syn}} : \mathrm{R}\Gamma'_{\mathrm{syn}}(X_{\bar{K},h}, r) \to \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r)) \tag{8}$$

that is compatible with the map ρ_{syn} via α_{syn}. To describe how it is constructed, recall that the crystalline period map of Beilinson [2013] induces compatible Hyodo–Kato and de Rham period maps

$$\begin{aligned} \rho_{\mathrm{HK}} &: \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\bar{K},h}) \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}^+ \to \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p) \otimes B_{\mathrm{st}}^+, \\ \rho_{\mathrm{dR}} &: \mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \otimes_K B_{\mathrm{dR}}^+ \to \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p) \otimes B_{\mathrm{dR}}^+. \end{aligned} \tag{9}$$

Applying them to the above homotopy limit, removing all the pluses from the period rings, reduces the homotopy limit to the complex

$$\left[\begin{array}{ccc} \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r)) \otimes B_{\mathrm{st}} & \xrightarrow{(1-\varphi_r, \iota_{\mathrm{dR}})} & \begin{array}{c}\mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r)) \otimes B_{\mathrm{st}} \\ \oplus(\mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r)) \otimes B_{\mathrm{dR}})/F^r\end{array} \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r)) \otimes B_{\mathrm{st}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r)) \otimes B_{\mathrm{st}} \end{array}\right]. \tag{10}$$

By the familiar fundamental exact sequence

$$0 \to \mathbb{Q}_p(r) \to B_{\mathrm{st}} \xrightarrow{(N, 1-\varphi_r, \iota)} B_{\mathrm{st}} \oplus B_{\mathrm{st}} \oplus B_{\mathrm{dR}}/F^r \xrightarrow{(1-\varphi_{r-1})-N} B_{\mathrm{st}} \to 0,$$

the above complex is quasi-isomorphic to $\mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r))$. This yields the syntomic period morphism from (8). We like to think of geometric syntomic cohomology as being represented by the complex from (7) and of geometric étale cohomology as represented by the complex (10).

From the above constructions we derive several of the properties mentioned in Theorem A. The quasi-isomorphisms (9) give that

$$\begin{aligned} H^i_{\mathrm{HK}}(X_{\bar{K},h}) &\simeq D_{\mathrm{pst}}(H^i(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r))), \\ H^i_{\mathrm{HK}}(X_h) &\simeq D_{\mathrm{st}}(H^i(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r))), \end{aligned}$$

where D_{pst} and D_{st} are the functors from [Fontaine and Perrin-Riou 1994]. This combined with the diagram (3) immediately yields the spectral sequence ${}^{\mathrm{syn}}E_t$ since the cohomology groups of the total complex of

$$\left[\begin{array}{ccc} H^j_{\mathrm{HK}}(X_h) & \xrightarrow{(1-\varphi_r, \iota_{\mathrm{dR}})} & H^j_{\mathrm{HK}}(X_h) \oplus H^j_{\mathrm{dR}}(X_K)/F^r \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ H^j_{\mathrm{HK}}(X_h) & \xrightarrow{1-\varphi_{r-1}} & H^j_{\mathrm{HK}}(X_h) \end{array}\right]$$

are equal to $H^*_{\mathrm{st}}(G_K, H^j(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r)))$. Moreover, the sequence of natural maps of diagrams (3) $\to$ (7) $\xrightarrow{\rho_{\mathrm{syn}}}$ (10) yields a compatibility of the syntomic descent spectral sequence with the Hochschild–Serre spectral sequence in étale cohomology (via the period maps). We remark that, in the case of proper varieties with semistable reduction, this fact was announced in [Nekovář 2000].

Looking again at the period map ρ_{syn} : (7)$\to$(10) we see that truncating all the complexes at level r will allow us to drop $+$ from the first diagram. Hence we have

$$\rho_{\mathrm{syn}} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r)).$$

To conclude that we have

$$\rho_{\mathrm{syn}} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r))$$

as well, we look at the map of spectral sequences ${}^{\mathrm{syn}}E \to {}^{\text{ét}}E$ and observe that, in the stated ranges of the Hodge–Tate filtration we have $H^*_{\mathrm{st}}(G_K, \cdot\,) = H^*(G_K, \cdot\,)$ (a fact that follows, for example, from the work of Berger [2002]).

1D. ***p-adic regulators.*** As an application of Theorem A, we look at the question of the image of Soulé's étale regulators

$$r^{\text{ét}}_{r,i} : K_{2r-i-1}(X)_0 \to H^1(G_K, H^i(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r))),$$

where $K_{2r-i-1}(X)_0 := \ker\big(c^{\text{ét}}_{r,i+1} : K_{2r-i-1}(X) \to H^{i+1}(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r))\big)$, inside the Galois cohomology group. We prove:

Theorem B. *The regulators $r^{\text{ét}}_{r,i}$ factor through the group $H^1_{\mathrm{st}}(G_K, H^i(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r)))$.*

As we explain in the article, this fact is known to follow from the work of Scholl [1993] on "geometric" extensions associated to K-theory classes. In our approach, this is a simple consequence of good properties of syntomic cohomology and the existence of the syntomic descent spectral sequence. Namely, as can be easily derived from the presentation (3), syntomic cohomology has a projective space theorem and homotopy property,[4] and hence admits Chern classes from higher K-theory. It can be easily shown that they are compatible with the étale Chern classes via the syntomic period maps. The factorization we want in the above theorem follows then from the compatibility of the two descent spectral sequences.

1E. ***Notation and conventions.*** Let V be a complete discrete valuation ring with fraction field K of characteristic 0, with perfect residue field k of characteristic p, and with maximal ideal $\mathfrak{m}_K$. Let v be the valuation on K normalized so that $v(p)=1$. Let $\overline{K}$ be an algebraic closure of K and let $\overline{V}$ denote the integral closure of V in $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of k with fraction field K_0 and denote by K_0^{nr} the maximal unramified extension of K_0. Denote by e_K the absolute ramification

[4] As explained in Appendix B, it follows that it is a Bloch–Ogus cohomology theory.

index of K, i.e., the degree of K over K_0. Set $G_K = \mathrm{Gal}(\overline{K}/K)$ and let I_K denote its inertia subgroup. Let φ be the absolute Frobenius on $W(\bar{k})$. We will denote by V, $V^\times$, and V^0 the scheme $\mathrm{Spec}(V)$ with the trivial, canonical (i.e., associated to the closed point), and ($\mathbb{N} \to V$, $1 \mapsto 0$) log-structure respectively. For a log-scheme X over $\mathscr{O}_K$, denote its reduction mod p^n by X_n and its special fiber by X_0.

Unless otherwise stated, we work in the category of integral quasi-coherent log-schemes. In general, we will not distinguish between simplicial abelian groups and complexes of abelian groups.

Let A be an abelian category with enough projective objects. In this paper A will be the category of abelian groups or $\mathbb{Z}_p$-, $\mathbb{Z}/p^n$-, or $\mathbb{Q}_p$-modules. Unless otherwise stated, we work in the (stable) ∞-category $\mathscr{D}(A)$, i.e., the stable ∞-category whose objects are (left-bounded) chain complexes of projective objects of A. For a readable introduction to such categories, the reader may consult [Groth 2010; Lurie 2016, Chapter 1]. The ∞-derived category is essential to us for two reasons: first, it allows us to work simply with the Beilinson–Hyodo–Kato complexes; second, it supplies functorial homotopy limits.

Many of our constructions will involve sheaves of objects from $\mathscr{D}(A)$. The reader may consult the notes of Illusie [2013] and Zheng [2013] for a brief introduction to the subject and [Lurie 2009; 2016] for a thorough treatment.

We will use a shorthand for certain homotopy limits. Namely, if $f : C \to C'$ is a map in the dg derived category of abelian groups, we set

$$[\, C \xrightarrow{f} C' \,] := \mathrm{holim}(C \to C' \leftarrow 0).$$

We also set

$$\begin{bmatrix} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{bmatrix} := \big[[C_1 \xrightarrow{f} C_2] \to [C_3 \xrightarrow{g} C_4]\big],$$

where the diagram in the brackets is a commutative diagram in the dg derived category.

2. Preliminaries

In this section we will do some preparation. In the first part, we will collect some relevant facts from the literature concerning period rings, derived log de Rham complexes and the h-topology. In the second part, we will prove vanishing results in Galois cohomology and a criterion comparing two spectral sequences that we will need to compare the syntomic descent spectral sequence with the étale Hochschild–Serre spectral sequence.

2A. ***The rings of periods.*** Let us recall briefly the definitions of the rings of periods B_{cr}, B_{dR}, B_{st} of [Fontaine 1994a]. As in 2.2 and 2.3 of that work, let A_{cr} denote Fontaine's ring of crystalline periods. This is a p-adically complete ring such that $A_{\mathrm{cr},n} := A_{\mathrm{cr}}/p^n$ is a universal PD-thickening of $\overline{V}_n$ over $W_n(k)$. Let $J_{\mathrm{cr},n}$ denote its PD-ideal, $A_{\mathrm{cr},n}/J_{\mathrm{cr},n} = \overline{V}_n$. We have

$$A_{\mathrm{cr},n} = H^0_{\mathrm{cr}}(\mathrm{Spec}(\overline{V}_n)/W_n(k)), \quad B^+_{\mathrm{cr}} := A_{\mathrm{cr}}[1/p], \quad B_{\mathrm{cr}} := B^+_{\mathrm{cr}}[t^{-1}],$$

where t is a certain element of B^+_{cr} (see [Fontaine 1994a] for a precise definition of t). The ring B^+_{cr} is a topological K_0-module equipped with a Frobenius φ coming from the crystalline cohomology and a natural G_K-action. We have that $\varphi(t) = pt$ and that G_K acts on t via the cyclotomic character.

Let

$$B^+_{\mathrm{dR}} := \varprojlim_r (\mathbb{Q} \otimes \varprojlim_n A_{\mathrm{cr},n}/J^{[r]}_{\mathrm{cr},n}), \quad B_{\mathrm{dR}} := B^+_{\mathrm{dR}}[t^{-1}].$$

The ring B^+_{dR} has a discrete valuation given by the powers of t. Its quotient field is B_{dR}. We set $F^n B_{\mathrm{dR}} = t^n B^+_{\mathrm{dR}}$. This defines a descending filtration on B_{dR}.

The period ring B_{st} lies between B_{cr} and B_{dR} [Fontaine 1994a, 3.1]. To define it, choose a sequence of elements $s = (s_n)_{n\geq 0}$ of $\overline{V}$ such that $s_0 = p$ and $s^p_{n+1} = s_n$. Fontaine associates to it an element u_s of B^+_{dR} that is transcendental over B^+_{cr}. Let B^+_{st} denote the subring of B_{dR} generated by B^+_{cr} and u_s. It is a polynomial algebra in one variable over B^+_{cr}. The ring B^+_{st} does not depend on the choice of s (because for another sequence $s' = (s'_n)_{n\geq 0}$ we have $u_s - u_{s'} \in \mathbb{Z}_p t \subset B^+_{\mathrm{cr}}$). The action of G_K on B^+_{dR} restricts well to B^+_{st}. The Frobenius φ extends to B^+_{st} by $\varphi(u_s) = pu_s$ and one defines the monodromy operator $N : B^+_{\mathrm{st}} \to B^+_{\mathrm{st}}$ as the unique B^+_{cr}-derivation such that $Nu_s = -1$. We have $N\varphi = p\varphi N$ and the short exact sequence

$$0 \to B^+_{\mathrm{cr}} \to B^+_{\mathrm{st}} \xrightarrow{N} B^+_{\mathrm{st}} \to 0. \tag{11}$$

Let $B_{\mathrm{st}} = B_{\mathrm{cr}}[u_s]$. We denote by ι the injection $\iota : B^+_{\mathrm{st}} \hookrightarrow B^+_{\mathrm{dR}}$. The topology on B_{st} is the one induced by B_{cr} and the inductive topology; the map ι is continuous (though the topology on B_{st} is not the one induced from B_{dR}).

2B. ***Derived log de Rham complex.*** In this subsection we collect a few facts about the relationship between crystalline cohomology and de Rham cohomology.

Let S be a log-PD-scheme on which p is nilpotent. For a log-scheme Z over S, let $\mathrm{L}\Omega^\bullet_{Z/S}$ denote the derived log de Rham complex (see [Beilinson 2012, 3.1] for a review). This is a commutative dg $\mathscr{O}_S$-algebra on $Z_{\text{ét}}$ equipped with a Hodge filtration F^m. There is a natural morphism of filtered commutative dg $\mathscr{O}_S$-algebras

$$\kappa : \mathrm{L}\Omega^\bullet_{Z/S} \to \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}), \tag{12}$$

where $u_{Z/S} : Z_{\mathrm{cr}} \to Z_{\text{ét}}$ is the projection from the log-crystalline to the étale topos [Beilinson 2013, (1.9.1)]. The following theorem was proved by Beilinson [2013, Theorem on p. 13] by direct computations of both sides.

Theorem 2.1. *Suppose that Z, S are fine and $f : Z \to S$ is an integral, locally complete intersection morphism. Then* (12) *yields quasi-isomorphisms*

$$\kappa_m : \mathrm{L}\Omega^\bullet_{Z/S}/F^m \xrightarrow{\sim} \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}/\mathscr{J}^{[m]}_{Z/S}).$$

Recall [Bhatt 2012, Definition 7.20] that a log-scheme is called G-log-syntomic if it is log-syntomic and the local log-smooth models can be chosen to be of Cartier type. The next theorem, finer than Theorem 2.1, was proved by Bhatt [2012, Theorem 7.22] by looking at the conjugate filtration of the left-hand side.

Theorem 2.2. *Suppose that $f : Z \to S$ is G-log-syntomic. Then we have a quasi-isomorphism*

$$\kappa : \mathrm{L}\Omega^\bullet_{Z/S} \xrightarrow{\sim} \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}).$$

Combining the two theorems above, we get a filtered version:

Corollary 2.3. *Suppose that $f : Z \to S$ is G-log-syntomic. Then we have a quasi-isomorphism*

$$F^m\mathrm{L}\Omega^\bullet_{Z/S} \xrightarrow{\sim} \mathrm{R}u_{Z/S*}(\mathscr{J}^{[m]}_{Z/S}).$$

Proof. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
F^m\mathrm{L}\Omega^\bullet_{Z/S} \arrow[r] \arrow[d] & \mathrm{L}\Omega^\bullet_{Z/S} \arrow[r] \arrow[d, "\wr"] & \mathrm{L}\Omega^\bullet_{Z/S}/F^m \arrow[d, "\wr"] \\
\mathrm{R}u_{Z/S*}(\mathscr{J}^{[m]}_{Z/S}) \arrow[r] & \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}) \arrow[r] & \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}/\mathscr{J}^{[m]}_{Z/S})
\end{tikzcd}$$

and use the above theorems of Bhatt and Beilinson. ☐

Let X be a fine, proper, log-smooth scheme over $V^\times$. Set

$$\mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/W(k)})\widehat{\otimes}\mathbb{Q}_p := \big(\operatorname{holim}_n \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X_n/W_n(k)})\big) \otimes \mathbb{Q}$$

and similarly for complexes over $V^\times$. Here the hat over the derived log de Rham complex refers to the completion with respect to the Hodge filtration (in the sense of prosystems). For $r \geq 0$, consider the sequence of maps

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r \xleftarrow{\sim} \mathrm{R}\Gamma(X, \mathrm{L}\Omega^\bullet_{X/V^\times}/F^r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^\bullet_{X/V^\times}/F^r)\widehat{\otimes}\mathbb{Q}_p \\ \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{O}_{X/V^\times}/\mathscr{J}^{[r]}_{X/V^\times})_{\mathbb{Q}} \leftarrow \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{O}_{X/W(k)}/\mathscr{J}^{[r]}_{X/W(k)})_{\mathbb{Q}}.\end{aligned} \tag{13}$$

The first quasi-isomorphism follows from the fact that since X_K is log-smooth over K_0, the natural map $\mathrm{L}\Omega^\bullet_{X_K/K_0}/F^r \xrightarrow{\sim} \Omega^\bullet_{X_K/K_0}/F^r$ is a quasi-isomorphism. The second quasi-isomorphism follows from X being proper and log-smooth over $V^\times$, and the third one from Theorem 2.1. Define the map

$$\gamma_r^{-1} : \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{O}_{X/W(k)}/\mathscr{J}^{[r]}_{X/W(k)})_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r$$

as the composition (13).

Corollary 2.4. *Let X be a fine, proper, log-smooth scheme over $V^\times$. Let $r \geq 0$. There exists a canonical quasi-isomorphism*

$$\gamma_r : \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{O}_{X/W(k)}/\mathscr{J}^{[r]}_{X/W(k)})_{\mathbb{Q}}.$$

Proof. It suffices to show that the last map in the composition (13) is also a quasi-isomorphism. By Theorem 2.1, this map is quasi-isomorphic to the map

$$(\mathrm{R}\Gamma(X_{\acute{e}t}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/W(k)})\widehat{\otimes}\mathbb{Q}_p)/F^r \to (\mathrm{R}\Gamma(X_{\acute{e}t}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/V^\times})\widehat{\otimes}\mathbb{Q}_p)/F^r.$$

Hence it suffices to show that the natural map

$$\mathrm{gr}^i_F \mathrm{R}\Gamma(X_{\acute{e}t}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/W(k)})\widehat{\otimes}\mathbb{Q}_p \to \mathrm{gr}^i_F \mathrm{R}\Gamma(X_{\acute{e}t}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/V^\times})\widehat{\otimes}\mathbb{Q}_p$$

is a quasi-isomorphism for all $i \geq 0$.

Fix $n \geq 1$ and $i \geq 0$ and recall [Beilinson 2012, 1.2] that we have a natural identification

$$\begin{aligned} \mathrm{gr}^i_F \mathrm{L}\Omega^{\bullet}_{X_n/W_n(k)} &\xrightarrow{\sim} L\Lambda^i_X(L_{X_n/W_n(k)})[-i], \\ \mathrm{gr}^i_F \mathrm{L}\Omega^{\bullet}_{X_n/V_n^\times} &\xrightarrow{\sim} L\Lambda^i_X(L_{X_n/V_n^\times})[-i], \end{aligned}$$

where $L_{Y/S}$ denotes the relative log cotangent complex [Beilinson 2012, 3.1] and $L\Lambda_X(\bullet)$ is the nonabelian left derived functor of the exterior power functor. The distinguished triangle

$$\mathscr{O}_X \otimes_V L_{V_n^\times/W_n(k)} \to L_{X_n/W_n(k)} \to L_{X_n/V_n^\times}$$

yields a distinguished triangle

$$L\Lambda^i_X(\mathscr{O}_X \otimes_V L_{V_n^\times/W_n(k)})[-i] \to \mathrm{gr}^i_F \mathrm{L}\Omega^{\bullet}_{X_n/W_n(k)} \to \mathrm{gr}^i_F \mathrm{L}\Omega^{\bullet}_{X_n/V_n^\times}.$$

Hence we have a distinguished triangle

$$\begin{aligned} &\mathrm{holim}_n\, \mathrm{R}\Gamma(X_{\acute{e}t}, L\Lambda^i_X(\mathscr{O}_X \otimes_V L_{V_n^\times/W_n(k)})) \otimes \mathbb{Q}[-i] \\ &\qquad \to \mathrm{gr}^i_F \mathrm{R}\Gamma(X_{\acute{e}t}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/W(k)})\widehat{\otimes}\mathbb{Q}_p \to \mathrm{gr}^i_F \mathrm{R}\Gamma(X_{\acute{e}t}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/V^\times})\widehat{\otimes}\mathbb{Q}_p. \end{aligned}$$

It suffices to show that the term on the left is zero. But this will follow as soon as we show that the cohomology groups of $L_{V_n^\times/W_n(k)}$ are annihilated by p^c, where c is a constant independent of n. To show this, recall that V is a log complete intersection over $W(k)$. If π is a generator of $V/W(k)$, and $f(t)$ is its minimal polynomial, then (see [Olsson 2005, 6.9]) $L_{V^\times/W(k)}$ is quasi-isomorphic to the cone of the multiplication by $f'(\pi)$ map on V. Hence $L_{V^\times/W(k)}$ is acyclic in nonzero degrees, $H^0 L_{V^\times/W(k)} = \Omega_{V^\times/W(k)}$ is a cyclic V-module and we have a short exact sequence

$$0 \to \Omega_{V/W(k)} \to \Omega_{V^\times/W(k)} \to V/\mathfrak{m}_K \to 0.$$

Since $\Omega_{V/W(k)} \simeq V/\mathscr{D}_{K/K_0}$, where $\mathscr{D}_{K/K_0}$ is the different, $p^c H^0 L_{V^\times/W(k)} = 0$ for a constant c independent of n. Since $L_{V^\times/W(k)} \simeq L_{V^\times/W(k)} \otimes^L_V V_n$, we are done. $\square$

Remark 2.5. Versions of the above corollary appear in various degrees of generality in the proofs of the p-adic comparison theorems (see [Kato and Messing 1992, Lemma 4.5; Langer 1999, Lemma 2.7]). They are proved using computations in crystalline cohomology. We find the above argument based on the Beilinson comparison theorem, Theorem 2.1, particularly conceptual and pleasing.

2C. *The h-topology.* In this subsection we review terminology connected with the h-topology from [Beilinson 2013; 2012; Bhatt 2012]; we will use it freely. Let $\mathscr{V}ar_K$ be the category of varieties (i.e., reduced and separated schemes of finite type) over a field K. An *arithmetic pair* over K is an open embedding $j : U \hookrightarrow \overline{U}$ with dense image of a K-variety U into a reduced proper flat V-scheme $\overline{U}$. A morphism $(U, \overline{U}) \to (T, \overline{T})$ of pairs is a map $\overline{U} \to \overline{T}$ which sends U to T. In the case that the pairs represent log-regular schemes, this is the same as a map of log-schemes. For a pair $(U, \overline{U})$, we set $V_U := \Gamma(\overline{U}, \mathscr{O}_{\overline{U}})$ and $K_U := \Gamma(\overline{U}_K, \mathscr{O}_{\overline{U}})$. K_U is a product of several finite extensions of K (labeled by the connected components of $\overline{U}$) and, if $\overline{U}$ is normal, V_U is the product of the corresponding rings of integers. We will denote by $\mathscr{P}^{\mathrm{ar}}_K$ the category of arithmetic pairs over K. A *semistable pair* (*ss-pair*) over K [Beilinson 2012, 2.2] is a pair of schemes $(U, \overline{U})$ over (K, V) such that

(i) $\overline{U}$ is regular and proper over V,

(ii) $\overline{U} \setminus U$ is a divisor with normal crossings on $\overline{U}$,

(iii) the closed fiber $\overline{U}_0$ of $\overline{U}$ is reduced.

The closed fiber is taken over the closed points of V_U. We will think of ss-pairs as log-schemes equipped with log-structure given by the divisor $\overline{U} \setminus U$. The closed fiber $\overline{U}_0$ has the induced log-structure. We will say that the log-scheme $(U, \overline{U})$ is *split* over V_U. We will denote by $\mathscr{P}^{\mathrm{ss}}_K$ the category of ss-pairs over K. A semistable pair is called *strict* if the irreducible components of the closed fiber are regular. We will often work with the larger category $\mathscr{P}^{\mathrm{log}}_K$ of log-schemes $(U, \overline{U}) \in \mathscr{P}^{\mathrm{ar}}_K$ log-smooth over $V_U^{\times}$.

A *semistable pair* (*ss-pair*) over $\overline{K}$ [Beilinson 2012, 2.2] is a pair of connected schemes $(T, \overline{T})$ over $(\overline{K}, \overline{V})$ such that there exists an ss-pair $(U, \overline{U})$ over K and a $\overline{K}$-point $\alpha : K_U \to \overline{K}$ such that $(T, \overline{T})$ is isomorphic to the base change $(U_{\overline{K}}, \overline{U}_{\overline{V}})$. We will denote by $\mathscr{P}^{\mathrm{ss}}_{\overline{K}}$ the category of ss-pairs over $\overline{K}$.

A *geometric pair* over K is a pair $(U, \overline{U})$ of varieties over K such that $\overline{U}$ is proper and $U \subset \overline{U}$ is open and dense. We say that the pair $(U, \overline{U})$ is an *nc-pair* if $\overline{U}$ is regular and $\overline{U} \setminus U$ is a divisor with normal crossings in $\overline{U}$; it is a strict nc-pair if the irreducible components of $U \setminus \overline{U}$ are regular. A morphism of pairs $f : (U_1, \overline{U}_1) \to (U, \overline{U})$ is a map $\overline{U}_1 \to \overline{U}$ that sends U_1 to U. We denote the category of nc-pairs over K by $\mathscr{P}^{\mathrm{nc}}_K$.

For a field K, the h-topology (see [Suslin and Voevodsky 2000; Beilinson 2012, 2.3]) on $\mathscr{V}ar_K$ is the coarsest topology finer than the Zariski and proper topologies.[5] It is stronger than the étale and proper topologies. It is generated by the pretopology whose coverings are finite families of maps $\{Y_i \to X\}$ such that $Y := \coprod Y_i \to X$ is a universal topological epimorphism (i.e., a subset of X is Zariski open if and only if its preimage in Y is open). We denote by $\mathscr{V}ar_{K,h}$ and X_h the corresponding h-sites. For any of the categories $\mathscr{P}$ mentioned above, let $\gamma : (U, \bar{U}) \to U$ denote the forgetful functor. Beilinson [2012, 2.5] proved that the categories $\mathscr{P}^{\mathrm{nc}}$, $(\mathscr{P}_K^{\mathrm{ar}}, \gamma)$ and $(\mathscr{P}_K^{\mathrm{ss}}, \gamma)$ form a base for $\mathscr{V}ar_{K,h}$. One can easily modify his argument to conclude the same about the categories $(\mathscr{P}_K^{\log}, \gamma)$.

2D. ***Galois cohomology.*** In this subsection we review the definition of (higher) semistable Selmer groups and prove that in stable ranges they are the same as Galois cohomology groups. Our main references are [Fontaine 1994b; 1994c; Colmez and Fontaine 2000; Bloch and Kato 1990; Fontaine and Perrin-Riou 1994; Nekovář 1993]. Recall [Fontaine 1994b, 1994c] that a p-adic representation V of G_K (i.e., a finite-dimensional continuous $\mathbb{Q}_p$-vector space representation) is called *semistable* (over K) if $\dim_{K_0}(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \dim_{\mathbb{Q}_p}(V)$.

It is called *potentially semistable* if there exists a finite extension K' of K such that $V|G_{K'}$ is semistable over K'. We denote by $\mathrm{Rep}_{\mathrm{st}}(G_K)$ and $\mathrm{Rep}_{\mathrm{pst}}(G_K)$ the categories of semistable and potentially semistable representations of G_K, respectively.

As in [Fontaine 1994c, 4.2], a φ-module over K_0 is a pair (D, φ), where D is a finite-dimensional K_0-vector space and $\varphi = \varphi_D$ is a φ-semilinear automorphism of D; a (φ, N)-module is a triple (D, φ, N), where (D, φ) is a φ-module and $N = N_V$ is a K_0-linear endomorphism of D such that $N\varphi = p\varphi N$ (hence N is nilpotent). A filtered (φ, N)-module is a tuple $(D, \varphi, N, F^\bullet)$, where (D, φ, N) is a (φ, N)-module and $F^\bullet$ is a decreasing finite filtration of D_K by K-vector spaces. There is a notion of a (*weakly*) *admissible* filtered (φ, N)-module [Colmez and Fontaine 2000]. Denote by $MF_K^{\mathrm{ad}}(\varphi, N) \subset MF_K(\varphi, N)$ the categories of admissible filtered (φ, N)-modules and filtered (φ, N)-modules, respectively. We know [Colmez and Fontaine 2000] that the pair of functors

$$D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \quad V_{\mathrm{st}}(D) = (B_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=\mathrm{Id}, N=0} \cap F^0(B_{\mathrm{dR}} \otimes_K D_K)$$

defines an equivalence of categories $MF_K^{\mathrm{ad}}(\varphi, N) \simeq \mathrm{Rep}_{\mathrm{st}}(G_K)$.

For $D \in MF_K(\varphi, N)$, set

$$C_{\mathrm{st}}(D) := \left[\begin{array}{ccc} D & \xrightarrow{(1-\varphi,\mathrm{can})} & D \oplus D_K/F^0 \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ D & \xrightarrow{1-p\varphi} & D \end{array}\right].$$

[5] The latter is generated by a pretopology whose coverings are proper surjective maps.

Here the brackets denote the total complex of the double complex inside the brackets. Consider also the complex

$$C^+(D) := \left[\begin{array}{ccc} D\otimes_{K_0} B_{\mathrm{st}}^+ & \xrightarrow{(1-\varphi,\mathrm{can}\otimes\iota)} & D\otimes_{K_0} B_{\mathrm{st}}^+ \oplus (D_K\otimes_K B_{\mathrm{dR}}^+)/F^0 \\ \downarrow N & & \downarrow (N,0) \\ D\otimes_{K_0} B_{\mathrm{st}}^+ & \xrightarrow{1-p\varphi} & D\otimes_{K_0} B_{\mathrm{st}}^+ \end{array}\right].$$

Define $C(D)$ by omitting the superscript $+$ in the above diagram. We have $C_{\mathrm{st}}(D) = C(D)^{G_K}$.

Remark 2.6. Recall [Nekovář 1993, 1.19; Fontaine and Perrin-Riou 1994, 3.3] that to every p-adic representation V of G_K we can associate a complex

$$C_{\mathrm{st}}(V) : D_{\mathrm{st}}(V) \xrightarrow{(N,1-\varphi,\iota)} D_{\mathrm{st}}(V)\oplus D_{\mathrm{st}}(V)\oplus t_V \xrightarrow{(1-p\varphi)-N} D_{\mathrm{st}}(V)\to 0,$$

where $t_V := (V\otimes_{\mathbb{Q}_p}(B_{\mathrm{dR}}/B_{\mathrm{dR}}^+))^{G_K}$ [Fontaine and Perrin-Riou 1994, I.2.2.1]. The cohomology of this complex is called $H^*_{\mathrm{st}}(G_K, V)$. If V is semistable then $C_{\mathrm{st}}(V) = C_{\mathrm{st}}(D_{\mathrm{st}}(V))$; hence $H^*(C_{\mathrm{st}}(D_{\mathrm{st}}(V))) = H^*_{\mathrm{st}}(G_K, V)$. If V is potentially semistable, the groups $H^*_{\mathrm{st}}(G_K, V)$ compute Yoneda extensions of $\mathbb{Q}_p$ by V in the category of potentially semistable representations [ibid., I.3.3.8]. In general [ibid., I.3.3.7], $H^0_{\mathrm{st}}(G_K, V) \xrightarrow{\sim} H^0(G_K, V)$ and $H^1_{\mathrm{st}}(G_K, V) \hookrightarrow H^1(G_K, V)$ computes st-extensions[6] of $\mathbb{Q}_p$ by V.

Remark 2.7. Let $D\in MF_K(\varphi, N)$. Note that:

(1) $H^0(C(D)) = V_{\mathrm{st}}(D)$.

(2) For $i\geq 2$, we have $H^i(C^+(D)) = H^i(C(D)) = 0$ (because N is surjective on B_{st}^+ and B_{st}).

(3) If $F^1D_K = 0$ then $F^0(D_K\otimes_K B_{\mathrm{dR}}^+) = F^0(D_K\otimes_K B_{\mathrm{dR}})$ (in particular, the map of complexes $C^+(D)\to C(D)$ is an injection).

(4) If $D = D_{\mathrm{st}}(V)$ is admissible then we have quasi-isomorphisms

$$C(D) \xleftarrow{\sim} V\otimes_{\mathbb{Q}_p}[B_{\mathrm{cr}} \xrightarrow{(1-\varphi,\mathrm{can})} B_{\mathrm{cr}}\oplus B_{\mathrm{dR}}/F^0] \xleftarrow{\sim} V\otimes_{\mathbb{Q}_p}(B_{\mathrm{cr}}^{\varphi=1}\cap F^0) = V$$

and the map of complexes $C_{\mathrm{st}}(D)\to C(D)$ represents the canonical map $H^i_{\mathrm{st}}(G_K, V)\to H^i(G_K, V)$.

Lemma 2.8 [Fontaine 1994a, Theorem II.5.3]. *If $X\subset B_{\mathrm{cr}}\cap B_{\mathrm{dR}}^+$ and $\varphi(X)\subset X$ then $\varphi^2(X)\subset B_{\mathrm{cr}}^+$.*

Proposition 2.9. *If $D\in MF_K(\varphi, N)$ and $F^1D_K = 0$ then $H^0(C(D)/C^+(D)) = 0$.*

[6]An extension $0\to V_1\to V_2\to V_3\to 0$ is called st if the sequence $0\to D_{\mathrm{st}}(V_1)\to D_{\mathrm{st}}(V_2)\to D_{\mathrm{st}}(V_3)\to 0$ is exact.

Proof. We will argue by induction on m such that $N^m = 0$. Assume first that $m = 1$ (hence $N = 0$). We have

$$
\begin{aligned}
&C(D)/C^+(D)\\
&\quad = \left[\begin{tikzcd} D\otimes_{K_0}(B_{\mathrm{st}}/B_{\mathrm{st}}^+) \arrow[r, "(1-\varphi,\mathrm{can}\otimes\iota)"] \arrow[d, "1\otimes N"] & D\otimes_{K_0}(B_{\mathrm{st}}/B_{\mathrm{st}}^+)\oplus D_K\otimes_K(B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \arrow[d, "(1\otimes N,0)"] \\ D\otimes_{K_0}(B_{\mathrm{st}}/B_{\mathrm{st}}^+) \arrow[r, "1-p\varphi"] & D\otimes_{K_0}(B_{\mathrm{st}}/B_{\mathrm{st}}^+) \end{tikzcd}\right]\\
&\quad \xleftarrow{\sim} \left[D\otimes_{K_0}(B_{\mathrm{cr}}/B_{\mathrm{cr}}^+) \xrightarrow{(1-\varphi,\mathrm{can})} D\otimes_{K_0}(B_{\mathrm{cr}}/B_{\mathrm{cr}}^+)\oplus D_K\otimes_K(B_{\mathrm{dR}}/B_{\mathrm{dR}}^+)\right]
\end{aligned}
$$

Write $D = \bigoplus_{i=1}^r K_0 d_i$ and, for $1 \le i \le r$, consider the maps

$$
\begin{aligned}
p_i : H^0(C(D)/C^+(D)) &= (D\otimes_{K_0}((B_{\mathrm{cr}}\cap B_{\mathrm{dR}}^+)/B_{\mathrm{cr}}^+))^{\varphi=1}\\
&\subset \bigoplus_{i=1}^r d_i\otimes((B_{\mathrm{cr}}\cap B_{\mathrm{dR}}^+)/B_{\mathrm{cr}}^+) \xrightarrow{\mathrm{pr}_i} (B_{\mathrm{cr}}\cap B_{\mathrm{dR}}^+)/B_{\mathrm{cr}}^+.
\end{aligned}
$$

Let Y_a, where $a \in H^0(C(D)/C^+(D))$, denote the K_0-subspace of $(B_{\mathrm{cr}}\cap B_{\mathrm{dR}}^+)/B_{\mathrm{cr}}^+$ spanned by $p_1(a), \dots, p_r(a)$. For $M \in \mathrm{GL}_r(K_0)$, we have $(p_1(a), \dots, p_r(a))^T = M\varphi(p_1(a), \dots, p_r(a))^T$. Hence $\varphi(Y_a) \subset Y_a$. Let $X_a \subset B_{\mathrm{cr}} \cap B_{\mathrm{dR}}^+$ be the inverse image of Y_a under the projection $B_{\mathrm{cr}}\cap B_{\mathrm{dR}}^+ \to (B_{\mathrm{cr}}\cap B_{\mathrm{dR}}^+)/B_{\mathrm{cr}}^+$ (naturally $B_{\mathrm{cr}}^+ \subset X_a$). Then $\varphi(X_a) \subset X_a + B_{\mathrm{cr}}^+ = X_a$. By the above lemma, $\varphi^2(X_a) \subset B_{\mathrm{cr}}^+$. Hence $\varphi^2(Y_a) = 0$ and (applying M^{-2}) $Y_a = 0$. This implies that $a = 0$ and $H^0(C(D)/C^+(D)) = 0$, as wanted.

For general $m > 0$, consider the filtration $D_1 \subset D$, where $D_1 := \ker(N)$ with induced structures. Set $D_2 := D/D_1$ with induced structures. Then $D_1, D_2 \in MF_K(\varphi, N)$; N^i is trivial on D_1 for $i = 1$ and on D_2 for $i = m - 1$. Clearly $F^1 D_{1,K} = F^1 D_{2,K} = 0$. Hence, by Remark 2.7.3, we have a short exact sequence

$$0 \to C(D_1)/C^+(D_1) \to C(D)/C^+(D) \to C(D_2)/C^+(D_2) \to 0.$$

By the inductive assumption, $H^0(C(D_1)/C^+(D_1)) = H^0(C(D_2)/C^+(D_2)) = 0$. Hence $H^0(C(D)/C^+(D)) = 0$, as wanted. □

Corollary 2.10. *If $D \in MF_K(\varphi, N)$ and $F^1 D_K = 0$ then*

$$H^0(C^+(D)) = H^0(C(D)) = V_{\mathrm{st}}(D)\ (\subset D\otimes_{K_0} B_{\mathrm{st}}^+)$$

and $H^1(C^+(D)) \hookrightarrow H^1(C(D))$.

Corollary 2.11. *If $D \in MF_K^{\mathrm{ad}}(\varphi, N)$ and $F^1 D_K = 0$ then*

$$H^i(C^+(D)) = H^i(C(D)) = \begin{cases} V_{\mathrm{st}}(D) & \text{if } i = 0,\\ 0 & \text{if } i \neq 0\end{cases}$$

(i.e., $C^+(D) \xrightarrow{\sim} C(D)$).

A filtered (φ, N, G_K)-module is a tuple $(D, \varphi, N, \rho, F^\bullet)$, where

(1) D is a finite-dimensional K_0^{nr}-vector space;

(2) $\varphi : D \to D$ is a Frobenius map;

(3) $N : D \to D$ is a K_0^{nr}-linear monodromy map such that $N\varphi = p\varphi N$;

(4) ρ is a K_0^{nr}-semilinear G_K-action on D (hence $\rho|I_K$ is linear) that is smooth, i.e., all vectors have open stabilizers, and that commutes with φ and N;

(5) $F^\bullet$ is a decreasing finite filtration of $D_K := (D \otimes_{K_0^{\mathrm{nr}}} \bar{K})^{G_K}$ by K-vector spaces.

Morphisms between filtered (φ, N, G_K)-modules are K_0^{nr}-linear maps preserving all structures. There is a notion of a (*weakly*) *admissible* filtered (φ, N, G_K)-module [Colmez and Fontaine 2000; Fontaine 1994b]. Denote by $MF_K^{\mathrm{ad}}(\varphi, N, G_K) \subset MF_K(\varphi, N, G_K)$ the categories of admissible filtered (φ, N, G_K)-modules and filtered (φ, N, G_K)-modules, respectively. We know [Colmez and Fontaine 2000] that the pair of functors $D_{\mathrm{pst}}(V) = \operatorname{inj\,lim}_H (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^H$, where $H \subset G_K$ is an open subgroup, and $V_{\mathrm{pst}}(D) = (B_{\mathrm{st}} \otimes_{K_0^{\mathrm{nr}}} D)^{\varphi=\mathrm{Id}, N=0} \cap F^0(B_{\mathrm{dR}} \otimes_K D_K)$ define an equivalence of categories $MF_K^{\mathrm{ad}}(\varphi, N, G_K) \simeq \operatorname{Rep}_{\mathrm{pst}}(G_K)$.

For $D \in MF_K(\varphi, N, G_K)$, set[7]

$$C_{\mathrm{pst}}(D) := \left[\begin{array}{ccc} D_{\mathrm{st}} & \xrightarrow{(1-\varphi,\mathrm{can})} & D_{\mathrm{st}} \oplus D_K/F^0 \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ D_{\mathrm{st}} & \xrightarrow{1-p\varphi} & D_{\mathrm{st}} \end{array}\right].$$

Here $D_{\mathrm{st}} := D^{G_K}$. Consider also the following complex (we set $D_{\bar{K}} := D \otimes_{K_0^{\mathrm{nr}}} \bar{K}$):

$$C^+(D) := \left[\begin{array}{ccc} D \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}^+ & \xrightarrow{(1-\varphi,\mathrm{can}\otimes\iota)} & (D \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}^+) \oplus (D_{\bar{K}} \otimes_{\bar{K}} B_{\mathrm{dR}}^+)/F^0 \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ D \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}^+ & \xrightarrow{1-p\varphi} & D \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}^+ \end{array}\right].$$

Define $C(D)$ by omitting the superscript $+$ in the above diagram. We have $C_{\mathrm{pst}}(D) = C(D)^{G_K}$.

Remark 2.12. If V is potentially semistable then $C_{\mathrm{st}}(V) = C_{\mathrm{pst}}(D_{\mathrm{pst}}(V))$; hence $H^*(C_{\mathrm{pst}}(D_{\mathrm{pst}}(V))) = H^*_{\mathrm{st}}(G_K, V)$.

Remark 2.13. If $D = D_{\mathrm{pst}}(V)$ is admissible then we have quasi-isomorphisms

$$C(D) \xleftarrow{\sim} V \otimes_{\mathbb{Q}_p} [B_{\mathrm{cr}} \xrightarrow{(1-\varphi,\mathrm{can})} B_{\mathrm{cr}} \oplus B_{\mathrm{dR}}/F^0] \xleftarrow{\sim} V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cr}}^{\varphi=1} \cap F^0) = V$$

[7] We hope that the notation below will not lead to confusion with the semistable case in general, but if in doubt we will add the data of the field K in the latter case.

and the map of complexes $C_{\mathrm{pst}}(D) \to C(D)$ represents the canonical map

$$H^i_{\mathrm{st}}(G_K, V) \to H^i(G_K, V).$$

Remark 2.14. Let $D = D_{\mathrm{pst}}(V)$ be admissible. The Bloch–Kato exponential

$$(Z^1C(D))^{G_K} \to H^1(G_K, V)$$

is given by the coboundary map arising from the exact sequence

$$0 \to V \to C^0(D) \to Z^1C(D) \to 0.$$

Its restriction to the de Rham part of $Z^1C(D)$ is the Bloch–Kato exponential

$$\exp_{\mathrm{BK}} : D_K/F^0 \to H^1(G_K, V).$$

It is also obtained by applying Rf, where $f(-) = (-)^{G_K}$, to the coboundary map $\partial : Z^1C(D) \to V[1]$ arising from the above exact sequence (see the proof of Theorem 4.8 for an appropriate formalism of continuous cohomology). Note that the composition of the canonical maps

$$Z^1C(D) \to (\sigma_{\geq 1}C(D))[1] \to C(D)[1] \xleftarrow{\sim} V[1]$$

is not equal to ∂, but to $-\partial$, by (18).

Corollary 2.15. *If $D \in MF_K^{\mathrm{ad}}(\varphi, N, G_K)$ and $F^1D_K = 0$ then*

$$H^i(C^+(D)) \xrightarrow{\sim} H^i(C(D)) = \begin{cases} V_{\mathrm{pst}}(D) & \text{if } i = 0, \\ 0 & \text{if } i \neq 0 \end{cases}$$

(i.e., $C^+(D) \xrightarrow{\sim} C(D)$).

Proof. By Remark 2.13 we have $C(D) \simeq V_{\mathrm{pst}}(D)[0]$. To prove the isomorphism $H^i(C^+(D)) \xrightarrow{\sim} H^i(C(D))$, $i \geq 0$, take a finite Galois extension K'/K such that D becomes semistable over K', i.e., $I_{K'}$ acts trivially on D. We have $(D', \varphi, N) \in MF_{K'}^{\mathrm{ad}}(\varphi, N)$, where $D' := D^{G_{K'}}$ and (compatibly) $D \simeq D' \otimes_{K'_0} K_0^{\mathrm{nr}}$ and $F^\bullet D'_{K'} \simeq F^\bullet D_K \otimes_K K'$. It easily follows that $C^+(D) = C^+(K', D')$ and $C(D) = C(K', D')$. Since $F^1D'_{K'} = 0$, our corollary is now a consequence of Corollary 2.11. □

Proposition 2.16. *If $D \in MF_K^{\mathrm{ad}}(\varphi, N, G_K)$ and $F^1D_K = 0$ then, for $i \geq 0$, the natural map*

$$H^i_{\mathrm{st}}(G_K, V_{\mathrm{pst}}(D)) \xrightarrow{\sim} H^i(G_K, V_{\mathrm{pst}}(D))$$

is an isomorphism.

Proof. Both sides satisfy Galois descent for finite Galois extensions. We can assume, therefore, that $D = D_{\mathrm{st}}(V)$ for a semistable representation V of G_K. For $i = 0$, we have (even without assuming $F^1D_K = 0$)

$$H^0(C_{\mathrm{st}}(D)) = H^0(C(D)^{G_K}) = H^0(C(D))^{G_K} = V^{G_K}.$$

For $i = 1$, the statement is proved in [Berger 2002, Théorème 6.2, Lemme 6.5]. For $i = 2$, it follows from the assumption $F^1 D_K = 0$ (by weak admissibility of D) that there is a $W(k)$-lattice $M \subset D$ such that $\varphi^{-1}(M) \subset p^2 M$, which implies that $1 - p\varphi = -p\varphi(1 - p^{-1}\varphi^{-1}) : D \to D$ is surjective, and hence $H^2(C_{\mathrm{st}}(D)) = 0$ (see the proof of [Berger 2002, Lemme 6.7]). The proof of the fact that $H^2(G_K, V) = 0$ if $F^1 D_K = 0$ was kindly communicated to us by L. Berger; it is reproduced in Appendix A (see Theorem A.1). For $i > 2$, both terms vanish. $\square$

2E. ***Comparison of spectral sequences.*** The purpose of this subsection is to prove a derived category theorem (Theorem 2.18) that will be used later to relate the syntomic descent spectral sequence with the étale Hochschild–Serre spectral sequence (see Theorem 4.8). Let D be a triangulated category and $H : D \to A$ a cohomological functor to an abelian category A. A finite collection of adjacent exact triangles (a "Postnikov system" in the language of [Gelfand and Manin 2003, IV.2, Exercise 2])

$$\begin{tikzcd}[column sep=small] & Y^0 \arrow[dr] & & Y^1 \arrow[dr] & & & & Y^n \arrow[dr] & \\ X = X^0 \arrow[ur] & & X^1 \arrow[ll, "{[1]}"] \arrow[ur] & & X^2 \arrow[ll, "{[1]}"] & \cdots & X^n \arrow[ur] & & X^{n+1} = 0 \arrow[ll, "{[1]}"] \end{tikzcd} \tag{14}$$

gives rise to an exact couple

$$D_1^{p,q} = H^q(X^p) = H(X^p[q]), \quad E_1^{p,q} = H^q(Y^p) \Rightarrow H^{p+q}(X).$$

The induced filtration on the abutment is given by

$$F^p H^{p+q}(X) = \mathrm{Im}\big(D_1^{p,q} = H^q(X^p) \to H^{p+q}(X)\big).$$

Remark 2.17. In the special case when A is the heart of a nondegenerate t-structure $(D^{\leq n}, D^{\geq n})$ on D and $H = \tau_{\leq 0}\tau_{\geq 0}$, the following conditions are equivalent:

(1) $E_2^{p,q} = 0$ for $p \neq 0$.

(2) $D_2^{p,q} = 0$ for all p, q.

(3) $D_r^{p,q} = 0$ for all p, q and $r > 1$.

(4) The sequence $0 \to H^q(X^p) \to H^q(Y^p) \to H^q(X^{p+1}) \to 0$ is exact for all p, q.

(5) The sequence $0 \to H^q(X) \to H^q(Y^0) \to H^q(Y^1) \to \cdots$ is exact for all q.

(6) The canonical map $H^q(X) \to E_1^{\bullet,q}$ is a quasi-isomorphism for all q.

(7) The triangle $\tau_{\leq q} X^p \to \tau_{\leq q} Y^p \to \tau_{\leq q} X^{p+1}$ is exact for all p, q.

From now on until the end of Section 2E assume that $D = D(A)$ is the derived category of A with the standard t-structure and that $X^i, Y^i \in D^+(A)$ for all i. Furthermore, assume that $f : A \to A'$ is a left exact functor to an abelian category A'

and that A admits a class of f-adapted objects (hence the derived functor $\mathrm{R}f : D^+(A) \to D^+(A')$ exists).

Applying $\mathrm{R}f$ to (14), we obtain another Postnikov system, this time in $D^+(A')$. The corresponding exact couple

$$ {}^{I}D_1^{p,q} = (\mathrm{R}^q f)(X^p), \quad {}^{I}E_1^{p,q} = (\mathrm{R}^q f)(Y^p) \Rightarrow (\mathrm{R}^{p+q} f)(X) \tag{15} $$

induces the filtration

$$ {}^{I}F^p(\mathrm{R}^{p+q} f)(X) = \mathrm{Im}\big({}^{I}D_1^{p,q} = (\mathrm{R}^q f)(X^p) \to (\mathrm{R}^{p+q} f)(X)\big). $$

Our goal is to compare (15), under the equivalent conditions in Remark 2.17, to the hypercohomology exact couple

$$ {}^{II}D_2^{p,q} = (\mathrm{R}^{p+q} f)(\tau_{\le q-1} X), \quad {}^{II}E_2^{p,q} = (\mathrm{R}^p f)(H^q(X)) \Rightarrow (\mathrm{R}^{p+q} f)(X) \tag{16} $$

for which

$$ {}^{II}F^p(\mathrm{R}^{p+q} f)(X) = \mathrm{Im}\big({}^{II}D_2^{p-1,q+1} = (\mathrm{R}^{p+q} f)(\tau_{\le q} X) \to (\mathrm{R}^{p+q} f)(X)\big). $$

Theorem 2.18. *Under the conditions in Remark 2.17, there is a natural morphism of exact couples*

$$ (u, v) : ({}^{I}D_2, {}^{I}E_2) \to ({}^{II}D_2, {}^{II}E_2). $$

Consequently, we have ${}^{I}F^p \subseteq {}^{II}F^p$ for all p and there is a natural morphism of spectral sequences ${}^{I}E_r^{*,*} \to {}^{II}E_r^{*,*}$ $(r > 1)$ compatible with the identity map on the common abutment.

Proof. Step 1: We begin by constructing a natural map $u : {}^{I}D_2 \to {}^{II}D_2$.

For each $p > 0$, there is a commutative diagram in $D^+(A')$

$$ \begin{tikzcd} (\mathrm{R}^{p+q} f)((\tau_{\le q} Y^{p-1})[-p]) \arrow[r] \arrow[d, "\wr"] & (\mathrm{R}^{p+q} f)((\tau_{\le q} X^p)[-p]) \arrow[r] \arrow[d, "\wr"] & (\mathrm{R}^{p+q} f)(\tau_{\le q} X) \arrow[d, "\alpha_{II}"] \\ {}^{I}E_1^{p-1,q} = (\mathrm{R}^{p+q} f)(Y^{p-1}[-p]) \arrow[r, "k_1"] & {}^{I}D_1^{p,q} = (\mathrm{R}^{p+q} f)(X^p[-p]) \arrow[r, "\alpha_I"] \arrow[ur, "u'"] & (\mathrm{R}^{p+q} f)(X) \end{tikzcd} $$

both of whose rows are complexes. This defines a map $u' : {}^{I}D_1^{p,q} \to {}^{II}D_2^{p-1,q+1}$ such that $u' k_1 = 0$ and $\alpha_{II} u' = \alpha_I$ (hence ${}^{I}F^p = \mathrm{Im}(\alpha_I) \subseteq \mathrm{Im}(\alpha_{II}) = {}^{II}F^p$). By construction, the diagram (with exact top row)

$$ \begin{tikzcd} {}^{I}E_1^{p,q-1} \arrow[r, "k_1"] \arrow[dr, "0"] & {}^{I}D_1^{p+1,q-1} \arrow[r, "i_1"] \arrow[d, "u'"] & {}^{I}D_1^{p,q} \arrow[d, "u'"] \\ & {}^{II}D_2^{p,q} \arrow[r, "i_2"] & {}^{II}D_2^{p-1,q+1} \end{tikzcd} $$

is commutative for each $p \geq 0$, which implies that the map

$$u = u' i_1^{-1} : {}^I D_2^{p,q} = i_1({}^I D_1^{p+1,q-1}) \to {}^{II} D_2^{p,q}$$

is well-defined and satisfies $u i_2 = i_2 u$.

Step 2: For all q, the canonical quasi-isomorphism $H^q(X) \to E_1^{\bullet,q}$ induces natural morphisms

$$\begin{aligned} v' : {}^I E_2^{p,q} &= H^p(i \mapsto (\mathrm{R}^q f)(Y^i)) \to H^p(i \mapsto f(H^q(Y^i))) \to (\mathrm{R}^p f)(i \mapsto H^q(Y^i)) \\ &= (\mathrm{R}^p f)(E_1^{\bullet,q}) \xleftarrow{\sim} (\mathrm{R}^p f)(H^q(X)) = {}^{II} E_2^{p,q}; \end{aligned}$$

set $v = (-1)^p v' : {}^I E_2^{p,q} \to {}^{II} E_2^{p,q}$.

It remains to show that u and v are compatible with the maps

$$ {}^? D_2^{p-1,q+1} \xrightarrow{j_2} {}^? E_2^{p,q} \xrightarrow{k_2} {}^? D_2^{p+1,q} \quad (? = I, II).$$

Step 3: For any complex $M^\bullet$ over A, denote by $Z^i(M^\bullet) = \mathrm{Ker}(\delta^i : M^i \to M^{i+1})$ the subobject of cycles in degree i.

If $M^\bullet$ is a resolution of an object M of A, then each exact sequence

$$0 \longrightarrow Z^p(M^\bullet) \longrightarrow M^p \xrightarrow{\delta^p} Z^{p+1}(M^\bullet) \longrightarrow 0 \quad (p \geq 0) \tag{17}$$

can be completed to an exact sequence of resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^p(M^\bullet) & \longrightarrow & M^p & \longrightarrow & Z^{p+1}(M^\bullet) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{can}} & & \big\downarrow{\scriptstyle \mathrm{can}} & & \big\downarrow{\scriptstyle -\mathrm{can}} & & \\ 0 & \to & (\sigma_{\geq p}(M^\bullet))[p] & \to & (\sigma_{\geq p}\mathrm{Cone}(M^\bullet \xrightarrow{\mathrm{id}} M^\bullet))[p] & \to & (\sigma_{\geq p+1}(M^\bullet))[p+1] & \to & 0 \end{array}$$

By induction, we obtain that the following diagram, whose top arrow is the composition of the natural maps $Z^i \to Z^{i-1}[1]$ induced by (17), commutes in $D^+(A)$:

$$\begin{array}{ccc} Z^p(M^\bullet) & \longrightarrow & Z^0(M^\bullet)[p] = M[p] \\ \big\downarrow{\scriptstyle \mathrm{can}} & & \big\downarrow{\scriptstyle (-1)^p\,\mathrm{can}} \\ (\sigma_{\geq p}(M^\bullet))[p] & \xrightarrow{\mathrm{can}} & M^\bullet[p] \end{array} \tag{18}$$

We are going to apply this statement to $M = H^q(X)$ and $M^\bullet = E_1^{\bullet,q}$, when $Z^p(M^\bullet) = D_1^{p,q} = H^q(X^p)$ and $Z^0(M^\bullet) = H^q(X)$.

Step 4: We are going to investigate ${}^I E_2^{p,q}$.

Complete the morphism $Y^p \to Y^{p+1}$ to an exact triangle $U^p \to Y^p \to Y^{p+1}$ in $D^+(A)$ and fix a lift $X^p \to U^p$ of the morphism $X^p \to Y^p$.

There are canonical epimorphisms

$$(\mathrm{R}^q f)(U^p) \twoheadrightarrow \mathrm{Ker}((\mathrm{R}^q f)(Y^p) \xrightarrow{j_1 k_1} (\mathrm{R}^q f)(Y^{p+1})) = Z^p({}^I E_1^{\bullet,q}) \twoheadrightarrow {}^I E_2^{p,q}, \tag{19}$$

and the map

$$k_2 : {}^I E_2^{p,q} \to {}^I D_2^{p+1,q} = \mathrm{Ker}({}^I D_1^{p+1,q} \xrightarrow{j_1} {}^I E_1^{p+1,q})$$

is induced by the restriction of $k_1 : {}^I E_1^{p,q} \to {}^I D_1^{p+1,q}$ to $Z^p({}^I E_1^{\bullet,q})$.

The octahedron (in which we have drawn only the four exact faces)

$$\begin{tikzcd}
X^{p+2} \arrow[dr, "{[1]}"'] & & Y^{p+1} \arrow[ll] \\
& X^{p+1} \arrow[ur] \arrow[dl] & \\
X^p[1] \arrow[rr, "{[1]}"] & & Y^p \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
X^{p+2} \arrow[dd, "{[1]}"'] & & Y^{p+1} \arrow[dl] \\
& U^p[1] \arrow[ul] \arrow[dr, "{[1]}"] & \\
X^p[1] \arrow[ur] & & Y^p \arrow[uu]
\end{tikzcd}$$

shows that the triangle $X^p \to U^p \to X^{p+2}[-1]$ is exact and the diagrams

$$\begin{tikzcd}
U^p[1] \arrow[r] \arrow[d] & Y^p[1] \arrow[d] \\
X^{p+2} \arrow[r] & X^{p+1}[1]
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{R}^q f)(U^p) \arrow[rr] \arrow[d] & & Z^p({}^I E_1^{\bullet,q}) \arrow[d, "k_1"'] \\
(\mathrm{R}^q f)(X^{p+2}[-1]) = {}^I D_2^{p+2,q-1} \arrow[rr, "i_1"] & & {}^I D_2^{p+1,q}
\end{tikzcd}$$

commute. The previous discussion implies that the composite map

$$\begin{aligned}
(\mathrm{R}^q f)(U^p) \twoheadrightarrow Z^p({}^I E_1^{\bullet,q}) \twoheadrightarrow {}^I E_2^{p,q} \xrightarrow{k_2} {}^I D_2^{p+1,q} \\
\xrightarrow{u} {}^{II} D_2^{p+1,q} = (\mathrm{R}^q f)((\tau_{\leq q-1} X)[p+1])
\end{aligned}$$

is obtained by applying $\mathrm{R}^q f$ to

$$\tau_{\leq q}\, U^p \to \tau_{\leq q}(X^{p+2}[-1]) = (\tau_{\leq q-1} X^{p+2})[-1] \to (\tau_{\leq q-1}\, X)[p+1]. \tag{20}$$

Step 5: All boundary maps $H^q(X^{p+2}[-1]) \to H^q(X^p)$ vanish by Remark 2.17, which means that the following triangles are exact:

$$\tau_{\leq q}\, X^p \to \tau_{\leq q}\, U^p \to \tau_{\leq q}(X^{p+2}[-1]) = (\tau_{\leq q-1}\, X^{p+2})[-1].$$

The commutative diagram

$$\begin{tikzcd}
\tau_{\leq q}\, U^p \arrow[r] & H^q(U^p)[-q] \arrow[r] & \mathrm{Ker}\big(H^q(Y^p) \to H^q(Y^{p+1})\big)[-q] \\
\tau_{\leq q}\, X^p \arrow[r] \arrow[u] & H^q(X^p)[-q] \arrow[u] \arrow[ur, equal] &
\end{tikzcd}$$

gives rise to an octahedron

$$
\begin{tikzcd}
V^p \arrow[dr, "{[1]}"'] & & H^q(X^p)[-q] \arrow[ll] \\
& \tau_{\le q} U^p \arrow[ur] \arrow[dl] & \\
\tau_{\le q}(X^{p+2}[-1]) \arrow[rr, "{[1]}"] & & \tau_{\le q} X^p \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
V^p \arrow[dd, "{[1]}"'] & & H^q(X^p)[-q] \arrow[dl] \\
& (\tau_{\le q-1} X^p)[1] \arrow[ul] \arrow[dr, "{[1]}"] & \\
X^p[1] \arrow[ur] & & Y^p \arrow[uu]
\end{tikzcd}
$$

In particular, the following diagram commutes:

$$
\begin{tikzcd}
\tau_{\le q} U^p \arrow[r] \arrow[d] & H^q(X^p)[-q] \arrow[d] \\
\tau_{\le q}(X^{p+2}[-1]) \arrow[r] & (\tau_{\le q-1} X^p)[1]
\end{tikzcd}
\tag{21}
$$

Step 6: The diagram (18) implies that the composition of $v : {}^I E_2^{p,q} \to {}^{II} E_2^{p,q}$ with the second epimorphism in (19) is equal to the composite map

$$
\begin{aligned}
Z^p({}^I E_1^{\bullet,q}) &= \mathrm{Ker}\big((\mathrm{R}^q f)(\tau_{\le q} Y^p) \to (\mathrm{R}^q f)(\tau_{\le q} Y^{p+1})\big) \\
&\to \mathrm{Ker}\big((\mathrm{R}^q f)(H^q(Y^p)[-q]) \to (\mathrm{R}^q f)(H^q(Y^{p+1})[-q])\big) \\
&= (\mathrm{R}^q f)(Z^p(E_1^{\bullet,q})[-q]) \to (\mathrm{R}^q f)(Z^0(E_1^{\bullet,q})[-q+p]) \\
&= (\mathrm{R}^p f)(H^q(X)) = {}^{II} E_2^{p,q}.
\end{aligned}
$$

As a result, the composition of v with (19) is obtained by applying $\mathrm{R}^q f$ to

$$\tau_{\le q} U^p \to H^q(X^p)[q] \to H^q(X)[-q+p]. \tag{22}$$

Consequently, the composite map

$${}^I D_1^{p,q} = (\mathrm{R}^q f)(\tau_{\le q} X^p) \xrightarrow{j_1} Z^p({}^I E_1^{\bullet,q}) \twoheadrightarrow {}^I E_2^{p,q} \xrightarrow{v} {}^{II} E_2^{p,q}$$

is given by applying $\mathrm{R}^q f$ to

$$\tau_{\le q} X^p \to H^q(X^p)[q] \to H^q(X)[-q+p],$$

and hence is equal to $j_2 u'$. It follows that $v j_2 = v j_1 i_1^{-1} = j_2 u' i_1^{-1} = j_2 u$.

Step 7: The diagram (21) implies that the map (20) coincides with the composition of (22) with the canonical map $H^q(X)[-q+p] \to (\tau_{\le q-1} X)[p+1]$; hence $u k_2 = k_2 v$. Thus the theorem is proved. ☐

Example 2.19. If $K^\bullet$ is a bounded-below filtered complex over A (with a finite filtration)

$$K^\bullet = F^0 K^\bullet \supset F^1 K^\bullet \supset \cdots \supset F^n K^\bullet \supset F^{n+1} K^\bullet = 0,$$

then the objects

$$X^p = F^p K^\bullet[p], \quad Y^p = (F^p K^\bullet / F^{p+1} K^\bullet)[p] = \mathrm{gr}_F^p(K^\bullet)[p] \in D^+(A)$$

form a Postnikov system of the kind considered in (14). The corresponding spectral sequences are equal to

$$\begin{aligned} E_1^{p,q} &= H^{p+q}(\mathrm{gr}_F^p(K^\bullet)) \Rightarrow H^{p+q}(K^\bullet), \\ {}^I E_1^{p,q} &= (\mathrm{R}^{p+q} f)(\mathrm{gr}_F^p(K^\bullet)) \Rightarrow (\mathrm{R}^{p+q} f)(K^\bullet). \end{aligned}$$

In the special case when $K^\bullet$ is the total complex associated to a first quadrant bicomplex $C^{\bullet,\bullet}$ and the filtration F^p is induced by the column filtration on $C^{\bullet,\bullet}$, then the complex $f(K^\bullet)$ over A' is equipped with a canonical filtration $(f F^p)(f(K^\bullet)) = f(F^p K^\bullet)$ satisfying

$$\mathrm{gr}^p_{f(F)}(f(K^\bullet)) = f(\mathrm{gr}_F^p(K^\bullet)).$$

Under the conditions in Remark 2.17, the corresponding exact couple

$$\begin{aligned} {}^f D_1^{p,q} &= H^{p+q}(f(F^p K^\bullet)), \\ {}^f E_1^{p,q} &= H^{p+q}(\mathrm{gr}^p_{f(F)}(f(K^\bullet))) = H^{p+q}(f(\mathrm{gr}_F^p(K^\bullet))) \Rightarrow H^{p+q}(f(K^\bullet)) \end{aligned}$$

then naturally maps to the exact couple (15), hence (beginning from (D_2, E_2)) to the exact couple (16), by Theorem 2.18.

3. Syntomic cohomology

In this section we will define the arithmetic and geometric syntomic cohomologies of varieties over K and $\overline{K}$, respectively, and study their basic properties.

3A. ***Hyodo–Kato morphism revisited.*** We will need to use the Hyodo–Kato morphism on the level of derived categories and vary it in the h-topology. Recall that the original morphism depends on the choice of a uniformizer and a change of such is encoded in a transition function involving the exponential of the monodromy. Since the fields of definition of semistable models in the bases for the h-topology change, we will need to use these transition functions. The problem though is that in the most obvious (i.e., crystalline) definition of the Hyodo–Kato complexes the monodromy is (at best) homotopically nilpotent — making the exponential in the transition functions impossible to define. Beilinson [2013] solves this problem by representing Hyodo–Kato complexes using modules with nilpotent monodromy. In this subsection we will summarize what we need from his approach.

We begin with a quick reminder. Let $(U, \overline{U})$ be a log-scheme, log-smooth over $V^\times$. For any $r \geq 0$, consider its absolute (meaning over $W(k)$) log-crystalline cohomology complexes

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n &:= \mathrm{R}\Gamma(\bar{U}_{\text{ét}},\mathrm{R}u_{U_n^\times/W_n(k)*}\mathscr{J}^{[r]}_{U_n^\times/W_n(k)}),\\ \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]}) &:= \operatorname{holim}_n \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n,\\ \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_{\mathbb{Q}} &:= \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})\otimes\mathbb{Q}_p,\end{aligned}$$

where $U^\times$ denotes the log-scheme $(U,\bar{U})$ and $u_{U_n^\times/W_n(k)}:(U_n^\times/W_n(k))_{\mathrm{cr}}\to\bar{U}_{\text{ét}}$ is the projection from the log-crystalline to the étale topos. For $r\geq 0$, we write $\mathscr{J}^{[r]}_{U_n^\times/W_n(k)}$ for the r-th divided power of the canonical PD-ideal $\mathscr{J}_{U_n^\times/W_n(k)}$; for $r\leq 0$, we set

$$\mathscr{J}^{[r]}_{U_n^\times/W_n(k)} := \mathscr{O}_{U_n^\times/W_n(k)}$$

and we will often omit it from the notation. The absolute log-crystalline cohomology complexes are filtered E_∞ algebras over $W_n(k)$, $W(k)$, or K_0, respectively. Moreover, the rational ones are filtered commutative dg algebras.

Remark 3.1. The canonical pullback map

$$\mathrm{R}\Gamma(\bar{U}_{\text{ét}},\mathrm{R}u_{U_n^\times/W_n(k)*}\mathscr{J}^{[r]}_{U_n^\times/W_n(k)}) \xrightarrow{\sim} \mathrm{R}u_{U_n^\times/\mathbb{Z}/p^n*}\mathscr{J}^{[r]}_{U_n^\times/\mathbb{Z}/p^n}$$

is a quasi-isomorphism. In what follows we will often call both the "absolute crystalline cohomology".

Let $W(k)\langle t_l\rangle$ be the divided-powers polynomial algebra generated by elements t_l, $l\in\mathfrak{m}_K/\mathfrak{m}_K^2\setminus\{0\}$, subject to the relations $t_{al}=[\bar{a}]t_l$ for $a\in V^*$, where $[\bar{a}]\in W(k)$ is the Teichmüller lift of $\bar{a}$ — the reduction mod $\mathfrak{m}_K$ of a. Let R_V (or simply R) be the p-adic completion of the subalgebra of $W(k)\langle t_l\rangle$ generated by t_l and $t_l^{ie_K}/i!$, $i\geq 1$. For a fixed l, the ring R is the following $W(k)$-subalgebra of $K_0[\![t_l]\!]$:

$$R=\left\{\sum_{i=0}^{\infty}a_i\frac{t_l^i}{\lfloor i/e_K\rfloor !}\;\middle|\; a_i\in W(k),\ \lim_{i\to\infty}a_i=0\right\}.$$

One extends the Frobenius φ_R (semilinearly) to R by setting $\varphi_R(t_l)=t_l^p$ and defines a monodromy operator N_R as a $W(k)$-derivation by setting $N_R(t_l)=-t_l$. Let $E:=\operatorname{Spec}(R)$ equipped with the log-structure generated by the t_l.

We have two exact closed embeddings

$$i_0: W(k)^0\hookrightarrow E,\quad i_\pi: V^\times\hookrightarrow E.$$

The first one is canonical and induced by $t_l\mapsto 0$. The second one depends on the choice of the class of the uniformizing parameter $\pi\in\mathfrak{m}_K/p\mathfrak{m}_K$ up to multiplication by Teichmüller elements. It is induced by $t_l\mapsto[\overline{l/\pi}]\pi$.

Assume $(U,\overline{U})$ is of Cartier type (i.e., the special fiber $\overline{U}_0$ is of Cartier type). Consider the log-crystalline and the Hyodo–Kato complexes (see [Beilinson 2013, 1.16])

$$\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,\mathscr{J}^{[r]})_n := \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})_n/R_n,\mathscr{J}^{[r]}_{\overline{U}_n/R_n}),$$
$$\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_n := \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})_0/W_n(k)^0).$$

Let $\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,\mathscr{J}^{[r]})$ and $\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})$ be their homotopy inverse limits. The last complex is called the *Hyodo–Kato complex*. The complex $\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)$ is R-perfect and

$$\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)\otimes^L_R R_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)\otimes^L \mathbb{Z}/p^n.$$

In general, we have $\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,\mathscr{J}^{[r]})_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,\mathscr{J}^{[r]})\otimes^L \mathbb{Z}/p^n$. The complex $\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})$ is $W(k)$-perfect and

$$\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_n \simeq \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})\otimes^L_{W(k)} W_n(k) \simeq \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})\otimes^L \mathbb{Z}/p^n.$$

We normalize the monodromy operators N on the rational complexes $\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}$ and $\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}}$ by replacing the standard N [Hyodo and Kato 1994, 3.6] by $N_R := e_K^{-1}N$. This makes them compatible with base change. The embedding $i_0 : (U,\overline{U})_0 \hookrightarrow (U,\overline{U})$ over $i_0 : W_n(k)^0 \hookrightarrow E_n$ yields compatible morphisms $i^*_{0,n} : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_n \to \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_n$. Completing, we get a morphism

$$i_0^* : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R) \to \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U}),$$

which induces a quasi-isomorphism $i_0^* : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)\otimes^L_R W(k) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})$. All the above objects have an action of Frobenius and these morphisms are compatible with Frobenius. The Frobenius action is invertible on $\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}$.

The map $i_0^* : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}$ admits a unique (in the classical derived category) $W(k)$-linear section ι_π [Beilinson 2013, 1.16; Tsuji 1999, Proposition 4.4.6] that commutes with φ and N. The map ι_π is functorial and its R-linear extension is a quasi-isomorphism

$$\iota_\pi : R\otimes_{W(k)} \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}}.$$

The composition (the *Hyodo–Kato map*)

$$\iota_{\mathrm{dR},\pi} := \gamma_r^{-1} i_\pi^* \cdot \iota_\pi : \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_K),$$

where

$$\gamma_r^{-1} : \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_K)/F^r$$

is the quasi-isomorphism from Corollary 2.4, induces a K-linear functorial quasi-isomorphism (the *Hyodo–Kato quasi-isomorphism*) [Tsuji 1999, Theorem 4.4.8, Corollary 4.4.13]

$$\iota_{\mathrm{dR},\pi} : \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})\otimes_{W(k)} K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_K). \tag{23}$$

We now describe the Beilinson–Hyodo–Kato morphism and provide a few examples. Let $S_n = \mathrm{Spec}(\mathbb{Z}/p^n)$ equipped with the trivial log-structure and let $S = \mathrm{Spf}(\mathbb{Z}_p)$ be the induced formal log-scheme. For any log-scheme $Y \to S_1$, let $D_\varphi((Y/S)_{\mathrm{cr}}, \mathscr{O}_{Y/S})$ denote the derived category of Frobenius $\mathscr{O}_{Y/S}$-modules and $D_\varphi^{\mathrm{pcr}}(Y/S)$ its thick subcategory of perfect F-crystals, i.e., those Frobenius modules that are perfect crystals [Beilinson 2013, 1.11]. We call a perfect F-crystal $(\mathscr{F}, \varphi)$ *nondegenerate* if the map $L\varphi^*(\mathscr{F}) \to \mathscr{F}$ is an isogeny. The corresponding derived category is denoted by $D_\varphi^{\mathrm{pcr}}(Y/S)^{\mathrm{nd}}$. It has a dg category structure [Beilinson 2013, 1.14] that we denote by $\mathscr{D}_\varphi^{\mathrm{pcr}}(Y/S)^{\mathrm{nd}}$. We will omit S if understood.

Suppose now that Y is a fine log-scheme that is affine. Assume also that there is a PD-thickening $P = \mathrm{Spf}\, R$ of Y that is formally smooth over S and such that R is a p-adically complete ring with no p-torsion. Let $f : Z \to Y$ be a log-smooth map of Cartier type with Z fine and proper over Y. Beilinson [2013, 1.11, 1.14] proves the following theorem.

Theorem 3.2. *The complex $\mathscr{F} := Rf_{\mathrm{cr}\,*}(\mathscr{O}_{Z/S})$ is a nondegenerate perfect F-crystal.*

Let $D_{\varphi,N}(K_0)$ denote the bounded derived category of (φ, N)-modules. By [Beilinson 2013, 1.15], it has a dg category structure that we will denote by $\mathscr{D}_{\varphi,N}(K_0)$. We call a (φ, N)-module *effective* if it contains a $W(k)$-lattice preserved by φ and N. Denote by $\mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \subset \mathscr{D}_{\varphi,N}(K_0)$ the bounded derived category of the abelian category of effective modules.

Let $f : Y \to k^0$ be a log-scheme. We think of k^0 as $W(k)_1^\times$. Then the map f is given by a k-structure on Y plus a section $l = f^*(\bar{p}) \in \Gamma(Y, M_Y)$ such that its image in $\Gamma(Y, \mathscr{O}_Y)$ equals 0. We will often write $f = f_l$, $l = l_f$.

Beilinson [2013, 1.15] proves the following theorem.

Theorem 3.3. (1) *There is a natural functor*

$$\varepsilon_f = \varepsilon_l : \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \to \mathscr{D}_\varphi^{\mathrm{pcr}}(Y)^{\mathrm{nd}} \otimes \mathbb{Q}. \tag{24}$$

(2) *ε_f is compatible with base change; i.e., for any $\theta : Y' \to Y$, one has a canonical identification $\varepsilon_{f\theta} \xrightarrow{\sim} L\theta_{\mathrm{cr}}^* \varepsilon_f$. For any $a \in k^*$, $m \in \mathbb{Z}_{>0}$, there is a canonical identification $\varepsilon_{al^m}(V, \varphi, N) \xrightarrow{\sim} \varepsilon_l(V, \varphi, mN)$.*

(3) *Suppose that Y is a local scheme with residue field k and nilpotent maximal ideal, $M_Y/\mathscr{O}_Y^* = \mathbb{Z}_{>0}$, and the map $f^* : M_{k^0}/k^* \to M_Y/\mathscr{O}_Y^*$ is injective. Then* (24) *is an equivalence of dg categories.*

In particular, we have an equivalence of dg categories

$$\varepsilon := \varepsilon_{\bar{p}} : \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \xrightarrow{\sim} \mathscr{D}_\varphi^{\mathrm{pcr}}(k^0)^{\mathrm{nd}} \otimes \mathbb{Q}$$

and a canonical identification $\varepsilon_f = Lf_{\mathrm{cr}}^* \varepsilon$.

On the level of sections, the functor (24) has a simple description [Beilinson 2013, 1.15.3]. Assume that $Y = \mathrm{Spec}(A/J)$, where A is a p-adic algebra and J is a PD-ideal in A, and that we have a PD-thickening $i : Y \hookrightarrow T = \mathrm{Spf}(A)$. Let $\lambda_{l,n}$ be the preimage of l under the map $\Gamma(T_n, M_{T_n}) \to i_*\Gamma(Y, M_Y)$. It is a trivial $(1 + J_n)^\times$-torsor. Set

$$\lambda_A := \varprojlim_n \Gamma(T_n, \lambda_{l,n}).$$

It is a $(1 + J)^\times$-torsor. Let $\tau_{A_\mathbb{Q}}$ be the *Fontaine–Hyodo–Kato torsor*, i.e., the $A_\mathbb{Q}$-torsor obtained from λ_A by the pushout by $(1 + J)^\times \xrightarrow{\log} J \to A_\mathbb{Q}$. We call the $\mathbb{G}_a$-torsor Spec $A^\tau_\mathbb{Q}$ over Spec $A_\mathbb{Q}$ with sections $\tau_{A_\mathbb{Q}}$ the same name. Denote by N_τ the $A_\mathbb{Q}$-derivation of $A^\tau_\mathbb{Q}$ given by the action of the generator of $\mathrm{Lie}_{\mathbb{G}_a}$.

Let M be an (φ, N)-module. Integrating the action of the monodromy N_M, we get an action of the group $\mathbb{G}_a$ on M. Denote by $M^\tau_{A_\mathbb{Q}}$ the $\tau_{A_\mathbb{Q}}$-twist of $M_{A_\mathbb{Q}} := M \otimes_{K_0} A_\mathbb{Q}$. It can be represented as the module of maps $v : \tau_{A_\mathbb{Q}} \to M_{A_\mathbb{Q}}$ that are $A_\mathbb{Q}$-equivariant, i.e., such that $v(\tau + a) = \exp(aN)(v(\tau))$, where $\tau \in \tau_{A_\mathbb{Q}}$, $a \in A_\mathbb{Q}$. We can also write

$$M^\tau_{A_\mathbb{Q}} = (M \otimes_{K_0} A^\tau_\mathbb{Q})^{\mathbb{G}_a} = (M \otimes_{K_0} A^\tau_\mathbb{Q})^{N=0},$$

where $N := N_M \otimes 1 + 1 \otimes N_\tau$. Now, by definition,

$$\varepsilon_f(M)(Y, T) = M^\tau_{A_\mathbb{Q}}. \tag{25}$$

The algebra $A^\tau_\mathbb{Q}$ has a concrete description. Take the natural map $a : \tau_{A_\mathbb{Q}} \to A^\tau_\mathbb{Q}$ of $A_\mathbb{Q}$-torsors which maps $\tau \in \tau_{A_\mathbb{Q}}$ to a function $a(\tau) \in A^\tau_\mathbb{Q}$ whose value on any $\tau' \in \tau_{A_\mathbb{Q}}$ is $\tau - \tau' \in A_\mathbb{Q}$. This map is compatible with the logarithm $\log : (1 + J)^\times \to A$. The algebra $A^\tau_{A_\mathbb{Q}}$ is freely generated over $A_\mathbb{Q}$ by $a(\tau)$ for any $\tau \in \tau_{A_\mathbb{Q}}$; the $A_\mathbb{Q}$-derivation N_τ is defined by $N_\tau(a(\tau)) = -1$. That is, for chosen $\tau \in \tau_{A_\mathbb{Q}}$, we can write

$$A^\tau_\mathbb{Q} = A_\mathbb{Q}[a(\tau)], \quad N_\tau(a(\tau)) = -1.$$

For every lifting φ_T of Frobenius to T, we have $\varphi_T^*\lambda_A = \lambda_A^p$. Hence φ_T extends canonically to a Frobenius φ_τ on $A^\tau_\mathbb{Q}$ in such a way that $N_\tau\varphi_\tau = p\varphi_\tau N_\tau$. The isomorphism (25) is compatible with Frobenius.

Example 3.4. As an example, consider the case when the pullback map

$$f^* : \mathbb{Q} = (M_{k^0}/k^*)^{\mathrm{gp}} \otimes \mathbb{Q} \xrightarrow{\sim} (\Gamma(Y, M_Y)/k^*)^{\mathrm{gp}} \otimes \mathbb{Q}$$

is an isomorphism. We have a surjection $v : (\Gamma(T, M_T)/k^*)^{\mathrm{gp}} \otimes \mathbb{Q} \to \mathbb{Q}$ with the kernel $\log : (1 + J)^\times_\mathbb{Q} \xrightarrow{\sim} J_\mathbb{Q} = A_\mathbb{Q}$. We obtain an identification of $A_\mathbb{Q}$-torsors $\tau_{A_\mathbb{Q}} \simeq v^{-1}(1)$. Hence every noninvertible $t \in \Gamma(T, M_T)$ yields an element $t^{1/v(t)} \in v^{-1}(1)$ and a trivialization of $\tau_{A_\mathbb{Q}}$.

For a fixed element $t^{1/v(t)} \in v^{-1}(1)$, we can write

$$A^\tau_{A_\mathbb{Q}} = A_\mathbb{Q}[a(t^{1/v(t)})], \quad N_\tau(a(t^{1/v(t)})) = -1.$$

For an (φ, N)-module M, the twist $M^{\tau}_{A_{\mathbb{Q}}}$ can be trivialized:

$$\beta_t : M\otimes_{K_0} A_{\mathbb{Q}} \xrightarrow{\sim} M^{\tau}_{A_{\mathbb{Q}}} = (M\otimes_{K_0} A_{\mathbb{Q}}[a(t^{1/v(t)})])^{N=0},$$
$$m \mapsto \exp(N_M(m)a(t^{1/v(t)})).$$

For a different choice $t_1^{1/v(t_1)} \in v^{-1}(1)$, the two trivializations β_t, β_{t_1} are related by the formula

$$\beta_{t_1} = \beta_t \exp(N_M(m)a(t_1,t)), \quad a(t_1,t) = a(t_1)/v(t_1) - a(t)/v(t).$$

Consider the map $f : V_1^{\times} \to k^0$. By Theorem 3.3, we have the equivalences of dg categories

$$\varepsilon : \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \xrightarrow{\sim} \mathscr{D}^{\mathrm{pcr}}_{\varphi}(k^0)^{\mathrm{nd}} \otimes \mathbb{Q},$$
$$\varepsilon_f = Lf^*_{\mathrm{cr}}\varepsilon : \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \xrightarrow{\sim} \mathscr{D}^{\mathrm{pcr}}_{\varphi}(V_1^{\times})^{\mathrm{nd}} \otimes \mathbb{Q}.$$

Let $Z_1 \to V_1^{\times}$ be a log-smooth map of Cartier type with Z_1 fine and proper over V_1. By Theorem 3.2, $Rf_{\mathrm{cr}*}(\mathscr{O}_{Z_1/\mathbb{Z}_p})$ is a nondegenerate perfect F-crystal on $V_{1,\mathrm{cr}}$. Set

$$\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(Z_1) := \varepsilon_f^{-1} Rf_{\mathrm{cr}*}(\mathscr{O}_{Z_1/\mathbb{Z}_p})_{\mathbb{Q}} \in \mathscr{D}_{\varphi,N}(K_0).$$

We will call it the *Beilinson–Hyodo–Kato complex* [Beilinson 2013, 1.16.1].

Example 3.5. To get familiar with the Beilinson–Hyodo–Kato complexes we will work out some examples.

(1) Let $g : X \to V^{\times}$ be a log-smooth log-scheme, proper, and of Cartier type. Adjunction yields a quasi-isomorphism

$$\varepsilon_f \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_1) = \varepsilon_f\varepsilon_f^{-1} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})_{\mathbb{Q}} \xrightarrow{\sim} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})_{\mathbb{Q}}. \tag{26}$$

Evaluating it on the PD-thickening $V_1^{\times} \hookrightarrow V^{\times}$ (here $A = V$, $J = pV$, $l = \bar{p}$, $\lambda_V = p(1+J)^{\times}$, and $\tau_K = p(1+J)^{\times} \times_{(1+J)^{\times}} K$), we get a map

$$\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_1)^{\tau}_K = \varepsilon_f \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_1)(V_1^{\times} \hookrightarrow V^{\times}) \xrightarrow{\sim} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})(V_1^{\times} \hookrightarrow V^{\times})_{\mathbb{Q}}$$
$$= \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/V^{\times})_{\mathbb{Q}} \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X/V^{\times})_{\mathbb{Q}} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X_K).$$

We will call it the *Beilinson–Hyodo–Kato map* [Beilinson 2013, 1.16.3]

$$\iota^{B}_{\mathrm{dR}} : \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_1)^{\tau}_K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_K). \tag{27}$$

Recall that

$$\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_1)^{\tau}_K = (\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_1)\otimes_{K_0} K[a(\tau)])^{N=0}, \quad \tau \in \tau_K.$$

This makes it clear that the Beilinson–Hyodo–Kato map is not only functorial for log-schemes over $V^{\times}$ but, by Theorem 3.3, it is also compatible with base change of $V^\times$. Moreover, if we use the canonical trivialization by p

$$\beta=\beta_p : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)_K \xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{HK}}(X)^\tau_K=(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)\otimes_{K_0} K[a(p)])^{N=0},$$
$$x\mapsto \exp(N(x)a(p)),$$

we get that the composition (which we also call the Beilinson–Hyodo–Kato map and denote by ι^B_{dR})

$$\iota^B_{\mathrm{dR}}=\iota^B_{\mathrm{dR}}\beta : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)\to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)$$

is functorial and compatible with base change.

(2) Evaluating the map (26) on the PD-thickening $V_1^\times \hookrightarrow E$ associated to a uniformizer π (here $A=R$, $l=\bar{p}$), we get a map

$$\kappa_R : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_{\mathbb{Q}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X/R)_{\mathbb{Q}} \tag{28}$$

as the composition

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_{\mathbb{Q}}}=\varepsilon_f\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)(V_1^\times\hookrightarrow E)\xrightarrow{\sim} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})(V_1^\times\hookrightarrow E)_{\mathbb{Q}}$$
$$=\mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{\mathbb{Q}}\simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X/R)_{\mathbb{Q}}.$$

We have

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_{\mathbb{Q}}}=(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)\otimes_{K_0} R_{\mathbb{Q}}[a(\tau)])^{N=0},\quad \tau\in\tau_{R_{\mathbb{Q}}}.$$

Since the map κ_R is compatible with the log-connection on R it is also compatible with the normalized monodromy operators. Specifically, if we define the monodromy on the left-hand side of (28) as

$$N : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_{\mathbb{Q}}}\to \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_{\mathbb{Q}}},$$
$$\sum_I m_{\tau_I}\otimes r_{\tau_I}a^{k_I}(\tau_I)\mapsto \sum_I (N_M(m_{\tau_I})\otimes r_{\tau_I}a^{k_I}(\tau_I)+m_{\tau_I}\otimes N_R(r_{\tau_I})a^{k_I}(\tau_I)),$$

the two operators will correspond under the map κ_R.

The exact immersion $i_\pi : V^\times\hookrightarrow E$ yields a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_{\mathbb{Q}}} \arrow[r, "\sim"] \arrow[d, "i_\pi^*"] & \mathrm{R}\Gamma_{\mathrm{cr}}(X/R)_{\mathbb{Q}} \arrow[d, "i_\pi^*"] \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_K \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{cr}}(X/V^\times)_{\mathbb{Q}}
\end{tikzcd}$$

If $p=u\pi^{e_K}$, $u\in V^\times$, we have $\lambda_R=\tilde{u}t_\pi^{e_K}(1+J)^\times$, where $\tilde{u}\in R$ is such that $\tilde{u}$ lifts u. Alternatively, $\lambda_R=[\bar{u}]t_\pi^{e_K}(1+J)^\times$. We have the associated trivialization

$$\beta_\pi : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)\otimes_{K_0} R_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_{\mathbb{Q}}}=(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)\otimes_{K_0} R_{\mathbb{Q}}[a(\tau_\pi)])^{N=0},$$
$$x\mapsto \exp(N(x)a(\tau_\pi)),$$

where $\tau_\pi:=[\bar{u}]t_\pi^{e_K}$.

(3) Consider the log-scheme k_1^0: the scheme $\mathrm{Spec}(k)$ with the log-structure induced by the exact closed immersion $i : k_1^0 \hookrightarrow V_1^\times$. We have the commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] \arrow[d, "g_0"] & X_1 \arrow[d, "g"] \\
k_1^0 \arrow[r, hook, "i"] \arrow[dr, "f_0"'] & V_1^\times \arrow[d, "f"] \\
& k^0
\end{tikzcd}$$

The morphisms f, f_0 map $\bar{p}$ to $\bar{p}$. By log-smooth base change we have a canonical quasi-isomorphism $Li^*Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p}) \simeq Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})$. By Theorem 3.3 we have the equivalence of dg categories

$$\varepsilon_{f_0} : \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \xrightarrow{\sim} \mathscr{D}_{\varphi}^{\mathrm{pcr}}(k_1^0)^{\mathrm{nd}} \otimes \mathbb{Q}, \quad \varepsilon_{f_0} = Li^*\varepsilon_f.$$

This implies the natural quasi-isomorphisms

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_1) &= \varepsilon_f^{-1} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})_{\mathbb{Q}} \simeq \varepsilon_{f_0}^{-1} Li^* Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})_{\mathbb{Q}} \\
&\simeq \varepsilon_{f_0}^{-1} Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})_{\mathbb{Q}}.
\end{aligned}$$

Hence, by adjunction,

$$\varepsilon_{f_0}\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_1) = \varepsilon_{f_0}\varepsilon_{f_0}^{-1} Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})_{\mathbb{Q}} \simeq Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})_{\mathbb{Q}}.$$

We will evaluate both sides on the PD-thickening $k_1^0 \hookrightarrow W(k)^0$. Here we write the log-structure on $W(k)^0$ as associated to the map $\Gamma(V^\times, M_{V^\times}) \to k \to W(k)$, $a \mapsto \bar{a}$. We take $A = W(k)$, $l = p$, $J = pW(k)$, $\lambda_{W(k)} = \bar{p}(1 + pW(k))^\times$ and $\tau_{K_0} = \bar{p}(1 + pW(k))^\times \times_{(1+pW(k))^\times} K_0$. We get a quasi-isomorphism

$$\kappa : \mathrm{R}\Gamma_{\mathrm{HK}}^B(X_1)_{K_0}^{\tau} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}$$

as the composition

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_1)_{K_0}^{\tau} &= \varepsilon_{f_0}\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_1)(k_1^0 \hookrightarrow W(k)^0) \simeq Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})(k_1^0 \hookrightarrow W(k)^0)_{\mathbb{Q}} \\
&= \mathrm{R}\Gamma_{\mathrm{cr}}(X_0/W(k)^0)_{\mathbb{Q}} = \mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}.
\end{aligned}$$

To compare the monodromy operators on both sides of the map κ, note that by Theorem 3.3, we have the canonical identification

$$Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})_{\mathbb{Q}} \simeq \varepsilon_{f_0}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_1), N) \simeq \varepsilon_{\bar{p}}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_1), e_K N).$$

Hence, from the description of the Hyodo–Kato monodromy [1994, 3.6], it follows easily that the map κ pairs the operator N on $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0}$ defined by

$$N\Big(\sum_I m_{\tau_I}\otimes r_{\tau_I}a^{k_I}(\tau_I)\Big)=\sum_I\big(N_M(m_{\tau_I})\otimes r_{\tau_I}a^{k_I}(\tau_I)+m_{\tau_I}\otimes N_R(r_{\tau_I})a^{k_I}(\tau_I)\big),$$

with the normalized Hyodo–Kato monodromy on $\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}$.

Composing the map κ with the trivialization

$$\begin{aligned}\beta=\beta_p:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)\xrightarrow{\sim}\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0}&=(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)[a(\bar{p})])^{N=0},\\ x\mapsto \exp(N(x)a(\bar{p})),&\end{aligned}$$

we get a quasi-isomorphism between Beilinson–Hyodo–Kato complexes and the (classical) Hyodo–Kato complexes:

$$\kappa=\beta\kappa:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}. \tag{29}$$

The trivialization above is compatible with Frobenius and the normalized monodromy; hence so is the quasi-isomorphism (29). It is clearly functorial and, by Theorem 3.3, compatible with base change.

By functoriality (Theorem 3.3), the morphism of PD-thickenings (exact closed immersion) $i_0:(k_1^0\hookrightarrow W(k)^0)\hookrightarrow(V_1^{\times}\hookrightarrow R)$ yields the right square in the diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}} \arrow[r, "\iota_\pi"] & \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{\mathbb{Q}} \arrow[r, "i_0^*"] & \mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}} \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0} \arrow[u, "\kappa"', "\wr"] \arrow[r, "\iota_\pi"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{R_{\mathbb{Q}}} \arrow[u, "\kappa_R"', "\wr"] \arrow[r, "i_0^*"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0} \arrow[u, "\kappa"', "\wr"]
\end{tikzcd} \tag{30}$$

In the left square, the bottom map ι_π is induced by the natural map $K_0\to R$ and by sending $a(\bar{p})\mapsto a(\tau_\pi)$. It is a (right) section to i_0^* and it (together with the vertical maps) commutes with Frobenius. By uniqueness of the top map ι_π this makes the left square commute in the classical derived category (of abelian groups).

It is easy to check that we have the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0} \arrow[r, "\iota_\pi"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{R_{\mathbb{Q}}} \arrow[r, "i_\pi^*"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K} \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) \arrow[u, "\beta_p", "\wr"'] \arrow[rr, "\mathrm{can}"] & & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)_K \arrow[u, "\beta_p", "\wr"']
\end{tikzcd}$$

and that the composition of maps on the top of it is equal to the map induced by the canonical map $K_0\to K$ and the map $\lambda_{W(k)^0}\to\lambda_V^{\times}$, $\bar{p}\to p$.

Combining the commutative diagrams in parts (2) and (3) of this example, we get the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}(X) \arrow[r, "\iota_\pi"] & \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{\mathbb{Q}} \arrow[r, "i_\pi^*"] & \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/V^\times)_{\mathbb{Q}} \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{K_0} \arrow[u, "\kappa", "\wr"'] \arrow[r, "\iota_\pi"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_{\mathbb{Q}}} \arrow[u, "\kappa_R", "\wr"'] \arrow[r, "i_\pi^*"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_K \arrow[u, "\iota^B_{\mathrm{dR}}", "\wr"'] \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) \arrow[u, "\beta_p", "\wr"'] \arrow[rr, "\mathrm{can}"] & & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)_K \arrow[u, "\beta_p", "\wr"']
\end{tikzcd}$$

Since the composition of the top maps is equal to the Hyodo–Kato map ι_{dR} and the bottom map is just the canonical map $\mathrm{R}\Gamma_{\mathrm{HK}}(X_1) \to \mathrm{R}\Gamma_{\mathrm{HK}}(X_1)_K$, we obtain that the Hyodo–Kato and the Beilinson–Hyodo–Kato maps are related by a natural quasi-isomorphism; i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}(X) \arrow[r, "\iota_{\mathrm{dR},\pi}"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X_K) \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) \arrow[u, "\kappa", "\wr"'] \arrow[ur, "\iota^B_{\mathrm{dR}}"']
\end{tikzcd} \tag{31}$$

The above examples can be generalized [Beilinson 2013, 1.16]. It turns out that the relative crystalline cohomology of all the base changes of the map f can be described using the Beilinson–Hyodo–Kato complexes [loc. cit., 1.16.2]. Namely, let $\theta : Y \to V_1^\times$ be an affine log-scheme and let T be a p-adic PD-thickening of Y, that is, $T = \mathrm{Spf}(A)$, $Y = \mathrm{Spec}(A/J)$. Denote by $f_Y : Z_{1Y} \to Y$ the θ-pullback of f. Beilinson [2013, 1.16.2] proved the following theorem.

Theorem 3.6. (1) *The A-complex $\mathrm{R}\Gamma_{\mathrm{cr}}(Z_{1Y}/T, \mathscr{O}_{Z_{1Y/T}})$ is perfect, and one has*

$$\mathrm{R}\Gamma_{\mathrm{cr}}(Z_{1Y}/T_n, \mathscr{O}_{Z_{1Y/T_n}}) = \mathrm{R}\Gamma_{\mathrm{cr}}(Z_{1Y}/T, \mathscr{O}_{Z_{1Y/T}}) \otimes^L \mathbb{Z}/p^n.$$

(2) *There exists a canonical Beilinson–Hyodo–Kato quasi-isomorphism of $A_{\mathbb{Q}}$-complexes*:

$$\kappa^B_{A_{\mathbb{Q}}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)^\tau_{A_{\mathbb{Q}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(Z_{1Y}/T, \mathscr{O}_{Z_{1Y/T}})_{\mathbb{Q}}.$$

If there is a Frobenius lifting φ_T, then $\kappa^B_{A_{\mathbb{Q}}}$ commutes with its action.

3B. *Log-syntomic cohomology.* We will study now (rational) log-syntomic cohomology. Let $(U, \overline{U})$ be log-smooth over $V^\times$. For $r \geq 0$, define the mod p^n, completed, and rational log-syntomic complexes

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_n &:= \mathrm{Cone}(\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{J}^{[r]})_n \xrightarrow{p^r-\varphi} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_n)[-1],\\ \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r) &:= \mathrm{holim}_n\, \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_n, \\ \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}} &:= \mathrm{Cone}(\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{J}^{[r]})_{\mathbb{Q}} \xrightarrow{1-\varphi_r} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_{\mathbb{Q}})[-1]. \end{aligned} \tag{32}$$

Here the Frobenius φ is defined by the composition

$$\begin{aligned} \varphi : \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{J}^{[r]})_n \to \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_n \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})_1/W(k))_n \\ \xrightarrow{\varphi} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})_1/W(k))_n \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_n \end{aligned}$$

and $\varphi_r := \varphi/p^r$. The mapping fibers are taken in the ∞-derived category of abelian groups. The direct sums

$$\bigoplus_{r\geq 0} \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_n, \quad \bigoplus_{r\geq 0} \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r), \quad \bigoplus_{r\geq 0} \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}}$$

are graded E_∞ algebras over $\mathbb{Z}/p^n$, $\mathbb{Z}_p$, and $\mathbb{Q}_p$, respectively [Hinich and Schechtman 1987, Theorem 1.6]. The rational log-syntomic complexes are moreover graded commutative dg algebras over $\mathbb{Q}_p$ [Hinich and Schechtman 1987, Theorem 4.1; Groth 2010, Perspective 3.22; Lurie 2016]. An explicit definition of syntomic product structure can be found in [Tsuji 1999, Section 2.2].

We have $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_n \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)\otimes^L \mathbb{Z}/p^n$. There is a canonical quasi-isomorphism of graded E_∞ algebras

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_n \xrightarrow{\sim} \mathrm{Cone}(\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_n \\ \xrightarrow{(p^r-\varphi,\mathrm{can})} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_n \oplus \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{O}/\mathscr{J}^{[r]})_n)[-1]. \end{aligned}$$

The completed and rational cases are similar.

Since, by Corollary 2.4, there is a quasi-isomorphism

$$\gamma_r^{-1} : \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_K)/F^r,$$

we have a very nice canonical description of rational log-syntomic cohomology:

$$\begin{aligned} &\mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}} \\ &\quad \xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\gamma_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_{\mathbb{Q}} \oplus \mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_K)/F^r)\big], \end{aligned}$$

where square brackets stand for mapping fiber.

Remark 3.7. In the above definition, one can replace the map $1-\varphi_r$ with any polynomial map $P \in 1 + XK[X]$ to obtain the analog of Besser's finite polynomial cohomology. This was studied in [Besser et al. 2016].

For arithmetic pairs $(U,\overline{U})$ that are log-smooth over $V^{\times}$ and of Cartier type, this can be simplified further by using Hyodo–Kato complexes (see Proposition 3.8 below). To do that, consider the following sequence of maps of homotopy limits. Homotopy limits are taken in the ∞-derived category (to do that we define the maps ι_π by the zigzag from diagram (30)). We will describe the coherence data only if they are nonobvious:

$$
\begin{aligned}
&\mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}}\\
&\xrightarrow{\sim}\ \Big[\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\gamma_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_{\mathbb{Q}}\oplus \mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_K)/F^r\Big]\\
&\xrightarrow{\sim}\ \left[\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}} & \xrightarrow{(1-\varphi_r,i_\pi^*\gamma_r^{-1})} & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}}\oplus \mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_K)/F^r\\
\downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)}\\
\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}}
\end{array}\right]\\
&\xleftarrow{\iota_\pi}\ \left[\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}} & \xrightarrow{(1-\varphi_r,\iota_{\mathrm{dR},\pi})} & \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}\oplus \mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_K)/F^r\\
\downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)}\\
\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}
\end{array}\right].
\end{aligned}
$$

The first map was described above. The second one is induced by the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})\to \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)\xrightarrow{N}\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R).$$

The third one is induced by the section $\iota_\pi:\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}\to \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}}$ (notice that $\iota_{\mathrm{dR},\pi}=\gamma_r^{-1}i_\pi^*\iota_\pi$). We will show below that the third map is a quasi-isomorphism.

Set $C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})\{r\})$ equal to the last homotopy limit in the above diagram.

Proposition 3.8. *Let $(U,\overline{U})$ be an arithmetic pair that is log-smooth over $V^{\times}$ and of Cartier type. Let $r\geq 0$. Then the above diagram defines a canonical quasi-isomorphism*:

$$\alpha_{\mathrm{syn},\pi}:\mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}}\xrightarrow{\sim} C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})\{r\}).$$

Proof. We need to show that the map ι_π in the above diagram is a quasi-isomorphism. Define complexes ($r\geq -1$)

$$
\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,r)&:=\mathrm{Cone}\big(\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}}\xrightarrow{1-\varphi_r}\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)_{\mathbb{Q}}\big)[-1],\\
\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U},r)&:=\mathrm{Cone}\big(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}\xrightarrow{1-\varphi_r}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}\big)[-1].
\end{aligned}
$$

It suffices to prove that the maps

$$
\begin{aligned}
i_0^* &: \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R,r) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r),\\
\iota_\pi &: \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R,r)
\end{aligned}
\tag{33}
$$

are quasi-isomorphisms. Since $i_0^*\iota_\pi=\mathrm{Id}$, it suffices to show that the map i_0^* is a quasi-isomorphism. Base-changing to $W(\bar{k})$, we may assume that the residue field of V is algebraically closed. It suffices to show that, for $i\geq 0$ and $t\geq -1$, in the commutative diagram

$$
\begin{tikzcd}
H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \arrow[r, "p^t-\varphi"] & H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \\
H^i_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}} \arrow[r, "p^t-\varphi"] \arrow[u, "i_0^*"] & H^i_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}} \arrow[u, "i_0^*"]
\end{tikzcd}
$$

the vertical maps induce isomorphisms between the kernels and cokernels of the horizontal maps.

Since the $W(k)$-linear map ι_π commutes with φ and its R-linear extension is a quasi-isomorphism

$$
\iota_\pi : R\otimes_{W(k)}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}},
$$

it suffices to show that in the commutative diagram

$$
\begin{tikzcd}
H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \arrow[r, "p^t-\varphi"] & H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \\
R\otimes_{W(k)} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \arrow[r, "p^t-\varphi"] \arrow[u, "i_0\otimes\mathrm{Id}"] & R\otimes_{W(k)} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \arrow[u, "i_0\otimes\mathrm{Id}"]
\end{tikzcd}
$$

the vertical maps induce isomorphisms between the kernels and cokernels of the horizontal maps. This will follow if we show that the map

$$
I\otimes_{W(k)} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \xrightarrow{p^t-\varphi} I\otimes_{W(k)} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}},
$$

for $I\subset R$, where I is the kernel of the projection $i_0: R_{\mathbb{Q}}\to K_0$, $t_l\mapsto 0$, is an isomorphism. We argue as in [Langer 1999, p. 210]. Let $M:=H^i_{\mathrm{HK}}(U,\bar{U})/\mathrm{tor}$. It is a lattice in $H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}$ that is stable under Frobenius. Consider the formal inverse $\psi:=\sum_{n\geq 0}(p^{-t}\varphi)^n$ of $1-p^{-t}\varphi$. It suffices to show that, for $y\in I\otimes_{W(k)}M$, $\psi(y)\in I\otimes_{W(k)}M$. Fix l and let $T^{\{k\}}:=t_l^k/\lfloor k/e_K\rfloor!$. We will show that, for any $m\in M$, we have $\psi(T^{\{k\}}\otimes m)\in I\otimes_{W(k)}M$ and the infinite series converges uniformly in k. We have

$$
(p^{-t}\varphi)^n(T^{\{k\}}\otimes m)=\frac{\lfloor kp^n/e_K\rfloor!}{\lfloor k/e_K\rfloor!\,p^{tn}}T^{\{kp^n\}}\otimes m'
$$

and $\mathrm{ord}_p(\lfloor kp^n/e_K\rfloor!/\lfloor k/e_K\rfloor!) \geq p^{n-1}$. Hence $\lfloor kp^n/e_K\rfloor!/(\lfloor k/e_K\rfloor! p^{tn})$ converges p-adically to zero, uniformly in k, as wanted. $\square$

Remark 3.9. It was Langer [1999, p. 193] (see [Nekovář 1998, Lemma 2.13] in the good reduction case) who observed the fact that while, in general, the crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U})$ behaves badly (it is "huge"), after taking "filtered Frobenius eigenspaces" we obtain syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}}$ that behaves well (it is "small"). In [Nekovář 2000, 3.5] this phenomenon is explained by relating syntomic cohomology to the complex $C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})\{r\})$.

Remark 3.10. The construction of the map $\alpha_{\mathrm{syn},\pi}$ depends on the choice of the uniformizer π, which makes the h-sheafification impossible. We will show now that there is a functorial and compatible-with-base-change quasi-isomorphism α'_{syn} between rational syntomic cohomology and certain complexes built from Hyodo–Kato cohomology and de Rham cohomology that h-sheafify well.

Set

$$\begin{aligned}\alpha'_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} &\xrightarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r) \xrightarrow{\gamma_r^{-1}} \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/F^r] \\ &\xrightarrow{\beta} [\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U}, r)^{N=0} \xrightarrow{\iota'_{\mathrm{dR}}} \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/F^r].\end{aligned}$$

Here the two morphisms β and ι'_{dR} are defined as the following compositions

$$\begin{aligned}\beta : \mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r) &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U_0, \overline{U}_0, r) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U}, r)^{N=0}, \\ \iota'_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U}, r)^{N=0} &\xleftarrow{\beta} \mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r) \xrightarrow{\gamma_r^{-1}} \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K),\end{aligned}$$

where $(\cdots)^{N=0}$ denotes the mapping fiber of the monodromy. The map β is a quasi-isomorphism because so is each of the intermediate maps. To see this, for the map $i_0^* : \mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r) \to \mathrm{R}\Gamma_{\mathrm{cr}}(U_0, \overline{U}_0, r)$, consider the factorization

$$F^m : \mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r) \xrightarrow{i_0^*} \mathrm{R}\Gamma_{\mathrm{cr}}(U_0, \overline{U}_0, r) \xrightarrow{\psi_m} \mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r)$$

of the m-th power of the Frobenius, where m is large enough. We also have $i_0^*\psi_m = F^m$. Because Frobenius is a quasi-isomorphism on $\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r)$ and $\mathrm{R}\Gamma_{\mathrm{cr}}(U_0, \overline{U}_0, r)$, both i_0^* and ψ_m are quasi-isomorphisms as well. The second morphism in the sequence defining β is a quasi-isomorphism by an argument similar to the one we used in the proof of Proposition 3.8.

Define the complex

$$C'_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})\{r\}) := [\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U}, r)^{N=0} \xrightarrow{\iota'_{\mathrm{dR}}} \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/F^r].$$

We have obtained a quasi-isomorphism

$$\alpha'_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} C'_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})\{r\}).$$

It is clearly functorial but it is also easy to check that it is compatible with base change (of the base V).

Define the complex

$$C_{\rm st}(\mathrm{R}\Gamma^B_{\rm HK}(U,\overline{U})\{r\}) := \big[\mathrm{R}\Gamma^B_{\rm HK}(U_1,\overline{U}_1,r)^{N=0}\xrightarrow{\iota^B_{\rm dR}}\mathrm{R}\Gamma_{\rm dR}(U,\overline{U}_K)/F^r\big].$$

From the commutative diagram (31) we obtain the natural quasi-isomorphisms

$$\gamma: C_{\rm st}(\mathrm{R}\Gamma^B_{\rm HK}(U,\overline{U})\{r\})\xrightarrow{\sim} C_{\rm st}(\mathrm{R}\Gamma_{\rm HK}(U,\overline{U})\{r\}),$$
$$\alpha^B_{{\rm syn},\pi}:=\gamma^{-1}\alpha_{{\rm syn},\pi}:\mathrm{R}\Gamma_{\rm syn}(U,\overline{U},r)_{\mathbb{Q}}\xrightarrow{\sim} C_{\rm st}(\mathrm{R}\Gamma^B_{\rm HK}(U,\overline{U})\{r\}).$$

We will show now that log-syntomic cohomology satisfies finite Galois descent. Let $(U,\overline{U})$ be a fine log-scheme, log-smooth over $V^\times$, and of Cartier type. Let $r\geq 0$. Let K' be a finite Galois extension of K and let $G=\mathrm{Gal}(K'/K)$. Let $(T,\overline{T})=(U\times_V V',\overline{U}\times_V V')$, where V' is the ring of integers in K', be the base change of $(U,\overline{U})$ to (K',V'), and let $f:(T,\overline{T})\to(U,\overline{U})$ be the canonical projection. Take $R=R_V$, N, e, π associated to V. Similarly, we define $R':=R_{V'}$, N', e', π'. Write the map $\alpha^B_{{\rm syn},\pi}$ as

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\rm syn}(U,\overline{U},r)_{\mathbb{Q}} \arrow[r, "\sim", "h"'] \arrow[d, "\alpha^B_{{\rm syn},\pi}", "\wr"'] & \big[\mathrm{R}\Gamma^{B,\tau}_{\rm HK}((U,\overline{U})_R,r)^{N=0} \arrow[r, "i^*_\pi"] & \mathrm{R}\Gamma_{\rm dR}(U,\overline{U}_K)/F^r\big] \arrow[d, equal] \\
C_{\rm st}(\mathrm{R}\Gamma^B_{\rm HK}(U,\overline{U})\{r\}) \arrow[r, "\sim"] & \big[\mathrm{R}\Gamma^B_{\rm HK}(U,\overline{U},r)^{N=0} \arrow[r, "\iota^B_{\rm dR}"] \arrow[u, "\iota_\pi\beta"', "\wr"] & \mathrm{R}\Gamma_{\rm dR}(U,\overline{U}_K)/F^r\big]
\end{tikzcd}$$

Here we defined the map h as the composition

$$\mathrm{R}\Gamma_{\rm syn}(U,\overline{U},r)_{\mathbb{Q}}\to\mathrm{R}\Gamma_{\rm cr}((U,\overline{U})/R)_{\mathbb{Q}}\xleftarrow{\sim}\mathrm{R}\Gamma^B_{\rm HK}(U_1,\overline{U}_1)^\tau_{R_{\mathbb{Q}}}. \tag{34}$$

From the construction of the Beilinson–Hyodo–Kato map

$$\iota^B_{\rm dR}:\mathrm{R}\Gamma^B_{\rm HK}(T_1,\overline{T}_1)\to\mathrm{R}\Gamma_{\rm dR}(T,\overline{T}_{K'}),$$

it follows that it is G-equivariant; hence the complex $C_{\rm st}(\mathrm{R}\Gamma^B_{\rm HK}(T,\overline{T})\{r\})$ is equipped with a natural G-action. We claim the map $\alpha^B_{{\rm syn},\pi'}$ induces a natural map

$$\tilde{\alpha}^B_{{\rm syn},\pi'}:\mathrm{R}\Gamma(G,\mathrm{R}\Gamma_{\rm syn}(T,\overline{T},r)_{\mathbb{Q}})\to\mathrm{R}\Gamma(G,C_{\rm st}(\mathrm{R}\Gamma^B_{\rm HK}(T,\overline{T})\{r\})),$$
$$\tilde{\alpha}^B_{{\rm syn},\pi'}:=(1/|G|)\sum_{g\in G}\alpha^B_{{\rm syn},g(\pi')}.$$

To see this it suffices to show that, for every $g\in G$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\rm syn}(T,\overline{T},r)_{\mathbb{Q}} \arrow[r, "\alpha^B_{{\rm syn},\pi'}"] \arrow[d, "g^*"] & C_{\rm st}(\mathrm{R}\Gamma^B_{\rm HK}(T,\overline{T})\{r\}) \arrow[d, "g^*"] \\
\mathrm{R}\Gamma_{\rm syn}(T,\overline{T},r)_{\mathbb{Q}} \arrow[r, "\alpha^B_{{\rm syn},g(\pi')}"] & C_{\rm st}(\mathrm{R}\Gamma^B_{\rm HK}(T,\overline{T})\{r\})
\end{tikzcd}$$

We accomplish this by constructing natural morphisms

$$g^*: \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R'_{\pi'}) \to \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R'_{g(\pi')}),$$
$$g^*: \mathrm{R}\Gamma^B_{\mathrm{HK}}(T_1,\overline{T}_1)^{\tau}_{R'_{\pi'}} \to \mathrm{R}\Gamma^B_{\mathrm{HK}}(T_1,\overline{T}_1)^{\tau}_{R'_{g(\pi')}}$$

that are compatible with the maps in (34) that define h, the maps $\iota_?$ and $i^*_?$, and the trivialization β. We define the pullbacks g^* from a map $g: R'_{\pi'} \to R'_{g(\pi')}$ constructed by lifting the action of g from V'_1 to R' by setting $g(t'_{\pi'}) = t'_{g(\pi')}$ and taking the induced action of g on $W(k')$. This map is compatible with Frobenius and monodromy. The induced pullbacks g^* are clearly compatible with the map i^*_0 and the maps $\iota_?$, the maps $i^*_{\pi'}$, $i^*_{g(\pi')}$, and the trivialization β. From the construction of the Beilinson–Hyodo–Kato map, the pullbacks g^* are also compatible with the maps $\kappa_{R'_?}$, and hence with the map h, as wanted.

Proposition 3.11. (1) *The following diagram commutes in the* (*classical*) *derived category*:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}} \arrow[r, "f^*"] \arrow[d, "\alpha^B_{\mathrm{syn},\pi}"] & \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{syn}}(T,\overline{T},r)_{\mathbb{Q}}) \arrow[d, "\tilde{\alpha}^B_{\mathrm{syn},\pi'}"] \\
C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\overline{U})\{r\}) \arrow[r, "f^*"] & \mathrm{R}\Gamma(G, C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(T,\overline{T})\{r\}))
\end{tikzcd}$$

(2) *The natural map*

$$f^*: \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma(G, \mathrm{R}\Gamma_{\mathrm{syn}}(T,\overline{T},r)_{\mathbb{Q}})$$

is a quasi-isomorphism.

Proof. The second claim of the proposition follows from the first one and the fact that the Hyodo–Kato and de Rham cohomologies satisfy finite Galois decent.

Since everything in sight is functorial and satisfies finite unramified Galois descent, we may assume that the extension K'/K is totally ramified. First, we will construct a G-equivariant (for the trivial action of G on R) map

$$f^*: \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,r)^{N=0} \to \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R',r)^{N'=0}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},r) \arrow[r, "f^*"] \arrow[d, "\wr"] & \mathrm{R}\Gamma_{\mathrm{cr}}(T,\overline{T},r) \arrow[d, "\wr"] \\
\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,r)^{N=0} \arrow[r, "f^*"] & \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R',r)^{N'=0} \\
\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U},r)^{N=0} \arrow[r, "f^*", "\sim"'] \arrow[u, "\iota_\pi", "\wr"'] & \mathrm{R}\Gamma_{\mathrm{HK}}(T,\overline{T},r)^{N'=0} \arrow[u, "\iota_{\pi'}", "\wr"']
\end{tikzcd} \tag{35}$$

Remark 3.12. Note that the bottom map is an isomorphism because f^* acts trivially on the Hyodo–Kato complexes. The commutativity of the above diagram and the quasi-isomorphisms (33) will imply that a totally ramified Galois extension does not change the log-crystalline complexes $\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r)$ and $\mathrm{R}\Gamma_{\mathrm{cr}}((U, \overline{U})/R, r)^{N=0}$.

Let e_1 be the ramification index of V'/V. Set $v = (\pi')^{e_1}\pi^{-1}$, and choose an integer s such that $(\pi')^{p^s} \in pV'$. Set $T := t_\pi$, $T' := t_{\pi'}$ and define the morphism $a : R \to R'$ by $T \mapsto (T')^{e_1}[\bar{v}]^{-1}$. Since V_1' and V_1 are defined by $pR + T^e R$ and by $pR' + (T')^{e'}R'$, respectively, a induces a morphism $a_1 : V_1 \to V_1'$. We have $F^s a_1 = F^s f_1$, where F is the absolute Frobenius on $\mathrm{Spec}(V_1)$. Notice that in general $f_1 \neq a_1$ if $v[\bar{v}]^{-1} \not\equiv 1 \bmod pV'$. The morphism $\varphi_R^s a : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is compatible with $F^s f_1 : \mathrm{Spec}(V_1') \to \mathrm{Spec}(V_1)$ and it commutes with the operators N and $p^s N'$. We have the following commutative diagram:

$$\begin{tikzcd}
(T, \overline{T})_1 \arrow[r, "F^s f_1"] \arrow[d] & (U, \overline{U})_1 \arrow[d] \\
\mathrm{Spec}(V_1') \arrow[r, "F^s a_1 = F^s f_1"] \arrow[d] & \mathrm{Spec}(V_1) \arrow[d] \\
\mathrm{Spec}(R') \arrow[r, "\varphi_R^s a"] & \mathrm{Spec}(R)
\end{tikzcd}$$

Hence we also have the commutative diagram of distinguished triangles

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U})_{\mathbb{Q}} \arrow[r] \arrow[d, "f^*F^s"] & \mathrm{R}\Gamma_{\mathrm{cr}}((U, \overline{U})/R)_{\mathbb{Q}} \arrow[r, "eN"] \arrow[d, "f^*F^s"] & \mathrm{R}\Gamma_{\mathrm{cr}}((U, \overline{U})/R)_{\mathbb{Q}} \arrow[d, "p^s e_1 f^* F^s"] \\
\mathrm{R}\Gamma_{\mathrm{cr}}(T, \overline{T})_{\mathbb{Q}} \arrow[r] & \mathrm{R}\Gamma_{\mathrm{cr}}((T, \overline{T})/R')_{\mathbb{Q}} \arrow[r, "e'N'"] & \mathrm{R}\Gamma_{\mathrm{cr}}((T, \overline{T})/R')_{\mathbb{Q}}
\end{tikzcd} \tag{36}$$

To see how this diagram arises, we may assume (by the usual Čech argument) that we have a fine affine log-scheme $X_n/V_n^\times$ that is log-smooth over $V_n^\times$. We can also assume that we have a lifting of $X_n \hookrightarrow Z_n$ over $\mathrm{Spec}(W_n(k)[T])$ (with the log-structure coming from T) and a lifting of Frobenius φ_Z on Z_n that is compatible with the Frobenius φ_R. Recall [Kato 1994, Lemma 4.2] that the horizontal distinguished triangles in the above diagram arise from an exact sequence of complexes of sheaves on $X_{n,\text{ét}}$

$$0 \to C_V'[-1] \xrightarrow{\wedge\, \mathrm{dlog}\, T} C_V \to C_V' \to 0, \tag{37}$$

where $C_V := R_n \otimes_{W_n(k)[T]} \Omega^\bullet_{Z_n/W_n(k)}$ and $C_V' := R_n \otimes_{W_n(k)[T]} \Omega^\bullet_{Z_n/W_n(k)[T]}$. Now consider the base change of $Z_n/W_n(k)[T]$ by the map $F^s a : \mathrm{Spec}(W_n(k)[T']) \to \mathrm{Spec}(W_n(k)[T])$ and the related complexes (37). We get a commutative diagram of complexes of sheaves on $X_{n,\text{ét}}$ (note that $X_{V',n,\text{ét}} = X_{n,\text{ét}}$)

$$\begin{tikzcd}
0 \arrow[r] & C'_{V'}[-1] \arrow[r, "\wedge \operatorname{dlog} T'"] & C_{V'} \arrow[r] & C'_{V'} \arrow[r] & 0 \\
0 \arrow[r] & C'_{V}[-1] \arrow[r, "\wedge \operatorname{dlog} T"] \arrow[u, "p^s e_1 a^* \varphi_Z^s"'] & C_{V} \arrow[r] \arrow[u, "a^* \varphi_Z^s"'] & C'_{V} \arrow[r] \arrow[u, "a^* \varphi_Z^s"'] & 0
\end{tikzcd}$$

Hence diagram (36) follows.

Combining diagram (36) with Frobenius, we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r) \arrow[d, "\wr"] & \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)f^*F^s \arrow[l, "F^s"'] \arrow[r] \arrow[d, "\wr"] & \mathrm{R}\Gamma_{\mathrm{cr}}(T,\bar{T},r) \arrow[d, "\wr"'] \\
\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R,r)^{N=0} \arrow[d, "i_0^*"] & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R,r)^{N=0} \arrow[l, "(F^s,p^sF^s)"'] \arrow[r, "(a^*F^s,p^sa^*F^s)"] \arrow[d, "i_0^*", "\wr"'] & \mathrm{R}\Gamma_{\mathrm{cr}}((T,\bar{T})/R',r)^{N'=0} \arrow[d, "i_0^*", "\wr"'] \\
\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r)^{N=0} & \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r)^{N=0} \arrow[l, "(F^s,p^sF^s)"', "\sim"] \arrow[r, "(F^s,p^sF^s)", "\sim"'] & \mathrm{R}\Gamma_{\mathrm{HK}}(T,\bar{T},r)^{N'=0}
\end{tikzcd}$$

It follows that all the maps in the above diagram are quasi-isomorphisms. We define the map

$$f^* : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R,r)^{N=0} \to \mathrm{R}\Gamma_{\mathrm{cr}}((T,\bar{T})/R',r)^{N'=0}$$

by the middle row. Since, for any $g \in G$, we have $v_{g(\pi')} = g(v_{\pi'})$, the map f^* is G-equivariant. In the (classical) derived category, this definition is independent of the constant s we have chosen. Since i_0^* is a quasi-isomorphism and $i_0^*\iota_{?*} = \mathrm{Id}$, the diagram (35) commutes as well, as wanted.

We define the map

$$f^* : \mathrm{R}\Gamma_{\mathrm{HK}}^{B,\tau}((U,\bar{U})/R,r)^{N=0} \to \mathrm{R}\Gamma_{\mathrm{HK}}^{B,\tau}((T,\bar{T})/R',r)^{N'=0} \tag{38}$$

in an analogous way. By the above diagram and by the compatibility of the Beilinson–Hyodo–Kato constructions with base change and with Frobenius, the two pullback maps f^* are compatible via the morphism h, i.e., the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r) \arrow[r] \arrow[d, "f^*"] & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R,r)^{N=0} \arrow[d, "f^*"] & \mathrm{R}\Gamma_{\mathrm{HK}}^{B,\tau}((U,\bar{U})/R,r)^{N=0} \arrow[l, "\kappa_R"'] \arrow[d, "f^*"] \\
\mathrm{R}\Gamma_{\mathrm{cr}}(T,\bar{T},r) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{cr}}((T,\bar{T})/R',r)^{N'=0} & \mathrm{R}\Gamma_{\mathrm{HK}}^{B,\tau}((T,\bar{T})/R',r)^{N'=0} \arrow[l, "\kappa_{R'}"']
\end{tikzcd}$$

From the analog of diagram (35) for the Beilinson–Hyodo–Kato complexes and by the universal nature of the trivialization at $\bar{p}$, we obtain that the pullback map f^* is compatible with the maps $\beta\iota_?$. It remains to show that we have a commutative diagram

$$\begin{tikzcd} \mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\overline{U},r)^{N=0} \arrow[r, "f^*", "\sim"'] \arrow[d, "\iota^B_{\mathrm{dR}}"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(T,\overline{T},r)^{N'=0} \arrow[d, "\iota^B_{\mathrm{dR}}"] \\ \mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_K)/F^r \arrow[r, "f^*"] & \mathrm{R}\Gamma_{\mathrm{dR}}(T,\overline{T}_{K'})/F^r \end{tikzcd}$$

But this follows since the Beilinson–Hyodo–Kato map is compatible with base change. $\square$

3C. ***Arithmetic syntomic cohomology.*** We are now ready to introduce and study arithmetic syntomic cohomology, i.e., syntomic cohomology over K. Let $\mathscr{J}^{[r]}_{\mathrm{cr}}$, $\mathscr{A}_{\mathrm{cr}}$, and $\mathscr{S}(r)$ for $r \geq 0$ be the h-sheafifications on $\mathscr{V}ar_K$ of the presheaves sending $(U,\overline{U}) \in \mathscr{P}^{\mathrm{ss}}_K$ to $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},J^{[r]})$, $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)$, respectively. Let $\mathscr{J}^{[r]}_{\mathrm{cr},n}$, $\mathscr{A}_{\mathrm{cr},n}$, and $\mathscr{S}_n(r)$ denote the h-sheafifications of the mod-p^n versions of the respective presheaves. We have

$$\mathscr{S}_n(r) \simeq \mathrm{Cone}\big(\mathscr{J}^{[r]}_{\mathrm{cr},n} \xrightarrow{p^r-\varphi} \mathscr{A}_{\mathrm{cr},n}\big)[-1], \quad \mathscr{S}(r) \simeq \mathrm{Cone}\big(\mathscr{J}^{[r]}_{\mathrm{cr}} \xrightarrow{p^r-\varphi} \mathscr{A}_{\mathrm{cr}}\big)[-1].$$

For $r \geq 0$, define $\mathscr{S}(r)_{\mathbb{Q}}$ as the h-sheafification of the presheaf sending ss-pairs $(U,\overline{U})$ to $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}}$. We have

$$\mathscr{S}(r)_{\mathbb{Q}} \simeq \mathrm{Cone}(\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}} \xrightarrow{1-\varphi_r} \mathscr{A}_{\mathrm{cr},\mathbb{Q}})[-1].$$

For $X \in \mathscr{V}ar_K$, set

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r)_n = \mathrm{R}\Gamma(X_h,\mathscr{S}_n(r)), \quad \mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r) := \mathrm{R}\Gamma(X_h,\mathscr{S}(r)_{\mathbb{Q}}).$$

We have

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r)_n \simeq \mathrm{Cone}(\mathrm{R}\Gamma(X_h,\mathscr{J}^{[r]}_{\mathrm{cr},n}) \xrightarrow{p^r-\varphi} \mathrm{R}\Gamma(X_h,\mathscr{A}_{\mathrm{cr},n}))[-1],$$
$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r) \simeq \mathrm{Cone}(\mathrm{R}\Gamma(X_h,\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}}) \xrightarrow{1-\varphi_r} \mathrm{R}\Gamma(X_h,\mathscr{A}_{\mathrm{cr},\mathbb{Q}}))[-1].$$

We will often write $\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)$ for $\mathrm{R}\Gamma(X_h,\mathscr{A}_{\mathrm{cr}})$ if this does not cause confusion.

Let $\mathscr{A}_{\mathrm{HK}}$ be the h-sheafification of the presheaf $(U,\overline{U}) \mapsto \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}$ on $\mathscr{P}^{\mathrm{ss}}_K$; this is an h-sheaf of E_∞ K_0-algebras on $\mathscr{V}ar_K$ equipped with a φ-action and a derivation N such that $N\varphi = p\varphi N$. For $X \in \mathscr{V}ar_K$, set $\mathrm{R}\Gamma_{\mathrm{HK}}(X_h) := \mathrm{R}\Gamma(X_h,\mathscr{A}_{\mathrm{HK}})$. Similarly, we define h-sheaves $\mathscr{A}^B_{\mathrm{HK}}$ and the complexes $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) := \mathrm{R}\Gamma(X_h,\mathscr{A}^B_{\mathrm{HK}})$. The maps $\kappa : \mathrm{R}\Gamma^B_{\mathrm{HK}}(U_1,\overline{U}_1) \to \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}$ h-sheafify and we obtain functorial quasi-isomorphisms

$$\kappa : \mathscr{A}^B_{\mathrm{HK}} \xrightarrow{\sim} \mathscr{A}_{\mathrm{HK}}, \quad \kappa : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X_h).$$

Remark 3.13. The complexes $\mathscr{J}^{[r]}_{\mathrm{cr},n}$ and $\mathscr{S}_n(r)$ (and their completions) have a concrete description. For the complexes $\mathscr{J}^{[r]}_{\mathrm{cr},n}$, we can represent the presheaves $(U,\overline{U}) \mapsto \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{J}^{[r]}_n)$ by Godement resolutions (on the crystalline site), sheafify them for the h-topology on $\mathscr{P}^{\mathrm{ss}}_K$, and then move them to $\mathscr{V}ar_K$. For the complexes $\mathscr{S}_n(r)$, the maps $p^r - \varphi$ can be lifted to the Godement resolutions and their mapping fiber (defining $\mathscr{S}_n(r)(U, \bar{U})$) can be computed in the abelian category of complexes of abelian groups. To get $\mathscr{S}_n(r)$, we h-sheafify on $\mathscr{P}^{\mathrm{ss}}_K$ and pass to $\mathscr{V}ar$.

Let, for a moment, K be any field of characteristic zero. Consider the presheaf $(U, \bar{U}) \mapsto \mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) := \mathrm{R}\Gamma(\bar{U}, \Omega^{\bullet}_{(U,\bar{U})})$ of filtered dg K-algebras on $\mathscr{P}^{\mathrm{nc}}_K$. Let $\mathscr{A}_{\mathrm{dR}}$ be its h-sheafification. It is a sheaf of filtered K-algebras on $\mathscr{V}ar_K$. For $X \in \mathscr{V}ar_K$, we have Deligne's de Rham complex of X equipped with Deligne's Hodge filtration: $\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) := \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{dR}})$. Beilinson proves the following comparison statement.

Proposition 3.14 [Beilinson 2012, 2.4]. (1) *For $(U, \bar{U}) \in \mathscr{P}^{\mathrm{nc}}_K$, the canonical map $\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U_h)$ is a filtered quasi-isomorphism.*

(2) *The cohomology groups $H^i_{\mathrm{dR}}(X_h) := H^i\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$ are K-vector spaces of dimension equal to the rank of $H^i(X_{\bar{K},\text{ét}}, \mathbb{Q}_p)$.*

Corollary 3.15. *For a geometric pair $(U, \bar{U})$ over K that is saturated and log-smooth, the canonical map*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U_h)$$

is a filtered quasi-isomorphism.

Proof. Recall [Nizioł 2006, Theorem 5.10] that there is a log-blow-up $(U, \bar{T}) \to (U, \bar{U})$ that resolves singularities of $(U, \bar{U})$, i.e., such that $(U, \bar{T}) \in \mathscr{P}^{\mathrm{nc}}_K$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{T}) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{dR}}(U_h) \\
\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) \arrow[u, "\wr"] \arrow[ur] &
\end{tikzcd}$$

The vertical map is a filtered quasi-isomorphism; the horizontal map is a filtered quasi-isomorphism by the above proposition. Our corollary follows. $\square$

Remark 3.16. Another proof of the above result (and a mild generalization) that does not use resolution of singularities can be found in [Beilinson 2013, 1.19] (where it is attributed to A. Ogus).

Return now to our p-adic field K.

Remark 3.17. By construction, we know the complexes $\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$, $\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)$, $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)$, $\mathrm{R}\Gamma(X_h, \mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}})$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ satisfy h-descent. In particular, since the h-topology is finer than the étale topology, they satisfy Galois descent for finite extensions. Hence, for any finite Galois extension K_1/K, the natural maps

$$\mathrm{R}\Gamma^*_?(X_h) \xrightarrow{\sim} \mathrm{R}\Gamma(G, \mathrm{R}\Gamma^*_?(X_{K_1,h})), \quad ? = \mathrm{cr}, \mathrm{syn}, \mathrm{HK}, \mathrm{dR}, \ * = B, \varnothing,$$

where $G = \mathrm{Gal}(K_1/K)$, are (filtered) quasi-isomorphisms. Since G is finite, it follows that the natural maps

$$\mathrm{R}\Gamma^*_{\mathrm{HK}}(X_h) \otimes_{K_0} K_{1,0} \xrightarrow{\sim} \mathrm{R}\Gamma^*_{\mathrm{HK}}(X_{K_1,h}), \quad \mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_K K_1 \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_{K_1,h})$$

are (filtered) quasi-isomorphisms as well.

Recall from [Beilinson 2013, 2.5] and Proposition 3.21, that for a fine, log-scheme X, log-smooth over $V^\times$, and of Cartier type we have a quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\overline{V}}, \mathscr{J}^{[r]}_{X_{\overline{V}}/W(k)})_{\mathbb{Q}} \simeq \mathrm{R}\Gamma(X_{\overline{K},h}, \mathscr{J}^{[r]}_{\mathrm{cr}})_{\mathbb{Q}}.$$

We can descend this result to K but on the level of rational log-syntomic cohomology; the key observation being that the field extensions introduced by the alterations are harmless since, by Proposition 3.11, log-syntomic cohomology satisfies finite Galois descent. Along the way we will get an analogous comparison quasi-isomorphism for the Hyodo–Kato cohomology.

Proposition 3.18. *For any arithmetic pair $(U, \overline{U})$ that is fine, log-smooth over $V^\times$, and of Cartier type, and $r \geq 0$, the canonical maps*

$$\mathrm{R}\Gamma^*_{\mathrm{HK}}(U, \overline{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma^*_{\mathrm{HK}}(U_h), \quad \mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{syn}}(U_h, r)$$

are quasi-isomorphisms.

Proof. It suffices to show that for any h-hypercovering $(U_\bullet, \overline{U}_\bullet) \to (U, \overline{U})$ by pairs from $\mathscr{P}^{\log}_K$, the natural maps

$$\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{HK}}(U_\bullet, \overline{U}_\bullet)_{\mathbb{Q}}, \quad \mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{syn}}(U_\bullet, \overline{U}_\bullet, r)_{\mathbb{Q}}$$

are (modulo taking a refinement of $(U_\bullet, \overline{U}_\bullet)$) quasi-isomorphisms. For the second map, since we have a canonical quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Cone}\big(\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, \mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}}\big)[-1],$$

it suffices to show that, up to a refinement of the hypercovering, we have quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, \mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet, \overline{U}_\bullet, \mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}},$$
$$\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet, \overline{U}_\bullet, r)_{\mathbb{Q}}.$$

For the first of these maps, by Corollary 2.4 this amounts to showing that the following map is a quasi-isomorphism:

$$\mathrm{R}\Gamma(\overline{U}_K, \Omega^\bullet_{(U,\overline{U}_K)})/F^r \xrightarrow{\sim} \mathrm{R}\Gamma(\overline{U}_{\bullet,K}, \Omega^\bullet_{(U_\bullet,\overline{U}_{\bullet,K})})/F^r.$$

But, by Corollary 3.15 this map is quasi-isomorphic to the map

$$\mathrm{R}\Gamma_{\mathrm{dR}}(U_h)/F^r \to \mathrm{R}\Gamma_{\mathrm{dR}}(U_{\bullet,h})/F^r,$$

which is clearly a quasi-isomorphism.

Hence it suffices to show that, up to a refinement of the hypercovering, we have quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(U_\bullet,\bar{U}_\bullet)_{\mathbb{Q}}, \quad \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet,\bar{U}_\bullet,r)_{\mathbb{Q}}.$$

Fix $t\geq 0$. To show that $H^t\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)_{\mathbb{Q}} \xrightarrow{\sim} H^t\mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet,\bar{U}_\bullet,r)_{\mathbb{Q}}$ is a quasi-isomorphism, we will often work with $(t+1)$-truncated h-hypercovers. This is because $\tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet,\bar{U}_\bullet,r)\simeq\tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\bar{U}_\bullet)_{\leq t+1},r)$, where $(U_\bullet,\bar{U}_\bullet)_{\leq t+1}$ denotes the $(t+1)$-truncation. Assume first that we have an h-hypercovering $(U_\bullet,\bar{U}_\bullet)\to(U,\bar{U})$ of arithmetic pairs over K, where each pair $(U_i,\bar{U}_i)$, $i\leq t+1$, is log-smooth over $V^\times$ and of Cartier type. We claim that then already the maps

$$\begin{aligned}\tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} &\xrightarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{HK}}((U_\bullet,\bar{U}_\bullet)_{\leq t+1})_{\mathbb{Q}},\\ \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}} &\xrightarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\bar{U}_\bullet)_{\leq t+1})_{\mathbb{Q}}\end{aligned}\tag{39}$$

are quasi-isomorphisms. To see the second quasi-isomorphism, consider the following commutative diagram of distinguished triangles ($R=R_V$):

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R) \arrow[r, "N"] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R) \arrow[d]\\
\mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\bar{U}_\bullet)_{\leq t+1}) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\bar{U}_\bullet)_{\leq t+1}/R) \arrow[r, "N"] & \mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\bar{U}_\bullet)_{\leq t+1}/R)
\end{tikzcd}$$

It suffices to show that the two right vertical arrows are rational quasi-isomorphisms in degrees less than or equal to t. But we have the R-linear quasi-isomorphisms

$$\begin{aligned}\iota: R\otimes_{W(k)}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} &\xrightarrow{\sim} \mathrm{R}\Gamma((U,\bar{U})/R)_{\mathbb{Q}},\\ \iota: R\otimes_{W(k)}\mathrm{R}\Gamma_{\mathrm{HK}}((U_\bullet,\bar{U}_\bullet)_{\leq t+1})_{\mathbb{Q}} &\xrightarrow{\sim} \mathrm{R}\Gamma((U_\bullet,\bar{U}_\bullet)_{\leq t+1}/R)_{\mathbb{Q}}.\end{aligned}$$

Hence to show both quasi-isomorphisms (39), it suffices to show that the map

$$\tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\to\tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{HK}}((U_\bullet,\bar{U}_\bullet)_{\leq t+1})_{\mathbb{Q}}$$

is a quasi-isomorphism.

Tensoring over K_0 with K and using the Hyodo–Kato quasi-isomorphism (23), we reduce to showing that the map

$$\tau_{\leq t}\mathrm{R}\Gamma(\bar{U}_K,\Omega^\bullet_{(U,\bar{U}_K)})\to\tau_{\leq t}\mathrm{R}\Gamma(\bar{U}_{\bullet K,\leq t+1},\Omega^\bullet_{(U_\bullet,\bar{U}_{\bullet,K})_{\leq t+1}})$$

is a quasi-isomorphism, and this we have done above.

To treat the general case, set $X=(U,\bar{U})$, $Y_\bullet=(U_\bullet,\bar{U}_\bullet)$. We will do a base change to reduce to the case discussed above. We may assume that all the fields $K_{n,i}$, $K_{U_n}\simeq\prod K_{n,i}$ are Galois over K. Choose a finite Galois extension $(V',K')/(V,K)$ for K' Galois over all the fields $K_{n,i}$, $n\leq t+1$. Write $N_X(X_{V'})$ for the "Čech nerve" of $X_{V'}/X$. The term $N_X(X_{V'})_n$ is defined as the $(n+1)$-fold fiber product of $X_{V'}$ over X: $N_X(X_{V'})_n=(U\times_K K'^{,n+1},(\bar{U}\times_V V'^{,n+1})^{\mathrm{norm}})$, where $V'^{,n+1}$, $K'^{,n+1}$ are defined as the $(n+1)$-fold product of V' over V and of K' over K, respectively. Normalization is taken with respect to the open regular subscheme $U \times_K K'^{,n+1}$. Note that $N_X(X_{V'})_n \simeq (U \times_K K' \times G^n, \bar{U} \times_V V' \times G^n)$, where $G = \mathrm{Gal}(K'/K)$. Hence it is a log-smooth scheme over $V'^{,\times}$ of Cartier type. The augmentation $N_X(X_{V'}) \to X$ is an h-hypercovering.

Consider the bisimplicial scheme $Y_\bullet \times_X N_X(X_{V'})_\bullet$,

$$\begin{aligned}(Y_\bullet \times_X N_X(X_{V'})_\bullet)_{n,m} &:= Y_n \times_X N_X(X_{V'})_m \\ &\simeq \big(U_n \times_U U \times_K K'^{,m+1}, (\bar{U}_n \times_{\bar{U}} (\bar{U} \times_V V'^{,m+1})^{\mathrm{norm}})^{\mathrm{norm}}\big) \\ &\simeq \coprod_i \big(U_n \times_{K_{n,i}} K_{n,i} \times_K K'^{,m+1}, \bar{U}_n \times_{V_{n,i}} (V_{n,i} \times_V V'^{,m+1})^{\mathrm{norm}}\big).\end{aligned}$$

Hence $(Y_\bullet \times_X N_X(X_{V'})_\bullet)_{n,m} \in \mathscr{P}_K^{\log}$. For $n, m \le t+1$, we have

$$(Y_\bullet \times_X N_X(X_{V'})_\bullet)_{n,m} \simeq \coprod_i (U_n \times_{K_{n,i}} K' \times G_{n,i} \times G^m, \bar{U}_n \times_{V_{n,i}} V' \times G_{n,i} \times G^m),$$

where $G_{n,i} = \mathrm{Gal}(K_{n,i}/K)$. It is a log-scheme log-smooth over $V'^{,\times}$ of Cartier type.

Consider now its diagonal $Y_\bullet \times_X N_X(X_{V'}) := \Delta(Y_\bullet \times_X N_X(X_{V'})_\bullet)$. It is an h-hypercovering of X refining $Y_\bullet$ such that $(Y_\bullet \times_X N_X(X_{V'}))_n$ is log-smooth over $V'^{,\times}$, of Cartier type, for $n \le t+1$. It suffices to show that the compositions

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{HK}}(Y_\bullet)_{\mathbb{Q}} &\xrightarrow{\mathrm{pr}_1^*} \mathrm{R}\Gamma_{\mathrm{HK}}(Y_\bullet \times_X N_X(X_{V'}))_{\mathbb{Q}}, \\ \mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(Y_\bullet,r)_{\mathbb{Q}} &\xrightarrow{\mathrm{pr}_1^*} \mathrm{R}\Gamma_{\mathrm{cr}}(Y_\bullet \times_X N_X(X_{V'}),r)_{\mathbb{Q}}\end{aligned} \tag{40}$$

are quasi-isomorphisms in degrees less than or equal to t. Using the commutative diagram of bisimplicial schemes

$$\begin{tikzcd} Y_\bullet \times_X N_X(X_{V'}) \arrow[r, "\Delta"] & Y_\bullet \times_X N_X(X_{V'})_\bullet \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & Y_\bullet \arrow[d] \\ & N_X(X_{V'}) \arrow[r, "f"] & X \end{tikzcd}$$

we can write the second composition as

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}} \xrightarrow{f^*} \mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'}),r)_{\mathbb{Q}} &\xrightarrow{\mathrm{pr}_2^*} \mathrm{R}\Gamma_{\mathrm{cr}}(Y_\bullet \times_X N_X(X_{V'})_\bullet,r)_{\mathbb{Q}} \\ &\xrightarrow{\Delta^*} \mathrm{R}\Gamma_{\mathrm{cr}}(Y_\bullet \times_X N_X(X_{V'}),r)_{\mathbb{Q}}.\end{aligned}$$

We claim that all of these maps are quasi-isomorphisms in degrees less than or equal to t. The map Δ^* is a quasi-isomorphism (in all degrees) by [Friedlander 1982, Proposition 2.5]. For the second map, fix $n \le t+1$ and consider the induced map $\mathrm{pr}_2 : (Y_\bullet \times_X N_X(X_{V'})_\bullet)_{\bullet,n} \to N_X(X_{V'})_n$. It is an h-hypercovering whose $(t+1)$-truncation is built from log-schemes, log-smooth over (V', K'), of Cartier type. It suffices to show that the induced map

$$\tau_{\le t}\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_n,r)_{\mathbb{Q}} \xrightarrow{\mathrm{pr}_2^*} \tau_{\le t}\mathrm{R}\Gamma_{\mathrm{cr}}((Y_\bullet \times_X N_X(X_{V'}))_{\bullet,n},r)_{\mathbb{Q}}$$

is a quasi-isomorphism. Since all maps are defined over K', this follows from the case considered at the beginning of the proof.

To prove that $f^*:\mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}}\to \mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'}),r)_{\mathbb{Q}}$ is a quasi-isomorphism, consider first the case when the extension V'/V is unramified. Then $\mathrm{R}\Gamma_{\mathrm{cr}}(X_{V'})\simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X)\otimes_{W(k)}W(k')$ and the map f^* is a quasi-isomorphism by finite étale descent for crystalline cohomology.

Assume now that the extension V'/V is totally ramified and let π and π' be uniformizers of V and V', respectively. Consider the target of f^* as a double complex. To show that f^* is a quasi-isomorphism, it suffices to show that, for each $s\geq 0$, the sequence

$$0\to H^s\mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}}\xrightarrow{f^*}H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_0,r)_{\mathbb{Q}}$$
$$\xrightarrow{d_0^*}H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_1,r)_{\mathbb{Q}}\xrightarrow{d_1^*}H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_2,r)_{\mathbb{Q}}\to\cdots$$

is exact. Embed it into the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^s\mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}} \arrow[r, "f^*"] \arrow[d, "\alpha^B_{\mathrm{syn},\pi}"', "\wr"] & H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_0,r)_{\mathbb{Q}} \arrow[r, "d_0^*"] \arrow[d, "\tilde{\alpha}^B_{\mathrm{syn},\pi'}"', "\wr"] & H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_1,r)_{\mathbb{Q}} \arrow[r] \arrow[d, "\tilde{\alpha}^B_{\mathrm{syn},\pi'}"', "\wr"] & {} \\
0 \arrow[r] & H^s\mathrm{R}\Gamma^B_{\mathrm{HK}}(X,r)^{N=0}_{\mathbb{Q}} \arrow[r, "f^*"] & H^s\mathrm{R}\Gamma^B_{\mathrm{HK}}(N_X(X_{V'})_0,r)^{N'=0}_{\mathbb{Q}} \arrow[r, "d_0^*"] & H^s\mathrm{R}\Gamma^B_{\mathrm{HK}}(N_X(X_{V'})_1,r)^{N'=0}_{\mathbb{Q}} \arrow[r] & {}
\end{tikzcd}$$

Note that, since all the maps d_i^* are induced from automorphisms of V'/V, by the proof of Proposition 3.11 (take the map f used there to be a given automorphism $g\in G=\mathrm{Gal}(K'/K)$ and π', $g(\pi')$ for the uniformizers of V') and the proof of Proposition 3.8, we get the vertical maps above that make all the squares commute.

Hence it suffices to show that the following sequence of Hyodo–Kato cohomology groups is exact:

$$0\to H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}\xrightarrow{f^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(N_X(X_{V'})_0)_{\mathbb{Q}}$$
$$\xrightarrow{d_0^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(N_X(X_{V'})_1)_{\mathbb{Q}}\xrightarrow{d_1^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(N_X(X_{V'})_2)_{\mathbb{Q}}\to\cdots.$$

But this sequence is isomorphic to the sequence

$$0\to H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}\xrightarrow{f^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})_{\mathbb{Q}}$$
$$\xrightarrow{d_0^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})_{\mathbb{Q}}\times G\xrightarrow{d_1^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})_{\mathbb{Q}}\times G^2\to\cdots$$

representing the (augmented) G-cohomology of $H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}$. Since G is finite, this complex is exact in degrees at least 1. It remains to show that

$$H^0(G,\,H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})_{\mathbb{Q}})\simeq H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}.$$

Since K'/K is totally ramified, we have $H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})\simeq H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)$. Hence the action of G on $H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})$ is trivial and we get the right H^0 as well. We have proved the second quasi-isomorphism from (40). Notice that along the way we have actually proved the first quasi-isomorphism. ☐

For $X \in \mathscr{V}ar_K$, we define a canonical K_0-linear map (*the Beilinson–Hyodo–Kato morphism*)

$$\iota^B_{\mathrm{dR}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$$

as the sheafification of the map $\iota^B_{\mathrm{dR}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(U_1, \overline{U}_1) \to \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)$. It follows from Proposition 3.22, which we prove in the next section, that the cohomology groups $H^i_{\mathrm{HK}}(X_h) := H^i\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)$ are finite-rank K_0-vector spaces and that they vanish for $i > 2\dim X$. This implies the following lemma.

Lemma 3.19. *The syntomic cohomology groups $H^i_{\mathrm{syn}}(X_h, r) := H^i\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ vanish for $i > 2\dim X + 2$.*

Proof. The map $\iota'_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U}, r)^{N=0} \to \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/F^r$ from Remark 3.10 sheafifies. The quasi-isomorphism $\alpha'_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} C'_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})\{r\})$ does as well. Hence $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ is quasi-isomorphic via α'_{syn} to the mapping fiber

$$C'_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\{r\}) := [\mathrm{R}\Gamma_{\mathrm{HK}}(X_h, r)^{N=0} \xrightarrow{\iota'_{\mathrm{dR}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)/F^r].$$

The statement of the lemma follows. $\square$

For $X \in \mathscr{V}ar_K$ and $r \geq 0$, define the complex

$$C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}) := \left[\begin{array}{ccc} \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) & \xrightarrow{(1-\varphi_r, \iota^B_{\mathrm{dR}})} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \oplus \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)/F^r \\ \big\downarrow N & & \big\downarrow (N,0) \\ \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \end{array}\right].$$

Proposition 3.20. *For $X \in \mathscr{V}ar_K$ and $r \geq 0$, there exists a canonical* (*in the classical derived category*) *quasi-isomorphism*

$$\alpha_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}).$$

Moreover, this morphism is compatible with finite base change (*of the field K*).

Proof. To construct the map α_{syn}, take a number $t \geq 2\dim X + 2$ and let $Y_\bullet \to X$, $Y_\bullet = (U_\bullet, \overline{U}_\bullet)$, be an h-hypercovering of X by ss-pairs over K. Choose a finite Galois extension $(V', K')/(V, K)$ and a uniformizer π' of V' as in the proof of Proposition 3.18. Keeping the notation from that proof, refine our hypercovering to the h-hypercovering $Y_\bullet \times_V V' \to X_{K'}$. Then the truncation $(Y_\bullet \times_V V')_{\leq t+1}$ is built from log-schemes log-smooth over $V'^{,\times}$ and of Cartier type. We have the sequence of quasi-isomorphisms

$$\begin{aligned}
\gamma_{\pi'}: \mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h}) &\xleftarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h}) \xrightarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{syn}}((U_\bullet \times_K K')_{\leq t+1,h})\\
&\xleftarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{syn}}((Y_\bullet \times_V V')_{\leq t+1})_{\mathbb{Q}}\\
&\xrightarrow{\sim} C_{\mathrm{st}}(\tau_{\leq t}\mathrm{R}\Gamma^B_{\mathrm{HK}}((Y_\bullet \times_V V')_{\leq t+1})\{r\})\\
&\xrightarrow{\sim} C_{\mathrm{st}}(\tau_{\leq t}\mathrm{R}\Gamma^B_{\mathrm{HK}}((U_\bullet \times_K K')_{\leq t+1,h})\{r\})\\
&\xleftarrow{\sim} C_{\mathrm{st}}(\tau_{\leq t}\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\}) \xrightarrow{\sim} C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\}).
\end{aligned}$$

The first quasi-isomorphism follows from Lemma 3.19. The third and fifth quasi-isomorphisms follow from Proposition 3.18. The fourth quasi-isomorphism (the map $\tilde{\alpha}^B_{\mathrm{syn},\pi'}$), since all the log-schemes involved are log-smooth over $V^{\prime,\times}$ and of Cartier type, follows from Proposition 3.8.

Now, set $G := \mathrm{Gal}(K'/K)$. Passing from $\gamma_{\pi'}$ to its G-fixed points, we obtain the map

$$\alpha_{\mathrm{syn}} := \alpha_{\mathrm{syn},\pi'}: \mathrm{R}\Gamma_{\mathrm{syn}}(X_h) \to C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$$

as the composition

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h) \to \mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h})^G \xrightarrow{\gamma_{\pi'}} C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\})^G \xleftarrow{\sim} C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K,h})\{r\}).$$

It remains to check that the so-defined map is independent of all choices. For that, it suffices to check that, in the above construction, for a finite Galois extension (V_1, K_1) of (V', K'), $H = \mathrm{Gal}(K_1/K')$, the corresponding maps

$$\alpha_{\mathrm{syn},?}: \mathrm{R}\Gamma_{\mathrm{syn}}(X_h) \to C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$$

are the same in the classical derived category (note that this includes trivial extensions). An easy diagram chase shows that this amounts to checking that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}((Y_\bullet \times_V V')_{\leq t+1})_{\mathbb{Q}} \arrow[r, "\sim", "\alpha_{\mathrm{syn},\pi'}"'] \arrow[d] & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}((Y_\bullet \times_V V')_{\leq t+1})\{r\}) \arrow[d]\\
\mathrm{R}\Gamma_{\mathrm{syn}}((Y_\bullet \times_V V_1)_{\leq t+1})^H_{\mathbb{Q}} \arrow[r, "\sim", "\alpha_{\mathrm{syn},\pi_1}"'] & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}((Y_\bullet \times_V V_1)_{\leq t+1})\{r\})^H
\end{tikzcd}$$

But this we have shown in Proposition 3.11.

For the compatibility with finite base change, consider a finite field extension L/K. We can choose in the above a Galois extension K'/K that works for both fields. We get the same maps $\gamma_{\pi'}$ for both L and K. Consider now the following commutative diagram. The top and bottom rows define the maps $\alpha^L_{\mathrm{syn},\pi'}$

and $\alpha^{K}_{\mathrm{syn},\pi'}$, respectively.

$$
\begin{array}{ccccccc}
\mathrm{R}\Gamma_{\mathrm{syn}}(X_{L,h}) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h})^{G_L} & \xrightarrow{\gamma_{\pi'}} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\})^{G_L} & \xleftarrow{\sim} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{L,h})\{r\}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{R}\Gamma_{\mathrm{syn}}(X_{h}) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h})^{G} & \xrightarrow{\gamma_{\pi'}} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\})^{G} & \xleftarrow{\sim} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K,h})\{r\})
\end{array}
$$

This proves the last claim of our proposition. □

3D. ***Geometric syntomic cohomology.*** We will now study geometric syntomic cohomology, i.e., syntomic cohomology over $\bar{K}$. Most of the constructions related to syntomic cohomology over K have their analogs over $\bar{K}$. We will summarize them briefly. For details, the reader should consult [Tsuji 1999; Beilinson 2013].

For $(U,\bar{U})\in\mathscr{P}^{\mathrm{ss}}_{\bar{K}}$, $r\geq 0$, we have the absolute crystalline cohomology complexes and their completions

$$
\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n &:= \mathrm{R}\Gamma_{\mathrm{cr}}(\bar{U}_{\text{ét}}, \mathrm{R}u_{\bar{U}_n/W_n(k)*}\mathscr{J}^{[r]}_{\bar{U}_n/W_n(k)}),\\
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]}) &:= \operatorname{holim}_n \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n,\\
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_{\mathbb{Q}} &:= \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})\otimes\mathbb{Q}_p.
\end{aligned}
$$

By [Beilinson 2013, Theorem 1.18], the complex $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})$ is a perfect A_{cr}-complex and

$$
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})\otimes^L_{A_{\mathrm{cr}}} A_{\mathrm{cr}}/p^n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})\otimes^L \mathbb{Z}/p^n.
$$

In general, we have $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n\simeq \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})\otimes^L\mathbb{Z}/p^n$. Moreover, $J^{[r]}_{\mathrm{cr}}=\mathrm{R}\Gamma_{\mathrm{cr}}(\operatorname{Spec}(\bar{K}),\operatorname{Spec}(\bar{V}),\mathscr{J}^{[r]})$ [Tsuji 1999, Lemmas 1.6.3 and 1.6.4]. The absolute log-crystalline cohomology complexes are filtered E_∞ algebras over $A_{\mathrm{cr},n}$, A_{cr}, or $A_{\mathrm{cr},\mathbb{Q}}$, respectively. Moreover, the rational ones are filtered commutative dg algebras.

For $r\geq 0$, the mod-p^n, completed, and rational log-syntomic complexes $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_n$, $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}}$ are defined by analogs of formulas (32). We have $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_n\simeq\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)\otimes^L\mathbb{Z}/p^n$. Let $\mathscr{J}^{[r]}_{\mathrm{cr}}$, $\mathscr{A}_{\mathrm{cr}}$, and $\mathscr{S}(r)$ be the h-sheafifications on $\mathscr{V}ar_{\bar{K}}$ of the presheaves sending $(U,\bar{U})\in\mathscr{P}^{\mathrm{ss}}_{\bar{K}}$ to $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})$, $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)$, respectively. Let $\mathscr{J}^{[r]}_{\mathrm{cr},n}$, $\mathscr{A}_{\mathrm{cr},n}$, and $\mathscr{S}_n(r)$ denote the h-sheafifications of the mod-p^n versions of the respective presheaves, and let $\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}}$, $\mathscr{A}_{\mathrm{cr},\mathbb{Q}}$, $\mathscr{S}(r)_{\mathbb{Q}}$ be the h-sheafification of the rational versions of the same presheaves.

For $X\in\mathscr{V}ar_{\bar{K}}$, set $\mathrm{R}\Gamma_{\mathrm{cr}}(X_h):=\mathrm{R}\Gamma(X_h,\mathscr{A}_{\mathrm{cr}})$. It is a filtered (by $\mathrm{R}\Gamma(X_h,\mathscr{J}^{[r]}_{\mathrm{cr}})$, $r\geq 0$) E_∞ A_{cr}-algebra equipped with the Frobenius action φ. The Galois group G_K acts on $\mathscr{V}ar_{\bar{K}}$ and it acts on $X\mapsto \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)$ by transport of structure. If X is defined over K then G_K acts naturally on $\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)$.

For $r \geq 0$, set $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_n = \mathrm{R}\Gamma(X_h, \mathscr{S}_n(r))$ and define $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) := \mathrm{R}\Gamma(X_h, \mathscr{S}(r)_{\mathbb{Q}})$. We have

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_n \simeq \mathrm{Cone}\big(\mathrm{R}\Gamma(X_h, \mathscr{J}^{[r]}_{\mathrm{cr},n}) \xrightarrow{p^r-\varphi} \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{cr},n})\big)[-1],$$
$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \simeq \mathrm{Cone}\big(\mathrm{R}\Gamma(X_h, \mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}}) \xrightarrow{1-\varphi_r} \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{cr},\mathbb{Q}})\big)[-1].$$

The direct sum $\bigoplus_{r\geq 0} \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ is a graded E_∞ algebra over $\mathbb{Z}_p$.

Let $\bar{f} : Z_1 \to \mathrm{Spec}(\overline{V}_1)^\times$ be an integral, quasi-coherent log-scheme. Suppose $\bar{f}$ is the base change of $\bar{f}_L : Z_{L,1} \to \mathrm{Spec}(\mathscr{O}_{L,1})^\times$ by $\theta_1 : \mathrm{Spec}(\overline{\mathscr{O}_{L,1}})^\times \to \mathrm{Spec}(\mathscr{O}_{L,1})^\times$ for a finite extension L/K. That is, we have a map $\theta_{L,1} : Z_1 \to Z_{L,1}$ such that the square $(\bar{f}, \bar{f}_L, \theta_1, \theta_{L,1})$ is Cartesian. Assume that $\bar{f}_L$ is log-smooth of Cartier type and that the underlying map of schemes is proper. Such data $(L, Z_1, \theta_{L,1})$ form a directed set Σ_1 and, for a morphism $(L', Z'_1, \theta'_{L',1}) \to (L, Z_1, \theta_{L,1})$, we have a canonical base change identification compatible with φ-action [Beilinson 2013, 1.18]

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_{L,1}) \otimes_{L_0} L'_0 \xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z'_{L',1}).$$

These identifications can be made compatible with respect to L, so we can set

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1) := \varinjlim_{\Sigma_1} \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_{L,1}).$$

It is a complex of (φ, N)-modules over K_0^{nr}, functorial with respect to morphisms of Z_1.

Consider the scheme $E_{\mathrm{cr}} := \mathrm{Spec}(A_{\mathrm{cr}})$. We have $E_{\mathrm{cr},1} = \mathrm{Spec}(\overline{V}_1)$ and we equip $E_{\mathrm{cr},1}$ with the induced log-structure. This log-structure extends uniquely to a log-structure on $E_{\mathrm{cr},n}$ and the PD-thickening $\mathrm{Spec}(\overline{V})_1^\times \hookrightarrow E_{\mathrm{cr},n}$ is universal over $\mathbb{Z}/p^n$. Set $E_{\mathrm{cr}} := \mathrm{Spec}(A_{\mathrm{cr}})$ with the limit log-structure. Since we have [Beilinson 2013, 1.18.1]

$$\mathrm{R}\Gamma_{\mathrm{cr}}(Z_1) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(Z_1/E_{\mathrm{cr}}),$$

Theorem 3.6 yields a canonical quasi-isomorphism of B^+_{cr}-complexes (called *the crystalline Beilinson–Hyodo–Kato quasi-isomorphism*)

$$\iota^B_{\mathrm{cr}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)^\tau_{B^+_{\mathrm{cr}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(Z_1)_{\mathbb{Q}}$$

compatible with the action of Frobenius. But we have

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)^\tau_{B^+_{\mathrm{cr}}} = (\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1) \otimes_{K_0^{\mathrm{nr}}} A^\tau_{\mathrm{cr},\mathbb{Q}})^{N=0}$$

and there is a canonical isomorphism $A^\tau_{\mathrm{cr},\mathbb{Q}} \xrightarrow{\sim} B^+_{\mathrm{st}}$ that is compatible with Frobenius and monodromy. This implies that the above quasi-isomorphism amounts to a quasi-isomorphism of B^+_{cr}-complexes

$$\iota^B_{\mathrm{cr}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)_{B^+_{\mathrm{st}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(Z_1) \otimes^L_{A_{\mathrm{cr}}} B^+_{\mathrm{st}}$$

compatible with the action of φ and N. The crystalline Beilinson–Hyodo–Kato map can be canonically trivialized at $[\tilde{p}]$, where $\tilde{p}$ is a sequence of p^n-th roots of p:

$$\beta=\beta_{[\tilde{p}]}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{cr}}\xrightarrow{\sim}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{cr}}[a([\tilde{p}])])^{N=0},$$
$$x\mapsto\exp(N(x)a([\tilde{p}])).$$

This trivialization is compatible with Frobenius and monodromy.

Suppose now that $\bar{f}_1:Z_1\to\operatorname{Spec}(\overline{V}_1)^\times$ is a reduction mod p of a log-scheme $\bar{f}:Z\to\operatorname{Spec}(\overline{V})^\times$. Suppose that $\bar{f}$ is the base change of $\bar{f}_L:Z_L\to\operatorname{Spec}(\mathscr{O}_L)^\times$ by $\theta:\operatorname{Spec}(\overline{\mathscr{O}_L})^\times\to\operatorname{Spec}(\mathscr{O}_L)^\times$ for a finite extension L/K. That is, we have a map $\theta_L:Z\to Z_L$ such that the square $(\bar{f},\bar{f}_L,\theta,\theta_L)$ is Cartesian. Assume that $\bar{f}_L$ is log-smooth of Cartier type and that the underlying map of schemes is proper. Such data (L,Z,θ_L) form a directed set Σ and the reduction mod p map $\Sigma\to\Sigma_1$ is cofinal. The Beilinson–Hyodo–Kato quasi-isomorphisms (27) are compatible with morphisms in Σ and their colimit yields a natural quasi-isomorphism (called again the *Beilinson–Hyodo–Kato quasi-isomorphism*)

$$\iota^B_{\mathrm{dR}}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)^\tau_{\overline{K}}\xrightarrow{\sim}\mathrm{R}\Gamma(Z_{\overline{K}},\Omega^\bullet_{Z/\overline{K}}).$$

The trivializations by p are also compatible with the maps in Σ; hence we obtain the Beilinson–Hyodo–Kato maps

$$\iota^B_{\mathrm{dR}}:=\iota^B_{\mathrm{dR}}\beta_p:\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)\to\mathrm{R}\Gamma(Z_{\overline{K}},\Omega^\bullet_{Z/\overline{K}}).$$

For an ss-pair $(U,\overline{U})$ over $\overline{K}$, set $\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\overline{U}):=\mathrm{R}\Gamma^B_{\mathrm{HK}}((U,\overline{U})_1)$. Let $\mathscr{A}^B_{\mathrm{HK}}$ be the h-sheafification of the presheaf $(U,\overline{U})\mapsto\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\overline{U})$ on $\mathscr{P}^{\mathrm{ss}}_{\overline{K}}$. This is an h-sheaf of E_∞ K_0^{nr}-algebras equipped with a φ-action and locally nilpotent derivation N such that $N\varphi=p\varphi N$. For $X\in\mathscr{V}ar_{\overline{K}}$, set $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h):=\mathrm{R}\Gamma(X_h,\mathscr{A}^B_{\mathrm{HK}})$.

Proposition 3.21. (1) *For any $(U,\overline{U})\in\mathscr{P}^{\mathrm{ss}}_{\overline{K}}$, the canonical maps*

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{J}^{[r]})_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{R}\Gamma(U_h,\mathscr{J}^{[r]}_{\mathrm{cr}})_{\mathbb{Q}},\quad \mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\overline{U})\xrightarrow{\sim}\mathrm{R}\Gamma^B_{\mathrm{HK}}(U_h)\tag{41}$$

are quasi-isomorphisms.

(2) *For every $X\in\mathscr{V}ar_{\overline{K}}$, the cohomology groups $H^n_{\mathrm{cr}}(X_h):=H^n\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}$ and $H^n_{\mathrm{HK}}(X_h):=H^n\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)$, are free B^+_{cr}-modules, resp. K_0^{nr}-modules, of rank equal to the rank of $H^n(X_{\text{ét}},\mathbb{Q}_p)$.*

Proof. Only the filtered statement in part (1) for $r>0$ requires argument since the rest has been proven by Beilinson [2013, 2.4]. Take $r>0$. To prove that we have a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{J}^{[r]})_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{R}\Gamma(U_h,\mathscr{J}^{[r]}_{\mathrm{cr}})_{\mathbb{Q}}$, it suffices to show that the map $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}}\to\mathrm{R}\Gamma(U_h,\mathscr{A}_{\mathrm{cr}}/\mathscr{J}^{[r]}_{\mathrm{cr}})_{\mathbb{Q}}$ is a quasi-isomorphism. Since, for an ss-pair $(T,\overline{T})$ over K, by Corollary 2.4 $\mathrm{R}\Gamma_{\mathrm{cr}}(T,\overline{T},\mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}}\simeq\mathrm{R}\Gamma(\overline{T}_K,\Omega^\bullet_{(T,\overline{T}_K)}/F^r)$, this is equivalent to showing that $\mathrm{R}\Gamma(\overline{U}_K,\Omega^\bullet_{(U,\overline{U}_K)}/F^r)\to\mathrm{R}\Gamma(U_h,\mathscr{A}_{\mathrm{dR}}/F^r)$ is a quasi-isomorphism, which follows from Proposition 3.14. $\square$

Proposition 3.22. *Let $X \in \mathscr{V}ar_K$. The natural projection $\varepsilon : X_{\bar{K},h} \to X_h$ defines pullback maps*

$$\varepsilon^* : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \to \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})^{G_K}, \quad \varepsilon^* : \mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})^{G_K}. \tag{42}$$

These are (filtered) quasi-isomorphisms.

Proof. Notice that the action of G_K on $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})$ is smooth, i.e., the stabilizer of every element is an open subgroup of G_K. We will prove only the first quasi-isomorphism — the proof of the second one being analogous. By Proposition 3.18, it suffices to show that for any ss-pair over K, the natural map

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(U_1, \bar{U}_1) \to \mathrm{R}\Gamma^B_{\mathrm{HK}}((U, \bar{U}) \otimes_K \bar{K})^{G_K}$$

is a quasi-isomorphism. Passing to a finite extension of K_U, if necessary, we may assume that $(U, \bar{U})$ is log-smooth of Cartier type over a finite Galois extension K_U of K. Then

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}((U, \bar{U}) \otimes_K \bar{K}) \simeq \mathrm{R}\Gamma^B_{\mathrm{HK}}(U_1, \bar{U}_1) \otimes_{K_{U,0}} K_0^{\mathrm{nr}} \times H, \quad H = \mathrm{Gal}(K_U/K).$$

Taking G_K-fixed points of this quasi-isomorphism, we get the first quasi-isomorphism of (42), as wanted. $\square$

Let $(U, \bar{U})$ be an ss-pair over $\bar{K}$. Set

$$\begin{aligned}
\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(U, \bar{U}) &:= \mathrm{R}\Gamma(\bar{U}_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{(U,\bar{U})/W(k)}),\\
\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(U, \bar{U})_n &:= \mathrm{R}\Gamma^\natural_{\mathrm{dR}}(U, \bar{U}) \otimes^{\mathbb{L}} \mathbb{Z}/p^n \simeq \mathrm{R}\Gamma(\bar{U}_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{(U,\bar{U})_n/W_n(k)}),\\
\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(U, \bar{U}) \widehat{\otimes} \mathbb{Z}_p &:= \mathrm{holim}_n \mathrm{R}\Gamma^\natural_{\mathrm{dR}}(U, \bar{U})_n,\\
\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(U, \bar{U}) \widehat{\otimes} \mathbb{Q}_p &:= (\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(U, \bar{U}) \widehat{\otimes} \mathbb{Z}_p) \otimes \mathbb{Q}.
\end{aligned}$$

These are F-filtered E_∞ algebras. Take the associated presheaves on $\mathscr{P}^{\mathrm{ss}}_{\bar{K}}$. Denote by $\mathscr{A}^\natural_{\mathrm{dR}}$, $\mathscr{A}^\natural_{\mathrm{dR},n}$, $\mathscr{A}^\natural_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p$, $\mathscr{A}^\natural_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p$ their sheafifications in the h-topology of $\mathscr{V}ar_{\bar{K}}$. These are sheaves of F-filtered E_∞ algebras (viewed as the projective system of quotients modulo F^i). Set $A_{\mathrm{dR}} := \mathrm{L}\Omega^{\bullet,\wedge}_{\bar{V}/V}$. By [Beilinson 2012, Lemma 3.2], $A_{\mathrm{dR}} = \mathscr{A}^\natural_{\mathrm{dR}}(\mathrm{Spec}(\bar{K})) = \mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\bar{K}, \bar{V})$. The corresponding F-filtered algebras $A_{\mathrm{dR},n}$, $A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p$, $A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p$ are acyclic in nonzero degrees and the projections $\cdot/F^{m+1} \to \cdot/F^m$ are surjective. Thus (we set $\lim_F := \mathrm{holim}_F$)

$$\begin{aligned}
A^\diamond_{\mathrm{dR},n} &:= \lim_F A_{\mathrm{dR},n} = \varprojlim_m H^0(A_{\mathrm{dR},n}/F^m),\\
A^\diamond_{\mathrm{dR}} &:= \lim_F (A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p) = \varprojlim_m H^0(A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p/F^m),\\
\lim_F A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p &= \varprojlim_m H^0(A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p/F^m) = B^+_{\mathrm{dR}},\\
A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p/F^m &= B^+_{\mathrm{dR}}/F^m.
\end{aligned}$$

For any $(U,\overline{U})$ over $\overline{K}$, the complex $\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\overline{U})$ is an F-filtered E_∞ filtered A_{dR}-algebra; hence $\lim_F \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\overline{U})_n$ is an $A^{\diamond}_{\mathrm{dR},n}$-algebra, $\lim_F(\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\overline{U})\mathbin{\widehat{\otimes}}\mathbb{Q}_p)$ is a B^+_{dR}-algebra, etc. We have canonical morphisms

$$\kappa'_{r,n}:\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_n\to\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_n/F^r\xrightarrow{\sim}\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\overline{U})_n/F^r.$$

In the case of $(\overline{K},\overline{V})$, from Theorem 2.1, we get isomorphisms

$$\kappa'_{r,n}=\kappa_r^{-1}:A_{\mathrm{cr},n}/J^{[r]}\xrightarrow{\sim}A_{\mathrm{dR},n}/F^r.$$

Hence $A^{\diamond}_{\mathrm{dR}}$ is the completion of A_{cr} with respect to the $J^{[r]}$-topology.

For $X\in\mathscr{V}ar_{\overline{K}}$, set $\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h):=\mathrm{R}\Gamma(X_h,\mathscr{A}^{\natural}_{\mathrm{dR}})$. Since $A_{\mathrm{dR},\mathbb{Q}}=\overline{K}$, for any variety X over $\overline{K}$, we have a filtered quasi-isomorphism of $\overline{K}$-algebras [Beilinson 2012, 3.2] $\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h)_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$ obtained by h-sheafification of the quasi-isomorphism

$$\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\overline{U})_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U}_{\mathbb{Q}}).\tag{43}$$

Concerning the p-adic coefficients, we have a quasi-isomorphism

$$\gamma_r:(\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}}B^+_{\mathrm{dR}})/F^r\xrightarrow{\sim}\mathrm{R}\Gamma(X_h,\mathscr{A}^{\natural}_{\mathrm{dR}}\mathbin{\widehat{\otimes}}\mathbb{Q}_p)/F^r.\tag{44}$$

To define it, consider, for any ss-pair $(U,\overline{U})$ over $\overline{K}$, the natural map $\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\overline{U})\to\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\overline{U})\widehat{\otimes}\mathbb{Z}_p$. It yields, by extension to $A_{\mathrm{dR}}\widehat{\otimes}\mathbb{Q}_p$ and by the quasi-isomorphism (43), a quasi-isomorphism of F-filtered $\overline{K}$-algebras [Beilinson 2013, 3.5]

$$\gamma:\mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U})_{\mathbb{Q}}\otimes_{\overline{K}}(A_{\mathrm{dR}}\mathbin{\widehat{\otimes}}\mathbb{Q}_p)\xrightarrow{\sim}\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\overline{U})\mathbin{\widehat{\otimes}}\mathbb{Q}_p.$$

Its (mod F^r)-version γ_r after h-sheafification yields the quasi-isomorphism

$$\gamma_r:(\mathscr{A}_{\mathrm{dR}}\otimes_{\overline{K}}B^+_{\mathrm{dR}})/F^r\xrightarrow{\sim}\mathscr{A}^{\natural}_{\mathrm{dR}}\mathbin{\widehat{\otimes}}\mathbb{Q}_p/F^r.$$

Passing to $\mathrm{R}\Gamma(X_h,\bullet)$ we get the quasi-isomorphism (44).

For $X\in\mathscr{V}ar_{\overline{K}}$, we have canonical quasi-isomorphisms

$$\iota^B_{\mathrm{cr}}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^{\tau}_{B^+_{\mathrm{cr}}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}},\quad\iota^B_{\mathrm{dR}}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^{\tau}_{\overline{K}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$$

compatible with the $\mathrm{Gal}(\overline{K}/K)$-action. Here ${}^{\tau}_{B^+_{\mathrm{cr}}}$ and ${}^{\tau}_{\overline{K}}$ denote the h-sheafification of the crystalline and de Rham Beilinson–Hyodo–Kato twists [Beilinson 2013, 2.5.1]. Trivializing the first map at $[\tilde{p}]$ and the second map at p, we get the Beilinson–Hyodo–Kato maps

$$\iota^B_{\mathrm{cr}}:=\iota^B_{\mathrm{cr}}\beta_{[\tilde{p}]}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{cr}}\to\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}},$$
$$\iota_{\mathrm{dR}}:=\iota_{\mathrm{dR}}\beta_p:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\to\mathrm{R}\Gamma_{\mathrm{dR}}(X_h).$$

Using the quasi-isomorphism

$$\kappa_r^{-1}:\mathscr{A}_{\mathrm{cr},\mathbb{Q}}/\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}}\xrightarrow{\sim}(\mathscr{A}^{\natural}_{\mathrm{dR}}\mathbin{\widehat{\otimes}}\mathbb{Q}_p)/F^r$$

from Theorem 2.1, we get the quasi-isomorphisms of complexes of sheaves on $X_{\bar{K},h}$

$$\mathscr{S}(r)_{\mathbb{Q}} \xrightarrow{\sim} [\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}} \xrightarrow{1-\varphi_r} \mathscr{A}_{\mathrm{cr},\mathbb{Q}}] \xrightarrow{\sim} [\mathscr{A}_{\mathrm{cr},\mathbb{Q}} \xrightarrow{(1-\varphi_r,\mathrm{can})} \mathscr{A}_{\mathrm{cr},\mathbb{Q}} \oplus \mathscr{A}_{\mathrm{cr},\mathbb{Q}}/\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}}]$$
$$\xleftarrow{\sim} [\mathscr{A}_{\mathrm{cr},\mathbb{Q}} \xrightarrow{(1-\varphi_r,\kappa_r^{-1})} \mathscr{A}_{\mathrm{cr},\mathbb{Q}} \oplus (\mathscr{A}^{\natural}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p)/F^r]\,.$$

Applying $\mathrm{R}\Gamma(X_h, \bullet)$ and the quasi-isomorphism $\gamma_r^{-1} : \mathrm{R}\Gamma(X_h, \mathscr{A}^{\natural}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p)/F^r \xrightarrow{\sim} (\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r$ from (44), we obtain the quasi-isomorphisms

$$\begin{aligned}
&\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)\\
&\quad\xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \oplus \mathrm{R}\Gamma(X_h, \mathscr{A}^{\natural}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p)/F^r\big]\\
&\quad\xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\gamma_r^{-1}\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r\big].
\end{aligned}\tag{45}$$

Corollary 3.23. *For any $(U, \bar{U}) \in \mathscr{P}^{\mathrm{ss}}_{\bar{K}}$, the canonical map*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(U, \bar{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{syn}}(U_h, r)$$

is a quasi-isomorphism.

Proof. Arguing as above, we find quasi-isomorphisms

$$\begin{aligned}
&\mathrm{R}\Gamma_{\mathrm{syn}}(U, \bar{U}, r)_{\mathbb{Q}}\\
&\quad\xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \oplus (\mathrm{R}\Gamma^{\natural}(U, \bar{U}) \widehat{\otimes} \mathbb{Q}_p)/F^r\big]\\
&\quad\xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\gamma_r^{-1}\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r\big].
\end{aligned}$$

Comparing them with quasi-isomorphisms (45), we see that it suffices to check that the natural maps

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U_h)_{\mathbb{Q}}, \quad \mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U_h)$$

are (filtered) quasi-isomorphisms, but this follows by Propositions 3.21 and 3.14. □

Consider the composition of morphisms

$$\begin{aligned}
&\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)\\
&\quad\xrightarrow{\sim} \big[\,\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\gamma_r^{-1}\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r\,\big]\\
&\quad\xleftarrow{\sim} \left[\begin{array}{ccc}
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} & \xrightarrow{(1-\varphi_r,\iota^B_{\mathrm{dR}}\otimes\iota)} & \begin{array}{c}\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}}\\ \oplus(\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r\end{array}\\
\downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)}\\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}}
\end{array}\right].
\end{aligned}\tag{46}$$

The second quasi-isomorphism uses the map

$$(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}})^{N=0} = \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^{\tau}_{B^+_{\mathrm{cr}}} \xrightarrow{\iota^B_{\mathrm{cr}}} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}$$

(that is compatible with the action of N and φ) and the following lemma.

Lemma 3.24. *The following diagrams commute*:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}\otimes_{B^+_{\mathrm{cr}}} B^+_{\mathrm{st}} \arrow[r, "\gamma_r^{-1}\kappa_r^{-1}\otimes\iota"] & (\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}} B^+_{\mathrm{dR}})/F^r \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} \arrow[u, "\iota^B_{cr}", "\wr"'] \arrow[ur, "\iota^B_{\mathrm{dR}}\otimes\iota"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}\otimes_{A_{\mathrm{cr}}} B_{\mathrm{dR}} \arrow[r, "\gamma_{\mathrm{dR}}", "\sim"'] & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}} B_{\mathrm{dR}} \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} \arrow[u, "\iota^B_{\mathrm{cr}}\otimes\iota"] \arrow[ur, "\iota^B_{\mathrm{dR}}\otimes\iota"'] &
\end{tikzcd}$$

(Here γ_{dR} is the map defined in [Beilinson 2013, 3.4.1].)

Proof. We will start with the top diagram. It suffices to show that it canonically commutes with X_h replaced by any ss-pair $\overline{Y}=(U,\overline{U})$ over $\overline{K}$ — a base change of an ss-pair Y split over (V,K). Proceeding as in Example 3.5, we obtain the following diagram in which all squares but the one in the top right clearly commute:

$$\begin{tikzcd}
\mathrm{R}\Gamma^B_{\mathrm{HK}}(Y_1)^{\tau}_K \arrow[r, "\mathrm{Id}\otimes 1"'] \arrow[d, "\iota^B_K"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^{\tau}_{\overline{K}}\otimes_{\overline{K}} B^+_{\mathrm{dR}} \arrow[d, "\iota^B_{\overline{K}}\otimes\kappa_r"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^{\tau}_{B^+_{\mathrm{cr}}}\otimes_{B^+_{\mathrm{cr}}} B^+_{\mathrm{st}} \arrow[l, "\delta"'] \arrow[d, "\iota^B_{\mathrm{cr}}\otimes\kappa_r\iota"] \\
\mathrm{R}\Gamma_{\mathrm{cr}}(Y_1/V^{\times})_{\mathbb{Q}}/F^r \arrow[r] & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/V^{\times})_{\mathbb{Q}}/F^r & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/A_{\mathrm{cr}})_{\mathbb{Q}}/F^r \arrow[l, "\sim"] \\
\mathrm{R}\Gamma(Y_{\mathrm{\acute{e}t}},\mathrm{L}\Omega^{\bullet,\wedge}_{Y/V^{\times}})\widehat{\otimes}\mathbb{Q}_p/F^r \arrow[r] \arrow[u, "\kappa_r", "\wr"'] & \mathrm{R}\Gamma(\overline{Y}_{\mathrm{\acute{e}t}},\mathrm{L}\Omega^{\bullet,\wedge}_{\overline{Y}/V^{\times}})\widehat{\otimes}\mathbb{Q}_p/F^r \arrow[u, "\kappa_r", "\wr"'] & (\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(\overline{Y})\widehat{\otimes}\mathbb{Q}_p)/F^r \arrow[l, "\sim"'] \arrow[u, "\kappa_r", "\wr"'] \\
\mathrm{R}\Gamma_{\mathrm{dR}}(Y_K)/F^r \arrow[r] \arrow[u, "\gamma_r", "\wr"'] & (\mathrm{R}\Gamma_{\mathrm{dR}}(\overline{Y}_K)\otimes_{\overline{K}} B^+_{\mathrm{dR}})/F^r \arrow[u, "\gamma_r", "\wr"'] \arrow[ur, "\gamma_r", "\sim"'] &
\end{tikzcd}$$

Here we have $B^+_{\mathrm{dR}}/F^m=(\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(\overline{K},\overline{V})\widehat{\otimes}\mathbb{Q}_p)/F^m$ and the map δ is defined as the composition

$$\begin{aligned}\delta:\mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^{\tau}_{B^+_{\mathrm{cr}}}\otimes_{B^+_{\mathrm{cr}}} B^+_{\mathrm{st}} &= (\mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)\otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}})^{N=0}\otimes_{B^+_{\mathrm{cr}}} B^+_{\mathrm{st}} \\ &\xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)\otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} \xrightarrow{\beta_p\otimes\iota} \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^{\tau}_{\overline{K}}\otimes_{\overline{K}} B^+_{\mathrm{dR}}.\end{aligned}$$

Recall that for the map $\iota_{\mathrm{dR}}^B : \mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_K^\tau \to \mathrm{R}\Gamma_{\mathrm{dR}}(Y_K)/F^r$, we have $\iota_{\mathrm{dR}}^B = \gamma_r^{-1}\kappa_r^{-1}\iota_K^B$. Everything in sight being compatible with change of the ss-pairs Y — more specifically with maps in the directed system Σ — if this diagram commutes so does its Σ colimit and the top diagram in the lemma for the pair $(U, \overline{U})$.

It remains to show that the top right square in the above diagram commutes. To do that, consider the ring $\widehat{A}_n$ defined as the PD-envelope of the closed immersion

$$\overline{V}_1^\times \hookrightarrow A_{\mathrm{cr},n} \times_{W_n(k)} V_n^\times .$$

That is, $\widehat{A}_n$ is the product of the PD-thickenings $(\overline{V}_1^\times \hookrightarrow A_{\mathrm{cr},n})$ and $(V_1^\times \hookrightarrow V_n^\times)$ over $(W_1(k) \hookrightarrow W_n(k))$. By [Beilinson 2013, Lemma 1.17], this makes $\overline{V}_1^\times \hookrightarrow \widehat{A}_{\mathrm{cr},n}$ into the universal PD-thickening in the log-crystalline site of $\overline{V}_1^\times$ over $V_n^\times$. Let $\widehat{A} := \operatorname{inj\,lim}_n \widehat{A}_{\mathrm{cr},n}$ with the limit log-structure. Set $\widehat{B}_{\mathrm{cr}}^+ := \widehat{A}_{\mathrm{cr}}[1/p]$.

Using Theorem 3.6, we obtain a canonical quasi-isomorphism

$$\iota_{\widehat{B}_{\mathrm{cr}}^+}^B : \mathrm{R}\Gamma_{\mathrm{HK}}^B(\overline{Y}_1)_{\widehat{B}_{\mathrm{cr}}^+}^\tau \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/\widehat{A}_{\mathrm{cr}})_{\mathbb{Q}}.$$

By construction, we have the maps of PD-thickenings

$$(V_1^\times \hookrightarrow V^\times) \xleftarrow{\mathrm{pr}_1} (\overline{V}_1^\times \hookrightarrow \widehat{A}_{\mathrm{cr}}) \xrightarrow{\mathrm{pr}_2} (\overline{V}_1^\times \hookrightarrow A_{\mathrm{cr}}).$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}^B(\overline{Y}_1)_{\widehat{B}_{\mathrm{cr}}^+}^\tau \arrow[dd, "\iota_{\widehat{B}_{\mathrm{cr}}^+}^B"'] & & \mathrm{R}\Gamma_{\mathrm{HK}}^B(\overline{Y}_1)_{\overline{K}}^\tau \otimes_{\overline{K}} B_{\mathrm{dR}}^+/F^r \arrow[ll, "\mathrm{pr}_1^*\otimes\mathrm{pr}_1^*\kappa_r"'] \arrow[dd, "\iota_{\overline{K}}^B\otimes\kappa_r"] \\
& \mathrm{R}\Gamma_{\mathrm{HK}}^B(\overline{Y}_1)_{B_{\mathrm{cr}}^+}^\tau \arrow[ul, "\mathrm{pr}_2^*"] \arrow[ur, "\delta"'] \arrow[d, "\iota_{\mathrm{cr}}^B"] & \\
& \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/A_{\mathrm{cr}})_{\mathbb{Q}}/F^r \arrow[dl, "\mathrm{pr}_2^*", "\sim"'] \arrow[dr, "\sim"] & \\
\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/\widehat{A}_{\mathrm{cr}})_{\mathbb{Q}}/F^r & & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/V^\times)_{\mathbb{Q}}/F^r \arrow[ll, "\sim", "\mathrm{pr}_1^*"']
\end{tikzcd}$$

The bottom triangle commutes since $\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/A_{\mathrm{cr}}) = \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/W(k))$. The pullback maps

$$\mathrm{pr}_1^* : \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/V^\times) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\widehat{A}_{\mathrm{cr}}),$$
$$\mathrm{pr}_2^* : \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/A_{\mathrm{cr}})_{\mathbb{Q}}/F^r \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\widehat{A}_{\mathrm{cr}})_{\mathbb{Q}}/F^r$$

are quasi-isomorphisms. Indeed, in the case of the first pullback this follows from the universal property of $\widehat{A}_{\mathrm{cr}}$; in the case of the second one, it follows from the commutativity of the bottom triangle since the right slanted map is a quasi-isomorphism as shown by the first diagram in our proof.

The left trapezoid and the big square commute by the definition of the Beilinson–Bloch–Kato maps. To see that the top triangle commutes, it suffices to show that for an element

$$x \in \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(\overline{Y}_1)^{\tau}_{B^+_{\mathrm{cr}}} = (\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(\overline{Y}_1) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}})^{N=0},$$
$$x = b \sum_{i \geq 0} N^i(m) a([\tilde{p}])^{[i]}, \quad m \in \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(\overline{Y}_1),\ b \in B^+_{\mathrm{cr}},$$

we have $\mathrm{pr}_2^*(x) = \mathrm{pr}_1^*\,\delta(x)$. Since $\iota(a([\tilde{p}])) = \log([\tilde{p}]/p)$ [Fontaine 1994a, 4.2.2], we calculate

$$\begin{aligned}\delta(x) = \delta\biggl(b \sum_{i \geq 0} N^i(m) a([\tilde{p}])^{[i]}\biggr) &= b \sum_{i \geq 0}\biggl(\sum_{j \geq 0} N^{i+j}(m) a(p)^{[j]}\biggr) \log([\tilde{p}]/p)^{[i]}\\ &= b \sum_{k \geq 0} N^k(m)\bigl(a(p) + \log([\tilde{p}]/p)\bigr)^{[k]}.\end{aligned}$$

Since in $\widehat{B}^+_{\mathrm{cr}}$ we have $[\tilde{p}] = ([\tilde{p}]/p)p$ and $[\tilde{p}]/p \in 1 + J_{\widehat{B}^+_{\mathrm{cr}}}$, it follows that $a([\tilde{p}]) = \log([\tilde{p}]/p) + a(p)$ and

$$\begin{aligned}\mathrm{pr}_1^*\,\delta(x) &= \mathrm{pr}_1^*\biggl(b \sum_{k \geq 0} N^k(m)(a(p) + \log([\tilde{p}]/p))^{[k]}\biggr)\\ &= b \sum_{k \geq 0} N^k(m) a([\tilde{p}])^{[k]} = \mathrm{pr}_2^*\biggl(b \sum_{k \geq 0} N^k(m) a([\tilde{p}])^{[k]}\biggr) = \mathrm{pr}_2^*(x),\end{aligned}$$

as wanted. It follows now that the right trapezoid in the above diagram commutes as well and that so does the top diagram in our lemma.

To check the commutativity of the bottom diagram, consider the following map obtained from the maps $\kappa'_{r,n}$ by passing to F-limit:

$$\kappa'_n : \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_n \otimes^L_{A_{\mathrm{cr},n}} A_{\mathrm{dR},n} \xrightarrow{\sim} \varprojlim_F \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_n / F^r.$$

By [Beilinson 2013, 3.6.2], this is a quasi-isomorphism. Beilinson [2013, 3.4.1] defines the map

$$\gamma_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} \otimes_{A_{\mathrm{cr}}} B^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\overline{Y}_K) \otimes_{\overline{K}} B^+_{\mathrm{dR}}$$

by B^+_{dR}-linearization of the composition $\varprojlim_r(\gamma_r^{-1}\kappa_r^{-1})\,\mathrm{holim}_n\,\kappa'_n$. We have

$$\gamma_{\mathrm{dR}} = \gamma_r^{-1}\kappa_r^{-1} : \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} \to (\mathrm{R}\Gamma_{\mathrm{dR}}(\overline{Y}_K) \otimes_{\overline{K}} B^+_{\mathrm{dR}})/F^r.$$

Hence the commutativity of the bottom diagram follows from that of the top one. $\square$

Let $C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$ denote the second homotopy limit in the diagram (46); denote by $C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$ the complex $C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$ with all the pluses removed. We have defined a map $\alpha_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r)\to C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$ and proved the following proposition.

Proposition 3.25. *There is a functorial G_K-equivariant quasi-isomorphism*

$$\alpha_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r)=\mathrm{R}\Gamma(X_h,\mathscr{S}(r)_{\mathbb{Q}})\simeq C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}).$$

Corollary 3.26. *For $(U,\bar{U})\in\mathscr{P}^{\mathrm{ss}}_K$, we have a long exact sequence*

$$\begin{aligned}\cdots\to H^i_{\mathrm{syn}}((U,\bar{U})_{\bar{K}},r)\to (H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\otimes_{K_0}B^+_{\mathrm{st}})^{\varphi=p^r,N=0}\\ \to (H^i_{\mathrm{dR}}(U,\bar{U})\otimes_K B^+_{\mathrm{dR}})/F^r\to H^{i+1}_{\mathrm{syn}}((U,\bar{U})_{\bar{K}},r)\to\cdots.\end{aligned}$$

Proof. By diagram (46), it suffices to show that

$$\begin{aligned}H^i[\mathrm{R}\Gamma^B_{\mathrm{HK}}((U,\bar{U})_1)\otimes_{K_0}B^+_{\mathrm{st}}]^{\varphi=p^r,N=0}\simeq(H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\otimes_{K_0}B^+_{st})^{\varphi=p^r,N=0},\\ H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U})\otimes_K B^+_{\mathrm{dR}})/F^r)\simeq(H^i_{\mathrm{dR}}(U,\bar{U})\otimes_K B^+_{\mathrm{dR}})/F^r.\end{aligned}$$

The second isomorphism is a consequence of the degeneration of the Hodge–de Rham spectral sequence. Keeping in mind that the Beilinson–Hyodo–Kato complexes $\mathrm{R}\Gamma^B_{\mathrm{HK}}((U,\bar{U})_1)$ are built from (φ,N)-modules, the first isomorphism follows from the short exact sequences (for a (φ,N)-module M)

$$\begin{aligned}0\to M\otimes_{K_0}B^+_{\mathrm{cr}}\to M\otimes_{K_0}B^+_{\mathrm{st}}\xrightarrow{N}M\otimes_{K_0}B^+_{\mathrm{st}}\to 0,\\ 0\to(M\otimes_{K_0}B^+_{\mathrm{cr}})^{\varphi=p^r}\to M\otimes_{K_0}B^+_{\mathrm{cr}}\xrightarrow{1-\varphi_r}M\otimes_{K_0}B^+_{\mathrm{cr}}\to 0.\end{aligned}$$

The first one follows, by induction on m such that $N^m=0$ on M, from the exact sequence (11) and the fact that $(M\otimes_{K_0}B^+_{\mathrm{st}})^{N=0}\simeq M\otimes_{K_0}B^+_{\mathrm{cr}}$. The second one follows from [Colmez and Niziol 2015, Remark 2.30]. □

4. Relation between syntomic cohomology and étale cohomology

In this section we will study the relationship between syntomic and étale cohomology in both the geometric and the arithmetic situation.

4A. *Geometric case.* We start with the geometric case. In this subsection, we will construct the geometric syntomic period map from syntomic to étale cohomology. We will prove that in the torsion case, on the level of h-sheaves it is a quasi-isomorphism modulo a universal constant; in the rational case it induces an isomorphism on cohomology groups in a stable range. Finally, we will construct the syntomic descent spectral sequence.

We will first recall the de Rham and crystalline Poincaré lemmas of Beilinson [2013; 2012] and Bhatt [2012].

Theorem 4.1 (de Rham Poincaré lemma [Beilinson 2012, 3.2]). *The maps*

$$A_{\mathrm{dR}} \otimes^L \mathbb{Z}/p^n \to \mathscr{A}^{\natural}_{\mathrm{dR}} \otimes^L \mathbb{Z}/p^n$$

are filtered quasi-isomorphisms of h-sheaves on $\mathscr{V}ar_{\overline{K}}$.

Theorem 4.2 (filtered crystalline Poincaré lemma [Beilinson 2013, 2.3, Bhatt 2012, Theorem 10.14]). *The map $J^{[r]}_{\mathrm{cr},n} \to \mathscr{J}^{[r]}_{\mathrm{cr},n}$ is a quasi-isomorphism of h-sheaves on $\mathscr{V}ar_{\overline{K}}$.*

Proof. We have the map of distinguished triangles

$$\begin{tikzcd}
J^{[r]}_{\mathrm{cr},n} \arrow[r] \arrow[d] & A_{\mathrm{cr},n} \arrow[r] \arrow[d, "\wr"] & A_{\mathrm{cr},n}/J^{[r]}_{\mathrm{cr},n} \arrow[d, "\wr"] \\
\mathscr{J}^{[r]}_{\mathrm{cr},n} \arrow[r] & \mathscr{A}_{\mathrm{cr},n} \arrow[r] & \mathscr{A}_{\mathrm{cr},n}/\mathscr{J}^{[r]}_{\mathrm{cr},n}
\end{tikzcd}$$

The middle map is a quasi-isomorphism by the crystalline Poincaré lemma proved in [Beilinson 2013, 2.3]. Hence it suffices to show that so is the rightmost map. But, by [Beilinson 2013, 1.9.2], this map is quasi-isomorphic to the map $A_{\mathrm{dR},n}/F^r \to \mathscr{A}^{\natural}_{\mathrm{dR},n}/F^r$. Since the last map is a quasi-isomorphism by the de Rham Poincaré lemma, Theorem 4.1, we are done. $\square$

We will now recall the definitions of the crystalline, Beilinson–Hyodo–Kato, and de Rham period maps [Beilinson 2013, 3.1; 2012, 3.5]. Let $X \in \mathscr{V}ar_{\overline{K}}$. To define the crystalline period map

$$\rho_{\mathrm{cr}} : \mathrm{R}\Gamma_{\mathrm{cr}}(X_h) \to \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Z}_p) \mathbin{\widehat{\otimes}} A_{\mathrm{cr}},$$

consider the natural map $\alpha_n : \mathrm{R}\Gamma_{\mathrm{cr}}(X_h) \to \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{cr},n})$ and the composition

$$\begin{aligned}\beta_n : \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Z}_p(r)) \otimes^L_{\mathbb{Z}_p} A_{\mathrm{cr},n} &\xrightarrow{\sim} \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, A_{\mathrm{cr},n}) \\ &\xrightarrow{\sim} \mathrm{R}\Gamma(X_h, A_{\mathrm{cr},n}) \xrightarrow{\sim} \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{cr},n}).\end{aligned}$$

Set $\rho_{\mathrm{cr},n} := \beta_n^{-1}\alpha_n$ and $\rho_{\mathrm{cr}} := \mathrm{holim}_n\, \rho_{\mathrm{cr},n}$. The Hyodo–Kato period map

$$\rho_{\mathrm{HK}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^{\tau}_{B^+_{\mathrm{cr}}} \to \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Q}_p) \otimes B^+_{\mathrm{cr}}, \quad \rho_{\mathrm{HK}} = \rho_{\mathrm{cr},\mathbb{Q}}\iota^B_{\mathrm{cr}},$$

is obtained by composing the map $\rho_{\mathrm{cr},\mathbb{Q}}$ with the quasi-isomorphism

$$\iota^B_{\mathrm{cr}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^{\tau}_{B^+_{\mathrm{cr}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}.$$

The maps ρ_{cr}, ρ_{HK} are morphisms of E_∞ A_{cr}- and B^+_{cr}-algebras equipped with a Frobenius action; they are compatible with the action of the Galois group G_K.

To define the de Rham period map $\rho_{\mathrm{dR}}:\mathrm{R}\Gamma_{dR}(X_h)\otimes_{\overline{K}}B^+_{\mathrm{dR}}\to\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B^+_{\mathrm{dR}}$ consider the compositions

$$\begin{aligned}\alpha &: \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\xrightarrow{\sim}\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h)\otimes\mathbb{Q}\to\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h)\,\widehat{\otimes}\,\mathbb{Q}_p,\\ \beta &: \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Z})\otimes^{\mathbb{L}}A_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma(X_{\text{ét}},A_{\mathrm{dR}})\to\mathrm{R}\Gamma(X_h,A_{\mathrm{dR}})\\ &\qquad\qquad\to\mathrm{R}\Gamma(X_h,\mathscr{A}^{\natural}_{\mathrm{dR}})=\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h).\end{aligned}$$

After tensoring the map β with $\mathbb{Z}/p^n$ and using the de Rham Poincaré lemma, we get a quasi-isomorphism

$$\beta_n:\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Z}/p^n)\otimes^{\mathbb{L}}A_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h)\otimes^{\mathbb{L}}\mathbb{Z}/p^n.$$

Set $\beta_{\mathbb{Q}}:=\operatorname{holim}_n\beta_n\otimes\mathbb{Q}$ and $\rho_{\mathrm{dR}}:=\beta^{-1}\alpha$. This is a morphism of filtered E_∞ B^+_{dR}-algebras, compatible with G_K-action.

Theorem 4.3 [Beilinson 2013, 3.2, 2012, 3.6]. *For $X\in\mathscr{V}ar_{\overline{K}}$, we have canonical quasi-isomorphisms*

$$\begin{aligned}\rho_{\mathrm{cr}}&:\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)\otimes_{A_{\mathrm{cr}}}B_{\mathrm{cr}}\xrightarrow{\sim}\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{cr}},\\ \rho_{\mathrm{HK}}&:\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_h)^{\tau}_{B_{\mathrm{cr}}}\xrightarrow{\sim}\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{cr}},\\ \rho_{\mathrm{dR}}&:\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}}B_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{dR}}.\end{aligned}$$

Pulling back ρ_{HK} to the Fontaine–Hyodo–Kato $\mathbb{G}_a$-torsor $\operatorname{Spec}(B_{\mathrm{st}})/\operatorname{Spec}(B_{\mathrm{cr}})$, we get a canonical quasi-isomorphism of B_{st}-complexes

$$\rho_{\mathrm{HK}}:\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}}B_{\mathrm{st}}\xrightarrow{\sim}\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{st}}\tag{47}$$

compatible with the (φ,N)-action and with the G_K-action on $\mathscr{V}ar_{\overline{K}}$.

Corollary 4.4. *The period morphisms are compatible*; *i.e.*, *the following diagrams commute*:

$$\begin{tikzcd}\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}}B_{\mathrm{st}} \arrow[r, "\iota^{B}_{\mathrm{dR}}\otimes\iota"] \arrow[d, "\rho_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}}B_{\mathrm{dR}} \arrow[d, "\rho_{\mathrm{dR}}"] \\ \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{st}} \arrow[r, "1\otimes\iota"] & \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{dR}}\end{tikzcd}$$

$$\begin{tikzcd}\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)\otimes_{A_{\mathrm{cr}}}B_{\mathrm{dR}} \arrow[d, "\rho_{\mathrm{cr}}\otimes\mathrm{Id}_{B_{\mathrm{dR}}}"'] & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}}B_{\mathrm{dR}} \arrow[l, "\gamma_{\mathrm{dR}}"', "\sim"] \arrow[dl, "\rho_{\mathrm{dR}}"] \\ \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{dR}} & \end{tikzcd}$$

Proof. The bottom diagram commutes by [Beilinson 2013, 3.4]. The commutativity of the top one can be reduced, by the equality $\rho_{\mathrm{HK}}=\rho_{\mathrm{cr}}\iota^{B}_{\mathrm{cr}}$ and the bottom diagram above, to the commutativity of the bottom diagram in Lemma 3.24. $\square$

We will now define the syntomic period map

$$\rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r)), \quad r \geq 0.$$

Set $\mathbb{Z}/p^n(r)' := (1/(p^a a!)\mathbb{Z}_p(r)) \otimes \mathbb{Z}/p^n$, where a is the largest integer $\leq r/(p-1)$. Recall that we have the fundamental exact sequence [Tsuji 1999, Theorem 1.2.4]

$$0 \to \mathbb{Z}/p^n(r)' \to J^{\langle r\rangle}_{\mathrm{cr},n} \xrightarrow{1-\varphi_r} A_{\mathrm{cr},n} \to 0,$$

where

$$J_n^{\langle r\rangle} := \{x \in J^{[r]}_{n+s} \mid \varphi(x) \in p^r A_{\mathrm{cr},n+s}\}/p^n$$

for some $s \geq r$. Set $S_n(r) := \mathrm{Cone}(J^{[r]}_{\mathrm{cr},n} \xrightarrow{p^r-\varphi} A_{\mathrm{cr},n})[-1]$. There is a natural morphism of complexes $S_n(r) \to \mathbb{Z}/p^n(r)'$ (induced by p^r on $J^{[r]}_{\mathrm{cr},n}$ and Id on $A_{\mathrm{cr},n}$), whose kernel and cokernel are annihilated by p^r.

The filtered crystalline Poincaré lemma implies easily the following syntomic Poincaré lemma.

Corollary 4.5. (1) *For* $0 \leq r \leq p-2$, *there is a unique quasi-isomorphism* $\mathbb{Z}/p^n(r) \xrightarrow{\sim} \mathscr{S}_n(r)$ *of complexes of sheaves on* $\mathscr{V}ar_{\overline{K},h}$ *that is compatible with the crystalline Poincaré lemma.*

(2) *There is a unique quasi-isomorphism* $S_n(r) \xrightarrow{\sim} \mathscr{S}_n(r)$ *of complexes of sheaves on* $\mathscr{V}ar_{\overline{K},h}$ *that is compatible with the crystalline Poincaré lemma.*

Proof. We will prove the second claim — the first one is proved in an analogous way. Consider the map of distinguished triangles

$$\begin{tikzcd}
\mathscr{S}_n(r) \arrow[r] & \mathscr{J}^{[r]}_{\mathrm{cr},n} \arrow[r, "p^r-\varphi"] & \mathscr{A}_{\mathrm{cr},n} \\
S_n(r) \arrow[r] \arrow[u, dashed] & J^{[r]}_{\mathrm{cr},n} \arrow[r, "p^r-\varphi"] \arrow[u, "\wr"] & A_{\mathrm{cr},n} \arrow[u, "\wr"]
\end{tikzcd}$$

The triangles are distinguished by definition. The vertical continuous arrows are quasi-isomorphisms by the crystalline Poincaré lemma. They induce the dashed arrow that is clearly a quasi-isomorphism. □

Consider the natural map $\alpha_n : \mathrm{R}\Gamma(X_h, \mathscr{S}(r)) \to \mathrm{R}\Gamma(X_h, \mathscr{S}_n(r))$ and the zig-zag

$$\beta_n : \mathrm{R}\Gamma(X_h, \mathscr{S}_n(r)) \leftarrow \mathrm{R}\Gamma(X_h, S_n(r)) \to \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n(r)') \xleftarrow{\sim} \mathrm{R}\Gamma(X_h, \mathbb{Z}/p^n(r)').$$

Set $\beta := (\mathrm{holim}_n\, \beta_n) \otimes \mathbb{Q}$; note that this is a quasi-isomorphism. Set

$$\rho_{\mathrm{syn}} := p^{-r}\beta\alpha : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \to \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r)),$$

where $\alpha := (\mathrm{holim}_n\, \alpha_n) \otimes \mathbb{Q}$. The period map ρ_{syn} induces a map of graded E_∞ algebras over $\mathbb{Q}_p$ compatible with the action of the Galois group G_K.

The syntomic period map has a different, more global definition that we find very useful. Define the map ρ'_{syn} by the diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r) \arrow[r, "\sim"] \arrow[d, "\rho'_{\mathrm{syn}}"] & \big[\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \arrow[r, "(1-\varphi_r,\gamma_r^{-1}\kappa_r^{-1})"] \arrow[d, "\rho_{\mathrm{cr}}"] & \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}\oplus \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)/F^r\big] \arrow[d, "\rho_{\mathrm{cr}}+\rho_{\mathrm{dR}}"] \\
\mathrm{R}\Gamma_{\text{ét}}(X,\mathbb{Q}_p(r)) \arrow[r, "\sim"] & \big[\mathrm{R}\Gamma_{\text{ét}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{cr}} \arrow[r, "(1-\varphi_r,\mathrm{can})"] & \mathrm{R}\Gamma_{\text{ét}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{cr}} \oplus \mathrm{R}\Gamma_{\text{ét}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{dR}}/F^r\big]
\end{tikzcd}$$

This definition makes sense since the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \arrow[r, "\gamma_r^{-1}\kappa_r^{-1}"] \arrow[d, "\rho_{\mathrm{cr}}"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)/F^r \arrow[d, "\rho_{\mathrm{dR}}"] \\
\mathrm{R}\Gamma_{\text{ét}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{cr}} \arrow[r, "\mathrm{can}"] & \mathrm{R}\Gamma_{\text{ét}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{dR}}/F^r
\end{tikzcd}$$

The syntomic period morphisms ρ_{syn} and ρ'_{syn} are homotopic by a homotopy compatible with the G_K-action (and, unless necessary, we will not distinguish them in what follows). These two facts follow easily from the definitions.

For $X\in\mathscr{V}ar_K$, we have a quasi-isomorphism

$$\alpha_{\text{ét}}:\mathrm{R}\Gamma(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r)) \xrightarrow{\sim} C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}) \tag{48}$$

that we define as the inverse of the following composition of quasi-isomorphisms (square brackets denote complex):

$$\begin{aligned}
&C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})\\
&\quad\xrightarrow[\sim]{\rho} \mathrm{R}\Gamma(X_{\bar{K},\text{ét}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}\big[B_{\mathrm{st}}\xrightarrow{(N,1-\varphi_r,\iota)} B_{\mathrm{st}}\oplus B_{\mathrm{st}}\oplus B_{\mathrm{dR}}/F^r \xrightarrow{(1-\varphi_{r-1})-N} B_{\mathrm{st}}\big]\\
&\quad\xleftarrow{\sim} \mathrm{R}\Gamma(X_{\bar{K},\text{ét}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} C(D_{\mathrm{st}}(\mathbb{Q}_p(r)))\xleftarrow{\sim}\mathrm{R}\Gamma(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r)).
\end{aligned}$$

The last quasi-isomorphism is by Remark 2.7. The map ρ is defined using the period morphisms ρ_{HK} and ρ_{dR} and their compatibility (Corollary 4.4). The map $\alpha_{\text{ét}}$ is compatible with the action of G_K.

Proposition 4.6. *For a variety $X\in\mathscr{V}ar_K$, we have a canonical, compatible with the action of G_K, quasi-isomorphism*

$$\rho_{\mathrm{syn}}:\tau_{\le r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h},r)\xrightarrow{\sim}\tau_{\le r}\mathrm{R}\Gamma(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r)).$$

Proof. The Bousfield–Kan spectral sequences associated to the homotopy limits defining the complexes $C^+(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$ and $C(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$ form the commutative diagram

$$\begin{tikzcd}
{}^{+}E_2^{i,j} = H^i(C^+(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \arrow[r, Rightarrow] \arrow[d, "\mathrm{can}"] & H^{i+j}(C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \arrow[d, "\mathrm{can}"] \\
E_2^{i,j} = H^i(C(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \arrow[r, Rightarrow] & H^{i+j}(C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}))
\end{tikzcd}$$

We have $D_j = H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\} \in MF^{\mathrm{ad}}_K(\varphi, N, G_K)$. For $j \leq r$,

$$F^1 D_{j,K} = F^{1-(r-j)} H^j_{\mathrm{dR}}(X_h)\{r\} = 0.$$

Hence, by Corollary 2.15, we have ${}^{+}E_2^{i,j} \xrightarrow{\sim} E_2^{i,j}$. This implies

$$\tau_{\leq r} C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}) \xrightarrow{\sim} \tau_{\leq r} C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}).$$

Since $\rho_{\mathrm{HK}} = \rho_{\mathrm{cr}}\iota^B_{\mathrm{cr}}$, we check easily that we have the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h}, r) \arrow[r, "\sim", "\alpha_{\mathrm{syn}}"'] \arrow[d, "\rho_{\mathrm{syn}}"'] & C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}) \arrow[d, "\mathrm{can}"'] \\
\mathrm{R}\Gamma(X_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)) \arrow[r, "\sim", "\alpha_{\mathrm{\acute{e}t}}"'] & C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})
\end{tikzcd} \tag{49}$$

It follows that

$$\rho_{\mathrm{syn}} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h}, r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)),$$

as wanted. □

Let $X \in \mathscr{V}ar_K$. The natural projection $\varepsilon : X_{\bar{K},h} \to X_h$ defines pullback maps

$$\varepsilon^* : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \to \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h}), \quad \varepsilon^* : \mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h}).$$

By construction they are compatible with the monodromy operator, Frobenius, the action of the Galois group G_K, and filtration. It is also clear that they are compatible with the Beilinson–Hyodo–Kato morphisms, i.e., that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \arrow[r, "\iota^B_{\mathrm{dR}}"] \arrow[d, "\varepsilon^*"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \arrow[d, "\varepsilon^*"] \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h}) \arrow[r, "\iota^B_{\mathrm{dR}}"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})
\end{tikzcd}$$

It follows that we can define a canonical pullback map

$$\varepsilon^* : C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}) \to C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}).$$

Lemma 4.7. *Let $r \geq 0$. The following diagram commutes in the derived category*:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \arrow[r, "\alpha_{\mathrm{syn}}"] \arrow[d, "\varepsilon^*"] & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}) \arrow[d, "\varepsilon^*"] \\
\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h}, r) \arrow[r, "\alpha_{\mathrm{syn}}"] & C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})
\end{tikzcd}$$

Proof. Take a number $t \geq 2\dim X + 2$ and choose a finite Galois extension $(V', K')/(V, K)$ (see the proof of Proposition 3.18) such that we have an h-hypercovering $Z_\bullet \to X_{K'}$ with $(Z_\bullet)_{\leq t+1}$ built from log-schemes log-smooth over $V'^{,\times}$ and of Cartier type. Since the top map α_{syn} is compatible with base change (see Proposition 3.20) it suffices to show that the diagram in the lemma commutes with X replaced by $(Z_\bullet)_{\leq t+1}$. By Propositions 3.21, 3.18, and 3.14, this reduces to showing that, for an ss-pair $(U, \overline{U})$ split over V, the following diagram commutes canonically in the ∞-derived category (we set $Y := (U, \overline{U})$, $\overline{Y} := Y_{\overline{V}}$, where π is a fixed uniformizer of V):

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}(Y, r)_{\mathbb{Q}} \arrow[r, "\alpha^B_{\mathrm{syn},\pi}"] \arrow[d, "\varepsilon^*"] & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(Y)\{r\}) \arrow[d, "\varepsilon^*"] \\
\mathrm{R}\Gamma_{\mathrm{syn}}(Y_{\bar{K}}, r)_{\mathbb{Q}} \arrow[r, "\alpha_{\mathrm{syn}}"] & C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(Y_{\bar{K}})\{r\})
\end{tikzcd}$$

From the uniqueness property of the homotopy fiber functor, it suffices to show that the following diagram commutes canonically in the ∞-derived category:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(Y)_{\mathbb{Q}} \arrow[r] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{cr}}(Y/R)^{N=0}_{\mathbb{Q}} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(Y_1)^{\tau,N=0}_{R_{\mathbb{Q}}} \arrow[l, "\iota_\pi"', "\sim"] & \mathrm{R}\Gamma^B_{\mathrm{HK}}(Y_1)^{N=0} \arrow[l, "\beta"', "\sim"] \arrow[dl] \\
\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^{\tau,N=0}_{B^+_{\mathrm{cr}}} \arrow[l, "\iota^B_{\mathrm{cr}}"', "\sim"] \arrow[r, equal] & (\mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}})^{N=0} &
\end{tikzcd}$$

To do that we will need the ring of periods $\widehat{A}_{\mathrm{st}}$ [Tsuji 1999, p. 253]. Set

$$\widehat{A}_{\mathrm{st},n} = H^0_{\mathrm{cr}}(\overline{V}^\times_n / R_n), \quad \widehat{A}_{\mathrm{st}} = \varprojlim_n H^0_{\mathrm{cr}}(\overline{V}^\times_n / R_n).$$

The ring $\widehat{A}_{\mathrm{st},n}$ has a natural action of G_K, Frobenius φ, and a monodromy operator N. It is also equipped with a PD-filtration $F^i \widehat{A}_{\mathrm{st},n} = H^0_{\mathrm{cr}}(\overline{V}^\times_n / R_n, \mathscr{J}^{[i]}_{\mathrm{cr},n})$. We have a morphism $A_{\mathrm{cr},n} \to \widehat{A}_{\mathrm{st},n}$ induced by the map $H^0_{\mathrm{cr}}(\overline{V}_n / W_n(k)) \to H^0_{\mathrm{cr}}(\overline{V}^\times_n / R_n)$. It is compatible with the Galois action, the Frobenius, and the filtration. The natural map $R_n \to \widehat{A}_{\mathrm{st},n}$ is compatible with all the structures. We can view $\widehat{A}_{\mathrm{st},n}$ as the PD-envelope of the closed immersion

$$\overline{V}^\times_n \hookrightarrow A_{\mathrm{cr},n} \times_{W_n(k)} W_n(k)[X]^\times$$

defined by the map $\theta : A_{\mathrm{cr},n} \to \overline{V}_n$ and the projection $W_n(k)[X] \to \overline{V}_n$, $X \mapsto \pi$. This makes $\overline{V}_1^\times \hookrightarrow \widehat{A}_{\mathrm{st},n}$ into a PD-thickening in the crystalline site of $\overline{V}_1$. Set $\widehat{B}_{\mathrm{st}}^+ := \widehat{A}_{\mathrm{st}}[1/p]$.

Commutativity of the last diagram will follow from the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(Y)_{\mathbb{Q}} \arrow[rr] \arrow[d] & & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} \arrow[dl, "\sim"] \\
\mathrm{R}\Gamma_{\mathrm{cr}}(Y/R)_{\mathbb{Q}}^{N=0} \arrow[r] & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\widehat{A}_{\mathrm{st}})_{\mathbb{Q}}^{N=0} & \\
\mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{R_{\mathbb{Q}}}^{\tau,N=0} \arrow[u, "\iota_\pi", "\wr"'] \arrow[r] & \mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{\widehat{B}_{\mathrm{st}}^+}^{\tau,N=0} \arrow[u, "\iota_{\widehat{B}_{\mathrm{st}}^+}^B", "\wr"'] & \mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{B_{\mathrm{cr}}^+}^{\tau,N=0} \arrow[l] \arrow[uu, "\iota_{\mathrm{cr}}^B", "\wr"'] \\
& \mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)^{N=0} \arrow[ul, "\beta", "\sim"'] \arrow[ur] &
\end{tikzcd}$$

as soon as we show that $\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\widehat{A}_{\mathrm{st}})_{\mathbb{Q}}^{N=0}$ is a quasi-isomorphism. Notice that the map $\iota_{\widehat{B}_{\mathrm{st}}^+}^B$ is a quasi-isomorphism by Theorem 3.6. Hence using the Beilinson–Hyodo–Kato maps $\iota_{\widehat{B}_{\mathrm{st}}^+}^B$ and ι_{cr}^B this reduces to proving that the canonical map

$$\mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{B_{\mathrm{cr}}^+}^{\tau,N=0} \to \mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{\widehat{B}_{\mathrm{st}}^+}^{\tau,N=0}$$

is a quasi-isomorphism. In fact, we claim that for any (φ, N)-module M we have an isomorphism $M_{B_{\mathrm{cr}}^+}^{\tau,N=0} \xrightarrow{\sim} M_{\widehat{B}_{\mathrm{st}}^+}^{\tau,N=0}$. Indeed, assume first that the monodromy N_M is trivial. We calculate

$$\begin{aligned}
M_{B_{\mathrm{cr}}^+}^{\tau} &= (M \otimes_{K_0} B_{\mathrm{cr}}^{+,\tau})^{N'=0} = M \otimes_{K_0} (B_{\mathrm{cr}}^{+,\tau})^{N_\tau=0} = M \otimes_{K_0} B_{\mathrm{cr}}^+,\\
M_{\widehat{B}_{\mathrm{st}}^+}^{\tau} &= (M \otimes_{K_0} \widehat{B}_{\mathrm{st}}^{+,\tau})^{N'=0} = M \otimes_{K_0} (\widehat{B}_{\mathrm{cr}}^{+,\tau})^{N_\tau=0} = M \otimes_{K_0} \widehat{B}_{\mathrm{st}}^+,\\
N' &= N_M \otimes 1 + 1 \otimes N_\tau = 1 \otimes N_\tau.
\end{aligned}$$

Hence

$$M_{B_{\mathrm{cr}}^+}^{\tau,N=0} = M \otimes_{K_0} B_{\mathrm{cr}}^+ \quad \text{and} \quad M_{\widehat{B}_{\mathrm{st}}^+}^{\tau,N=0} = M \otimes_{K_0} (\widehat{B}_{\mathrm{st}}^+)^{N=0} = M \otimes_{K_0} B_{\mathrm{cr}}^+,$$

where the last equality is proved in [Tsuji 1999, Lemma 1.6.5]. We are done in this case.

In general, we can write $M \otimes_{K_0} B_{\mathrm{st}}^+ \xleftarrow{\sim} M' \otimes_{K_0} B_{\mathrm{st}}^+$ for a (φ, N)-module M' such that $N_{M'} = 0$ (take for M' the image of the map $M \to M \otimes_{K_0} B_{\mathrm{st}}^+$, $m \mapsto \exp(N_M(m)u)$ for $u \in B_{\mathrm{st}}^+$ such that $B_{\mathrm{st}}^+ = B_{\mathrm{cr}}^+[u]$, $N_\tau(u) = -1$). Similarly, using the fact that the ring B_{st}^+ is canonically (and compatibly with all the structures) isomorphic to the elements of $\widehat{B}_{\mathrm{st}}^+$ annihilated by a power of the monodromy operator

[Kato 1994, 3.7], we can write in a compatible way $M\otimes_{K_0}B_{\mathrm{st}}^+ \xleftarrow{\sim} M'\otimes_{K_0}\widehat{B}_{\mathrm{st}}^+$ for the same module M'. We obtain a commutative diagram

$$\begin{tikzcd}
M_{B_{\mathrm{cr}}^+}^{\tau,N=0} \arrow[r] \arrow[d, "\wr"] & M_{\widehat{B}_{\mathrm{st}}^+}^{\tau,N=0} \arrow[d, "\wr"] \\
M'^{\tau,N=0}_{B_{\mathrm{cr}}^+} \arrow[r, "\sim"] & M'^{\tau,N=0}_{\widehat{B}_{\mathrm{st}}^+}
\end{tikzcd}$$

that reduces the general case to the case of trivial monodromy on M that we treated above. $\square$

Let $X\in \mathscr{V}ar_K$, $r\geq 0$. Set

$$C_{\mathrm{pst}}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})\{r\}) := \left[\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})^{G_K} \arrow[r, "(1-\varphi_r,\iota_{\mathrm{dR}}^B)"] \arrow[d, "N"] & \mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})^{G_K}\oplus(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})/F^r)^{G_K} \arrow[d, "(N,0)"] \\
\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})^{G_K} \arrow[r, "1-\varphi_{r-1}"] & \mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})^{G_K}
\end{tikzcd}\right].$$

The above makes sense since the action of G_K on $\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})\{r\}$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})$ is smooth. In particular, we have

$$H^j(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})\{r\}^{G_K})\simeq H^j(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})\{r\})^{G_K},$$
$$H^j(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})^{G_K})\simeq H^j(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h}))^{G_K}.$$

Consider the canonical pullback map

$$\varepsilon^*: C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_h)\{r\}) \xrightarrow{\sim} C_{\mathrm{pst}}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})\{r\}).$$

By Proposition 3.22, this is a quasi-isomorphism. This allows us to construct a canonical spectral sequence (the *syntomic descent spectral sequence*)

$$^{\mathrm{syn}}E_2^{i,j} = H^i_{\mathrm{st}}(G_K, H^j(X_{\bar{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))) \Longrightarrow H^{i+j}_{\mathrm{syn}}(X_h,r)\ . \tag{50}$$

Indeed, the Bousfield–Kan spectral sequences associated to the homotopy limits defining complexes $C_{\mathrm{pst}}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})\{r\})$ and $C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_h)\{r\})$ give us the commutative diagram

$$\begin{tikzcd}
{}^{\mathrm{pst}}E_2^{i,j} = H^i(C_{\mathrm{pst}}(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \arrow[r, Rightarrow] & H^{i+j}(C_{\mathrm{pst}}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\bar{K},h})\{r\})) \\
{}^{\mathrm{syn}}E_2^{i,j} = H^i(C_{\mathrm{st}}(H^j_{\mathrm{HK}}(X_h)\{r\})) \arrow[r, Rightarrow] \arrow[u, "\wr", "\varepsilon^*"'] & H^{i+j}(C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_h)\{r\})) \arrow[u, "\wr", "\varepsilon^*"']
\end{tikzcd}$$

Since, by Proposition 3.20, we have $\alpha_{\mathrm{syn}}: H^{i+j}_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} H^{i+j}(C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}))$, we have obtained a spectral sequence

$$E_2^{i,j} = H^i(C_{\mathrm{pst}}(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \Longrightarrow H^{i+j}_{\mathrm{syn}}(X_h, r).$$

It remains to show that there is a canonical isomorphism

$$H^i(C_{\mathrm{pst}}(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \simeq H^i_{\mathrm{st}}(G_K, H^j(X_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r))). \tag{51}$$

But, we have $D_j = H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\} \in MF^{\mathrm{ad}}_K(\varphi, N, G_K)$,

$$V_{\mathrm{pst}}(D_j) \simeq H^j(X_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Q}(r)) \quad \text{and} \quad D_{\mathrm{pst}}(H^j(X_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Q}(r))) \simeq D_j.$$

Hence isomorphism (51) follows from Remark 2.12 and we have obtained the spectral sequence (50).

4B. ***Arithmetic case.*** In this subsection, we define the arithmetic syntomic period map by Galois descent from the geometric case. Then we show that, via this period map, the syntomic descent spectral sequence and the étale Hochschild–Serre spectral sequence are compatible. Finally, we show that this implies that the arithmetic syntomic cohomology and étale cohomology are isomorphic in a stable range.

Let $X \in \mathscr{V}ar_K$. For $r \geq 0$, we define the canonical syntomic period map

$$\rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \to \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Q}_p(r))$$

as the composition

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) = \mathrm{R}\Gamma(X_h, \mathscr{S}(r))_{\mathbb{Q}} &\to \operatorname{holim}_n \mathrm{R}\Gamma(X_h, \mathscr{S}_n(r))_{\mathbb{Q}} \\ &\xrightarrow{\varepsilon^*} \operatorname{holim}_n \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma(X_{\bar{K},h}, \mathscr{S}_n(r)))_{\mathbb{Q}} \\ &\xrightarrow{p^{-r}\beta} \operatorname{holim}_n \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma(X_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Z}/p^n(r)'))_{\mathbb{Q}} \\ &\xleftarrow{\simeq} \operatorname{holim}_n \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Z}/p^n(r)')_{\mathbb{Q}} = \mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)). \end{aligned}$$

It induces a morphism of graded E_∞ algebras over $\mathbb{Q}_p$.

The syntomic period map ρ_{syn} is compatible with the syntomic descent and the Hochschild–Serre spectral sequences.

Theorem 4.8. *For $X \in \mathscr{V}ar_K$, $r \geq 0$, there is a canonical map of spectral sequences*

$$\begin{array}{ccc} {}^{\mathrm{syn}}E_2^{i,j} = H^i_{\mathrm{st}}(G_K, H^j(X_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r))) & \Longrightarrow & H^{i+j}_{\mathrm{syn}}(X_h, r) \\ \Big\downarrow{\scriptstyle \mathrm{can}} & & \Big\downarrow{\scriptstyle \rho_{\mathrm{syn}}} \\ {}^{\mathrm{\acute{e}t}}E_2^{i,j} = H^i(G_K, H^j(X_{\bar{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r))) & \Longrightarrow & H^{i+j}(X_{\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)) \end{array}$$

Proof. We work in the (classical) derived category. The Bousfield–Kan spectral sequences associated to the homotopy limits defining complexes $C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$

and $C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$, and Theorem 2.18 give us the commutative diagram of spectral sequences

$$\begin{array}{ccc} {}^{II}E_2^{i,j} = H^i(G_K, C(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) & \Longrightarrow & H^{i+j}(G_K, C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \\ \delta \uparrow & & \delta \uparrow \\ {}^{\mathrm{pst}}E_2^{i,j} = H^i(C_{\mathrm{pst}}(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) & \Longrightarrow & H^{i+j}(C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \end{array}$$

More specifically, in the language of Section 2E, set $X = C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$ (hopefully, the notation will not be too confusing). Filtering complex X in the direction of the homotopy limit, we obtain a Postnikov system (14) with $Y^i = 0$, $i \geq 3$, and

$$\begin{aligned} Y^0 &= \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\} \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}, \\ Y^1 &= \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r-1\} \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} \\ &\qquad \oplus \big(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\} \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}}) \otimes_{\bar{K}} B_{\mathrm{dR}})/F^r\big), \\ Y^2 &= \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r-1\} \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}. \end{aligned}$$

Still in the setting of Section 2E, take for A the abelian category of sheaves of abelian groups on the pro-étale site $\mathrm{Spec}(K)_{\mathrm{proét}}$ of Scholze [2013, Section 3].

Remark 4.9. We work with the pro-étale site to make sense of the continuous cohomology $\mathrm{R}\Gamma(G_K, \cdot)$. If the reader is willing to accept that this is possible then he can skip the tedious parts of the proof involving passage to the pro-étale site (and existence of continuous sections).

Recall that there is a projection map $\nu : \mathrm{Spec}(K)_{\mathrm{proét}} \to \mathrm{Spec}(K)_{\mathrm{ét}}$ such that, for an étale sheaf $\mathscr{F}$, we have the quasi-isomorphism $\nu^* : \mathscr{F} \xrightarrow{\sim} \mathrm{R}\nu_*\nu^*\mathscr{F}$ [Bhatt and Scholze 2015, Proposition 5.2.6]. More generally, for a topological G_K-module M, we get a sheaf νM on $\mathrm{Spec}(K)_{\mathrm{proét}}$ by setting $\nu M(S) = \mathrm{Hom}_{\mathrm{cont},G_K}(S, M)$ for a profinite G_K-set S, and Scholze [2013, Proposition 3.7(iii); 2016] showed that there is a canonical quasi-isomorphism

$$H^*(\mathrm{Spec}(K)_{\mathrm{proét}}, \nu M) \simeq H^*_{\mathrm{cont}}(G_K, M).$$

In this proof we will need this kind of quasi-isomorphism for complexes M as well and this will require extra arguments. For that, observe that the functor ν is left exact. To study right exactness, it suffices to look at the global sections on profinite sets S with a free G_K-action of the form $S = S' \times G_K$ for a profinite set S' with trivial G_K-action.[8] Then, for any G_K-module T, we have $\Gamma(S, \nu T) = \mathrm{Hom}_{\mathrm{cont}}(S', T)$. It follows that, for a surjective map $T_1 \to T_2$ of G_K-modules, the pullback map

[8] To see this, for a profinite G_K-set S', use the covering $S' \times G_K \to S'$, where the first S' has trivial G_K-action, induced from the G_K-action on S'.

$\nu T_1 \to \nu T_2$ is also surjective if the original map had a continuous set-theoretical section. This is a criterion familiar from continuous cohomology and we will use it often.

We will see the complex X as a complex of sheaves on the site $\mathrm{Spec}(K)_{\text{proét}}$ in the following way: represent $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}})$ by (filtered) perfect complexes of K_0^{nr}- and $\bar{K}$-modules respectively, think of X as νX, and work on the pro-étale site. This makes sense, i.e., functor ν transfers (filtered) quasi-isomorphisms of representatives of $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}})$ to quasi-isomorphisms of the corresponding sheaves νX. To see this, look at the Postnikov system of sheaves on $\mathrm{Spec}(K)_{\text{proét}}$ obtained by pulling back by ν the above Postnikov system. Now, look at the global sections on profinite sets $S = S' \times G_K$ as above and note that we have $\Gamma(S, \nu Y^0) = \mathrm{Hom}_{\mathrm{cont}}(S', Y^0)$. Conclude that, by perfectness of the Beilinson–Hyodo–Kato complexes, quasi-isomorphisms of representatives of $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})$ yield quasi-isomorphisms of the sheaves νY^0. By a similar argument, we get the analogous statement for Y^2. For Y^1, we just have to show that filtered quasi-isomorphisms of representatives of $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}})$ yield quasi-isomorphisms of the sheaves $\nu((\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}}) \otimes_{\bar{K}} B_{\mathrm{dR}})/F^r)$. Again, we look at the global section on $S = S' \times G_K$ as above. By compactness of S', we may replace $(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}}) \otimes_{\bar{K}} B_{\mathrm{dR}})/F^r$ by $(t^{-i}\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}}) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r$ for some $i \geq 0$, where, using devissage, we can again argue by (filtered) perfection of $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}})$. Observe that the same argument shows that $\mathscr{H}^j(\nu Y^i) \simeq \nu H^j(Y^i)$ for $i = 0, 1, 2$.

The above Postnikov system gives rise to an exact couple

$$D_1^{i,j} = \mathscr{H}^j(X^i), \quad E_1^{i,j} = \mathscr{H}^j(Y^i) \Rightarrow \mathscr{H}^{i+j}(X).$$

This is the Bousfield–Kan spectral sequence associated to X.

Consider now the complex $X_{\mathrm{pst}} := C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$. We claim that the canonical map

$$C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}) \xrightarrow{\sim} C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})^{G_K}$$

is a quasi-isomorphism (recall that taking G_K-fixed points corresponds to taking global sections on the pro-étale site), and, in particular, that the term on the right-hand side makes sense. To see this, it suffices to show that the canonical maps

$$(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})/F^r)^{G_K} \xrightarrow{\sim} ((\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h}) \otimes_{\bar{K}} B_{\mathrm{dR}})/F^r)^{G_K},$$
$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})^{G_K} \xrightarrow{\sim} (\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h}) \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}})^{G_K}$$

are quasi-isomorphisms and to use the fact that the action of G_K on $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})$ is smooth. The fact that the first map is a quasi-isomorphism follows from the filtered quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_K \bar{K} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})$ and the fact that $B_{\mathrm{dR}}^{G_K} = K$. Similarly, the second map is a quasi-isomorphism because, by [Fontaine 1994a, 4.2.4], $\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_{\bar{K},h})$ is the subcomplex of those elements of $\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_{\bar{K},h})\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}$ whose stabilizers in G_K are open.

Taking the G_K-fixed points of the above Postnikov system we get an exact couple

$$\begin{aligned}{}^{\mathrm{pst}}D_1^{i,j} &= H^j(X^i_{\mathrm{pst}}),\\ {}^{\mathrm{pst}}E_1^{i,j} &= H^j(Y^i_{\mathrm{pst}}) \Rightarrow H^{i+j}(X_{\mathrm{pst}})\end{aligned}$$

corresponding to the Bousfield–Kan filtration of the complex X_{pst}. On the other hand, applying $\mathrm{R}\Gamma(\mathrm{Spec}(K)_{\mathrm{proét}},\cdot)$ to the same Postnikov system, we obtain an exact couple

$$\begin{aligned}{}^{I}D_1^{i,j} &= H^j(\mathrm{Spec}(K)_{\mathrm{proét}}, X^i),\\ {}^{I}E_1^{i,j} &= H^j(\mathrm{Spec}(K)_{\mathrm{proét}}, Y^i) \Rightarrow H^{i+j}(\mathrm{Spec}(K)_{\mathrm{proét}}, X)\end{aligned}$$

together with a natural map of exact couples $({}^{\mathrm{pst}}D_1^{i,j}, {}^{\mathrm{pst}}E_1^{i,j}) \to ({}^{I}D_1^{i,j}, {}^{I}E_1^{i,j})$.

We also have the hypercohomology exact couple

$$\begin{aligned}{}^{II}D_2^{i,j} &= H^{i+j}(\mathrm{Spec}(K)_{\mathrm{proét}}, \tau_{\leq j-1}X),\\ {}^{II}E_2^{i,j} &= H^{i}(\mathrm{Spec}(K)_{\mathrm{proét}}, \mathscr{H}^j(X)) \Rightarrow H^{i+j}(\mathrm{Spec}(K)_{\mathrm{proét}}, X).\end{aligned}$$

Theorem 2.18 gives us a natural morphism of exact couples $({}^{I}D_2^{i,j}, {}^{I}E_2^{i,j}) \to ({}^{II}D_2^{i,j}, {}^{II}E_2^{i,j})$ — hence a natural morphism of spectral sequences ${}^{I}E_2^{i,j} \to {}^{II}E_2^{i,j}$ compatible with the identity map on the common abutment — if our original Postnikov system satisfies the equivalent conditions in Remark 2.17. We will check the condition (4), i.e., that the following long sequence is exact for all j:

$$0 \to \mathscr{H}^j(X) \to \mathscr{H}^j(Y^0) \to \mathscr{H}^j(Y^1) \to \mathscr{H}^j(Y^2) \to 0.$$

For that it is enough to show that

(1) $\mathscr{H}^j(\nu Y^i) \simeq \nu H^j(Y^i)$ for $i = 0, 1, 2$;

(2) $\mathscr{H}^j(\nu X) \simeq \nu H^j(X)$;

(3) the following long sequence of G_K-modules

$$0 \to H^j(X) \to H^j(Y^0) \to H^j(Y^1) \to H^j(Y^2) \to 0$$

is exact;

(4) the pullback ν preserves its exactness.

The assertion in (1) was shown above. The sequence in (3) is equal to the top sequence in the following commutative diagram (where we set $M = H^j_{\mathrm{HK}}(X_{\bar{K},h})$, $M_{\mathrm{dR}} = H^j_{\mathrm{dR}}(X_{\bar{K},h})$, and $E = H^j(X_{\bar{K},\mathrm{ét}}, \mathbb{Q}_p)$):

$$\begin{array}{ccccccc}
H^j(X) & \longrightarrow & M\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} & \xrightarrow{(N,1-\varphi_r,\iota)} & \begin{array}{c} M\otimes_{K_0^{\mathrm{nr}}}(B_{\mathrm{st}}\oplus B_{\mathrm{st}}) \\ \oplus(M_{\mathrm{dR}}\otimes_{\overline{K}} B_{\mathrm{dR}})/F^r \end{array} & \xrightarrow{(1-\varphi_{r-1})-N} & M\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} \\
\wr\downarrow\alpha_{\mathrm{\acute{e}t}}^{-1} & & \wr\downarrow\rho_{\mathrm{HK}} & & \wr\downarrow\rho_{\mathrm{HK}}+\rho_{\mathrm{HK}}+\rho_{\mathrm{dR}} & & \wr\downarrow\rho_{\mathrm{HK}} \\
E(r) & \hookrightarrow & E\otimes B_{\mathrm{st}} & \xrightarrow{(N,1-\varphi_r,\iota)} & \begin{array}{c} E\otimes(B_{\mathrm{st}}\oplus B_{\mathrm{st}}) \\ \oplus E\otimes B_{\mathrm{dR}}/F^r \end{array} & \xrightarrow{(1-\varphi_{r-1})-N} & E\otimes B_{\mathrm{st}}
\end{array}$$

Since the bottom sequence is just a fundamental exact sequence of p-adic Hodge theory, the top sequence is exact, as wanted.

To prove assertion (4), we pass to the bottom exact sequence above and apply ν to it. It is easy to see that it enough now to show that the following surjections have continuous $\mathbb{Q}_p$-linear sections:

$$B_{\mathrm{st}} \xrightarrow{N} B_{\mathrm{st}}, \qquad B_{\mathrm{cr}} \xrightarrow{(1-\varphi_r,\mathrm{can})} B_{\mathrm{cr}}\oplus B_{\mathrm{dR}}/F^r.$$

For the monodromy, write $B_{\mathrm{st}} = B_{\mathrm{cr}}[u_s]$ and take for a continuous section the map induced by $bu_s^i \mapsto -(b/(i+1))u_s^{i+1}$, $b\in B_{\mathrm{cr}}$. For the second map, the existence of continuous section was proved in [Bloch and Kato 1990, 1.18]. For a different argument: observe that an analogous statement was proved in [Colmez 1998, Proposition II.3.1] with $B_{\max}$ in place of B_{cr} as a consequence of the general theory of p-adic Banach spaces. We will just modify it here. Write $A_i = t^{-i}B_{\mathrm{cr}}^+$ and $B_i = t^{-i}B_{\mathrm{cr}}^+\oplus t^{-i}B_{\mathrm{dR}}^+/t^r$ for $i\ge 1$. These are p-adic Banach spaces. Observe that $B_i\subset B_{i+1}$ is closed. Indeed, it is enough to show that $tB_{\mathrm{cr}}^+\subset B_{\mathrm{cr}}^+$ is closed. But we have $tB_{\mathrm{cr}}^+ = \bigcap_{n\ge 0}\ker(\theta\circ\varphi^n)$.

It follows [Colmez 1998, Proposition I.1.5] that we can find a closed complement C_{i+1} of B_i in B_{i+1}. Set $f=(1-\varphi_r,\mathrm{can}) : B_{\mathrm{cr}}\to B_{\mathrm{cr}}\oplus B_{\mathrm{dR}}/F^r$. We know that f maps A_i onto B_i. Write $t^{-i}B_{\mathrm{cr}}^+\oplus t^{-i}B_{\mathrm{dR}}^+/t^r = B_1\oplus\bigl(\bigoplus_{j=2}^{i-1}C_j\bigr)$. By [Colmez 1998, Proposition I.1.5], we can find a continuous section $s_1 : B_1\to A_1$ of f and, if $i\ge 2$, a continuous section $s_i : C_i\to A_i$ of f. Define the map $s : t^{-i}B_{\mathrm{cr}}^+\oplus t^{-i}B_{\mathrm{dR}}^+/t^r\to B_{\mathrm{cr}}$ by s_1 on B_1 and by s_i on C_i for $i\ge 2$. Taking the inductive limit over i, we get our section of f.

Finally, to prove assertion (2), take a perfect representative of the complex $\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Z}_p(r))$. Consider the complex $Z=\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))$ as a complex of sheaves on $\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}}$. As before, we see that this makes sense and we easily find that (canonically) $\mathscr{H}^j(Z)\simeq \nu H^j(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))$. To prove (2), it is enough to show that we can also pass with the map $\alpha_{\mathrm{\acute{e}t}} : \mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))\xrightarrow{\sim} C(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\overline{K},h})\{r\})$ to the site $\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}}$. Looking at its definition (see (48)), we see that we need to show that the period quasi-isomorphisms $\rho_{\mathrm{cr}}, \rho_{\mathrm{HK}}, \rho_{\mathrm{dR}}$ as well as the quasi-isomorphism

$$\mathbb{Q}_p(r)\xrightarrow{\sim}\bigl[B_{\mathrm{st}}\xrightarrow{(N,1-\varphi_r,\iota)} B_{\mathrm{st}}\oplus B_{\mathrm{st}}\oplus B_{\mathrm{dR}}/F^r\xrightarrow{(1-\varphi_{r-1})-N} B_{\mathrm{st}}\bigr]$$

can be lifted to the pro-étale site. The last fact we have just shown. For the crystalline period map ρ_{cr}, this follows from the fact that it is defined integrally and all the relevant complexes are perfect. For the Hyodo–Kato period map ρ_{HK}, it follows from the case of ρ_{cr} and from perfection of complexes involved in the definition of the Beilinson–Hyodo–Kato map. For the de Rham period map ρ_{dR}, this follows from perfection of the involved complexes as well as from the exactness of holim_n (in the definition of ρ_{dR}) on the pro-étale site of K (see [Scholze 2013, Lemma 3.18]).

We define the map of spectral sequences $\delta := (\delta_D, \delta) := ({}^{\mathrm{pst}}D_2^{i,j}, {}^{\mathrm{pst}}E_2^{i,j}) \to ({}^{II}D_2^{i,j}, {}^{II}E_2^{i,j})$ — which we stated at the beginning of the proof — as the composition of the two maps constructed above:

$$\delta : ({}^{\mathrm{pst}}D_2^{i,j}, {}^{\mathrm{pst}}E_2^{i,j}) \to ({}^{I}D_2^{i,j}, {}^{I}E_2^{i,j}) \to ({}^{II}D_2^{i,j}, {}^{II}E_2^{i,j}).$$

To get the spectral sequence from the theorem, we need to pass from ${}^{II}E_2$ to the Hochschild–Serre spectral sequence. To do that, consider the hypercohomology exact couple

$$\begin{aligned} {}^{\text{ét}}D_2^{i,j} &= H^{i+j}(\operatorname{Spec}(K)_{\text{proét}}, \tau_{\leq j-1}Z), \\ {}^{\text{ét}}E_2^{i,j} &= H^{i}(\operatorname{Spec}(K)_{\text{proét}}, \mathscr{H}^j(Z)) \Rightarrow H^{i+j}(\operatorname{Spec}(K)_{\text{proét}}, Z) \end{aligned}$$

and, via $\alpha_{\text{ét}}^{-1}$, a natural morphism of exact couples $({}^{II}D_2^{i,j}, {}^{II}E_2^{i,j}) \to ({}^{\text{ét}}D_2^{i,j}, {}^{\text{ét}}E_2^{i,j})$, and hence a natural morphism of spectral sequences ${}^{II}E_2^{i,j} \to {}^{\text{ét}}E_2^{i,j}$ compatible with the map $\alpha_{\text{ét}}^{-1}$ on the abutment. We have a quasi-isomorphism

$$\psi : \mathrm{R}\Gamma(\operatorname{Spec}(K)_{\text{proét}}, Z) \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r))$$

defined as the composition

$$\begin{aligned} \psi : \mathrm{R}\Gamma(\operatorname{Spec}(K)_{\text{proét}}, \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r))) \\ \xrightarrow{\sim} \mathbb{Q} \otimes \operatorname{holim}_n \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Z}/p^n(r))) \\ = \mathbb{Q} \otimes \operatorname{holim}_n \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n(r)) = \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}(r)). \end{aligned}$$

We have obtained the natural maps of spectral sequences

$$\begin{tikzcd} {}^{\mathrm{syn}}E_2^{i,j} = H^i_{\mathrm{st}}(G_K, H^j(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r))) \arrow[r, Rightarrow] \arrow[d, "\wr"'] & H^{i+j}_{\mathrm{syn}}(X_h, r) \arrow[d, "\wr"', "\alpha_{\mathrm{syn}}"] \\ E_2^{i,j} = H^i(C_{\mathrm{st}}(H^j_{\mathrm{HK}}(X_h)\{r\})) \arrow[r, Rightarrow] \arrow[d, "\alpha_{\text{ét}}^{-1}\delta\varepsilon^*"] & H^{i+j}(C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})) \arrow[d, "\psi\alpha_{\text{ét}}^{-1}\delta\varepsilon^*"] \\ {}^{\text{ét}}E_2^{i,j} = H^i(G_K, H^j(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r))) \arrow[r, Rightarrow] & H^{i+j}(X_{\text{ét}}, \mathbb{Q}_p(r)) \end{tikzcd}$$

It remains to show that the right vertical composition

$$\gamma : H^{i+j}_{\mathrm{syn}}(X_h, r) \to H^{i+j}(X_{\text{ét}}, \mathbb{Q}_p(r))$$

is equal to the map ρ_{syn}. Since we have the equality $\alpha_{\mathrm{syn}} = \rho_{\mathrm{syn}}\alpha_{\mathrm{\acute{e}t}}$ (in the derived category) from (49) and, by Lemma 4.7, $\varepsilon^*\alpha_{\mathrm{syn}} = \alpha_{\mathrm{syn}}\varepsilon^*$, the map γ can be written as the composition

$$\begin{aligned}\tilde{\rho}_{\mathrm{syn}} : H^{i+j}_{\mathrm{syn}}(X_h, r) &\xrightarrow{\varepsilon^*} H^{i+j}(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}}, \nu\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, r))\\ &\xrightarrow{\rho_{\mathrm{syn}}} H^{i+j}(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}}, \nu\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)))\\ &\xrightarrow{\psi} H^{i+j}(X_{\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)),\end{aligned}$$

where the period map ρ_{syn} is understood to be on sheaves on $\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}}$. There is no problem with that since we care only about the induced map on cohomology groups. It is easy now to see that $\tilde{\rho}_{\mathrm{syn}} = \rho_{\mathrm{syn}}$, as wanted. □

Remark 4.10. If X is proper and smooth, it is known that the étale Hochschild–Serre spectral sequence degenerates, i.e., ${}^{\mathrm{\acute{e}t}}E_2 = {}^{\mathrm{\acute{e}t}}E_\infty$. It is very likely that so does the syntomic descent spectral sequence in this case, i.e., ${}^{\mathrm{syn}}E_2 = {}^{\mathrm{syn}}E_\infty$.[9]

Corollary 4.11. *For $X \in \mathscr{V}ar_K$, we have a canonical quasi-isomorphism*

$$\rho_{\mathrm{syn}} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_{\mathbb{Q}} \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)).$$

Proof. By Theorem 4.8, the syntomic descent and the Hochschild–Serre spectral sequence are compatible. We have $D_j = H^j_{\mathrm{HK}}(X_{\overline{K},h})\{r\} \in MF^{\mathrm{ad}}_K(\varphi, N, G_K)$. For $j \leq r$, we know $F^1 D_{j,K} = F^{1-(r-j)} H^j_{\mathrm{dR}}(X_h) = 0$. Hence, by Proposition 2.16, we have ${}^{\mathrm{syn}}E_2^{i,j} \xrightarrow{\sim} {}^{\mathrm{\acute{e}t}}E_2^{i,j}$. This implies $\rho_{\mathrm{syn}} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}}, \mathbb{Q}_p(r))$, as wanted. □

Remark 4.12. All of the above automatically extends to finite diagrams of K-varieties, and hence to essentially finite diagrams of K-varieties (i.e., the diagrams for which every truncation of their cohomology $\tau_{\leq n}$ is computed by truncating the cohomology of some finite diagram). This includes, in particular, simplicial and cubical varieties.

Proposition 4.13. *Let $X \in \mathscr{V}ar_K$ and $i \geq 0$. The composition*

$$H^q_{\mathrm{dR}}(X)/F^r \xrightarrow{\partial} H^{q+1}_{\mathrm{syn}}(X_h, r) \xrightarrow{\rho_{\mathrm{syn}}} H^{q+1}_{\mathrm{\acute{e}t}}(X, \mathbb{Q}_p(r)) \to H^{q+1}_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p(r))$$

is the zero map. The map induced by the syntomic descent spectral sequence

$$H^q_{\mathrm{dR}}(X)/F^r \to H^1(G_K, H^q_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p(r)))$$

is equal to the Bloch–Kato exponential associated with the Galois representation $V^q(r) = H^q_{\mathrm{\acute{e}t}}(X_{\overline{K}}, \mathbb{Q}_p(r))$.

[9] This was, in fact, shown in [Déglise and Nizioł 2015].

Proof. In what follows, we will omit the passage to the pro-étale site. Consider the Postnikov system from the proof of Theorem 4.8, which arises from the complex $X = C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$; then $Y^p = C^p(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$. The discussion from Example 2.19 then applies to the functor $f(-) = (-)^{G_K}$ and yields the following four exact couples.

(1) $D_1^{p,q} = H^q(X^p)$ and $E_1^{p,q} = H^q(Y^p) = C^p(H^q_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) = C^p(H^q_{\mathrm{HK}}\{r\})$. The corresponding quasi-isomorphism $H^q(X) \xrightarrow{\sim} E_1^{\bullet,q}$ is then identified, via the various period maps, with
$$V^q(r) \xrightarrow{\sim} C(H^q_{\mathrm{HK}}\{r\}) = C(D_{\mathrm{pst}}(V^q(r))).$$

(2) ${}^fD_1^{p,q} = H^q(f(X^p))$ and ${}^fE_1^{p,q} = H^q(f(Y^p)) = f(H^q(Y^p)) = C^p_{\mathrm{st}}(H^q_{\mathrm{HK}}\{r\}) = f(E_1^{p,q})$.

(3) ${}^ID_1^{p,q} = (\mathrm{R}^qf)(X^p)$ and ${}^IE_1^{p,q} = (\mathrm{R}^qf)(Y^p)$.

(4) ${}^{II}D_2^{p,q} = (\mathrm{R}^{p+q}f)(\tau_{\leq q-1}X)$ and ${}^{II}E_2^{p,q} = (\mathrm{R}^pf)(H^q(X)) = H^p(G_K, V^q(r))$.

There is a canonical morphism of exact couples (2) → (3) and a morphism (3) → (4) given by the maps (u, v) from the proof of Theorem 2.18. As observed in Remark 2.14, the Bloch–Kato exponential for $V = V^q(r)$ is obtained by applying R^0f to
$$\begin{aligned} Z^1C(H^q_{\mathrm{HK}}\{r\}) = Z^1(E_1^{\bullet,q}) &\xrightarrow{\mathrm{can}} (\sigma_{\geq 1}C(H^q_{\mathrm{HK}}\{r\}))[1] = (\sigma_{\geq 1}C(E_1^{\bullet,q}))[1] \\ &\xrightarrow{-\mathrm{can}} C(H^q_{\mathrm{HK}}\{r\})[1] = E_1^{\bullet,q}[1] \\ &\xleftarrow{\sim} V^q(r)[1] = H^q(X)[1], \end{aligned}$$
and hence is equal to the composite map
$$f(Z^1(E_1^{\bullet,q})) = Z^1({}^fE_1^{\bullet,q}) \to {}^fE_2^{1,q} \xrightarrow{\mathrm{can}} {}^IE_2^{1,q} \xrightarrow{-v'=v} (R^1f)(E_1^{\bullet,q}) = {}^{II}E_2^{p,q},$$
which coincides, in turn, with
$$Z^1C_{\mathrm{st}}(H^q_{\mathrm{HK}}\{r\}) \xrightarrow{\mathrm{can}} H^1_{\mathrm{st}}(G_K, V^q(r)) \to H^1(G_K, V^q(r)).$$
After restricting to the de Rham part of $Z^1C(H^q_{\mathrm{HK}}\{r\})$, we obtain the desired statement about $H^q_{\mathrm{dR}}(X)/F^r$. $\square$

In more concrete terms, the above proposition says that the following diagram commutes:
$$\begin{tikzcd} H^{q+1}_{\mathrm{syn}}(X_h, r)_0 \arrow[r, "\rho_{\mathrm{syn}}"] & H^{q+1}_{\mathrm{\acute{e}t}}(X, \mathbb{Q}_p(r))_0 \arrow[d] \\ H^q_{\mathrm{dR}}(X)/F^r \arrow[u, "\partial"] \arrow[r, "\exp_{\mathrm{BK}}"] & H^1(G_K, H^q_{\mathrm{\acute{e}t}}(X_{\bar{K}}, \mathbb{Q}_p(r))) \end{tikzcd}$$
where the subscript 0 refers to the classes that vanish in $H^{q+1}_{\mathrm{syn}}(X_{\bar{K},h}, r)$ and $H^{q+1}_{\mathrm{\acute{e}t}}(X_{\bar{K}}, \mathbb{Q}_p(r))$, respectively.

Remark 4.14. Assume that $r > q$. Then in the above diagram all the maps are isomorphisms. Indeed, we have $F^r H^q_{\mathrm{dR}}(X) = 0$. By [Berger 2002, Theorem 6.8], the map $\exp_{\mathrm{BK}}$ is an isomorphism. By Proposition 4.6 and Corollary 4.11, so is the period map ρ_{syn}. Since, by Theorem A.1,

$$H^2(G_K, H^q_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))) = H^2(G_K, H^{q-1}_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))) = 0,$$

the vertical map is an isomorphism as well. Hence so is the map ∂.

5. Syntomic regulators

In this section, we prove that Soulé's étale regulators land in the semistable Selmer groups. This will be done by constructing syntomic regulators that are compatible with the étale ones via the period map and by exploiting the syntomic descent spectral sequence.

5A. ***Construction of syntomic Chern classes.*** We start with the construction of syntomic Chern classes. This will be standard once we prove that syntomic cohomology satisfies the projective space theorem and homotopy property.

In this subsection we will work in the (classical) derived category. For a fine log-scheme (X, M), log-smooth over $V^\times$, we have the log-crystalline and log-syntomic first Chern class maps of complexes of sheaves on $X_{\text{ét}}$ [Tsuji 1999, (2.2.3)]

$$\begin{aligned}
c_1^{\mathrm{cr}} &: j_*\mathscr{O}^*_{X_{\mathrm{tr}}} \xrightarrow{\sim} M^{\mathrm{gp}} \to M_n^{\mathrm{gp}} \to R\varepsilon_*\mathscr{J}^{[1]}_{X_n/W_n(k)}[1],\\
c_1^{\mathrm{st}} &: j_*\mathscr{O}^*_{X_{\mathrm{tr}}} \xrightarrow{\sim} M^{\mathrm{gp}} \to M_n^{\mathrm{gp}} \to R\varepsilon_*\mathscr{J}^{[1]}_{X_n/R_n}[1],\\
c_1^{\mathrm{HK}} &: j_*\mathscr{O}^*_{X_{\mathrm{tr}}} \xrightarrow{\sim} M^{\mathrm{gp}} \to M_0^{\mathrm{gp}} \to R\varepsilon_*\mathscr{J}^{[1]}_{X_0/W_n(k)^0}[1],\\
c_1^{\mathrm{syn}} &: j_*\mathscr{O}^*_{X_{\mathrm{tr}}} \xrightarrow{\sim} M^{\mathrm{gp}} \to \mathscr{S}(1)_{X,\mathbb{Q}}[1].
\end{aligned}$$

Here ε is the projection from the corresponding crystalline site to the étale site. The maps c_1^{cr}, c_1^{st}, and c_1^{syn} are clearly compatible. So are the maps c_1^{st} and c_1^{HK}. For ss-pairs $(U, \bar{U})$ over K, we get the induced functorial maps

$$\begin{aligned}
c_1^{\mathrm{cr}} &: \Gamma(U, \mathscr{O}_U^*) \xleftarrow{\sim} \Gamma(\bar{U}, j_*\mathscr{O}_U^*) \to \mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U}, \mathscr{J}^{[1]})[1],\\
c_1^{\mathrm{st}} &: \Gamma(U, \mathscr{O}_U^*) \to \mathrm{R}\Gamma_{\mathrm{cr}}((U, \bar{U})/R, \mathscr{J}^{[1]})[1],\\
c_1^{\mathrm{HK}} &: \Gamma(U, \mathscr{O}_U^*) \to \mathrm{R}\Gamma_{\mathrm{cr}}((U, \bar{U})_0/W_n(k)^0, \mathscr{J}^{[1]})[1],\\
c_1^{\mathrm{syn}} &: \Gamma(U, \mathscr{O}_U^*) \to \mathrm{R}\Gamma_{\mathrm{syn}}(U, \bar{U}, 1)_{\mathbb{Q}}[1].
\end{aligned}$$

For $X \in \mathscr{V}ar_K$, we can glue the absolute log-crystalline and log-syntomic classes to obtain the absolute crystalline and syntomic first Chern class maps

$$c_1^{\mathrm{cr}} : \mathscr{O}^*_{X_h} \to \mathscr{J}_{\mathrm{cr},X}[1], \quad c_1^{\mathrm{syn}} : \mathscr{O}^*_{X_h} \to \mathscr{S}(1)_{X,\mathbb{Q}}[1].$$

They induce (compatible) maps

$$c_1^{\mathrm{cr}} : \mathrm{Pic}(X) = H^1(X_{\text{ét}}, \mathscr{O}_X^*) \to H^1(X_h, \mathscr{O}_X^*) \xrightarrow{c_1^{\mathrm{cr}}} H^2(X_h, \mathscr{J}_{\mathrm{cr}}),$$
$$c_1^{\mathrm{syn}} : \mathrm{Pic}(X) = H^1(X_{\text{ét}}, \mathscr{O}_X^*) \to H^1(X_h, \mathscr{O}_X^*) \xrightarrow{c_1^{\mathrm{syn}}} H^2_{\mathrm{syn}}(X_h, 1).$$

Recall that, for a log-scheme (X, M) as above, we also have the log de Rham first Chern class map

$$c_1^{\mathrm{dR}} : j_*\mathscr{O}^*_{X_{\mathrm{tr}}} \xrightarrow{\sim} M^{\mathrm{gp}} \to M_n^{\mathrm{gp}} \xrightarrow{\mathrm{dlog}} \Omega^\bullet_{(X,M)_n/V_n^\times}[1].$$

For ss-pairs $(U, \bar{U})$ over K, it induces maps

$$c_1^{\mathrm{dR}} : \Gamma(U, \mathscr{O}_U^*) \xleftarrow{\sim} \Gamma(\bar{U}, j_*\mathscr{O}_U^*) \to \mathrm{R}\Gamma(\bar{U}, \Omega^\bullet_{(U,\bar{U})/V^\times})[1].$$

By the map $\mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U}, \mathscr{J}^{[1]}) \to \mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U}) \to \mathrm{R}\Gamma(\bar{U}, \Omega^\bullet_{(U,\bar{U})/V^\times})$, they are compatible with the absolute log-crystalline and log-syntomic classes [Tsuji 1999, (2.2.3)].

Lemma 5.1. *For strict ss-pairs $(U, \bar{U})$ over K, the Hyodo–Kato map and the Hyodo–Kato isomorphism*

$$\iota : H^2_{\mathrm{HK}}(U, \bar{U})_{\mathbb{Q}} \to H^2_{\mathrm{cr}}((U, \bar{U})/R)_{\mathbb{Q}},$$
$$\iota_{\mathrm{dR},\pi} : H^2_{\mathrm{HK}}(U, \bar{U})_{\mathbb{Q}} \otimes_{K_0} K \xrightarrow{\sim} H^2(\bar{U}_K, \Omega^\bullet_{(U,\bar{U}_K)/K})$$

are compatible with first Chern class maps.

Proof. Since $\iota_{\mathrm{dR},\pi} = i_\pi^*\iota \otimes \mathrm{Id}$ and the map i_π^* is compatible with first Chern classes, it suffices to show the compatibility for the Hyodo–Kato map ι. Let $\mathscr{L}$ be a line bundle on U. Since the map ι is a section of the map $i_0^* : H^2_{\mathrm{cr}}((U, \bar{U})/R)_{\mathbb{Q}} \to H^2_{\mathrm{HK}}(U, \bar{U})_{\mathbb{Q}}$ and the map i_0^* is compatible with first Chern classes, we have that the element $\zeta \in H^2_{\mathrm{cr}}((U, \bar{U})/R)_{\mathbb{Q}}$ defined as $\zeta = \iota(c_1^{\mathrm{HK}}(\mathscr{L})) - c_1^{\mathrm{st}}(\mathscr{L})$ lies in $TH^2_{\mathrm{cr}}((U, \bar{U})/R)_{\mathbb{Q}}$. Hence $\zeta = T\gamma$. Since the map ι is compatible with Frobenius and $\varphi(c_1^{\mathrm{HK}}(\mathscr{L})) = pc_1^{\mathrm{HK}}(\mathscr{L})$, $\varphi(c_1^{\mathrm{st}}(\mathscr{L})) = pc_1^{\mathrm{st}}(\mathscr{L})$, we have $\varphi(\zeta) = p\zeta$. Since $\varphi(T\gamma) = T^p\varphi(\gamma)$, this implies that $\gamma \in \bigcap_{n=1}^\infty T^n H^2_{\mathrm{cr}}((U, \bar{U})/R)_{\mathbb{Q}}$, which is not possible unless γ (and hence ζ) are zero. But this is what we wanted to show. $\square$

We have the following projective space theorem for syntomic cohomology.

Proposition 5.2. *Let $\mathscr{E}$ be a locally free sheaf of rank $d+1$, $d \geq 0$, on a scheme $X \in \mathscr{V}ar_K$. Consider the associated projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to X$. Then we have the quasi-isomorphism of complexes of sheaves on X_h*

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{syn}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} \mathscr{S}(r-i)_{X,\mathbb{Q}}[-2i] \xrightarrow{\sim} R\pi_*\mathscr{S}(r)_{\mathbb{P}(\mathscr{E}),\mathbb{Q}}, \quad 0 \leq d \leq r.$$

Here, the class $c_1^{\mathrm{syn}}(\mathscr{O}(1)) \in H^2_{\mathrm{syn}}(\mathbb{P}(\mathscr{E})_h, 1)$ refers to the class of the tautological bundle on $\mathbb{P}(\mathscr{E})$.

Proof. By (tedious) checking of many compatibilities, we will reduce the above projective space theorem to the projective space theorems for the Hyodo–Kato and the filtered de Rham cohomologies.

To prove our proposition it suffices to show that for any ss-pair $(U, \overline{U})$ over K and the projective space $\pi : \mathbb{P}^d_{\overline{U}} \to \overline{U}$ of dimension d over $\overline{U}$ we have a projective space theorem for syntomic cohomology ($a \geq 0$):

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{syn}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{syn}}(U_h, r-i) \xrightarrow{\sim} H^a_{\mathrm{syn}}(\mathbb{P}^d_{U,h}, r), \quad 0 \leq d \leq r.$$

By Proposition 3.18 and the compatibility of the maps

$$H^*_{\mathrm{syn}}(U, \overline{U}, j)_{\mathbb{Q}} \xrightarrow{\sim} H^*_{\mathrm{syn}}(U_h, j)_{\mathbb{Q}}$$

with products and first Chern classes, this reduces to proving a projective space theorem for log-syntomic cohomology, i.e., a quasi-isomorphism of complexes

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{syn}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{syn}}(U, \overline{U}, r-i)_{\mathbb{Q}} \xrightarrow{\sim} H^a_{\mathrm{syn}}(\mathbb{P}^d_U, \mathbb{P}^d_{\overline{U}}, r)_{\mathbb{Q}}, \quad 0 \leq d \leq r,$$

where the class $c_1^{\mathrm{syn}}(\mathscr{O}(1)) \in H^2_{\mathrm{syn}}(\mathbb{P}^d_U, \mathbb{P}^d_{\overline{U}}, 1)$ refers to the class of the tautological bundle on $\mathbb{P}^d_{\overline{U}}$.

By the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/F^r$$

and its compatibility with the action of c_1^{syn}, it suffices to prove the following two quasi-isomorphisms for the twisted absolute log-crystalline complexes and for the filtered log de Rham complexes ($0 \leq d \leq r$):

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{cr}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{cr}}(U, \overline{U}, r-i)_{\mathbb{Q}} \xrightarrow{\sim} H^a_{\mathrm{cr}}(\mathbb{P}^d_U, \mathbb{P}^d_{\overline{U}}, r)_{\mathbb{Q}},$$

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{dR}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} F^{r-i} H^{a-2i}_{\mathrm{dR}}(U, \overline{U}_K) \xrightarrow{\sim} F^r H^a_{\mathrm{dR}}(\mathbb{P}^d_U, \mathbb{P}^d_{\overline{U}_K}).$$

For the log de Rham cohomology, notice that the above map is quasi-isomorphic to the map [Beilinson 2012, 3.2]

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{dR}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} F^{r-i} H^{a-2i}_{\mathrm{dR}}(U) \xrightarrow{\sim} F^r H^a_{\mathrm{dR}}(\mathbb{P}^d_U),$$

and hence well-known to be a quasi-isomorphism.

For the twisted log-crystalline cohomology, notice that since Frobenius behaves well with respect to c_1^{cr}, it suffices to prove a projective space theorem for the absolute log-crystalline cohomology $H^*_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}}$:

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{cr}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}} \xrightarrow{\sim} H^a_{\mathrm{cr}}(\mathbb{P}^d_U, \mathbb{P}^d_{\bar{U}})_{\mathbb{Q}}.$$

Without loss of generality, we may assume that the pair $(U,\bar{U})$ is split over K. By the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U}) \to \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R) \xrightarrow{N} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)$$

and its compatibility with the action of $c_1^{\mathrm{cr}}(\mathscr{O}(1))$ (see [Tsuji 1999, Lemma 4.3.7]), it suffices to prove a projective space theorem for the log-crystalline cohomology $H^*_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}$. Since the R-linear isomorphism $\iota : H^*_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \otimes R_{\mathbb{Q}} \xrightarrow{\sim} H^*_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}$ is compatible with products [Tsuji 1999, Proposition 4.4.9] and first Chern classes (see Lemma 5.1), we reduce the problem to showing the projective space theorem for the Hyodo–Kato cohomology:

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{HK}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \xrightarrow{\sim} H^a_{\mathrm{HK}}(\mathbb{P}^d_U, \mathbb{P}^d_{\bar{U}})_{\mathbb{Q}}.$$

Tensoring by K and using the isomorphism

$$\iota_{\mathrm{dR},\pi} : H^*_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \otimes_{K_0} K \xrightarrow{\sim} H^*_{\mathrm{dR}}(U,\bar{U}_K)$$

that is compatible with products [Tsuji 1999, Corollary 4.4.13] and first Chern classes (see Lemma 5.1), we reduce to checking the projective space theorem for the log de Rham cohomology $H^*_{\mathrm{dR}}(U,\bar{U}_K)$, and we have done this above. □

The above proof proves also the projective space theorem for the absolute crystalline cohomology.

Corollary 5.3. *Let $\mathscr{E}$ be a locally free sheaf of rank $d+1$, $d \geq 0$, on a scheme $X \in \mathscr{V}ar_K$. Consider the associated projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to X$. Then we have the following quasi-isomorphism of complexes of sheaves on X_h*

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{cr}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} \mathscr{J}^{[r-i]}_{X,\mathbb{Q}}[-2i] \xrightarrow{\sim} R\pi_* \mathscr{J}^{[r]}_{\mathbb{P}(\mathscr{E}),\mathbb{Q}}, \quad 0 \leq d \leq r.$$

Here, the class $c_1^{\mathrm{cr}}(\mathscr{O}(1)) \in H^2(\mathbb{P}(\mathscr{E})_h, \mathscr{J}_{\mathrm{cr}})$ refers to the class of the tautological bundle on $\mathbb{P}(\mathscr{E})$.

For $X \in \mathscr{V}ar_K$, using the projective space theorem (see Proposition 5.2) and the Chern classes

$$c_0^{\mathrm{syn}} : \mathbb{Q}_p \xrightarrow{\mathrm{can}} \mathscr{S}(0)_{X_{\mathbb{Q}}}, \quad c_1^{\mathrm{syn}} : \mathscr{O}^*_{X_h} \to \mathscr{S}(1)_{X_{\mathbb{Q}}}[1],$$

we obtain syntomic Chern classes $c_i^{\mathrm{syn}}(\mathscr{E})$ for any locally free sheaf $\mathscr{E}$ on X.

Syntomic cohomology has the homotopy invariance property.

Proposition 5.4. *Let $X \in \mathscr{V}ar_K$ and $f : \mathbb{A}^1_X \to X$ be the natural projection from the affine line over X to X. Then, for all $r \geq 0$, the pullback map*

$$f^* : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} \sim\mathrm{R}\Gamma_{\mathrm{syn}}(\mathbb{A}^1_{X,h}, r)$$

is a quasi-isomorphism.

Proof. Localizing in the h-topology of X, we may assume that $X = U$, the open set of an ss-pair $(U, \overline{U})$ over K. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \arrow[r, "f^*"] \arrow[d, "\wr"] & \mathrm{R}\Gamma_{\mathrm{syn}}(\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}}, r)_{\mathbb{Q}} \arrow[d, "\wr"] \\
\mathrm{R}\Gamma_{\mathrm{syn}}(U_h, r) \arrow[r, "f^*"] & \mathrm{R}\Gamma_{\mathrm{syn}}(\mathbb{A}^1_{U,h}, r)
\end{tikzcd}$$

The vertical maps are quasi-isomorphisms by Proposition 3.18. It suffices thus to show that the top horizontal map is a quasi-isomorphism. By Proposition 3.8, this reduces to showing that the map

$$C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})_{\mathbb{Q}}\{r\}) \xrightarrow{f^*} C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}})_{\mathbb{Q}}\{r\})$$

is a quasi-isomorphism, or, that the map $f : (\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}}) \to (U, \overline{U})$ induces a quasi-isomorphism on the Hyodo–Kato cohomology and a filtered quasi-isomorphism on the log de Rham cohomology:

$$\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})_{\mathbb{Q}} \xrightarrow{f^*} \mathrm{R}\Gamma_{\mathrm{HK}}(\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}})_{\mathbb{Q}}, \quad \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K) \xrightarrow{f^*} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}_K}).$$

Without loss of generality, we may assume that the pair $(U, \overline{U})$ is split over K. Tensoring with K and using the Hyodo–Kato quasi-isomorphism, we reduce the Hyodo–Kato case to the log de Rham one. The latter follows easily from the projective space theorem and the existence of the Gysin sequence in log de Rham cohomology. □

Remark 5.5. The above implies that syntomic cohomology is a Bloch–Ogus theory. A proof of this fact was kindly communicated to us by Frédéric Déglise and is contained in Appendix B, Proposition B.4.

Proposition 5.6. *For a scheme X, let $K_*(X)$ denote Quillen's higher K-theory groups of X. For $X \in \mathscr{V}ar_K$, $i, j \geq 0$, there are functorial syntomic Chern class maps*

$$c^{\mathrm{syn}}_{i,j} : K_j(X) \to H^{2i-j}_{\mathrm{syn}}(X_h, i).$$

Proof. Recall the construction of the classes $c^{\mathrm{syn}}_{i,j}$. First, one constructs universal classes $C^{\mathrm{syn}}_{i,l} \in H^{2i}_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, i)$. By a standard argument, the projective space theorem and the homotopy property show that

$$H^*_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, *) \simeq H^*_{\mathrm{syn}}(K, *)[x_1^{\mathrm{syn}}, \dots, x_l^{\mathrm{syn}}],$$

where the classes $x_i^{\mathrm{syn}} \in H^{2i}_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, i)$ are the syntomic Chern classes of the universal locally free sheaf on $B_\bullet \mathrm{GL}_l$ (defined via a projective space theorem). For $l \geq i$, we define

$$C_{i,l}^{\mathrm{syn}} = x_i^{\mathrm{syn}} \in H^{2i}_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, i).$$

The classes $C_{i,l}^{\mathrm{syn}} \in H^{2i}_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, i)$ yield compatible universal classes (see [Gillet 1981, p. 221]) $C_{i,l}^{\mathrm{syn}} \in H^{2i}_{\mathrm{syn}}(X, \mathrm{GL}_l(\mathscr{O}_X), i)$, and hence a natural map of pointed simplicial sheaves on X_{ZAR}, $C_i^{\mathrm{syn}} : B_\bullet \mathrm{GL}(\mathscr{O}_X) \to \mathscr{K}(2i, \mathscr{S}'(i)_X)$, where $\mathscr{K}$ is the Dold–Puppe functor of $\tau_{\geq 0}\mathscr{S}'(i)_X[2i]$ and $\mathscr{S}'(i)_X$ is an injective resolution of $\mathscr{S}(i)_X := R\varepsilon_*\mathscr{S}(i)_{\mathbb{Q}}$, $\varepsilon : X_h \to X_{\mathrm{ZAR}}$. The characteristic classes $c_{i,j}^{\mathrm{syn}}$ are now defined [Gillet 1981, Definition 2.22] as the composition

$$\begin{aligned} K_j(X) &\to H^{-j}(X, \mathbb{Z} \times B_\bullet \mathrm{GL}(\mathscr{O}_X)^+) \to H^{-j}(X, B_\bullet \mathrm{GL}(\mathscr{O}_X)^+) \\ &\xrightarrow{C_i^{\mathrm{syn}}} H^{-j}(X, \mathscr{K}(2i, \mathscr{S}'(i)_X)) \xrightarrow{h_j} H^{2i-j}_{\mathrm{syn}}(X_h, i), \end{aligned}$$

where $B_\bullet \mathrm{GL}(\mathscr{O}_X)^+$ is the (pointed) simplicial sheaf on X associated to the $+$-construction [Soulé 1982, 4.2]. Here, for a (pointed) simplicial sheaf $\mathscr{E}_\bullet$ on X_{ZAR}, we know $H^{-j}(X, \mathscr{E}_\bullet) = \pi_j(\mathrm{R}\Gamma(X_{\mathrm{ZAR}}, \mathscr{E}_\bullet))$ is the generalized sheaf cohomology of $\mathscr{E}_\bullet$ [Gillet 1981, Definition 1.7]. The map h_j is the Hurewicz map:

$$\begin{aligned} H^{-j}(X, \mathscr{K}(2i, \mathscr{S}'(i)_X)) = \pi_j(\mathscr{K}(2i, \mathscr{S}'(i)(X))) &\xrightarrow{h_j} H_j(\mathscr{K}(2i, \mathscr{S}'(i)(X))) \\ &= H_j(\mathscr{S}'(i)(X)[2i]) = H^{2i-j}_{\mathrm{syn}}(X_h, i). \qquad \square \end{aligned}$$

Proposition 5.7. *The syntomic and the étale Chern classes are compatible, i.e., for* $X \in \mathscr{V}ar_K$, $j \geq 0$, $2i - j \geq 0$, *the following diagram commutes*:

$$\begin{tikzcd} & K_j(X) \arrow[dl, "c_{i,j}^{\mathrm{syn}}"'] \arrow[dr, "c_{i,j}^{\text{ét}}"] & \\ H^{2i-j}_{\mathrm{syn}}(X_h, i) \arrow[rr, "\rho_{\mathrm{syn}}"] & & H^{2i-j}_{\text{ét}}(X, \mathbb{Q}_p(i)) \end{tikzcd}$$

Proof. We can pass to the universal case ($X = B_\bullet \mathrm{GL}_l := B_\bullet \mathrm{GL}_l / K$, $l \geq 1$). We have

$$H^*_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, *) \simeq H^*_{\mathrm{syn}}(K, *)[x_1^{\mathrm{syn}}, \dots, x_l^{\mathrm{syn}}],$$
$$H^*_{\text{ét}}(B_\bullet \mathrm{GL}_l, *) \simeq H^*_{\text{ét}}(K, *)[x_1^{\text{ét}}, \dots, x_l^{\text{ét}}].$$

By the projective space theorem and the fact that the syntomic period map commutes with products, it suffices to check that $\rho_{\mathrm{syn}}(x_1^{\mathrm{syn}}) = x_1^{\text{ét}}$ and that the syntomic period map ρ_{syn} commutes with the classes $c_0^{\mathrm{syn}} : \mathbb{Q}_p \to \mathscr{S}(0)_{\mathbb{Q}}$ and $c_0^{\text{ét}} : \mathbb{Q}_p \to \mathbb{Q}_p(0)$. The statement about c_0 is clear from the definition of ρ_{cr}; for c_1, consider the canonical map $f : B_\bullet \mathrm{GL}_l \to B_\bullet \mathrm{GL}_{l,\bar{K}}$ and the induced pullback map

$$f^*_{\text{ét}} : H^*_{\text{ét}}(B_\bullet \mathrm{GL}_l, *) = H^*_{\text{ét}}(K, *)[x_1, \dots, x_l] \to H^*_{\text{ét}}(B_\bullet \mathrm{GL}_{l,\bar{K}}, *) = \mathbb{Q}_p[\bar{x}_1, \dots, \bar{x}_l]$$

that sends the Chern classes $x_i^{\text{ét}}$ of the universal vector bundle to the classes $\bar{x}_i^{\text{ét}}$ of its pullback. It suffices to show that $f^*_{\text{ét}}\rho_{\mathrm{syn}}(C^{\mathrm{syn}}_{1,1}) = C^{\text{ét}}_{1,1}$. But, by definition, $f^*_{\text{ét}}\rho_{\mathrm{syn}} = \rho_{\mathrm{syn}} f^*_{\mathrm{syn}}$ and, by construction, we have the commutative diagram

$$\begin{array}{ccc}
H^2_{\mathrm{syn}}(B_\bullet \mathbb{G}_{m,h}, 1) & \xrightarrow{\quad\text{can}\quad} & H^2_{\mathrm{cr}}(B_\bullet \mathbb{G}_{m,\bar{K},h}) \\
\downarrow{\scriptstyle \rho_{\mathrm{syn}}} & & \downarrow{\scriptstyle \rho_{cr}} \\
H^2_{\text{ét}}(B_\bullet \mathbb{G}_{m,\bar{K}}, \mathbb{Q}_p(1)) & \longrightarrow & H^2_{\text{ét}}(B_\bullet \mathbb{G}_{m,\bar{K}}, B^+_{\mathrm{cr}}) = H^2_{\text{ét}}(B_\bullet \mathbb{G}_{m,\bar{K}}, \mathbb{Q}_p(1)) \otimes B^+_{\mathrm{cr}}
\end{array}$$

where the bottom map sends the generator of $\mathbb{Q}_p(1)$ to the element $t \in B^+_{\mathrm{cr}}$ associated to it. Since the syntomic and the crystalline Chern classes are compatible, it suffices to show that, for a line bundle $\mathscr{L}$, we have $\rho_{\mathrm{cr}}(c_1^{\mathrm{cr}}(\mathscr{L})) = c_1^{\text{ét}}(\mathscr{L}) \otimes t$. But this is [Beilinson 2013, 3.2]. □

Remark 5.8. If $\mathscr{X}$ is a scheme over V and $X = \mathscr{X}_K$, we can consider the syntomic Chern classes $c^{\mathrm{syn}}_{i,j} : K_j(\mathscr{X}) \to H^{2i-j}_{\mathrm{syn}}(X_h, i)$ defined as the composition

$$K_j(\mathscr{X}) \to K_j(X) \xrightarrow{c^{\mathrm{syn}}_{i,j}} H^{2i-j}_{\mathrm{syn}}(X_h, i).$$

By the above proposition, these classes are compatible with the étale Chern classes. Recall that analogous results were proved earlier for $\mathscr{X}$ smooth and projective [Niziol 1997], for $\mathscr{X}$ a complement of a divisor with relative normal crossings in such, and for $\mathscr{X}$ a semistable scheme over V [Niziol 2016b].

5B. *Image of étale regulators.* In this subsection we show that Soulé's étale regulators factor through the semistable Selmer groups.

Let $X \in \mathscr{V}ar_K$. For $2r - i - 1 \geq 0$, set

$$K_{2r-i-1}(X)_0 := \ker\big(K_{2r-i-1}(X) \xrightarrow{c^{\text{ét}}_{r,i+1}} H^0(G_K, H^{i+1}_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r)))\big).$$

Write $r^{\text{ét}}_{r,i}$ for the map

$$r^{\text{ét}}_{r,i} : K_{2r-i-1}(X)_0 \to H^1(G_K, H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r)))$$

induced by the Chern class map $c^{\text{ét}}_{r,i+1}$ and the Hochschild–Serre spectral sequence map $\delta : H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r))_0 \to H^1(G_K, H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r)))$, where we set

$$H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r))_0 := \ker(H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r)) \to H^{i+1}_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))).$$

Theorem 5.9. *The map $r^{\text{ét}}_{r,i}$ factors through the subgroup*
$$H^1_{\text{st}}(G_K, H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))) \subset H^1(G_K, H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))).$$

Proof. By Proposition 5.7, we have the commutative diagram

$$\begin{tikzcd}
K_{2r-i-1}(X) \arrow[d, "c^{\text{syn}}_{r,i+1}"] \arrow[dr, "c^{\text{ét}}_{r,i+1}"] & & \\
H^{i+1}_{\text{syn}}(X_h, r) \arrow[r, "\rho_{\text{syn}}"] & H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r)) \arrow[r] & H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))
\end{tikzcd}$$

Hence the Chern class map $c^{\text{syn}}_{r,i+1} : K_{2r-i-1}(X)_0 \to H^{i+1}_{\text{syn}}(X_h, r)$ factors through $H^{i+1}_{\text{syn}}(X_h, r)_0 := \ker(H^{i+1}_{\text{syn}}(X_h, r) \xrightarrow{\rho_{\text{syn}}} H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)))$. Compatibility of the syntomic descent and the Hochschild–Serre spectral sequences (see Theorem 4.8) yields the commutative diagram

$$\begin{tikzcd}
K_{2r-i-1}(X)_0 \arrow[d, "c^{\text{syn}}_{r,i+1}"] \arrow[dr, "c^{\text{ét}}_{r,i+1}"] & \\
H^{i+1}_{\text{syn}}(X_h, r)_0 \arrow[r, "\rho_{\text{syn}}"] \arrow[d, "\delta"] & H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r))_0 \arrow[d, "\delta"] \\
H^1_{\text{st}}(G_K, H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))) \arrow[r, "\text{can}"] & H^1(G_K, H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)))
\end{tikzcd}$$

Our theorem follows. □

Remark 5.10. The question of the image of Soulé's regulators $r^{\text{ét}}_{r,i}$ was raised by Bloch and Kato [1990] in connection with their Tamagawa number conjecture. Theorem 5.9 is known to follow from the constructions of Scholl [1993]. The argument goes as follows. Recall that for a class $y \in K_{2r-i-1}(X)_0$, he constructs an explicit extension $E_y \in \operatorname{Ext}^1_{\mathscr{M}\mathscr{M}_K}(\mathbb{Q}(-r), h^i(X))$ in the category of mixed motives over K. The association $y \mapsto E_y$ is compatible with the étale cycle class and realization maps. By the de Rham comparison theorem, the étale realization $r^{\text{ét}}_{r,i}(y)$ of the extension class E_y in
$$\operatorname{Ext}^1_{G_K}(\mathbb{Q}_p(-r), H^i(X_{\overline{K}}, \mathbb{Q}_p)) = H^1(G_K, H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)))$$
is de Rham, hence potentially semistable by [Berger 2002], as wanted.

Appendix A: Vanishing of $H^2(G_K, V)$

by Laurent Berger

Let V be a $\mathbb{Q}_p$-linear representation of G_K. In this appendix we prove the following theorem.

Theorem A.1. *If V is semistable and all its Hodge–Tate weights are ≥ 2, then $H^2(G_K, V) = 0$.*

Let $\mathrm{D}(V)$ be Fontaine's (φ, Γ)-module [1990] attached to V. It comes with a Frobenius map φ and an action of Γ_K. Let $H_K = \mathrm{Gal}(\bar{K}/K(\mu_{p^\infty}))$ and let $I_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{nr}})$. The injectivity of the restriction map $H^2(G_K, V) \to H^2(G_L, V)$ for L/K finite allows us to replace K by a finite extension, so that we can assume that $H_K I_K = G_K$ and that $\Gamma_K \simeq \mathbb{Z}_p$. Let γ be a topological generator of Γ_K. Recall [Cherbonnier and Colmez 1999, §I.5] that we have a map $\psi : \mathrm{D}(V) \to \mathrm{D}(V)$.

Ideally, our proof of this theorem would go as follows. We use the Hochschild–Serre spectral sequence

$$H^i(G_K/I_K, H^j(I_K, V|_{I_K})) \Rightarrow H^{i+j}(G_K, V)$$

and, interpreting Galois cohomology in terms of (φ, Γ)-modules, we compute that $H^2(I_K, V|_{I_K}) = 0$ and $H^1(I_K, V|_{I_K}) = \widehat{K}^{\mathrm{nr}} \otimes_K D_{\mathrm{dR}}(V)$. We conclude since, by Hilbert 90, $H^1(G_K/I_K, H^1(I_K, V|_{I_K})) = 0$. However, we do not, in general, have Hochschild–Serre spectral sequences for continuous cohomology. We mimic thus the above argument with direct computations on continuous cocycles (again using (φ, Γ)-modules). Laurent Berger is grateful to Kevin Buzzard for discussions related to the above spectral sequence.

Lemma A.2. (1) *If V is a representation of G_K, then there is an exact sequence*

$$0 \to \mathrm{D}(V)^{\psi=1}/(\gamma - 1) \to H^1(G_K, V) \to (\mathrm{D}(V)/(\psi - 1))^{\Gamma_K} \to 0.$$

(2) *We have $H^2(G_K, V) = \mathrm{D}(V)/(\psi - 1, \gamma - 1)$.*

Proof. See I.5.5 and II.3.2 of [Cherbonnier and Colmez 1999]. □

Lemma A.3. *We have* $\mathrm{D}(V|_{I_K})/(\psi - 1) = 0$.

Proof. Since $V|_{I_K}$ corresponds to the case when k is algebraically closed, see the proof of Lemma VI.7 of [Berger 2001]. □

Let γ_I denote a generator of $\Gamma_{\widehat{K}^{\mathrm{nr}}}$.

Lemma A.4. *The natural map* $\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I - 1) \to (\mathrm{D}(V|_{I_K})/(\gamma_I - 1))^{\psi=1}$ *is an isomorphism if $V^{I_K} = 0$.*

Proof. This map is part of the six-term exact sequence that comes from the map $\gamma_I - 1$ applied to $0 \to \mathrm{D}(V|_{I_K})^{\psi=1} \to \mathrm{D}(V|_{I_K}) \xrightarrow{\psi-1} \mathrm{D}(V|_{I_K}) \to 0$. Its kernel is included in $\mathrm{D}(V|_{I_K})^{\gamma_I=1}$, which is 0 since $V^{I_K} = 0$ (note that the inclusion $(\widehat{K}^{\mathrm{nr}} \otimes V)^{G_K} \subseteq (\widehat{\mathscr{E}}^{\mathrm{nr}} \otimes V)^{G_K} = \mathrm{D}(V)^{G_K}$ is an isomorphism). □

Suppose that $x \in \mathrm{D}(V)/(\psi - 1, \gamma - 1)$. If $\tilde{x} \in \mathrm{D}(V)$ lifts x, then Lemma A.3 gives us an element $y \in \mathrm{D}(V|_{I_K})$ such that $(\psi - 1)y = \tilde{x}$. Define a cocycle $\delta(x) \in Z^1(G_K/I_K, \mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I - 1))$ by $\delta(x) : \bar{g} \mapsto (g - 1)(y)$ if $g \in G_K$ lifts $\bar{g} \in G_K/I_K$.

Proposition A.5. *If* $V^{I_K} = 0$, *then the map*

$$\delta : \mathrm{D}(V)/(\psi - 1, \gamma - 1) \to H^1\big(G_K/I_K, (\mathrm{D}(V|_{I_K})/(\gamma_I - 1))^{\psi=1}\big)$$

is well-defined and injective.

Proof. We first check that

$$\delta(x)(g) \in (\mathrm{D}(V|_{I_K})/(\gamma_I - 1))^{\psi=1}.$$

We have $(\psi - 1)(g - 1)(y) = (g - 1)(x)$. If we write $g = ih \in I_K H_K$, then $(g - 1)x = (ih - 1)x = (i - 1)x \in (\gamma_I - 1)\mathrm{D}(V|_{I_K})$ since $\gamma_I - 1$ divides the image of $i - 1$ in $\mathbb{Z}_p[\![\Gamma_{\widehat{K}^{\mathrm{nr}}}]\!]$. This implies $\delta(x)(g) \in (\mathrm{D}(V|_{I_K})/(\gamma_I - 1))^{\psi=1}$.

We now check that $\delta(x)$ does not depend on the choices. If we choose another lift $g' \in G_K$ of $\bar{g} \in G_K/I_K$, then $g' = ig$ for some $i \in I_K$ and $(g' - 1)y - (g - 1)y = (i-1)gy \in (\gamma_I - 1)\mathrm{D}(V|_{I_K})$ since $\gamma_I - 1$ divides the image of $i - 1$ in $\mathbb{Z}_p[\![\Gamma_{\widehat{K}^{\mathrm{nr}}}]\!]$. If we choose another y' such that $(\psi - 1)y' = \tilde{x}$, then $y - y' \in \mathrm{D}(V|_{I_K})^{\psi=1}$ so that δ and δ' are cohomologous. Finally, if $\tilde{x}'$ is another lift of x, then $\tilde{x}' - \tilde{x} = (\gamma - 1)a + (\psi - 1)b$ with $a, b \in \mathrm{D}(V)$. We can then take $y' = y + b + (\gamma_G - 1)c$, where $(\psi - 1)c = a$. We then have $(g - 1)y' = (g - 1)y + (g - 1)b + (\gamma_G - 1)(g - 1)c$. Since $G_K = I_K H_K$, we can write $g = ih$ and $(g - 1)b = (i - 1)b$. Using $G_K = I_K H_K$ once again, we see that $I_K \to G_K/H_K$ is surjective, so that we can identify γ_I and γ_G. The resulting cocycle is then cohomologous to $\delta(x)$. This proves that δ is well-defined.

We now prove that δ is injective. If $\delta(x) = 0$, then using Lemma A.4 there exists $z \in \mathrm{D}(V|_{I_K})^{\psi=1}$ such that $\delta(x)(\bar{g})$ is the image of $(g - 1)(z)$ in $\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I - 1)$. This implies that $(g - 1)(y - z) \in (\gamma_I - 1)\mathrm{D}(V|_{I_K})^{\psi=1}$. Applying $\psi - 1$ gives $(g - 1)\tilde{x} = 0$ so that $\tilde{x} \in \mathrm{D}(V)^{G_K} \subset V^{I_K} = 0$. The map δ is therefore injective. □

Lemma A.6. *If* V *is semistable and the weights of* V *are all* ≥ 2, *then*

$$\exp_V : \mathrm{D}_{\mathrm{dR}}(V|_{I_K}) \to H^1(I_K, V)$$

is an isomorphism.

Proof. Apply Theorem 6.8 of [Berger 2002] to $V|_{I_K}$. □

Proof of Theorem A.1. We can replace K by K_n for $n \gg 0$ and use the fact that if $H^2(G_{K_n}, V) = 0$, then $H^2(G_K, V) = 0$ since the restriction map is injective. In particular, we can assume that $H_K I_K = G_K$ and that Γ_K is isomorphic to $\mathbb{Z}_p$. By item (2) of Lemma A.2, we have $H^2(G_K, V) = \mathrm{D}(V)/(\psi - 1, \gamma - 1)$, and so by Proposition A.5 above, it is enough to prove that

$$H^1(G_K/I_K, (\mathrm{D}(V|_{I_K})/(\gamma_I - 1))^{\psi=1}) = 0.$$

Lemma A.4 tells us that $(\mathrm{D}(V|_{I_K})/(\gamma_I - 1))^{\psi=1} = \mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I - 1)$. Since $\mathrm{D}(V|_{I_K})/(\psi - 1) = 0$ by Lemma A.3, item (1) of Lemma A.2 tells us that $\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma - 1) = H^1(I_K, V)$.

The map $\exp_V : \mathrm{D}_{\mathrm{dR}}(V|_{I_K}) \to H^1(I_K, V)$ is an isomorphism by Lemma A.6, and this isomorphism commutes with the action of G_K since it is a natural map. We therefore have $H^1(I_K, V) = \widehat{K}^{\mathrm{nr}} \otimes_K \mathrm{D}_{\mathrm{dR}}(V)$ as G_K-modules. It remains to observe that the cocycle $\delta(x) \in Z^1(G_K/I_K, \widehat{K}^{\mathrm{nr}} \otimes_K \mathrm{D}_{\mathrm{dR}}(V))$ is continuous and that $H^1(G_K/I_K, \widehat{K}^{\mathrm{nr}}) = 0$ by taking a lattice, reducing modulo a uniformizer of K, and applying Hilbert 90. □

Appendix B: The syntomic ring spectrum

by Frédéric Déglise

In this appendix, we explain why syntomic cohomology as defined in this paper is representable by a motivic ring spectrum in the sense of Morel and Voevodsky's homotopy theory. More precisely, we will exhibit a monoid object $\mathscr{S}$ of the triangulated category of motives with $\mathbb{Q}_p$-coefficients (see below), DM, such that for any variety X and any pair of integers (i, r),

$$H^i_{\mathrm{syn}}(X_h, r) = \mathrm{Hom}_{DM}(M(X), \mathscr{S}(r)[i]).$$

In fact, it is possible to apply directly [Déglise and Mazzari 2015, Theorem 1.4.10] to the graded commutative dg-algebra $\mathrm{R}\Gamma_{\mathrm{syn}}(X, *)$ of Theorem A in view of the existence of Chern classes established in Section 5A. However, the use of the h-topology in this paper makes the construction of $\mathbb{E}_{synt}$ much more straightforward and that is what we explain in this appendix. Reformulating slightly the original definition of Voevodsky [1996], we introduce:

Definition B.1. Let $\mathrm{PSh}(K, \mathbb{Q}_p)$ be the category of presheaves of $\mathbb{Q}_p$-modules over the category of varieties.

Let C be a complex in $\mathrm{PSh}(K, \mathbb{Q}_p)$. We say

(1) C is h-local if for any h-hypercovering $\pi : Y_\bullet \to X$, the induced map

$$C(X) \to \pi_* \mathrm{Tot}^{\oplus}(C(Y_\bullet))$$

is a quasi-isomorphism;

(2) C is $\mathbb{A}^1$-local if for any variety X, the map induced by the projection

$$H^i(X_h, C) \to H^i(\mathbb{A}^1_{X,h}, C)$$

is an isomorphism.

We define the triangulated category $DM^{\mathrm{eff}}_h(K, \mathbb{Q}_p)$ of effective h-motives as the full subcategory of the derived category $D(\mathrm{PSh}(K, \mathbb{Q}_p))$ made by the complexes which are h-local and $\mathbb{A}^1$-local.

Equivalently, we can define this category as the $\mathbb{A}^1$-localization of the derived category of h-sheaves on K-varieties (see Section 5.2 of [Cisinski and Déglise 2009], and more precisely Proposition 5.2.10 and Example 5.2.17(2)). Recall also from [loc. cit.] that there are derived tensor products and internal Hom on $DM^{\mathrm{eff}}_h(K, \mathbb{Q}_p)$.

For any integer $r \geq 0$, the *syntomic sheaf* $\mathscr{S}(r)$ is both h-local (by definition) and $\mathbb{A}^1$-local (Proposition 5.4). Thus it defines an object of $DM_h^{\mathrm{eff}}(K, \mathbb{Q}_p)$ and for any variety X, one has an isomorphism

$$\mathrm{Hom}_{DM_h^{\mathrm{eff}}(K,\mathbb{Q}_p)}(\mathbb{Q}_p(X), \mathscr{S}(r)[i]) \\ = \mathrm{Hom}_{D(\mathrm{PSh}(K,\mathbb{Q}_p))}(\mathbb{Q}_p(X), \mathscr{S}(r)[i]) = H^i_{\mathrm{syn}}(X_h, r),$$

where $\mathbb{Q}_p(X)$ is the presheaf of $\mathbb{Q}_p$-vector spaces represented by X. Thus, the representability assertion for syntomic cohomology is obvious in the effective setting.

Recall that one defines the Tate motive in $DM_h^{\mathrm{eff}}(K, \mathbb{Q}_p)$ as the object $\mathbb{Q}_p(1) := \mathbb{Q}_p(\mathbb{P}^1_K)/\mathbb{Q}_p(\{\infty\})[-2]$. Given any complex object C of $DM_h^{\mathrm{eff}}(K, \mathbb{Q}_p)$, we put $C(n) := C \otimes \mathbb{Q}_p(1)^{\otimes,n}$. One should be careful that this notation is in conflict with that of $\mathscr{S}(r)$ considered as an effective h-motive, as the natural twist on syntomic cohomology is unrelated to the twist of h-motives. To solve this matter, we are led to consider the following notion of Tate spectrum, borrowed from algebraic topology according to Morel and Voevodsky.

Definition B.2. A *Tate h-spectrum* (over K with coefficients in $\mathbb{Q}_p$) is a sequence $\mathbb{E} = (E_i, \sigma_i)_{i \in \mathbb{N}}$ such that:

- For each $i \in \mathbb{N}$, E_i is a complex of $\mathrm{PSh}(K, \mathbb{Q}_p)$ equipped with an action of the symmetric group Σ_i of the set with i-element.
- For each $i \in \mathbb{N}$, $\sigma_i : E_i(1) \to E_{i+1}$ is a morphism of complexes called the *suspension map* in degree i.
- For any integers $i \geq 0$, $r > 0$, the map induced by the morphisms $\sigma_i, \dots, \sigma_{i+r}$

  $$E_i(r) \to E_{i+r}$$

  is compatible with the action of $\Sigma_i \times \Sigma_r$, given on the left by the structural Σ_i-action on E_i and the action of Σ_r via the permutation isomorphism of the tensor structure on $C(\mathrm{PSh}(K, \mathbb{Q}_p))$, and on the right via the embedding $\Sigma_i \times \Sigma_r \to \Sigma_{i+r}$.

A morphism of Tate h-spectra $f : \mathbb{E} \to \mathbb{F}$ is a sequence of Σ_i-equivariant maps $(f_i : E_i \to F_i)_{i \in \mathbb{N}}$ compatible with the suspension maps. The corresponding category will be denoted by $\mathrm{Sp_h}(K, \mathbb{Q}_p)$.

There is an adjunction of categories

$$\Sigma^\infty : C\big(\mathrm{PSh}(K, \mathbb{Q}_p)\big) \leftrightarrows \mathrm{Sp_h}(K, \mathbb{Q}_p) : \Omega^\infty \tag{52}$$

such that for any complex K of h-sheaves, $\Sigma^\infty C$ is the Tate spectrum equal in degree n to $C(n)$, equipped with the obvious action of Σ_n induced by the symmetric structure on tensor product and with the obvious suspension maps.

Definition B.3. A morphism of Tate spectra $(f_i : E_i \to F_i)_{i\in\mathbb{N}}$ is a level quasi-isomorphism if for any i, we have f_i is a quasi-isomorphism.

A Tate spectrum $\mathbb{E}$ is called a Ω-spectrum if for any i, we have E_i is h-local and $\mathbb{A}^1$-local and the map of complexes

$$E_i \to \underline{\mathrm{Hom}}(\mathbb{Q}_p(1), E_{i+1})$$

is a quasi-isomorphism.

We define the triangulated category $DM_h(K, \mathbb{Q}_p)$ of h-motives over K with coefficients in $\mathbb{Q}_p$ as the category of Tate Ω-spectra localized by the level quasi-isomorphisms.

The category of h-motives notably enjoys the following properties:

(DM1) The adjunction of categories (52) induces an adjunction of triangulated categories

$$\Sigma^\infty : DM_h^{\mathrm{eff}}(K, \mathbb{Q}_p) \leftrightarrows DM_h(K, \mathbb{Q}_p) : \Omega^\infty$$

such that for a Tate Ω-spectrum $\mathbb{E}$, and any integer $r \geq 0$, we have $\Omega^\infty(\mathbb{E}(r)) = E_r$ (see [Cisinski and Déglise 2009, Section 5.3.d, and Example 5.3.31(2)]).

Given any variety X, we define the (stable) h-motive of X as $M(X) := \Sigma^\infty\mathbb{Q}_p(X)$.

(DM2) There exists a symmetric closed monoidal structure on $DM(K, \mathbb{Q}_p)$ such that Σ^∞ is monoidal and such that $\Sigma^\infty\mathbb{Q}_p(1)$ admits a tensor inverse (see [Cisinski and Déglise 2009, Section 5.3, Example 5.3.31(2)]). By abuse of notations, we put $\mathbb{Q}_p = \Sigma^\infty\mathbb{Q}_p$.

(DM3) The triangulated monoidal category $DM_h(K, \mathbb{Q}_p)$ is equivalent to all known versions of triangulated categories of mixed motives over $\mathrm{Spec}(K)$ with coefficients in $\mathbb{Q}_p$ (see [Cisinski and Déglise 2009, Section 16, and Theorem 16.1.2]). In particular, it contains as a full subcategory the category $DM_{gm}(K) \otimes \mathbb{Q}_p$ obtained from the category of Voevodsky geometric motives ([Voevodsky et al. 2000, Chapter 5]) by tensoring Hom-groups with $\mathbb{Q}_p$ (see [Cisinski and Déglise 2009, Corollary 16.1.6, 15.2.5]).

With that definition, the construction of a Tate spectrum representing syntomic cohomology is almost obvious. In fact, we consider the sequence of presheaves

$$\mathscr{S} := (\mathscr{S}(r), r \in \mathbb{N}),$$

where each $\mathscr{S}(r)$ is equipped with with the trivial action of Σ_r. According to the first paragraph of Section 5A, we can consider the first Chern class of the canonical invertible sheaf $\mathbb{P}^1$: $\bar{c} \in H^2_{\mathrm{syn}}(\mathbb{P}^1_K, 1) = H^2(\mathbb{P}^1_{K,h}, \mathscr{S}(1))$. Take any lift $c : \mathbb{Q}_p(\mathbb{P}^1_K) \to \mathscr{S}(1)[2]$ of this class. By the definition of the Tate twist, it defines an element $\mathbb{Q}_p(1) \to \mathscr{S}(1)$ still denoted by c. We define the suspension map

$$\mathscr{S}(r) \otimes \mathbb{Q}_p(1) \xrightarrow{\mathrm{Id}\otimes c} \mathscr{S}(r) \otimes \mathscr{S}(1) \xrightarrow{\mu} \mathscr{S}(r+1),$$

where μ is the multiplication coming from the graded dg-structure on $\mathscr{S}(*)$. Because this dg-structure is commutative, we obtain that these suspension maps induce structures of a Tate spectrum on $\mathscr{S}$. Moreover, $\mathscr{S}$ is a Tate Ω-spectrum because each $\mathscr{S}(r)$ is h-local and $\mathbb{A}^1$-local, and the map obtained by adjunction from σ_r is a quasi-isomorphism because of the projective bundle theorem for $\mathbb{P}^1$ (an easy case of Proposition 5.2).

Now, by definition of $DM_h(K, \mathbb{Q}_p)$ and because of property (DM1) above, for any variety X, and any integers (i, r), we get

$$\begin{aligned}\operatorname{Hom}_{DM_h(K,\mathbb{Q}_p)}&(M(X), \mathscr{S}(r)[i])\\ &= \operatorname{Hom}_{DM_h^{\mathrm{eff}}(K,\mathbb{Q}_p)}(\mathbb{Q}_p(X), \Omega^\infty(\mathscr{S}(r))[i]) = H^i_{\mathrm{syn}}(X_h, r).\end{aligned}$$

Moreover, the commutative dg-structure on the complex $\mathscr{S}(*)$ induces a monoid structure on the associated Tate spectrum. In other words, $\mathscr{S}$ is a ring spectrum (strict and commutative). This construction is completely analogous to the proof of [Déglise and Mazzari 2015, Proposition 1.4.10]. In particular, we can apply all the constructions of [Déglise and Mazzari 2015, Section 3] to the ring spectrum $\mathscr{S}$. Let us summarize this briefly:

Proposition B.4. (1) *Syntomic cohomology is covariant with respect to projective morphisms of smooth varieties* (Gysin morphisms in the terminology of [Déglise and Mazzari 2015]). *More precisely, to a projective morphism of smooth K-varieties $f : Y \to X$ one can associate a Gysin morphism in syntomic cohomology*

$$f_* : H^n_{\mathrm{syn}}(Y_h, i) \to H^{n-2d}_{\mathrm{syn}}(X_h, i-d),$$

where d is the dimension of f.

(2) *The syntomic regulator over $\mathbb{Q}_p$ is induced by the unit $\eta : \mathbb{Q}_p \to \mathscr{S}$ of the ring spectrum $\mathscr{S}$:*

$$\begin{aligned}r_{\mathrm{syn}} : H^{r,i}_M(X) \otimes \mathbb{Q}_p &= \operatorname{Hom}_{DM_h(K,\mathbb{Q}_p)}(M(X), \mathbb{Q}_p(r)[i])\\ &\longrightarrow \operatorname{Hom}_{DM_h(K,\mathbb{Q}_p)}(M(X), \mathscr{S}(r)[i]) = H^i_{\mathrm{syn}}(X_h, r).\end{aligned}$$

It is compatible with product, pullbacks and pushforwards.

(3) *The syntomic cohomology has a natural extension to h-motives*[10]

$$DM_h(K, \mathbb{Q}_p)^{op} \to D(\mathbb{Q}_p), \quad M \mapsto \operatorname{Hom}_{DM_h(K,\mathbb{Q}_p)}(M, \mathscr{S})$$

and the syntomic regulator r_{syn} can be extended to motives.

(4) *There exists a canonical syntomic Borel-Moore homology $H^{\mathrm{syn}}_*(-, *)$ such that the pair of functors $(H^*_{\mathrm{syn}}(-, *), H^{\mathrm{syn}}_*(-, *))$ defines a Bloch–Ogus theory.*

[10] And in particular to the usual Voevodsky geometrical motives by (DM3) above.

(5) *To the ring spectrum $\mathscr{S}$ there is associated a cohomology with compact support satisfying the usual properties.*

For points (1) and (2), we refer the reader to [Déglise and Mazzari 2015, Section 3.1] and for the remaining ones to Section 3.2 of the same paper.

Remark B.5. Note that the construction of the syntomic ring spectrum $\mathscr{S}$ in $DM_h(K, \mathbb{Q}_p)$ automatically yields the general projective bundle theorem (already obtained in Proposition 5.2). More generally, the ring spectrum $\mathscr{S}$ is *oriented* in the terminology of motivic homotopy theory. Thus, besides the theory of Gysin morphisms, this gives various constructions — symbols, residue morphisms — and yields various formulas — excess intersection formula, blow-up formulas (see [Déglise 2008] for more details).

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jan.nekovar@imj-prg.fr *Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie (Paris 6), 75252 Paris Cedex 05, France*

wieslawa.niziol@ens-lyon.fr *École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, 69364 Lyon Cedex 07, France*

laurent.berger@ens-lyon.fr *École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, 69364 Lyon Cedex 07, France*

frederic.deglise@ens-lyon.fr *École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, 69364 Lyon Cedex 07, France*

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Steinberg groups as amalgams

Daniel Allcock

For any root system and any commutative ring, we give a relatively simple presentation of a group related to its Steinberg group $\mathfrak{St}$. This includes the case of infinite root systems used in Kac–Moody theory, for which the Steinberg group was defined by Tits and Morita–Rehmann. In most cases, our group equals $\mathfrak{St}$, giving a presentation with many advantages over the usual presentation of $\mathfrak{St}$. This equality holds for all spherical root systems, all irreducible affine root systems of rank > 2, and all 3-spherical root systems. When the coefficient ring satisfies a minor condition, the last condition can be relaxed to 2-sphericity.

Our presentation is defined in terms of the Dynkin diagram rather than the full root system. It is concrete, with no implicit coefficients or signs. It makes manifest the exceptional diagram automorphisms in characteristics 2 and 3, and their generalizations to Kac–Moody groups. And it is a Curtis–Tits style presentation: it is the direct limit of the groups coming from 1- and 2-node subdiagrams of the Dynkin diagram. Over nonfields this description as a direct limit is new and surprising. Our main application is that many Steinberg and Kac–Moody groups over finitely generated rings are finitely presented.

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1. Introduction

In this paper we give a presentation for a Steinberg-like group, over any commutative ring, for any root system, finite or not. For many root systems, including all finite ones, it is the same as the Steinberg group $\mathfrak{St}$. This is the case of interest, for then it gives a new presentation of $\mathfrak{St}$ and associated Chevalley and Kac–Moody groups. Our presentation

(i) is defined in terms of the Dynkin diagram rather than the set of all (real) roots (Sections 2 and 7);

(ii) is concrete, with no coefficients or signs left implicit;

(iii) generalizes the Curtis–Tits presentation of Chevalley groups to rings other than fields (Corollary 1.3);

(iv) is rewritable as a finite presentation when R is finitely generated as an abelian group (Theorem 1.4);

(v) is often rewritable as a finite presentation when R is merely finitely generated as a ring (Theorem 1.4);

(vi) allows one to prove that many Kac–Moody groups are finitely presented (Theorem 1.5); and

(vii) makes manifest the exceptional diagram automorphisms that lead to the Suzuki and Ree groups, and allows one to construct similar automorphisms of Kac–Moody groups in characteristic 2 or 3 (Section 3).

More precisely, given any generalized Cartan matrix A, in Section 7 we give two definitions of a new group functor. We call it the pre-Steinberg group $\mathfrak{PSt}_A$ because it has a natural map to $\mathfrak{St}_A$. This will be obvious from the first definition, which mimics Tits' definition [1987] of the Steinberg group $\mathfrak{St}_A$, as refined by Morita and Rehmann [1990]. The difference is that we leave out most of the relations. If the root system is finite then both $\mathfrak{PSt}_A$ and $\mathfrak{St}_A$ coincide with Steinberg's original group functor, so they coincide with each other too. Our perspective is that $\mathfrak{PSt}_A(R)$ is interesting if and only if $\mathfrak{PSt}_A(R) \to \mathfrak{St}_A(R)$ is an isomorphism, when our second definition of $\mathfrak{PSt}_A$ provides a new and useful presentation of $\mathfrak{St}_A$. We will discuss this second definition after listing some cases in which

$$\mathfrak{PSt}_A(R) \cong \mathfrak{St}_A(R).$$

Theorem 1.1 (coincidence of Steinberg and pre-Steinberg groups). *Suppose R is a commutative ring and A is a generalized Cartan matrix. Then the natural map $\mathfrak{PSt}_A(R) \to \mathfrak{St}_A(R)$ is an isomorphism in any of the following cases*:

(i) *if A is spherical*; *or*

(ii) *if A is irreducible affine of rank > 2*; *or*

(iii) *if A is* 3*-spherical*; *or*

(iv) *if A is* 2*-spherical and* (*if A has a multiple bond*) *R has no quotient* $\mathbb{F}_2$ *and* (*if A has a triple bond*) *R has no quotient* $\mathbb{F}_3$.

Language. We pass between Cartan matrices and Dynkin diagrams whenever convenient. The rank $\operatorname{rk} A$ of A means the number of nodes of the Dynkin diagram. A is called spherical if its Weyl group is finite; this is equivalent to every component of the Dynkin diagram being one of the classical *ABCDEFG* diagrams. A is called k-spherical if every subdiagram with $\leq k$ nodes is spherical.

As mentioned above, case (i) in Theorem 1.1 is obvious once $\mathfrak{PSt}$ is defined. Cases (iii)–(iv) are proven in Section 11. By considering the list of affine Dynkin diagrams, one sees that these cases imply case (ii) except in rank 3 when R has a forbidden $\mathbb{F}_2$ or $\mathbb{F}_3$ quotient. Proving (ii) requires removing this restriction on R, for which we refer to [Allcock 2016]. An early version of the present paper was used in [Allcock and Carbone 2016] to establish Theorem 1.1 for certain hyperbolic Dynkin diagrams. Those diagrams are now covered by case (iv).

Our second "definition" of $\mathfrak{PSt}_A(R)$ is the following theorem, giving a presentation for it. It is a restatement of Theorem 7.12, whose proof occupies Sections 7–9. The proof relies on an understanding of root stabilizers under a certain extension of the Weyl group, which appears to be a new ingredient in Lie theory. To give the flavor of the result, the full presentation appears in Table 1.1 if A is simply laced without A_1 components. In this case we have $\mathfrak{PSt}_A(R) = \mathfrak{St}_A(R)$ by the previous theorem, so it is a new presentation for $\mathfrak{St}_A(R)$.

Theorem 1.2 (presentation of pre-Steinberg groups). *For any commutative ring R and any generalized Cartan matrix A, $\mathfrak{PSt}_A(R)$ has a presentation with generators S_i and $X_i(t)$, where i varies over the simple roots and t varies over R, and relators* (7-1)–(7-26).

Table 1.1 shows that the presentation is less intimidating than a list of 26 relations would suggest. See Section 2 for the B_2 and G_2 cases. Each relator (7-1)–(7-26) involves at most two distinct subscripts. This proves the following.

Corollary 1.3 (Curtis–Tits presentation for pre-Steinberg groups). *Let A be a generalized Cartan matrix and R a commutative ring. Consider the groups $\mathfrak{PSt}_B(R)$ and the obvious maps between them, as B varies over the* 1×1 *and* 2×2 *submatrices of A coming from singletons and pairs of nodes of the Dynkin diagram. The direct limit of this family of groups equals $\mathfrak{PSt}_A(R)$.* □

In any of the cases in Theorem 1.1, we may replace $\mathfrak{PSt}_A$ by $\mathfrak{St}_A$ everywhere in Corollary 1.3, yielding a Curtis–Tits style presentation for $\mathfrak{St}_A$. This is the source of our title *Steinberg groups as amalgams*. We learned after writing this paper that Dennis and Stein [1974, Theorem B] announced Corollary 1.3 for finite root

all i	$X_i(t)X_i(u) = X_i(t+u)$ $[S_i^2, X_i(t)] = 1$ $S_i = X_i(1)S_iX_i(1)S_i^{-1}X_i(1)$
all (i, j) with $i \neq j$ unjoined	$S_iS_j = S_jS_i$ $[S_i, X_j(t)] = 1$ $[X_i(t), X_j(u)] = 1$
all (i, j) with $i \neq j$ joined	$S_iS_jS_i = S_jS_iS_j$ $S_i^2S_jS_i^{-2} = S_j^{-1}$ $X_i(t)S_jS_i = S_jS_iX_j(t)$ $S_i^2X_j(t)S_i^{-2} = X_j(t)^{-1}$ $[X_i(t), S_iX_j(u)S_i^{-1}] = 1$ $[X_i(t), X_j(u)] = S_iX_j(tu)S_i^{-1}$

Table 1.1. Our defining relations for the Steinberg group $\mathfrak{St}_A(R)$, when A is any simply laced generalized Cartan matrix, without A_1 components, and R is any commutative ring. The generators are $X_i(t)$ and S_i, where i varies over the nodes of the Dynkin diagram and t over R.

systems. They did not publish a proof, and from their announcement it appears that their approach was not via our Theorem 1.2.

In the A_1, A_2, B_2 and G_2 cases, we write out our presentation of $\mathfrak{PSt}_A(R) = \mathfrak{St}_A(R)$ explicitly in Section 2. We do this to make our results as accessible as possible, and to show in Section 3 that our presentation makes manifest the exceptional diagram automorphisms in characteristics 2 and 3. Namely, the arrow-reversing diagram automorphism of the B_2 or G_2 Dynkin diagram yields a self-homomorphism of the corresponding Steinberg group if the coefficient ring R has characteristic 2 or 3, respectively. If R is a perfect field then this self-homomorphism is the famous outer automorphism that leads to the Suzuki and (small) Ree groups.

Because of the direct limit property (Corollary 1.3), one obtains the corresponding self-homomorphisms of F_4 in characteristic 2 with no more work. That is, the defining relations for $\mathfrak{St}_{F_4}$ are those for $\mathfrak{St}_{B_2}$, two copies of $\mathfrak{St}_{A_2}$ and three copies of

$$\mathfrak{St}_{A_1^2} = \mathfrak{St}_{A_1} \times \mathfrak{St}_{A_1}.$$

The diagram automorphism transforms the B_2 relations as in the previous paragraph and sends the other relations into each other. The same argument applies to many Kac–Moody groups. By work of Hée, this leads to Kac–Moody-like analogues of the Suzuki and Ree groups, discussed briefly in Section 3.

An application of the theory we have described is that Steinberg groups and Kac–Moody groups are finitely presented under quite weak hypotheses on their Dynkin diagrams and coefficient rings. We state the Steinberg group result in terms of $\mathfrak{PSt}_A(R)$, keeping in mind that the interesting case is when $\mathfrak{PSt}_A(R)$ coincides with $\mathfrak{St}_A(R)$. See Section 12 for the proof.

Theorem 1.4 (finite presentation of pre-Steinberg groups). *Let R be a commutative ring and A a generalized Cartan matrix. Then $\mathfrak{PSt}_A(R)$ is finitely presented in any of the following cases*:

(i) *if R is finitely generated as an abelian group*; *or*

(ii) *if A is* 2*-spherical without A_1 components*, *and R is finitely generated as a module over a subring generated by finitely many units*; *or*

(iii) *if R is finitely generated as a ring*, *and any two nodes of A lie in an irreducible spherical diagram of rank ≥ 3.*

Many authors have studied the finite presentation of Steinberg groups and related groups. Our Theorem 1.4 is inspired by work of Splitthoff [1986]. See [Kiralis et al. 1996; Zhang 1991; Li 1989] for some additional results.

The Kac–Moody group version of Theorem 1.4 concerns the group functors $\mathfrak{G}_D$ constructed by Tits [1987] (he wrote $\widetilde{\mathfrak{G}}_D$). They were his motivation for generalizing the Steinberg groups beyond the case of spherical Dynkin diagrams. He defined the “simply connected” Kac–Moody groups as certain quotients of Steinberg groups, and arbitrary Kac–Moody groups are only slightly more general. Specifying a Kac–Moody group requires specifying a root datum D, which is slightly more refined information than D’s associated generalized Cartan matrix A. But the choice of D doesn’t affect any of our results.

Our final theorem shows that a great many Kac–Moody groups over rings are finitely presented. This is surprising because one thinks of Kac–Moody groups over (say) $\mathbb{R}$ as infinite-dimensional Lie groups, so the same groups over (say) $\mathbb{Z}$ should be some sort of discrete subgroups. There is no obvious reason why a discrete subgroup of an infinite-dimensional Lie group should be finitely presented. See Section 12 for the definition of the Kac–Moody groups, and the proof of the following theorem.

Theorem 1.5 (finite presentation of Kac–Moody groups). *Suppose A is a generalized Cartan matrix and R is a commutative ring whose group of units R^* is finitely generated. Let D be any root datum with generalized Cartan matrix A. Then Tits’ Kac–Moody group $\mathfrak{G}_D(R)$ is finitely presented if $\mathfrak{St}_A(R)$ is.*

In particular, *this holds if one of* (i)–(iv) *from Theorem 1.1 holds and one of* (i)–(iii) *from Theorem 1.4 holds.*

The paper is organized as follows. Sections 2 and 3 are expository and not essential for later sections. Section 2 is really a continuation of the introduction, writing down the essential cases of our presentation of $\mathfrak{PSt}_A(R)$. These can be understood independently of the rest of the paper. Section 3 treats the exceptional diagram automorphisms: their existence is hardly even an exercise.

Sections 4–6 give necessary background. Section 4 gives a little background on the Kac–Moody algebra $\mathfrak{g}_A$. Section 5 is mostly a review of results of Tits about a certain extension $W^* \subseteq \mathrm{Aut}(\mathfrak{g}_A)$ of the Weyl group W. But we also use a more recent result of Brink [1996] on Coxeter groups to describe generators for root stabilizers in W^*, and how they act on the corresponding root spaces (Theorem 5.7). Section 6 reviews Tits' definition of $\mathfrak{St}_A$ and its refinement by Morita and Rehmann.

Sections 7–9 are the technical heart of the paper, establishing Theorem 1.2. In Section 7 we define $\mathfrak{PSt}_A$ and then establish a presentation for it. We do this by defining a group functor $\mathfrak{G}_4$ by a presentation and proving $\mathfrak{PSt}_A \cong \mathfrak{G}_4$. As the notation suggests, this is the last in a chain of group functors $\mathfrak{G}_1, \dots, \mathfrak{G}_4$ that give successively better approximations to $\mathfrak{PSt}_A$. Lemma 7.4 and Theorems 7.5, 7.11 and 7.12 give "intrinsic" descriptions of $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ and $\mathfrak{G}_4$, the last one being the same as Theorem 1.2 above. See Section 2 for a quick overview of the meanings of these intermediate groups. The proof for $\mathfrak{G}_1$ is trivial, the proofs for $\mathfrak{G}_2$ and $\mathfrak{G}_3$ occupy Sections 8 and 9, and the proof for $\mathfrak{G}_4$ appears in Section 7.

Section 10 reviews work of Rémy [2002] on the adjoint representation of a Kac–Moody group, regarded as a representation of the corresponding Steinberg group. The definition of $\mathfrak{St}$ is as the direct limit of a family of unipotent groups, and we use the adjoint representation to show that the natural maps from these groups to $\mathfrak{St}$ are embeddings. This is necessary for the proof of Theorem 1.1 in Section 11. Finally, in Section 12 we discuss finite presentability of pre-Steinberg groups and Kac–Moody groups. In particular, we prove Theorems 1.4 and 1.5. The result for pre-Steinberg groups relies heavily on work of Splitthoff.

2. Examples

In this section we give our presentation of $\mathfrak{PSt}_A(R) = \mathfrak{St}_A(R)$ when R is a commutative ring and $A = A_1, A_2, B_2$ or G_2. It is mostly a writing-out of the general construction in Section 7. Because of the direct limit property of the pre-Steinberg group (Corollary 1.3), understanding these cases, together with

$$\mathfrak{PSt}_{A_1^2} = \mathfrak{PSt}_{A_1} \times \mathfrak{PSt}_{A_1},$$

is enough to present $\mathfrak{PSt}_A$ whenever A is 2-spherical. As usual, we are mainly interested in the presentation when $\mathfrak{PSt}$ and $\mathfrak{St}$ coincide. This happens in any of the cases of Theorem 1.1.

For generators we take formal symbols S, S', $X(t)$ and $X'(t)$, with t varying over R. The primed generators should be omitted in the A_1 case. We divide the relations into batches 0 through 4, with several intermediate groups having useful descriptions. At the end of the section we give an overview of these descriptions. For now we make only brief remarks. The batch 0 relations make the S's generate something like the Weyl group. The batch 1 relations make the $X(t)$'s additive in t. The batch 2 relations describe the interaction between the S's and the $X(t)$'s. These are the essentially new component of our approach to Steinberg groups. The batch 3 relations are Chevalley relations, describing commutators of conjugates of the $X(t)$'s by various words in the S's. Finally, the batch 4 relations are Steinberg's A_1-specific relations, and relations identifying the S's with the generators of the "Weyl group" inside the Steinberg group.

In the presentations we write $x \rightleftarrows y$ to indicate that x and y commute. The notation "(& primed)" next to a relation means to also impose the relation got from it by the typographical substitution $S \leftrightarrow S'$ and $X(t) \leftrightarrow X'(t)$.

Example 2.1 (A_1). We take generators S and $X(t)$, with t varying over R. There are no batch 0 or batch 3 relations:

Batch 1:
$$X(t)X(u) = X(t+u) \tag{2-1}$$
Batch 2:
$$S^2 \rightleftarrows X(t) \tag{2-2}$$
Batch 4:
$$S = \tilde{s}(1) \tag{2-3}$$
$$\tilde{h}(r) \cdot X(t) \cdot \tilde{h}(r)^{-1} = X(r^2 t) \tag{2-4}$$
$$\tilde{h}(r) \cdot SX(t)S^{-1} \cdot \tilde{h}(r)^{-1} = SX(r^{-2}t)S^{-1} \tag{2-5}$$

These relations hold for all $t, u \in R$ and all r in the unit group R^* of R, where

$$\tilde{s}(r) := X(r) \cdot SX(1/r)S^{-1} \cdot X(r),$$
$$\tilde{h}(r) := \tilde{s}(r)\tilde{s}(-1).$$

This is essentially Steinberg's original presentation (the group G' on page 78 of [Steinberg 1968]), with a slightly different generating set.

Example 2.2 (A_2). We take generators S, S', $X(t)$ and $X'(t)$, with t varying over R:

Batch 0:
$$SS'S = S'SS' \tag{2-6}$$
$$S^2 \cdot S' \cdot S^{-2} = S'^{-1} \qquad \text{(\& primed)} \tag{2-7}$$
Batch 1:
$$X(t)X(u) = X(t+u) \qquad \text{(\& primed)} \tag{2-8}$$
Batch 2:
$$S^2 \rightleftarrows X(t) \qquad \text{(\& primed)} \tag{2-9}$$
$$S^2 \cdot X'(t) \cdot S^{-2} = X'(-t) \qquad \text{(\& primed)} \tag{2-10}$$
$$SS'X(t) = X'(t)SS' \qquad \text{(\& primed)} \tag{2-11}$$

Batch 3: $$[X(t), X'(u)] = SX'(tu)S^{-1} \qquad \text{(\& primed)} \quad (2\text{-}12)$$

$$X(t) \rightleftarrows SX'(u)S^{-1} \qquad \text{(\& primed)} \quad (2\text{-}13)$$

Batch 4: $$S = X(1)SX(1)S^{-1}X(1) \qquad \text{(\& primed)} \quad (2\text{-}14)$$

As before, these relations hold for all $t, u \in R$. The diagram automorphism is given by $S \leftrightarrow S'$ and $X(t) \leftrightarrow X'(t)$.

Example 2.3 (B_2). We take generators S, S', $X(t)$ and $X'(t)$, with t varying over R. Unprimed letters correspond to the short simple root and primed letters to the long one:

Batch 0: $$SS'SS' = S'SS'S \quad (2\text{-}15)$$

$$S^2 \rightleftarrows S' \quad (2\text{-}16)$$

$$S'^2 \cdot S \cdot S'^{-2} = S^{-1} \quad (2\text{-}17)$$

Batch 1: $$X(t)X(u) = X(t+u) \qquad \text{(\& primed)} \quad (2\text{-}18)$$

Batch 2: $$S^2 \rightleftarrows X(t) \qquad \text{(\& primed)} \quad (2\text{-}19)$$

$$S^2 \rightleftarrows X'(t) \quad (2\text{-}20)$$

$$S'^2 \cdot X(t) \cdot S'^{-2} = X(-t) \quad (2\text{-}21)$$

$$SS'S \rightleftarrows X'(t) \qquad \text{(\& primed)} \quad (2\text{-}22)$$

Batch 3: $$SX'(t)S^{-1} \rightleftarrows S'X(u)S'^{-1} \quad (2\text{-}23)$$

$$X'(t) \rightleftarrows SX'(u)S^{-1} \quad (2\text{-}24)$$

$$[X(t), S'X(u)S'^{-1}] = SX'(-2tu)S^{-1} \quad (2\text{-}25)$$

$$[X(t), X'(u)] = S'X(-tu)S'^{-1} \cdot SX'(t^2u)S^{-1} \quad (2\text{-}26)$$

Batch 4: $$S = X(1)SX(1)S^{-1}X(1) \qquad \text{(\& primed)} \quad (2\text{-}27)$$

Example 2.4 (G_2). We take generators S, S', $X(t)$ and $X'(t)$ as in the B_2 case:

Batch 0: $$SS'SS'SS' = S'SS'SS'S \quad (2\text{-}28)$$

$$S^2 \cdot S' \cdot S^{-2} = S'^{-1} \qquad \text{(\& primed)} \quad (2\text{-}29)$$

Batch 1: $$X(t)X(u) = X(t+u) \qquad \text{(\& primed)} \quad (2\text{-}30)$$

Batch 2: $$S^2 \rightleftarrows X(t) \qquad \text{(\& primed)} \quad (2\text{-}31)$$

$$S^2 \cdot X'(t) \cdot S^{-2} = X'(-t) \qquad \text{(\& primed)} \quad (2\text{-}32)$$

$$SS'SS'S \rightleftarrows X'(t) \qquad \text{(\& primed)} \quad (2\text{-}33)$$

Batch 3: $$X'(t) \rightleftarrows S'SX'(u)S^{-1}S'^{-1} \quad (2\text{-}34)$$

$$SS'X(t)S'^{-1}S^{-1} \rightleftarrows S'SX'(u)S^{-1}S'^{-1} \quad (2\text{-}35)$$

$$SX'(t)S^{-1} \rightleftarrows S'X(u)S'^{-1} \tag{2-36}$$

$$[X'(t),\, SX'(u)S^{-1}] = S'SX'(tu)S^{-1}S'^{-1} \tag{2-37}$$

$$[X(t),\, SS'X(u)S'^{-1}S^{-1}] = SX'(3tu)S^{-1} \tag{2-38}$$

$$\begin{aligned}[X(t),\, S'X(u)S'^{-1}] = {} & SS'X(-2tu)S'^{-1}S^{-1} \\ & \cdot SX'(-3t^2u)S^{-1} \\ & \cdot S'SX'(-3tu^2)S^{-1}S'^{-1}\end{aligned} \tag{2-39}$$

$$\begin{aligned}[X(t),\, X'(u)] = {} & SS'X(t^2u)S'^{-1}S^{-1} \\ & \cdot S'X(-tu)S'^{-1} \cdot SX'(t^3u)S^{-1} \\ & \cdot S'SX'(-t^3u^2)S^{-1}S'^{-1}\end{aligned} \tag{2-40}$$

Batch 4: $$S = X(1)SX(1)S^{-1}X(1) \qquad \text{(\& primed)} \tag{2-41}$$

Now we explain the meaning of the batches. The group with generators S and S', modulo the batch 0 relations, is what we call $\widehat{W}$ in Section 7. It is an extension of the Weyl group W, slightly "more extended" than a better-known extension of W introduced by Tits [1966a]. We write W^* for Tits' extension and discuss it in Section 5. "More extended" means that $\widehat{W} \to W$ factors through W^*. The kernel of $W^* \to W$ is an elementary abelian 2-group, while the kernel of $\widehat{W} \to W$ can be infinite and nilpotent of class 2. These details are not needed for a general understanding.

The group with generators $X(t)$ and $X'(t)$, modulo the batch 1 relations, is what we call $\mathfrak{G}_1(R)$ in Section 7. It is just a free product of copies of the additive group of R, one for each simple root.

The group generated by S, S' and the $X(t)$ and $X'(t)$, modulo the relations from batches 0 through 2, is what we call $\mathfrak{G}_2(R)$ in Section 7. It is isomorphic to $(*_{\alpha\in\Phi} R) \rtimes \widehat{W}$ by Theorem 7.5, where Φ is the set of all roots. In fact, this theorem applies to any generalized Cartan matrix A. This is the main technical result of the paper, and the batch 2 relations are the main new ingredient in our treatment of the Steinberg groups. Furthermore, Theorem 7.5 generalizes to groups with a root group datum in the sense of [Tits 1992; Caprace and Rémy 2009]; see Remark 7.6. This should lead to generalizations of our results with such groups in place of Kac–Moody groups.

The batch 3 relations are a few of the Chevalley relations, written in a manner due to Demazure; see Section 7 for discussion and references. No batch 3 relations are present in the A_1 case. In the A_2, B_2 and G_2 cases, adjoining them yields $\mathfrak{St}(R) \rtimes \widehat{W}$, by Theorem 7.11. For any generalized Cartan matrix A, the corresponding presentation is called $\mathfrak{G}_3(R)$ in Section 7, and Theorem 7.11 asserts that it is isomorphic to $\mathfrak{P}\mathfrak{St}^{\mathrm{Tits}}(R) \rtimes \widehat{W}$. Here $\mathfrak{P}\mathfrak{St}^{\mathrm{Tits}}$ is the "pre-" version of Tits' version of the Steinberg group. See Section 7 for more details.

Adjoining the batch 4 relations yields the group called $\mathfrak{G}_4(R)$ in Section 7. In all four examples this coincides with $\mathfrak{St}_A(R)$. This result is really the concatenation of Theorem 7.12, that $\mathfrak{G}_4$ equals $\mathfrak{PSt}_A$ (for any A), with the isomorphism $\mathfrak{PSt}_A = \mathfrak{St}_A$ when A is spherical.

3. Diagram automorphisms

In this section we specialize our presentations of $\mathfrak{St}_{B_2}(R)$ and $\mathfrak{St}_{G_2}(R)$ when the ground ring R has characteristic 2 or 3, respectively. The exceptional diagram automorphisms are then visible. These results are not needed later in the paper.

We begin with the B_2 case, so assume $2 = 0$ in R. Then $X(t) = X(-t)$ for all t. In particular, the right side of (2-27) is its own inverse, so S and S' have order 2. The relations involving S^2 or S'^2 are therefore trivial and may be omitted. Also, the right side of (2-25) is the identity, so that (2-25) is the primed version of (2-24). In summary, the defining relations for $\mathfrak{St}$ are now the following, with t and u varying over R:

$$SS'SS' = S'SS'S \tag{3-1}$$

$$X(t)X(u) = X(t+u) \qquad \text{(\& primed)} \tag{3-2}$$

$$SS'S \rightleftarrows X'(t) \qquad \text{(\& primed)} \tag{3-3}$$

$$SX'(t)S^{-1} \rightleftarrows S'X(u)S'^{-1} \tag{3-4}$$

$$X'(t) \rightleftarrows SX'(u)S^{-1} \qquad \text{(\& primed)} \tag{3-5}$$

$$[X(t), X'(u)] = S'X(-tu)S'^{-1} \cdot SX'(t^2u)S^{-1} \tag{3-6}$$

$$S = X(1)SX(1)S^{-1}X(1) \qquad \text{(\& primed)} \tag{3-7}$$

Theorem 3.1. *Suppose R is a ring of characteristic 2. Then the map $S \leftrightarrow S'$, $X'(t) \mapsto X(t) \mapsto X'(t^2)$ extends to an endomorphism ϕ of $\mathfrak{St}_{B_2}(R)$. If R is a perfect field then ϕ is an automorphism.*

Proof. One must check that each relation (3-1)–(3-7) remains true after the substitution $S \leftrightarrow S'$, $X'(t) \mapsto X(t) \mapsto X'(t^2)$. It is easy to check that every relation maps to its primed form (except that some t's and u's are replaced by their squares). The relations (3-1), (3-4) and (3-6) are their own primed forms. Only (3-6) deserves any comment: we must check the identity

$$[X'(t^2), X(u)] = SX'(t^2u^2)S^{-1} \cdot S'X(t^2u)S'^{-1}$$

in $\mathfrak{St}$. The left side equals $[X(u), X'(t^2)]^{-1}$. The identity follows by expanding the commutator using (3-6).

Now suppose R is a perfect field. By a similar argument, one can check that there is an endomorphism ψ of $\mathfrak{St}$ that fixes S and S', and for each $t \in R$ sends $X(t)$

to $X(\sqrt{t})$ and $X'(t)$ to $X'(\sqrt{t})$. (Because R is a perfect field of characteristic 2, square roots exist and are unique, and $t \mapsto \sqrt{t}$ is a field automorphism.) Since $\psi \circ \phi \circ \phi$ sends each generator to itself, ϕ and ψ must be isomorphisms. □

Now we consider the G_2 case, so suppose $3 = 0$ in R. The main simplifications of Section 2's presentation of $\mathfrak{St}$ are that the right side of (2-38) is the identity, so (2-38) is the primed version of (2-34), and that the last two terms on the right of (2-39) are trivial, so that (2-39) is the primed version of (2-37). So the relations simplify to:

$$SS'SS'SS' = S'SS'SS'S \tag{3-8}$$

$$S^2 \cdot S' \cdot S^{-2} = S'^{-1} \qquad \text{(\& primed)} \tag{3-9}$$

$$X(t)X(u) = X(t+u) \qquad \text{(\& primed)} \tag{3-10}$$

$$S^2 \rightleftarrows X(t) \qquad \text{(\& primed)} \tag{3-11}$$

$$S^2 \cdot X'(t) \cdot S^{-2} = X'(-t) \qquad \text{(\& primed)} \tag{3-12}$$

$$SS'SS'S \rightleftarrows X'(t) \qquad \text{(\& primed)} \tag{3-13}$$

$$X'(t) \rightleftarrows S'SX'(u)S^{-1}S'^{-1} \qquad \text{(\& primed)} \tag{3-14}$$

$$SS'X(t)S'^{-1}S^{-1} \rightleftarrows S'SX'(u)S^{-1}S'^{-1} \tag{3-15}$$

$$SX'(t)S^{-1} \rightleftarrows S'X(u)S'^{-1} \tag{3-16}$$

$$[X'(t), SX'(u)S^{-1}] = S'SX'(tu)S^{-1}S'^{-1} \qquad \text{(\& primed)} \tag{3-17}$$

$$\begin{aligned}[X(t), X'(u)] = SS'X(t^2u)S'^{-1}S^{-1} &\\ \cdot S'X(-tu)S'^{-1} \cdot SX'(t^3u)S^{-1} &\\ \cdot S'SX'(-t^3u^2)S^{-1}S'^{-1} &\end{aligned} \tag{3-18}$$

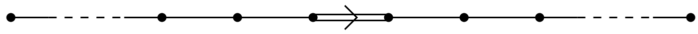
$$S = X(1)SX(1)S^{-1}X(1) \qquad \text{(\& primed)} \tag{3-19}$$

The following theorem is proven just like the previous one.

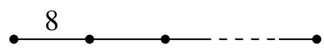
Theorem 3.2. *Suppose R is a ring of characteristic 3. Then the map $S \leftrightarrow S'$, $X'(t) \mapsto X(t) \mapsto X'(t^3)$ extends to an endomorphism ϕ of $\mathfrak{St}_{G_2}(R)$. If R is a perfect field then ϕ is an automorphism.* □

The exceptional diagram automorphisms lead to the famous Suzuki and Ree groups. If R is the finite field $\mathbb{F}_q$ where $q = 2^{\text{odd}}$, then the Frobenius automorphism of R (namely squaring) is the square of a field automorphism ξ. Writing ξ also for the induced automorphism of $\mathfrak{St}_{B_2}(R)$, the Suzuki group is defined as the subgroup where ξ agrees with ϕ. The same construction with F_4 in place of B_2 yields the large Ree groups, and in characteristic 3 with G_2 yields the small Ree groups. These groups are "like" groups of Lie type in that they admit root group data in the sense of [Tits 1992] or [Caprace and Rémy 2009], but they are not algebraic groups.

Hée generalized this [2008]. He showed that when a group with a root group datum admits two automorphisms that permute the simple roots' root groups, and satisfy some other natural conditions, then the subgroup where they coincide also admits a root group datum. Furthermore, the Weyl group for the subgroup may be computed in a simple way from the Weyl group for the containing group. For example, over $\mathbb{F}_q$ with $q = 2^{\text{odd}}$, the Kac–Moody group



contains a Kac–Moody-like analogue of the Suzuki groups. By Hée's theorem, its Weyl group is



Hée [1990] constructs diagram automorphisms in a different way than we do, and discusses the case "G_4" in some detail.

4. The Kac–Moody algebra

In this section we begin the technical part of the paper, by recalling the Kac–Moody algebra and some notation from [Tits 1987]. All group actions are on the left. We will use the following general notation:

$\langle\, ,\, \rangle$	a bilinear pairing
$\langle \cdot \rangle$	group generated by the elements enclosed
$\langle \cdot \mid \cdot \rangle$	a group presentation
$[x, y]$	$xyx^{-1}y^{-1}$ if x and y are group elements
$*$	free product of groups (possibly with amalgamation)

The Steinberg group is built from a generalized Cartan matrix A, for which we will use the following notation:

I	an index set (the nodes of the Dynkin diagram)
i, j	will always indicate elements of I
$A=(A_{ij})$	a generalized Cartan matrix: an integer matrix satisfying $A_{ii} = 2$, $A_{ij} \leq 0$ if $i \neq j$, and $A_{ij} = 0 \iff A_{ji} = 0$
m_{ij}	numerical edge labels of the Dynkin diagram: $m_{ij} = 2, 3, 4, 6$ or ∞, according to whether $A_{ij}A_{ji} = 0, 1, 2, 3$ or ≥ 4, except that $m_{ii} = 1$
W	the Coxeter group $\langle s_{i\in I} \mid (s_i s_j)^{m_{ij}} = 1 \text{ if } m_{ij} \neq \infty \rangle$
$\mathbb{Z}^I$	the free abelian group with basis $\alpha_{i\in I}$, called the simple roots; W acts on $\mathbb{Z}^I$ by $s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i$ (this action is faithful by the theory of the Tits cone [Bourbaki 2002, Chapter V, §4.4])
Φ	the set of (real) roots: all $w\alpha_i$ with $w \in W$ and $i \in I$

The Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}_A$ associated to A means the complex Lie algebra with generators $e_{i\in I}$, $f_{i\in I}$, $\bar{h}_{i\in I}$ and defining relations

$$[\bar{h}_i, e_j] = A_{ij} e_j, \quad [\bar{h}_i, f_i] = -A_{ij} f_j, \quad [\bar{h}_i, \bar{h}_j] = 0, \quad [e_i, f_i] = -\bar{h}_i,$$

and, for $i \neq j$,

$$[e_i, f_j] = 0, \quad (\operatorname{ad} e_i)^{1-A_{ij}}(e_j) = (\operatorname{ad} f_i)^{1-A_{ij}}(f_j) = 0.$$

(Note: $(\operatorname{ad} x)(y)$ means $[x, y]$. Also, Tits' generators differ from Kac's generators [1990] by a sign on f_i.) For any i the linear span of e_i, f_i and $\bar{h}_i$ is isomorphic to $\mathfrak{sl}_2\mathbb{C}$, via

$$e_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_i = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \bar{h}_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4-1}$$

We equip $\mathfrak{g}$ with a grading by $\mathbb{Z}^I$, with $\bar{h}_i \in \mathfrak{g}_0$, $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$. For $\alpha \in \mathbb{Z}^I$ we refer to $\mathfrak{g}_\alpha$ as its root space, and abbreviate $\mathfrak{g}_{\alpha_i}$ to $\mathfrak{g}_i$. We follow [Tits 1987] in saying "root" for "real root" (meaning an element of Φ). Imaginary roots play no role in this paper.

5. The extension $W^* \subseteq \operatorname{Aut}\mathfrak{g}$ of the Weyl group

The Weyl group W does not necessarily act on $\mathfrak{g}$, but a certain extension of it called W^* does. In this section we review its basic properties. The results through Theorem 5.5 are due to Tits. The last result is new: it describes the root stabilizers in W^*. The proof relies on Brink's study [1996] of reflection centralizers in Coxeter groups, in the form given in [Allcock 2013].

It is standard [Kac 1990, Lemma 3.5] that $\operatorname{ad} e_i$ and $\operatorname{ad} f_i$ are locally nilpotent on $\mathfrak{g}$, so their exponentials are automorphisms of $\mathfrak{g}$. Furthermore,

$$(\exp\operatorname{ad} e_i)(\exp\operatorname{ad} f_i)(\exp\operatorname{ad} e_i) = (\exp\operatorname{ad} f_i)(\exp\operatorname{ad} e_i)(\exp\operatorname{ad} f_i). \tag{5-1}$$

We write s_i^* for this element of $\operatorname{Aut}\mathfrak{g}$ and W^* for $\langle s^*_{i\in I}\rangle \subseteq \operatorname{Aut}\mathfrak{g}$. One shows [Kac 1990, Lemma 3.8] that $s_i^*(\mathfrak{g}_\alpha) = \mathfrak{g}_{s_i(\alpha)}$ for all $\alpha \in \mathbb{Z}^I$. This defines a W^*-action on $\mathbb{Z}^I$, with s_i^* acting as s_i. Since W acts faithfully on $\mathbb{Z}^I$ this yields a homomorphism $W^* \to W$. Using W^*, the general theory [Kac 1990, Proposition 5.1] shows that $\mathfrak{g}_\alpha$ is 1-dimensional for any $\alpha \in \Phi$.

Let $\mathbb{Z}^{I\vee}$ be the free abelian group with basis the formal symbols $\alpha^\vee_{i\in I}$ and define a bilinear pairing $\mathbb{Z}^{I\vee} \times \mathbb{Z}^I \to \mathbb{Z}$ by $\langle \alpha_i^\vee, \alpha_j\rangle = A_{ij}$. We define an action of W on $\mathbb{Z}^{I\vee}$ by $s_i(\alpha_j^\vee) = \alpha_j^\vee - A_{ji}\alpha_i^\vee$. One can check that this action satisfies $\langle w\alpha^\vee, w\beta\rangle = \langle \alpha^\vee, \beta\rangle$. There is a homomorphism $\operatorname{Ad} : \mathbb{Z}^{I\vee} \to \operatorname{Aut}\mathfrak{g}$, with $\operatorname{Ad}(\alpha^\vee)$ acting on $\mathfrak{g}_\beta$ by $(-1)^{\langle\alpha^\vee,\beta\rangle}$, where $\beta \in \mathbb{Z}^I$. The proof of the next lemma is easy and standard.

Lemma 5.1. $\mathrm{Ad} : \mathbb{Z}^{I\vee} \to \mathrm{Aut}\,\mathfrak{g}$ *is W^*-equivariant in the sense that*

$$w^* \cdot \mathrm{Ad}(\alpha^\vee) \cdot w^{*-1} = \mathrm{Ad}(w\alpha^\vee),$$

where $\alpha^\vee \in \mathbb{Z}^{I\vee}$ and w is the image in W of $w^* \in W^*$. □

Lemma 5.2. *The following identities hold in* $\mathrm{Aut}\,\mathfrak{g}$:

(i) $s_i^{*2} = \mathrm{Ad}(\alpha_i^\vee)$.

(ii) $s_i^*(s_j^*)^2 s_i^{*-1} = (s_j^*)^2(s_i^*)^{-2A_{ji}}$.

Proof sketch. (i) Identifying the span of $e_i, f_i, \bar{h}_i$ with $\mathfrak{sl}_2\mathbb{C}$ as in (4-1) identifies s_i^{*2} with $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2\,\mathbb{C}$. One uses the representation theory of $\mathrm{SL}_2\,\mathbb{C}$ to see how this acts on $\mathfrak{g}$'s weight spaces.

(ii) Use (i) to identify s_j^{*2} with $\mathrm{Ad}(\alpha_j^\vee)$, then Lemma 5.1 to identify $s_i^*\,\mathrm{Ad}(\alpha_j^\vee)s_i^{*-1}$ with $\mathrm{Ad}(s_i(\alpha_j^\vee))$, then the formula defining $s_i(\alpha_j^\vee)$, and finally (i) again to convert back to s_i^{*2} and s_j^{*2}. □

To understand the relations satisfied by the s_i^* it will be useful to have a characterization of them in terms of the choice of e_i (together with the grading on $\mathfrak{g}$). This is part of Tits' "trijection" [1966b, §1.1]. In the notation of the next lemma, s_i^* is $s_{e_i}^*$ (or equally well $s_{f_i}^*$).

Lemma 5.3. *If $\alpha \in \Phi$ and $e \in \mathfrak{g}_\alpha - \{0\}$ then there exists a unique $f \in \mathfrak{g}_{-\alpha}$ such that*

$$s_e^* := (\exp \mathrm{ad}\, e)(\exp \mathrm{ad}\, f)(\exp \mathrm{ad}\, e)$$

exchanges $\mathfrak{g}_{\pm\alpha}$. Furthermore, s_e^* coincides with s_f^* and exchanges e and f. Finally, if $\phi \in \mathrm{Aut}\,\mathfrak{g}$ permutes the $\mathfrak{g}_{\beta\in\Phi}$ then $\phi s_e^* \phi^{-1} = s_{\phi(e)}^*$. □

Lemma 5.4. (i) *If $m_{ij} = 3$ then $s_j^* s_i^*(e_j) = e_i$.*

(ii) *If $m_{ij} = 2, 4$ or 6 then e_j is fixed by s_i^*, $s_i^*s_j^*s_i^*$ or $s_i^*s_j^*s_i^*s_j^*s_i^*$, respectively.*

Proof. Part (i) follows from direct calculation in $\mathfrak{sl}_3\mathbb{C}$. In the $m_{ij} = 2$ case of (ii) we have $(\mathrm{ad}\, e_i)(e_j) = (\mathrm{ad}\, f_i)(e_j) = 0$, and $s_i^*(e_j) = e_j$ follows immediately. The remaining cases involve careful tracking of signs. We will write $(\mathfrak{sl}_2\mathbb{C})_i$ for the span of $e_i, f_i, \bar{h}_i$.

If $m_{ij} = 4$ then $\{A_{ij}, A_{ji}\} = \{-1, -2\}$ and α_i and α_j are simple roots for a B_2 root system. Using Lemma 5.3,

$$s_i^*s_j^*s_i^*(e_j) = s_i^*s_{e_j}^*s_i^{*-1}s_i^{*2}(e_j) = s_{s_i^*(e_j)}^*\big((\mathrm{Ad}\,\alpha_i^\vee)(e_j)\big) = (-1)^{A_{ij}} s_{s_i^*(e_j)}^*(e_j). \quad (5\text{-}2)$$

Suppose first that $A_{ij} = -2$. Then α_i is the short simple root, α_j the long one, and $s_i(\alpha_j)$ is a long root orthogonal to α_j. We have

$$s_{s_i^*(e_j)}^* = (\exp \mathrm{ad}\, s_i^*(e_j))(\exp \mathrm{ad}\, s_i^*(f_j))(\exp \mathrm{ad}\, s_i^*(e_j)) \in \exp \mathrm{ad}\big(s_i^*((\mathfrak{sl}_2\mathbb{C})_j)\big).$$

Now, $s_i^*((\mathfrak{sl}_2\mathbb{C})_j)$ annihilates $\mathfrak{g}_j$ because its root string through α_j has length 1. So $s^*_{s_i^*(e_j)}$ fixes e_j and (5-2) becomes

$$s_i^* s_j^* s_i^*(e_j) = (-1)^{A_{ij}} e_j = (-1)^{-2} e_j = e_j.$$

On the other hand, if $A_{ij} = -1$ then α_j and $s_i(\alpha_j)$ are orthogonal short roots. Now the root string through α_j for $s_i^*((\mathfrak{sl}_2\mathbb{C})_j)$ has length 3, so the $s_i^*((\mathfrak{sl}_2\mathbb{C})_j)$-module generated by e_j is a copy of the adjoint representation. In particular,

$$s^*_{s_i^*(e_j)} = s_i^* s_j^* s_i^{*-1}$$

acts on $\mathfrak{g}_j$ by the same scalar as on the Cartan subalgebra $s_i^*(\mathbb{C}\bar{h}_j)$ of $s_i^*((\mathfrak{sl}_2\mathbb{C})_j)$. This is the same scalar by which s_j^* acts on $\mathbb{C}\bar{h}_j$, which is -1. So $s^*_{s_i^*(e_j)}$ negates e_j and (5-2) reads

$$s_i^* s_j^* s_i^*(e_j) = (-1)^{A_{ij}}(-e_j) = (-1)^{-1}(-e_j) = e_j.$$

Now suppose $m_{ij} = 6$, so that $\{A_{ij}, A_{ji}\} = \{-1, -3\}$, α_i and α_j are simple roots for a G_2 root system, and $s_i s_j(\alpha_i) \perp \alpha_j$. Then

$$\begin{aligned} s_i^* s_j^* s_i^* s_j^* s_i^*(e_j) &= (s_i^* s_j^* s_{e_i}^* s_j^{*-1} s_i^{*-1}) s_i^* s_j^* s_j^* s_i^*(e_j) \\ &= s^*_{s_i^* s_j^*(e_i)} \circ (s_i^* s_j^{*2} s_i^{*-1}) \circ s_i^{*2}(e_j) \\ &= s^*_{s_i^* s_j^*(e_i)} \circ s_j^{*2} s_i^{*-2A_{ji}} \circ s_i^{*2}(e_j) \\ &= s^*_{s_i^* s_j^*(e_i)} \circ s_j^{*2} s_i^{*4 \text{ or } 8}(e_j) \\ &= s^*_{s_i^* s_j^*(e_i)}(e_j). \end{aligned}$$

The root string through α_j for $s_i^* s_j^*((\mathfrak{sl}_2\mathbb{C})_i)$ has length 1, so arguing as in the B_2 case shows that $s^*_{s_i^* s_j^*(e_i)}$ fixes e_j. □

Theorem 5.5 [Tits 1966a, §4.6]. *The s_i^* satisfy the Artin relations of M. That is, if $m_{ij} \neq \infty$ then $s_i^* s_j^* \cdots = s_j^* s_i^* \cdots$, where there are m_{ij} factors on each side, alternately s_i^* and s_j^*.*

Proof. For $m_{ij} = 3$ we start with $e_j = s_i^* s_j^*(e_i)$ from Lemma 5.4(i). Using Lemma 5.3 yields

$$s_j^* = s_{e_j}^* = s^*_{s_i^* s_j^*(e_i)} = s_i^* s_j^* s_{e_i}^* s_j^{*-1} s_i^{*-1} = s_i^* s_j^* s_i^* s_j^{*-1} s_i^{*-1}.$$

The other cases are the same. □

We will need to understand the W^*-stabilizer of a simple root α_i and how it acts on $\mathfrak{g}_i$. The first step is to quote from [Allcock 2013] a refinement of a theorem of Brink [1996] on reflection centralizers in Coxeter groups. Then we will "lift" this result to W^* by keeping track of signs.

Both theorems refer to the "odd Dynkin diagram" Δ^{odd}, which means the graph with vertex set I, where vertices i and j are joined just if $m_{ij}=3$. For γ an edge path in Δ^{odd}, with $i_0,\ldots,i_n$ the vertices along it, we define

$$p_\gamma := (s_{i_{n-1}}s_{i_n})(s_{i_{n-2}}s_{i_{n-1}})\cdots(s_{i_1}s_{i_2})(s_{i_0}s_{i_1}). \tag{5-3}$$

(If γ has length 0 then we set $p_\gamma=1$.) For $i\in I$ we write Δ_i^{odd} for its component of Δ^{odd}.

Theorem 5.6 [Allcock 2013, Corollary 8]. *Suppose $i\in I$, Z is a set of closed edge paths based at i that generate $\pi_1(\Delta_i^{\text{odd}},i)$, and δ_j is an edge path in Δ_i^{odd} from i to j, for each vertex j of Δ_i^{odd}. For each such j and each $k\in I$ with m_{jk} finite and even, define*

$$r_{jk} := p_{\delta_j}^{-1}\cdot\begin{Bmatrix} s_k \\ s_ks_js_k \\ s_ks_js_ks_js_k\end{Bmatrix}\cdot p_{\delta_j} \tag{5-4}$$

according to whether $m_{jk}=2$, 4 or 6. Then the W-stabilizer of the simple root α_i is generated by the r_{jk} and the $p_{z\in Z}$. □

It is easy to see that the r_{jk} and p_z stabilize α_i. In fact, this is the "image under $W^*\to W$" of the corresponding part of the next theorem.

Theorem 5.7. *Suppose i, Z and the δ_j are as in Theorem 5.6. Define p_γ^* and r_{jk}^* by attaching $*$ to each s, p and r in* (5-3) *and* (5-4)*. Then the $p_{z\in Z}^*$ and r_{jk}^* fix e_i, and together with the $s_{l\in I}^{*2}$ they generate the W^*-stabilizer of α_i.* (*By Lemma 5.2*(i), *s_l^{*2} acts on e_i by $(-1)^{A_{li}}$*).

Proof. The W^*-stabilizer of α_i is generated by $\ker(W^*\to W)$ and any set of elements of W^* whose projections to W generate the W-stabilizer of α_i. Now, the s_i^{*2} normally generate the kernel because of the Artin relations. Lemma 5.2(ii) shows that the subgroup they generate is normal, hence equal to this kernel. Since the p^*'s and r^*'s project to the p's and r's of Theorem 5.6, our generation claim follows from that theorem. To see that the p_z^*'s fix e_i, apply Lemma 5.4(i) repeatedly. The same argument proves $p_{\delta_j}^*(e_i)=e_j$. Then using Lemma 5.4(ii) shows that e_j is fixed by s_k^*, $s_k^*s_j^*s_k^*$ or $s_k^*s_j^*s_k^*s_j^*s_k^*$ according to whether m_{jk} is 2, 4 or 6. Applying $p_{\delta_j}^{*-1}$ sends e_j back to e_i, proving $r_{jk}^*(e_i)=e_i$. □

6. The Steinberg group $\mathfrak{St}$

In this section we give an overview of the Steinberg group $\mathfrak{St}_A$, as defined by Tits [1987] and refined by Morita and Rehmann [1990]. The purpose is to be able to compare the pre-Steinberg group $\mathfrak{PSt}_A$ (see Section 7) with $\mathfrak{St}_A$. For example, Theorem 1.1 gives many cases in which the natural map $\mathfrak{PSt}_A(R)\to\mathfrak{St}_A(R)$ is an isomorphism.

The Morita–Rehmann definition is got from Tits' definition by imposing some additional relations. These are also due to Tits, but he imposed them only later in his construction, when defining Kac–Moody groups in terms of $\mathfrak{St}_A$. In the few places where we need to distinguish between the definitions, we will write $\mathfrak{St}_A^{\mathrm{Tits}}$ for Tits' version and $\mathfrak{St}_A$ for the Morita–Rehmann version. In the rest of this section we will regard A as fixed and omit it from the subscripts.

$\mathfrak{Add}$ denotes the additive group, regarded as a group scheme over $\mathbb{Z}$. That is, it is the functor assigning to each commutative ring R its underlying abelian group. The Lie algebra of $\mathfrak{Add}$ is canonically isomorphic to $\mathbb{Z}$.

For each $\alpha \in \Phi$, $\mathfrak{g}_\alpha \cap W^*(\{e_{i\in I}\})$ consists of either one vector or two antipodal vectors. This is [Tits 1987, (3.3.2)] and its following paragraph, which relies on [Tits 1974, §13.31]. Alternately, it follows from our Theorem 5.7. We write $\mathfrak{g}_{\alpha,\mathbb{Z}}$ for the $\mathbb{Z}$-span in $\mathfrak{g}_\alpha$ of this element or antipodal pair, and E_α for the set of its generators (a set of size 2). The symbol e will always indicate an element of some E_α. We define $\mathfrak{U}_\alpha$ as the group scheme over $\mathbb{Z}$ which is isomorphic to $\mathfrak{Add}$ and has Lie algebra $\mathfrak{g}_{\alpha,\mathbb{Z}}$. That is, $\mathfrak{U}_\alpha$ is the functor assigning to each commutative ring R the abelian group $\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes R \cong R$. For $i \in I$ we abbreviate $\mathfrak{U}_{\pm\alpha_i}$ to $\mathfrak{U}_{\pm i}$.

If $\alpha \in \Phi$ and $e \in E_\alpha$ then we define $\mathfrak{x}_e$ as the isomorphism $\mathfrak{Add} \to \mathfrak{U}_\alpha$, whose corresponding Lie algebra isomorphism identifies $1 \in \mathbb{Z}$ with $e \in \mathfrak{g}_{\alpha,\mathbb{Z}}$. For fixed R this amounts to

$$\mathfrak{x}_e(t) := e \otimes t \in \mathfrak{g}_{\alpha,\mathbb{Z}} \otimes R = \mathfrak{U}_\alpha.$$

If $R = \mathbb{R}$ or $\mathbb{C}$ then one may think of $\mathfrak{x}_e(t)$ as $\exp(te)$. For $i \in I$ we abbreviate $\mathfrak{x}_{e_i}$ to $\mathfrak{x}_i$ and $\mathfrak{x}_{f_i}$ to $\mathfrak{x}_{-i}$.

Tits calls a set of roots $\Psi \subseteq \Phi$ prenilpotent if some chamber in the open Tits cone lies on the positive side of all their mirrors and some other chamber lies on the negative side of all of them. (Equivalently, some element of W sends Ψ into the set of positive roots and some other element of W sends Ψ into the set of negative roots.) It follows that Ψ is finite. If Ψ is also closed under addition then it is called nilpotent. In this case $\mathfrak{g}_\Psi := \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ is a nilpotent Lie algebra [Tits 1987, p. 547].

Lemma 6.1 [Tits 1987, §3.4]. *If $\Psi \subseteq \Phi$ is a nilpotent set of roots, then there is a unique unipotent group scheme $\mathfrak{U}_\Psi$ over $\mathbb{Z}$ with these properties*:

(i) *$\mathfrak{U}_\Psi$ contains all the $\mathfrak{U}_{\alpha\in\Psi}$.*

(ii) *$\mathfrak{U}_\Psi(\mathbb{C})$ has Lie algebra $\mathfrak{g}_\Psi$.*

(iii) *For any ordering on Ψ, the product morphism $\prod_{\alpha\in\Psi} \mathfrak{U}_\alpha \to \mathfrak{U}_\Psi$ is an isomorphism of the underlying schemes.* □

Tits' version $\mathfrak{St}^{\mathrm{Tits}}$ of the Steinberg group functor is defined as follows. For each prenilpotent pair α, β of roots, $\theta(\alpha, \beta)$ is defined as $(\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Phi$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Consider the groups $\mathfrak{U}_{\theta(\alpha,\beta)}$ with $\{\alpha, \beta\}$ varying over all prenilpotent pairs. If $\gamma \in \theta(\alpha, \beta)$ then there is a natural injection $\mathfrak{U}_\gamma \to \mathfrak{U}_{\theta(\alpha,\beta)}$, yielding a diagram of inclusions of group functors. $\mathfrak{St}^{\mathrm{Tits}}$ is defined as the direct limit of this diagram. Every automorphism of $\mathfrak{g}$ that permutes the subgroups $\mathfrak{g}_{\alpha,\mathbb{Z}}$ induces an automorphism of the diagram of inclusions of group functors, hence an automorphism of $\mathfrak{St}^{\mathrm{Tits}}$. In particular, W^* acts on $\mathfrak{St}^{\mathrm{Tits}}$.

As Tits points out, a helpful but less canonical way to think about $\mathfrak{St}^{\mathrm{Tits}}(R)$ is to begin with the free product $*_{\alpha\in\Phi}\,\mathfrak{U}_\alpha(R)$ and impose relations of the form

$$[\mathfrak{x}_{e_\alpha}(t), \mathfrak{x}_{e_\beta}(u)] = \prod_{\gamma=m\alpha+n\beta} \mathfrak{x}_{e_\gamma}(C_{\alpha\beta\gamma} t^m u^n) \tag{6-1}$$

for each prenilpotent pair $\alpha, \beta \in \Phi$. Here $\gamma = m\alpha + n\beta$ runs over $\theta(\alpha, \beta) - \{\alpha, \beta\}$, so in particular m and n are positive integers. Also, e_α, e_β and the various e_γ lie in E_α, E_β and the various E_γ, and must be chosen before the relation can be written down explicitly. The $C_{\alpha\beta\gamma}$ are integers that depend on the position of γ relative to α and β, the choices of e_α, e_β and the e_γ, and the ordering of the product; compare (3) of [Tits 1987]. Usually (6-1) is called "the Chevalley relation of α and β". It is really a family of relations parametrized by t and u, and (strictly speaking) not defined without the various choices being fixed.

Unfortunately, Tits' version of the Steinberg group is different from Steinberg's original group when the Dynkin diagram has A_1 components. Therefore, we follow [Morita and Rehmann 1990] in defining the Steinberg group functor $\mathfrak{St}$. That is, we impose the additional relations (6-5), which correspond to the relations (B′) in [Steinberg 1968] or [Morita and Rehmann 1990]. These relations make the "maximal torus" and "Weyl group" act on the root groups $\mathfrak{U}_\alpha$ in the expected manner. If A is 2-spherical without A_1 components then the Morita–Rehmann relations already hold in $\mathfrak{St}^{\mathrm{Tits}}$ and this part of the construction can be skipped, by [Tits 1987, (a$_4$), p. 550].

The relators involve the following elements of $\mathfrak{St}^{\mathrm{Tits}}$. If $\alpha \in \Phi$ and $e \in E_\alpha$ then recall from Lemma 5.3 that there is a distinguished $f \in E_{-\alpha}$. As the notation suggests, if $e = e_i$ then $f = f_i$. For any $r \in R^*$ we define

$$\tilde{s}_e(r) := \mathfrak{x}_e(r)\mathfrak{x}_f(1/r)\mathfrak{x}_e(r), \tag{6-2}$$

$$\tilde{h}_e(r) := \tilde{s}_e(r)\tilde{s}_e(-1). \tag{6-3}$$

We abbreviate special cases in the usual way: $\tilde{h}_{\pm i}(r)$ for $\tilde{h}_{e_i}(r)$ and $\tilde{h}_{f_i}(r)$, $\tilde{s}_{\pm i}(r)$ for $\tilde{s}_{e_i}(r)$ and $\tilde{s}_{f_i}(r)$, $\tilde{s}_{\pm i}$ for $\tilde{s}_{\pm i}(1)$, and $\tilde{s}_e$ for $\tilde{s}_e(1)$. It is useful to note several immediate consequences of the definitions: $\tilde{s}_e(-r) = \tilde{s}_e(r)^{-1}$, $\tilde{h}_e(1) = 1$, and

$$\tilde{s}_e(r)\tilde{s}_e(r')^{-1} = \tilde{h}_e(r)\tilde{h}_e(r')^{-1}. \tag{6-4}$$

Conceptually, the relations we will impose on $\mathfrak{St}^{\mathrm{Tits}}$ to get $\mathfrak{St}$ force the conjugation maps of the various $\tilde{s}_e(r)$ to be the same as certain automorphisms of $\mathfrak{St}^{\mathrm{Tits}}$. So we will describe these automorphisms and then state the relations.

Recall from Lemma 5.1 and its preceding remarks that $\mathbb{Z}^{I\vee}$ is the free abelian group generated by formal symbols $\alpha^\vee_{i\in I}$. Also, the bilinear pairing $\mathbb{Z}^{I\vee} \times \mathbb{Z}^I \to \mathbb{Z}$ given by $\langle \alpha_i^\vee, \alpha_j\rangle = A_{ij}$ is W-invariant. We defined a map $\mathrm{Ad} : \mathbb{Z}^{I\vee} \to \mathrm{Aut}\,\mathfrak{g}$, which we generalize to $\mathrm{Ad} : (R^* \otimes \mathbb{Z}^{I\vee}) \to \mathrm{Aut}(\ast_{\alpha\in\Phi}\,\mathfrak{U}_\alpha)$ as follows. For any $\alpha^\vee \in \mathbb{Z}^{I\vee}$, $r \in R^*$ and $\beta \in \Phi$, $\mathrm{Ad}(r \otimes \alpha^\vee)$ acts on $\mathfrak{U}_\beta \cong R$ by multiplication by $r^{\langle \alpha^\vee, \beta\rangle} \in R^*$. One recovers the original Ad by taking $r = -1$.

The Chevalley relations have a homogeneity property, namely that $\mathrm{Ad}(r \otimes \alpha^\vee)$ permutes them. This is most visible when they are stated in the form (6-1). Therefore, the action Ad of $R^* \otimes \mathbb{Z}^{I\vee}$ on $\ast_{\alpha\in\Phi}\,\mathfrak{U}_\alpha$ descends to an action on $\mathfrak{St}^{\mathrm{Tits}}(R)$.

It is standard that there is a W-equivariant bijection $\alpha \mapsto \alpha^\vee$ from the roots $\Phi \subseteq \mathbb{Z}^I$ to their corresponding coroots in $\mathbb{Z}^{I\vee}$. As the notation suggests, the coroots corresponding to the simple roots α_i are our basis $\alpha_i^\vee$ for $\mathbb{Z}^{I\vee}$. In view of W-equivariance this determines the bijection uniquely. For $\alpha \in \Phi$ and $r \in R^*$ we define $h_\alpha(r) \in \mathrm{Aut}\,\mathfrak{St}^{\mathrm{Tits}}(R)$ as $\mathrm{Ad}(r \otimes \alpha^\vee)$. As usual, we abbreviate $h_{\alpha_i}(r)$ to $h_i(r)$.

We define the Steinberg group functor $\mathfrak{St}$ as follows. Informally, $\mathfrak{St}(R)$ is the quotient of $\mathfrak{St}^{\mathrm{Tits}}(R)$ got by forcing every $\tilde{s}_e(r)$ to act on every $\mathfrak{U}_\beta(R)$ by $h_\alpha(r) \circ s_e^*$, where α is the root with $e \in E_\alpha$. Formally, it is the quotient by the subgroup normally generated by the elements

$$\tilde{s}_e(r) u \tilde{s}_e(r)^{-1} \cdot \big((h_\alpha(r) \circ s_e^*)(u)\big)^{-1} \tag{6-5}$$

as α, β vary over Φ, e over E_α, r over R^*, and u over $\mathfrak{U}_\beta(R)$. This set of relators is visibly W^*-invariant, so W^* acts on $\mathfrak{St}$.

Remark 6.2. Because $\tilde{s}_e(r) = \tilde{h}_e(r)\tilde{s}_e$, an equivalent way to impose the relations (6-5) is by quotienting by the subgroup of $\mathfrak{St}^{\mathrm{Tits}}(R)$ normally generated by all

$$\tilde{s}_e u \tilde{s}_e^{-1} \cdot s_e^*(u)^{-1} \tag{6-6}$$

$$\tilde{h}_e(r) u \tilde{h}_e(r)^{-1} \cdot (h_\alpha(r)(u))^{-1} \tag{6-7}$$

Remark 6.3. Our relations differ slightly from the relations (B′) of [Morita and Rehmann 1990], because we follow Tits' convention for the presentation of $\mathfrak{g}$, while they follow Kac's convention (see Section 4). Our relations also differ from Tits' relations [1987, §3.6] in the definition of his Kac–Moody group functor, even taking into account that our $\tilde{h}_i(r)$ corresponds to his r^{h_i}. This is because Rémy observed [2002, §8.3.3] that Tits' relator (6), namely $\tilde{s}_i(r)^{-1} \cdot \tilde{s}_i \cdot r^{h_i}$, is in error. Rémy fixed it by replacing the first r by $1/r$. Our repair, by exchanging the last two terms, is equivalent.

Theorem 6.4 (alternative defining relations for $\mathfrak{St}$). *The kernel of the natural map $\mathfrak{St}^{\mathrm{Tits}}(R) \to \mathfrak{St}(R)$ is the smallest normal subgroup containing the elements*

$$\tilde{h}_i(r)\mathfrak{x}_j(t)\tilde{h}_i(r)^{-1} \cdot \mathfrak{x}_j(r^{A_{ij}}t)^{-1} \tag{6-8}$$

$$\tilde{h}_i(r)\tilde{s}_j\mathfrak{x}_j(t)\tilde{s}_j^{-1}\tilde{h}_i(r)^{-1} \cdot \bigl(\tilde{s}_j\mathfrak{x}_j(r^{-A_{ij}}t)\tilde{s}_j^{-1}\bigr)^{-1} \tag{6-9}$$

$$\tilde{s}_i u \tilde{s}_i^{-1} \cdot s_i^*(u)^{-1} \tag{6-10}$$

for all $i, j \in I$, $r \in R^*$, $t \in R$ and $u \in \mathfrak{U}_\beta$, where β may be any root. Furthermore, the identities

$$\tilde{s}_i\tilde{h}_j(r)\tilde{s}_i^{-1} = \tilde{h}_i(r^{A_{ji}})^{-1}\tilde{h}_j(r), \tag{6-11}$$

$$[\tilde{h}_i(r), \tilde{h}_j(r')] = \tilde{h}_j(r^{A_{ij}}r')\tilde{h}_j(r^{A_{ij}})^{-1}\tilde{h}_j(r')^{-1} \tag{6-12}$$

hold in $\mathfrak{St}(R)$, for all $i, j \in I$, $r, r' \in R^*$.

Remark 6.5 (applicability to $\mathfrak{PSt}$). The proof below does not use the relations defining $\mathfrak{St}^{\mathrm{Tits}}$. So it shows that the subgroup of $*_{\alpha\in\Phi}\,\mathfrak{U}_\alpha(R)$ normally generated by the relators (6-5) is the same as the one normally generated by (6-8)–(6-10), and that (6-11)–(6-12) hold in the quotient. This is useful because we will use the same relations when defining the pre-Steinberg group $\mathfrak{PSt}$ in the next section.

Proof. We begin by showing that (6-8)–(6-10) are trivial in $\mathfrak{St}(R)$. First, (6-10) is got from (6-5) by taking $e = e_i$ and $r = 1$. Next, recall the definition of $\tilde{h}_i(r)$ as $\tilde{s}_i(r)\tilde{s}_i(-1)$ in (6-3), and that the defining relations (6-5) for $\mathfrak{St}(R)$ say how $\tilde{s}_i(r)$ acts on every $\mathfrak{U}_\beta$. So $\tilde{h}_i(r)$ acts on every $\mathfrak{U}_\beta$ as

$$h_i(r)\circ s_i^*\circ h_i(-1)\circ s_i^* = h_i(r)\circ h_i(-1)\circ (s_i^*)^2 = h_i(r)\circ h_i(-1)\circ h_i(-1) = h_i(r).$$

Taking $\beta = \alpha_j$ gives (6-8). For (6-9), take $\beta = -\alpha_j$ and use the fact that $\tilde{s}_j$ swaps $\mathfrak{U}_{\pm\alpha_j}$ (since it acts as s_j^*). This finishes the proof that (6-8)–(6-10) are trivial in $\mathfrak{St}(R)$.

Now we write N for the smallest normal subgroup of $\mathfrak{St}^{\mathrm{Tits}}(R)$ containing (6-8)–(6-10) and $\equiv$ for equality modulo N. We will show that (6-11)–(6-12) hold modulo N and that the relators (6-6)–(6-7) are trivial modulo N. We will use relator (6-10) without explicit mention: modulo N, each $\tilde{s}_i$ acts on every $\mathfrak{U}_\beta$ as s_i^*.

First we establish (6-11)–(6-12). Starting from the definition of $\tilde{s}_j(r')$, we have

$$\tilde{s}_j(r') = \mathfrak{x}_j(r')\mathfrak{x}_{-j}(1/r')\mathfrak{x}_j(r') \equiv \mathfrak{x}_j(r')\cdot\tilde{s}_j\mathfrak{x}_j(1/r')\tilde{s}_j^{-1}\cdot\mathfrak{x}_j(r').$$

Now the relators (6-8)–(6-9) give

$$\tilde{h}_i(r)\tilde{s}_j(r')\tilde{h}_i(r)^{-1} \equiv \tilde{s}_j(r^{A_{ij}}r'). \tag{6-13}$$

Taking $r' = 1$, left-multiplying by $\tilde{h}_i(r)^{-1}$, right-multiplying by $\tilde{s}_j^{-1}$, and then inverting both sides and using (6-4), gives

$$\tilde{s}_j \tilde{h}_i(r) \tilde{s}_j^{-1} \equiv \tilde{s}_j(1) \tilde{s}_j(r^{A_{ij}})^{-1} \tilde{h}_i(r) \equiv \tilde{h}_j(r^{A_{ij}})^{-1} \tilde{h}_i(r). \tag{6-14}$$

Exchanging i and j establishes (6-11). Also, (6-13), (6-3) and (6-4) show that

$$\tilde{h}_i(r) \tilde{h}_j(r') \tilde{h}_i(r)^{-1} \equiv \tilde{s}_j(r^{A_{ij}} r') \tilde{s}_j(r^{A_{ij}})^{-1} = \tilde{h}_j(r^{A_{ij}} r') \tilde{h}_j(r^{A_{ij}})^{-1}.$$

Right-multiplication by $\tilde{h}_j(r')^{-1}$ gives (6-12).

Now we will prove (6-7) for all e_i. That is: modulo N, $\tilde{h}_i(r)$ acts on every $\mathfrak{U}_\beta$ by $h_i(r)$. To prove this, write E for $\bigcup_{\beta \in \Phi} E_\beta$ and consider for any $e \in E$ the condition

$$\tilde{h}_i(r) \mathfrak{x}_e(t) \tilde{h}_i(r)^{-1} \equiv \mathfrak{x}_e(r^{\langle \alpha_i^\vee, \beta \rangle} t) \quad \text{for all } i \in I, r \in R^* \text{ and } t \in R, \tag{6-15}$$

where β is the root with $e \in E_\beta$. The set of $e \in E$ satisfying this condition is closed under negation, because $\mathfrak{x}_{-e}(t) = \mathfrak{x}_e(-t)$. This set contains $e_j \in E_{\alpha_j}$ and $f_j \in E_{-\alpha_j}$, for every $j \in I$, by relations (6-8)–(6-9). The next paragraph shows that it is closed under the action of W^*. Therefore, all $e \in E$ satisfy (6-15), establishing (6-7) for all $e = e_i$.

Here is the calculation that if $e \in E$ satisfies (6-15), and j is any element of I, then $s_j^*(e)$ also satisfies (6-15). We must establish it for all i, so fix some $i \in I$. We have

$$\begin{aligned}
&\tilde{h}_i(r) \mathfrak{x}_{s_j^*(e)}(t) \tilde{h}_i(r)^{-1} \\
&\quad = \tilde{h}_i(r) \mathfrak{x}_{s_j^{*-1} \circ h_j(-1)(e)}(t) \tilde{h}_i(r)^{-1} && \text{by } (s_j^*)^2 = h_j(-1) \\
&\quad \equiv \tilde{h}_i(r) \tilde{s}_j^{-1} \mathfrak{x}_e((-1)^{\langle \alpha_j^\vee, \beta \rangle} t) \tilde{s}_j \tilde{h}_i(r)^{-1} \\
&\quad = \tilde{s}_j^{-1} (\tilde{s}_j \tilde{h}_i(r) \tilde{s}_j^{-1}) \mathfrak{x}_e((-1)^{\langle \alpha_j^\vee, \beta \rangle} t) (\tilde{s}_j \tilde{h}_i(r)^{-1} \tilde{s}_j^{-1}) \tilde{s}_j \\
&\quad \equiv \tilde{s}_j^{-1} (\tilde{h}_j(r^{A_{ij}})^{-1} \tilde{h}_i(r)) \mathfrak{x}_e((-1)^{\langle \alpha_j^\vee, \beta \rangle} t) (\tilde{h}_i(r)^{-1} \tilde{h}_j(r^{A_{ij}})) \tilde{s}_j && \text{by (6-14)} \\
&\quad \equiv \tilde{s}_j^{-1} \mathfrak{x}_e((-1)^{\langle \alpha_j^\vee, \beta \rangle} r^{\langle \alpha_i^\vee, \beta \rangle} r^{-A_{ij} \langle \alpha_j^\vee, \beta \rangle} t) \tilde{s}_j && \text{by (6-15) for } e \\
&\quad \equiv \mathfrak{x}_{s_j^{*-1}(e)}((-1)^{\langle \alpha_j^\vee, \beta \rangle} r^{\langle \alpha_i^\vee, \beta \rangle} r^{-A_{ij} \langle \alpha_j^\vee, \beta \rangle} t) \\
&\quad = \mathfrak{x}_{s_j^* \circ h_j(-1)(e)}((-1)^{\langle \alpha_j^\vee, \beta \rangle} r^{\langle \alpha_i^\vee, \beta \rangle} r^{-A_{ij} \langle \alpha_j^\vee, \beta \rangle} t) \\
&\quad = \mathfrak{x}_{s_j^*(e)}(r^{\langle \alpha_i^\vee, \beta \rangle} r^{-A_{ij} \langle \alpha_j^\vee, \beta \rangle} t).
\end{aligned}$$

The right side of (6-15) for $s_j^*(e)$ has a similar form. Establishing equality amounts to showing $\langle \alpha_i^\vee - A_{ij} \alpha_j^\vee, \beta \rangle = \langle \alpha_i^\vee, s_j(\beta) \rangle$. This follows from $s_j(\beta) = \beta - \langle \alpha_j^\vee, \beta \rangle \alpha_j$, finishing the proof of (6-15) for all $e \in E$.

For e equal to any $\pm e_i$, we were given (6-6) and we have proven (6-7). The same results for all e follow by W^* symmetry. More precisely, we claim that, for all $j \in I$, if (6-6) and (6-7) hold for some $e \in E$ then they hold for $s_j^*(e)$ too. We give the details for (6-7), and the argument is the same for (6-6). Suppose $r \in R^*$ and $u \in \bigcup_{\beta\in\Phi} \mathfrak{U}_\beta$. Then the left and right "sides" of the known relation (6-7) for e lie in $\bigcup_{\beta\in\Phi} \mathfrak{U}_\beta$, so conjugating the left by $\tilde{s}_j$ has the same result as applying s_j^* to the right. That is,

$$\begin{aligned}\tilde{s}_j\tilde{s}_e u \tilde{s}_e^{-1}\tilde{s}_j^{-1} &\equiv s_j^* \circ s_e^*(u),\\ (\tilde{s}_j\tilde{s}_e\tilde{s}_j^{-1})(\tilde{s}_j u \tilde{s}_j^{-1})(\tilde{s}_j\tilde{s}_e^{-1}\tilde{s}_j^{-1}) &\equiv s_j^* \circ s_e^* \circ s_j^{*-1} \circ s_j^*(u),\\ s^*_{s_j^*(e)} s_j^*(u) (s^*_{s_j^*(e)})^{-1} &\equiv s^*_{s_j^*(e)}(s_j^*(u)).\end{aligned}$$

As u varies over all of $\bigcup_{\beta\in\Phi} \mathfrak{U}_\beta$, so does $s_j^*(u)$. This verifies relation (6-7) for $s_j^*(e)$. □

7. The pre-Steinberg group $\mathfrak{PSt}$

In this section we define the pre-Steinberg group functor $\mathfrak{PSt}_A$ in the same way as $\mathfrak{St}_A$, but omitting some of its Chevalley relations. So it has a natural map to $\mathfrak{St}_A$. Then we will write down another group functor as a concrete presentation, and show in Theorem 7.12 that it equals $\mathfrak{PSt}_A$. Since $\mathfrak{PSt}_A \to \mathfrak{St}_A$ is often an isomorphism (Theorem 1.1), this often gives a new presentation for $\mathfrak{St}_A$. As discussed in the Introduction, it is simpler and more explicit than previous presentations, and special cases of it appear in Table 1.1 and Section 2. In the rest of this section we suppress the subscript A.

We call two roots α, β classically prenilpotent if $(\mathbb{Q}\alpha + \mathbb{Q}\beta) \cap \Phi$ is finite and $\alpha + \beta \neq 0$. Then they are prenilpotent, and lie in some A_1, A_1^2, A_2, B_2 or G_2 root system. We define the pre-Steinberg group functor $\mathfrak{PSt}$ exactly as we did the Steinberg functor $\mathfrak{St}$ (Section 6), except that when imposing the Chevalley relations we only vary α, β over the classically prenilpotent pairs rather than all prenilpotent pairs. We still impose the relations (6-5) of Morita–Rehmann, or equivalently (6-6)–(6-7) or (6-8)–(6-10). (See Remark 6.5 for why Theorem 6.4 applies with $\mathfrak{PSt}$ in place of $\mathfrak{St}$.) Just as for $\mathfrak{St}$, W^* acts on $\mathfrak{PSt}$ because it permutes the defining relators.

There is an obvious natural map $\mathfrak{PSt} \to \mathfrak{St}$, got by imposing the remaining Chevalley relations, coming from prenilpotent pairs that are not classically prenilpotent. If Φ is finite then every prenilpotent pair is classically prenilpotent, so $\mathfrak{PSt} \to \mathfrak{St}$ is an isomorphism.

The rest of this section is devoted to writing down a presentation for $\mathfrak{PSt}$. We start by defining an analogue $\widehat{W}$ of the Weyl group. It is the quotient of the free group on formal symbols $S_{i\in I}$ by the subgroup normally generated by the words

$$(S_iS_j\cdots)\cdot(S_jS_i\cdots)^{-1} \qquad \text{if } m_{ij}\neq\infty, \tag{7-1}$$

$$S_i^2S_jS_i^{-2}\cdot S_j^{-1} \qquad \text{if } A_{ij} \text{ is even}, \tag{7-2}$$

$$S_i^2S_jS_i^{-2}\cdot S_j \qquad \text{if } A_{ij} \text{ is odd}, \tag{7-3}$$

where i, j vary over I, and (7-1) has m_{ij} terms inside each pair of parentheses, alternating between S_i and S_j. These are called the Artin relators, for example, $S_iS_jS_i\cdot(S_jS_iS_j)^{-1}$ if $m_{ij}=3$.

Remark 7.1. We chose these defining relations so that $\widehat{W}$ would have four properties. First, it maps naturally to W^*, so that it acts on $\mathfrak{g}$ and $*_{\alpha\in\Phi}\,\mathfrak{U}_\alpha$. Second, the kernel of $\widehat{W}\to W$ is generated (not just normally) by the S_i^2. This plays a key role in the proof of Theorem 7.5 below. Third, each relation involves just two subscripts, which is needed for the Curtis–Tits property of $\mathfrak{PSt}$ (Corollary 1.3). And fourth, the $\tilde{s}_i\in\mathfrak{St}$, defined in (7-27), satisfy the same relations. (Formally: $S_i\to\tilde{s}_i$ extends to a homomorphism $\widehat{W}\to\mathfrak{St}$.) The first two properties are established in the next lemma, the third is obvious, and the fourth is part of Theorem 7.12.

Lemma 7.2 (basic properties of $\widehat{W}$). (i) *$S_i\mapsto s_i^*$ defines a surjection $\widehat{W}\to W^*$.*

(ii) *$S_jS_i^2S_j^{-1}=S_i^2$ (resp. $S_j^2S_i^2$) if A_{ij} is even (resp. odd).*

(iii) *The S_i^2 generate the kernel of the composition $\widehat{W}\to W^*\to W$.*

Proof. We saw in Theorem 5.5 that the s_i^* satisfy the Artin relations. Rewriting Lemma 5.2(ii)'s relation in W^* with i and j reversed gives

$$s_j^*(s_i^*)^2s_j^{*-1}=(s_i^*)^2(s_j^*)^{-2A_{ij}}.$$

Multiplying on the left by s_j^{*-1} and on the right by $(s_i^*)^{-2}$, then inverting, gives

$$(s_i^*)^2s_j^*(s_i^*)^{-2}=(s_i^*)^2(s_j^*)^{2A_{ij}}(s_i^*)^{-2}s_j^*=(s_j^*)^{1+2A_{ij}}.$$

In the second step we used the fact that s_i^{*2} and s_j^{*2} commute. Using $s_j^{*4}=1$, the right side is s_j^* if A_{ij} is even and s_j^{*-1} if A_{ij} is odd. This shows that $S_i\mapsto s_i^*$ sends the relators (7-2)–(7-3) to the trivial element of W^*, proving (i).

One can manipulate (7-2)–(7-3) in a similar way, yielding (ii). It follows immediately that the subgroup generated by the S_i^2 is normal. Because of the Artin relations, this is the kernel of $\widehat{W}\to W$. So we have proven (iii). □

Remark 7.3. Though we don't need them, the following relations in $\widehat{W}$ show that $\widehat{W}$ is "not much larger" than W^*. First (7-2)–(7-3) imply the centrality of every S_i^4. Second, if some A_{ij} is odd then (7-3) shows that $S_j^{\pm4}$ are conjugate; since both are central they must be equal, so $S_j^8=1$. Third, the relation obtained at the end of the proof implies $[S_j^2, S_i^2] = 1$ or S_j^4, according to whether A_{ij} is even or odd. In particular, these commutators are central. Finally, we can use this twice:

$$\begin{Bmatrix} 1 & \text{if } A_{ij} \text{ is even} \\ S_j^4 & \text{if } A_{ij} \text{ is odd} \end{Bmatrix} = [S_j^2, S_i^2] = [S_i^2, S_j^2]^{-1} = \begin{Bmatrix} 1 & \text{if } A_{ji} \text{ is even} \\ S_i^{-4} & \text{if } A_{ji} \text{ is odd} \end{Bmatrix}.$$

In particular, if both A_{ij} and A_{ji} are odd then S_i^4 and S_j^4 are equal. If A_{ij} is even while A_{ji} is odd then we get $S_i^4 = 1$.

Now we begin our presentation in earnest. Ultimately, $\mathfrak{PSt}(R)$ will have generators S_i and $X_i(t)$, with i varying over I and t varying over R, and relators (7-1)–(7-26).

We first define a group functor $\mathfrak{G}_1$ by declaring that $\mathfrak{G}_1(R)$ is the quotient of the free group on the formal symbols $X_i(t)$, by the subgroup normally generated by the relators

$$X_i(t)X_i(u) \cdot X_i(t+u)^{-1} \tag{7-4}$$

for all $i \in I$ and $t, u \in R$. The following description of $\mathfrak{G}_1$ is obvious.

Lemma 7.4. *$\mathfrak{G}_1 \cong *_{i \in I} \mathfrak{U}_i$, via the correspondence $X_i(t) \leftrightarrow \mathfrak{x}_i(t)$.* □

Next we define a group functor $\mathfrak{G}_2$ as a certain quotient of the free product $\mathfrak{G}_1 * \widehat{W}$. Namely, $\mathfrak{G}_2(R)$ is the quotient of $\mathfrak{G}_1(R) * \widehat{W}$ by the subgroup normally generated by the following relators, with i and j varying over I and t over R:

$$S_i^2 X_j(t) S_i^{-2} \cdot \big(X_j((-1)^{A_{ij}} t)\big)^{-1} \tag{7-5}$$

$$[S_i, X_j(t)] \qquad \text{if } m_{ij} = 2, \tag{7-6}$$

$$S_j S_i X_j(t) \cdot \big(X_i(t) S_j S_i\big)^{-1} \qquad \text{if } m_{ij} = 3, \tag{7-7}$$

$$[S_i S_j S_i, X_j(t)] \qquad \text{if } m_{ij} = 4, \tag{7-8}$$

$$[S_i S_j S_i S_j S_i, X_j(t)] \qquad \text{if } m_{ij} = 6. \tag{7-9}$$

The next theorem is the key step in our development; see Section 8 for the proof. Although it is not at all obvious, we have presented $(*_{\alpha \in \Phi} \mathfrak{U}_\alpha) \rtimes \widehat{W}$. Therefore, we "have" the root groups $\mathfrak{U}_\alpha$ for all α, not just simple α. This sets us up for imposing the Chevalley relations in the next step.

Theorem 7.5. *$\mathfrak{G}_2$ is the semidirect product of $*_{\alpha \in \Phi} \mathfrak{U}_\alpha$ by $\widehat{W}$, where $\widehat{W}$ acts on the free product via its homomorphism to W^* and W^*'s action on $*_{\alpha \in \Phi} \mathfrak{U}_\alpha$ is induced by its action on $\bigcup_{\alpha \in \Phi} \mathfrak{g}_{\alpha, \mathbb{Z}}$.*

Remark 7.6 (groups with a root group datum). A Kac–Moody group over a field is an example of a group G with a "root group datum". This means: a generating set of subgroups $\mathfrak{U}_\alpha$ parametrized by the roots α of a root system, permuted by (some extension $\widetilde{W}$ of) the Weyl group W of that root system, and satisfying some additional hypotheses. See [Tits 1992] or [Caprace and Rémy 2009] for details.

Examples include the Suzuki and Ree groups and isotropic forms of algebraic groups (or Kac–Moody groups) over fields. In many of these cases, some of the root groups are noncommutative. The heart of the proof of Theorem 7.5 is our understanding of root stabilizers in W^* (Theorem 5.7), which would still apply in this more general setting. So there should be an analogous presentation of $(*_{\alpha\in\Phi}\mathfrak{U}_\alpha)\rtimes\widetilde{W}$. The main change would be to replace (7-4) by defining relations for $\mathfrak{U}_i$, and interpret the parameter t of $X_i(t)$ as varying over some fixed copy of $\mathfrak{U}_i$, rather than over R. Since G is a quotient of $(*_{\alpha\in\Phi}\mathfrak{U}_\alpha)\rtimes\widetilde{W}$, analogues of the rest of this section presumably yield a presentation of G.

Next we adjoin Chevalley relations corresponding to finite edges in the Dynkin diagram. That is, we define $\mathfrak{G}_3(R)$ as the quotient of $\mathfrak{G}_2(R)$ by the subgroup normally generated by the relators (7-10)–(7-23) below, for all $t,u\in R$. These are particular cases of the standard Chevalley relators, written in a form due to Demazure (see Remark 7.8 below).

When $i,j\in I$ with $m_{ij}=2$,

$$[X_i(t),X_j(u)] \tag{7-10}$$

When $i,j\in I$ with $m_{ij}=3$,

$$[X_i(t),S_iX_j(u)S_i^{-1}] \tag{7-11}$$

$$[X_i(t),X_j(u)]\cdot S_iX_j(-tu)S_i^{-1} \tag{7-12}$$

When $s,l\in I$, $m_{sl}=4$ and s is the shorter root of the B_2,

$$[S_sX_l(t)S_s^{-1},S_lX_s(u)S_l^{-1}] \tag{7-13}$$

$$[X_l(t),S_sX_l(u)S_s^{-1}] \tag{7-14}$$

$$[X_s(t),S_lX_s(u)S_l^{-1}]\cdot S_sX_l(2tu)S_s^{-1} \tag{7-15}$$

$$[X_s(t),X_l(u)]\cdot S_sX_l(-t^2u)S_s^{-1}\cdot S_lX_s(tu)S_l^{-1} \tag{7-16}$$

When $s,l\in I$, $m_{sl}=6$ and s is the shorter root of the G_2,

$$[X_l(t),S_lS_sX_l(u)S_s^{-1}S_l^{-1}] \tag{7-17}$$

$$[S_sS_lX_s(t)S_l^{-1}S_s^{-1},S_lS_sX_l(u)S_s^{-1}S_l^{-1}] \tag{7-18}$$

$$[S_sX_l(t)S_s^{-1},S_lX_s(u)S_l^{-1}] \tag{7-19}$$

$$[X_l(t),S_sX_l(u)S_s^{-1}]\cdot S_lS_sX_l(-tu)S_s^{-1}S_l^{-1} \tag{7-20}$$

$$[X_s(t),S_sS_lX_s(u)S_l^{-1}S_s^{-1}]\cdot S_sX_l(-3tu)S_s^{-1} \tag{7-21}$$

$$[X_s(t),\, S_l X_s(u) S_l^{-1}] \cdot S_l S_s X_l(3tu^2) S_s^{-1} S_l^{-1} \cdot S_s X_l(3t^2u) S_s^{-1} \cdot S_s S_l X_s(2tu) S_l^{-1} S_s^{-1} \tag{7-22}$$

$$[X_s(t),\, X_l(u)] \cdot S_l S_s X_l(t^3u^2) S_s^{-1} S_l^{-1} \cdot S_s X_l(-t^3u) S_s^{-1} \cdot S_l X_s(tu) S_l^{-1} \cdot S_s S_l X_s(-t^2u) S_l^{-1} S_s^{-1} \tag{7-23}$$

Remark 7.7 (asymmetry in the A_2 relators). The relators (7-11)–(7-12) are not symmetric in i and j. Since $m_{ji} = 3$ whenever $m_{ij} = 3$, we are using both these relators and the ones got from them by exchanging i and j.

Remark 7.8 (Demazure's form of the Chevalley relations). Our relators are written in a form due to Demazure (Propositions 3.2.1, 3.3.1 and 3.4.1 in [SGA 3_{III} 1970, Exposé XXIII]). They appear more complicated than the more usual one (for example, [Carter 1972, Theorem 5.2.2]), but have two important advantages. First, there are no implicit signs to worry about, and second, the presentation refers only to the Dynkin diagram, rather than the full root system.

One can convert (7-10)–(7-23) to a more standard form by working out which root groups contain the terms on the "right-hand sides" of the relators. For example, the term $S_l X_s(tu) S_l^{-1}$ of (7-23) lies in $S_l \mathfrak{U}_s S_l^{-1} = \mathfrak{U}_{\alpha_s+\alpha_l}$ because reflection in α_l sends α_s to $\alpha_s + \alpha_l$. Applying the same reasoning to the other terms, (7-23) equals $[X_s(t),\, X_l(u)]$ times a particular element of $\mathfrak{U}_{3\alpha_s+2\alpha_l} \cdot \mathfrak{U}_{2\alpha_s+\alpha_l} \cdot \mathfrak{U}_{\alpha_s+\alpha_l} \cdot \mathfrak{U}_{2\alpha_s+\alpha_l}$. The advantages of Demazure's form of the relators come from the fact that no identifications of these root groups with R is required. We simply use the already-fixed identifications of the simple root groups with R, and transfer them to these other root groups by conjugation by S_s and S_l.

Remark 7.9 (diagram automorphisms in characteristics 2 and 3). Some of the relators can be written in simpler but less symmetric ways. For example, (7-13) is the Chevalley relator for the roots $s_s(\alpha_l)$ and $s_l(\alpha_s)$ of B_2, which make angle $\pi/4$. As we will see in the proof of Theorem 7.11, one could replace this pair of roots by any other pair of roots in the span of α_s, α_l that make this angle. So, for example, one could replace (7-13) by the simpler relator $[S_s X_l(t) S_s^{-1},\, X_s(u)]$. We prefer (7-13) because it maps to itself under the exceptional diagram automorphism in characteristic 2; see Section 3 for details. Similar considerations informed our choice of relators (7-18)–(7-19), and the ordering of the last four terms of (7-23).

Remark 7.10 (redundant relations). In practice, most of the relators coming from absent and single bonds in the Dynkin diagram, i.e., (7-10)–(7-12), can be omitted. Usually this reduces the size of the presentation greatly. See Propositions 9.1 and 9.2.

In Section 9 we prove the following more conceptual description of $\mathfrak{G}_3$. To be able to state it we use the temporary notation $\mathfrak{PSt}^{\text{Tits}}$ for the group functor defined in the same way as $\mathfrak{St}^{\mathrm{Tits}}$ (see Section 6), but only using classically prenilpotent pairs rather than all prenilpotent pairs. So $\mathfrak{PSt}^{\mathrm{Tits}}$ is related to $\mathfrak{St}^{\mathrm{Tits}}$ in the same way that $\mathfrak{PSt}$ is related to $\mathfrak{St}$. $\widehat{W}$ acts on $\mathfrak{PSt}^{\mathrm{Tits}}$ for the same reason it acts on $\mathfrak{St}^{\mathrm{Tits}}$.

Theorem 7.11. *The group functor $\mathfrak{PSt}^{\mathrm{Tits}} \rtimes \widehat{W}$ coincides with $\mathfrak{G}_3$. More precisely, under the identification $\mathfrak{G}_2 \cong (*_{\alpha\in\Phi}\mathfrak{U}_\alpha) \rtimes \widehat{W}$ of Theorem 7.5, the kernels of $\mathfrak{G}_2 \to \mathfrak{G}_3$ and $(*_{\alpha\in\Phi}\mathfrak{U}_\alpha) \rtimes \widehat{W} \to \mathfrak{PSt}^{\mathrm{Tits}} \rtimes \widehat{W}$ coincide.*

Finally, we define $\mathfrak{G}_4$ as the quotient of $\mathfrak{G}_3$ by the smallest normal subgroup containing the relators

$$\tilde{h}_i(r)X_j(t)\tilde{h}_i(r)^{-1} \cdot X_j(r^{A_{ij}}t)^{-1} \tag{7-24}$$

$$\tilde{h}_i(r)S_jX_j(t)S_j^{-1}\tilde{h}_i(r)^{-1} \cdot S_jX_j(r^{-A_{ij}}t)^{-1}S_j^{-1} \tag{7-25}$$

$$S_i \cdot \tilde{s}_i(1)^{-1} \tag{7-26}$$

where r varies over R^*, t over R and i, j over I. We are using the definitions

$$\tilde{s}_i(r) := X_i(r)S_iX_i(1/r)S_i^{-1}X_i(r), \tag{7-27}$$

$$\tilde{h}_i(r) := \tilde{s}_i(r)\tilde{s}_i(-1). \tag{7-28}$$

Note that this definition of $\tilde{s}_i(r)$ is compatible with the one in Section 6, because $X_i(r) \in \mathfrak{G}_3$ corresponds to $\mathfrak{x}_{e_i}(r) \in \mathfrak{PSt}^{\mathrm{Tits}}$ under the isomorphism of Lemma 7.4, while $S_iX_i(1/r)S_i^{-1}$ corresponds to $s_i^*(\mathfrak{x}_{e_i}(1/r)) = \mathfrak{x}_{f_i}(1/r)$. As before, we will abbreviate $\tilde{s}_i(1)$ to $\tilde{s}_i$.

The following theorem is the main result of this section and a restatement of Theorem 1.2 from the Introduction.

Theorem 7.12 (presentation of the pre-Steinberg group $\mathfrak{PSt}$). *The group functor $\mathfrak{PSt}$ coincides with $\mathfrak{G}_4$. In particular, for any commutative ring R, $\mathfrak{PSt}(R)$ has a presentation with generators S_i and $X_i(t)$ for $i \in I$ and $t \in R$, and relators* (7-1)–(7-26).

Proof. By definition, $\mathfrak{G}_4$ is the quotient of $\mathfrak{G}_3$ by the relations (7-24)–(7-26). Because S_i acts on each $\mathfrak{U}_\beta$ by s_i^* (Theorem 7.5), imposing (7-26) forces $\tilde{s}_i$ to also act this way. We consider the intermediate group $\mathfrak{G}_{3.5}$, of fleeting interest, got from $\mathfrak{G}_3$ by imposing (7-24)–(7-25) and the relations that $\tilde{s}_i$ acts on every $\mathfrak{U}_\beta$ as s_i^* does. In other words, we are imposing on $\mathfrak{PSt}^{\mathrm{Tits}} \subseteq \mathfrak{PSt}^{\mathrm{Tits}} \rtimes \widehat{W} = \mathfrak{G}_3$ the relations (6-8)–(6-10). Theorem 6.4 and Remark 6.5 show that this reduces $\mathfrak{G}_3$ to $\mathfrak{PSt} \rtimes \widehat{W}$.

So $\mathfrak{G}_4$ is the quotient of $\mathfrak{G}_{3.5} = \mathfrak{PSt} \rtimes \widehat{W}$ by the relations $S_i = \tilde{s}_i$. We use Tietze transformations to eliminate the S_i from the presentation, in favor of the $\tilde{s}_i$. So $\mathfrak{G}_4$ is the quotient of $\mathfrak{PSt}$ by the subgroup normally generated by the words got by replacing S_i by $\tilde{s}_i$ in each of the relators (7-1)–(7-25). All of these relators are already trivial in $\mathfrak{PSt}$, so $\mathfrak{G}_4 = \mathfrak{PSt}$.

In more detail, (7-1) requires the $\tilde{s}_i$ to satisfy the Artin relations, which they do in $\mathfrak{PSt}$ by [Tits 1987, (d) on p. 551]. The remaining relations (7-2)–(7-25) involve the S_i only by their conjugacy action. For example, (7-17) says that $X_l(t)$ commutes with the conjugate of $X_l(u)$ by a certain word in S_s and S_l. Since S_i acts as s_i^* by Theorem 7.5 and $\tilde{s}_i$ acts the same way by the definition of $\mathfrak{PSt}$, these relations still hold after replacing each S_i by the corresponding $\tilde{s}_i$. (When defining $\widehat{W}$, we were careful not to impose any relations on the S_i except those which are also satisfied by the $\tilde{s}_i$.) □

Remark 7.13 (redundant relators). In most cases of interest, A is 2-spherical without A_1 components. Then one can forget the relators (7-24)–(7-25) because they follow from previous relations. More specifically, suppose m_{ij} is 3, 4 or 6. Then the relators (7-24)–(7-25) are already trivial in $\mathfrak{G}_3$. The same holds if $i = j$ and there exists some $k \in I$ with $m_{ik} \in \{3, 4, 6\}$. See [Tits 1987, (a_4), p. 550] for details.

Remark 7.14 (more redundant relators). One need only impose the relators (7-26) for a single i in each component Ω of the "odd Dynkin diagram" Δ^{odd} considered in Section 5. This is because if $m_{ij} = 3$ then $S_i S_j$ conjugates S_i to S_j and $X_i(t)$ to $X_j(t)$. This uses relators (7-1) and (7-7).

Remark 7.15 (precautions against typographical errors). We found explicit matrices for our generators, in standard representations of the A_1^2, A_2, B_2 and G_2 Chevalley groups over $\mathbb{Z}[r^{\pm 1}, t, u]$. Then we checked on the computer that they satisfy the defining relations (7-1)–(7-26).

8. The isomorphism $\mathfrak{G}_2 \cong (*_{\alpha\in\Phi}\, \mathfrak{U}_\alpha) \rtimes \widehat{W}$

In this section we will suppress the dependence of group functors on the base ring R, always meaning the group of points over R. Our goal is to prove Theorem 7.5, namely that the group $\mathfrak{G}_2$ with generators S_i and $X_i(t)$, $i \in I$ and $t \in R$, modulo the subgroup normally generated by the relators (7-1)–(7-9), is $(*_{\alpha\in\Phi}\, \mathfrak{U}_\alpha) \rtimes \widehat{W}$. The genesis of the theorem is the following elementary principle. It seems unlikely to be new, but I have not seen it before.

Lemma 8.1. *Suppose $G = (*_{\alpha\in\Phi}\, U_\alpha) \rtimes H$, where Φ is some index set, the U_α are groups isomorphic to each other, and H is a group whose action on the free product permutes the displayed factors transitively. Then $G \cong (U_\infty \rtimes H_\infty) *_{H_\infty} H$, where ∞ is some element of Φ and H_∞ is its H-stabilizer.*

Proof. The idea is that $U_\infty \rtimes H_\infty \mapsto (U_\infty \rtimes H_\infty) *_{H_\infty} H$ is a sort of free-product analogue of inducing a representation from H_∞ to H. We suppress the subscript ∞

from U_∞. Take a set Z of left coset representatives for H_∞ in H, and for $u \in U$ and $z \in Z$ define $u_z := zuz^{-1} \in G$. The u_z for fixed z form the free factor $zUz^{-1} = U_{z(\infty)}$ of $(*_{\alpha\in\Phi} U_\alpha) \subseteq G$. Assuming $U \neq 1$, every displayed free factor occurs exactly once this way, since H's action on Φ is the same as on H_∞'s left cosets. So the maps $u_z \mapsto zuz^{-1} \in (U \rtimes H_\infty) *_{H_\infty} H$ define a homomorphism $(*_{\alpha\in\Phi} U_\alpha) \to (U \rtimes H_\infty) *_{H_\infty} H$. This homomorphism is obviously H-equivariant, so it extends to a homomorphism $G \to (U \rtimes H_\infty) *_{H_\infty} H$. It is easy to see that this is inverse to the obvious homomorphism $(U \rtimes H_\infty) *_{H_\infty} H \to G$. □

Now we begin proving Theorem 7.5 by reducing it to Lemma 8.2 below, which is an analogue of Theorem 7.5 for a single component of the "odd Dynkin diagram" Δ^{odd} introduced in Section 5. It is well-known that two generators s_i, s_j of W $(i, j \in I)$ are conjugate in W if and only if i and j lie in the same component of Δ^{odd}. (If $m_{ij} = 3$ then $s_i s_j s_i = s_j s_i s_j$ implies the conjugacy of s_i and s_j, while distinct components of Δ^{odd} correspond to different elements of the abelianization of W.)

Let Ω be one of these components, and write $\Phi(\Omega) \subseteq \Phi$ for the roots whose reflections are conjugate to some (hence any) $s_{i\in\Omega}$. Because $\Phi(\Omega)$ is a W-invariant subset of Φ, we may form the group $(*_{\alpha\in\Phi(\Omega)} \mathfrak{U}_\alpha) \rtimes \widehat{W}$ just as we did $(*_{\alpha\in\Phi} \mathfrak{U}_\alpha) \rtimes \widehat{W}$. We will write $\mathfrak{G}_{2,\Omega}$ for the group having generators S_i, with $i \in I$, and $X_i(t)$, with $i \in \Omega$ and $t \in R$, modulo the subgroup normally generated by the relators (7-1)–(7-3), and those relators (7-4)–(7-9) with $i \in \Omega$. Note that (7-7) is relevant only if $m_{ij} = 3$, in which case $i \in \Omega$ if and only if $j \in \Omega$, so the relator makes sense. *Caution:* the subscripts on S vary over all of I, while those on X vary only over $\Omega \subseteq I$.

Lemma 8.2. *For any component Ω of Δ^{odd},*

$$\mathfrak{G}_{2,\Omega} \cong \left(*_{\alpha\in\Phi(\Omega)} \mathfrak{U}_\alpha\right) \rtimes \widehat{W}.$$

Proof of Theorem 7.5, given Lemma 8.2. An examination of the presentation of $\mathfrak{G}_2$ reveals that the X's corresponding to different components of Δ^{odd} don't interact. Precisely: $\mathfrak{G}_2$ is the amalgamated free product of the $\mathfrak{G}_{2,\Omega}$, where Ω varies over the components of Δ^{odd} and the amalgamation is that the copies of $\widehat{W}$ in the $\mathfrak{G}_{2,\Omega}$ are identified in the obvious way. Lemma 8.2 shows that $\mathfrak{G}_{2,\Omega} = (*_{\alpha\in\Phi(\Omega)} \mathfrak{U}_\alpha) \rtimes \widehat{W}$ for each Ω. Taking their free product, amalgamated along their copies of $\widehat{W}$, obviously yields $(*_{\alpha\in\Phi} \mathfrak{U}_\alpha) \rtimes \widehat{W}$. □

The rest of the section is devoted to proving Lemma 8.2. So we fix a component Ω of Δ^{odd} and phrase our problem in terms of the free product $F := (*_{j\in\Omega} \mathfrak{U}_j) * \widehat{W}$. This is the group with generators $S_{i\in I}$ and $X_{j\in\Omega}(t)$, whose relations are (7-1)–(7-3) and those cases of (7-4) with $i \in \Omega$. The heart of the proof of Lemma 8.2 is to define normal subgroups M, N of F and show they are equal. M turns out to be normally generated by the relators from (7-5)–(7-9) for which $i \in \Omega$. Given this,

$\mathfrak{G}_{2,\Omega} = F/M$ by definition. The other group F/N has a presentation like the one in Lemma 8.1. But it requires some preparation even to define, so we begin with an informal overview.

Start with the presentation of $\mathfrak{G}_{2,\Omega}$, and distinguish some point ∞ of Ω and a spanning tree T for Ω. We will use the relators (7-7) coming from the edges of T to rewrite the $X_{j\in\Omega-\{\infty\}}(t)$ in terms of $X_\infty(t)$, and then eliminate the $X_{j\in\Omega-\{\infty\}}(t)$ from the presentation. This "uses up" those relators and makes the other relators messier because each $X_{j\neq\infty}(t)$ must be replaced by a word in $X_\infty(t)$ and elements of $\widehat{W}$. We studied the W^*-stabilizer of α_∞ in Theorem 5.7, and how it acts on $\mathfrak{g}_\infty$, hence on $\mathfrak{U}_\infty$. It turns out that the remaining relations in $\mathfrak{G}_{2,\Omega}$ are exactly the relations that the $\widehat{W}$-stabilizer $\widehat{W}_\infty$ of α_∞ acts on $\mathfrak{U}_\infty$ via $\widehat{W}_\infty \to \widehat{W} \to W^* \subseteq \operatorname{Aut}\mathfrak{g}$. That is, $\mathfrak{G}_{2,\Omega} \cong (\mathfrak{U}_\infty \rtimes \widehat{W}_\infty) *_{\widehat{W}_\infty} \widehat{W}$. Then Lemma 8.1 identifies this with $(*_{\alpha\in\Phi(\Omega)} \mathfrak{U}_\alpha) \rtimes \widehat{W}$.

Now we proceed to the formal proof, beginning by defining some elements of F. For γ an edge path in Ω, with $i_0, \ldots, i_n$ the vertices along it, define $\alpha(\gamma) = i_0$ and $\omega(\gamma) = i_n$ as its initial and final endpoints, and define P_γ by (5-3) with S's in place of s's. For $k \in I$ evenly joined to the end of γ (i.e., $m_{k\omega(\gamma)}$ finite and even), define

$$R_{\gamma,k} = P_\gamma^{-1} \cdot \left\{ \begin{matrix} S_k \\ S_k S_{\omega(\gamma)} S_k \\ S_k S_{\omega(\gamma)} S_k S_{\omega(\gamma)} S_k \end{matrix} \right\} \cdot P_\gamma$$

according to whether $m_{k\omega(\gamma)} = 2, 4$ or 6. (We get $R_{\gamma,k}$ from (5-4) by replacing s's and p's by S's and P's, and j by $\omega(\gamma)$.) Next, for $t \in R$ we define

$$C_\gamma(t) := P_\gamma X_{\alpha(\gamma)}(t) \cdot \bigl(X_{\omega(\gamma)}(t) P_\gamma\bigr)^{-1},$$

and for $k \in I$ evenly joined to $\omega(\gamma)$ we define

$$D_{\gamma,k}(t) := [R_{\gamma,k}, X_{\alpha(\gamma)}(t)].$$

For ease of reference we will also give the name

$$B_{ij}(t) := S_i^2 X_j(t) S_i^{-2} \cdot X_j\bigl((-1)^{A_{ij}} t\bigr)^{-1}$$

to the word (7-5), where $i \in I$ and $j \in \Omega$. We will suppress the dependence of the X_j, B_{ij}, C_γ and $D_{\gamma,k}$ on t except where it plays a role.

The following formally meaningless intuition may help the reader; Lemma 8.3 below gives it some support. The relation $C_\gamma = 1$ declares that the path γ conjugates the X "at" the beginning of γ to the X "at" the end. And the relation $D_{\gamma,k} = 1$ declares that the X "at" the beginning of γ commutes with a certain word that corresponds to going along γ, going around some sort of "loop based at the endpoint of γ", and then retracing γ.

Our first normal subgroup M of F is defined as the subgroup normally generated by all the B_{ij}, the C_γ for all γ of length 1, and the $D_{\gamma,k}$ for all γ of length 0.

Unwinding the definitions shows that these elements of F are exactly the ones we used in defining $\mathfrak{G}_{2,\Omega}$. For example, if γ is the length-1 path from one vertex j of Ω to an adjacent vertex i then $P_\gamma = S_j S_i$ and C_γ is the word (7-7). And if $i \in \Omega$ is evenly joined to $j \in I$ then we take γ to be the zero-length path at i, and $D_{\gamma,j}$ turns out to be the relator (7-6), (7-8) or (7-9). Which one of these applies depends on $m_{ij} \in \{2, 4, 6\}$. So $F/M \cong \mathfrak{G}_{2,\Omega}$.

Before defining the other normal subgroup N, we explain how to work with the C's and D's by thinking in terms of paths rather than complicated words.

Lemma 8.3. *Suppose γ_1 and γ_2 are paths in Ω with $\omega(\gamma_1) = \alpha(\gamma_2)$, and let γ be the path which traverses γ_1 and then γ_2:*

(i) *Any normal subgroup of F containing two of C_{γ_1}, C_{γ_2} and C_γ contains the third.*

(ii) *Suppose $k \in I$ is evenly joined to $\omega(\gamma_2)$. Then any normal subgroup of F containing C_{γ_1} and one of $D_{\gamma_2,k}$ and $D_{\gamma,k}$ contains the other as well.*

Proof. Both identities

$$C_\gamma = (P_{\gamma_2} C_{\gamma_1} P_{\gamma_2}^{-1}) C_{\gamma_2},$$
$$D_{\gamma,k} = P_{\gamma_1}^{-1}((R_{\gamma_2,k} C_{\gamma_1} R_{\gamma_2,k}^{-1}) D_{\gamma_2,k} C_{\gamma_1}^{-1}) P_{\gamma_1}$$

unravel to tautologies, using $P_\gamma = P_{\gamma_2} P_{\gamma_1}$. These imply (i) and (ii), respectively. □

To define N we refer to the base vertex ∞ and spanning tree T that we introduced above. For each $j \in \Omega$ we take δ_j to be the backtracking-free path in T from ∞ to j. For each edge of Ω not in T, choose an orientation of it, and define $\mathcal{E}$ as the corresponding set of paths of length 1. For $\gamma \in \mathcal{E}$ we write $z(\gamma)$ for the corresponding loop in Ω based at ∞. That is, $z(\gamma)$ is $\delta_{\alpha(\gamma)}$ followed by γ followed by the reverse of $\delta_{\omega(\gamma)}$. We define Z as $\{z(\gamma) \mid \gamma \in \mathcal{E}\}$, which is a free basis for the fundamental group $\pi_1(\Omega, \infty)$. We define N as the subgroup of F normally generated by all $B_{i\infty}$ with $i \in I$, all $C_{z \in Z}$, the C_{δ_j} with $j \in \Omega$, and all $D_{\delta_j,k}$ where $j \in \Omega$ and $k \in I$ are evenly joined. We will show $M = N$; one direction is easy:

Lemma 8.4. *M contains N.*

Proof. Since M contains C_γ for every length-1 path γ, repeated applications of Lemma 8.3(i) show that it contains the C_{δ_i} and $C_{z \in Z}$. Since M contains $D_{\gamma,k}$ for every γ of length 0, part (ii) of the same lemma shows that M also contains the $D_{\delta_j,k}$. Since M contains all the B_{ij}, not just the $B_{i\infty}$, the proof is complete. □

Now we set about proving the reverse inclusion. For convenience we use $\equiv$ to mean "equal modulo N". We must show that each generator of M is $\equiv 1$.

Lemma 8.5. *$C_\gamma \equiv 1$ for every length-1 subpath γ of every δ_j.*

Proof. This follows from Lemma 8.3(i) because $\delta_{\alpha(\gamma)}$ followed by γ is $\delta_{\omega(\gamma)}$. □

Lemma 8.6. $B_{ik} \equiv 1$ *for all* $i \in I$ *and* $k \in \Omega$.

Proof. We claim that if γ is a length-1 path in Ω, such that $C_\gamma \equiv 1$ and $B_{i\alpha(\gamma)} \equiv 1$ for every $i \in I$, then also $B_{i\omega(\gamma)} \equiv 1$ for every $i \in I$. Assuming this, we use the fact that $B_{i\infty} \equiv 1$ for all $i \in I$ and also $C_\gamma \equiv 1$ for every length-1 subpath γ of every δ_k (Lemma 8.5). Since every $k \in \Omega$ is the end of chain of such γ's starting at ∞, the lemma follows by induction.

So now we prove the claim, writing i for some element of I and j and k for the initial and final endpoints of γ. We use $C_\gamma \equiv 1$, i.e., $S_j S_k X_j(t) \equiv X_k(t) S_j S_k$, to get

$$\begin{aligned} S_i^2 X_k(t) S_i^{-2} &\equiv S_i^2 S_j S_k X_j(t) S_k^{-1} S_j^{-1} S_i^{-2} \\ &= S_j S_k[(S_k^{-1} S_j^{-1} S_i^2 S_j S_k) X_j(t) (S_k^{-1} S_j^{-1} S_i^{-2} S_j S_k)] S_k^{-1} S_j^{-1}. \end{aligned} \tag{8-1}$$

We rewrite the relation from Lemma 7.2(ii) as $S_j^{-1} S_i^2 S_j = S_j^{(-1)^{A_{ij}}-1} S_i^2$. Then we use it and its analogues with subscripts permuted to simplify the first parenthesized term in (8-1). We also use $A_{jk} = -1$, which holds since j and k are joined. The result is

$$S_k^{-1} S_j^{-1} S_i^2 S_j S_k = S_k^{1-(-1)^{A_{ij}}} S_j^{(-1)^{A_{ij}}-1} S_k^{-1+(-1)^{A_{ik}}} S_i^2.$$

Note that each exponent is 0 or ± 2.

The bracketed term in (8-1) is the conjugate of $X_j(t)$ by this. We work this out in four steps, using our assumed relations $B_{ij} \equiv B_{jj} \equiv B_{kj} \equiv 1$. Conjugation by S_i^2 changes $X_j(t)$ to $X_j((-1)^{A_{ij}} t)$. Because $A_{kj} = -1$, conjugating $X_j((-1)^{A_{ij}} t)$ by $S_k^{(-1)^{A_{ik}}-1}$ sends it to

$$\begin{cases} \text{itself} & \text{if } A_{ik} \text{ is even, because } (-1)^{A_{ik}} - 1 = 0, \\ X_j(-(-1)^{A_{ij}} t) & \text{if } A_{ik} \text{ is odd, because } (-1)^{A_{ik}} - 1 = -2. \end{cases}$$

We write this as $X_j\big((-1)^{A_{ik}} (-1)^{A_{ij}} t\big)$. In the third step we conjugate by an even power of S_j, which does nothing. The fourth step is like the second, and introduces a second factor $(-1)^{A_{ij}}$. The net result is that the bracketed term of (8-1) equals $X_j((-1)^{A_{ik}} t)$ modulo N.

Plugging this into (8-1) and then using the conjugacy relation $C_\gamma \equiv 1$ between X_j and X_k yields

$$S_i^2 X_k(t) S_i^{-2} \equiv S_j S_k X_j((-1)^{A_{ik}} t) S_k^{-1} S_j^{-1} \equiv X_k((-1)^{A_{ik}} t).$$

We have established the desired relation $B_{ik} \equiv 1$. □

Lemma 8.7. *Suppose* γ *is a length-*1 *path in* Ω *with* $C_\gamma \equiv 1$. *Then* $C_{\mathrm{reverse}(\gamma)} \equiv 1$ *also.*

Proof. Suppose γ goes from j to k. We begin with our assumed relation $C_\gamma \equiv 1$, i.e., $S_j S_k X_j(t) \equiv X_k(t) S_j S_k$, rearrange and apply the relation from Lemma 7.2(ii) with $A_{jk} = \text{odd}$:

$$X_k(t) \equiv S_j S_k X_j(t) S_k^{-1} S_j^{-1},$$
$$S_k S_j X_k(t) \equiv (S_k S_j^2 S_k^{-1}) S_k^2 X_j(t) S_k^{-1} S_j^{-1} = (S_k^2 S_j^2) S_k^2 X_j(t) S_k^{-1} S_j^{-1}.$$

Now we simplify the right side using Lemma 8.6's $B_{jj} \equiv B_{kj} \equiv 1$ with $A_{kj} = \text{odd}$:

$$\begin{aligned}(S_k^2 S_j^2) S_k^2 X_j(t) S_k^{-1} S_j^{-1} &\equiv S_k^2 S_j^2 X_j(-t) S_k^2 S_k^{-1} S_j^{-1} \\ &\equiv S_k^2 X_j(-t) S_j^2 \cdot S_k^2 S_k^{-1} S_j^{-1} \\ &\equiv X_j(t) S_k^2 S_j^2 \cdot S_k^2 S_k^{-1} S_j^{-1} \\ &= X_j(t) S_k S_j^2 S_k^{-1} \cdot S_k^2 S_k^{-1} S_j^{-1} \\ &= X_j(t) S_k S_j.\end{aligned}$$

We have shown $C_{\text{reverse}(\gamma)} \equiv 1$, as desired. □

Lemma 8.8. *$M = N$. In particular, $\mathfrak{G}_{2,\Omega}$ is the quotient of $F = (*_{j\in\Omega} \mathfrak{U}_j) * \widehat{W}$ by N.*

Proof. We showed $N \subseteq M$ in Lemma 8.4. To show the reverse inclusion, recall that M is normally generated by all B_{ij}, the C_γ for all γ of length 1, and the $D_{\gamma,k}$ for all γ of length 0. We must show that each of these is $\equiv 1$. We showed $B_{ij} \equiv 1$ in Lemma 8.6.

Next we show that $C_\gamma \equiv 1$ for every length-1 path γ in T. If γ is part of one of the paths δ_j in T based at ∞, then $C_\gamma \equiv 1$ by Lemma 8.5, and then $C_{\text{reverse}(\gamma)} \equiv 1$ by Lemma 8.7.

Lemma 8.3(i) now shows $C_\gamma \equiv 1$ for every path γ in T.

Next we show $C_\gamma \equiv 1$ for every length-1 path γ not in T. Recall that we chose a set $\mathcal{E}$ of length-1 paths, one traversing each edge of Ω not in T. For $\gamma \in \mathcal{E}$ we wrote $z(\gamma)$ for the corresponding loop in Ω based at ∞, namely $\delta_{\alpha(\gamma)}$ followed by γ followed by reverse$(\delta_{\omega(\gamma)})$. Recall that N contains $C_{z(\gamma)}$ by definition, and contains $C_{\delta_{\alpha(\gamma)}}$ and $C_{\text{reverse}(\delta_{\omega(\gamma)})}$ by the previous paragraph. So a double application of Lemma 8.3(i) proves $C_\gamma \in N$. And another use of Lemma 8.7 shows that N also contains $C_{\text{reverse}(\gamma)}$. This finishes the proof that $C_\gamma \equiv 1$ for all length-1 paths γ in Ω.

It remains only to show $D_{\gamma,k} \equiv 1$ for every length-0 path γ in Ω and each $k \in I$ joined evenly to the unique point of γ, say j. Since N contains C_{δ_j} and $D_{\delta_j,k}$ by definition, and δ_j followed by γ is trivially equal to δ_j, Lemma 8.3(ii) shows that N contains $D_{\gamma,k}$ also. □

We now review the general form of the description F/N of $\mathfrak{G}_{2,\Omega}$ that we have just established. The generators are the $S_{i\in I}$ and the $X_{j\in\Omega}(t)$, with $t \in R$. The relations are the addition rules defining the $\mathfrak{U}_j$, the relations on the S_i defining $\widehat{W}$, and the $B_{i\infty}$, $C_{z\in Z}$, C_{δ_j} and $D_{\delta_j,k}$, where i varies over I, j over Ω, and $k \in I$ is evenly joined to j. The relations $B_{i\infty} \equiv 1$ say that S_i^2 centralizes or inverts every

$X_\infty(t)$. Each relation $C_z \equiv 1$ says that a certain word in $\widehat{W}$ conjugates every $X_\infty(t)$ to itself. The relations $D_{\delta_j,k} \equiv 1$ say that certain other words in $\widehat{W}$ also commute with every $X_\infty(t)$. Finally, for each j, the relations $C_{\delta_j} \equiv 1$ express the $X_j(t)$ as conjugates of the $X_\infty(t)$ by still more words in $\widehat{W}$. The obvious way to simplify the presentation is to use this last batch of relations to eliminate the $X_{j\neq\infty}(t)$ from the presentation. We make this precise in the following lemma.

Lemma 8.9. *Define $F_\infty = \mathfrak{U}_\infty * \widehat{W}$ and let N_∞ be the subgroup normally generated by the $B_{i\infty}$ ($i \in I$), the C_z ($z \in Z$), and the $D_{\delta_j,k}$ ($j \in \Omega$ and $k \in I$ evenly joined). Then the natural map $F_\infty/N_\infty \to F/N$ is an isomorphism.*

Proof. We begin with the presentation F/N from the previous paragraph and apply Tietze transformations. The relation $C_{\delta_j}(t) \equiv 1$ reads

$$X_j(t) \equiv P_{\delta_j} X_\infty(t) P_{\delta_j}^{-1}.$$

For $j = \infty$ this is the trivial relation $X_\infty(t) = X_\infty(t)$, which we may discard. For $j \neq \infty$ we use it to replace $X_j(t)$ by $P_{\delta_j} X_\infty(t) P_{\delta_j}^{-1}$ everywhere else in the presentation, and then discard $X_j(t)$ from the generators and $C_{\delta_j}(t)$ from the relators.

The only other occurrences of $X_{j\neq\infty}(t)$ in the presentation are in the relators defining $\mathfrak{U}_j$. After the replacement of the previous paragraph, these relations read

$$P_{\delta_j} X_\infty(t) P_{\delta_j}^{-1} \cdot P_{\delta_j} X_\infty(u) P_{\delta_j}^{-1} \equiv P_{\delta_j} X_\infty(t+u) P_{\delta_j}^{-1}.$$

These relations can be discarded because they are the P_{δ_j}-conjugates of the relations $X_\infty(t) X_\infty(u) \equiv X_\infty(t+u)$. What remains is the presentation F_∞/N_∞. □

Proof of Lemma 8.2. The previous lemma shows $\mathfrak{G}_{2,\Omega} \cong F_\infty/N_\infty$. So $\mathfrak{G}_{2,\Omega}$ is the quotient of $\mathfrak{U}_\infty * \widehat{W}$ by relations asserting that certain elements of $\widehat{W}$ act on $\mathfrak{U}_\infty$ by certain automorphisms. The relations $B_{i\infty} = 1$ make S_i^2 act on $\mathfrak{U}_\infty$ by $(-1)^{A_{i\infty}}$. The relations $C_z = D_{\delta_j,k} = 1$ make the words P_z and $R_{\delta_j,k}$ centralize $\mathfrak{U}_\infty$.

By Lemma 7.2(iii), the S_i^2 generate the kernel of $\widehat{W} \to W$. By Theorem 5.7, the images of the P_z and $R_{\delta_j,k}$ in W generate the W-stabilizer of the simple root $\infty \in I$. Therefore, the S_i^2, P_z and $R_{\delta_j,k}$ generate the $\widehat{W}$-stabilizer $\widehat{W}_\infty$ of ∞. Their actions on $\mathfrak{U}_\infty$ are the same as the ones given by the homomorphism $\widehat{W} \to W^*$, by Theorem 5.7. Therefore, $\mathfrak{G}_{2,\Omega} = (\mathfrak{U}_\infty \rtimes \widehat{W}_\infty) *_{\widehat{W}_\infty} \widehat{W}$. And Lemma 8.1 identifies this with $(*_{\alpha\in\Phi(\Omega)} \mathfrak{U}_\alpha) \rtimes \widehat{W}$, as desired. □

9. The isomorphism $\mathfrak{G}_3 \cong \mathfrak{PSt}^{\mathrm{Tits}} \rtimes \widehat{W}$

We have two goals in this section. The first is to start from Theorem 7.5, that $\mathfrak{G}_2 \cong (*_{\alpha\in\Phi} \mathfrak{U}_\alpha) \rtimes \widehat{W}$, and prove Theorem 7.11, that $\mathfrak{G}_3 \cong \mathfrak{PSt}^{\mathrm{Tits}} \rtimes \widehat{W}$. The second is to explain how one may discard many of the Chevalley relations; for example, for $E_{n\geq 6}$ one can get away with imposing the relations for a single unjoined pair of nodes of the Dynkin diagram, and for a single joined pair. The latter material is not necessary for our main results.

Proof of Theorem 7.11. First we show that the relators (7-10)–(7-23), regarded as elements of $\mathfrak{G}_2 \cong (*_{\alpha\in\Phi}\,\mathfrak{U}_\alpha) \rtimes \widehat{W}$, become trivial in $\mathfrak{St}^{\mathrm{Tits}} \rtimes \widehat{W}$. Then we will show that they normally generate the whole kernel of $\mathfrak{G}_2 \to \mathfrak{St}^{\mathrm{Tits}} \rtimes \widehat{W}$.

If α, β are a prenilpotent pair of roots with $\theta(\alpha, \beta) = \{\alpha, \beta\}$, then the Chevalley relation for α and β is $[\mathfrak{U}_\alpha, \mathfrak{U}_\beta] = 1$. This shows that relators (7-10), (7-11), (7-13), (7-14), (7-17), (7-18) and (7-19) become trivial in $\mathfrak{St}^{\mathrm{Tits}} \rtimes \widehat{W}$. Careful calculation verifies that the remaining relators are equivalent to those given by Demazure in [SGA 3$_{\text{III}}$ 1970, Exposé XXIII]. Here are some remarks on the correspondence between his notation and ours. In the A_2 case (his Proposition 3.2.1), his α and β correspond to our α_j and α_i, his X_α and X_β to our e_j and e_i, his $X_{-\alpha}$ and $X_{-\beta}$ to our $-f_j$ and $-f_i$, and his $p_\alpha(t)$ and $p_\beta(t)$ to our $X_j(t)$ and $X_i(t)$. His w_α and w_β are not the same as our S_j and S_i (which are not even elements of $*_{\gamma\in\Phi}\,\mathfrak{U}_\gamma$), but their actions on the $\mathfrak{U}_\gamma$ are the same, so his $p_{\alpha+\beta}(t) := w_\beta p_\alpha(t) w_\beta^{-1}$ corresponds to our $S_i X_j(t) S_i^{-1}$. One can now check that our (7-12) is equivalent to his Proposition 3.2.1(iii).

In the B_2 case (his Proposition 3.3.1), his α and β correspond to our α_s and α_l, his X_α and X_β to our e_s and e_l, his $X_{-\alpha}$ and $X_{-\beta}$ to our $-f_s$ and $-f_l$, and his $p_\alpha(t)$ and $p_\beta(t)$ to our $X_s(t)$ and $X_l(t)$. His w_α and w_β correspond to our S_s and S_l in the same sense as above. It follows that his $p_{\alpha+\beta}(t)$ and $p_{2\alpha+\beta}(t)$ correspond to our $S_l X_s(t) S_l^{-1}$ and $S_s X_l(t) S_s^{-1}$. Then our (7-15) and (7-16) are equivalent to his Proposition 3.3.1. The G_2 case is the same (his Proposition 3.4.1), except that his $p_{\alpha+\beta}(t)$, $p_{2\alpha+\beta}(t)$, $p_{3\alpha+\beta}(t)$ and $p_{3\alpha+2\beta}(t)$ correspond to our

$$S_l X_s(t) S_l^{-1}, \quad S_s S_l X_s(t) S_l^{-1} S_s^{-1}, \quad S_s X_l(-t) S_s^{-1} \quad \text{and} \quad S_l S_s X_l(-t) S_s^{-1} S_l^{-1}.$$

Then our (7-20)–(7-23) are among the relations in his Proposition 3.4.1(iii).

As a check (indeed a second proof that our relations are the Chevalley relations) we constructed our elements of the various root groups in explicit representations of the Chevalley groups $\mathrm{SL}_2 \times \mathrm{SL}_2$, SL_3, Sp_4 and G_2 over $R = \mathbb{Z}[t, u]$, faithful on the unipotent subgroups of their Borel subgroups. As mentioned in Remark 7.15, we used a computer to check that our relators map to the identity. By functoriality, the same holds with R replaced by any ring. In addition to our relations, the root groups satisfy the Chevalley relations, by construction. By the isomorphism $\mathfrak{U}_{\theta(\alpha,\beta)} \cong \prod_{\gamma\in\theta(\alpha,\beta)} \mathfrak{U}_\gamma$ of underlying schemes (Lemma 6.1), the only relations having the form of the Chevalley relations that can hold are the Chevalley relations themselves. So our relations are among them.

It remains to prove that the Chevalley relators of any classically prenilpotent pair α', $\beta' \in \Phi$ become trivial in $\mathfrak{G}_3$. By classical prenilpotency, $\Phi_0' := (\mathbb{Q}\alpha' + \mathbb{Q}\beta') \cap \Phi$

is an A_1, A_1^2, A_2, B_2 or G_2 root system. In the A_1 case we have $\alpha' = \beta'$ and the Chevalley relations amount to the commutativity of $\mathfrak{U}_{\alpha'}$. This follows from $\mathfrak{U}_{\alpha'} \cong R$. So we consider the other cases. There exists $w \in W$ sending Φ_0' to the root system $\Phi_0 \subseteq \Phi$ generated by some pair of simple roots. (Choose simple roots for Φ_0'. Then choose a chamber in the Tits cone which has two of its facets lying in the mirrors of those roots, and which lies on the positive sides of these mirrors. Choose w to send this chamber to the standard one.)

We choose a pair of roots $\alpha, \beta \in \Phi_0$ as follows. First, they should have the same relative configuration as α', β' have. (That is, they should have the same short/long root status, and make the same angle.) And second, their Chevalley relators should appear among (7-10)–(7-23). Such α, β can always be chosen. For example, in the G_2 case, (7-17)–(7-23) are, respectively, the Chevalley relations for two long roots with angle $\pi/3$, a short and a long root with angle $\pi/6$, two orthogonal roots, two long roots with angle $2\pi/3$, two short roots with angle $\pi/3$, two short roots with angle $2\pi/3$, and a short and a long root with angle $5\pi/6$. The other cases are similarly exhaustive. By refining the choice of w, we may suppose that it sends $\{\alpha', \beta'\}$ to $\{\alpha, \beta\}$. Now choose $\hat{w} \in \widehat{W}$ lying over w. The Chevalley relators for α', β' are the $\hat{w}^{-1}$-conjugates of the Chevalley relators for α, β. Since the latter become trivial in $\mathfrak{G}_3$, so do the former. □

The proof of Theorem 7.11 exploited the $\widehat{W}$-action on $*_{\alpha\in\Phi}\,\mathfrak{U}_\alpha$ to obtain the Chevalley relators for all classically prenilpotent pairs from those listed explicitly in (7-10)–(7-23). One can further exploit this idea to omit many of the relators coming from the cases $m_{ij} = 2$ or 3. Our method derives from the notion of an ordered pair of simple roots being associate to another pair, due to [Brink and Howlett 1999] and [Borcherds 1998]. But we need very little of their machinery, so we will argue directly. There does not seem to be any similar simplification possible if $m_{ij} = 4$ or 6.

Proposition 9.1. *Suppose $i, j, k \in I$ form an A_1A_2 diagram, with j and k joined. Then imposing the relation $[\mathfrak{U}_i, \mathfrak{U}_j] = 1$ on $\mathfrak{G}_2 \cong (*_{\alpha\in\Phi}\,\mathfrak{U}_\alpha) \rtimes \widehat{W}$ also imposes $[\mathfrak{U}_i, \mathfrak{U}_k] = 1$. More formally, the normal closure of the relators* (7-10) *in $\mathfrak{G}_2$ contains the relators got from them by replacing j by k.*

Proof. Some element of the copy of $W(A_2)$ generated by s_j and s_k sends α_j to α_k, and of course it fixes α_i. Choose any lift of it to $\widehat{W}$. Conjugation by it in $\mathfrak{G}_2$ fixes $\mathfrak{U}_i$ and sends $\mathfrak{U}_j$ to $\mathfrak{U}_k$. So it sends the relators (7-10) to the relators got from them by replacing j by k. □

The lemma shows that imposing on $\mathfrak{G}_2$ the relations (7-10) for a few well-chosen unordered pairs $\{i, j\}$ in I with $m_{ij} = 2$ automatically imposes the corresponding relations for all such pairs. As examples, for spherical Dynkin diagrams it suffices to impose these relations for

$$\begin{cases} \text{3 such pairs (that is, all of them)} & \text{for } D_4, \\ \text{2 such pairs} & \text{for } B_{n\geq 4}, C_{n\geq 4} \text{ or } D_{n\geq 5}, \\ \text{1 such pair} & \text{for } A_{n\geq 3}, B_3, C_3, E_n \text{ or } F_4. \end{cases}$$

Proposition 9.2. *Suppose $i, j, k \in I$ form an A_3 diagram, with i and k unjoined. Then the normal closure of the relators* (7-11)–(7-12) *in $\mathfrak{G}_2 \cong (*_{\alpha\in\Phi}\,\mathfrak{U}_\alpha) \rtimes \widehat{W}$ contains the relators got from them by replacing i and j by j and k, respectively.*

Proof. The argument is the same as for Proposition 9.1, using an element of $W(A_3)$ that sends α_i and α_j to α_j and α_k. An example of such an element is the "fundamental element" (or "long word") of $\langle s_i, s_j\rangle$, followed by the fundamental element of $\langle s_i, s_j, s_k\rangle$. The first transformation sends α_i and α_j to $-\alpha_j$ and $-\alpha_i$. The second sends α_i, α_j and α_k to $-\alpha_k$, $-\alpha_j$ and $-\alpha_i$. □

Similarly to the $m_{ij} = 2$ case, imposing on $\mathfrak{G}_2$ the relations (7-11)–(7-12) for some well-chosen ordered pairs (i, j) in I with $m_{ij} = 3$ automatically imposes the corresponding relations for all such pairs. For spherical diagrams, it suffices to impose these relations for

$$\begin{cases} \text{4 such pairs (that is, all of them)} & \text{for } F_4, \\ \text{2 such pairs} & \text{for } A_{n\geq 2}, B_{n\geq 3} \text{ or } C_{n\geq 3}, \\ \text{1 such pair} & \text{for } D_{n\geq 4} \text{ or } E_n. \end{cases}$$

10. The adjoint representation

A priori, it is conceivable that for some commutative ring $R \neq 0$ and some generalized Cartan matrix A, the Steinberg group $\mathfrak{St}_A(R)$ might collapse to the trivial group. That this doesn't happen follows from work of Tits [1987, §4] and Rémy [2002, Chapter 9] on the "adjoint representation" of $\mathfrak{St}_A$. We will improve their results slightly by proving that the unipotent group scheme $\mathfrak{U}_\Psi$ embeds in the Steinberg group functor $\mathfrak{St}_A$, for any nilpotent set of roots Ψ. We need this result in the next section, in our proof that $\mathfrak{PSt}_A(R) \to \mathfrak{St}_A(R)$ is often an isomorphism.

Recall that Lemma 6.1 associates to Ψ a unipotent group scheme $\mathfrak{U}_\Psi$ over $\mathbb{Z}$. Furthermore, there are natural homomorphisms $\mathfrak{U}_\gamma \to \mathfrak{U}_\Psi$ for all $\gamma \in \Psi$, and the product map $\prod_{\gamma\in\Psi} \mathfrak{U}_\gamma \to \mathfrak{U}_\Psi$ is an isomorphism of the underlying schemes, for any ordering of the factors.

Also in Section 6, we defined Tits' Steinberg functor $\mathfrak{St}_A^{\text{Tits}}$ as the direct limit of the group schemes $\mathfrak{U}_\gamma$ and $\mathfrak{U}_\Psi$, where γ varies over Φ, and Ψ varies over the nilpotent subsets of Φ of the form $\Psi = \theta(\alpha, \beta)$, with α, β a prenilpotent pair of roots. Composing with $\mathfrak{St}_A^{\text{Tits}} \to \mathfrak{St}_A$, we have natural maps $\mathfrak{U}_\Psi \to \mathfrak{St}_A$ for such Ψ. A special case of the following theorem is that these maps are embeddings. We would like to say that the same holds for Ψ an arbitrary nilpotent set of roots. But "the same holds" doesn't quite have meaning, because the definition of $\mathfrak{St}_A$ doesn't provide a natural map $\mathfrak{U}_\Psi \to \mathfrak{St}_A$ for general Ψ. So we phrase the result as follows.

Theorem 10.1 (injection of unipotent subgroups into $\mathfrak{St}_A$). *Suppose A is a generalized Cartan matrix and Ψ is a nilpotent set of roots. Then there is a unique homomorphism $\mathfrak{U}_\Psi \to \mathfrak{St}_A$ whose restriction to each $\mathfrak{U}_{\alpha\in\Psi}$ is the natural map to $\mathfrak{St}_A$, and it is an embedding.*

Uniqueness is trivial, by the isomorphism of underlying schemes $\mathfrak{U}_\Psi \cong \prod_{\alpha\in\Psi} \mathfrak{U}_\alpha$. Existence is easy: every pair of roots in Ψ is prenilpotent, their Chevalley relations hold in $\mathfrak{St}$, and these relations suffice to define $\mathfrak{U}_\Psi$ as a quotient of $*_{\beta\in\Psi} \mathfrak{U}_\beta$. So we must show that this homomorphism is an embedding. Our proof below relies on a linear representation of $\mathfrak{St}_A$, functorial in R, called the adjoint representation. Its essential properties are developed in [Rémy 2002, Chapter 9], relying on a $\mathbb{Z}$-form of the universal enveloping algebra of $\mathfrak{g}$ introduced in [Tits 1987, §4].

Following Tits and Rémy we will indicate all ground rings other than $\mathbb{Z}$ explicitly, in particular writing $\mathfrak{g}_\mathbb{C}$ for the Kac–Moody algebra $\mathfrak{g}$. We write $\mathcal{U}_\mathbb{C}$ for its universal enveloping algebra. Recall from Section 6 that for each root $\alpha \in \Phi$ we distinguished a subgroup $\mathfrak{g}_{\alpha,\mathbb{Z}} \cong \mathbb{Z}$ of $\mathfrak{g}_{\alpha,\mathbb{C}}$ and the set E_α consisting of the two generators for $\mathfrak{g}_{\alpha,\mathbb{Z}}$.

Generalizing work of Kostant [1966] and Garland [1978], Tits defined an integral form of $\mathcal{U}_\mathbb{C}$, meaning a subring $\mathcal{U}$ with the property that the natural map $\mathcal{U}\otimes\mathbb{C}\to\mathcal{U}_\mathbb{C}$ is an isomorphism. It is the subring generated by the divided powers $e_i^n/n!$ and $f_i^n/n!$, as i varies over I, together with the "binomial coefficients"

$$\binom{h}{n} := h(h-1)\cdots(h-n+1)/n!,$$

where h varies over the $\mathbb{Z}$-submodule of $\mathfrak{g}_{0,\mathbb{C}}$ with basis $\bar{h}_i$.

Remark 10.2 (the role of the root datum). Although it isn't strictly necessary, we mention that lurking behind the scenes is a choice of root datum. It is the one which Rémy calls simply connected [2002, §7.1.2] and Tits calls "simply connected in the strong sense" [1987, Remark 3.7(c)]. A choice of root datum is necessary to define $\mathcal{U}$, hence the adjoint representation, and the choice does matter. For example, SL_2 and PGL_2 have the same Cartan matrix, but different root data. Their adjoint representations are distinct in characteristic 2, when we compare them by regarding both as representations of SL_2 via the central isogeny $SL_2 \to PGL_2$. Similarly, they provide distinct representations of $\mathfrak{St}_{A_1}$. For us the essential fact is that each $\bar{h}_i$ generates a $\mathbb{Z}$-module summand of $\mathcal{U}$, as explained in the next paragraph. As an example of what could go wrong, using the root datum for PGL_2 would lead to $\bar{h}_i/2 \in \mathcal{U}$ and spoil the proof of Theorem 10.1 in characteristic 2.

In the sense Tits used, an integral form of a $\mathbb{C}$-algebra need not be free as a $\mathbb{Z}$-module. For example, $\mathbb{Q}$ is a $\mathbb{Z}$-form of $\mathbb{C}$ since $\mathbb{Q}\otimes_\mathbb{Z}\mathbb{C}\to\mathbb{C}$ is an isomorphism.

But $\mathcal{U}$ is free as a $\mathbb{Z}$-module. To see this, one uses the following ingredients from [Tits 1987, §4.4]. First, the $\mathbb{Z}^I$-grading makes it easy to see that

$$\mathcal{U}_+ := \langle \{e_i^n/n! \mid i \in I \text{ and } n \geq 0\}\rangle \quad \text{and} \quad \mathcal{U}_- := \langle \{f_i^n/n! \mid i \in I \text{ and } n \geq 0\}\rangle$$

are free as $\mathbb{Z}$-modules, and that $\{e_{i\in I}\}$ and $\{f_{i\in I}\}$ extend to bases of them. Second, the universal enveloping algebra $\mathcal{U}_{0,\mathbb{C}}$ of the Cartan algebra $\mathfrak{g}_{0,\mathbb{C}}$ is a polynomial ring. This makes it easy to see that

$$\mathcal{U}_0 := \left\langle \left\{ \tbinom{h}{n} \mid h \in \textstyle\bigoplus_i \mathbb{Z}\bar{h}_i \text{ and } n \geq 0 \right\} \right\rangle$$

is free as a $\mathbb{Z}$-module. Indeed, Proposition 2 of [Bourbaki 1975, Chapter VIII, §12.4] extends $\{\bar{h}_{i\in I}\}$ to a $\mathbb{Z}$-basis for $\mathcal{U}_0$. Finally, $\mathcal{U}_- \otimes \mathcal{U}_0 \otimes \mathcal{U}_+ \to \mathcal{U}$ is an isomorphism by [Tits 1987, Proposition 2]. One can obtain a $\mathbb{Z}$-basis for $\mathcal{U}$ by tensoring together members of bases for $\mathcal{U}_-$, $\mathcal{U}_0$ and $\mathcal{U}_+$.

A key property of $\mathcal{U}$ is its stability under $(\operatorname{ad} e_i)^n/n!$ and $(\operatorname{ad} f_i)^n/n!$ for all $n \geq 0$ (see [Tits 1987, equation (12)]). The local nilpotence of $\operatorname{ad} e_i$ and $\operatorname{ad} f_i$ on $\mathfrak{g}_\mathbb{C}$ implies their local nilpotence on $\mathcal{U}_\mathbb{C}$. As exponentials of locally nilpotent derivations, $\exp \operatorname{ad} e_i$ and $\exp \operatorname{ad} f_i$ are automorphisms of $\mathcal{U}_\mathbb{C}$. Since they preserve its subring $\mathcal{U}$, they are automorphisms of it. Since the generators s_i^* for W^* are defined in terms of them by (5-1), W^* also acts on $\mathcal{U}$.

Because $\mathcal{U}$ is free as a $\mathbb{Z}$-module, $\mathcal{U}_R := \mathcal{U} \otimes R$ is free as an R-module. It is the R-module underlying the adjoint representation of $\mathfrak{St}_A(R)$ in Theorem 10.3 below, which we will now develop. For each root α we define an exponential map $\exp : \mathfrak{U}_\alpha(R) \to \operatorname{Aut}(\mathcal{U}_R)$ as follows. Recall that $\mathfrak{U}_\alpha(R)$ was defined as $\mathfrak{g}_{\alpha,\mathbb{Z}} \otimes R$. If x is an element of this, then we choose $e \in E_\alpha$ and define $t \in R$ by $x = te$. Then we define $\exp x$ to be the R-module endomorphism of $\mathcal{U}_R$ given by $\sum_{n=0}^\infty t^n (\operatorname{ad} e)^n/n!$. The apparent dependence on the choice of e is no dependence at all, because if one makes the other choice $-e$ then one must also replace t by $-t$. As shown in [Rémy 2002, §9.4], $\exp x$ is an R-algebra automorphism of $\mathcal{U}_R$, not merely an R-module endomorphism.

Theorem 10.3 (adjoint representation). *For any commutative ring R, there exists a homomorphism* $\operatorname{Ad} : \mathfrak{St}_A(R) \to \operatorname{Aut} \mathcal{U}_R$, *functorial in R and characterized by the following property. For every root α the exponential map* $\exp : \mathfrak{U}_\alpha(R) \to \operatorname{Aut} \mathcal{U}_R$ *factors as the natural map* $\mathfrak{U}_\alpha(R) \to \mathfrak{St}_A(R)$ *followed by* Ad.

Proof. This is from Sections 9.5.2–9.5.3 of [Rémy 2002]. We remark that he used Tits' version of the Steinberg functor (what we call $\mathfrak{St}_A^{\text{Tits}}$) rather than the Morita–Rehmann version (what we call $\mathfrak{St}_A$). But his Theorem 9.5.2 states that Ad is a representation of Tits' Kac–Moody group $\tilde{\mathcal{G}}_D(R)$. Since the extra relations in the Morita–Rehmann version of the Steinberg group are among those defining $\tilde{\mathcal{G}}_D(R)$, we may regard Ad as a representation of $\mathfrak{St}_A(R)$.

A few comments are required to identify our relations with (some of) his. $\tilde{\mathcal{G}}_D(R)$ is defined in [Rémy 2002, §8.3.3] as a quotient of the free product of $\mathfrak{PSt}_A^{\mathrm{Tits}}(R)$ with a certain torus $\mathcal{T}$. Rémy's third relation identifies our $\tilde{h}_i(r)$ from (6-3) with the element of $\mathcal{T}$ that Rémy calls r^{h_i}. Rémy's first relation says how $\mathcal{T}$ acts on each $\mathfrak{U}_j$, and amounts to our (6-8). Rémy's fourth relation is our (6-10), saying that each $\tilde{s}_i$ acts as s_i^* on every $\mathfrak{U}_\beta$. Rémy's second relation says how each $\tilde{s}_i$ acts on $\mathcal{T}$, and in particular describes $\tilde{s}_j r^{h_i} \tilde{s}_j^{-1}$. Together with the known action of $\tilde{h}_i(r)$ on $\mathfrak{U}_j$ and the fact that $\tilde{s}_j$ exchanges $\mathfrak{U}_{\pm j}$, this describes how $\tilde{h}_i(r)$ acts on $\mathfrak{U}_{-j}$, and recovers our relation (6-9). By Theorem 6.4, this shows that all the relations in our $\mathfrak{St}(R)$ hold in $\tilde{\mathcal{G}}_D(R)$. □

Proof of Theorem 10.1. By induction on $|\Psi|$. The base case, with $\Psi = \varnothing$, is trivial. So suppose $|\Psi| > 0$. Since Ψ is nilpotent, there is some chamber pairing positively with every member of Ψ and another one pairing negatively with every member. It follows that there is a chamber pairing positively with one member and negatively with all the others. In other words, after applying an element of W^* we may suppose that Ψ contains exactly one positive root. We may even suppose that this root is simple, say α_i. Write Ψ_0 for $\Psi - \{\alpha_i\}$.

Consider the adjoint representation $\mathfrak{U}_\Psi(R) \to \mathfrak{St}(R) \to \operatorname{Aut}\mathcal{U}_R$, in particular the action of $x \in \mathfrak{U}_\Psi(R)$ on $f_i \in \mathcal{U}_R$. If $x \in \mathfrak{U}_{\Psi_0}(R)$ then the component of $x(f_i)$ in the subspace of $\mathcal{U}_R$ graded by $0 \in \mathbb{Z}^I$ is trivial, since f_i and the $\beta \in \Psi_0$ are all negative roots. On the other hand, we can work out the action of $\mathfrak{x}_i(t)$ as follows. A computation in $\mathcal{U}$ shows

$$(\operatorname{ad} e_i)(f_i) = -\bar{h}_i, \quad \tfrac{1}{2}(\operatorname{ad} e_i)^2(f_i) = e_i, \quad \text{and} \quad \tfrac{1}{n!}(\operatorname{ad} e_i)^n(f_i) = 0$$

for $n > 2$. Therefore, we have

$$\operatorname{Ad}(\mathfrak{x}_i(t))(f_i) = \sum_{n=0}^{\infty} t^n \frac{(\operatorname{ad} e_i)^n}{n!}(f_i) = f_i - t\bar{h}_i + t^2 e_i.$$

Recall that f_i, $\bar{h}_i$ and e_i are three members of a $\mathbb{Z}$-basis for $\mathcal{U}$. So their images in $\mathcal{U}_R$ are members of an R-basis. If $t \neq 0$ then the component of $\operatorname{Ad}(\mathfrak{x}_i(t))(f_i)$ graded by $0 \in \mathbb{Z}^I$ is the nonzero element $-t\bar{h}_i$ of $\mathcal{U}_R$.

Therefore, only the trivial element of $\mathfrak{U}_i(R)$ maps into the image of $\mathfrak{U}_{\Psi_0}(R)$ in $\operatorname{Aut}\mathcal{U}_R$. So the same is true with $\mathfrak{St}(R)$ in place of $\operatorname{Aut}\mathcal{U}_R$. From induction and the bijectivity of the product map $\mathfrak{U}_i(R) \times \mathfrak{U}_{\Psi_0}(R) \to \mathfrak{U}_\Psi(R)$ it follows that $\mathfrak{U}_\Psi(R)$ embeds in $\mathfrak{St}(R)$. □

11. $\mathfrak{PSt} \to \mathfrak{St}$ is often an isomorphism

The purpose of this section is to prove parts (iii)–(iv) of Theorem 1.1, showing that the natural map $\mathfrak{PSt}_A(R) \to \mathfrak{St}_A(R)$ is an isomorphism for many choices of generalized Cartan matrix A and commutative ring R. These cases include most of part (ii) of the same theorem; see [Allcock 2016] for the complete result. And part (i) of the theorem is the case that A is spherical. As remarked in Section 7, in this case $\mathfrak{PSt}_A$ and $\mathfrak{St}_A$ are the same group by definition.

In the case that R is a field, Abramenko and Mühlherr [1997] proved our (iv) with Kac–Moody groups in place of Steinberg groups. Our proof of (iv) derives from the proof of their Theorem A; with the following preparatory lemma, the argument goes through in our setting. For (iii) we use a more elaborate form of the idea, with Lemma 11.2 as preparation.

Lemma 11.1 (generators for unipotent groups in rank 2). *Let R be a commutative ring, Φ be a rank-2 spherical root system equipped with a choice of simple roots, and Φ^+ be the set of positive roots. If Φ has type A_1^2 or A_2 then $\mathfrak{U}_{\Phi^+}(R)$ is generated by the root groups of the simple roots.*

If Φ has type B_2 then write α_s and α_l for the short and long simple roots, and $\alpha_{s'}$ (resp. $\alpha_{l'}$) for the image of α_s (resp. α_l) under reflection in α_l (resp. α_s). Then $\mathfrak{U}_{\Phi^+}(R)$ is generated by $\mathfrak{U}_s(R)$, $\mathfrak{U}_l(R)$ and either one of $\mathfrak{U}_{s'}(R)$ and $\mathfrak{U}_{l'}(R)$. If R has no quotient $\mathbb{F}_2$ then $\mathfrak{U}_s(R)$ and $\mathfrak{U}_l(R)$ suffice.

If Φ has type G_2 then, using notation as for B_2, $\mathfrak{U}_{\Phi^+}(R)$ is generated by $\mathfrak{U}_s(R)$, $\mathfrak{U}_l(R)$ and $\mathfrak{U}_{s'}(R)$. If R has no quotient $\mathbb{F}_2$ or $\mathbb{F}_3$ then $\mathfrak{U}_s(R)$ and $\mathfrak{U}_l(R)$ suffice.

Proof. We will suppress the dependence of group functors on R, always meaning groups of points over R. The A_1^2 case is trivial because the simple roots are the only positive roots.

In the A_2 case we write α_i and α_j for the simple roots. The only other positive root is $\alpha_i + \alpha_j$. As in Section 6, we choose $e_i \in E_i$ and $e_j \in E_j$. Then we can use the notation $X_i(t)$, $X_j(t)$ for the elements of $\mathfrak{U}_i$ and $\mathfrak{U}_j$, where t varies over R. The Chevalley relation (7-12) is $[X_i(t), X_j(u)] = S_i X_j(tu) S_i^{-1}$. Therefore, every element of $S_i \mathfrak{U}_j(R) S_i^{-1}$ lies in $\langle \mathfrak{U}_i(R), \mathfrak{U}_j(R)\rangle$. Since $S_i \mathfrak{U}_j S_i^{-1} = \mathfrak{U}_{\alpha_i+\alpha_j}$, the proof is complete.

In the B_2 and G_2 cases we choose $e_s \in E_s$ and $e_l \in E_l$, so we may speak of $X_s(t) \in \mathfrak{U}_s$ and $X_l(u) \in \mathfrak{U}_l$. We write $X_{s'}(t)$ for $S_l X_s(t) S_l^{-1}$ and $X_{l'}(t)$ for $S_s X_l(t) S_s^{-1}$. In the G_2 case we also define

$$X_{s''}(t) = S_s S_l X_s(t) S_l^{-1} S_s^{-1} \quad \text{and} \quad X_{l''}(t) = S_l S_s X_l(t) S_s^{-1} S_l^{-1}.$$

Rather than mimicking the direct computation of the A_2 case, we use the well-known fact that a subset of a nilpotent group generates that group if and only if its image in the abelianization generates the abelianization. We will apply this to the subgroup of $\mathfrak{U}_{\Phi^+}$ generated by $\mathfrak{U}_s \cup \mathfrak{U}_l$. Namely, we write Q for the quotient of the abelianization of $\mathfrak{U}_{\Phi^+}$ by the image of $\langle \mathfrak{U}_s, \mathfrak{U}_l \rangle$. Under the hypotheses about R having no tiny fields as quotients, we will prove $Q = 0$. In this case it follows that

$\langle \mathfrak{U}_s, \mathfrak{U}_l \rangle$ maps onto the abelianization and is therefore all of $\mathfrak{U}_{\Phi^+}$. We must also prove, this time with no hypotheses on R, that $\mathfrak{U}_{\Phi^+} = \langle \mathfrak{U}_s, \mathfrak{U}_l, \mathfrak{U}_{s'} \rangle$ and (in the B_2 case) that $\mathfrak{U}_{\Phi^+} = \langle \mathfrak{U}_s, \mathfrak{U}_l, \mathfrak{U}_{l'} \rangle$. This uses the same argument, with calculations so much simpler that we omit them.

First consider the B_2 case. Among the Chevalley relators defining $\mathfrak{U}_{\Phi^+}$ are (7-15) and (7-16), namely

$$[X_s(t), X_{s'}(u)] \cdot X_{l'}(2tu)$$
$$[X_s(t), X_l(u)] \cdot X_{l'}(-t^2u) X_{s'}(tu)$$

for all $t, u \in R$. The remaining Chevalley relations say that various root groups commute with various other root groups. Therefore, the abelianization of $\mathfrak{U}_{\Phi^+}$ is the quotient of the abelian group

$$\mathfrak{U}_s \times \mathfrak{U}_l \times \mathfrak{U}_{s'} \times \mathfrak{U}_{l'} \cong R^4$$

by the images of the displayed relators. We obtain Q by killing the image of $\mathfrak{U}_s \times \mathfrak{U}_l$.

So, changing to additive notation, Q is the quotient of $\mathfrak{U}_{s'} \oplus \mathfrak{U}_{l'} \cong R^2$ by the subgroup generated by $0 \oplus 2R$ and all $(tu, -t^2u)$, where t, u vary over R. Taking $t = 1$ in the latter shows that $2R \oplus 0$ also dies in Q. So Q is the quotient of $(R/2R)^2$ by the subgroup generated by all $(tu, -t^2u)$. That is, Q is (the abelian group underlying) the quotient of $(R/2R)^2$ by the submodule(!) generated by all $(t, -t^2)$. This submodule contains $(1, -1)$, so it is equally well-generated by it and all $(t, -t^2) - t(1, -1) = (0, t - t^2)$. We may discard the first summand $R/2R$ from the generators and $(1, -1)$ from the relators. So Q is the (abelian group underlying) the quotient of $R/2R$ by the ideal I generated by all $t - t^2$. To prove $Q = 0$ we will suppose $Q \neq 0$ and derive a contradiction. As a nonzero ring with identity, R/I has some field as a quotient, in which $t = t^2$ holds identically. The only field with this property is $\mathbb{F}_2$, which is a contradiction since we supposed that R has no such quotient.

For the G_2 case the Chevalley relators include

$$[X_l(t), X_{l'}(u)] \cdot X_{l''}(-tu)$$
$$[X_s(t), X_{s''}(u)] \cdot X_{l'}(-3tu)$$
$$[X_s(t), X_{s'}(u)] \cdot X_{l''}(3tu^2) X_{l'}(3t^2u) X_{s''}(2tu)$$
$$[X_s(t), X_l(u)] \cdot X_{l''}(t^3u^2) X_{l'}(-t^3u) X_{s'}(tu) X_{s''}(-t^2u)$$
$$[X_{s'}(t), X_{s''}(u)] \cdot X_{l''}(-3tu)$$

for all $t, u \in R$. The first four relations are from (7-20)–(7-23). The fifth is the conjugate of (7-21) by S_l, which commutes with $\mathfrak{U}_{s''}$ and sends $X_s(t)$ to $X_{s'}(t)$ and $X_{l'}(-3tu)$ to $X_{l''}(-3tu)$, by their definitions. All the remaining Chevalley relations say that various root groups commute with each other.

Proceeding as in the B_2 case, we see that Q is the quotient of the abelian group $\mathfrak{U}_{l'} \oplus \mathfrak{U}_{s''} \oplus \mathfrak{U}_{s'} \oplus \mathfrak{U}_{l''}$ by the subgroup generated by the relators $(0, 0, 0, -tu)$, $(-3tu, 0, 0, 0)$, $(3t^2u, 2tu, 0, 3tu^2)$, $(-t^3u, -t^2u, tu, t^3u^2)$ and $(0, 0, 0, -3tu)$, where t, u vary over R. Because of the first relator, we may discard the $\mathfrak{U}_{l''}$ summand. This leads to the following description of Q: the quotient of R^3 by the R-submodule spanned by the relators $(-3t, 0, 0)$, $(3t^2, 2t, 0)$ and $(-t^3, -t^2, t)$, where t varies over R. Using $(-1, -1, 1)$ in the same way we used $(1, -1)$ in the B_2 case shows that Q is the quotient of R^2 by the submodule generated by all $(-3t, 0)$, $(3t^2, 2t)$ and $(t^3 - t, t^2 - t)$. This is the same as the quotient of $R/3R \oplus R/2R$ by the submodule generated by all $(t^3 - t, t^2 - t)$. Now, $R/3R \oplus R/2R$ is isomorphic to $R/6R$ by $(a, b) \leftrightarrow 2a + 3b$. So Q is the quotient of $R/6R$ by the ideal I generated by $2(t^3 - t) + 3(t^2 - t)$ for all t. As in the B_2 case, if $Q \neq 0$ then it has a further quotient that is a field F, obviously of characteristic 2 or 3. In F, either $t^2 = t$ holds identically or $t^3 = t$ holds identically, according to these two possibilities. So $F = \mathbb{F}_2$ or $\mathbb{F}_3$, a contradiction. □

Lemma 11.2 (generators for unipotent groups in rank 3). *Let R be a commutative ring, Φ be a spherical root system of rank 3, $\{\beta_{i\in I}\}$ be simple roots for it, and Φ^+ be the corresponding set of positive roots. Write s_i for the reflection in β_i, and for each ordered pair (i, j) of distinct elements of I write $\gamma_{i,j}$ for $s_i(\beta_j)$. Then $\mathfrak{U}_{\Phi^+}(R)$ is generated by the $\mathfrak{U}_{\beta_i}(R)$ and the $\mathfrak{U}_{\gamma_{i,j}}(R)$.*

Proof. As in the previous proof, we suppress the dependence of group functors on R. If Φ is reducible then we apply the previous lemma. So it suffices to treat the cases $\Phi = A_3$, B_3 and C_3. We write U for the subgroup of $\mathfrak{U}_{\Phi^+}$ generated by the $\mathfrak{U}_{\beta_i}$ and $\mathfrak{U}_{\gamma_{i,j}}$. We must show that it is all of $\mathfrak{U}_{\Phi^+}$.

For type A_3 we describe Φ by using four coordinates summing to zero, and take the simple roots β_i to be $(+{-}00)$, $(0{+}{-}0)$ and $(00{+}{-})$, where $\pm$ are short for ± 1. The $\gamma_{i,j}$ are the roots $(+0{-}0)$ and $(0{+}0{-})$. The only remaining positive root is $(+00{-})$. This is the sum of $(+0{-}0)$ and $(00{+}{-})$. So the A_2 case of Lemma 11.1 shows that its root group lies in the U.

For type B_3 we take the simple roots β_i to be $(+{-}0)$, $(0{+}{-})$ and $(00{+})$. The $\gamma_{i,j}$ are $(+0{-})$ and $(0{+}0)$. The remaining positive roots are $(+00)$, $(++0)$, $(+0+)$ and $(0++)$. First, $(00+)$, $(0{+}0)$ and $(0{+}{-})$ are three of the four positive roots of a B_2 root system in Φ, including a pair of simple roots for it. Since U contains $\mathfrak{U}_{00+}$, $\mathfrak{U}_{0+0}$ and $\mathfrak{U}_{0+-}$, Lemma 11.1 shows that U also contains a root group corresponding to the fourth positive root, namely $(0++)$. Second, applying the A_2 case of that lemma to $\mathfrak{U}_{0++}, \mathfrak{U}_{+-0} \subseteq U$ shows that U also contains $\mathfrak{U}_{+0+}$. Third, repeating this using $\mathfrak{U}_{+0+}, \mathfrak{U}_{0+-} \subseteq U$ shows that U contains $\mathfrak{U}_{++0}$. Finally, using the B_2 case again, the fact that U contains $\mathfrak{U}_{00+}$, $\mathfrak{U}_{+0-}$ and $\mathfrak{U}_{+0+}$ shows that U contains $\mathfrak{U}_{+00}$. We have shown that U contains all the positive root groups, so $U = \mathfrak{U}_{\Phi^+}$, as desired.

The C_3 case is the same: replacing the short roots $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ by $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$ does not affect the proof. $\square$

The next proof uses the geometric language of the Tits cone (or Coxeter complex), its subdivision into chambers, and the combinatorial distance between chambers. Here is minimal background; see [Rémy 2002, Chapter 5] for more. The root system Φ lies in $\mathbb{Z}^I \subseteq \mathbb{R}^I$. The fundamental (open) chamber is the set of elements in $\mathrm{Hom}(\mathbb{R}^I, \mathbb{R})$ having positive pairing with all simple roots. We defined an action of the Weyl group W on $\mathbb{Z}^I$ in Section 4, so W also acts on this dual space. A chamber means a W-translate of the fundamental chamber, and the Tits cone means the union of the closures of the chambers. It is tiled by them. W's action is properly discontinuous on the interior of this cone. A gallery of length n means a sequence of chambers $C_0, \dots, C_n$, each C_i sharing a facet with C_{i-1} for $i = 1, \dots, n$. The gallery is called minimal if there is no shorter gallery from C_0 to C_n.

To each root $\alpha \in \Phi$ corresponds a halfspace in the Tits cone, namely those points in it having positive pairing with α. We write the boundary of this halfspace as $\alpha^\perp$. We will identify each root with its halfspace, so we may speak of roots containing chambers. In this language, a set of roots is prenilpotent if there is some chamber lying in all of them, and some chamber lying in none of them.

Proof of Theorem 1.1(iii)–(iv). We suppress the dependence of group functors on R, always meaning groups of points over R. Recall that $\mathfrak{St}$ is obtained from $\mathfrak{PSt}$ by adjoining the Chevalley relations for the prenilpotent pairs of roots that are not classically prenilpotent. So we must show that these relations already hold in $\mathfrak{PSt}$. For Ψ any nilpotent set of roots we will write G_Ψ for the subgroup of $\mathfrak{PSt}$ generated by the $\mathfrak{U}_{\alpha\in\Psi}$. Theorem 10.1 shows that the subgroup of $\mathfrak{St}$ generated by these $\mathfrak{U}_\alpha$ is a copy of $\mathfrak{U}_\Psi$, so we will just write $\mathfrak{U}_\Psi$ for it.

We will prove by induction the following assertion $(N_{n\geq 1})$: Suppose $C_0, \dots, C_n$ is a minimal gallery, for each $k = 1, \dots, n$ let α_k be the root which contains C_k but not C_{k-1}, and define $\Psi = \{\alpha_1, \dots, \alpha_n\}$ and $\Psi_0 = \Psi - \{\alpha_n\}$. Then $\mathfrak{U}_{\alpha_n}$ normalizes G_{Ψ_0} in G_Ψ. (The $\boldsymbol{N}$ stands for "normalizes". Also, it is easy to see that Ψ is the set of all roots containing C_n but not C_0, so it is nilpotent, and similarly for Ψ_0. So G_Ψ and G_{Ψ_0} are defined.)

Assuming (N_n) for all $n \geq 1$, it follows that, for Ψ of this form, the multiplication map $\mathfrak{U}_{\alpha_1} \times \cdots \times \mathfrak{U}_{\alpha_n} \to G_\Psi$ in $\mathfrak{PSt}$ is surjective. We know from Lemma 6.1 and Theorem 10.1 that the corresponding multiplication map in $\mathfrak{St}$, namely $\mathfrak{U}_{\alpha_1} \times \cdots \times \mathfrak{U}_{\alpha_n} \to \mathfrak{U}_\Psi$, is bijective. Since $G_\Psi \to \mathfrak{U}_\Psi$ is surjective, it must also be bijective, hence an isomorphism. Now, if α and β are a prenilpotent pair of roots then we may choose a chamber in neither of them and a chamber in both of them. We join these chambers by a minimal gallery $(C_0, \dots, C_n)$. As mentioned above, the corresponding nilpotent set Ψ of roots consists of all roots which contain C_n but not C_0. In particular, Ψ contains α and β. We have shown that $G_\Psi \to \mathfrak{U}_\Psi$ is an isomorphism. Since the Chevalley relation of α and β holds in $\mathfrak{U}_\Psi$ (by the definition of $\mathfrak{U}_\Psi$), it holds in G_Ψ too. This shows that the Chevalley relations of all prenilpotent pairs hold in $\mathfrak{PSt}$, so $\mathfrak{PSt} \to \mathfrak{St}$ is an isomorphism, finishing the proof.

It remains to prove $(\boldsymbol{N}_n)$. First we treat a special case that does not require induction. By hypothesis, A is S-spherical, where S is 2 (resp. 3) for part (iv) (resp. (iii)) of the theorem. To avoid degeneracies we suppose $\operatorname{rk} A > S$; the case $\operatorname{rk} A \le S$ is trivial because then A is spherical and the isomorphism $\mathfrak{PSt} \to \mathfrak{St}$ is tautological. Suppose that all the chambers in some minimal gallery $(C_0, \dots, C_n)$ have a face F with codimension $\le S$ in common. By S-sphericity, the mirrors $\alpha^\perp$ of only finitely many $\alpha \in \Phi$ contain F. Therefore, any pair from $\alpha_1, \dots, \alpha_n$ is classically prenilpotent. Their Chevalley relations hold in $\mathfrak{PSt}$ by definition. The fact that $\mathfrak{U}_{\alpha_n}$ normalizes G_{Ψ_0} in G_Ψ follows from these relations.

Now, for any minimal gallery of length $n \le S$, its chambers have a face of codimension $n \le S$ in common. (It is a subset of $\alpha_1^\perp \cap \dots \cap \alpha_n^\perp$.) So the previous paragraph applies. This proves $(\boldsymbol{N}_n)$ for $n \le S$, which we take as the base case of our induction. For the inductive step we take $n > S$, assume $(\boldsymbol{N}_1), \dots, (\boldsymbol{N}_{n-1})$, and suppose $(C_0, \dots, C_n)$ is a minimal gallery. For $1 \le k \le l \le n$ we write $G_{k,l}$ for

$$\langle \mathfrak{U}_{\alpha_k}, \dots, \mathfrak{U}_{\alpha_l} \rangle \subseteq \mathfrak{PSt}.$$

We must show that $\mathfrak{U}_{\alpha_n}$ normalizes $G_{1,n-1}$.

Consider the subgallery $(C_{n-S}, \dots, C_n)$ of length S. These chambers have a codimension-S face F in common. Write W_F for its W-stabilizer, which is finite by S-sphericity. Among all chambers having F as a face, let D be the one closest to C_0. By [Abramenko and Brown 2008, Proposition 5.34] it is unique and there is a minimal gallery from C_0 to C_{n-1} having D as one of its terms, such that every chamber from D to C_{n-1} contains F. By replacing the subgallery $(C_0, \dots, C_{n-1})$ of our original minimal gallery with this one, we may suppose without loss of generality that $D = C_m$ for some $0 \le m \le n - S$ and that $C_m, \dots, C_n$ all contain F. (This replacement may change the ordering on $\Psi_0 = \{\alpha_1, \dots, \alpha_{n-1}\}$, which is harmless.) The special case shows that $\mathfrak{U}_{\alpha_n}$ normalizes $G_{m+1,n-1}$. So it suffices to show that $\mathfrak{U}_{\alpha_n}$ also normalizes $G_{1,m}$.

At this point we specialize to proving part (iv) of the theorem. In this case F has codimension 2. There are two chambers adjacent to C_m that contain F. One is C_{m+1} and we call the other one C'_{m+1}. We write α'_{m+1} for the root that contains C'_{m+1} but not C_m. Recall that C_m was the unique chamber closest to C_0, of all those containing F. It follows that $(C_0, \dots, C_m, C'_{m+1})$ is a minimal gallery. By a double application of $(\boldsymbol{N}_{m+1})$, which we may use because $m \le n - S = n - 2$, both $\mathfrak{U}_{\alpha_{m+1}}$ and $\mathfrak{U}_{\alpha'_{m+1}}$ normalize $G_{1,m}$. Since α_{m+1} and α'_{m+1} are simple roots for W_F, and α_n is positive with respect to them, Lemma 11.1 shows that $\mathfrak{U}_{\alpha_n}$ lies in $\langle \mathfrak{U}_{\alpha_{m+1}}, \mathfrak{U}_{\alpha'_{m+1}} \rangle$. This uses the hypotheses on R to deal with the possibility that W_F has type B_2 or G_2. Therefore, $\mathfrak{U}_{\alpha_n}$ normalizes $G_{1,m}$, completing the proof of part (iv).

Now we prove part (iii). F has codimension 3. So there are three chambers adjacent to C_m that contain F. Write C'_{m+1} for any one of them (possibly C_{m+1}) and define β as the root containing C'_{m+1} but not C_m. The three possibilities for β form a system Σ of simple roots for W_F. With respect to Σ, the positive roots of W_F are exactly the ones that do not contain C_m, for example, α_n.

There are two chambers adjacent to C'_{m+1} that contain F, besides C_m. Write C'_{m+2} for either of them and γ for the root containing C'_{m+2} but not C'_{m+1}. Because C_m is the unique chamber containing F that is closest to C_0, $(C_0, \dots, C_m, C'_{m+1}, C'_{m+2})$ is a minimal gallery. In particular, γ is a positive root with respect to Σ.

We claim that $\mathfrak{U}_\beta$ and $\mathfrak{U}_\gamma$ normalize $G_{1,m}$. For β this is just induction using $(\boldsymbol{N}_{m+1})$. For γ, we appeal to $(\boldsymbol{N}_{m+2})$, but all this tells us is that $\mathfrak{U}_\gamma$ normalizes $\langle \mathfrak{U}_\beta, G_{1,m} \rangle$. In particular, it conjugates $G_{1,m}$ into this larger group. To show that $\mathfrak{U}_\gamma$ normalizes $G_{1,m}$ it suffices to show for every $k = 1, \dots, m$ that the Chevalley relation for γ and α_k has no $\mathfrak{U}_\beta$ term. That is, it suffices to show that $\beta \notin \theta(\alpha_k, \gamma)$. Suppose to the contrary. Then β is an $\mathbb{N}$-linear combination of α_k and γ. So α_k is a $\mathbb{Q}$-linear combination of β and γ, and in particular its mirror contains F. Of the Weyl chambers for W_F, the one containing C_0 is the same as the one containing C_m, since C_m is as close as possible to C_0. Since α_k does not contain C_0, it does not contain C_m either. So, as a root of W_F, it is positive with respect to Σ. Now we have the contradiction that the simple root β of W_F is an $\mathbb{N}$-linear combination of the positive roots α_k and γ. This proves $\beta \notin \theta(\alpha_k, \gamma)$, so $\mathfrak{U}_\gamma$ normalizes $G_{1,m}$.

We have proven that $\mathfrak{U}_\beta$ and $\mathfrak{U}_\gamma$ normalize $G_{1,m}$. Letting β and γ vary over all possibilities gives all the roots called β_i and $\gamma_{i,j}$ in Lemma 11.2. By that lemma, the group generated by these root groups contains the root groups of all positive roots of W_F. In particular, $\mathfrak{U}_{\alpha_n}$ normalizes $G_{1,m}$, as desired. This completes the proof of (iii). □

12. Finite presentations

In this section we prove Theorems 1.4 and 1.5: pre-Steinberg groups, Steinberg groups and Kac–Moody groups are finitely presented under various hypotheses. Our strategy is to first prove parts (ii)–(iii) of Theorem 1.4, and then prove part (i) together with Theorem 1.5.

For use in the proof of Theorem 1.4(ii)–(iii), we recall the following result of Splitthoff, which grew from earlier work of Rehmann and Soulé [1976]. Then we prove Theorem 12.2, addressing finite generation rather than finite presentation, using his methods. Then we will prove Theorem 1.4(ii)–(iii).

Theorem 12.1 [Splitthoff 1986, Theorem I]. *Suppose R is a commutative ring and A is one of the ABCDEFG Dynkin diagrams. If either*

(i) $\operatorname{rk} A \geq 3$ *and R is finitely generated as a ring, or*

(ii) $\operatorname{rk} A \geq 2$ *and R is finitely generated as a module over a subring generated by finitely many units,*

then $\mathfrak{St}_A(R)$ is finitely presented. □

Theorem 12.2. *Suppose R is a commutative ring and A is one of the ABCDEFG Dynkin diagrams. If either*

(i) $\operatorname{rk} A \geq 2$ *and R is finitely generated as a ring, or*

(ii) $\operatorname{rk} A \geq 1$ *and R is finitely generated as a module over a subring generated by finitely many units,*

then $\mathfrak{St}_A(R)$ is finitely generated.

Proof. In light of Splitthoff's theorem, it suffices to treat the cases $A = A_2, B_2, G_2$ in (i) and the case $A = A_1$ in (ii). For (i) it suffices to treat the case $R = \mathbb{Z}[z_1, \ldots, z_n]$, since $\mathfrak{St}_A(R) \to \mathfrak{St}_A(R/I)$ is surjective for any ideal I. In the rest of the proof we abbreviate $\mathfrak{St}_A(R)$ to $\mathfrak{St}$. Keeping our standard notation, Φ is the root system, and $\mathfrak{St}$ is generated by groups $\mathfrak{U}_\alpha \cong R$ with α varying over Φ. As discussed in Section 6, writing down elements $X_\alpha(t)$ of $\mathfrak{U}_\alpha$ requires choosing one of the two elements of E_α, but the sign issues coming from this choice will not affect the proof. For each $p \geq 1$ we write $\mathfrak{U}_{\alpha,p}$ for the subgroup of $\mathfrak{U}_\alpha$ consisting of all $X_\alpha(t)$ where $t \in R$ is a polynomial of degree $\leq p$.

A_2 case: If $\alpha, \beta \in \Phi$ make angle $2\pi/3$ then their Chevalley relation reads

$$[X_\alpha(t), X_\beta(u)] = X_{\alpha+\beta}(\pm tu),$$

where the unimportant sign depends on the choices of elements of E_α, E_β and $E_{\alpha+\beta}$. It follows that $[\mathfrak{U}_{\alpha,p}, \mathfrak{U}_{\beta,q}]$ contains $\mathfrak{U}_{\alpha+\beta,p+q}$. An easy induction shows that $\mathfrak{St}$ is generated by the $\mathfrak{U}_{\alpha,1} \cong \mathbb{Z}^{n+1}$, with α varying over Φ.

B_2 case: We write $\mathfrak{U}_{S,p}$ (resp. $\mathfrak{U}_{L,p}$) for the subgroup of $\mathfrak{St}$ generated by all $\mathfrak{U}_{\alpha,p}$ with α a short (resp. long root). If σ, λ are short and long roots with angle $3\pi/4$, then we recall their Chevalley relation from (7-16) as

$$[X_\sigma(t), X_\lambda(u)] = X_{\lambda+\sigma}(-tu)X_{\lambda+2\sigma}(t^2u). \tag{12-1}$$

Here we have implicitly chosen some elements of E_σ, E_λ, $E_{\lambda+\sigma}$ and $E_{\lambda+2\sigma}$ so that one can write down the relation explicitly. Note that the first term on the right lies in a short root group and the second lies in a long root group. Recall that n is the number of variables in the polynomial ring R. We claim that $\mathfrak{St}$ equals $\langle \mathfrak{U}_{S,n}, \mathfrak{U}_{L,n+2} \rangle$ and is therefore finitely generated. The case $n = 0$ is trivial, so suppose $n > 0$. Our claim follows from induction using the following two ingredients.

First, for any $p \geq 1$, $\langle \mathfrak{U}_{S,p}, \mathfrak{U}_{L,p+2} \rangle$ contains $\mathfrak{U}_{S,p+1}$. To see this let $g \in R$ be any monomial of degree $p+1$ and write it as tu for monomials $t, u \in R$ of degrees 1 and p. Then (12-1) yields

$$X_{\lambda+\sigma}(g) = X_{\lambda+2\sigma}(t^2u)[X_\lambda(u), X_\sigma(t)] \in \mathfrak{U}_{L,p+2} \cdot [\mathfrak{U}_{L,p}, \mathfrak{U}_{S,1}].$$

Letting g vary shows that $\mathfrak{U}_{\lambda+\sigma,p+1} \subseteq \langle \mathfrak{U}_{S,p}, \mathfrak{U}_{L,p+2} \rangle$. Then letting σ, λ vary over all pairs of roots making angle $3\pi/4$, so that $\lambda+\sigma$ varies over all short roots, shows that $\mathfrak{U}_{S,p+1} \subseteq \langle \mathfrak{U}_{S,p}, \mathfrak{U}_{L,p+2} \rangle$, as desired.

Second, for any $p \geq n$, $\langle \mathfrak{U}_{S,p+1}, \mathfrak{U}_{L,p+2} \rangle$ contains $\mathfrak{U}_{L,p+3}$. To see this let $g \in R$ be any monomial of degree $p+3$ and write it as t^2u for monomials $t, u \in R$ of degrees 2 and $p-1$. This is possible because $p+3$ is at least 3 more than the number of variables in the polynomial ring R. Then (12-1) can be written

$$X_{\lambda+2\sigma}(g) = X_{\lambda+\sigma}(tu)[X_\sigma(t), X_\lambda(u)] \in \mathfrak{U}_{S,p+1} \cdot [\mathfrak{U}_{S,2}, \mathfrak{U}_{L,p-1}].$$

Varying g and the pair (σ, λ) as in the previous paragraph establishes

$$\mathfrak{U}_{L,p+3} \subseteq \langle \mathfrak{U}_{S,p+1}, \mathfrak{U}_{L,p+2} \rangle.$$

G_2 case: Defining $\mathfrak{U}_{S,p}$ and $\mathfrak{U}_{L,p}$ as in the B_2 case, it suffices to show that $\mathfrak{St}$ equals $\langle \mathfrak{U}_{L,1}, \mathfrak{U}_{S,n} \rangle$. The A_2 case shows that $\mathfrak{U}_{L,1}$ equals the union $\mathfrak{U}_{L,\infty}$ of all the $\mathfrak{U}_{L,p}$. So it suffices to prove that if $p \geq n$ then $\langle \mathfrak{U}_{L,\infty}, \mathfrak{U}_{S,p} \rangle$ contains $\mathfrak{U}_{S,p+1}$. If $\sigma, \lambda \in \Phi$ are short and long simple roots then their Chevalley relation (7-23) can be written

$$[X_\sigma(t), X_\lambda(u)] = X_{\sigma''}(t^2u) X_{\sigma'}(-tu) \cdot (\text{long-root-group elements}),$$

where σ', σ'' are the short roots $\sigma+\lambda$ and $2\sigma+\lambda$. As before, we have implicitly chosen elements of E_σ, E_λ, $E_{\sigma'}$ and $E_{\sigma''}$. Given any monomial $g \in R$ of degree $p+1$, by using $p+1 > n$ we may write it as t^2u, where t has degree 1 and u has degree $p-1$. So every term in the Chevalley relation except $X_{\sigma''}(t^2u)$ lies in $\mathfrak{U}_{S,p}$ or $\mathfrak{U}_{L,\infty}$. Therefore, $\langle \mathfrak{U}_{S,p}, \mathfrak{U}_{L,\infty} \rangle$ contains $X_{\sigma''}(g)$, hence $\mathfrak{U}_{\sigma'',p+1}$ (by varying g), hence $\mathfrak{U}_{S,p+1}$ (by varying σ and λ so that σ'' varies over the short roots).

A_1 case: in this case we are assuming there exist units $x_1, \dots, x_n$ of R and a finite set Y of generators for R as a module over $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm}]$. We suppose without loss that Y contains 1. We use the description of $\mathfrak{St}_{A_1}$ from Section 2, and write G for the subgroup generated by S and the $X(x_1^{m_1} \cdots x_n^{m_n} y)$ with $m_1, \dots, m_n \in \{0, \pm 1\}$ and $y \in Y$. By construction, G contains the $\tilde{s}(x_k^{\pm 1})$, and it contains $\tilde{s}(-1)$ since Y contains 1. Therefore, G contains every $\tilde{h}(x_k^{\pm 1})$. Relation (2-4) shows that if G contains $X(u)$ for some u, then it also contains every $X(x_k^{\pm 2}u)$. It follows that G contains every $X(x_1^{m_1} \cdots x_n^{m_n} y)$ with $m_1, \dots, m_n \in \mathbb{Z}$. Therefore, $G = \mathfrak{St}$. $\square$

Proof of Theorem 1.4(ii)–(iii). We abbreviate $\mathfrak{PSt}_A(R)$ to $\mathfrak{PSt}_A$. We begin with (ii), so A is assumed 2-spherical without A_1 components, and R is finitely generated as a module over a subring generated by finitely many units. We must show that $\mathfrak{PSt}_A$ is finitely presented. Let G be the direct limit of the groups $\mathfrak{PSt}_B$ with B varying over the singletons and *irreducible* rank-2 subdiagrams. By 2-sphericity, each $\mathfrak{PSt}_B$ is isomorphic to the corresponding $\mathfrak{St}_B$. Also G is generated by the images of the $\mathfrak{St}_B$ with $|B| = 2$, because every singleton lies in some irreducible rank-2 diagram. By Splitthoff's theorem, each of these $\mathfrak{St}_B$ is finitely presented. And Theorem 12.2 shows that each $\mathfrak{St}_B$ with $|B| = 1$ is finitely generated. Therefore, the direct limit G is finitely presented.

Now we consider all A_1A_1 subdiagrams $\{i, j\}$ of A. For each of them we impose on G the relations that (the images in G of) $\mathfrak{St}_{\{i\}}$ and $\mathfrak{St}_{\{j\}}$ commute. Because these two groups are finitely generated (Theorem 12.2 again), this can be done with finitely many relations. This finitely presented quotient of G is then the direct limit of the groups $\mathfrak{St}_B$ with B varying over *all* subdiagrams of A of rank ≤ 2. Again using 2-sphericity, we can replace the $\mathfrak{St}_B$'s by $\mathfrak{PSt}_B$'s. Then Corollary 1.3 says that the direct limit is $\mathfrak{PSt}_A$. This finishes the proof of (ii).

Now we prove (iii), in which we are assuming R is a finitely generated ring. Consider the direct limit of the groups $\mathfrak{PSt}_B$ with B varying over the irreducible spherical subdiagrams of rank ≥ 2. Because every node and every pair of nodes lies in such a subdiagram, this direct limit is the same as $\mathfrak{PSt}_A$. Because every B is spherical, we may replace the groups $\mathfrak{PSt}_B$ by $\mathfrak{St}_B$. By hypothesis on A, G is generated by the $\mathfrak{St}_B$ with $|B| > 2$, which are finitely presented by Splitthoff's theorem. And Theorem 12.2 shows that those with $|B| = 2$ are finitely generated. So the direct limit is finitely presented. $\square$

Now we turn to Kac–Moody groups. For our purposes, Tits' Kac–Moody group $\mathfrak{G}_A(R)$ may be defined as the quotient of $\mathfrak{St}_A(R)$ by the subgroup normally generated by the relators

$$\tilde{h}_i(u)\tilde{h}_i(v) \cdot \tilde{h}_i(uv)^{-1} \tag{12-2}$$

with $i \in I$ and $u, v \in R^*$. See [Rémy 2002, §8.3.3] or [Tits 1987, §3.6] for the more general construction of $\mathfrak{G}_D(R)$ from a root datum D. In the rest of this section, R^* will be finitely generated, and under this hypothesis the choice of root datum has no effect on whether $\mathfrak{G}_D(R)$ is finitely presented. (We are using the root datum which Rémy calls simply connected [2002, §7.1.2] and Tits calls "simply connected in the strong sense" [1987, Remark 3.7(c)].)

The following technical lemma shows that when R^* is finitely generated, killing a finite set of relators (12-2) kills all the rest too. The reason it assumes only some of the relations present in $\mathfrak{PSt}_A(R)$ is so we can use it in the proof of Theorem 1.4(i).

There, the goal is to deduce the full presentation of $\mathfrak{PSt}_A(R)$ from just some of its relations.

Lemma 12.3. *Suppose R is a commutative ring and $r_1, \dots, r_m$ are generators for R^*, closed under inversion. Suppose G is the group with generators S and $X(t)$, with $t \in R$, subject to the relations*

$$\tilde{h}(r)X(t)\tilde{h}(r)^{-1} = X(r^2 t), \tag{12-3}$$

$$\tilde{h}(r)SX(t)S^{-1}\tilde{h}(r)^{-1} = SX(t/r^2)S^{-1}, \tag{12-4}$$

for all $r = r_1, \dots, r_m$ and all $t \in R$, where

$$\tilde{h}(r) := \tilde{s}(r)\tilde{s}(1)^{-1} \quad \text{and} \quad \tilde{s}(r) := X(r)SX(1/r)S^{-1}X(r).$$

Then all $\mathcal{P}_{u,v} := \tilde{h}(uv)\tilde{h}(u)^{-1}\tilde{h}(v)^{-1}$, with $u, v \in R^*$, lie in the subgroup of G normally generated by some finite set of them.

Proof. Define N as the subgroup of G normally generated by the following finite set of $\mathcal{P}_{u,v}$:

$$\tilde{h}(r_k r_1^{p_1} \cdots r_m^{p_m}) \cdot \tilde{h}(r_1^{p_1} \cdots r_m^{p_m})^{-1}\tilde{h}(r_k)^{-1},$$

with $k = 1, \dots, m$ and $p_1, \dots, p_m \in \{0, 1\}$. We write $\equiv$ to indicate equality modulo N. As special cases we have $[\tilde{h}(r_k), \tilde{h}(r_l)] \equiv 1$, $\tilde{h}(r_k^2) \equiv \tilde{h}(r_k)^2$, and that if $p_1, \dots, p_m \in \{0, 1\}$ then $\tilde{h}(r_1^{p_1} \cdots r_m^{p_m})$ lies in the abelian subgroup Y of G/N generated by $\tilde{h}(r_1), \dots, \tilde{h}(r_m)$.

We claim that every $\mathcal{P}_{u,v}$ lies in Y. Since Y is finitely generated abelian, we may therefore kill all the $\mathcal{P}_{u,v}$'s by killing some finite set of them, proving the theorem. To prove the claim it suffices to show that every $\tilde{h}(u)$ lies in Y, which we do by induction. That is, supposing $\tilde{h}(u) \in Y$ we will prove $\tilde{h}(r_k^2 u) \in Y$ for each $k = 1, \dots, m$. The following calculations in G mimic the proof of (6-12), paying close attention to which relations are used. First, (12-3)–(12-4) imply

$$\tilde{h}(r_k)\tilde{s}(u)\tilde{h}(r_k)^{-1} = \tilde{s}(r_k^2 u).$$

From the definition of $\tilde{h}(u)$ we get

$$\tilde{h}(r_k)\tilde{h}(u)\tilde{h}(r_k)^{-1} = \tilde{h}(r_k^2 u)\tilde{h}(r_k^2)^{-1}.$$

Right-multiplying by $\tilde{h}(u)^{-1}$ yields $[\tilde{h}(r_k), \tilde{h}(u)] = \mathcal{P}_{r_k^2,u}$. Now, $\tilde{h}(u) \in Y$ implies $[\tilde{h}(r_k), \tilde{h}(u)] \equiv 1$, so $\mathcal{P}_{r_k^2,u} \equiv 1$, so $\tilde{h}(r_k^2 u) \equiv \tilde{h}(u)\tilde{h}(r_k^2) \in Y$, as desired. □

Corollary 12.4. *Suppose R is a commutative ring with finitely generated unit group R^*, and A is any generalized Cartan matrix. Then the subgroup of $\mathfrak{PSt}_A(R)$ normally generated by all relators* (12-2) *is normally generated by finitely many of them.* □

Proof of Theorem 1.5. We must show that $\mathfrak{G}_D(R)$ is finitely presented, assuming that $\mathfrak{St}_A(R)$ is and that R^* is finitely generated. For $\mathfrak{G}_A(R)$ this is immediate from Corollary 12.4. Also, its subgroup H generated by the images of the $\tilde{h}_i(r)$ with $i \in I$ and $r \in R^*$ is finitely generated abelian. For a general root datum D, one obtains $\mathfrak{G}_D(R)$ by the following construction. First one quotients $\mathfrak{G}_A(R)$ by a subgroup of H. Then one takes the semidirect product of this by a torus $\mathcal{T}$ (a copy of $(R^*)^n$). Then one identifies the generators of H with certain elements of $\mathcal{T}$. Since R^* is finitely generated, none of these steps affects finite presentability. □

Proof of Theorem 1.4(i). We must show that if R is finitely generated as an abelian group, then $\mathfrak{PSt}_A(R)$ is finitely presented for any generalized Cartan matrix A. Suppose R is generated as an abelian group by $t_1, \dots, t_n$. Then $\mathfrak{PSt}_A(R)$ is generated by the S_i and $X_i(t_k)$, so it is finitely generated. Because R is finitely generated as an abelian group, its multiplicative group R^* is also. At its heart, this is the Dirichlet unit theorem. See [Lang 1983, Corollary 7.5] for the full result. Let $r_1, \dots, r_m$ be a set of generators for R^*, closed under inversion.

Let N be the central subgroup of $\mathfrak{PSt}_A(R)$ normally generated by all relators (12-2). It is elementary and well-known that if a group is finitely generated, and a central quotient of it is finitely presented, then it is itself finitely presented. (See [Johnson 1997, §10.2] for the required background.) Therefore, the finite presentability of $\mathfrak{PSt}_A(R)$ will follow from that of $\mathfrak{PSt}_A(R)/N$. The relators defining the latter group are (7-1)–(7-26) and (12-2). We will show that finitely many of them imply all the others.

In the definition of $\widehat{W}$, there are only finitely many relations (7-1)–(7-3). The addition rules (7-4) in $\mathfrak{U}_i \cong R$ can be got by imposing finitely many relations on the $X_i(t_k)$. Relations (7-5)–(7-9) describe how certain words in the S_i conjugate arbitrary $X_j(t)$. By the additivity of $X_j(t)$ in t, it suffices to impose only those with t among $t_1, \dots, t_n$. The Chevalley relations (7-10)–(7-23) may be imposed using only finitely many relations, because the Borel subgroup of any rank-2 Chevalley group over R is polycyclic (since R is).

Now for the tricky step: we impose relations (7-24)–(7-25) for $r = r_1, \dots, r_m$ and $t = t_1, \dots, t_n$. The additivity of $X_j(t)$ in t implies these relations for $r = r_1, \dots, r_m$ and arbitrary $t \in R$. These are exactly the relations (12-3)–(12-4) assumed in the statement of Lemma 12.3. That lemma shows that we may impose all the relations (12-2) by imposing some finite number of them. Working modulo these, $\tilde{h}_i(r)$ is multiplicative in r, for each i. Therefore, our relations (7-24)–(7-25) for $r = r_1, \dots, r_m$ imply the same relations for arbitrary r.

Starting with the generators S_i, $X_i(t)$, with $i \in I$ and $t = t_1, \dots, t_n$, we have found finitely many relations from (7-1)–(7-26) and (12-2) that imply all the others. Therefore, $\mathfrak{PSt}_A(R)/N$ is finitely presented, so the same holds for $\mathfrak{PSt}_A(R)$ itself. □

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allcock@math.utexas.edu *Department of Mathematics, University of Texas at Austin, RLM 8.100, 2515 Speedway Stop C1200, Austin, TX 78712, United States*

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