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Algebraicity of normal analytic compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity

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We present an effective criterion to determine if a normal analytic compactification of $\mathbb{C}^2$ with one irreducible curve at infinity is algebraic or not. As a byproduct we establish a correspondence between normal algebraic compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity and algebraic curves contained in $\mathbb{C}^2$ with one place at infinity. Using our criterion we construct pairs of homeomorphic normal analytic surfaces with minimally elliptic singularities such that one of the surfaces is algebraic and the other is not. Our main technical tool is the sequence of *key forms* — a "global" variant of the sequence of *key polynomials* introduced by MacLane [1936] to study valuations in the "local" setting — which also extends the notion of *approximate roots* of polynomials considered by Abhyankar and Moh [1973].

1. Introduction

Algebraic compactifications of $\mathbb{C}^2$ (i.e., compact algebraic surfaces containing $\mathbb{C}^2$) are in a sense the simplest compact algebraic surfaces. The simplest among these are the *primitive compactifications*, i.e., those for which the complement of $\mathbb{C}^2$ (a.k.a. the *curve at infinity*) is irreducible. It follows from a famous result of and Van de Ven that up to isomorphism, $\mathbb{P}^2$ is the only *nonsingular* primitive compactification of $\mathbb{C}^2$. In some sense a more natural category than nonsingular algebraic surfaces is the category of *normal* algebraic surfaces[1]. In this article we tackle the problem of understanding the simplest normal algebraic compactifications of $\mathbb{C}^2$:

Question 1.1. What are the normal primitive algebraic compactifications of $\mathbb{C}^2$?

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[1]This is true for example from the perspective of valuation theory: the irreducible components of the curve at infinity of a normal compactification $\bar{X}$ of $\mathbb{C}^2$ correspond precisely to the discrete valuations on $\mathbb{C}[x, y]$ which are *centered at infinity* with positive dimensional center on $\bar{X}$. Therefore $\bar{X}$ is primitive and normal if and only if $\bar{X}$ corresponds to precisely one discrete valuation centered at infinity on $\mathbb{C}[x, y]$.

We give a complete answer to this question; in particular, we characterize both algebraic and *nonalgebraic* primitive compactifications of $\mathbb{C}^2$. Our answer is equivalent to an *explicit* criterion for determining *algebraicity* of (analytic) contractions of a class of curves: indeed, it follows from well-known results of Kodaira, and independently of Morrow, that any normal analytic compactification $\bar{X}$ of $\mathbb{C}^2$ is the result of contraction of a (possibly reducible) curve E from a nonsingular surface constructed from $\mathbb{P}^2$ by a sequence of blow-ups. On the other hand, a well-known result of Grauert completely and *effectively* characterizes all curves on a nonsingular analytic surface which can be *analytically* contracted: namely it is necessary and sufficient that the matrix of intersection numbers of the irreducible components of E is negative definite. It follows that the question of understanding algebraicity of analytic compactifications of $\mathbb{C}^2$ is equivalent to the following question.

Question 1.2. Let $\pi : Y \to \mathbb{P}^2$ be a birational morphism of nonsingular complex algebraic surfaces and $L \subseteq \mathbb{P}^2$ be a line. Assume π restricts to an isomorphism on $\pi^{-1}(\mathbb{P}^2 \setminus L)$. Let E be the *exceptional divisor* of π (i.e., E is the union of curves on Y which map to points in $\mathbb{P}^2$) and $E_1, \dots, E_N$ be irreducible curves contained in E. Let E' be the union of the strict transform L' (on Y) of L and all components of E *excluding* $E_1, \dots, E_N$. Assume E' is analytically contractible; let $\pi' : Y \to Y'$ be the contraction of E'. When is Y' algebraic?

Question 1.1 is equivalent to the $N = 1$ case of Question 1.2. We give a complete solution (Theorem 4.1) to this case of Question 1.2. Our answer is in particular *effective*, i.e., given a description of Y (e.g., if we know a sequence of blow ups which construct Y from $\mathbb{P}^2$, or if we know precisely the discrete valuation ν on $\mathbb{C}(x, y)$ associated to the unique curve on $Y \setminus \mathbb{C}^2$ which does *not* get contracted), our algorithm determines in finite time if the contraction is algebraic. In fact the algorithm is a one-liner: "Compute the *key forms* of ν. Y' is algebraic if and only if the last key form is a polynomial." The only previously known effective criteria for determining the algebraicity of contraction of curves on surfaces was the well-known criteria of Artin [1962] which states that a normal surface is algebraic if all its singularities are *rational*. We refer the reader to [Morrow and Rossi 1975; Brenton 1977; Franco and Lascu 1999; Schröer 2000; Bădescu 2001; Palka 2013] for other criteria — some of these are more general, but none is effective in the above sense. Moreover, as opposed to Artin's criterion, ours is *not* numerical, i.e., it is not determined by numerical invariants of the associated singularities. We give an example (in Section 2) which shows that in fact there is *no topological*, let alone numerical, answer to Question 1.2 even for $N = 1$.

As a corollary of our criterion, we establish a new correspondence between normal primitive algebraic compactifications of $\mathbb{C}^2$ and algebraic curves in $\mathbb{C}^2$ with *one place at infinity*[2] (Theorem 4.3). Curves with one place at infinity have been extensively studied in affine algebraic geometry (see, e.g., [Abhyankar and Moh 1973; 1975; Ganong 1979; Russell 1980; Nakazawa and Oka 1997; Suzuki 1999; Wightwick 2007]), and we believe the connection we found between these and compactifications of $\mathbb{C}^2$ will be useful for the study of both[3].

Our main technical tool is the sequence of *key forms*, which is a direct analogue of the sequence *key polynomials* introduced by MacLane [1936]. The key polynomials were introduced (and have been extensively used — see, e.g., [Moyls 1951; Favre and Jonsson 2004; Vaquié 2007; Herrera Govantes et al. 2007]) to study valuations in a *local* setting. However, our criterion shows how they retain information about the *global geometry* when computed in "global coordinates."

The example in Section 2 shows that algebraicity of Y' from Question 1.2 can not be determined only from the (weighted) *dual graph* (Definition 3.25) of E'. However, at least when $N = 1$, it is possible to completely characterize the weighted dual graphs (more precisely, *augmented and marked* weighted dual graphs — see Definition 3.26) which correspond to *only algebraic* contractions, those which correspond to *only nonalgebraic* contractions, and those which correspond to *both* types of contractions (Theorem 4.4). The characterization involves two sets of *semigroup conditions* (S1-k) and (S2-k). We note that the first set of semigroup conditions (S1-k) are equivalent to the semigroup conditions that appear in the theory of plane curves with one place at infinity developed in [Abhyankar and Moh 1973; Abhyankar 1977; 1978; Sathaye and Stenerson 1994].

Finally we would like to point that Question 1.1 is equivalent to a two dimensional *Cousin-type problem at infinity*: let $O_1, \ldots, O_N \in \mathbb{P}^2 \setminus \mathbb{C}^2$ be *points at infinity*. Let (u_j, v_j) be coordinates near O_j, $\psi_j(u_j)$ be a *Puiseux series* (Definition 3.2) in u_j, and r_j be a positive rational number, $1 \leq j \leq N$.

Question 1.3. Determine if there exists a polynomial $f \in \mathbb{C}[x, y]$ such that for each analytic branch C of the curve $f = 0$ at infinity, there exists j, $1 \leq j \leq N$, such that

- C intersects L_∞ at O_j,
- C has a Puiseux expansion $v_j = \theta(u_j)$ at O_j such that $\operatorname{ord}_{u_j}(\theta - \psi_j) \geq r_j$.

On our way to understand normal primitive compactifications of $\mathbb{C}^2$, we solve the $N = 1$ case of Question 1.3 (Theorem 4.7).

Remark 1.4. We use *Puiseux series* in an essential way in this article. However, instead of the usual Puiseux series, from Section 3 onward, we almost exclusively

[2]Let $C \subseteq \mathbb{C}^2$ be an algebraic curve, and let $\bar{C}'$ be the closure of C in $\mathbb{P}^2$ and $\sigma : \bar{C} \to \bar{C}'$ be the desingularization of $\bar{C}'$. C has *one place at infinity* if and only if $|\sigma^{-1}(\bar{C}' \setminus C)| = 1$.

[3]For example, we use this connection in [Mondal 2013b] to solve completely the main problem studied in [Campillo et al. 2002].

work with *descending* Puiseux series (a descending Puiseux series in x is simply a meromorphic Puiseux series in x^{-1} — see Definition 3.4). The choice was enforced on us "naturally" from the context — while key polynomials and Puiseux series are natural tools in the study of valuations in the local setting, when we need to study the relation of valuations corresponding to curves at infinity (on a compactification of $\mathbb{C}^2$) to global properties of the surface, key forms and descending Puiseux series are sometimes more convenient.

1A. ***Organization.*** We start with an example in Section 2 to illustrate that the answer to Question 1.2 can not be numerical or topological. The construction also serves as an example of nonalgebraic normal *Moishezon surfaces*[4] with the "simplest possible" singularities (see Remark 2.1). In Section 3 we recall some background material and in Section 4 we state our results. The rest of the article is devoted to the proof of the results of Section 4. In Section 5 we recall some more background material needed for the proof; in particular in Section 5A we state the algorithm to compute key forms of a valuation from the associated descending Puiseux series, and illustrate the algorithm via an elaborate example (we note that this algorithm is essentially the same as the algorithm used in [Makar-Limanov 2015] for a different purpose). In Section 6 we build some tools for dealing with descending Puiseux series and in Section 7 we use these tools to prove the results from Section 4. The appendices contain proof of two lemmas from Section 6 — the proofs were relegated to the appendix essentially because of their length.

2. Algebraic and nonalgebraic compactifications with homeomorphic singularities

Let (u, v) be a system of "affine" coordinates near a point $O \in \mathbb{P}^2$ ("affine" means that both $u = 0$ and $v = 0$ are lines on $\mathbb{P}^2$) and L be the line $\{u = 0\}$. Let C_1 and C_2 be curve-germs at O defined respectively by $f_1 := v^5 - u^3$ and $f_2 := (v - u^2)^5 - u^3$. For each i, let Y_i be the surface constructed by resolving the singularity of C_i at O and then blowing up 8 more times the point of intersection of the (successive) strict transform of C_i with the exceptional divisor. Let E_i^* be the *last* exceptional curve, and $E'^{(i)}$ be the union of the strict transform L'_i (on Y_i) of L and (the strict transforms of) all exceptional curves except E_i^*.

Note that the pairs of germs (C_1, L) and (C_2, L) are *isomorphic* via the map $(u, v) \mapsto (u, v + u^2)$. It follows that "weighted dual graphs" (Definition 3.25) of the $E'^{(i)}$ are *identical*; they are depicted in Figure 1, left (we labeled the vertices according to the order of appearance of the corresponding curves in the sequence of blow-ups). It is straightforward to compute that the matrices of intersection

[4] *Moishezon surfaces* are analytic surfaces for which the fields of meromorphic functions have transcendence degree 2 over $\mathbb{C}$.

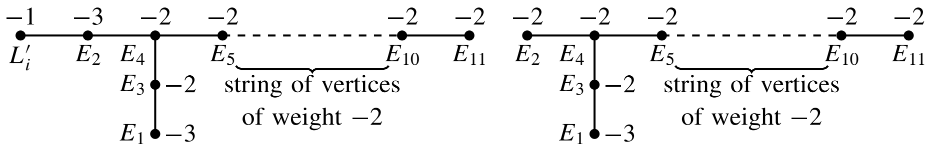

Figure 1. Singularity of Y_i'. Left: weighted dual graph of $E'^{(i)}$. Right: weighted dual graph of the minimal resolution of the singularity of Y_i'.

numbers of the components of the $E'^{(i)}$ are negative definite, so that there is a bimeromorphic analytic map $Y_i \to Y_i'$ contracting $E'^{(i)}$. Note that each Y_i' is a normal analytic surface with one singular point P_i. It follows from the construction that the weighted dual graphs of the minimal resolution of singularities of Y_i' are identical (see Figure 1, right), so that the numerical invariants of the singularities of the Y_i' are also *identical*.

In fact it follows (from, e.g., [Neumann 1981, Section 8]) that the singularities of the Y_i' are also *homeomorphic*. However, Theorem 4.1 and Example 3.19 imply that Y_1' is algebraic, but Y_2' is *not*.

Remark 2.1. It is straightforward to verify that the weighted dual graph presented in Figure 1, right, is precisely the graph labeled $D_{9,*,0}$ in [Laufer 1977]. It then follows from [Laufer 1977] that the singularities at P_i are *Gorenstein hypersurface singularities* of multiplicity 2 and geometric genus 1, which are also *minimally elliptic* (in the sense of [Laufer 1977]). Minimally elliptic Gorenstein singularities have been extensively studied (see, e.g., [Yau 1979; Ohyanagi 1981; Némethi 1999]), and in a sense they form the simplest class of nonrational singularities[5]. Since having only rational singularities implies algebraicity of the surface [Artin 1962], it follows that the surface Y_2' we constructed above is a normal nonalgebraic Moishezon surface with the "simplest possible" singularity.

It follows from [Laufer 1977, Table 2] that the singularity at the origin of $z^2 = x^5 + xy^5$ (Figure 2) is of the same type as the singularity of each Y_i', $1 \le i \le 2$.

3. Background I

Here we compile the background material needed to state the results. In Section 5 we compile further background material that we use for the proof.

Notation 3.1. Throughout the rest of the article we use X to denote $\mathbb{C}^2$ with coordinate ring $\mathbb{C}[x, y]$ and $\bar{X}_{(x,y)}$ to denote the copy of $\mathbb{P}^2$ such that X is embedded

[5]Indeed, every connected proper subvariety of the exceptional divisor of the minimal resolution of a minimally elliptic singularity is the exceptional divisor of the minimal resolution of a rational singularity [Laufer 1977].

Figure 2. The singularity of $z^2 = x^5 + xy^5$ at the origin (*whirling dervish*).

into $\bar{X}_{(x,y)}$ via the map $(x, y) \mapsto [1 : x : y]$. We also denote by L_∞ the *line at infinity* $\bar{X}_{(x,y)} \setminus X$, and by Q_y the point of intersection of L_∞ and (the closure of) the y-axis. Finally, if $\omega_0, \dots, \omega_n$ are positive integers, we denote by $\mathbb{P}^n(\omega_0, \dots, \omega_n)$ the complex n-dimensional weighted projective space corresponding to weights $\omega_0, \dots, \omega_n$.

3A. *Meromorphic and descending Puiseux series.*

Definition 3.2 (meromorphic Puiseux series). A *meromorphic Puiseux series* in a variable u is a fractional power series of the form $\sum_{m \ge M} a_m u^{m/p}$ for some $m, M \in \mathbb{Z}$, $p \ge 1$ and $a_m \in \mathbb{C}$ for all $m \in \mathbb{Z}$. If all exponents of u appearing in a meromorphic Puiseux series are positive, then it is simply called a *Puiseux series* (in u). Given a meromorphic Puiseux series $\phi(u)$ in u, write it in the form

$$\phi(u) = \cdots + a_1 u^{q_1/p_1} + \cdots + a_2 u^{q_2/(p_1 p_2)} + \cdots + a_l u^{q_l/(p_1 p_2 \cdots p_l)} + \cdots,$$

where q_1/p_1 is the smallest noninteger exponent, and for each k, $1 \le k \le l$, we have $a_k \ne 0$, $p_k \ge 2$, $\gcd(p_k, q_k) = 1$, and the exponents of all terms with order between $q_k/(p_1 \cdots p_k)$ and $q_k/(p_1 \cdots p_{k+1})$ (or, if $k = l$, all terms of order above $1/(p_1 \cdots p_l)$) belong to $1/(p_1 \cdots p_k)\mathbb{Z}$. Then the pairs $(q_1, p_1), \dots, (q_l, p_l)$, are called the *Puiseux pairs* of ϕ and the exponents $q_k/(p_1 \cdots p_k)$, $1 \le k \le l$, are called *characteristic exponents* of ϕ. The *polydromy order* [Casas-Alvero 2000, Chapter 1] of ϕ is $p := p_1 \cdots p_l$, i.e., the polydromy order of ϕ is the smallest p such that $\phi \in \mathbb{C}((u^{1/p}))$. Let ζ be a primitive p-th root of unity. The *conjugates* of ϕ are

$$\phi_j(u) := \cdots + a_1 \zeta^{j q_1 p_2 \cdots p_l} u^{q_1/p_1} + \cdots + a_2 \zeta^{j q_2 p_3 \cdots p_l} u^{q_2/(p_1 p_2)} + \cdots + a_l \zeta^{j q_l} u^{q_l/(p_1 p_2 \cdots p_l)} + \cdots$$

for $1 \le j \le p$ (i.e., ϕ_j is constructed by multiplying the coefficients of terms of ϕ with order n/p by ζ^{jn}).

We recall the standard fact that the field of meromorphic Puiseux series in u is the algebraic closure of the field $\mathbb{C}((u))$ of Laurent polynomials in u:

Theorem 3.3. *Let $f \in \mathbb{C}((u))[v]$ be an irreducible monic polynomial in v of degree d. Then there exists a meromorphic Puiseux series $\phi(u)$ in u of polydromy order d such that*

$$f = \prod_{i=1}^{d} (v - \phi_i(u)),$$

where the ϕ_i are conjugates of ϕ.

Definition 3.4 (descending Puiseux series). A *descending Puiseux series* in x is a meromorphic Puiseux series in x^{-1}. The notions regarding meromorphic Puiseux series defined in Definition 3.2 extend naturally to the setting of descending Puiseux series. In particular, if $\phi(x)$ is a descending Puiseux series and the Puiseux pairs of $\phi(1/x)$ are $(q_1, p_1), \dots, (q_l, p_l)$, then ϕ has Puiseux pairs $(-q_1, p_1), \dots, (-q_l, p_l)$, polydromy order $p := p_1 \cdots p_l$, and characteristic exponents $-q_k/(p_1 \cdots p_k)$ for $1 \le k \le l$.

Notation 3.5. We use $\mathbb{C}\langle\langle x \rangle\rangle$ to denote the field of descending Puiseux series in x. For $\phi \in \mathbb{C}\langle\langle x \rangle\rangle$ and $r \in \mathbb{R}$, we denote by $[\phi]_{>r}$ the *descending Puiseux polynomial* (i.e., descending Puiseux series with finitely many terms) consisting of all terms of ϕ of degree $> r$. If ψ is also in $\mathbb{C}\langle\langle x \rangle\rangle$, then we write

$$\phi \equiv_r \psi \iff [\phi]_{>r} = [\psi]_{>r} \iff \deg_x(\phi - \psi) \le r.$$

The following is an immediate Corollary of Theorem 3.3:

Theorem 3.6. *Let $f \in \mathbb{C}[x, x^{-1}, y]$. Then there are (up to conjugacy) unique descending Puiseux series $\phi_1, \dots, \phi_k$ in x, a unique nonnegative integer m and $c \in \mathbb{C}^*$ such that*

$$f = c x^m \prod_{i=1}^{k} \prod_{\substack{\phi_{ij} \text{ is a} \\ \text{conjugate} \\ \text{of } \phi_i}} (y - \phi_{ij}(x)).$$

3B. ***Divisorial discrete valuation and semidegree.*** Let $\sigma : \tilde{Y} \dashrightarrow Y$ be a birational correspondence of normal complex algebraic surfaces and C be an irreducible analytic curve on $\tilde{Y}$. Then the local ring $\mathcal{O}_{\tilde{Y},C}$ of C on $\tilde{Y}$ is a discrete valuation ring. Let ν be the associated valuation on the field $\mathbb{C}(Y)$ of rational functions on Y; in other words ν is the order of vanishing along C. We say that ν is a *divisorial discrete valuation* on $\mathbb{C}(Y)$; the *center* of ν on Y is $\sigma(C \setminus S)$, where S is the set of points of indeterminacy of σ (the normality of Y ensures that S is a discrete set, so that $C \setminus S \neq \varnothing$). Moreover, if U is an open subset of Y, we say that ν is *centered at infinity* with respect to U if and only if $\sigma(C \setminus S) \subseteq Y \setminus U$.

Definition 3.7 (semidegree). Let U be an affine variety and ν be a divisorial discrete valuation on the ring $\mathbb{C}[U]$ of regular functions on U which is centered at infinity with respect to U. Then we say that $\delta := -\nu$ is a *semidegree* on $\mathbb{C}[U]$.

The following result, which connects semidegrees on $\mathbb{C}[x, y]$ with descending Puiseux series in x, is a reformulation of [Favre and Jonsson 2004, Proposition 4.1].

Theorem 3.8. *Let δ be a semidegree on $\mathbb{C}[x, y]$. Assume that $\delta(x) > 0$. Then there is a* descending Puiseux polynomial (*i.e., a descending Puiseux series with finitely many terms*) $\phi_\delta(x)$ (*unique up to conjugacy*) *in x and a* (*unique*) *rational number $r_\delta < \operatorname{ord}_x(\phi_\delta)$ such that for every $f \in \mathbb{C}[x, y]$,*

$$\delta(f) = \delta(x)\deg_x(f(x, \phi_\delta(x) + \xi x^{r_\delta})), \tag{3-1}$$

where ξ is an indeterminate.

Definition 3.9. In the situation of Theorem 3.8, we say that $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ is the *generic descending Puiseux series* associated to δ. Moreover, if $\bar{X}$ is an analytic compactification of $X = \mathbb{C}^2$ and $Z \subseteq \bar{X} \setminus \mathbb{C}^2$ is a curve at infinity such that δ is the order of pole along Z, then we also say that $\tilde{\phi}_\delta(x, \xi)$ is the *generic descending Puiseux series associated to Z*.

Example 3.10. If δ is a weighted degree in (x, y)-coordinates corresponding to weights p for x and q for y with p, q positive integers, then the generic descending Puiseux series corresponding to δ is $\tilde{\phi}_\delta = \xi x^{q/p}$. Note that if we embed $\mathbb{C}^2 = \operatorname{Spec}\mathbb{C}[x, y]$ into the weighted projective space $\mathbb{P}^2(1, p, q)$ via $(x, y) \mapsto [1 : x : y]$, then δ is precisely the order of the pole along the curve at infinity.

Example 3.11. Recall the setup of the example from Section 2. Then the C_i have Puiseux expansions $v = \psi_i(u)$ at O, where

$$\psi_1(u) = u^{3/5}, \quad \psi_2(u) = u^{3/5} + u^2.$$

Now note that $(x, y) := (1/u, v/u)$ are coordinates on $\mathbb{P}^2 \setminus L \cong \mathbb{C}^2$, and with respect to (x, y) coordinates the C_1 has a *descending* Puiseux expansion of the form $y = x\psi_1(1/x) = x^{2/5}$. Similarly, C_2 has a descending Puiseux expansion of the form $y = x\psi_2(1/x) = x^{2/5} + x^{-1}$. Let δ_i be the order of pole along E_i^*, $1 \le i \le 2$. Then the generic descending Puiseux series corresponding to δ_1 and δ_2 are respectively of the form

$$\tilde{\phi}_{\delta_1}(x, \xi_1) = x^{2/5} + \xi_1 x^{-6/5}, \quad \tilde{\phi}_{\delta_2}(x, \xi_2) = x^{2/5} + x^{-1} + \xi_2 x^{-6/5}. \tag{3-2}$$

Definition 3.12 (formal Puiseux pairs of generic descending Puiseux series). Let δ and $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ be as in Definition 3.9. Let the Puiseux pairs of ϕ_δ be $(q_1, p_1), \dots, (q_l, p_l)$. Express r_δ as $q_{l+1}/(p_1 \cdots p_l p_{l+1})$ where $p_{l+1} \geq 1$ and $\gcd(q_{l+1}, p_{l+1}) = 1$. The *formal Puiseux pairs* of $\tilde{\phi}_\delta$ are $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$, with (q_{l+1}, p_{l+1}) being the *generic* formal Puiseux pair. Note that

(1) $\delta(x) = p_1 \cdots p_{l+1}$,

(2) it is possible that $p_{l+1} = 1$ (whereas every other p_k is ≥ 2).

3C. ***Geometric interpretation of generic descending Puiseux series.*** In this subsection we recall from [Mondal 2016] the geometric interpretation of generic descending Puiseux series. We keep the conventions introduced in Notation 3.1.

Definition 3.13. An *irreducible analytic curve germ at infinity* on X is the image γ of an analytic map h from a punctured neighborhood Δ' of the origin in $\mathbb{C}$ to X such that $|h(s)| \to \infty$ as $|s| \to 0$ (in other words, h is analytic on Δ' and has a pole at the origin). If $\bar{X}$ is an analytic compactification of X, then there is a unique point $P \in \bar{X} \setminus X$ such that $|h(s)| \to P$ as $|s| \to 0$. We call P the *center* of γ on $\bar{X}$, and write $P = \lim_{\bar{X}} \gamma$.

Let $\bar{X}$ be a primitive normal analytic compactification of X with an irreducible curve C_∞ at infinity. Let $\sigma : \bar{X}_{(x,y)} \dashrightarrow \bar{X}$ be the natural bimeromorphic map, and let Y be a *resolution of indeterminacies* of σ, i.e., Y is a nonsingular rational surface equipped with analytic maps $\pi : Y \to \bar{X}_{(x,y)}$ and $\pi' : Y \to \bar{X}$ such that $\pi' = \sigma \circ \pi$. Let L'_∞ be the strict transform of $L_\infty \subseteq \bar{X}_{(x,y)}$ on Y and $Q'_y \in L'_\infty$ be (the unique point) such that $\pi(Q'_y) = Q_y$. Let $P_\infty := \pi'(Q'_y) \in C_\infty$.

Proposition 3.14 [Mondal 2016, Proposition 3.5]. *Let δ be the order of pole along C_∞, $\tilde{\phi}_\delta(x, \xi)$ be the generic descending Puiseux series associated to δ and γ be an irreducible analytic curve germ on X. Then* $\lim_{\bar{X}} \gamma \in C_\infty \setminus \{P_\infty\}$ *if and only if γ has a parametrization of the form*

$$t \mapsto (t, \tilde{\phi}_\delta(t, \xi)|_{\xi=c} + \text{l.d.t.}) \quad \textit{for } |t| \gg 0$$

for some $c \in \mathbb{C}$, where l.d.t. *means* lower degree terms (*in t*).

Remark-Definition 3.15. We call P_∞ *a center of* $\mathbb{P}^2$*-infinity on* $\bar{X}$. P_∞ is in fact *unique* in the case of "generic" primitive normal compactifications of $\mathbb{C}^2$ (we do not use this uniqueness in this article, so we state it without a proof):

- If $\bar{X} \cong \mathbb{P}^2(1, 1, q)$ for some $q > 0$, then *every* point of C_∞ is a center of $\mathbb{P}^2$-infinity on $\bar{X}$.
- If $\bar{X} \cong \mathbb{P}^2(1, p, q)$ for some $p, q > 1$, then $\bar{X}$ has *two* singular points, and these are precisely the centers of $\mathbb{P}^2$-infinity on $\bar{X}$.

- In all other cases, there is a *unique* center of $\mathbb{P}^2$-infinity on $\bar{X}$ — it is precisely the unique point on $\bar{X}$ which has a nonquotient singularity.

3D. ***Key forms of a semidegree.*** Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Pick $k > 1$ such that $\delta(y/x^k) < 0$. Set $(u, v) := (1/x, y/x^k)$. Then $\nu := -\delta$ is a discrete valuation on $\mathbb{C}[u, v]$ which is *centered at the origin*. It follows that ν can be completely described in terms of a finite sequence of *key polynomials* in (u, v) [MacLane 1936]. The *key forms* of δ that we introduce in this section are precisely the analogue of key polynomials of ν. We refer to [Favre and Jonsson 2004, Chapter 2] for the properties of key polynomials that we used as a model for our definition of key forms.

Definition 3.16 (key forms). Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. A sequence of elements $g_0, g_1, \ldots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ is called the sequence of *key forms* for δ if the following properties are satisfied with $\eta_j := \delta(g_j)$, $0 \le j \le n+1$:

(P0) $\eta_{j+1} < \alpha_j \eta_j = \sum_{i=0}^{j-1} \beta_{j,i} \eta_i$ for $1 \le j \le n$, where
   - (a) $\alpha_j = \min\{\alpha \in \mathbb{Z}_{>0} : \alpha \eta_j \in \mathbb{Z}\eta_0 + \cdots + \mathbb{Z}\eta_{j-1}\}$ for $1 \le j \le n$,
   - (b) the $\beta_{j,i}$ are integers such that $0 \le \beta_{j,i} < \alpha_i$ for $1 \le i < j \le n$ (in particular, only the $\beta_{j,0}$ are allowed to be negative).

(P1) $g_0 = x$, $g_1 = y$.

(P2) For $1 \le j \le n$, there exists $\theta_j \in \mathbb{C}^*$ such that

$$g_{j+1} = g_j^{\alpha_j} - \theta_j g_0^{\beta_{j,0}} \cdots g_{j-1}^{\beta_{j,j-1}}.$$

(P3) Let $z_1, \ldots, z_{n+1}$ be indeterminates and η be the *weighted degree* on $B := \mathbb{C}[x, x^{-1}, z_1, \ldots, z_{n+1}]$ corresponding to weights η_0 for x and η_j for z_j, where $1 \le j \le n+1$ (i.e., the value of η on a polynomial is the maximum "weight" of its monomials). Then for every polynomial $g \in \mathbb{C}[x, x^{-1}, y]$,

$$\delta(g) = \min\{\eta(G) : G(x, z_1, \ldots, z_{n+1}) \in B,\ G(x, g_1, \ldots, g_{n+1}) = g\}. \tag{3-3}$$

The properties of key forms of semidegrees compiled in the following theorem are straightforward analogues of corresponding (standard) properties of key polynomials of valuations.

Theorem 3.17. (1) *Every semidegree δ on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$ has a unique and finite sequence of key forms.*

(2) *Conversely, given $g_0, \ldots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ and integers $\eta_0, \ldots, \eta_{l+1}$ with $\eta_0 > 0$ which satisfy properties* (P0)–(P2), *there is a unique semidegree δ on $\mathbb{C}[x, y]$ such that the g_j are key forms of δ and $\eta_j = \delta(g_j)$, $0 \le j \le n+1$.*

(3) (Recall Notation 3.1.) *Assume $\sigma : \bar{X}^* \to \bar{X}_{(x,y)}$ is a composition of point blow-ups and $E^* \subseteq \bar{X}^*$ is an exceptional curve of σ. Let δ be the order of pole*

along E^*. Assume $\delta(x) > 0$. Then the following data are equivalent: given any one of them, there is an explicit algorithm to construct the others in finite time.

(a) *A minimal sequence of points on successive blow-ups of $\bar{X}_{(x,y)}$ such that σ factors through the composition of these blow-ups and E^* is the strict transform of the exceptional curve of the last blow-up.*

(b) *A generic descending Puiseux series of δ.*

(c) *The sequence of key forms of δ.*

(4) *Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Let*

$$\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$$

be the generic descending Puiseux series and let g_{n+1} be the last key form of δ. Then the descending Puiseux factorization of g_{n+1} is of the form

$$g_{n+1} = \prod_{\substack{\psi_j \text{ is a} \\ \text{conjugate} \\ \text{of } \psi}} (y - \psi_j(x))$$

for some $\psi \in \mathbb{C}\langle\langle x \rangle\rangle$ such that $\psi \equiv_{r_\delta} \phi_\delta$ (see Notation 3.5).

Example 3.18. Let δ be the weighted degree from Example 3.10. The key forms of δ are $g_0 = x$ and $g_1 = y$.

Example 3.19. Let δ_1 and δ_2 be the semidegrees from Example 3.11. Then the key forms of δ_1 are $x, y, y^5 - x^2$. On the other hand the key forms of δ_2 are $x, y, y^5 - x^2, y^5 - x^2 - 5x^{-1}y^4$ (see Algorithm 5.1 for the general algorithm to compute key forms from generic descending Puiseux series).

Definition 3.20 (essential key forms). Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$, and let $g_0, \dots, g_{n+1}$ be the key forms of δ. Pick the subsequence $j_1, j_2, \dots, j_m$ of $1, \dots, n$ consisting of all j_k such that $\alpha_{j_k} > 1$ (where α_{j_k} is as in property (P0) of Definition 3.16). Set

$$f_k := \begin{cases} g_0 = x & \text{if } k = 0, \\ g_{j_k} & \text{if } 1 \le k \le m, \\ g_{n+1} & \text{if } k = m + 1. \end{cases}$$

We say that $f_0, \dots, f_{m+1}$ are the *essential key forms* of δ.

The following properties of essential key forms follow in a straightforward manner from the defining properties of key forms.

Proposition 3.21. (Let the notation be as in Definition 3.20.) *Let $\tilde{\phi}_\delta(x, \xi)$ be the generic descending Puiseux series of δ and $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$ be the formal Puiseux pairs of $\tilde{\phi}_\delta$. Then:*

(1) *$l = m$, i.e., the number of essential key forms of δ is precisely $l + 1$.*

(2) *Set* $\omega_k := \delta(f_k)$, $0 \le k \le l+1$. *Then the sequence* $\omega_0, \dots, \omega_{l+1}$ *depends only on the formal Puiseux pairs of* $\tilde{\phi}_\delta$. *More precisely, with* $p_0 := q_0 := 1$, *we have*

$$\omega_k = \begin{cases} p_1 \cdots p_{l+1} & \text{if } k = 0, \\ p_{k-1}\omega_{k-1} + (q_k - q_{k-1}p_k)p_{k+1} \cdots p_{l+1} & \text{if } 1 \le k \le l+1. \end{cases} \tag{3-4}$$

(3) *Let* $\alpha_1, \dots, \alpha_{n+1}$ *be as in property* (P0) *of key forms. Then*

$$\alpha_j = \begin{cases} p_k & \text{if } j = j_k, \quad 1 \le k \le l+1, \\ 1 & \text{otherwise.} \end{cases}$$

(4) *Pick* j, $0 \le j \le n+1$. *Assume* $j_k < j < j_{k+1}$ *for some* k, $0 \le k \le l$. *Then* $\delta(g_j)$ *is in the group generated by* $\omega_0, \dots, \omega_k$.

Definition 3.22. We call $\omega_0, \dots, \omega_{l+1}$ of Proposition 3.21 the sequence of *essential key values* of δ.

Example 3.23. Let δ_1, δ_2 be as in Examples 3.11 and 3.19. Then all the key forms of δ_1 are essential, and the essential key values are $\omega_0 = \delta_1(x) = 5$, $\omega_1 = \delta_1(y) = 2$, $\omega_2 = \delta_1(y^5 - x^2) = 2$. The key forms of δ_2 are x, y, $y^5 - x^2 - 5x^{-1}y^4$. The sequence of essential key values of δ_2 is the *same* as that of δ_1.

3E. *Resolution of singularities of primitive normal compactifications.* Given two birational algebraic surfaces Y_1, Y_2, we say that Y_1 *dominates* Y_2 if the birational map $Y_1 \dashrightarrow Y_2$ is in fact a morphism. Let $\bar{X}$ be a primitive normal analytic compactification of $X := \mathbb{C}^2$ and $\pi : Y \to \bar{X}$ be a resolution of singularities of $\bar{X}$. We say that π or Y is $\mathbb{P}^2$-*dominating* if Y dominates $\mathbb{P}^2$. The resolution π is a *minimal* $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}$ if up to isomorphism (of algebraic varieties) Y is the only $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}$ which is dominated by Y.

Theorem 3.24. *Every primitive normal analytic compactification of* $\mathbb{C}^2$ *has a unique minimal* $\mathbb{P}^2$*-dominating resolution of singularities.*

We have not found any proof of Theorem 3.24 in the literature. We give a proof in [Mondal 2013a] (using Theorem 4.1 of this article). In this section we recall from [Mondal 2016] a description of the *dual graphs* of minimal $\mathbb{P}^2$-dominating resolutions of singularities of primitive normal analytic compactifications of $\mathbb{C}^2$.

Definition 3.25. Let $E_1, \dots, E_k$ be nonsingular curves on a (nonsingular) surface such that for each $i \neq j$, either $E_i \cap E_j = \varnothing$, or E_i and E_j intersect transversally at a single point. Then $E = E_1 \cup \cdots \cup E_k$ is called a *simple normal crossing curve*. The *(weighted) dual graph* of E is a weighted graph with k vertices $V_1, \dots, V_k$ such that

- there is an edge between V_i and V_j if and only if $E_i \cap E_j \neq \varnothing$,
- the weight of V_i is the self intersection number of E_i.

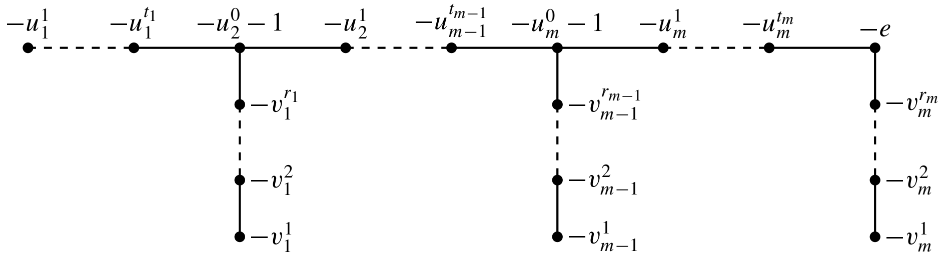

Figure 3. $\tilde{\Gamma}_{\vec{\tilde{q}},\vec{\tilde{p}},m,e}$.

Usually we will abuse the notation, and label V_i also by E_i.

Definition 3.26. Let $\bar{X}$ be a primitive normal analytic compactification of $X := \mathbb{C}^2$ and $\pi : Y \to \bar{X}$ be a resolution of singularities of $\bar{X}$ such that $Y \setminus X$ is a simple normal crossing curve. The *augmented dual graph* of π is the dual graph (Definition 3.25) of $Y \setminus X$. If Y is $\mathbb{P}^2$-dominating, we define the *augmented and marked dual graph* of π to be its augmented dual graph with the strict transforms of the curves at infinity on $\mathbb{P}^2$ and $\bar{X}$ marked (e.g., by different colors or labels).

Given a sequence $(\tilde{q}_1, \tilde{p}_1), \dots, (\tilde{q}_n, \tilde{p}_n)$ of pairs of relatively prime integers, and positive integers m, e such that $1 \le m \le n$, we denote by $\tilde{\Gamma}_{\vec{\tilde{q}},\vec{\tilde{p}},m,e}$ the weighted graph in Figure 3, where the right-most vertex in the top row has weight $-e$, and the other weights satisfy: $u_i^0, v_i^0 \ge 1$ and $u_i^j, v_i^j \ge 2$ for $j > 0$, and are uniquely determined from the continued fractions

$$\frac{\tilde{p}_i}{q'_i} = u_i^0 - \cfrac{1}{u_i^1 - \cfrac{1}{\ddots - \cfrac{1}{u_i^{t_i}}}}, \qquad \frac{q'_i}{\tilde{p}_i} = v_i^0 - \cfrac{1}{v_i^1 - \cfrac{1}{\ddots - \cfrac{1}{v_i^{r_i}}}}, \tag{3-5}$$

where $q'_1 := q_1$ and $q'_i := \tilde{q}_i - \tilde{q}_{i-1}\tilde{p}_i$ if $i \ne 1$.

Remark 3.27. $\tilde{\Gamma}_{\vec{\tilde{q}},\vec{\tilde{p}},m,1}$ is the weighted dual graph of the exceptional divisor of the minimal resolution of an irreducible plane curve singularity with Puiseux pairs $(\tilde{q}_1, \tilde{p}_1), \dots, (\tilde{q}_m, \tilde{p}_m)$ (see, e.g., [Mendris and Némethi 2005, Section 2.2]).

Theorem 3.28 [Mondal 2016, Proposition 4.2, Corollary 6.3]. *Let $\bar{X}$ be a primitive normal compactification of* $X := \operatorname{Spec}\mathbb{C}[x, y] \cong \mathbb{C}^2$.

(1) *If $\bar{X}$ is nonsingular, then $\bar{X} \cong \mathbb{P}^2$.*

(2) *Assume $\bar{X}$ is singular. Let $\tilde{\phi}_\delta(x, \xi)$ be the generic descending Puiseux series* (Definition 3.9) *associated to $E^* := \bar{X} \setminus X$ and* $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$

be the formal Puiseux pairs *of* $\tilde{\phi}(x,\xi)$ (Definition 3.12). *Define* $(\tilde{q}_i, \tilde{p}_i) := (p_1 \cdots p_i - q_i, p_i)$, $1 \le i \le l+1$.

(a) *After a (polynomial) change of coordinates of* $\mathbb{C}^2$ *if necessary, we may assume that* $q_1 < p_1$ *and either* $l = 0$ *or* $q_1 > 1$.

(b) *Assume* (2a) *holds. If* $p_{l+1} > 1$, *then the augmented and marked dual graph of the minimal* $\mathbb{P}^2$*-dominating resolution of singularities of* $\bar{X}$ *is as in Figure 4, left, where the strict transform of the curve at infinity on* $\mathbb{P}^2$ (*resp.* $\bar{X}$) *is marked by* L (*resp.* E^*).

(c) *Assume* (2a) *holds. If* $p_{l+1} = 1$, *then* ($l \ge 1$, *and*) *the augmented and marked dual graph of the minimal* $\mathbb{P}^2$*-dominating resolution of singularities of* $\bar{X}$ *is as in Figure 4, right, where the strict transform of the curve at infinity on* $\mathbb{P}^2$ (*resp.* $\bar{X}$) *is marked by* L (*resp.* E^*).

(3) *Conversely, let* $0 \le l$, *and* $(q_1, p_1), \ldots, (q_{l+1}, p_{l+1})$ *be pairs of integers such that*

(a) $p_k \ge 2$, $1 \le k \le l$,

(b) $p_{l+1} \ge 1$,

(c) $\tilde{q}_k := p_1 \cdots p_k - q_k > 0$, $1 \le k \le l+1$,

(d) $\gcd(p_k, q_k) = 1$, $1 \le k \le l+1$.

Assume moreover that (2a) *holds, i.e., either* $l = 0$ *or* $q_{l+1} > 1$. *Define* $\omega_0, \ldots, \omega_{l+1}$ *as in* (3-4). *Let*

$$\Gamma_{\vec{p},\vec{q}} := \begin{cases} \text{the graph from Figure 4, left} & \text{if } p_{l+1} > 1, \\ \text{the graph from Figure 4, right} & \text{if } p_{l+1} = 1. \end{cases} \tag{3-6}$$

Then $\Gamma_{\vec{p},\vec{q}}$ *is the augmented and marked dual graph of the minimal* $\mathbb{P}^2$*-dominating resolution of singularities of a primitive normal analytic compactification of* $\mathbb{C}^2$ *if and only if* $\omega_{l+1} > 0$.

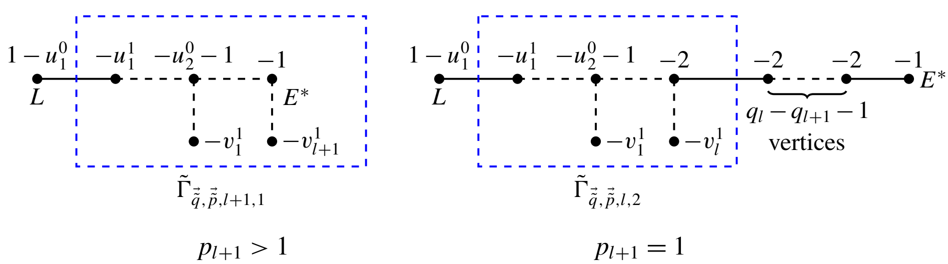

Figure 4. Augmented and marked dual graph for the minimal $\mathbb{P}^2$-dominating resolutions of singularities of primitive normal analytic compactifications of $\mathbb{C}^2$.

Remark 3.29. Let $\bar{X}$ be a primitive normal analytic compactification of $\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[x, y]$ and Γ be the augmented and marked dual graph for the minimal $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}$. Theorem 3.28 and identity (3-5) imply that Γ determines, and is determined by, the formal Puiseux pairs of the generic descending Puiseux series associated to the curve E^* at infinity on $\bar{X}$. Let δ be the semidegree on $\mathbb{C}[x, y]$ corresponding to E^*. Proposition 3.21 then implies that the δ-value of essential key forms of δ are also uniquely determined by Γ; we call these the *essential key values* of Γ.

4. Main results

Consider the setup of Question 1.2. Assume $N = 1$. Choose coordinates (x, y) on $\mathbb{P}^2 \setminus L$. Let δ be the semidegree on $\mathbb{C}[x, y]$ associated to E_1 (i.e., δ is the order of pole along E_1) and let $g_0, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ be the key forms of δ.

Theorem 4.1 (answering Question 1.2 in the case $N = 1$). *The following are equivalent*:

(1) *Y' is algebraic.*

(2) *g_j is a polynomial*, $0 \le j \le n + 1$.

(3) *g_{n+1} is a polynomial.*

If any of these conditions holds, *then Y' is isomorphic to the closure of the image of $\mathbb{C}^2$ in the weighted projective variety $\mathbb{P}^{n+2}(1, \delta(g_0), \dots, \delta(g_{n+1}))$ under the mapping* $(x, y) \mapsto [1 : g_0 : \cdots : g_{n+1}]$.

Remark 4.2. To ask Question 1.2 we need to determine if the given curve E' is analytically contractible. We would like to point out that in addition to the direct application of Grauert's criterion, the contractibility of E' can be determined in terms of the semidegrees associated to $E_1, \dots, E_N$ [Mondal 2016, Theorem 1.4]. In particular, in the $N = 1$ case, E' of Question 1.2 is analytically contractible if and only if $\delta(g_{n+1}) > 0$ (where δ and g_{n+1} are as above).

We now state the correspondence between primitive normal algebraic compactifications of $\mathbb{C}^2$ and algebraic curves in $\mathbb{C}^2$ with one place at infinity.

Theorem 4.3. *Let $\bar{X}$ be a primitive normal analytic compactification of $\mathbb{C}^2$. Let $P \in \bar{X} \setminus \mathbb{C}^2$ be a center of a $\mathbb{P}^2$-infinity on $\bar{X}$* (Remark-Definition 3.15). *Then the following are equivalent*:

(1) *$\bar{X}$ is algebraic.*

(2) *There is an algebraic curve C in $\mathbb{C}^2$ with* one place at infinity *such that P is not on the closure of C in $\bar{X}$.*

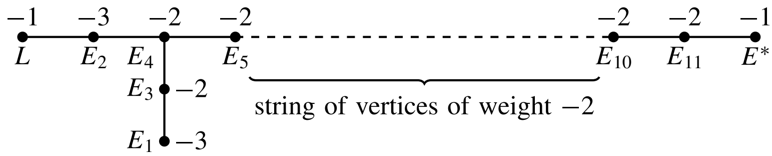

Figure 5. Augmented and marked dual graph of the minimal $\mathbb{P}^2$-dominating resolution of singularities of Y_i' from Section 2.

Let δ be the semidegree on $\mathbb{C}[x, y]$ corresponding to the curve at infinity on $\bar{X}$, and $g_0, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ be the sequence of key forms of δ. If either (1) or (2) is true, then g_{n+1} is a polynomial, and defines a curve C as in (2).

Now we come to the question of characterization of augmented and marked dual graphs of the resolution of singularities of primitive normal analytic compactifications of $\mathbb{C}^2$. For a primitive normal analytic compactification $\bar{X}$ of $\mathbb{C}^2$, let $\Gamma_{\bar{X}}$ be the *augmented and marked* dual graph (from Theorem 3.28) associated to the minimal $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}$. Let $\mathcal{G}$ be the collection of $\Gamma_{\bar{X}}$ as $\bar{X}$ varies over all primitive normal analytic compactifications of $\mathbb{C}^2$; note that assertions (2) and (3) of Theorem 3.28 give a complete description of $\mathcal{G}$. Pick $\Gamma \in \mathcal{G}$. Let $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$ be the formal Puiseux pairs, and $\omega_0, \dots, \omega_{l+1}$ be the sequence of *essential key values* of Γ (Remark 3.29). Fix k, $1 \le k \le l$. The *semigroup conditions* for k are:

$$p_k\omega_k \in \mathbb{Z}_{\ge 0}\langle \omega_0, \dots, \omega_{k-1}\rangle, \tag{S1-k}$$

$$(\omega_{k+1}, p_k\omega_k) \cap \mathbb{Z}\langle \omega_0, \dots, \omega_k\rangle = (\omega_{k+1}, p_k\omega_k) \cap \mathbb{Z}_{\ge 0}\langle \omega_0, \dots, \omega_k\rangle, \tag{S2-k}$$

where $(\omega_{k+1}, p_k\omega_k) := \{a \in \mathbb{R} : \omega_{k+1} < a < p_k\omega_k\}$ and $\mathbb{Z}_{\ge 0}\langle \omega_0, \dots, \omega_k\rangle$ (respectively, $\mathbb{Z}\langle \omega_0, \dots, \omega_k\rangle$) denotes the semigroup (respectively, group) generated by linear combinations of $\omega_0, \dots, \omega_k$ with nonnegative integer (respectively, integer) coefficients.

Theorem 4.4. (1) $\Gamma = \Gamma_{\bar{X}}$ *for some primitive normal* algebraic *compactification* $\bar{X}$ *of* $\mathbb{C}^2$ *if and only if the semigroup conditions* (S1-k) *hold for all* k, $1 \le k \le l$.

(2) $\Gamma = \Gamma_{\bar{X}}$ *for some primitive normal* nonalgebraic *compactification* $\bar{X}$ *of* $\mathbb{C}^2$ *if and only if either* (S1-k) *or* (S2-k) fails *for some* k, $1 \le k \le l$.

Remark-Example 4.5. Note that if (S1-k) holds for all k, $1 \le k \le l$, but (S2-k) *fails* for some k, $1 \le k \le l$, then Theorem 4.4 implies that there exist primitive normal analytic compactifications $\bar{X}_1, \bar{X}_2$ of $\mathbb{C}^2$ such that $\bar{X}_1$ is algebraic, $\bar{X}_2$ is *not* algebraic, and $\Gamma = \Gamma_{\bar{X}_1} = \Gamma_{\bar{X}_2}$. Indeed, that is precisely what happens in the setup of Section 2: let Γ be the augmented and marked dual graph corresponding to the minimal $\mathbb{P}^2$-dominating resolution of singularities of the Y_i' (Figure 5). It follows from (3-2) that the formal Puiseux pairs associated to Γ are $(2, 5)$, $(-6, 1)$;

in particular $l = 1$. Example 3.23 implies that the sequence of essential key values of Γ is $(5, 2, 2)$. It is straightforward to verify that (S1-k) is satisfied for $k = 1$. On the other hand,

$$3 \in (2, 10) \cap \mathbb{Z}\langle 5, 2\rangle \setminus \mathbb{Z}_{\geq 0}\langle 5, 2\rangle$$

so that (S2-k) is violated for $k = 1$. This implies that Γ corresponds to both algebraic and nonalgebraic normal compactifications of $\mathbb{C}^2$, as we have already seen in Section 2.

Remark-Example 4.6. We state some straightforward corollaries of Theorem 4.4 and of the fact — a special case of [Herzog 1970, Proposition 2.1] — that if p, q are relatively prime positive integers, then the greatest integer not belonging to $\mathbb{Z}_{\geq 0}\langle p, q\rangle$ is $pq - p - q$.

(1) Pick relatively prime positive integers p, q such that $p > q$. Then $\Gamma_{p,q}$ (defined as in (3-6)) corresponds to only algebraic compactifications of $\mathbb{C}^2$.

(2) Pick integers p, q, r such that p, q are relatively prime, $p > q > 0$ and $q > r$. Set $l := 1$, $(q_1, p_1) := (q, p)$, $(q_2, p_2) := (r, 1)$. Then (3-4) implies that $\omega_0 = p$, $\omega_1 = q$ and $\omega_2 = (p-1)q + r$. Assertion (3) of Theorem 3.28 therefore implies that $\Gamma_{\vec{p},\vec{q}}$ corresponds to a compactification of $\mathbb{C}^2$ if and only if $(p-1)q+r > 0$. So assume $q > r > -(p-1)q$.

   (a) If $r \geq -p$, then $\Gamma_{\vec{p},\vec{q}}$ corresponds to only algebraic compactifications of $\mathbb{C}^2$.

   (b) If $-p > r > -(p-1)q$, then $\Gamma_{\vec{p},\vec{q}}$ corresponds to both algebraic and nonalgebraic compactifications of $\mathbb{C}^2$.

(3) Let p_1, q_1, p_2 be integers such that $p_1 > q_1 > 1$, $p_2 \geq 2$, p_1 is relatively prime to q_1, and p_2 is relatively prime to $p_1q_1 - p_1 - q_1$. Set

$$q_2 := p_1q_1 - p_1 - q_1 - q_1(p_1 - 1)p_2, \quad q_3 := q_2 - 1, \quad p_3 := 1.$$

In this case $\omega_0 = p_1p_2$, $\omega_1 = q_1p_2$, $\omega_2 = p_1q_1 - p_1 - q_1$ and $\omega_3 = p_2\omega_2 - 1$. It follows that (S1-k) fails for $k = 2$ and therefore $\Gamma_{\vec{p},\vec{q}}$ corresponds to *only nonalgebraic* compactifications of $\mathbb{C}^2$.

Finally we formulate our answer to Question 1.3 in the case $N = 1$. Consider $O \in L_\infty := \mathbb{P}^2 \setminus \mathbb{C}^2$. Let (u, v) be coordinates near O, $\psi(u)$ be a Puiseux series in u, and r be a positive rational number. After a change of coordinates near O if necessary, we may assume that the coordinate of O is $(0, 0)$ with respect to the (u, v)-coordinates, and $(x, y) := (1/u, v/u)$ is a system of coordinates on $\mathbb{P}^2 \setminus L_\infty \cong \mathbb{C}^2$. Let

$$\phi(x) := x\psi(1/x).$$

Note that $\phi(x)$ is a descending Puiseux series in x. Let ξ be an indeterminate, and define, following Notation 3.5,

$$\tilde{\phi}(x,\xi) := [\phi(x)]_{>1-r} + \xi x^{1-r}.$$

Let δ be the semidegree on $\mathbb{C}[x, y]$ with generic descending Puiseux series $\tilde{\phi}$, and let $g_0, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ be the key forms of δ (see Algorithm 5.1 for the algorithm to determine key forms of δ from $\tilde{\phi}$).

Theorem 4.7 (answering Question 1.3 in the case $N = 1$). *The following are equivalent*:

(1) *There exists a polynomial $f \in \mathbb{C}[x, y]$ such that for each analytic branch C of the curve $f = 0$ at infinity,*
   - *C intersects L_∞ at O,*
   - *C has a Puiseux expansion $v = \theta(u)$ at O such that* $\operatorname{ord}_u(\theta - \psi) \geq r$.

(2) *g_j is a polynomial, $0 \leq j \leq n+1$.*

(3) *g_{n+1} is a polynomial.*

If any of these conditions holds, g_{n+1} satisfies the properties of f from condition (1).

5. Background II: notions required for the proof

In this section we collect more background material we use in the proof of the results stated in Section 4.

5A. ***Key forms from descending Puiseux series.*** Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Assume the generic descending Puiseux series for δ is

$$\tilde{\phi}_\delta(x,\xi) := \sum_{j=1}^{k'_0} a_{0j} x^{q_{0j}} + a_1 x^{q_1/p_1} + \cdots + a_2 x^{q_2/(p_1 p_2)} + \cdots$$
$$+ a_l x^{q_l/(p_1 p_2 \cdots p_l)} + \xi x^{q_{l+1}/(p_1 p_2 \cdots p_{l+1})},$$

where $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$ are the *formal* Puiseux pairs of $\tilde{\phi}_\delta$ (Definition 3.12), $k'_0 \geq 0$, and $q_{01} > \cdots > q_{0k'_0}$ are integers greater than q_1/p_1. Let

$$g_0 = x,\ g_1 = y, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$$

be the key forms of δ. Recall from Proposition 3.21 that precisely $l+2$ of the key forms of δ are *essential*. Let $0 = j_0 < \cdots < j_{l+1} = n+1$ be the subsequence of $(0, \dots, n)$ consisting of indices of essential key forms of δ.

Algorithm 5.1 (construction of key forms from descending Puiseux series; cf. the algorithm in [Makar-Limanov 2015]).

1. *Base step.* Set $j_0 := 0,\ g_0 := x,\ g_1 := y$. Also define $p_0 := 1$. Now assume

(i) $g_0, \dots, g_s$ have been calculated, $s \ge 1$,

(ii) $j_0, \dots, j_k$ have been calculated, $k \ge 0$,

(iii) $j_k < s \le j_{k+1}$.

2. *Inductive step for* (s,k). Let

$$\tilde{\omega}_s := \deg_x(g_s|_{y=\tilde{\phi}_\delta(x,\xi)}), \quad \tilde{c}_s := \text{coefficient of } x^{\tilde{\omega}_s} \text{ in } g_s|_{y=\tilde{\phi}_\delta(x,\xi)}.$$

Case 2.1: If $\tilde{c}_s \in \mathbb{C}[\xi] \setminus \mathbb{C}$, then set $n := s-1,\ j_{k+1} := s$, and stop the process.

Case 2.2: Otherwise if $\tilde{\omega}_s \in 1/(p_0 \cdots p_k)\mathbb{Z}$, then there are unique integers $\beta_0^s, \cdots, \beta_k^s$ and unique $c \in \mathbb{C}^*$ such that

(1) $0 \le \beta_i^s < p_i$ for $1 \le i \le k$,

(2) $\sum_{i=0}^k \beta_i^s \tilde{\omega}_{j_i} = \tilde{\omega}_s$, and

(3) the coefficient of $x^{\tilde{\omega}_s}$ in $cg_{j_0}^{\beta_0^s} \cdots g_{j_k}^{\beta_k^s}|_{y=\tilde{\phi}_\delta(x,\xi)}$ is $\tilde{c}_s$.

Then set $g_{s+1} := g_s - cg_{j_0}^{\beta_0^s} \cdots g_{j_k}^{\beta_k^s}$, and repeat the inductive step for $(s+1,k)$.

Case 2.3: Otherwise

$$\tilde{\omega}_s \in \frac{1}{p_0 \cdots p_{k+1}}\mathbb{Z} \setminus \frac{1}{p_0 \cdots p_k}\mathbb{Z},$$

and there are unique integers $\beta_0^s, \cdots, \beta_k^s$ and unique $c \in \mathbb{C}^*$ such that

(1) $0 \le \beta_i^s < p_i$ for $1 \le i \le k$,

(2) $\sum_{i=0}^k \beta_i^s \tilde{\omega}_{j_i} = p_{k+1}\tilde{\omega}_s$, and

(3) the coefficient of $x^{\tilde{\omega}_s}$ in $cg_{j_0}^{\beta_0^s} \cdots g_{j_k}^{\beta_k^s}|_{y=\tilde{\phi}_\delta(x,\xi)}$ is $(\tilde{c}_s)^{p_{k+1}}$.

Then set $j_{k+1} := s,\ g_{s+1} := g_s^{p_{k+1}} - cg_{j_0}^{\beta_0^s} \cdots g_{j_k}^{\beta_k^s}$, and repeat the inductive step for $(s+1, k+1)$.

Example 5.2. Let $\tilde{\phi}_\delta(x,\xi) := x^3 + x^2 + x^{5/3} + x + x^{-13/6} + x^{-7/3} + \xi x^{-8/3}$. The formal Puiseux pairs of $\tilde{\phi}_\delta$ are $(5,3), (-13,2), (-16,1)$. We compute the key forms of δ following Algorithm 5.1: by definition we have $g_0 = x,\ g_1 = y,\ j_0 = 0$. Since the exponents of x in the first two terms of $\tilde{\psi}_\delta$ are integers, subsequent applications of Case 2.2 of Algorithm 5.1 implies that the next two key forms are $g_2 = y - x^3$ and $g_3 = y - x^3 - x^2$. Note that

$$g_3|_{y=\tilde{\psi}_\delta} = x^{5/3} + x + x^{-13/6} + x^{-7/3} + \xi x^{-8/3}, \tag{5-1}$$

In the notation of Algorithm 5.1, we have $\tilde{\omega}_3 = 5/3 \notin \mathbb{Z}$. It follows that $j_1 = 3$. Since

$$g_3^3|_{y=\tilde{\psi}_\delta} = x^5 + 3x^{13/3} + 3x^{11/3} + x^3 + 3x^{7/6} + 3x + 3\xi x^{2/3} + \text{l.d.t.}, \tag{5-2}$$

(where l.d.t. denotes terms with smaller degree in x), Case 2.3 of Algorithm 5.1 implies that $g_4 = g_3^3 - x^5$. Now note that $13/3 = 1 + 2 \cdot (5/3)$ and $11/3 = 2 + 5/3$, so that (5-1) and (5-2) imply that

$$\begin{aligned} g_4|_{y=\tilde{\psi}_\delta} &= 3x\big(g_3|_{y=\tilde{\psi}_\delta} - x - x^{-13/6} - x^{-7/3} - \xi x^{-8/3}\big)^2 \\ &\qquad + 3x^2\big(g_3|_{y=\tilde{\psi}_\delta} - x - x^{-13/6} - x^{-7/3} - \xi x^{-8/3}\big) \\ &\qquad\qquad + x^3 + 3x^{7/6} + 3x + 3\xi x^{2/3} + \text{l.d.t.} \\ &= 3xg_3^2|_{y=\tilde{\psi}_\delta} - 3x^2 g_3|_{y=\tilde{\psi}_\delta} + x^3 + 3x^{7/6} + 3x + 3\xi x^{2/3} + \text{l.d.t.} \end{aligned}$$

Repeated applications of Case 2.2 of Algorithm 5.1 then imply that

$$g_5 = g_4 - 3xg_3^2, \quad g_6 = g_4 - 3xg_3^2 - 3x^2 g_3, \quad g_7 = g_4 - 3xg_3^2 - 3x^2 g_3 - x^3.$$

Note that

$$g_7|_{y=\tilde{\psi}_\delta} = 3x^{7/6} + 3x + 3\xi x^{2/3} + \text{l.d.t.} \tag{5-3}$$

Since $\tilde{\omega}_7 = \frac{7}{6} \notin \frac{1}{3}\mathbb{Z}$, following Case 2.3 of Algorithm 5.1 we have $j_2 = 7$. Since $p_2 = 2$, we compute

$$g_7^2|_{y=\tilde{\psi}_\delta} = 9x^{7/3} + 18x^{13/6} + 18\xi x^{11/6} + \text{l.d.t.},$$

Since $\frac{7}{3} = -1 + 2 \cdot \frac{5}{3} + 0 \cdot \frac{7}{6}$ and $\frac{13}{6} = 1 + 0 \cdot \frac{5}{3} + \frac{7}{6}$, identities (5-1) and (5-3) imply that

$$\begin{aligned} g_7^2|_{y=\tilde{\psi}_\delta} &= 9x^{-1}\big(g_3|_{y=\tilde{\psi}_\delta} - x - x^{-13/6} - x^{-5/2} - \xi x^{-8/3}\big)^2 \\ &\qquad + 6x\big(g_7|_{y=\tilde{\psi}_\delta} - 3x - 3\xi x^{2/3} - \text{l.d.t.}\big) + 18\xi x^{11/6} + \text{l.d.t.} \\ &= 9x^{-1}g_3^2|_{y=\tilde{\psi}_\delta} + 6xg_7|_{y=\tilde{\psi}_\delta} - 18x^2 + 18\xi x^{11/6} + \text{l.d.t.} \end{aligned}$$

Cases 2.3 and 2.2 of Algorithm 5.1 then imply that the next key forms are

$$g_8 = g_7^2 - 9x^{-1}g_3^2, \quad g_9 = g_7^2 - 9x^{-1}g_3^2 - 6xg_7, \quad g_{10} = g_7^2 - 9x^{-1}g_3^2 - 6xg_7 + 18x^2.$$

Since

$$g_{10}|_{y=\tilde{\psi}_\delta} = 18\xi x^{11/6} + \text{l.d.t.},$$

Case 2.1 of Algorithm 5.1 implies that g_{10} is the last key form of δ, and $n = 9$, $j_3 = 10$. In particular, note that there are precisely 4 essential key forms (namely g_0, g_3, g_7, g_{10}) of δ, as predicted by Proposition 3.21.

The assertions of the following proposition are straightforward implications of Algorithm 5.1.

Proposition 5.3. *Let δ be a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Let $g_0, \dots, g_{n+1}$ be key forms and $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ be the generic descending Puiseux series of δ,*

(1) *Let* $n_* \le n$ *and let* δ_* *be the unique semidegree such that the key forms of* δ_* *are* $g_0, \dots, g_{n_*+1}$ *and* $\delta^*(g_j) = \delta(g_j)/e$, $0 \le j \le n_* + 1$, *where* $e := \gcd\bigl(\delta(g_0), \dots, \delta(g_{n_*+1})\bigr)$. *Then* δ_* *has a generic descending Puiseux series of the form*

$$\tilde{\phi}_{\delta_*}(x, \xi) = \phi_*(x) + \xi x^{r_*},$$

where

(a) $r_* \ge r_\delta$, *and*

(b) $\phi_*(x) = [\phi_\delta(x)]_{>r_*}$.

(2) *Let* α_i, $1 \le i \le n$, *be the smallest positive integer such that* $\alpha_i \delta(g_i)$ *is in the* (*abelian*) *group generated by* $\delta(g_0), \dots, \delta(g_{i-1})$. *Fix* m, $0 \le m \le n$. *Recall that each* g_i *is an element in* $\mathbb{C}[x, x^{-1}, y]$ *which is monic in* y. *The following are equivalent*:

(a) g_i *is a polynomial*, $0 \le i \le m + 1$.

(b) *For each* i, $1 \le i \le m$, *the semigroup generated by* $\delta(g_0), \dots, \delta(g_{i-1})$ *contains* $\alpha_i \delta(g_i)$.

5B. ***Degree-like functions and compactifications.*** In this subsection we recall from [Mondal 2014b] the basic facts of compactifications of affine varieties via *degree-like functions*. Recall that $X = \mathbb{C}^2$ in our notation; however the results in this subsection remain valid if X is an arbitrary affine variety.

Definition 5.4. A map $\delta : \mathbb{C}[X] \setminus \{0\} \to \mathbb{Z}$ is called a *degree-like function* if

(1) $\delta(f + g) \le \max\{\delta(f), \delta(g)\}$ for all $f, g \in \mathbb{C}[X]$, with $<$ in the preceding inequality implying $\delta(f) = \delta(g)$.

(2) $\delta(fg) \le \delta(f) + \delta(g)$ for all $f, g \in \mathbb{C}[X]$.

Every degree-like function δ on $\mathbb{C}[X]$ defines an *ascending filtration* $\{\mathcal{F}^\delta_d\}_{d \ge 0}$ on $\mathbb{C}[X]$, where $\mathcal{F}^\delta_d := \{f \in \mathbb{C}[X] : \delta(f) \le d\}$. Define

$$\mathbb{C}[X]^\delta := \bigoplus_{d \ge 0} \mathcal{F}^\delta_d, \quad \operatorname{gr} \mathbb{C}[X]^\delta := \bigoplus_{d \ge 0} \mathcal{F}^\delta_d / \mathcal{F}^\delta_{d-1}.$$

Remark 5.5. For every $f \in \mathbb{C}[X]$, there are infinitely many “copies” of f in $\mathbb{C}[X]^\delta$, namely the copy of f in $\mathcal{F}^\delta_d$ for *each* $d \ge \delta(f)$; we denote the copy of f in $\mathcal{F}^\delta_d$ by $(f)_d$. If t is a new indeterminate, then

$$\mathbb{C}[X]^\delta \cong \sum_{d \ge 0} \mathcal{F}^\delta_d t^d,$$

via the isomorphism $(f)_d \mapsto f t^d$. Note that t corresponds to $(1)_1$ under this isomorphism.

We say that δ is *finitely generated* if $\mathbb{C}[X]^\delta$ is a finitely generated algebra over $\mathbb{C}$ and that δ is *projective* if in addition $\mathcal{F}_0^\delta = \mathbb{C}$. The motivation for the terminology comes from the following straightforward result.

Proposition 5.6 [Mondal 2014b, Proposition 2.8]. *If δ is a projective degree-like function on $\mathbb{C}[X]$, then $\bar{X}^\delta := \operatorname{Proj} \mathbb{C}[X]^\delta$ is a projective compactification of X. The* hypersurface at infinity *$\bar{X}^\delta_\infty := \bar{X}^\delta \setminus X$ is the zero set of the $\mathbb{Q}$-Cartier divisor defined by $(1)_1$ and is isomorphic to $\operatorname{Proj} \operatorname{gr} \mathbb{C}[X]^\delta$. Conversely, if $\bar{X}$ is any projective compactification of X such that $\bar{X} \setminus X$ is the support of an effective ample divisor, then there is a projective degree-like function δ on $\mathbb{C}[X]$ such that $\bar{X}^\delta \cong \bar{X}$.*

Remark 5.7. A *semidegree*, which we already defined in Section 3B, is a degree-like function which always satisfies property (2) of Definition 5.4 with an equality.

The following proposition (which is straightforward to prove) is a special case of [Mondal 2014b, Theorem 4.1].

Proposition 5.8. *Let δ be a projective degree-like function on $\mathbb{C}[X]$, and $\bar{X}^\delta$ be the corresponding projective compactification from Proposition 5.6. Assume δ is a semidegree. Then $\bar{X}^\delta$ is a normal variety and $\bar{X}^\delta_\infty := \bar{X}^\delta \setminus X$ is an irreducible codimension-one subvariety. Moreover, there is a positive integer d_δ such that δ agrees with d_δ times the order of* pole *along $\bar{X}^\delta_\infty$.*

6. Some preparatory results

In this section we develop some preliminary results to be used in Section 7 for the proofs of our main results.

Convention 6.1. Let $y_1, \dots, y_k$ be indeterminates. From now on we write A_k, $\tilde{A}_k$, R, $\tilde{R}$ to denote respectively $\mathbb{C}[x, x^{-1}, y_1, \dots, y_k]$, $\mathbb{C}\langle\langle x\rangle\rangle[y_1, \dots, y_k]$, $\mathbb{C}[x, x^{-1}, y]$, $\mathbb{C}\langle\langle x\rangle\rangle[y]$. Below we frequently deal with maps $A_k \to R$. We always (unless there is a misprint!) use upper-case letters $F, G, \dots$ for elements in A_k and corresponding lower-case letters $f, g, \dots$ for their images in R.

6A. *The "star action" on descending Puiseux series.*

Definition 6.2. Let $\phi = \sum_j a_j x^{q_j/p} \in \mathbb{C}\langle\langle x\rangle\rangle$ be a descending Puiseux series with polydromy order p and r be a multiple of p. Then for all $c \in \mathbb{C}$ we define

$$c \star_r \phi := \sum_j a_j c^{q_j r/p} x^{q_j/p}.$$

For $\Phi = \sum_{\alpha \in \mathbb{Z}^k_{\geq 0}} \phi_\alpha(x) y_1^{\alpha_1} \cdots y_k^{\alpha_k} \in \tilde{A}_k$, the *polydromy order* of Φ is the lowest common multiple of the polydromy orders of all the nonzero ϕ_α. Let r be a multiple of the polydromy order Φ. Then we define

$$c \star_r \Phi := \sum_\alpha (c \star_r \phi_\alpha) y_1^{\alpha_1} \cdots y_k^{\alpha_k}.$$

Remark 6.3. It is straightforward to see that in the case that c is an r-th root of unity (and r is a multiple of the polydromy order of ϕ), $c \star_r \phi$ is a *conjugate* of ϕ (cf. Remark-Notation 6.5).

The following properties of the $\star_r$ operator are straightforward to see:

Lemma 6.4. (1) *Let p be the polydromy order of $\Phi \in \tilde{A}_k$, d and e be positive integers, and $c \in \mathbb{C}$. Then $c \star_{pde} \Phi = c^e \star_{pd} \Phi = c^{de} \star_p \Phi$.*

(2) *Let $\Phi_j = \sum_j \phi_{j,\alpha}(x) y_1^{\alpha_1} \cdots y_k^{\alpha_k} \in \tilde{A}_k$ for $j = 1, 2$, and r be a multiple of the polydromy order of each nonzero $\phi_{j,\alpha}$. Then*

$$c \star_r (\Phi_1 + \Phi_2) = (c \star_r \Phi_1) + (c \star_r \Phi_2),$$
$$c \star_r (\Phi_1 \Phi_2) = (c \star_r \Phi_1)(c \star_r \Phi_2).$$

(3) *Let $\pi : \tilde{A}_k \to \tilde{R}$ be a $\mathbb{C}$-algebra homomorphism defined by $x \mapsto x$ and*

$$y_j \mapsto f_j \in R \quad \textit{for } 1 \le j \le k.$$

Let $\Phi = \sum_\alpha \phi_\alpha(x) y_1^{\alpha_1} \cdots y_k^{\alpha_k} \in \tilde{A}_k$, let r be a multiple of the polydromy order of each nonzero ϕ_α, and μ be a (not necessarily primitive) r-th root of unity. Then $\pi(\mu \star_r \Phi) = \mu \star_r \pi(\Phi)$. □

Remark-Notation 6.5. If ϕ is a descending Puiseux series in x with polydromy order p, then we write

$$f_\phi := \prod_{\substack{\phi_j \text{ is a} \\ \text{conjugate} \\ \text{of } \phi}} \bigl(y - \phi_j(x)\bigr) = \prod_{j=0}^{p-1} \bigl(y - \zeta^j \star_p \phi(x)\bigr) \in \tilde{R}, \tag{6-1}$$

where ζ is a primitive p-th root of unity. If $f \in \mathbb{C}[x, y]$, then its descending Puiseux factorization (Theorem 3.6) can be described as follows: There are unique (up to conjugacy) descending Puiseux series $\phi_1, \dots, \phi_k$, a unique nonnegative integer m, and $c \in \mathbb{C}^*$ such that

$$f = c x^m \prod_{i=1}^{k} f_{\phi_i}.$$

Let $(q_1, p_1), \dots, (q_l, p_l)$ be Puiseux pairs of ϕ. Set $p_0 := 1$. For each k, $0 \le k \le l$, we write

$$f_\phi^{(k)} := \prod_{j=0}^{p_0 p_1 \cdots p_k - 1} (y - \zeta^j \star_p \phi(x)), \tag{6-2}$$

where ζ is a primitive $(p_1 \cdots p_l)$-th root of unity. Note that $f_\phi^{(l)} = f_\phi$, and for each m, n, $0 \le m < n \le l$,

$$\begin{aligned}
f_\phi^{(n)} &= \prod_{j=0}^{p_0 p_1 \cdots p_n - 1} \bigl(y - \zeta^j \star_p \phi(x)\bigr) \\
&= \prod_{i=0}^{p_{m+1} \cdots p_n - 1} \prod_{j=0}^{p_0 p_1 \cdots p_m - 1} \bigl(y - \zeta^{i p_0 p_1 \cdots p_m + j} \star_p \phi(x)\bigr) \\
&= \prod_{i=0}^{p_{m+1} \cdots p_n - 1} \zeta^{i p_0 p_1 \cdots p_m} \star_p \Biggl(\prod_{j=0}^{p_0 p_1 \cdots p_m - 1} \bigl(y - \zeta^j \star_p \phi(x)\bigr)\Biggr) \\
&= \prod_{i=0}^{p_{m+1} \cdots p_n - 1} \zeta^{i p_0 p_1 \cdots p_m} \star_p (f_\phi^{(m)}). \qquad (6\text{-}3)
\end{aligned}$$

6B. *“Canonical” preimages of polynomials and their comparison.*

Lemma 6.6 (“canonical” preimages of elements in $\mathbb{C}\langle\langle x \rangle\rangle[y]$). *Let $p_0 := 1$, and $p_1, \dots, p_{k-1}$ be positive integers, and $\pi : A_k \to R$ be a ring homomorphism which sends $x \mapsto x$ and $y_j \mapsto f_j$, where f_j is monic in y of degree $p_0 \cdots p_{j-1}$, $1 \le j \le k$. Then π induces a homomorphism $\tilde{A}_k \to \tilde{R}$ which we also denote by π. If f is a nonzero element in $\tilde{R}$, then there is a unique $F_f^\pi \in \tilde{A}_k$ such that*

(1) *$\pi(F_f^\pi) = f$, and*

(2) *$\deg_{y_j}(F_f^\pi) < p_j$ for all j, $1 \le j \le k-1$.*

Moreover, if f is monic in y of degree $p_1 \cdots p_{k-1} d$ for some integer d, then

(3) *F_f^π is monic in y_k of degree d,*

(4) *if the coefficient of $x^\alpha y_1^{\beta_1} \cdots y_k^{\beta_k}$ in $F_f^\pi - y_k^d$ is nonzero, then*

$$\sum_{i=1}^{j} p_0 \cdots p_{i-1} \beta_i < p_1 \cdots p_j \quad \text{for all } j,\ 1 \le j \le k-1,$$

and

$$\sum_{i=1}^{k} p_0 \cdots p_{i-1} \beta_i < p_1 \cdots p_{k-1} d.$$

Finally,

(5) *if each of $f, f_1, \dots, f_k$ is in $\mathbb{C}[x, y]$ (resp. R), then F_f^π is in $\mathbb{C}[x, y_1, \dots, y_k]$ (resp. A_k).*

Proof. This results from an application of Theorem 2.13 of [Abhyankar 1977]. □

Now assume δ is a semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Assume the generic descending Puiseux series for δ is

$$\begin{aligned}\tilde{\phi}_\delta(x,\xi) &:= \phi_\delta(x) + \xi x^{r_\delta}\\ &= \cdots + a_1 x^{q_1/p_1} + \cdots + a_2 x^{q_2/(p_1 p_2)} + \cdots + a_l x^{q_l/(p_1\cdots p_l)} + \xi x^{q_{l+1}/(p_1\cdots p_{l+1})},\end{aligned}$$

where $(q_1, p_1), \ldots, (q_{l+1}, p_{l+1})$ are the formal Puiseux pairs of $\tilde{\phi}_\delta$. Let $g_0 = x$, $g_1 = y, \ldots, g_{n+1} \in R$ be the sequence of key forms of δ and $g_{j_0}, \ldots, g_{j_{l+1}}$ be the subsequence of *essential* key forms. For $0 \le k \le l+1$, define

$$f_k := g_{j_k}, \qquad \omega_k := \delta(f_k). \tag{6-4}$$

Lemma 6.7. *f_1 has the form y − a polynomial in x. If $1 \le k \le l$, one can write*

$$f_{k+1} = f_k^{p_k} - \sum_{i=0}^{m_k} c_{k,i} f_0^{\beta^i_{k,0}} \cdots f_k^{\beta^i_{k,k}}$$

where

(1) $m_k \ge 0$,

(2) $c_{k,i} \in \mathbb{C}^*$ *for all i, $0 \le i \le m_k$,*

(3) *the $\beta^i_{k,j}$ are integers such that* $0 \le \beta^i_{k,j} < p_j$ *for* $1 \le j \le k$ *and* $0 \le i \le m_k$,

(4) $\beta^0_{k,k} = 0$,

(5) $p_k\omega_k = \sum_{j=0}^{k-1} \beta^0_{k,j}\omega_j > \sum_{j=0}^{k} \beta^1_{k,j}\omega_j > \cdots > \sum_{j=0}^{k} \beta^{m_k}_{k,j}\omega_j > \omega_{k+1}$.

Proof. Combine property (P2) of key forms, assertion (3) of Proposition 3.21, and the defining property of essential key forms (Definition 3.20). □

Let $\pi : A_{l+1} \to R$ be the $\mathbb{C}$-algebra homomorphism which maps $x \mapsto x$ and $y_k \to f_k$, $1 \le k \le l+1$, and let $\pi_k := \pi|_{A_k} : A_k \to R$, $1 \le k \le l+1$. Let ω be the weighted degree on A_{l+1} corresponding to weights ω_0 for x and ω_k for y_k, $1 \le k \le l+1$. We will often abuse the notation and write π and ω respectively for π_k and $\omega|_{A_k}$ for each k, $1 \le k \le l+1$. Define

$$F_{k+1} := \begin{cases} y_1 & \text{if } k = 0,\\ y_k^{p_k} - \sum_{i=0}^{m_k} c_{k,i} x^{\beta^i_{k,0}} y_1^{\beta^i_{k,1}} \cdots y_k^{\beta^i_{k,k}} & \text{if } 1 \le k \le l,\end{cases} \tag{6-5}$$

where the $c_{k,i}$ and $\beta^i_{k,j}$ are as in Lemma 6.7. Note that $F_1 \in A_1$ and $F_k \in A_{k-1}$ for $2 \le k \le l+1$. Moreover, $\pi(F_k) = f_k$ for each k, $1 \le k \le l+1$.

Lemma 6.8. *Fix k, $1 \le k \le l+1$.*

(1) *Let H_1, H_2 be two monomials in A_k such that* $\deg_{y_j}(H) < p_j$ *for all j, $1 \le j \le k$. Then $\omega(H_1) \ne \omega(H_2)$.*

(2) *Suppose $H \in A_k$ is such that* $\deg_{y_j}(H) < p_j$ *for all j, $1 \le j \le k$. Then $\delta(\pi(H)) = \omega(H)$.*

Proof. Assertion (3) of Proposition 3.21 implies that for each j, $1 \le j \le k$, p_j is the smallest positive integer such that $p_j \omega_j$ is in the group generated by $\omega_0, \dots, \omega_{j-1}$. This immediately implies assertion (1). For assertion (2), write $H = \sum_{i \ge 1} H_i$, where the H_i are monomials in A_k. By assertion (1) we may assume w.l.o.g. that $\omega(H) = \omega(H_1) > \omega(H_2) > \cdots$. Since $\delta(\pi(y_j)) = \delta(f_j) = \omega_j = \omega(y_j)$ for each j, $1 \le j \le k$, it follows that $\delta(\pi(H_i)) = \omega(H_i)$ for all i. It then follows from the definition of degree-like functions (Definition 5.4) that $\delta(\pi(H)) = \omega(H_1) = \omega(H)$. □

Lemma 6.9. *For each* k, $1 \le k \le l+1$, *define*

$$r_k := \frac{q_k}{p_1 p_2 \cdots p_k}, \tag{6-6}$$

$$\phi_k := [\phi_\delta]_{>r_k}. \tag{6-7}$$

Define $f_{\phi_k} \in \tilde{R}$ *as in* (6-1). *Also define*

$$F_{\phi_k} := \begin{cases} F^{\pi_1}_{f_{\phi_1}} \in \tilde{A}_1 & \text{for } k = 1, \\ F^{\pi_{k-1}}_{f_{\phi_k}} \in \tilde{A}_{k-1} & \text{for } 2 \le k \le l+1. \end{cases}$$

Then:

(a) $\delta(f_{\phi_k}) = \omega_k$.

(b) $F_1 = F_{\phi_1} = y_1$.

(c) *For* $k \ge 1$, F_{k+1} *is precisely the sum of all monomial terms* T (*in* $x, y_1, \dots, y_k$) *of* $F_{\phi_{k+1}}$ *such that* $\omega(T) > \omega_{k+1}$.

Proof. We compute $\delta(f_{\phi_k})$ using (3-1). Let $\tilde{p}_k := p_1 \cdots p_{k-1}$. It is straightforward to see that ϕ_k has precisely $\tilde{p}_k$ conjugates $\phi_{k,1}, \dots, \phi_{k,\tilde{p}_k}$, and $\deg_x(\tilde{\phi}_\delta(x,\xi) - \phi_{k,j}(x))$ equals r_1 for $(p_1 - 1)p_2 \cdots p_{k-1}$ of the $\phi_{k,j}$, equals r_2 for $(p_2 - 1)p_3 \cdots p_{k-1}$ of the $\phi_{k,j}$, and so on. Identity (3-1) then implies that

$$\begin{aligned}\delta(f_{\phi_k}) &= \delta(x) \sum_{j=1}^{\tilde{p}_k} \deg_x(\tilde{\phi}_\delta(x,\xi) - \phi_{k,j}(x)) \\ &= p_1 \cdots p_{l+1} \biggl((p_1 - 1) p_2 \cdots p_{k-1} \frac{q_1}{p_1} + (p_2 - 1) p_3 \cdots p_{k-1} \frac{q_2}{p_1 p_2} + \cdots \\ &\qquad\qquad + (p_{k-1} - 1) \frac{q_{k-1}}{p_1 \cdots p_{k-1}} + \frac{q_k}{p_1 \cdots p_k} \biggr).\end{aligned}$$

A straightforward induction on k then yields that

$$\delta(f_{\phi_k}) = p_{k-1} \delta(f_{\phi_{k-1}}) + (q_k - q_{k-1} p_k) p_{k+1} \cdots p_{l+1}.$$

Identity (3-4) then implies that $\delta(f_{\phi_k}) = \omega_k$, which proves assertion (a). Assertion (b) follows immediately from the definitions. We now prove assertion (c). Fix k, $1 \le k \le l$. Let $\tilde{F}$ be the sum of all monomial terms T (in $x, y_1, \dots, y_k$) of $F_{\phi_{k+1}}$

such that $\omega(T) > \omega_{k+1}$, i.e., $F_{\phi_{k+1}} = \tilde{F} + \tilde{G}$ for some $\tilde{G} \in \tilde{A}_k$ with $\omega(\tilde{G}) \le \omega_{k+1}$. It follows that

$$\begin{aligned}\delta(\pi(\tilde{F})) = \delta(\pi(F_{\phi_{k+1}}) - \pi(\tilde{G})) &\le \max\{\delta(\pi(F_{\phi_{k+1}})), \delta(\pi(\tilde{G}))\} \\ &\le \max\{\delta(f_{\phi_{k+1}}), \omega(\tilde{G})\} \le \omega_{k+1}.\end{aligned}$$

On the other hand, $\delta(\pi(F_{k+1})) = \delta(f_{k+1}) = \omega_{k+1}$. It follows that

$$\delta(\pi(\tilde{F}-F_{k+1}))=\delta(\pi(\tilde{F})-\pi(F_{k+1}))\le\max\{\delta(\pi(\tilde{F})),\delta(\pi(F_{k+1}))\}=\omega_{k+1}. \tag{6-8}$$

Now, (6-5) and the defining properties of F_{ϕ_k} in Lemma 6.6 imply that $H := \tilde{F} - F_{k+1}$ satisfies the hypothesis of assertion (2) of Lemma 6.8, so that $\delta(\pi(\tilde{F} - F_{k+1})) = \omega(\tilde{F} - F_{k+1})$. Inequality (6-8) then implies that

$$\omega(\tilde{F} - F_{k+1}) \le \omega_{k+1}. \tag{6-9}$$

But the construction of $\tilde{F}$ and assertion (5) of Lemma 6.7 imply that all the monomials that appear in $\tilde{F}$ or F_{k+1} have ω-value *greater than* ω_{k+1}. Therefore (6-9) implies that $\tilde{F} = F_{k+1}$, as required to complete the proof. □

The proof of the next lemma is long, and we put it in Appendix A.

Lemma 6.10. *Fix k, $0 \le k \le l$. Pick $\psi \in \mathbb{C}\langle\langle x\rangle\rangle$ such that $\psi \equiv_{r_{k+1}} \phi_\delta$; in particular, the first k Puiseux pairs of ψ are $(q_1, p_1), \dots, (q_k, p_k)$. As in* (6-2), *define*

$$f_\psi^{(k)} := \prod_{j=0}^{p_0 p_1 \cdots p_k - 1} \big(y - \zeta^j \star_q \psi(x)\big),$$

where q is the polydromy order of ψ and ζ is a primitive q-th root of unity. Define

$$F_\psi^{(k)} := \begin{cases} F^{\pi_1}_{f_\psi^{(0)}} \in \tilde{A}_1 & \text{for } k = 0, \\ F^{\pi_k}_{f_\psi^{(k)}} \in \tilde{A}_k & \text{for } 1 \le k \le l. \end{cases} \tag{6-10}$$

Then

$$\omega(F_\psi^{(k)} - F_{k+1}) \le \omega_{k+1}.$$

6C. *Implications of polynomial key forms.* We continue with the notation of Section 6B. Let $\xi_1, \dots, \xi_{l+1}$ be new indeterminates, and for each k, $1 \le k \le l+1$, let δ_k be the semidegree on $\mathbb{C}[x, y]$ corresponding to the generic degreewise Puiseux series

$$\tilde{\phi}_k(x, \xi_k) := \phi_k(x) + \xi_k x^{r_k},$$

i.e., $\delta_k(x) = p_1 \cdots p_k$ and for each $f \in \mathbb{C}[x, y] \setminus \{0\}$,

$$\delta_k(f(x, y)) = \delta_k(x) \deg_x\big(f(x, \tilde{\phi}_k(x, \xi_k))\big). \tag{6-11}$$

The following lemma follows from a straightforward examination of Algorithm 5.1.

Lemma 6.11. *For each k, $1 \leq k \leq l+1$, the following hold*:

(1) *The key forms of δ_k are $g_0, g_1, \dots, g_{j_k}$.*

(2) *The essential key forms of δ_k are $f_0, \dots, f_k$.*

(3) $\delta_k(g_j) = \frac{\delta(g_j)}{p_{k+1}\cdots p_{l+1}}$, $0 \leq j \leq j_k$. □

Fix k, $1 \leq k \leq l+1$. In this subsection we assume condition (Polynomial$_k$) below is satisfied, and examine some of its implications.

$$\text{All the key forms of } \delta_k \text{ are polynomials.} \tag{Polynomial$_k$}$$

Lemma 6.12. *Assume* (Polynomial$_k$) *holds. Then $\delta(g_j) \geq 0$ for $0 \leq j \leq j_k - 1$.*

Proof. This follows from combining assertion (2) of Proposition 5.3 with assertion (3) of Lemma 6.11. □

Let $C_k := \mathbb{C}[x, y_1, \dots, y_k] \subseteq A_k$. Since the g_j are polynomial for $0 \leq j \leq j_k$, Algorithm 5.1 implies that the F_j (defined in (6-5)) are also polynomial for $0 \leq j \leq k$; in particular, $F_1 \in C_1$ and $F_{j+1} \in C_j$, $1 \leq j \leq k-1$. For $1 \leq j \leq k-1$, let H_{j+1} be the *leading form* of F_{j+1} with respect to ω, i.e.,

$$H_{j+1} := y_j^{p_j} - c_{j,0} x^{\beta^0_{j,0}} y_1^{\beta^0_{j,1}} \cdots y_{j-1}^{\beta^0_{j,j-1}}, \quad 1 \leq j \leq k-1. \tag{6-12}$$

Let $\prec$ be the reverse lexicographic order on C_k, i.e., $x^{\beta_0} y_1^{\beta_1} \cdots y_k^{\beta_k} \prec x^{\beta'_0} y_1^{\beta'_1} \cdots y_k^{\beta'_k}$ if and only if the right-most nonzero entry of $(\beta_0 - \beta'_0, \dots, \beta_k - \beta'_k)$ is negative.

The following lemma is the main result of this subsection. Its proof is long, and we put it in Appendix B.

Lemma 6.13. *Assume* (Polynomial$_{l+1}$) *holds. Then*

(1) (Recall the notation of Section 5B.) *Define*

$$S^\delta := \bigoplus_{d \in \mathbb{Z}} \mathcal{F}^\delta_d \supseteq \mathbb{C}[x, y]^\delta.$$

Then S^δ is generated as a $\mathbb{C}$-algebra by $(1)_1$, $(x)_{\omega_0}$, $(y_1)_{\omega_1}, \dots, (y_{l+1})_{\omega_{l+1}}$.

(2) *Let J_{l+1} be the ideal in C_{l+1} generated by the* leading weighted homogeneous forms (*with respect to ω*) *of polynomials $F \in C_{l+1}$ such that $\delta(\pi(F)) < \omega(F)$. Then $\mathcal{B}_{l+1} := (H_{l+1}, \dots, H_2)$ is a Gröbner basis of J_{l+1} with respect to $\prec$.*

7. Proof of the main results

In this section we give proofs of Theorems 4.1, 4.3, 4.4 and 4.7.

Proof of Theorem 4.7. The implication (2) ⇒ (3) is obvious. We prepare the ground for the rest with an easily seen reformulation:

Lemma 7.1. *Assertion* (1) *of Theorem 4.7 is equivalent to the following assertion*:

(1′) *There exists a polynomial* $f \in \mathbb{C}[x, y]$ *such that for each analytic branch* C *of the curve* $f = 0$ *at infinity*,

- C *intersects* L_∞ *at* O, *and*
- C *has a* ***descending*** *Puiseux expansion* $y = \theta(x)$ *at* O *such that* $\deg_x(\theta - \phi)$ *is at most* $1 - r$. □

Assertion (4) of Theorem 3.17 implies that if g_{n+1} is a polynomial, then g_{n+1} satisfies the properties of f from (1′); in particular (3) ⇒ (1′). To finish the proof of Theorem 4.7 it remains to prove that (1′) ⇒ (2). So assume (1′) holds. We proceed by contradiction, i.e., we also assume that there exists m, $1 \le m \le n$, such that g_{m+1} is not a polynomial, and show that this leads to a contradiction. By assertion (1) of Proposition 5.3, we may (and will) assume that $m = n$.

We adopt the notation of Sections 6B and 6C. In particular, we write $\tilde{\phi}_\delta(x, \xi)$ and $\phi_\delta(x)$, r_δ for $\tilde{\phi}(x, \xi)$ and $[\phi(x)]_{>1-r}$, $1 - r$, respectively, and we denote by $(q_1, p_1), \dots, (q_{l+1}, p_{l+1})$ the formal Puiseux pairs of $\tilde{\phi}_\delta$. We also denote by $g_{j_0}, \dots, g_{j_{l+1}}$ the sequence of essential key forms, and set $f_k := g_{j_k}$, $0 \le k \le l + 1$.

Let $f \in \mathbb{C}[x, y]$ be as in (1′). By assumption f has a descending Puiseux factorization of the form

$$f = a \prod_{m=1}^{M} f_{\psi_m} \tag{7-1}$$

for some $a \in \mathbb{C}^*$ and $\psi_1, \dots, \psi_m \in \mathbb{C}\langle\langle x \rangle\rangle$ such that

$$\psi_m \equiv_{r_\delta} \phi_\delta, \quad \text{for each } m,\ 1 \le m \le M, \tag{7-2}$$

where the f_{ψ_m} are defined as in (6-1). Without loss of generality we may (and will) assume that $a = 1$.

At first we claim that $l \ge 1$. Indeed, assume to the contrary that $l = 0$. Then

$$\tilde{\phi}_\delta(x, \xi) = h(x) + \xi x^{r_\delta}$$

for some $h \in \mathbb{C}[x, x^{-1}]$. Since g_{n+1} is not a polynomial, assertion (2) of Proposition 5.3 implies that $h(x) = h_1(x) + h_2(x)$, where $h_1 \in \mathbb{C}[x]$, $h_2 \in \mathbb{C}[x^{-1}] \setminus \mathbb{C}$, and $0 > \deg_x(h_2(x)) > r_\delta$. Let $e := -\deg_x(h_2(x)) > 0$ and $y' := y - h_1(x)$. Then (7-1) implies that f is a product of elements in $\mathbb{C}\langle\langle x \rangle\rangle[y']$ of the form $y' - \psi_{m,i}(x)$ for $\psi_{m,i} \in \mathbb{C}\langle\langle x \rangle\rangle$ such that each $\psi_{m,i}(x) = h_2(x) + \text{l.d.t.}$, where l.d.t. denotes terms with degree smaller than $\operatorname{ord}_x(h_2) < -e$. It is then straightforward to see that $f \notin \mathbb{C}[x, y'] = \mathbb{C}[x, y]$, which contradicts our choice of f. It follows that $l \ge 1$, as claimed.

Let F_k, $1 \leq k \leq l+1$, be as in (6-5). Fix m, $1 \leq m \leq M$. Then (7-2) and Lemma 6.10 imply that

$$F^{(l)}_{\psi_m} = F_{l+1} + \tilde{F}_m, \tag{7-3}$$

where $\tilde{F}_m \in \tilde{A}_l := \mathbb{C}\langle\langle x \rangle\rangle[y_1, \dots, y_l]$ and $\omega(\tilde{F}_m) \leq \omega_{l+1}$. Let s_m denote the polydromy order of ψ_m and μ_m be a primitive s_m-th root of unity. Identity (7-2) implies that $t_m := s_m/(p_1 p_2 \cdots p_l)$ is a positive integer. Therefore (6-3) and assertion (3) of Lemma 6.4 imply that

$$f_{\psi_m} = \prod_{j=0}^{t_m - 1} \mu_k^{j p_1 \cdots p_l} \star_{s_m} (f^{(l)}_{\psi_m}) = \prod_{j=0}^{t_m - 1} \mu_m^{j p_1 \cdots p_l} \star_{s_m} (\pi_l(F_{l+1} + \tilde{F}_m)) = \pi_l(G_m), \tag{7-4}$$

where

$$G_m := \prod_{j=0}^{t_m - 1} \bigl(F_{l+1} + \mu_m^{j p_1 \cdots p_l} \star_{s_m} (\tilde{F}_m)\bigr) \in \tilde{B}_l. \tag{7-5}$$

Recall that $F_{l+1} = y_l^{p_l} - \sum_{i=1}^{m_l} c_{l,i} x^{\beta^i_{l,0}} y_1^{\beta^i_{l,1}} \cdots y_l^{\beta^i_{l,l}}$. Since by our assumption all the key forms but the last one are polynomials, it follows from assertion (2) of Proposition 5.3 that $\beta^i_{l,0} \geq 0$ for all $i < m_l$, but $\beta^i_{m_l,0} < 0$; set

$$\omega'_{l+1} := \omega(x^{\beta^{m_l}_{l,0}} y_1^{\beta^{m_l}_{l,1}} \cdots y_l^{\beta^{m_l}_{l,l}}) = \sum_{i=0}^{l} \beta^{m_l}_{l,i} \omega_i. \tag{7-6}$$

Then $\omega'_{l+1} > \omega_{l+1}$ and therefore we may express G_m as

$$G_m = \prod_{j=0}^{t_m - 1} \Biggl(y_l^{p_l} - \sum_{i=0}^{m_l} c_{l,i} x^{\beta^i_{l,0}} y_1^{\beta^i_{l,1}} \cdots y_l^{\beta^i_{l,l}} - G_{m,j} \Biggr), \tag{7-7}$$

for some $G_{m,j} \in \tilde{B}_l$ with $\omega(G_{m,j}) < \omega'_{l+1}$. Identities (7-1), (7-4) and (7-7) imply that $f = \pi_l(F)$ for some $F \in \tilde{A}_l$ of the form

$$F = \prod_{m'=1}^{M'} \Biggl(y_l^{p_l} - \sum_{i=0}^{m_l} c_{l,i} x^{\beta^i_{l,0}} y_1^{\beta^i_{l,1}} \cdots y_l^{\beta^i_{l,l}} - G'_{m'} \Biggr), \tag{7-8}$$

where $\omega(G'_{m'}) < \omega'_{l+1}$ for all m', $1 \leq m' \leq M'$. Let

$$G := \begin{cases} F - y_l^{M' p_l} & \text{if } m_l = 0, \\ F - \bigl(y_l^{p_l} - \sum_{i=0}^{m_l - 1} c_{l,i} x^{\beta^i_{l,0}} y_1^{\beta^i_{l,1}} \cdots y_l^{\beta^i_{l,l}}\bigr)^{M'} & \text{otherwise.} \end{cases}$$

Recall from assertion (4) of Lemma 6.7 that $\beta^0_{l,l} = 0$. It follows that the *leading weighted homogeneous form* of G with respect to ω is

$$\mathfrak{L}_\omega(G) = \begin{cases} -c_{l,0}x^{\beta^0_{l,0}}y_1^{\beta^0_{l,1}}\cdots y_{l-1}^{\beta^0_{l,l-1}} & \text{if } m_l=0,\ M'=1,\\ \sum_{i=1}^{M'}\binom{M'}{i}(-c_{l,0})^i y_l^{(M'-i)p_l}x^{i\beta^0_{l,0}}y_1^{i\beta^0_{l,1}}\cdots y_{l-1}^{i\beta^0_{l,l-1}} & \text{if } m_l=0,\ M'>1,\\ M'c_{l,m_l}(y_l^{p_l}-c_{l,0}x^{\beta^0_{l,0}}y_1^{\beta^0_{l,1}}\cdots y_{l-1}^{\beta^0_{l,l-1}})^{M'-1}x^{\beta^{m_l}_{l,0}}y_1^{\beta^{m_l}_{l,1}}\cdots y_l^{\beta^{m_l}_{l,l}} & \text{otherwise.}\end{cases} \tag{7-9}$$

Since $\pi_l(F)=f\in\mathbb{C}[x,y]$, it follows that $g:=\pi_l(G)$ is also a polynomial in x and y. Assertion (1) of Lemma 6.13 then implies that there is a polynomial $\tilde{G}\in C_l:=\mathbb{C}[x,y_1,\dots,y_l]$ such that $\pi_l(\tilde{G})=g$ and $\omega(\tilde{G})=\delta_l(g)$. In particular, $\omega(\tilde{G})\le\omega(G)$.

Claim 7.2. $\omega(\tilde{G})=\omega(G)$.

Proof. Let $\prec$ be the reverse lexicographic monomial ordering on C_l from Section 6C and let α be the smallest positive integer such that $x^\alpha\mathfrak{L}_\omega(G)$ is a polynomial. Then (7-9) implies that the *leading term* of $x^\alpha\mathfrak{L}_\omega(G)$ with respect to $\prec$ is

$$\mathrm{LT}_\prec(x^\alpha\mathfrak{L}_\omega(G)) = \begin{cases} -c_{l,0}y_1^{\beta^0_{l,1}}\cdots y_{l-1}^{\beta^0_{l,l-1}} & \text{if } m_l=0,\ M'=1,\\ -c_{l,0}M'y_l^{(M'-1)p_l}x^{(M'-1)|\beta^0_{l,0}|}y_1^{\beta^0_{l,1}}\cdots y_{l-1}^{\beta^0_{l,l-1}} & \text{if } m_l=0,\ M'>1,\\ M'c_{l,m_l}y_l^{(M'-1)p_l+\beta^{m_l}_{l,l}}y_1^{\beta^{m_l}_{l,1}}\cdots y_{l-1}^{\beta^{m_l}_{l,l-1}} & \text{otherwise.}\end{cases} \tag{7-10}$$

Assume contrary to the claim that $\omega(G)>\omega(\tilde{G})=\delta_l(g)$. Then $x^\alpha\mathfrak{L}_\omega(G)\in J_l$, where J_l is the ideal from assertion (2) of Lemma 6.13. Assertion (2) of Lemma 6.13 then implies that there exists j, $1\le j\le l-1$, such that $y_j^{p_j}=\mathrm{LT}_\prec(H_{j+1})$ divides $\mathrm{LT}_\prec(x^\alpha\mathfrak{L}_\omega(G))$. But this contradicts the fact that $\beta^{m_l}_{l,j'}<p_{j'}$ for all j', $1\le j'\le l-1$ (assertion (3) of Lemma 6.7) and completes the proof of the claim. □

Let J_l and α be as in the proof of Claim 7.2. Note that $\mathfrak{L}_\omega(x^\alpha\tilde{G})\notin J_l$ by our choice of $\tilde{G}$. Therefore, after "dividing out" $\tilde{G}$ by the Gröbner basis $\mathcal{B}_l$ of Lemma 6.13 (which does not change $\omega(\tilde{G})$) if necessary, we may (and will) assume that

$$y_j^{p_j} \text{ does } \textit{not} \text{ divide any of the monomial terms of } \mathfrak{L}_\omega(x^\alpha\tilde{G}) \text{ for any } j,\ 1\le j\le l-1. \tag{7-11}$$

Since $\pi_l(x^\alpha G-x^\alpha\tilde{G})=0$, it follows that $\mathfrak{L}_\omega(x^\alpha G-x^\alpha\tilde{G})\in J_l$. Since $\omega(G)=\omega(\tilde{G})$ by Claim 7.2, it follows that $H^*:=\mathfrak{L}_\omega(x^\alpha G)-\mathfrak{L}_\omega(x^\alpha\tilde{G})\in J_l$. Let

$$H:=\mathrm{LT}_\prec(\mathfrak{L}_\omega(x^\alpha G)) \quad\text{and}\quad \tilde{H}:=\mathrm{LT}_\prec(\mathfrak{L}_\omega(x^\alpha\tilde{G})).$$

Since $\tilde{G}\in\mathbb{C}[x,y_1,\dots,y_l]$, it follows that $\deg_x(\tilde{H})\ge\alpha$. On the other hand, (7-10) implies that $\deg_x(H)=\alpha+\beta^0_{m_l,0}<\alpha$. It follows in particular that $H\ne\tilde{H}$ and $\mathrm{LT}_\prec(H^*)=\max_\prec\{H,-\tilde{H}\}$. Then (7-10) and (7-11) imply that $y_j^{p_j}=\mathrm{LT}_\prec(H_j)$ does *not* divide $\mathrm{LT}_\prec(H^*)$ for any j, $1\le j\le l-1$. This contradicts assertion (2) of Lemma 6.13 and finishes the proof of the implication $(1') \Rightarrow (2)$, as required to complete the proof of Theorem 4.7. $\square$

Proof of Theorem 4.1. Theorem 4.7 implies that $(2) \Leftrightarrow (3)$. Now assume (2) is true. Note that $\delta(f) > 0$ for each nonconstant $f \in \mathbb{C}[x, y]$ (since such an f must have a pole at the irreducible curve $E'_1 := \pi'(E_1) \subseteq Y'$); so that the ring S^δ defined in Lemma 6.13 is precisely the ring $\mathbb{C}[x, y]^\delta$ from Section 5B. Assertion (1) of Lemma 6.13 and Proposition 5.8 then imply that Y' is isomorphic to the closure of the image of $\mathbb{C}^2$ in the weighted projective variety $\mathbb{P}^{l+2}(1, \delta(f_0), \dots, \delta(f_{l+1}))$ under the mapping $(x, y) \mapsto [1 : f_0 : \cdots : f_{l+1}]$. In particular this shows $(2) \Rightarrow (1)$.

It remains to show that $(1) \Rightarrow (2)$. So assume that Y' is algebraic. Recall the setup of Proposition 3.14. We can identify Y' with $\bar{X}$ and E'_1 with C_∞ (where $\bar{X}$ and C_∞ are as in Proposition 3.14). Let $P_\infty \in C_\infty$ be as in Proposition 3.14. Since Y' is algebraic, there exists a compact algebraic curve C on Y' such that $P_\infty \notin C$. Let $f \in \mathbb{C}[x, y]$ be the polynomial that generates the ideal of C in $\mathbb{C}[x, y]$. Proposition 3.14 then implies that f satisfies the condition of property $(1')$ from Lemma 7.1. Theorem 4.7 and Lemma 7.1 then show that (2) is true, as required. $\square$

Proof of Theorem 4.3. Let δ be the semidegree on $\mathbb{C}[x, y]$ corresponding to the curve at infinity on $\bar{X}$, $\tilde{\phi}_\delta(x, \xi)$ be the associated generic descending Puiseux series, and $g_0, \dots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ be the key forms. If $\bar{X}$ is algebraic, then Theorem 4.1 implies that g_{n+1} is a polynomial. Proposition 3.14 and assertion (4) of Theorem 3.17 then imply that $g_{n+1} = 0$ defines a curve C as in assertion (2) of Theorem 4.3. This proves the implication $(1) \Rightarrow (2)$, and also the last assertion of Theorem 4.3. It remains to prove the implication $(2) \Rightarrow (1)$. So assume there exists $f \in \mathbb{C}[x, y]$ such that $C := \{f = 0\}$ is as in (2). Proposition 3.14 implies that f satisfies the condition of property $(1')$ from Lemma 7.1. Then Lemma 7.1, Theorem 4.7 and Theorem 4.1 imply that $\bar{X}$ is algebraic, as required. $\square$

Proof of Theorem 4.4. The $(\Rightarrow)$ direction of assertion (1) follows from Theorem 4.1 and assertion (2) of Proposition 5.3. For the $(\Leftarrow)$ implication, note that assertion (3) of Proposition 3.21 and Property (P0) of key forms imply for each k, $1 \le k \le l$, that

$$p_k \omega_k = \sum_{j=0}^{k-1} \beta'_{k,j} \omega_j,$$

where the $\beta'_{k,j}$ are integers such that $0 \le \beta'_{k,j} < p_j$ for $1 \le j < k$. Define g_k^0, $0 \le k \le l+1$, by

$$g_k^0 = \begin{cases} x & \text{if } k = 0, \\ y & \text{if } k = 1, \\ (g_{k-1}^0)^{p_{k-1}} - \prod_{j=0}^{k-2} (g_j^0)^{\beta'_{k-1,j}} & \text{if } 2 \le k \le l+1. \end{cases}$$

Assertion (2) of Theorem 3.17 implies that there is a unique semidegree δ^0 on $\mathbb{C}[x, y]$ such that its key forms are $g_0^0, \ldots, g_{l+1}^0$ and $\delta^0(g_k^0) = \omega_k$, $0 \le k \le l+1$. Since $\omega_{l+1} > 0$ (assertion (3) of Theorem 3.28) it follows that δ^0 defines a primitive normal compactification $\bar{X}^0$ of $\mathbb{C}^2$ (Remark 4.2). It follows from Proposition 3.21 that (q_k, p_k), $1 \le k \le l+1$, are uniquely determined in terms of $\omega_0, \ldots, \omega_{l+1}$. Therefore Γ is precisely the augmented and marked dual graph associated to the minimal $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}^0$. Now, if (S1-k) holds for each k, $1 \le k \le l$, then each $\beta'_{k,0}$ is nonnegative, so that each g_k^0 is a polynomial. Theorem 4.1 then implies that $\bar{X}^0$ is algebraic, which proves the ($\Leftarrow$) implication of assertion (1).

Now we prove assertion (2). For the ($\Rightarrow$) implication, pick a nonalgebraic normal primitive compactification $\bar{X}$ of $\mathbb{C}^2$ such that $\Gamma = \Gamma_{\bar{X}}$. Let δ be the order of pole along the curve at infinity on $\bar{X}$. Theorem 4.1 implies that at least one of the key forms of δ is not a polynomial. Assertions (2) and (4) of Proposition 3.21 and assertion (2) of Proposition 5.3 now imply that either (S1-k) or (S2-k) fails, as required. It remains to prove the ($\Leftarrow$) implication of assertion (2). Let g_k^0, $0 \le k \le l+1$, be as in the preceding paragraph. If (S1-k) fails for some k, $1 \le k \le l$, take the smallest such k. Then by construction g_k^0 is *not* a polynomial, so that $\bar{X}^0$ is not algebraic (Theorem 4.1), as required. Now assume that (S1-k) holds for all k, $1 \le k \le l$, but there exists k, $1 \le k \le l$, such that (S2-k) fails; let k be the smallest such integer. Pick $\tilde{\omega} \in (\omega_{k+1}, p_k\omega_k) \cap \mathbb{Z}\langle \omega_0, \ldots, \omega_k\rangle \setminus \mathbb{Z}_{\ge 0}\langle \omega_0, \ldots, \omega_k\rangle$. Then it is straightforward to see that there exist integers $\tilde{\beta}_0, \ldots, \tilde{\beta}_k$ such that $\tilde{\beta}_0 < 0$, $0 \le \tilde{\beta}_j < p_j$, $1 \le j < k$, and

$$\tilde{\omega} = \sum_{j=0}^{k-1} \tilde{\beta}_j \omega_j.$$

Define g_i^1, where $0 \le i \le l+2$, by

$$g_i^1 = \begin{cases} g_i^0 & \text{if } 0 \le i \le k+1, \\ g_{k+1}^0 - \prod_{j=0}^{k} (g_j^0)^{\tilde{\beta}_j} & \text{if } i = k+2, \\ (g_{i-1}^1)^{p_{i-2}} - \prod_{j=0}^{k} (g_j^1)^{\beta'_{i-2,j}} \prod_{j=k+2}^{i-2} (g_j^1)^{\beta'_{i-2,j-1}} & \text{if } k+3 \le i \le l+2. \end{cases}$$

The same arguments as in the proof of assertion (1) imply that there is a primitive normal compactification $\bar{X}^1$ of $\mathbb{C}^2$ such that:

- $g_0^1, \ldots, g_{l+2}^1$ are the key forms of the semidegree δ^1 corresponding to its curve at infinity, and

$$\delta^1(g_i^1) = \begin{cases} \omega_i & \text{if } 0 \le i \le k, \\ \tilde{\omega} & \text{if } i = k+1, \\ \omega_{i-1} & \text{if } k+2 \le i \le l+2. \end{cases}$$

- Γ is the augmented and marked dual graph associated to the minimal $\mathbb{P}^2$-dominating resolution of singularities of $\bar{X}^1$.

Since g^1_{k+2} is not a polynomial, $\bar{X}^1$ is not algebraic (Theorem 4.1), as required to complete the proof of assertion (2). $\square$

Appendix A: Proof of Lemma 6.10

Notation A.1. Fix k, $1 \leq k \leq l+1$. For $F \in \tilde{A}_k$ and $\mu \in \mathbb{R}$, we write $[F]_{>\mu}$ for the sum of all monomial terms H of F such that $\omega(H) > \mu$.

Lemma A.2. *Fix k, $1 \leq k \leq l$. Pick $\psi_1, \psi_2 \in \mathbb{C}\langle\langle x \rangle\rangle$ and $\mu \leq \omega_k \in \mathbb{R}$. Assume*

(1) *the first k Puiseux pairs of each ψ_j are $(q_1, p_1), \dots, (q_k, p_k)$.*

Assumption (1) implies that we can define $F^{(k-1)}_{\psi_j}$, $F^{(k)}_{\psi_j}$, $1 \leq j \leq 2$, as in Lemma 6.10. Assume

(2) $[F^{(k-1)}_{\psi_1}]_{>\mu} = [F^{(k-1)}_{\psi_2}]_{>\mu}$, *and*

(3) $[F^{(k-1)}_{\psi_j}]_{>\omega_k} = [F_k]_{>\omega_k}$ *for each j, $1 \leq j \leq 2$.*

Then $[F^{(k)}_{\psi_1}]_{>(p_k-1)\omega_k+\mu} = [F^{(k)}_{\psi_2}]_{>(p_k-1)\omega_k+\mu}$.

Proof. Let

$$\tilde{A} := \begin{cases} \tilde{A}_1 & \text{if } k = 1, \\ \tilde{A}_{k-1} & \text{otherwise.} \end{cases}$$

Assumptions (2) and (3) imply that there exists $G \in \tilde{A}$ with $\omega(G) \leq \omega_k$ such that for both j, $1 \leq j \leq 2$,

$$F^{(k-1)}_{\psi_j} = F_k + G + G_j$$

for some $G_j \in \tilde{A}$ with $\omega(G_j) \leq \mu$. Fix j, $1 \leq j \leq 2$. Let m_j be the polydromy order of ψ_j and μ_j be a primitive m_j-th root of unity. Then identity (6-3) and assertion (3) of Lemma 6.4 imply that

$$f^{(k)}_{\psi_j} = \prod_{i=0}^{p_k-1} \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} (f^{(k-1)}_{\psi_j}) = \pi_{k-1}(G^*_j),$$

where

$$\begin{aligned} G^*_j &:= \prod_{i=0}^{p_k-1} \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} (F^{(k-1)}_{\psi_j}) = \prod_{i=0}^{p_k-1} \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} (F_k + G + G_j) \\ &= \prod_{i=0}^{p_k-1} \big(F_k + \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} G + \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} G_j\big) \\ &= \prod_{i=0}^{p_k-1} \big(F_k + \mu^{ip_1\cdots p_{k-1}} \star_{m} G + \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} G_j\big), \end{aligned}$$

m is the *polydromy order* of G (Definition 6.2), and μ is a primitive m-th root of unity (the last equality is an implication of assertion (1) of Lemma 6.4). Let

$$G_{j,0} := \prod_{i=0}^{p_k-1} \left(y_k + \mu^{ip_1\cdots p_{k-1}} \star_m G + \mu_j^{ip_1\cdots p_{k-1}} \star_{m_j} G_j\right) \in \tilde{A}_k. \tag{A-1}$$

Note that $\pi_k(G_{j,0}) = f_{\psi_j}^{(k)} = \pi_k(F_{\psi_j}^{(k)})$. Now we construct $F_{\psi_j}^{(k)}$ from $G_{j,0}$ via constructing a sequence of elements $G_{j,0}, G_{j,1}, \ldots$ as follows:

- For $\beta := (\beta_1, \ldots, \beta_k) \in \mathbb{Z}_{\geq 0}^k$, define

$$|\beta|_{k-1} := \sum_{j=1}^{k-1} p_0 \cdots p_{j-1}\beta_j.$$

  Consider the well order $\prec_{k-1}^*$ on $\mathbb{Z}_{\geq 0}^k$ defined as follows: $\beta \prec_{k-1}^* \beta'$ if and only if

  (1) $|\beta|_{k-1} < |\beta'|_{k-1}$, or

  (2) $|\beta|_{k-1} = |\beta'|_{k-1}$ and the left-most nonzero entry of $\beta - \beta'$ is negative.

- Assume $G_{j,N}$ has been constructed, $N \geq 0$. Express $G_{j,N}$ as

$$G_{j,N} = \sum_{\beta \in \mathbb{Z}_{\geq 0}^k} g_{j,N,\beta}(x) y_1^{\beta_1} \cdots y_k^{\beta_k}$$

  and define

$$\mathcal{E}_{j,N} := \left\{\beta \in \mathbb{Z}_{\geq 0}^k : g_{j,N,\beta} \neq 0 \text{ and } \beta_i \geq p_i \text{ for some } i,\ 1 \leq i \leq k-1\right\}.$$

- If $\mathcal{E}_{j,N} = \varnothing$, then **stop.**
- Otherwise pick the maximal element $\beta^* = (\beta_1^*, \ldots, \beta_k^*)$ of $\mathcal{E}_{j,N}$ with respect to $\prec_{k-1}^*$, and the maximal i^*, $1 \leq i^* \leq k-1$, such that $\beta_{i^*}^* \geq p_{i^*}$, and set

$$\begin{aligned} G_{j,N+1} = \sum_{\beta \neq \beta^*} g_{j,N,\beta}(x) y_1^{\beta_1} \cdots y_k^{\beta_k} + g_{j,N,\beta^*}(x) \\ \times \prod_{i \neq i^*} (y_i)^{\beta_i^*} (y_i)^{\beta_{i^*}^* - p_{i^*}} (y_{i+1} - (F_{i+1} - y_{i^*}^{p_{i^*}})). \end{aligned} \tag{A-2}$$

Assertion (c) of Lemma 6.9 and assertion (4) of Lemma 6.6 imply that all the "new" exponents of $(y_1, \ldots, y_k)$ that appear in $G_{j,N+1}$ are smaller (with respect to $\prec_{k-1}^*$) than β^*, and it follows that the sequence of $G_{j,N}$'s stops at some finite value N^* of N.

Claim A.2.1. $G_{j,N^*} = F_{\psi_j}^{(k)}.$

Proof. Indeed, (A-2) implies that $\pi_k(G_{j,N^*}) = \pi_k(G_{j,0}) = f_{\psi_j}^{(k)}$. Since we must have $\mathcal{E}_{j,N^*} = \varnothing$ for G_{j,N^*} to be the last element of the sequence of $G_{j,N}$'s, G_{j,N^*} satisfies the characterizing properties of $F_{\psi_j}^{(k)} = F_{f_{\psi_j}^{(k)}}^{\pi_k}$ from Lemma 6.6. □

Now note that, for each i, $1 \le i \le k-1$, every monomial term in $y_{i+1}-(F_{i+1}-y_i^{p_i})$ has ω-value smaller than or equal to $\omega(y_i^{p_i})$ (assertion (5), Lemma 6.7). It then follows from (A-1) and (A-2) that G_j has no effect on $[G_{j,N}]_{>(p_k-1)\omega_k+\mu}$ for any N, i.e., $[G_{1,N}]_{>(p_k-1)\omega_k+\mu} = [G_{2,N}]_{>(p_k-1)\omega_k+\mu}$ for all N. Claim A.2.1 then implies the lemma. $\square$

Corollary A.3. *Let*

$$\omega_{i,j} := \omega_i + q_j p_{j+1} \cdots p_{l+1} - q_i p_{i+1} \cdots p_{l+1} \quad \textit{for } 1 \le i \le j \le l+1.$$

Fix j, $0 \le j \le l$. Let $\psi \in \mathbb{C}\langle\langle x \rangle\rangle$ be such that $\psi \equiv_{r_{j+1}} \phi_{j+1}$ (where $r_1, \dots, r_{l+1}$ and $\phi_1, \dots, \phi_{l+1}$ are as in (6-6) *and* (6-7)*, respectively). Then for all i such that $0 \le i \le j$,*

$$[F_\psi^{(i)}]_{>\omega_{i+1,j+1}} = [F_{\phi_{j+1}}^{(i)}]_{>\omega_{i+1,j+1}}.$$

Proof. At first we consider the $i=0$ case. Equation (6-1) implies that $f_\psi^{(0)} = y - \psi(x)$ and $f_{\phi_{k+1}}^{(0)} = y - \phi_{k+1}(x)$. Then (6-10) implies that

$$F_\psi^{(0)} = y_1 + \phi_1(x) - \psi(x), \quad F_{\phi_{k+1}}^{(0)} = y_1 + \phi_1(x) - \phi_{j+1}(x).$$

It follows that

$$\begin{aligned}\omega(F_\psi^{(0)} - F_{\phi_{j+1}}^{(0)}) &= \omega_0 \deg_x(\phi_{j+1}(x) - \psi(x)) \\ &\le p_1 \cdots p_{l+1} r_{j+1} = q_{j+1} p_{j+2} \cdots p_{l+1} = \omega_{1,j+1}.\end{aligned}$$

It follows that the corollary is true for $i = 0$ and all j, $0 \le j \le l$.

Now we start the proof of the general case. We proceed by induction on j. It follows from the preceding discussion that the corollary is true for $j = 0$. So assume it holds for $0 \le j \le j' \le l-1$. To show that it holds for $j = j'+1$, we proceed by induction on i. By the same reasoning, we may assume that it also holds for $j = j'+1$ and $0 \le i \le i' \le j'$. Pick ψ such that $\psi \equiv_{r_{j'+2}} \phi_{j'+2}$. Then applying the induction hypothesis with $j = j'+1$ and $i = i'$, we have

$$[F_\psi^{(i')}]_{>\omega_{i'+1,j'+2}} = [F_{\phi_{j'+2}}^{(i')}]_{>\omega_{i'+1,j'+2}}. \tag{A-3}$$

On the other hand, since $\psi \equiv_{r_{i'+1}} \phi_{i'+1}$, we can apply the induction hypothesis with $j = i'$ and $i = i'$ to obtain

$$[F_\psi^{(i')}]_{>\omega_{i'+1,i'+1}} = [F_{\phi_{i'+1}}^{(i')}]_{>\omega_{i'+1,i'+1}}.$$

Similarly, since $\phi_{j'+2} \equiv_{r_{i'+1}} \phi_{i'+1}$, we have

$$[F_{\phi_{j'+2}}^{(i')}]_{>\omega_{i'+1,i'+1}} = [F_{\phi_{i'+1}}^{(i')}]_{>\omega_{i'+1,i'+1}}.$$

Since $\omega_{i'+1,i'+1} = \omega_{i'+1}$, it follows that

$$[F_\psi^{(i')}]_{>\omega_{i'+1}} = [F_{\phi_{j'+2}}^{(i')}]_{>\omega_{i'+1}} = [F_{\phi_{i'+1}}^{(i')}]_{>\omega_{i'+1}}. \tag{A-4}$$

Identities (A-3), (A-4) and assertion (c) of Lemma 6.9 imply that ψ and $\phi_{j'+2}$ satisfy the hypotheses of Lemma A.2 with $\mu = \omega_{i'+1,j'+2}$ and $k = i'+1$. Lemma A.2 therefore implies that

$$[F_{\psi}^{(i'+1)}]_{>\mu'} = [F_{\phi_{j'+2}}^{(i'+1)}]_{>\mu'},$$

where $\mu' = (p_{i'+1}-1)\omega_{i'+1} + \omega_{i'+1,j'+2}$. It is straightforward to check using (3-4) that $\mu' = \omega_{i'+2,j'+2}$, as required to complete the induction. $\square$

Proof of Lemma 6.10. Since $\omega_{k+1} = \omega_{k+1,k+1}$ and $F_{\phi_{k+1}}^{(k)} = F_{\phi_{k+1}}$, Lemma 6.10 is simply a special case of Corollary A.3. $\square$

Appendix B: Proof of Lemma 6.13

We freely use the notation of Section 6C.

Proof of assertion (1) of Lemma 6.13. Since $f_0 = x$ and each f_j, $1 \le j \le l+1$, is monic in y with $\deg_y(f_j) = p_0 \cdots p_{j-1}$ (where $p_0 := 1$), it is straightforward to see that each polynomial $f \in \mathbb{C}[x, y]$ can be represented as a finite sum of the form

$$f = \sum_{\beta \in \mathbb{Z}_{\ge 0}^{l+2}} a_\beta f_0^{\beta_0} \cdots f_{l+1}^{\beta_{l+1}},$$

where for each $\beta = (\beta_0, \dots, \beta_{l+1})$, we have $a_\beta \in \mathbb{C}$ and $\beta_j < p_j$, $1 \le j \le l$. It suffices to show that

$$\delta(f) = \max\{\delta(f_0^{\beta_0} \cdots f_{l+1}^{\beta_{l+1}}) : c_\beta \ne 0\}.$$

We compute $\delta(f)$ via identity (3-1). Assertion (4) of Theorem 3.17 implies that

$$f_j|_{y=\tilde{\phi}_\delta(x,\xi)} = \begin{cases} c_j^* x^{\omega_j/\omega_0} + \text{l.d.t.} & \text{for } 0 \le j \le l, \\ (c_{l+1}^*\xi + c_{l+1}) x^{\omega_{l+1}/\omega_0} + \text{l.d.t.} & \text{for } j = l+1, \end{cases} \tag{B-1}$$

where $c_j^* \in \mathbb{C}^*$, $0 \le j \le l$, $c_{l+1} \in \mathbb{C}$, and l.d.t. denotes terms with lower degree in x. Let $d := \max\{\delta(f_0^{\beta_0} \cdots f_{l+1}^{\beta_{l+1}}) : a_\beta \ne 0\}$ and $\mathcal{B} := \{\beta : a_\beta \ne 0,\ \delta(f_0^{\beta_0} \cdots f_{l+1}^{\beta_{l+1}}) = d\}$. It follows that

$$f|_{y=\tilde{\phi}_\delta(x,\xi)} = c(\xi) x^{d/\omega_0} + \text{l.d.t.}, \tag{B-2}$$

where

$$c(\xi) := \sum_{\beta \in \mathcal{B}} a_\beta (c_{l+1}^*\xi + c_{l+1})^{\beta_{l+1}} \prod_{j=0}^{l} (c_j^*)^{\beta_j}. \tag{B-3}$$

Now, assertion (1) of Lemma 6.8 implies that for two distinct elements β, β' of $\mathcal{B}$, $\beta_{l+1} \ne \beta'_{l+1}$. Identity (B-3) then implies that $c(\xi) \ne 0$, so that (B-2) implies that $\delta(f) = d$, as required to complete the proof of assertion (1). $\square$

For each j, $0 \le j \le l+1$, let $\Omega_j \subseteq \mathbb{Z}$ be the semigroup generated by $\omega_0, \dots, \omega_j$; recall that for $j \ge 1$, condition (Polynomial$_j$) implies that $\Omega_{j-1} \subseteq \mathbb{Z}_{\ge 0}$ (Lemma 6.12).

Lemma B.1. *Assume* (Polynomial$_{l+1}$) *holds. Fix j, $1 \le j \le l$. Let $\bar{J}_{j+1}$ be the ideal in C_j generated by $H_2, \dots, H_{j+1}$. Let t be an indeterminate. Then*

$$C_j/\bar{J}_{j+1} \cong \mathbb{C}[\Omega_j] \cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_j}],$$

via the mapping $x \mapsto t^{\omega_0}$ and $y_i \mapsto b_i t^{\omega_i}$, $1 \le i \le j$, for some $b_1, \dots, b_j \in \mathbb{C}^*$.

Proof. We proceed by induction on j. For $j = 1$, identity (6-12) and assertions (4) and (5) of Lemma 6.7 imply that

$$C_1/\bar{J}_2 = \mathbb{C}[x, y_1]/\langle y_1^{p_1} - c_{1,0}x^{q_1}\rangle \cong \mathbb{C}[t^{p_1}, t^{q_1}],$$

where t is an indeterminate and the isomorphism maps $x \mapsto t^{p_1}$ and $y_1 \mapsto c_{1,0}^{1/p_1} t^{q_1}$, where $c_{1,0}^{1/p_1}$ is a p_1-th root of $c_{1,0} \in \mathbb{C}^*$. Since $\omega_0 = p_1 p_2 \cdots p_l$ and $\omega_1 = q_1 p_2 \cdots p_l$, this proves the lemma for $j = 1$. Now assume that the lemma is true for $j - 1$, $2 \le j \le l$, i.e., there exists an isomorphism

$$C_{j-1}/\bar{J}_j \cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_{j-1}}]$$

which maps $x \mapsto t^{\omega_0}$ and $y_i \mapsto b_i t^{\omega_i}$, $1 \le i \le j-1$ for some $b_1, \cdots, b_{j-1} \in \mathbb{C}^*$. It follows that

$$\begin{aligned} C_j/\bar{J}_{j+1} &= C_{j-1}[y_j]/\langle \bar{J}_j, y_j^{p_j} - c_{j,0}x^{\beta^0_{j,0}} y_1^{\beta^0_{j,1}} \cdots y_{j-1}^{\beta^0_{j,j-1}}\rangle \\ &\cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_{j-1}}, y_j]/\langle y_j^{p_j} - \tilde{c}t^{p_j\omega_j}\rangle \end{aligned}$$

for some $\tilde{c} \in \mathbb{C}^*$ (the last isomorphism uses assertion (5) of Lemma 6.7). Since $p_j = \min\{\alpha \in \mathbb{Z}_{>0};\, \alpha\omega_j \in \mathbb{Z}\omega_0 + \cdots + \mathbb{Z}\omega_{j-1}\}$ (assertion (3) of Proposition 3.21), it follows that

$$\mathbb{C}[t^{\omega_0}, \dots, t^{\omega_{j-1}}, y_j]/\langle y_j^{p_j} - \tilde{c}t^{p_j\omega_j}\rangle \cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_j}]$$

via a map which sends $y_j \mapsto (\tilde{c})^{1/p_j} t^{\omega_j}$ (where $(\tilde{c})^{1/p_j}$ is a p_j-th root of $\tilde{c}$), which completes the induction. □

Let z be an indeterminate and $\hat{C}_{l+1} := C_{l+1}[z] = \mathbb{C}[z, x, y_1, \dots, y_{l+1}]$. Let $\hat{\omega}$ be the weighted degree on $\hat{C}_{l+1}$ such that $\hat{\omega}(z) = 1$ and $\hat{\omega}|_{C_{l+1}} = \omega$. Equip $\hat{C}^{l+1}$ with the grading determined by $\hat{\omega}$. Let S^δ be as in assertion (1) of Lemma 6.13 and $\hat{\pi} : \hat{C}_{l+1} \to S^\delta$ be the map which sends $z \mapsto (1)_1$, $x \mapsto (x)_{\omega_0}$, and $y_j \mapsto (f_j)_{\omega_j}$, $1 \le j \le l+1$. Assertion (1) implies that $\hat{\pi}$ is a surjective homomorphism of graded rings. Let I be the ideal generated by $(1)_1$ in S^δ and $\hat{J}_{l+1} := \hat{\pi}^{-1}(I) \subseteq \hat{C}_{l+1}$.

Claim B.2. *$\hat{J}_{l+1}$ is generated by $\hat{\mathcal{B}}_{l+1} := (H_{l+1}, \dots, H_2, z)$.*

Proof. Let $\bar{J}_{l+1}$ be the ideal of C_l as defined in Lemma B.1, and $\hat{J}'_{l+1}$ be the ideal of $\hat{C}_{l+1}$ generated by $\bar{J}_{l+1}$ and z. It is straightforward to see that $\hat{J}'_{l+1} \subseteq \hat{J}_{l+1}$. Lemma B.1 implies that

$$\hat{C}_{l+1}/\hat{J}'_{l+1} \cong \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_l}, y_{l+1}].$$

Let $R := \mathbb{C}[t^{\omega_0}, \dots, t^{\omega_l}, y_{l+1}]$. Then $S^\delta/I \cong \hat{C}_{l+1}/\hat{J}_{l+1} \cong R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R. Now, it follows from the construction of S^δ that $\dim(S^\delta) = 3$. Since I is the principal ideal generated by a nonzero divisor in S^δ, it follows that $\dim(R/\mathfrak{p}) = \dim(S^\delta/I) = 2$. Since R is an integral domain of dimension 2, we must have $\mathfrak{p} = 0$, which implies the claim. $\square$

Proof of assertion (2) of Lemma 6.13. Since $J_{l+1} = \hat{J}_{l+1} \cap C_{l+1}$, Claim B.2 shows that $\mathcal{B}_{l+1}$ generates J_{l+1}. Therefore, to show that $\mathcal{B}_k$ is a Gröbner basis of J_k with respect to $\prec_k$, it suffices to show that running a step of *Buchberger's algorithm* with $\mathcal{B}_{l+1}$ as input leaves $\mathcal{B}_{l+1}$ unchanged. We follow Buchberger's algorithm as described in [Cox et al. 1997, Section 2.7], which consists of performing the following steps for each pair of $H_i, H_j \in \mathcal{B}_{l+1}$, $2 \le i < j \le l+1$:

Step 1: Compute the *S-polynomial* $S(H_i, H_j)$ of H_i and H_j. The leading terms of H_i and H_j with respect to $\prec$ are respectively $\mathrm{LT}_\prec(H_i) = y_{i-1}^{p_{i-1}}$ and $\mathrm{LT}_\prec(H_j) = y_{j-1}^{p_{j-1}}$, so that the *S-polynomial* of H_i and H_j is

$$\begin{aligned} S(H_i, H_j) &:= y_{j-1}^{p_{j-1}} H_i - y_{i-1}^{p_{i-1}} H_j \\ &= -(c_{i-1,0} x^{\beta^0_{i-1,0}} y_1^{\beta^0_{i-1,1}} \cdots y_{i-2}^{\beta^0_{i-1,i-2}}) y_{j-1}^{p_{j-1}} \\ &\qquad + (c_{j-1,0} x^{\beta^0_{j-1,0}} y_1^{\beta^0_{j-1,1}} \cdots y_{j-2}^{\beta^0_{j-1,j-2}}) y_{i-1}^{p_{i-1}}. \end{aligned}$$

Step 2: Divide $S(H_i, H_j)$ by $\mathcal{B}_k$ and if the remainder is nonzero, then adjoin it to $\mathcal{B}_{l+1}$. Since $i < j$, the leading term of $S(H_i, H_j)$ is

$$\mathrm{LT}_\prec(S(H_i, H_j)) = -(c_{i-1,0} x^{\beta^0_{i-1,0}} y_1^{\beta^0_{i-1,1}} \cdots y_{i-2}^{\beta^0_{i-1,i-2}}) y_{j-1}^{p_{j-1}}.$$

Since $\beta^0_{i-1,j'} < p_{j'}$ for all j', $1 \le j' \le i-1$ (assertion (3) of Lemma 6.7), it follows that H_j is the only element of $\mathcal{B}_{l+1}$ such that $\mathrm{LT}_\prec(H_j)$ divides $\mathrm{LT}_\prec(S(H_i, H_j))$. The remainder of the division of $S(H_i, H_j)$ by H_j is

$$\begin{aligned} S_1 &:= S(H_i, H_j) + (c_{i-1,0} x^{\beta^0_{i-1,0}} y_1^{\beta^0_{i-1,1}} \cdots y_{i-2}^{\beta^0_{i-1,i-2}}) H_j \\ &= (c_{j-1,0} x^{\beta^0_{j-1,0}} y_1^{\beta^0_{j-1,1}} \cdots y_{j-2}^{\beta^0_{j-1,j-2}}) H_i, \end{aligned}$$

so that the leading term of S_1 is

$$\mathrm{LT}_\prec(S_1) = (c_{j-1,0} x^{\beta^0_{j-1,0}} y_1^{\beta^0_{j-1,1}} \cdots y_{j-2}^{\beta^0_{j-1,j-2}}) y_{i-1}^{p_{i-1}}.$$

It follows as in the case of $S(H_i, H_j)$ that H_i is the only element of $\mathcal{B}_{l+1}$ whose leading term divides $\mathrm{LT}_\prec(S_1)$. Since H_i divides S_1, the remainder of the division of S_1 by H_i is zero, and it follows that the remainder of the division of $S(H_i, H_k)$ by $\mathcal{B}_k$ is zero. Consequently *Step 2 concludes without changing* $\mathcal{B}_{l+1}$.

It follows from the preceding paragraphs that running one step of Buchberger's algorithm keeps $\mathcal{B}_{l+1}$ unchanged, and consequently $\mathcal{B}_{l+1}$ is a Gröbner basis of J_{l+1} with respect to $\prec$ [Cox et al. 1997, Theorem 2.7.2]. This completes the proof of assertion (2) of Lemma 6.13. □

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