Algebra & Number Theory

Volume 10

2016

No. 8

Syntomic cohomology and p-adic regulators for varieties over p-adic fields

Jan Nekovář and Wiesława Nizioł

With appendices by Laurent Berger and Frédéric Déglise

msp

ALGEBRA AND NUMBER THEORY 10:8 (2016)
dx.doi.org/10.2140/ant.2016.10.1695

Syntomic cohomology and p-adic regulators for varieties over p-adic fields

Jan Nekovář and Wiesława Nizioł

With appendices by Laurent Berger and Frédéric Déglise

We show that the logarithmic version of the syntomic cohomology of Fontaine and Messing for semistable varieties over p-adic rings extends uniquely to a cohomology theory for varieties over p-adic fields that satisfies h-descent. This new cohomology — syntomic cohomology — is a Bloch–Ogus cohomology theory, admits a period map to étale cohomology, and has a syntomic descent spectral sequence (from an algebraic closure of the given field to the field itself) that is compatible with the Hochschild–Serre spectral sequence on the étale side and is related to the Bloch–Kato exponential map. In relative dimension zero we recover the potentially semistable Selmer groups and, as an application, we prove that Soulé's étale regulators land in the potentially semistable Selmer groups.

Our construction of syntomic cohomology is based on new ideas and techniques developed by Beilinson and Bhatt in their recent work on p-adic comparison theorems.

Nekovář's research was supported in part by the grant ANR-BLAN-0114 and Nizioł's by the NSF grant DMS0703696.

MSC2010: primary 14F30; secondary 11G25.

Keywords: syntomic cohomology, regulators.

1. Introduction

In this article we define syntomic cohomology for varieties over p-adic fields, relate it to the Bloch–Kato exponential map, and use it to study the images of Soulé's étale regulators. Contrary to all the previous constructions of syntomic cohomology (see below for a brief review), we do not restrict ourselves to varieties coming with a nice model over the integers. Hence our syntomic regulators make no integrality assumptions on the K-theory classes in the domain.

1A. ***Statement of the main result.*** Recall that, for varieties proper and smooth over a p-adic ring of mixed characteristic, syntomic cohomology (or its nonproper variant: syntomic-étale cohomology) was introduced by Fontaine and Messing [1987] in their proof of the crystalline comparison theorem as a natural bridge between crystalline cohomology and étale cohomology. It was generalized to log-syntomic cohomology for semistable varieties by Kato [1994]. For a log-smooth scheme $\mathscr{X}$ over a complete discrete valuation ring V of mixed characteristic $(0, p)$ and a perfect residue field, and for any $r \geq 0$, rational log-syntomic cohomology of $\mathscr{X}$ can be defined as the "filtered Frobenius eigenspace" in log-crystalline cohomology, i.e., as the mapping fiber

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, r) := \mathrm{Cone}\big(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}, \mathscr{J}^{[r]}) \xrightarrow{1-\varphi_r} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\big)[-1], \tag{1}$$

where $\mathrm{R}\Gamma_{\mathrm{cr}}(\cdot, \mathscr{J}^{[r]})$ denotes the absolute rational log-crystalline cohomology (i.e., over $\mathbb{Z}_p$) of the r-th Hodge filtration sheaf $\mathscr{J}^{[r]}$ and φ_r is the crystalline Frobenius divided by p^r. This definition suggested that the log-syntomic cohomology could be the sought-for p-adic analog of Deligne–Beilinson cohomology. Recall that, for a complex manifold X, the latter can be defined as the cohomology $\mathrm{R}\Gamma(X, \mathbb{Z}(r)_{\mathscr{D}})$ of Deligne complex $\mathbb{Z}(r)_{\mathscr{D}}$:

$$0 \to \mathbb{Z}(r) \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{r-1}_X \to 0.$$

And, indeed, since its introduction, log-syntomic cohomology has been used with some success in the study of special values of p-adic L-functions and in formulating p-adic Beilinson conjectures (see [Besser et al. 2009] for a review).

The syntomic cohomology theory with $\mathbb{Q}_p$-coefficients $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ $(r \geq 0)$ for arbitrary varieties — more generally, for arbitrary essentially finite diagrams of varieties — over the p-adic field K (the fraction field of V) that we construct in this article is a generalization of Fontaine–Messing(–Kato) log-syntomic cohomology. That is, for a semistable scheme[1] $\mathscr{X}$ over V, we have $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, r) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$, where X is the largest subvariety of $\mathscr{X}_K$ with trivial log-structure. An analogous theory $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h}, r)$ $(r \geq 0)$ exists for (diagrams of) varieties over $\bar{K}$, where $\bar{K}$ is an algebraic closure of K.

[1] Throughout the Introduction, the divisors at infinity of semistable schemes have no multiplicities.

Our main result can be stated as follows.

Theorem A. *For any variety X over K, there is a canonical graded commutative dg $\mathbb{Q}_p$-algebra $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,*)$ such that*:

(1) *It is the unique extension of log-syntomic cohomology to varieties over K that satisfies h-descent; i.e., for any hypercovering $\pi : Y_\bullet \to X$ in the h-topology, we have a quasi-isomorphism*

$$\pi^* : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h,*) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{syn}}(Y_{\bullet,h},*).$$

(2) *It is a Bloch–Ogus cohomology theory* [1974].

(3) *For $X = \mathrm{Spec}(K)$, we have*

$$H^*_{\mathrm{syn}}(X_h,r) \simeq H^*_{\mathrm{st}}(G_K,\mathbb{Q}_p(r)),$$

where $H^i_{\mathrm{st}}(G_K,-)$ denotes the Ext-*group* $\mathrm{Ext}^i(\mathbb{Q}_p,-)$ *in the category of (potentially) semistable representations of $G_K = \mathrm{Gal}(\bar{K}/K)$.*

(4) *There are functorial syntomic period morphisms*

$$\rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r) \to \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p(r)),$$
$$\rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h},r) \to \mathrm{R}\Gamma(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r))$$

compatible with products which induce quasi-isomorphisms

$$\tau_{\le r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r) \xrightarrow{\sim} \tau_{\le r}\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p(r)),$$
$$\tau_{\le r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h},r) \xrightarrow{\sim} \tau_{\le r}\mathrm{R}\Gamma(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r)).$$

(5) *The Hochschild–Serre spectral sequence for étale cohomology*

$${}^{\text{ét}}E_2^{i,j} = H^i(G_K, H^j(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r))) \Rightarrow H^{i+j}(X_{\text{ét}},\mathbb{Q}_p(r))$$

has a syntomic analog

$${}^{\mathrm{syn}}E_2^{i,j} = H^i_{\mathrm{st}}(G_K, H^j(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r))) \Rightarrow H^{i+j}_{\mathrm{syn}}(X_h,r).$$

(6) *There is a canonical morphism of spectral sequences ${}^{\mathrm{syn}}E_t \to {}^{\text{ét}}E_t$ compatible with the syntomic period map.*

(7) *There are syntomic Chern classes*

$$c^{\mathrm{syn}}_{i,j} : K_j(X) \to H^{2i-j}_{\mathrm{syn}}(X_h,i)$$

compatible with étale Chern classes via the syntomic period map.

As is shown in [Déglise and Nizioł 2015], syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,*)$ can be interpreted as an absolute p-adic Hodge cohomology. That is, it is a derived Hom in the category of admissible (φ, N, G_K)-modules between the trivial module and a complex of such modules canonically associated to a variety. Alternatively, it is a derived Hom in the category of potentially semistable representations between the trivial representation and a complex of such representations canonically associated to a variety. A particularly simple construction of such a complex, using Beilinson's basic lemma, was proposed by Beilinson (and is presented in [Déglise and Nizioł 2015]). The category of modules over the syntomic cohomology algebra $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, *)$ (taken in a motivic sense) yields a category of p-adic Galois representations that better approximates the category of geometric representations than the category of potentially semistable representations [Déglise and Nizioł 2015]. For further applications of the syntomic cohomology algebra, we refer the interested reader to [loc. cit.].

Similarly, as is shown in [Nizioł 2016a], geometric syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, *)$ is a derived Hom in the category of effective φ-gauges (with one paw) [Fargues 2015] between the trivial gauge and a complex of such gauges canonically associated to a variety. In particular, geometric syntomic cohomology group is a finite-dimensional Banach–Colmez space [Colmez 2002], and hence has a very rigid structure.

The syntomic descent spectral sequence and its compatibility with the Hochschild–Serre spectral sequence in étale cohomology imply the following proposition.

Proposition 1.1. *Let $i \geq 0$. The composition*

$$H^{i-1}_{\mathrm{dR}}(X)/F^r \xrightarrow{\partial} H^i_{\mathrm{syn}}(X_h, r) \xrightarrow{\rho_{\mathrm{syn}}} H^i_{\text{ét}}(X, \mathbb{Q}_p(r)) \longrightarrow H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))$$

is the zero map. The induced (*from the syntomic descent spectral sequence*) *map*

$$H^{i-1}_{\mathrm{dR}}(X)/F^r \to H^1(G_K, H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)))$$

is equal to the Bloch–Kato exponential associated with the Galois representation $H^{i-1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))$.

This yields a comparison between p-adic étale regulators, syntomic regulators, and the Bloch–Kato exponential (which was proved in the good reduction case in [Nekovář 1998] and [Nizioł 2001, Theorem 5.2][2]) that is of fundamental importance for the theory of special values of L-functions, both complex valued and p-adic. The point is that syntomic regulators can be thought of as an abstract p-adic integration theory. The comparison results stated above then relate certain p-adic integrals to the values of the p-adic étale regulator via the Bloch–Kato exponential map. A modification of syntomic cohomology developed in [Besser 2000] in the good reduction case (resp. in [Besser et al. 2016] — using the techniques of the present article — in the case of arbitrary reduction) can be used to perform explicit computations. For example, the formulas from [Besser et al. 2016, §3] were applied to a calculation of certain p-adic regulators in [Bertolini et al. 2015; Darmon and Rotger 2016].

[2] The Bloch–Kato exponential is called l there.

1B. ***Construction of syntomic cohomology.*** We will now sketch the proof of Theorem A. Recall first that a little bit after log-syntomic cohomology had appeared on the scene, Selmer groups of Galois representations — describing extensions in certain categories of Galois representations — were introduced by Bloch and Kato [1990] and linked to special values of L-functions. And a syntomic cohomology (in the good reduction case), a priori different than that of Fontaine and Messing, was defined in [Nizioł 2001] and by Besser [2000] as a higher-dimensional analog of the complexes computing these groups. The guiding idea here was that just as Selmer groups classify extensions in certain categories of "geometric" Galois representations, their higher-dimensional analogs — syntomic cohomology groups — should classify extensions in a category of "p-adic motivic sheaves". This was shown to be the case for H^1 by Bannai [2002], who has also shown that Besser's (rigid) syntomic cohomology is a p-adic analog of Beilinson's absolute Hodge cohomology [1986].

Complexes computing the semistable and potentially semistable Selmer groups were introduced in [Nekovář 1993; Fontaine and Perrin-Riou 1994]. For a semistable scheme $\mathscr{X}$ over V, their higher-dimensional analog can be written as the homotopy limit[3]

$$\mathrm{R}\Gamma'_{\mathrm{syn}}(\mathscr{X},r) := \left[\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) & \xrightarrow{(1-\varphi_r,\iota_{\mathrm{dR}})} & \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)\oplus \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)/F^r \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \end{array}\right], \qquad (2)$$

where $\mathscr{X}_0$ is the special fiber of $\mathscr{X}$, $\mathrm{R}\Gamma_{\mathrm{HK}}(\cdot)$ is the Hyodo–Kato cohomology, N denotes the Hyodo–Kato monodromy, and $\mathrm{R}\Gamma_{\mathrm{dR}}(\cdot)$ is the logarithmic de Rham cohomology. The map ι_{dR} is the Hyodo–Kato morphism that induces a quasi-isomorphism $\iota_{\mathrm{dR}}:\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)\otimes_{K_0}K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)$ for K_0 — the fraction field of Witt vectors of the residue field of V.

Using Dwork's trick, we prove (see Proposition 3.8) that the two definitions of log-syntomic cohomology are the same, i.e., that there is a quasi-isomorphism

$$\alpha_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},r) \xrightarrow{\sim} \mathrm{R}\Gamma'_{\mathrm{syn}}(\mathscr{X},r).$$

It follows that log-syntomic cohomology groups vanish in degrees strictly higher than $2\dim X_K+2$ and that, if $\mathscr{X}=\mathrm{Spec}(V)$, then $H^i\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},r)\simeq H^i_{\mathrm{st}}(G_K,\mathbb{Q}_p(r))$.

The syntomic cohomology for varieties over p-adic fields that we introduce in this article is a generalization of the log-syntomic cohomology of Fontaine and Messing. Observe that it is clear how one can try to use log-syntomic cohomology

[3] See Section 1E for an explanation of the notation we use for certain homotopy limits.

to define syntomic cohomology for varieties over fields that satisfies h-descent. Namely, for a variety X over K, consider the h-topology of X and recall that (using alterations) one can show that it has a basis consisting of semistable models over finite extensions of V [Beilinson 2012]. By h-sheafifying the complexes $Y \mapsto \mathrm{R}\Gamma_{\mathrm{syn}}(Y, r)$ (for a semistable model Y) we get syntomic complexes $\mathscr{S}(r)$. We define the (*arithmetic*) syntomic cohomology as

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) := \mathrm{R}\Gamma(X_h, \mathscr{S}(r)).$$

A priori it is not clear that the so-defined syntomic cohomology behaves well: the finite ramified field extensions introduced by alterations are in general a problem for log-crystalline cohomology. For example, the related complexes $\mathrm{R}\Gamma_{\mathrm{cr}}(X_h, \mathscr{J}^{[r]})$ are huge. However, taking Frobenius eigenspaces cuts off the "noise" and the resulting syntomic complexes do indeed behave well. To get an idea why this is the case, h-sheafify the complexes $Y \mapsto \mathrm{R}\Gamma'_{\mathrm{syn}}(Y, r)$ and imagine that you can sheafify the maps α_{syn} as well. We get sheaves $\mathscr{S}'(r)$ and quasi-isomorphisms $\alpha_{\mathrm{syn}} : \mathscr{S}(r) \xrightarrow{\sim} \mathscr{S}'(r)$. Setting $\mathrm{R}\Gamma'_{\mathrm{syn}}(X_h, r) := \mathrm{R}\Gamma(X_h, \mathscr{S}'(r))$, we obtain the quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \simeq \mathrm{R}\Gamma'_{\mathrm{syn}}(X_h, r) \simeq \begin{bmatrix} \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) & \xrightarrow{(1-\varphi_r, \iota_{\mathrm{dR}})} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) \oplus \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r \\ \big\downarrow N & & \big\downarrow (N,0) \\ \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_h) \end{bmatrix}, \quad (3)$$

where $\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)$ denotes the Hyodo–Kato cohomology (defined as h-cohomology of the presheaf: $Y \mapsto \mathrm{R}\Gamma_{\mathrm{HK}}(Y_0)$) and $\mathrm{R}\Gamma_{\mathrm{dR}}(\cdot)$ is Deligne's de Rham cohomology [1974]. The Hyodo–Kato map ι_{dR} is the h-sheafification of the logarithmic Hyodo–Kato map. It is well-known that Deligne's de Rham cohomology groups are finite-rank K-vector spaces; it turns out that the Hyodo–Kato cohomology groups are finite-rank K_0-vector spaces: we have a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{HK}}(X_h) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\bar{K},h})^{G_K}$, and the geometric Hyodo–Kato groups $H^*\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\bar{K},h})$ are finite-rank K_0^{nr}-vector spaces, where K_0^{nr} is the maximal unramified extension of K_0 (see (4) below).

It follows that syntomic cohomology groups vanish in degrees higher than $2\dim X_K + 2$ and that syntomic cohomology is, in fact, a generalization of the classical log-syntomic cohomology; i.e., for a semistable scheme $\mathscr{X}$ over V, we have $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, r) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$, where X is the largest subvariety of $\mathscr{X}_K$ with trivial log-structure. This follows from the quasi-isomorphism α_{syn}: logarithmic Hyodo–Kato and de Rham cohomologies (over a fixed base) satisfy proper descent and the finite field extensions that appear as the "noise" in alterations do not destroy anything since logarithmic Hyodo–Kato and de Rham cohomologies satisfy finite Galois descent.

Alas, we were not able to sheafify the map $\alpha_{\rm syn}$. The reason for that is that the construction of $\alpha_{\rm syn}$ uses a twist by a high power of Frobenius — a power depending on the field K. And alterations are going to introduce a finite extension of K — hence a need for higher and higher powers of Frobenius. So instead we construct directly the map

$$\alpha_{\rm syn}: \mathrm{R}\Gamma_{\rm syn}(X_h, r) \to \mathrm{R}\Gamma'_{\rm syn}(X_h, r).$$

To do that, we show first that the syntomic cohomological dimension of X is finite. Then we take a semistable h-hypercovering of X, truncate it at an appropriate level, extend the base field K to K', and base-change everything to K'. There we can work with one field and use the map $\alpha_{\rm syn}$ defined earlier. Finally, we show that we can descend.

1C. ***Syntomic period maps.*** We pass now to the construction of the period maps from syntomic to étale cohomology that appear in Theorem A. They are easier to define over $\overline{K}$, i.e., from the *geometric* syntomic cohomology. In this setting, things go smoother with h-sheafification since going all the way up to $\overline{K}$ before completing kills a lot of "noise" in log-crystalline cohomology. More precisely, for a semistable scheme $\mathscr{X}$ over V, we have the canonical quasi-isomorphisms [Beilinson 2013]

$$\iota_{\rm cr}: \mathrm{R}\Gamma_{\rm HK}(\mathscr{X}_{\overline{V}})^{\tau}_{B^+_{\rm cr}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm cr}(\mathscr{X}_{\overline{V}}), \quad \iota_{\rm dR}: \mathrm{R}\Gamma_{\rm HK}(\mathscr{X}_{\overline{V}})^{\tau}_{\overline{K}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm dR}(\mathscr{X}_{\overline{K}}), \tag{4}$$

where $\overline{V}$ is the integral closure of V in $\overline{K}$, $B^+_{\rm cr}$ is the crystalline period ring, and τ denotes certain twist. These quasi-isomorphisms h-sheafify well: for a variety X over K, they induce the quasi-isomorphisms [Beilinson 2013]

$$\iota_{\rm cr}: \mathrm{R}\Gamma_{\rm HK}(X_{\overline{K},h})^{\tau}_{B^+_{\rm cr}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm cr}(X_{\overline{K},h}), \quad \iota_{\rm dR}: \mathrm{R}\Gamma_{\rm HK}(X_{\overline{K},h})^{\tau}_{\overline{K}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\rm dR}(X_{\overline{K}}), \tag{5}$$

where the terms have obvious meaning. Since Deligne's de Rham cohomology has proper descent (by definition), it follows that h-crystalline cohomology behaves well. That is, if we define crystalline sheaves $\mathscr{J}^{[r]}_{\rm cr}$ and $\mathscr{A}_{\rm cr}$ on $X_{\overline{K},h}$ by h-sheafifying the complexes $Y \mapsto \mathrm{R}\Gamma_{\rm cr}(Y, \mathscr{J}^{[r]})$ and $Y \mapsto \mathrm{R}\Gamma_{\rm cr}(Y)$, respectively, for Y which are a base change to $\overline{V}$ of a semistable scheme over a finite extension of V (such schemes Y form a basis of $X_{\overline{K},h}$) then the complexes $\mathrm{R}\Gamma(X_{\overline{K},h}, \mathscr{J}^{[r]})$ and $\mathrm{R}\Gamma_{\rm cr}(X_{\overline{V},h}) := \mathrm{R}\Gamma(X_{\overline{K},h}, \mathscr{A}_{\rm cr})$ generalize log-crystalline cohomology (in the sense described above) and the latter one is a perfect complex of $B^+_{\rm cr}$-modules.

We obtain syntomic complexes $\mathscr{S}(r)$ on $X_{\overline{K},h}$ by h-sheafifying the complexes $Y \mapsto \mathrm{R}\Gamma_{\rm syn}(Y, r)$ and (geometric) syntomic cohomology by setting $\mathrm{R}\Gamma_{\rm syn}(X_{\overline{K},h}, r) :=$

$\mathrm{R}\Gamma(X_{\overline{K},h},\mathscr{S}(r))$. They fit into an analog of the exact sequence (1) and, by the above, generalize log-syntomic cohomology.

To construct the syntomic period maps

$$\begin{aligned}\rho_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h},r)&\to\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r)),\\ \rho_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X_{h},r)&\to\mathrm{R}\Gamma(X_{\mathrm{\acute{e}t}},\mathbb{Q}_p(r)),\end{aligned}\tag{6}$$

consider the syntomic complexes $\mathscr{S}_n(r)$: the mod-p^n version of the syntomic complexes $\mathscr{S}(r)$ on $X_{\overline{K},h}$. We have the distinguished triangle

$$\mathscr{S}_n(r)\to\mathscr{J}^{[r]}_{\mathrm{cr},n}\xrightarrow{p^r-\varphi}\mathscr{A}_{\mathrm{cr},n}.$$

Recall that the filtered Poincaré lemma of Beilinson [2013] and Bhatt [2012] yields a quasi-isomorphism $\rho_{\mathrm{cr}}:J^{[r]}_{\mathrm{cr},n}\xrightarrow{\sim}\mathscr{J}^{[r]}_{\mathrm{cr},n}$, where $J^{[r]}_{\mathrm{cr}}\subset A_{\mathrm{cr}}$ is the r-th filtration level of the period ring A_{cr}. Using the fundamental sequence of p-adic Hodge theory,

$$0\to\mathbb{Z}/p^n(r)'\to J^{\langle r\rangle}_{\mathrm{cr},n}\xrightarrow{1-\varphi_r}A_{\mathrm{cr},n}\to 0,$$

where $\mathbb{Z}/p^n(r)':=(1/(p^a a!)\mathbb{Z}_p(r))\otimes\mathbb{Z}/p^n$ and a denotes the largest integer $\le r/(p-1)$, we obtain the syntomic period map $\rho_{\mathrm{syn}}:\mathscr{S}_n(r)\to\mathbb{Z}/p^n(r)'$. It is a quasi-isomorphism modulo a universal constant. It induces the geometric syntomic period map in (6), and, by Galois descent, its arithmetic analog.

To study the descent spectral sequences from Theorem A, we need to consider the other version of syntomic cohomology, i.e., the complexes

$$\mathrm{R}\Gamma'_{\mathrm{syn}}(X_{\overline{K},h},r):=\left[\begin{array}{ccc}\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K},h})\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{st}} & \xrightarrow{(1-\varphi_r,\iota_{\mathrm{dR}})} & \begin{array}{c}\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K},h})\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{st}}\\ \oplus(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})\otimes_{\overline{K}}B^+_{\mathrm{dR}})/F^r\end{array}\\ \downarrow N & & \downarrow (N,0)\\ \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K},h})\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{st}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K},h})\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{st}}\end{array}\right],\tag{7}$$

where B^+_{st} and B^+_{dR} are the semistable and de Rham p-adic period rings, respectively. We deduce a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h},r)\xrightarrow{\sim}\mathrm{R}\Gamma'_{\mathrm{syn}}(X_{\overline{K},h},r)$.

Remark 1.2. This quasi-isomorphism yields, for a semistable scheme $\mathscr{X}$ over V, the exact sequence

$$\begin{aligned}\cdots\to H^i_{\mathrm{syn}}(\mathscr{X}_{\overline{K}},r)\to(H^i_{\mathrm{HK}}(\mathscr{X})_{\mathbb{Q}}\otimes_{K_0}B^+_{\mathrm{st}})^{\varphi=p^r,N=0}&\\ \to(H^i_{\mathrm{dR}}(\mathscr{X}_K)\otimes_K B^+_{\mathrm{dR}})/F^r\to H^{i+1}_{\mathrm{syn}}(\mathscr{X}_{\overline{K}},r)\to\cdots.&\end{aligned}$$

It is a sequence of finite-dimensional Banach–Colmez Spaces [Colmez 2002] and as such is a key in the proof of the semistable comparison theorem for formal schemes in [Colmez and Niziol 2015].

We also have a syntomic period map

$$\rho'_{\mathrm{syn}}:\mathrm{R}\Gamma'_{\mathrm{syn}}(X_{\overline{K},h},r)\to \mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r)) \tag{8}$$

that is compatible with the map ρ_{syn} via α_{syn}. To describe how it is constructed, recall that the crystalline period map of Beilinson [2013] induces compatible Hyodo–Kato and de Rham period maps

$$\begin{aligned}\rho_{\mathrm{HK}}:\mathrm{R}\Gamma_{\mathrm{HK}}(X_{\overline{K},h})\otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}}&\to \mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p)\otimes B^+_{\mathrm{st}},\\ \rho_{\mathrm{dR}}:\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)\otimes_K B^+_{\mathrm{dR}}&\to \mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p)\otimes B^+_{\mathrm{dR}}.\end{aligned} \tag{9}$$

Applying them to the above homotopy limit, removing all the pluses from the period rings, reduces the homotopy limit to the complex

$$\left[\begin{array}{ccc}\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))\otimes B_{\mathrm{st}} & \xrightarrow{(1-\varphi_r,\iota_{\mathrm{dR}})} & \mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))\otimes B_{\mathrm{st}} \oplus(\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))\otimes B_{\mathrm{dR}})/F^r\\ \downarrow N & & \downarrow (N,0)\\ \mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))\otimes B_{\mathrm{st}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))\otimes B_{\mathrm{st}}\end{array}\right]. \tag{10}$$

By the familiar fundamental exact sequence

$$0\to\mathbb{Q}_p(r)\to B_{\mathrm{st}}\xrightarrow{(N,1-\varphi_r,\iota)} B_{\mathrm{st}}\oplus B_{\mathrm{st}}\oplus B_{\mathrm{dR}}/F^r\xrightarrow{(1-\varphi_{r-1})-N} B_{\mathrm{st}}\to 0,$$

the above complex is quasi-isomorphic to $\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))$. This yields the syntomic period morphism from (8). We like to think of geometric syntomic cohomology as being represented by the complex from (7) and of geometric étale cohomology as represented by the complex (10).

From the above constructions we derive several of the properties mentioned in Theorem A. The quasi-isomorphisms (9) give that

$$\begin{aligned}H^i_{\mathrm{HK}}(X_{\overline{K},h})&\simeq D_{\mathrm{pst}}(H^i(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))),\\ H^i_{\mathrm{HK}}(X_h)&\simeq D_{\mathrm{st}}(H^i(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))),\end{aligned}$$

where D_{pst} and D_{st} are the functors from [Fontaine and Perrin-Riou 1994]. This combined with the diagram (3) immediately yields the spectral sequence ${}^{\mathrm{syn}}E_t$ since the cohomology groups of the total complex of

$$\left[\begin{array}{ccc}H^j_{\mathrm{HK}}(X_h) & \xrightarrow{(1-\varphi_r,\iota_{\mathrm{dR}})} & H^j_{\mathrm{HK}}(X_h)\oplus H^j_{\mathrm{dR}}(X_K)/F^r\\ \downarrow N & & \downarrow (N,0)\\ H^j_{\mathrm{HK}}(X_h) & \xrightarrow{1-\varphi_{r-1}} & H^j_{\mathrm{HK}}(X_h)\end{array}\right]$$

are equal to $H^*_{\mathrm{st}}(G_K, H^j(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r)))$. Moreover, the sequence of natural maps of diagrams (3) $\to$ (7) $\xrightarrow{\rho_{\mathrm{syn}}}$ (10) yields a compatibility of the syntomic descent spectral sequence with the Hochschild–Serre spectral sequence in étale cohomology (via the period maps). We remark that, in the case of proper varieties with semistable reduction, this fact was announced in [Nekovář 2000].

Looking again at the period map ρ_{syn} : (7)$\to$(10) we see that truncating all the complexes at level r will allow us to drop $+$ from the first diagram. Hence we have

$$\rho_{\mathrm{syn}} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r)).$$

To conclude that we have

$$\rho_{\mathrm{syn}} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r))$$

as well, we look at the map of spectral sequences ${}^{\mathrm{syn}}E \to {}^{\text{ét}}E$ and observe that, in the stated ranges of the Hodge–Tate filtration we have $H^*_{\mathrm{st}}(G_K, \cdot) = H^*(G_K, \cdot)$ (a fact that follows, for example, from the work of Berger [2002]).

1D. ***p-adic regulators.*** As an application of Theorem A, we look at the question of the image of Soulé's étale regulators

$$r^{\text{ét}}_{r,i} : K_{2r-i-1}(X)_0 \to H^1(G_K, H^i(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r))),$$

where $K_{2r-i-1}(X)_0 := \ker\bigl(c^{\text{ét}}_{r,i+1} : K_{2r-i-1}(X) \to H^{i+1}(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r))\bigr)$, inside the Galois cohomology group. We prove:

Theorem B. *The regulators $r^{\text{ét}}_{r,i}$ factor through the group $H^1_{\mathrm{st}}(G_K, H^i(X_{\overline{K},\text{ét}}, \mathbb{Q}_p(r)))$.*

As we explain in the article, this fact is known to follow from the work of Scholl [1993] on "geometric" extensions associated to K-theory classes. In our approach, this is a simple consequence of good properties of syntomic cohomology and the existence of the syntomic descent spectral sequence. Namely, as can be easily derived from the presentation (3), syntomic cohomology has a projective space theorem and homotopy property,[4] and hence admits Chern classes from higher K-theory. It can be easily shown that they are compatible with the étale Chern classes via the syntomic period maps. The factorization we want in the above theorem follows then from the compatibility of the two descent spectral sequences.

1E. ***Notation and conventions.*** Let V be a complete discrete valuation ring with fraction field K of characteristic 0, with perfect residue field k of characteristic p, and with maximal ideal $\mathfrak{m}_K$. Let v be the valuation on K normalized so that $v(p)=1$. Let $\overline{K}$ be an algebraic closure of K and let $\overline{V}$ denote the integral closure of V in $\overline{K}$. Let $W(k)$ be the ring of Witt vectors of k with fraction field K_0 and denote by K_0^{nr} the maximal unramified extension of K_0. Denote by e_K the absolute ramification

[4]As explained in Appendix B, it follows that it is a Bloch–Ogus cohomology theory.

index of K, i.e., the degree of K over K_0. Set $G_K = \mathrm{Gal}(\bar{K}/K)$ and let I_K denote its inertia subgroup. Let φ be the absolute Frobenius on $W(\bar{k})$. We will denote by V, $V^\times$, and V^0 the scheme $\mathrm{Spec}(V)$ with the trivial, canonical (i.e., associated to the closed point), and $(\mathbb{N} \to V, 1 \mapsto 0)$ log-structure respectively. For a log-scheme X over $\mathscr{O}_K$, denote its reduction mod p^n by X_n and its special fiber by X_0.

Unless otherwise stated, we work in the category of integral quasi-coherent log-schemes. In general, we will not distinguish between simplicial abelian groups and complexes of abelian groups.

Let A be an abelian category with enough projective objects. In this paper A will be the category of abelian groups or $\mathbb{Z}_p$-, $\mathbb{Z}/p^n$-, or $\mathbb{Q}_p$-modules. Unless otherwise stated, we work in the (stable) ∞-category $\mathscr{D}(A)$, i.e., the stable ∞-category whose objects are (left-bounded) chain complexes of projective objects of A. For a readable introduction to such categories, the reader may consult [Groth 2010; Lurie 2016, Chapter 1]. The ∞-derived category is essential to us for two reasons: first, it allows us to work simply with the Beilinson–Hyodo–Kato complexes; second, it supplies functorial homotopy limits.

Many of our constructions will involve sheaves of objects from $\mathscr{D}(A)$. The reader may consult the notes of Illusie [2013] and Zheng [2013] for a brief introduction to the subject and [Lurie 2009; 2016] for a thorough treatment.

We will use a shorthand for certain homotopy limits. Namely, if $f : C \to C'$ is a map in the dg derived category of abelian groups, we set

$$[\, C \xrightarrow{f} C'\,] := \mathrm{holim}(C \to C' \leftarrow 0).$$

We also set

$$\begin{bmatrix} C_1 & \xrightarrow{f} & C_2 \\ \downarrow & & \downarrow \\ C_3 & \xrightarrow{g} & C_4 \end{bmatrix} := \big[[C_1 \xrightarrow{f} C_2] \to [C_3 \xrightarrow{g} C_4]\big],$$

where the diagram in the brackets is a commutative diagram in the dg derived category.

2. Preliminaries

In this section we will do some preparation. In the first part, we will collect some relevant facts from the literature concerning period rings, derived log de Rham complexes and the h-topology. In the second part, we will prove vanishing results in Galois cohomology and a criterion comparing two spectral sequences that we will need to compare the syntomic descent spectral sequence with the étale Hochschild–Serre spectral sequence.

2A. ***The rings of periods.*** Let us recall briefly the definitions of the rings of periods B_{cr}, B_{dR}, B_{st} of [Fontaine 1994a]. As in 2.2 and 2.3 of that work, let A_{cr} denote Fontaine's ring of crystalline periods. This is a p-adically complete ring such that $A_{\mathrm{cr},n} := A_{\mathrm{cr}}/p^n$ is a universal PD-thickening of $\overline{V}_n$ over $W_n(k)$. Let $J_{\mathrm{cr},n}$ denote its PD-ideal, $A_{\mathrm{cr},n}/J_{\mathrm{cr},n} = \overline{V}_n$. We have

$$A_{\mathrm{cr},n} = H^0_{\mathrm{cr}}(\mathrm{Spec}(\overline{V}_n)/W_n(k)), \quad B^+_{\mathrm{cr}} := A_{\mathrm{cr}}[1/p], \quad B_{\mathrm{cr}} := B^+_{\mathrm{cr}}[t^{-1}],$$

where t is a certain element of B^+_{cr} (see [Fontaine 1994a] for a precise definition of t). The ring B^+_{cr} is a topological K_0-module equipped with a Frobenius φ coming from the crystalline cohomology and a natural G_K-action. We have that $\varphi(t) = pt$ and that G_K acts on t via the cyclotomic character.

Let

$$B^+_{\mathrm{dR}} := \varprojlim_r (\mathbb{Q} \otimes \varprojlim_n A_{\mathrm{cr},n}/J^{[r]}_{\mathrm{cr},n}), \quad B_{\mathrm{dR}} := B^+_{\mathrm{dR}}[t^{-1}].$$

The ring B^+_{dR} has a discrete valuation given by the powers of t. Its quotient field is B_{dR}. We set $F^n B_{\mathrm{dR}} = t^n B^+_{\mathrm{dR}}$. This defines a descending filtration on B_{dR}.

The period ring B_{st} lies between B_{cr} and B_{dR} [Fontaine 1994a, 3.1]. To define it, choose a sequence of elements $s = (s_n)_{n\geq 0}$ of $\overline{V}$ such that $s_0 = p$ and $s^p_{n+1} = s_n$. Fontaine associates to it an element u_s of B^+_{dR} that is transcendental over B^+_{cr}. Let B^+_{st} denote the subring of B_{dR} generated by B^+_{cr} and u_s. It is a polynomial algebra in one variable over B^+_{cr}. The ring B^+_{st} does not depend on the choice of s (because for another sequence $s' = (s'_n)_{n\geq 0}$ we have $u_s - u_{s'} \in \mathbb{Z}_p t \subset B^+_{\mathrm{cr}}$). The action of G_K on B^+_{dR} restricts well to B^+_{st}. The Frobenius φ extends to B^+_{st} by $\varphi(u_s) = pu_s$ and one defines the monodromy operator $N : B^+_{\mathrm{st}} \to B^+_{\mathrm{st}}$ as the unique B^+_{cr}-derivation such that $Nu_s = -1$. We have $N\varphi = p\varphi N$ and the short exact sequence

$$0 \to B^+_{\mathrm{cr}} \to B^+_{\mathrm{st}} \xrightarrow{N} B^+_{\mathrm{st}} \to 0. \tag{11}$$

Let $B_{\mathrm{st}} = B_{\mathrm{cr}}[u_s]$. We denote by ι the injection $\iota : B^+_{\mathrm{st}} \hookrightarrow B^+_{\mathrm{dR}}$. The topology on B_{st} is the one induced by B_{cr} and the inductive topology; the map ι is continuous (though the topology on B_{st} is not the one induced from B_{dR}).

2B. ***Derived log de Rham complex.*** In this subsection we collect a few facts about the relationship between crystalline cohomology and de Rham cohomology.

Let S be a log-PD-scheme on which p is nilpotent. For a log-scheme Z over S, let $\mathrm{L}\Omega^\bullet_{Z/S}$ denote the derived log de Rham complex (see [Beilinson 2012, 3.1] for a review). This is a commutative dg $\mathscr{O}_S$-algebra on $Z_{\text{ét}}$ equipped with a Hodge filtration F^m. There is a natural morphism of filtered commutative dg $\mathscr{O}_S$-algebras

$$\kappa : \mathrm{L}\Omega^\bullet_{Z/S} \to \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}), \tag{12}$$

where $u_{Z/S} : Z_{\mathrm{cr}} \to Z_{\text{ét}}$ is the projection from the log-crystalline to the étale topos [Beilinson 2013, (1.9.1)]. The following theorem was proved by Beilinson [2013, Theorem on p. 13] by direct computations of both sides.

Theorem 2.1. *Suppose that Z, S are fine and $f : Z \to S$ is an integral, locally complete intersection morphism. Then (12) yields quasi-isomorphisms*

$$\kappa_m : \mathrm{L}\Omega^\bullet_{Z/S}/F^m \xrightarrow{\sim} \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}/\mathscr{J}^{[m]}_{Z/S}).$$

Recall [Bhatt 2012, Definition 7.20] that a log-scheme is called G-log-syntomic if it is log-syntomic and the local log-smooth models can be chosen to be of Cartier type. The next theorem, finer than Theorem 2.1, was proved by Bhatt [2012, Theorem 7.22] by looking at the conjugate filtration of the left-hand side.

Theorem 2.2. *Suppose that $f : Z \to S$ is G-log-syntomic. Then we have a quasi-isomorphism*

$$\kappa : \mathrm{L}\Omega^\bullet_{Z/S} \xrightarrow{\sim} \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}).$$

Combining the two theorems above, we get a filtered version:

Corollary 2.3. *Suppose that $f : Z \to S$ is G-log-syntomic. Then we have a quasi-isomorphism*

$$F^m\mathrm{L}\Omega^\bullet_{Z/S} \xrightarrow{\sim} \mathrm{R}u_{Z/S*}(\mathscr{J}^{[m]}_{Z/S}).$$

Proof. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccc} F^m\mathrm{L}\Omega^\bullet_{Z/S} & \longrightarrow & \mathrm{L}\Omega^\bullet_{Z/S} & \longrightarrow & \mathrm{L}\Omega^\bullet_{Z/S}/F^m \\ \downarrow & & \downarrow \wr & & \downarrow \wr \\ \mathrm{R}u_{Z/S*}(\mathscr{J}^{[m]}_{Z/S}) & \longrightarrow & \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}) & \longrightarrow & \mathrm{R}u_{Z/S*}(\mathscr{O}_{Z/S}/\mathscr{J}^{[m]}_{Z/S}) \end{array}$$

and use the above theorems of Bhatt and Beilinson. ☐

Let X be a fine, proper, log-smooth scheme over $V^\times$. Set

$$\mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/W(k)})\widehat{\otimes}\mathbb{Q}_p := \big(\mathrm{holim}_n\, \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X_n/W_n(k)})\big) \otimes \mathbb{Q}$$

and similarly for complexes over $V^\times$. Here the hat over the derived log de Rham complex refers to the completion with respect to the Hodge filtration (in the sense of prosystems). For $r \geq 0$, consider the sequence of maps

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r \xleftarrow{\sim} \mathrm{R}\Gamma(X, \mathrm{L}\Omega^\bullet_{X/V^\times}/F^r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^\bullet_{X/V^\times}/F^r)\widehat{\otimes}\mathbb{Q}_p \\ \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{O}_{X/V^\times}/\mathscr{J}^{[r]}_{X/V^\times})_{\mathbb{Q}} \leftarrow \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{O}_{X/W(k)}/\mathscr{J}^{[r]}_{X/W(k)})_{\mathbb{Q}}. \end{aligned} \tag{13}$$

The first quasi-isomorphism follows from the fact that since X_K is log-smooth over K_0, the natural map $\mathrm{L}\Omega^\bullet_{X_K/K_0}/F^r \xrightarrow{\sim} \Omega^\bullet_{X_K/K_0}/F^r$ is a quasi-isomorphism. The second quasi-isomorphism follows from X being proper and log-smooth over $V^\times$, and the third one from Theorem 2.1. Define the map

$$\gamma_r^{-1} : \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{O}_{X/W(k)}/\mathscr{J}^{[r]}_{X/W(k)})_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r$$

as the composition (13).

Corollary 2.4. *Let X be a fine, proper, log-smooth scheme over $V^{\times}$. Let $r \geq 0$. There exists a canonical quasi-isomorphism*

$$\gamma_r : \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X, \mathscr{O}_{X/W(k)}/\mathscr{J}^{[r]}_{X/W(k)})_{\mathbb{Q}}.$$

Proof. It suffices to show that the last map in the composition (13) is also a quasi-isomorphism. By Theorem 2.1, this map is quasi-isomorphic to the map

$$(\mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/W(k)})\widehat{\otimes}\mathbb{Q}_p)/F^r \to (\mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/V^{\times}})\widehat{\otimes}\mathbb{Q}_p)/F^r.$$

Hence it suffices to show that the natural map

$$\mathrm{gr}^i_F\, \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/W(k)})\widehat{\otimes}\mathbb{Q}_p \to \mathrm{gr}^i_F\, \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/V^{\times}})\widehat{\otimes}\mathbb{Q}_p$$

is a quasi-isomorphism for all $i \geq 0$.

Fix $n \geq 1$ and $i \geq 0$ and recall [Beilinson 2012, 1.2] that we have a natural identification

$$\mathrm{gr}^i_F\, \mathrm{L}\Omega^{\bullet}_{X_n/W_n(k)} \xrightarrow{\sim} L\Lambda^i_X(L_{X_n/W_n(k)})[-i],$$
$$\mathrm{gr}^i_F\, \mathrm{L}\Omega^{\bullet}_{X_n/V_n^{\times}} \xrightarrow{\sim} L\Lambda^i_X(L_{X_n/V_n^{\times}})[-i],$$

where $L_{Y/S}$ denotes the relative log cotangent complex [Beilinson 2012, 3.1] and $L\Lambda_X(\,\bullet\,)$ is the nonabelian left derived functor of the exterior power functor. The distinguished triangle

$$\mathscr{O}_X \otimes_V L_{V_n^{\times}/W_n(k)} \to L_{X_n/W_n(k)} \to L_{X_n/V_n^{\times}}$$

yields a distinguished triangle

$$L\Lambda^i_X(\mathscr{O}_X \otimes_V L_{V_n^{\times}/W_n(k)})[-i] \to \mathrm{gr}^i_F\, \mathrm{L}\Omega^{\bullet}_{X_n/W_n(k)} \to \mathrm{gr}^i_F\, \mathrm{L}\Omega^{\bullet}_{X_n/V_n^{\times}}.$$

Hence we have a distinguished triangle

$$\begin{aligned}&\mathrm{holim}_n\, \mathrm{R}\Gamma(X_{\text{ét}}, L\Lambda^i_X(\mathscr{O}_X \otimes_V L_{V_n^{\times}/W_n(k)})) \otimes \mathbb{Q}[-i]\\ &\qquad \to \mathrm{gr}^i_F\, \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/W(k)})\widehat{\otimes}\mathbb{Q}_p \to \mathrm{gr}^i_F\, \mathrm{R}\Gamma(X_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{X/V^{\times}})\widehat{\otimes}\mathbb{Q}_p.\end{aligned}$$

It suffices to show that the term on the left is zero. But this will follow as soon as we show that the cohomology groups of $L_{V_n^{\times}/W_n(k)}$ are annihilated by p^c, where c is a constant independent of n. To show this, recall that V is a log complete intersection over $W(k)$. If π is a generator of $V/W(k)$, and $f(t)$ is its minimal polynomial, then (see [Olsson 2005, 6.9]) $L_{V^{\times}/W(k)}$ is quasi-isomorphic to the cone of the multiplication by $f'(\pi)$ map on V. Hence $L_{V^{\times}/W(k)}$ is acyclic in nonzero degrees, $H^0 L_{V^{\times}/W(k)} = \Omega_{V^{\times}/W(k)}$ is a cyclic V-module and we have a short exact sequence

$$0 \to \Omega_{V/W(k)} \to \Omega_{V^{\times}/W(k)} \to V/\mathfrak{m}_K \to 0.$$

Since $\Omega_{V/W(k)} \simeq V/\mathscr{D}_{K/K_0}$, where $\mathscr{D}_{K/K_0}$ is the different, $p^c H^0 L_{V^{\times}/W(k)} = 0$ for a constant c independent of n. Since $L_{V^{\times}/W(k)} \simeq L_{V^{\times}/W(k)} \otimes^L_V V_n$, we are done. □

Remark 2.5. Versions of the above corollary appear in various degrees of generality in the proofs of the p-adic comparison theorems (see [Kato and Messing 1992, Lemma 4.5; Langer 1999, Lemma 2.7]). They are proved using computations in crystalline cohomology. We find the above argument based on the Beilinson comparison theorem, Theorem 2.1, particularly conceptual and pleasing.

2C. ***The h-topology.*** In this subsection we review terminology connected with the h-topology from [Beilinson 2013; 2012; Bhatt 2012]; we will use it freely. Let $\mathscr{V}ar_K$ be the category of varieties (i.e., reduced and separated schemes of finite type) over a field K. An *arithmetic pair* over K is an open embedding $j : U \hookrightarrow \overline{U}$ with dense image of a K-variety U into a reduced proper flat V-scheme $\overline{U}$. A morphism $(U, \overline{U}) \to (T, \overline{T})$ of pairs is a map $\overline{U} \to \overline{T}$ which sends U to T. In the case that the pairs represent log-regular schemes, this is the same as a map of log-schemes. For a pair $(U, \overline{U})$, we set $V_U := \Gamma(\overline{U}, \mathscr{O}_{\overline{U}})$ and $K_U := \Gamma(\overline{U}_K, \mathscr{O}_{\overline{U}})$. K_U is a product of several finite extensions of K (labeled by the connected components of $\overline{U}$) and, if $\overline{U}$ is normal, V_U is the product of the corresponding rings of integers. We will denote by $\mathscr{P}_K^{\mathrm{ar}}$ the category of arithmetic pairs over K. A *semistable pair* (*ss-pair*) over K [Beilinson 2012, 2.2] is a pair of schemes $(U, \overline{U})$ over (K, V) such that

(i) $\overline{U}$ is regular and proper over V,

(ii) $\overline{U} \setminus U$ is a divisor with normal crossings on $\overline{U}$,

(iii) the closed fiber $\overline{U}_0$ of $\overline{U}$ is reduced.

The closed fiber is taken over the closed points of V_U. We will think of ss-pairs as log-schemes equipped with log-structure given by the divisor $\overline{U} \setminus U$. The closed fiber $\overline{U}_0$ has the induced log-structure. We will say that the log-scheme $(U, \overline{U})$ is *split* over V_U. We will denote by $\mathscr{P}_K^{\mathrm{ss}}$ the category of ss-pairs over K. A semistable pair is called *strict* if the irreducible components of the closed fiber are regular. We will often work with the larger category $\mathscr{P}_K^{\log}$ of log-schemes $(U, \overline{U}) \in \mathscr{P}_K^{\mathrm{ar}}$ log-smooth over $V_U^{\times}$.

A *semistable pair* (*ss-pair*) over $\overline{K}$ [Beilinson 2012, 2.2] is a pair of connected schemes $(T, \overline{T})$ over $(\overline{K}, \overline{V})$ such that there exists an ss-pair $(U, \overline{U})$ over K and a $\overline{K}$-point $\alpha : K_U \to \overline{K}$ such that $(T, \overline{T})$ is isomorphic to the base change $(U_{\overline{K}}, \overline{U}_{\overline{V}})$. We will denote by $\mathscr{P}_{\overline{K}}^{\mathrm{ss}}$ the category of ss-pairs over $\overline{K}$.

A *geometric pair* over K is a pair $(U, \overline{U})$ of varieties over K such that $\overline{U}$ is proper and $U \subset \overline{U}$ is open and dense. We say that the pair $(U, \overline{U})$ is an *nc-pair* if $\overline{U}$ is regular and $\overline{U} \setminus U$ is a divisor with normal crossings in $\overline{U}$; it is a strict nc-pair if the irreducible components of $U \setminus \overline{U}$ are regular. A morphism of pairs $f : (U_1, \overline{U}_1) \to (U, \overline{U})$ is a map $\overline{U}_1 \to \overline{U}$ that sends U_1 to U. We denote the category of nc-pairs over K by $\mathscr{P}_K^{\mathrm{nc}}$.

For a field K, the h-topology (see [Suslin and Voevodsky 2000; Beilinson 2012, 2.3]) on $\mathscr{V}ar_K$ is the coarsest topology finer than the Zariski and proper topologies.[5] It is stronger than the étale and proper topologies. It is generated by the pretopology whose coverings are finite families of maps $\{Y_i \to X\}$ such that $Y := \coprod Y_i \to X$ is a universal topological epimorphism (i.e., a subset of X is Zariski open if and only if its preimage in Y is open). We denote by $\mathscr{V}ar_{K,h}$ and X_h the corresponding h-sites. For any of the categories $\mathscr{P}$ mentioned above, let $\gamma : (U, \bar{U}) \to U$ denote the forgetful functor. Beilinson [2012, 2.5] proved that the categories $\mathscr{P}^{\mathrm{nc}}$, $(\mathscr{P}_K^{\mathrm{ar}}, \gamma)$ and $(\mathscr{P}_K^{\mathrm{ss}}, \gamma)$ form a base for $\mathscr{V}ar_{K,h}$. One can easily modify his argument to conclude the same about the categories $(\mathscr{P}_K^{\log}, \gamma)$.

2D. ***Galois cohomology.*** In this subsection we review the definition of (higher) semistable Selmer groups and prove that in stable ranges they are the same as Galois cohomology groups. Our main references are [Fontaine 1994b; 1994c; Colmez and Fontaine 2000; Bloch and Kato 1990; Fontaine and Perrin-Riou 1994; Nekovář 1993]. Recall [Fontaine 1994b, 1994c] that a p-adic representation V of G_K (i.e., a finite-dimensional continuous $\mathbb{Q}_p$-vector space representation) is called *semistable* (over K) if $\dim_{K_0}(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K} = \dim_{\mathbb{Q}_p}(V)$.

It is called *potentially semistable* if there exists a finite extension K' of K such that $V|G_{K'}$ is semistable over K'. We denote by $\mathrm{Rep}_{\mathrm{st}}(G_K)$ and $\mathrm{Rep}_{\mathrm{pst}}(G_K)$ the categories of semistable and potentially semistable representations of G_K, respectively.

As in [Fontaine 1994c, 4.2], a φ-module over K_0 is a pair (D, φ), where D is a finite-dimensional K_0-vector space and $\varphi = \varphi_D$ is a φ-semilinear automorphism of D; a (φ, N)-module is a triple (D, φ, N), where (D, φ) is a φ-module and $N = N_V$ is a K_0-linear endomorphism of D such that $N\varphi = p\varphi N$ (hence N is nilpotent). A filtered (φ, N)-module is a tuple $(D, \varphi, N, F^\bullet)$, where (D, φ, N) is a (φ, N)-module and $F^\bullet$ is a decreasing finite filtration of D_K by K-vector spaces. There is a notion of a (*weakly*) *admissible* filtered (φ, N)-module [Colmez and Fontaine 2000]. Denote by $MF_K^{\mathrm{ad}}(\varphi, N) \subset MF_K(\varphi, N)$ the categories of admissible filtered (φ, N)-modules and filtered (φ, N)-modules, respectively. We know [Colmez and Fontaine 2000] that the pair of functors

$$D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}, \quad V_{\mathrm{st}}(D) = (B_{\mathrm{st}} \otimes_{K_0} D)^{\varphi=\mathrm{Id}, N=0} \cap F^0(B_{\mathrm{dR}} \otimes_K D_K)$$

defines an equivalence of categories $MF_K^{\mathrm{ad}}(\varphi, N) \simeq \mathrm{Rep}_{\mathrm{st}}(G_K)$.

For $D \in MF_K(\varphi, N)$, set

$$C_{\mathrm{st}}(D) := \left[\begin{array}{ccc} D & \xrightarrow{(1-\varphi,\mathrm{can})} & D \oplus D_K/F^0 \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle (N,0)} \\ D & \xrightarrow{1-p\varphi} & D \end{array}\right].$$

[5] The latter is generated by a pretopology whose coverings are proper surjective maps.

Here the brackets denote the total complex of the double complex inside the brackets. Consider also the complex

$$C^+(D) := \left[\begin{array}{ccc} D\otimes_{K_0} B_{\mathrm{st}}^+ & \xrightarrow{(1-\varphi,\mathrm{can}\otimes\iota)} & D\otimes_{K_0} B_{\mathrm{st}}^+ \oplus (D_K\otimes_K B_{\mathrm{dR}}^+)/F^0 \\ \downarrow N & & \downarrow (N,0) \\ D\otimes_{K_0} B_{\mathrm{st}}^+ & \xrightarrow{1-p\varphi} & D\otimes_{K_0} B_{\mathrm{st}}^+ \end{array}\right].$$

Define $C(D)$ by omitting the superscript $+$ in the above diagram. We have $C_{\mathrm{st}}(D) = C(D)^{G_K}$.

Remark 2.6. Recall [Nekovář 1993, 1.19; Fontaine and Perrin-Riou 1994, 3.3] that to every p-adic representation V of G_K we can associate a complex

$$C_{\mathrm{st}}(V) : D_{\mathrm{st}}(V) \xrightarrow{(N,1-\varphi,\iota)} D_{\mathrm{st}}(V)\oplus D_{\mathrm{st}}(V)\oplus t_V \xrightarrow{(1-p\varphi)-N} D_{\mathrm{st}}(V) \to 0,$$

where $t_V := (V\otimes_{\mathbb{Q}_p}(B_{\mathrm{dR}}/B_{\mathrm{dR}}^+))^{G_K}$ [Fontaine and Perrin-Riou 1994, I.2.2.1]. The cohomology of this complex is called $H^*_{\mathrm{st}}(G_K, V)$. If V is semistable then $C_{\mathrm{st}}(V) = C_{\mathrm{st}}(D_{\mathrm{st}}(V))$; hence $H^*(C_{\mathrm{st}}(D_{\mathrm{st}}(V))) = H^*_{\mathrm{st}}(G_K, V)$. If V is potentially semistable, the groups $H^*_{\mathrm{st}}(G_K, V)$ compute Yoneda extensions of $\mathbb{Q}_p$ by V in the category of potentially semistable representations [ibid., I.3.3.8]. In general [ibid., I.3.3.7], $H^0_{\mathrm{st}}(G_K, V) \xrightarrow{\sim} H^0(G_K, V)$ and $H^1_{\mathrm{st}}(G_K, V) \hookrightarrow H^1(G_K, V)$ computes st-extensions[6] of $\mathbb{Q}_p$ by V.

Remark 2.7. Let $D \in MF_K(\varphi, N)$. Note that:

(1) $H^0(C(D)) = V_{\mathrm{st}}(D)$.

(2) For $i \geq 2$, we have $H^i(C^+(D)) = H^i(C(D)) = 0$ (because N is surjective on B_{st}^+ and B_{st}).

(3) If $F^1 D_K = 0$ then $F^0(D_K\otimes_K B_{\mathrm{dR}}^+) = F^0(D_K\otimes_K B_{\mathrm{dR}})$ (in particular, the map of complexes $C^+(D)\to C(D)$ is an injection).

(4) If $D = D_{\mathrm{st}}(V)$ is admissible then we have quasi-isomorphisms

$$C(D) \xleftarrow{\sim} V\otimes_{\mathbb{Q}_p}[B_{\mathrm{cr}} \xrightarrow{(1-\varphi,\mathrm{can})} B_{\mathrm{cr}}\oplus B_{\mathrm{dR}}/F^0] \xleftarrow{\sim} V\otimes_{\mathbb{Q}_p}(B_{\mathrm{cr}}^{\varphi=1}\cap F^0) = V$$

and the map of complexes $C_{\mathrm{st}}(D)\to C(D)$ represents the canonical map $H^i_{\mathrm{st}}(G_K, V)\to H^i(G_K, V)$.

Lemma 2.8 [Fontaine 1994a, Theorem II.5.3]. *If $X\subset B_{\mathrm{cr}}\cap B_{\mathrm{dR}}^+$ and $\varphi(X)\subset X$ then $\varphi^2(X)\subset B_{\mathrm{cr}}^+$.*

Proposition 2.9. *If $D\in MF_K(\varphi, N)$ and $F^1 D_K = 0$ then $H^0(C(D)/C^+(D)) = 0$.*

---

[6] An extension $0\to V_1\to V_2\to V_3\to 0$ is called st if the sequence $0\to D_{\mathrm{st}}(V_1)\to D_{\mathrm{st}}(V_2)\to D_{\mathrm{st}}(V_3)\to 0$ is exact.

Proof. We will argue by induction on m such that $N^m = 0$. Assume first that $m = 1$ (hence $N = 0$). We have

$$C(D)/C^+(D) = \left[\begin{array}{ccc} D\otimes_{K_0}(B_{\rm st}/B_{\rm st}^+) & \xrightarrow{(1-\varphi,\,{\rm can}\otimes\iota)} & D\otimes_{K_0}(B_{\rm st}/B_{\rm st}^+)\oplus D_K\otimes_K(B_{\rm dR}/B_{\rm dR}^+) \\ \downarrow{\scriptstyle 1\otimes N} & & \downarrow{\scriptstyle (1\otimes N,0)} \\ D\otimes_{K_0}(B_{\rm st}/B_{\rm st}^+) & \xrightarrow{1-p\varphi} & D\otimes_{K_0}(B_{\rm st}/B_{\rm st}^+) \end{array}\right]$$

$$\xleftarrow{\sim} \left[D\otimes_{K_0}(B_{\rm cr}/B_{\rm cr}^+)\xrightarrow{(1-\varphi,\,{\rm can})} D\otimes_{K_0}(B_{\rm cr}/B_{\rm cr}^+)\oplus D_K\otimes_K(B_{\rm dR}/B_{\rm dR}^+)\right]$$

Write $D = \bigoplus_{i=1}^r K_0 d_i$ and, for $1 \le i \le r$, consider the maps

$$p_i : H^0(C(D)/C^+(D)) = (D\otimes_{K_0}((B_{\rm cr}\cap B_{\rm dR}^+)/B_{\rm cr}^+))^{\varphi=1} \subset \bigoplus_{i=1}^r d_i\otimes((B_{\rm cr}\cap B_{\rm dR}^+)/B_{\rm cr}^+) \xrightarrow{{\rm pr}_i} (B_{\rm cr}\cap B_{\rm dR}^+)/B_{\rm cr}^+.$$

Let Y_a, where $a \in H^0(C(D)/C^+(D))$, denote the K_0-subspace of $(B_{\rm cr}\cap B_{\rm dR}^+)/B_{\rm cr}^+$ spanned by $p_1(a), \ldots, p_r(a)$. For $M \in {\rm GL}_r(K_0)$, we have $(p_1(a), \ldots, p_r(a))^T = M\varphi(p_1(a), \ldots, p_r(a))^T$. Hence $\varphi(Y_a) \subset Y_a$. Let $X_a \subset B_{\rm cr}\cap B_{\rm dR}^+$ be the inverse image of Y_a under the projection $B_{\rm cr}\cap B_{\rm dR}^+ \to (B_{\rm cr}\cap B_{\rm dR}^+)/B_{\rm cr}^+$ (naturally $B_{\rm cr}^+ \subset X_a$). Then $\varphi(X_a) \subset X_a + B_{\rm cr}^+ = X_a$. By the above lemma, $\varphi^2(X_a) \subset B_{\rm cr}^+$. Hence $\varphi^2(Y_a) = 0$ and (applying M^{-2}) $Y_a = 0$. This implies that $a = 0$ and $H^0(C(D)/C^+(D)) = 0$, as wanted.

For general $m > 0$, consider the filtration $D_1 \subset D$, where $D_1 := \ker(N)$ with induced structures. Set $D_2 := D/D_1$ with induced structures. Then $D_1, D_2 \in MF_K(\varphi, N)$; N^i is trivial on D_1 for $i = 1$ and on D_2 for $i = m-1$. Clearly $F^1D_{1,K} = F^1D_{2,K} = 0$. Hence, by Remark 2.7.3, we have a short exact sequence

$$0 \to C(D_1)/C^+(D_1) \to C(D)/C^+(D) \to C(D_2)/C^+(D_2) \to 0.$$

By the inductive assumption, $H^0(C(D_1)/C^+(D_1)) = H^0(C(D_2)/C^+(D_2)) = 0$. Hence $H^0(C(D)/C^+(D)) = 0$, as wanted. □

Corollary 2.10. *If* $D \in MF_K(\varphi, N)$ *and* $F^1D_K = 0$ *then*

$$H^0(C^+(D)) = H^0(C(D)) = V_{\rm st}(D)\ (\subset D\otimes_{K_0} B_{\rm st}^+)$$

and $H^1(C^+(D)) \hookrightarrow H^1(C(D))$.

Corollary 2.11. *If* $D \in MF_K^{\rm ad}(\varphi, N)$ *and* $F^1D_K = 0$ *then*

$$H^i(C^+(D)) = H^i(C(D)) = \begin{cases} V_{\rm st}(D) & \text{if } i = 0, \\ 0 & \text{if } i \neq 0 \end{cases}$$

(*i.e.*, $C^+(D) \xrightarrow{\sim} C(D)$).

A filtered (φ, N, G_K)-module is a tuple $(D, \varphi, N, \rho, F^\bullet)$, where

(1) D is a finite-dimensional K_0^{nr}-vector space;

(2) $\varphi : D \to D$ is a Frobenius map;

(3) $N : D \to D$ is a K_0^{nr}-linear monodromy map such that $N\varphi = p\varphi N$;

(4) ρ is a K_0^{nr}-semilinear G_K-action on D (hence $\rho|I_K$ is linear) that is smooth, i.e., all vectors have open stabilizers, and that commutes with φ and N;

(5) $F^\bullet$ is a decreasing finite filtration of $D_K := (D \otimes_{K_0^{\mathrm{nr}}} \bar{K})^{G_K}$ by K-vector spaces.

Morphisms between filtered (φ, N, G_K)-modules are K_0^{nr}-linear maps preserving all structures. There is a notion of a (*weakly*) *admissible* filtered (φ, N, G_K)-module [Colmez and Fontaine 2000; Fontaine 1994b]. Denote by $MF_K^{\mathrm{ad}}(\varphi, N, G_K) \subset MF_K(\varphi, N, G_K)$ the categories of admissible filtered (φ, N, G_K)-modules and filtered (φ, N, G_K)-modules, respectively. We know [Colmez and Fontaine 2000] that the pair of functors $D_{\mathrm{pst}}(V) = \operatorname{inj\,lim}_H (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^H$, where $H \subset G_K$ is an open subgroup, and $V_{\mathrm{pst}}(D) = (B_{\mathrm{st}} \otimes_{K_0^{\mathrm{nr}}} D)^{\varphi=\mathrm{Id}, N=0} \cap F^0(B_{\mathrm{dR}} \otimes_K D_K)$ define an equivalence of categories $MF_K^{\mathrm{ad}}(\varphi, N, G_K) \simeq \mathrm{Rep}_{\mathrm{pst}}(G_K)$.

For $D \in MF_K(\varphi, N, G_K)$, set[7]

$$C_{\mathrm{pst}}(D) := \left[\begin{array}{ccc} D_{\mathrm{st}} & \xrightarrow{(1-\varphi,\mathrm{can})} & D_{\mathrm{st}} \oplus D_K/F^0 \\ \downarrow N & & \downarrow (N,0) \\ D_{\mathrm{st}} & \xrightarrow{1-p\varphi} & D_{\mathrm{st}} \end{array}\right].$$

Here $D_{\mathrm{st}} := D^{G_K}$. Consider also the following complex (we set $D_{\bar{K}} := D \otimes_{K_0^{\mathrm{nr}}} \bar{K}$):

$$C^+(D) := \left[\begin{array}{ccc} D \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}^+ & \xrightarrow{(1-\varphi,\mathrm{can}\otimes\iota)} & (D \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}^+) \oplus (D_{\bar{K}} \otimes_{\bar{K}} B_{\mathrm{dR}}^+)/F^0 \\ \downarrow N & & \downarrow (N,0) \\ D \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}^+ & \xrightarrow{1-p\varphi} & D \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}^+ \end{array}\right].$$

Define $C(D)$ by omitting the superscript $+$ in the above diagram. We have $C_{\mathrm{pst}}(D) = C(D)^{G_K}$.

Remark 2.12. If V is potentially semistable then $C_{\mathrm{st}}(V) = C_{\mathrm{pst}}(D_{\mathrm{pst}}(V))$; hence $H^*(C_{\mathrm{pst}}(D_{\mathrm{pst}}(V))) = H^*_{\mathrm{st}}(G_K, V)$.

Remark 2.13. If $D = D_{\mathrm{pst}}(V)$ is admissible then we have quasi-isomorphisms

$$C(D) \xleftarrow{\sim} V \otimes_{\mathbb{Q}_p} [B_{\mathrm{cr}} \xrightarrow{(1-\varphi,\mathrm{can})} B_{\mathrm{cr}} \oplus B_{\mathrm{dR}}/F^0] \xleftarrow{\sim} V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cr}}^{\varphi=1} \cap F^0) = V$$

[7] We hope that the notation below will not lead to confusion with the semistable case in general, but if in doubt we will add the data of the field K in the latter case.

and the map of complexes $C_{\mathrm{pst}}(D)\to C(D)$ represents the canonical map

$$H^i_{\mathrm{st}}(G_K,V)\to H^i(G_K,V).$$

Remark 2.14. Let $D=D_{\mathrm{pst}}(V)$ be admissible. The Bloch–Kato exponential

$$(Z^1C(D))^{G_K}\to H^1(G_K,V)$$

is given by the coboundary map arising from the exact sequence

$$0\to V\to C^0(D)\to Z^1C(D)\to 0.$$

Its restriction to the de Rham part of $Z^1C(D)$ is the Bloch–Kato exponential

$$\exp_{\mathrm{BK}}:D_K/F^0\to H^1(G_K,V).$$

It is also obtained by applying Rf, where $f(-)=(-)^{G_K}$, to the coboundary map $\partial : Z^1C(D)\to V[1]$ arising from the above exact sequence (see the proof of Theorem 4.8 for an appropriate formalism of continuous cohomology). Note that the composition of the canonical maps

$$Z^1C(D)\to(\sigma_{\geq 1}C(D))[1]\to C(D)[1]\xleftarrow{\sim} V[1]$$

is not equal to ∂, but to $-\partial$, by (18).

Corollary 2.15. *If $D\in MF^{\mathrm{ad}}_K(\varphi,N,G_K)$ and $F^1D_K=0$ then*

$$H^i(C^+(D))\xrightarrow{\sim} H^i(C(D))=\begin{cases}V_{\mathrm{pst}}(D) & \text{if } i=0,\\ 0 & \text{if } i\neq 0\end{cases}$$

(*i.e.*, $C^+(D)\xrightarrow{\sim} C(D)$).

Proof. By Remark 2.13 we have $C(D)\simeq V_{\mathrm{pst}}(D)[0]$. To prove the isomorphism $H^i(C^+(D))\xrightarrow{\sim} H^i(C(D))$, $i\geq 0$, take a finite Galois extension K'/K such that D becomes semistable over K', i.e., $I_{K'}$ acts trivially on D. We have $(D',\varphi,N)\in MF^{\mathrm{ad}}_{K'}(\varphi,N)$, where $D':=D^{G_{K'}}$ and (compatibly) $D\simeq D'\otimes_{K'_0}K_0^{\mathrm{nr}}$ and $F^\bullet D'_{K'}\simeq F^\bullet D_K\otimes_K K'$. It easily follows that $C^+(D)=C^+(K',D')$ and $C(D)=C(K',D')$. Since $F^1D'_{K'}=0$, our corollary is now a consequence of Corollary 2.11. □

Proposition 2.16. *If $D\in MF^{\mathrm{ad}}_K(\varphi,N,G_K)$ and $F^1D_K=0$ then, for $i\geq 0$, the natural map*

$$H^i_{\mathrm{st}}(G_K,V_{\mathrm{pst}}(D))\xrightarrow{\sim} H^i(G_K,V_{\mathrm{pst}}(D))$$

is an isomorphism.

Proof. Both sides satisfy Galois descent for finite Galois extensions. We can assume, therefore, that $D=D_{\mathrm{st}}(V)$ for a semistable representation V of G_K. For $i=0$, we have (even without assuming $F^1D_K=0$)

$$H^0(C_{\mathrm{st}}(D))=H^0(C(D)^{G_K})=H^0(C(D))^{G_K}=V^{G_K}.$$

For $i = 1$, the statement is proved in [Berger 2002, Théorème 6.2, Lemme 6.5]. For $i = 2$, it follows from the assumption $F^1 D_K = 0$ (by weak admissibility of D) that there is a $W(k)$-lattice $M \subset D$ such that $\varphi^{-1}(M) \subset p^2 M$, which implies that $1 - p\varphi = -p\varphi(1 - p^{-1}\varphi^{-1}) : D \to D$ is surjective, and hence $H^2(C_{\mathrm{st}}(D)) = 0$ (see the proof of [Berger 2002, Lemme 6.7]). The proof of the fact that $H^2(G_K, V) = 0$ if $F^1 D_K = 0$ was kindly communicated to us by L. Berger; it is reproduced in Appendix A (see Theorem A.1). For $i > 2$, both terms vanish. □

2E. *Comparison of spectral sequences.* The purpose of this subsection is to prove a derived category theorem (Theorem 2.18) that will be used later to relate the syntomic descent spectral sequence with the étale Hochschild–Serre spectral sequence (see Theorem 4.8). Let D be a triangulated category and $H : D \to A$ a cohomological functor to an abelian category A. A finite collection of adjacent exact triangles (a "Postnikov system" in the language of [Gelfand and Manin 2003, IV.2, Exercise 2])

$$
\begin{array}{ccccccccc}
 & Y^0 & & Y^1 & & & Y^n & & \\
 \nearrow & & \searrow \quad \nearrow & & \searrow & & \nearrow \quad & \searrow & \\
X = X^0 & \xleftarrow[{[1]}]{} & X^1 & \xleftarrow[{[1]}]{} & X^2 & \cdots & X^n & \xleftarrow[{[1]}]{} & X^{n+1} = 0
\end{array}
\tag{14}
$$

gives rise to an exact couple

$$D_1^{p,q} = H^q(X^p) = H(X^p[q]), \quad E_1^{p,q} = H^q(Y^p) \Rightarrow H^{p+q}(X).$$

The induced filtration on the abutment is given by

$$F^p H^{p+q}(X) = \mathrm{Im}\big(D_1^{p,q} = H^q(X^p) \to H^{p+q}(X)\big).$$

Remark 2.17. In the special case when A is the heart of a nondegenerate t-structure $(D^{\leq n}, D^{\geq n})$ on D and $H = \tau_{\leq 0}\tau_{\geq 0}$, the following conditions are equivalent:

(1) $E_2^{p,q} = 0$ for $p \neq 0$.

(2) $D_2^{p,q} = 0$ for all p, q.

(3) $D_r^{p,q} = 0$ for all p, q and $r > 1$.

(4) The sequence $0 \to H^q(X^p) \to H^q(Y^p) \to H^q(X^{p+1}) \to 0$ is exact for all p, q.

(5) The sequence $0 \to H^q(X) \to H^q(Y^0) \to H^q(Y^1) \to \cdots$ is exact for all q.

(6) The canonical map $H^q(X) \to E_1^{\bullet,q}$ is a quasi-isomorphism for all q.

(7) The triangle $\tau_{\leq q} X^p \to \tau_{\leq q} Y^p \to \tau_{\leq q} X^{p+1}$ is exact for all p, q.

From now on until the end of Section 2E assume that $D = D(A)$ is the derived category of A with the standard t-structure and that $X^i, Y^i \in D^+(A)$ for all i. Furthermore, assume that $f : A \to A'$ is a left exact functor to an abelian category A'

and that A admits a class of f-adapted objects (hence the derived functor $\mathrm{R}f : D^+(A) \to D^+(A')$ exists).

Applying $\mathrm{R}f$ to (14), we obtain another Postnikov system, this time in $D^+(A')$. The corresponding exact couple

$$ {}^{I}D_1^{p,q} = (\mathrm{R}^q f)(X^p), \quad {}^{I}E_1^{p,q} = (\mathrm{R}^q f)(Y^p) \Rightarrow (\mathrm{R}^{p+q} f)(X) \tag{15} $$

induces the filtration

$$ {}^{I}F^p(\mathrm{R}^{p+q} f)(X) = \mathrm{Im}\big({}^{I}D_1^{p,q} = (\mathrm{R}^q f)(X^p) \to (\mathrm{R}^{p+q} f)(X)\big). $$

Our goal is to compare (15), under the equivalent conditions in Remark 2.17, to the hypercohomology exact couple

$$ {}^{II}D_2^{p,q} = (\mathrm{R}^{p+q} f)(\tau_{\leq q-1} X), \quad {}^{II}E_2^{p,q} = (\mathrm{R}^p f)(H^q(X)) \Rightarrow (\mathrm{R}^{p+q} f)(X) \tag{16} $$

for which

$$ {}^{II}F^p(\mathrm{R}^{p+q} f)(X) = \mathrm{Im}\big({}^{II}D_2^{p-1,q+1} = (\mathrm{R}^{p+q} f)(\tau_{\leq q} X) \to (\mathrm{R}^{p+q} f)(X)\big). $$

Theorem 2.18. *Under the conditions in Remark 2.17, there is a natural morphism of exact couples*

$$ (u, v) : ({}^{I}D_2, {}^{I}E_2) \to ({}^{II}D_2, {}^{II}E_2). $$

Consequently, we have ${}^{I}F^p \subseteq {}^{II}F^p$ for all p and there is a natural morphism of spectral sequences ${}^{I}E_r^{*,*} \to {}^{II}E_r^{*,*}$ $(r > 1)$ compatible with the identity map on the common abutment.

Proof. Step 1: We begin by constructing a natural map $u : {}^{I}D_2 \to {}^{II}D_2$.

For each $p > 0$, there is a commutative diagram in $D^+(A')$

$$ \begin{array}{ccccc} (\mathrm{R}^{p+q} f)((\tau_{\leq q} Y^{p-1})[-p]) & \longrightarrow & (\mathrm{R}^{p+q} f)((\tau_{\leq q} X^{p})[-p]) & \longrightarrow & (\mathrm{R}^{p+q} f)(\tau_{\leq q} X) \\ \downarrow \wr & & \downarrow \wr & \overset{u'}{\nearrow} & \downarrow \alpha_{II} \\ {}^{I}E_1^{p-1,q} = (\mathrm{R}^{p+q} f)(Y^{p-1}[-p]) & \xrightarrow{k_1} & {}^{I}D_1^{p,q} = (\mathrm{R}^{p+q} f)(X^{p}[-p]) & \xrightarrow{\alpha_I} & (\mathrm{R}^{p+q} f)(X) \end{array} $$

both of whose rows are complexes. This defines a map $u' : {}^{I}D_1^{p,q} \to {}^{II}D_2^{p-1,q+1}$ such that $u' k_1 = 0$ and $\alpha_{II} u' = \alpha_I$ (hence ${}^{I}F^p = \mathrm{Im}(\alpha_I) \subseteq \mathrm{Im}(\alpha_{II}) = {}^{II}F^p$). By construction, the diagram (with exact top row)

$$ \begin{array}{ccccc} {}^{I}E_1^{p,q-1} & \xrightarrow{k_1} & {}^{I}D_1^{p+1,q-1} & \xrightarrow{i_1} & {}^{I}D_1^{p,q} \\ & \overset{0}{\searrow} & \downarrow u' & & \downarrow u' \\ & & {}^{II}D_2^{p,q} & \xrightarrow{i_2} & {}^{II}D_2^{p-1,q+1} \end{array} $$

is commutative for each $p \geq 0$, which implies that the map

$$u = u' i_1^{-1} : {}^I D_2^{p,q} = i_1({}^I D_1^{p+1,q-1}) \to {}^{II} D_2^{p,q}$$

is well-defined and satisfies $u i_2 = i_2 u$.

Step 2: For all q, the canonical quasi-isomorphism $H^q(X) \to E_1^{\bullet,q}$ induces natural morphisms

$$\begin{aligned} v' : {}^I E_2^{p,q} &= H^p(i \mapsto (\mathrm{R}^q f)(Y^i)) \to H^p(i \mapsto f(H^q(Y^i))) \to (\mathrm{R}^p f)(i \mapsto H^q(Y^i)) \\ &= (\mathrm{R}^p f)(E_1^{\bullet,q}) \xleftarrow{\sim} (\mathrm{R}^p f)(H^q(X)) = {}^{II} E_2^{p,q}; \end{aligned}$$

set $v = (-1)^p v' : {}^I E_2^{p,q} \to {}^{II} E_2^{p,q}$.

It remains to show that u and v are compatible with the maps

$$ {}^? D_2^{p-1,q+1} \xrightarrow{j_2} {}^? E_2^{p,q} \xrightarrow{k_2} {}^? D_2^{p+1,q} \quad (? = I, II).$$

Step 3: For any complex $M^\bullet$ over A, denote by $Z^i(M^\bullet) = \mathrm{Ker}(\delta^i : M^i \to M^{i+1})$ the subobject of cycles in degree i.

If $M^\bullet$ is a resolution of an object M of A, then each exact sequence

$$0 \longrightarrow Z^p(M^\bullet) \longrightarrow M^p \xrightarrow{\delta^p} Z^{p+1}(M^\bullet) \longrightarrow 0 \quad (p \geq 0) \tag{17}$$

can be completed to an exact sequence of resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^p(M^\bullet) & \longrightarrow & M^p & \longrightarrow & Z^{p+1}(M^\bullet) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle -\mathrm{can}} & & \\ 0 & \to & (\sigma_{\geq p}(M^\bullet))[p] & \to & (\sigma_{\geq p}\mathrm{Cone}(M^\bullet \xrightarrow{\mathrm{id}} M^\bullet))[p] & \to & (\sigma_{\geq p+1}(M^\bullet))[p+1] & \to & 0 \end{array}$$

By induction, we obtain that the following diagram, whose top arrow is the composition of the natural maps $Z^i \to Z^{i-1}[1]$ induced by (17), commutes in $D^+(A)$:

$$\begin{array}{ccc} Z^p(M^\bullet) & \longrightarrow & Z^0(M^\bullet)[p] = M[p] \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle (-1)^p\,\mathrm{can}} \\ (\sigma_{\geq p}(M^\bullet))[p] & \xrightarrow{\mathrm{can}} & M^\bullet[p] \end{array} \tag{18}$$

We are going to apply this statement to $M = H^q(X)$ and $M^\bullet = E_1^{\bullet,q}$, when $Z^p(M^\bullet) = D_1^{p,q} = H^q(X^p)$ and $Z^0(M^\bullet) = H^q(X)$.

Step 4: We are going to investigate ${}^I E_2^{p,q}$.

Complete the morphism $Y^p \to Y^{p+1}$ to an exact triangle $U^p \to Y^p \to Y^{p+1}$ in $D^+(A)$ and fix a lift $X^p \to U^p$ of the morphism $X^p \to Y^p$.

There are canonical epimorphisms

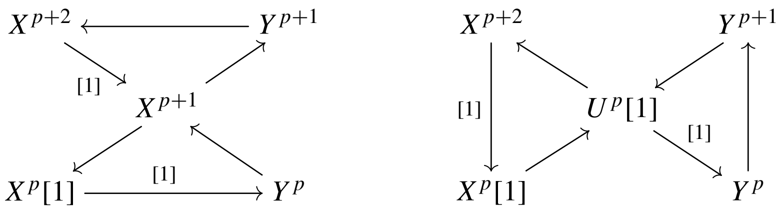
$$(\mathrm{R}^q f)(U^p) \twoheadrightarrow \mathrm{Ker}((\mathrm{R}^q f)(Y^p) \xrightarrow{j_1 k_1} (\mathrm{R}^q f)(Y^{p+1})) = Z^p({}^I E_1^{\bullet,q}) \twoheadrightarrow {}^I E_2^{p,q}, \tag{19}$$

and the map

$$k_2 : {}^I E_2^{p,q} \to {}^I D_2^{p+1,q} = \mathrm{Ker}({}^I D_1^{p+1,q} \xrightarrow{j_1} {}^I E_1^{p+1,q})$$

is induced by the restriction of $k_1 : {}^I E_1^{p,q} \to {}^I D_1^{p+1,q}$ to $Z^p({}^I E_1^{\bullet,q})$.

The octahedron (in which we have drawn only the four exact faces)



shows that the triangle $X^p \to U^p \to X^{p+2}[-1]$ is exact and the diagrams

$$\begin{array}{ccc} U^p[1] & \longrightarrow & Y^p[1] \\ \downarrow & & \downarrow \\ X^{p+2} & \longrightarrow & X^{p+1}[1] \end{array} \qquad \begin{array}{ccc} (\mathrm{R}^q f)(U^p) & \longrightarrow & Z^p({}^I E_1^{\bullet,q}) \\ \downarrow & & \downarrow{\scriptstyle k_1} \\ (\mathrm{R}^q f)(X^{p+2}[-1]) = {}^I D_2^{p+2,q-1} & \xrightarrow{i_1} & {}^I D_2^{p+1,q} \end{array}$$

commute. The previous discussion implies that the composite map

$$(\mathrm{R}^q f)(U^p) \twoheadrightarrow Z^p({}^I E_1^{\bullet,q}) \twoheadrightarrow {}^I E_2^{p,q} \xrightarrow{k_2} {}^I D_2^{p+1,q} \\ \xrightarrow{u} {}^{II} D_2^{p+1,q} = (\mathrm{R}^q f)((\tau_{\leq q-1} X)[p+1])$$

is obtained by applying $\mathrm{R}^q f$ to

$$\tau_{\leq q}\, U^p \to \tau_{\leq q}(X^{p+2}[-1]) = (\tau_{\leq q-1} X^{p+2})[-1] \to (\tau_{\leq q-1}\, X)[p+1]. \tag{20}$$

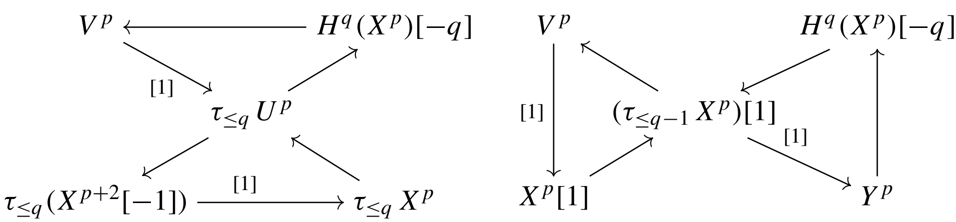
<u>Step 5</u>: All boundary maps $H^q(X^{p+2}[-1]) \to H^q(X^p)$ vanish by Remark 2.17, which means that the following triangles are exact:

$$\tau_{\leq q}\, X^p \to \tau_{\leq q}\, U^p \to \tau_{\leq q}(X^{p+2}[-1]) = (\tau_{\leq q-1}\, X^{p+2})[-1].$$

The commutative diagram

$$\begin{array}{ccccc} \tau_{\leq q}\, U^p & \longrightarrow & H^q(U^p)[-q] & \longrightarrow & \mathrm{Ker}\big(H^q(Y^p) \to H^q(Y^{p+1})\big)[-q] \\ \uparrow & & \uparrow & \diagup\!\!\diagup & \\ \tau_{\leq q}\, X^p & \longrightarrow & H^q(X^p)[-q] & & \end{array}$$

gives rise to an octahedron



In particular, the following diagram commutes:

$$\begin{array}{ccc} \tau_{\leq q} U^p & \longrightarrow & H^q(X^p)[-q] \\ \downarrow & & \downarrow \\ \tau_{\leq q}(X^{p+2}[-1]) & \longrightarrow & (\tau_{\leq q-1} X^p)[1] \end{array} \tag{21}$$

Step 6: The diagram (18) implies that the composition of $v : {}^I E_2^{p,q} \to {}^{II} E_2^{p,q}$ with the second epimorphism in (19) is equal to the composite map

$$\begin{aligned} Z^p({}^I E_1^{\bullet,q}) &= \mathrm{Ker}\big((\mathrm{R}^q f)(\tau_{\leq q} Y^p) \to (\mathrm{R}^q f)(\tau_{\leq q} Y^{p+1})\big) \\ &\to \mathrm{Ker}\big((\mathrm{R}^q f)(H^q(Y^p)[-q]) \to (\mathrm{R}^q f)(H^q(Y^{p+1})[-q])\big) \\ &= (\mathrm{R}^q f)(Z^p(E_1^{\bullet,q})[-q]) \to (\mathrm{R}^q f)(Z^0(E_1^{\bullet,q})[-q+p]) \\ &= (\mathrm{R}^p f)(H^q(X)) = {}^{II} E_2^{p,q}. \end{aligned}$$

As a result, the composition of v with (19) is obtained by applying $\mathrm{R}^q f$ to

$$\tau_{\leq q} U^p \to H^q(X^p)[q] \to H^q(X)[-q+p]. \tag{22}$$

Consequently, the composite map

$${}^I D_1^{p,q} = (\mathrm{R}^q f)(\tau_{\leq q} X^p) \xrightarrow{j_1} Z^p({}^I E_1^{\bullet,q}) \twoheadrightarrow {}^I E_2^{p,q} \xrightarrow{v} {}^{II} E_2^{p,q}$$

is given by applying $\mathrm{R}^q f$ to

$$\tau_{\leq q} X^p \to H^q(X^p)[q] \to H^q(X)[-q+p],$$

and hence is equal to $j_2 u'$. It follows that $v j_2 = v j_1 i_1^{-1} = j_2 u' i_1^{-1} = j_2 u$.

Step 7: The diagram (21) implies that the map (20) coincides with the composition of (22) with the canonical map $H^q(X)[-q+p] \to (\tau_{\leq q-1} X)[p+1]$; hence $uk_2 = k_2 v$. Thus the theorem is proved. □

Example 2.19. If $K^\bullet$ is a bounded-below filtered complex over A (with a finite filtration)

$$K^\bullet = F^0 K^\bullet \supset F^1 K^\bullet \supset \cdots \supset F^n K^\bullet \supset F^{n+1} K^\bullet = 0,$$

then the objects

$$X^p = F^pK^\bullet[p], \quad Y^p = (F^pK^\bullet/F^{p+1}K^\bullet)[p] = \mathrm{gr}^p_F(K^\bullet)[p] \in D^+(A)$$

form a Postnikov system of the kind considered in (14). The corresponding spectral sequences are equal to

$$E_1^{p,q} = H^{p+q}(\mathrm{gr}^p_F(K^\bullet)) \Rightarrow H^{p+q}(K^\bullet),$$
$${}^I E_1^{p,q} = (\mathrm{R}^{p+q}f)(\mathrm{gr}^p_F(K^\bullet)) \Rightarrow (\mathrm{R}^{p+q}f)(K^\bullet).$$

In the special case when $K^\bullet$ is the total complex associated to a first quadrant bicomplex $C^{\bullet,\bullet}$ and the filtration F^p is induced by the column filtration on $C^{\bullet,\bullet}$, then the complex $f(K^\bullet)$ over A' is equipped with a canonical filtration $(fF^p)(f(K^\bullet)) = f(F^pK^\bullet)$ satisfying

$$\mathrm{gr}^p_{f(F)}(f(K^\bullet)) = f(\mathrm{gr}^p_F(K^\bullet)).$$

Under the conditions in Remark 2.17, the corresponding exact couple

$${}^f D_1^{p,q} = H^{p+q}(f(F^pK^\bullet)),$$
$${}^f E_1^{p,q} = H^{p+q}(\mathrm{gr}^p_{f(F)}(f(K^\bullet))) = H^{p+q}(f(\mathrm{gr}^p_F(K^\bullet))) \Rightarrow H^{p+q}(f(K^\bullet))$$

then naturally maps to the exact couple (15), hence (beginning from (D_2, E_2)) to the exact couple (16), by Theorem 2.18.

3. Syntomic cohomology

In this section we will define the arithmetic and geometric syntomic cohomologies of varieties over K and $\overline{K}$, respectively, and study their basic properties.

3A. ***Hyodo–Kato morphism revisited.*** We will need to use the Hyodo–Kato morphism on the level of derived categories and vary it in the h-topology. Recall that the original morphism depends on the choice of a uniformizer and a change of such is encoded in a transition function involving the exponential of the monodromy. Since the fields of definition of semistable models in the bases for the h-topology change, we will need to use these transition functions. The problem though is that in the most obvious (i.e., crystalline) definition of the Hyodo–Kato complexes the monodromy is (at best) homotopically nilpotent — making the exponential in the transition functions impossible to define. Beilinson [2013] solves this problem by representing Hyodo–Kato complexes using modules with nilpotent monodromy. In this subsection we will summarize what we need from his approach.

We begin with a quick reminder. Let $(U, \overline{U})$ be a log-scheme, log-smooth over $V^\times$. For any $r \geq 0$, consider its absolute (meaning over $W(k)$) log-crystalline cohomology complexes

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n &:= \mathrm{R}\Gamma(\bar{U}_{\text{ét}},\mathrm{R}u_{U_n^\times/W_n(k)*}\mathscr{J}^{[r]}_{U_n^\times/W_n(k)}),\\ \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]}) &:= \mathrm{holim}_n\,\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n,\\ \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_{\mathbb{Q}} &:= \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})\otimes\mathbb{Q}_p, \end{aligned}$$

where $U^\times$ denotes the log-scheme $(U,\bar{U})$ and $u_{U_n^\times/W_n(k)}:(U_n^\times/W_n(k))_{\mathrm{cr}}\to\bar{U}_{\text{ét}}$ is the projection from the log-crystalline to the étale topos. For $r\geq 0$, we write $\mathscr{J}^{[r]}_{U_n^\times/W_n(k)}$ for the r-th divided power of the canonical PD-ideal $\mathscr{J}_{U_n^\times/W_n(k)}$; for $r\leq 0$, we set

$$\mathscr{J}^{[r]}_{U_n^\times/W_n(k)} := \mathscr{O}_{U_n^\times/W_n(k)}$$

and we will often omit it from the notation. The absolute log-crystalline cohomology complexes are filtered E_∞ algebras over $W_n(k)$, $W(k)$, or K_0, respectively. Moreover, the rational ones are filtered commutative dg algebras.

Remark 3.1. The canonical pullback map

$$\mathrm{R}\Gamma(\bar{U}_{\text{ét}},\mathrm{R}u_{U_n^\times/W_n(k)*}\mathscr{J}^{[r]}_{U_n^\times/W_n(k)}) \xrightarrow{\sim} \mathrm{R}u_{U_n^\times/\mathbb{Z}/p^n*}\mathscr{J}^{[r]}_{U_n^\times/\mathbb{Z}/p^n}$$

is a quasi-isomorphism. In what follows we will often call both the "absolute crystalline cohomology".

Let $W(k)\langle t_l\rangle$ be the divided-powers polynomial algebra generated by elements t_l, $l\in\mathfrak{m}_K/\mathfrak{m}_K^2\setminus\{0\}$, subject to the relations $t_{al}=[\bar{a}]t_l$ for $a\in V^*$, where $[\bar{a}]\in W(k)$ is the Teichmüller lift of $\bar{a}$ — the reduction mod $\mathfrak{m}_K$ of a. Let R_V (or simply R) be the p-adic completion of the subalgebra of $W(k)\langle t_l\rangle$ generated by t_l and $t_l^{ie_K}/i!$, $i\geq 1$. For a fixed l, the ring R is the following $W(k)$-subalgebra of $K_0[\![t_l]\!]$:

$$R=\left\{\sum_{i=0}^{\infty}a_i\frac{t_l^i}{\lfloor i/e_K\rfloor!}\;\middle|\;a_i\in W(k),\ \lim_{i\to\infty}a_i=0\right\}.$$

One extends the Frobenius φ_R (semilinearly) to R by setting $\varphi_R(t_l)=t_l^p$ and defines a monodromy operator N_R as a $W(k)$-derivation by setting $N_R(t_l)=-t_l$. Let $E:=\mathrm{Spec}(R)$ equipped with the log-structure generated by the t_l.

We have two exact closed embeddings

$$i_0:W(k)^0\hookrightarrow E,\quad i_\pi:V^\times\hookrightarrow E.$$

The first one is canonical and induced by $t_l\mapsto 0$. The second one depends on the choice of the class of the uniformizing parameter $\pi\in\mathfrak{m}_K/p\mathfrak{m}_K$ up to multiplication by Teichmüller elements. It is induced by $t_l\mapsto[\overline{l/\pi}]\pi$.

Assume $(U, \bar{U})$ is of Cartier type (i.e., the special fiber $\bar{U}_0$ is of Cartier type). Consider the log-crystalline and the Hyodo–Kato complexes (see [Beilinson 2013, 1.16])

$$\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R, \mathscr{J}^{[r]})_n := \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})_n/R_n, \mathscr{J}^{[r]}_{\bar{U}_n/R_n}),$$
$$\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_n := \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})_0/W_n(k)^0).$$

Let $\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R, \mathscr{J}^{[r]})$ and $\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})$ be their homotopy inverse limits. The last complex is called the *Hyodo–Kato complex*. The complex $\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)$ is R-perfect and

$$\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)\otimes^L_R R_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)\otimes^L \mathbb{Z}/p^n.$$

In general, we have $\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R, \mathscr{J}^{[r]})_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R, \mathscr{J}^{[r]})\otimes^L \mathbb{Z}/p^n$. The complex $\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})$ is $W(k)$-perfect and

$$\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_n \simeq \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})\otimes^L_{W(k)} W_n(k) \simeq \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})\otimes^L \mathbb{Z}/p^n.$$

We normalize the monodromy operators N on the rational complexes $\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}$ and $\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}$ by replacing the standard N [Hyodo and Kato 1994, 3.6] by $N_R := e_K^{-1}N$. This makes them compatible with base change. The embedding $i_0 : (U,\bar{U})_0 \hookrightarrow (U,\bar{U})$ over $i_0 : W_n(k)^0 \hookrightarrow E_n$ yields compatible morphisms $i^*_{0,n} : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_n \to \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_n$. Completing, we get a morphism

$$i_0^* : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R) \to \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U}),$$

which induces a quasi-isomorphism $i_0^* : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)\otimes^L_R W(k) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})$. All the above objects have an action of Frobenius and these morphisms are compatible with Frobenius. The Frobenius action is invertible on $\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}$.

The map $i_0^* : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}$ admits a unique (in the classical derived category) $W(k)$-linear section ι_π [Beilinson 2013, 1.16; Tsuji 1999, Proposition 4.4.6] that commutes with φ and N. The map ι_π is functorial and its R-linear extension is a quasi-isomorphism

$$\iota_\pi : R\otimes_{W(k)} \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}.$$

The composition (the *Hyodo–Kato map*)

$$\iota_{\mathrm{dR},\pi} := \gamma_r^{-1} i_\pi^* \cdot \iota_\pi : \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K),$$

where

$$\gamma_r^{-1} : \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r$$

is the quasi-isomorphism from Corollary 2.4, induces a K-linear functorial quasi-isomorphism (the *Hyodo–Kato quasi-isomorphism*) [Tsuji 1999, Theorem 4.4.8, Corollary 4.4.13]

$$\iota_{\mathrm{dR},\pi} : \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})\otimes_{W(k)} K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K). \tag{23}$$

We now describe the Beilinson–Hyodo–Kato morphism and provide a few examples. Let $S_n = \mathrm{Spec}(\mathbb{Z}/p^n)$ equipped with the trivial log-structure and let $S = \mathrm{Spf}(\mathbb{Z}_p)$ be the induced formal log-scheme. For any log-scheme $Y \to S_1$, let $D_\varphi((Y/S)_{\mathrm{cr}}, \mathscr{O}_{Y/S})$ denote the derived category of Frobenius $\mathscr{O}_{Y/S}$-modules and $D_\varphi^{\mathrm{pcr}}(Y/S)$ its thick subcategory of perfect F-crystals, i.e., those Frobenius modules that are perfect crystals [Beilinson 2013, 1.11]. We call a perfect F-crystal $(\mathscr{F}, \varphi)$ *nondegenerate* if the map $L\varphi^*(\mathscr{F}) \to \mathscr{F}$ is an isogeny. The corresponding derived category is denoted by $D_\varphi^{\mathrm{pcr}}(Y/S)^{\mathrm{nd}}$. It has a dg category structure [Beilinson 2013, 1.14] that we denote by $\mathscr{D}_\varphi^{\mathrm{pcr}}(Y/S)^{\mathrm{nd}}$. We will omit S if understood.

Suppose now that Y is a fine log-scheme that is affine. Assume also that there is a PD-thickening $P = \mathrm{Spf}\, R$ of Y that is formally smooth over S and such that R is a p-adically complete ring with no p-torsion. Let $f : Z \to Y$ be a log-smooth map of Cartier type with Z fine and proper over Y. Beilinson [2013, 1.11, 1.14] proves the following theorem.

Theorem 3.2. *The complex $\mathscr{F} := Rf_{\mathrm{cr}*}(\mathscr{O}_{Z/S})$ is a nondegenerate perfect F-crystal.*

Let $D_{\varphi,N}(K_0)$ denote the bounded derived category of (φ, N)-modules. By [Beilinson 2013, 1.15], it has a dg category structure that we will denote by $\mathscr{D}_{\varphi,N}(K_0)$. We call a (φ, N)-module *effective* if it contains a $W(k)$-lattice preserved by φ and N. Denote by $\mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \subset \mathscr{D}_{\varphi,N}(K_0)$ the bounded derived category of the abelian category of effective modules.

Let $f : Y \to k^0$ be a log-scheme. We think of k^0 as $W(k)_1^\times$. Then the map f is given by a k-structure on Y plus a section $l = f^*(\bar{p}) \in \Gamma(Y, M_Y)$ such that its image in $\Gamma(Y, \mathscr{O}_Y)$ equals 0. We will often write $f = f_l,\ l = l_f$.

Beilinson [2013, 1.15] proves the following theorem.

Theorem 3.3. (1) *There is a natural functor*

$$\varepsilon_f = \varepsilon_l : \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \to \mathscr{D}_\varphi^{\mathrm{pcr}}(Y)^{\mathrm{nd}} \otimes \mathbb{Q}. \tag{24}$$

(2) *ε_f is compatible with base change; i.e., for any $\theta : Y' \to Y$, one has a canonical identification $\varepsilon_{f\theta} \xrightarrow{\sim} L\theta_{\mathrm{cr}}^*\varepsilon_f$. For any $a \in k^*$, $m \in \mathbb{Z}_{>0}$, there is a canonical identification $\varepsilon_{al^m}(V, \varphi, N) \xrightarrow{\sim} \varepsilon_l(V, \varphi, mN)$.*

(3) *Suppose that Y is a local scheme with residue field k and nilpotent maximal ideal, $M_Y/\mathscr{O}_Y^* = \mathbb{Z}_{>0}$, and the map $f^* : M_{k^0}/k^* \to M_Y/\mathscr{O}_Y^*$ is injective. Then* (24) *is an equivalence of dg categories.*

In particular, we have an equivalence of dg categories

$$\varepsilon := \varepsilon_{\bar{p}} : \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \xrightarrow{\sim} \mathscr{D}_\varphi^{\mathrm{pcr}}(k^0)^{\mathrm{nd}} \otimes \mathbb{Q}$$

and a canonical identification $\varepsilon_f = Lf_{\mathrm{cr}}^*\varepsilon$.

On the level of sections, the functor (24) has a simple description [Beilinson 2013, 1.15.3]. Assume that $Y = \mathrm{Spec}(A/J)$, where A is a p-adic algebra and J is a PD-ideal in A, and that we have a PD-thickening $i : Y \hookrightarrow T = \mathrm{Spf}(A)$. Let $\lambda_{l,n}$ be the preimage of l under the map $\Gamma(T_n, M_{T_n}) \to i_*\Gamma(Y, M_Y)$. It is a trivial $(1+J_n)^\times$-torsor. Set

$$\lambda_A := \varprojlim_n \Gamma(T_n, \lambda_{l,n}).$$

It is a $(1+J)^\times$-torsor. Let $\tau_{A_\mathbb{Q}}$ be the *Fontaine–Hyodo–Kato torsor*, i.e., the $A_\mathbb{Q}$-torsor obtained from λ_A by the pushout by $(1+J)^\times \xrightarrow{\log} J \to A_\mathbb{Q}$. We call the $\mathbb{G}_a$-torsor Spec $A^\tau_\mathbb{Q}$ over Spec $A_\mathbb{Q}$ with sections $\tau_{A_\mathbb{Q}}$ the same name. Denote by N_τ the $A_\mathbb{Q}$-derivation of $A^\tau_\mathbb{Q}$ given by the action of the generator of $\mathrm{Lie}_{\mathbb{G}_a}$.

Let M be an (φ, N)-module. Integrating the action of the monodromy N_M, we get an action of the group $\mathbb{G}_a$ on M. Denote by $M^\tau_{A_\mathbb{Q}}$ the $\tau_{A_\mathbb{Q}}$-twist of $M_{A_\mathbb{Q}} := M \otimes_{K_0} A_\mathbb{Q}$. It can be represented as the module of maps $v : \tau_{A_\mathbb{Q}} \to M_{A_\mathbb{Q}}$ that are $A_\mathbb{Q}$-equivariant, i.e., such that $v(\tau + a) = \exp(aN)(v(\tau))$, where $\tau \in \tau_{A_\mathbb{Q}}$, $a \in A_\mathbb{Q}$. We can also write

$$M^\tau_{A_\mathbb{Q}} = (M \otimes_{K_0} A^\tau_\mathbb{Q})^{\mathbb{G}_a} = (M \otimes_{K_0} A^\tau_\mathbb{Q})^{N=0},$$

where $N := N_M \otimes 1 + 1 \otimes N_\tau$. Now, by definition,

$$\varepsilon_f(M)(Y, T) = M^\tau_{A_\mathbb{Q}}. \tag{25}$$

The algebra $A^\tau_\mathbb{Q}$ has a concrete description. Take the natural map $a : \tau_{A_\mathbb{Q}} \to A^\tau_\mathbb{Q}$ of $A_\mathbb{Q}$-torsors which maps $\tau \in \tau_{A_\mathbb{Q}}$ to a function $a(\tau) \in A^\tau_\mathbb{Q}$ whose value on any $\tau' \in \tau_{A_\mathbb{Q}}$ is $\tau - \tau' \in A_\mathbb{Q}$. This map is compatible with the logarithm $\log : (1+J)^\times \to A$. The algebra $A^\tau_{A_\mathbb{Q}}$ is freely generated over $A_\mathbb{Q}$ by $a(\tau)$ for any $\tau \in \tau_{A_\mathbb{Q}}$; the $A_\mathbb{Q}$-derivation N_τ is defined by $N_\tau(a(\tau)) = -1$. That is, for chosen $\tau \in \tau_{A_\mathbb{Q}}$, we can write

$$A^\tau_\mathbb{Q} = A_\mathbb{Q}[a(\tau)], \quad N_\tau(a(\tau)) = -1.$$

For every lifting φ_T of Frobenius to T, we have $\varphi_T^*\lambda_A = \lambda_A^p$. Hence φ_T extends canonically to a Frobenius φ_τ on $A^\tau_\mathbb{Q}$ in such a way that $N_\tau\varphi_\tau = p\varphi_\tau N_\tau$. The isomorphism (25) is compatible with Frobenius.

Example 3.4. As an example, consider the case when the pullback map

$$f^* : \mathbb{Q} = (M_{k^0}/k^*)^{\mathrm{gp}} \otimes \mathbb{Q} \xrightarrow{\sim} (\Gamma(Y, M_Y)/k^*)^{\mathrm{gp}} \otimes \mathbb{Q}$$

is an isomorphism. We have a surjection $v : (\Gamma(T, M_T)/k^*)^{\mathrm{gp}} \otimes \mathbb{Q} \to \mathbb{Q}$ with the kernel $\log : (1+J)^\times_\mathbb{Q} \xrightarrow{\sim} J_\mathbb{Q} = A_\mathbb{Q}$. We obtain an identification of $A_\mathbb{Q}$-torsors $\tau_{A_\mathbb{Q}} \simeq v^{-1}(1)$. Hence every noninvertible $t \in \Gamma(T, M_T)$ yields an element $t^{1/v(t)} \in v^{-1}(1)$ and a trivialization of $\tau_{A_\mathbb{Q}}$.

For a fixed element $t^{1/v(t)} \in v^{-1}(1)$, we can write

$$A^\tau_{A_\mathbb{Q}} = A_\mathbb{Q}[a(t^{1/v(t)})], \quad N_\tau(a(t^{1/v(t)})) = -1.$$

For an (φ, N)-module M, the twist $M^{\tau}_{A_{\mathbb{Q}}}$ can be trivialized:

$$\beta_t : M\otimes_{K_0} A_{\mathbb{Q}} \xrightarrow{\sim} M^{\tau}_{A_{\mathbb{Q}}} = (M\otimes_{K_0} A_{\mathbb{Q}}[a(t^{1/v(t)})])^{N=0}, \\ m \mapsto \exp(N_M(m)a(t^{1/v(t)})).$$

For a different choice $t_1^{1/v(t_1)} \in v^{-1}(1)$, the two trivializations β_t, β_{t_1} are related by the formula

$$\beta_{t_1} = \beta_t \exp(N_M(m)a(t_1, t)), \quad a(t_1, t) = a(t_1)/v(t_1) - a(t)/v(t).$$

Consider the map $f : V_1^{\times} \to k^0$. By Theorem 3.3, we have the equivalences of dg categories

$$\varepsilon : \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \xrightarrow{\sim} \mathscr{D}^{\mathrm{pcr}}_{\varphi}(k^0)^{\mathrm{nd}} \otimes \mathbb{Q}, \\ \varepsilon_f = Lf^*_{\mathrm{cr}}\varepsilon : \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \xrightarrow{\sim} \mathscr{D}^{\mathrm{pcr}}_{\varphi}(V_1^{\times})^{\mathrm{nd}} \otimes \mathbb{Q}.$$

Let $Z_1 \to V_1^{\times}$ be a log-smooth map of Cartier type with Z_1 fine and proper over V_1. By Theorem 3.2, $Rf_{\mathrm{cr}*}(\mathscr{O}_{Z_1/\mathbb{Z}_p})$ is a nondegenerate perfect F-crystal on $V_{1,\mathrm{cr}}$. Set

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1) := \varepsilon_f^{-1} Rf_{\mathrm{cr}*}(\mathscr{O}_{Z_1/\mathbb{Z}_p})_{\mathbb{Q}} \in \mathscr{D}_{\varphi,N}(K_0).$$

We will call it the *Beilinson–Hyodo–Kato complex* [Beilinson 2013, 1.16.1].

Example 3.5. To get familiar with the Beilinson–Hyodo–Kato complexes we will work out some examples.

(1) Let $g : X \to V^{\times}$ be a log-smooth log-scheme, proper, and of Cartier type. Adjunction yields a quasi-isomorphism

$$\varepsilon_f \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) = \varepsilon_f \varepsilon_f^{-1} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})_{\mathbb{Q}} \xrightarrow{\sim} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})_{\mathbb{Q}}. \tag{26}$$

Evaluating it on the PD-thickening $V_1^{\times} \hookrightarrow V^{\times}$ (here $A = V$, $J = pV$, $l = \bar{p}$, $\lambda_V = p(1+J)^{\times}$, and $\tau_K = p(1+J)^{\times} \times_{(1+J)^{\times}} K$), we get a map

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_K = \varepsilon_f \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)(V_1^{\times} \hookrightarrow V^{\times}) \xrightarrow{\sim} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})(V_1^{\times} \hookrightarrow V^{\times})_{\mathbb{Q}} \\ = \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/V^{\times})_{\mathbb{Q}} \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X/V^{\times})_{\mathbb{Q}} \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X_K).$$

We will call it the *Beilinson–Hyodo–Kato map* [Beilinson 2013, 1.16.3]

$$\iota^B_{\mathrm{dR}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_K). \tag{27}$$

Recall that

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_K = (\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) \otimes_{K_0} K[a(\tau)])^{N=0}, \quad \tau \in \tau_K.$$

This makes it clear that the Beilinson–Hyodo–Kato map is not only functorial for log-schemes over $V^{\times}$ but, by Theorem 3.3, it is also compatible with base change of $V^\times$. Moreover, if we use the canonical trivialization by p

$$\beta = \beta_p : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)_K \xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{HK}}(X)^\tau_K = (\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) \otimes_{K_0} K[a(p)])^{N=0},$$
$$x \mapsto \exp(N(x)a(p)),$$

we get that the composition (which we also call the Beilinson–Hyodo–Kato map and denote by ι^B_{dR})

$$\iota^B_{\mathrm{dR}} = \iota^B_{\mathrm{dR}}\beta : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_K)$$

is functorial and compatible with base change.

(2) Evaluating the map (26) on the PD-thickening $V_1^\times \hookrightarrow E$ associated to a uniformizer π (here $A = R$, $l = \bar{p}$), we get a map

$$\kappa_R : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X/R)_\mathbb{Q} \tag{28}$$

as the composition

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_\mathbb{Q}} = \varepsilon_f \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)(V_1^\times \hookrightarrow E) \xrightarrow{\sim} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})(V_1^\times \hookrightarrow E)_\mathbb{Q}$$
$$= \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_\mathbb{Q} \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X/R)_\mathbb{Q}.$$

We have

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_\mathbb{Q}} = (\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) \otimes_{K_0} R_\mathbb{Q}[a(\tau)])^{N=0}, \quad \tau \in \tau_{R_\mathbb{Q}}.$$

Since the map κ_R is compatible with the log-connection on R it is also compatible with the normalized monodromy operators. Specifically, if we define the monodromy on the left-hand side of (28) as

$$N : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_\mathbb{Q}} \to \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_\mathbb{Q}},$$
$$\sum_I m_{\tau_I} \otimes r_{\tau_I} a^{k_I}(\tau_I) \mapsto \sum_I (N_M(m_{\tau_I}) \otimes r_{\tau_I} a^{k_I}(\tau_I) + m_{\tau_I} \otimes N_R(r_{\tau_I}) a^{k_I}(\tau_I)),$$

the two operators will correspond under the map κ_R.

The exact immersion $i_\pi : V^\times \hookrightarrow E$ yields a commutative diagram

$$\begin{array}{ccc}
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{R}\Gamma_{\mathrm{cr}}(X/R)_\mathbb{Q} \\
\downarrow{\scriptstyle i^*_\pi} & & \downarrow{\scriptstyle i^*_\pi} \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_K & \xrightarrow{\sim} & \mathrm{R}\Gamma_{\mathrm{cr}}(X/V^\times)_\mathbb{Q}
\end{array}$$

If $p = u\pi^{e_K}$, $u \in V^\times$, we have $\lambda_R = \tilde{u} t_\pi^{e_K}(1+J)^\times$, where $\tilde{u} \in R$ is such that $\tilde{u}$ lifts u. Alternatively, $\lambda_R = [\bar{u}] t_\pi^{e_K}(1+J)^\times$. We have the associated trivialization

$$\beta_\pi : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) \otimes_{K_0} R_\mathbb{Q} \xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^\tau_{R_\mathbb{Q}} = (\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) \otimes_{K_0} R_\mathbb{Q}[a(\tau_\pi)])^{N=0},$$
$$x \mapsto \exp(N(x)a(\tau_\pi)),$$

where $\tau_\pi := [\bar{u}] t_\pi^{e_K}$.

(3) Consider the log-scheme k_1^0: the scheme $\mathrm{Spec}(k)$ with the log-structure induced by the exact closed immersion $i: k_1^0 \hookrightarrow V_1^\times$. We have the commutative diagram

$$\begin{array}{ccc} X_0 & \stackrel{i}{\hookrightarrow} & X_1 \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} \\ k_1^0 & \stackrel{i}{\hookrightarrow} & V_1^\times \\ & \searrow^{f_0} & \downarrow{\scriptstyle f} \\ & & k^0 \end{array}$$

The morphisms f, f_0 map $\bar{p}$ to $\bar{p}$. By log-smooth base change we have a canonical quasi-isomorphism $Li^* Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p}) \simeq Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})$. By Theorem 3.3 we have the equivalence of dg categories

$$\varepsilon_{f_0}: \mathscr{D}_{\varphi,N}(K_0)^{\mathrm{eff}} \xrightarrow{\sim} \mathscr{D}_{\varphi}^{\mathrm{pcr}}(k_1^0)^{\mathrm{nd}} \otimes \mathbb{Q}, \quad \varepsilon_{f_0} = Li^* \varepsilon_f.$$

This implies the natural quasi-isomorphisms

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{HK}}^{B}(X_1) = \varepsilon_f^{-1} Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})_{\mathbb{Q}} &\simeq \varepsilon_{f_0}^{-1} Li^* Rg_{\mathrm{cr}*}(\mathscr{O}_{X_1/\mathbb{Z}_p})_{\mathbb{Q}} \\ &\simeq \varepsilon_{f_0}^{-1} Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})_{\mathbb{Q}}. \end{aligned}$$

Hence, by adjunction,

$$\varepsilon_{f_0} \mathrm{R}\Gamma_{\mathrm{HK}}^{B}(X_1) = \varepsilon_{f_0} \varepsilon_{f_0}^{-1} Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})_{\mathbb{Q}} \simeq Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})_{\mathbb{Q}}.$$

We will evaluate both sides on the PD-thickening $k_1^0 \hookrightarrow W(k)^0$. Here we write the log-structure on $W(k)^0$ as associated to the map $\Gamma(V^\times, M_{V^\times}) \to k \to W(k)$, $a \mapsto \bar{a}$. We take $A = W(k)$, $l = p$, $J = pW(k)$, $\lambda_{W(k)} = \bar{p}(1 + pW(k))^\times$ and $\tau_{K_0} = \bar{p}(1 + pW(k))^\times \times_{(1+pW(k))^\times} K_0$. We get a quasi-isomorphism

$$\kappa: \mathrm{R}\Gamma_{\mathrm{HK}}^{B}(X_1)_{K_0}^{\tau} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}$$

as the composition

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{HK}}^{B}(X_1)_{K_0}^{\tau} &= \varepsilon_{f_0} \mathrm{R}\Gamma_{\mathrm{HK}}^{B}(X_1)(k_1^0 \hookrightarrow W(k)^0) \simeq Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})(k_1^0 \hookrightarrow W(k)^0)_{\mathbb{Q}} \\ &= \mathrm{R}\Gamma_{\mathrm{cr}}(X_0/W(k)^0)_{\mathbb{Q}} = \mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}. \end{aligned}$$

To compare the monodromy operators on both sides of the map κ, note that by Theorem 3.3, we have the canonical identification

$$Rg_{0\,\mathrm{cr}*}(\mathscr{O}_{X_0/\mathbb{Z}_p})_{\mathbb{Q}} \simeq \varepsilon_{f_0}(\mathrm{R}\Gamma_{\mathrm{HK}}^{B}(X_1), N) \simeq \varepsilon_{\bar{p}}(\mathrm{R}\Gamma_{\mathrm{HK}}^{B}(X_1), e_K N).$$

Hence, from the description of the Hyodo–Kato monodromy [1994, 3.6], it follows easily that the map κ pairs the operator N on $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0}$ defined by

$$N\Big(\sum_I m_{\tau_I}\otimes r_{\tau_I}a^{k_I}(\tau_I)\Big)=\sum_I\big(N_M(m_{\tau_I})\otimes r_{\tau_I}a^{k_I}(\tau_I)+m_{\tau_I}\otimes N_R(r_{\tau_I})a^{k_I}(\tau_I)\big),$$

with the normalized Hyodo–Kato monodromy on $\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}$.

Composing the map κ with the trivialization

$$\begin{aligned}\beta=\beta_p:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)\xrightarrow{\sim}\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0}=(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)[a(\bar{p})])^{N=0},\\ x\mapsto\exp(N(x)a(\bar{p})),\end{aligned}$$

we get a quasi-isomorphism between Beilinson–Hyodo–Kato complexes and the (classical) Hyodo–Kato complexes:

$$\kappa=\beta\kappa:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}. \tag{29}$$

The trivialization above is compatible with Frobenius and the normalized monodromy; hence so is the quasi-isomorphism (29). It is clearly functorial and, by Theorem 3.3, compatible with base change.

By functoriality (Theorem 3.3), the morphism of PD-thickenings (exact closed immersion) $i_0:(k^0_1\hookrightarrow W(k)^0)\hookrightarrow(V_1^{\times}\hookrightarrow R)$ yields the right square in the diagram

$$\begin{array}{ccccc}\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}} & \xrightarrow{\iota_\pi} & \mathrm{R}\Gamma_{\mathrm{cr}}(X_1/R)_{\mathbb{Q}} & \xrightarrow{i_0^*} & \mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}\\ \wr\uparrow\kappa & & \wr\uparrow\kappa_R & & \wr\uparrow\kappa\\ \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0} & \xrightarrow{\iota_\pi} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{R_{\mathbb{Q}}} & \xrightarrow{i_0^*} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0}\end{array} \tag{30}$$

In the left square, the bottom map ι_π is induced by the natural map $K_0\to R$ and by sending $a(\bar{p})\mapsto a(\tau_\pi)$. It is a (right) section to i_0^* and it (together with the vertical maps) commutes with Frobenius. By uniqueness of the top map ι_π this makes the left square commute in the classical derived category (of abelian groups).

It is easy to check that we have the commutative diagram

$$\begin{array}{ccccc}\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K_0} & \xrightarrow{\iota_\pi} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{R_{\mathbb{Q}}} & \xrightarrow{i_\pi^*} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)^{\tau}_{K}\\ \beta_p\uparrow\wr & & & & \beta_p\uparrow\wr\\ \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1) & & \xrightarrow{\quad\mathrm{can}\quad} & & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_1)_K\end{array}$$

and that the composition of maps on the top of it is equal to the map induced by the canonical map $K_0\to K$ and the map $\lambda_{W(k)^0}\to\lambda_V^{\times}$, $\bar{p}\to p$.

Combining the commutative diagrams in parts (2) and (3) of this example, we get the commutative diagram

$$\begin{array}{ccccc}
R\Gamma_{\mathrm{HK}}(X) & \xrightarrow{\iota_\pi} & R\Gamma_{\mathrm{cr}}(X_1/R)_{\mathbb{Q}} & \xrightarrow{i_\pi^*} & R\Gamma_{\mathrm{cr}}(X_1/V^{\times})_{\mathbb{Q}} \\
\wr\uparrow\kappa & & \wr\uparrow\kappa_R & & \wr\uparrow\iota_{\mathrm{dR}}^B \\
R\Gamma_{\mathrm{HK}}^B(X_1)_{K_0}^{\tau} & \xrightarrow{\iota_\pi} & R\Gamma_{\mathrm{HK}}^B(X_1)_{R_{\mathbb{Q}}}^{\tau} & \xrightarrow{i_\pi^*} & R\Gamma_{\mathrm{HK}}^B(X_1)_{K}^{\tau} \\
\beta_p\uparrow\wr & & & & \beta_p\uparrow\wr \\
R\Gamma_{\mathrm{HK}}^B(X_1) & & \xrightarrow{\quad\mathrm{can}\quad} & & R\Gamma_{\mathrm{HK}}^B(X_1)_K
\end{array}$$

Since the composition of the top maps is equal to the Hyodo–Kato map ι_{dR} and the bottom map is just the canonical map $R\Gamma_{\mathrm{HK}}(X_1) \to R\Gamma_{\mathrm{HK}}(X_1)_K$, we obtain that the Hyodo–Kato and the Beilinson–Hyodo–Kato maps are related by a natural quasi-isomorphism; i.e., the following diagram commutes:

$$\begin{array}{ccc}
R\Gamma_{\mathrm{HK}}(X) & \xrightarrow{\iota_{\mathrm{dR},\pi}} & R\Gamma_{\mathrm{dR}}(X_K) \\
\wr\uparrow\kappa & \nearrow_{\iota_{\mathrm{dR}}^B} & \\
R\Gamma_{\mathrm{HK}}^B(X_1) & &
\end{array} \tag{31}$$

The above examples can be generalized [Beilinson 2013, 1.16]. It turns out that the relative crystalline cohomology of all the base changes of the map f can be described using the Beilinson–Hyodo–Kato complexes [loc. cit., 1.16.2]. Namely, let $\theta : Y \to V_1^{\times}$ be an affine log-scheme and let T be a p-adic PD-thickening of Y, that is, $T = \mathrm{Spf}(A)$, $Y = \mathrm{Spec}(A/J)$. Denote by $f_Y : Z_{1Y} \to Y$ the θ-pullback of f. Beilinson [2013, 1.16.2] proved the following theorem.

Theorem 3.6. (1) *The A-complex $R\Gamma_{\mathrm{cr}}(Z_{1Y}/T, \mathscr{O}_{Z_{1Y/T}})$ is perfect, and one has*

$$R\Gamma_{\mathrm{cr}}(Z_{1Y}/T_n, \mathscr{O}_{Z_{1Y/T_n}}) = R\Gamma_{\mathrm{cr}}(Z_{1Y}/T, \mathscr{O}_{Z_{1Y/T}}) \otimes^L \mathbb{Z}/p^n.$$

(2) *There exists a canonical Beilinson–Hyodo–Kato quasi-isomorphism of $A_{\mathbb{Q}}$-complexes*:

$$\kappa_{A_{\mathbb{Q}}}^B : R\Gamma_{\mathrm{HK}}^B(Z_1)_{A_{\mathbb{Q}}}^{\tau} \xrightarrow{\sim} R\Gamma_{\mathrm{cr}}(Z_{1Y}/T, \mathscr{O}_{Z_{1Y/T}})_{\mathbb{Q}}.$$

If there is a Frobenius lifting φ_T, then $\kappa_{A_{\mathbb{Q}}}^B$ commutes with its action.

3B. *Log-syntomic cohomology.* We will study now (rational) log-syntomic cohomology. Let $(U, \bar{U})$ be log-smooth over $V^{\times}$. For $r \geq 0$, define the mod p^n, completed, and rational log-syntomic complexes

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_n &:= \mathrm{Cone}(\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n \xrightarrow{p^r-\varphi} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_n)[-1],\\ \mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r) &:= \mathrm{holim}_n\,\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_n,\\ \mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}} &:= \mathrm{Cone}(\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_{\mathbb{Q}} \xrightarrow{1-\varphi_r} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}})[-1]. \end{aligned} \tag{32}$$

Here the Frobenius φ is defined by the composition

$$\begin{aligned} \varphi:\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n \to \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_n \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})_1/W(k))_n&\\ \xrightarrow{\varphi} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})_1/W(k))_n \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_n& \end{aligned}$$

and $\varphi_r := \varphi/p^r$. The mapping fibers are taken in the ∞-derived category of abelian groups. The direct sums

$$\bigoplus_{r\geq 0}\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_n,\quad \bigoplus_{r\geq 0}\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r),\quad \bigoplus_{r\geq 0}\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}}$$

are graded E_∞ algebras over $\mathbb{Z}/p^n$, $\mathbb{Z}_p$, and $\mathbb{Q}_p$, respectively [Hinich and Schechtman 1987, Theorem 1.6]. The rational log-syntomic complexes are moreover graded commutative dg algebras over $\mathbb{Q}_p$ [Hinich and Schechtman 1987, Theorem 4.1; Groth 2010, Perspective 3.22; Lurie 2016]. An explicit definition of syntomic product structure can be found in [Tsuji 1999, Section 2.2].

We have $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_n \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)\otimes^L \mathbb{Z}/p^n$. There is a canonical quasi-isomorphism of graded E_∞ algebras

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_n \xrightarrow{\sim} \mathrm{Cone}(\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_n&\\ \xrightarrow{(p^r-\varphi,\mathrm{can})} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_n \oplus \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{O}/\mathscr{J}^{[r]})_n)[-1].& \end{aligned}$$

The completed and rational cases are similar.

Since, by Corollary 2.4, there is a quasi-isomorphism

$$\gamma_r^{-1}:\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r,$$

we have a very nice canonical description of rational log-syntomic cohomology:

$$\begin{aligned} &\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}}\\ &\qquad \xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\gamma_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}} \oplus \mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r)\big], \end{aligned}$$

where square brackets stand for mapping fiber.

Remark 3.7. In the above definition, one can replace the map $1-\varphi_r$ with any polynomial map $P\in 1+XK[X]$ to obtain the analog of Besser's finite polynomial cohomology. This was studied in [Besser et al. 2016].

For arithmetic pairs $(U,\bar{U})$ that are log-smooth over $V^{\times}$ and of Cartier type, this can be simplified further by using Hyodo–Kato complexes (see Proposition 3.8 below). To do that, consider the following sequence of maps of homotopy limits. Homotopy limits are taken in the ∞-derived category (to do that we define the maps ι_π by the zigzag from diagram (30)). We will describe the coherence data only if they are nonobvious:

$$\begin{aligned}
&\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}}\\
&\quad\xrightarrow{\sim}\Big[\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}}\xrightarrow{(1-\varphi_r,\gamma_r^{-1})}\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}}\oplus\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r\Big]\\
&\quad\xrightarrow{\sim}\left[\begin{array}{ccc}\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}} & \xrightarrow{(1-\varphi_r,i_\pi^*\gamma_r^{-1})} & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}\oplus\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r\\ \downarrow N & & \downarrow (N,0)\\ \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}\end{array}\right]\\
&\quad\xleftarrow{\iota_\pi}\left[\begin{array}{ccc}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} & \xrightarrow{(1-\varphi_r,\iota_{\mathrm{dR},\pi})} & \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\oplus\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r\\ \downarrow N & & \downarrow (N,0)\\ \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\end{array}\right].
\end{aligned}$$

The first map was described above. The second one is induced by the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})\to\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)\xrightarrow{N}\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R).$$

The third one is induced by the section $\iota_\pi:\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\to\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}$ (notice that $\iota_{\mathrm{dR},\pi}=\gamma_r^{-1}i_\pi^*\iota_\pi$). We will show below that the third map is a quasi-isomorphism.

Set $C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})\{r\})$ equal to the last homotopy limit in the above diagram.

Proposition 3.8. *Let $(U,\bar{U})$ be an arithmetic pair that is log-smooth over $V^{\times}$ and of Cartier type. Let $r\geq 0$. Then the above diagram defines a canonical quasi-isomorphism*:

$$\alpha_{\mathrm{syn},\pi}:\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}}\xrightarrow{\sim}C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})\{r\}).$$

Proof. We need to show that the map ι_π in the above diagram is a quasi-isomorphism. Define complexes $(r\geq -1)$

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R,r)&:=\mathrm{Cone}\big(\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}\xrightarrow{1-\varphi_r}\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}\big)[-1],\\
\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r)&:=\mathrm{Cone}\big(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\xrightarrow{1-\varphi_r}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\big)[-1].
\end{aligned}$$

It suffices to prove that the maps

$$\begin{aligned} i_0^* &: \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R,r) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r),\\ \iota_\pi &: \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R,r) \end{aligned} \tag{33}$$

are quasi-isomorphisms. Since $i_0^*\iota_\pi = \mathrm{Id}$, it suffices to show that the map i_0^* is a quasi-isomorphism. Base-changing to $W(\bar{k})$, we may assume that the residue field of V is algebraically closed. It suffices to show that, for $i \geq 0$ and $t \geq -1$, in the commutative diagram

$$\begin{array}{ccc} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} & \xrightarrow{p^t-\varphi} & H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \\ \uparrow{\scriptstyle i_0^*} & & \uparrow{\scriptstyle i_0^*} \\ H^i_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}} & \xrightarrow{p^t-\varphi} & H^i_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}} \end{array}$$

the vertical maps induce isomorphisms between the kernels and cokernels of the horizontal maps.

Since the $W(k)$-linear map ι_π commutes with φ and its R-linear extension is a quasi-isomorphism

$$\iota_\pi : R\otimes_{W(k)} \mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}},$$

it suffices to show that in the commutative diagram

$$\begin{array}{ccc} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} & \xrightarrow{p^t-\varphi} & H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \\ \uparrow{\scriptstyle i_0\otimes\mathrm{Id}} & & \uparrow{\scriptstyle i_0\otimes\mathrm{Id}} \\ R\otimes_{W(k)} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} & \xrightarrow{p^t-\varphi} & R\otimes_{W(k)} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \end{array}$$

the vertical maps induce isomorphisms between the kernels and cokernels of the horizontal maps. This will follow if we show that the map

$$I\otimes_{W(k)} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}} \xrightarrow{p^t-\varphi} I\otimes_{W(k)} H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}},$$

for $I\subset R$, where I is the kernel of the projection $i_0 : R_{\mathbb{Q}} \to K_0$, $t_l \mapsto 0$, is an isomorphism. We argue as in [Langer 1999, p. 210]. Let $M := H^i_{\mathrm{HK}}(U,\bar{U})/\mathrm{tor}$. It is a lattice in $H^i_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}$ that is stable under Frobenius. Consider the formal inverse $\psi := \sum_{n\geq 0}(p^{-t}\varphi)^n$ of $1-p^{-t}\varphi$. It suffices to show that, for $y\in I\otimes_{W(k)} M$, $\psi(y)\in I\otimes_{W(k)} M$. Fix l and let $T^{\{k\}} := t_l^k/\lfloor k/e_K\rfloor!$. We will show that, for any $m\in M$, we have $\psi(T^{\{k\}}\otimes m)\in I\otimes_{W(k)} M$ and the infinite series converges uniformly in k. We have

$$(p^{-t}\varphi)^n(T^{\{k\}}\otimes m) = \frac{\lfloor kp^n/e_K\rfloor!}{\lfloor k/e_K\rfloor!\,p^{tn}} T^{\{kp^n\}}\otimes m'$$

and $\mathrm{ord}_p(\lfloor kp^n/e_K\rfloor!/\lfloor k/e_K\rfloor!)\geq p^{n-1}$. Hence $\lfloor kp^n/e_K\rfloor!/(\lfloor k/e_K\rfloor!p^{tn})$ converges p-adically to zero, uniformly in k, as wanted. □

Remark 3.9. It was Langer [1999, p. 193] (see [Nekovář 1998, Lemma 2.13] in the good reduction case) who observed the fact that while, in general, the crystalline cohomology $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})$ behaves badly (it is "huge"), after taking "filtered Frobenius eigenspaces" we obtain syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}}$ that behaves well (it is "small"). In [Nekovář 2000, 3.5] this phenomenon is explained by relating syntomic cohomology to the complex $C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})\{r\})$.

Remark 3.10. The construction of the map $\alpha_{\mathrm{syn},\pi}$ depends on the choice of the uniformizer π, which makes the h-sheafification impossible. We will show now that there is a functorial and compatible-with-base-change quasi-isomorphism α'_{syn} between rational syntomic cohomology and certain complexes built from Hyodo–Kato cohomology and de Rham cohomology that h-sheafify well.

Set

$$\begin{aligned}\alpha'_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}}&\xrightarrow{\sim}[\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)\xrightarrow{\gamma_r^{-1}}\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r]\\&\xrightarrow{\beta}[\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r)^{N=0}\xrightarrow{\iota'_{\mathrm{dR}}}\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r].\end{aligned}$$

Here the two morphisms β and ι'_{dR} are defined as the following compositions

$$\begin{aligned}\beta:\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)&\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(U_0,\bar{U}_0,r)\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r)^{N=0},\\ \iota'_{\mathrm{dR}}:\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r)^{N=0}&\xleftarrow{\beta}\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)\xrightarrow{\gamma_r^{-1}}\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K),\end{aligned}$$

where $(\cdots)^{N=0}$ denotes the mapping fiber of the monodromy. The map β is a quasi-isomorphism because so is each of the intermediate maps. To see this, for the map $i_0^*:\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)\to\mathrm{R}\Gamma_{\mathrm{cr}}(U_0,\bar{U}_0,r)$, consider the factorization

$$F^m:\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)\xrightarrow{i_0^*}\mathrm{R}\Gamma_{\mathrm{cr}}(U_0,\bar{U}_0,r)\xrightarrow{\psi_m}\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)$$

of the m-th power of the Frobenius, where m is large enough. We also have $i_0^*\psi_m=F^m$. Because Frobenius is a quasi-isomorphism on $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},r)$ and $\mathrm{R}\Gamma_{\mathrm{cr}}(U_0,\bar{U}_0,r)$, both i_0^* and ψ_m are quasi-isomorphisms as well. The second morphism in the sequence defining β is a quasi-isomorphism by an argument similar to the one we used in the proof of Proposition 3.8.

Define the complex

$$C'_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})\{r\}):=[\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U},r)^{N=0}\xrightarrow{\iota'_{\mathrm{dR}}}\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r].$$

We have obtained a quasi-isomorphism

$$\alpha'_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}}\xrightarrow{\sim}C'_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})\{r\}).$$

It is clearly functorial but it is also easy to check that it is compatible with base change (of the base V).

Define the complex

$$C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\bar{U})\{r\}) := \big[\mathrm{R}\Gamma^B_{\mathrm{HK}}(U_1,\bar{U}_1,r)^{N=0} \xrightarrow{\iota^B_{\mathrm{dR}}} \mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r\big].$$

From the commutative diagram (31) we obtain the natural quasi-isomorphisms

$$\gamma : C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\bar{U})\{r\}) \xrightarrow{\sim} C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U,\bar{U})\{r\}),$$
$$\alpha^B_{\mathrm{syn},\pi} := \gamma^{-1}\alpha_{\mathrm{syn},\pi} : \mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}} \xrightarrow{\sim} C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\bar{U})\{r\}).$$

We will show now that log-syntomic cohomology satisfies finite Galois descent. Let $(U,\bar{U})$ be a fine log-scheme, log-smooth over $V^\times$, and of Cartier type. Let $r\geq 0$. Let K' be a finite Galois extension of K and let $G=\mathrm{Gal}(K'/K)$. Let $(T,\bar{T})=(U\times_V V',\bar{U}\times_V V')$, where V' is the ring of integers in K', be the base change of $(U,\bar{U})$ to (K',V'), and let $f:(T,\bar{T})\to(U,\bar{U})$ be the canonical projection. Take $R=R_V$, N, e, π associated to V. Similarly, we define $R':=R_{V'}$, N', e', π'. Write the map $\alpha^B_{\mathrm{syn},\pi}$ as

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}} & \xrightarrow[h]{\sim} & \big[\mathrm{R}\Gamma^{B,\tau}_{\mathrm{HK}}((U,\bar{U})_R,r)^{N=0} \xrightarrow{i^*_\pi} \mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r\big] \\
\wr\downarrow \alpha^B_{\mathrm{syn},\pi} & & \iota_\pi\beta \uparrow\wr \qquad\qquad\qquad \Vert \\
C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\bar{U})\{r\}) & \xrightarrow{\sim} & \big[\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\bar{U},r)^{N=0} \xrightarrow{\iota^B_{\mathrm{dR}}} \mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_K)/F^r\big]
\end{array}$$

Here we defined the map h as the composition

$$\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}} \xleftarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{HK}}(U_1,\bar{U}_1)^\tau_{R_{\mathbb{Q}}}. \tag{34}$$

From the construction of the Beilinson–Hyodo–Kato map

$$\iota^B_{\mathrm{dR}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(T_1,\bar{T}_1) \to \mathrm{R}\Gamma_{\mathrm{dR}}(T,\bar{T}_{K'}),$$

it follows that it is G-equivariant; hence the complex $C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(T,\bar{T})\{r\})$ is equipped with a natural G-action. We claim the map $\alpha^B_{\mathrm{syn},\pi'}$ induces a natural map

$$\tilde{\alpha}^B_{\mathrm{syn},\pi'} : \mathrm{R}\Gamma(G,\mathrm{R}\Gamma_{\mathrm{syn}}(T,\bar{T},r)_{\mathbb{Q}}) \to \mathrm{R}\Gamma(G,C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(T,\bar{T})\{r\})),$$
$$\tilde{\alpha}^B_{\mathrm{syn},\pi'} := (1/|G|)\sum_{g\in G}\alpha^B_{\mathrm{syn},g(\pi')}.$$

To see this it suffices to show that, for every $g\in G$, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{syn}}(T,\bar{T},r)_{\mathbb{Q}} & \xrightarrow{\alpha^B_{\mathrm{syn},\pi'}} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(T,\bar{T})\{r\}) \\
\downarrow g^* & & \downarrow g^* \\
\mathrm{R}\Gamma_{\mathrm{syn}}(T,\bar{T},r)_{\mathbb{Q}} & \xrightarrow{\alpha^B_{\mathrm{syn},g(\pi')}} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(T,\bar{T})\{r\})
\end{array}$$

We accomplish this by constructing natural morphisms

$$g^*: \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R'_{\pi'}) \to \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R'_{g(\pi')}),$$
$$g^*: \mathrm{R}\Gamma^B_{\mathrm{HK}}(T_1,\overline{T}_1)^{\tau}_{R'_{\pi'}} \to \mathrm{R}\Gamma^B_{\mathrm{HK}}(T_1,\overline{T}_1)^{\tau}_{R'_{g(\pi')}}$$

that are compatible with the maps in (34) that define h, the maps $\iota_?$ and $i^*_?$, and the trivialization β. We define the pullbacks g^* from a map $g: R'_{\pi'} \to R'_{g(\pi')}$ constructed by lifting the action of g from V'_1 to R' by setting $g(t'_{\pi'}) = t'_{g(\pi')}$ and taking the induced action of g on $W(k')$. This map is compatible with Frobenius and monodromy. The induced pullbacks g^* are clearly compatible with the map i^*_0 and the maps $\iota_?$, the maps $i^*_{\pi'}$, $i^*_{g(\pi')}$, and the trivialization β. From the construction of the Beilinson–Hyodo–Kato map, the pullbacks g^* are also compatible with the maps $\kappa_{R'_?}$, and hence with the map h, as wanted.

Proposition 3.11. (1) *The following diagram commutes in the* (*classical*) *derived category*:

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}} & \xrightarrow{f^*} & \mathrm{R}\Gamma(G,\mathrm{R}\Gamma_{\mathrm{syn}}(T,\overline{T},r)_{\mathbb{Q}}) \\
\downarrow{\scriptstyle \alpha^B_{\mathrm{syn},\pi}} & & \downarrow{\scriptstyle \tilde{\alpha}^B_{\mathrm{syn},\pi'}} \\
C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\overline{U})\{r\}) & \xrightarrow{f^*} & \mathrm{R}\Gamma(G, C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(T,\overline{T})\{r\}))
\end{array}$$

(2) *The natural map*

$$f^*: \mathrm{R}\Gamma_{\mathrm{syn}}(U,\overline{U},r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma(G,\mathrm{R}\Gamma_{\mathrm{syn}}(T,\overline{T},r)_{\mathbb{Q}})$$

is a quasi-isomorphism.

Proof. The second claim of the proposition follows from the first one and the fact that the Hyodo–Kato and de Rham cohomologies satisfy finite Galois decent.

Since everything in sight is functorial and satisfies finite unramified Galois descent, we may assume that the extension K'/K is totally ramified. First, we will construct a G-equivariant (for the trivial action of G on R) map

$$f^*: \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,r)^{N=0} \to \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R',r)^{N'=0}$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},r) & \xrightarrow{f^*} & \mathrm{R}\Gamma_{\mathrm{cr}}(T,\overline{T},r) \\
\downarrow\wr & & \downarrow\wr \\
\mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,r)^{N=0} & \xrightarrow{f^*} & \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R',r)^{N'=0} \\
\wr\uparrow{\scriptstyle \iota_\pi} & & \wr\uparrow{\scriptstyle \iota_{\pi'}} \\
\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U},r)^{N=0} & \xrightarrow[\sim]{f^*} & \mathrm{R}\Gamma_{\mathrm{HK}}(T,\overline{T},r)^{N'=0}
\end{array} \tag{35}$$

Remark 3.12. Note that the bottom map is an isomorphism because f^* acts trivially on the Hyodo–Kato complexes. The commutativity of the above diagram and the quasi-isomorphisms (33) will imply that a totally ramified Galois extension does not change the log-crystalline complexes $\mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U}, r)$ and $\mathrm{R}\Gamma_{\mathrm{cr}}((U, \bar{U})/R, r)^{N=0}$.

Let e_1 be the ramification index of V'/V. Set $v = (\pi')^{e_1}\pi^{-1}$, and choose an integer s such that $(\pi')^{p^s} \in pV'$. Set $T := t_\pi$, $T' := t_{\pi'}$ and define the morphism $a : R \to R'$ by $T \mapsto (T')^{e_1}[\bar{v}]^{-1}$. Since V_1' and V_1 are defined by $pR + T^e R$ and by $pR' + (T')^{e'} R'$, respectively, a induces a morphism $a_1 : V_1 \to V_1'$. We have $F^s a_1 = F^s f_1$, where F is the absolute Frobenius on $\mathrm{Spec}(V_1)$. Notice that in general $f_1 \neq a_1$ if $v[\bar{v}]^{-1} \not\equiv 1 \bmod pV'$. The morphism $\varphi_R^s a : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is compatible with $F^s f_1 : \mathrm{Spec}(V_1') \to \mathrm{Spec}(V_1)$ and it commutes with the operators N and $p^s N'$. We have the following commutative diagram:

$$\begin{array}{ccc}
(T, \bar{T})_1 & \xrightarrow{F^s f_1} & (U, \bar{U})_1 \\
\downarrow & & \downarrow \\
\mathrm{Spec}(V_1') & \xrightarrow{F^s a_1 = F^s f_1} & \mathrm{Spec}(V_1) \\
\downarrow & & \downarrow \\
\mathrm{Spec}(R') & \xrightarrow{\varphi_R^s a} & \mathrm{Spec}(R)
\end{array}$$

Hence we also have the commutative diagram of distinguished triangles

$$\begin{array}{ccccc}
\mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}((U, \bar{U})/R)_{\mathbb{Q}} & \xrightarrow{eN} & \mathrm{R}\Gamma_{\mathrm{cr}}((U, \bar{U})/R)_{\mathbb{Q}} \\
\downarrow{\scriptstyle f^* F^s} & & \downarrow{\scriptstyle f^* F^s} & & \downarrow{\scriptstyle p^s e_1 f^* F^s} \\
\mathrm{R}\Gamma_{\mathrm{cr}}(T, \bar{T})_{\mathbb{Q}} & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}((T, \bar{T})/R')_{\mathbb{Q}} & \xrightarrow{e'N'} & \mathrm{R}\Gamma_{\mathrm{cr}}((T, \bar{T})/R')_{\mathbb{Q}}
\end{array} \tag{36}$$

To see how this diagram arises, we may assume (by the usual Čech argument) that we have a fine affine log-scheme $X_n/V_n^\times$ that is log-smooth over $V_n^\times$. We can also assume that we have a lifting of $X_n \hookrightarrow Z_n$ over $\mathrm{Spec}(W_n(k)[T])$ (with the log-structure coming from T) and a lifting of Frobenius φ_Z on Z_n that is compatible with the Frobenius φ_R. Recall [Kato 1994, Lemma 4.2] that the horizontal distinguished triangles in the above diagram arise from an exact sequence of complexes of sheaves on $X_{n,\text{ét}}$

$$0 \to C_V'[-1] \xrightarrow{\wedge \,\mathrm{dlog}\, T} C_V \to C_V' \to 0, \tag{37}$$

where $C_V := R_n \otimes_{W_n(k)[T]} \Omega^\bullet_{Z_n/W_n(k)}$ and $C_V' := R_n \otimes_{W_n(k)[T]} \Omega^\bullet_{Z_n/W_n(k)[T]}$. Now consider the base change of $Z_n/W_n(k)[T]$ by the map $F^s a : \mathrm{Spec}(W_n(k)[T']) \to \mathrm{Spec}(W_n(k)[T])$ and the related complexes (37). We get a commutative diagram of complexes of sheaves on $X_{n,\text{ét}}$ (note that $X_{V',n,\text{ét}} = X_{n,\text{ét}}$)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_{V'}[-1] & \xrightarrow{\wedge \operatorname{dlog} T'} & C_{V'} & \longrightarrow & C'_{V'} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle p^s e_1 a^* \varphi_Z^s} & & \uparrow{\scriptstyle a^*\varphi_Z^s} & & \uparrow{\scriptstyle a^*\varphi_Z^s} & & \\ 0 & \longrightarrow & C'_{V}[-1] & \xrightarrow{\wedge \operatorname{dlog} T} & C_{V} & \longrightarrow & C'_{V} & \longrightarrow & 0 \end{array}$$

Hence diagram (36) follows.

Combining diagram (36) with Frobenius, we obtain the commutative diagram

$$\begin{array}{ccccc} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},r) & \xleftarrow{F^s} & \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},r)f^*F^s & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}(T,\overline{T},r) \\ \downarrow\wr & & \downarrow\wr & & \wr\downarrow \\ \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,r)^{N=0} & \xleftarrow{(F^s,p^sF^s)} & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,r)^{N=0} & \xrightarrow{(a^*F^s,p^sa^*F^s)} & \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R',r)^{N'=0} \\ \downarrow{\scriptstyle i_0^*} & & \wr\downarrow{\scriptstyle i_0^*} & & \wr\downarrow{\scriptstyle i_0^*} \\ \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U},r)^{N=0} & \underset{\sim}{\xleftarrow{(F^s,p^sF^s)}} & \mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U},r)^{N=0} & \underset{\sim}{\xrightarrow{(F^s,p^sF^s)}} & \mathrm{R}\Gamma_{\mathrm{HK}}(T,\overline{T},r)^{N'=0} \end{array}$$

It follows that all the maps in the above diagram are quasi-isomorphisms. We define the map

$$f^* : \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,r)^{N=0} \to \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R',r)^{N'=0}$$

by the middle row. Since, for any $g \in G$, we have $v_{g(\pi')} = g(v_{\pi'})$, the map f^* is G-equivariant. In the (classical) derived category, this definition is independent of the constant s we have chosen. Since i_0^* is a quasi-isomorphism and $i_0^*\iota_{?*} = \mathrm{Id}$, the diagram (35) commutes as well, as wanted.

We define the map

$$f^* : \mathrm{R}\Gamma_{\mathrm{HK}}^{B,\tau}((U,\overline{U})/R,r)^{N=0} \to \mathrm{R}\Gamma_{\mathrm{HK}}^{B,\tau}((T,\overline{T})/R',r)^{N'=0} \tag{38}$$

in an analogous way. By the above diagram and by the compatibility of the Beilinson–Hyodo–Kato constructions with base change and with Frobenius, the two pullback maps f^* are compatible via the morphism h, i.e., the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},r) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R,r)^{N=0} & \xleftarrow{\kappa_R} & \mathrm{R}\Gamma_{\mathrm{HK}}^{B,\tau}((U,\overline{U})/R,r)^{N=0} \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{R}\Gamma_{\mathrm{cr}}(T,\overline{T},r) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}((T,\overline{T})/R',r)^{N'=0} & \xleftarrow{\kappa_{R'}} & \mathrm{R}\Gamma_{\mathrm{HK}}^{B,\tau}((T,\overline{T})/R',r)^{N'=0} \end{array}$$

From the analog of diagram (35) for the Beilinson–Hyodo–Kato complexes and by the universal nature of the trivialization at $\bar{p}$, we obtain that the pullback map f^* is compatible with the maps $\beta\iota_?$. It remains to show that we have a commutative diagram

$$\begin{array}{ccc} R\Gamma^B_{\rm HK}(U,\overline{U},r)^{N=0} & \xrightarrow[\sim]{f^*} & R\Gamma^B_{\rm HK}(T,\overline{T},r)^{N'=0} \\ \downarrow{\scriptstyle \iota^B_{\rm dR}} & & \downarrow{\scriptstyle \iota^B_{\rm dR}} \\ R\Gamma_{\rm dR}(U,\overline{U}_K)/F^r & \xrightarrow{f^*} & R\Gamma_{\rm dR}(T,\overline{T}_{K'})/F^r \end{array}$$

But this follows since the Beilinson–Hyodo–Kato map is compatible with base change. □

3C. ***Arithmetic syntomic cohomology.*** We are now ready to introduce and study arithmetic syntomic cohomology, i.e., syntomic cohomology over K. Let $\mathscr{J}^{[r]}_{\rm cr}$, $\mathscr{A}_{\rm cr}$, and $\mathscr{S}(r)$ for $r \geq 0$ be the h-sheafifications on $\mathscr{V}ar_K$ of the presheaves sending $(U,\overline{U}) \in \mathscr{P}^{\rm ss}_K$ to $R\Gamma_{\rm cr}(U,\overline{U},J^{[r]})$, $R\Gamma_{\rm cr}(U,\overline{U})$, and $R\Gamma_{\rm syn}(U,\overline{U},r)$, respectively. Let $\mathscr{J}^{[r]}_{{\rm cr},n}$, $\mathscr{A}_{{\rm cr},n}$, and $\mathscr{S}_n(r)$ denote the h-sheafifications of the mod-p^n versions of the respective presheaves. We have

$$\mathscr{S}_n(r) \simeq \mathrm{Cone}\big(\mathscr{J}^{[r]}_{{\rm cr},n} \xrightarrow{p^r-\varphi} \mathscr{A}_{{\rm cr},n}\big)[-1], \quad \mathscr{S}(r) \simeq \mathrm{Cone}\big(\mathscr{J}^{[r]}_{\rm cr} \xrightarrow{p^r-\varphi} \mathscr{A}_{\rm cr}\big)[-1].$$

For $r \geq 0$, define $\mathscr{S}(r)_{\mathbb{Q}}$ as the h-sheafification of the presheaf sending ss-pairs $(U,\overline{U})$ to $R\Gamma_{\rm syn}(U,\overline{U},r)_{\mathbb{Q}}$. We have

$$\mathscr{S}(r)_{\mathbb{Q}} \simeq \mathrm{Cone}(\mathscr{J}^{[r]}_{{\rm cr},\mathbb{Q}} \xrightarrow{1-\varphi_r} \mathscr{A}_{{\rm cr},\mathbb{Q}})[-1].$$

For $X \in \mathscr{V}ar_K$, set

$$R\Gamma_{\rm syn}(X_h,r)_n = R\Gamma(X_h,\mathscr{S}_n(r)), \quad R\Gamma_{\rm syn}(X_h,r) := R\Gamma(X_h,\mathscr{S}(r)_{\mathbb{Q}}).$$

We have

$$R\Gamma_{\rm syn}(X_h,r)_n \simeq \mathrm{Cone}(R\Gamma(X_h,\mathscr{J}^{[r]}_{{\rm cr},n}) \xrightarrow{p^r-\varphi} R\Gamma(X_h,\mathscr{A}_{{\rm cr},n}))[-1],$$
$$R\Gamma_{\rm syn}(X_h,r) \simeq \mathrm{Cone}(R\Gamma(X_h,\mathscr{J}^{[r]}_{{\rm cr},\mathbb{Q}}) \xrightarrow{1-\varphi_r} R\Gamma(X_h,\mathscr{A}_{{\rm cr},\mathbb{Q}}))[-1].$$

We will often write $R\Gamma_{\rm cr}(X_h)$ for $R\Gamma(X_h,\mathscr{A}_{\rm cr})$ if this does not cause confusion.

Let $\mathscr{A}_{\rm HK}$ be the h-sheafification of the presheaf $(U,\overline{U}) \mapsto R\Gamma_{\rm HK}(U,\overline{U})_{\mathbb{Q}}$ on $\mathscr{P}^{\rm ss}_K$; this is an h-sheaf of E_∞ K_0-algebras on $\mathscr{V}ar_K$ equipped with a φ-action and a derivation N such that $N\varphi = p\varphi N$. For $X \in \mathscr{V}ar_K$, set $R\Gamma_{\rm HK}(X_h) := R\Gamma(X_h,\mathscr{A}_{\rm HK})$. Similarly, we define h-sheaves $\mathscr{A}^B_{\rm HK}$ and the complexes $R\Gamma^B_{\rm HK}(X_h) := R\Gamma(X_h,\mathscr{A}^B_{\rm HK})$. The maps $\kappa : R\Gamma^B_{\rm HK}(U_1,\overline{U}_1) \to R\Gamma_{\rm HK}(U,\overline{U})_{\mathbb{Q}}$ h-sheafify and we obtain functorial quasi-isomorphisms

$$\kappa : \mathscr{A}^B_{\rm HK} \xrightarrow{\sim} \mathscr{A}_{\rm HK}, \quad \kappa : R\Gamma^B_{\rm HK}(X_h) \xrightarrow{\sim} R\Gamma_{\rm HK}(X_h).$$

Remark 3.13. The complexes $\mathscr{J}^{[r]}_{{\rm cr},n}$ and $\mathscr{S}_n(r)$ (and their completions) have a concrete description. For the complexes $\mathscr{J}^{[r]}_{{\rm cr},n}$, we can represent the presheaves $(U,\overline{U}) \mapsto R\Gamma_{\rm cr}(U,\overline{U},\mathscr{J}^{[r]}_n)$ by Godement resolutions (on the crystalline site), sheafify them for the h-topology on $\mathscr{P}_K^{\mathrm{ss}}$, and then move them to $\mathscr{V}ar_K$. For the complexes $\mathscr{S}_n(r)$, the maps $p^r - \varphi$ can be lifted to the Godement resolutions and their mapping fiber (defining $\mathscr{S}_n(r)(U, \bar{U})$) can be computed in the abelian category of complexes of abelian groups. To get $\mathscr{S}_n(r)$, we h-sheafify on $\mathscr{P}_K^{\mathrm{ss}}$ and pass to $\mathscr{V}ar$.

Let, for a moment, K be any field of characteristic zero. Consider the presheaf $(U, \bar{U}) \mapsto \mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) := \mathrm{R}\Gamma(\bar{U}, \Omega^{\bullet}_{(U,\bar{U})})$ of filtered dg K-algebras on $\mathscr{P}_K^{\mathrm{nc}}$. Let $\mathscr{A}_{\mathrm{dR}}$ be its h-sheafification. It is a sheaf of filtered K-algebras on $\mathscr{V}ar_K$. For $X \in \mathscr{V}ar_K$, we have Deligne's de Rham complex of X equipped with Deligne's Hodge filtration: $\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) := \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{dR}})$. Beilinson proves the following comparison statement.

Proposition 3.14 [Beilinson 2012, 2.4]. (1) *For $(U, \bar{U}) \in \mathscr{P}_K^{\mathrm{nc}}$, the canonical map $\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U_h)$ is a filtered quasi-isomorphism.*

(2) *The cohomology groups $H^i_{\mathrm{dR}}(X_h) := H^i\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$ are K-vector spaces of dimension equal to the rank of $H^i(X_{\bar{K},\text{ét}}, \mathbb{Q}_p)$.*

Corollary 3.15. *For a geometric pair $(U, \bar{U})$ over K that is saturated and log-smooth, the canonical map*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U_h)$$

is a filtered quasi-isomorphism.

Proof. Recall [Nizioł 2006, Theorem 5.10] that there is a log-blow-up $(U, \bar{T}) \to (U, \bar{U})$ that resolves singularities of $(U, \bar{U})$, i.e., such that $(U, \bar{T}) \in \mathscr{P}_K^{\mathrm{nc}}$. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{T}) & \xrightarrow{\sim} & \mathrm{R}\Gamma_{\mathrm{dR}}(U_h) \\
\wr\uparrow & \nearrow & \\
\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) & &
\end{array}$$

The vertical map is a filtered quasi-isomorphism; the horizontal map is a filtered quasi-isomorphism by the above proposition. Our corollary follows. □

Remark 3.16. Another proof of the above result (and a mild generalization) that does not use resolution of singularities can be found in [Beilinson 2013, 1.19] (where it is attributed to A. Ogus).

Return now to our p-adic field K.

Remark 3.17. By construction, we know the complexes $\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$, $\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)$, $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)$, $\mathrm{R}\Gamma(X_h, \mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}})$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ satisfy h-descent. In particular, since the h-topology is finer than the étale topology, they satisfy Galois descent for finite extensions. Hence, for any finite Galois extension K_1/K, the natural maps

$$\mathrm{R}\Gamma^*_?(X_h) \xrightarrow{\sim} \mathrm{R}\Gamma(G, \mathrm{R}\Gamma^*_?(X_{K_1,h})), \quad ? = \mathrm{cr}, \mathrm{syn}, \mathrm{HK}, \mathrm{dR}, \ * = B, \varnothing,$$

where $G = \mathrm{Gal}(K_1/K)$, are (filtered) quasi-isomorphisms. Since G is finite, it follows that the natural maps

$$\mathrm{R}\Gamma^*_{\mathrm{HK}}(X_h)\otimes_{K_0} K_{1,0} \xrightarrow{\sim} \mathrm{R}\Gamma^*_{\mathrm{HK}}(X_{K_1,h}), \quad \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_K K_1 \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_{K_1,h})$$

are (filtered) quasi-isomorphisms as well.

Recall from [Beilinson 2013, 2.5] and Proposition 3.21, that for a fine, log-scheme X, log-smooth over $V^\times$, and of Cartier type we have a quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{cr}}(X_{\overline{V}}, \mathscr{J}^{[r]}_{X_{\overline{V}}/W(k)})_{\mathbb{Q}} \simeq \mathrm{R}\Gamma(X_{\bar{K},h}, \mathscr{J}^{[r]}_{\mathrm{cr}})_{\mathbb{Q}}.$$

We can descend this result to K but on the level of rational log-syntomic cohomology; the key observation being that the field extensions introduced by the alterations are harmless since, by Proposition 3.11, log-syntomic cohomology satisfies finite Galois descent. Along the way we will get an analogous comparison quasi-isomorphism for the Hyodo–Kato cohomology.

Proposition 3.18. *For any arithmetic pair $(U, \overline{U})$ that is fine, log-smooth over $V^\times$, and of Cartier type, and $r \geq 0$, the canonical maps*

$$\mathrm{R}\Gamma^*_{\mathrm{HK}}(U, \overline{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma^*_{\mathrm{HK}}(U_h), \quad \mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{syn}}(U_h, r)$$

are quasi-isomorphisms.

Proof. It suffices to show that for any h-hypercovering $(U_\bullet, \overline{U}_\bullet) \to (U, \overline{U})$ by pairs from $\mathscr{P}^{\log}_K$, the natural maps

$$\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{HK}}(U_\bullet, \overline{U}_\bullet)_{\mathbb{Q}}, \quad \mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{syn}}(U_\bullet, \overline{U}_\bullet, r)_{\mathbb{Q}}$$

are (modulo taking a refinement of $(U_\bullet, \overline{U}_\bullet)$) quasi-isomorphisms. For the second map, since we have a canonical quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Cone}\big(\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, \mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}}\big)[-1],$$

it suffices to show that, up to a refinement of the hypercovering, we have quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, \mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet, \overline{U}_\bullet, \mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}},$$
$$\mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet, \overline{U}_\bullet, r)_{\mathbb{Q}}.$$

For the first of these maps, by Corollary 2.4 this amounts to showing that the following map is a quasi-isomorphism:

$$\mathrm{R}\Gamma(\overline{U}_K, \Omega^\bullet_{(U,\overline{U}_K)})/F^r \xrightarrow{\sim} \mathrm{R}\Gamma(\overline{U}_{\bullet,K}, \Omega^\bullet_{(U_\bullet,\overline{U}_{\bullet,K})})/F^r.$$

But, by Corollary 3.15 this map is quasi-isomorphic to the map

$$\mathrm{R}\Gamma_{\mathrm{dR}}(U_h)/F^r \to \mathrm{R}\Gamma_{\mathrm{dR}}(U_{\bullet,h})/F^r,$$

which is clearly a quasi-isomorphism.

Hence it suffices to show that, up to a refinement of the hypercovering, we have quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(U_\bullet,\overline{U}_\bullet)_{\mathbb{Q}}, \quad \mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet,\overline{U}_\bullet,r)_{\mathbb{Q}}.$$

Fix $t \geq 0$. To show that $H^t\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},r)_{\mathbb{Q}} \xrightarrow{\sim} H^t\mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet,\overline{U}_\bullet,r)_{\mathbb{Q}}$ is a quasi-isomorphism, we will often work with $(t+1)$-truncated h-hypercovers. This is because $\tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{cr}}(U_\bullet,\overline{U}_\bullet,r)\simeq\tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\overline{U}_\bullet)_{\leq t+1},r)$, where $(U_\bullet,\overline{U}_\bullet)_{\leq t+1}$ denotes the $(t+1)$-truncation. Assume first that we have an h-hypercovering $(U_\bullet,\overline{U}_\bullet)\to(U,\overline{U})$ of arithmetic pairs over K, where each pair $(U_i,\overline{U}_i)$, $i\leq t+1$, is log-smooth over $V^\times$ and of Cartier type. We claim that then already the maps

$$\begin{aligned}\tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}} &\xrightarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{HK}}((U_\bullet,\overline{U}_\bullet)_{\leq t+1})_{\mathbb{Q}},\\ \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U})_{\mathbb{Q}} &\xrightarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\overline{U}_\bullet)_{\leq t+1})_{\mathbb{Q}}\end{aligned}\tag{39}$$

are quasi-isomorphisms. To see the second quasi-isomorphism, consider the following commutative diagram of distinguished triangles ($R=R_V$):

$$\begin{array}{ccccc}\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U}) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R) & \xrightarrow{N} & \mathrm{R}\Gamma_{\mathrm{cr}}((U,\overline{U})/R)\\ \downarrow & & \downarrow & & \downarrow\\ \mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\overline{U}_\bullet)_{\leq t+1}) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\overline{U}_\bullet)_{\leq t+1}/R) & \xrightarrow{N} & \mathrm{R}\Gamma_{\mathrm{cr}}((U_\bullet,\overline{U}_\bullet)_{\leq t+1}/R)\end{array}$$

It suffices to show that the two right vertical arrows are rational quasi-isomorphisms in degrees less than or equal to t. But we have the R-linear quasi-isomorphisms

$$\iota: R\otimes_{W(k)}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma((U,\overline{U})/R)_{\mathbb{Q}},$$

$$\iota: R\otimes_{W(k)}\mathrm{R}\Gamma_{\mathrm{HK}}((U_\bullet,\overline{U}_\bullet)_{\leq t+1})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma((U_\bullet,\overline{U}_\bullet)_{\leq t+1}/R)_{\mathbb{Q}}.$$

Hence to show both quasi-isomorphisms (39), it suffices to show that the map

$$\tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}} \to \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{HK}}((U_\bullet,\overline{U}_\bullet)_{\leq t+1})_{\mathbb{Q}}$$

is a quasi-isomorphism.

Tensoring over K_0 with K and using the Hyodo–Kato quasi-isomorphism (23), we reduce to showing that the map

$$\tau_{\leq t}\mathrm{R}\Gamma(\overline{U}_K,\Omega^\bullet_{(U,\overline{U}_K)}) \to \tau_{\leq t}\mathrm{R}\Gamma(\overline{U}_{\bullet K,\leq t+1},\Omega^\bullet_{(U_\bullet,\overline{U}_{\bullet,K})_{\leq t+1}})$$

is a quasi-isomorphism, and this we have done above.

To treat the general case, set $X=(U,\overline{U})$, $Y_\bullet=(U_\bullet,\overline{U}_\bullet)$. We will do a base change to reduce to the case discussed above. We may assume that all the fields $K_{n,i}$, $K_{U_n}\simeq\prod K_{n,i}$ are Galois over K. Choose a finite Galois extension $(V',K')/(V,K)$ for K' Galois over all the fields $K_{n,i}$, $n\leq t+1$. Write $N_X(X_{V'})$ for the "Čech nerve" of $X_{V'}/X$. The term $N_X(X_{V'})_n$ is defined as the $(n+1)$-fold fiber product of $X_{V'}$ over X: $N_X(X_{V'})_n=(U\times_K K'^{,n+1},(\overline{U}\times_V V'^{,n+1})^{\mathrm{norm}})$, where $V'^{,n+1}$, $K'^{,n+1}$ are defined as the $(n+1)$-fold product of V' over V and of K' over K, respectively. Normalization is taken with respect to the open regular subscheme $U \times_K K'^{,n+1}$. Note that $N_X(X_{V'})_n \simeq (U \times_K K' \times G^n, \bar{U} \times_V V' \times G^n)$, where $G = \mathrm{Gal}(K'/K)$. Hence it is a log-smooth scheme over $V'^{,\times}$ of Cartier type. The augmentation $N_X(X_{V'}) \to X$ is an h-hypercovering.

Consider the bisimplicial scheme $Y_\bullet \times_X N_X(X_{V'})_\bullet$,

$$\begin{aligned}(Y_\bullet \times_X N_X(X_{V'})_\bullet)_{n,m} &:= Y_n \times_X N_X(X_{V'})_m \\ &\simeq \big(U_n \times_U U \times_K K'^{,m+1}, (\bar{U}_n \times_{\bar{U}} (\bar{U} \times_V V'^{,m+1})^{\mathrm{norm}})^{\mathrm{norm}}\big) \\ &\simeq \coprod_i \big(U_n \times_{K_{n,i}} K_{n,i} \times_K K'^{,m+1}, \bar{U}_n \times_{V_{n,i}} (V_{n,i} \times_V V'^{,m+1})^{\mathrm{norm}}\big).\end{aligned}$$

Hence $(Y_\bullet \times_X N_X(X_{V'})_\bullet)_{n,m} \in \mathscr{P}_K^{\log}$. For $n, m \le t+1$, we have

$$(Y_\bullet \times_X N_X(X_{V'})_\bullet)_{n,m} \simeq \coprod_i (U_n \times_{K_{n,i}} K' \times G_{n,i} \times G^m, \bar{U}_n \times_{V_{n,i}} V' \times G_{n,i} \times G^m),$$

where $G_{n,i} = \mathrm{Gal}(K_{n,i}/K)$. It is a log-scheme log-smooth over $V'^{,\times}$ of Cartier type.

Consider now its diagonal $Y_\bullet \times_X N_X(X_{V'}) := \Delta(Y_\bullet \times_X N_X(X_{V'})_\bullet)$. It is an h-hypercovering of X refining $Y_\bullet$ such that $(Y_\bullet \times_X N_X(X_{V'}))_n$ is log-smooth over $V'^{,\times}$, of Cartier type, for $n \le t+1$. It suffices to show that the compositions

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{HK}}(Y_\bullet)_{\mathbb{Q}} &\xrightarrow{\mathrm{pr}_1^*} \mathrm{R}\Gamma_{\mathrm{HK}}(Y_\bullet \times_X N_X(X_{V'}))_{\mathbb{Q}}, \\ \mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(Y_\bullet,r)_{\mathbb{Q}} &\xrightarrow{\mathrm{pr}_1^*} \mathrm{R}\Gamma_{\mathrm{cr}}(Y_\bullet \times_X N_X(X_{V'}),r)_{\mathbb{Q}}\end{aligned} \tag{40}$$

are quasi-isomorphisms in degrees less than or equal to t. Using the commutative diagram of bisimplicial schemes

$$\begin{array}{ccccc} Y_\bullet \times_X N_X(X_{V'}) & \xrightarrow{\Delta} & Y_\bullet \times_X N_X(X_{V'})_\bullet & \xrightarrow{\mathrm{pr}_1} & Y_\bullet \\ & & \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow \\ & & N_X(X_{V'}) & \xrightarrow{f} & X \end{array}$$

we can write the second composition as

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}} \xrightarrow{f^*} \mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'}),r)_{\mathbb{Q}} &\xrightarrow{\mathrm{pr}_2^*} \mathrm{R}\Gamma_{\mathrm{cr}}(Y_\bullet \times_X N_X(X_{V'})_\bullet,r)_{\mathbb{Q}} \\ &\xrightarrow{\Delta^*} \mathrm{R}\Gamma_{\mathrm{cr}}(Y_\bullet \times_X N_X(X_{V'}),r)_{\mathbb{Q}}.\end{aligned}$$

We claim that all of these maps are quasi-isomorphisms in degrees less than or equal to t. The map Δ^* is a quasi-isomorphism (in all degrees) by [Friedlander 1982, Proposition 2.5]. For the second map, fix $n \le t+1$ and consider the induced map $\mathrm{pr}_2 : (Y_\bullet \times_X N_X(X_{V'})_\bullet)_{\bullet,n} \to N_X(X_{V'})_n$. It is an h-hypercovering whose $(t+1)$-truncation is built from log-schemes, log-smooth over (V', K'), of Cartier type. It suffices to show that the induced map

$$\tau_{\le t}\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_n,r)_{\mathbb{Q}} \xrightarrow{\mathrm{pr}_2^*} \tau_{\le t}\mathrm{R}\Gamma_{\mathrm{cr}}((Y_\bullet \times_X N_X(X_{V'}))_{\bullet,n},r)_{\mathbb{Q}}$$

is a quasi-isomorphism. Since all maps are defined over K', this follows from the case considered at the beginning of the proof.

To prove that $f^*:\mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}}\to \mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'}),r)_{\mathbb{Q}}$ is a quasi-isomorphism, consider first the case when the extension V'/V is unramified. Then $\mathrm{R}\Gamma_{\mathrm{cr}}(X_{V'})\simeq \mathrm{R}\Gamma_{\mathrm{cr}}(X)\otimes_{W(k)}W(k')$ and the map f^* is a quasi-isomorphism by finite étale descent for crystalline cohomology.

Assume now that the extension V'/V is totally ramified and let π and π' be uniformizers of V and V', respectively. Consider the target of f^* as a double complex. To show that f^* is a quasi-isomorphism, it suffices to show that, for each $s\geq 0$, the sequence

$$\begin{aligned}0\to H^s\mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}}&\xrightarrow{f^*}H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_0,r)_{\mathbb{Q}}\\ &\xrightarrow{d_0^*}H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_1,r)_{\mathbb{Q}}\xrightarrow{d_1^*}H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_2,r)_{\mathbb{Q}}\to\cdots\end{aligned}$$

is exact. Embed it into the diagram

$$\begin{array}{ccccccc}0\longrightarrow H^s\mathrm{R}\Gamma_{\mathrm{cr}}(X,r)_{\mathbb{Q}} & \xrightarrow{f^*} & H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_0,r)_{\mathbb{Q}} & \xrightarrow{d_0^*} & H^s\mathrm{R}\Gamma_{\mathrm{cr}}(N_X(X_{V'})_1,r)_{\mathbb{Q}}\longrightarrow \\ \downarrow{\scriptstyle\alpha^B_{\mathrm{syn},\pi}}\wr & & \downarrow{\scriptstyle\tilde{\alpha}^B_{\mathrm{syn},\pi'}}\wr & & \downarrow{\scriptstyle\tilde{\alpha}^B_{\mathrm{syn},\pi'}}\wr \\ 0\longrightarrow H^s\mathrm{R}\Gamma^B_{\mathrm{HK}}(X,r)^{N=0}_{\mathbb{Q}} & \xrightarrow{f^*} & H^s\mathrm{R}\Gamma^B_{\mathrm{HK}}(N_X(X_{V'})_0,r)^{N'=0}_{\mathbb{Q}} & \xrightarrow{d_0^*} & H^s\mathrm{R}\Gamma^B_{\mathrm{HK}}(N_X(X_{V'})_1,r)^{N'=0}_{\mathbb{Q}}\longrightarrow\end{array}$$

Note that, since all the maps d_i^* are induced from automorphisms of V'/V, by the proof of Proposition 3.11 (take the map f used there to be a given automorphism $g\in G=\mathrm{Gal}(K'/K)$ and π', $g(\pi')$ for the uniformizers of V') and the proof of Proposition 3.8, we get the vertical maps above that make all the squares commute.

Hence it suffices to show that the following sequence of Hyodo–Kato cohomology groups is exact:

$$\begin{aligned}0\to H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}&\xrightarrow{f^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(N_X(X_{V'})_0)_{\mathbb{Q}}\\ &\xrightarrow{d_0^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(N_X(X_{V'})_1)_{\mathbb{Q}}\xrightarrow{d_1^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(N_X(X_{V'})_2)_{\mathbb{Q}}\to\cdots.\end{aligned}$$

But this sequence is isomorphic to the sequence

$$\begin{aligned}0\to H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}&\xrightarrow{f^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})_{\mathbb{Q}}\\ &\xrightarrow{d_0^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})_{\mathbb{Q}}\times G\xrightarrow{d_1^*}H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})_{\mathbb{Q}}\times G^2\to\cdots\end{aligned}$$

representing the (augmented) G-cohomology of $H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}$. Since G is finite, this complex is exact in degrees at least 1. It remains to show that

$$H^0(G,H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})_{\mathbb{Q}})\simeq H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)_{\mathbb{Q}}.$$

Since K'/K is totally ramified, we have $H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})\simeq H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X)$. Hence the action of G on $H^s\mathrm{R}\Gamma_{\mathrm{HK}}(X_{V'})$ is trivial and we get the right H^0 as well. We have proved the second quasi-isomorphism from (40). Notice that along the way we have actually proved the first quasi-isomorphism. □

For $X \in \mathcal{V}ar_K$, we define a canonical K_0-linear map (*the Beilinson–Hyodo–Kato morphism*)

$$\iota^B_{\mathrm{dR}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$$

as the sheafification of the map $\iota^B_{\mathrm{dR}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(U_1, \overline{U}_1) \to \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)$. It follows from Proposition 3.22, which we prove in the next section, that the cohomology groups $H^i_{\mathrm{HK}}(X_h) := H^i\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)$ are finite-rank K_0-vector spaces and that they vanish for $i > 2\dim X$. This implies the following lemma.

Lemma 3.19. *The syntomic cohomology groups $H^i_{\mathrm{syn}}(X_h, r) := H^i\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ vanish for $i > 2\dim X + 2$.*

Proof. The map $\iota'_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U}, r)^{N=0} \to \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/F^r$ from Remark 3.10 sheafifies. The quasi-isomorphism $\alpha'_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} C'_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})\{r\})$ does as well. Hence $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ is quasi-isomorphic via α'_{syn} to the mapping fiber

$$C'_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(X_h)\{r\}) := [\mathrm{R}\Gamma_{\mathrm{HK}}(X_h, r)^{N=0} \xrightarrow{\iota'_{\mathrm{dR}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)/F^r].$$

The statement of the lemma follows. □

For $X \in \mathcal{V}ar_K$ and $r \geq 0$, define the complex

$$C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}) := \left[\begin{array}{ccc} \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) & \xrightarrow{(1-\varphi_r,\iota^B_{\mathrm{dR}})} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \oplus \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)/F^r \\ \downarrow N & & \downarrow (N,0) \\ \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \end{array}\right].$$

Proposition 3.20. *For $X \in \mathcal{V}ar_K$ and $r \geq 0$, there exists a canonical (in the classical derived category) quasi-isomorphism*

$$\alpha_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}).$$

Moreover, this morphism is compatible with finite base change (of the field K).

Proof. To construct the map α_{syn}, take a number $t \geq 2\dim X + 2$ and let $Y_\bullet \to X$, $Y_\bullet = (U_\bullet, \overline{U}_\bullet)$, be an h-hypercovering of X by ss-pairs over K. Choose a finite Galois extension $(V', K')/(V, K)$ and a uniformizer π' of V' as in the proof of Proposition 3.18. Keeping the notation from that proof, refine our hypercovering to the h-hypercovering $Y_\bullet \times_V V' \to X_{K'}$. Then the truncation $(Y_\bullet \times_V V')_{\leq t+1}$ is built from log-schemes log-smooth over $V'^{,\times}$ and of Cartier type. We have the sequence of quasi-isomorphisms

$$\begin{aligned}\gamma_{\pi'}: \mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h}) &\xleftarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h}) \xrightarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{syn}}((U_\bullet \times_K K')_{\leq t+1,h})\\ &\xleftarrow{\sim} \tau_{\leq t}\mathrm{R}\Gamma_{\mathrm{syn}}((Y_\bullet \times_V V')_{\leq t+1})_{\mathbb{Q}}\\ &\xrightarrow{\sim} C_{\mathrm{st}}(\tau_{\leq t}\mathrm{R}\Gamma^B_{\mathrm{HK}}((Y_\bullet \times_V V')_{\leq t+1})\{r\})\\ &\xrightarrow{\sim} C_{\mathrm{st}}(\tau_{\leq t}\mathrm{R}\Gamma^B_{\mathrm{HK}}((U_\bullet \times_K K')_{\leq t+1,h})\{r\})\\ &\xleftarrow{\sim} C_{\mathrm{st}}(\tau_{\leq t}\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\}) \xrightarrow{\sim} C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\}).\end{aligned}$$

The first quasi-isomorphism follows from Lemma 3.19. The third and fifth quasi-isomorphisms follow from Proposition 3.18. The fourth quasi-isomorphism (the map $\tilde{\alpha}^B_{\mathrm{syn},\pi'}$), since all the log-schemes involved are log-smooth over $V'^{,\times}$ and of Cartier type, follows from Proposition 3.8.

Now, set $G := \mathrm{Gal}(K'/K)$. Passing from $\gamma_{\pi'}$ to its G-fixed points, we obtain the map

$$\alpha_{\mathrm{syn}} := \alpha_{\mathrm{syn},\pi'}: \mathrm{R}\Gamma_{\mathrm{syn}}(X_h) \to C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$$

as the composition

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h) \to \mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h})^G \xrightarrow{\gamma_{\pi'}} C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\})^G \xleftarrow{\sim} C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K,h})\{r\}).$$

It remains to check that the so-defined map is independent of all choices. For that, it suffices to check that, in the above construction, for a finite Galois extension (V_1, K_1) of (V', K'), $H = \mathrm{Gal}(K_1/K')$, the corresponding maps

$$\alpha_{\mathrm{syn},?}: \mathrm{R}\Gamma_{\mathrm{syn}}(X_h) \to C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$$

are the same in the classical derived category (note that this includes trivial extensions). An easy diagram chase shows that this amounts to checking that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{syn}}((Y_\bullet \times_V V')_{\leq t+1})_{\mathbb{Q}} & \xrightarrow[\alpha_{\mathrm{syn},\pi'}]{\sim} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}((Y_\bullet \times_V V')_{\leq t+1})\{r\}) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{\mathrm{syn}}((Y_\bullet \times_V V_1)_{\leq t+1})^H_{\mathbb{Q}} & \xrightarrow[\alpha_{\mathrm{syn},\pi_1}]{\sim} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}((Y_\bullet \times_V V_1)_{\leq t+1})\{r\})^H \end{array}$$

But this we have shown in Proposition 3.11.

For the compatibility with finite base change, consider a finite field extension L/K. We can choose in the above a Galois extension K'/K that works for both fields. We get the same maps $\gamma_{\pi'}$ for both L and K. Consider now the following commutative diagram. The top and bottom rows define the maps $\alpha^L_{\mathrm{syn},\pi'}$

and $\alpha^K_{\mathrm{syn},\pi'}$, respectively.

$$\begin{array}{ccccccc}
\mathrm{R}\Gamma_{\mathrm{syn}}(X_{L,h}) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h})^{G_L} & \xrightarrow{\gamma_{\pi'}} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\})^{G_L} & \xleftarrow{\sim} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{L,h})\{r\}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{R}\Gamma_{\mathrm{syn}}(X_h) & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{syn}}(X_{K',h})^{G} & \xrightarrow{\gamma_{\pi'}} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K',h})\{r\})^{G} & \xleftarrow{\sim} & C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{K,h})\{r\})
\end{array}$$

This proves the last claim of our proposition. □

3D. ***Geometric syntomic cohomology.*** We will now study geometric syntomic cohomology, i.e., syntomic cohomology over $\bar{K}$. Most of the constructions related to syntomic cohomology over K have their analogs over $\bar{K}$. We will summarize them briefly. For details, the reader should consult [Tsuji 1999; Beilinson 2013].

For $(U,\bar{U})\in\mathscr{P}^{\mathrm{ss}}_{\bar{K}}$, $r\geq 0$, we have the absolute crystalline cohomology complexes and their completions

$$\begin{aligned}
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n &:= \mathrm{R}\Gamma_{\mathrm{cr}}(\bar{U}_{\text{ét}}, \mathrm{R}u_{\bar{U}_n/W_n(k)*}\mathscr{J}^{[r]}_{\bar{U}_n/W_n(k)}),\\
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]}) &:= \operatorname{holim}_n \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n,\\
\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_{\mathbb{Q}} &:= \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})\otimes\mathbb{Q}_p.
\end{aligned}$$

By [Beilinson 2013, Theorem 1.18], the complex $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})$ is a perfect A_{cr}-complex and

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})\otimes^L_{A_{\mathrm{cr}}} A_{\mathrm{cr}}/p^n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})\otimes^L \mathbb{Z}/p^n.$$

In general, we have $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})_n \simeq \mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})\otimes^L\mathbb{Z}/p^n$. Moreover, $J^{[r]}_{\mathrm{cr}} = \mathrm{R}\Gamma_{\mathrm{cr}}(\operatorname{Spec}(\bar{K}),\operatorname{Spec}(\bar{V}),\mathscr{J}^{[r]})$ [Tsuji 1999, Lemmas 1.6.3 and 1.6.4]. The absolute log-crystalline cohomology complexes are filtered E_∞ algebras over $A_{\mathrm{cr},n}$, A_{cr}, or $A_{\mathrm{cr},\mathbb{Q}}$, respectively. Moreover, the rational ones are filtered commutative dg algebras.

For $r\geq 0$, the mod-p^n, completed, and rational log-syntomic complexes $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_n$, $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_{\mathbb{Q}}$ are defined by analogs of formulas (32). We have $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)_n\simeq\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)\otimes^L\mathbb{Z}/p^n$. Let $\mathscr{J}^{[r]}_{\mathrm{cr}}$, $\mathscr{A}_{\mathrm{cr}}$, and $\mathscr{S}(r)$ be the h-sheafifications on $\mathscr{V}ar_{\bar{K}}$ of the presheaves sending $(U,\bar{U})\in\mathscr{P}^{\mathrm{ss}}_{\bar{K}}$ to $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[r]})$, $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})$, and $\mathrm{R}\Gamma_{\mathrm{syn}}(U,\bar{U},r)$, respectively. Let $\mathscr{J}^{[r]}_{\mathrm{cr},n}$, $\mathscr{A}_{\mathrm{cr},n}$, and $\mathscr{S}_n(r)$ denote the h-sheafifications of the mod-p^n versions of the respective presheaves, and let $\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}}$, $\mathscr{A}_{\mathrm{cr},\mathbb{Q}}$, $\mathscr{S}(r)_{\mathbb{Q}}$ be the h-sheafification of the rational versions of the same presheaves.

For $X\in\mathscr{V}ar_{\bar{K}}$, set $\mathrm{R}\Gamma_{\mathrm{cr}}(X_h):=\mathrm{R}\Gamma(X_h,\mathscr{A}_{\mathrm{cr}})$. It is a filtered (by $\mathrm{R}\Gamma(X_h,\mathscr{J}^{[r]}_{\mathrm{cr}})$, $r\geq 0$) E_∞ A_{cr}-algebra equipped with the Frobenius action φ. The Galois group G_K acts on $\mathscr{V}ar_{\bar{K}}$ and it acts on $X\mapsto\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)$ by transport of structure. If X is defined over K then G_K acts naturally on $\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)$.

For $r \geq 0$, set $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_n = \mathrm{R}\Gamma(X_h, \mathscr{S}_n(r))$ and define $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) := \mathrm{R}\Gamma(X_h, \mathscr{S}(r)_{\mathbb{Q}})$. We have

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_n \simeq \mathrm{Cone}\big(\mathrm{R}\Gamma(X_h, \mathscr{J}^{[r]}_{\mathrm{cr},n}) \xrightarrow{p^r-\varphi} \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{cr},n})\big)[-1],$$
$$\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \simeq \mathrm{Cone}\big(\mathrm{R}\Gamma(X_h, \mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}}) \xrightarrow{1-\varphi_r} \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{cr},\mathbb{Q}})\big)[-1].$$

The direct sum $\bigoplus_{r\geq 0} \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)$ is a graded E_∞ algebra over $\mathbb{Z}_p$.

Let $\bar{f} : Z_1 \to \mathrm{Spec}(\overline{V}_1)^\times$ be an integral, quasi-coherent log-scheme. Suppose $\bar{f}$ is the base change of $\bar{f}_L : Z_{L,1} \to \mathrm{Spec}(\mathscr{O}_{L,1})^\times$ by $\theta_1 : \mathrm{Spec}(\overline{\mathscr{O}_{L,1}})^\times \to \mathrm{Spec}(\mathscr{O}_{L,1})^\times$ for a finite extension L/K. That is, we have a map $\theta_{L,1} : Z_1 \to Z_{L,1}$ such that the square $(\bar{f}, \bar{f}_L, \theta_1, \theta_{L,1})$ is Cartesian. Assume that $\bar{f}_L$ is log-smooth of Cartier type and that the underlying map of schemes is proper. Such data $(L, Z_1, \theta_{L,1})$ form a directed set Σ_1 and, for a morphism $(L', Z'_1, \theta'_{L',1}) \to (L, Z_1, \theta_{L,1})$, we have a canonical base change identification compatible with φ-action [Beilinson 2013, 1.18]

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_{L,1}) \otimes_{L_0} L'_0 \xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z'_{L',1}).$$

These identifications can be made compatible with respect to L, so we can set

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1) := \varinjlim_{\Sigma_1} \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_{L,1}).$$

It is a complex of (φ, N)-modules over K_0^{nr}, functorial with respect to morphisms of Z_1.

Consider the scheme $E_{\mathrm{cr}} := \mathrm{Spec}(A_{\mathrm{cr}})$. We have $E_{\mathrm{cr},1} = \mathrm{Spec}(\overline{V}_1)$ and we equip $E_{\mathrm{cr},1}$ with the induced log-structure. This log-structure extends uniquely to a log-structure on $E_{\mathrm{cr},n}$ and the PD-thickening $\mathrm{Spec}(\overline{V})_1^\times \hookrightarrow E_{\mathrm{cr},n}$ is universal over $\mathbb{Z}/p^n$. Set $E_{\mathrm{cr}} := \mathrm{Spec}(A_{\mathrm{cr}})$ with the limit log-structure. Since we have [Beilinson 2013, 1.18.1]

$$\mathrm{R}\Gamma_{\mathrm{cr}}(Z_1) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(Z_1/E_{\mathrm{cr}}),$$

Theorem 3.6 yields a canonical quasi-isomorphism of B^+_{cr}-complexes (called *the crystalline Beilinson–Hyodo–Kato quasi-isomorphism*)

$$\iota^B_{\mathrm{cr}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)^\tau_{B^+_{\mathrm{cr}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(Z_1)_{\mathbb{Q}}$$

compatible with the action of Frobenius. But we have

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)^\tau_{B^+_{\mathrm{cr}}} = (\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1) \otimes_{K_0^{\mathrm{nr}}} A^\tau_{\mathrm{cr},\mathbb{Q}})^{N=0}$$

and there is a canonical isomorphism $A^\tau_{\mathrm{cr},\mathbb{Q}} \xrightarrow{\sim} B^+_{\mathrm{st}}$ that is compatible with Frobenius and monodromy. This implies that the above quasi-isomorphism amounts to a quasi-isomorphism of B^+_{cr}-complexes

$$\iota^B_{\mathrm{cr}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)_{B^+_{\mathrm{st}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(Z_1) \otimes^L_{A_{\mathrm{cr}}} B^+_{\mathrm{st}}$$

compatible with the action of φ and N. The crystalline Beilinson–Hyodo–Kato map can be canonically trivialized at $[\tilde{p}]$, where $\tilde{p}$ is a sequence of p^n-th roots of p:

$$\beta=\beta_{[\tilde{p}]}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{cr}}\xrightarrow{\sim}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{cr}}[a([\tilde{p}])])^{N=0},$$
$$x\mapsto\exp(N(x)a([\tilde{p}])).$$

This trivialization is compatible with Frobenius and monodromy.

Suppose now that $\bar{f}_1:Z_1\to\operatorname{Spec}(\overline{V}_1)^\times$ is a reduction mod p of a log-scheme $\bar{f}:Z\to\operatorname{Spec}(\overline{V})^\times$. Suppose that $\bar{f}$ is the base change of $\bar{f}_L:Z_L\to\operatorname{Spec}(\mathscr{O}_L)^\times$ by $\theta:\operatorname{Spec}(\overline{\mathscr{O}_L})^\times\to\operatorname{Spec}(\mathscr{O}_L)^\times$ for a finite extension L/K. That is, we have a map $\theta_L:Z\to Z_L$ such that the square $(\bar{f},\bar{f}_L,\theta,\theta_L)$ is Cartesian. Assume that $\bar{f}_L$ is log-smooth of Cartier type and that the underlying map of schemes is proper. Such data (L,Z,θ_L) form a directed set Σ and the reduction mod p map $\Sigma\to\Sigma_1$ is cofinal. The Beilinson–Hyodo–Kato quasi-isomorphisms (27) are compatible with morphisms in Σ and their colimit yields a natural quasi-isomorphism (called again the *Beilinson–Hyodo–Kato quasi-isomorphism*)

$$\iota^B_{\mathrm{dR}}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)^\tau_{\overline{K}}\xrightarrow{\sim}\mathrm{R}\Gamma(Z_{\overline{K}},\Omega^\bullet_{Z/\overline{K}}).$$

The trivializations by p are also compatible with the maps in Σ; hence we obtain the Beilinson–Hyodo–Kato maps

$$\iota^B_{\mathrm{dR}}:=\iota^B_{\mathrm{dR}}\beta_p:\mathrm{R}\Gamma^B_{\mathrm{HK}}(Z_1)\to\mathrm{R}\Gamma(Z_{\overline{K}},\Omega^\bullet_{Z/\overline{K}}).$$

For an ss-pair $(U,\overline{U})$ over $\overline{K}$, set $\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\overline{U}):=\mathrm{R}\Gamma^B_{\mathrm{HK}}((U,\overline{U})_1)$. Let $\mathscr{A}^B_{\mathrm{HK}}$ be the h-sheafification of the presheaf $(U,\overline{U})\mapsto\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\overline{U})$ on $\mathscr{P}^{\mathrm{ss}}_{\overline{K}}$. This is an h-sheaf of E_∞ K_0^{nr}-algebras equipped with a φ-action and locally nilpotent derivation N such that $N\varphi=p\varphi N$. For $X\in\mathscr{V}ar_{\overline{K}}$, set $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h):=\mathrm{R}\Gamma(X_h,\mathscr{A}^B_{\mathrm{HK}})$.

Proposition 3.21. (1) *For any $(U,\overline{U})\in\mathscr{P}^{\mathrm{ss}}_{\overline{K}}$, the canonical maps*

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{J}^{[r]})_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{R}\Gamma(U_h,\mathscr{J}^{[r]}_{\mathrm{cr}})_{\mathbb{Q}},\quad\mathrm{R}\Gamma^B_{\mathrm{HK}}(U,\overline{U})\xrightarrow{\sim}\mathrm{R}\Gamma^B_{\mathrm{HK}}(U_h)\tag{41}$$

are quasi-isomorphisms.

(2) *For every $X\in\mathscr{V}ar_{\overline{K}}$, the cohomology groups $H^n_{\mathrm{cr}}(X_h):=H^n\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}$ and $H^n_{\mathrm{HK}}(X_h):=H^n\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)$, are free B^+_{cr}-modules, resp. K_0^{nr}-modules, of rank equal to the rank of $H^n(X_{\text{ét}},\mathbb{Q}_p)$.*

Proof. Only the filtered statement in part (1) for $r>0$ requires argument since the rest has been proven by Beilinson [2013, 2.4]. Take $r>0$. To prove that we have a quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{J}^{[r]})_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{R}\Gamma(U_h,\mathscr{J}^{[r]}_{\mathrm{cr}})_{\mathbb{Q}}$, it suffices to show that the map $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\overline{U},\mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}}\to\mathrm{R}\Gamma(U_h,\mathscr{A}_{\mathrm{cr}}/\mathscr{J}^{[r]}_{\mathrm{cr}})_{\mathbb{Q}}$ is a quasi-isomorphism. Since, for an ss-pair $(T,\overline{T})$ over K, by Corollary 2.4 $\mathrm{R}\Gamma_{\mathrm{cr}}(T,\overline{T},\mathscr{O}/\mathscr{J}^{[r]})_{\mathbb{Q}}\simeq\mathrm{R}\Gamma(\overline{T}_K,\Omega^\bullet_{(T,\overline{T}_K)}/F^r)$, this is equivalent to showing that $\mathrm{R}\Gamma(\overline{U}_K,\Omega^\bullet_{(U,\overline{U}_K)}/F^r)\to\mathrm{R}\Gamma(U_h,\mathscr{A}_{\mathrm{dR}}/F^r)$ is a quasi-isomorphism, which follows from Proposition 3.14. □

Proposition 3.22. *Let $X \in \mathscr{V}ar_K$. The natural projection $\varepsilon : X_{\bar{K},h} \to X_h$ defines pullback maps*

$$\varepsilon^* : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \to \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})^{G_K}, \quad \varepsilon^* : \mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})^{G_K}. \quad (42)$$

These are (filtered) quasi-isomorphisms.

Proof. Notice that the action of G_K on $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})$ is smooth, i.e., the stabilizer of every element is an open subgroup of G_K. We will prove only the first quasi-isomorphism — the proof of the second one being analogous. By Proposition 3.18, it suffices to show that for any ss-pair over K, the natural map

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(U_1, \bar{U}_1) \to \mathrm{R}\Gamma^B_{\mathrm{HK}}((U, \bar{U}) \otimes_K \bar{K})^{G_K}$$

is a quasi-isomorphism. Passing to a finite extension of K_U, if necessary, we may assume that $(U, \bar{U})$ is log-smooth of Cartier type over a finite Galois extension K_U of K. Then

$$\mathrm{R}\Gamma^B_{\mathrm{HK}}((U, \bar{U}) \otimes_K \bar{K}) \simeq \mathrm{R}\Gamma^B_{\mathrm{HK}}(U_1, \bar{U}_1) \otimes_{K_{U,0}} K_0^{\mathrm{nr}} \times H, \quad H = \mathrm{Gal}(K_U/K).$$

Taking G_K-fixed points of this quasi-isomorphism, we get the first quasi-isomorphism of (42), as wanted. □

Let $(U, \bar{U})$ be an ss-pair over $\bar{K}$. Set

$$\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U, \bar{U}) := \mathrm{R}\Gamma(\bar{U}_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{(U,\bar{U})/W(k)}),$$
$$\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U, \bar{U})_n := \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U, \bar{U}) \otimes^{\mathbb{L}} \mathbb{Z}/p^n \simeq \mathrm{R}\Gamma(\bar{U}_{\text{ét}}, \mathrm{L}\Omega^{\bullet,\wedge}_{(U,\bar{U})_n/W_n(k)}),$$
$$\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U, \bar{U}) \widehat{\otimes} \mathbb{Z}_p := \mathrm{holim}_n \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U, \bar{U})_n,$$
$$\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U, \bar{U}) \widehat{\otimes} \mathbb{Q}_p := (\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U, \bar{U}) \widehat{\otimes} \mathbb{Z}_p) \otimes \mathbb{Q}.$$

These are F-filtered E_∞ algebras. Take the associated presheaves on $\mathscr{P}^{\mathrm{ss}}_{\bar{K}}$. Denote by $\mathscr{A}^{\natural}_{\mathrm{dR}}$, $\mathscr{A}^{\natural}_{\mathrm{dR},n}$, $\mathscr{A}^{\natural}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p$, $\mathscr{A}^{\natural}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p$ their sheafifications in the h-topology of $\mathscr{V}ar_{\bar{K}}$. These are sheaves of F-filtered E_∞ algebras (viewed as the projective system of quotients modulo F^i). Set $A_{\mathrm{dR}} := \mathrm{L}\Omega^{\bullet,\wedge}_{\bar{V}/V}$. By [Beilinson 2012, Lemma 3.2], $A_{\mathrm{dR}} = \mathscr{A}^{\natural}_{\mathrm{dR}}(\mathrm{Spec}(\bar{K})) = \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(\bar{K}, \bar{V})$. The corresponding F-filtered algebras $A_{\mathrm{dR},n}$, $A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p$, $A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p$ are acyclic in nonzero degrees and the projections $\cdot/F^{m+1} \to \cdot/F^m$ are surjective. Thus (we set $\lim_F := \mathrm{holim}_F$)

$$A^{\diamond}_{\mathrm{dR},n} := \lim_F A_{\mathrm{dR},n} = \varprojlim_m H^0(A_{\mathrm{dR},n}/F^m),$$
$$A^{\diamond}_{\mathrm{dR}} := \lim_F (A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p) = \varprojlim_m H^0(A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Z}_p/F^m),$$
$$\lim_F A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p = \varprojlim_m H^0(A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p/F^m) = B^+_{\mathrm{dR}},$$
$$A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p/F^m = B^+_{\mathrm{dR}}/F^m.$$

For any $(U,\bar{U})$ over $\bar{K}$, the complex $\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\bar{U})$ is an F-filtered E_∞ filtered A_{dR}-algebra; hence $\lim_F \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\bar{U})_n$ is an $A^{\diamond}_{\mathrm{dR},n}$-algebra, $\lim_F(\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\bar{U})\,\widehat{\otimes}\,\mathbb{Q}_p)$ is a B^+_{dR}-algebra, etc. We have canonical morphisms

$$\kappa'_{r,n}:\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_n\to\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})_n/F^r\xrightarrow{\sim}\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\bar{U})_n/F^r.$$

In the case of $(\bar{K},\bar{V})$, from Theorem 2.1, we get isomorphisms

$$\kappa'_{r,n}=\kappa_r^{-1}:A_{\mathrm{cr},n}/J^{[r]}\xrightarrow{\sim}A_{\mathrm{dR},n}/F^r.$$

Hence $A^{\diamond}_{\mathrm{dR}}$ is the completion of A_{cr} with respect to the $J^{[r]}$-topology.

For $X\in\mathcal{V}ar_{\bar{K}}$, set $\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h):=\mathrm{R}\Gamma(X_h,\mathscr{A}^{\natural}_{\mathrm{dR}})$. Since $A_{\mathrm{dR},\mathbb{Q}}=\bar{K}$, for any variety X over $\bar{K}$, we have a filtered quasi-isomorphism of $\bar{K}$-algebras [Beilinson 2012, 3.2] $\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h)_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$ obtained by h-sheafification of the quasi-isomorphism

$$\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\bar{U})_{\mathbb{Q}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U}_{\mathbb{Q}}).\tag{43}$$

Concerning the p-adic coefficients, we have a quasi-isomorphism

$$\gamma_r:(\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\bar{K}}B^+_{\mathrm{dR}})/F^r\xrightarrow{\sim}\mathrm{R}\Gamma(X_h,\mathscr{A}^{\natural}_{\mathrm{dR}}\,\widehat{\otimes}\,\mathbb{Q}_p)/F^r.\tag{44}$$

To define it, consider, for any ss-pair $(U,\bar{U})$ over $\bar{K}$, the natural map $\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\bar{U})\to\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\bar{U})\widehat{\otimes}\mathbb{Z}_p$. It yields, by extension to $A_{\mathrm{dR}}\widehat{\otimes}\mathbb{Q}_p$ and by the quasi-isomorphism (43), a quasi-isomorphism of F-filtered $\bar{K}$-algebras [Beilinson 2013, 3.5]

$$\gamma:\mathrm{R}\Gamma_{\mathrm{dR}}(U,\bar{U})_{\mathbb{Q}}\otimes_{\bar{K}}(A_{\mathrm{dR}}\,\widehat{\otimes}\,\mathbb{Q}_p)\xrightarrow{\sim}\mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(U,\bar{U})\,\widehat{\otimes}\,\mathbb{Q}_p.$$

Its (mod F^r)-version γ_r after h-sheafification yields the quasi-isomorphism

$$\gamma_r:(\mathscr{A}_{\mathrm{dR}}\otimes_{\bar{K}}B^+_{\mathrm{dR}})/F^r\xrightarrow{\sim}\mathscr{A}^{\natural}_{\mathrm{dR}}\,\widehat{\otimes}\,\mathbb{Q}_p/F^r.$$

Passing to $\mathrm{R}\Gamma(X_h,\bullet)$ we get the quasi-isomorphism (44).

For $X\in\mathcal{V}ar_{\bar{K}}$, we have canonical quasi-isomorphisms

$$\iota^B_{\mathrm{cr}}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^{\tau}_{B^+_{\mathrm{cr}}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}},\quad \iota^B_{\mathrm{dR}}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^{\tau}_{\bar{K}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)$$

compatible with the $\mathrm{Gal}(\bar{K}/K)$-action. Here ${}^{\tau}_{B^+_{\mathrm{cr}}}$ and ${}^{\tau}_{\bar{K}}$ denote the h-sheafification of the crystalline and de Rham Beilinson–Hyodo–Kato twists [Beilinson 2013, 2.5.1]. Trivializing the first map at $[\tilde{p}]$ and the second map at p, we get the Beilinson–Hyodo–Kato maps

$$\iota^B_{\mathrm{cr}}:=\iota^B_{\mathrm{cr}}\beta_{[\tilde{p}]}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}}B^+_{\mathrm{cr}}\to\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}},$$
$$\iota_{\mathrm{dR}}:=\iota_{\mathrm{dR}}\beta_p:\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\to\mathrm{R}\Gamma_{\mathrm{dR}}(X_h).$$

Using the quasi-isomorphism

$$\kappa_r^{-1}:\mathscr{A}_{\mathrm{cr},\mathbb{Q}}/\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}}\xrightarrow{\sim}(\mathscr{A}^{\natural}_{\mathrm{dR}}\,\widehat{\otimes}\,\mathbb{Q}_p)/F^r$$

from Theorem 2.1, we get the quasi-isomorphisms of complexes of sheaves on $X_{\bar{K},h}$

$$\mathscr{S}(r)_{\mathbb{Q}} \xrightarrow{\sim} [\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}} \xrightarrow{1-\varphi_r} \mathscr{A}_{\mathrm{cr},\mathbb{Q}}] \xrightarrow{\sim} [\mathscr{A}_{\mathrm{cr},\mathbb{Q}} \xrightarrow{(1-\varphi_r,\mathrm{can})} \mathscr{A}_{\mathrm{cr},\mathbb{Q}} \oplus \mathscr{A}_{\mathrm{cr},\mathbb{Q}}/\mathscr{J}^{[r]}_{\mathrm{cr},\mathbb{Q}}]$$
$$\xleftarrow{\sim} [\mathscr{A}_{\mathrm{cr},\mathbb{Q}} \xrightarrow{(1-\varphi_r,\kappa_r^{-1})} \mathscr{A}_{\mathrm{cr},\mathbb{Q}} \oplus (\mathscr{A}^{\natural}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p)/F^r].$$

Applying $\mathrm{R}\Gamma(X_h, \bullet)$ and the quasi-isomorphism $\gamma_r^{-1} : \mathrm{R}\Gamma(X_h, \mathscr{A}^{\natural}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p)/F^r \xrightarrow{\sim} (\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r$ from (44), we obtain the quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{syn}}&(X_h, r)\\ &\xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \oplus \mathrm{R}\Gamma(X_h, \mathscr{A}^{\natural}_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p)/F^r\big]\\ &\xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\gamma_r^{-1}\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r\big].\end{aligned} \tag{45}$$

Corollary 3.23. *For any $(U, \bar{U}) \in \mathscr{P}^{\mathrm{ss}}_{\bar{K}}$, the canonical map*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(U, \bar{U}, r)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{syn}}(U_h, r)$$

is a quasi-isomorphism.

Proof. Arguing as above, we find quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{syn}}&(U, \bar{U}, r)_{\mathbb{Q}}\\ &\xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \oplus (\mathrm{R}\Gamma^{\natural}(U, \bar{U}) \widehat{\otimes} \mathbb{Q}_p)/F^r\big]\\ &\xrightarrow{\sim} \big[\mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\gamma_r^{-1}\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r\big].\end{aligned}$$

Comparing them with quasi-isomorphisms (45), we see that it suffices to check that the natural maps

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(U_h)_{\mathbb{Q}}, \quad \mathrm{R}\Gamma_{\mathrm{dR}}(U, \bar{U}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(U_h)$$

are (filtered) quasi-isomorphisms, but this follows by Propositions 3.21 and 3.14. □

Consider the composition of morphisms

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{syn}}&(X_h, r)\\ &\xrightarrow{\sim} \big[\,\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \xrightarrow{(1-\varphi_r,\gamma_r^{-1}\kappa_r^{-1})} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r\,\big]\\ &\xleftarrow{\sim} \left[\begin{array}{ccc} \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} & \xrightarrow{(1-\varphi_r,\iota^B_{\mathrm{dR}}\otimes\iota)} & \begin{array}{c}\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}}\\ \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_{\bar{K}} B^+_{\mathrm{dR}})/F^r\end{array}\\ \downarrow N & & \downarrow (N,0)\\ \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} \end{array}\right].\end{aligned} \tag{46}$$

The second quasi-isomorphism uses the map

$$(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}})^{N=0} = \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^\tau_{B^+_{\mathrm{cr}}} \xrightarrow{\iota^B_{\mathrm{cr}}} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}$$

(that is compatible with the action of N and φ) and the following lemma.

Lemma 3.24. *The following diagrams commute*:

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}\otimes_{B^+_{\mathrm{cr}}} B^+_{\mathrm{st}} & \xrightarrow{\gamma_r^{-1}\kappa_r^{-1}\otimes\iota} & (\mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}} B^+_{\mathrm{dR}})/F^r \\
\iota^B_{cr}\uparrow\wr & \nearrow_{\iota^B_{\mathrm{dR}}\otimes\iota} & \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} & &
\end{array}$$

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}\otimes_{A_{\mathrm{cr}}} B_{\mathrm{dR}} & \xrightarrow[\sim]{\gamma_{\mathrm{dR}}} & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}} B_{\mathrm{dR}} \\
\iota^B_{\mathrm{cr}}\otimes\iota\uparrow & \nearrow_{\iota^B_{\mathrm{dR}}\otimes\iota} & \\
\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} & &
\end{array}$$

(Here γ_{dR} is the map defined in [Beilinson 2013, 3.4.1].)

Proof. We will start with the top diagram. It suffices to show that it canonically commutes with X_h replaced by any ss-pair $\overline{Y}=(U,\overline{U})$ over $\overline{K}$ — a base change of an ss-pair Y split over (V,K). Proceeding as in Example 3.5, we obtain the following diagram in which all squares but the one in the top right clearly commute:

$$\begin{array}{ccccc}
\mathrm{R}\Gamma^B_{\mathrm{HK}}(Y_1)^\tau_K & \xrightarrow[\mathrm{Id}\otimes 1]{} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^\tau_{\overline{K}}\otimes_{\overline{K}} B^+_{\mathrm{dR}} & \xleftarrow{\delta} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^\tau_{B^+_{\mathrm{cr}}}\otimes_{B^+_{\mathrm{cr}}} B^+_{\mathrm{st}} \\
\iota^B_K\downarrow & & \iota^B_{\overline{K}}\otimes\kappa_r\downarrow & & \iota^B_{\mathrm{cr}}\otimes\kappa_r\iota\downarrow \\
\mathrm{R}\Gamma_{\mathrm{cr}}(Y_1/V^\times)_{\mathbb{Q}}/F^r & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/V^\times)_{\mathbb{Q}}/F^r & \xleftarrow[\sim]{} & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/A_{\mathrm{cr}})_{\mathbb{Q}}/F^r \\
\kappa_r\uparrow\wr & & \kappa_r\uparrow\wr & & \kappa_r\uparrow\wr \\
\mathrm{R}\Gamma(Y_{\text{ét}},\mathrm{L}\Omega^{\bullet,\wedge}_{Y/V^\times})\widehat{\otimes}\mathbb{Q}_p/F^r & \longrightarrow & \mathrm{R}\Gamma(\overline{Y}_{\text{ét}},\mathrm{L}\Omega^{\bullet,\wedge}_{\overline{Y}/V^\times})\widehat{\otimes}\mathbb{Q}_p/F^r & \xleftarrow{\sim} & (\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\overline{Y})\widehat{\otimes}\mathbb{Q}_p)/F^r \\
\gamma_r\uparrow\wr & & \gamma_r\uparrow\wr & \nearrow^{\gamma_r}_{\sim} & \\
\mathrm{R}\Gamma_{\mathrm{dR}}(Y_K)/F^r & \longrightarrow & (\mathrm{R}\Gamma_{\mathrm{dR}}(\overline{Y}_K)\otimes_{\overline{K}} B^+_{\mathrm{dR}})/F^r & &
\end{array}$$

Here we have $B^+_{\mathrm{dR}}/F^m=(\mathrm{R}\Gamma^\natural_{\mathrm{dR}}(\overline{K},\overline{V})\widehat{\otimes}\mathbb{Q}_p)/F^m$ and the map δ is defined as the composition

$$\begin{aligned}
\delta:\mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^\tau_{B^+_{\mathrm{cr}}}\otimes_{B^+_{\mathrm{cr}}} B^+_{\mathrm{st}} &= (\mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)\otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}})^{N=0}\otimes_{B^+_{\mathrm{cr}}} B^+_{\mathrm{st}} \\
&\xrightarrow{\sim} \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)\otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}} \xrightarrow{\beta_p\otimes\iota} \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^\tau_{\overline{K}}\otimes_{\overline{K}} B^+_{\mathrm{dR}}.
\end{aligned}$$

Recall that for the map $\iota^B_{\mathrm{dR}}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(Y_1)^\tau_K\to \mathrm{R}\Gamma_{\mathrm{dR}}(Y_K)/F^r$, we have $\iota^B_{\mathrm{dR}}=\gamma_r^{-1}\kappa_r^{-1}\iota^B_K$. Everything in sight being compatible with change of the ss-pairs Y — more specifically with maps in the directed system Σ — if this diagram commutes so does its Σ colimit and the top diagram in the lemma for the pair $(U,\overline{U})$.

It remains to show that the top right square in the above diagram commutes. To do that, consider the ring $\widehat{A}_n$ defined as the PD-envelope of the closed immersion

$$\overline{V}_1^\times \hookrightarrow A_{\mathrm{cr},n}\times_{W_n(k)} V_n^\times .$$

That is, $\widehat{A}_n$ is the product of the PD-thickenings $(\overline{V}_1^\times\hookrightarrow A_{\mathrm{cr},n})$ and $(V_1^\times\hookrightarrow V_n^\times)$ over $(W_1(k)\hookrightarrow W_n(k))$. By [Beilinson 2013, Lemma 1.17], this makes $\overline{V}_1^\times\hookrightarrow\widehat{A}_{\mathrm{cr},n}$ into the universal PD-thickening in the log-crystalline site of $\overline{V}_1^\times$ over $V_n^\times$. Let $\widehat{A}:=\operatorname{inj\,lim}_n\widehat{A}_{\mathrm{cr},n}$ with the limit log-structure. Set $\widehat{B}^+_{\mathrm{cr}}:=\widehat{A}_{\mathrm{cr}}[1/p]$.

Using Theorem 3.6, we obtain a canonical quasi-isomorphism

$$\iota^B_{\widehat{B}^+_{\mathrm{cr}}}:\mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^\tau_{\widehat{B}^+_{\mathrm{cr}}}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/\widehat{A}_{\mathrm{cr}})_{\mathbb{Q}}.$$

By construction, we have the maps of PD-thickenings

$$(V_1^\times\hookrightarrow V^\times)\xleftarrow{\mathrm{pr}_1}(\overline{V}_1^\times\hookrightarrow\widehat{A}_{\mathrm{cr}})\xrightarrow{\mathrm{pr}_2}(\overline{V}_1^\times\hookrightarrow A_{\mathrm{cr}}).$$

Consider the diagram

$$\begin{array}{ccccc}
\mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^\tau_{\widehat{B}^+_{\mathrm{cr}}} & & \xleftarrow{\ \mathrm{pr}_1^*\otimes\mathrm{pr}_1^*\kappa_r\ } & & \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^\tau_{\overline{K}}\otimes_{\overline{K}}B^+_{\mathrm{dR}}/F^r \\
 & \nwarrow^{\mathrm{pr}_2^*} & & \nearrow^{\delta} & \\
\Big\downarrow{\scriptstyle \iota^B_{\widehat{B}^+_{\mathrm{cr}}}} & & \mathrm{R}\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^\tau_{B^+_{\mathrm{cr}}} & & \Big\downarrow{\scriptstyle \iota^B_{\overline{K}}\otimes\kappa_r} \\
 & & \Big\downarrow{\scriptstyle \iota^B_{\mathrm{cr}}} & & \\
 & & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/A_{\mathrm{cr}})_{\mathbb{Q}}/F^r & & \\
 & \swarrow^{\mathrm{pr}_2^*}_{\sim} & & \searrow^{\sim} & \\
\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/\widehat{A}_{\mathrm{cr}})_{\mathbb{Q}}/F^r & & \xleftarrow[\mathrm{pr}_1^*]{\sim} & & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/V^\times)_{\mathbb{Q}}/F^r
\end{array}$$

The bottom triangle commutes since $\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/A_{\mathrm{cr}})=\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/W(k))$. The pullback maps

$$\mathrm{pr}_1^*:\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}_1/V^\times)\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\widehat{A}_{\mathrm{cr}}),$$
$$\mathrm{pr}_2^*:\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/A_{\mathrm{cr}})_{\mathbb{Q}}/F^r\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\widehat{A}_{\mathrm{cr}})_{\mathbb{Q}}/F^r$$

are quasi-isomorphisms. Indeed, in the case of the first pullback this follows from the universal property of $\widehat{A}_{\mathrm{cr}}$; in the case of the second one, it follows from the commutativity of the bottom triangle since the right slanted map is a quasi-isomorphism as shown by the first diagram in our proof.

The left trapezoid and the big square commute by the definition of the Beilinson–Bloch–Kato maps. To see that the top triangle commutes, it suffices to show that for an element

$$x \in \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(\overline{Y}_1)^{\tau}_{B^+_{\mathrm{cr}}} = (\mathrm{R}\Gamma^{B}_{\mathrm{HK}}(\overline{Y}_1) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}})^{N=0},$$
$$x = b\sum_{i\geq 0} N^i(m)a([\tilde{p}])^{[i]}, \quad m \in \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(\overline{Y}_1),\ b \in B^+_{\mathrm{cr}},$$

we have $\mathrm{pr}_2^*(x) = \mathrm{pr}_1^*\,\delta(x)$. Since $\iota(a([\tilde{p}])) = \log([\tilde{p}]/p)$ [Fontaine 1994a, 4.2.2], we calculate

$$\delta(x) = \delta\Big(b\sum_{i\geq 0} N^i(m)a([\tilde{p}])^{[i]}\Big) = b\sum_{i\geq 0}\Big(\sum_{j\geq 0} N^{i+j}(m)a(p)^{[j]}\Big)\log([\tilde{p}]/p)^{[i]}$$
$$= b\sum_{k\geq 0} N^k(m)\big(a(p)+\log([\tilde{p}]/p)\big)^{[k]}.$$

Since in $\widehat{B}^+_{\mathrm{cr}}$ we have $[\tilde{p}] = ([\tilde{p}]/p)p$ and $[\tilde{p}]/p \in 1 + J_{\widehat{B}^+_{\mathrm{cr}}}$, it follows that $a([\tilde{p}]) = \log([\tilde{p}]/p) + a(p)$ and

$$\mathrm{pr}_1^*\,\delta(x) = \mathrm{pr}_1^*\Big(b\sum_{k\geq 0} N^k(m)(a(p)+\log([\tilde{p}]/p))^{[k]}\Big)$$
$$= b\sum_{k\geq 0} N^k(m)a([\tilde{p}])^{[k]} = \mathrm{pr}_2^*\Big(b\sum_{k\geq 0} N^k(m)a([\tilde{p}])^{[k]}\Big) = \mathrm{pr}_2^*(x),$$

as wanted. It follows now that the right trapezoid in the above diagram commutes as well and that so does the top diagram in our lemma.

To check the commutativity of the bottom diagram, consider the following map obtained from the maps $\kappa'_{r,n}$ by passing to F-limit:

$$\kappa'_n : \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_n \otimes^{L}_{A_{\mathrm{cr},n}} A_{\mathrm{dR},n} \xrightarrow{\sim} \varprojlim_{F} \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_n/F^r.$$

By [Beilinson 2013, 3.6.2], this is a quasi-isomorphism. Beilinson [2013, 3.4.1] defines the map

$$\gamma_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} \otimes_{A_{\mathrm{cr}}} B^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\overline{Y}_K) \otimes_{\overline{K}} B^+_{\mathrm{dR}}$$

by B^+_{dR}-linearization of the composition $\varprojlim_r(\gamma_r^{-1}\kappa_r^{-1})\,\mathrm{holim}_n\,\kappa'_n$. We have

$$\gamma_{\mathrm{dR}} = \gamma_r^{-1}\kappa_r^{-1} : \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} \to (\mathrm{R}\Gamma_{\mathrm{dR}}(\overline{Y}_K) \otimes_{\overline{K}} B^+_{\mathrm{dR}})/F^r.$$

Hence the commutativity of the bottom diagram follows from that of the top one. □

Let $C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$ denote the second homotopy limit in the diagram (46); denote by $C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$ the complex $C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$ with all the pluses removed. We have defined a map $\alpha_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r)\to C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$ and proved the following proposition.

Proposition 3.25. *There is a functorial G_K-equivariant quasi-isomorphism*

$$\alpha_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r)=\mathrm{R}\Gamma(X_h,\mathscr{S}(r)_{\mathbb{Q}})\simeq C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}).$$

Corollary 3.26. *For $(U,\overline{U})\in\mathscr{P}^{\mathrm{ss}}_K$, we have a long exact sequence*

$$\cdots\to H^i_{\mathrm{syn}}((U,\overline{U})_{\overline{K}},r)\to(H^i_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}\otimes_{K_0}B^+_{\mathrm{st}})^{\varphi=p^r,N=0}$$
$$\to(H^i_{\mathrm{dR}}(U,\overline{U})\otimes_K B^+_{\mathrm{dR}})/F^r\to H^{i+1}_{\mathrm{syn}}((U,\overline{U})_{\overline{K}},r)\to\cdots.$$

Proof. By diagram (46), it suffices to show that

$$H^i[\mathrm{R}\Gamma^B_{\mathrm{HK}}((U,\overline{U})_1)\otimes_{K_0}B^+_{\mathrm{st}}]^{\varphi=p^r,N=0}\simeq(H^i_{\mathrm{HK}}(U,\overline{U})_{\mathbb{Q}}\otimes_{K_0}B^+_{st})^{\varphi=p^r,N=0},$$
$$H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(U,\overline{U})\otimes_K B^+_{\mathrm{dR}})/F^r)\simeq(H^i_{\mathrm{dR}}(U,\overline{U})\otimes_K B^+_{\mathrm{dR}})/F^r.$$

The second isomorphism is a consequence of the degeneration of the Hodge–de Rham spectral sequence. Keeping in mind that the Beilinson–Hyodo–Kato complexes $\mathrm{R}\Gamma^B_{\mathrm{HK}}((U,\overline{U})_1)$ are built from (φ,N)-modules, the first isomorphism follows from the short exact sequences (for a (φ,N)-module M)

$$0\to M\otimes_{K_0}B^+_{\mathrm{cr}}\to M\otimes_{K_0}B^+_{\mathrm{st}}\xrightarrow{N}M\otimes_{K_0}B^+_{\mathrm{st}}\to 0,$$
$$0\to(M\otimes_{K_0}B^+_{\mathrm{cr}})^{\varphi=p^r}\to M\otimes_{K_0}B^+_{\mathrm{cr}}\xrightarrow{1-\varphi_r}M\otimes_{K_0}B^+_{\mathrm{cr}}\to 0.$$

The first one follows, by induction on m such that $N^m=0$ on M, from the exact sequence (11) and the fact that $(M\otimes_{K_0}B^+_{\mathrm{st}})^{N=0}\simeq M\otimes_{K_0}B^+_{\mathrm{cr}}$. The second one follows from [Colmez and Niziol 2015, Remark 2.30]. □

4. Relation between syntomic cohomology and étale cohomology

In this section we will study the relationship between syntomic and étale cohomology in both the geometric and the arithmetic situation.

4A. ***Geometric case.*** We start with the geometric case. In this subsection, we will construct the geometric syntomic period map from syntomic to étale cohomology. We will prove that in the torsion case, on the level of h-sheaves it is a quasi-isomorphism modulo a universal constant; in the rational case it induces an isomorphism on cohomology groups in a stable range. Finally, we will construct the syntomic descent spectral sequence.

We will first recall the de Rham and crystalline Poincaré lemmas of Beilinson [2013; 2012] and Bhatt [2012].

Theorem 4.1 (de Rham Poincaré lemma [Beilinson 2012, 3.2]). *The maps*

$$A_{\mathrm{dR}} \otimes^L \mathbb{Z}/p^n \to \mathscr{A}^{\natural}_{\mathrm{dR}} \otimes^L \mathbb{Z}/p^n$$

are filtered quasi-isomorphisms of h-sheaves on $\mathscr{V}ar_{\bar{K}}$.

Theorem 4.2 (filtered crystalline Poincaré lemma [Beilinson 2013, 2.3, Bhatt 2012, Theorem 10.14]). *The map $J^{[r]}_{\mathrm{cr},n} \to \mathscr{J}^{[r]}_{\mathrm{cr},n}$ is a quasi-isomorphism of h-sheaves on $\mathscr{V}ar_{\bar{K}}$.*

Proof. We have the map of distinguished triangles

$$\begin{array}{ccccc} J^{[r]}_{\mathrm{cr},n} & \longrightarrow & A_{\mathrm{cr},n} & \longrightarrow & A_{\mathrm{cr},n}/J^{[r]}_{\mathrm{cr},n} \\ \downarrow & & \downarrow \wr & & \downarrow \wr \\ \mathscr{J}^{[r]}_{\mathrm{cr},n} & \longrightarrow & \mathscr{A}_{\mathrm{cr},n} & \longrightarrow & \mathscr{A}_{\mathrm{cr},n}/\mathscr{J}^{[r]}_{\mathrm{cr},n} \end{array}$$

The middle map is a quasi-isomorphism by the crystalline Poincaré lemma proved in [Beilinson 2013, 2.3]. Hence it suffices to show that so is the rightmost map. But, by [Beilinson 2013, 1.9.2], this map is quasi-isomorphic to the map $A_{\mathrm{dR},n}/F^r \to \mathscr{A}^{\natural}_{\mathrm{dR},n}/F^r$. Since the last map is a quasi-isomorphism by the de Rham Poincaré lemma, Theorem 4.1, we are done. □

We will now recall the definitions of the crystalline, Beilinson–Hyodo–Kato, and de Rham period maps [Beilinson 2013, 3.1; 2012, 3.5]. Let $X \in \mathscr{V}ar_{\bar{K}}$. To define the crystalline period map

$$\rho_{\mathrm{cr}} : \mathrm{R}\Gamma_{\mathrm{cr}}(X_h) \to \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Z}_p) \widehat{\otimes} A_{\mathrm{cr}},$$

consider the natural map $\alpha_n : \mathrm{R}\Gamma_{\mathrm{cr}}(X_h) \to \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{cr},n})$ and the composition

$$\begin{aligned} \beta_n : \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Z}_p(r)) \otimes^L_{\mathbb{Z}_p} A_{\mathrm{cr},n} &\xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}}, A_{\mathrm{cr},n}) \\ &\xrightarrow{\sim} \mathrm{R}\Gamma(X_h, A_{\mathrm{cr},n}) \xrightarrow{\sim} \mathrm{R}\Gamma(X_h, \mathscr{A}_{\mathrm{cr},n}). \end{aligned}$$

Set $\rho_{\mathrm{cr},n} := \beta_n^{-1}\alpha_n$ and $\rho_{\mathrm{cr}} := \mathrm{holim}_n\, \rho_{\mathrm{cr},n}$. The Hyodo–Kato period map

$$\rho_{\mathrm{HK}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^\tau_{B^+_{\mathrm{cr}}} \to \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p) \otimes B^+_{\mathrm{cr}}, \quad \rho_{\mathrm{HK}} = \rho_{\mathrm{cr},\mathbb{Q}}\iota^B_{\mathrm{cr}},$$

is obtained by composing the map $\rho_{\mathrm{cr},\mathbb{Q}}$ with the quasi-isomorphism

$$\iota^B_{\mathrm{cr}} : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)^\tau_{B^+_{\mathrm{cr}}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}.$$

The maps $\rho_{\mathrm{cr}}, \rho_{\mathrm{HK}}$ are morphisms of E_∞ A_{cr}- and B^+_{cr}-algebras equipped with a Frobenius action; they are compatible with the action of the Galois group G_K.

To define the de Rham period map $\rho_{\mathrm{dR}}:\mathrm{R}\Gamma_{dR}(X_h)\otimes_{\overline{K}} B_{\mathrm{dR}}^+ \to \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{dR}}^+$ consider the compositions

$$\begin{aligned}\alpha &: \mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \xrightarrow{\sim} \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h)\otimes\mathbb{Q} \to \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h)\,\widehat{\otimes}\,\mathbb{Q}_p,\\ \beta &: \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Z})\otimes^{\mathbb{L}} A_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}},A_{\mathrm{dR}}) \to \mathrm{R}\Gamma(X_h,A_{\mathrm{dR}})\\ &\qquad\qquad \to \mathrm{R}\Gamma(X_h,\mathscr{A}^{\natural}_{\mathrm{dR}}) = \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h).\end{aligned}$$

After tensoring the map β with $\mathbb{Z}/p^n$ and using the de Rham Poincaré lemma, we get a quasi-isomorphism

$$\beta_n : \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Z}/p^n)\otimes^{\mathbb{L}} A_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma^{\natural}_{\mathrm{dR}}(X_h)\otimes^{\mathbb{L}}\mathbb{Z}/p^n.$$

Set $\beta_{\mathbb{Q}} := \operatorname{holim}_n \beta_n\otimes\mathbb{Q}$ and $\rho_{\mathrm{dR}} := \beta^{-1}\alpha$. This is a morphism of filtered E_∞ B_{dR}^+-algebras, compatible with G_K-action.

Theorem 4.3 [Beilinson 2013, 3.2, 2012, 3.6]. *For $X\in\mathscr{V}ar_{\overline{K}}$, we have canonical quasi-isomorphisms*

$$\begin{aligned}\rho_{\mathrm{cr}} &: \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)\otimes_{A_{\mathrm{cr}}} B_{\mathrm{cr}} \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{cr}},\\ \rho_{\mathrm{HK}} &: \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_h)^{\tau}_{B_{\mathrm{cr}}} \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{cr}},\\ \rho_{\mathrm{dR}} &: \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}} B_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{dR}}.\end{aligned}$$

Pulling back ρ_{HK} to the Fontaine–Hyodo–Kato $\mathbb{G}_a$-torsor $\operatorname{Spec}(B_{\mathrm{st}})/\operatorname{Spec}(B_{\mathrm{cr}})$, we get a canonical quasi-isomorphism of B_{st}-complexes

$$\rho_{\mathrm{HK}} : \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{st}} \tag{47}$$

compatible with the (φ,N)-action and with the G_K-action on $\mathscr{V}ar_{\overline{K}}$.

Corollary 4.4. *The period morphisms are compatible*; i.e., *the following diagrams commute*:

$$\begin{array}{ccc} \mathrm{R}\Gamma^{B}_{\mathrm{HK}}(X_h)\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} & \xrightarrow{\iota^{B}_{\mathrm{dR}}\otimes\iota} & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}} B_{\mathrm{dR}}\\ \downarrow{\scriptstyle \rho_{\mathrm{HK}}} & & \downarrow{\scriptstyle \rho_{\mathrm{dR}}}\\ \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{st}} & \xrightarrow{1\otimes\iota} & \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{dR}} \end{array}$$

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)\otimes_{A_{\mathrm{cr}}} B_{\mathrm{dR}} & \xleftarrow[\sim]{\gamma_{\mathrm{dR}}} & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)\otimes_{\overline{K}} B_{\mathrm{dR}}\\ \downarrow{\scriptstyle \rho_{\mathrm{cr}}\otimes\mathrm{Id}_{B_{\mathrm{dR}}}} & \swarrow{\scriptstyle \rho_{\mathrm{dR}}} & \\ \mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p)\otimes B_{\mathrm{dR}} & & \end{array}$$

Proof. The bottom diagram commutes by [Beilinson 2013, 3.4]. The commutativity of the top one can be reduced, by the equality $\rho_{\mathrm{HK}} = \rho_{\mathrm{cr}}\iota^{B}_{\mathrm{cr}}$ and the bottom diagram above, to the commutativity of the bottom diagram in Lemma 3.24. □

We will now define the syntomic period map

$$\rho_{\mathrm{syn}} : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r)), \quad r \geq 0.$$

Set $\mathbb{Z}/p^n(r)' := (1/(p^a a!)\mathbb{Z}_p(r)) \otimes \mathbb{Z}/p^n$, where a is the largest integer $\leq r/(p-1)$. Recall that we have the fundamental exact sequence [Tsuji 1999, Theorem 1.2.4]

$$0 \to \mathbb{Z}/p^n(r)' \to J_{\mathrm{cr},n}^{\langle r\rangle} \xrightarrow{1-\varphi_r} A_{\mathrm{cr},n} \to 0,$$

where

$$J_n^{\langle r\rangle} := \{x \in J_{n+s}^{[r]} \mid \varphi(x) \in p^r A_{\mathrm{cr},n+s}\}/p^n$$

for some $s \geq r$. Set $S_n(r) := \mathrm{Cone}(J_{\mathrm{cr},n}^{[r]} \xrightarrow{p^r-\varphi} A_{\mathrm{cr},n})[-1]$. There is a natural morphism of complexes $S_n(r) \to \mathbb{Z}/p^n(r)'$ (induced by p^r on $J_{\mathrm{cr},n}^{[r]}$ and Id on $A_{\mathrm{cr},n}$), whose kernel and cokernel are annihilated by p^r.

The filtered crystalline Poincaré lemma implies easily the following syntomic Poincaré lemma.

Corollary 4.5. (1) *For $0 \leq r \leq p-2$, there is a unique quasi-isomorphism $\mathbb{Z}/p^n(r) \xrightarrow{\sim} \mathscr{S}_n(r)$ of complexes of sheaves on $\mathscr{V}ar_{\bar{K},h}$ that is compatible with the crystalline Poincaré lemma.*

(2) *There is a unique quasi-isomorphism $S_n(r) \xrightarrow{\sim} \mathscr{S}_n(r)$ of complexes of sheaves on $\mathscr{V}ar_{\bar{K},h}$ that is compatible with the crystalline Poincaré lemma.*

Proof. We will prove the second claim — the first one is proved in an analogous way. Consider the map of distinguished triangles

$$\begin{array}{ccccc} \mathscr{S}_n(r) & \longrightarrow & \mathscr{J}_{\mathrm{cr},n}^{[r]} & \xrightarrow{p^r-\varphi} & \mathscr{A}_{\mathrm{cr},n} \\ \uparrow & & \wr\uparrow & & \wr\uparrow \\ S_n(r) & \longrightarrow & J_{\mathrm{cr},n}^{[r]} & \xrightarrow{p^r-\varphi} & A_{\mathrm{cr},n} \end{array}$$

The triangles are distinguished by definition. The vertical continuous arrows are quasi-isomorphisms by the crystalline Poincaré lemma. They induce the dashed arrow that is clearly a quasi-isomorphism. □

Consider the natural map $\alpha_n : \mathrm{R}\Gamma(X_h, \mathscr{S}(r)) \to \mathrm{R}\Gamma(X_h, \mathscr{S}_n(r))$ and the zig-zag

$$\beta_n : \mathrm{R}\Gamma(X_h, \mathscr{S}_n(r)) \leftarrow \mathrm{R}\Gamma(X_h, S_n(r)) \to \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n(r)') \xleftarrow{\sim} \mathrm{R}\Gamma(X_h, \mathbb{Z}/p^n(r)').$$

Set $\beta := (\mathrm{holim}_n\, \beta_n) \otimes \mathbb{Q}$; note that this is a quasi-isomorphism. Set

$$\rho_{\mathrm{syn}} := p^{-r}\beta\alpha : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \to \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r)),$$

where $\alpha := (\mathrm{holim}_n\, \alpha_n) \otimes \mathbb{Q}$. The period map ρ_{syn} induces a map of graded E_∞ algebras over $\mathbb{Q}_p$ compatible with the action of the Galois group G_K.

The syntomic period map has a different, more global definition that we find very useful. Define the map ρ'_{syn} by the diagram

$$\begin{array}{ccccc}
\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r) & \xrightarrow{\sim} & \big[\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} & \xrightarrow{(1-\varphi_r,\gamma_r^{-1}\kappa_r^{-1})} & \mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}}\oplus \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)/F^r\big] \\
\downarrow{\scriptstyle \rho'_{\mathrm{syn}}} & & \downarrow{\scriptstyle \rho_{\mathrm{cr}}} & & \downarrow{\scriptstyle \rho_{\mathrm{cr}}+\rho_{\mathrm{dR}}} \\
\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbb{Q}_p(r)) & \xrightarrow{\sim} & \Big[\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{cr}} & \xrightarrow{(1-\varphi_r,\mathrm{can})} & \begin{matrix}\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{cr}}\\ \oplus\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{dR}}/F^r\end{matrix}\Big]
\end{array}$$

This definition makes sense since the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{R}\Gamma_{\mathrm{cr}}(X_h)_{\mathbb{Q}} & \xrightarrow{\gamma_r^{-1}\kappa_r^{-1}} & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)/F^r \\
\downarrow{\scriptstyle \rho_{\mathrm{cr}}} & & \downarrow{\scriptstyle \rho_{\mathrm{dR}}} \\
\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{cr}} & \xrightarrow{\mathrm{can}} & \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X,\mathbb{Q}_p(r))\otimes B_{\mathrm{dR}}/F^r
\end{array}$$

The syntomic period morphisms ρ_{syn} and ρ'_{syn} are homotopic by a homotopy compatible with the G_K-action (and, unless necessary, we will not distinguish them in what follows). These two facts follow easily from the definitions.

For $X\in\mathscr{V}ar_K$, we have a quasi-isomorphism

$$\alpha_{\mathrm{\acute{e}t}}:\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r))\xrightarrow{\sim} C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})\{r\}) \tag{48}$$

that we define as the inverse of the following composition of quasi-isomorphisms (square brackets denote complex):

$$\begin{aligned}
&C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})\{r\})\\
&\quad\xrightarrow[\sim]{\rho}\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}\big[B_{\mathrm{st}}\xrightarrow{(N,1-\varphi_r,\iota)}B_{\mathrm{st}}\oplus B_{\mathrm{st}}\oplus B_{\mathrm{dR}}/F^r\xrightarrow{(1-\varphi_{r-1})-N}B_{\mathrm{st}}\big]\\
&\quad\xleftarrow{\sim}\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}C(D_{\mathrm{st}}(\mathbb{Q}_p(r)))\xleftarrow{\sim}\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r)).
\end{aligned}$$

The last quasi-isomorphism is by Remark 2.7. The map ρ is defined using the period morphisms ρ_{HK} and ρ_{dR} and their compatibility (Corollary 4.4). The map $\alpha_{\mathrm{\acute{e}t}}$ is compatible with the action of G_K.

Proposition 4.6. *For a variety $X\in\mathscr{V}ar_K$, we have a canonical, compatible with the action of G_K, quasi-isomorphism*

$$\rho_{\mathrm{syn}}:\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\overline{K},h},r)\xrightarrow{\sim}\tau_{\leq r}\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r)).$$

Proof. The Bousfield–Kan spectral sequences associated to the homotopy limits defining the complexes $C^+(H^j_{\mathrm{HK}}(X_{\overline{K},h})\{r\})$ and $C(H^j_{\mathrm{HK}}(X_{\overline{K},h})\{r\})$ form the commutative diagram

$$\begin{array}{ccc} {}^{+}E_2^{i,j} = H^i(C^+(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) & \Longrightarrow & H^{i+j}(C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \\ \big\downarrow \mathrm{can} & & \big\downarrow \mathrm{can} \\ E_2^{i,j} = H^i(C(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) & \Longrightarrow & H^{i+j}(C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \end{array}$$

We have $D_j = H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\} \in MF^{\mathrm{ad}}_K(\varphi, N, G_K)$. For $j \leq r$,

$$F^1 D_{j,K} = F^{1-(r-j)} H^j_{\mathrm{dR}}(X_h)\{r\} = 0.$$

Hence, by Corollary 2.15, we have ${}^{+}E_2^{i,j} \xrightarrow{\sim} E_2^{i,j}$. This implies

$$\tau_{\leq r} C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}) \xrightarrow{\sim} \tau_{\leq r} C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}).$$

Since $\rho_{\mathrm{HK}} = \rho_{\mathrm{cr}}\iota^B_{\mathrm{cr}}$, we check easily that we have the commutative diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h}, r) & \xrightarrow[\alpha_{\mathrm{syn}}]{\sim} & C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}) \\ \rho_{\mathrm{syn}} \big\downarrow & & \mathrm{can} \big\downarrow \\ \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r)) & \xrightarrow[\alpha_{\text{ét}}]{\sim} & C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}) \end{array} \tag{49}$$

It follows that

$$\rho_{\mathrm{syn}} : \tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h}, r) \xrightarrow{\sim} \tau_{\leq r}\mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r)),$$

as wanted. □

Let $X \in \mathcal{V}ar_K$. The natural projection $\varepsilon : X_{\bar{K},h} \to X_h$ defines pullback maps

$$\varepsilon^* : \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) \to \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h}), \quad \varepsilon^* : \mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h}).$$

By construction they are compatible with the monodromy operator, Frobenius, the action of the Galois group G_K, and filtration. It is also clear that they are compatible with the Beilinson–Hyodo–Kato morphisms, i.e., that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h) & \xrightarrow{\iota^B_{\mathrm{dR}}} & \mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \\ \big\downarrow \varepsilon^* & & \big\downarrow \varepsilon^* \\ \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h}) & \xrightarrow{\iota^B_{\mathrm{dR}}} & \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h}) \end{array}$$

It follows that we can define a canonical pullback map

$$\varepsilon^* : C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}) \to C^+(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}).$$

Lemma 4.7. *Let $r \geq 0$. The following diagram commutes in the derived category*:

$$\begin{array}{ccc} R\Gamma_{\mathrm{syn}}(X_h, r) & \xrightarrow{\alpha_{\mathrm{syn}}} & C_{\mathrm{st}}(R\Gamma^B_{\mathrm{HK}}(X_h)\{r\}) \\ \downarrow{\scriptstyle \varepsilon^*} & & \downarrow{\scriptstyle \varepsilon^*} \\ R\Gamma_{\mathrm{syn}}(X_{\overline{K},h}, r) & \xrightarrow{\alpha_{\mathrm{syn}}} & C^+(R\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})\{r\}) \end{array}$$

Proof. Take a number $t \geq 2\dim X + 2$ and choose a finite Galois extension $(V', K')/(V, K)$ (see the proof of Proposition 3.18) such that we have an h-hypercovering $Z_\bullet \to X_{K'}$ with $(Z_\bullet)_{\leq t+1}$ built from log-schemes log-smooth over $V'^{,\times}$ and of Cartier type. Since the top map α_{syn} is compatible with base change (see Proposition 3.20) it suffices to show that the diagram in the lemma commutes with X replaced by $(Z_\bullet)_{\leq t+1}$. By Propositions 3.21, 3.18, and 3.14, this reduces to showing that, for an ss-pair $(U, \overline{U})$ split over V, the following diagram commutes canonically in the ∞-derived category (we set $Y := (U, \overline{U})$, $\overline{Y} := Y_{\overline{V}}$, where π is a fixed uniformizer of V):

$$\begin{array}{ccc} R\Gamma_{\mathrm{syn}}(Y, r)_{\mathbb{Q}} & \xrightarrow{\alpha^B_{\mathrm{syn},\pi}} & C_{\mathrm{st}}(R\Gamma^B_{\mathrm{HK}}(Y)\{r\}) \\ \downarrow{\scriptstyle \varepsilon^*} & & \downarrow{\scriptstyle \varepsilon^*} \\ R\Gamma_{\mathrm{syn}}(Y_{\overline{K}}, r)_{\mathbb{Q}} & \xrightarrow{\alpha_{\mathrm{syn}}} & C^+(R\Gamma^B_{\mathrm{HK}}(Y_{\overline{K}})\{r\}) \end{array}$$

From the uniqueness property of the homotopy fiber functor, it suffices to show that the following diagram commutes canonically in the ∞-derived category:

$$\begin{array}{ccccccc} R\Gamma_{\mathrm{cr}}(Y)_{\mathbb{Q}} & \longrightarrow & R\Gamma_{\mathrm{cr}}(Y/R)^{N=0}_{\mathbb{Q}} & \underset{\sim}{\xleftarrow{\iota_\pi}} & R\Gamma^B_{\mathrm{HK}}(Y_1)^{\tau,N=0}_{R_{\mathbb{Q}}} & \underset{\sim}{\xleftarrow{\beta}} & R\Gamma^B_{\mathrm{HK}}(Y_1)^{N=0} \\ \downarrow & & & & & \swarrow & \\ R\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} & \underset{\sim}{\xleftarrow{\iota^B_{\mathrm{cr}}}} & R\Gamma^B_{\mathrm{HK}}(\overline{Y}_1)^{\tau,N=0}_{B^+_{\mathrm{cr}}} & = & (R\Gamma^B_{\mathrm{HK}}(\overline{Y}_1) \otimes_{K_0^{\mathrm{nr}}} B^+_{\mathrm{st}})^{N=0} & & \end{array}$$

To do that we will need the ring of periods $\widehat{A}_{\mathrm{st}}$ [Tsuji 1999, p. 253]. Set

$$\widehat{A}_{\mathrm{st},n} = H^0_{\mathrm{cr}}(\overline{V}^\times_n/R_n), \qquad \widehat{A}_{\mathrm{st}} = \varprojlim_n H^0_{\mathrm{cr}}(\overline{V}^\times_n/R_n).$$

The ring $\widehat{A}_{\mathrm{st},n}$ has a natural action of G_K, Frobenius φ, and a monodromy operator N. It is also equipped with a PD-filtration $F^i\widehat{A}_{\mathrm{st},n} = H^0_{\mathrm{cr}}(\overline{V}^\times_n/R_n, \mathscr{J}^{[i]}_{\mathrm{cr},n})$. We have a morphism $A_{\mathrm{cr},n} \to \widehat{A}_{\mathrm{st},n}$ induced by the map $H^0_{\mathrm{cr}}(\overline{V}_n/W_n(k)) \to H^0_{\mathrm{cr}}(\overline{V}^\times_n/R_n)$. It is compatible with the Galois action, the Frobenius, and the filtration. The natural map $R_n \to \widehat{A}_{\mathrm{st},n}$ is compatible with all the structures. We can view $\widehat{A}_{\mathrm{st},n}$ as the PD-envelope of the closed immersion

$$\overline{V}^\times_n \hookrightarrow A_{\mathrm{cr},n} \times_{W_n(k)} W_n(k)[X]^\times$$

defined by the map $\theta : A_{\mathrm{cr},n} \to \overline{V}_n$ and the projection $W_n(k)[X] \to \overline{V}_n$, $X \mapsto \pi$. This makes $\overline{V}_1^{\times} \hookrightarrow \widehat{A}_{\mathrm{st},n}$ into a PD-thickening in the crystalline site of $\overline{V}_1$. Set $\widehat{B}_{\mathrm{st}}^+ := \widehat{A}_{\mathrm{st}}[1/p]$.

Commutativity of the last diagram will follow from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{R}\Gamma_{\mathrm{cr}}(Y)_{\mathbb{Q}} & & \longrightarrow & & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} \\
\downarrow & & & \swarrow\sim & \\
\mathrm{R}\Gamma_{\mathrm{cr}}(Y/R)_{\mathbb{Q}}^{N=0} & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\widehat{A}_{\mathrm{st}})_{\mathbb{Q}}^{N=0} & & \iota_{\mathrm{cr}}^B \uparrow \wr \\
\iota_\pi \uparrow \wr & & \iota_{\widehat{B}_{\mathrm{st}}^+}^B \uparrow \wr & & \\
\mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{R_{\mathbb{Q}}}^{\tau,N=0} & \longrightarrow & \mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{\widehat{B}_{\mathrm{st}}^+}^{\tau,N=0} & \longleftarrow & \mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{B_{\mathrm{cr}}^+}^{\tau,N=0} \\
 & \nwarrow^{\sim}_{\beta} & & \nearrow & \\
 & & \mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)^{N=0} & &
\end{array}$$

as soon as we show that $\mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y})_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(\overline{Y}/\widehat{A}_{\mathrm{st}})_{\mathbb{Q}}^{N=0}$ is a quasi-isomorphism. Notice that the map $\iota_{\widehat{B}_{\mathrm{st}}^+}^B$ is a quasi-isomorphism by Theorem 3.6. Hence using the Beilinson–Hyodo–Kato maps $\iota_{\widehat{B}_{\mathrm{st}}^+}^B$ and ι_{cr}^B this reduces to proving that the canonical map

$$\mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{B_{\mathrm{cr}}^+}^{\tau,N=0} \to \mathrm{R}\Gamma_{\mathrm{HK}}^B(Y_1)_{\widehat{B}_{\mathrm{st}}^+}^{\tau,N=0}$$

is a quasi-isomorphism. In fact, we claim that for any (φ, N)-module M we have an isomorphism $M_{B_{\mathrm{cr}}^+}^{\tau,N=0} \xrightarrow{\sim} M_{\widehat{B}_{\mathrm{st}}^+}^{\tau,N=0}$. Indeed, assume first that the monodromy N_M is trivial. We calculate

$$\begin{aligned}
M_{B_{\mathrm{cr}}^+}^{\tau} &= (M \otimes_{K_0} B_{\mathrm{cr}}^{+,\tau})^{N'=0} = M \otimes_{K_0} (B_{\mathrm{cr}}^{+,\tau})^{N_\tau=0} = M \otimes_{K_0} B_{\mathrm{cr}}^+, \\
M_{\widehat{B}_{\mathrm{st}}^+}^{\tau} &= (M \otimes_{K_0} \widehat{B}_{\mathrm{st}}^{+,\tau})^{N'=0} = M \otimes_{K_0} (\widehat{B}_{\mathrm{cr}}^{+,\tau})^{N_\tau=0} = M \otimes_{K_0} \widehat{B}_{\mathrm{st}}^+, \\
N' &= N_M \otimes 1 + 1 \otimes N_\tau = 1 \otimes N_\tau.
\end{aligned}$$

Hence

$$M_{B_{\mathrm{cr}}^+}^{\tau,N=0} = M \otimes_{K_0} B_{\mathrm{cr}}^+ \quad \text{and} \quad M_{\widehat{B}_{\mathrm{st}}^+}^{\tau,N=0} = M \otimes_{K_0} (\widehat{B}_{\mathrm{st}}^+)^{N=0} = M \otimes_{K_0} B_{\mathrm{cr}}^+,$$

where the last equality is proved in [Tsuji 1999, Lemma 1.6.5]. We are done in this case.

In general, we can write $M \otimes_{K_0} B_{\mathrm{st}}^+ \xleftarrow{\sim} M' \otimes_{K_0} B_{\mathrm{st}}^+$ for a (φ, N)-module M' such that $N_{M'} = 0$ (take for M' the image of the map $M \to M \otimes_{K_0} B_{\mathrm{st}}^+$, $m \mapsto \exp(N_M(m)u)$ for $u \in B_{\mathrm{st}}^+$ such that $B_{\mathrm{st}}^+ = B_{\mathrm{cr}}^+[u]$, $N_\tau(u) = -1$). Similarly, using the fact that the ring B_{st}^+ is canonically (and compatibly with all the structures) isomorphic to the elements of $\widehat{B}_{\mathrm{st}}^+$ annihilated by a power of the monodromy operator

[Kato 1994, 3.7], we can write in a compatible way $M\otimes_{K_0}B_{\mathrm{st}}^+\xleftarrow{\sim}M'\otimes_{K_0}\widehat{B}_{\mathrm{st}}^+$ for the same module M'. We obtain a commutative diagram

$$\begin{array}{ccc} M^{\tau,N=0}_{B_{\mathrm{cr}}^+} & \longrightarrow & M^{\tau,N=0}_{\widehat{B}_{\mathrm{st}}^+} \\ \downarrow\wr & & \downarrow\wr \\ M'^{\tau,N=0}_{B_{\mathrm{cr}}^+} & \xrightarrow{\sim} & M'^{\tau,N=0}_{\widehat{B}_{\mathrm{st}}^+} \end{array}$$

that reduces the general case to the case of trivial monodromy on M that we treated above. □

Let $X\in\mathscr{V}ar_K$, $r\geq 0$. Set

$$C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}) := \left[\begin{array}{ccc} \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})^{G_K} & \xrightarrow{(1-\varphi_r,\iota^B_{\mathrm{dR}})} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})^{G_K}\oplus(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})/F^r)^{G_K} \\ \downarrow N & & \downarrow (N,0) \\ \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})^{G_K} & \xrightarrow{1-\varphi_{r-1}} & \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})^{G_K} \end{array}\right].$$

The above makes sense since the action of G_K on $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})$ is smooth. In particular, we have

$$H^j(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}^{G_K})\simeq H^j(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})^{G_K},$$
$$H^j(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h})^{G_K})\simeq H^j(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K},h}))^{G_K}.$$

Consider the canonical pullback map

$$\varepsilon^*: C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})\xrightarrow{\sim} C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\}).$$

By Proposition 3.22, this is a quasi-isomorphism. This allows us to construct a canonical spectral sequence (the *syntomic descent spectral sequence*)

$$^{\mathrm{syn}}E_2^{i,j}=H^i_{\mathrm{st}}(G_K,H^j(X_{\bar{K},\mathrm{\acute{e}t}},\mathbb{Q}_p(r)))\Longrightarrow H^{i+j}_{\mathrm{syn}}(X_h,r)\ . \tag{50}$$

Indeed, the Bousfield–Kan spectral sequences associated to the homotopy limits defining complexes $C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$ and $C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})$ give us the commutative diagram

$$\begin{array}{ccc} ^{\mathrm{pst}}E_2^{i,j}=H^i(C_{\mathrm{pst}}(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) & \Longrightarrow & H^{i+j}(C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \\ \wr\uparrow\varepsilon^* & & \wr\uparrow\varepsilon^* \\ ^{\mathrm{syn}}E_2^{i,j}=H^i(C_{\mathrm{st}}(H^j_{\mathrm{HK}}(X_h)\{r\})) & \Longrightarrow & H^{i+j}(C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})) \end{array}$$

Since, by Proposition 3.20, we have $\alpha_{\mathrm{syn}}\colon H^{i+j}_{\mathrm{syn}}(X_h,r)\xrightarrow{\sim} H^{i+j}(C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\}))$, we have obtained a spectral sequence

$$E_2^{i,j}=H^i(C_{\mathrm{pst}}(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\}))\Longrightarrow H^{i+j}_{\mathrm{syn}}(X_h,r).$$

It remains to show that there is a canonical isomorphism

$$H^i(C_{\mathrm{pst}}(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\}))\simeq H^i_{\mathrm{st}}(G_K,H^j(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r))). \tag{51}$$

But, we have $D_j=H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\}\in MF^{\mathrm{ad}}_K(\varphi,N,G_K)$,

$$V_{\mathrm{pst}}(D_j)\simeq H^j(X_{\bar{K},\text{ét}},\mathbb{Q}(r))\quad\text{and}\quad D_{\mathrm{pst}}(H^j(X_{\bar{K},\text{ét}},\mathbb{Q}(r)))\simeq D_j.$$

Hence isomorphism (51) follows from Remark 2.12 and we have obtained the spectral sequence (50).

4B. ***Arithmetic case.*** In this subsection, we define the arithmetic syntomic period map by Galois descent from the geometric case. Then we show that, via this period map, the syntomic descent spectral sequence and the étale Hochschild–Serre spectral sequence are compatible. Finally, we show that this implies that the arithmetic syntomic cohomology and étale cohomology are isomorphic in a stable range.

Let $X\in\mathscr{V}ar_K$. For $r\geq 0$, we define the canonical syntomic period map

$$\rho_{\mathrm{syn}}:\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r)\to\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p(r))$$

as the composition

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,r)=\mathrm{R}\Gamma(X_h,\mathscr{S}(r))_{\mathbb{Q}}&\to\operatorname{holim}_n\mathrm{R}\Gamma(X_h,\mathscr{S}_n(r))_{\mathbb{Q}}\\ &\xrightarrow{\varepsilon^*}\operatorname{holim}_n\mathrm{R}\Gamma(G_K,\mathrm{R}\Gamma(X_{\bar{K},h},\mathscr{S}_n(r)))_{\mathbb{Q}}\\ &\xrightarrow{p^{-r}\beta}\operatorname{holim}_n\mathrm{R}\Gamma(G_K,\mathrm{R}\Gamma(X_{\bar{K},\text{ét}},\mathbb{Z}/p^n(r)'))_{\mathbb{Q}}\\ &\xleftarrow{\sim}\operatorname{holim}_n\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Z}/p^n(r)')_{\mathbb{Q}}=\mathrm{R}\Gamma(X_{\text{ét}},\mathbb{Q}_p(r)).\end{aligned}$$

It induces a morphism of graded E_∞ algebras over $\mathbb{Q}_p$.

The syntomic period map ρ_{syn} is compatible with the syntomic descent and the Hochschild–Serre spectral sequences.

Theorem 4.8. *For $X\in\mathscr{V}ar_K$, $r\geq 0$, there is a canonical map of spectral sequences*

$$\begin{array}{ccc}{}^{\mathrm{syn}}E_2^{i,j}=H^i_{\mathrm{st}}(G_K,H^j(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r))) & \Longrightarrow & H^{i+j}_{\mathrm{syn}}(X_h,r)\\ \Big\downarrow{\scriptstyle\mathrm{can}} & & \Big\downarrow{\scriptstyle\rho_{\mathrm{syn}}}\\ {}^{\text{ét}}E_2^{i,j}=H^i(G_K,H^j(X_{\bar{K},\text{ét}},\mathbb{Q}_p(r))) & \Longrightarrow & H^{i+j}(X_{\text{ét}},\mathbb{Q}_p(r))\end{array}$$

Proof. We work in the (classical) derived category. The Bousfield–Kan spectral sequences associated to the homotopy limits defining complexes $C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$

and $C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$, and Theorem 2.18 give us the commutative diagram of spectral sequences

$$\begin{array}{ccc} {}^{II}E_2^{i,j} = H^i(G_K, C(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) & \Longrightarrow & H^{i+j}(G_K, C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \\ \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\ {}^{\mathrm{pst}}E_2^{i,j} = H^i(C_{\mathrm{pst}}(H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) & \Longrightarrow & H^{i+j}(C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})) \end{array}$$

More specifically, in the language of Section 2E, set $X = C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\})$ (hopefully, the notation will not be too confusing). Filtering complex X in the direction of the homotopy limit, we obtain a Postnikov system (14) with $Y^i = 0$, $i \geq 3$, and

$$\begin{aligned} Y^0 &= \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\} \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}, \\ Y^1 &= \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r-1\} \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} \\ &\qquad\qquad \oplus \big(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r\} \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} \oplus (\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\bar{K}}) \otimes_{\bar{K}} B_{\mathrm{dR}})/F^r\big), \\ Y^2 &= \mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\{r-1\} \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}. \end{aligned}$$

Still in the setting of Section 2E, take for A the abelian category of sheaves of abelian groups on the pro-étale site $\mathrm{Spec}(K)_{\text{proét}}$ of Scholze [2013, Section 3].

Remark 4.9. We work with the pro-étale site to make sense of the continuous cohomology $\mathrm{R}\Gamma(G_K, \cdot)$. If the reader is willing to accept that this is possible then he can skip the tedious parts of the proof involving passage to the pro-étale site (and existence of continuous sections).

Recall that there is a projection map $\nu : \mathrm{Spec}(K)_{\text{proét}} \to \mathrm{Spec}(K)_{\text{ét}}$ such that, for an étale sheaf $\mathscr{F}$, we have the quasi-isomorphism $\nu^* : \mathscr{F} \xrightarrow{\sim} \mathrm{R}\nu_*\nu^*\mathscr{F}$ [Bhatt and Scholze 2015, Proposition 5.2.6]. More generally, for a topological G_K-module M, we get a sheaf νM on $\mathrm{Spec}(K)_{\text{proét}}$ by setting $\nu M(S) = \mathrm{Hom}_{\mathrm{cont},G_K}(S, M)$ for a profinite G_K-set S, and Scholze [2013, Proposition 3.7(iii); 2016] showed that there is a canonical quasi-isomorphism

$$H^*(\mathrm{Spec}(K)_{\text{proét}}, \nu M) \simeq H^*_{\mathrm{cont}}(G_K, M).$$

In this proof we will need this kind of quasi-isomorphism for complexes M as well and this will require extra arguments. For that, observe that the functor ν is left exact. To study right exactness, it suffices to look at the global sections on profinite sets S with a free G_K-action of the form $S = S' \times G_K$ for a profinite set S' with trivial G_K-action.[8] Then, for any G_K-module T, we have $\Gamma(S, \nu T) = \mathrm{Hom}_{\mathrm{cont}}(S', T)$. It follows that, for a surjective map $T_1 \to T_2$ of G_K-modules, the pullback map

[8] To see this, for a profinite G_K-set S', use the covering $S' \times G_K \to S'$, where the first S' has trivial G_K-action, induced from the G_K-action on S'.

$\nu T_1 \to \nu T_2$ is also surjective if the original map had a continuous set-theoretical section. This is a criterion familiar from continuous cohomology and we will use it often.

We will see the complex X as a complex of sheaves on the site $\operatorname{Spec}(K)_{\text{proét}}$ in the following way: represent $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})$ by (filtered) perfect complexes of K_0^{nr}- and $\overline{K}$-modules respectively, think of X as νX, and work on the pro-étale site. This makes sense, i.e., functor ν transfers (filtered) quasi-isomorphisms of representatives of $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})$ and $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})$ to quasi-isomorphisms of the corresponding sheaves νX. To see this, look at the Postnikov system of sheaves on $\operatorname{Spec}(K)_{\text{proét}}$ obtained by pulling back by ν the above Postnikov system. Now, look at the global sections on profinite sets $S = S' \times G_K$ as above and note that we have $\Gamma(S, \nu Y^0) = \operatorname{Hom}_{\text{cont}}(S', Y^0)$. Conclude that, by perfectness of the Beilinson–Hyodo–Kato complexes, quasi-isomorphisms of representatives of $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})$ yield quasi-isomorphisms of the sheaves νY^0. By a similar argument, we get the analogous statement for Y^2. For Y^1, we just have to show that filtered quasi-isomorphisms of representatives of $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})$ yield quasi-isomorphisms of the sheaves $\nu((\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} B_{\mathrm{dR}})/F^r)$. Again, we look at the global section on $S = S' \times G_K$ as above. By compactness of S', we may replace $(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} B_{\mathrm{dR}})/F^r$ by $(t^{-i}\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} B^+_{\mathrm{dR}})/F^r$ for some $i \geq 0$, where, using devissage, we can again argue by (filtered) perfection of $\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K}})$. Observe that the same argument shows that $\mathscr{H}^j(\nu Y^i) \simeq \nu H^j(Y^i)$ for $i = 0, 1, 2$.

The above Postnikov system gives rise to an exact couple

$$D_1^{i,j} = \mathscr{H}^j(X^i), \quad E_1^{i,j} = \mathscr{H}^j(Y^i) \Rightarrow \mathscr{H}^{i+j}(X).$$

This is the Bousfield–Kan spectral sequence associated to X.

Consider now the complex $X_{\mathrm{pst}} := C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})\{r\})$. We claim that the canonical map

$$C_{\mathrm{pst}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})\{r\}) \xrightarrow{\sim} C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})\{r\})^{G_K}$$

is a quasi-isomorphism (recall that taking G_K-fixed points corresponds to taking global sections on the pro-étale site), and, in particular, that the term on the right-hand side makes sense. To see this, it suffices to show that the canonical maps

$$(\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K},h})/F^r)^{G_K} \xrightarrow{\sim} ((\mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K},h}) \otimes_{\overline{K}} B_{\mathrm{dR}})/F^r)^{G_K},$$
$$\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})^{G_K} \xrightarrow{\sim} (\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h}) \otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}})^{G_K}$$

are quasi-isomorphisms and to use the fact that the action of G_K on $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})$ is smooth. The fact that the first map is a quasi-isomorphism follows from the filtered quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_K \overline{K} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_{\overline{K},h})$ and the fact that $B_{\mathrm{dR}}^{G_K} = K$. Similarly, the second map is a quasi-isomorphism because, by [Fontaine 1994a, 4.2.4], $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})$ is the subcomplex of those elements of $\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\bar{K},h})\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}}$ whose stabilizers in G_K are open.

Taking the G_K-fixed points of the above Postnikov system we get an exact couple

$$\begin{aligned}{}^{\mathrm{pst}}D_1^{i,j}&=H^j(X^i_{\mathrm{pst}}),\\ {}^{\mathrm{pst}}E_1^{i,j}&=H^j(Y^i_{\mathrm{pst}})\Rightarrow H^{i+j}(X_{\mathrm{pst}})\end{aligned}$$

corresponding to the Bousfield–Kan filtration of the complex X_{pst}. On the other hand, applying $\mathrm{R}\Gamma(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}},\cdot)$ to the same Postnikov system, we obtain an exact couple

$$\begin{aligned}{}^{I}D_1^{i,j}&=H^j(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}},X^i),\\ {}^{I}E_1^{i,j}&=H^j(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}},Y^i)\Rightarrow H^{i+j}(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}},X)\end{aligned}$$

together with a natural map of exact couples $({}^{\mathrm{pst}}D_1^{i,j},{}^{\mathrm{pst}}E_1^{i,j})\to({}^{I}D_1^{i,j},{}^{I}E_1^{i,j})$.

We also have the hypercohomology exact couple

$$\begin{aligned}{}^{II}D_2^{i,j}&=H^{i+j}(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}},\tau_{\leq j-1}X),\\ {}^{II}E_2^{i,j}&=H^{i}(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}},\mathscr{H}^j(X))\Rightarrow H^{i+j}(\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}},X).\end{aligned}$$

Theorem 2.18 gives us a natural morphism of exact couples $({}^{I}D_2^{i,j},{}^{I}E_2^{i,j})\to({}^{II}D_2^{i,j},{}^{II}E_2^{i,j})$ — hence a natural morphism of spectral sequences ${}^{I}E_2^{i,j}\to{}^{II}E_2^{i,j}$ compatible with the identity map on the common abutment — if our original Postnikov system satisfies the equivalent conditions in Remark 2.17. We will check the condition (4), i.e., that the following long sequence is exact for all j:

$$0\to\mathscr{H}^j(X)\to\mathscr{H}^j(Y^0)\to\mathscr{H}^j(Y^1)\to\mathscr{H}^j(Y^2)\to 0.$$

For that it is enough to show that

(1) $\mathscr{H}^j(\nu Y^i)\simeq\nu H^j(Y^i)$ for $i=0,1,2$;

(2) $\mathscr{H}^j(\nu X)\simeq\nu H^j(X)$;

(3) the following long sequence of G_K-modules

$$0\to H^j(X)\to H^j(Y^0)\to H^j(Y^1)\to H^j(Y^2)\to 0$$

is exact;

(4) the pullback ν preserves its exactness.

The assertion in (1) was shown above. The sequence in (3) is equal to the top sequence in the following commutative diagram (where we set $M=H^j_{\mathrm{HK}}(X_{\bar{K},h})$, $M_{\mathrm{dR}}=H^j_{\mathrm{dR}}(X_{\bar{K},h})$, and $E=H^j(X_{\bar{K},\mathrm{\acute{e}t}},\mathbb{Q}_p)$):

$$\begin{array}{ccccccc} H^j(X) & \longrightarrow & M\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} & \xrightarrow{(N,1-\varphi_r,\iota)} & \begin{array}{c} M\otimes_{K_0^{\mathrm{nr}}}(B_{\mathrm{st}}\oplus B_{\mathrm{st}}) \\ \oplus (M_{\mathrm{dR}}\otimes_{\overline{K}} B_{\mathrm{dR}})/F^r \end{array} & \xrightarrow{(1-\varphi_{r-1})-N} & M\otimes_{K_0^{\mathrm{nr}}} B_{\mathrm{st}} \\ \wr\downarrow \alpha_{\mathrm{\acute{e}t}}^{-1} & & \wr\downarrow \rho_{\mathrm{HK}} & & \wr\downarrow \rho_{\mathrm{HK}}+\rho_{\mathrm{HK}}+\rho_{\mathrm{dR}} & & \wr\downarrow \rho_{\mathrm{HK}} \\ E(r) & \hookrightarrow & E\otimes B_{\mathrm{st}} & \xrightarrow{(N,1-\varphi_r,\iota)} & \begin{array}{c} E\otimes(B_{\mathrm{st}}\oplus B_{\mathrm{st}}) \\ \oplus E\otimes B_{\mathrm{dR}}/F^r \end{array} & \xrightarrow{(1-\varphi_{r-1})-N} & E\otimes B_{\mathrm{st}} \end{array}$$

Since the bottom sequence is just a fundamental exact sequence of p-adic Hodge theory, the top sequence is exact, as wanted.

To prove assertion (4), we pass to the bottom exact sequence above and apply ν to it. It is easy to see that it enough now to show that the following surjections have continuous $\mathbb{Q}_p$-linear sections:

$$B_{\mathrm{st}} \xrightarrow{N} B_{\mathrm{st}}, \qquad B_{\mathrm{cr}} \xrightarrow{(1-\varphi_r,\mathrm{can})} B_{\mathrm{cr}} \oplus B_{\mathrm{dR}}/F^r.$$

For the monodromy, write $B_{\mathrm{st}} = B_{\mathrm{cr}}[u_s]$ and take for a continuous section the map induced by $bu_s^i \mapsto -(b/(i+1))u_s^{i+1}$, $b \in B_{\mathrm{cr}}$. For the second map, the existence of continuous section was proved in [Bloch and Kato 1990, 1.18]. For a different argument: observe that an analogous statement was proved in [Colmez 1998, Proposition II.3.1] with $B_{\max}$ in place of B_{cr} as a consequence of the general theory of p-adic Banach spaces. We will just modify it here. Write $A_i = t^{-i}B_{\mathrm{cr}}^+$ and $B_i = t^{-i}B_{\mathrm{cr}}^+ \oplus t^{-i}B_{\mathrm{dR}}^+/t^r$ for $i \geq 1$. These are p-adic Banach spaces. Observe that $B_i \subset B_{i+1}$ is closed. Indeed, it is enough to show that $tB_{\mathrm{cr}}^+ \subset B_{\mathrm{cr}}^+$ is closed. But we have $tB_{\mathrm{cr}}^+ = \bigcap_{n\geq 0}\ker(\theta\circ\varphi^n)$.

It follows [Colmez 1998, Proposition I.1.5] that we can find a closed complement C_{i+1} of B_i in B_{i+1}. Set $f = (1-\varphi_r, \mathrm{can}) : B_{\mathrm{cr}} \to B_{\mathrm{cr}} \oplus B_{\mathrm{dR}}/F^r$. We know that f maps A_i onto B_i. Write $t^{-i}B_{\mathrm{cr}}^+ \oplus t^{-i}B_{\mathrm{dR}}^+/t^r = B_1 \oplus \big(\bigoplus_{j=2}^{i-1} C_j\big)$. By [Colmez 1998, Proposition I.1.5], we can find a continuous section $s_1 : B_1 \to A_1$ of f and, if $i \geq 2$, a continuous section $s_i : C_i \to A_i$ of f. Define the map $s : t^{-i}B_{\mathrm{cr}}^+ \oplus t^{-i}B_{\mathrm{dR}}^+/t^r \to B_{\mathrm{cr}}$ by s_1 on B_1 and by s_i on C_i for $i \geq 2$. Taking the inductive limit over i, we get our section of f.

Finally, to prove assertion (2), take a perfect representative of the complex $\mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}}, \mathbb{Z}_p(r))$. Consider the complex $Z = \mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r))$ as a complex of sheaves on $\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}}$. As before, we see that this makes sense and we easily find that (canonically) $\mathscr{H}^j(Z) \simeq \nu H^j(X_{\overline{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r))$. To prove (2), it is enough to show that we can also pass with the map $\alpha_{\mathrm{\acute{e}t}} : \mathrm{R}\Gamma(X_{\overline{K},\mathrm{\acute{e}t}}, \mathbb{Q}_p(r)) \xrightarrow{\sim} C(\mathrm{R}\Gamma_{\mathrm{HK}}^B(X_{\overline{K},h})\{r\})$ to the site $\mathrm{Spec}(K)_{\mathrm{pro\acute{e}t}}$. Looking at its definition (see (48)), we see that we need to show that the period quasi-isomorphisms ρ_{cr}, ρ_{HK}, ρ_{dR} as well as the quasi-isomorphism

$$\mathbb{Q}_p(r) \xrightarrow{\sim} \big[B_{\mathrm{st}} \xrightarrow{(N,1-\varphi_r,\iota)} B_{\mathrm{st}} \oplus B_{\mathrm{st}} \oplus B_{\mathrm{dR}}/F^r \xrightarrow{(1-\varphi_{r-1})-N} B_{\mathrm{st}}\big]$$

can be lifted to the pro-étale site. The last fact we have just shown. For the crystalline period map ρ_{cr}, this follows from the fact that it is defined integrally and all the relevant complexes are perfect. For the Hyodo–Kato period map ρ_{HK}, it follows from the case of ρ_{cr} and from perfection of complexes involved in the definition of the Beilinson–Hyodo–Kato map. For the de Rham period map ρ_{dR}, this follows from perfection of the involved complexes as well as from the exactness of holim_n (in the definition of ρ_{dR}) on the pro-étale site of K (see [Scholze 2013, Lemma 3.18]).

We define the map of spectral sequences $\delta := (\delta_D, \delta) := ({}^{\mathrm{pst}}D_2^{i,j}, {}^{\mathrm{pst}}E_2^{i,j}) \to ({}^{II}D_2^{i,j}, {}^{II}E_2^{i,j})$ — which we stated at the beginning of the proof — as the composition of the two maps constructed above:

$$\delta : ({}^{\mathrm{pst}}D_2^{i,j}, {}^{\mathrm{pst}}E_2^{i,j}) \to ({}^{I}D_2^{i,j}, {}^{I}E_2^{i,j}) \to ({}^{II}D_2^{i,j}, {}^{II}E_2^{i,j}).$$

To get the spectral sequence from the theorem, we need to pass from ${}^{II}E_2$ to the Hochschild–Serre spectral sequence. To do that, consider the hypercohomology exact couple

$$\begin{aligned} {}^{\text{ét}}D_2^{i,j} &= H^{i+j}(\mathrm{Spec}(K)_{\text{proét}}, \tau_{\leq j-1}Z), \\ {}^{\text{ét}}E_2^{i,j} &= H^{i}(\mathrm{Spec}(K)_{\text{proét}}, \mathscr{H}^j(Z)) \Rightarrow H^{i+j}(\mathrm{Spec}(K)_{\text{proét}}, Z) \end{aligned}$$

and, via $\alpha_{\text{ét}}^{-1}$, a natural morphism of exact couples $({}^{II}D_2^{i,j}, {}^{II}E_2^{i,j}) \to ({}^{\text{ét}}D_2^{i,j}, {}^{\text{ét}}E_2^{i,j})$, and hence a natural morphism of spectral sequences ${}^{II}E_2^{i,j} \to {}^{\text{ét}}E_2^{i,j}$ compatible with the map $\alpha_{\text{ét}}^{-1}$ on the abutment. We have a quasi-isomorphism

$$\psi : \mathrm{R}\Gamma(\mathrm{Spec}(K)_{\text{proét}}, Z) \xrightarrow{\sim} \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r))$$

defined as the composition

$$\begin{aligned} \psi : \mathrm{R}\Gamma(\mathrm{Spec}(K)_{\text{proét}}, &\mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r))) \\ &\xrightarrow{\sim} \mathbb{Q} \otimes \mathrm{holim}_n \mathrm{R}\Gamma(G_K, \mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Z}/p^n(r))) \\ &= \mathbb{Q} \otimes \mathrm{holim}_n \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n(r)) = \mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}(r)). \end{aligned}$$

We have obtained the natural maps of spectral sequences

$$\begin{array}{ccc} {}^{\mathrm{syn}}E_2^{i,j} = H^i_{\mathrm{st}}(G_K, H^j(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r))) & \Longrightarrow & H^{i+j}_{\mathrm{syn}}(X_h, r) \\ \downarrow \wr & & \wr \downarrow \alpha_{\mathrm{syn}} \\ E_2^{i,j} = H^i(C_{\mathrm{st}}(H^j_{\mathrm{HK}}(X_h)\{r\})) & \Longrightarrow & H^{i+j}(C_{\mathrm{st}}(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_h)\{r\})) \\ \downarrow \alpha_{\text{ét}}^{-1}\delta\varepsilon^* & & \downarrow \psi\alpha_{\text{ét}}^{-1}\delta\varepsilon^* \\ {}^{\text{ét}}E_2^{i,j} = H^i(G_K, H^j(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r))) & \Longrightarrow & H^{i+j}(X_{\text{ét}}, \mathbb{Q}_p(r)) \end{array}$$

It remains to show that the right vertical composition

$$\gamma : H^{i+j}_{\mathrm{syn}}(X_h, r) \to H^{i+j}(X_{\text{ét}}, \mathbb{Q}_p(r))$$

is equal to the map ρ_{syn}. Since we have the equality $\alpha_{\mathrm{syn}} = \rho_{\mathrm{syn}}\alpha_{\text{ét}}$ (in the derived category) from (49) and, by Lemma 4.7, $\varepsilon^*\alpha_{\mathrm{syn}} = \alpha_{\mathrm{syn}}\varepsilon^*$, the map γ can be written as the composition

$$\begin{aligned}\tilde{\rho}_{\mathrm{syn}} : H^{i+j}_{\mathrm{syn}}(X_h, r) &\xrightarrow{\varepsilon^*} H^{i+j}(\mathrm{Spec}(K)_{\text{proét}}, \nu\mathrm{R}\Gamma_{\mathrm{syn}}(X_{\bar{K},h}, r))\\ &\xrightarrow{\rho_{\mathrm{syn}}} H^{i+j}(\mathrm{Spec}(K)_{\text{proét}}, \nu\mathrm{R}\Gamma(X_{\bar{K},\text{ét}}, \mathbb{Q}_p(r)))\\ &\xrightarrow{\psi} H^{i+j}(X_{\text{ét}}, \mathbb{Q}_p(r)),\end{aligned}$$

where the period map ρ_{syn} is understood to be on sheaves on $\mathrm{Spec}(K)_{\text{proét}}$. There is no problem with that since we care only about the induced map on cohomology groups. It is easy now to see that $\tilde{\rho}_{\mathrm{syn}} = \rho_{\mathrm{syn}}$, as wanted. □

Remark 4.10. If X is proper and smooth, it is known that the étale Hochschild–Serre spectral sequence degenerates, i.e., ${}^{\text{ét}}E_2 = {}^{\text{ét}}E_\infty$. It is very likely that so does the syntomic descent spectral sequence in this case, i.e., ${}^{\mathrm{syn}}E_2 = {}^{\mathrm{syn}}E_\infty$.[9]

Corollary 4.11. *For $X \in \mathscr{V}ar_K$, we have a canonical quasi-isomorphism*

$$\rho_{\mathrm{syn}} : \tau_{\le r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r)_{\mathbb{Q}} \xrightarrow{\sim} \tau_{\le r}\mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r)).$$

Proof. By Theorem 4.8, the syntomic descent and the Hochschild–Serre spectral sequence are compatible. We have $D_j = H^j_{\mathrm{HK}}(X_{\bar{K},h})\{r\} \in MF^{\mathrm{ad}}_K(\varphi, N, G_K)$. For $j \le r$, we know $F^1 D_{j,K} = F^{1-(r-j)} H^j_{\mathrm{dR}}(X_h) = 0$. Hence, by Proposition 2.16, we have ${}^{\mathrm{syn}}E_2^{i,j} \xrightarrow{\sim} {}^{\text{ét}}E_2^{i,j}$. This implies $\rho_{\mathrm{syn}} : \tau_{\le r}\mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} \tau_{\le r}\mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{Q}_p(r))$, as wanted. □

Remark 4.12. All of the above automatically extends to finite diagrams of K-varieties, and hence to essentially finite diagrams of K-varieties (i.e., the diagrams for which every truncation of their cohomology $\tau_{\le n}$ is computed by truncating the cohomology of some finite diagram). This includes, in particular, simplicial and cubical varieties.

Proposition 4.13. *Let $X \in \mathscr{V}ar_K$ and $i \ge 0$. The composition*

$$H^q_{\mathrm{dR}}(X)/F^r \xrightarrow{\partial} H^{q+1}_{\mathrm{syn}}(X_h, r) \xrightarrow{\rho_{\mathrm{syn}}} H^{q+1}_{\text{ét}}(X, \mathbb{Q}_p(r)) \to H^{q+1}_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))$$

is the zero map. The map induced by the syntomic descent spectral sequence

$$H^q_{\mathrm{dR}}(X)/F^r \to H^1(G_K, H^q_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r)))$$

is equal to the Bloch–Kato exponential associated with the Galois representation $V^q(r) = H^q_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))$.

[9] This was, in fact, shown in [Déglise and Nizioł 2015].

Proof. In what follows, we will omit the passage to the pro-étale site. Consider the Postnikov system from the proof of Theorem 4.8, which arises from the complex $X = C(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})\{r\})$; then $Y^p = C^p(\mathrm{R}\Gamma^B_{\mathrm{HK}}(X_{\overline{K},h})\{r\})$. The discussion from Example 2.19 then applies to the functor $f(-) = (-)^{G_K}$ and yields the following four exact couples.

(1) $D_1^{p,q} = H^q(X^p)$ and $E_1^{p,q} = H^q(Y^p) = C^p(H^q_{\mathrm{HK}}(X_{\overline{K},h})\{r\})) = C^p(H^q_{\mathrm{HK}}\{r\})$. The corresponding quasi-isomorphism $H^q(X) \xrightarrow{\sim} E_1^{\bullet,q}$ is then identified, via the various period maps, with

$$V^q(r) \xrightarrow{\sim} C(H^q_{\mathrm{HK}}\{r\}) = C(D_{\mathrm{pst}}(V^q(r))).$$

(2) ${}^fD_1^{p,q} = H^q(f(X^p))$ and ${}^fE_1^{p,q} = H^q(f(Y^p)) = f(H^q(Y^p)) = C^p_{\mathrm{st}}(H^q_{\mathrm{HK}}\{r\}) = f(E_1^{p,q})$.

(3) ${}^ID_1^{p,q} = (\mathrm{R}^q f)(X^p)$ and ${}^IE_1^{p,q} = (\mathrm{R}^q f)(Y^p)$.

(4) ${}^{II}D_2^{p,q} = (\mathrm{R}^{p+q} f)(\tau_{\leq q-1}X)$ and ${}^{II}E_2^{p,q} = (\mathrm{R}^p f)(H^q(X)) = H^p(G_K, V^q(r))$.

There is a canonical morphism of exact couples (2) → (3) and a morphism (3) → (4) given by the maps (u, v) from the proof of Theorem 2.18. As observed in Remark 2.14, the Bloch–Kato exponential for $V = V^q(r)$ is obtained by applying $R^0 f$ to

$$\begin{aligned} Z^1C(H^q_{\mathrm{HK}}\{r\}) = Z^1(E_1^{\bullet,q}) &\xrightarrow{\text{can}} (\sigma_{\geq 1}C(H^q_{\mathrm{HK}}\{r\}))[1] = (\sigma_{\geq 1}C(E_1^{\bullet,q}))[1] \\ &\xrightarrow{-\text{can}} C(H^q_{\mathrm{HK}}\{r\})[1] = E_1^{\bullet,q}[1] \\ &\xleftarrow{\sim} V^q(r)[1] = H^q(X)[1], \end{aligned}$$

and hence is equal to the composite map

$$f(Z^1(E_1^{\bullet,q})) = Z^1({}^fE_1^{\bullet,q}) \to {}^fE_2^{1,q} \xrightarrow{\text{can}} {}^IE_2^{1,q} \xrightarrow{-v'=v} (R^1 f)(E_1^{\bullet,q}) = {}^{II}E_2^{p,q},$$

which coincides, in turn, with

$$Z^1C_{\mathrm{st}}(H^q_{\mathrm{HK}}\{r\}) \xrightarrow{\text{can}} H^1_{\mathrm{st}}(G_K, V^q(r)) \to H^1(G_K, V^q(r)).$$

After restricting to the de Rham part of $Z^1C(H^q_{\mathrm{HK}}\{r\})$, we obtain the desired statement about $H^q_{\mathrm{dR}}(X)/F^r$. □

In more concrete terms, the above proposition says that the following diagram commutes:

$$\begin{array}{ccc} H^{q+1}_{\mathrm{syn}}(X_h, r)_0 & \xrightarrow{\rho_{\mathrm{syn}}} & H^{q+1}_{\text{ét}}(X, \mathbb{Q}_p(r))_0 \\ \uparrow{\scriptstyle\partial} & & \downarrow \\ H^q_{\mathrm{dR}}(X)/F^r & \xrightarrow{\exp_{\mathrm{BK}}} & H^1(G_K, H^q_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))) \end{array}$$

where the subscript 0 refers to the classes that vanish in $H^{q+1}_{\mathrm{syn}}(X_{\overline{K},h}, r)$ and $H^{q+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))$, respectively.

Remark 4.14. Assume that $r > q$. Then in the above diagram all the maps are isomorphisms. Indeed, we have $F^r H^q_{\mathrm{dR}}(X) = 0$. By [Berger 2002, Theorem 6.8], the map $\exp_{\mathrm{BK}}$ is an isomorphism. By Proposition 4.6 and Corollary 4.11, so is the period map ρ_{syn}. Since, by Theorem A.1,

$$H^2(G_K, H^q_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))) = H^2(G_K, H^{q-1}_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))) = 0,$$

the vertical map is an isomorphism as well. Hence so is the map ∂.

5. Syntomic regulators

In this section, we prove that Soulé's étale regulators land in the semistable Selmer groups. This will be done by constructing syntomic regulators that are compatible with the étale ones via the period map and by exploiting the syntomic descent spectral sequence.

5A. ***Construction of syntomic Chern classes.*** We start with the construction of syntomic Chern classes. This will be standard once we prove that syntomic cohomology satisfies the projective space theorem and homotopy property.

In this subsection we will work in the (classical) derived category. For a fine log-scheme (X, M), log-smooth over $V^\times$, we have the log-crystalline and log-syntomic first Chern class maps of complexes of sheaves on $X_{\text{ét}}$ [Tsuji 1999, (2.2.3)]

$$\begin{aligned}
c_1^{\mathrm{cr}} &: j_*\mathcal{O}^*_{X_{\mathrm{tr}}} \xrightarrow{\sim} M^{\mathrm{gp}} \to M_n^{\mathrm{gp}} \to R\varepsilon_*\mathcal{J}^{[1]}_{X_n/W_n(k)}[1],\\
c_1^{\mathrm{st}} &: j_*\mathcal{O}^*_{X_{\mathrm{tr}}} \xrightarrow{\sim} M^{\mathrm{gp}} \to M_n^{\mathrm{gp}} \to R\varepsilon_*\mathcal{J}^{[1]}_{X_n/R_n}[1],\\
c_1^{\mathrm{HK}} &: j_*\mathcal{O}^*_{X_{\mathrm{tr}}} \xrightarrow{\sim} M^{\mathrm{gp}} \to M_0^{\mathrm{gp}} \to R\varepsilon_*\mathcal{J}^{[1]}_{X_0/W_n(k)^0}[1],\\
c_1^{\mathrm{syn}} &: j_*\mathcal{O}^*_{X_{\mathrm{tr}}} \xrightarrow{\sim} M^{\mathrm{gp}} \to \mathcal{S}(1)_{X,\mathbb{Q}}[1].
\end{aligned}$$

Here ε is the projection from the corresponding crystalline site to the étale site. The maps c_1^{cr}, c_1^{st}, and c_1^{syn} are clearly compatible. So are the maps c_1^{st} and c_1^{HK}. For ss-pairs $(U, \bar{U})$ over K, we get the induced functorial maps

$$\begin{aligned}
c_1^{\mathrm{cr}} &: \Gamma(U, \mathcal{O}^*_U) \xleftarrow{\sim} \Gamma(\bar{U}, j_*\mathcal{O}^*_U) \to \mathrm{R}\Gamma_{\mathrm{cr}}(U, \bar{U}, \mathcal{J}^{[1]})[1],\\
c_1^{\mathrm{st}} &: \Gamma(U, \mathcal{O}^*_U) \to \mathrm{R}\Gamma_{\mathrm{cr}}((U, \bar{U})/R, \mathcal{J}^{[1]})[1],\\
c_1^{\mathrm{HK}} &: \Gamma(U, \mathcal{O}^*_U) \to \mathrm{R}\Gamma_{\mathrm{cr}}((U, \bar{U})_0/W_n(k)^0, \mathcal{J}^{[1]})[1],\\
c_1^{\mathrm{syn}} &: \Gamma(U, \mathcal{O}^*_U) \to \mathrm{R}\Gamma_{\mathrm{syn}}(U, \bar{U}, 1)_{\mathbb{Q}}[1].
\end{aligned}$$

For $X \in \mathcal{V}ar_K$, we can glue the absolute log-crystalline and log-syntomic classes to obtain the absolute crystalline and syntomic first Chern class maps

$$c_1^{\mathrm{cr}} : \mathcal{O}^*_{X_h} \to \mathcal{J}_{\mathrm{cr},X}[1], \quad c_1^{\mathrm{syn}} : \mathcal{O}^*_{X_h} \to \mathcal{S}(1)_{X,\mathbb{Q}}[1].$$

They induce (compatible) maps

$$c_1^{\mathrm{cr}}:\mathrm{Pic}(X)=H^1(X_{\text{ét}},\mathscr{O}_X^*)\to H^1(X_h,\mathscr{O}_X^*)\xrightarrow{c_1^{\mathrm{cr}}}H^2(X_h,\mathscr{J}_{\mathrm{cr}}),$$
$$c_1^{\mathrm{syn}}:\mathrm{Pic}(X)=H^1(X_{\text{ét}},\mathscr{O}_X^*)\to H^1(X_h,\mathscr{O}_X^*)\xrightarrow{c_1^{\mathrm{syn}}}H^2_{\mathrm{syn}}(X_h,1).$$

Recall that, for a log-scheme (X, M) as above, we also have the log de Rham first Chern class map

$$c_1^{\mathrm{dR}}:j_*\mathscr{O}_{X_{\mathrm{tr}}}^*\xrightarrow{\sim}M^{\mathrm{gp}}\to M_n^{\mathrm{gp}}\xrightarrow{\mathrm{dlog}}\Omega^\bullet_{(X,M)_n/V_n^\times}[1].$$

For ss-pairs $(U,\bar{U})$ over K, it induces maps

$$c_1^{\mathrm{dR}}:\Gamma(U,\mathscr{O}_U^*)\xleftarrow{\sim}\Gamma(\bar{U},j_*\mathscr{O}_U^*)\to \mathrm{R}\Gamma(\bar{U},\Omega^\bullet_{(U,\bar{U})/V^\times})[1].$$

By the map $\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U},\mathscr{J}^{[1]})\to\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})\to\mathrm{R}\Gamma(\bar{U},\Omega^\bullet_{(U,\bar{U})/V^\times})$, they are compatible with the absolute log-crystalline and log-syntomic classes [Tsuji 1999, (2.2.3)].

Lemma 5.1. *For strict ss-pairs $(U,\bar{U})$ over K, the Hyodo–Kato map and the Hyodo–Kato isomorphism*

$$\iota: H^2_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\to H^2_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}},$$
$$\iota_{\mathrm{dR},\pi}: H^2_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\otimes_{K_0}K\xrightarrow{\sim}H^2(\bar{U}_K,\Omega^\bullet_{(U,\bar{U}_K)/K})$$

are compatible with first Chern class maps.

Proof. Since $\iota_{\mathrm{dR},\pi}=i_\pi^*\iota\otimes\mathrm{Id}$ and the map i_π^* is compatible with first Chern classes, it suffices to show the compatibility for the Hyodo–Kato map ι. Let $\mathscr{L}$ be a line bundle on U. Since the map ι is a section of the map $i_0^*: H^2_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}\to H^2_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}$ and the map i_0^* is compatible with first Chern classes, we have that the element $\zeta\in H^2_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}$ defined as $\zeta=\iota(c_1^{\mathrm{HK}}(\mathscr{L}))-c_1^{\mathrm{st}}(\mathscr{L})$ lies in $TH^2_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}$. Hence $\zeta=T\gamma$. Since the map ι is compatible with Frobenius and $\varphi(c_1^{\mathrm{HK}}(\mathscr{L}))=pc_1^{\mathrm{HK}}(\mathscr{L})$, $\varphi(c_1^{\mathrm{st}}(\mathscr{L}))=pc_1^{\mathrm{st}}(\mathscr{L})$, we have $\varphi(\zeta)=p\zeta$. Since $\varphi(T\gamma)=T^p\varphi(\gamma)$, this implies that $\gamma\in\bigcap_{n=1}^\infty T^nH^2_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}$, which is not possible unless γ (and hence ζ) are zero. But this is what we wanted to show. □

We have the following projective space theorem for syntomic cohomology.

Proposition 5.2. *Let $\mathscr{E}$ be a locally free sheaf of rank $d+1$, $d\geq 0$, on a scheme $X\in\mathscr{V}ar_K$. Consider the associated projective bundle $\pi:\mathbb{P}(\mathscr{E})\to X$. Then we have the quasi-isomorphism of complexes of sheaves on X_h*

$$\bigoplus_{i=0}^{d}c_1^{\mathrm{syn}}(\mathscr{O}(1))^i\cup\pi^*:\bigoplus_{i=0}^{d}\mathscr{S}(r-i)_{X,\mathbb{Q}}[-2i]\xrightarrow{\sim}R\pi_*\mathscr{S}(r)_{\mathbb{P}(\mathscr{E}),\mathbb{Q}},\quad 0\leq d\leq r.$$

Here, the class $c_1^{\mathrm{syn}}(\mathscr{O}(1))\in H^2_{\mathrm{syn}}(\mathbb{P}(\mathscr{E})_h,1)$ refers to the class of the tautological bundle on $\mathbb{P}(\mathscr{E})$.

Proof. By (tedious) checking of many compatibilities, we will reduce the above projective space theorem to the projective space theorems for the Hyodo–Kato and the filtered de Rham cohomologies.

To prove our proposition it suffices to show that for any ss-pair $(U, \overline{U})$ over K and the projective space $\pi : \mathbb{P}^d_{\overline{U}} \to \overline{U}$ of dimension d over $\overline{U}$ we have a projective space theorem for syntomic cohomology ($a \geq 0$):

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{syn}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{syn}}(U_h, r-i) \xrightarrow{\sim} H^a_{\mathrm{syn}}(\mathbb{P}^d_{U,h}, r), \quad 0 \leq d \leq r.$$

By Proposition 3.18 and the compatibility of the maps

$$H^*_{\mathrm{syn}}(U, \overline{U}, j)_{\mathbb{Q}} \xrightarrow{\sim} H^*_{\mathrm{syn}}(U_h, j)_{\mathbb{Q}}$$

with products and first Chern classes, this reduces to proving a projective space theorem for log-syntomic cohomology, i.e., a quasi-isomorphism of complexes

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{syn}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{syn}}(U, \overline{U}, r-i)_{\mathbb{Q}} \xrightarrow{\sim} H^a_{\mathrm{syn}}(\mathbb{P}^d_U, \mathbb{P}^d_{\overline{U}}, r)_{\mathbb{Q}}, \quad 0 \leq d \leq r,$$

where the class $c_1^{\mathrm{syn}}(\mathscr{O}(1)) \in H^2_{\mathrm{syn}}(\mathbb{P}^d_U, \mathbb{P}^d_{\overline{U}}, 1)$ refers to the class of the tautological bundle on $\mathbb{P}^d_{\overline{U}}$.

By the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{cr}}(U, \overline{U}, r)_{\mathbb{Q}} \to \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K)/F^r$$

and its compatibility with the action of c_1^{syn}, it suffices to prove the following two quasi-isomorphisms for the twisted absolute log-crystalline complexes and for the filtered log de Rham complexes ($0 \leq d \leq r$):

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{cr}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{cr}}(U, \overline{U}, r-i)_{\mathbb{Q}} \xrightarrow{\sim} H^a_{\mathrm{cr}}(\mathbb{P}^d_U, \mathbb{P}^d_{\overline{U}}, r)_{\mathbb{Q}},$$

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{dR}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} F^{r-i} H^{a-2i}_{\mathrm{dR}}(U, \overline{U}_K) \xrightarrow{\sim} F^r H^a_{\mathrm{dR}}(\mathbb{P}^d_U, \mathbb{P}^d_{\overline{U}_K}).$$

For the log de Rham cohomology, notice that the above map is quasi-isomorphic to the map [Beilinson 2012, 3.2]

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{dR}}(\mathscr{O}(1))^i \cup \pi^* : \bigoplus_{i=0}^{d} F^{r-i} H^{a-2i}_{\mathrm{dR}}(U) \xrightarrow{\sim} F^r H^a_{\mathrm{dR}}(\mathbb{P}^d_U),$$

and hence well-known to be a quasi-isomorphism.

For the twisted log-crystalline cohomology, notice that since Frobenius behaves well with respect to c_1^{cr}, it suffices to prove a projective space theorem for the absolute log-crystalline cohomology $H^*_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}}$:

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{cr}}(\mathscr{O}(1))^i\cup\pi^*:\bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{cr}}(U,\bar{U})_{\mathbb{Q}}\xrightarrow{\sim} H^a_{\mathrm{cr}}(\mathbb{P}^d_U,\mathbb{P}^d_{\bar{U}})_{\mathbb{Q}}.$$

Without loss of generality, we may assume that the pair $(U,\bar{U})$ is split over K. By the distinguished triangle

$$\mathrm{R}\Gamma_{\mathrm{cr}}(U,\bar{U})\to\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)\xrightarrow{N}\mathrm{R}\Gamma_{\mathrm{cr}}((U,\bar{U})/R)$$

and its compatibility with the action of $c_1^{\mathrm{cr}}(\mathscr{O}(1))$ (see [Tsuji 1999, Lemma 4.3.7]), it suffices to prove a projective space theorem for the log-crystalline cohomology $H^*_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}$. Since the R-linear isomorphism $\iota:H^*_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\otimes R_{\mathbb{Q}}\xrightarrow{\sim} H^*_{\mathrm{cr}}((U,\bar{U})/R)_{\mathbb{Q}}$ is compatible with products [Tsuji 1999, Proposition 4.4.9] and first Chern classes (see Lemma 5.1), we reduce the problem to showing the projective space theorem for the Hyodo–Kato cohomology:

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{HK}}(\mathscr{O}(1))^i\cup\pi^*:\bigoplus_{i=0}^{d} H^{a-2i}_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\xrightarrow{\sim} H^a_{\mathrm{HK}}(\mathbb{P}^d_U,\mathbb{P}^d_{\bar{U}})_{\mathbb{Q}}.$$

Tensoring by K and using the isomorphism

$$\iota_{\mathrm{dR},\pi}:H^*_{\mathrm{HK}}(U,\bar{U})_{\mathbb{Q}}\otimes_{K_0}K\xrightarrow{\sim} H^*_{\mathrm{dR}}(U,\bar{U}_K)$$

that is compatible with products [Tsuji 1999, Corollary 4.4.13] and first Chern classes (see Lemma 5.1), we reduce to checking the projective space theorem for the log de Rham cohomology $H^*_{\mathrm{dR}}(U,\bar{U}_K)$, and we have done this above. □

The above proof proves also the projective space theorem for the absolute crystalline cohomology.

Corollary 5.3. *Let $\mathscr{E}$ be a locally free sheaf of rank $d+1$, $d\geq 0$, on a scheme $X\in\mathscr{V}ar_K$. Consider the associated projective bundle $\pi:\mathbb{P}(\mathscr{E})\to X$. Then we have the following quasi-isomorphism of complexes of sheaves on X_h*

$$\bigoplus_{i=0}^{d} c_1^{\mathrm{cr}}(\mathscr{O}(1))^i\cup\pi^*:\bigoplus_{i=0}^{d}\mathscr{J}^{[r-i]}_{X,\mathbb{Q}}[-2i]\xrightarrow{\sim} R\pi_*\mathscr{J}^{[r]}_{\mathbb{P}(\mathscr{E}),\mathbb{Q}},\quad 0\leq d\leq r.$$

Here, the class $c_1^{\mathrm{cr}}(\mathscr{O}(1))\in H^2(\mathbb{P}(\mathscr{E})_h,\mathscr{J}_{\mathrm{cr}})$ refers to the class of the tautological bundle on $\mathbb{P}(\mathscr{E})$.

For $X\in\mathscr{V}ar_K$, using the projective space theorem (see Proposition 5.2) and the Chern classes

$$c_0^{\mathrm{syn}}:\mathbb{Q}_p\xrightarrow{\mathrm{can}}\mathscr{S}(0)_{X_{\mathbb{Q}}},\qquad c_1^{\mathrm{syn}}:\mathscr{O}^*_{X_h}\to\mathscr{S}(1)_{X_{\mathbb{Q}}}[1],$$

we obtain syntomic Chern classes $c_i^{\mathrm{syn}}(\mathscr{E})$ for any locally free sheaf $\mathscr{E}$ on X.

Syntomic cohomology has the homotopy invariance property.

Proposition 5.4. *Let $X \in \mathscr{V}ar_K$ and $f : \mathbb{A}^1_X \to X$ be the natural projection from the affine line over X to X. Then, for all $r \geq 0$, the pullback map*

$$f^* : \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, r) \xrightarrow{\sim} \sim\mathrm{R}\Gamma_{\mathrm{syn}}(\mathbb{A}^1_{X,h}, r)$$

is a quasi-isomorphism.

Proof. Localizing in the h-topology of X, we may assume that $X = U$, the open set of an ss-pair $(U, \overline{U})$ over K. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathrm{syn}}(U, \overline{U}, r)_{\mathbb{Q}} & \xrightarrow{f^*} & \mathrm{R}\Gamma_{\mathrm{syn}}(\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}}, r)_{\mathbb{Q}} \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{R}\Gamma_{\mathrm{syn}}(U_h, r) & \xrightarrow{f^*} & \mathrm{R}\Gamma_{\mathrm{syn}}(\mathbb{A}^1_{U,h}, r) \end{array}$$

The vertical maps are quasi-isomorphisms by Proposition 3.18. It suffices thus to show that the top horizontal map is a quasi-isomorphism. By Proposition 3.8, this reduces to showing that the map

$$C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})_{\mathbb{Q}}\{r\}) \xrightarrow{f^*} C_{\mathrm{st}}(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}})_{\mathbb{Q}}\{r\})$$

is a quasi-isomorphism, or, that the map $f : (\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}}) \to (U, \overline{U})$ induces a quasi-isomorphism on the Hyodo–Kato cohomology and a filtered quasi-isomorphism on the log de Rham cohomology:

$$\mathrm{R}\Gamma_{\mathrm{HK}}(U, \overline{U})_{\mathbb{Q}} \xrightarrow{f^*} \mathrm{R}\Gamma_{\mathrm{HK}}(\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}})_{\mathbb{Q}}, \quad \mathrm{R}\Gamma_{\mathrm{dR}}(U, \overline{U}_K) \xrightarrow{f^*} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathbb{A}^1_U, \mathbb{P}^1_{\overline{U}_K}).$$

Without loss of generality, we may assume that the pair $(U, \overline{U})$ is split over K. Tensoring with K and using the Hyodo–Kato quasi-isomorphism, we reduce the Hyodo–Kato case to the log de Rham one. The latter follows easily from the projective space theorem and the existence of the Gysin sequence in log de Rham cohomology. □

Remark 5.5. The above implies that syntomic cohomology is a Bloch–Ogus theory. A proof of this fact was kindly communicated to us by Frédéric Déglise and is contained in Appendix B, Proposition B.4.

Proposition 5.6. *For a scheme X, let $K_*(X)$ denote Quillen's higher K-theory groups of X. For $X \in \mathscr{V}ar_K$, $i, j \geq 0$, there are functorial syntomic Chern class maps*

$$c^{\mathrm{syn}}_{i,j} : K_j(X) \to H^{2i-j}_{\mathrm{syn}}(X_h, i).$$

Proof. Recall the construction of the classes $c^{\mathrm{syn}}_{i,j}$. First, one constructs universal classes $C^{\mathrm{syn}}_{i,l} \in H^{2i}_{\mathrm{syn}}(B_{\bullet}\mathrm{GL}_{l,h}, i)$. By a standard argument, the projective space theorem and the homotopy property show that

$$H^*_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, *) \simeq H^*_{\mathrm{syn}}(K, *)[x_1^{\mathrm{syn}}, \dots, x_l^{\mathrm{syn}}],$$

where the classes $x_i^{\mathrm{syn}} \in H^{2i}_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, i)$ are the syntomic Chern classes of the universal locally free sheaf on $B_\bullet \mathrm{GL}_l$ (defined via a projective space theorem). For $l \geq i$, we define

$$C^{\mathrm{syn}}_{i,l} = x_i^{\mathrm{syn}} \in H^{2i}_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, i).$$

The classes $C^{\mathrm{syn}}_{i,l} \in H^{2i}_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, i)$ yield compatible universal classes (see [Gillet 1981, p. 221]) $C^{\mathrm{syn}}_{i,l} \in H^{2i}_{\mathrm{syn}}(X, \mathrm{GL}_l(\mathscr{O}_X), i)$, and hence a natural map of pointed simplicial sheaves on X_{ZAR}, $C_i^{\mathrm{syn}} : B_\bullet \mathrm{GL}(\mathscr{O}_X) \to \mathscr{K}(2i, \mathscr{S}'(i)_X)$, where $\mathscr{K}$ is the Dold–Puppe functor of $\tau_{\geq 0}\mathscr{S}'(i)_X[2i]$ and $\mathscr{S}'(i)_X$ is an injective resolution of $\mathscr{S}(i)_X := R\varepsilon_* \mathscr{S}(i)_{\mathbb{Q}}$, $\varepsilon : X_h \to X_{\mathrm{ZAR}}$. The characteristic classes $c^{\mathrm{syn}}_{i,j}$ are now defined [Gillet 1981, Definition 2.22] as the composition

$$\begin{aligned} K_j(X) \to H^{-j}(X, \mathbb{Z} \times B_\bullet \mathrm{GL}(\mathscr{O}_X)^+) &\to H^{-j}(X, B_\bullet \mathrm{GL}(\mathscr{O}_X)^+) \\ \xrightarrow{C_i^{\mathrm{syn}}} H^{-j}(X, \mathscr{K}(2i, \mathscr{S}'(i)_X)) &\xrightarrow{h_j} H^{2i-j}_{\mathrm{syn}}(X_h, i), \end{aligned}$$

where $B_\bullet \mathrm{GL}(\mathscr{O}_X)^+$ is the (pointed) simplicial sheaf on X associated to the $+$-construction [Soulé 1982, 4.2]. Here, for a (pointed) simplicial sheaf $\mathscr{E}_\bullet$ on X_{ZAR}, we know $H^{-j}(X, \mathscr{E}_\bullet) = \pi_j(\mathrm{R}\Gamma(X_{\mathrm{ZAR}}, \mathscr{E}_\bullet))$ is the generalized sheaf cohomology of $\mathscr{E}_\bullet$ [Gillet 1981, Definition 1.7]. The map h_j is the Hurewicz map:

$$\begin{aligned} H^{-j}(X, \mathscr{K}(2i, \mathscr{S}'(i)_X)) = \pi_j(\mathscr{K}(2i, \mathscr{S}'(i)(X))) &\xrightarrow{h_j} H_j(\mathscr{K}(2i, \mathscr{S}'(i)(X))) \\ &= H_j(\mathscr{S}'(i)(X)[2i]) = H^{2i-j}_{\mathrm{syn}}(X_h, i). \end{aligned}$$ □

Proposition 5.7. *The syntomic and the étale Chern classes are compatible, i.e., for $X \in \mathscr{V}ar_K$, $j \geq 0$, $2i - j \geq 0$, the following diagram commutes:*

$$\begin{array}{ccc} & K_j(X) & \\ {}^{c^{\mathrm{syn}}_{i,j}}\swarrow & & \searrow^{c^{\text{ét}}_{i,j}} \\ H^{2i-j}_{\mathrm{syn}}(X_h, i) & \xrightarrow{\rho_{\mathrm{syn}}} & H^{2i-j}_{\text{ét}}(X, \mathbb{Q}_p(i)) \end{array}$$

Proof. We can pass to the universal case ($X = B_\bullet \mathrm{GL}_l := B_\bullet \mathrm{GL}_l /K$, $l \geq 1$). We have

$$H^*_{\mathrm{syn}}(B_\bullet \mathrm{GL}_{l,h}, *) \simeq H^*_{\mathrm{syn}}(K, *)[x_1^{\mathrm{syn}}, \dots, x_l^{\mathrm{syn}}],$$
$$H^*_{\text{ét}}(B_\bullet \mathrm{GL}_l, *) \simeq H^*_{\text{ét}}(K, *)[x_1^{\text{ét}}, \dots, x_l^{\text{ét}}].$$

By the projective space theorem and the fact that the syntomic period map commutes with products, it suffices to check that $\rho_{\mathrm{syn}}(x_1^{\mathrm{syn}}) = x_1^{\text{ét}}$ and that the syntomic period map ρ_{syn} commutes with the classes $c_0^{\mathrm{syn}} : \mathbb{Q}_p \to \mathscr{S}(0)_{\mathbb{Q}}$ and $c_0^{\text{ét}} : \mathbb{Q}_p \to \mathbb{Q}_p(0)$. The statement about c_0 is clear from the definition of ρ_{cr}; for c_1, consider the canonical map $f : B_\bullet \mathrm{GL}_l \to B_\bullet \mathrm{GL}_{l,\bar{K}}$ and the induced pullback map

$$f^*_{\text{ét}} : H^*_{\text{ét}}(B_\bullet \mathrm{GL}_l, *) = H^*_{\text{ét}}(K, *)[x_1, \dots, x_l] \to H^*_{\text{ét}}(B_\bullet \mathrm{GL}_{l,\bar{K}}, *) = \mathbb{Q}_p[\bar{x}_1, \dots, \bar{x}_l]$$

that sends the Chern classes $x_i^{\text{ét}}$ of the universal vector bundle to the classes $\bar{x}_i^{\text{ét}}$ of its pullback. It suffices to show that $f^*_{\text{ét}}\rho_{\mathrm{syn}}(C^{\mathrm{syn}}_{1,1}) = C^{\text{ét}}_{1,1}$. But, by definition, $f^*_{\text{ét}}\rho_{\mathrm{syn}} = \rho_{\mathrm{syn}} f^*_{\mathrm{syn}}$ and, by construction, we have the commutative diagram

$$\begin{array}{ccc} H^2_{\mathrm{syn}}(B_\bullet \mathbb{G}_{m,h}, 1) & \xrightarrow{\quad\text{can}\quad} & H^2_{\mathrm{cr}}(B_\bullet \mathbb{G}_{m,\bar{K},h}) \\ \downarrow{\scriptstyle \rho_{\mathrm{syn}}} & & \downarrow{\scriptstyle \rho_{cr}} \\ H^2_{\text{ét}}(B_\bullet \mathbb{G}_{m,\bar{K}}, \mathbb{Q}_p(1)) & \longrightarrow & H^2_{\text{ét}}(B_\bullet \mathbb{G}_{m,\bar{K}}, B^+_{\mathrm{cr}}) = H^2_{\text{ét}}(B_\bullet \mathbb{G}_{m,\bar{K}}, \mathbb{Q}_p(1)) \otimes B^+_{\mathrm{cr}} \end{array}$$

where the bottom map sends the generator of $\mathbb{Q}_p(1)$ to the element $t \in B^+_{\mathrm{cr}}$ associated to it. Since the syntomic and the crystalline Chern classes are compatible, it suffices to show that, for a line bundle $\mathscr{L}$, we have $\rho_{\mathrm{cr}}(c_1^{\mathrm{cr}}(\mathscr{L})) = c_1^{\text{ét}}(\mathscr{L}) \otimes t$. But this is [Beilinson 2013, 3.2]. □

Remark 5.8. If $\mathscr{X}$ is a scheme over V and $X = \mathscr{X}_K$, we can consider the syntomic Chern classes $c^{\mathrm{syn}}_{i,j} : K_j(\mathscr{X}) \to H^{2i-j}_{\mathrm{syn}}(X_h, i)$ defined as the composition

$$K_j(\mathscr{X}) \to K_j(X) \xrightarrow{c^{\mathrm{syn}}_{i,j}} H^{2i-j}_{\mathrm{syn}}(X_h, i).$$

By the above proposition, these classes are compatible with the étale Chern classes. Recall that analogous results were proved earlier for $\mathscr{X}$ smooth and projective [Niziol 1997], for $\mathscr{X}$ a complement of a divisor with relative normal crossings in such, and for $\mathscr{X}$ a semistable scheme over V [Niziol 2016b].

5B. ***Image of étale regulators.*** In this subsection we show that Soulé's étale regulators factor through the semistable Selmer groups.

Let $X \in \mathscr{V}ar_K$. For $2r - i - 1 \geq 0$, set

$$K_{2r-i-1}(X)_0 := \ker\big(K_{2r-i-1}(X) \xrightarrow{c^{\text{ét}}_{r,i+1}} H^0(G_K, H^{i+1}_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r)))\big).$$

Write $r^{\text{ét}}_{r,i}$ for the map

$$r^{\text{ét}}_{r,i} : K_{2r-i-1}(X)_0 \to H^1(G_K, H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r)))$$

induced by the Chern class map $c^{\text{ét}}_{r,i+1}$ and the Hochschild–Serre spectral sequence map $\delta : H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r))_0 \to H^1(G_K, H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r)))$, where we set

$$H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r))_0 := \ker(H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r)) \to H^{i+1}_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p(r))).$$

Theorem 5.9. *The map $r_{r,i}^{\text{ét}}$ factors through the subgroup*

$$H^1_{\text{st}}(G_K, H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))) \subset H^1(G_K, H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))).$$

Proof. By Proposition 5.7, we have the commutative diagram

$$\begin{array}{ccccc} K_{2r-i-1}(X) & & & & \\ \downarrow{\scriptstyle c^{\text{syn}}_{r,i+1}} & \searrow^{c^{\text{ét}}_{r,i+1}} & & & \\ H^{i+1}_{\text{syn}}(X_h, r) & \xrightarrow{\rho_{\text{syn}}} & H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r)) & \longrightarrow & H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)) \end{array}$$

Hence the Chern class map $c^{\text{syn}}_{r,i+1} : K_{2r-i-1}(X)_0 \to H^{i+1}_{\text{syn}}(X_h, r)$ factors through $H^{i+1}_{\text{syn}}(X_h, r)_0 := \ker(H^{i+1}_{\text{syn}}(X_h, r) \xrightarrow{\rho_{\text{syn}}} H^{i+1}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)))$. Compatibility of the syntomic descent and the Hochschild–Serre spectral sequences (see Theorem 4.8) yields the commutative diagram

$$\begin{array}{ccc} K_{2r-i-1}(X)_0 & & \\ \downarrow{\scriptstyle c^{\text{syn}}_{r,i+1}} & \searrow^{c^{\text{ét}}_{r,i+1}} & \\ H^{i+1}_{\text{syn}}(X_h, r)_0 & \xrightarrow{\rho_{\text{syn}}} & H^{i+1}_{\text{ét}}(X, \mathbb{Q}_p(r))_0 \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\ H^1_{\text{st}}(G_K, H^{i}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))) & \xrightarrow{\text{can}} & H^1(G_K, H^{i}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r))) \end{array}$$

Our theorem follows. □

Remark 5.10. The question of the image of Soulé's regulators $r^{\text{ét}}_{r,i}$ was raised by Bloch and Kato [1990] in connection with their Tamagawa number conjecture. Theorem 5.9 is known to follow from the constructions of Scholl [1993]. The argument goes as follows. Recall that for a class $y \in K_{2r-i-1}(X)_0$, he constructs an explicit extension $E_y \in \text{Ext}^1_{\mathcal{M}\mathcal{M}_K}(\mathbb{Q}(-r), h^i(X))$ in the category of mixed motives over K. The association $y \mapsto E_y$ is compatible with the étale cycle class and realization maps. By the de Rham comparison theorem, the étale realization $r^{\text{ét}}_{r,i}(y)$ of the extension class E_y in

$$\text{Ext}^1_{G_K}(\mathbb{Q}_p(-r), H^i(X_{\overline{K}}, \mathbb{Q}_p)) = H^1(G_K, H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(r)))$$

is de Rham, hence potentially semistable by [Berger 2002], as wanted.

Appendix A: Vanishing of $H^2(G_K, V)$

by Laurent Berger

Let V be a $\mathbb{Q}_p$-linear representation of G_K. In this appendix we prove the following theorem.

Theorem A.1. *If V is semistable and all its Hodge–Tate weights are ≥ 2, then $H^2(G_K, V) = 0$.*

Let $\mathrm{D}(V)$ be Fontaine's (φ, Γ)-module [1990] attached to V. It comes with a Frobenius map φ and an action of Γ_K. Let $H_K = \mathrm{Gal}(\bar{K}/K(\mu_{p^\infty}))$ and let $I_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{nr}})$. The injectivity of the restriction map $H^2(G_K, V) \to H^2(G_L, V)$ for L/K finite allows us to replace K by a finite extension, so that we can assume that $H_K I_K = G_K$ and that $\Gamma_K \simeq \mathbb{Z}_p$. Let γ be a topological generator of Γ_K. Recall [Cherbonnier and Colmez 1999, §I.5] that we have a map $\psi : \mathrm{D}(V) \to \mathrm{D}(V)$.

Ideally, our proof of this theorem would go as follows. We use the Hochschild–Serre spectral sequence

$$H^i(G_K/I_K, H^j(I_K, V|_{I_K})) \Rightarrow H^{i+j}(G_K, V)$$

and, interpreting Galois cohomology in terms of (φ, Γ)-modules, we compute that $H^2(I_K, V|_{I_K}) = 0$ and $H^1(I_K, V|_{I_K}) = \widehat{K}^{\mathrm{nr}} \otimes_K D_{\mathrm{dR}}(V)$. We conclude since, by Hilbert 90, $H^1(G_K/I_K, H^1(I_K, V|_{I_K})) = 0$. However, we do not, in general, have Hochschild–Serre spectral sequences for continuous cohomology. We mimic thus the above argument with direct computations on continuous cocycles (again using (φ, Γ)-modules). Laurent Berger is grateful to Kevin Buzzard for discussions related to the above spectral sequence.

Lemma A.2. (1) *If V is a representation of G_K, then there is an exact sequence*

$$0 \to \mathrm{D}(V)^{\psi=1}/(\gamma - 1) \to H^1(G_K, V) \to (\mathrm{D}(V)/(\psi - 1))^{\Gamma_K} \to 0.$$

(2) *We have $H^2(G_K, V) = \mathrm{D}(V)/(\psi - 1, \gamma - 1)$.*

Proof. See I.5.5 and II.3.2 of [Cherbonnier and Colmez 1999]. □

Lemma A.3. *We have* $\mathrm{D}(V|_{I_K})/(\psi - 1) = 0$.

Proof. Since $V|_{I_K}$ corresponds to the case when k is algebraically closed, see the proof of Lemma VI.7 of [Berger 2001]. □

Let γ_I denote a generator of $\Gamma_{\widehat{K}^{\mathrm{nr}}}$.

Lemma A.4. *The natural map* $\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I - 1) \to (\mathrm{D}(V|_{I_K})/(\gamma_I - 1))^{\psi=1}$ *is an isomorphism if $V^{I_K} = 0$.*

Proof. This map is part of the six-term exact sequence that comes from the map $\gamma_I - 1$ applied to $0 \to \mathrm{D}(V|_{I_K})^{\psi=1} \to \mathrm{D}(V|_{I_K}) \xrightarrow{\psi-1} \mathrm{D}(V|_{I_K}) \to 0$. Its kernel is included in $\mathrm{D}(V|_{I_K})^{\gamma_I=1}$, which is 0 since $V^{I_K} = 0$ (note that the inclusion $(\widehat{K}^{\mathrm{nr}} \otimes V)^{G_K} \subseteq (\widehat{\mathscr{E}}^{\mathrm{nr}} \otimes V)^{G_K} = \mathrm{D}(V)^{G_K}$ is an isomorphism). □

Suppose that $x \in \mathrm{D}(V)/(\psi - 1, \gamma - 1)$. If $\tilde{x} \in \mathrm{D}(V)$ lifts x, then Lemma A.3 gives us an element $y \in \mathrm{D}(V|_{I_K})$ such that $(\psi - 1)y = \tilde{x}$. Define a cocycle $\delta(x) \in Z^1(G_K/I_K, \mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I - 1))$ by $\delta(x) : \bar{g} \mapsto (g - 1)(y)$ if $g \in G_K$ lifts $\bar{g} \in G_K/I_K$.

Proposition A.5. *If $V^{I_K} = 0$, then the map*

$$\delta : \mathrm{D}(V)/(\psi-1, \gamma-1) \to H^1\big(G_K/I_K, (\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1}\big)$$

is well-defined and injective.

Proof. We first check that

$$\delta(x)(g) \in (\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1}.$$

We have $(\psi-1)(g-1)(y) = (g-1)(x)$. If we write $g = ih \in I_K H_K$, then $(g-1)x = (ih-1)x = (i-1)x \in (\gamma_I-1)\mathrm{D}(V|_{I_K})$ since $\gamma_I - 1$ divides the image of $i-1$ in $\mathbb{Z}_p[\![\Gamma_{\widehat{K}^{\mathrm{nr}}}]\!]$. This implies $\delta(x)(g) \in (\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1}$.

We now check that $\delta(x)$ does not depend on the choices. If we choose another lift $g' \in G_K$ of $\bar{g} \in G_K/I_K$, then $g' = ig$ for some $i \in I_K$ and $(g'-1)y-(g-1)y = (i-1)gy \in (\gamma_I-1)\mathrm{D}(V|_{I_K})$ since γ_I-1 divides the image of $i-1$ in $\mathbb{Z}_p[\![\Gamma_{\widehat{K}^{\mathrm{nr}}}]\!]$. If we choose another y' such that $(\psi-1)y' = \tilde{x}$, then $y-y' \in \mathrm{D}(V|_{I_K})^{\psi=1}$ so that δ and δ' are cohomologous. Finally, if $\tilde{x}'$ is another lift of x, then $\tilde{x}'-\tilde{x} = (\gamma-1)a+(\psi-1)b$ with $a, b \in \mathrm{D}(V)$. We can then take $y' = y+b+(\gamma_G-1)c$, where $(\psi-1)c = a$. We then have $(g-1)y' = (g-1)y+(g-1)b+(\gamma_G-1)(g-1)c$. Since $G_K = I_K H_K$, we can write $g = ih$ and $(g-1)b = (i-1)b$. Using $G_K = I_K H_K$ once again, we see that $I_K \to G_K/H_K$ is surjective, so that we can identify γ_I and γ_G. The resulting cocycle is then cohomologous to $\delta(x)$. This proves that δ is well-defined.

We now prove that δ is injective. If $\delta(x) = 0$, then using Lemma A.4 there exists $z \in \mathrm{D}(V|_{I_K})^{\psi=1}$ such that $\delta(x)(\bar{g})$ is the image of $(g-1)(z)$ in $\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I-1)$. This implies that $(g-1)(y-z) \in (\gamma_I-1)\mathrm{D}(V|_{I_K})^{\psi=1}$. Applying $\psi-1$ gives $(g-1)\tilde{x} = 0$ so that $\tilde{x} \in \mathrm{D}(V)^{G_K} \subset V^{I_K} = 0$. The map δ is therefore injective. □

Lemma A.6. *If V is semistable and the weights of V are all ≥ 2, then*

$$\exp_V : \mathrm{D}_{\mathrm{dR}}(V|_{I_K}) \to H^1(I_K, V)$$

is an isomorphism.

Proof. Apply Theorem 6.8 of [Berger 2002] to $V|_{I_K}$. □

Proof of Theorem A.1. We can replace K by K_n for $n \gg 0$ and use the fact that if $H^2(G_{K_n}, V) = 0$, then $H^2(G_K, V) = 0$ since the restriction map is injective. In particular, we can assume that $H_K I_K = G_K$ and that Γ_K is isomorphic to $\mathbb{Z}_p$. By item (2) of Lemma A.2, we have $H^2(G_K, V) = \mathrm{D}(V)/(\psi-1, \gamma-1)$, and so by Proposition A.5 above, it is enough to prove that

$$H^1(G_K/I_K, (\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1}) = 0.$$

Lemma A.4 tells us that $(\mathrm{D}(V|_{I_K})/(\gamma_I-1))^{\psi=1} = \mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma_I-1)$. Since $\mathrm{D}(V|_{I_K})/(\psi-1) = 0$ by Lemma A.3, item (1) of Lemma A.2 tells us that $\mathrm{D}(V|_{I_K})^{\psi=1}/(\gamma-1) = H^1(I_K, V)$.

The map $\exp_V : \mathrm{D}_{\mathrm{dR}}(V|_{I_K}) \to H^1(I_K, V)$ is an isomorphism by Lemma A.6, and this isomorphism commutes with the action of G_K since it is a natural map. We therefore have $H^1(I_K, V) = \widehat{K}^{\mathrm{nr}} \otimes_K \mathrm{D}_{\mathrm{dR}}(V)$ as G_K-modules. It remains to observe that the cocycle $\delta(x) \in Z^1(G_K/I_K, \widehat{K}^{\mathrm{nr}} \otimes_K \mathrm{D}_{\mathrm{dR}}(V))$ is continuous and that $H^1(G_K/I_K, \widehat{K}^{\mathrm{nr}}) = 0$ by taking a lattice, reducing modulo a uniformizer of K, and applying Hilbert 90. □

Appendix B: The syntomic ring spectrum
by Frédéric Déglise

In this appendix, we explain why syntomic cohomology as defined in this paper is representable by a motivic ring spectrum in the sense of Morel and Voevodsky's homotopy theory. More precisely, we will exhibit a monoid object $\mathscr{S}$ of the triangulated category of motives with $\mathbb{Q}_p$-coefficients (see below), DM, such that for any variety X and any pair of integers (i, r),

$$H^i_{\mathrm{syn}}(X_h, r) = \mathrm{Hom}_{DM}(M(X), \mathscr{S}(r)[i]).$$

In fact, it is possible to apply directly [Déglise and Mazzari 2015, Theorem 1.4.10] to the graded commutative dg-algebra $\mathrm{R}\Gamma_{\mathrm{syn}}(X, *)$ of Theorem A in view of the existence of Chern classes established in Section 5A. However, the use of the h-topology in this paper makes the construction of $\mathbb{E}_{synt}$ much more straightforward and that is what we explain in this appendix. Reformulating slightly the original definition of Voevodsky [1996], we introduce:

Definition B.1. Let $\mathrm{PSh}(K, \mathbb{Q}_p)$ be the category of presheaves of $\mathbb{Q}_p$-modules over the category of varieties.

Let C be a complex in $\mathrm{PSh}(K, \mathbb{Q}_p)$. We say

(1) C is h-local if for any h-hypercovering $\pi : Y_\bullet \to X$, the induced map
$$C(X) \to \pi_*\mathrm{Tot}^{\oplus}(C(Y_\bullet))$$
is a quasi-isomorphism;

(2) C is $\mathbb{A}^1$-local if for any variety X, the map induced by the projection
$$H^i(X_h, C) \to H^i(\mathbb{A}^1_{X,h}, C)$$
is an isomorphism.

We define the triangulated category $DM^{\mathrm{eff}}_h(K, \mathbb{Q}_p)$ of effective h-motives as the full subcategory of the derived category $D(\mathrm{PSh}(K, \mathbb{Q}_p))$ made by the complexes which are h-local and $\mathbb{A}^1$-local.

Equivalently, we can define this category as the $\mathbb{A}^1$-localization of the derived category of h-sheaves on K-varieties (see Section 5.2 of [Cisinski and Déglise 2009], and more precisely Proposition 5.2.10 and Example 5.2.17(2)). Recall also from [loc. cit.] that there are derived tensor products and internal Hom on $DM^{\mathrm{eff}}_h(K, \mathbb{Q}_p)$.

For any integer $r \geq 0$, the *syntomic sheaf* $\mathscr{S}(r)$ is both h-local (by definition) and $\mathbb{A}^1$-local (Proposition 5.4). Thus it defines an object of $DM_h^{\text{eff}}(K, \mathbb{Q}_p)$ and for any variety X, one has an isomorphism

$$\operatorname{Hom}_{DM_h^{\text{eff}}(K,\mathbb{Q}_p)}(\mathbb{Q}_p(X), \mathscr{S}(r)[i]) \\ = \operatorname{Hom}_{D(\operatorname{PSh}(K,\mathbb{Q}_p))}(\mathbb{Q}_p(X), \mathscr{S}(r)[i]) = H^i_{\text{syn}}(X_h, r),$$

where $\mathbb{Q}_p(X)$ is the presheaf of $\mathbb{Q}_p$-vector spaces represented by X. Thus, the representability assertion for syntomic cohomology is obvious in the effective setting.

Recall that one defines the Tate motive in $DM_h^{\text{eff}}(K, \mathbb{Q}_p)$ as the object $\mathbb{Q}_p(1) := \mathbb{Q}_p(\mathbb{P}^1_K)/\mathbb{Q}_p(\{\infty\})[-2]$. Given any complex object C of $DM_h^{\text{eff}}(K, \mathbb{Q}_p)$, we put $C(n) := C \otimes \mathbb{Q}_p(1)^{\otimes, n}$. One should be careful that this notation is in conflict with that of $\mathscr{S}(r)$ considered as an effective h-motive, as the natural twist on syntomic cohomology is unrelated to the twist of h-motives. To solve this matter, we are led to consider the following notion of Tate spectrum, borrowed from algebraic topology according to Morel and Voevodsky.

Definition B.2. A *Tate h-spectrum* (over K with coefficients in $\mathbb{Q}_p$) is a sequence $\mathbb{E} = (E_i, \sigma_i)_{i \in \mathbb{N}}$ such that:

- For each $i \in \mathbb{N}$, E_i is a complex of $\operatorname{PSh}(K, \mathbb{Q}_p)$ equipped with an action of the symmetric group Σ_i of the set with i-element.
- For each $i \in \mathbb{N}$, $\sigma_i : E_i(1) \to E_{i+1}$ is a morphism of complexes called the *suspension map* in degree i.
- For any integers $i \geq 0$, $r > 0$, the map induced by the morphisms $\sigma_i, \ldots, \sigma_{i+r}$

$$E_i(r) \to E_{i+r}$$

  is compatible with the action of $\Sigma_i \times \Sigma_r$, given on the left by the structural Σ_i-action on E_i and the action of Σ_r via the permutation isomorphism of the tensor structure on $C(\operatorname{PSh}(K, \mathbb{Q}_p))$, and on the right via the embedding $\Sigma_i \times \Sigma_r \to \Sigma_{i+r}$.

A morphism of Tate h-spectra $f : \mathbb{E} \to \mathbb{F}$ is a sequence of Σ_i-equivariant maps $(f_i : E_i \to F_i)_{i \in \mathbb{N}}$ compatible with the suspension maps. The corresponding category will be denoted by $\operatorname{Sp}_{\text{h}}(K, \mathbb{Q}_p)$.

There is an adjunction of categories

$$\Sigma^\infty : C\big(\operatorname{PSh}(K, \mathbb{Q}_p)\big) \leftrightarrows \operatorname{Sp}_{\text{h}}(K, \mathbb{Q}_p) : \Omega^\infty \tag{52}$$

such that for any complex K of h-sheaves, $\Sigma^\infty C$ is the Tate spectrum equal in degree n to $C(n)$, equipped with the obvious action of Σ_n induced by the symmetric structure on tensor product and with the obvious suspension maps.

Definition B.3. A morphism of Tate spectra $(f_i : E_i \to F_i)_{i\in\mathbb{N}}$ is a level quasi-isomorphism if for any i, we have f_i is a quasi-isomorphism.

A Tate spectrum $\mathbb{E}$ is called a Ω-spectrum if for any i, we have E_i is h-local and $\mathbb{A}^1$-local and the map of complexes

$$E_i \to \underline{\mathrm{Hom}}(\mathbb{Q}_p(1), E_{i+1})$$

is a quasi-isomorphism.

We define the triangulated category $DM_h(K, \mathbb{Q}_p)$ of h-motives over K with coefficients in $\mathbb{Q}_p$ as the category of Tate Ω-spectra localized by the level quasi-isomorphisms.

The category of h-motives notably enjoys the following properties:

(DM1) The adjunction of categories (52) induces an adjunction of triangulated categories

$$\Sigma^\infty : DM_h^{\mathrm{eff}}(K, \mathbb{Q}_p) \leftrightarrows DM_h(K, \mathbb{Q}_p) : \Omega^\infty$$

such that for a Tate Ω-spectrum $\mathbb{E}$, and any integer $r \geq 0$, we have $\Omega^\infty(\mathbb{E}(r)) = E_r$ (see [Cisinski and Déglise 2009, Section 5.3.d, and Example 5.3.31(2)]).

Given any variety X, we define the (stable) h-motive of X as $M(X) := \Sigma^\infty\mathbb{Q}_p(X)$.

(DM2) There exists a symmetric closed monoidal structure on $DM(K, \mathbb{Q}_p)$ such that Σ^∞ is monoidal and such that $\Sigma^\infty\mathbb{Q}_p(1)$ admits a tensor inverse (see [Cisinski and Déglise 2009, Section 5.3, Example 5.3.31(2)]). By abuse of notations, we put $\mathbb{Q}_p = \Sigma^\infty\mathbb{Q}_p$.

(DM3) The triangulated monoidal category $DM_h(K, \mathbb{Q}_p)$ is equivalent to all known versions of triangulated categories of mixed motives over $\mathrm{Spec}(K)$ with coefficients in $\mathbb{Q}_p$ (see [Cisinski and Déglise 2009, Section 16, and Theorem 16.1.2]). In particular, it contains as a full subcategory the category $DM_{gm}(K) \otimes \mathbb{Q}_p$ obtained from the category of Voevodsky geometric motives ([Voevodsky et al. 2000, Chapter 5]) by tensoring Hom-groups with $\mathbb{Q}_p$ (see [Cisinski and Déglise 2009, Corollary 16.1.6, 15.2.5]).

With that definition, the construction of a Tate spectrum representing syntomic cohomology is almost obvious. In fact, we consider the sequence of presheaves

$$\mathscr{S} := (\mathscr{S}(r), r \in \mathbb{N}),$$

where each $\mathscr{S}(r)$ is equipped with with the trivial action of Σ_r. According to the first paragraph of Section 5A, we can consider the first Chern class of the canonical invertible sheaf $\mathbb{P}^1$: $\bar{c} \in H^2_{\mathrm{syn}}(\mathbb{P}^1_K, 1) = H^2(\mathbb{P}^1_{K,h}, \mathscr{S}(1))$. Take any lift $c : \mathbb{Q}_p(\mathbb{P}^1_K) \to \mathscr{S}(1)[2]$ of this class. By the definition of the Tate twist, it defines an element $\mathbb{Q}_p(1) \to \mathscr{S}(1)$ still denoted by c. We define the suspension map

$$\mathscr{S}(r) \otimes \mathbb{Q}_p(1) \xrightarrow{\mathrm{Id}\otimes c} \mathscr{S}(r) \otimes \mathscr{S}(1) \xrightarrow{\mu} \mathscr{S}(r+1),$$

where μ is the multiplication coming from the graded dg-structure on $\mathscr{S}(*)$. Because this dg-structure is commutative, we obtain that these suspension maps induce structures of a Tate spectrum on $\mathscr{S}$. Moreover, $\mathscr{S}$ is a Tate Ω-spectrum because each $\mathscr{S}(r)$ is h-local and $\mathbb{A}^1$-local, and the map obtained by adjunction from σ_r is a quasi-isomorphism because of the projective bundle theorem for $\mathbb{P}^1$ (an easy case of Proposition 5.2).

Now, by definition of $DM_h(K, \mathbb{Q}_p)$ and because of property (DM1) above, for any variety X, and any integers (i, r), we get

$$\begin{aligned}\operatorname{Hom}_{DM_h(K,\mathbb{Q}_p)}(M(X), \mathscr{S}(r)[i])\\ = \operatorname{Hom}_{DM_h^{\mathrm{eff}}(K,\mathbb{Q}_p)}(\mathbb{Q}_p(X), \Omega^\infty(\mathscr{S}(r))[i]) = H^i_{\mathrm{syn}}(X_h, r).\end{aligned}$$

Moreover, the commutative dg-structure on the complex $\mathscr{S}(*)$ induces a monoid structure on the associated Tate spectrum. In other words, $\mathscr{S}$ is a ring spectrum (strict and commutative). This construction is completely analogous to the proof of [Déglise and Mazzari 2015, Proposition 1.4.10]. In particular, we can apply all the constructions of [Déglise and Mazzari 2015, Section 3] to the ring spectrum $\mathscr{S}$. Let us summarize this briefly:

Proposition B.4. (1) *Syntomic cohomology is covariant with respect to projective morphisms of smooth varieties* (Gysin morphisms in the terminology of [Déglise and Mazzari 2015]). *More precisely, to a projective morphism of smooth K-varieties $f : Y \to X$ one can associate a Gysin morphism in syntomic cohomology*

$$f_* : H^n_{\mathrm{syn}}(Y_h, i) \to H^{n-2d}_{\mathrm{syn}}(X_h, i-d),$$

where d is the dimension of f.

(2) *The syntomic regulator over $\mathbb{Q}_p$ is induced by the unit $\eta : \mathbb{Q}_p \to \mathscr{S}$ of the ring spectrum $\mathscr{S}$*:

$$\begin{aligned} r_{\mathrm{syn}} : H^{r,i}_M(X) \otimes \mathbb{Q}_p = \operatorname{Hom}_{DM_h(K,\mathbb{Q}_p)}(M(X), \mathbb{Q}_p(r)[i])\\ \longrightarrow \operatorname{Hom}_{DM_h(K,\mathbb{Q}_p)}(M(X), \mathscr{S}(r)[i]) = H^i_{\mathrm{syn}}(X_h, r).\end{aligned}$$

It is compatible with product, pullbacks and pushforwards.

(3) *The syntomic cohomology has a natural extension to h-motives*[10]

$$DM_h(K, \mathbb{Q}_p)^{op} \to D(\mathbb{Q}_p), \quad M \mapsto \operatorname{Hom}_{DM_h(K,\mathbb{Q}_p)}(M, \mathscr{S})$$

and the syntomic regulator r_{syn} can be extended to motives.

(4) *There exists a canonical syntomic Borel-Moore homology $H^{\mathrm{syn}}_*(-, *)$ such that the pair of functors $(H^*_{\mathrm{syn}}(-, *), H^{\mathrm{syn}}_*(-, *))$ defines a Bloch–Ogus theory.*

[10] And in particular to the usual Voevodsky geometrical motives by (DM3) above.

(5) *To the ring spectrum $\mathscr{S}$ there is associated a cohomology with compact support satisfying the usual properties.*

For points (1) and (2), we refer the reader to [Déglise and Mazzari 2015, Section 3.1] and for the remaining ones to Section 3.2 of the same paper.

Remark B.5. Note that the construction of the syntomic ring spectrum $\mathscr{S}$ in $DM_h(K, \mathbb{Q}_p)$ automatically yields the general projective bundle theorem (already obtained in Proposition 5.2). More generally, the ring spectrum $\mathscr{S}$ is *oriented* in the terminology of motivic homotopy theory. Thus, besides the theory of Gysin morphisms, this gives various constructions — symbols, residue morphisms — and yields various formulas — excess intersection formula, blow-up formulas (see [Déglise 2008] for more details).

Acknowledgments

Parts of this article were written during our visits to the Fields Institute in Toronto in spring 2012. Nizioł worked on this article also at BICMR, Beijing, and at the University of Padova. We would like to thank these institutions for their support and hospitality.

This article was inspired by the work of Beilinson on p-adic comparison theorems. We would like to thank him for discussions related to his work. Luc Illusie and Weizhe Zheng helped us understand the ∞-category theory involved in Beilinson's work and made their notes [Illusie 2013; Zheng 2013] available to us — we would like to thank them for that. We have also profited from conversations with Laurent Berger, Amnon Besser, Bharghav Bhatt, Bruno Chiarellotto, Pierre Colmez, Frédéric Déglise, Luc Illusie, Tony Scholl, and Weizhe Zheng — we are grateful for these exchanges. Moreover, we would like to thank Pierre Colmez for reading and correcting parts of this article. Special thanks go to Laurent Berger and Frédéric Déglise for writing the appendices.

References

[Bannai 2002] K. Bannai, "Syntomic cohomology as a p-adic absolute Hodge cohomology", *Math. Z.* **242**:3 (2002), 443–480. MR Zbl

[Beilinson 1986] A. A. Beilinson, "Notes on absolute Hodge cohomology", pp. 35–68 in *Applications of algebraic K-theory to algebraic geometry and number theory, I* (Boulder, CO, 1983), edited by S. J. Bloch et al., Contemp. Math. **55**, Amer. Math. Soc., Providence, RI, 1986. MR Zbl

[Beilinson 2012] A. Beilinson, "p-adic periods and derived de Rham cohomology", *J. Amer. Math. Soc.* **25**:3 (2012), 715–738. MR Zbl

[Beilinson 2013] A. Beilinson, "On the crystalline period map", extended version of article published in *Camb. J. Math.* **1**:1 (2013), 1–51, 2013. arXiv

[Berger 2001] L. Berger, *Représentations p-adiques et équations différentielles*, Ph.D. thesis, Université Pierre et Marie Curie, 2001, http://perso.ens-lyon.fr/laurent.berger/autrestextes/these.pdf.

[Berger 2002] L. Berger, "Représentations p-adiques et équations différentielles", *Invent. Math.* **148**:2 (2002), 219–284. MR Zbl

[Bertolini et al. 2015] M. Bertolini, H. Darmon, and V. Rotger, "Beilinson–Flach elements and Euler systems, II: The Birch–Swinnerton-Dyer conjecture for Hasse–Weil–Artin L-series", *J. Algebraic Geom.* **24**:3 (2015), 569–604. MR Zbl

[Besser 2000] A. Besser, "Syntomic regulators and p-adic integration, I: Rigid syntomic regulators", *Israel J. Math.* **120**:2 (2000), 291–334. MR Zbl

[Besser et al. 2009] A. Besser, P. Buckingham, R. de Jeu, and X.-F. Roblot, "On the p-adic Beilinson conjecture for number fields", *Pure Appl. Math. Q.* **5**:1 (2009), 375–434. MR Zbl

[Besser et al. 2016] A. Besser, D. Loeffler, and S. L. Zerbes, "Finite polynomial cohomology for general varieties", *Ann. Math. Qué.* **40**:1 (2016), 203–220. MR

[Bhatt 2012] B. Bhatt, "p-adic derived de Rham cohomology", preprint, 2012. arXiv

[Bhatt and Scholze 2015] B. Bhatt and P. Scholze, "The pro-étale topology for schemes", pp. 99–201 in *De la géométrie algébrique aux formes automorphes, I*, edited by J. B. Bost et al., Astérisque **369**, Société Mathématique de France, Paris, 2015. MR Zbl

[Bloch and Kato 1990] S. Bloch and K. Kato, "L-functions and Tamagawa numbers of motives", pp. 333–400 in *The Grothendieck Festschrift, I*, Progr. Math. **86**, Birkhäuser, Boston, 1990. MR Zbl

[Bloch and Ogus 1974] S. Bloch and A. Ogus, "Gersten's conjecture and the homology of schemes", *Ann. Sci. École Norm. Sup.* (4) **7** (1974), 181–201. MR Zbl

[Cherbonnier and Colmez 1999] F. Cherbonnier and P. Colmez, "Théorie d'Iwasawa des représentations p-adiques d'un corps local", *J. Amer. Math. Soc.* **12**:1 (1999), 241–268. MR Zbl

[Cisinski and Déglise 2009] D.-C. Cisinski and F. Déglise, "Triangulated categories of mixed motives", preprint, 2009. arXiv

[Colmez 1998] P. Colmez, "Théorie d'Iwasawa des représentations de de Rham d'un corps local", *Ann. of Math.* (2) **148**:2 (1998), 485–571. MR Zbl

[Colmez 2002] P. Colmez, "Espaces de Banach de dimension finie", *J. Inst. Math. Jussieu* **1**:3 (2002), 331–439. MR Zbl

[Colmez and Fontaine 2000] P. Colmez and J.-M. Fontaine, "Construction des représentations p-adiques semi-stables", *Invent. Math.* **140**:1 (2000), 1–43. MR Zbl

[Colmez and Niziol 2015] P. Colmez and W. Niziol, "Syntomic complexes and p-adic nearby cycles", preprint, 2015. arXiv

[Darmon and Rotger 2016] H. Darmon and V. Rotger, "Diagonal cycles and Euler systems, II: The Birch and Swinnerton-Dyer conjecture for Hasse–Weil–Artin L-functions", *J. Amer. Math. Soc.* (online publication June 2016).

[Déglise 2008] F. Déglise, "Around the Gysin triangle, II", *Doc. Math.* **13** (2008), 613–675. MR Zbl

[Déglise and Mazzari 2015] F. Déglise and N. Mazzari, "The rigid syntomic ring spectrum", *J. Inst. Math. Jussieu* **14**:4 (2015), 753–799. MR Zbl

[Déglise and Nizioł 2015] F. Déglise and W. Nizioł, "On p-adic absolute Hodge cohomology and syntomic coefficients, I", preprint, 2015. arXiv

[Deligne 1974] P. Deligne, "Théorie de Hodge, III", *Inst. Hautes Études Sci. Publ. Math.* 44 (1974), 5–77. MR Zbl

[Fargues 2015] L. Fargues, "Quelques résultats et conjectures concernant la courbe", pp. 325–374 in *De la géométrie algébrique aux formes automorphes, I*, edited by J. B. Bost et al., Astérisque **369**, Société Mathématique de France, Paris, 2015. MR Zbl

[Fontaine 1990] J.-M. Fontaine, "Représentations p-adiques des corps locaux, I", pp. 249–309 in *The Grothendieck Festschrift, II*, edited by P. Cartier et al., Progr. Math. **87**, Birkhäuser, Boston, 1990. MR Zbl

[Fontaine 1994a] J.-M. Fontaine, "Le corps des périodes p-adiques", pp. 59–111 in *Périodes p-adiques* (Bures-sur-Yvette, 1988), Astérisque **223**, Société Mathématique de France, 1994. MR Zbl

[Fontaine 1994b] J.-M. Fontaine, "Représentations l-adiques potentiellement semi-stables", pp. 321–347 in *Périodes p-adiques* (Bures-sur-Yvette, 1988), Astérisque **223**, Société Mathématique de France, Paris, 1994. MR Zbl

[Fontaine 1994c] J.-M. Fontaine, "Représentations p-adiques semi-stables", pp. 113–184 in *Périodes p-adiques* (Bures-sur-Yvette, 1988), Astérisque **223**, Société Mathématique de France, Paris, 1994. MR Zbl

[Fontaine and Messing 1987] J.-M. Fontaine and W. Messing, "p-adic periods and p-adic étale cohomology", pp. 179–207 in *Current trends in arithmetical algebraic geometry* (Arcata, CA, 1985), Contemp. Math. **67**, Amer. Math. Soc., Providence, RI, 1987. MR Zbl

[Fontaine and Perrin-Riou 1994] J.-M. Fontaine and B. Perrin-Riou, "Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L", pp. 599–706 in *Motives* (Seattle, WA, 1991), vol. 1, edited by U. Jannsen et al., Proc. Sympos. Pure Math. **55**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Friedlander 1982] E. M. Friedlander, *Étale homotopy of simplicial schemes*, Annals of Mathematics Studies **104**, Princeton University Press, 1982. MR Zbl

[Gelfand and Manin 2003] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, 2nd ed., Springer, Berlin, 2003. MR Zbl

[Gillet 1981] H. Gillet, "Riemann–Roch theorems for higher algebraic K-theory", *Adv. in Math.* **40**:3 (1981), 203–289. MR Zbl

[Groth 2010] M. Groth, "A short course on ∞-categories", preprint, 2010. arXiv

[Hinich and Schechtman 1987] V. A. Hinich and V. V. Schechtman, "On homotopy limit of homotopy algebras", pp. 240–264 in *K-theory, arithmetic and geometry* (Moscow, 1984–1986), Lecture Notes in Math. **1289**, Springer, Berlin, 1987. MR Zbl

[Hyodo and Kato 1994] O. Hyodo and K. Kato, "Semi-stable reduction and crystalline cohomology with logarithmic poles", pp. 221–268 in *Périodes p-adiques* (Bures-sur-Yvette, 1988), Astérisque **223**, Société Mathématique de France, Paris, 1994. MR Zbl

[Illusie 2013] L. Illusie, "On the category of sheaves of objects of $\mathscr{D}(R)$ (after Beilinson and Lurie)", notes, 2013. Available upon request from luc.illusie@math.u-psud.fr.

[Kato 1994] K. Kato, "Semi-stable reduction and p-adic étale cohomology", pp. 269–293 in *Périodes p-adiques* (Bures-sur-Yvette, 1988), Astérisque **223**, Société Mathématique de France, Paris, 1994. MR Zbl

[Kato and Messing 1992] K. Kato and W. Messing, "Syntomic cohomology and p-adic étale cohomology", *Tohoku Math. J.* (2) **44**:1 (1992), 1–9. MR Zbl

[Langer 1999] A. Langer, "Local points of motives in semistable reduction", *Compositio Math.* **116**:2 (1999), 189–217. MR Zbl

[Lurie 2009] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies **170**, Princeton University Press, 2009. MR Zbl

[Lurie 2016] J. Lurie, "Higher algebra", preprint, 2016, http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[Nekovář 1993] J. Nekovář, "On p-adic height pairings", pp. 127–202 in *Séminaire de Théorie des Nombres* (Paris, 1990–91), Progr. Math. **108**, Birkhäuser, Boston, 1993. MR Zbl

[Nekovář 2000] J. Nekovář, "p-adic Abel-Jacobi maps and p-adic heights", pp. 367–379 in *The arithmetic and geometry of algebraic cycles* (Banff, 1998), edited by B. B. Gordon et al., CRM Proc. Lecture Notes **24**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[Nekovář 1998] J. Nekovář, "Syntomic cohomology and p-adic regulators", preprint, 1998, https://webusers.imj-prg.fr/~jan.nekovar/pu/syn.pdf.

[Niziol 1997] W. Niziol, "On the image of p-adic regulators", *Invent. Math.* **127**:2 (1997), 375–400. MR Zbl

[Nizioł 2001] W. Nizioł, "Cohomology of crystalline smooth sheaves", *Compositio Math.* **129**:2 (2001), 123–147. MR Zbl

[Nizioł 2006] W. Nizioł, "Toric singularities: log-blow-ups and global resolutions", *J. Algebraic Geom.* **15**:1 (2006), 1–29. MR Zbl

[Nizioł 2016a] W. Nizioł, "Geometric syntomic cohomology and vector bundles on the Fargues–Fontaine curve", preprint, 2016. arXiv

[Niziol 2016b] W. Niziol, "On syntomic regulators, I: Constructions", preprint, 2016. arXiv

[Olsson 2005] M. C. Olsson, "The logarithmic cotangent complex", *Math. Ann.* **333**:4 (2005), 859–931. MR Zbl

[Scholl 1993] A. J. Scholl, "Extensions of motives, higher Chow groups and special values of L-functions", pp. 279–292 in *Séminaire de Théorie des Nombres* (Paris, 1991–92), edited by S. David, Progr. Math. **116**, Birkhäuser, Boston, 1993. MR

[Scholze 2013] P. Scholze, "p-adic Hodge theory for rigid-analytic varieties", *Forum Math. Pi* **1** (2013), e1, 77. MR Zbl

[Scholze 2016] P. Scholze, "Erratum to 'p-adic Hodge theory for rigid-analytic varieties'", 2016, http://www.math.uni-bonn.de/people/scholze/pAdicHodgeErratum.pdf.

[Soulé 1982] C. Soulé, "Operations on étale K-theory: applications", pp. 271–303 in *Algebraic K-theory, I* (Oberwolfach, 1980), edited by R. K. Dennis, Lecture Notes in Math. **966**, Springer, Berlin, 1982. MR Zbl

[Suslin and Voevodsky 2000] A. Suslin and V. Voevodsky, "Bloch–Kato conjecture and motivic cohomology with finite coefficients", pp. 117–189 in *The arithmetic and geometry of algebraic cycles* (Banff, 1998), NATO Sci. Ser. C Math. Phys. Sci. **548**, Kluwer Acad. Publ., Dordrecht, 2000. MR Zbl

[Tsuji 1999] T. Tsuji, "p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case", *Invent. Math.* **137**:2 (1999), 233–411. MR Zbl

[Voevodsky 1996] V. Voevodsky, "Homology of schemes", *Selecta Math.* (*N.S.*) **2**:1 (1996), 111–153. MR Zbl

[Voevodsky et al. 2000] V. Voevodsky, A. Suslin, and E. M. Friedlander, *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies **143**, Princeton University Press, 2000. MR Zbl

[Zheng 2013] W. Zheng, "Note on derived ∞-categories and monoidal structures", notes, 2013. Available upon request from wzheng@math.ac.cn.

Communicated by Hélène Esnault
Received 2016-02-26 Accepted 2016-07-05

jan.nekovar@imj-prg.fr *Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie (Paris 6), 75252 Paris Cedex 05, France*

wieslawa.niziol@ens-lyon.fr *École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, 69364 Lyon Cedex 07, France*

laurent.berger@ens-lyon.fr *École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, 69364 Lyon Cedex 07, France*

frederic.deglise@ens-lyon.fr *École Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, 69364 Lyon Cedex 07, France*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2016 is US $290/year for the electronic version, and $485/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscribers address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 10 No. 8 2016