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We give a criterion for the vanishing of the weight-one syzygies associated to a line bundle B in a sufficiently positive embedding of a smooth complex projective variety of arbitrary dimension.

Introduction

Inspired by the methods of Voisin [2002; 2005], Ein and Lazarsfeld [2015] recently proved the gonality conjecture of [Green and Lazarsfeld 1986], asserting that one can read off the gonality of an algebraic curve C from the syzygies of its ideal in any one embedding of sufficiently large degree. They deduced this as a special case of a vanishing theorem for the asymptotic syzygies associated to an arbitrary line bundle B on C, and conjectured that an analogous statement should hold on a smooth projective variety of any dimension. The purpose of this note is to prove the conjecture in question.

Turning to details, let X be a smooth complex projective variety of dimension n, and set

$$L_d = dA + P,$$

where A is ample and P is arbitrary. We always assume that d is sufficiently large so that L_d is very ample, defining an embedding

$$X \subseteq \mathbb{P} H^0(X, L_d) = \mathbb{P}^{r_d}.$$

Given an arbitrary line bundle B on X, we wish to study the weight-one syzygies of B with respect to L_d for $d \gg 0$. More precisely, let $S = \operatorname{Sym} H^0(X, L_d)$ be the homogeneous coordinate ring of $\mathbb{P} H^0(X, L_d)$, and put

$$R = R(X, B; L_d) = \bigoplus_m H^0(X, B + mL_d).$$

Thus R is a finitely generated graded S-module, and hence has a minimal graded free resolution $E_\bullet = E_\bullet(B; L_d)$:

$$0 \longrightarrow E_{r_d} \longrightarrow \cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow R \longrightarrow 0,$$

where $E_p = \bigoplus S(-a_{p,j})$. As customary, denote by $K_{p,q}(X, B; L_d)$ the finite-dimensional vector space of degree $p + q$ minimal generators of E_p, so that

$$E_p(B; L_d) = \bigoplus_q K_{p,q}(X, B; L_d) \otimes_\mathbb{C} S(-p - q).$$

We refer to elements of this group as p-th syzygies of B with respect to L_d of weight q. When $B = \mathcal{O}_X$ we write simply $K_{p,q}(X; L_d)$, which — provided that d is large enough so that L_d is normally generated — are the vector spaces describing the syzygies of the homogeneous ideal $I_X \subset S$ of X in $\mathbb{P}H^0(X, L_d)$.

The question we address involves fixing B and asking when it happens that

$$K_{p,q}(X, B; L_d) = 0 \quad \text{for } d \gg 0.$$

When $q = 0$ or $q \geq 2$ the situation is largely understood thanks to results of Green [1984a; 1984b] and Ein and Lazarsfeld [1993; 2012]. (See Remark 1.10 for a summary.) Moreover in this range the statements are uniform in nature, in that they don't depend on the geometry of X or B. However as suggested in [Ein and Lazarsfeld 2012, Problem 7.1], for $K_{p,1}(X, B; L_d)$ one can anticipate more precise asymptotic results that do involve geometry. This is what we establish here.

Recall that a line bundle B on a smooth projective variety X is said to be *p-jet very ample* if for every effective zero-cycle

$$w = a_1 x_1 + \cdots + a_s x_s$$

of degree $p + 1 = \sum a_i$ on X, the natural map

$$H^0(X, B) \to H^0(X, B \otimes \mathcal{O}_X / \mathfrak{m}_1^{a_1} \cdots \mathfrak{m}_s^{a_s})$$

is surjective, where $\mathfrak{m}_i \subseteq \mathcal{O}_X$ is the ideal sheaf of x_i. So for example if $p = 1$ this is simply asking that B be very ample. When $\dim X = 1$ the condition is the same as requiring that B be p-very ample — i.e., that every subscheme $\xi \subset X$ of length $p + 1$ imposes independent conditions in $H^0(X, B)$ — but in higher dimensions it is a stronger condition.

Our main result is

Theorem A. *If B is p-jet very ample, then*

$$K_{p,1}(X, B; L_d) = 0 \quad \text{for } d \gg 0.$$

The statement was conjectured in [Ein and Lazarsfeld 2015, Conjecture 2.4], where the case dim $X = 1$ was established.

It is not clear whether one should expect that p-jet amplitude is equivalent to the vanishing of $K_{p,1}(X, B; L_d)$ for $d \gg 0$. However we prove:

Theorem B. *Suppose that there is a* reduced *$(p + 1)$-cycle w on X that fails to impose independent conditions on $H^0(X, B)$. Then*

$$K_{p,1}(X, B; L_d) \neq 0 \quad for \ d \gg 0.$$

In general, the proof of Theorem A will show that if $H^1(X, B) = 0$, then the jet amplitude hypothesis on B is equivalent when $d \gg 0$ to the vanishing of a group that contains $K_{p,1}(X, B; L_d)$ as a subspace (Remark 1.8).

When $B = K_X$ is the canonical bundle of X, Theorem A translates under a mild additional hypothesis into a statement involving the syzygies of L_d itself.

Corollary C. *Assume that $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$, or equivalently that $X \subseteq \mathbb{P}^{r_d}$ is projectively Cohen–Macaulay for $d \gg 0$.*

(i) *The canonical bundle K_X of X is very ample if and only if*

$$K_{r_d-n-1,n}(X; L_d) = 0 \quad for \ d \gg 0.$$

(ii) *If K_X is p-jet very ample, then*

$$K_{r_d-n-p,n}(X; L_d) = 0 \quad for \ d \gg 0.$$

When $n = \dim X = 1$, this (together with Theorem B) implies $K_{r_d-c,1}(X; L_d) \neq 0$ for $d \gg 0$ if and only if X admits a branched covering $X \to \mathbb{P}^1$ of degree $\leq c$, which is the statement of the gonality conjecture established in [Ein and Lazarsfeld 2015].

The proof of Theorem A occupies Section 1. It follows very closely the strategy of [Ein and Lazarsfeld 2015], which in turn was inspired by the ideas of Voisin [2002; 2005]. However instead of working on a Hilbert scheme or symmetric product, we work on a Cartesian product of X, using an idea that goes back in a general way to Green [1984b]. For the benefit of nonexperts, we outline now the approach in some detail in the toy case $p = 0$.[1]

Keeping notation as above, it follows from the definition that $K_{0,1}(X, B; L_d) = 0$ if and only if the multiplication map

$$H^0(X, B) \otimes H^0(X, L_d) \to H^0(X, B \otimes L_d) \tag{$*$}$$

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[1]This was in fact the train of thought that led us to the arguments here and in [Ein and Lazarsfeld 2015].

is surjective: in fact, $K_{0,1}$ is its cokernel. A classical way to study such maps is to pass to the product $X \times X$ and then restrict to the diagonal. Specifically, $(*)$ is identified with the homomorphism

$$H^0(X \times X, \mathrm{pr}_1^* B \otimes \mathrm{pr}_2^* L_d) \to H^0(X \times X, \mathrm{pr}_1^* B \otimes \mathrm{pr}_2^* L_d \otimes \mathcal{O}_\Delta) \qquad (**)$$

arising from this restriction. Thus the vanishing $K_{0,1}(X, B; L_d) = 0$ is implied by the surjectivity of $(**)$. Green [1984b] observed that there is a similar way to tackle the $K_{p,1}$ for $p \geq 1$: one works on the $(p+2)$-fold product $X^{p+2} = X \times X^{p+1}$ and restricts to a suitable union of pairwise diagonals (Proposition 1.1). This is explained in Section 1, and forms the starting point of our argument. Although not strictly necessary we give a new proof of Green's result here that clarifies its relation to other approaches.

There remains the issue of actually proving the surjectivity of $(**)$ for $d \gg 0$ provided that B is 0-jet very ample, i.e., globally generated. For this one starts with the restriction

$$\mathrm{pr}_1^* B \to \mathrm{pr}_1^* B \otimes \mathcal{O}_\Delta$$

of sheaves on $X \times X$ and pushes down to X via pr_2. There results a map of vector bundles

$$\mathrm{ev}_B : H^0(X, B) \otimes_{\mathbb{C}} \mathcal{O}_X \to B$$

on X which is given by evaluation of sections of B. Note that ev_B is surjective as a map of bundles if and only if B is globally generated. The surjectivity in $(*)$ or $(**)$ is then equivalent to the surjectivity on global sections of the map

$$H^0(X, B) \otimes L_d \to B \otimes L_d$$

obtained from twisting ev_B by L_d.

Suppose now that B is 0-jet very ample. The setting $M_B = \ker(\mathrm{ev}_B)$, we get an exact sequence

$$0 \to M_B \to H^0(X, B) \otimes_{\mathbb{C}} \mathcal{O}_X \to B \to 0$$

of sheaves on X. Serre vanishing implies that

$$H^1(X, M_B \otimes L_d) = 0$$

for $d \gg 0$, and by what we have just said this means that $K_{0,1}(X, B; L_d) = 0$. The proof of Theorem A in general proceeds along analogous lines. We construct a torsion-free sheaf $\mathcal{E}_B = \mathcal{E}_{p+1,B}$ of rank $p+1$ on X^{p+1} whose fiber at $(x_1, \ldots, x_{p+1})$ is identified with

$$H^0(X, B \otimes \mathcal{O}_X / \mathfrak{m}_1 \cdots \mathfrak{m}_{p+1}),$$

where $\mathfrak{m}_i \subseteq \mathcal{O}_X$ is the ideal sheaf of x_i. This comes with an evaluation map

$$\mathrm{ev}_{p+1,B} : H^0(X, B) \otimes_{\mathbb{C}} \mathcal{O}_{X^{p+1}} \to \mathcal{E}_B$$

which is surjective (as a map of sheaves) if and only if B is p-jet very ample.[2] Green's criterion for the vanishing of $K_{p,1}(X, B; L_d)$ turns out to be equivalent to the surjectivity of the map on global sections resulting from twisting $\mathrm{ev}_{p+1,B}$ by a suitable ample divisor $\mathcal{N}_d$ on X^{p+1} deduced from L_d, and this again follows from Serre vanishing.

Returning to the case $p = 0$, the argument just sketched actually proves more. Namely for arbitrary B one has an exact sequence

$$0 \to \ker(\mathrm{ev}_B) \to H^0(X, B) \otimes_{\mathbb{C}} \mathcal{O}_X \to B \to \mathrm{coker}(\mathrm{ev}_B) \to 0,$$

and so Serre vanishing shows conversely that if B is not 0-jet very ample then

$$K_{p,1}(X, B; L_d) = H^0(X, \mathrm{coker}(\mathrm{ev}_B) \otimes L_d) \neq 0 \qquad (***)$$

for $d \gg 0$. Unfortunately this does not generalize when $p \geq 1$ because the computations on X^{p+1} lead to groups that contain $K_{p,1}(X, B; L_d)$ as summands, but may contain other terms as well. (Said differently, Green's criterion is sufficient but not necessary for the vanishing of $K_{p,1}$.) To prove a nonvanishing statement such as Theorem B, one needs a geometric interpretation of $K_{p,1}$ itself. Voisin [2002; 2005] achieves this by working on a Hilbert scheme — which has the advantage of being smooth when $\dim X = 2$ — while Yang [2014] passes in effect to the symmetric product.[3] We follow the latter approach for Theorem B: we exhibit a sheaf on $\mathrm{Sym}^{p+1}(X)$ whose twisted global sections compute $K_{p,1}(X, B; L_d)$, and we show that it is nonzero provided that there is a reduced cycle that fails to impose independent conditions on $H^0(X, B)$. Then we can argue much as in the case $p = 0$ just described. This is the content of Section 2.

1. Proof of Theorem A

This section is devoted to the proof of Theorem A from the introduction.

We start by describing the set-up. As above, X is a smooth complex projective variety of dimension n, and we consider the $(p + 2)$-fold product

$$Y \overset{\mathrm{def}}{=} X \times X^{p+1}$$

of X with itself. For $0 \leq i < j \leq p + 1$ denote by

$$\pi_{i,j} : Y \to X \times X$$

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[2]Note that the fiber of B at a point $x \in X$ is identified with $B \otimes \mathcal{O}_X/\mathfrak{m}_x$, i.e., $\mathcal{E}_{1,B} = B$.

[3]Roughly speaking, one is picking out $K_{p,1}$ inside Green's construction as the space of invariants under a suitable action of the symmetric group.

the projection of Y onto the product of the i and j factors. We write $\Delta_{i,j} \subseteq Y$ for the pull-back of the diagonal $\Delta \subseteq X \times X$ under $\pi_{i,j}$, so that $\Delta_{i,j}$ consists of those points $y = (x_0, x_1, \ldots, x_{p+1}) \in Y$ with $x_i = x_j$.

The basic idea — which goes back to Green [1984b] and has been used repeatedly since (e.g., [Inamdar 1997; Bertram et al. 1991; Lazarsfeld et al. 2011; Hwang and To 2013; Yang 2014]) — is to relate syzygies on X to a suitable union of pairwise diagonals on Y. Specifically, let

$$Z = Z_{p+1} = \Delta_{0,1} \cup \cdots \cup \Delta_{0,p+1} \subseteq X \times X^{p+1} \tag{1-1}$$

be the union of the indicated pairwise diagonals, considered as a reduced subscheme. We denote by

$$q : Z \to X, \quad \sigma : Z \to X^{p+1} \tag{1-2}$$

the indicated projections.

The importance of this construction for us is given by:

Proposition 1.1. *Let L and B be respectively base-point-free and arbitrary line bundles on X, and assume (for simplicity) that $H^1(X, L) = 0$. If the restriction homomorphism*

$$H^0(Y, B \boxtimes L^{\boxtimes p+1}) \to H^0(Y, (B \boxtimes L^{\boxtimes p+1}) \mid Z) \tag{1-3}$$

is surjective, then

$$K_{p,1}(X, B; L) = 0.$$

The Proposition was essentially established for instance in [Yang 2014], but it is instructive to give a direct argument. We start with a lemma that will also be useful later:

Lemma 1.2. *Writing $I_{Z/Y}$ for the ideal sheaf of Z in Y, one has*

$$I_{Z/Y} = I_{\Delta_{0,1}/Y} \cdot I_{\Delta_{0,2}/Y} \cdots I_{\Delta_{0,p+1}/Y} = \bigotimes_{j=1}^{p+1} \pi_{0,j}^* I_{\Delta/X \times X}.$$

Sketch of Proof. This is implicit in [Li 2009, Theorem 1.3], but does not appear there explicitly so we very briefly indicate an argument. The statement is étale local, so we can assume $X = \mathbb{A}^n$. By looking at a suitable subtraction map, as in [Lazarsfeld et al. 2011, (1-3)], it then suffices to prove the analogous statement for $Y = X^{p+1}$ with Z being the union of the "coordinate planes"

$$L_i = \{(x_1, \ldots, x_i, \ldots, x_{p+1}) \mid x_i = 0 \in \mathbb{A}^n\} = \{\mathbb{A}^n\} \times \ldots \times \{0\} \times \ldots \times \{\mathbb{A}^n\} \subseteq Y$$

$(1 \le i \le p+1)$. For this one can proceed by induction on p, writing out explicitly the equations defining each L_i. $\qquad\square$

Remark 1.3. This is the essential place where we use the hypothesis that X is smooth. We do not know whether the statement of the Lemma remains true for singular X.

Proof of Proposition 1.1. To begin with, it is well known (see [Green 1984a]) that $K_{p,1}(X, B; L)$ is the cohomology of the Koszul-type complex

$$\Lambda^{p+1} H^0(L) \otimes H^0(B) \to \Lambda^p H^0(L) \otimes H^0(B+L) \to \Lambda^{p-1} H^0(L) \otimes H^0(B+2L).$$

Moreover this cohomology can in turn be interpreted geometrically in terms of the vector bundle M_L on X defined (as in the introduction) as the kernel of the evaluation map

$$\mathrm{ev}_L : H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_X \to L.$$

Specifically, M_L sits in an exact sequence of vector bundles

$$0 \to M_L \to H^0(L) \otimes \mathcal{O}_X \to L \to 0 \qquad (\star)$$

on X, and then $K_{p,1}(X, B; L) = 0$ if and only if the sequence

$$0 \to \Lambda^{p+1} M_L \otimes B \to \Lambda^{p+1} H^0(L) \otimes B \to \Lambda^p M_L \otimes L \otimes B \to 0 \qquad (\star\star)$$

deduced from $(\star)$ is exact on global sections. (See for instance [Green and Lazarsfeld 1986, Lemma 1.10] or [Lazarsfeld 1989].)

On the other hand, consider on $X \times X$ the exact sequence

$$0 \to I_\Delta \otimes \mathrm{pr}_2^* L \to \mathrm{pr}_2^* L \to L \otimes \mathcal{O}_\Delta \to 0.$$

As in the introduction, this pushes down via pr_1 to $(\star)$. Therefore one finds from Lemma 1.2 and the Künneth formula that

$$q_*(I_{Z/Y} \otimes B \boxtimes L^{\boxtimes p+1}) = \left(\bigotimes^{p+1} M_L\right) \otimes B,$$

and moreover the $R^1 q_*$ vanishes thanks to our hypothesis that $H^1(L) = 0$.[4]

Writing

$$\mathcal{N} = \mathrm{coker}\left(\bigotimes^{p+1} M_L \to \bigotimes^{p+1} H^0(L)\right),$$

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[4]The Künneth theorem in play here is the following: let $V_1 \to S, \ldots, V_r \to S$ be mappings of schemes over a field, and suppose that F_i is a quasicoherent sheaf on V_i that is flat over S. Write

$$p_i : V_1 \times_S \cdots \times_S V_r \to V_i, \quad p : V_1 \times_S \cdots \times_S V_r \to S$$

for the natural maps. Then

$$p_*(p_1^* F_1 \otimes \cdots \otimes p_r^* F_r) = p_{1,*} F_1 \otimes \cdots \otimes p_{r,*} F_r$$

as sheaves on S, with analogous Künneth-type computations of the $R^j p_*$. See for instance [Kempf 1980, Theorem 14] for a simple proof when S is affine, and [EGA III$_2$ 1963, Theorem 6.7.8] for the general case.

it follows that

$$q_*((B \boxtimes L^{\boxtimes p+1}) \mid Z) = \mathcal{N} \otimes B,$$

and hence the surjectivity of (1-3) is equivalent to asking that

$$0 \to \left(\bigotimes^{p+1} M_L\right) \otimes B \to \left(\bigotimes^{p+1} H^0(L)\right) \otimes B \to \mathcal{N} \otimes B \to 0 \qquad (\star\star\star)$$

be exact on global sections. But since we are in characteristic zero, the exact sequence $(\star\star)$ is a summand of this, and the lemma follows. $\qquad\square$

Remark 1.4. The argument just completed shows that if in addition $H^1(X, B) = 0$ then (1-3) is surjective if and only if

$$H^1\left(X, \left(\bigotimes^{p+1} M_L\right) \otimes B\right) = 0.$$

It remains to relate these considerations to the jet-amplitude of B. To this end, keeping notation as in (1-2), set

$$\mathcal{E}_B = \mathcal{E}_{p+1,B} = \sigma_*(q^* B).$$

This is a torsion-free sheaf of rank $p + 1$ on X^{p+1} (since it is the push forward of a line bundle under a finite mapping of degree $p + 1$), and one has:

Lemma 1.5. (i) *Fix a point*

$$\xi = (x_1, \ldots, x_{p+1}) \in X^{p+1},$$

the x_i being (possibly nondistinct) points of X, and denote by $\mathcal{E}_B|\xi$ the fiber of $\mathcal{E}_B$ at ξ. Then there is a natural identification

$$\mathcal{E}_B|\xi = H^0(X, B \otimes \mathcal{O}_X/\mathfrak{m}_1 \cdots \mathfrak{m}_{p+1}),$$

where $\mathfrak{m}_i \subseteq \mathcal{O}_X$ is the maximal ideal of x_i.

 (ii) *There is a canonical injection*

$$H^0(X, B) \hookrightarrow H^0(X^{p+1}, \mathcal{E}_B),$$

giving rise to a homomorphism

$$\mathrm{ev}_B = \mathrm{ev}_{p+1,B} : H^0(X, B) \otimes_{\mathbb{C}} \mathcal{O}_{X^{p+1}} \to \mathcal{E}_B.$$

of sheaves on X^{p+1}. Under the identification in (i), ev_B is given fiberwise by the natural map

$$H^0(X, B) \to H^0\big(X, B \otimes \mathcal{O}_X/(\mathfrak{m}_1 \cdots \mathfrak{m}_{p+1})\big).$$

(iii) *The homomorphism (1-3) is identified with the map on global sections arising from the sheaf homomorphism*

$$H^0(X, B) \otimes_{\mathbb{C}} L^{\boxtimes p+1} \to \mathcal{E}_B \otimes L^{\boxtimes p+1}. \tag{1-4}$$

on X^{p+1} determined by twisting ev_B by $L^{\boxtimes p+1}$.

(iv) *The mapping $\mathrm{ev}_{p+1,B}$ is surjective as a homomorphism of sheaves on X^{p+1} if and only if B is p-jet very ample.*

Proof. For (i), consider the diagram

$$\tag{1-5}$$

and fix $\xi = (x_1, \ldots, x_{p+1}) \in X^{p+1}$. The scheme-theoretic fiber $\sigma^{-1}(\xi)$ lives naturally as a subscheme of X, and Lemma 1.2 implies that it is in fact the scheme defined by the ideal sheaf

$$\mathfrak{m}_1 \cdots \mathfrak{m}_{p+1} \subseteq \mathcal{O}_X.$$

Therefore, thanks to the projection formula, the fiber $\mathcal{E}_B|\xi$ of $\mathcal{E}_B$ at ξ is identified with

$$\mathrm{pr}_{2,*}\big(B \otimes \mathcal{O}_X / (\mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_{p+1})\big) = H^0\big(X, B \otimes \mathcal{O}_X / (\mathfrak{m}_1 \cdots \mathfrak{m}_{p+1})\big),$$

as claimed. For (ii), note that in any event

$$H^0(X^{p+1}, \mathcal{E}_B) = H^0(X^{p+1}, \sigma_* q^* B) = H^0(Z, q^* B).$$

On the other hand, each of the irreducible components of Z maps via projection onto X, and this gives an inclusion

$$q^* : H^0(X, B) \to H^0(Z, q^* B) = H^0\big(X \times X^{p+1}, (\mathrm{pr}_1^* B)|Z\big).$$

It is evident from the construction that fiber by fiber ev_B is as described, and (iv) is then a consequence of the fact that a morphism of sheaves is surjective if and only it is so on each fiber. Finally, statement (iii) follows from the construction of $\mathcal{E}_B$ and ev_B. $\qquad\square$

Remark 1.6. Using the resolution of $\mathcal{O}_Z$ appearing in [Yang 2014, p. 4], one can show that in fact

$$H^0(X^{p+1}, \mathcal{E}_B) = H^0(X, B).$$

However this isn't necessary for the argument.

Remark 1.7. The reader familiar with [Ein and Lazarsfeld 2015] or Voisin's Hilbert schematic approach to syzygies will recognize that $Z \to X^{p+1}$ plays the role of the universal family over the Hilbert scheme, and that Proposition 1.1 is the analogue of [Voisin 2002, Lemma 1, p. 369]. The sheaf $\mathcal{E}_{p+1,B}$ plays the role of the vector bundle $E_{p+1,B}$ on the symmetric product appearing in [Ein and Lazarsfeld 2015].

Just as in [Ein and Lazarsfeld 2015], the main result now follows immediately from Serre vanishing.

Proof of Theorem A. Assuming that B is p-jet very ample, so that $\mathrm{ev}_{p+1,B}$ is surjective, let $\mathcal{M}_{p+1,B}$ denote its kernel:

$$0 \to \mathcal{M}_{p+1,B} \to H^0(B) \otimes \mathcal{O}_{X^{p+1}} \to \mathcal{E}_B \to 0.$$

To show that $K_{p,1}(X, B; L_d) = 0$ it suffices, thanks to Proposition 1.1 and its interpretation in terms of (1-4), to prove that

$$H^1(X^{p+1}, \mathcal{M}_{p+1,B} \otimes L_d^{\boxtimes p+1}) = 0$$

for $d \gg 0$. But this follows immediately from Serre vanishing. □

Remark 1.8. It follows from the argument just completed that if $L = L_d$ then the surjectivity in Proposition 1.1 holds for $d \gg 0$ if and *only if* B is p-jet very ample. In particular, in view of Remark 1.4 this means that if $H^1(X, B) = 0$ then the p-jet very amplitude of B is equivalent to the vanishing

$$H^1\left(X, \left(\bigotimes^{p+1} M_{L_d}\right) \otimes B\right) = 0 \quad \text{for } d \gg 0.$$

Proof of Corollary C. Under the stated hypothesis on X, the groups in question are Serre dual to $K_{p,1}(X, B; L_d)$ for $d \gg 0$ (see [Green 1984a, §2]). If B fails to be very ample, then a simple argument as in [Eisenbud et al. 2006, Theorem 1.1] shows that $K_{1,1}(X, B; L_d) \neq 0$ for $d \gg 0$, and therefore the corollary follows from the main theorem. □

Remark 1.9. In the case of curves, Rathmann [2016] has given a very interesting argument that leads to an essentially optimal effective version of the asymptotic results of [Ein and Lazarsfeld 2015]: in fact, it suffices that $H^1(L) = H^1(L - B) = 0$. In this spirit, it would be very interesting to find an effective estimate for the positivity of L to guarantee the vanishing of $K_{p,1}(X, B; L)$ when B is p-jet very ample.

Remark 1.10 (other Koszul cohomology groups). To conclude this section, we briefly summarize what is known about the groups $K_{p,q}(X, B; L_d)$ for $q \neq 1$. Specifically, fix B. Then for $d \gg 0$:

(i) $K_{p,q}(X, B; L_d) = 0$ for $q \geq n + 2$.

(ii) One has

$$K_{p,0}(X, B; L_d) \neq 0 \Longleftrightarrow 0 \leq p \leq r(B), \text{ and}$$
$$K_{p,n+1}(X, B; L_d) \neq 0 \Longleftrightarrow r_d - n - r(K_X - B) \leq p \leq r_d - n.$$

(iii) If $q \geq 2$, then $K_{p,q}(X, B; L_d) = 0$ when $p \leq O(d)$.

(iv) For $1 \leq q \leq n$, $K_{p,q}(X, B; L_d) \neq 0$ for

$$O(d^{q-1}) \leq p \leq r_d - O(d^{n-1}).$$

Statement (i) is a consequence of Castelnuovo–Mumford regularity, while (ii) is due to Green and others. (See [Green 1984a, §3; Ein and Lazarsfeld 2012, Corollary 3.3, §5].) Assertion (ii) follows for instance from [Ein and Lazarsfeld 1993], while (iv) is the main result of [Ein and Lazarsfeld 2012]. Furthermore, it is conjectured in [Ein and Lazarsfeld 2012] that if $q \geq 2$, then $K_{p,q}(X, B; L_d) = 0$ for $p \leq O(d^{q-1})$.

2. A nonvanishing theorem

This section is devoted to the proof of Theorem B from the introduction. Recall the statement:

Theorem 2.1. *Assume that the nonsingular projective variety X carries an effective $(p+1)$-cycle $w = x_1 + \cdots + x_{p+1}$ consisting of $p+1$ distinct points $x_1, \ldots, x_{p+1} \in X$ that fail to impose independent conditions on $H^0(X, B)$. Then*

$$K_{p,1}(X, B; L_d) \neq 0 \quad \text{for } d \gg 0.$$

The argument is somewhat technical, so before launching into it we would like to outline the rough strategy. As in the case $p = 0$ discussed in the introduction, in principle we would like to find a map of sheaves on $\text{Sym}^{p+1}(X)$ depending on B — say $a_B : \mathcal{A}_1 \to \mathcal{A}_2$ — having the property that

$$K_{p,1}(X, B; L_d) = \text{coker}\big(H^0(\mathcal{A}_1 \otimes \mathcal{N}(L_d)) \to H^0(\mathcal{A}_2 \otimes \mathcal{N}(L_d))\big), \quad (2\text{-}1)$$

where $\mathcal{N}(L_d)$ is a line bundle whose positivity grows suitably with d. Ideally — as in equation (∗∗∗) from the introduction — we would be able to see that a_B cannot be surjective as a map of sheaves if B is not p-jet very ample, and then one could hope to apply Serre vanishing to conclude that $K_{p,1}(X, B; L_d)$ cannot vanish for $d \gg 0$. Unfortunately we do now know whether such a construction is possible. Instead, what we do in effect is to use the ideas of Yang [2014] to construct a map α_B, and show that the nonvanishing of $K_{p,1}$ is implied by the nonvanishing of a certain quotient sheaf of $\mathcal{A}_2$. We show that a *reduced $(p + 1)$-cycle* that fails to impose independent conditions on $H^0(X, B)$ must appear in the support of this quotient, and this leads to the stated nonvanishing.

We start by recalling the results of Yang [2014] interpreting $K_{p,1}$ as an equivariant cohomology group. Consider then a very ample line bundle L on the smooth complex projective variety X. Then the symmetric group S_{p+1} acts in two ways on the bundle $L^{\boxtimes p+1}$ on X^{p+1}, namely via the symmetric and the alternating characters. Denote these S_{p+1}-bundles on X^{p+1} by

$$L^{\boxtimes p+1,\text{sym}} \quad \text{and} \quad L^{\boxtimes p+1,\text{alt}}, \tag{2-2}$$

respectively. Now let S_{p+1} act on $X \times X^{p+1}$ via the trivial action on the first factor, so that the union of pairwise diagonals $Z \subseteq X \times X^{p+1}$ defined in (1-1) becomes an S_{p+1}-subspace. It is established in [Yang 2014, Theorem 3] that if

$$H^i(X, mL) = H^i(X, B + mL) = 0 \quad \text{for } i, m > 0, \tag{2-3}$$

then $K_{p,1}(X, B; L)$ is identified with the cokernel of the restriction mapping

$$H^0_{S_{p+1}}(X \times X^{p+1}, \text{pr}_1^* B \otimes \text{pr}_2^* L^{\boxtimes p+1,\text{alt}}) \to H^0_{S_{p+1}}(Z, \text{pr}_1^* B \otimes \text{pr}_2^* L^{\boxtimes p+1,\text{alt}} \otimes \mathcal{O}_Z) \tag{2-4}$$

on S_{p+1}-equivariant cohomology groups.[5] One can think of this as a precision and strengthening of Proposition 1.1. Following the line of attack of Section 1, the plan is to study these groups by modding out by the symmetric group and pushing down to the symmetric product.

To this end, denote by $\text{Sym}^{p+1}(X)$ the $(p+1)$-st symmetric product of X, which we view as parametrizing zero-cycles of degree $p+1$, and write

$$\pi : X^{p+1} \to \text{Sym}^{p+1}(X)$$

---

[5]For the theory of equivariant cohomology groups and pushforwards, see [Grothendieck 1957, Chapter 5]. What we need can be summarized as follows. Let G be a finite group acting on a complex projective variety V, and suppose a coherent sheaf F on X together with an action of G on F are given. Then one can define equivariant cohomology groups $H^j_G(V, F)$. While this isn't how they are initially constructed, one can show that

$$H^j_G(V, F) = H^j(V, F)^G,$$

the group on the right being the G-invariant subspace of $H^j(V, F)$ under the natural action of G on this cohomology group [Grothendieck 1957, p. 202], and for practical purposes one can take this as the definition. Writing

$$\pi : V \to V/G \stackrel{\text{def}}{=} W,$$

one also has an action of G on $\pi_* F$. The G-equivariant direct image of F can be interpreted as

$$\pi_*^G F = (\pi_*(F))^G,$$

and one can show that

$$H^j_G(V, F) = H^j(W, \pi_*^G F).$$

for the quotient map. The equivariant pushforward of the line bundles in (2-2) determine respectively a line bundle and torsion-free sheaf of rank one

$$\mathcal{S}_{p+1}(L) \overset{\text{def}}{=} \pi_*^{S_{p+1}}(L^{\boxtimes p+1, \text{sym}}), \quad \mathcal{N}_{p+1}(L) \overset{\text{def}}{=} \pi_*^{S_{p+1}}(L^{\boxtimes p+1, \text{alt}})$$

on $\text{Sym}^{p+1}(X)$. One has

$$H^0\big(\text{Sym}^{p+1}(X), \mathcal{S}_{p+1}(L)\big) = \text{Sym}^{p+1} H^0(X, L),$$
$$H^0\big(\text{Sym}^{p+1}(X), \mathcal{N}_{p+1}(L)\big) = \Lambda^{p+1} H^0(X, L),$$

and for any line bundle A on X:

$$\mathcal{S}(L \otimes A) = \mathcal{S}(L) \otimes \mathcal{S}(A), \quad \mathcal{N}(L \otimes A) = \mathcal{N}(L) \otimes \mathcal{S}(A). \tag{2-5}$$

Moreover $\mathcal{S}(A)$ is ample if A is.

Proof of Theorem 2.1. Note to begin with that the symmetric group S_{p+1} acts on each of the spaces appearing in diagram (1-5). Taking the quotients yields the diagram

$$\bar{Z} = X \times \text{Sym}^p(X) \overset{\subseteq}{\hookrightarrow} X \times \text{Sym}^{p+1}(X)$$

$$\bar{\sigma} \searrow \quad \swarrow p_2$$

$$\text{Sym}^{p+1}(X) \tag{2-6}$$

where $\bar{\sigma}$ is the addition map,

$$p_1 : X \times \text{Sym}^{p+1}(X) \to X, \quad p_2 : X \times \text{Sym}^{p+1}(X) \to \text{Sym}^{p+1}(X)$$

are the projections, and the inclusion on the top line is given by $(x, w) \mapsto (x, x+w)$. One has

$$(1 \times \pi)_*^{S_{p+1}}(\text{pr}_1^* B \otimes \text{pr}_2^* L^{\boxtimes p+1, \text{alt}}) = p_1^* B \otimes p_2^* \mathcal{N}(L).$$

Now define

$$\mathcal{G}(B; L) = (1 \times \pi)_*^{S_{p+1}}(\text{pr}_1^* B \otimes \text{pr}_2^* L^{\boxtimes p+1, \text{alt}} \otimes \mathcal{O}_Z).$$

Pushing forward the restriction to Z gives rise to a natural surjective mapping

$$\varepsilon(B; L) : p_1^* B \otimes p_2^* \mathcal{N}(L) \to \mathcal{G}(B; L) \tag{2-7}$$

of sheaves on $X \times \text{Sym}^{p+1}(X)$. Thanks to [Grothendieck 1957, §5.2], the groups appearing in (2-4) are given by the global sections of the sheaves in (2-7), and hence under the vanishing hypothesis (2-3), $K_{p,1}(X, B; L)$ is computed as the cokernel

$$K_{p,1}(X, B; L) = \text{coker}\big(H^0(\varepsilon(B; L))\big)$$

on global sections determined by $\varepsilon(B; L)$. Hence we are reduced to showing that under the hypothesis of the theorem, $\varepsilon(B; L_d)$ cannot be surjective on global sections when $d \gg 0$.

The next step is to form and study the push-forward of (2-7) to $\mathrm{Sym}^{p+1}(X)$. To begin with, define $\mathcal{F}(B; L) = p_{2,*}\mathcal{G}(B; L)$ and $\delta(B; L) = p_{2,*}\varepsilon(B; L)$. This gives rise to a morphism

$$\delta(B; L) : H^0(X, B) \otimes_{\mathbb{C}} \mathcal{N}(L) \to \mathcal{F}(B; L)$$

of sheaves on $\mathrm{Sym}^{p+1}(X)$ with the property that

$$K_{p,1}(X, B; L) = \mathrm{coker}\big(H^0(\delta(B, L))\big).$$

We wish to study the geometry of this mapping assuming that B does not impose independent conditions on all reduced cycles. We assert that there is a natural homomorphism

$$t : \big(p_1^* B \otimes p_2^* \mathcal{N}(L) \otimes \mathcal{O}_{\bar{Z}}\big) \to \mathcal{G}(B; L) \tag{2-8}$$

which is an isomorphism on the smooth locus of $\bar{Z}$. Grant this for the time being. By the projection formula one has

$$p_{2,*}\big(p_1^* B \otimes p_2^* \mathcal{N}(L) \otimes \mathcal{O}_{\bar{Z}}\big) = \bar{\sigma}_*(p_1^* B) \otimes \mathcal{N}(L),$$

and then taking direct images in (2-7) and (2-8), one arrives at a diagram

$$\begin{array}{ccc}
H^0(X, B) \otimes_{\mathbb{C}} \mathcal{N}(L) & \xrightarrow{\ e \otimes 1\ } & \bar{\sigma}_*(p_1^* B) \otimes \mathcal{N}(L) \\
& \searrow{\scriptstyle \delta(B;L)} & \Big\downarrow{\scriptstyle s} \\
& & \mathcal{F}(B; L)
\end{array} \tag{2-9}$$

where s is an isomorphism over the smooth locus of $\mathrm{Sym}^{p+1}(X)$.

Now fix a reduced zero-cycle $w \in \mathrm{Sym}^{p+1}(X)$ that fails to impose independent conditions on $H^0(X, B)$, and let $\xi \subseteq X$ be the corresponding subscheme of length $p + 1$. We can identify the fiber of the morphism

$$e : H^0(X, B) \otimes \mathcal{O}_{\mathrm{Sym}^{p+1}(X)} \to \bar{\sigma}(p_1^* B)$$

appearing in (2-9) at w with the evaluation $H^0(X, B) \to H^0(X, B \otimes \mathcal{O}_{\xi})$. Writing $\mathcal{K} = \mathrm{coker}(e)$, so that

$$\mathrm{coker}(e \otimes 1) = \mathcal{K} \otimes \mathcal{N}(L),$$

it follows that $w \in \mathrm{supp}(\mathcal{K} \otimes \mathcal{N}(L))$. But thanks to (2-5) and Serre vanishing, $\bar{\sigma}_*(p_1^* B) \otimes \mathcal{N}(L_d)$ is globally generated and

$$H^0\big(\bar{\sigma}(p_1^* B) \otimes \mathcal{N}(L_d)\big) \to H^0\big(\mathcal{K} \otimes \mathcal{N}(L_d)\big)$$

is surjective when $d \gg 0$. On the other hand, s is an isomorphism in a neighborhood of w since w is reduced, and it then follows that the map

$$H^0\big(\mathcal{K} \otimes \mathcal{N}(L_d)\big) \to H^0\big(\mathrm{coker}(\delta(B; L_d))\big)$$

is nonzero when $d \gg 0$. In other words, we have a commutative diagram

$$H^0(B) \otimes H^0(\mathcal{N}(L_d)) \longrightarrow H^0\big(\bar{\sigma}(p_1^*B) \otimes \mathcal{N}(L_d)\big) \relbar\joinrel\twoheadrightarrow H^0\big(\mathcal{K} \otimes \mathcal{N}(L_d)\big)$$

with exact top row, in which the right-hand diagonal mapping, and hence also the bottom homomorphism, are nonzero. Therefore $\delta(B; L_d)$ cannot be surjective on global sections when $d \gg 0$, as required.

It remains to construct the homomorphism t appearing in (2-8). To this end, let

$$\widehat{Z} = \bar{Z} \times_{X \times \mathrm{Sym}^{p+1}(X)} (X \times X^{p+1}).$$

The projection formula gives an isomorphism

$$(1 \times \pi)_*\big((\mathrm{pr}_1^*B \otimes \mathrm{pr}_2^*L^{\boxtimes p+1,\mathrm{alt}}) \otimes \mathcal{O}_{\widehat{Z}}\big) = \big((1 \times \pi)_*(\mathrm{pr}_1^*B \otimes \mathrm{pr}_2^*L^{\boxtimes p+1,\mathrm{alt}})\big) \otimes \mathcal{O}_{\bar{Z}},$$

which, upon taking S_{p+1} invariants, yields

$$(1 \times \pi)_*^{S_{p+1}}\big(\mathrm{pr}_1^*B \otimes \mathrm{pr}_2^*L^{\boxtimes p+1,\mathrm{alt}} \otimes \mathcal{O}_{\widehat{Z}}\big) = \big(p_1^*B \otimes p_2^*\mathcal{N}(L)\big) \otimes \mathcal{O}_{\bar{Z}}.$$

On the other hand, as a set $\widehat{Z}$ consists of those points

$$(x_0, (x_1, \ldots, x_{p+1})) \in X \times X^{p+1}$$

having the property that x_0 appears in the cycle $x_1 + \cdots + x_{p+1}$. In other words, $\widehat{Z}$ and Z coincide set-theoretically. Since Z is reduced this implies that $Z = \widehat{Z}_{\mathrm{red}}$, and in particular Z is a subscheme of $\widehat{Z}$. Thus there is a natural surjective map

$$(\mathrm{pr}_1^*B \otimes \mathrm{pr}_2^*L^{\boxtimes p+1,\mathrm{alt}}) \otimes \mathcal{O}_{\widehat{Z}} \to (\mathrm{pr}_1^*B \otimes \mathrm{pr}_2^*L^{\boxtimes p+1,\mathrm{alt}}) \otimes \mathcal{O}_Z$$

which is an isomorphism over the smooth locus of Z, and taking direct images gives (2-8). □

Remark 2.2. It would be very interesting to give a necessary and sufficient condition for the nonvanishing of $K_{p,1}(X, B; L_d)$ for $d \gg 0$. Keeping the notation of the previous proof, the issue is to determine when $\delta(B; L)$ has a nonzero cokernel. It is conceivable that the failure of B to be p-jet very ample suffices, but the question seems somewhat difficult to analyze. Already the case $\dim X = 2$ would be interesting.

Remark 2.3. Recall that a line bundle B on a smooth variety X is said to be p-very ample if every finite subscheme of length $p + 1$ imposes independent conditions on $H^0(X, B)$. When $\dim X = 1$ this is the same as jet-amplitude, but when $\dim X \geq 2$ it is a strictly weaker condition in general. A quick way to see this is to recall that if A is an ample line bundle on a smooth surface X, then $B_p = K_X + (p + 3)A$ is always p-very ample thanks to a theorem of Beltrametti, Francia and Sommese [1989]. On the other hand, the p-jet amplitude of B_p for $p \gg 0$ would imply that the Seshadri constant $\varepsilon(A; x)$ is very close to 1 for every point $x \in X$ (see [Lazarsfeld 1997, Proposition 5.10]). Hence any line bundle A for which there exist points with small Seshadri constant gives rise to examples of the required sort.

Acknowledgements

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