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A tropical approach to nonarchimedean Arakelov geometry

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Chambert-Loir and Ducros have recently introduced a theory of real valued differential forms and currents on Berkovich spaces. In analogy to the theory of forms with logarithmic singularities, we enlarge the space of differential forms by so called δ-forms on the nonarchimedean analytification of an algebraic variety. This extension is based on an intersection theory for tropical cycles with smooth weights. We prove a generalization of the Poincaré–Lelong formula which allows us to represent the first Chern current of a formally metrized line bundle by a δ-form. We introduce the associated Monge–Ampère measure μ as a wedge-power of this first Chern δ-form and we show that μ is equal to the corresponding Chambert-Loir measure. The $*$-product of Green currents is a crucial ingredient in the construction of the arithmetic intersection product. Using the formalism of δ-forms, we obtain a nonarchimedean analogue at least in the case of divisors. We use it to compute nonarchimedean local heights of proper varieties.

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0. Introduction

Weil's adelic point of view was to compactify the ring of integers $\mathcal{O}_K$ of a number field K by the archimedean primes. Arakelov's brilliant idea was to add metrics on the "fibre at infinity" of a surface over $\mathcal{O}_K$ which gave a good intersection theory for arithmetic divisors. This Arakelov theory became popular after Faltings used it to prove Mordell's conjecture. A higher dimensional arithmetic intersection theory was developed by Gillet and Soulé. Their theory combines algebraic intersection theory on a regular model $\mathscr{X}$ over $\mathcal{O}_K$ with differential geometry on the associated complex manifold X^{an} of the generic fibre X of $\mathscr{X}$. Roughly speaking, an arithmetic cycle on $\mathscr{X}$ is given by a pair $(\mathscr{Z}, g_Z)$, where $\mathscr{Z}$ is a cycle on $\mathscr{X}$ with generic fibre Z and g_Z is a current on X^{an} satisfying the equation

$$dd^c g_Z = [\omega_Z] - \delta_Z$$

for a smooth differential form ω_Z and the current of integration δ_Z over Z^{an}. The arithmetic intersection product uses the algebraic intersection product for algebraic cycles in the first component and the $*$-product of Green currents in the second component. This arithmetic intersection theory is nowadays called Arakelov theory. It found many nice applications such as Faltings's proof of the Mordell–Lang conjecture for abelian varieties and the proof of Ullmo and Zhang of the Bogomolov conjecture for abelian varieties.

It is an old dream to handle archimedean and nonarchimedean places in a similar way. This means that we are looking for a description in terms of currents for the contributions of the nonarchimedean places to Arakelov theory. Such a nonarchimedean Arakelov theory at finite places was developed by Bloch–Gillet–Soulé relying strongly on the conjectured existence of resolution of singularities for models in mixed characteristics. The use of models also has another disadvantage since they are not suitable to describe canonical metrics as for line bundles on abelian varieties with bad reduction. A more analytic nonarchimedean Arakelov theory was developed by Chinburg and Rumely, and Zhang in the case of curves. A crucial role is played here by the reduction graph of the curve. Without any doubt, the latter should be replaced by the Berkovich analytic space associated to the curve and this was done by Thuillier in his thesis introducing a nonarchimedean potential theory. Chambert-Loir and Ducros [2012] recently introduced differential forms and currents on Berkovich spaces. These provide us with a new tool to give an analytic description of nonarchimedean Arakelov theory in higher dimensions.

We recall the definition of differential forms given in [loc. cit.]. We restrict here to the algebraic case. Let U be an n-dimensional very affine open variety which means that U has a closed embedding into a multiplicative torus $T = \mathbb{G}_m^r$ over a nonarchimedean field K. By definition, such a field K is endowed with a complete nonarchimedean absolute value $|\cdot|$. Let $t_1, \dots, t_r$ be the torus coordinates. Then we have the tropicalization map

$$\mathrm{trop} : T^{\mathrm{an}} \to \mathbb{R}^r, \quad t \mapsto (-\log|t_1|, \dots, -\log|t_r|).$$

By the Bieri–Groves theorem, the tropical variety $\mathrm{Trop}(U) := \mathrm{trop}(U^{\mathrm{an}})$ is a finite union of n-dimensional polyhedra. More precisely, $\mathrm{Trop}(U)$ is an n-dimensional tropical cycle which means that $\mathrm{Trop}(U)$ is a polyhedral complex endowed with canonical weights satisfying a balancing condition. Let $x_1, \dots, x_r$ be the coordinates on $\mathbb{R}^r$. Then Lagerberg's superforms on $\mathbb{R}^r$ are formally given by

$$\alpha = \sum_{\substack{|I|=p,\\ |J|=q}} \alpha_{IJ} d'x_{i_1} \wedge \cdots \wedge d'x_{i_p} \wedge d''x_{j_1} \wedge \cdots \wedge d''x_{j_q},$$

where I (resp. J) consists of $i_1 < \cdots < i_p$ (resp. $j_1 < \cdots < j_q$), $\alpha_{IJ} \in C^\infty(\mathbb{R}^r)$. We have differential operators d' and d'' on the space of superforms given by

$$d'\alpha := \sum_{\substack{|I|=p,\\ |J|=q}} \sum_{i=1}^{r} \frac{\partial \alpha_{IJ}}{\partial x_i} d'x_i \wedge d'x_{i_1} \wedge \cdots \wedge d'x_{i_p} \wedge d''x_{j_1} \wedge \cdots \wedge d''x_{j_q}$$

and

$$d''\alpha := \sum_{\substack{|I|=p,\\ |J|=q}} \sum_{j=1}^{r} \frac{\partial \alpha_{IJ}}{\partial x_j} d''x_j \wedge d'x_{i_1} \wedge \cdots \wedge d'x_{i_p} \wedge d''x_{j_1} \wedge \cdots \wedge d''x_{j_q}.$$

They are the analogues of the differential operators ∂ and $\bar{\partial}$ in complex analysis. The space of superforms on $\mathbb{R}^r$ with the usual wedge product is a differential bigraded $\mathbb{R}$-algebra with respect to d' and d''. The space of supercurrents on $\mathbb{R}^r$ is given as the topological dual of the space of superforms.

Every superform α induces a differential form on U^{an} and two superforms α, α' induce the same form if and only if they restrict to the same superform on $\mathrm{Trop}(U)$. In general, a differential form on an n-dimensional variety X is given locally for the Berkovich analytic topology on very affine open subsets by Lagerberg's superforms which agree on common intersections (see [Gubler 2016] for more details). The wedge product and the differential operators can be carried over to X^{an} leading to a sheaf $A^{\cdot,\cdot}$ of differential forms on X^{an}. Integration of superforms leads to integration of compactly supported (n, n)-forms on X^{an}. The space of currents $D^{\cdot,\cdot}(X^{\mathrm{an}})$ is defined as the topological dual of the space of compactly supported forms.

A major result of Chambert-Loir and Ducros is the Poincaré–Lelong formula for the meromorphic section of a line bundle endowed with a continuous metric $\|\cdot\|$. Note that in this situation, $c_1(L, \|\cdot\|)$ is only a current, while a smooth metric allows one to define the first Chern form in $A^{1,1}(X^{\mathrm{an}})$. For a smooth metric, $c_1(L, \|\cdot\|)^n$

is a form of top degree and hence defines a signed measure called the Monge–Ampère measure of $(L, \|\cdot\|)$. In arithmetic, metrics are often induced by proper algebraic models over the valuation ring. Such metrics are called algebraic. They are continuous on X^{an}, but not smooth. This makes it difficult to define the Monge–Ampère measure as a wedge product of currents. In the complex situation, one needs Bedford–Taylor theory to define such a wedge product. In the nonarchimedean situation, Chambert-Loir and Ducros use an approximation process by smooth metrics to define this top-dimensional wedge product of first Chern currents.

The main theorem in [Chambert-Loir and Ducros 2012] shows that the Monge–Ampère measure of a line bundle endowed with a formal metric is equal to the Chambert-Loir measure. The latter was introduced in [Chambert-Loir 2006] before a definition of first Chern current was available. It is defined as a discrete measure on the Berkovich space using degrees of the irreducible components of the special fibre. Chambert-Loir measures play a prominent role in nonarchimedean equidistribution results. For example, they occur in the nonarchimedean version of Yuan's equidistribution theorem, which has applications to the geometric Bogomolov conjecture.

In the thesis of Christensen [2013] a different approach to a first Chern form was given. Christensen studied the example E^2 for a Tate elliptic curve E and he defined the first Chern form as a tropical divisor on the skeleton of E^2. Then he showed that the 2-fold tropical self-intersection of this divisor gives the Chambert-Loir measure.

In this paper, we combine both approaches. We enrich the theory of differential forms given in [Chambert-Loir and Ducros 2012] by enlarging the space of smooth forms to the space of δ-forms. They behave as forms and they have the advantage that we can define a first Chern δ-form for a line bundle endowed with a formal metric. This leads to a direct definition of the Monge–Ampère measure as a wedge product of δ-forms and to an approach to nonarchimedean Arakelov theory.

This will be explained in more detail now. Throughout this paper K denotes an algebraically closed field endowed with a nontrivial nonarchimedean complete absolute value. Note that this is no restriction of generality as for many problems including the ones discussed in this paper such a setup can always be achieved by base change. This is similar to the archimedean case where analysis is usually performed over the complex numbers. For sake of simplicity, we assume in the introduction that tropical cycles have constant weights as usual in tropical geometry (see Section 1 for details and for a generalization to smooth weights). A δ-preform on $\mathbb{R}^r$ is a supercurrent α on $\mathbb{R}^r$ of the form

$$\alpha = \sum_{i \in I} \alpha_i \wedge \delta_{C_i} \tag{0.0.1}$$

for finitely many superforms α_i and tropical cycles C_i on $\mathbb{R}^r$. Using the wedge product of superforms and the stable intersection product of tropical cycles, we get a wedge product of δ-preforms. Since the supercurrents of integration δ_{C_i} are d'-closed and d''-closed, we can extend the differential operators d' and d'' to δ-preforms leading to a differential bigraded $\mathbb{R}$-algebra. We refer to Section 2 for precise definitions and generalizations allowing smooth tropical weights.

In Section 3, we extend all these notions from $\mathbb{R}^r$ to a fixed tropical cycle C of $\mathbb{R}^r$. The balancing condition is equivalent to closedness of the supercurrent δ_C, which means that C is boundaryless in the sense that no boundary integral shows up in the theorem of Stokes over C. Therefore we may view a tropical cycle as a combinatorial analogue of a complex analytic space. Using integration over C, we will see that a piecewise smooth form η on the support of C induces a supercurrent $[\eta]$ on C. We apply this to a piecewise smooth function ϕ on C. In tropical geometry, ϕ plays the role of a Cartier divisor on C and has an associated tropical Weil divisor $\phi \cdot C$. The latter is also called the corner locus of ϕ as it is a tropical cycle of codimension 1 with support equal to the singular locus of ϕ. We show in Corollary 3.19 the following *tropical Poincaré–Lelong formula*:

Theorem 0.1. *Let ϕ be a piecewise smooth function on C and let $\delta_{\phi\cdot C}$ be the supercurrent of integration over the corner locus $\phi \cdot C$. Then we have*

$$d'd''[\phi] - [d'd''\phi] = \delta_{\phi\cdot C}$$

as supercurrents on C.

This is a statement about integration of superforms on tropical currents and its proof relies on Stokes theorem. In fact, we prove a more general statement in Theorem 3.16 involving integration of δ-preforms on C.

Let X be an n-dimensional algebraic variety over K. We now define δ-forms on X^{an} similarly as differential forms, but replacing superforms by the more general δ-preforms. This means that a δ-form is given locally with respect to the Berkovich analytic topology on very affine open subsets by pull-backs of δ-preforms with respect to the tropicalization maps. The δ-preforms have to agree on overlaps which involves a quite complicated restriction process which is explained in Section 4. Moreover, we will show that δ-forms are bigraded, have a wedge product and differential operators d', d'' extending the corresponding structures for differential forms on X^{an}. There is also a pull-back with respect to morphisms and so we see that δ-forms behave as differential forms on complex manifolds.

In Section 5, we study integration of compactly supported δ-forms of bidegree (n,n) on X^{an}. To define the integral of such a δ-form α, we choose a dense open subset U of X with a closed embedding $U \hookrightarrow \mathbb{G}_m^r$ such that α is given on U^{an} by the pull-back of a δ-preform α_U on $\mathbb{R}^r$ with respect to the tropicalization map $\mathrm{trop}_U : U^{\mathrm{an}} \to \mathbb{R}^r$. Using the corresponding tropical variety $\mathrm{Trop}(U)$, we set

$$\int_{X^{\mathrm{an}}} \alpha := \int_{|\mathrm{Trop}(U)|} \alpha_U .$$

In Section 6, we introduce δ-currents as continuous linear functionals on the space of compactly supported δ-forms. By integration, every δ-form α induces a δ-current $[\alpha]$. Similarly, we get a current of integration δ_Z for every cycle Z on X. As a major result, we show in Corollary 6.15 that $[\alpha]$ is a signed Radon measure on X^{an} for every α of bidegree (n,n). We deduce in Proposition 6.16 that every continuous real function g on X^{an} induces a δ-current $[g]$ on X^{an} which is defined at a compactly supported δ-form α of bidegree (n,n) by integrating g with respect to the corresponding Radon measure.

Now let f be a rational function on X which is not identically zero. By integration again, we will get a δ-current $[-\log|f|]$ on X^{an}.

Theorem 0.2. *Let* $\mathrm{cyc}(f)$ *be the Weil divisor associated to* f. *Then we have the Poincaré–Lelong equation*

$$\delta_{\mathrm{cyc}(f)} = d'd''[\log|f|]$$

of δ*-currents on* X^{an}.

This is demonstrated as Theorem 7.2. The Poincaré–Lelong equation of Chambert-Loir and Ducros is the special case of our formula where one evaluates the δ-currents at differential forms. The generalization to δ-forms is not obvious and needs a more tropical adaptation of their beautiful arguments. In Section 7, we introduce the first Chern δ-current $[c_1(L,\|\cdot\|)]$ of a continuously metrized line bundle $(L,\|\cdot\|)$ on X. As usual, we mean here continuity with respect to the Berkovich topology on X^{an}. In Corollary 7.8, we deduce from Theorem 0.2 that a nonzero meromorphic section s of L satisfies the Poincaré–Lelong equation

$$d'd''[-\log\|s\|] = [c_1(L,\|\cdot\|)] - \delta_{\mathrm{cyc}(s)} \tag{0.2.1}$$

for δ-currents on X^{an}.

In Section 8, we define piecewise smooth and piecewise linear metrics on L. We show in Proposition 8.11 that a metric is piecewise linear if and only if it is induced by a formal model of the line bundle. In Section 9, we introduce piecewise smooth forms on X^{an}. For a piecewise smooth metric $\|\cdot\|$ on L, the first Chern δ-current $[c_1(L,\|\cdot\|)]$ has a canonical decomposition into a sum of a piecewise smooth form and a residual current. If $\|\cdot\|$ is smooth, then $c_1(L,\|\cdot\|)$ is a differential form on X^{an}. We say that a piecewise smooth metric $\|\cdot\|$ is a δ-metric if the first Chern δ-current $[c_1(L,\|\cdot\|)]$ is induced by a δ-form $c_1(L,\|\cdot\|)$ (see Definition 9.9 for a more precise definition). In this situation, we call $c_1(L,\|\cdot\|)$ the first Chern δ-form of $(L,\|\cdot\|)$. We will see in Remark 9.16 that every piecewise linear metric is a δ-metric. Canonical metrics on line bundles exist on line bundles on abelian varieties, on line bundles which are algebraically equivalent to zero and on line bundles on toric varieties. It follows from our considerations in Section 8 that all these canonical metrics are δ-metrics (see Example 9.17).

In Section 10, we consider a proper algebraic variety X over K of dimension n with a line bundle L endowed with an algebraic metric $\|\cdot\|$. This means that the metric is induced by an algebraic model of L. Based on the formal GAGA principle, we show in Proposition 8.13 that an algebraic metric is the same as a formal metric and hence this is also the same as a piecewise linear metric. As a consequence, we note that $\|\cdot\|$ is a δ-metric and hence $c_1(L, \|\cdot\|)$ is a well-defined δ-form. We deduce that $c_1(L, \|\cdot\|)^n$ is a δ-form of bidegree (n, n) on X^{an}, which we may view as a signed Radon measure on X^{an} by the above. We call it the Monge–Ampère measure associated to $(L, \|\cdot\|)$. Our Theorem 10.5 can be expressed as follows:

Theorem 0.3. *Under the assumptions above, the Monge–Ampère measure associated to $(L, \|\cdot\|)$ is equal to the Chambert-Loir measure associated to $(L, \|\cdot\|)$.*

As mentioned before, this theorem was first proved by Chambert-Loir and Ducros in a slightly different setting (for discrete valuations, but their method works also for algebraically closed fields). However, they have a different construction of the Monge–Ampère measure. Since algebraic metrics are usually not smooth, they have only a first Chern current $c_1(L, \|\cdot\|)$ available. In general, the wedge product of currents is not well defined. In the present situation, they can use a rather complicated approximation process by smooth metrics to make sense of the wedge product $c_1(L_1, \|\cdot\|_1)^n$ as a current leading to their Monge–Ampère measure. Our Monge–Ampère measure is defined directly as a wedge product of δ-forms based on tropical intersection theory instead of the approximation process. This means that our proof is more influenced by tropical methods.

In Section 11, we define a Green current for a cycle Z on the algebraic variety X over K as a δ-current g_Z such that

$$d'd''g_Z = [\omega_Z] - \delta_Z$$

for a δ-form ω_Z on X^{an}. By the Poincaré–Lelong equation (0.2.1), a nonzero meromorphic section s of L induces a Green current $g_Y := -\log\|s\|$ for the Weil divisor Y of s. Here, we assume that $\|\cdot\|$ is a δ-metric on the line bundle L of X. In case of proper intersection, we define $g_Y * g_Z := g_Y \wedge \delta_Z + \omega_Y \wedge g_Z$ as in the archimedean theory of Gillet–Soulé. It is an easy consequence of the Poincaré–Lelong equation that $g_Y * g_Z$ is a Green current for the cycle $Y \cdot Z$. We show the usual properties for such $*$-products. Most difficult is the proof of the commutativity of the $*$-product of two Green currents for properly intersecting divisors. It relies on the study of piecewise smooth forms and the tropical Poincaré–Lelong formula in Theorem 0.1.

In Section 12, we define the local height of a proper n-dimensional variety X over K with respect to properly intersecting Cartier divisors $D_0, \dots, D_n$ endowed with δ-metrics on $O(D_0), \dots, O(D_n)$ as follows: Let g_{Y_j} be the Green current for the Weil divisor Y_j associated to D_j as above; then the local height is given by

$$\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X) := g_{Y_0} * \cdots * g_{Y_n}(1).$$

We show that these local heights are multilinear and symmetric in the metrized Cartier divisors $\hat{D}_0, \dots, \hat{D}_n$, functorial with respect to morphisms and satisfy an induction formula useful to decrease the dimension of X. For algebraic metrics, local heights of proper varieties are also defined using intersection theory on a suitable proper model (see [Gubler 1998, §9]).

Theorem 0.4. *Suppose that the metrics on $O(D_0), \dots, O(D_n)$ are all algebraic. Then the local height $\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X)$ based on the $*$-product of Green currents is equal to the local height of X given by intersection theory of divisors on K°-models.*

The proof uses the observation that the induction formula holds for both definitions of local heights and then Theorem 0.3 gives the claim (see Remark 12.7 for more details and the proof).

In the introduction, we have presented the whole theory of δ-forms based on δ-preforms as in (0.0.1) using tropical cycles with constant weights. However, the theory can be extended to δ-forms locally given by δ-preforms allowing tropical cycles with smooth weights. This will be done throughout the whole paper which leads to slightly more complications, but it increases the class of δ-metrics at the end which makes it worthwhile. Observe that tropical cycles which arise as tropicalizations from varieties always have integer weights. Therefore tropical cycles are always considered with constant weights when they serve, as in Section 3, as underlying spaces for supercurrents and δ-preforms.

Notation and terminology. Throughout this paper K denotes an algebraically closed field endowed with a complete nontrivial nonarchimedean absolute value $|\cdot|$, valuation ring K°, and corresponding valuation $v = -\log|\cdot|$. Let $\Gamma := v(K^\times)$ be the value group.

In $A \subset B$, A is strictly smaller than B. The complement of A in B is denoted by $B \setminus A$. The zero is included in $\mathbb{N}$ and in $\mathbb{R}_+$.

The group of multiplicative units in a ring A with 1 is denoted by $A^\times$. An (algebraic) variety over a field is an irreducible separated reduced scheme of finite type. The terminology from convex geometry is explained in the Appendix.

1. Tropical intersection theory with smooth weights

In tropical geometry, a tropical cycle is given by a polyhedral complex whose maximal faces are weighted by integers satisfying a balancing condition along the faces of codimension 1. In this section, we generalize the notion of a tropical cycle allowing smooth real functions on the maximal faces as weights. This is similar to the tropical fans with polynomial weights introduced by Esterov [2012] and François [2013]. We generalize basic facts from stable tropical intersection theory and introduce the corner locus of a piecewise smooth function.

Throughout this section N and N' denote free $\mathbb{Z}$-modules of finite rank r and r'. We write $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $N'_{\mathbb{R}} = N' \otimes_{\mathbb{Z}} \mathbb{R}$. Every integral $\mathbb{R}$-affine polyhedron σ of dimension n in the $\mathbb{R}$-vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ determines an affine subspace $\mathbb{A}_\sigma$ with underlying vector space $\mathbb{L}_\sigma$ and a lattice $N_\sigma = \mathbb{L}_\sigma \cap N$ in $\mathbb{L}_\sigma$ (see A.2 in the Appendix). A smooth function $f : \sigma \to \mathbb{R}$ on a polyhedron σ in $N_{\mathbb{R}}$ is the restriction of a smooth function on some open neighbourhood of σ in $\mathbb{A}_\sigma$. For further notation borrowed from convex geometry, we refer to the Appendix.

Definition 1.1. (i) A polyhedral complex $\mathscr{C}$ of pure dimension n is called *weighted (with smooth weights)* if each polyhedron $\sigma \in \mathscr{C}_n$ is endowed with a smooth weight function $m_\sigma : \sigma \to \mathbb{R}$. If all m_σ are constant functions, then we call them *constant weights*. The *support* $|(\mathscr{C}, m)|$ of a weighted polyhedral complex $(\mathscr{C}, m)$ of pure dimension n is the closed set

$$|(\mathscr{C}, m)| = \bigcup_{\sigma \in \mathscr{C}_n} \operatorname{supp}(m_\sigma).$$

The support $|\mathscr{C}|$ of a polyhedral complex $\mathscr{C}$ is the support of $(\mathscr{C}, m)$, where $m = 1$ is the trivial weight function. We have $|(\mathscr{C}, m)| \subseteq |\mathscr{C}|$.

(ii) Let $C = (\mathscr{C}, m)$ be an integral $\mathbb{R}$-affine polyhedral complex of pure dimension n with smooth weights in $N_{\mathbb{R}}$. For each codimension-one face τ of a polyhedron $\sigma \in \mathscr{C}_n$ we choose a representative $\omega_{\sigma,\tau} \in N_\sigma$ of the generator of the one-dimensional lattice N_σ / N_τ pointing in the direction of σ. Then we say that the weighted polyhedral complex C satisfies the *balancing condition* if we have

$$\sum_{\substack{\sigma \in \mathscr{C}_n \\ \sigma \succ \tau}} m_\sigma(\omega)\omega_{\sigma,\tau} \in \mathbb{L}_\tau \tag{1.1.1}$$

for all $\tau \in \mathscr{C}_{n-1}$ and all $\omega \in \tau$, where $\sigma \succ \tau$ means that τ is a face of σ. This is a straightforward generalization of the balancing condition for polyhedral complexes with integer weights [Gubler 2013, 13.9].

(iii) A *tropical cycle* $C = (\mathscr{C}, m)$ *of dimension* n in $N_{\mathbb{R}}$ is a weighted integral $\mathbb{R}$-affine polyhedral complex of pure dimension n which satisfies the balancing condition (1.1.1). In the following, we identify two tropical cycles $(\mathscr{C}, m)$ and $(\mathscr{C}', m')$ of dimension n if $|(\mathscr{C}, m)| = |(\mathscr{C}', m')|$ and if $m_\sigma = m_{\sigma'}$ on the intersection of the relative interiors of σ and σ' for all $\sigma \in \mathscr{C}_n$ and $\sigma' \in \mathscr{C}'_n$. A tropical cycle

$(\mathscr{C}, m)$ whose underlying polyhedral complex is a rational polyhedral fan in the vector space $N_{\mathbb{R}}$ is called a *tropical fan*.

(iv) Let $C = (\mathscr{C}, m)$ be a tropical cycle of dimension n. Given an integral $\mathbb{R}$-affine subdivision $\mathscr{C}'$ of $\mathscr{C}$, there is a unique family of weight functions m' such that $(\mathscr{C}', m')$ is a tropical cycle and $m_\sigma|_{\sigma'} = m'_{\sigma'}$ holds for all $\sigma' \in \mathscr{C}'_n$ and $\sigma \in \mathscr{C}_n$ such that $\sigma' \subseteq \sigma$. If a tropical cycle C in $N_{\mathbb{R}}$ is defined by a weighted integral $\mathbb{R}$-affine polyhedral complex $(\mathscr{C}, m)$, we call $\mathscr{C}$ a *polyhedral complex of definition for the tropical cycle C*.

(v) The set of tropical cycles with smooth weights of pure dimension n in $N_{\mathbb{R}}$ defines an abelian group $\mathrm{TZ}_n(N_{\mathbb{R}})$ where the group law is given by the addition of multiplicity functions on a common refinement of the integral $\mathbb{R}$-affine polyhedral complexes. We denote by $\mathrm{TZ}^k(N_{\mathbb{R}}) = \mathrm{TZ}_{r-k}(N_{\mathbb{R}})$ the *group of tropical cycles of codimension k*.

Remark 1.2 (reduction from smooth to constant weight functions). In tropical geometry, one usually considers tropical cycles with integer weights. However it causes no problems to work instead with tropical cycles with constant but not necessarily integer weights.

Many properties of these tropical cycles with integer or constant weights extend even to tropical cycles with smooth weights by the following local argument in $\omega \in |\mathscr{C}|$. We replace $\mathscr{C}$ by the rational polyhedral fan of local cones in ω (see A.6) and we endow the local cone of $\sigma \in \mathscr{C}_n$ by the constant weight $m_\sigma(\omega)$. By definition, these constant weights on the rational cones satisfy the balancing condition. We illustrate the use of this reduction process in Remark 1.4(ii).

1.3. In tropical geometry, there is a *stable tropical intersection product* of tropical cycles with integer weights. The astonishing fact is that this product is well-defined as a tropical cycle in contrast to algebraic intersection theory or homology, where an equivalence relation is needed. Constructions of a stable tropical intersection product of tropical cycles with integer weights have been given by Mikhalkin [2006] and Allermann and Rau [2010]. In both cases the construction is reduced to the case of tropical fans. For tropical fans with integer weights, Mikhalkin uses the fan displacement rule from [Fulton and Sturmfels 1997], whereas Allermann and Rau use reduction to the diagonal and intersections with tropical Cartier divisors. It is shown in [Katz 2012, §5; Rau 2009, Theorem 1.5.17] that both definitions agree. This is based on a result of Fulton and Sturmfels [1997, Theorem 3.1] which shows that the space of tropical fans, with integer weights and with a given complete rational polyhedral fan Σ as a polyhedral complex of definition, is canonically isomorphic to the Chow cohomology ring of the complete toric variety Y_Σ associated to Σ. Then the product in Chow cohomology leads to the stable intersection product of tropical fans with integer weights and the usual properties in Chow cohomology lead to corresponding properties in stable tropical intersection theory. By passing to a smooth rational polyhedral fan subdividing Σ, which means that Y_Σ is smooth, we may use the usual Chow groups instead of the Chow cohomology groups from [Fulton 1984, Chapter 17].

Remark 1.4 (stable tropical intersection theory). As an application of the reduction principle described in Remark 1.2, we get a stable tropical intersection theory for tropical cycles with smooth weights. The reduction process leads to tropical fans with constant weight functions. These weights are not necessarily integers, but it is still possible to apply 1.3 by using Chow cohomology with real coefficients. We list here the main properties:

(i) There exists a natural bilinear pairing

$$\mathrm{TZ}^k(N_\mathbb{R}) \times \mathrm{TZ}^l(N_\mathbb{R}) \to \mathrm{TZ}^{k+l}(N_\mathbb{R}), \quad (C_1, C_2) \mapsto C_1 \cdot C_2$$

which is called the *stable intersection product for tropical cycles*. It is associative and commutative and respects supports in the sense that we have $|C_1 \cdot C_2| \subseteq |C_1| \cap |C_2|$.

(ii) The concrete construction of the stable intersection product for tropical cycles C_1 and C_2 of codimension l_1 and l_2 in $N_\mathbb{R}$ is based on the fan displacement rule (see [Fulton and Sturmfels 1997, §4]). We choose a common polyhedral complex of definition $\mathscr{C}$ for C_1 and C_2 and write $C_i = (\mathscr{C}, m_i)$ $(i = 1, 2)$ for suitable families of weight functions $m_i = (m_{i,\sigma})_{\sigma \in \mathscr{C}^{l_i}}$. Let $\mathscr{D}$ denote the polyhedral subcomplex of $\mathscr{C}$ which is generated by $\mathscr{C}^{l_1+l_2}$. We choose a generic vector $v \in N_\mathbb{R}$ for $\mathscr{D}$, a small $\varepsilon > 0$, and equip $\mathscr{D}$ with the family of weight functions $m = (m_\tau)_{\tau \in \mathscr{C}^{l_1+l_2}}$, where $m_\tau : \tau \to \mathbb{R}$ is given by

$$m_\tau(\omega) = \sum_{\substack{(\sigma_1,\sigma_2) \in \mathscr{C}^{l_1} \times \mathscr{C}^{l_2} \\ \tau = \sigma_1 \cap \sigma_2 \\ \sigma_1 \cap (\sigma_2 + \varepsilon v) \neq \varnothing}} [N : N_{\sigma_1} + N_{\sigma_2}]\, m_{1,\sigma_1}(\omega)\, m_{2,\sigma_2}(\omega). \tag{1.4.1}$$

We will show that $D = (\mathscr{D}, m)$ is a tropical cycle whose construction is independent of the choice of the generic vector v and a sufficiently small $\varepsilon > 0$. We use D as the definition of the stable intersection product $C_1 \cdot C_2$.

The proof illustrates the reduction to constant weights given in Remark 1.2. For $\omega \in |\mathscr{C}|$, let $\mathscr{C}_\omega$ be the rational polyhedral fan of local cones in ω of the polyhedra in $\mathscr{C}$. First, we note that $\sigma \mapsto \rho := \mathrm{LC}_\sigma(\omega)$ is a bijective map from the set of polyhedra in $\mathscr{C}$ containing ω onto $\mathscr{C}_\omega$. For $i = 1, 2$ and $\sigma \in \mathscr{C}_n$ with $\omega \in \sigma$, we endow the local cone $\rho = \mathrm{LC}_\sigma(\omega)$ with the constant weight $m_{i,\rho}(\omega) := m_{i,\sigma}(\omega)$. Since the weight functions $m_{i,\sigma}$ pointwise satisfy the balancing condition, we get a tropical fan $(\mathscr{C}_\omega, m_i(\omega))$ with real weights.

We claim that $m_\tau(\omega)$ from (1.4.1) is the same as the weight of the stable intersection product $(\mathscr{C}_\omega, m_1(\omega)) \cdot (\mathscr{C}_\omega, m_2(\omega))$ in $\tau \in \mathscr{C}^{l_1+l_2}$ obtained from Chow cohomology as in 1.3. To see this, note that for a generic vector $v \in N_R$, we choose $\varepsilon > 0$ so small that the condition $\sigma_1 \cap (\sigma_2 + \varepsilon v) \neq \varnothing$ is equivalent to $\rho_1 \cap (\rho_2 + v) \neq \varnothing$ for the corresponding cones ρ_1, ρ_2. Then (1.4.1) agrees with the formula in [Fulton and Sturmfels 1997, Theorem on p. 336] for the product in Chow cohomology of proper toric varieties. By definition, the same formula is used for the stable intersection product of tropical fans with constant weights, proving our local claim. It is well known in tropical geometry, and follows from the comparison with Chow cohomology in [Fulton and Sturmfels 1997], that the definition of the stable tropical intersection product of tropical fans with real weights is independent of the choice of generic vector v, and hence the definition of $D = (\mathscr{D}, m)$ is independent of the choice of generic vector $v \in N_{\mathbb{R}}$ and sufficiently small $\varepsilon > 0$.

It is easily seen that the definition of D is compatible with subdivisions and hence $C_1 \cdot C_2$ is a well-defined tropical cycle. The properties in (i) follow from the corresponding properties of the stable tropical intersection product of tropical fans with real weights.

(iii) Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. Let $C' = (\mathscr{C}', m')$ be a weighted integral $\mathbb{R}$-affine polyhedral complex in $N'_{\mathbb{R}}$ of pure dimension n. After a suitable refinement we can assume that

$$F_* \mathscr{C}' := \{F(\tau') \mid \exists \sigma' \in \mathscr{C}'_n \text{ such that } \tau' \preccurlyeq \sigma' \text{ and } F|_{\sigma'} \text{ is injective}\} \tag{1.4.2}$$

is a polyhedral complex in $N_{\mathbb{R}}$. We equip $F_* \mathscr{C}'$ with the family of weight functions

$$m_\nu : \nu \to \mathbb{R}, \quad m_\nu(\omega) = \sum_{\substack{\sigma' \in \mathscr{C}'_n \\ F(\sigma') = \nu}} [N_\nu : \mathbb{L}_F(N'_{\sigma'})] m'_{\sigma'}((F|_{\sigma'})^{-1}(\omega)) \tag{1.4.3}$$

for ν in $(F_* \mathscr{C}')_n$, where $\mathbb{L}_F$ denotes the linear morphism defined by the affine morphism F. The weighted integral $\mathbb{R}$-affine polyhedral complex

$$F_* C' = (F_* \mathscr{C}', m)$$

in $N_{\mathbb{R}}$ of pure dimension n is called the *direct image of C' under F*.

(iv) Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. There is a natural push-forward morphism

$$F_* : \mathrm{TZ}_n(N'_{\mathbb{R}}) \to \mathrm{TZ}_n(N_{\mathbb{R}}), \quad C' \mapsto F_* C',$$

which satisfies $|F_* C'| \subseteq F\big(|C'|\big)$. Given a tropical cycle C' in $\mathrm{TZ}_n(N'_{\mathbb{R}})$, we write $C' = (\mathscr{C}', m')$ for a polyhedral complex of definition $\mathscr{C}'$ such that $F_* \mathscr{C}'$ from (1.4.2) is a polyhedral complex in $N_{\mathbb{R}}$. One defines the direct image $F_* C' = (F_* \mathscr{C}', m)$ as in (iii) and verifies that $F_* C'$ is again a tropical cycle. The formation of F_* is functorial in F. For further details see [Allermann and Rau 2010, §7] or [Gubler 2013, 13.16].

(v) Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. There is a natural pull-back

$$F^* : \mathrm{TZ}^l(N_{\mathbb{R}}) \to \mathrm{TZ}^l(N'_{\mathbb{R}}), \quad C \mapsto F^*(C)$$

which satisfies $|F^*C| \subseteq F^{-1}(|C|)$. The formation of F^* is functorial in F. For a tropical cycle C in $\mathrm{TZ}^l(N_{\mathbb{R}})$, there is a complete polyhedral complex of definition $\mathscr{C}$ and we write $C = (\mathscr{C}, m)$. After passing to a subdivision, there is a complete, integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}'$ of $N'_{\mathbb{R}}$ such that for every $\gamma' \in \mathscr{C}'$, there is a $\sigma \in \mathscr{C}$ with $F(\gamma') \subseteq \sigma$. We choose a generic vector $v \in N_{\mathbb{R}}$ and a sufficiently small $\varepsilon > 0$. For $\gamma' \in (\mathscr{C}')^l$, $\sigma \in \mathscr{C}^l$ with $F(\gamma') \subseteq \sigma$ and $\sigma' \in (\mathscr{C}')^0$ with $\gamma' \subseteq \sigma'$, we define

$$m^{\gamma'}_{\sigma',\sigma} := \begin{cases} [N : \mathbb{L}_F(N') + N_\sigma] & \text{if } F(\sigma') \text{ meets } \sigma + \varepsilon v, \\ 0 & \text{otherwise.} \end{cases}$$

These coefficients may depend on the choice of the generic vector v, but the following smooth weight function $m_{\gamma'}$ on $\gamma' \in (\mathscr{C}')^l$ does not:

$$m_{\gamma'}(\omega') := \sum_{\sigma',\sigma} m^{\gamma'}_{\sigma',\sigma} m_\sigma(F(\omega')), \tag{1.4.4}$$

where (σ', σ) ranges over all pairs in $(\mathscr{C}')^0 \times \mathscr{C}^l$ with $\gamma' \subseteq \sigma'$, $F(\gamma') \subseteq \sigma$ and where $\omega' \in \gamma'$. By [Fulton and Sturmfels 1997, 4.5–4.7], $(\mathscr{C}')^{\geq l}$ equipped with the smooth weight functions $m_{\gamma'}$ is a tropical cycle in $\mathrm{TZ}^l(N'_{\mathbb{R}})$, which we define as $F^*(C)$.

Let p_1 (resp. p_2) be the projection of $N'_{\mathbb{R}} \times N_{\mathbb{R}}$ to $N'_{\mathbb{R}}$ (resp. $N_{\mathbb{R}}$) and let Γ_F be the graph of F in $N'_{\mathbb{R}} \times N_{\mathbb{R}}$. Using the stable tropical intersection product from (ii) and [Fulton and Sturmfels 1997, 4.5–4.7], we deduce

$$F^*(C) = (p_1)_*(p_2^*(C) \cdot \Gamma_F). \tag{1.4.5}$$

Proposition 1.5. *Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map.*

(i) *For tropical cycles C and D on $N_{\mathbb{R}}$ we have*

$$F^*(C \cdot D) = F^*(C) \cdot F^*(D).$$

(ii) *For tropical cycles C on $N_{\mathbb{R}}$ and C' on $N'_{\mathbb{R}}$ we have*

$$F_*\big(F^*(C) \cdot C'\big) = C \cdot F_*(C').$$

Proof. We reduce as in Remark 1.2 to the case where our tropical cycles are tropical fans with constant weight functions. Since both sides of the claims are linear in the weights of the tropical fans, we may assume that the weights are integers. In this situation, the claims were proven by L. Allermann [2012, Theorem 3.3]. □

Definition 1.6. Let Ω be an open subset of an integral $\mathbb{R}$-affine polyhedral set P in $N_{\mathbb{R}}$. We call $\phi : \Omega \to \mathbb{R}$ *piecewise smooth* if there is an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ with support P and smooth functions $\phi_\sigma : \sigma \cap \Omega \to \mathbb{R}$ for every $\sigma \in \mathscr{C}$ such that $\phi|_\sigma = \phi_\sigma$ on $\sigma \cap \Omega$. In this situation, we call $\mathscr{C}$ a *polyhedral complex of definition for the piecewise smooth function* ϕ. We call ϕ *piecewise linear* if each ϕ_σ extends to an integral $\mathbb{R}$-affine function on $\mathbb{A}_\sigma$.

Remark 1.7. The balancing condition (1.1.1) for smooth weights shows easily that a tropical cycle of codimension 0 in $N_{\mathbb{R}}$ is the same as a piecewise smooth function defined on the whole space $N_{\mathbb{R}}$.

Proposition 1.8. *Let ϕ be a piecewise smooth function on the open subset Ω of the integral $\mathbb{R}$-affine polyhedral set P in $N_{\mathbb{R}}$ and let $\widetilde{\Omega}$ be any open subset of $N_{\mathbb{R}}$ with $\widetilde{\Omega} \cap P = \Omega$. Then there is a piecewise smooth function on $\widetilde{\Omega}$ which restricts to ϕ on Ω.*

Proof. We first show the claim in the special case when $\Omega = P$ is the support of a rational polyhedral fan $\mathscr{C}$ of definition for ϕ and $\widetilde{\Omega} = N_{\mathbb{R}}$. After passing to a subdivision of $\mathscr{C}$, we can easily find a complete rational polyhedral fan $\mathscr{C}'$ in $N_{\mathbb{R}}$ which contains $\mathscr{C}$. After suitable subdivisions of $\mathscr{C}'$ (and $\mathscr{C}$) we may furthermore assume that all cones in $\mathscr{C}'$ are simplicial. Now we will extend ϕ inductively by ascending dimension from the cones in $\mathscr{C}$ to the cones in $\mathscr{C}'$.

Let σ be a cone in $\mathscr{C}'$ of dimension m. We are looking for an extension $\tilde{\phi}$ of ϕ to σ. By our inductive procedure, we can assume that ϕ is defined already on all faces of codimension one of σ. After a linear change of coordinates, we may assume that σ is the standard cone $\mathbb{R}_+^m$ in $\mathbb{R}^m$. Let us assume that ϕ is given on the face $\{x_i = 0\}$ of σ by the smooth function $\phi_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$. For any $1 \leq i_1 < \cdots < i_k \leq m$, the restriction of ϕ to the face $\{x_{i_1} = \cdots = x_{i_k} = 0\}$ of σ is given by a smooth function $\phi_{i_1 \cdots i_k}$ depending only on the coordinates x_j with $j \notin \{i_1, \ldots, i_k\}$ which agrees with the restrictions of the functions $\phi_{i_1}, \ldots, \phi_{i_k}$ to this face of codimension k. We consider all these functions $\phi_{i_1 \cdots i_k}$ as functions on σ depending only on the coordinates x_j. Then an elementary combinatorial argument shows that

$$\tilde{\phi} := \sum_i \phi_i - \sum_{i<j} \phi_{ij} + \cdots + (-1)^{k+1} \sum_{i_1 < \cdots < i_k} \phi_{i_1 \cdots i_k} \pm \cdots + (-1)^{n+1} \phi_{1 \cdots n}$$

is a smooth extension of ϕ to σ.

Finally, we prove the claim in general. There is a finite open covering $(\Omega_i)_{i \in I}$ of Ω such that $\Omega_i = \Omega_i \cap (\mathrm{LC}_{\omega_i}(P) + \omega_i)$ for the local cone $\mathrm{LC}_{\omega_i}(P)$ of P at a suitable ω_i. Let us choose an open covering $(\widetilde{\Omega}_i)_{i \in I}$ of $\widetilde{\Omega}$ such that $\widetilde{\Omega}_i \cap P = \Omega_i$. There is a partition of unity $(\rho_j)_{j \in J}$ on $\widetilde{\Omega}$ such that every ρ_j has compact support in $\widetilde{\Omega}_{i(j)}$ for a suitable $i(j) \in I$. We choose $\nu_j \in C^\infty(N_{\mathbb{R}})$ with compact support in $\widetilde{\Omega}_{i(j)}$ such that $\nu_j \equiv 1$ on $\operatorname{supp}(\rho_j)$. Then the special case above shows that the piecewise smooth function $\nu_j\phi$ on P has a piecewise smooth extension $\tilde{\phi}_j$ to $N_{\mathbb{R}}$. Even if J is infinite, we note that only finitely many rational fans of definition occur and the above construction gives piecewise smooth extensions $\tilde{\phi}_j$ with finitely many integral $\mathbb{R}$-affine polyhedral complexes of definition. By passing to a common refinement, we may assume that they are all equal to a complete integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}$. We conclude that $\tilde{\phi} = \sum_{j\in J} \tilde{\rho}_j\tilde{\phi}_j$ is a piecewise smooth extension of ϕ to $\widetilde{\Omega}$ with $\mathscr{D}$ as a polyhedral complex of definition. $\square$

Remark 1.9. Let Σ be a rational polyhedral fan of $N_{\mathbb{R}}$ and let $\phi : |\Sigma| \to \mathbb{R}$ be a piecewise linear function with polyhedral complex of definition Σ. Then ϕ is the restriction of a piecewise linear function on $N_{\mathbb{R}}$ with a complete rational polyhedral fan of definition. The argument is a little different: By toric resolution of singularities, one can subdivide Σ until we get a subcomplex of a smooth rational polyhedral fan Σ' of $N_{\mathbb{R}}$ (see A.7 for the connection to toric varieties). We may assume that $\phi(0) = 0$. Let λ be a primitive lattice vector contained in an edge of Σ' with $\lambda \notin |\Sigma|$ and let $\phi(\lambda) \in \mathbb{Z}$. Then there is a unique piecewise linear function ϕ' on $N_{\mathbb{R}}$ with $\phi' = \phi$ on $|\Sigma|$ and $\phi'(\lambda) = \phi(\lambda)$ for all primitive lattice vectors λ as above.

Similarly to [Esterov 2012; François 2013], we introduce the corner locus of a piecewise smooth function.

Definition 1.10 (corner locus). Let $C = (\mathscr{C}, m)$ be a tropical cycle with smooth weights of dimension n. We consider a piecewise smooth function $\phi : |\mathscr{C}| \to \mathbb{R}$ with polyhedral complex of definition $\mathscr{C}$. Given $\tau \in \mathscr{C}_{n-1}$ we choose for each $\sigma \in \mathscr{C}_n$ with $\tau \prec \sigma$ an $\omega_{\sigma,\tau} \in N_\sigma$ as in Definition 1.1(ii). For ω in τ, we define

$$\omega_\tau := \sum_{\substack{\sigma\in\mathscr{C}_n \\ \tau\prec\sigma}} m_\sigma(\omega)\omega_{\sigma,\tau} \in \mathbb{L}_\tau.$$

Note that ω_τ depends on the choice of ω. Viewing $\omega_{\sigma,\tau}$ and ω_τ as tangential vectors at ω, we denote the corresponding derivatives by

$$\frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}} := \langle d\phi_\sigma, \omega_{\sigma,\tau}\rangle, \quad \text{and} \quad \frac{\partial\phi_\tau}{\partial\omega_\tau} := \langle d\phi_\tau, \omega_\tau\rangle,$$

respectively. It is straightforward to check that the definition of the weight function

$$m_\tau : \tau \to \mathbb{R}, \quad m_\tau(\omega) := \left(\sum_{\substack{\sigma\in\mathscr{C}_n \\ \tau\prec\sigma}} m_\sigma(\omega)\frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}}(\omega)\right) - \frac{\partial\phi_\tau}{\partial\omega_\tau}(\omega) \tag{1.10.1}$$

does not depend on the choice of the $\omega_{\sigma,\tau}$. The *corner locus $\phi \cdot C$ of ϕ* is by definition the weighted polyhedral subcomplex $\mathscr{C}'$ of $\mathscr{C}$ generated by $\mathscr{C}_{n-1}$ endowed with the smooth weight functions m_τ defined in (1.10.1).

Remark 1.11. Let $\phi : |\mathscr{C}| \to \mathbb{R}$ be a piecewise linear function on a tropical cycle $C = (\mathscr{C}, m)$ with integral weights. Then the corner locus $\phi \cdot \mathscr{C}$ is a tropical cycle with integral weights which is the tropical Weil divisor of ϕ on C in the sense of Allermann and Rau [2010, 6.5].

Esterov [2012, Theorem 2.7] showed that the corner locus of a piecewise polynomial function on a tropical cycle with polynomial weights is again a tropical cycle of the same kind. We have here a similar result for tropical cycles with smooth weights:

Proposition 1.12. *The corner locus $\phi \cdot C$ of a piecewise smooth function $\phi : |\mathscr{C}| \to \mathbb{R}$ on a tropical cycle $C = (\mathscr{C}, m)$ of dimension n is a tropical cycle with smooth weights of dimension $n - 1$. The corner locus is defined independently of the choice of the polyhedral complex $\mathscr{C}$ and $\phi \cdot C$ depends only on the function $\phi|_{|C|}$.*

Proof. This follows from Remark 1.11 as explained in Remark 1.2. □

Proposition 1.13. *Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. Let ϕ be a piecewise smooth function on an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ on $N_{\mathbb{R}}$. Suppose that $C' = (\mathscr{C}', m')$ is a tropical cycle on $N'_{\mathbb{R}}$ with smooth weights m' such that $F(|\mathscr{C}'|) \subseteq |\mathscr{C}|$. Then we have the projection formula $F_*(F^*(\phi) \cdot C') = \phi \cdot F_*(C')$, where $F^*(\phi)$ is the piecewise smooth function on $|\mathscr{C}'|$ obtained by $\phi \circ F$.*

Proof. This follows locally as in [Allermann and Rau 2010, Proposition 4.8] using Remarks 1.2 and 1.11, and a linearization procedure (see proof of Proposition 1.14). □

Proposition 1.14. *Let C and C' be tropical cycles on $N_{\mathbb{R}}$ with smooth weights. Let $\mathscr{C}$ be a polyhedral complex of definition for C and $\phi : |\mathscr{C}| \to \mathbb{R}$ and $\psi : |\mathscr{C}| \to \mathbb{R}$ piecewise smooth functions. Then we have the associativity law*

$$\phi \cdot (C \cdot C') = (\phi \cdot C) \cdot C' \tag{1.14.1}$$

and the commutativity law

$$\phi \cdot (\psi \cdot C) = \psi \cdot (\phi \cdot C) \tag{1.14.2}$$

as identities of tropical cycles on $N_{\mathbb{R}}$.

Proof. Using Remark 1.11 it is shown in [Allermann and Rau 2010, Lemma 9.7, Proposition 6.7] that (1.14.1) and (1.14.2) hold for tropical cycles C, C' with integral weights and piecewise linear functions $\phi, \psi : |\mathscr{C}| \to \mathbb{R}$ with integral slopes. As both sides of (1.14.1) and (1.14.2) are linear in weights and slopes, both formulas extend by linearity to tropical cycles with constant weight functions and piecewise linear functions with arbitrary real slopes.

To reduce to the above situation, we use the procedure described in Remark 1.2. We may assume that C and C' are tropical cycles of pure dimension n and n'

respectively. Let $\mathscr{C}$ be an integral $\mathbb{R}$-affine polyhedral complex such that $\mathscr{C}_{\leq n}$ and $\mathscr{C}_{\leq n'}$ are polyhedral complexes of definition for C and C'. We write $C = (\mathscr{C}_{\leq n}, m)$, $C' = (\mathscr{C}'_{\leq n'}, m')$, and $C \cdot C' = (\mathscr{C}_{\leq l}, m'')$ with $l := n + n' - r$. Given $\omega \in |\mathscr{C}|$ we denote by $\mathscr{C}_\omega$ the rational polyhedral fan of local cones of $\mathscr{C}$ in ω. There is a bijective correspondence between the polyhedra $\sigma \in \mathscr{C}$ with $\omega \in \sigma$ and the cones σ_ω in $\mathscr{C}_\omega$. Each $\sigma \in \mathscr{C}$ with $\omega \in \sigma$ determines a canonical isomorphism of affine spaces $I_\omega : \mathbb{L}_{\sigma_\omega} \xrightarrow{\sim} \mathbb{A}_\sigma$ with $I_\omega(0) = \omega$. We obtain tropical fans with constant weight functions $C_\omega = (\mathscr{C}_{\omega,\leq n}, m(\omega))$, $C'_\omega = (\mathscr{C}_{\omega,\leq n'}, m'(\omega))$, and $(C \cdot C')_\omega = (\mathscr{C}_{\omega,\leq n}, m''(\omega))$. We have $C_\omega \cdot C'_\omega = (C \cdot C')_\omega$ by our construction of the stable tropical intersection product with smooth weights. There is a unique piecewise linear function $\phi_\omega : |\mathscr{C}_\omega| \to \mathbb{R}$ such that for all $\sigma_\omega \in \mathscr{C}_\omega$ the $\mathbb{R}$-linear function ϕ_{σ_ω} on $\mathbb{L}_{\sigma_\omega}$ determined by $\phi_\omega|_{\sigma_\omega} = \phi_{\sigma_\omega}|_{\sigma_\omega}$ satisfies

$$(d\phi_{\sigma_\omega})(0) = (I_\omega^* d\phi)(0)$$

in $\mathbb{L}^*_{\sigma_\omega}$. We write

$$\phi \cdot (C \cdot C') = (\mathscr{C}_{\leq l-1}, m_1), \quad \phi_\omega \cdot (C_\omega \cdot C'_\omega) = (\mathscr{C}_{\omega,\leq l-1}, m_{\omega,1}),$$
$$(\phi \cdot C) \cdot C' = (\mathscr{C}_{\leq l-1}, m_2), \quad (\phi_\omega \cdot C_\omega) \cdot C'_\omega = (\mathscr{C}_{\omega,\leq l-1}, m_{\omega,2}).$$

The local nature of our definitions yields

$$m_{i,\sigma}(\omega) = m_{\omega,i,\sigma_\omega}(0)$$

for $i = 1, 2$ and all $\sigma \in \mathscr{C}_{\leq n+n'-r-1}$ with $\omega \in \sigma$. Formula (1.14.1) for constant weight functions and piecewise linear functions with arbitrary real slopes gives $m_{\omega,1,\sigma_\omega}(0) = m_{\omega,2,\sigma_\omega}(0)$. Hence $m_1 = m_2$ and (1.14.1) is proven in general. The reduction of (1.14.2) to the case of constant weight functions and piecewise linear functions proceeds in exactly the same way. □

Corollary 1.15. *Let $F : N'_\mathbb{R} \to N_\mathbb{R}$ be an integral $\mathbb{R}$-affine map. We consider a tropical cycle $C = (\mathscr{C}, m)$ with smooth weights on $N_\mathbb{R}$ and a piecewise smooth function $\phi : |\mathscr{C}| \to \mathbb{R}$. We write $F^*C = (\mathscr{C}', m')$, where $F(|\mathscr{C}'|) \subseteq |\mathscr{C}|$. Then ϕ induces a piecewise smooth function $F^*(\phi) : |\mathscr{C}'| \to \mathbb{R}$ and we have*

$$F^*(\phi) \cdot F^*(C) = F^*(\phi \cdot C),$$

i.e., the formation of the corner locus is compatible with pull-back.

Proof. Using (1.4.5) giving pull-back as a stable intersection with the graph, the claim follows by applying Proposition 1.13 and (1.14.1) in Proposition 1.14. □

2. The algebra of delta-preforms

In this section we define polyhedral supercurrents on an open subset $\widetilde{\Omega}$ in $N_\mathbb{R}$ for some free $\mathbb{Z}$-module N of finite rank. The polyhedral supercurrents are special supercurrents in the sense of Lagerberg. We show that an analogue of Stokes' theorem holds for polyhedral supercurrents with respect to the polyhedral derivatives d'_{P} and d''_{P}. Then we introduce the algebra of δ-preforms on $\widetilde{\Omega}$ which is going to play a central role in this paper. These δ-preforms are special polyhedral supercurrents defined by tropical cycles and superforms. We show that δ-preforms admit products and pull-back morphisms, satisfy a projection formula and that the polyhedral derivative of a δ-preform coincides with its derivative in the sense of supercurrents.

Throughout this section N and N' denote free $\mathbb{Z}$-modules of finite rank r and r'. We write $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ and $N'_{\mathbb{R}} = N' \otimes_{\mathbb{Z}} \mathbb{R}$. We refer to the Appendix for the notation from convex geometry.

2.1. Given an open subset $\widetilde{\Omega}$ in $N_{\mathbb{R}}$, we denote by $A^{p,q}(\widetilde{\Omega})$ the space of superforms of type (p, q) on $\widetilde{\Omega}$, by $A_c^{p,q}(\widetilde{\Omega})$ the space of superforms with compact support of type (p, q) on $\widetilde{\Omega}$, and by $D_{k,l}(\widetilde{\Omega}) = D^{r-k,r-l}(\widetilde{\Omega})$ the space of supercurrents of type (k, l) on $\widetilde{\Omega}$ in the sense of Lagerberg [2012] (see also [Chambert-Loir and Ducros 2012; Gubler 2016]). We have seen in the introduction that $A := \bigoplus_{p,q} A^{p,q}$ defines a sheaf of differential bigraded $\mathbb{R}$-algebras with respect to the differentials d' and d''. The bigraded sheaf $D := \bigoplus_{p,q} D^{p,q}$ contains A as a bigraded subsheaf and has canonical differentials d' and d'' extending those of A.

The sheaf $A^{p,q}$ comes with a natural operator $J^{p,q} : A^{p,q} \to A^{q,p}$ which extends to $J^{p,q} : D^{p,q} \to D^{q,p}$. The first one induces an involution $J := \bigoplus_{p,q} J^{p,q}$ on A which is determined by the fact that it is an endomorphism of sheaves of $A^{0,0}$-algebras and that $d' \circ J = J \circ d''$. The extension of J to supercurrents is determined by

$$\langle J(T), \alpha\rangle = (-1)^r \langle T, J(\alpha)\rangle$$

for $\alpha \in A^{r-p,r-q}(\widetilde{\Omega})$ and $T \in D^{p,q}(\widetilde{\Omega})$. Sections of $A^{p,p}$ (resp. $D^{p,p}$) which are invariant under the action of $(-1)^p J^{p,p}$ are called *symmetric superforms* (resp. *symmetric supercurrents*). Sections of $A^{p,p}$ (resp. $D^{p,p}$) which are invariant under the action of $(-1)^{p+1} J^{p,p}$ are called *antisymmetric superforms* (resp. *antisymmetric supercurrents*).

2.2. Let $\widetilde{\Omega}$ be an open subset of $N_{\mathbb{R}}$. An integral $\mathbb{R}$-affine polyhedron Δ of dimension n in $N_{\mathbb{R}}$ determines a canonical calibration

$$\mu_\Delta \in \left|\bigwedge^n \mathbb{L}_\Delta\right| = \mathrm{Or}(\mathbb{A}_\Delta) \times^{\pm 1} \bigwedge^n \mathbb{L}_\Delta$$

as in [Chambert-Loir and Ducros 2012, (1.3.5)]. Given a superform $\alpha \in A_c^{n,n}(\widetilde{\Omega})$ the integral

$$\int_\Delta \alpha = \int_{N_{\mathbb{R}}} \langle \alpha, \mu_\Delta\rangle$$

was defined in [Chambert-Loir and Ducros 2012, §1.5] (see also [Gubler 2016, §3]). The polyhedron Δ determines a continuous functional

$$A_c^{n,n}(\widetilde{\Omega}) \to \mathbb{R}, \quad \alpha \mapsto \int_\Delta \alpha \tag{2.2.1}$$

and a symmetric supercurrent $\delta_\Delta \in D_{n,n}(\widetilde{\Omega})$.

For $\Omega := \widetilde{\Omega} \cap \Delta$, we define $A_\Delta^{p,q}(\Omega)$ as the space of superforms on the open subset $\widetilde{\Omega} \cap \operatorname{relint}(\Delta)$ of the affine space $\mathbb{A}_\Delta$ given by restriction of elements in $A^{p,q}(\widetilde{\Omega})$. A partition of unity argument shows that this definition is independent of the choice of $\widetilde{\Omega}$.

For a superform $\alpha \in A_\Delta^{p,q}(\widetilde{\Omega} \cap \Delta)$, the supercurrent

$$\alpha \wedge \delta_\Delta \in D_{n-p,n-q}(\widetilde{\Omega})$$

is defined by $\langle \alpha \wedge \delta_\Delta, \beta \rangle = \langle \delta_\Delta, \alpha \wedge \beta \rangle$ for all $\beta \in A_c^{n-p,n-q}(\widetilde{\Omega})$.

Definition 2.3 (polyhedral supercurrents). Let $\widetilde{\Omega}$ be an open subset of $N_\mathbb{R}$. A supercurrent $\alpha \in D(\widetilde{\Omega})$ is called *polyhedral* if there exists an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ in $N_\mathbb{R}$ and a family $(\alpha_\Delta)_{\Delta \in \mathscr{C}}$ of superforms $\alpha_\Delta \in A_\Delta(\widetilde{\Omega} \cap \Delta)$ such that

$$\alpha = \sum_{\Delta \in \mathscr{C}} \alpha_\Delta \wedge \delta_\Delta \tag{2.3.1}$$

holds in $D(\widetilde{\Omega})$. In this case we say that $\mathscr{C}$ is *a polyhedral complex of definition for α*. The *polyhedral derivatives* $d'_{\mathrm{P}}(\alpha)$ and $d''_{\mathrm{P}}(\alpha)$ of a polyhedral supercurrent (2.3.1) are the polyhedral supercurrents defined by the formulas

$$d'_{\mathrm{P}}(\alpha) = \sum_{\Delta \in \mathscr{C}} d'(\alpha_\Delta) \wedge \delta_\Delta, \quad d''_{\mathrm{P}}(\alpha) = \sum_{\Delta \in \mathscr{C}} d''(\alpha_\Delta) \wedge \delta_\Delta.$$

Remark 2.4. (i) We observe that the family of forms $(\alpha_\Delta)_{\Delta \in \mathscr{C}}$ in (2.3.1) is uniquely determined by α and $\mathscr{C}$. Furthermore the support $\operatorname{supp}(\alpha)$ of a polyhedral supercurrent α is the union of the supports of the forms α_Δ for all $\Delta \in \mathscr{C}$.

(ii) It is straightforward to check that the definitions of the polyhedral derivatives $d'_{\mathrm{P}}(\alpha)$ and $d''_{\mathrm{P}}(\alpha)$ do not depend on the choice of the polyhedral complex of definition $\mathscr{C}$.

(iii) We do not claim that the polyhedral derivatives of a polyhedral supercurrent α coincide with derivative of a α in the sense of supercurrents. In fact the derivatives of a polyhedral supercurrent in the sense of supercurrents are in general not even polyhedral.

Definition 2.5. Let $\widetilde{\Omega}$ denote an open subset of $N_\mathbb{R}$. Let $P \subseteq \widetilde{\Omega}$ be an integral $\mathbb{R}$-affine polyhedral set in $N_\mathbb{R}$. We choose an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ in $N_\mathbb{R}$ whose support is P.

(i) Let $\alpha \in D_{0,0}(\widetilde{\Omega})$ be a polyhedral supercurrent such that $\operatorname{supp}(\alpha) \cap P$ is compact. After suitable refinements, we may assume that α admits a polyhedral complex of definition $\mathscr{D}$ such that $\mathscr{D}$ is a subcomplex of $\mathscr{C}$. In this situation we write

$$\alpha = \sum_{\Delta \in \mathscr{D}} \alpha_\Delta \wedge \delta_\Delta \tag{2.5.1}$$

as in (2.3.1) and define the integral of α over P as

$$\int_P \alpha = \sum_{\Delta \in \mathscr{D}} \int_\Delta \alpha_\Delta. \tag{2.5.2}$$

(ii) Let $\beta \in D_{1,0}(\widetilde{\Omega})$ be a polyhedral supercurrent with $\operatorname{supp}(\beta) \cap P$ compact. Proceeding as in (i), we get $\beta = \sum_{\Delta \in \mathscr{D}} \beta_\Delta \wedge \delta_\Delta$ for a suitable subcomplex $\mathscr{D}$ of $\mathscr{C}$ and we define the integral of β over the boundary of P as

$$\int_{\partial P} \beta = \sum_{\Delta \in \mathscr{D}} \int_{\partial \Delta} \beta_\Delta, \tag{2.5.3}$$

where the boundary integrals on the right are defined as in [Chambert-Loir and Ducros 2012, §1.5; Gubler 2016, 2.6]. We define the boundary integral (2.5.3) for a polyhedral supercurrent $\beta \in D_{0,1}(\widetilde{\Omega})$ with $\operatorname{supp}(\beta) \cap P$ compact by the same formula.

Remark 2.6. (i) The definitions in (2.5.2) and (2.5.3) do not depend on the choice of the polyhedral complex $\mathscr{D}$.

(ii) On the Borel algebra $\mathbb{B}(P)$, we get signed measures

$$\mu_{P,\alpha} : \mathbb{B}(P) \to \mathbb{R}, \quad \mu_{P,\alpha}(M) = \sum_{\Delta \in \mathscr{D}} \int_{M \cap \Delta} \alpha_\Delta$$

and

$$\mu_{\partial P,\beta} : \mathbb{B}(P) \to \mathbb{R}, \quad \mu_{\partial P,\beta}(M) = \sum_{\Delta \in \mathscr{D}} \int_{M \cap \partial \Delta} \beta_\Delta.$$

(iii) We recall from A.5 that $\operatorname{relint}(P)$ denotes the set of regular points of a polyhedral set P. Then $\operatorname{supp}(\beta) \cap P \subseteq \operatorname{relint}(P)$ implies

$$\int_{\partial P} \beta = 0 \tag{2.6.1}$$

as an immediate consequence of the definitions.

Proposition 2.7 (Stokes' formula for polyhedral supercurrents). *Let $\widetilde{\Omega}$ denote an open subset and P an integral $\mathbb{R}$-affine polyhedral subset in $N_\mathbb{R}$ with $P \subseteq \widetilde{\Omega}$. Then we have*

$$\int_P d'_{\mathrm{P}} \alpha = \int_{\partial P} \alpha, \quad \int_P d''_{\mathrm{P}} \beta = \int_{\partial P} \beta$$

for all polyhedral supercurrents $\alpha \in D_{1,0}(\widetilde{\Omega})$ *and* $\beta \in D_{0,1}(\widetilde{\Omega})$ *with* $\mathrm{supp}(\alpha) \cap P$ *and* $\mathrm{supp}(\beta) \cap P$ *compact.*

Proof. We choose a polyhedral complex of definition $\mathscr{C}$ for α such that a subcomplex $\mathscr{D}$ has support P. By linearity it is sufficient to treat the case $\alpha = \alpha_\Delta \wedge \delta_\Delta$ for a superform $\alpha_\Delta \in A_c^{n-1,n}(\widetilde{\Omega} \cap \Delta)$ and $\Delta \in \mathscr{D}_n$. We get

$$\int_P d'_{\mathrm{P}}(\alpha) = \int_P d'(\alpha_\Delta) \wedge \delta_\Delta = \int_\Delta d'(\alpha_\Delta) = \int_{\partial \Delta} \alpha_\Delta = \int_{\partial P} \alpha,$$

using Stokes' formula for superforms on polyhedra (see [Chambert-Loir and Ducros 2012, (1.5.7)] or [Gubler 2016, 2.9]). The formula for β follows in the same way. □

Remark 2.8. Let $\widetilde{\Omega}$ be an open subset of $N_{\mathbb{R}}$. An integral $\mathbb{R}$-affine polyhedral complex $C = (\mathscr{C}, m)$ with smooth weights of pure dimension n and a superform $\alpha \in A^{p,q}(\widetilde{\Omega})$ determine a polyhedral supercurrent

$$\alpha \wedge \delta_C = \sum_{\Delta \in \mathscr{C}_n} (m_\Delta \cdot \alpha|_\Delta) \wedge \delta_\Delta \in D_{n-p,n-q}(\widetilde{\Omega}).$$

In particular we get the polyhedral supercurrents $[\alpha] = \alpha \wedge \delta_{N_{\mathbb{R}}} \in D_{r-p,r-q}(\widetilde{\Omega})$ and $\delta_C = 1 \wedge \delta_C \in D_{n,n}(\widetilde{\Omega})$.

Definition 2.9 (δ-preforms). (i) Let $\widetilde{\Omega}$ be an open subset of $N_{\mathbb{R}}$. A supercurrent $\alpha \in D_{r-p,r-q}(\widetilde{\Omega})$ is called a *δ-preform of type* (p, q) if there exist a finite set I, a family $(C_i)_{i \in I}$ of tropical cycles with smooth weights $C_i = (\mathscr{C}_i, m_i)$ of codimension n_i in $N_{\mathbb{R}}$, and a family $(\alpha_i)_{i \in I}$ of superforms $\alpha_i \in A^{p_i,q_i}(\widetilde{\Omega})$ such that $p_i + n_i = p$ and $q_i + n_i = q$ for all $i \in I$ and

$$\alpha = \sum_{i \in I} \alpha_i \wedge \delta_{C_i} \tag{2.9.1}$$

holds in $D_{r-p,r-q}(\widetilde{\Omega})$. The *support* of a δ-preform is the support of its underlying supercurrent.

(ii) The δ-preforms define a subspace $P^{p,q}(\widetilde{\Omega})$ in $D_{r-p,r-q}(\widetilde{\Omega})$. We put

$$P^n(\widetilde{\Omega}) = \bigoplus_{p+q=n} P^{p,q}(\widetilde{\Omega})$$

and $P(\widetilde{\Omega}) = \bigoplus_{n \in \mathbb{N}} P^n(\widetilde{\Omega})$. We denote by $P_c(\widetilde{\Omega})$ the subspace of $P(\widetilde{\Omega})$ given by the δ-preforms with compact support. A δ-preform $\alpha \in P^{p,p}(\widetilde{\Omega})$ of type (p, p) is called *symmetric* (resp. *antisymmetric*), if the underlying supercurrent of α is symmetric (resp. antisymmetric).

(iii) We say that a δ-preform α has *codimension* l, if it admits a presentation (2.9.1) where all the tropical cycles $\mathscr{C}_i$ are of pure codimension l. The δ-preforms of type $(p + l, q + l)$ of codimension l define a subspace of $D^{p+l,q+l}(\widetilde{\Omega})$ which we denote by $P^{p,q,l}(\widetilde{\Omega})$. As an immediate consequence of our definitions, we have the direct sum

$$P^n(\widetilde{\Omega}) = \bigoplus_{p+q+2l=n} P^{p,q,l}(\widetilde{\Omega}).$$

Example 2.10. It follows from Remark 1.7 that a δ-preform of codimension 0 on $\widetilde{\Omega}$ is the same as a superform on $\widetilde{\Omega}$ with piecewise smooth coefficients.

Remark 2.11. Let

$$\alpha = \sum_{i \in I} \alpha_i \wedge \delta_{C_i} \in P^{p,q,l}(\widetilde{\Omega})$$

be a δ-preform as in (2.9.1). Let $\mathscr{C}$ be a common polyhedral complex of definition for the tropical cycles $(C_i)_{i\in I}$. Then the supercurrent α is polyhedral and $\mathscr{C}$ is a polyhedral complex of definition for α. In fact we have $C_i = (\mathscr{C}, m_i)$ for suitable families of weight functions $m_{i,\Delta}$ on polyhedra Δ in $\mathscr{C}_{r-l}$ and define

$$\alpha_\Delta := \sum_{i\in I} m_{i,\Delta} \cdot (\alpha_i|_\Delta) \in A_\Delta^{p,q}(\widetilde{\Omega} \cap |\Delta|).$$

Then we get

$$\delta_{C_i} = \sum_{\Delta \in \mathscr{C}_{r-l}} m_{i,\Delta} \wedge \delta_\Delta$$

and

$$\alpha = \sum_{\Delta \in \mathscr{C}_{r-l}} \alpha_\Delta \wedge \delta_\Delta.$$

In order to compare δ-preforms in $P^{p,q,l}(\widetilde{\Omega})$, presented as in (2.9.1),

$$\alpha = \sum_{i\in I} \alpha_i \wedge \delta_{C_i}, \quad \beta = \sum_{j \in J} \beta_j \wedge \delta_{D_j},$$

we choose a common polyhedral complex of definition $\mathscr{C}$ for the finite families $(C_i)_{i\in I}$ and $(D_j)_{j\in J}$ of tropical cycles and obtain

$$\alpha = \beta \iff \alpha_\Delta = \beta_\Delta \quad \text{for all } \Delta \in \mathscr{C}_{r-l}. \tag{2.11.1}$$

Proposition 2.12. *Let $\widetilde{\Omega}$ denote an open subset of $N_{\mathbb{R}}$. Presenting δ-preforms as in* (2.9.1), *we can perform the following constructions*:

(i) *We have a canonical $C^\infty(\widetilde{\Omega})$-linear map*

$$A^{p,q}(\widetilde{\Omega}) \to P^{p,q,0}(\widetilde{\Omega}), \quad \alpha \mapsto \alpha \wedge \delta_{N_{\mathbb{R}}}$$

and a $C^\infty(N_{\mathbb{R}})$-linear isomorphism

$$\mathrm{TZ}^l(N_{\mathbb{R}}) \xrightarrow{\sim} P^{0,0,l}(N_{\mathbb{R}}), \quad C \mapsto 1 \wedge \delta_C.$$

(ii) *There are well-defined $C^\infty(\widetilde{\Omega})$-bilinear products*

$$\wedge : P^{p,q,l}(\widetilde{\Omega}) \otimes_{\mathbb{R}} P^{p',q',l'}(\widetilde{\Omega}) \to P^{p+p',q+q',l+l'}(\widetilde{\Omega}),$$
$$\Big(\sum_{i\in I}\alpha_i \wedge \delta_{C_i}\Big) \wedge \Big(\sum_{j\in J}\beta_j \wedge \delta_{D_j}\Big) = \sum_{(i,j)\in I\times J}(\alpha_i \wedge \beta_j)\wedge \delta_{C_i\cdot D_j}.$$

(iii) *An integral $\mathbb{R}$-affine map $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ induces a natural pull-back*

$$F^* : P^{p,q,k}(\widetilde{\Omega}) \to P^{p,q,k}(\widetilde{\Omega}'), \quad \sum_{i\in I}\alpha_i \wedge \delta_{C_i} \mapsto \sum_{i\in I}(F^*\alpha_i)\wedge\delta_{F^*C_i}$$

for any open subset $\widetilde{\Omega}'$ of $F^{-1}(\widetilde{\Omega})$.

(iv) *The pull-back morphism F^* in (iii) satisfies*

$$F^*(\alpha\wedge\beta) = (F^*\alpha)\wedge(F^*\beta)$$

for all $\alpha, \beta \in P(\widetilde{\Omega})$.

Proof. The proof of (i) is straightforward. For (ii), we have to show that the definition

$$\alpha\wedge\beta := \sum_{(i,j)\in I\times J}(\alpha_i\wedge\beta_j)\wedge\delta_{C_i\cdot D_j} \tag{2.12.1}$$

is independent of the presentations

$$\alpha = \sum_{i\in I}\alpha_i\wedge\delta_{C_i} \in P^{p,q,l}(\widetilde{\Omega}), \quad \beta = \sum_{j\in J}\beta_j\wedge\delta_{D_j} \in P^{p',q',l'}(\widetilde{\Omega})$$

given as in (2.9.1). We choose a common polyhedral complex of definition $\mathscr{C}$ for all tropical cycles C_i and D_j. Using Remark 2.11, we represent the δ-preforms as polyhedral supercurrents

$$\alpha = \sum_{\sigma\in\mathscr{C}^l}\alpha_\sigma\wedge\delta_\sigma, \quad \beta = \sum_{\sigma'\in\mathscr{C}^{l'}}\beta_{\sigma'}\wedge\delta_{\sigma'}. \tag{2.12.2}$$

We choose a generic vector v and $\varepsilon > 0$ as in 1.4(ii). From (2.12.1) and (1.4.1), we deduce

$$\alpha\wedge\beta = \sum_{\tau\in\mathscr{C}^{l+l'}}\sum_{\sigma,\sigma'}[N : N_\sigma + N_{\sigma'}]\cdot\alpha_\sigma\wedge\beta_{\sigma'}\wedge\delta_\tau, \tag{2.12.3}$$

where σ, σ' ranges over all pairs in $\mathscr{C}^l\times\mathscr{C}^{l'}$ with $\sigma\cap\sigma' = \tau$ and $\sigma\cap(\sigma'+\varepsilon v)\neq\varnothing$. Then (ii) follows from (2.12.3) and from the uniqueness of the representations in (2.12.2). Bilinearity is obvious.

Similarly we show (iii). Given a δ-preform α as above, we have to prove that

$$F^*(\alpha) := \sum_{i\in I}(F^*\alpha_i)\wedge\delta_{F^*C_i} \tag{2.12.4}$$

is independent of the representation of α in (2.12.2). There is a complete, integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}'$ of $N'_{\mathbb{R}}$ and a complete, common polyhedral complex of definition $\mathscr{C}$ for all tropical cycles C_i satisfying the following compatibility property: for every $\sigma' \in \mathscr{C}'$, there is a $\sigma \in \mathscr{C}$ with $F(\sigma') \subseteq \sigma$. Using the coefficients $m_{\sigma',\sigma}^{\gamma'}$ from Remark 1.4(v), we deduce from (2.12.4) and (1.4.4) that

$$F^*(\alpha) = \sum_{\gamma' \in (\mathscr{C}')^l} \sum_{\sigma',\sigma} m_{\sigma',\sigma}^{\gamma'} \cdot F^* \alpha_\sigma \wedge \delta_{\gamma'}, \tag{2.12.5}$$

where σ', σ ranges over all pairs in $(\mathscr{C}')^0 \times \mathscr{C}^l$ with $\gamma' \subseteq \sigma'$, $F(\gamma') \subseteq \sigma$. Then (iii) follows from (2.12.5) and uniqueness of the representation (2.12.2).

Note that (iv) is a direct consequence of our definitions. □

Remark 2.13. Let P be an integral $\mathbb{R}$-affine polyhedral subset in $N_{\mathbb{R}}$ of dimension n. Let $C = (\mathscr{C}, m)$ be a tropical cycle with $|C| = P$ or $|\mathscr{C}| = P$ and $\alpha \in P_c^{\cdot}(N_{\mathbb{R}})$. Observe that $\int_P \alpha$ is in general different from $\int_{N_{\mathbb{R}}} \alpha \wedge \delta_C$ as the latter integral takes the multiplicities of C into account.

Proposition 2.14 (projection formula). *Let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map and C' a tropical cycle of dimension n on $N'_{\mathbb{R}}$. Let P be an integral $\mathbb{R}$-affine polyhedral subset and $\widetilde{\Omega}$ an open subset of $N_{\mathbb{R}}$ with $P \subseteq \widetilde{\Omega}$. Let $\alpha \in P(\widetilde{\Omega})$ be a δ-preform such that* $\operatorname{supp}(F^*(\alpha) \wedge \delta_{C'}) \cap F^{-1}(P)$ *is compact. Then* $\operatorname{supp}(\alpha \wedge \delta_{F_*(C')}) \cap P$ *is compact. If $\alpha \in P^{n,n}(\widetilde{\Omega})$, then*

$$\int_P \alpha \wedge \delta_{F_*(C')} = \int_{F^{-1}(P)} F^*(\alpha) \wedge \delta_{C'}. \tag{2.14.1}$$

If $\alpha \in P^{n-1,n}(\widetilde{\Omega})$, then

$$\int_{\partial P} \alpha \wedge \delta_{F_*(C')} = \int_{\partial F^{-1}(P)} F^*(\alpha) \wedge \delta_{C'}. \tag{2.14.2}$$

Proof. We consider first the case where $\alpha \in P^{n,n}(\widetilde{\Omega})$. We may assume without loss of generality that $\alpha \in P^{p,p,l}(\widetilde{\Omega})$, where $n = p + l$. We write

$$\alpha = \sum_{i \in I} \alpha_i \wedge \delta_{C_i}$$

for suitable $\alpha_i \in A^{p,p}(\widetilde{\Omega})$ and $C_i \in \mathrm{TZ}^l(N_{\mathbb{R}})$ as in (2.9.1). We get

$$\alpha \wedge \delta_{F_*(C')} = \sum_{i \in I} \alpha_i \wedge \delta_{F_*(F^* C_i \cdot C')}, \qquad F^*(\alpha) \wedge \delta_{C'} = \sum_{i \in I} F^*(\alpha_i) \wedge \delta_{F^* C_i \cdot C'}$$

by the projection formula, Proposition 1.5(ii). We choose common polyhedral complexes of definition $\mathscr{C}'$ in $N'_{\mathbb{R}}$ for C' and $F^* C_i$ for all $i \in I$ and $\mathscr{C}$ in $N_{\mathbb{R}}$ for $F_* C'$ and C_i for all $i \in I$. We may assume that $F_*(\mathscr{C}')$ is a subcomplex of $\mathscr{C}$. After further refinements we can find polyhedral subcomplexes $\mathscr{D}'$ of $\mathscr{C}'$ with support $F^{-1}(P)$ and $\mathscr{D}$ of $\mathscr{C}$ with support P. Then $F_*\mathscr{D}'$ is a subcomplex of $\mathscr{D}$ and we write

$$\sum_{i\in I} \alpha_i \wedge \delta_{F_*(F^*C_i \cdot C')} = \sum_{\sigma\in\mathscr{C}_p} \alpha_\sigma \wedge \delta_\sigma, \tag{2.14.3}$$

$$\sum_{i\in I} F^*(\alpha_i) \wedge \delta_{F^*C_i \cdot C'} = \sum_{\sigma'\in\mathscr{C}'_p} \alpha_{\sigma'} \wedge \delta_{\sigma'}. \tag{2.14.4}$$

Consider $\sigma \in \mathscr{C}_p$. Given $\sigma' \in \mathscr{C}'_p$ with $F(\sigma') = \sigma$ there is a unique form $\tilde{\alpha}_{\sigma'} \in A_\sigma(\sigma)$ such that $F^*(\tilde{\alpha}_{\sigma'}) = \alpha_{\sigma'}$ in $A_{\sigma'}(\sigma')$. From (1.4.3), (2.14.3) and (2.14.4) we get

$$\alpha_\sigma = \sum_{\substack{\sigma'\in\mathscr{C}'_p \\ F(\sigma')=\sigma}} [N_\sigma : \mathbb{L}_F(N'_{\sigma'})] \cdot \tilde{\alpha}_{\sigma'} \tag{2.14.5}$$

and $\alpha_{\sigma'} = 0$ for all $\sigma' \in \mathscr{C}'_p$ with $\dim F(\sigma') < p$. If $\sigma \in \mathscr{D}_p$, which means $\sigma \subseteq P$, then only the $\sigma' \in \mathscr{D}'_p$ contribute to the sum in (2.14.5). Since $\sigma' \in \mathscr{D}'_p$ is equivalent to $\sigma' \subseteq F^{-1}(P)$, we deduce from (2.14.5) and compactness of $\mathrm{supp}(F^*(\alpha) \wedge \delta_{C'}) \cap F^{-1}(P)$ that $\mathrm{supp}(\alpha \wedge \delta_{F_*(C')}) \cap P$ is compact. The above formulas show that

$$\int_P \alpha \wedge \delta_{F_*(C')} = \sum_{\sigma\in\mathscr{D}_p} \int_\sigma \alpha_\sigma = \sum_{\sigma\in\mathscr{D}_p} \sum_{\substack{\sigma'\in\mathscr{C}'_p \\ F(\sigma')=\sigma}} [N_\sigma : \mathbb{L}_F(N'_{\sigma'})] \int_\sigma \tilde{\alpha}_{\sigma'}$$

and hence the transformation formula of integration theory (see [Chambert-Loir and Ducros 2012, (1.5.8); Gubler 2016, Proposition 3.10]) gives

$$\int_P \alpha \wedge \delta_{F_*(C')} = \sum_{\sigma\in\mathscr{D}_p} \sum_{\substack{\sigma'\in\mathscr{C}'_p \\ F(\sigma')=\sigma}} \int_{\sigma'} \alpha_{\sigma'} = \sum_{\sigma'\in\mathscr{D}'_p} \int_{\sigma'} \alpha_{\sigma'} = \int_{F^{-1}(P)} F^*(\alpha) \wedge \delta_{C'}.$$

This proves (2.14.1). Formula (2.14.2) is proved in exactly the same way using the transformation formula for boundary integrals in [Chambert-Loir and Ducros 2012, (1.5.8)]. □

2.15. Given a tropical cycle $C = (\mathscr{C}, m)$ with constant weight functions, it follows from Stokes' theorem that the supercurrent δ_C is closed under d' and d'' [Gubler 2016, Proposition 3.8]. The following proposition shows that this is no longer true for tropical cycles with smooth weights.

Proposition 2.16. *Let $C = (\mathscr{C}, m)$ be a tropical cycle with smooth weights of pure dimension n in $N_\mathbb{R}$. Then we have*

$$d'\delta_C = d'm \wedge \delta_\mathscr{C}, \quad d''\delta_C = d''m \wedge \delta_\mathscr{C}$$

in $D_{\cdot}(N_{\mathbb{R}})$, where the polyhedral supercurrent $d'm \wedge \delta_{\mathscr{C}}$ is defined by

$$\langle d'm \wedge \delta_{\mathscr{C}}, \alpha \rangle = \sum_{\sigma \in \mathscr{C}_n} \int_\sigma d'm_\sigma \wedge \alpha$$

and the supercurrent $d''m \wedge \delta_{\mathscr{C}}$ is defined analogously.

Proof. This is a direct consequence of Stokes' formula for superforms on polyhedra [Chambert-Loir and Ducros 2012, lemme (1.5.7)], [Gubler 2016, 2.9] and the balancing condition (1.1.1). □

Remark 2.17. It follows from Proposition 2.16 that the subspace $P^{\cdot}(N_{\mathbb{R}})$ of $D^{\cdot}(N_{\mathbb{R}})$ of δ-preforms is not closed under the differentials d' and d'' in the sense of supercurrents. We will address this problem again in 4.6.

Proposition 2.18. *Let $\widetilde{\Omega}$ denote an open subset of $N_{\mathbb{R}}$. Then we have*

$$d'(\beta) = d'_{\mathrm{P}}(\beta), \quad d''(\beta) = d''_{\mathrm{P}}(\beta)$$

for all δ-preforms $\beta \in P(\widetilde{\Omega})$.

Proof. It is sufficient to treat the case $\beta = \alpha \wedge \delta_C$ for a superform $\alpha \in A^{p,q}(\widetilde{\Omega})$ and a tropical cycle $C = (\mathscr{C}, m)$ of pure dimension n on $N_{\mathbb{R}}$. We have

$$\beta = \sum_{\sigma \in \mathscr{C}_n} (m_\sigma \cdot \alpha|_\sigma) \wedge \delta_\sigma.$$

From Proposition 2.16 we get

$$d'\beta = d'\alpha \wedge \delta_C + (-1)^{p+q} \alpha \wedge d'm \wedge \delta_{\mathscr{C}} = \sum_{\sigma \in \mathscr{C}_n} \big(m_\sigma \cdot d'\alpha|_\sigma + d'm_\sigma \wedge \alpha|_\sigma\big) \wedge \delta_\sigma.$$

Then Leibniz's rule shows

$$d'\beta = \sum_{\sigma \in \mathscr{C}_n} d'(m_\sigma \cdot \alpha|_\sigma) \wedge \delta_\sigma = d'_{\mathrm{P}}(\beta)$$

which proves the first equality. The second claim is proved similarly. □

3. Supercurrents and delta-preforms on tropical cycles

In this section, we introduce supercurrents and δ-preforms on a given tropical cycle $C = (\mathscr{C}, m)$ of pure dimension n with *constant weight functions*. Similarly to complex manifolds, such tropical cycles have no boundary as $d'\delta_C = d''\delta_C = 0$. In the applications, C will be the tropical variety of a closed subvariety of a multiplicative torus. We build upon the results in Section 2. We will obtain the formulas of Stokes and Green. The main result is the tropical Poincaré–Lelong equation which will be used in Section 9 for the first Chern δ-current of a metrized line bundle.

3.1. The space $A^{p,q}_{\mathscr{C}}(\Omega)$ of (p,q)-superforms on an open subset Ω in $|\mathscr{C}|$ is defined as follows. We choose an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ such that $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$. Elements in $A^{p,q}_{\mathscr{C}}(\Omega)$ are represented by elements in $A^{p,q}(\widetilde{\Omega})$ where two such elements are identified if they induce the same element in $A^{p,q}_{\Delta}(\Omega \cap \Delta)$ (see 2.2) for all maximal polyhedra Δ in $\mathscr{C}$. A partition of unity argument shows that this definition is independent of the choice of $\tilde{\Omega}$. Observe furthermore that $A^{p,q}_{\mathscr{C}}(\Omega)$ depends only on the support $|\mathscr{C}|$ of $\mathscr{C}$. We will often omit the polyhedral complex $\mathscr{C}$ from our notation and write simply $A^{p,q}(\Omega)$ instead of $A^{p,q}_{\mathscr{C}}(\Omega)$ when $\mathscr{C}$ or at least $|\mathscr{C}|$ is clear from the context. The spaces $A^{p,q}(\Omega)$ define a sheaf on $|\mathscr{C}|$. Hence the support of a superform in $A^{p,q}(\Omega)$ is defined as a closed subset of Ω. We denote by $A^{p,q}_c(\Omega)$ the space of superforms on Ω with compact support.

Definition 3.2. We define the *space of supercurrents* $D^{\mathscr{C}}_{p,q}(\Omega)$ of type (p,q) on an open subset Ω in $|\mathscr{C}|$ as follows. An element in $D^{\mathscr{C}}_{p,q}(\Omega)$ is given by a linear form $T \in \mathrm{Hom}_{\mathbb{R}}(A^{p,q}_c(\Omega), \mathbb{R})$ such that we can find an open set $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ and a supercurrent $T' \in D_{p,q}(\widetilde{\Omega})$ such that $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$ and $T(\eta|_{\Omega}) = T'(\eta)$ for all $\eta \in A^{p,q}_c(\widetilde{\Omega})$. As in 3.1 we often omit $\mathscr{C}$ from the notation and write $D_{p,q}(\Omega)$ instead of $D^{\mathscr{C}}_{p,q}(\Omega)$. We also use the grading $D^{p,q}(\Omega) := D_{n-p,n-q}(\Omega)$.

Remark 3.3. In the situation of Definition 3.2 we fix an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$. It follows from a partition of unity argument that in the definition of $D_{p,q}(\Omega)$ we may use this $\widetilde{\Omega}$. We may identify $D_{p,q}(\Omega)$ with a subspace of $D_{p,q}(\tilde{\Omega})$ using the canonical map $T \mapsto T'$. Indeed, this map is well defined and injective since $T(\eta|_{\Omega}) = T'(\eta)$ holds for all $\eta \in A^{p,q}_c(\widetilde{\Omega})$. Furthermore the differentials d' and d'' on $D(\widetilde{\Omega})$ induce well-defined differentials d' and d'' on $D(\Omega)$.

A polyhedral supercurrent α' on $\widetilde{\Omega}$ is in $D(\Omega)$ if and only if $\mathrm{supp}(\alpha')$ is contained in Ω. The corresponding element α in $D(\Omega)$ is called a *polyhedral supercurrent on* Ω. Using Definition 2.3, the polyhedral derivatives $d'_{\mathrm{P}}\alpha$ and $d''_{\mathrm{P}}\alpha$ are again polyhedral supercurrents on Ω. Definition 2.5 yields integrals $\int_P \alpha = \int_P \alpha'$ and boundary integrals $\int_{\partial P} \alpha = \int_{\partial P} \alpha'$ of polyhedral supercurrents α in $D_0(\Omega)$ and $D_1(\Omega)$, respectively, over an integral $\mathbb{R}$-affine polyhedral subset P of Ω, provided that $\mathrm{supp}(\alpha) \cap P$ is compact.

Definition 3.4. Let Ω be an open subset of $|\mathscr{C}|$ and consider an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$. For any δ-preform $\tilde{\alpha} \in P(\widetilde{\Omega})$ on $\widetilde{\Omega}$, the supercurrent $\tilde{\alpha} \wedge \delta_C$ on $\widetilde{\Omega}$ lies in the subspace $D(\Omega)$ of $D(\widetilde{\Omega})$. We will denote the corresponding element in $D(\Omega)$ by $\tilde{\alpha}|_{\Omega}$. A supercurrent $\alpha \in D(\Omega)$ is called a *δ-preform on* Ω if there is an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$ and a $\tilde{\alpha} \in P(\widetilde{\Omega})$ with $\alpha = \tilde{\alpha} \wedge \delta_C$. The space of δ-preforms on Ω is denoted by $P(\Omega)$ and the subspace of compactly supported δ-preforms is denoted by $P_c(\Omega)$. Note that these spaces depend also on the weights m of the tropical cycle $C = (\mathscr{C}, m)$ and not only on the open subset Ω of $|\mathscr{C}|$.

Remark 3.5. (i) A partition of unity argument again shows that $P(\Omega)$ is the image of the natural morphism

$$P(\widetilde{\Omega}) \to D(\Omega), \quad \tilde{\alpha} \mapsto \tilde{\alpha}|_\Omega := \tilde{\alpha} \wedge \delta_C$$

for any open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$. We give $P(\Omega)$ the unique structure as a bigraded algebra such that the surjective map $P(\widetilde{\Omega}) \to P(\Omega)$ is a homomorphism of bigraded algebras. Similarly, we define the grading by codimension on $P(\Omega)$. For δ-preforms $\alpha = \tilde{\alpha} \wedge \delta_C$ and $\alpha' = \tilde{\alpha}' \wedge \delta_C$ on Ω, their product is given by the formula

$$\alpha \wedge \alpha' = \tilde{\alpha} \wedge \tilde{\alpha}' \wedge \delta_C.$$

(ii) By Remarks 2.11 and 3.3, $\alpha = \tilde{\alpha} \wedge \delta_C \in P(\Omega)$ is a polyhedral supercurrent on Ω. After possibly passing to a subdivision of $\mathscr{C}$, we have

$$\alpha = \sum_{\Delta \in \mathscr{C}} \alpha_\Delta \wedge \delta_\Delta \in D(\Omega),$$

with $\alpha_\Delta \in A_\Delta(\Omega \cap \Delta)$. It follows from Proposition 2.18 that

$$d'_{\mathrm{P}}\alpha = d'\alpha \quad \text{and} \quad d''_{\mathrm{P}}\alpha = d''\alpha, \tag{3.5.1}$$

where we use the polyhedral derivative introduced in Definition 2.3 on the left-hand sides, and the derivative of currents in $D(\Omega)$ on the right-hand sides.

(iii) Now we assume that $\alpha \in P^{n,n}(\Omega)$ and that P is an integral $\mathbb{R}$-affine polyhedral subset of Ω such that $\operatorname{supp}(\alpha) \cap P$ is compact. By passing again to a subdivision, we may assume that $\mathscr{C}$ has a subcomplex $\mathscr{D}$ with $|\mathscr{D}| = P$. Using the definition of the integral of polyhedral supercurrents on Ω in Remark 3.3 and a decomposition of α as above, (2.5.2) gives

$$\int_P \alpha = \sum_{\Delta \in \mathscr{D}} \int_\Delta \alpha_\Delta.$$

A similar formula holds for the boundary integral $\int_{\partial P} \alpha$ for $\alpha \in P^{n-1,n}(\Omega)$ or $\alpha \in P^{n,n-1}(\Omega)$.

Proposition 3.6 (Stokes' formula for δ-preforms). *Let P be an integral $\mathbb{R}$-affine polyhedral subset of the open subset Ω of $|\mathscr{C}|$. Then we have*

$$\int_P d'\alpha = \int_{\partial P} \alpha, \quad \int_P d''\beta = \int_{\partial P} \beta$$

for all δ-preforms $\alpha \in P^{n-1,n}(\Omega)$ and $\beta \in P^{n,n-1}(\Omega)$ with $\operatorname{supp}(\alpha) \cap P$ *and* $\operatorname{supp}(\beta) \cap P$ *compact.*

Proof. This follows from Proposition 2.7 and (3.5.1). □

The following result will be important in the construction of δ-forms on algebraic varieties.

Lemma 3.7. *Let Ω be an open subset of $|\mathscr{C}|$. Given d'-closed (resp. d''-closed) δ-preforms γ and γ' on Ω, their product $\gamma \wedge \gamma'$ is again d'-closed (resp. d''-closed).*

Proof. Consider an open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap |\mathscr{C}|$ and d'-closed δ-preforms γ and γ' on Ω. We may assume that γ (resp. γ') is of codimension l (resp. l') and that γ has degree $k+2l$ (resp. $k'+2l$). We choose δ-preforms $\tilde{\gamma} = \sum_i \alpha_i \wedge \delta_{C_i}$ and $\tilde{\gamma}' := \sum_j \alpha'_j \wedge \delta_{C'_j}$ for superforms $\alpha_i \in A^k(\widetilde{\Omega})$, $\alpha'_j \in A^{k'}(\widetilde{\Omega})$ and tropical cycles $C_i = (\mathscr{C}_i, m_i)$, $C'_j = (\mathscr{C}'_j, m'_j)$ of codimension l, l' with smooth weight functions such that $\gamma = \tilde{\gamma} \wedge \delta_C$ and $\gamma' = \tilde{\gamma}' \wedge \delta_C$. We have to show that the supercurrent

$$\gamma \wedge \gamma' = \tilde{\gamma} \wedge \tilde{\gamma}' \wedge \delta_C \in D(\Omega)$$

is d'-closed. After suitable refinements we may assume that the polyhedral complexes $\mathscr{C}_i, \mathscr{C}'_j$ and $\mathscr{C}$ are all subcomplexes of a complete integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}$ in $N_{\mathbb{R}}$. We choose generic vectors $v, w \in N_{\mathbb{R}}$ in order to compute stable tropical intersection products as in Remark 1.4 for tropical cycles with polyhedral complex of definition $\mathscr{D}$. We have $C'_j \cdot C = (\mathscr{D}_{\leq n-l'}, m''_j)$. For $\rho \in \mathscr{D}_{n-l'}$ and $\omega \in \rho$, we have

$$m''_{j\rho}(\omega) = \sum_{\rho = \sigma' \cap \Delta} c_{\sigma'\Delta}\, m'_{j\sigma'}(\omega)\, m_\Delta$$

for small $\varepsilon > 0$, where (σ', Δ) ranges over $\mathscr{D}^{l'} \times \mathscr{D}_n$ and $c_{\sigma'\Delta} = [N : N_{\sigma'} + N_\Delta]$ if $\sigma' \cap (\Delta + \varepsilon v) \neq \varnothing$ and $c_{\sigma'\Delta} = 0$ otherwise. In the same way we write

$$C_i \cdot C'_j \cdot C = (\mathscr{D}_{\leq n-l-l'}, m'''_{ij}).$$

For $\tau \in \mathscr{D}_{n-l-l'}$ and $\omega \in \tau$, we have

$$m'''_{ij\tau}(\omega) = \sum_{\tau = \sigma \cap \rho} c_{\sigma\rho}\, m_{i\sigma}(\omega)\, m''_{j\rho}(\omega)$$

for small $\varepsilon > 0$, where (σ, ρ) ranges over $\mathscr{D}^l \times \mathscr{D}_{n-l'}$ and $c_{\sigma\rho} = [N : N_\sigma + N_\rho]$ if $\sigma \cap (\rho + \varepsilon w) \neq \varnothing$ and $c_{\sigma\rho} = 0$ otherwise. Combining the last two formulas, we get

$$m'''_{ij\tau}(\omega) = \sum_{\tau = \sigma \cap \sigma' \cap \Delta} c_{\sigma\sigma'\Delta}\, m_{i\sigma}(\omega)\, m'_{j\sigma'}(\omega)\, m_\Delta \tag{3.7.1}$$

where $(\sigma, \sigma', \Delta)$ ranges over $\mathscr{D}^l \times \mathscr{D}^{l'} \times \mathscr{D}_n$ and

$$c_{\sigma\sigma'\Delta} = c_{\sigma,\sigma'\cap\Delta} \cdot c_{\sigma'\Delta}. \tag{3.7.2}$$

We observe that by associativity and commutativity $C_i \cdot (C'_j \cdot C) = C'_j \cdot (C_i \cdot C)$. This implies

$$c_{\sigma\sigma'\Delta} = c_{\sigma',\sigma\cap\Delta} \cdot c_{\sigma\Delta}. \tag{3.7.3}$$

Now we use $d'_{\mathrm{P}}(\tilde{\gamma} \wedge \delta_C) = d'\gamma = 0$ in $D(\Omega)$. For every $\rho \in \mathscr{D}_{n-l}$, we get

$$\sum_i \sum_{\rho=\sigma\cap\Delta} c_{\sigma\Delta}\, d'(m_{i\sigma} m_\Delta \alpha_i) = 0 \tag{3.7.4}$$

on $\Omega \cap \rho$. Similarly, we use $d'_{\mathrm{P}}(\tilde{\gamma}' \wedge \delta_C) = d'\gamma' = 0$. For every $\rho' \in \mathscr{D}_{n-l'}$, this gives

$$\sum_j \sum_{\rho'=\sigma'\cap\Delta} c_{\sigma'\Delta}\, d'(m'_{j\sigma'} m_\Delta \alpha'_j) = 0 \tag{3.7.5}$$

on $\Omega \cap \rho'$. We have to show that

$$d'(\gamma \wedge \gamma') = d'(\tilde{\gamma} \wedge \tilde{\gamma}' \wedge \delta_C) = \sum_{ij} d'(\alpha_i \wedge \alpha'_j \wedge \delta_{C_i \cdot C'_j \cdot C}) \tag{3.7.6}$$

vanishes in $D(\Omega)$. Since d' agrees with d'_{P} on δ-preforms, we deduce

$$d'(\alpha_i \wedge \alpha'_j \wedge \delta_{C_i \cdot C'_j \cdot C}) = \sum_\tau d'(m'''_{ij\tau} \alpha_i \wedge \alpha'_j) \wedge \delta_\tau. \tag{3.7.7}$$

By (3.7.1) and Leibniz's rule, we can split this into the sum of

$$\sum_\tau \sum_{\tau=\sigma\cap\sigma'\cap\Delta} c_{\sigma\sigma'\Delta} m_\Delta\, d'(m_{i\sigma}\alpha_i) \wedge m'_{j\sigma'}\alpha'_j \wedge \delta_\tau \tag{3.7.8}$$

and

$$(-1)^k \sum_\tau \sum_{\tau=\sigma\cap\sigma'\cap\Delta} c_{\sigma\sigma'\Delta} m_\Delta m_{i\sigma}\alpha_i \wedge d'(m'_{j\sigma'}\alpha'_j) \wedge \delta_\tau. \tag{3.7.9}$$

Note that here and in the following, we use our standing assumption that the weight m_Δ of C is constant. From (3.7.3) and (3.7.4) we get

$$\begin{aligned}\sum_{ij} \sum_\tau \sum_{\tau=\sigma\cap\sigma'\cap\Delta} & c_{\sigma\sigma'\Delta} m_\Delta\, d'(m_{i\sigma}\alpha_i) \wedge m'_{j\sigma'}\alpha'_j \\ &= \sum_j \sum_\tau \sum_{\tau=\sigma'\cap\rho} c_{\sigma'\rho} \Bigl(\sum_{\rho=\sigma\cap\Delta} \sum_i c_{\sigma\Delta} m_\Delta\, d'(m_{i\sigma}\alpha_i) \Bigr) \wedge m'_{j\sigma'}\alpha'_j = 0.\end{aligned}$$

In the same way we get

$$\sum_{ij} \sum_\tau \sum_{\tau=\sigma\cap\sigma'\cap\Delta} c_{\sigma\sigma'\Delta} m_\Delta m_{i\sigma}\alpha_i \wedge d'(m'_{j\sigma'}\alpha'_j) = 0$$

from (3.7.2) and (3.7.5). These two equations and (3.7.6)–(3.7.9) prove the vanishing of $d'(\gamma \wedge \gamma')$. In the same way, one derives $d''(\gamma \wedge \gamma') = 0$ from the vanishing of $d''(\gamma)$ and $d''(\gamma')$. □

Corollary 3.8. *Let Ω be an open subset of $|\mathscr{C}|$. We consider $\beta = \eta \wedge \gamma \in P^k(\Omega)$ and $\beta' = \eta' \wedge \gamma' \in P^{k'}(\Omega)$ for superforms $\eta, \eta' \in A(\Omega)$ and δ-preforms $\gamma, \gamma' \in P(\Omega)$. If $d'\gamma = d'\gamma' = 0$, then $d'\beta$ is again a δ-preform with*

$$d'\beta = d'\eta \wedge \gamma \quad \textit{and} \quad d'(\beta \wedge \beta') = d'\beta \wedge \beta' + (-1)^k \beta \wedge d'\beta'.$$

If $d''\gamma = d''\gamma' = 0$, then $d''\beta$ is again a δ-preform with

$$d''\beta = d''\eta \wedge \gamma \quad \text{and} \quad d''(\beta \wedge \beta') = d''\beta \wedge \beta' + (-1)^k \beta \wedge d''\beta'.$$

Proof. Given a superform $\eta \in A^p(\Omega)$ and a supercurrent $T \in D(\Omega)$, we have

$$d'(\eta \wedge T) = d'\eta \wedge T + (-1)^p \eta \wedge d'T. \tag{3.8.1}$$

This implies the first formula and hence $d'\beta$ is a preform. Combined with Lemma 3.7, we deduce the second formula as well. Similarly, we prove the corresponding claims for d''. □

Proposition 3.9 (Green's formula for δ-preforms). *Let Ω be an open subset of $|\mathscr{C}|$ and let P be an integral $\mathbb{R}$-affine polyhedral subset of Ω. We consider symmetric δ-preforms $\beta_i \in P^{p_i,p_i}(\Omega)$ for $i = 1, 2$ with $p_1 + p_2 = n - 1$ such that $\beta_i = \eta_i \wedge \gamma_i$ for superforms $\eta_i \in A(\Omega)$ and δ-preforms $\gamma_i \in P(\Omega)$ with $d'\gamma = d'\gamma' = d''\gamma = d''\gamma' = 0$. Then we have*

$$\int_P \big(\beta_1 \wedge d'd''\beta_2 - \beta_2 \wedge d'd''\beta_1\big) = \int_{\partial P} \big(\beta_1 \wedge d''\beta_2 - \beta_2 \wedge d''\beta_1\big),$$

if we assume furthermore that $\operatorname{supp}(\beta_1) \cap \operatorname{supp}(\beta_2) \cap P$ *is compact.*

Proof. As in [Chambert-Loir and Ducros 2012, lemme (1.3.8)], the formula is obtained as a direct consequence of Proposition 3.6 and the Leibniz formula in Corollary 3.8. □

Definition 3.10. Let P be an integral $\mathbb{R}$-affine polyhedral subset in $N_{\mathbb{R}}$. A *piecewise smooth superform* α on an open subset Ω of P is given by an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}$ with support P and smooth superforms $\alpha_\Delta \in A_\Delta(\Omega \cap \Delta)$ for every $\Delta \in \mathscr{D}$ such that α_Δ restricts to α_ρ for every closed face ρ of Δ. In this case we call $\mathscr{D}$ a *polyhedral complex of definition* for α. The *support* of a piecewise smooth superform α as above is the union of the supports of the forms α_Δ for all Δ in $\mathscr{D}$. We identify two superforms α, α' on Ω if they have the same support and if $\alpha_\Delta = \alpha'_{\Delta'}$ on $\Delta \cap \Delta' \cap \Omega$ for all polyhedra Δ, Δ' of the underlying polyhedral complexes $\mathscr{D}, \mathscr{D}'$.

Remark 3.11 (properties of piecewise smooth superforms). Let Ω denote an open subset of an integral $\mathbb{R}$-affine polyhedral subset P in $N_{\mathbb{R}}$.

(i) The space of piecewise smooth superforms on Ω is denoted by $\mathrm{PS}(\Omega)$. It comes with a natural bigrading and has a natural wedge product. We conclude that $\mathrm{PS}^{\cdot,\cdot}(\Omega)$ is a bigraded $\mathbb{R}$-algebra which contains $A^{\cdot,\cdot}(\Omega)$ as a subalgebra. We denote by $\mathrm{PS}_c^{\cdot,\cdot}(\Omega)$ the subspace of $\mathrm{PS}^{\cdot,\cdot}(\Omega)$ given by piecewise smooth superforms with compact support.

(ii) Let N' be also a free abelian group of finite rank and let $F : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ be an integral $\mathbb{R}$-affine map. Suppose that Ω' is an open subset of an integral $\mathbb{R}$-affine polyhedral subset Q in $N'_{\mathbb{R}}$ with $F(Q) \subseteq P$ and $F(\Omega') \subseteq \Omega$. For a piecewise smooth superform α on Ω, there is an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}'$ with $|\mathscr{D}'| = Q$ and a polyhedral complex of definition $\mathscr{D}$ for α such that for every $\Delta' \in \mathscr{D}'$, there is a $\Delta \in \mathscr{D}$ with $F(\Delta') \subseteq \Delta$. Then we define a piecewise smooth superform $F^*(\alpha) = \alpha'$ on Ω' with $\mathscr{D}'$ as a polyhedral complex of definition by setting $\alpha'_{\Delta'} := F^*(\alpha_\Delta) \in A_{\Delta'}(\Omega' \cap \Delta')$ for every $\Delta' \in \mathscr{D}'$. In this way, we get a well-defined graded $\mathbb{R}$-algebra homomorphism

$$F^* : \mathrm{PS}^{\cdot,\cdot}(\Omega) \to \mathrm{PS}^{\cdot,\cdot}(\Omega').$$

In particular, we can restrict α to an open subset of an integral $\mathbb{R}$-affine polyhedral subset of P.

(iii) Let $\alpha \in \mathrm{PS}^{p,q}(\Omega)$ be given by an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{D}$ and smooth superforms $\alpha_\Delta \in A^{p,q}(\Omega \cap \Delta)$ for every $\Delta \in \mathscr{D}$. Then the superforms $d'\alpha_\Delta \in A^{p+1,q}_\Delta(\Omega \cap \Delta)$, with Δ ranging over $\mathscr{D}$, define an element in $\mathrm{PS}^{p+1,q}(\Omega)$ which we denote by $d'_{\mathrm{P}}\alpha$. Similarly, we define $d''_{\mathrm{P}}\alpha \in \mathrm{PS}^{p,q+1}(\Omega)$. One verifies immediately that $\mathrm{PS}^{\cdot,\cdot}(W)$ is a differential graded $\mathbb{R}$-algebra. with respect to the differentials

$$d'_{\mathrm{P}} : \mathrm{PS}^{p,q}(\Omega) \to \mathrm{PS}^{p+1,q}(\Omega), \quad d''_{\mathrm{P}} : \mathrm{PS}^{p,q}(\Omega) \to \mathrm{PS}^{p,q+1}(\Omega). \tag{3.11.1}$$

(iv) The elements of $\mathrm{PS}^{0,0}(\Omega)$ are the piecewise smooth functions on the open subset Ω of P from Definition 1.6.

3.12. Now we apply the above to an open subset Ω of the polyhedral set $P := |\mathscr{C}|$ for the given tropical cycle $C = (\mathscr{C}, m)$ with constant weight functions. A piecewise smooth superform α as above defines a polyhedral supercurrent

$$[\alpha] := \sum_{\Delta \in \mathscr{C}_n} \alpha_\Delta \wedge \delta_\Delta$$

and the derivatives in (3.11.1) coincide — as suggested by the notation — with the polyhedral derivatives introduced in Definition 2.3. Note that these differentials of piecewise smooth superforms are not compatible with the corresponding differentials of the associated supercurrents. We define the *d'-residue of* α by

$$\mathrm{Res}_{d'}(\alpha) := d'[\alpha] - [d'_{\mathrm{P}}\alpha].$$

Similarly, we define residues with respect to the differential operators d'' and $d'd''$.

3.13. Given $\alpha \in \mathrm{PS}(\Omega)$ and a polyhedral supercurrent β on the open subset Ω of $|\mathscr{C}|$, there is natural bilinear product $\alpha \wedge \beta$ which is defined as a polyhedral supercurrent on Ω as follows. After passing to a subdivision of $\mathscr{C}$, we may assume that $\mathscr{C}$ is a polyhedral complex of definition for α and β. Then $\mathscr{C}$ is a polyhedral complex of definition for $\alpha \wedge \beta$ and for every $\Delta \in \mathscr{C}$ we set

$$(\alpha \wedge \beta)_\Delta := \alpha_\Delta \wedge \beta_\Delta \in A_\Delta(\Omega \wedge \Delta),$$

where α, β are given on Ω by $\alpha_\Delta, \beta_\Delta \in A_\Delta(\Omega \wedge \Delta)$. For $\alpha \in \mathrm{PS}^k(\Omega)$, the Leibniz-type formula

$$d'_{\mathrm{P}}(\alpha \wedge \beta) = d'_{\mathrm{P}}\alpha \wedge \beta + (-1)^k \alpha \wedge d'_{\mathrm{P}}\beta \tag{3.13.1}$$

is a direct consequence of our definitions. An analogous formula holds for d''_{P}.

There is no obvious product on the space of polyhedral currents which extends the given products on the subspaces $P(\Omega)$ and $\mathrm{PS}(\Omega)$. The next remark shows that such a product exists for a canonical subspace $\mathrm{PSP}(\Omega)$ of the space of polyhedral currents.

Remark 3.14. The linear subspace $\mathrm{PSP}(\Omega)$ of $D(\Omega)$, generated by currents of the form $\alpha \wedge \beta$ with $\alpha \in \mathrm{PS}(\Omega)$ and with $\beta \in P(\Omega)$, will play a role later. Note that $\mathrm{PSP}(\Omega)$ has a unique structure as a bigraded differential $\mathbb{R}$-algebra with respect d'_{P} and d''_{P} extending the corresponding structures on $\mathrm{PS}(\Omega)$ and $P(\Omega)$. To see that the wedge product is well defined, we can use the same arguments as for $P(\Omega)$. The crucial point is that for a piecewise smooth form α as in 3.12 and $\tau \preccurlyeq \Delta \in \mathscr{C}$, the restriction of α_Δ to τ is α_τ. This allows us to use the arguments in Proposition 2.12 which show that $\wedge$ is well defined on $\mathrm{PSP}(\Omega)$.

If $F : N'_{\mathbb{R}} \to N_R$ is an integral $\mathbb{R}$-affine map and if $\widetilde{\Omega}'$ is an open subset of the preimage of the open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$, then we have a unique pull-back $F^* : \mathrm{PSP}(\widetilde{\Omega}) \to \mathrm{PSP}(\widetilde{\Omega}')$ which extends the pull-back maps on piecewise smooth forms and on δ-preforms and which is compatible with the bigrading and the wedge product. Again, the argument is the same as in the proof of Proposition 2.12. Moreover, it is clear that the projection formulas in Proposition 2.14 hold more generally for $\alpha \in \mathrm{PSP}(\widetilde{\Omega})$.

3.15. Recall that $C = (\mathscr{C}, m)$ is a tropical cycle on $N_{\mathbb{R}}$ of pure dimension n and with constant weight functions. Let ϕ be a piecewise smooth function on $|\mathscr{C}|$. We have seen in Proposition 1.12 that the corner locus $\phi \cdot C$ is again a tropical cycle. It induces a polyhedral supercurrent $\delta_{\phi\cdot C} \in D_{n-1,n-1}(|\mathscr{C}|)$ on $|\mathscr{C}|$. By Proposition 1.8, there is a piecewise smooth function $\tilde{\phi}$ on $N_{\mathbb{R}}$ extending ϕ. We have

$$\delta_{\phi\cdot C} = \delta_{\tilde{\phi}\cdot N_{\mathbb{R}}} \wedge \delta_C$$

and hence $\delta_{\phi\cdot C}$ is a δ-preform in $P^{1,1}(|\mathscr{C}|)$. By Remark 3.5, we obtain a δ-preform $\delta_{\phi\cdot C} \wedge \beta \in P^{p,q,l+1}(|\mathscr{C}|)$ for any $\beta \in P^{p,q,l}(|\mathscr{C}|)$.

The following *tropical Poincaré–Lelong formula* and its Corollary 3.19 compute the $d'd''$-residue of ϕ.

Theorem 3.16. *We consider a δ-preform $\omega \in P^{p,q,l}(|\mathscr{C}|)$ such that $d'\omega = 0 = d''\omega$. Let $\eta \in A^{n-p-l-1,n-q-l-1}(|\mathscr{C}|)$ be a superform such that $\beta = \eta \wedge \omega$ has compact support. Then we have*

$$\int_{|\mathscr{C}|} \phi\, d'd''\beta - \int_{|\mathscr{C}|} d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta = \int_{|\mathscr{C}|} \delta_{\phi\cdot C} \wedge \beta, \tag{3.16.1}$$

where we use the integral of polyhedral supercurrents on $|\mathscr{C}|$ defined in Remark 3.3.

Proof. We may assume after suitable refinements that $\mathscr{C}$ is also a polyhedral complex of definition for ϕ and ω. From (3.13.1) and (3.5.1), we get

$$d''_{\mathrm{P}}(\phi\, d'\beta) + d'_{\mathrm{P}}(d'_{\mathrm{P}}\phi \wedge \beta) = \phi\, d''d'\beta + d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta = d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta - \phi\, d'd''\beta.$$

Let P denote the polyhedral set $|\mathscr{C}|$. Stokes' formula for polyhedral supercurrents, Proposition 2.7, yields

$$\int_P \left(d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta - \phi\, d'd''\beta\right) = \int_{\partial P} \phi \wedge d'\beta + \int_{\partial P} d''_{\mathrm{P}}\phi \wedge \beta. \tag{3.16.2}$$

We write

$$\omega = \sum_{i\in I} \omega_i \wedge \delta_{C_i}$$

for tropical cycles $C_i = (\mathscr{C}_{\le n-l}, m_i)$ with suitable smooth weight functions m_i and superforms ω_i. Then we have

$$\begin{aligned}\int_{\partial P} \phi \wedge d'\beta &= \sum_{i\in I} \int_{\partial P} \phi \wedge d'\eta \wedge \omega_i \wedge \delta_{C_i}\\ &= \sum_{i\in I} \sum_{\sigma\in\mathscr{C}_{n-l}} \int_{\partial\sigma} m_{i\sigma}\phi_\sigma\, d'\eta \wedge \omega_i.\end{aligned}$$

For each $\sigma \in \mathscr{C}_{n-l}$ and each face $\tau \in \mathscr{C}_{n-l-1}$ we choose an element $\omega_{\sigma,\tau}$ as in (1.1.1). We observe that the elements $\omega_{\tau,\sigma}$ used in [Gubler 2016, 2.8] to define the boundary integrals $\int_{\partial\sigma}$ satisfy $\omega_{\tau,\sigma} = -\omega_{\sigma,\tau}$. The definition of the boundary integral uses the contraction $\langle\,\cdot\,, \omega_{\tau,\sigma}\rangle_{\{n-l\}}$ of the involved superform of type $(n-l, n-l)$ given by inserting $\omega_{\tau,\sigma}$ for the $(n-l)$-th variable and leads to

$$\int_{\partial P} \phi \wedge d'\beta = -\sum_{i\in I} \sum_{\tau\in\mathscr{C}_{n-l-1}} \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} \int_\tau \langle m_{i\sigma}\phi_\sigma\, d'\eta \wedge \omega_i, \omega_{\sigma,\tau}\rangle_{\{n-l\}}.$$

Given $i \in I$ and $\tau \in \mathscr{C}_{n-l-1}$, the balancing condition (1.1.1) for C_i gives us the vector field

$$\omega_{i\tau} := \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} m_{i\sigma}\omega_{\sigma,\tau} : \tau \to \mathbb{L}_\tau.$$

We observe that $\phi_\sigma|_\tau = \phi_\tau$ for all $\tau \prec \sigma$ yielding

$$\int_{\partial P} \phi \wedge d'\beta = -\sum_{i\in I}\sum_{\tau\in\mathscr{C}_{n-1}} \int_\tau \langle \phi_\tau\, d'\eta \wedge \omega_i, \omega_{i\tau}\rangle_{\{n-l\}} = 0$$

as a superform contracted with a vector field with values in $\mathbb{L}_\tau$ restricts to zero on τ. Using this in (3.16.2), we obtain

$$\int_{|\mathscr{C}|} \phi\, d'd''\beta - \int_{|\mathscr{C}|} d'_{\mathrm{P}}d''_{\mathrm{P}}\phi \wedge \beta = -\int_{\partial|\mathscr{C}|} d''_{\mathrm{P}}\phi \wedge \beta. \tag{3.16.3}$$

Our claim is then a consequence of the following lemma. □

Lemma 3.17. *Let ϕ be a piecewise smooth function on $|\mathscr{C}|$. For any δ-preform $\beta \in P_c^{n-1,n-1}(|\mathscr{C}|)$ with compact support, we have*

$$\int_{\partial|\mathscr{C}|} d''_{\mathrm{P}}\phi \wedge \beta = -\int_{|\mathscr{C}|} \delta_{\phi\cdot C} \wedge \beta, \quad \int_{\partial|\mathscr{C}|} d'_{\mathrm{P}}\phi \wedge \beta = \int_{|\mathscr{C}|} \delta_{\phi\cdot C} \wedge \beta. \tag{3.17.1}$$

Proof. We prove only the first formula. The second formula follows by applying the first one to $J^*(\beta)$ and using the symmetry of the supercurrent of integration. We use the notation introduced in the proof of Theorem 3.16. We may assume that $\beta \in P^{n-l-1,n-l-1,l}(|\mathscr{C}|)$ and that

$$\beta = \sum_{i\in I} \eta_i \wedge \delta_{C_i}$$

for tropical cycles $C_i = (\mathscr{C}_{\le n-l}, m_i)$ with suitable smooth weight functions m_i and superforms $\eta_i \in A^{n-l-1,n-l-1}(|\mathscr{C}|)$. Since β is a δ-preform on C, we may assume that there is a tropical cycle $\widetilde{C}_i$ of codimension l in $N_{\mathbb{R}}$ such that $C_i = \widetilde{C}_i.C$ for every $i \in I$. Recall that $\partial/\partial\omega_{\sigma,\tau}$ denotes the partial derivative along the tangential vector $\omega_{\sigma,\tau}$. An exercise in linear algebra gives

$$\langle d''\phi_\sigma \wedge \eta_i, \omega_{\sigma,\tau}\rangle_{\{2n-2l-1\}} = \frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}} \wedge \eta_i + d''\phi_\sigma \wedge \langle \eta_i, \omega_{\sigma,\tau}\rangle_{\{2n-2l-2\}}$$

for all $i \in I$, $\sigma \in \mathscr{C}_{n-l}$ and all faces τ of σ of codimension one. Furthermore one sees easily that

$$\int_\tau d''\phi_\tau \wedge \langle \eta_i, \omega_{i\tau}\rangle_{\{2n-2l-2\}} = -\int_\tau \frac{\partial\phi_\tau}{\partial\omega_{i\tau}} \wedge \eta_i.$$

Let $\phi \cdot C_i = (\mathscr{C}_{\le n-l-1}, m_i)$ denote the corner locus of ϕ on C_i introduced in Definition 1.10. Using the last two formulas and the definition of the weight functions $m_{i\tau}$ of the corner locus in (1.10.1), we get

$$\begin{aligned}
\sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} \int_\tau \langle m_{i\sigma}\, d''\phi_\sigma \wedge \eta_i\,,\, \omega_{\sigma,\tau}\rangle_{\{2n-2l-1\}}
&= \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} \int_\tau \Bigl(m_{i\sigma} \frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}} \wedge \eta_i + d''\phi_\tau \wedge \Bigl\langle \eta_i\,,\, \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} m_{i\sigma}\omega_{\sigma,\tau} \Bigr\rangle_{\{2n-2l-2\}} \Bigr) \\
&= \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} \int_\tau \Bigl(m_{i\sigma} \frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}} \wedge \eta_i - \frac{\partial\phi_\tau}{\partial\omega_{i\tau}} \wedge \eta_i \Bigr) \\
&= \int_\tau \Bigl(\sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} m_{i\sigma} \frac{\partial\phi_\sigma}{\partial\omega_{\sigma,\tau}} - \frac{\partial\phi_\tau}{\partial\omega_{i\tau}} \Bigr) \wedge \eta_i \\
&= \int_\tau m_{i\tau}\eta_i
\end{aligned}$$

for all $i \in I$ and $\tau \in \mathscr{C}_{n-l-1}$. For the polyhedral set $P := |\mathscr{C}|$, we have

$$\begin{aligned}
\int_{\partial P} d''_{\mathrm{P}}\phi \wedge \beta &= \sum_{i\in I} \sum_{\sigma\in\mathscr{C}_{n-l}} \int_{\partial\sigma} m_{i\sigma}\, d''\phi_\sigma \wedge \eta_i \\
&= -\sum_{i\in I} \sum_{\tau\in\mathscr{C}_{n-l-1}} \sum_{\substack{\sigma\in\mathscr{C}_{n-l}\\ \tau\prec\sigma}} \int_\tau \bigl\langle m_{i\sigma}\, d''\phi_\sigma \wedge \eta_i\,,\, \omega_{\sigma,\tau} \bigr\rangle_{\{2n-2l-1\}} \\
&= -\sum_{i\in I} \sum_{\tau\in\mathscr{C}_{n-l-1}} \int_\tau m_{i\tau}\eta_i \\
&= -\sum_{i\in I} \int_P \eta_i \wedge \delta_{\phi\cdot C_i}.
\end{aligned}$$

We get $\delta_{\phi\cdot C_i} = \delta_{\phi\cdot\widetilde{C}_i\cdot C} = \delta_{\widetilde{C}_i} \wedge \delta_{\phi\cdot C}$ from Proposition 1.14. Hence

$$\sum_{i\in I} \int_P \eta_i \wedge \delta_{\phi\cdot C_i} = \int_P \Bigl(\sum_{i\in I} \eta_i \wedge \delta_{\widetilde{C}_i} \Bigr) \wedge \delta_{\phi\cdot C} = \int_P \delta_{\phi\cdot C} \wedge \beta$$

yields our claim. □

Remark 3.18. In the situation of Lemma 3.17 we consider a δ-preform $\beta \in P^{n-1,n-1}(|\mathscr{C}|)$ on C. However we do no longer assume that β has compact support. Instead we make the weaker assumption that the polyhedral supercurrents $d''_{\mathrm{P}}\phi \wedge \beta \in D_{1,0}(|\mathscr{C}|)$ (resp. $d'_{\mathrm{P}}\phi \wedge \beta \in D_{0,1}(|\mathscr{C}|)$) and $\delta_{\phi\cdot C} \wedge \beta \in D_{0,0}(|\mathscr{C}|)$ have compact support. Then the first (resp. second) formula in (3.17.1) still hold for β. In order to prove this, one chooses a function $f \in A^0_c(|\mathscr{C}|)$ which is equal to 1 on the above compact supports and applies Lemma 3.17 to $f \cdot \beta$.

Corollary 3.19. *Let $C = (\mathscr{C}, m)$ be a tropical cycle with constant weight functions of pure dimension n on $N_{\mathbb{R}}$ and $\phi : |\mathscr{C}| \to \mathbb{R}$ a piecewise smooth function with corner locus $\phi \cdot C$. Then we have*

$$d'd''[\phi] - [d'_{\mathrm{P}} d''_{\mathrm{P}} \phi] = \delta_{\phi \cdot C} \tag{3.19.1}$$

in $D^{\mathscr{C}}_{n-1,n-1}(|\mathscr{C}|)$.

Proof. Both sides of (3.19.1) have support in $|\mathscr{C}|$. Hence it suffices to show that

$$\big(d'd''[\phi] - [d'_{\mathrm{P}} d''_{\mathrm{P}} \phi]\big)(\alpha) = \delta_{\phi \cdot C}(\alpha)$$

holds for all $\alpha \in A_c^{n-1,n-1}(|\mathscr{C}|)$, and this is a special case of Theorem 3.16. □

Corollary 3.20. *Let $\phi : |\mathscr{C}| \to \mathbb{R}$ a piecewise linear function on C. Then we have*

$$d'd''[\phi] = \delta_{\phi \cdot C} \tag{3.20.1}$$

in $D^{\mathscr{C}}_{n-1,n-1}(N_{\mathbb{R}})$.

Proof. This follows from Corollary 3.19. □

4. Delta-forms on algebraic varieties

Let X be an algebraic variety over K of dimension n and X^{an} the associated Berkovich space.

We introduce the algebra $B(W)$ of δ-forms on an open subset W of X^{an}. We use tropicalizations as in [Chambert-Loir and Ducros 2012] and [Gubler 2016] to pull-back algebras of δ-preforms to X^{an}. After a suitable sheafification process we obtain the sheaves of algebras B and P of δ-forms and generalized δ-forms. We show that B is a sheaf of bigraded differential $\mathbb{R}$-algebras with respect to natural differentials d' and d''.

4.1. Consider a *tropical chart* (V, φ_U) on X as in [Gubler 2016, 4.15]. It consists of a very affine Zariski open U in X. Recall that U is called *very affine* if U has a closed immersion into a multiplicative torus. Then there is a canonical torus T_U with cocharacter group

$$N_U = \mathrm{Hom}_{\mathbb{Z}}(\mathscr{O}(U)^\times / K^\times, \mathbb{Z}),$$

and a canonical closed embedding $\varphi_U : U \to T_U$, unique up to translation (see [Gubler 2016, 4.12, 4.13] for details). We get a tropicalization map

$$\mathrm{trop}_U : U^{\mathrm{an}} \xrightarrow{\varphi_U^{\mathrm{an}}} T_U^{\mathrm{an}} \xrightarrow{\mathrm{trop}} N_{U,\mathbb{R}}$$

associated with φ_U. The second ingredient of a tropical chart is an open subset $V \subseteq U^{\mathrm{an}}$ for which there is an open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ with $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$.

The set $\mathrm{trop}_U(U^{\mathrm{an}})$ is the support of a canonical tropical cycle $\mathrm{Trop}(U) = (\mathrm{Trop}(U), m_U)$ with integral weights. It is the *tropical variety* associated to the closed subvariety U of T_U equipped with its canonical tropical weights (see [Gubler 2013, §3, §13]). Note that $V = \mathrm{trop}_U^{-1}(\Omega)$ for the open subset $\Omega := \widetilde{\Omega} \cap \mathrm{Trop}(U)$ of $\mathrm{Trop}(U)$.

Definition 4.2. Let $f : X' \to X$ be a morphism of algebraic varieties over K. We say that charts (V, φ_U) and $(V', \varphi_{U'})$ of X and X' respectively are *compatible with respect to* f, if we have $f(U') \subseteq U$ and $f^{\mathrm{an}}(V') \subseteq V$.

4.3. Let $f : X' \to X$ be a morphism of algebraic varieties over K. Given compatible charts (V, φ_U) and $(V', \varphi_{U'})$ of X and X', we obtain a commutative diagram

$$\begin{tikzcd} V' \arrow[r, hook] \arrow[d, "f^{\mathrm{an}}|_{V'}"] & (U')^{\mathrm{an}} \arrow[r, "\varphi_{U'}"] \arrow[d, "f^{\mathrm{an}}|_{U'^{\mathrm{an}}}"] & T_{U'} \arrow[r, "\mathrm{trop}"] \arrow[d, "\psi"] & N_{U',\mathbb{R}} \arrow[d, "F"] \\ V \arrow[r, hook] & U^{\mathrm{an}} \arrow[r, "\varphi_U"] & T_U \arrow[r, "\mathrm{trop}"] & N_{U,\mathbb{R}} \end{tikzcd}$$

where $\psi : T_{U'} \to T_U$ is the canonical affine homomorphism of tori induced by $\mathscr{O}^\times(U) \to \mathscr{O}^\times(U')$ and $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ is the induced canonical integral Γ-affine map. These maps are unique up to translation, but this ambiguity will never play a role. If Ω' is the open subset of $\mathrm{Trop}(U')$ with $\mathrm{trop}_{U'}^{-1}(\Omega') = V'$, then $\Omega' \subseteq F^{-1}(\Omega) \cap \mathrm{Trop}(U')$.

We define $\deg(f) = [K(X') : K(X)]$ if f is dominant and the extension of function fields is finite. Otherwise we set $\deg(f) = 0$. Let Y be the schematic image of f and $f' : X' \to Y$ the induced morphism. Then a formula of Sturmfels and Tevelev [2008] which was generalized by Baker, Payne and Rabinoff [2016, Section 7] to the present setting gives

$$F_* \,\mathrm{Trop}(U') = \deg(f') \cdot \mathrm{Trop}(\overline{f(U')}) \tag{4.3.1}$$

as an equality of tropical cycles (see [Gubler 2013, Theorem 13.17]).

Definition 4.4. Let us consider a tropical chart (V, φ_U) of X. As above, we consider the open subset $\Omega := \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. We choose an open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap \mathrm{Trop}(U)$ and a δ-preform $\tilde{\alpha} \in P^{p,q}(\widetilde{\Omega})$. For any morphism $f : X' \to X$ of varieties over K and a tropical chart $(V', \varphi_{U'})$ of X' compatible with (V, φ_U), we define $\Omega' := \mathrm{trop}_{U'}(V')$. We choose an open subset $\widetilde{\Omega}'$ of $F^{-1}(\widetilde{\Omega})$ with $\widetilde{\Omega}' \cap \mathrm{Trop}(U') = \Omega'$. By Proposition 2.12, we have $F^*(\tilde{\alpha}) \in P^{p,q}(\widetilde{\Omega}')$. We denote by $N^{p,q}(V, \varphi_U)$ the subspace given by elements $\tilde{\alpha} \in P^{p,q}(\widetilde{\Omega})$ such that we have $F^*(\tilde{\alpha})|_{\Omega'} = 0 \in P^{p,q}(\Omega')$ for all compatible pairs of charts as above (see Definition 3.4 for the definition of the restriction). We define

$$P^{p,q}(V, \varphi_U) := P^{p,q}(\widetilde{\Omega}) / N^{p,q}(V, \varphi_U).$$

A partition of unity argument shows that this definition is independent of the choice of $\widetilde{\Omega}$. We call an element in $P^{p,p}(V,\varphi_U)$ *symmetric* (resp. *antisymmetric*) if it can be represented by a symmetric (resp. antisymmetric) δ-preform in $P^{p,p}(\widetilde{\Omega})$. We define

$$P^{p,q,l}(V,\varphi_U) := P^{p,q,l}(\widetilde{\Omega})\big/\big(P^{p,q,l}(\widetilde{\Omega})\cap N^{p,q}(V,\varphi_U)\big)$$

using the δ-preforms on $\widetilde{\Omega}$ of codimension l from Definition 2.9.

Remark 4.5. (i) The $\wedge$-product descends to the space

$$P(V,\varphi_U) := \bigoplus_{p,q\geq 0} P^{p,q}(V,\varphi_U)$$

and we get a bigraded anticommutative $\mathbb{R}$-algebra which contains $A(\Omega)$ as a bigraded subalgebra.

(ii) If $(V',\varphi_{U'})$ and (V,φ_U) are compatible charts with respect to $f : X' \to X$ as in Definition 4.2, then we get a canonical bigraded homomorphism

$$f^* : P(V,\varphi_U) \to P(V',\varphi_{U'})$$

of bigraded $\mathbb{R}$-algebras which is defined for $\alpha \in P^{p,q}(V,\varphi_U)$ as follows: By definition, α is represented by some $\tilde{\alpha} \in P^{p,q}(\widetilde{\Omega})$. Let $\Omega' := \operatorname{trop}_{U'}(V')$ and choose an open subset $\widetilde{\Omega}'$ of $F^{-1}(\widetilde{\Omega})$ with $\Omega' = \widetilde{\Omega}' \cap \operatorname{Trop}(U')$. Then we define $f^*(\alpha) \in P^{p,q}(V',\varphi_{U'})$ as the class of $F^*(\tilde{\alpha}) \in P^{p,q}(\widetilde{\Omega}')$. If $X = X'$ and $f = \mathrm{id}$, then $(V',\varphi_{U'})$ is a tropical subchart of (V,φ_U) and we write $\alpha|_{V'}$ for the pull-back of $\alpha \in P^{p,q}(V,\varphi_U)$.

Note that the definition of $f^*(\alpha)$ does not depend on the choice of the representative $\tilde{\alpha}$.

However, the elements of $P^{p,q}(V,\varphi_U)$ do not only depend on the *restriction*

$$\alpha|_\Omega := \tilde{\alpha}|_\Omega = \tilde{\alpha} \wedge \delta_{\operatorname{Trop}(U)} \in P^{p,q}(\Omega) \subseteq D^{p,q}(\Omega) \qquad (4.5.1)$$

to Ω as Example 4.22 below shows that it might happen that two different elements $\alpha, \beta \in P^{p,q}(V,\varphi_U)$ satisfy $\alpha|_\Omega = \beta|_\Omega \in P^{p,q}(\Omega)$. The purpose of our definition of $P(V,\varphi_U)$ is to have a pull-back as above at hand. Here we use the fact that we always have a pull-back from tropical cycles on $N_{U,\mathbb{R}}$ to tropical cycles on $N_{U',\mathbb{R}}$, but there is a pull-back available from tropical cycles on $\operatorname{Trop}(U)$ to tropical cycles on $\operatorname{Trop}(U')$ only if these tropical varieties are smooth (see [François and Rau 2013]). To have a pull-back available, we consider all morphisms $f : X' \to X$ of varieties over K in the definition of $N^{p,q}(V,\varphi_U)$ and not only open immersions.

4.6. As mentioned already in Remark 2.17, we have the problem that the differential operators d' and d'' are not defined on the algebra $P(V,\varphi_U)$. For α in $P^{p,q}(V,\varphi_U)$ and every compatible tropical chart $(V',\varphi_{U'})$ with respect to $f : X' \to X$, we use the above notation. We get a δ-preform $f^*(\alpha)|_{\Omega'} = F^*(\tilde{\alpha})|_{\Omega'} \in P^{p,q}(\Omega')$. Recall that

$f^*(\alpha)|_{\Omega'}$ is a supercurrent on Ω'. We differentiate it in the sense of supercurrents to get $d'[f^*(\alpha)|_{\Omega'}] \in D(\Omega')$, but it need not be a δ-preform on Ω'. In the following construction, we pass to a convenient subalgebra of $P(V, \varphi_U)$ which is invariant under d' and d''.

As an initial step, we consider the elements ω of $P^{p,q}(V, \varphi_U)$ and $P^{p,q,l}(V, \varphi_U)$, respectively, satisfying the *closedness condition*

$$d'[f^*(\omega)|_{\Omega'}] = d''[f^*(\omega)|_{\Omega'}] = 0 \in D(\Omega') \tag{4.6.1}$$

for every tropical chart $(V', \varphi_{U'})$ which is compatible with (V, φ_U) with respect to $f : X' \to X$. These elements form subspaces $Z^{p,q}(V, \varphi_U)$ of $P^{p,q}(V, \varphi_U)$ and $Z^{p,q,l}(V, \varphi_U)$ of $P^{p,q,l}(V, \varphi_U)$, respectively, and we define

$$Z(V, \varphi_U) := \bigoplus_{p,q \geq 0} Z^{p,q}(V, \varphi_U) = \bigoplus_{p,q,l \geq 0} Z^{p,q,l}(V, \varphi_U)$$

as usual.

Proposition 4.7. *Using the notation above*, $Z(V, \varphi_U)$ *is a bigraded* $\mathbb{R}$-*subalgebra of* $P(V, \varphi_U)$.

Proof. The only nontrivial point is that $Z(V, \varphi_U)$ is closed under the $\wedge$-product. This is a direct consequence of Lemma 3.7 applied to δ-preforms on the tropical cycle $\mathrm{Trop}(U')$ for any tropical chart $(V', \varphi_{U'})$ compatible with (V, φ_U). □

Example 4.8. Every tropical cycle $C = (\mathscr{C}, m)$ on $N_{U,\mathbb{R}}$ with *constant weight functions* induces an element in $Z(V, \varphi_U)$. Indeed, if $(V', \varphi_{U'})$ is a tropical chart on X' compatible with (V, φ_U) as above, then $F^*(\delta_C)|_{\Omega'}$ is given by the restriction of $\delta_{F^*(C)\cdot \mathrm{Trop}(U')}$ to Ω'. Since $F^*(C) \cdot \mathrm{Trop}(U')$ is a tropical cycle with constant weight functions, the associated current is d'- and d''-closed [Gubler 2016, Proposition 3.8].

Definition 4.9. Let $\mathrm{AZ}(V, \varphi_U)$ be the subalgebra of $P(V, \varphi_U)$ generated by $A(\Omega)$ and $Z(V, \varphi_U)$. An element $\beta \in \mathrm{AZ}(V, \varphi_U)$ has the form

$$\beta = \sum_{i \in I} \alpha_i \wedge \omega_i \tag{4.9.1}$$

for a finite set I with all $\alpha_i \in A(\Omega)$ and $\omega_i \in Z(V, \varphi_U)$. We define

$$d'\beta := \sum_{i \in I} d'(\alpha_i) \wedge \omega_i, \quad d''\beta := \sum_{i \in I} d''(\alpha_i) \wedge \omega_i.$$

It follows from the closedness condition (4.6.1) that $d'\beta$ and $d''\beta$ are well-defined elements in $\mathrm{AZ}(V, \varphi_U)$. By definition, we have

$$Z(V, \varphi_U) = \{\alpha \in \mathrm{AZ}(V, \varphi_U) \mid d'(\alpha) = d''(\alpha) = 0\}.$$

An element in $\mathrm{AZ}(V, \varphi_U)$ is called *symmetric* (resp. *antisymmetric*) if it is symmetric (resp. antisymmetric) in $P(V, \varphi_U)$.

The following result shows that $\mathrm{AZ}(V, \varphi_U)$ is a good analogue of the algebra of complex differential forms.

Proposition 4.10. *The space* $\mathrm{AZ}(V, \varphi_U)$ *is a bigraded differential* $\mathbb{R}$*-algebra with respect to the differentials* d' *and* d''.

Proof. This follows easily from Leibniz's rule 3.8(ii) and Proposition 4.7. □

Proposition 4.11. *Let* $f : X' \to X$ *be a morphism of varieties over* K*. Let* (V, φ_U) *and* $(V', \varphi_{U'})$ *be tropical charts of* X *and* X' *respectively which are compatible with respect to* f*. Then the pull-back homomorphism* $f^* : P(V, \varphi_U) \to P(V', \varphi_{U'})$ *maps* $Z(V, \varphi_U)$ *to* $Z(V', \varphi_{U'})$ *and* $\mathrm{AZ}(V, \varphi_U)$ *to* $\mathrm{AZ}(V', \varphi_{U'})$.

Proof. This follows directly from the definitions. We leave the details to the reader. □

Proposition 4.12. *Let* (V, φ_U) *be a tropical chart of* X *and* $\Omega := \mathrm{trop}_U(V)$*. Let* $(\Omega_i)_{i\in I}$ *be a finite open covering of* Ω*. For* $i \in I$*, let* $V_i := \mathrm{trop}_U^{-1}(\Omega_i)$ *and let* $\alpha_i \in P(V_i, \varphi_U)$*. For all* $i, j \in I$*, we assume that* $\alpha_i|_{V_i\cap V_j} = \alpha_j|_{V_i\cap V_j}$*. Then there is a unique* $\alpha \in P(V, \varphi_U)$ *with* $\alpha|_{V_i} = \alpha_i$ *for every* $i \in I$*. If* $\alpha_i \in \mathrm{AZ}(V_i, \varphi_U)$ *for every* $i \in I$ *then* $\alpha \in \mathrm{AZ}(V, \varphi_U)$.

Proof. It is a straightforward consequence of our definitions that α is unique. In order to construct α we choose for each $i \in I$ an open subset $\widetilde{\Omega}_i$ in $N_{U,\mathbb{R}}$ such that $\widetilde{\Omega}_i \cap \mathrm{Trop}(U) = \Omega_i$ and a δ-preform $\tilde{\alpha}_i \in P(\widetilde{\Omega}_i)$ which represents α_i. Let $(\phi_i)_{i\in I}$ be a smooth partition of unity on $\widetilde{\Omega} = \bigcup_{i\in I} \widetilde{\Omega}_i$ with respect to the covering $(\widetilde{\Omega}_i)_{i\in I}$. Observe that we may choose the same index set I as we do not require that the ϕ_i have compact support. Then by our assumptions $\tilde{\alpha} := \sum_{i\in I} \phi_i \tilde{\alpha}_i \in P(\widetilde{\Omega})$ induces the desired element α in $P(V, \varphi_U)$. If

$$\alpha_i = \sum_{j\in I_i} \beta_{ij} \wedge \omega_{ij} \in \mathrm{AZ}(V_i, \varphi_U)$$

as in (4.9.1), we choose representatives $\tilde{\beta}_{ij} \in A(\widetilde{\Omega}_i)$ of $\beta_{ij} \in A(\Omega_i)$ and $\tilde{\omega}_{ij}$ in $P(\widetilde{\Omega}_i)$ of $\omega_{ij} \in Z(V, \varphi_U)$. Then we may choose $\tilde{\alpha}_i$ as $\sum_{j\in I_i} \phi_i \beta_{ij} \wedge \tilde{\omega}_{ij}$ and

$$\tilde{\alpha} = \sum_{i\in I} \phi_i \tilde{\alpha}_i = \sum_{i\in I} \sum_{j\in I_i} \phi_i \beta_{ij} \wedge \tilde{\omega}_{ij}$$

shows $\alpha \in \mathrm{AZ}(V, \varphi_U)$, using the finiteness of I. □

Recall that the tropical charts (V, φ_U) of X form a basis for X^{an} [Gubler 2016, Proposition 4.16]. Hence we can use the algebras $P(V, \varphi_U)$ and $\mathrm{AZ}(V, \varphi_U)$ to define sheaves on X^{an} as follows:

Definition 4.13. For a fixed open subset W in X^{an}, the set of all tropical charts (V, φ_U) on X with $W \subseteq V$ is ordered with respect to compatibility and forms a directed set. Then we get presheaves

$$W \mapsto \varinjlim P(V, \varphi_U), \quad W \mapsto \varinjlim \mathrm{AZ}(V, \varphi_U) \tag{4.13.1}$$

of real vector spaces on X^{an}, where the limit is taken over this directed set with respect to the pull-back maps considered in Proposition 4.11. The associated sheaves P and B on X^{an} are by definition the *sheaf of generalized δ-forms* and the *subsheaf of δ-forms*. On an open subset W of X^{an} the space of δ-forms

$$B(W) = \bigoplus_{p,q \geq 0} B^{p,q}(W) = \bigoplus_{p,q,l \geq 0} B^{p,q,l}(W)$$

and the space of generalized δ-forms

$$P(W) = \bigoplus_{p,q \geq 0} P^{p,q}(W) = \bigoplus_{p,q,l \geq 0} P^{p,q,l}(W)$$

carry natural gradings by the (p, q)-type of the underlying currents and the codimension of the underlying tropical cycles (as defined in Definitions 2.9 and 4.4). The wedge product on the spaces $\mathrm{AZ}(V, \varphi_U)$ (resp. $P(V, \varphi_U)$) induces a product on $B(W)$ (resp. $P(W)$). Moreover, the differential operators d', d'' on $\mathrm{AZ}(V, \varphi_U)$ induce differential operators d', d'' on $B(W)$. The symmetric and antisymmetric elements in $P(V, \varphi_U)$ define subsheaves of (generalized) symmetric and antisymmetric δ-forms in $B^{p,q}$ and $P^{p,q}$ for all $p, q \geq 0$.

4.14. We conclude that a *δ-form β* of bidegree (p, q) on an open subset W of X^{an} is given by a covering $(V_i)_{i \in I}$ of W by tropical charts (V_i, φ_{U_i}) of X^{an} and elements $\beta_i \in \mathrm{AZ}^{p,q}(V_i, \varphi_{U_i})$ such that

$$\beta_i|_{V_i \cap V_j} = \beta_j|_{V_i \cap V_j}$$

holds for all $i, j \in I$. If β' is another δ-form of bidegree (p, q) on W given by $\beta'_j \in \mathrm{AZ}^{p,q}(V'_j, \varphi_{U'_j})$ with respect to the tropical charts $(V'_j, \varphi_{U'_j})_{j \in J}$ covering W, then β and β' define the same δ-forms if and only if

$$\beta_i|_{V_i \cap V'_j} = \beta'_j|_{V_i \cap V'_j}$$

holds for all $i \in I$ and $j \in J$. A similar description holds for generalized δ-forms.

Proposition 4.15. (i) *The sheaves P and B are sheaves of bigraded anticommutative $\mathbb{R}$-algebras.*

(ii) *We have natural monomorphisms of sheaves of bigraded $\mathbb{R}$-algebras $A \to B$ and $B \to P$.*

(iii) *The differentials $d', d'' : B \to B$ turn (B, d', d'') into a sheaf of bigraded differential $\mathbb{R}$-algebras.*

Proof. Only the injectivity of the natural morphism $A \to B$ does not follow directly from what we have shown before. The injectivity of $A \to B$ can be checked on the presheaves (4.13.1). For each tropical chart (V, φ_U) of X the natural map from $A(V)$ to $\mathrm{AZ}(V, \varphi_U)$ is injective as the associated map $A(\Omega) \to \mathrm{AZ}(V, \varphi_U) \to D(\Omega)$ for $\Omega = \mathrm{trop}_U(V)$ is injective. This directly yields our claim. □

4.16. Let $f : X' \to X$ be a morphism of varieties over K. For an open subset W of X^{an} and an open subset W' of $f^{-1}(W)$, we have a canonical pull-back morphism $f^* : P(W) \to P(W')$ which respects products and the bigrading. Furthermore it induces a homomorphism $f^* : B(W) \to B(W')$ of bigraded $\mathbb{R}$-algebras which commutes with the differentials d' and d'' on B. They are induced by the pull-back $f^* : P(V, \varphi_U) \to P(V', \varphi_{U'})$ for compatible charts $(V', \varphi_{U'})$ on W' and (V, φ_U) on W given in Proposition 4.11.

Lemma 4.17. *Let (V, φ_U) be a tropical chart on X. Let $(V_i)_{i \in I}$ be an open covering of V by tropical charts (V_i, φ_{U_i}) on X which are compatible with (V, φ_U). There are canonical integral Γ-affine morphisms $F_i : N_{U_i,\mathbb{R}} \to N_{U,\mathbb{R}}$ such that $\mathrm{trop}_U = F_i \circ \mathrm{trop}_{U_i}$. We choose open subsets $\widetilde{\Omega}$ in $N_{U,\mathbb{R}}$ and $\widetilde{\Omega}_i$ in $F_i^{-1}(\widetilde{\Omega})$ such that $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$ and $V_i = \mathrm{trop}_U^{-1}(\widetilde{\Omega}_i)$ for all $i \in I$. Let $\tilde{\alpha}_U \in P(\widetilde{\Omega})$ be a δ-preform. Then $\tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)}$ vanishes in $D(\widetilde{\Omega})$ if $F_i^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U_i)}$ vanishes in $D(\widetilde{\Omega}_i)$ for every $i \in I$.*

Proof. We write $\tilde{\alpha}_U = \sum_{j \in J} \alpha_j \wedge \delta_{C_j}$ for suitable superforms $\alpha_j \in A(\widetilde{\Omega})$ and tropical cycles C_j. We have $F_{i*}\,\mathrm{Trop}(U_i) = \mathrm{Trop}(U)$ by (4.3.1). The projection formula (Proposition 1.5) gives

$$F_{i*}(F_i^* C_j \cdot \mathrm{Trop}(U_i)) = C_j \cdot \mathrm{Trop}(U).$$

By the same arguments as in the proof of Proposition 2.14, the vanishing of

$$F_i^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U_i)} = \sum_{j \in J} F_i^*(\alpha_j) \wedge \delta_{F_i^* C_j \cdot \mathrm{Trop}(U_i)}$$

in $D(\widetilde{\Omega}_i)$ for all $i \in I$ yields that

$$\tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)} = \sum_{j \in J} \alpha_j \wedge \delta_{F_{i*}(F_i^* C_j \cdot \mathrm{Trop}(U_i))}$$

vanishes in $D(\widetilde{\Omega})$. □

Proposition 4.18. *Given a tropical chart (V, φ_U) on X, we have by construction natural algebra homomorphisms*

$$\mathrm{trop}_U^* : P^{p,q}(V, \varphi_U) \to P^{p,q}(V), \quad \mathrm{trop}_U^* : \mathrm{AZ}^{p,q}(V, \varphi_U) \to B^{p,q}(V)$$

for all $p, q \geq 0$. *These maps are injective.*

Proof. We extend the argument in [Chambert-Loir and Ducros 2012, lemme (3.2.2)]. It suffices to show that the first map is injective. Let $\mathrm{trop}_U^*(\alpha_U)$ vanish for some $\alpha_U \in P(V, \varphi_U)$. We obtain an open covering $(V_i)_{i\in I}$ of V by tropical charts (V_i, φ_{U_i}) compatible with (V, φ_U) such that $\alpha_U|_{V_i} = 0$ in $P(V_i, \varphi_{U_i})$ for all $i \in I$. Let α_U be induced by $\tilde{\alpha}_U \in P(\widetilde{\Omega})$ for some open subset $\widetilde{\Omega} \in N_{U,\mathbb{R}}$ with $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$. We have to show that $\tilde{\alpha}_U \in N(V, \varphi_U)$. Let $f : X' \to X$ be a morphism of varieties and $(V', \varphi_{U'})$ a tropical chart on X' which is compatible with (V, φ_U). We obtain a canonical integral Γ-affine morphism $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ such that $\mathrm{trop}_U = F \circ \mathrm{trop}_{U'}$. We choose an open subset $\widetilde{\Omega}'$ in $F^{-1}(\widetilde{\Omega})$ such that $V' = \mathrm{trop}_U^{-1}(\widetilde{\Omega}')$. We have to show that $F^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U')}$ vanishes in $D(\widetilde{\Omega}')$.

For every $i \in I$ we choose an open covering $(V'_{ij})_{j\in J_i}$ of $(f^{\mathrm{an}})^{-1}(V_i) \cap V'$ by tropical charts $(V'_{ij}, \varphi_{U'_{ij}})$ on X' which are compatible with $(V', \varphi_{U'})$ and (V_i, φ_{U_i}). For all $i \in I$ and $j \in J_i$ we obtain a commutative diagram

$$\begin{tikzcd}
(U'_{ij})^{\mathrm{an}} \arrow[r, hook] \arrow[d, "f|_{U'_{ij}}"'] \arrow[rr, "\mathrm{trop}_{U'_{ij}}"] & (U')^{\mathrm{an}} \arrow[d, "f|_{U'}"'] \arrow[rr, "\mathrm{trop}_{U'}"] & N_{U'_{ij},\mathbb{R}} \arrow[r, "F'_{ij}"'] \arrow[d, "F_{ij}"] & N_{U',\mathbb{R}} \arrow[d, "F"] \\
U_i^{\mathrm{an}} \arrow[r, hook] \arrow[rr, "\mathrm{trop}_{U_i}"'] & U^{\mathrm{an}} \arrow[rr, "\mathrm{trop}_U"'] & N_{U_i,\mathbb{R}} \arrow[r, "F_i"'] & N_{U,\mathbb{R}}
\end{tikzcd}$$

of canonical maps. We choose an open subset $\widetilde{\Omega}'_{ij}$ in $(F'_{ij})^{-1}(\widetilde{\Omega}') \cap (F_{ij})^{-1}(\widetilde{\Omega}_i)$ such that $V'_{ij} = \mathrm{trop}_{U'_{ij}}^{-1}(\widetilde{\Omega}'_{ij})$. We have $(F'_{ij})^* F^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U'_{ij})} = 0$ in $D(\widetilde{\Omega}'_{ij})$ by the commutativity of the above diagram and the fact that $\alpha_U|_{V_i} = 0$ in $P(V_i, \varphi_{U_i})$. Now Lemma 4.17 applied to $F^*(\tilde{\alpha}_U)$ on $(V', \varphi_{U'})$ and the covering $(V'_{ij})_{ij}$ of V' yields the vanishing of $F^*(\tilde{\alpha}_U) \wedge \delta_{\mathrm{Trop}(U')}$ in $D(\widetilde{\Omega}')$. $\square$

4.19. Let W be an open subset of X^{an}. By construction, the algebra $A^{\cdot,\cdot}(W)$ of differential forms on W is a bigraded subalgebra of the algebra $B^{\cdot,\cdot}(W)$ of δ-forms. In general, $A^{p,q}(W)$ is a proper subspace of $B^{p,q}(W)$. The situation in degree zero is quite different as we may identify δ-forms of degree 0 with functions. We will show that

$$A^{0,0}(W) = B^{0,0}(W). \tag{4.19.1}$$

Clearly, this is a local statement and so we may consider a tropical chart (V, φ_U) on W. It is enough to show

$$A^{0,0}(\Omega) = \mathrm{AZ}^{0,0}(V, \varphi_U) \tag{4.19.2}$$

for the open subset $\Omega := \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. Let $\widetilde{\Omega}$ be any open subset of $N_{U,\mathbb{R}}$. Since pull-back of functions is always well defined, we may identify the elements of $P^{0,0}(V, \varphi_U)$ with some continuous functions on Ω and a partition of unity argument together with Example 2.10 shows

$$P^{0,0}(V, \varphi_U) = \{\phi|_\Omega \mid \phi \in P^{0,0}(\widetilde{\Omega})\} = \{\phi|_\Omega \mid \phi \in \mathrm{PS}^{0,0}(\widetilde{\Omega})\}. \tag{4.19.3}$$

To prove (4.19.2), it is enough to show that the elements of $Z(V, \varphi_U)$ are precisely the locally constant functions on Ω. By (4.19.3), we have to show that $\phi|_\Omega$ is locally constant for any $\phi \in \mathrm{PS}^{0,0}(\widetilde{\Omega})$ with $\phi|_\Omega \in Z(V, \varphi_U)$. This means that ϕ is a continuous function on $\widetilde{\Omega}$ with an integral $\mathbb{R}$-affine complete polyhedral complex $\mathscr{C}$ on $N_\mathbb{R}$ such that $\phi|_{\Omega\cap\Delta}$ is smooth for every $\Delta \in \mathscr{C}$. By refinement, we may assume that a subcomplex $\mathscr{D}$ of $\mathscr{C}$ has support equal to $\mathrm{Trop}(U)$. Then the closedness condition (4.6.1) yields that $[\phi|_\Omega]$ is d'- and d''-closed. We conclude that $\phi|_{\Omega\cap\Delta}$ is constant on every $\Delta \in \mathscr{D}$. By continuity, we deduce that $\phi|_\Omega$ is locally constant proving the claim.

4.20. Let (V, φ_U) be a tropical chart on X and $\Omega = \mathrm{trop}_U(V)$.

(i) If Ω_0 is an open subset of Ω, then $V_0 := \mathrm{trop}_U^{-1}(\Omega_0)$ is an open subset of V and (V_0, φ_U) is a tropical chart of X. We say that $\alpha_U \in P(V, \varphi_U)$ vanishes on the open subset Ω_0 if we have $\alpha_U|_{V_0} = 0$ in $P(V_0, \varphi_U)$ (see Remark 4.5). We define $\mathrm{supp}(\alpha_U)$, the *support of* $\alpha_U \in P(V, \varphi_U)$, as

$$\bigl\{\omega \in \Omega \mid \alpha_U \text{ does not vanish on any open neighbourhood } \Omega_0 \text{ of } \omega \text{ in } \Omega\bigr\},$$

which is a closed subset of Ω.

(ii) A (generalized) δ-form α on an open subset W of X^{an} has a well defined support as a section of the sheaf $B^{p,q}$ (resp. $P^{p,q}$). We denote by $B_c^{p,q}$ (resp. by $P_c^{p,q}$) the subsheaves of forms with compact support.

(iii) Observe that compact support always implies proper support in the sense of [Chambert-Loir and Ducros 2012, (4.2.1)] as our assumptions imply that we have $\partial W = \varnothing$ for each open subset W of X^{an} (using that X^{an} is closed, meaning that it has no boundary, see [Berkovich 1990, Theorem 3.4.1]).

Proposition 4.21. *Let (V, φ_U) be a tropical chart on X. Suppose that a generalized δ-form $\alpha \in P(V)$ is given by $\alpha_U \in P(V, \varphi_U)$. Then α_U is uniquely determined and we have* $\mathrm{trop}_U(\mathrm{supp}(\alpha)) = \mathrm{supp}(\alpha_U)$. *Furthermore α has compact support if and only if α_U has compact support.*

Proof. This uniqueness follows from Proposition 4.18. The second statement follows from Proposition 4.18 by the same arguments as in [Chambert-Loir and Ducros 2012, corollaire (3.2.3)]. The last statement is a direct consequence of the continuity and properness of the tropicalization map trop_U (see [Baker et al. 2016, Remark in 2.3]). □

Example 4.22. We construct a tropical chart (V, φ_U) and a nonzero δ-form $\alpha \in \mathrm{AZ}(V, \varphi_U) \setminus \{0\}$ with $\alpha|_\Omega = 0$ for $\Omega := \mathrm{trop}_U(V)$. This example announced in Remark 4.5 justifies the functorial definition of (generalized) delta-forms in Definition 4.4.

We work over the ground field $K = \mathbb{C}_p$ for some prime number $p \neq 2, 3$ and consider the affine curve X in $\mathbb{A}^2_K$ defined by the affine equation

$$f(x, y) = xy + px^3 + py^3.$$

We consider the very affine open subset $U = X \setminus (\{x = 1\} \cup \{y = 1\})$. The only singularity of the rational cubic X is the origin $0 = (0, 0)$, which is an ordinary double point. The normalization of X may be seen as an open subset of $\mathbb{P}^1_K$ and can be obtained as the blowup of X in $(0, 0)$, as in [Hartshorne 1977, Example I.4.9.1]. This description leads to a surjective morphism

$$\varphi\colon \mathbb{P}^1_K \setminus \{\xi_1, \xi_2, \xi_3\} \to X, \quad u \mapsto \left(x = \frac{-u}{p(1+u^3)}, y = \frac{-u^2}{p(1+u^3)}\right)$$

for a suitable affine coordinate u on $\mathbb{P}^1_K$, where ξ_i are the roots of $u^3 + 1 = 0$. It is clear that all ξ_i have absolute value 1 and we may choose $\xi_1 = -1$. Note that $\varphi^{-1}(\{x = 1\}) = \{\rho_1, \rho_2, \rho_3\}$ for the roots ρ_i of $pu^3 + u + p = 0$ in K and $\varphi^{-1}(X \setminus \{y = 1\}) = \{\rho_1^{-1}, \rho_2^{-1}, \rho_3^{-1}\}$. Moreover, we have $\varphi^{-1}(0) = \{0, \infty\}$.

The method of the Newton polygon [Neukirch 1999, Proposition II.6.3] shows that $pu^3 + u + p = 0$ has one root ρ_1 of absolute value $|p|$, and two roots ρ_2, ρ_3 of absolute value $|p|^{-\frac{1}{2}}$. We put

$$(\lambda_0, \lambda_1, \dots, \lambda_8) = (\xi_1, \xi_2, \xi_3, \rho_1, \rho_2, \rho_3, \rho_1^{-1}, \rho_2^{-1}, \rho_3^{-1})$$

and get

$$W := \varphi^{-1}(U) = \mathbb{P}^1_K \setminus \{(\lambda_i : 1) \mid i = 0, \dots, 8\}.$$

The abelian group $\mathcal{O}(W)^\times/K^\times$ is free of rank eight with generators $b_i = \dfrac{u-\lambda_i}{u+1}$, $i = 1, \dots, 8$. We deduce from [Liu 2002, Proposition 7.5.15] that

$$\mathcal{O}(U)^\times = \{f \in \mathcal{O}(W)^\times \mid f(0) = f(\infty)\}.$$

We conclude that $M_U := \mathcal{O}(U)^\times/K^\times$ is a free abelian group of rank seven.

In the following, we would like to describe the canonical tropicalization $\mathrm{Trop}(U)$ in the euclidean space $\mathbb{R}^7$ given by choosing a basis in M_U. This is rather complicated and so we compute the tropicalization $\mathrm{trop}_{x-1,y-1}(U)$ in $\mathbb{R}^2$ using the tropicalization map

$$\mathrm{trop}_{x-1,y-1}\colon U^{\mathrm{an}} \to \mathbb{R}^2, \quad q \mapsto \big(-\log|(x-1)(q)|, -\log|(y-1)(q)|\big).$$

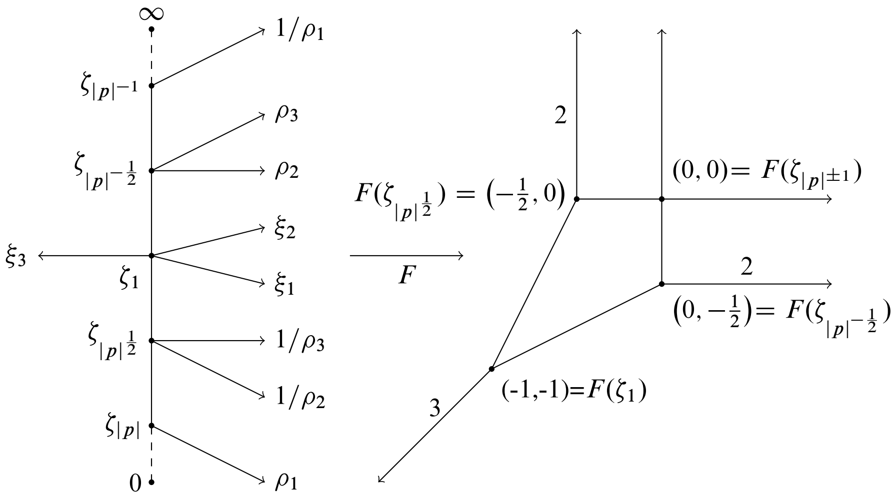

Figure 1. Minimal skeleton $S(W)$ and $\mathrm{Trop}_{x-1,y-1}(U)$.

This will be not enough for our purpose, but we will use the minimal skeleton $S(W)$ of W for the computation and as $S(W)$ also covers $\mathrm{Trop}(U)$, we get a very good picture of the latter. This method to compute tropicalizations is due to [Baker et al. 2013; 2016] and we will refer to these papers for details of the following construction. Skeleta are discussed in [Baker et al. 2013] and we refer to [Baker et al. 2013, Corollary 4.23] for existence and uniqueness of the minimal skeleton $S(W)$ of the smooth curve W. We recall that the skeleton $S(W)$ has a canonical retraction $\tau\colon (\mathbb{P}^1_K)^{\mathrm{an}} \to S(W)$ and hence $S(W)$ is a compact subset of $(\mathbb{P}^1_K)^{\mathrm{an}}$. Similarly as in the examples in [Baker et al. 2016, Section 2], we describe the minimal skeleton $S(W)$ and the tropicalization $\mathrm{Trop}_{x-1,y-1}(U) := \mathrm{trop}_{x-1,y-1}(U^{\mathrm{an}})$ in Figure 1[1]. Using [Gubler et al. 2016, Section 5], there is a map $F : S(W) \to \mathrm{Trop}_{x-1,y-1}(U)$ with $F \circ \tau = \mathrm{trop}_{x-1,y-1} \circ \varphi^{\mathrm{an}}$ such that F maps each segment (resp. leaf) of $S(W)$ by an integral $\mathbb{Q}$-affine map onto a segment (resp. leaf) of $\mathrm{Trop}_{x-1,y-1}(U)$. One computes easily that these affine maps are all integral $\mathbb{Q}$-affine isomorphisms. The polyhedral set $\mathrm{Trop}_{x-1,y-1}(U)$ carries a natural structure of a tropical cycle [Gubler 2013, Theorem 13.11]. All weights are one if not indicated otherwise in Figure 1. For $r > 0$, let $\zeta_r \in (\mathbb{P}^1_K)^{\mathrm{an}}$ be the supremum norm on the closed ball $\{|u| \leq r\}$, where u denotes our distinguished affine coordinate on $\mathbb{P}^1_K$.

Let
$$\widetilde{\Omega} := \left\{(x, y) \in \mathbb{R}^2 \mid x > -\tfrac{1}{2},\ y > -\tfrac{1}{2}\right\}$$
and $\Omega := \widetilde{\Omega} \cap \mathrm{Trop}_{x-1,y-1}(U)$. Let $H\colon N_{U,\mathbb{R}} \to \mathbb{R}^2$ be the canonical affine map with N_U the dual of M_U. Moreover, we have a canonical surjective map G from

[1]Thanks to Christian Vilsmeier for drawing the figure.

the minimal skeleton $S(W)$ onto the canonical tropicalization $\mathrm{Trop}(U)$ which is affine on every segment and every leaf of the minimal skeleton and such that $\mathrm{trop}_U \circ \varphi^{\mathrm{an}} = G \circ \tau$ for the canonical retraction τ onto the skeleton $S(W)$ (see [Gubler et al. 2016, Section 5]). Using our description of $\mathcal{O}(U)^\times$, we deduce that

$$G(\zeta_{|p|^{-1}}) = G \circ \tau(\infty) = \mathrm{trop}_U \circ \varphi^{\mathrm{an}}(\infty) \quad \text{and} \quad G(\zeta_{|p|}) = G \circ \tau(0) = \mathrm{trop}_U \circ \varphi^{\mathrm{an}}(0)$$

are equal as the right-hand sides are given in terms of units on U.

Using the fact that $F = H \circ G$, we conclude that the fibre of the surjective map $H\colon \mathrm{Trop}(U) \to \mathrm{trop}_{x-1,y-1}(U)$ over $(0, 0)$ is one single point and that H maps $\Omega' := H^{-1}(\widetilde{\Omega}) \cap \mathrm{Trop}(U)$ homeomorphically and isometrically with respect to lattice length onto Ω. We express this fact by saying that Ω' is unimodular to Ω. This is all we need in the following.

Now we consider the tropical chart (V, φ_U) around the ordinary double point $0 = (0, 0)$ of U, where $V := \mathrm{trop}_U^{-1}(\Omega)$. We consider the unique function $\tilde{\phi}$ on $\mathbb{R}^2$ which is linear on each quadrant with $\tilde{\phi}(1, 0) = 1$, $\tilde{\phi}(0, 1) = -1$ and which is zero in the third quadrant. Let ϕ be the restriction of $\tilde{\phi}$ to Ω. Let $\phi' := \phi \circ H$ as a real function on Ω'. It follows from the tropical Poincaré–Lelong formula in Theorem 0.1 that $d'd''[\phi]$ is the supercurrent on Ω given by $\delta_{\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)}$, where $\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)$ is the corner locus of ϕ. Similarly, $d'd''[\phi'] = \delta_{\phi' \cdot \mathrm{Trop}(U)}$. It is clear that $\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)$ is zero on $\Omega \setminus \{(0, 0)\}$ as ϕ is linear there. By definition, the multiplicity of $\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)$ in $(0, 0)$ is the sum of the four outgoing slopes, which is zero as well. We conclude that the corner locus $\phi \cdot \mathrm{Trop}_{x-1,y-1}(U)$ is zero. Since we have shown that Ω' is unimodular to Ω, we conclude that the corner locus $\phi' \cdot \mathrm{Trop}(U)$ is zero on Ω' as well.

We note that the corner locus $\tilde{\phi}' \cdot N_{U,\mathbb{R}}$ of the function $\tilde{\phi}' := \tilde{\phi} \circ H$ on $N_{U,\mathbb{R}}$ induces a δ-preform $\delta_{\tilde{\phi}' \cdot N_{U,\mathbb{R}}}$ on $N_{U,\mathbb{R}}$ which represents a δ-form α on the tropical chart (V, φ_U). We have $\alpha \in \mathrm{AZ}^{1,1}(V, \varphi_U) \subset P^{1,1}(V, \varphi_U)$. It follows from Proposition 1.14 that

$$\alpha|_\Omega = \delta_{\tilde{\phi}' \cdot N_{U,\mathbb{R}}} \wedge \delta_{\mathrm{Trop}(U)} = \delta_{\phi' \cdot \mathrm{Trop}(U)} = 0. \tag{4.22.1}$$

Now let us consider the open ball $B := \{|u| < |p|^{\frac{1}{2}}\}$ in $\mathbb{P}^1_K$. It is clear that $V'' := B \setminus \{\rho_1\}$ is mapped by F to $\Omega \cap \{x = 0\}$. The coordinate $w := u - \rho_1$ on $U'' := \mathbb{P}^1_K \setminus \{\rho_1, \infty\}$ induces an isomorphism $\varphi_{U''}\colon U'' \to \mathbb{G}_m$. Note that $(V'', \varphi_{U''})$ is a tropical chart. Indeed, we have $V'' = \mathrm{trop}_{U''}^{-1}(\Omega'')$ and $\mathrm{trop}_{U''}(V'') = \Omega''$ for $\Omega'' := \left(\frac{1}{2}, \infty\right)$. The tropical charts $(V'', \varphi_{U''})$ and (V, φ_U) are compatible with respect to the morphism φ and hence there is a canonical affine map $E : \mathbb{R} \to N_{U,\mathbb{R}}$ with $\mathrm{trop}_U \circ \varphi^{\mathrm{an}} = E \circ \mathrm{trop}_w$. We have

$$\varphi^*(\alpha)|_{\Omega''} = E^*(\delta_{\tilde{\phi}' \cdot N_{U,\mathbb{R}}})|_{\Omega''} = \delta_{E^*(\tilde{\phi}') \cdot N_{U'',\mathbb{R}}}|_{\Omega''}.$$

It is clear that $\phi'' := E^*(\tilde{\phi}')$ is a piecewise linear function on Ω'' which is identically zero on $\left(\frac{1}{2}, 1\right]$ and which has slope 1 on $[1, \infty)$. It follows that $\varphi^*(\alpha)|_{\Omega''} = \delta_1$. We conclude that $\alpha \in \mathrm{AZ}^{1,1}(V, \varphi_U)$ is an example with $\alpha|_\Omega = 0$, but $\alpha \neq 0$ as $\varphi^*(\alpha)|_{\Omega''} \neq 0$.

5. Integration of delta-forms

We keep the notation and the hypotheses from the previous section. Our goal is to introduce integration of generalized δ-forms of top degree with compact support. We proceed as in [Gubler 2016, 5.13]. A crucial ingredient in our definition of the integral is Lemma 5.5 which shows that the support of a generalized δ-form of high degree is always concentrated in points of high local dimensions. This allows us to compute the integral with a single chart of integration. We obtain a well defined integral for generalized δ-forms which satisfies a projection formula and the theorem of Stokes.

5.1. Let (V, φ_U) be a tropical chart of X. As before we write $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$ for some open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ and $\Omega = \widetilde{\Omega} \cap \mathrm{Trop}(U)$. Recall $n := \dim(X)$.

(i) An element α_U in $P(V, \varphi_U)$ is represented by a δ-preform $\tilde{\alpha}_U$ in $P(\widetilde{\Omega})$ and determines a δ-preform

$$\alpha_U|_\Omega = \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)} \in P(\Omega) \subseteq D(\Omega)$$

on Ω as in (4.5.1) which does neither depend on the choice of $\tilde{\alpha}_U$ nor on the choice of $\widetilde{\Omega}$. Often, it is convenient to use the notation $\alpha_U|_{\mathrm{Trop}(U)}$ for $\alpha_U|_\Omega$.

(ii) Given α_U in $P^{n,n}(V, \varphi_U)$ and an integral $\mathbb{R}$-affine polyhedral subset P of Ω such that $P \cap \mathrm{supp}(\alpha_U|_\Omega)$ is compact, we define

$$\int_P \alpha_U := \int_P \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)},$$

where the right-hand side is defined as in Remark 3.5. As usual, we extend the integral by 0 to the α_U of other bidegrees.

(iii) If α_U in $P^{n,n}(V, \varphi_U)$ and if the support of $\alpha_U|_\Omega$ is compact, then we can consider $\alpha_U|_\Omega$ as a δ-preform on $\mathrm{Trop}(U)$ with compact support and we write

$$\int_\Omega \alpha_U := \int_{|\mathrm{Trop}(U)|} \alpha_U|_\Omega,$$

again using Remark 3.5.

(iv) Given α_U in $P^{n-1,n}(V, \varphi_U)$ or $P^{n,n-1}(V, \varphi_U)$ and an integral $\mathbb{R}$-affine polyhedral subset P of Ω such that $P \cap \mathrm{supp}(\alpha_U|_\Omega)$ is compact, we define

$$\int_{\partial P} \alpha_U := \int_{\partial P} \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)},$$

where the right-hand side is defined in Remark 3.5. We extend the boundary integral by 0 to the α_U of other bidegrees.

5.2. In the next result, we look at functoriality of the above integrals with respect to a morphism $f: X' \to X$ of algebraic varieties over K. Let (V, φ_U) be a tropical chart of X. Let U' be a very affine open subset of X' with $f(U') \subseteq U$. Recall that there is a canonical integral Γ-affine morphism $F: N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ such that $\mathrm{trop}_U = F \circ \mathrm{trop}_{U'}$. Letting $V' := (f^{\mathrm{an}})^{-1}(V) \cap (U')^{\mathrm{an}}$, we deduce easily that $(V', \varphi_{U'})$ is a tropical chart of X' which is compatible with the tropical chart (V, φ_U). Let P be an integral $\mathbb{R}$-affine polyhedral subset of $\Omega := \mathrm{trop}_U(V)$ and let $Q := F^{-1}(P) \cap \mathrm{Trop}(U')$. We consider $\alpha_U \in P(V, \varphi_U)$ and its pull-back $f^*(\alpha_U) \in P(V', \varphi_{U'})$ (see Remark 4.5). In the following, we will use the degree of a morphism as introduced in 4.3.

Proposition 5.3. *Under the hypothesis of 5.2 and with $n := \dim(X)$, we assume additionally that $Q \cap \mathrm{supp}(f^*(\alpha_U)|_{\mathrm{Trop}(U')})$ is compact. Then the following properties hold*:

(i) *The set $P \cap \mathrm{supp}(\alpha_U|_{\mathrm{Trop}(U)})$ is compact.*

(ii) *If α_U is of bidegree (n, n), then*

$$\deg(f) \cdot \int_P \alpha_U = \int_Q f^*(\alpha_U). \tag{5.3.1}$$

(iii) *If α_U is of bidegree $(n-1, n)$ or $(n, n-1)$, then*

$$\deg(f) \cdot \int_{\partial P} \alpha_U = \int_{\partial Q} f^*(\alpha_U). \tag{5.3.2}$$

Proof. We choose an open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ with $\Omega = \widetilde{\Omega} \cap \mathrm{Trop}(U)$. We write $V' = \mathrm{trop}_U^{-1}(\widetilde{\Omega}')$ for some open subset $\widetilde{\Omega}'$ of $N'_{\mathbb{R}}$. Replacing $\widetilde{\Omega}'$ by $\widetilde{\Omega}' \cap F^{-1}(\widetilde{\Omega})$, we may assume that $\widetilde{\Omega}'$ is contained in $F^{-1}(\widetilde{\Omega})$. We write $\Omega' = \widetilde{\Omega}' \cap \mathrm{Trop}(U')$. If $\alpha_U \in P(V, \varphi_U)$ is represented by some element $\tilde{\alpha}_U \in P(\widetilde{\Omega})$, then $f^*(\alpha_U)$ is represented by the element $F^*(\tilde{\alpha}_U)$ in $P(\widetilde{\Omega}')$. We obtain from (4.3.1) and (2.14.1) that $P \cap \mathrm{supp}(\alpha_U|_{\mathrm{Trop}(U)})$ is compact. This proves (i).

If $\alpha_U \in P^{n,n}(V, \varphi_U)$, then we obtain

$$\deg(f) \int_P \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)} = \int_{F^{-1}(P)} F^* \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U')} \tag{5.3.3}$$

if we combine (4.3.1) with the projection formula (2.14.1). By definition, (5.3.1) is a direct consequence of (5.3.3). Equation (5.3.2) is derived in the same way from (4.3.1) and (2.14.2) □

Let W denote an open subset of X^{an}. Note that a generalized δ-form on W is locally given by elements of $P(V, \varphi_U)$ for tropical charts (V, φ_U). The following corollary will be crucial for the definition of the integral of generalized δ-forms.

Corollary 5.4. *We consider very affine open subsets $U' \subseteq U$ in X. Let $\alpha = \mathrm{trop}_U^*(\alpha_U)$ for some $\alpha_U \in P(U^{\mathrm{an}}, \varphi_U)$. Then there is a unique $\alpha_{U'} \in P((U')^{\mathrm{an}}, \varphi_{U'})$ with $\alpha|_{(U')^{\mathrm{an}}} = \mathrm{trop}_{U'}^*(\alpha_{U'})$. If α is of bidegree (n,n) and has compact support in $(U')^{\mathrm{an}}$, then we have*

$$\int_{|\mathrm{Trop}(U)|} \alpha_U = \int_{|\mathrm{Trop}(U')|} \alpha_{U'}.$$

Proof. Let $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ be the canonical affine map with $\mathrm{trop}_U = F \circ \mathrm{trop}_{U'}$ on $(U')^{\mathrm{an}}$. Then $\alpha|_{(U')^{\mathrm{an}}}$ is given by $\alpha_{U'} := F^*(\alpha_U) \in P((U')^{\mathrm{an}}, \varphi_{U'})$. This proves existence, and uniqueness follows from Proposition 4.18. To prove the last claim, we use (5.3.1) for $f = \mathrm{id}$, $P = |\mathrm{Trop}(U)|$ and $Q = |\mathrm{Trop}(U')|$. □

In the following result, we need the local invariant $d(x)$ for $x \in X^{\mathrm{an}}$ [Gubler 2016, 4.2]. This invariant was introduced in [Berkovich 1990, Chapter 9] and was extensively studied in [Ducros 2012]. We note that $d(x) \leq m$ if x belongs to a Zariski closed subset of dimension m [Berkovich 1990, Proposition 9.1.3].

Lemma 5.5. *Let W be an open subset of X^{an} and let $\alpha \in P^{p,q}(W)$. If $x \in W$ satisfies $d(x) < \max(p,q)$, then $x \notin \mathrm{supp}(\alpha)$.*

Proof. The proof relies on a result of Ducros [2012, théorème 3.4] which says roughly that in a sufficiently small analytic neighbourhood of x, the dimension of the tropical variety is bounded by $d(x)$. The details are as follows. We choose a tropical chart (V, φ_U) around x such that α is induced by a δ-preform $\sum_{i \in I} \alpha_i \wedge \delta_{C_i}$ on $N_{U,\mathbb{R}}$. By linearity, we may assume that α is induced by $\alpha_1 \wedge \delta_{C_1}$ for a superform α_1 in $A^{p',q'}(N_{U,\mathbb{R}})$ and a tropical cycle C_1 of codimension $c := p - p' = q - q' \geq 0$ in $N_{U,\mathbb{R}}$. By definition of a tropical chart, there is an open subset $\widetilde{\Omega}$ of $N_{U,\mathbb{R}}$ such that $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$. By the mentioned result of Ducros (see also [Gubler 2016, Proposition 4.14]), there is a compact neighbourhood V_x of x in V such that $\mathrm{trop}_U(V_x)$ is a polyhedral subset of $N_{U,\mathbb{R}}$ with

$$\dim(\mathrm{trop}_U(V_x)) \leq d(x) < \max(p,q). \tag{5.5.1}$$

We will show that $\alpha|_{V_x} = 0$. Let $f : X' \to X$ be a morphism of algebraic varieties over K and $(V', \varphi_{U'})$ a tropical chart of X' with $f^{\mathrm{an}}(V') \subseteq V_x$. By definition, we have $V' = \mathrm{trop}_{U'}^{-1}(\Omega')$ for the open subset $\Omega' := \mathrm{trop}_{U'}(V')$ of $\mathrm{Trop}(U')$. In this situation, we get a commutative diagram

$$\begin{tikzcd}
V' \arrow[r, hook] \arrow[d, "f^{\mathrm{an}}|_{V'}"] & (U')^{\mathrm{an}} \arrow[r, "\varphi_{U'}"] \arrow[d, "f^{\mathrm{an}}|_{(U')^{\mathrm{an}}}"] & T_{U'} \arrow[r] \arrow[d] & N_{U',\mathbb{R}} \arrow[d, "F"] \\
V \arrow[r, hook] & U^{\mathrm{an}} \arrow[r, "\varphi_U"] & T_U \arrow[r] & N_{U,\mathbb{R}}
\end{tikzcd} \tag{5.5.2}$$

as before. To prove the claim, it is enough to show that

$$f^*(\alpha)|_{\Omega'} := F^*(\alpha_1) \wedge \delta_{F^*(C_1)}|_{\Omega'} = 0,$$

or equivalently

$$F^*(\alpha_1) \wedge \delta_{C'} = 0 \in D(\Omega') \tag{5.5.3}$$

for the tropical cycle $C' := F^*(C_1) \cdot \mathrm{Trop}(U')$ of codimension c in $\mathrm{Trop}(U')$. We note that $\Omega' = \mathrm{trop}_{U'}(V') \subseteq F^{-1}(\mathrm{trop}_U(V_x))$. Let Δ' be a maximal polyhedron from C' with $\Delta' \cap \Omega' \neq \varnothing$. Then $F(\Delta' \cap \Omega') \subseteq \mathrm{trop}_U(V_x)$ and hence

$$F^*(\alpha_1)|_{\Delta' \cap \Omega'} = (F|_{\Delta' \cap \Omega'})^*(\alpha_1|_{F(\Delta') \cap \mathrm{trop}_U(V_x)}). \tag{5.5.4}$$

We will show below that

$$\mathrm{codim}\big(F(\Delta' \cap \Omega'), \mathrm{trop}_U(V_x)\big) \geq c. \tag{5.5.5}$$

Then (5.5.3) follows from (5.5.4) by using (5.5.5) and (5.5.1). This proves $x \notin \mathrm{supp}(\alpha)$.

It remains to prove (5.5.5). By definition of the stable tropical intersection product in Remark 1.4(ii), there are maximal polyhedra Δ'_0 and Δ'_1 of $\mathrm{Trop}(U')$ and $F^*(C_1)$, respectively, such that $\Delta' = \Delta'_0 \cap \Delta'_1$. Moreover, $N_{\Delta'_0,\mathbb{R}}$ and $N_{\Delta'_1,\mathbb{R}}$ intersect transversely in $N_{U',\mathbb{R}}$ which means that

$$N_{\Delta'_0,\mathbb{R}} + N_{\Delta'_1,\mathbb{R}} = N_{U',\mathbb{R}}. \tag{5.5.6}$$

Similarly, the definition of pull-back of tropical cycles in Remark 1.4(v) shows that there is a maximal polyhedron Δ_1 of C_1 with $F(\Delta'_1) \subseteq \Delta_1$ and such that

$$N_{\Delta_1,\mathbb{R}} + \mathbb{L}_F(N_{U',\mathbb{R}}) = N_{U,\mathbb{R}}. \tag{5.5.7}$$

It follows from (5.5.6) and (5.5.7) that $\mathbb{L}_F(N_{\Delta'_0,\mathbb{R}})$ intersects $N_{\Delta_1,\mathbb{R}}$ transversely in $N_{U,\mathbb{R}}$. Since the codimension is decreasing under a surjective linear map, we easily get

$$\mathbb{L}_F(N_{\Delta',\mathbb{R}}) = \mathbb{L}_F(N_{\Delta'_0,\mathbb{R}} \cap N_{\Delta'_1,\mathbb{R}}) = \mathbb{L}_F(N_{\Delta'_0,\mathbb{R}}) \cap N_{\Delta_1,\mathbb{R}}$$

and hence

$$\mathrm{codim}\big(F(\Delta' \cap \Omega'), F(\Delta'_0 \cap \Omega')\big) = \mathrm{codim}\big(\mathbb{L}_F(N_{\Delta',\mathbb{R}}), \mathbb{L}_F(N_{\Delta'_0,\mathbb{R}})\big) = c$$

by transversality. Using $F(\Omega') \subseteq \mathrm{trop}_U(V_x)$, this proves (5.5.5). □

Corollary 5.6. *Let W be an open subset of X^{an} and let U be an open subset of X. If $\alpha \in P^{p,q}(W)$ with $\dim(X \setminus U) < \max(p,q)$, then $\mathrm{supp}(\alpha) \subseteq W \cap U^{\mathrm{an}}$.*

Proof. If $x \in W \setminus U^{\mathrm{an}}$, then the assumptions yield $d(x) \leq \dim(X \setminus U) < \max(p,q)$ and hence the claim follows from Lemma 5.5. □

Proposition 5.7. *Let $\alpha \in P^{p,q}(X^{\mathrm{an}})$ with compact support in the open subset W of X^{an}.*

(a) *There is a nonempty tropical chart (V, φ_U) with* $\mathrm{supp}(\alpha) \cap U^{\mathrm{an}} \subseteq V \subseteq U^{\mathrm{an}} \cap W$ *and $\alpha_U \in P^{p,q}(U^{\mathrm{an}}, \varphi_U)$ such that $\alpha = \mathrm{trop}_U^*(\alpha_U)$ on U^{an}.*

(b) *Given U, the element α_U in* (a) *is unique.*

(c) *If α is a δ-form, then we may choose $\alpha_U \in \mathrm{AZ}^{p,q}(U^{\mathrm{an}}, \varphi_U)$.*

(d) *If* $\max(p,q) = \dim(X)$, *then any nonempty very affine open subset U of X with $\alpha|_{U^{\mathrm{an}}} = \mathrm{trop}_U^*(\alpha_U)$ for some $\alpha_U \in P^{p,q}(U^{\mathrm{an}}, \varphi_U)$ satisfies automatically* $\mathrm{supp}(\alpha) \subseteq U^{\mathrm{an}}$. *Moreover, α_U has always compact support in* $\mathrm{Trop}(U)$.

Explicitly, if $\mathrm{supp}(\alpha)$ *is covered by nonempty tropical charts $(V_i, \varphi_{U_i})_{i=1,\dots,s}$ in W and if α is given on V_i by $\alpha_i \in P^{p,q}(V_i, \varphi_{U_i})$, then any nonempty very affine open subset U of $U_1 \cap \dots \cap U_s$ and $V = (V_1 \cup \dots \cup V_s) \cap U^{\mathrm{an}}$ fit in (a).*

Proof. Since the support of α is a compact subset of W, it is covered by tropical charts $(V_i, \varphi_{U_i})_{i=1,\dots,s}$ describing α as above. Compactness again shows that for any $i = 1, \dots, s$, there is a relatively compact open subset Ω_i' of Ω_i with corresponding open subset $V_i' := \mathrm{trop}_{U_i}^{-1}(\Omega_i')$ of V_i such that $\mathrm{supp}(\alpha) \subseteq V_1' \cup \dots \cup V_s'$. Let us consider a nonempty very affine open subset U of $U_1 \cap \dots \cap U_s$ of X and the open subsets

$$V' := U^{\mathrm{an}} \cap \bigcup_{i=1}^{s} V_i' \subseteq V := U^{\mathrm{an}} \cap \bigcup_{i=1}^{s} V_i$$

of $W \cap U^{\mathrm{an}}$. We have to show that V and U satisfy (a). Let $F_i : N_{U,\mathbb{R}} \to N_{U_i,\mathbb{R}}$ be the canonical integral Γ-affine map induced by the inclusion $U \subseteq U_i$ (see 4.3). Then the open subsets

$$\Omega' := \mathrm{Trop}(U) \cap \bigcup_{i=1}^{s} F_i^{-1}(\Omega_i') \subseteq \Omega := \mathrm{Trop}(U) \cap \bigcup_{i=1}^{s} F_i^{-1}(\Omega_i)$$

of $\mathrm{Trop}(U)$ satisfy $V = \mathrm{trop}_U^{-1}(\Omega)$ and $V' = \mathrm{trop}_U^{-1}(\Omega')$ which means that (V', φ_U) and (V, φ_U) are compatible tropical charts of X contained in W. Note that the tropical chart $(V_i \cap U^{\mathrm{an}}, \varphi_U)$ is compatible with (V_i, φ_{U_i}) and hence α is given on $(V_i \cap U^{\mathrm{an}}, \varphi_U)$ by $\alpha_i' = \alpha_i|_{V_i \cap U^{\mathrm{an}}} \in P(V_i \cap U^{\mathrm{an}}, \varphi_U)$. Using that Ω_i' is relatively compact in Ω_i, we deduce that the closure S of Ω' in $\mathrm{Trop}(U)$ is contained in Ω. We set $V'' := \mathrm{trop}_U^{-1}(\mathrm{Trop}(U) \setminus S)$ leading to the tropical chart (V'', φ_U). Since α has compact support in W, we may view α as an element of $P^{p,q}(X^{\mathrm{an}})$. By construction, we have $\mathrm{supp}(\alpha) \cap U^{\mathrm{an}} \subseteq V'$. Using that V' and V'' are disjoint, we deduce that α is given on the tropical chart (V'', φ_U) by $0 \in P^{p,q}(V'', \varphi_U)$. We note that the tropical charts $(V_i \cap U^{\mathrm{an}}, \varphi_U)_{i=1,\dots,s}$ and (V', φ_U) cover U^{an} and hence we may apply the glueing from Proposition 4.12 to get the desired $\alpha_U \in P^{p,q}(U^{\mathrm{an}}, \varphi_U)$ from (a).

Uniqueness in (b) follows from Proposition 4.18. If $\alpha \in B_c^{p,q}(X^{\mathrm{an}})$, then we may choose $\alpha_i \in \mathrm{AZ}^{p,q}(V_i, \varphi_{U_i})$ and hence we get (c).

If $\max(p,q) = \dim(X)$, then (d) follows from Corollary 5.6 and Proposition 4.21. □

Definition 5.8. Let W be an open subset of X^{an} and let $\alpha \in P_c^{n,n}(W)$, where $n := \dim(X)$. We may view α as a generalized δ-form on X^{an} with compact support contained in W. A nonempty very affine open subset U of X is called a *very affine chart of integration for* α if $\alpha|_{U^{\mathrm{an}}} = \mathrm{trop}_U^*(\alpha_U)$ for some $\alpha_U \in P^{n,n}(U^{\mathrm{an}}, \varphi_U)$. By Proposition 5.7, a chart of integration exists, and α_U is unique and has compact support in $\mathrm{Trop}(U)$. We define *the integral of* α *over* W by

$$\int_W \alpha := \int_{|\mathrm{Trop}(U)|} \alpha_U ,$$

where the right-hand side is defined in 5.1. As usual, we extend the integral by 0 to generalized δ-forms of other bidegrees.

Proposition 5.9. *Let* W *be an open subset of* X^{an} *and* $\alpha \in P_c^{n,n}(W)$ *as above.*

(i) *If* $\mathrm{supp}(\alpha)$ *is covered by finitely many nonempty tropical charts* (V_i, φ_{U_i}) *such that* α *is given on any* V_i *by* $\alpha_i \in P^{n,n}(V_i, \varphi_{U_i})$*, then* $U := \bigcap_i U_i$ *is a very affine chart of integration for* α*.*

(ii) *The definition of the integral* $\int_W \alpha$ *given in Definition 5.8 does not depend on the choice of the very affine chart of integration for* α*.*

(iii) *The integral defines a linear map* $\int_W : P_c^{n,n}(W) \to \mathbb{R}$*.*

(iv) *If* $f : X' \to X$ *is a proper morphism of degree* $\deg(f)$ *then the projection formula*

$$\deg(f) \int_W \alpha = \int_{(f^{\mathrm{an}})^{-1}(W)} f^*\alpha \tag{5.9.1}$$

holds for all $\alpha \in P_c^{n,n}(W)$*.*

Proof. The explicit description of U in Proposition 5.7 proves (i). We show (ii). Let U be a very affine chart of integration for α. Then every nonempty very affine open subset U' of U is a very affine chart of integration and it is enough to show that U' leads to the same integral. By uniqueness in Proposition 5.7, the pull-back of α_U with respect to the canonical affine map $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ is equal to $\alpha_{U'}$ and the claim follows from Corollary 5.4.

Claim (iii) is a direct consequence of our definitions. To prove (iv), we may assume that $\dim(X') = \dim(X) = n$. We choose a very affine chart of integration U for α and a nonempty very affine open subset U' of X' with $f(U') \subseteq U$. Note that $f^*(\alpha)$ is given on $(U')^{\mathrm{an}}$ by $f^*(\alpha_U) \in P^{n,n}(U', \varphi_{U'})$ constructed in Remark 4.5.

Since f^{an} is proper as well, the support of $f^*(\alpha)$ is compact. We conclude that U' is a very affine chart of integration for $f^*(\alpha)$ and

$$\int_{(f^{\mathrm{an}})^{-1}(W)} f^*\alpha = \int_{|\mathrm{Trop}(U')|} f^*(\alpha_U).$$

The projection formula in (iv) is now a direct consequence of (5.3.1). □

In our setting, we have the following version of the *theorem of Stokes*.

Theorem 5.10. *For $\alpha \in B_c^{2n-1}(X)$ we have*

$$\int_{X^{\mathrm{an}}} d'\alpha = \int_{X^{\mathrm{an}}} d''\alpha = 0.$$

Proof. By Proposition 5.7, there is a nonempty very affine open subset U of X such that $\mathrm{supp}(\alpha) \subseteq U^{\mathrm{an}}$ and $\alpha_U \in \mathrm{AZ}_c^{2n-1}(U^{\mathrm{an}}, \varphi_U)$ such that $\alpha|_{U^{\mathrm{an}}} = \mathrm{trop}_U^*(\alpha_U)$. Then U is a chart of integration for $d'\alpha$ and $d''\alpha$ using $d'\alpha_U$ and $d''\alpha_U$ on the tropical side for integration. The claim follows from Stokes' formula for δ-preforms on $\mathrm{Trop}(U)$ (see Proposition 3.6) by observing that boundary integrals $\int_{\partial|\mathrm{Trop}(U)|}$ vanish as $\mathrm{Trop}(U)$ satisfies the balancing condition. □

6. Delta-currents

In this section, we define δ-currents on an open subset W of X^{an} for an n-dimensional algebraic variety X over K. We proceed analogously to the case of manifolds in differential geometry, endowing some specific subspaces of the space $B_c(W)$ of δ-forms with compact support in W with the structure of a locally convex topological vector space. Then we define a δ-current as a linear functional on $B_c(W)$ with continuous restrictions to all these subspaces.

6.1. Let (V, φ_U) be a tropical chart of X with $V \subseteq W$ and let $\Omega := \mathrm{trop}_U(V)$ be as usual. We recall from Definition 4.9 that an element $\beta \in \mathrm{AZ}(V, \varphi_U)$ has the form

$$\beta = \sum_{j \in J} \alpha_j \wedge \omega_j \in P(V, \varphi_U) \tag{6.1.1}$$

for a finite set J, $\alpha_j \in A(\Omega)$ and $\omega_j \in Z(V, \varphi_U)$.

Now we *fix* the family $\omega_J := (\omega_j)_{j \in J}$ and define $\mathrm{AZ}(V, \varphi_U, \omega_J)$ as the subspace of $\mathrm{AZ}(V, \varphi_U)$ given by all elements β with a decomposition (6.1.1) for suitable $\alpha_j \in A(\Omega)$. For every $s \in \mathbb{N}$ and every compact subset C of Ω, we have the usual seminorms $p_{C,s}$ on $A(\Omega)$ measuring uniform convergence on C of the derivatives of the coefficients of the superforms up to order s (see for example [Dieudonné 1972, (17.3.1)]). We get seminorms p_{C,s,ω_J} on $\mathrm{AZ}(V, \varphi_U, \omega_J)$ by defining

$$p_{C,s,\omega_J}(\beta) := \inf\left\{ \max_{j \in J} p_{C,s}(\alpha_j) \;\middle|\; \beta = \sum_{j \in J} \alpha_j \wedge \omega_j,\ \alpha_j \in A(\Omega) \right\}.$$

Letting $s \in \mathbb{N}$ and the compact subset C of W vary, we get a structure of a locally convex topological vector space on $\mathrm{AZ}(V, \varphi_U, \omega_J)$.

6.2. A δ-form β on W is given by a covering $(V_i, \varphi_{U_i})_{i\in I}$ of W by tropical charts and by $\beta_i \in \mathrm{AZ}(V_i, \varphi_{U_i})$ such that $\beta|_{V_i} = \mathrm{trop}_{U_i}^*(\beta_i)$ for every $i \in I$. Using 6.1, we have a finite tuple ω_{J_i} of elements in $\mathrm{AZ}(V_i, \varphi_{U_i})$ such that $\beta_i \in \mathrm{AZ}(V_i, \varphi_{U_i}, \omega_{J_i})$ for every $i \in I$. Now we fix the covering by tropical charts and all ω_{J_i} and we define $B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ to be the subspace of $B(W)$ given by the elements β such that $\beta|_{V_i} = \mathrm{trop}_{U_i}^*(\beta_i)$ for some $\beta_i \in \mathrm{AZ}(V_i, \varphi_{U_i}, \omega_{J_i})$ and for every $i \in I$. We endow $B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ with the coarsest structure of a locally convex topological vector space such that the canonical linear maps

$$B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I) \to \mathrm{AZ}(V_i, \varphi_{U_i}, \omega_{J_i})$$

are continuous for every $i \in I$. An element $\beta \in B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ given as above is mapped to β_i, which is well defined by Proposition 4.18.

For a compact subset C of W, let $B_C(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ be the subspace of $B(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ given by the δ-forms with compact support in C. We endow it with the induced structure of a locally convex topological vector space.

Definition 6.3. A *δ-current* on W is a real linear functional T on $B_c(W)$ such that the restriction of T to $B_C(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ is continuous for every compact subset C of W, for every covering $(V_i, \varphi_{U_i})_{i\in I}$ of W by tropical charts and for every finite tuple ω_{J_i} of elements in $Z(V_i, \varphi_{U_i})$. We denote the space of δ-currents on W by $E(W)$. A δ-current is called *symmetric* (resp. *antisymmetric*) if it vanishes on the subspace of antisymmetric (resp. symmetric) δ-forms in $B_c(W)$.

6.4. Let W be an open subset of X^{an}. Using that $B_c(W) = \bigoplus_{p,q} B_c^{p,q}(W)$ is bigraded, we get $E(W) = \bigoplus_{r,s} E_{r,s}(W)$ as a bigraded $\mathbb{R}$-vector space, where a δ-current in $E_{r,s}(W)$ acts trivially on every $B_c^{p,q}(W)$ with $(p,q) \neq (r,s)$. We set $E^{p,q}(W) := E_{n-p,n-q}(W)$. The definition of δ-currents in 6.3 is local and hence $E_{\cdot,\cdot}$ is a sheaf of bigraded real vector spaces on X^{an}. This follows from standard arguments using partition of unity if W is paracompact, and follows in general from the fact that every compact subset C of W has a paracompact open neighbourhood in W by [Chambert-Loir and Ducros 2012, lemme (2.1.6)]. The argument is similar to that in [Chambert-Loir and Ducros 2012, lemme (4.2.5)] and we leave the details to the reader.

There is a product

$$B^{p,q}(W) \times E^{p',q'}(W) \to E^{p+p',q+q'}(W), \quad (\alpha, T) \mapsto \alpha \wedge T \tag{6.4.1}$$

such that

$$\langle \alpha \wedge T, \beta \rangle = (-1)^{(p+q)(p'+q')} T(\alpha \wedge \beta)$$

for each $\beta \in B_c^{n-p-p',n-q-q'}(W)$.

Proposition 6.5. *Let U be a Zariski open subset of X and let W be an open subset of X^{an}. If $\operatorname{codim}(X \setminus U, X) > \min(p,q)$, then $E^{p,q}(W \cap U^{\mathrm{an}}) = E^{p,q}(W)$.*

Proof. Corollary 5.6 shows that every δ-form on W of bidegree $(n-p, n-q)$ has support in $W \cap U^{\mathrm{an}}$. We conclude that every δ-current T in $E^{p,q}(W \cap U^{\mathrm{an}})$ is a linear functional on $B_c^{n-p,n-q}(W)$. It remains to prove that the restriction of T to $B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ is continuous for every compact subset C of W, for every covering $(V_i, \varphi_{U_i})_{i \in I}$ of W by tropical charts and for every finite tuple ω_{J_i} of elements in $Z(V_i, \varphi_{U_i})$.

We consider the set S of $x \in W$ for which there is $i \in I$ and a compact neighbourhood V_x of x in $W \cap V_i$ with

$$\dim(\operatorname{trop}_{U_i}(V_x)) < \max(n-p, n-q). \tag{6.5.1}$$

Note that $\operatorname{trop}_{U_i}(V_x)$ is a polyhedral subset of $N_{U,\mathbb{R}}$ by [Ducros 2012, théorème 3.2]. Obviously, S is an open subset of W. It follows from the proof of Lemma 5.5 that S is disjoint from the support of any δ-form in $B^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$. We conclude that

$$B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I) = B_D^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I) \tag{6.5.2}$$

for the compact subset $D := C \setminus S$ of C. By the proof of Lemma 5.5 again, every $x \in X^{\mathrm{an}} \setminus U^{\mathrm{an}}$ satisfies

$$d(x) \leq \dim(X \setminus U) < \max(n-p, n-q)$$

and has a compact neighbourhood V_x contained in some V_i and satisfying (6.5.1). This proves $D \subseteq W \cap U^{\mathrm{an}}$. Using (6.5.2) and $T \in E^{p,q}(W \cap U^{\mathrm{an}})$, we get the continuity of the restriction of T to $B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$. □

Proposition 6.6. *A generalized δ-form $\eta \in P^{p,q}(W)$ determines a δ-current $[\eta] \in E^{p,q}(W)$ such that*

$$\langle [\eta], \beta \rangle = \int_W \eta \wedge \beta$$

for each $\beta \in B_c^{n-p,n-q}(W)$.

Proof. We have to show that the restriction of $[\eta]$ to every subspace

$$B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$$

as in Definition 6.3 is continuous. By passing to a refinement of the covering by tropical charts, we may assume that η is given on V_i by $\eta_i \in P^{p,q}(V_i, \varphi_{U_i})$ for every $i \in I$. Since C is compact, there is a finite subset I_0 of I such that $\bigcup_{i \in I_0} V_i$

covers C. By Proposition 5.9(i), we may use $U := \bigcap_{i \in I_0} U_i$ as a very affine chart of integration for any $\gamma \in B_C^{n,n}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$.

Similarly to the proof of Proposition 6.5, we consider the set S of $x \in W$ for which there is an $i \in I$ and a compact neighbourhood V_x of x in $W \cap V_i$ with

$$\dim(\mathrm{trop}_{U_i}(V_x)) < n. \tag{6.6.1}$$

It follows again from the proof of Lemma 5.5 that the open subset S of W is disjoint from the support of $\eta \wedge \beta \in P^{n,n}(W)$ for any $\beta \in B^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ and that the compact set $D := C \setminus S$ is contained in $W \cap U^{\mathrm{an}}$.

By definition, $\beta \in B_C^{n-p,n-q}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ is given on V_i by $\beta_i = \sum_{j \in J_i} \alpha_{ij} \wedge \omega_{ij}$ with $\alpha_{ij} \in A(\Omega_i)$ and $\omega_{ij} \in Z(V_i, \varphi_{U_i})$, where $\Omega_i := \mathrm{trop}_{U_i}(V_i)$. For $i \in I_0$, let $F_i : N_{U,\mathbb{R}} \to N_{U_i,\mathbb{R}}$ be the canonical affine map with $\mathrm{trop}_{U_i} = F_i \circ \mathrm{trop}_U$ on U^{an} and let $\Omega_i' := F_i^{-1}(\Omega_i) \cap \mathrm{Trop}(U) = \mathrm{trop}_U(V_i \cap U^{\mathrm{an}})$. The definition of $\int_W \eta \wedge \beta$ uses that $\eta \wedge \beta$ is given on U^{an} by a unique $\gamma_U \in P^{n,n}(U^{\mathrm{an}}, \varphi_U)$ (see Definition 5.8). Moreover, Proposition 5.7 shows that γ_U has compact support in $\bigcup_{i \in I_0} \Omega_i'$ and that γ_U is characterized by the restrictions

$$\gamma_U|_{V_i \cap U^{\mathrm{an}}} = \sum_{j \in J_i} \eta_i \wedge \alpha_{ij} \wedge \omega_{ij}|_{V_i \cap U^{\mathrm{an}}} \in P^{n,n}(V_i, \varphi_{U_i})$$

for every $i \in I_0$. Recall that D is a compact subset of $W \cap U^{\mathrm{an}}$ with $\mathrm{supp}(\gamma) \subseteq D$. By Proposition 4.21, $\mathrm{trop}_U(D)$ is a compact set of $\mathrm{Trop}(U)$ containing the support of γ_U. Then there is an integral $\mathbb{R}$-affine polyhedral subset P of $\mathrm{Trop}(U)$ with $\mathrm{trop}_U(D) \subseteq P$ and hence we have

$$\langle [\eta], \beta \rangle = \int_{X^{\mathrm{an}}} \eta \wedge \beta = \int_{|\mathrm{Trop}(U)|} \gamma_U = \int_P \gamma_U. \tag{6.6.2}$$

We use now that P is independent of the choice of $\beta \in B_C^{n,n}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$. If all the α_{ij} are small with respect to the supremum-norm (of the coefficients), then a partition of unity argument on $\mathrm{Trop}(U)$ shows that (6.6.2) is small, proving the desired continuity. □

Remark 6.7. The maps $P^{p,q}(W) \to E^{p,q}(W)$ induce a map of sheaves $P^{p,q} \to E^{p,q}$, $\alpha \mapsto [\alpha]$ which fits into a commutative diagram

$$\begin{tikzcd} A^{p,q} \arrow[r, hook] \arrow[d, hook, "{[\cdot]_D}"] & B^{p,q} \arrow[r, hook] \arrow[d, "{[\cdot]}"] & P^{p,q} \arrow[dl, "{[\cdot]}"] \\ D^{p,q} & E^{p,q} \arrow[l] & \end{tikzcd} \tag{6.7.1}$$

There is an induced map $P^{p,q}(W) \to D^{p,q}(W)$. For $\beta \in P^{p,q}(W)$, we denote the associated current in $D^{p,q}(W)$ by $[\beta]_D$.

There is no a priori reason that the canonical map from δ-forms to currents or δ-currents is injective. However, we have the following functorial criterion:

Proposition 6.8. *Let W be an open subset of X^{an} and let $\alpha, \beta \in P^{p,q}(W)$. Then $\alpha = \beta$ if and only if $[f^*(\alpha)]_D = [f^*(\beta)]_D \in D^{p,q}(W')$ for all morphisms $f : X' \to X$ from algebraic varieties X' over K and for all open subsets W' of $(X')^{\mathrm{an}}$ with $f(W') \subseteq W$.*

Proof. If $\alpha = \beta$, then all pull-backs and also their associated currents are the same. Conversely, we assume that the associated currents of all pull-backs are the same for α and β. There is an open covering $(V_i)_{i \in I}$ of X^{an} by tropical charts (V_i, φ_{U_i}) such that α, β are given on V_i by $\alpha_i, \beta_i \in P^{p,q}(V_i, \varphi_{U_i})$. Let $f : X' \to X$ be a morphism of varieties over K and let $(V', \varphi_{U'})$ be a tropical chart of X' which is compatible with (V_i, φ_{U_i}). Let Ω' denote the open subset $\mathrm{trop}_{U'}(V')$ of $\mathrm{Trop}(U')$. It follows from our definitions that $\alpha_i = \beta_i$ in $P(V_i, \varphi_{U_i})$ if we show $f^*(\alpha_i)|_{\Omega'} = f^*(\beta_i)|_{\Omega'} \in D^{p,q}(\Omega')$ for all morphisms f and all charts $(V', \varphi_{U'})$ compatible with (V_i, φ_{U_i}). By assumption, we have $[f^*(\alpha)]_D = [f^*(\beta)]_D$ in $D^{p,q}(V')$. We conclude that $f^*(\alpha_i)|_{\Omega'} = f^*(\beta_i)|_{\Omega'} \in P^{p,q}(\Omega') \subseteq D^{p,q}(\Omega')$ and get $\alpha_i = \beta_i \in P(V_i, \varphi_{U_i})$ proving the claim. $\square$

6.9. As usual, we define the linear differential operators $d' : E^{p,q}(W) \to E^{p+1,q}(W)$ and $d'' : E^{p,q} \to E^{p,q+1}(W)$ by

$$\langle d'T, \beta \rangle := (-1)^{p+q+1} \langle T, d'\beta \rangle, \quad \langle d''T, \beta \rangle := (-1)^{p+q+1} \langle T, d''\beta \rangle.$$

Note that d' and d'' induce continuous linear maps on the locally convex topological vector spaces introduced in 6.2 and hence it is easy to check that d' and d'' are well-defined on δ-currents. Moreover, the natural maps from Remark 6.7 fit into commutative diagrams

$$\begin{tikzcd} B^{p,q} \arrow[r, "{[\,\cdot\,]}"] \arrow[d, "d'"] & E^{p,q} \arrow[d, "d'"] \\ B^{p+1,q} \arrow[r, "{[\,\cdot\,]}"] & E^{p+1,q} \end{tikzcd} \qquad \begin{tikzcd} B^{p,q} \arrow[r, "{[\,\cdot\,]}"] \arrow[d, "d''"] & E^{p,q} \arrow[d, "d''"] \\ B^{p,q+1} \arrow[r, "{[\,\cdot\,]}"] & E^{p,q+1} \end{tikzcd} \tag{6.9.1}$$

of sheaves. As usual, we define $d := d' + d''$ also on E.

6.10. If $f : X' \to X$ is a proper morphism of algebraic varieties, then we get a *push-forward* $f_* : E_{r,s}(f^{-1}(W)) \to E_{r,s}(W)$ as follows: For $T' \in E_{r,s}(f^{-1}(W))$, the push-forward is the δ-current on W given by

$$\langle f_*(T), \beta \rangle := \langle T, f^*(\beta) \rangle$$

for $\beta \in B_c^{r,s}(W)$. It is easy to see that pull-back of δ-forms induces continuous linear maps between appropriate locally convex topological vector spaces defined in 6.2 and hence the proper push-forward of δ-currents is well defined.

Example 6.11. In Definition 5.8, we introduced $\int_{X^{\mathrm{an}}} \beta$ for $\beta \in P_c^{n,n}(X^{\mathrm{an}})$. Setting $\langle \delta_X, \beta \rangle := \int_{X^{\mathrm{an}}} \beta$, we get the δ-current $\delta_X = [1] \in E^{0,0}(X^{\mathrm{an}})$. We call it the *$\delta$-current of integration* along X. Using linearity in the components and 6.10, we get a δ-current of integration δ_Z for every algebraic cycle Z on X.

Proposition 6.12. *Let $f : X' \to X$ be a proper morphism of algebraic varieties and let Z' be a p-dimensional algebraic cycle on X'. Then we have the equality $f_* \delta_{Z'} = \delta_{f_* Z'}$ in $E_{p,p}(X^{\mathrm{an}})$.*

Proof. This is a direct consequence of the projection formula (5.9.1). □

Proposition 6.13. *Let W be an open subset of X^{an}. We equip the space $C_c(W)$ of continuous functions $f : W \to \mathbb{R}$ with compact support with the supremum norm $|\cdot|_W$ and its subspace $A_c^0(W)$ of smooth functions with compact support with the induced norm. Then for each $\alpha \in P_c^{n,n}(W)$ the map*

$$A_c^0(W) \to \mathbb{R}, \quad f \mapsto \int_W f \cdot \alpha$$

is continuous and extends in a unique way to a continuous map $C_c(W) \to \mathbb{R}$.

Proof. We may assume that α is of codimension l. We observe that the Stone–Weierstraß theorem [Chambert-Loir and Ducros 2012, proposition (3.3.5)] implies that $A_c^0(W)$ is a dense subspace of $C_c(W)$. Consider $f \in A_c^0(W)$ and $\alpha \in P_c^{n,n}(W)$. Our claims are obvious once we have obtained a bound C_α such that the inequality

$$\left| \int_W f \cdot \alpha \right| \le C_\alpha \cdot |f|_W \tag{6.13.1}$$

holds. We are going to prove this inequality in four steps.

First step: The definition of the bound C_α. We fix a very affine chart of integration U for α which means that there is $\alpha_U \in P_c^{n,n}(U^{\mathrm{an}}, \varphi_U)$ with $\mathrm{trop}_U^*(\alpha_U) = \alpha$ and we set $N := N_U$. Then α_U is represented by a δ-preform $\tilde{\alpha}_U \in P_c^{n,n}(N_{\mathbb{R}})$ of the form

$$\tilde{\alpha}_U = \sum_\sigma \alpha_\sigma \wedge \delta_\sigma \tag{6.13.2}$$

as a polyhedral supercurrent, where σ ranges over $\mathscr{C}^l$ for a complete integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ of N and where $\alpha_\sigma \in A_c^{n-l,n-l}(\sigma)$. The definition of the bound C_α will depend on the choice of U and of the lift $\tilde{\alpha}_U$, but not on the choice of $\mathscr{C}$. The restriction $\alpha_{\sigma\tau}$ of α_σ to an $(n-l)$-dimensional face τ of σ is an element of $A_c^{n-l,n-l}(\tau)$. As this is a superform of top-degree, we have a well-defined compactly supported superform $|\alpha_{\sigma\tau}|$ of degree $(n-l, n-l)$ with continuous coefficient on τ. This single coefficient is independent of the choice of an integral base of $\mathbb{L}_\tau$ and it is given by the absolute value of the coefficient of α.

After passing to a refinement, we may assume that $\mathrm{Trop}(U)$ is given by the tropical cycle $(\mathscr{C}_{\leq n}, m)$. Then we define

$$C_\alpha := \sum_{(\Delta,\sigma)} [N : N_\Delta + N_\sigma] m_\Delta \int_\tau |\alpha_{\sigma\tau}|, \tag{6.13.3}$$

where (Δ, σ) ranges over all elements of $\mathscr{C}_n \times \mathscr{C}^l$ such that $\mathbb{L}_\Delta + \mathbb{L}_\sigma = N_\mathbb{R}$ and such that $\tau := \Delta \cap \sigma$ is $(n-l)$-dimensional. Here, the integral of a superform of top-degree with continuous coefficient is defined as in [Chambert-Loir and Ducros 2012, (1.2.2), (1.4.1)].

Second step: A first estimate for the integral. By definition of a smooth function, there is a covering of W by tropical charts $(V'_j, \varphi_{U'_j})_{j\in J}$ such that $f|_{V'_j} = \mathrm{trop}^*_{U'_j}(\phi'_j)$ for smooth functions ϕ'_j on the open subsets $\Omega'_j = \mathrm{trop}_{U'_j}(V'_j)$ of $\mathrm{Trop}(U'_j)$. Any given compact subset C of W containing the support of α will be covered by $(V'_j)_{j\in J_0}$ for a finite subset J_0 of J. By Proposition 5.9, $U' := U \cap \bigcap_{j\in J_0} U'_j$ is a very affine chart of integration for α and for $f\alpha$. Let $N' := N_{U'}$ and let $F : N'_\mathbb{R} \to N_\mathbb{R}$ be the canonical integral $\mathbb{R}$-affine map. Since the restriction map $\mathscr{O}(U)^\times \to \mathscr{O}(U')^\times$ is injective, it follows that F is surjective. After refining $\mathscr{C}$, there is a complete integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}'$ on $N'_\mathbb{R}$ such that $\mathrm{Trop}(U') = (\mathscr{C}'_{\leq n}, m')$ and such that $\Delta := F(\Delta') \in \mathscr{C}$ for every $\Delta' \in \mathscr{C}'$.

For $V' := \bigcup_{j\in J_0} V_j \cap (U')^{\mathrm{an}}$, note that by Corollary 5.6, $(V', \varphi_{U'})$ is a tropical chart of W containing $C \cap (U')^{\mathrm{an}}$ and the support of α. The pull-backs of the functions ϕ'_j with respect to the canonical affine maps $F_j : N'_\mathbb{R} \to N_{U'_j,\mathbb{R}}$ glue to a well-defined smooth function $f_{U'}$ on $\Omega' := \mathrm{trop}_{U'}(V')$. By definition, we have

$$\int_W f\alpha = \int_{|\mathrm{Trop}(U')|} f_{U'} F^*(\tilde{\alpha}_U)|_{\mathrm{Trop}(U')}. \tag{6.13.4}$$

Using that F is surjective, we deduce from (6.13.2) and (2.12.5) that

$$F^*(\tilde{\alpha}_U) = \sum_{\sigma'} [N : \mathbb{L}_F(N') + N_\sigma] \cdot F^*\alpha_\sigma \wedge \delta_{\sigma'},$$

where σ' ranges over all elements of $(\mathscr{C}')^l$ such that $\sigma := F(\sigma')$ is of codimension l in N. We choose a generic vector $v' \in N'_\mathbb{R}$. It follows from (2.12.3) that

$$F^*(\tilde{\alpha}_U)|_{\mathrm{Trop}(U')} = \sum_{\tau'} \sum_{(\Delta',\sigma')} [N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] m_{\Delta'} F^*\alpha_\sigma \wedge \delta_{\tau'},$$

where τ' ranges over $\mathscr{C}'_{n-l}$ and (Δ', σ') ranges over all pairs in $\mathscr{C}'_n \times (\mathscr{C}')^l$ such that $\tau' = \Delta' \cap \sigma'$ and such that $\Delta' \cap (\sigma' + \varepsilon v') \neq \varnothing$ for all sufficiently small $\varepsilon > 0$. Additionally, we assume that $\sigma := F(\sigma')$ is of codimension l in $N_\mathbb{R}$ as above. For degree reasons, we may restrict the sum to those τ' with $\tau := F(\tau')$ of dimension $n-l$. Note that this is equivalent to restricting our attention to those Δ' with $\Delta := F(\Delta')$

of dimension n. Since α has support in V', the restriction of $F^*\alpha_\sigma$ to σ' has support in $\Omega' \cap \sigma'$. By (6.13.4), we have

$$\int_W f\alpha = \sum_{\tau'} \sum_{(\Delta',\sigma')} [N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] m_{\Delta'} \int_{\tau'} f_{U'} F^*\alpha_\sigma .$$

We deduce the following bound:

$$\left| \int_W f\alpha \right| \\ \leq |f|_W \sum_{\tau'} \sum_{(\Delta',\sigma')} [N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] m_{\Delta'} \int_{\tau'} |F^*\alpha_{\sigma\tau}|. \quad (6.13.5)$$

The transformation formula shows

$$\int_{\tau'} |F^*\alpha_{\sigma\tau}| = [N_\tau : \mathbb{L}_F(N'_{\tau'})] \int_\tau |\alpha_{\sigma\tau}|$$

and hence the sum in (6.13.5) is equal to

$$\sum_{\tau'} \sum_{(\Delta',\sigma')} [N_\tau : \mathbb{L}_F(N'_{\tau'})][N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] m_{\Delta'} \int_\tau |\alpha_{\sigma\tau}|. \quad (6.13.6)$$

Third step: The following basic lattice index identity holds:

$$[N_\tau : \mathbb{L}_F(N'_{\tau'})][N' : N'_{\Delta'} + N'_{\sigma'}][N : \mathbb{L}_F(N') + N_\sigma] \\ = [N : N_\Delta + N_\sigma][N_\Delta : \mathbb{L}_F(N_{\Delta'})]. \quad (6.13.7)$$

In the basic lattice index identity (6.13.7), (Δ', σ') is a pair in $\mathscr{C}'_n \times (\mathscr{C}')^l$ such that $\Delta' \cap (\sigma' + \varepsilon v') \neq \varnothing$ for $\varepsilon > 0$ sufficiently small and such that $\sigma := F(\sigma')$ is of codimension l in N. We have also used $\Delta := F(\Delta')$, $\tau' := \Delta' \cap \sigma'$ and $\tau := F(\tau')$. Since F is a surjective integral $\mathbb{R}$-affine map, all lattice indices in the claim of the third step are finite. Setting $P' := N'_{\Delta'}$ and $Q := N_\sigma$, the basic lattice identity (6.13.7) follows from the projection formula for lattices in Lemma 6.14 below.

Fourth step: The desired inequality (6.13.1) *holds.* To prove (6.13.1), we note that $v := F(v')$ is a generic vector for $\mathscr{C}$. We have $\tau = \Delta \cap \sigma$ and $\Delta \cap (\sigma + \varepsilon v) \neq \varnothing$. The basic lattice index identity (6.13.7) yields that the sum in (6.13.6) is equal to

$$\sum_{\tau'} \sum_{(\Delta',\sigma')} [N : N_\Delta + N_\sigma][N_\Delta : \mathbb{L}_F(N'_{\Delta'})] m_{\Delta'} \int_\tau |\alpha_{\sigma\tau}|. \quad (6.13.8)$$

The Sturmfels–Tevelev multiplicity formula (4.3.1) gives

$$\sum_{\Delta'} [N_\Delta : \mathbb{L}_F(N'_{\Delta'})] m_{\Delta'} = m_\Delta,$$

where Δ' ranges over all elements of $\mathscr{C}_n'$ mapping onto a given $\Delta \in \mathscr{C}_n$. Using this, one can show that (6.13.8) is equal to

$$\sum_{\tau} \sum_{(\Delta,\sigma)} [N : N_\Delta + N_\sigma] m_\Delta \int_\tau |\alpha_{\sigma\tau}|, \tag{6.13.9}$$

where the sum is over all pairs $(\Delta, \sigma) \in \mathscr{C}_n \times \mathscr{C}^l$ such that $\Delta \cap (\sigma + \varepsilon v) \neq \varnothing$ and $\tau = \Delta \cap \sigma$. Now (6.13.1) follows from (6.13.3)–(6.13.9). □

The basic lattice index identity (6.13.7) is a special case of the following *projection formula for lattices.* Note that it is stronger than the projection formula for tropical cycles in Proposition 1.5. The latter would not give the required bound in the fourth step above.

Lemma 6.14. *Let $F : N' \to N$ be a homomorphism of free abelian groups of finite rank and let $P' \subseteq N'$, $Q \subseteq N$ be subgroups. We assume that* $\mathrm{rk}(F(N')) = \mathrm{rk}(N) = \mathrm{rk}(F(P') + Q)$. *Then we have the equality*

$$\begin{aligned}[F(P')_{\mathbb{R}} \cap Q : F(P' \cap F^{-1}(Q))][N' : P' + F^{-1}(Q)][N : F(N') + Q] \\ = [N : F(P')_{\mathbb{R}} \cap N + Q][F(P')_{\mathbb{R}} \cap N : F(P')],\end{aligned} \tag{6.14.1}$$

where all involved lattice indices are finite.

Proof. The assumptions show easily that all lattice indices are finite. Using

$$F(P' \cap F^{-1}(Q)) = F(P') \cap Q$$

and the isomorphism theorem $A/(A \cap B) \cong (A + B)/B$ for abelian groups, we get

$$(F(P')_{\mathbb{R}} \cap Q)/F(P' \cap F^{-1}(Q)) \cong (F(P')_{\mathbb{R}} \cap Q + F(P'))/F(P').$$

Similarly, $F(P')_{\mathbb{R}} \cap Q + F(P') = F(P')_{\mathbb{R}} \cap (F(P') + Q)$ yields

$$(F(P')_{\mathbb{R}} \cap N)/(F(P')_{\mathbb{R}} \cap Q + F(P')) \cong (F(P')_{\mathbb{R}} \cap N + Q)/(F(P') + Q).$$

Multiplying (6.14.1) by $[F(P')_{\mathbb{R}} \cap N + Q : F(P') + Q]$, the above two isomorphisms show that the claim is equivalent to

$$[N' : P' + F^{-1}(Q)][N : F(N') + Q] = [N : F(P') + Q]. \tag{6.14.2}$$

Using $F(P') + Q \cap F(N') = (F(P') + Q) \cap F(N')$, we have

$$N'/(P' + F^{-1}(Q)) \cong F(N')/(F(P') + Q \cap F(N')) \cong (F(N') + Q)/(F(P') + Q)$$

and hence (6.14.2) holds. This proves the claim. □

We recall that on a locally compact Hausdorff space Y, the Riesz representation theorem gives a bijective correspondence between positive (resp. signed) Radon measures on Y and positive (resp. bounded) linear functionals on the space of continuous real functions with compact support on Y, endowed with the supremum norm.

Corollary 6.15. *Let W be an open subset of X^{an}. For each $\alpha \in P_c^{n,n}(W)$ there is a unique signed Radon measure μ_α on W such that*

$$\int_W f \cdot \alpha = \int_W f \, d\mu_\alpha \tag{6.15.1}$$

for all smooth functions f on W with compact support.

Proof. This is a consequence of Proposition 6.13 and Riesz's representation theorem. □

Proposition 6.16. *Let W be an open subset of X^{an} and let f be a continuous function on W. Then the map*

$$[f] : B_c^{n,n}(W) \to \mathbb{R}, \quad \alpha \mapsto \int_W f \, d\mu_\alpha$$

is a δ-current in $E^{0,0}(W)$.

Proof. The integral is well defined by Corollary 6.15 using that $\mathrm{supp}(\alpha)$ is compact. Obviously, $[f]$ is a linear map. We have to show that the restriction of $[f]$ to any subspace $B_C^{n,n}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$ as in 6.2 is continuous. For $i \in I$, let $\Omega_i := \mathrm{trop}_{U_i}(V_i)$. For every $x \in C$, there is an $i(x) \in I$ with $x \in V_{i(x)}$. We choose a polytopal neighbourhood $P_{i(x)}$ of $\mathrm{trop}_{U_{i(x)}}(x)$ in $N_{U_{i(x)},\mathbb{R}}$ such that $P_{i(x)} \cap \mathrm{Trop}(U_{i(x)}) \subseteq \Omega_{i(x)}$ and we denote the interior of $P_{i(x)}$ by $Q_{i(x)}$. There is a finite set Y of X such that the open sets $\mathrm{trop}_{U_{i(x)}}^{-1}(Q_{i(x)})$, $x \in Y$, cover the compact set C. By Proposition 5.9, $U := \bigcap_{x \in Y} U_{i(x)}$ works as a very affine chart of integration for every $\alpha \in B_C^{n,n}(W; V_i, \varphi_{U_i}, \omega_{J_i} : i \in I)$. Then we have $\alpha_U \in \mathrm{AZ}_c^{n,n}(U, \varphi_U)$ with $\mathrm{trop}_U^*(\alpha_U) = \alpha$. By the Sturmfels–Tevelev multiplicity formula (4.3.1) and by degree reasons, one can show that α_U has support in the compact subset

$$C_U = \bigcup_{x \in Y} \bigcup_{\Delta_{i(x)}} \Delta_{i(x)} \cap F_{i(x)}^{-1}(P_{i(x)})$$

of $\mathrm{Trop}(U)$, where $\Delta_{i(x)}$ ranges over all n-dimensional faces of $\mathrm{Trop}(U)$ such that $\Delta_{i(x)} \cap F_{i(x)}^{-1}(P_{i(x)})$ is mapped onto an n-dimensional face of $\mathrm{Trop}(U_{i(x)})$ by the canonical affine map $F_{i(x)} : N_{U,\mathbb{R}} \to N_{U_{i(x)},\mathbb{R}}$. Using the supremum seminorm $|f|_C$ on C, we get

$$\left| \int_W f \, d\mu_\alpha \right| \le C_\alpha \cdot |f|_C. \tag{6.16.1}$$

To see this, we note first that $\mathrm{supp}(\mu_\alpha) \subseteq C$. There is a smooth function g on W with $0 \le g \le 1$, with $g \equiv 1$ on C and with compact support in a sufficiently small neighbourhood of C [Chambert-Loir and Ducros 2012, corollaire (3.3.4)].

Then (6.16.1) follows from applying (6.13.1) to compactly supported smooth approximations of fg using the Stone–Weierstraß theorem in [Chambert-Loir and Ducros 2012, corollaire (3.3.4)].

Now we deduce the claim from the definition of the bound C_α in (6.13.3): We set $i := i(x)$ for $x \in Y$ and we may assume that $\alpha|_{V_i}$ is given by

$$\sum_{j \in J_i} \alpha_{ij} \wedge \omega_{ij}$$

for $\alpha_{ij} \in A(\Omega_i)$ with coefficients of small supremum seminorm $p_{P_i \cap \mathrm{Trop}(U_i),0}(\alpha_{ij})$. Noting that the ω_{ij} are fixed, this yields that every $\alpha_{\sigma\tau}$ in (6.13.3) has small coefficient. Using that only the compact subset $C_U \cap \tau$ matters for integration, we deduce that C_α is small and hence (6.16.1) shows that $[f]$ is continuous. □

7. The Poincaré–Lelong formula and first Chern delta-currents

The Poincaré–Lelong formula in complex analysis is of fundamental importance for Arakelov theory. Chambert-Loir and Ducros [2012, §4.6] have shown that the Poincaré–Lelong formula holds as an identity between currents on Berkovich spaces while Theorem 7.2 below enhances the Poincaré–Lelong formula as an equality of δ-currents. We use the Poincaré–Lelong formula to define the first Chern δ-current of a continuously metrized line bundle.

7.1. Let X be a variety over K of dimension n and let $f \in K(X) \setminus \{0\}$. In Example 6.11, we introduced the δ-current of integration δ_X leading to the definition of the δ-current δ_Z for any cycle Z on X. Using that for the Weil divisor $\mathrm{cyc}(f)$ of f, we get a δ-current $\delta_{\mathrm{cyc}(f)}$ on X^{an}.

On the other hand, the complement U of the support of the principal Cartier divisor $\mathrm{div}(f)$ is an open dense subset of X. By Proposition 6.5, we get the δ-current $[\log|f|] \in E^{0,0}(U^{\mathrm{an}}) = E^{0,0}(X^{\mathrm{an}})$.

Theorem 7.2. *For a nonzero rational function f on X, the Poincaré–Lelong equation*

$$\delta_{\mathrm{cyc}(f)} = d'd''[\log|f|]$$

holds in $E^{1,1}(X^{\mathrm{an}})$.

Proof. The proof is similar to that in [Chambert-Loir and Ducros 2012, §4.6], but it is more on the tropical side as we do not have integrals of δ-forms over analytic subdomains at hand. We will first do some reduction steps and then we will introduce some notation which allows us to use results from [Chambert-Loir and Ducros 2012]. The claim is local on X^{an} and so we may assume that $X = \mathrm{Spec}(A)$ and $f \in A$. The latter induces a morphism $f : X \to \mathbb{A}^1$. We may assume that the morphism is not constant as otherwise all terms are 0. Since A is a domain, the property S_1 of Serre is satisfied.

Let us recall some results from [Chambert-Loir and Ducros 2012] before we start the actual proof. Let W be an affinoid subdomain of X^{an} and let $g : W \to T^{\mathrm{an}}$ be an analytic map for $T = \mathbb{G}_m^r$. Following [Chambert-Loir and Ducros 2012], we call such a map to a torus an *analytic moment map*. We obtain a continuous map

$$g_{\mathrm{trop}} := \mathrm{trop} \circ g : W \to \mathbb{R}^r.$$

We get an analytic map $h := (f, g) : W \to (\mathbb{A}^1)^{\mathrm{an}} \times T^{\mathrm{an}}$. We denote the fibre of W over $t \in (\mathbb{A}^1)^{\mathrm{an}}$ by W_t with respect to the restriction of f to W. If I is an interval in $(0, \infty)$, then $W_I := |f|^{-1}(I) \cap W$. We observe that W_t and W_I carry natural structures of analytic spaces of dimension $n-1$ and n respectively. It follows from general results of Ducros [2012, théorème 3.2] that the sets $g_{\mathrm{trop}}(W_t)$ and $h_{\mathrm{trop}}(W_I)$ are integral $\mathbb{R}$-affine polyhedral sets of dimension less or equal to $n-1$ and n respectively. These polyhedral sets can be equipped with natural integral weights. A construction of these so called tropical weights can be found in [Gubler 2016, §7] or in [Chambert-Loir and Ducros 2012, §3.5] in the language of calibrations. We observe that the tropical weights take the multiplicities of irreducible components into account. The k-skeleton of a polyhedral set P of dimension at most k is by definition the union of all k-dimensional polyhedra contained in P. By [Chambert-Loir and Ducros 2012, proposition (4.6.6)], there exist a real number $r > 0$ and an integral $\mathbb{R}$-affine polyhedral complex $\mathscr{C}$ in $\mathbb{R}^r$ of pure dimension $n-1$ with integer weights m such that all polyhedra in $\mathscr{C}$ are polytopes with the following properties:

(a) For every t in the closed ball in $(\mathbb{A}^1)^{\mathrm{an}}$ with centre 0 and radius r, the $(n-1)$-skeleton of $g_{\mathrm{trop}}(W_t)$ endowed with the canonical tropical weights is equal to $(\mathscr{C}, m)$.

(b) For every closed interval $I \subset (0, r]$, the n-skeleton of $h_{\mathrm{trop}}(W_I)$ endowed with the canonical analytic tropical weights is equal to $(-\log(I), 1) \times (\mathscr{C}, m)$ as a product of weighted polyhedral complexes.

In fact, Chambert-Loir and Ducros formulated this crucial result in terms of canonical calibrations instead of analytic tropical weights. We refer to [Gubler 2016, §7] for the definition and translation of these equivalent notions. The analytic space W_0 coincides with the closed analytic subspace of W determined by the effective Cartier divisor $\mathrm{div}(f|_W)$. Using (a) for $t = 0$, we see that $(\mathscr{C}, m)$ is equal to the $(n-1)$-skeleton of $g_{\mathrm{trop}}(\mathrm{div}(f|_W))$ as a weighted polyhedral complex.

Having recalled these preliminary results, we proceed with the proof. Since the δ-currents $\delta_{\mathrm{cyc}(f)}$ and $d'd''[\log|f|]$ are symmetric, it is enough to check the Poincaré–Lelong equation by evaluating at a symmetric $\alpha \in B_c^{n-1,n-1}(X^{\mathrm{an}})$. The δ-form α is given by tropical charts $(V_i, \varphi_{U_i})_{i \in I}$ covering X^{an} and symmetric

$\alpha_i \in \mathrm{AZ}^{n-1,n-1}(V_i, \varphi_{U_i})$. Since α has compact support, there are finitely many i such that the corresponding V_i's cover $\mathrm{supp}(\alpha)$. In the following, we restrict our attention to these finitely many i's and we number them by $i = 1, \dots, m$.

Let us consider the very affine open subset $U := U_1 \cap \cdots \cap U_m \setminus \mathrm{supp}(\mathrm{div}(f))$ of X. Let $G_i : N_{U,\mathbb{R}} \to N_{U_i,\mathbb{R}}$ (resp. $F : N_{U,\mathbb{R}} \to \mathbb{R}$) be the canonical affine map compatible with trop_U and trop_{U_i} (resp. $-\log|f|$). Let x_0 be the coordinate on $\mathbb{R}$ and let $H_i := (F, G_i) : N_{U,\mathbb{R}} \to \mathbb{R} \times N_{U_i,\mathbb{R}}$.

For every $x \in \mathrm{supp}(\alpha)$, there is an $i \in \{1, \dots, m\}$ such that $x \in V_i$. We choose an integral Γ-affine polytope Δ_i of maximal dimension in $N_{U_i,\mathbb{R}}$ containing $\mathrm{trop}_{U_i}(x)$ in its interior. We may assume that $\Delta_i \cap \mathrm{Trop}(U_i) \subseteq \mathrm{trop}_{U_i}(V_i)$. Then $W_i := \mathrm{trop}_{U_i}^{-1}(\Delta_i)$ is an affinoid subdomain of X^{an} with $x \in \mathrm{Int}(W_i)$. Renumbering the covering and using again compactness of $\mathrm{supp}(\alpha)$, we may assume that i does not depend on x, which means that the interiors of the affinoid subdomains $W_1, \dots, W_m$ cover $\mathrm{supp}(\alpha)$. Note that $W := \bigcup_{i=1}^m W_i$ is a compact analytic subdomain of X^{an}.

For every nonempty subset E of $\{1, \dots, m\}$, the set $W_E := \bigcap_{i \in E} W_i$ is affinoid (using that X^{an} is separated). Note that $U_E := \bigcap_{i \in E} U_i$ is very affine and we set $V_E := \bigcap_{i \in E} V_i$. We choose $r > 0$ sufficiently small such that (a) and (b) above hold for every W_E and moment map $g_E := \varphi_{U_E}$. Note that the union of the integral Γ-affine polyhedral sets

$$\mathrm{trop}_U(W_i \cap U^{\mathrm{an}}) = \mathrm{Trop}(U) \cap G_i^{-1}(\Delta_i) \quad (i = 1, \dots, m) \tag{7.2.1}$$

is equal to $\mathrm{trop}_U(W \cap U^{\mathrm{an}})$. For every subset E of $\{1, \dots, m\}$, we have a integral Γ-affine polyhedral set

$$\mathrm{trop}_U(W_E \cap U^{\mathrm{an}}) = \mathrm{Trop}(U) \cap \bigcap_{i \in E} G_i^{-1}(\Delta_i) = \bigcap_{i \in E} \mathrm{trop}_U(W_i \cap U^{\mathrm{an}}). \tag{7.2.2}$$

For $V := \bigcup_i V_i \cap U^{\mathrm{an}}$, it follows from Corollary 5.6 that (V, φ_U) is a tropical chart containing the support of $d''\alpha$. The δ-form α is represented on V by $\alpha_U \in \mathrm{AZ}^{n-1,n-1}(V, \varphi_U)$, i.e., $\alpha = \mathrm{trop}_U^*(\alpha_U)$ on V. In fact, we have seen in Proposition 5.7 that α_U extends by 0 to an element of $\mathrm{AZ}^{n-1,n-1}(U^{\mathrm{an}}, \varphi_U)$, but the support of this extension is not necessarily compact. We conclude that U is a very affine chart of integration for $\log|f|\, d'd''\alpha$ and that

$$\langle d'd''[\log|f|], \alpha\rangle = -\int_{\mathrm{trop}_U(V)} F^*(x_0)\, d'd''\alpha_U. \tag{7.2.3}$$

The minus sign comes from the tropical coordinates $\mathrm{trop}_U^*(F^*(x_0)) = -\log|f|$ as remarked above. Corollary 5.6 shows that the support of $d''\alpha$ does not meet $f^{-1}(0)$. Since the support of $d''\alpha$ is compact, there is a positive $s < r$ such that $|f(x)| > s$ for every $x \in \mathrm{supp}(d''\alpha)$. We consider the analytic subdomain of W,

$W(s) := \{x \in W \mid |f(x)| \geq s\}$, and the affinoid subdomains of W_i and W_E,

$$W_i(s) := \{x \in W_i \mid |f(x)| \geq s\} \quad \text{and} \quad W_E(s) := \{x \in W_E \mid |f(x)| \geq s\}.$$

It follows from (7.2.1) and (7.2.2) that their tropicalizations are integral $\mathbb{R}$-affine polyhedral sets such that the union of all

$$\operatorname{trop}_U(W_i(s) \cap U^{\mathrm{an}}) = \operatorname{Trop}(U) \cap G_i^{-1}(\Delta_i) \cap F^{-1}((-\infty, -\log s]) \tag{7.2.4}$$

for $i = 1, \ldots, m$ is equal to $\operatorname{trop}_U(W(s) \cap U^{\mathrm{an}})$ and such that

$$\operatorname{trop}_U(W_E(s) \cap U^{\mathrm{an}}) = \operatorname{Trop}(U) \cap \bigcap_{i \in E} G_i^{-1}(\Delta_i) \cap F^{-1}((-\infty, -\log s]). \tag{7.2.5}$$

In the following, we use integrals and boundary integrals of δ-preforms over integral $\mathbb{R}$-affine polyhedral sets as introduced in Definition 2.5, Remark 3.5 and 5.1. By the choice of s, we have $\operatorname{supp}(d''\alpha) \subseteq W(s) \cap U^{\mathrm{an}}$. We conclude that $\operatorname{supp}(d''\alpha_U) \subseteq \operatorname{trop}_U(W(s) \cap U^{\mathrm{an}})$ and hence

$$\int_{\operatorname{trop}_U(V)} F^*(x_0)\, d'd''\alpha_U = \int_{\operatorname{trop}_U(W(s)\cap U^{\mathrm{an}})} F^*(x_0)\, d'd''\alpha_U. \tag{7.2.6}$$

By Green's formula (see Proposition 3.9) and using $d'd''F^*(x_0) = 0$, the integrals in (7.2.6) are equal to

$$\int_{\partial(\operatorname{trop}_U(W(s)\cap U^{\mathrm{an}}))} \big(F^*(x_0)\, d''\alpha_U - d''(F^*(x_0)) \wedge \alpha_U\big). \tag{7.2.7}$$

By construction and (7.2.1), we have

$$\operatorname{supp}(\alpha_U) \subseteq \operatorname{relint}(\operatorname{trop}_U(W \cap U^{\mathrm{an}})).$$

By the choice of s, it follows that $\operatorname{supp}(d''\alpha_U) \subseteq \operatorname{relint}(\operatorname{trop}_U(W(s) \cap U^{\mathrm{an}}))$. Applying Remark 2.6(iii) to the integral $\mathbb{R}$-affine polyhedral set $\operatorname{trop}_U(W(s) \cap U^{\mathrm{an}})$, it follows that

$$\int_{\partial(\operatorname{trop}_U(W(s)\cap U^{\mathrm{an}}))} F^*(x_0)\, d''\alpha_U = 0. \tag{7.2.8}$$

Combining (7.2.3) and (7.2.6)–(7.2.8) with (7.3.1) below, we get

$$\langle d'd''[\log|f|], \alpha\rangle = \langle \delta_{\operatorname{cyc}(f)}, \alpha\rangle, \tag{7.2.9}$$

proving the claim. □

Lemma 7.3. *In the situation of the proof of Theorem 7.2 above, we have*

$$\int_{\partial(\operatorname{trop}_U(W(s)\cap U^{\mathrm{an}}))} d''(F^*(x_0)) \wedge \alpha_U = \langle \delta_{\operatorname{cyc}(f)}, \alpha\rangle. \tag{7.3.1}$$

Proof. For integers $\ell \geq 1$, there are $\varphi_\ell \in C^\infty(\mathbb{R})$ with $0 \leq \varphi_\ell \leq 1$, $\varphi_\ell(t) = 1$ for $t \leq -\log(s) - 1/\ell$ and $\varphi_\ell(t) = 0$ for $t \geq -\log(s) - 1/(2\ell)$. By construction, $\operatorname{supp}(\varphi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U$ is contained in the relative interior of $\operatorname{trop}_U(W(s) \cap U^{\mathrm{an}})$ and hence

$$\int_{\partial(\operatorname{trop}_U(W(s)\cap U^{\mathrm{an}}))} \varphi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U = 0 \tag{7.3.2}$$

as above. Setting $\psi_\ell := 1 - \varphi_\ell$, it follows from (7.3.2) that the left-hand side in (7.3.1) is equal to

$$\int_{\partial(\operatorname{trop}_U(W(s)\cap U^{\mathrm{an}}))} \psi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U. \tag{7.3.3}$$

Now we use the additivity of measures from Remark 2.6(ii). The decomposition (7.2.4) of the polyhedral set $\operatorname{trop}_U(W(s) \cap U^{\mathrm{an}})$ and Equation (7.2.5) show that (7.3.3) is equal to

$$\sum_{j=1}^{m} (-1)^{j+1} \sum_{|E|=j} \int_{\partial(\operatorname{trop}_U(W_E(s)\cap U^{\mathrm{an}}))} \psi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U. \tag{7.3.4}$$

We fix $i \in E$. Let $G_E : N_{U,\mathbb{R}} \to N_{U_E,\mathbb{R}}$ and $G_{E,i} : N_{U_E,\mathbb{R}} \to N_{U_i,\mathbb{R}}$ be the canonical affine maps which are compatible with the given moment maps. Let us consider the closed embedding

$$h_E := (f, g_E) = (f, \varphi_{U_E}) : U_E \setminus \operatorname{div}(f) \to \mathbb{G}_m \times T_{U_E}$$

inducing the tropical variety $h_{E,\mathrm{trop}}(U_E \setminus \operatorname{div}(f))$, which we view as a tropical cycle on $\mathbb{R} \times N_{U_E,\mathbb{R}}$. The affine maps $H_E := (F, G_E) : N_{U,\mathbb{R}} \to \mathbb{R} \times N_{U_E,\mathbb{R}}$ (resp. $H_{E,i} := \mathrm{id}_{\mathbb{R}} \times G_{E,i} : \mathbb{R} \times N_{U_E,\mathbb{R}} \to \mathbb{R} \times N_{U_i,\mathbb{R}}$) are compatible with the moment maps φ_U and h_E (resp. h_E and h_i). The Sturmfels–Tevelev multiplicity formula shows that

$$h_{E,\mathrm{trop}}(U_E \setminus \operatorname{div}(f)) = (H_E)_*(\operatorname{Trop}(U)) \tag{7.3.5}$$

(see [Gubler 2016, Proposition 4.11] for the required generalization of (4.3.1)). For $\alpha_E := \alpha_i|_{V_E} \in \mathrm{AZ}^{n-1,n-1}(V_E, \varphi_{U_E})$, we have $\alpha|_{V_E} = \operatorname{trop}_{U_E}^*(\alpha_E)$ and the definition of α_E does not depend on the choice of $i \in E$. In the following, the weighted integral $\mathbb{R}$-affine polyhedral complex $\Sigma_E(s) := h_{E,\mathrm{trop}}(W_E(s))$ in $\mathbb{R} \times N_{U_E,\mathbb{R}}$ plays a crucial role. Note that we have

$$\Sigma_E(s) = h_{E,\mathrm{trop}}(U_E \setminus \operatorname{div}(f)) \cap \bigcap_{i\in E} H_{E,i}^{-1}\big((-\infty, -\log s] \times \Delta_i\big). \tag{7.3.6}$$

Let $P_E : \mathbb{R} \times N_{U_E,\mathbb{R}} \to N_{U_E,\mathbb{R}}$ denote the canonical projection. By definition, the element α_E of $\mathrm{AZ}^{n-1,n-1}(V_E, \varphi_{U_E})$ is represented by a δ-preform $\tilde{\alpha}_E$ on an open subset $\widetilde{\Omega}_E$ of $N_{U_E,\mathbb{R}}$ with $\widetilde{\Omega}_E \cap \operatorname{Trop}(U_E) = \operatorname{trop}_{U_E}(V_E)$. Recall from (4.5.1), that

$\alpha_U|_\Omega = \tilde{\alpha}_U \wedge \delta_{\mathrm{Trop}(U)}$ denotes the δ-preform on $\Omega := \mathrm{Trop}_U(V)$ induced by α_U. Using $\alpha_U = G_E^*(\alpha_E)$, we get

$$\alpha_U|_\Omega = G_E^*(\alpha_E)|_\Omega = G_E^*(\tilde{\alpha}_E) \wedge \delta_{\mathrm{Trop}(U)} = H_E^* P_E^*(\tilde{\alpha}_E) \wedge \delta_{\mathrm{Trop}(U)}.$$

We consider the coordinate x_0 on $\mathbb{R}$ also as a function on $\mathbb{R} \times N_{U_E,\mathbb{R}}$. Using $\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}}) = H_E^{-1}(\Sigma_E(s)) \cap \mathrm{Trop}(U)$ and (7.3.5), the projection formula (2.14.2) shows that

$$\begin{aligned}\int_{\partial(\mathrm{trop}_U(W_E(s)\cap U^{\mathrm{an}}))} & \psi_\ell(F^*(x_0))\, d''(F^*(x_0)) \wedge \alpha_U \\ &= \int_{\partial(\mathrm{trop}_U(W_E(s)\cap U^{\mathrm{an}}))} H_E^*(\psi_\ell(x_0) d'' x_0) \wedge H_E^* P_E^*(\tilde{\alpha}_E) \wedge \delta_{\mathrm{Trop}(U)} \\ &= \int_{\partial(\Sigma_E(s))} \psi_\ell(x_0)\, d'' x_0 \wedge P_E^*(\tilde{\alpha}_E) \wedge \delta_{h_{E,\mathrm{trop}}(U_E \setminus \mathrm{div}(f))}. \qquad (7.3.7)\end{aligned}$$

By construction of the functions φ_ℓ, we have

$$\begin{aligned}\lim_{\ell\to\infty} \int_{\partial(\Sigma_E(s))} & \psi_\ell(x_0)\, d'' x_0 \wedge P_E^*(\tilde{\alpha}_E) \wedge \delta_{h_{E,\mathrm{trop}}(U_E \setminus \mathrm{div}(f))} \\ &= \int_{\Sigma_E(s)\cap\{x_0=-\log|s|\}} P_E^*(\tilde{\alpha}_E) \wedge \delta_{h_{E,\mathrm{trop}}(U_E \setminus \mathrm{div}(f))}. \qquad (7.3.8)\end{aligned}$$

By (7.3.6) and [Gubler 2016, §7], the analytic tropical weights on the n-skeleton of the tropicalization $\Sigma_E(s)$ of the affinoid domain $W_E(s)$ are the same as the tropical weights induced by $h_{E,\mathrm{trop}}(U_E \setminus \mathrm{div}(f))$. Using that $s < r$ and $I := [s, r]$, it follows from (a) and (b) that the n-skeletons of $\Sigma_E(I) := \{\omega \in \Sigma_E(s) \mid x_0(\omega) \in -\log(I)\}$ and $-\log(I) \times \mathrm{trop}_{U_E}(\mathrm{div}(f) \cap W_E)$ are equal even as a product of weighted polyhedral complexes if we endow $-\log(I)$ with weight 1. Note that these tropicalizations can differ from the n-skeletons only inside the relative boundary. As we have some flexibility in the choice of the polyhedra Δ_i and in the choice of s, we may assume that $\Sigma_E(I) = -\log(I) \times \mathrm{trop}_{U_E}(\mathrm{div}(f) \cap W_E)$ and that this is of pure dimension n. We conclude that (7.3.8) is equal to

$$\int_{\mathrm{trop}_{U_E}(\mathrm{div}(f)\cap W_E \cap U_E^{\mathrm{an}})} \alpha_E. \qquad (7.3.9)$$

Using (7.3.3)–(7.3.9), it follows that the left-hand side of (7.3.1) is equal to

$$\sum_{j=1}^m (-1)^{j+1} \sum_{|E|=j} \int_{\mathrm{trop}_{U_E}(\mathrm{div}(f)\cap W_E \cap U_E^{\mathrm{an}})} \alpha_E.$$

Let Y be an irreducible component of $\mathrm{div}(f)$ and let

$$E_Y := \{i \in \{1, \ldots, m\} \mid U_i \cap Y \neq \varnothing\}.$$

Then we use the very affine open subset U_{E_Y} to compute the following integrals over Y by performing the above steps backwards:

$$\sum_{j=1}^{m}(-1)^{j+1}\sum_{|E|=j}\int_{\mathrm{trop}_{U_E}(Y\cap W_E\cap U_E^{\mathrm{an}})}\alpha_E=\int_{\mathrm{trop}_{U_{E_Y}}(Y\cap W\cap U_{E_Y}^{\mathrm{an}})}\alpha_{U_{E_Y}}=\int_Y\alpha, \quad (7.3.10)$$

where we have used in the last step that W covers $\mathrm{supp}(\alpha)$. Using linearity in the irreducible components Y (see [Gubler 2013, Remark 13.12]), we get Equation (7.3.1). □

Remark 7.4. Let f denote a regular function on the affine variety X. The proof of Lemma 7.3 given above shows that Equation (7.3.1) holds more generally for any generalized δ-forms α on X^{an} with compact support. If we permute the roles of d' and d'', we obtain by the same argument that

$$-\int_{\partial(\mathrm{trop}_U(W(s)\cap U^{\mathrm{an}}))} d'F^*(x_0)\wedge\alpha_U=\langle\delta_{\mathrm{cyc}(f)},\alpha\rangle \quad (7.4.1)$$

holds for all generalized δ-forms $\alpha\in P_c^{n-1,n-1}(X^{\mathrm{an}})$. An elegant way to deduce (7.4.1) is to apply (7.3.1) for $J^*(\alpha)$ and to use the symmetry of the δ-current of integration.

7.5. Let φ denote an invertible analytic function on some open subset W of X^{an}. Given $x\in W$ there exists by [Gubler 2016, Proposition 7.2] an open subset U of X, an algebraic moment map $f:U\to\mathbb{G}_m$ and an open neighbourhood V of x in $U^{\mathrm{an}}\cap W$ such that $-\log|\varphi|$ and $-\log|f|$ agree on V. It follows that the function $-\log|\varphi|$ belongs to $A^0(W)$ and we get

$$d'd''[-\log|\varphi|]=-[d'd''\log|\varphi|]=0 \quad (7.5.1)$$

from (6.9.1) and the trivial case of the Poincaré–Lelong formula where f is invertible.

7.6. Let L be a line bundle on X and let W be an open subset of X^{an}. We fix an open covering $(U_i)_{i\in I}$ of X, a family $(s_i)_{i\in I}$ of nowhere vanishing sections $s_i\in\Gamma(U_i,L)$, and the 1-cocycle (h_{ij}) with values in $\mathscr{O}_X^\times$ determined by $s_j=h_{ij}s_i$. Recall that a *continuous metric* $\|\cdot\|$ *on* L over W is given by a family $(\rho_i)_{i\in I}$ of continuous functions $\rho_i:U_i^{\mathrm{an}}\cap W\to\mathbb{R}$ such that $\rho_j=|h_{ij}|\rho_i$ on $(U_i\cap U_j)^{\mathrm{an}}\cap W$ for all $i,j\in I$. An analytic section $s\in\Gamma(V,L^{\mathrm{an}})$ on some open subset V of W determines as follows a continuous function $\|s\|:V\to\mathbb{R}$. We write $s=f_is_i$ for some analytic function f_i on $V\cap U_i^{\mathrm{an}}$ and define $\|s\|=|f_i|\cdot\rho_i$ on $V\cap U_i^{\mathrm{an}}$. Observe that we have $\rho_i=\|s_i\|$ on $U_i^{\mathrm{an}}\cap W$.

7.7. Let L be a line bundle on X endowed with a continuous metric $\|\cdot\|$ over the open subset W of X^{an}. Then we define the *first Chern current associated to the metrized line bundle* $(L|_W,\|\cdot\|)$ as the δ-current $[c_1(L|_W,\|\cdot\|)]\in E^{1,1}(W)$ given locally on $W \cap U^{\mathrm{an}}$ by $d'd''[-\log\|s|_{U^{\mathrm{an}}\cap W}\|]$ for any trivialization U of L with nowhere vanishing section $s \in \Gamma(U, L)$. Here, we have used that a continuous function defines a δ-current as explained in Proposition 6.16. Since $d'd''[-\log|\varphi|] = 0$ for an invertible analytic function φ, the δ-current $[c_1(L|_W, \|\cdot\|)]$ is well-defined on W and we may even use analytic trivializations in the definition. Obviously, the formation of the first Chern current is compatible with tensor products of metrized line bundles as usual.

If the metric is smooth then $[c_1(L, \|\cdot\|)]$ is the current associated to the first Chern form $c_1(L|_W, \|\cdot\|)$ defined in [Chambert-Loir and Ducros 2012]. In general, the notion $c_1(L|_W, \|\cdot\|)$ has no meaning as a form and we use brackets to emphasize that $[c_1(L|_W, \|\cdot\|)]$ is a δ-current. In Section 9, we will introduce metrics for which $c_1(L|_W, \|\cdot\|)$ has a meaning as a δ-form.

Corollary 7.8. *Let L be a line bundle on X endowed with a continuous metric $\|\cdot\|$ over the open subset W of X^{an}. For every nontrivial meromorphic section s of L with associated Weil divisor Y, the equality*

$$\big[c_1(L|_W, \|\cdot\|)\big] = d'd''\big[-\log\big\|s|_W\big\|\big] + \delta_Y|_W$$

holds in $E^{1,1}(W)$.

Proof. This can be checked locally on a trivialization U of L with a nowhere vanishing $s_U \in \Gamma(U, L)$. Then there is a rational function f on X with $s = f s_U$ and hence

$$d'd''\big[-\log\big\|s|_{W\cap U^{\mathrm{an}}}\big\|\big] = d'd''\big[-\log\big\|s_U|_{W\cap U^{\mathrm{an}}}\big\|\big] + d'd''\big[-\log\big|f|_{W\cap U^{\mathrm{an}}}\big|\big].$$

Using the definition of $c_1(L|_{W\cap U^{\mathrm{an}}}, \|\cdot\|)$ for the first summand and Theorem 7.2 for the second summand, we get the claim. $\square$

8. Piecewise smooth and formal metrics on line bundles

In this section, X is an algebraic variety over K. In the following, we consider an open subset W of X^{an}.

We first introduce piecewise smooth functions and piecewise linear functions on W. This leads to corresponding notions for metrics on line bundles. We prove that a piecewise linear metric is the same as a formal metric. We show that canonical metrics in various situations are piecewise smooth.

In Definition 1.6, we have defined piecewise smooth functions on an open subset of an integral $\mathbb{R}$-affine polyhedral set. Using tropicalizations and viewing tropical varieties as polyhedral sets, we will define piecewise smooth functions on W as follows:

Definition 8.1. A function $f : W \to \mathbb{R}$ is called *piecewise smooth* if for every $x \in W$ there is a tropical chart (V, φ_U) such that V is an open neighbourhood of

x in W and such that there is a piecewise smooth function ϕ on $\mathrm{trop}_U(V)$ with $f = \phi \circ \mathrm{trop}_U$ on V.

In a similar way, we will define a piecewise linear function on W. We recall from Definition 1.6 that we have defined piecewise linear functions on integral $\mathbb{R}$-affine polyhedral complexes. As we are working with a variety over a valued field, we will take the value group Γ into account and in the definition of piecewise linear functions we will additionally require that the underlying polyhedral complex and the restriction of the functions are both integral Γ-affine. Note however that in Definition 8.1, the underlying polyhedral complex for ϕ is only assumed to be integral $\mathbb{R}$-affine.

Definition 8.2. A function $f : W \to \mathbb{R}$ is called *piecewise linear* if for every $x \in W$ there is a tropical chart (V, φ_U) such that V is an open neighbourhood of x in W and a real function ϕ on $\mathrm{trop}_U(V)$ with $f = \phi \circ \mathrm{trop}_U$ on V. We require that there is an integral Γ-affine polyhedral complex Σ in $N_{U,\mathbb{R}}$ with $\mathrm{trop}_U(V) \subseteq |\Sigma|$ such that ϕ is the restriction of a function on $|\Sigma|$ with integral Γ-affine restrictions to all faces of Σ.

8.3. The space of piecewise smooth functions on W is an $\mathbb{R}$-subalgebra of the $\mathbb{R}$-algebra of continuous functions on W. It contains all smooth functions on W. The space of piecewise linear functions on W is closed under forming max and min. Moreover, it is a subgroup of the space of piecewise smooth functions on W with respect to addition. If $\varphi : X' \to X$ is a morphism and W' is an open subset of $(\varphi^{\mathrm{an}})^{-1}(W)$, then for every piecewise smooth (resp. piecewise linear) function f on W, the restriction of $f \circ \varphi$ to W' is a piecewise smooth (resp. piecewise linear) function on W'.

In the following result, we need the G-topology on W. It is a Grothendieck topology build up from analytic subdomains of W and it is closely related to the Grothendieck topology of the underlying rigid analytic space ([Berkovich 1993, §1.3, §1.6]).

Proposition 8.4. *Let $f : W \to \mathbb{R}$ be a continuous function. Then f is piecewise smooth* (*resp. piecewise linear*) *if and only if there is a* G*-covering $(W_i)_{i \in I}$ by analytic* (*resp. strict analytic*) *subdomains W_i of W and analytic moment maps $\varphi_i : W_i \to (T_i)^{\mathrm{an}}$ to tori $T_i := \mathrm{Spec}(K[M_i])$ such that $f = \phi_i \circ \varphi_{i,\mathrm{trop}}$ on W_i for a smooth* (*resp. integral Γ-affine*) *function $\phi_i : N_{i,\mathbb{R}} \to \mathbb{R}$, where $N_i := \mathrm{Hom}(M_i, \mathbb{Z})$ as usual.*

Proof. First, we assume that f is piecewise smooth (resp. piecewise linear). For any $x \in W$, there is a tropical chart (V, φ_U) in W containing x such that $f = \phi \circ \mathrm{trop}_U$ on V for a piecewise smooth (resp. integral Γ-affine function) ϕ on the open subset $\Omega := \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. There are finitely many integral $\mathbb{R}$-affine (resp. Γ-affine) polytopes Δ_i in $N_{U,\mathbb{R}}$ containing $\mathrm{trop}_U(x)$ such that $\bigcup_i \Delta_i$ is a neighbourhood of $\mathrm{trop}_U(x)$ in Ω and such that $\phi|_{\Delta_i} = \phi_i|_{\Delta_i}$ for a smooth (resp. integral Γ-affine) function $\phi_i : N_{U,\mathbb{R}} \to \mathbb{R}$. Note that the affinoid (resp. strictly affinoid) subdomains $W_i(x) := \mathrm{trop}_U^{-1}(\Delta_i)$ of W contain x and cover a neighbourhood of x. Letting x vary over W, we get a G-covering of W with the desired properties.

To prove the converse, we assume that f is given on a G-covering $(W_i)_{i\in I}$ of W by smooth (resp. integral Γ-affine) functions $\phi_i : N_{i,\mathbb{R}} \to \mathbb{R}$ with respect to analytic moment maps $\varphi_i : W_i \to (T_i)^{\mathrm{an}}$. Piecewise smoothness (resp. piecewise linearity) is a local condition and so we have to check that f is piecewise smooth in a neighbourhood of $x \in X^{\mathrm{an}}$. There is a finite $I_0 \subseteq I$ such that the sets $(W_i)_{i\in I_0}$ cover a sufficiently small strict affinoid neighbourhood W' of x in W. By shrinking W, we may assume that $x \in W_i$ for every $i \in I_0$. In the following, we restrict our attention to elements $i \in I_0$. The definition of an analytic (resp. of a strict analytic) domain shows that we may assume that all $W_i' := W_i \cap W'$ are affinoid (resp. strict affinoid) subdomains of W. Any analytic function on a neighbourhood of x in W_i' can be approximated uniformly on a sufficiently small neighbourhood of x by rational functions on X. By shrinking W again, this shows that we may assume that $\varphi_i|_{W_i'}$ is induced by the restriction of an algebraic moment map $\varphi_i' : U_i \to T_i$ for a dense open subset U_i of X with $W_i' \subseteq (U_i)^{\mathrm{an}}$ (see [Gubler 2016, Proposition 7.2] for a similar argument). Similarly, we may assume that there are affinoid coordinates $(x_{ij})_{j\in J_i}$ on W_i' which extend to rational functions on X. Clearly, we may assume that $|x_{ij}(x)| = 1$ for $i \in I_0$ and $j \in J_i$. There is a tropical chart (V, φ_U) with $x \in V \subseteq W'$, $U \subseteq \bigcap_{i\in I_0} U_i$ and such that all the functions x_{ij} are in $\mathscr{O}(U)^\times$. We may assume that $\mathrm{trop}_U(x) = 0$ and hence there is an open neighbourhood $\widetilde{\Omega}$ of 0 in $N_{U,\mathbb{R}}$ with $V = \mathrm{trop}_U^{-1}(\widetilde{\Omega})$. By [Gubler 2016, 4.12, Proposition 4.16], $\varphi_i'|_U$ is the composition of an affine homomorphism $\psi_i : T_U \to T_i$ with φ_U. By shrinking V and using the Bieri–Groves theorem [Gubler 2013, Theorem 3.3], we may assume that there are finitely many rational cones $(\Delta_j)_{j\in J}$ in $N_{U,\mathbb{R}}$ such that

$$\mathrm{trop}_U(V) = \widetilde{\Omega} \cap \bigcup_{j\in J} \Delta_j. \tag{8.4.1}$$

For every $i \in I_0$ and every $j \in J_i$, there is a linear form $u_{ij} \in M_U$ with $-\log|x_{ij}| = u_{ij} \circ \mathrm{trop}_U$ on U^{an}. The definition of affinoid coordinates yields

$$W_i' \cap U^{\mathrm{an}} = \mathrm{trop}_U^{-1}(\sigma_i) \tag{8.4.2}$$

for

$$\sigma_i := \{\omega \in N_{U,\mathbb{R}} \mid u_{ij}(\omega) \geq r_{ij}\ \forall j \in J_i\}$$

and suitable $r_{ij} \in \mathbb{R}$. Note that σ_i is an integral $\mathbb{R}$-affine polyhedron. In the piecewise linear case, we may choose always $r_{ij} = 0$ and hence σ_i is a rational cone. Using that the sets $W_i' \cap U^{\mathrm{an}}$ cover V and equations (8.4.1), (8.4.2), we get the decomposition

$(\sigma_i \cap \Delta_j \cap \widetilde{\Omega})_{i\in I_0, j\in J}$ of $\mathrm{trop}_U(V)$. On $\sigma_i \cap \Delta_j \cap \widetilde{\Omega}$, we choose the smooth (resp. integral Γ-affine) function $\phi'_{ij} := \phi_i \circ \psi_i$. Using (8.4.2), we see that these functions paste to a continuous piecewise smooth (resp. continuous piecewise linear) function ϕ' on $\mathrm{trop}_U(V)$ with $\phi' \circ \mathrm{trop}_U = f$ on V. This proves easily that f is piecewise smooth (resp. piecewise linear) on W. □

Definition 8.5. Let L be a line bundle on X and let W be an open subset of X^{an}. A metric $\|\cdot\|$ on $L|_W$ is called *piecewise smooth* (resp. *piecewise linear*) if for every $x \in W$, there is a tropical chart (V, φ_U) with $x \in V \subseteq W$ and a nowhere vanishing section $s \in \Gamma(U, L)$ such that $-\log\|s|_V\|$ is piecewise smooth (resp. piecewise linear) on V.

8.6. Since $-\log|f|$ is smooth for an invertible regular function f and even the pull-back of a linear function with respect to a suitable tropicalization, the last definition does neither depend on the choice of the trivialization s nor on the choice of the tropical chart (V, φ_U). Moreover, we may also use analytic trivializations in the definition. By Proposition 8.4, the definition of a piecewise linear metric agrees with the definition of PL-metrics in [Chambert-Loir and Ducros 2012, §6.2].

Note that every piecewise linear metric is piecewise smooth. It follows from 8.3 that every piecewise smooth metric is continuous, that the tensor product of piecewise linear (resp. piecewise smooth) metrics is again a piecewise linear (resp. piecewise smooth) metric and that the dual metric of a piecewise linear (resp. piecewise smooth) metric is piecewise linear (resp. piecewise smooth). Moreover, the pull-back of a piecewise linear (resp. piecewise smooth) metric on $L|_W$ with respect to a morphism $\varphi : X' \to X$ is a piecewise linear (resp. piecewise smooth) metric on $\varphi^*(L)|_{W'}$ for any open subset W' of $\varphi^{-1}(W)$.

8.7. Recall that K° is the valuation ring of the given nonarchimedean absolute value $|\ |$ on K. Raynaud introduced an *admissible formal scheme over* K° as a formal scheme $\mathscr{X}$ over the valuation ring K° which is locally isomorphic to $\mathrm{Spf}(A)$ for a flat K°-algebra A of topologically finite type over K° (see [Bosch and Lütkebohmert 1993, §1] for details). For simplicity, we require additionally that $\mathscr{X}$ has a locally finite atlas of admissible affine formal schemes over K°. Then $\mathscr{X}$ has a generic fibre $\mathscr{X}_\eta$ (resp. a special fibre $\mathscr{X}_s$) which is a paracompact strictly analytic Berkovich space over K (resp. an algebraic scheme over the residue field $\tilde{K}$) locally isomorphic to $\mathscr{M}(\mathscr{A})$ (resp. $\mathrm{Spec}(A \otimes_{K^\circ} \tilde{K})$) for the strict affinoid algebra $\mathscr{A} := A \otimes_{K^\circ} K$ (see [Berkovich 1993, §1.6] for the equivalence to rigid analytic spaces over K with an affinoid covering of finite type).

A *formal K°-model of X* is an admissible formal scheme $\mathscr{X}$ over K° with an isomorphism $\mathscr{X}_\eta \cong X^{\mathrm{an}}$. For a line bundle L on X, we define a *formal K°-model of L* as a line bundle $\mathscr{L}$ on a formal K°-model $\mathscr{X}$ of X with an isomorphism $\mathscr{L}_\eta \cong L^{\mathrm{an}}$ over $\mathscr{X}_\eta \cong X^{\mathrm{an}}$. For simplicity, we usually identify $\mathscr{L}_\eta$ with L^{an}.

8.8. Let L be a line bundle on X. A *formal metric* on L is a metric $\|\cdot\|_{\mathscr{L}}$ associated to a formal K°-model $\mathscr{L}$ of L in the following way: If $\mathscr{L}$ admits a formal trivialization over $\mathscr{U}$ and if $s \in \Gamma(\mathscr{U}, \mathscr{L})$ corresponds under this trivialization to the function $\gamma \in \mathscr{O}_{\mathscr{X}}(\mathscr{U})$, then $\|s(x)\| = |\gamma(x)|$ for all $x \in \mathscr{U}_\eta$. This definition is independent of the choice of the trivialization and shows immediately that formal metrics are continuous. The tensor product and the pull-back of formal metrics are again formal metrics.

Proposition 8.9. *Every line bundle L on X has a formal K°-model and hence a formal metric.*

Proof. This follows as in [Gubler 1998, Proposition 7.6] based on the theorem of Raynaud that every paracompact analytic space has a formal K°-model (see [Bosch 2014, Theorem 8.4.3]). The argument for paracompact strictly K-analytic spaces was first given in [Chambert-Loir and Ducros 2012, proposition (6.2.13)]. □

Proposition 8.10. *Let $\|\cdot\|$ be a formal metric on the line bundle L on X. Then there is an admissible formal K°-model $\mathscr{X}$ of X with reduced special fibre $\mathscr{X}_s$ and a K°-model $\mathscr{L}$ of L on $\mathscr{X}$ such that $\|\cdot\| = \|\cdot\|_{\mathscr{L}}$. Moreover, the invertible sheaf associated to $\mathscr{L}$ is always canonically isomorphic to the sheaf on $\mathscr{X}$ given by $\{s \in \Gamma(L, \mathscr{U}_\eta) \mid \|s(s)\| \leq 1 \ \forall x \in \mathscr{U}_\eta\}$ on a formal open subset $\mathscr{U}$ of $\mathscr{X}$.*

Proof. This follows as in [Gubler 1998, Lemma 7.4 and Proposition 7.5]. □

Proposition 8.11. *Let $\|\cdot\|$ be a metric on the line bundle L on X. Then the following properties are equivalent*:

(a) *$\|\cdot\|$ is a formal metric*;

(b) *$\|\cdot\|$ is a piecewise linear metric*;

(c) *there is a* G*-covering $(W_i)_{i\in I}$ of X^{an} by strict analytic subdomains W_i of X^{an} and trivializations $s_i \in \Gamma(W_i, L^{\mathrm{an}})$ with $\|s_i(x)\| = 1$ for all $x \in W_i$, $i \in I$.*

Proof. If we use again Raynaud's theorem to generalize to paracompact X^{an}, the equivalence of (a) and (c) is proved as in [Gubler 1998, Lemma 7.4 and Proposition 7.5]. The implication (a) $\Rightarrow$ (c) can also be found in [Chambert-Loir and Ducros 2012, exemple (6.2.10)]. It remains to see the equivalence of (b) and (c). Suppose that (b) holds. Then there is a locally finite covering of X by trivializations U_i of L such that $-\log\|s_i\|$ is piecewise linear on $(U_i)^{\mathrm{an}}$ for every $i \in I$. By Proposition 8.4, there is a G-covering W_{ij} of $(U_i)^{\mathrm{an}}$ and analytic moment maps $\varphi_{ij} : W_{ij} \to (T_{ij})^{\mathrm{an}}$ such that $-\log\|s_i\| = \phi_{ij} \circ \varphi_{ij,\mathrm{trop}}$ on W_{ij} for an integral Γ-affine function ϕ_{ij} on $N_{ij,\mathbb{R}}$. The definition of integral Γ-affine functions shows that there is an invertible analytic function γ_{ij} on W_{ij} such that $\|s_i\| = |\gamma_{ij}|$ on W_{ij}. Using the trivialization $\gamma_{ij}^{-1} s_i$ on W_{ij}, we get (b) $\Rightarrow$ (c). The converse is an immediate application of Proposition 8.4. □

8.12. If X is proper over K, then an *algebraic K°-model* of X is an integral scheme $\mathfrak{X}$ which is of finite type, flat and proper over K° and with a fixed isomorphism between the generic fibre $\mathscr{X}_\eta$ and X. We use the isomorphism to identify $\mathfrak{X}_\eta$ and X. An *algebraic K°-model* of L is a line bundle $\mathfrak{L}$ on an algebraic K°-model $\mathfrak{X}$ of X together with a fixed isomorphism between $\mathfrak{L}_\eta$ and L. We define an *algebraic metric* on L as in 8.8 by using an algebraic K°-model $\mathfrak{L}$ of L.

Proposition 8.13. *On a line bundle on a proper variety over K, a metric is algebraic if and only if it is formal.*

Proof. Passing to the formal completion along the special fibre, it is clear that every algebraic metric is a formal metric. Using [Gubler 2003, Proposition 10.5], the converse is true if X is projective. The same argument shows that the converse is also true for proper X if the formal GAGA theorem in [EGA III_1 1961, Theorem 5.1.4] holds over K° and if X has an algebraic K°-model. In [EGA III_1 1961, Theorem 5.1.4], the base has to be noetherian and hence it applies only for discrete valuation rings. The required generalization is now given in [Fujiwara and Kato 2014, Theorem I.10.1.2]. The existence of an algebraic K°-model follows from Nagata's compactification theorem. This was proved by Nagata in the noetherian case and proved by Conrad in general (based on notes of Deligne, see [Temkin 2011] for another proof and references). □

Corollary 8.14. *Let L be a line bundle on a proper variety over K. Then L has an algebraic metric.*

Proof. This follows from Proposition 8.9 and Proposition 8.13. □

We will show now that many important metrics are piecewise smooth.

Example 8.15. Let L be a line bundle on the abelian variety A over K. Choosing a rigidification of L at $0 \in A$ and assuming L symmetric (resp. odd), the theorem of the cube allows one to identify $[m]^*(L)$ with $L^{\otimes m^2}$ (resp. $L^{\otimes m}$). There is a unique continuous metric $\|\cdot\|_{\text{can}}$ on L^{an} with $[m]^*\|\cdot\|_{\text{can}} = \|\cdot\|_{\text{can}}^{\otimes m^2}$ (resp. $[m]^*\|\cdot\|_{\text{can}} = \|\cdot\|_{\text{can}}^{\otimes m}$) for all $m \in \mathbb{Z}$. In general, $L^{\otimes 2}$ is the tensor product of a symmetric and an odd line bundle, unique up to 2-torsion in $\text{Pic}(A)$, and hence we get a *canonical metric* $\|\cdot\|_{\text{can}}$ on L which is unique up to multiples from $|K^\times|$ if we vary rigidifications. We claim that $\|\cdot\|_{\text{can}}$ is locally on X^{an} the tensor product of a smooth metric and a piecewise linear metric. In particular, we deduce that $\|\cdot\|_{\text{can}}$ is a piecewise smooth metric.

To prove the claim, we use the Raynaud extension of A to describe the canonical metric on L (see [Gubler 2010, §4] for details). The Raynaud extension is an exact sequence

$$1 \to T^{\text{an}} \to E \xrightarrow{q} B^{\text{an}} \to 0 \qquad (8.15.1)$$

of commutative analytic groups, where $T = \mathrm{Spec}(K[M])$ is a multiplicative torus of rank r and B is an abelian variety of good reduction. Moreover, there is a lattice P in E with $E/P = A^{\mathrm{an}}$. More precisely P is a discrete subgroup of $E(K)$ which is mapped by a canonical map, $\mathrm{val} : E \to N_{\mathbb{R}}$, isomorphically onto a complete lattice Λ of $N_{\mathbb{R}}$, where N is the dual of M. The map val is locally over B a tropicalization. Note that the Raynaud extension is algebraizable, but the quotient homomorphism $p : E \to A^{\mathrm{an}}$ is only defined in the analytic category.

Let $\mathscr{B}$ be the abelian scheme over K° with generic fibre B. By [loc. cit.] there exists a line bundle $\mathscr{H}$ on $\mathscr{B}$ such that $q^*((\mathscr{H}_\eta)^{\mathrm{an}}) = p^*(L^{\mathrm{an}})$. Here, and in the following, we use rigidified line bundles to identify isomorphic line bundles. Then $q^*\|\cdot\|_{\mathscr{H}}$ is a formal metric on $p^*(L^{\mathrm{an}})$. On $p^*(L^{\mathrm{an}})$, we have a canonical P-action α over the canonical action of P on E by translation. By [loc. cit.] there is a 1-cocycle Z in $Z^1(P, (\mathbb{R}^\times)^E)$ such that

$$(q^*\|\alpha_\gamma(w)\|_{\mathscr{H}})_{\gamma\cdot x} = Z_\gamma(x)^{-1} \cdot (q^*\|w\|_{\mathscr{H}})_x \tag{8.15.2}$$

for all $\gamma \in P,\ x \in E$ and $w \in (p^*L^{\mathrm{an}})_x$. The cocycle Z depends only on the map val, which means that there is a unique function $z_\lambda : N_{\mathbb{R}} \to \mathbb{R}$ with

$$z_\lambda(\mathrm{val}(x)) = -\log(Z_\gamma(x)) \quad (\gamma \in P,\ x \in E,\ \lambda = \mathrm{val}(\gamma)).$$

Moreover, there is a canonical symmetric bilinear form $b : \Lambda \times \Lambda \to \mathbb{Z}$ associated to L such that

$$z_\lambda(\omega) = z_\lambda(0) + b(\omega, \lambda) \quad (\omega \in N_{\mathbb{R}},\ \lambda \in \Lambda).$$

The cocycle condition

$$Z_{\rho\gamma}(x) = Z_\rho(\gamma x) Z_\gamma(x) \quad (\rho, \gamma \in P,\ x \in E)$$

shows that

$$z_{\lambda+\mu}(0) = z_\lambda(0) + z_\mu(0) + b(\lambda, \mu) \quad (\lambda, \mu \in \Lambda),$$

which means that $\lambda \mapsto z_\lambda(0)$ is a quadratic function on Λ. There is a unique extension to a quadratic function $q_0 : N_{\mathbb{R}} \to \mathbb{R}$. We define a metric $\|\cdot\|$ on $p^*(L^{\mathrm{an}})$ by $\|\cdot\| := e^{-q_0\circ\mathrm{val}} q^*\|\cdot\|_{\mathscr{H}}$. Using (8.15.2) and that q_0 is a quadratic function with associated bilinear form b, it follows easily that $\|\cdot\|$ descends to the canonical metric on L. We conclude from the descent with respect to the local isomorphism p that the canonical metric on L is locally on A^{an} the tensor product of a smooth metric with a piecewise linear metric. This proves the claim.

Example 8.16. Let L be a line bundle on a proper smooth algebraic variety over K which is algebraically equivalent to zero. Let A denote the Albanese variety of X (see [Grothendieck 1966, théorème 2.1, corollaire 3.2]). We fix some $x \in X(K)$ and obtain a universal morphism $\psi : X \to A$ from X to the abelian variety A with

$\psi(x) = 0$. Furthermore L is in a canonical way the pull-back of an odd line bundle on A along ψ. It follows that L carries a canonical metric $\|\cdot\|_{\mathrm{can}}$, unique up to multiples from $|K^\times|$. By [Gubler 2010, Example 3.7], there is an integer $N \geq 1$ such that $\|\cdot\|_{\mathrm{can}}^{\otimes N}$ is an algebraic metric and hence piecewise linear. We conclude that $\|\cdot\|_{\mathrm{can}}$ is a piecewise smooth metric.

Example 8.17. Let L be a line bundle on a complete toric variety X over K. Similarly as in the case of abelian varieties and using rigidifications, we have $[m]^*(L) = L^{\otimes m}$ and there is a unique metric $\|\cdot\|_{\mathrm{can}}$ on L with $[m]^*\|\cdot\|_{\mathrm{can}} = \|\cdot\|_{\mathrm{can}}^{\otimes m}$ for all integers $m \in \mathbb{Z}$ (see [Maillot 2000, Section 3]). There is a canonical algebraic K°-model $\mathscr{X}$ of K° and a canonical algebraic K°-model $\mathscr{L}$ by using the same rational polyhedral fan and the same piecewise linear function. Since $\|\cdot\|_{\mathrm{can}} = \|\cdot\|_{\mathscr{L}}$, the canonical metric on L is algebraic and hence a piecewise linear metric.

8.18. Finally, we consider the case where our variety X is defined over a ground field F which is equipped with the trivial valuation. If L is a line bundle on X, then we choose an algebraically closed extension field K endowed with a nontrivial complete absolute value extending the trivial absolute value of F. Then $F \subseteq K^\circ$ and the line bundle $L \otimes_F K^\circ$ on $X \otimes_F K^\circ$ is a canonical algebraic K°-model of the line bundle L_K on X_K. We conclude that L has a canonical metric $\|\cdot\|_{\mathrm{can}}$.

The metric $\|\cdot\|_{\mathrm{can}}$ has the following intrinsic description. Let $U = \mathrm{Spec}(A)$ be an affine open subset of X which is a trivialization of L given by the nowhere vanishing section $s \in \Gamma(U, L)$. We consider the formal affinoid subdomain $U^\circ := \{x \in U^{\mathrm{an}} \mid |f(x)| \leq 1 \ \forall f \in A\}$ of X^{an}. Note that U° is the set of points in U^{an} with reduction contained in U (see [Gubler 2013, §4] for more details). It follows that $\|s(x)\|_{\mathrm{can}} = 1$ for all $x \in U^\circ$. Since X is proper, such trivializations U° cover X^{an} leading to a description of $\|\cdot\|_{\mathrm{can}}$ which is independent of K.

For simplicity, we have considered only varieties in this paper. We may also consider continuous metrics on L^{an} for a line bundle over a separated scheme X of finite type over the ground field F. For such schemes X, the intrinsic description above shows in particular that we still have a canonical metric $\|\cdot\|_{\mathrm{can}}$ on L in the case of a trivially valued F.

9. Piecewise smooth forms and delta-metrics

We consider again an algebraic variety X over K of dimension n. In this section, we first study piecewise smooth forms on an open subset W of X^{an}. This leads to a decomposition of the first Chern current of a piecewise smoothly metrized line bundle $(L|_W, \|\cdot\|)$ into the sum of a piecewise smooth form and a residual current. We show that the residual current is induced by a generalized δ-form. If the first Chern current of $(L|_W, \|\cdot\|)$ is induced by a δ-form on W, then $\|\cdot\|$ is called a δ-metric and the δ-form is called the first Chern δ-form. We show that many important metrics are δ-metrics. In the following sections, we will use δ-metrics for our approach to nonarchimedean Arakelov theory.

9.1. In Definition 3.10, we defined the space $\mathrm{PS}(\Omega)$ of piecewise smooth superforms on an open subset Ω of a polyhedral subset. If (V, φ_U) is a tropical chart, then we apply this definition for the open subset $\Omega := \mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$. If $\alpha \in \mathrm{PS}(\Omega)$ and $(V', \varphi_{U'})$ is a tropical chart with $V' \subseteq V$ and $U' \subseteq U$, then we define $\alpha|_{V'}$ as the piecewise smooth form on $\Omega' := \mathrm{trop}_{U'}(V')$ given by pull-back of α with respect to the canonical affine map $N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$.

Definition 9.2. A *piecewise smooth form* on an open subset W of X^{an} may be defined in a similar way as a differential form in $A(W)$: A piecewise smooth form α is given by an open covering $(V_i, \varphi_{U_i})_{i \in I}$ of W by tropical charts and piecewise smooth superforms α_i on $\Omega_i := \mathrm{trop}_{U_i}(V_i)$ such that $\alpha_i|_{V_i \cap V_j} = \alpha_j|_{V_i \cap V_j}$ for all $i, j \in I$. A superform α' given by the covering $(V'_j, \varphi_{U'_j})_{j \in J}$ and piecewise smooth superforms α'_j on $\Omega'_j := \mathrm{trop}_{U'_j}(V'_j)$ will be identified with α if and only if $\alpha_i|_{V_i \cap V'_j} = \alpha'_j|_{V_i \cap V'_j}$ for every $i \in I$ and every $j \in J$.

9.3. We denote the space of piecewise smooth forms on W by $\mathrm{PS}(W)$. It comes with a bigrading and is canonically equipped with a $\wedge$-product. We conclude easily that $\mathrm{PS}^{\cdot,\cdot}(W)$ is a bigraded $A^{\cdot,\cdot}(W)$-algebra on X^{an}. It is clear that $\mathrm{PS}^{0,0}(W)$ is the space of piecewise smooth functions on W. It coincides with the space $P^{0,0}(W)$ of generalized δ-preforms of degree zero. The equality

$$\mathrm{PS}^{0,0}(W) = P^{0,0}(W) \tag{9.3.1}$$

is in fact a direct consequence of (4.19.3).

If $\varphi : X' \to X$ is a morphism of algebraic varieties over K, then the pull-back of piecewise smooth superforms from Definition 3.10 carries over to define a *pull-back* $f^* : \mathrm{PS}^{p,q}(W) \to \mathrm{PS}^{p,q}(W')$ for any open subset W' of $(X')^{\mathrm{an}}$ with $f(W') \subseteq W$. In the special case of $X = X'$, $f = \mathrm{id}$ and W' an open subset of W, we denote the pull-back by $\alpha|_{W'}$ and call it the *restriction of* α *to* W'.

9.4. In (3.11.1), we introduced differentials of piecewise smooth forms on open subsets of polyhedral sets. If $\alpha \in \mathrm{PS}^{p,q}(W)$ is given as in Definition 9.2, then the polyhedral differential $d'_{\mathrm{P}}\alpha \in \mathrm{PS}^{p+1,q}(W)$ is locally defined by $d'_{\mathrm{P}}\alpha_i \in \mathrm{PS}^{p+1,q}(\Omega_i)$. Similarly, we define $d''_{\mathrm{P}}\alpha \in \mathrm{PS}^{p,q+1}(W)$. Then $\mathrm{PS}^{\cdot,\cdot}(W)$ is a differential graded $\mathbb{R}$-algebra with respect to the polyhedral differentials d'_{P} and d''_{P}.

9.5. The bigraded differential $\mathbb{R}$-algebras $\mathrm{PS}(W)$ of piecewise smooth forms and $P(W)$ of generalized δ-forms are not directly comparable except that they both contain $A(W)$ as a bigraded differential $\mathbb{R}$-subalgebra. We construct a bigraded differential $\mathbb{R}$-algebra $\mathrm{PSP}(W)$ containing both spaces as follows.

Recall from Remark 3.14 that we have obtained a bigraded differential $\mathbb{R}$-algebra $\mathrm{PSP}(\widetilde{\Omega})$ with respect to $d'_{\mathrm{P}}, d''_{\mathrm{P}}$ for any open subset $\widetilde{\Omega}$ of $N_{\mathbb{R}}$. We repeat now the construction of generalized δ-forms in Section 4 building upon the spaces $\mathrm{PSP}(\widetilde{\Omega})$ instead of $P(\widetilde{\Omega})$. This leads first to spaces $\mathrm{PSP}(V, \varphi_U)$ for tropical charts (V, φ_U) of X and then to the desired space $\mathrm{PSP}(W)$. Note that $\mathrm{PSP}(W)$ is a differential bigraded $\mathbb{R}$-algebra with respect to the polyhedral differential operators d'_{P} and d''_{P} which extends the corresponding structure on the subalgebra $P(W)$. To see that $\mathrm{PS}(W)$ is a graded subalgebra of $\mathrm{PSP}(W)$, we use the obvious generalization of Proposition 1.8 from piecewise smooth functions to piecewise smooth forms. Obviously, $\mathrm{PSP}(W)$ is generated by the subalgebras $\mathrm{PS}(W)$ and $P(W)$. Moreover, the polyhedral differentials d'_{P} and d''_{P} agree with the corresponding differential operators on $\mathrm{PS}(W)$.

All properties of generalized δ-forms from Section 4 and Section 5 extend immediately to the sheaves PSP. Hence we have an integral $\int_W \alpha$ for any $\alpha \in \mathrm{PSP}^{n,n}_c(W)$. As a special case, we obtain such an integral for a piecewise smooth form with compact support on W. As in 6.4, this leads to a δ-current $[\alpha] \in E^{p,q}(W)$ for any $\alpha \in \mathrm{PSP}^{p,q}(W)$. In particular, this applies to a piecewise smooth α.

Remark 9.6. Note that the polyhedral differential $d'_{\mathrm{P}}\alpha$ of a piecewise smooth form α, or more generally of any $\alpha \in \mathrm{PSP}(W)$, is not compatible with the corresponding differential of the associated δ-current. We define the *d'-residue* by

$$\mathrm{Res}_{d'}(\alpha) := d'[\alpha] - [d'_{\mathrm{P}}\alpha].$$

Similarly, we define residues with respect to d'' and $d'd''$.

9.7. Now we consider a line bundle L on X endowed with a piecewise smooth metric $\|\cdot\|$ over the open subset W of X^{an}. We are going to obtain a canonical decomposition of the Chern current $[c_1(L|_W, \|\cdot\|)] \in E^{1,1}(X^{\mathrm{an}})$ (see 7.7) into a *piecewise smooth part* $c_1(L|_W, \|\cdot\|)_{\mathrm{ps}} \in \mathrm{PS}^{1,1}(W)$ and a *residual part* $[c_1(L|_W, \|\cdot\|)]_{\mathrm{res}} \in E^{1,1}(W)$.

Let (U, s) be a trivialization of L, i.e., U is an open subset of X and s is a nowhere vanishing section in $\Gamma(U, L)$. Then $-\log\|s\|$ is a piecewise smooth function on $U^{\mathrm{an}} \cap W$ and hence $-d'_{\mathrm{P}}d''_{\mathrm{P}} \log\|s|_{U^{\mathrm{an}}\cap W}\| \in \mathrm{PS}^{1,1}(U^{\mathrm{an}} \cap W)$. Note that this piecewise smooth form is independent of the choice of s by the same argument as in 7.7 and hence we obtain a globally defined element of $\mathrm{PS}^{1,1}(W)$ which we denote by $c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}$. Recall from 9.5 that we denote the associated δ-current on W by $[c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}]$. The same argument shows that the residues $\mathrm{Res}_{d'd''}(-\log\|s|_{U^{\mathrm{an}}\cap W}\|)$ paste together to give a global δ-current $[c_1(L|_W, \|\cdot\|)]_{\mathrm{res}} \in E^{1,1}(W)$ and we have

$$[c_1(L|_W, \|\cdot\|)] = [c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}] + [c_1(L|_W, \|\cdot\|)]_{\mathrm{res}}. \tag{9.7.1}$$

Proposition 9.8. *Let $\|\cdot\|$ be a piecewise smooth metric on $L|_W$. Then there is a unique $\beta \in P^{1,1}(W)$ with*

$$[\varphi^*(\beta)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)]_{\mathrm{res}} \in E^{1,1}(W')$$

for every morphism $\varphi : X' \to X$ from any algebraic variety X' over K and for every open subset W' of $\varphi^{-1}(W)$. The generalized δ-form β has codimension 1 (see Definition 4.13) *and will be denoted by $c_1(L|_W, \|\cdot\|)_{\mathrm{res}}$.*

Proof. Note that uniqueness follows from Proposition 6.8. By definition of a piecewise smooth metric, there is an open covering $(V_i)_{i\in I}$ of W by tropical charts (V_i, φ_{U_i}), nowhere vanishing sections $s_i \in \Gamma(V_i, L^{\mathrm{an}})$ and piecewise smooth functions ϕ_i on $\Omega_i := \mathrm{trop}_{U_i}(V_i)$ with $-\log\|s_i\| = \phi_i \circ \mathrm{trop}_{U_i}$ on V_i. Passing to a refinement of the open covering, we may assume that ϕ_i is defined on $\mathrm{Trop}(U_i)$. By Proposition 1.8, there is a piecewise smooth function $\tilde{\phi}_i$ on $N_{U_i,\mathbb{R}}$ restricting to ϕ_i. By Proposition 1.12, the corner locus $C_i := \tilde{\phi}_i \cdot N_{U_i,\mathbb{R}}$ of $\tilde{\phi}_i$ is a tropical cycle of codimension 1.

The δ-preform δ_{C_i} represents an element $\beta_i \in P(V_i, \varphi_{U_i})$ of codimension 1 (see Definition 4.4). We have seen in Remark 4.5 that there is a pull-back $f^*(\beta_i) \in P^{1,1}(V', \varphi_{U'})$ for every morphism $f : X' \to X$ of algebraic varieties over K and every tropical chart $(V', \varphi_{U'})$ of X' compatible with (V_i, φ_{U_i}). For the open subset $\Omega' := \mathrm{trop}_{U'}(V')$ of $\mathrm{Trop}(U')$, we have $f^*(\beta_i)|_{\Omega'} \in P^{1,1}(\Omega') \subseteq D^{1,1}(\Omega')$ (see (4.5.1)). Let $F : N_{U',\mathbb{R}} \to N_{U_i,\mathbb{R}}$ be the canonical affine map with $\mathrm{trop}_{U_i} = F \circ \mathrm{trop}_{U'}$ on $(U')^{\mathrm{an}}$. By Proposition 1.14 and Corollary 1.15, $F^*(C_i) \cdot \mathrm{Trop}(U')$ is the corner locus of $\phi' := \phi_i \circ F|_{\mathrm{Trop}(U')}$ and hence we get

$$f^*(\beta_i)|_{\Omega'} = F^*(\delta_{C_i}) \wedge \delta_{\mathrm{Trop}(U')} = \delta_{\phi' \cdot \mathrm{Trop}(U')} \in P^{1,1}(\Omega').$$

Together with the tropical Poincaré–Lelong formula (Corollary 3.19), we get

$$f^*(\beta_i)|_{\Omega'} + [d'_{\mathrm{P}} d''_{\mathrm{P}} \phi'] = d'd''[\phi'] \in D^{1,1}(\Omega'). \tag{9.8.1}$$

It follows from (9.8.1) that $f^*(\beta_i)|_{\Omega'}$ is independent of all choices. This yields that $\beta_i|_{V_i\cap V_j} = \beta_j|_{V_i\cap V_j}$ for all $i, j \in I$. We get a well-defined generalized δ-form $\beta \in P^{1,1}(W)$ of codimension 1 given by $\beta_i \in P^{1,1}(V_i, \varphi_{U_i})$ on V_i for every $i \in I$.

It remains to check that $[\varphi^*(\beta)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)]_{\mathrm{res}}$ for every morphism $\varphi : X' \to X$ and every open subset W' of $\varphi^{-1}(W)$. This has to be tested on $\alpha \in B_c^{n-1,n-1}(W')$. The claim is local and a partition of unity argument in a paracompact open neighbourhood of $\mathrm{supp}(\alpha)$ shows that we may assume $\mathrm{supp}(\alpha) \subseteq \varphi^{-1}(V_i)$ for some $i \in I$.

There are finitely many tropical charts $(V'_j, \varphi_{U'_j})_{j\in J}$ within W' which cover $\mathrm{supp}(\alpha)$ such that α is given on every V'_j by $\alpha_j \in \mathrm{AZ}^{n-1,n-1}(V'_j, \varphi_{U'_j})$. We choose a nonempty very affine open subset U' of X' contained in every U'_j and in $\varphi^{-1}(U_i)$.

By Proposition 5.9, U' is a very affine chart of integration for both $\varphi^*(\beta) \wedge \alpha$ and $d'd''\alpha$. By construction, $V' := \bigcup_{j \in J} V_j' \cap \varphi^{-1}(V_i) \cap (U')^{\mathrm{an}}$ and $\varphi_{U'}$ form a tropical chart in W'. By Proposition 5.7, α is given on $\Omega' := \mathrm{trop}_{U'}(V')$ by $\alpha_{U'} \in \mathrm{AZ}^{n-1,n-1}(V', \varphi_{U'})$. In the following, we will use only the δ-preform $\alpha' \in P^{n-1,n-1}(\Omega')$ induced by $\alpha_{U'}$. For the tropical cycle $C' := \mathrm{Trop}(U')$ and the canonical affine map $F : N_{U',\mathbb{R}} \to N_{U_i,\mathbb{R}}$, it follows as above that $\varphi^*(\beta)$ is given on V' by the element in $P^{1,1}(V', \varphi_{U'})$ represented by $\delta_{(\phi_i \circ F) \cdot N_{U',\mathbb{R}}} \in P^{1,1}(N_{U',\mathbb{R}})$. For $\phi' := \phi_i \circ F|_{\mathrm{Trop}(U')}$, we have seen that

$$\varphi^*(\beta_i)|_{\Omega'} = (\delta_{(\phi_i \circ F) \cdot N_{U',\mathbb{R}}})|_{\Omega'} = \delta_{\phi' \cdot C'} \in P^{1,1}(\Omega').$$

Note that $\mathrm{supp}(\alpha) \subseteq \bigcup_{j \in J} V_j' \cap \varphi^{-1}(V_i)$. We deduce from the generalizations of Corollary 5.6 and Proposition 4.21 to PSP-forms (see 9.5) that the currents $d_{\mathrm{P}}'\phi' \wedge \alpha'$, $d_{\mathrm{P}}''\phi' \wedge \alpha'$, $d'd''\alpha'$, $\alpha' \wedge \delta_{\phi' \cdot C'}$ have compact support in Ω'. We write $C' = (\mathscr{C}', m')$ for an integral Γ-affine polyhedral complex $\mathscr{C}'$ and a family of integral weights m'. To prove $[c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)]_{\mathrm{res}} = [\varphi^*(\beta)]$, we have to show that

$$\int_{|\mathscr{C}'|} \phi' \wedge d'd''\alpha' = \int_{|\mathscr{C}'|} \delta_{\phi' \cdot C'} \wedge \alpha' + \int_{|\mathscr{C}'|} d_{\mathrm{P}}'d_{\mathrm{P}}''\phi' \wedge \alpha' \tag{9.8.2}$$

holds. If α' has compact support in Ω', then this follows from the tropical Poincaré–Lelong formula (9.8.1). In general, we still can deduce from the proof of the tropical Poincaré–Lelong formula in Theorem 3.16 the formula (3.16.3) which here reads as

$$\int_{|\mathscr{C}'|} \phi' \wedge d'd''\alpha' = -\int_{\partial|\mathscr{C}'|} d_{\mathrm{P}}''\phi' \wedge \alpha' + \int_{|\mathscr{C}'|} d_{\mathrm{P}}'d_{\mathrm{P}}''\phi' \wedge \alpha'$$

as we have used only that $d'\alpha'$ and $d_{\mathrm{P}}''\phi' \wedge \alpha'$ have compact support. Now (9.8.2) follows from Lemma 3.17 and Remark 3.18 using additionally that $\alpha' \wedge \delta_{\phi' \cdot C'}$ has compact support. □

Definition 9.9. A metric $\|\cdot\|$ on $L|_W$ is called a *δ-metric* if for every $x \in W$, there are a tropical chart (V, φ_U) such that $x \in V \subseteq W$ and a piecewise smooth function ϕ on $\mathrm{Trop}(U)$ satisfying the following properties:

(i) There is a nowhere vanishing section s of L over U such that $\phi \circ \mathrm{trop}_U = -\log\|s\|$ on V.

(ii) There is a superform γ on $N_{U,\mathbb{R}}$ of bidegree $(1,1)$ with piecewise smooth coefficients such that $d_{\mathrm{P}}'d_{\mathrm{P}}''\phi$ and $\gamma|_{\mathrm{Trop}(U)}$ agree on the open subset $\mathrm{trop}_U(V)$ of $\mathrm{Trop}(U)$.

Remark 9.10. Condition (i) just means that the metric is piecewise smooth. Note that a superform on $N_{U,\mathbb{R}}$ with piecewise smooth coefficients is the same as a δ-preform on $N_{U,\mathbb{R}}$ of codimension 0 (see Example 2.10). Using 9.7, we deduce easily that (ii) is equivalent to the condition that $[c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}]$ is the δ-current associated to a generalized δ-form on W (of codimension 0).

Proposition 9.11. *Let $\|\cdot\|$ be a piecewise smooth metric on $L|_W$. Then $\|\cdot\|$ is a δ-metric if and only if there is a $\beta \in B^{1,1}(W)$ with*

$$[\varphi^*(\beta)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)] \in E^{1,1}(W')$$

for every morphism $\varphi : X' \to X$ from any algebraic variety X' over K and for every open subset W' of $\varphi^{-1}(W)$.

Proof. Suppose that $\|\cdot\|$ is a δ-metric. By Remark 9.10, there is $\gamma \in P^{1,1}(W)$ of codimension 0 such that $[c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}] = [\gamma]$. Since γ is of codimension 0, we may handle γ as a piecewise smooth form and hence we get

$$[\varphi^*(\gamma)] = [\varphi^*(c_1(L|_W, \|\cdot\|)_{\mathrm{ps}})] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)_{\mathrm{ps}}] \in E^{1,1}(W').$$

Proposition 9.8 yields that $\beta := c_1(L, \|\cdot\|)_{\mathrm{res}} + \gamma \in P^{1,1}(W)$ and that

$$[\varphi^*(\beta)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)_{\mathrm{res}}] + [\varphi^*(\gamma)] = [c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|)] \in E^{1,1}(W')$$

as claimed. It remains to show that $\beta \in B^{1,1}(W)$. Let (V, φ_U) be a tropical chart in W and let ϕ be a piecewise smooth function on $\mathrm{Trop}(U)$ as in Definition 9.9 such that $\beta|_V$ is given by $\beta_V \in P^{1,1}(V, \varphi_U)$. For every tropical chart $(U', \varphi_{U'})$ of an algebraic variety X' over K compatible with (V, φ_U) with respect to the morphism $f : X' \to X$ and for $\Omega' := \mathrm{trop}_{U'}(V')$, the last display yields

$$[f^*(\beta_V)|_{\Omega'}] = d'd''[\phi \circ F] \in D^{1,1}(\Omega'), \tag{9.11.1}$$

where $F : N_{U',\mathbb{R}} \to N_{U,\mathbb{R}}$ is the canonical affine map. Since this supercurrent is d'-closed and d''-closed on Ω', we conclude that β is given on V by an element of $Z(V, \varphi_U)$. This shows $\beta \in B^{1,1}(W)$.

To prove the converse, we use that $[c_1(L|_W, \|\cdot\|)] = [\beta]$ for some $\beta \in P^{1,1}(W)$. By Proposition 9.8, the δ-current associated to $\beta - c_1(L|_W, \|\cdot\|)_{\mathrm{res}} \in P^{1,1}(W)$ is $[c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}]$. By Remark 9.10, $\|\cdot\|$ is a δ-metric. $\square$

Definition 9.12. Let $\|\cdot\|$ be a δ-metric on $L|_W$. By Proposition 6.8, the δ-form β in Proposition 9.11 is unique. We call it the *first Chern δ-form* of $(L|_W, \|\cdot\|)$ and we denote it by $c_1(L|_W, \|\cdot\|)$.

9.13. We summarize the above constructions and definitions. A metric $\|\cdot\|$ on $L|_W$ is a δ-metric if and only if every $x \in W$ is contained in a tropical chart (V, φ_U) in W with a piecewise smooth function ϕ on $N_{U,\mathbb{R}}$ and a nowhere vanishing section s of L over U such that

$$-\log\|s\| = \phi \circ \mathrm{trop}_U$$

on V and such that

$$d'_{\mathrm{P}}d''_{\mathrm{P}}(\phi|_{\mathrm{trop}_U(V)}) = \gamma|_{\mathrm{trop}_U(V)}$$

for a superform γ on $N_{\mathbb{R}}$ of bidegree (1, 1) with piecewise smooth coefficients. Then the restriction of $c_1(L|_W, \|\cdot\|)_{\mathrm{res}}$ to V is represented by the δ-preform $\delta_{\phi \cdot N_{U,\mathbb{R}}}$ on $N_{U,\mathbb{R}}$, $c_1(L|_W, \|\cdot\|)_{\mathrm{ps}}|_V$ is given by γ and $c_1(L|_W, \|\cdot\|)|_V$ is represented by the δ-preform $\gamma + \delta_{\phi \cdot N_{U,\mathbb{R}}}$ on $N_{U,\mathbb{R}}$. A piecewise linear metric is a δ-metric as we can choose ϕ integral Γ-affine (use Remark 1.9) and $\gamma = 0$.

9.14. By construction, the δ-current associated to $c_1(L|_W, \|\cdot\|)$ is equal to the first Chern current $[c_1(L|_W, \|\cdot\|)]$ defined in 7.7 which explains the notation used there. It is an immediate consequence of (9.11.1) that the first Chern δ-form $c_1(L|_W, \|\cdot\|)$ is d'-closed and d''-closed.

To be a δ-metric is a local property and respects isometry. The tensor product of δ-metrics is again a δ-metric and the dual metric of a δ-metric is also a δ-metric. If a positive tensor power of a metric $\|\cdot\|$ on $L|_W$ is a δ-metric, then $\|\cdot\|$ is a δ-metric. It is easy to see that the first Chern δ-form $c_1(L|_W, \|\cdot\|)$ is additive in terms of isometry classes $(L|_W, \|\cdot\|)$ for δ-metrics $\|\cdot\|$.

Proposition 9.15. *Let $\varphi : X' \to X$ be a morphism of algebraic varieties and let L be a line bundle on X endowed with a δ-metric $\|\cdot\|$ over the open subset W of X^{an}. Then $\varphi^*\|\cdot\|$ is a δ-metric on $\varphi^*(L)|_{W'}$ and we have*

$$c_1(\varphi^*(L)|_{W'}, \varphi^*\|\cdot\|) = \varphi^* c_1(L|_W, \|\cdot\|) \in B^{1,1}(W') \tag{9.15.1}$$

for any open subset W' of $\varphi^{-1}(W)$.

Proof. This follows from 8.6 and Proposition 9.11. □

Remark 9.16. Smooth metrics and piecewise linear metrics are δ-metrics, which is clear from the definitions. It follows from Proposition 8.11 that every formal metric is a δ-metric. In particular, every algebraic metric on a line bundle of a proper variety is a δ-metric.

Example 9.17. All the canonical metrics in Examples 8.15, 8.16 and 8.17 are δ-metrics. Indeed, a positive tensor power of such a metric is locally the tensor product of a formal metric with a smooth metric and hence the claim follows from Remark 9.16.

10. Monge–Ampère measures

We have seen in the previous section that formal metrics are δ-metrics giving rise to a first Chern δ-form. The formalism of δ-forms allows us to define the Monge–Ampère measure as a wedge product of first Chern δ-forms. We recall that Chambert-Loir has introduced discrete measures for formally metrized line bundles on a proper variety which are important for nonarchimedean equidistribution. The main result of this section shows that the Monge–Ampère measure is equal to the Chambert-Loir measure.

In this section X is a proper algebraic variety over K of dimension n.

10.1. Let $\overline{L}_1, \dots, \overline{L}_n$ be line bundles on X endowed with δ-metrics. Then the wedge product $c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n)$ of the first Chern δ-forms is a δ-form of bidegree (n, n). By Corollary 6.15, the δ-current associated to a δ-form on X^{an} of type (n, n) extends to a bounded linear functional on the space of continuous functions and defines a signed Radon measure on X^{an}. The *Monge-Ampère measure* is the signed measure associated to $c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_n)$; it is denoted by

$$\mathrm{MA}\big(c_1(\overline{L}_1), \dots, c_1(\overline{L}_n)\big).$$

Proposition 10.2. *If $\varphi : X' \to X$ is a morphism of n-dimensional proper varieties over K, then the following projection formula holds*:

$$\varphi_* \,\mathrm{MA}\big(c_1(\varphi^*\overline{L}_1), \dots, c_1(\varphi^*\overline{L}_n)\big) = \deg(\varphi)\,\mathrm{MA}\big(c_1(\overline{L}_1), \dots, c_1(\overline{L}_n)\big).$$

Proof. The Stone–Weierstraß theorem [Chambert-Loir and Ducros 2012, proposition (3.3.5)] implies that $A^0(X^{\mathrm{an}})$ is a dense subspace of $C(X^{\mathrm{an}})$. For functions in $A^0(X^{\mathrm{an}})$ the desired equality follows from Proposition 9.15 and from the projection formula for δ-forms (5.9.1). This yields our claim. □

Proposition 10.3. *If X is a proper variety of dimension n, then the total mass of* $\mathrm{MA}\big(c_1(L_1, \|\cdot\|_1), \dots, c_1(L_n, \|\cdot\|_n)\big)$ *is equal to* $\deg_{L_1,\dots,L_n}(X)$.

Proof. This follows as in [Chambert-Loir and Ducros 2012, proposition (6.4.3)]. They handled there only the case of smooth metrics, but our formalism of δ-forms allows us to obtain this result more generally for δ-metrics. □

We recall the crucial properties of Chambert-Loir's measures. They were introduced in a slightly different setting by Chambert-Loir [2006].

Proposition 10.4. *There is a unique way to associate to any n-dimensional proper variety X over K and to any family of formally metrized line bundles $\overline{L}_1, \dots, \overline{L}_n$ on X a signed Radon measure $\mu = \mu_{\overline{L}_1,\dots,\overline{L}_n}$ on X^{an} such that the following properties hold*:

(a) *The measure μ is multilinear and symmetric in $\overline{L}_1, \dots, \overline{L}_n$.*

(b) *If $\varphi : Y \to X$ is a morphism of n-dimensional proper varieties over K, then the following projection formula holds*:

$$\varphi_*(\mu_{\varphi^*\overline{L}_1,\dots,\varphi^*\overline{L}_n}) = \deg(\varphi)\mu_{\overline{L}_1,\dots,\overline{L}_n}.$$

(c) *If $\mathscr{X}$ is a formal K°-model of X with reduced special fibre $\mathscr{X}_s$ and if the metric of $\overline{L}_j$ is induced by a formal K°-model $\mathscr{L}_j$ of L_j on $\mathscr{X}$ for every $j = 1, \dots, n$, then*

$$\mu = \sum_Y \deg_{\mathscr{L}_1,\dots,\mathscr{L}_n}(Y)\delta_{\xi_Y},$$

where Y ranges over the irreducible components of $\mathscr{X}_s$ and δ_{ξ_Y} is the Dirac measure in the unique point ξ_Y of X^{an} which reduces to the generic point of Y (see [Berkovich 1990, Proposition 2.4.4]).

(d) *The total mass is given by $\mu(X^{\mathrm{an}}) = \deg_{L_1,\dots,L_n}(X)$.*

Proof. For existence, we refer to [Gubler 2007, §3]. Uniqueness follows from (c) alone as the existence of a simultaneous formal K°-model with reduced special fibre is a consequence of [Gubler 1998, Proposition 7.5]. □

Theorem 10.5. *For formally metrized line bundles $\overline{L}_1, \dots, \overline{L}_n$ on the proper variety X of dimension n, the Monge–Ampère measure $\mathrm{MA}(c_1(\overline{L}_1), \dots, c_1(\overline{L}_n))$ agrees with the Chambert-Loir measure $\mu_{\overline{L}_1,\dots,\overline{L}_n}$.*

This theorem was first proven by Chambert-Loir and Ducros [2012, §6.9] for their Monge–Ampère measures defined by a tricky approximation process with smooth metrics. Their argument uses Zariski–Riemann spaces, while we use here a more tropical approach related to our δ-forms.

10.6. In Lemma 10.8, we will consider a closed subvariety $\mathscr{U}$ of a torus $\mathbb{T} = \mathrm{Spec}(K^\circ[M])$ over K°. We will use the following notation: N is the dual of the free abelian group M of finite rank. Let U be the generic fibre of $\mathscr{U}$ and let $\mathscr{U}_s$ be the special fibre.

The tropicalization $\mathrm{trop} : (\mathbb{T}_K)^{\mathrm{an}} \to N_{\mathbb{R}}$ (resp. $\mathrm{trop} : \mathbb{T}_s^{\mathrm{an}} \to N_{\mathbb{R}}$) with respect to the valuation v on K (resp. the trivial valuation on $\tilde{K}$) leads to the tropical variety $\mathrm{Trop}(U)$ (resp. $\mathrm{Trop}(\mathscr{U}_s)$).

The local cone $\mathrm{LC}_0(\mathrm{Trop}(U))$ at 0 is defined as the cone in $N_{\mathbb{R}}$ which agrees with $\mathrm{Trop}(U)$ in a neighbourhood of 0. We endow it with the weights induced by the canonical tropical weights on $\mathrm{Trop}(U)$.

10.7. For the proof of Theorem 10.5, we need a preparatory result. Let L be a line bundle on the proper variety X over K. We consider an algebraic K°-model $(\mathscr{X}, \mathscr{L})$ of (X, L). Then we get an algebraic metric $\|\cdot\|_{\mathscr{L}}$ on L. We have seen in 8.18 that the restriction $\mathscr{L}_s$ of $\mathscr{L}$ to the special fibre $\mathscr{X}_s$ has a canonical metric $\|\cdot\|_{\mathrm{can}}$. Note that the first metric is continuous on the Berkovich space X^{an} with respect to the given valuation v while $\|\cdot\|_{\mathrm{can}}$ is continuous on the Berkovich space $\mathscr{X}_s^{\mathrm{an}}$ with respect to the trivial valuation on the residue field $\tilde{K}$. Since $\mathscr{X}$ is assumed to be proper, we have a reduction map $\pi : X^{\mathrm{an}} \to \mathscr{X}_s$. For $x \in X^{\mathrm{an}}$, $\pi(x)$ is a scheme theoretic point of $\mathscr{X}_s$. Using the trivial valuation on the residue field of $\pi(x)$, we will view $\pi(x)$ as a point of $\mathscr{X}_s^{\mathrm{an}}$.

In the next lemma, we will show that $\|\cdot\|_{\mathrm{can}}$ is piecewise linear in an neighbourhood of $\pi(x)$ in $\mathscr{X}_s^{\mathrm{an}}$. This means that using a trivialization and a tropicalization, the canonical metric is induced by a piecewise linear function on the tropical variety. It will be crucial in the proof of Theorem 10.5 that we can use tropically the same piecewise linear function to describe the formal metric $\|\cdot\|_{\mathscr{L}}$ in a neighbourhood of x in X^{an}. We now make this precise:

Lemma 10.8. *Under the setup given in 10.7, we fix an element $x \in X^{\mathrm{an}}$ and an open neighbourhood $\mathscr{V}$ of $\pi(x)$ in $\mathscr{X}$. Then there is an open neighbourhood $\mathscr{U}$ of $\pi(x)$ in $\mathscr{V}$ and a closed embedding $\mathscr{U} \hookrightarrow \mathbb{T}$ into a torus $\mathbb{T} = \mathrm{Spec}(K^\circ[M])$ with the following properties* (using the notation from 10.6):

(a) *We have $0 = \mathrm{trop}(x)$ and the weighted local cone in 0 satisfies*

$$\mathrm{LC}_0(\mathrm{Trop}(U)) = \mathrm{Trop}(\mathscr{U}_s).$$

(b) *There is an open neighbourhood $\widetilde{\Omega}$ of 0 in $N_{\mathbb{R}}$ with*

$$\mathrm{LC}_0(\mathrm{Trop}(U)) \cap \widetilde{\Omega} = \mathrm{Trop}(U) \cap \widetilde{\Omega}.$$

(c) *There exist a complete rational polyhedral fan Σ on $N_{\mathbb{R}}$ and a continuous function $\phi : N_{\mathbb{R}} \to \mathbb{R}$ which is piecewise linear with respect to Σ (i.e., for every $\sigma \in \Sigma$, there is $u_\sigma \in M$ with $\phi = u_\sigma$ on σ).*

(d) *$\mathscr{U}$ is a trivialization of $\mathscr{L}$ with respect to a nowhere vanishing section $s \in \Gamma(\mathscr{U}, \mathscr{L})$.*

(e) *We have $-\log\|s\|_{\mathscr{L}} = \phi \circ \mathrm{trop}$ on a neighbourhood of x in X^{an}.*

(f) *We have $-\log\|s\|_{\mathrm{can}} = \phi \circ \mathrm{trop}$ on a neighbourhood of $\pi(x)$ in $\mathscr{X}_s^{\mathrm{an}}$.*

If $\pi(x)$ is the generic point of an irreducible component of $\mathscr{X}_s$, then there is a $\mathscr{U}$ as above with (a)–(d) *and the following stronger properties*:

(e′) *We have $-\log\|s\|_{\mathscr{L}} = \phi \circ \mathrm{trop}$ on $\mathrm{trop}^{-1}(\widetilde{\Omega}) \subseteq U^{\mathrm{an}}$.*

(f′) *The identity $-\log\|s\|_{\mathrm{can}} = \phi \circ \mathrm{trop}$ holds on $\mathscr{U}_s^{\mathrm{an}}$.*

Proof. Let $(\mathscr{U}_i)_{i \in I}$ be a finite affine open covering of $\mathscr{X}$ such that $\mathscr{L}$ is trivial over any $\mathscr{U}_i$. The generic (resp. special) fibre of $\mathscr{U}_i$ is denoted by U_i (resp. $\mathscr{U}_{i,s}$). For every $i \in I$, we choose a nowhere vanishing section $s_i \in \Gamma(\mathscr{U}_i, \mathscr{L})$. Let $I(x) := \{i \in I \mid \pi(x) \in \mathscr{U}_{i,s}\}$. For $i \in I(x)$, let $(x_{ij})_{j \in J_i}$ be a finite set of generators of the K°-algebra $\mathscr{O}(\mathscr{U}_i)$. Replacing x_{ij} by $1 + x_{ij}$ if necessary, we may assume that these generators are invertible in $\pi(x)$. For $i \in I$, we have an affinoid subdomain

$$U_i^\circ := \{z \in U_i^{\mathrm{an}} \mid |a(z)| \leq 1 \ \forall a \in \mathscr{O}(\mathscr{U}_i)\} = \{z \in U_i^{\mathrm{an}} \mid \pi(z) \in \mathscr{U}_{i,s}\}$$

of X^{an}. Using the trivial valuation on $\tilde{K}$, we similarly get an affinoid subdomain $\mathscr{U}_{i,s}^\circ := \{z \in \mathscr{U}_{i,s}^{\mathrm{an}} \mid |a(z)| \leq 1 \ \forall a \in \mathscr{O}(\mathscr{U}_{i,s})\}$ of $\mathscr{X}_s^{\mathrm{an}}$. We consider $\pi(x)$ as a point of $\mathscr{X}_s^{\mathrm{an}}$ by using the trivial absolute value on the residue field of $\pi(x)$ and hence we have $I(x) = \{i \in I \mid \pi(x) \in \mathscr{U}_{i,s}^\circ\}$.

It is easy to see that $\pi(x)$ has a very affine open neighbourhood $\mathscr{U}$ in $\mathscr{X}$ such that $\mathscr{U}$ is contained in $\mathscr{U}_i$ for every $i \in I(x)$. Very affine means that there is a closed embedding $\varphi : \mathscr{U} \hookrightarrow \mathbb{T}$ into a torus $\mathbb{T} = \operatorname{Spec}(K^\circ[M])$. By shrinking $\mathscr{U}$ and by adding new invertible functions to φ, we obtain the following properties:

(i) For every $i, k \in I(x)$, the invertible meromorphic function s_i/s_k on $\mathscr{U}$ is the restriction of a character $\chi^{u_{ik}}$ associated to some $u_{ik} \in M$.

(ii) For every $i \in I(x)$ and every $j \in J_i$, the generator x_{ij} is invertible on $\mathscr{U}$ and equal to the restriction of a character $\chi^{u'_{ij}}$ associated to some $u'_{ij} \in M$.

Note that we have $0 = \operatorname{trop}(\pi(x)) \in \operatorname{Trop}(\mathscr{U}_s)$ since we use the trivial valuation on the residue field of $\pi(x)$. It follows from $\pi(x) \in \mathscr{U}_s$ that $\operatorname{trop}(x) = 0$. By definition, $\mathscr{U}_s$ is the initial degeneration of U at 0 and hence (a) follows from [Gubler 2013, Propositions 10.15, 13.7]. By definition of the local cone, we find an open neighbourhood $\widetilde{\Omega}$ of 0 in $N_{\mathbb{R}}$ with (b).

By construction, $\mathscr{L}$ is trivial over $\mathscr{U}$ and we choose $s := s_k$ for a fixed $k \in I(x)$ in (d). For $i \in I(x)$, we define the rational cone $\sigma_i := \{\omega \in N_{\mathbb{R}} \mid \langle \omega, u'_{ij} \rangle \geq 0 \, \forall j \in J_i\}$ in $N_{\mathbb{R}}$. Then (ii) yields

$$\mathscr{U}_{i,s}^\circ \cap \mathscr{U}_s^{\mathrm{an}} = \operatorname{trop}^{-1}(\sigma_i) \cap \mathscr{U}_s^{\mathrm{an}}. \tag{10.8.1}$$

By the Bieri–Groves theorem, $\operatorname{Trop}(\mathscr{U}_s)$ is the support of a rational polyhedral fan in $N_{\mathbb{R}}$ (see [Gubler 2013, Remark 3.4]). We conclude that there is a complete rational polyhedral fan Σ on $N_{\mathbb{R}}$ and a rational polyhedral subfan Σ_x with

$$|\Sigma_x| = \bigcup_{i \in I(x)} \sigma_i \cap \operatorname{Trop}(\mathscr{U}_s)$$

such that every cone $\sigma \in \Sigma_x$ is contained in σ_i for some $i \in I(x)$. Note that $\|s_i\|_{\mathrm{can}} = 1$ on $\mathscr{U}_{i,s}^\circ$ and hence (i) shows that

$$-\log\|s\|_{\mathrm{can}} = -\log|s_k/s_i| = u_{ki} \circ \operatorname{trop} \tag{10.8.2}$$

on $\mathscr{U}_{i,s}^\circ \cap \mathscr{U}_s^{\mathrm{an}}$. By (10.8.1), there is a continuous function $\phi : |\Sigma_x| \to \mathbb{R}$ with $\phi = u_{ki}$ on every σ. Using Remark 1.9 and passing to a refinement of Σ, we easily extend ϕ to a continuous function on $N_{\mathbb{R}}$ satisfying (c). Since $\mathscr{X}_s$ is proper over $\tilde{K}$, the sets $\mathscr{U}_{i,s}^\circ$, $i \in I$, form an open covering of $\mathscr{X}_s^{\mathrm{an}}$. It follows from (10.8.1) and (10.8.2) that (f) holds in the neighbourhood

$$W := \mathscr{U}_s^{\mathrm{an}} \setminus \bigcup_{i \in I \setminus I(x)} \mathscr{U}_{i,s}^\circ$$

of $\pi(x)$ in $\mathscr{X}_s^{\mathrm{an}}$.

Again (ii) shows

$$U_i^\circ \cap U^{\mathrm{an}} = \operatorname{trop}^{-1}(\sigma_i) \cap U^{\mathrm{an}} \tag{10.8.3}$$

for every $i \in I(x)$. Note that $\|s_i\|_{\mathscr{L}} = 1$ on U_i° and hence (i) shows that

$$-\log\|s\|_{\mathscr{L}} = -\log|s_k/s_i| = u_{ki} \circ \operatorname{trop} \tag{10.8.4}$$

on $U_i^\circ \cap U^{\mathrm{an}}$. Since X is proper, the sets U_i°, $i \in I$, form a compact covering of X^{an}. It follows from (a), (b), (10.8.3) and (10.8.4) that (e) holds in the neighbourhood $\mathrm{trop}^{-1}(\widetilde{\Omega}) \setminus \bigcup_{i \in I \setminus I(x)} U_i^\circ$ of x in X^{an}. This proves (e).

We assume now that $\pi(x)$ is the generic point of an irreducible component Y of $\mathscr{X}_s$. Then we may assume that $\mathscr{U}_s \subseteq Y$. Let $i \in I$ with $\mathscr{U}_{i,s} \cap Y \neq \varnothing$. Since we use the trivial valuation on the residue field of $\pi(x)$, we deduce easily that $\pi(x) \in \mathscr{U}_{i,s}^\circ$. By construction, we get $W = \mathscr{U}_s^{\mathrm{an}}$, proving (f′). It remains to show (e′). Let $i \in I \setminus I(x)$. By construction, we have $\mathscr{U}_{i,s} \cap Y = \varnothing$. For $y \in U_i^\circ \cap U^{\mathrm{an}}$, we have $\pi(y) \in \mathscr{U}_{i,s}$ and hence $\pi(y) \notin Y$. In particular, we have $y \notin U^\circ$. Using $\mathrm{trop}^{-1}(0) \cap U^{\mathrm{an}} = U^\circ$, we see that $\mathrm{trop}(U_i^\circ \cap U^{\mathrm{an}})$ is a closed subset of $\mathrm{Trop}(U)$ not containing 0. By shrinking $\widetilde{\Omega}$, we may assume that $\widetilde{\Omega}$ is a neighbourhood of 0 which is disjoint from $\mathrm{trop}(U_i^\circ \cap U^{\mathrm{an}})$ for every $i \in I \setminus I(x)$. Then the above proof of (e) shows that (e′) holds. □

Proof of Theorem 10.5. Let $\mu^{\mathrm{MA}} := \mathrm{MA}(c_1(\overline{L_1}), \ldots, c_1(\overline{L_n}))$. For simplicity, we assume that $L = L_1 = \cdots = L_n$ and that all metrics are induced by the same K°-model $\mathscr{L}$ on $\mathscr{X}$. The general case follows either by the same arguments or by multilinearity. It is more convenient for us to work algebraically and so we use Proposition 8.13 to assume that $\mathscr{X}$ and $\mathscr{L}$ are algebraic K°-models. There is a generically finite surjective morphism $\mathscr{X}' \to \mathscr{X}$ from a proper flat variety $\mathscr{X}'$ over K° with reduced special fibre. This is a consequence of de Jong's pluristable alteration theorem which works over any Henselian valuation ring (see [Berkovich 1999, Lemma 9.2]). Since both sides of the claim satisfy the projection formula, we may prove the claim for $\mathscr{X}'$. This shows that we may assume that $\mathscr{X}$ is an algebraic K°-model of X with reduced special fibre.

We will analyse μ^{MA} in a neighbourhood of $x \in X^{\mathrm{an}}$. Let $\pi(x) \in \mathscr{X}_s$ be the reduction of x. We choose a very affine open neighbourhood $\mathscr{U}$ of $\pi(x)$ in $\mathscr{X}$ as in Lemma 10.8. We will use the closed embedding $\mathscr{U} \hookrightarrow \mathbb{T}$ into the torus $\mathbb{T}$ and the notation from there.

It follows from a theorem of Ducros [2012, théorème 3.4] that x has a compact analytic neighbourhood V such that the germ of $\mathrm{trop}(V)$ in $\mathrm{trop}(x)$ (considering polytopal neighbourhoods) agrees with the germ of $\mathrm{trop}(W)$ in $\mathrm{trop}(x)$ for every compact analytic neighbourhood $W \subseteq V$ of x. Using that $\mathrm{trop}(x) = 0$, we deduce from [Ducros 2012, théorème 3.4 item 1)] that the dimension of the germ is equal to the transcendence degree of $\tilde{K}(\pi(x))$ over $\tilde{K}$.

We first assume that $\pi(x)$ is not the generic point of an irreducible component of $\mathscr{X}_s$. Then the transcendence degree of $\tilde{K}(\pi(x))$ over $\tilde{K}$ is less than $n = \dim(X)$. Using the theorem of Ducros, there is a compact analytic neighbourhood V of x in X^{an} such that $\mathrm{trop}(V)$ has dimension $< n$ and such that Lemma 10.8(e) holds on V. We choose a tropical chart $(V', \varphi_{U'})$ in x which is contained in V and with U' contained in the generic fibre U of $\mathscr{U}$. We describe $c_1(\bar{L})|_{V'}$ using the function ϕ constructed in Lemma 10.8 and the canonical affine map $F : N_{U',\mathbb{R}} \to N_{\mathbb{R}}$. Using that ϕ is piecewise linear, it follows from 9.13 that $c_1(\bar{L})|_{V'} = \mathrm{trop}_{U'}^*(\beta)$ for $\beta \in \mathrm{AZ}^{1,1}(V', \varphi_{U'})$ represented by the δ-preform $\delta_{F^*(\phi)\cdot N_{U',\mathbb{R}}}$ on $N_{U',\mathbb{R}}$. By our construction of products and Corollary 1.15, μ^{MA} is given on V' by $\beta^{\wedge n} \in \mathrm{AZ}^{n,n}(V', \varphi_{U'})$ represented by the δ-preform

$$(\delta_{F^*(\phi)\cdot N_{U',\mathbb{R}}})^{\wedge n} = \delta_{F^*(C)} \tag{10.8.5}$$

on $N_{U',\mathbb{R}}$, where C is the n-codimensional tropical cycle of $N_{\mathbb{R}}$ obtained by the n-fold self-intersection of the tropical divisor $\phi \cdot N_{\mathbb{R}}$. Since $V' \subseteq V$, we have $F(\mathrm{trop}_{U'}(V')) \subseteq \mathrm{trop}(V)$ and hence $\dim(F(\mathrm{trop}_{U'}(V'))) < n$. It follows from the definition of the pull-back and the local nature of stable tropical intersection that $\delta_{F^*(C)\cdot \mathrm{Trop}(U')}$ does not meet the open subset $\mathrm{trop}_{U'}(V')$ of $\mathrm{Trop}(U')$. A similar argument applies to any tropical chart compatible with $(V', \varphi_{U'})$ and hence (10.8.5) yields $\beta^{\wedge n} = 0$. We conclude that the support of μ^{MA} does not meet V'.

Now we assume that $\pi(x)$ is the generic point of an irreducible component Y of $\mathscr{X}_s$. Then x is the unique point of X^{an} with reduction $\pi(x)$ (see [Berkovich 1990, Proposition 2.4.4]) and we write $x = \xi_Y$. We may assume that the very affine open neighbourhood $\mathscr{U}$ of $\pi(x)$ in $\mathscr{X}$ from Lemma 10.8 has special fibre $\mathscr{U}_s$ disjoint from all other irreducible components Y' of $\mathscr{X}_s$. We conclude that $\pi(\xi_{Y'}) \notin \mathscr{U}_s$ and hence $\mathrm{trop}(\xi_{Y'}) \neq 0 = \mathrm{trop}(x)$. We may choose the neighbourhood $\widetilde{\Omega}$ of 0 in $N_{\mathbb{R}}$ disjoint from all points $\mathrm{trop}(\xi_{Y'})$. We will use in the following that Lemma 10.8(e$'$) holds on the open subset $V := \mathrm{trop}^{-1}(\widetilde{\Omega})$ of X^{an}. Since no $\xi_{Y'}$ is contained in V, the nongeneric case above shows that the restriction of μ^{MA} to V is supported in ξ_Y.

Now we choose a very affine open subset U' contained in the generic fibre U of $\mathscr{U}$ with $x \in (U')^{\mathrm{an}}$. Let $F : N_{U',\mathbb{R}} \to N_{\mathbb{R}}$ be the canonical affine map. Then

$$V' := \mathrm{trop}_{U'}^{-1}(F^{-1}(\widetilde{\Omega})) = (U')^{\mathrm{an}} \cap V.$$

is an open neighbourhood of x in X^{an} and $(V', \varphi_{U'})$ is a tropical chart. Similarly to the above, μ^{MA} is given on V' by $\beta^{\wedge n} \in \mathrm{AZ}^{n,n}(V', \varphi_{U'})$ represented by the δ-preform in (10.8.5). Since $\mu^{\mathrm{MA}}|_{V'}$ is supported in the single point $x = \xi_Y$, we conclude that the 0-dimensional tropical cycle $F^*(C) \cdot \mathrm{Trop}(U')$ has only one point ω' contained in the open subset $\mathrm{trop}_{U'}(V')$ of $\mathrm{Trop}(U')$. In fact, we have $\omega' = \mathrm{trop}_{U'}(x)$ with multiplicity $\mu^{\mathrm{MA}}(V')$. The tropical projection formula in Proposition 1.5 and the Sturmfels–Tevelev multiplicity formula [Gubler 2013, Theorem 13.17] give the identity

$$F_*(F^*(C) \cdot \mathrm{Trop}(U')) = C \cdot \mathrm{Trop}(U)$$

of tropical cycles on $N_{\mathbb{R}}$. Using that $\mathrm{trop}_{U'}(V') = F^{-1}(\mathrm{trop}(V)) \cap \mathrm{Trop}(U')$, we deduce that $\mu^{\mathrm{MA}}(V')$ is equal to the multiplicity of $0 = \mathrm{trop}(x) = F(\omega')$ in $C \cdot \mathrm{Trop}(U)$.

By Lemma 10.8, we conclude that $\mu^{\mathrm{MA}}(V')$ is equal to the tropical intersection number of C with $\mathrm{LC}_0(\mathrm{Trop}(U)) = \mathrm{Trop}(\mathscr{U}_s)$.

We recall that C is the n-fold self-intersection of the tropical divisor $\phi \cdot N_{\mathbb{R}}$ and we note that these objects are weighted tropical fans. Now we use Lemma 10.8(f′). This shows that $\mathscr{U}_s$ is a very affine chart of integration for $c_1(\mathscr{L}_s|_Y, \|\cdot\|_{\mathrm{can}})^n$, where this δ-form is represented by the pull-back of δ_C with respect to the canonical affine map $N_{\mathscr{U}_s,\mathbb{R}} \to N_{\mathbb{R}}$. Note that we may perform a base change to omit the trivial valuation which was excluded for simplicity in our paper. The tropical projection formula and the Sturmfels–Tevelev multiplicity formula show

$$\int_{Y^{\mathrm{an}}} c_1(\mathscr{L}_s|_Y, \|\cdot\|_{\mathrm{can}})^n = \deg(C \cdot \mathrm{Trop}(\mathscr{U}_s))$$

as above. By Proposition 10.3, the left-hand side is equal to $\deg_{\mathscr{L}}(Y)$. We have seen above that the right-hand side equals $\mu^{\mathrm{MA}}(V')$. This proves that $\mu^{\mathrm{MA}}|_V$ is a point measure concentrated in $x = \xi_Y$ with total mass $\deg_{\mathscr{L}}(Y)$. By Proposition 10.4(c), the Chambert-Loir measure $\mu_{\overline{L}_1,\dots,\overline{L}_n}$ is equal to μ^{MA}. □

11. Green currents

In this section X is an algebraic variety over K of dimension n. We introduce Green currents for cycles on X. We define the product $g_Y * g_Z$ for a divisor Y and a cycle Z on X which intersect properly. This operation has the expected properties.

Definition 11.1. Let Z be a cycle of X of codimension p and let g be any δ-current in $E^{p-1,p-1}(X^{\mathrm{an}})$. Then we define

$$\omega(Z, g) := d'd''g + \delta_Z \in E^{p,p}(X^{\mathrm{an}}).$$

If there is a δ-form $\omega_{Z,g} \in B^{p,p}(X^{\mathrm{an}})$ with $\omega(Z, g) = [\omega_{Z,g}]$, then we call g a *Green current* for the cycle Z. We will use often the notation g_Z for such a current and then we set $\omega(g_Z) := \omega(Z, g_Z)$ and $\omega_Z := \omega_{Z,g_Z}$ for simplicity.

11.2. Let $(L, \|\cdot\|)$ be a line bundle on X endowed with a δ-metric and let Z be a cycle of codimension p in X with any current $g_Z \in E^{p-1,p-1}(X^{\mathrm{an}})$. We assume that s is a meromorphic section of L with Cartier divisor D intersecting Z properly. By the Poincaré–Lelong equation in Corollary 7.8 and by the definition of the first Chern δ-form in Definition 9.12, $g_Y := [-\log\|s\|]$ is a Green current for the Weil divisor Y associated to D with $\omega_Y = c_1(L, \|\cdot\|)$.

If Z is a prime cycle of codimension p, then we define $g_Y \wedge \delta_Z \in E^{p,p}(X^{\mathrm{an}})$ as the push-forward of $[-\log\|s\||_Z]$ with respect to the inclusion $i_Z : Z \to X$. In general, we proceed by linearity in the prime components of Z to define $g_Y \wedge \delta_Z \in E^{p,p}(X^{\mathrm{an}})$. This leads to the definition of the $*$-*product*

$$g_Y * g_Z := g_Y \wedge \delta_Z + \omega_Y \wedge g_Z \in E^{p,p}(X^{\mathrm{an}}).$$

Lemma 11.3. *Under the hypothesis above and if Z is prime, then we have the identity*

$$d'd''[-\log\|s\||_Z] = [\omega_Y|_Z] - \delta_{D\cdot Z} \tag{11.3.1}$$

of δ-currents on Z^{an}.

Proof. This follows immediately from the Poincaré–Lelong equation for $s|_Z$ (see Corollary 7.8). We use here $c_1(L|_Z, \|\cdot\|) = c_1(L, \|\cdot\|)|_Z$, which follows from Proposition 9.15. □

Proposition 11.4. *Under the hypothesis in 11.2, we have*

$$\omega(D\cdot Z, g_Y * g_Z) = \omega_Y \wedge \omega(g_Z).$$

If g_Z is a Green current for Z, then $g_Y * g_Z$ is a Green current for $D\cdot Z$.

Proof. Using Lemma 11.3 and linearity in the prime components of Z, we get

$$d'd''[-\log\|s\| \wedge \delta_Z] = \omega_Y \wedge \delta_Z - \delta_{D\cdot Z} \tag{11.4.1}$$

and hence 9.14 and Proposition 4.15(iii) give

$$\omega(D\cdot Z, g_Y * g_Z) = d'd''[-\log\|s\| \wedge \delta_Z] + d'd''(\omega_Y \wedge g_Z) + \delta_{D\cdot Z} = \omega_Y \wedge \omega(g_Z),$$

proving the claim. □

Proposition 11.5. *For $i = 1, 2$, let L_i be a line bundle on X with a δ-metric $\|\cdot\|_i$ and nonzero meromorphic section s_i. We assume that the associated Cartier divisors D_1 and D_2 intersect properly. Let $\eta_{Y_i} := -\log\|s_i\|_i$ and let $g_{Y_i} = [\eta_{Y_i}]$ be the induced Green current for the Weil divisor Y_i of D_i. Then we have the identity*

$$g_{Y_1} * g_{Y_2} - g_{Y_2} * g_{Y_1} = d'[d''_{\mathrm{P}}\eta_{Y_1} \wedge \eta_{Y_2}] + d''[\eta_{Y_1} \wedge d'_{\mathrm{P}}\eta_{Y_2}]$$

of δ-currents on X^{an}.

Note that the piecewise smooth forms $d''_{\mathrm{P}}\eta_{Y_1} \wedge \eta_{Y_2}$ and $\eta_{Y_1} \wedge d'_{\mathrm{P}}\eta_{Y_2}$ of degree 1 are defined on the analytification of a Zariski open and dense subset of X. By 9.5 and Proposition 6.5, they define δ-currents on X^{an}.

Proof. The claim can be checked locally on X. Hence we may assume that X is affine and $L_1 = L_2 = O_X$. For $i = 1, 2$, we may view s_i as a rational function f_i and we have

$$\eta_{Y_i} = -\log|f_i| - \log\|1\|_i.$$

The usual partition of unity argument shows that it is enough to test the claim by evaluating at $\alpha \in B_c^{n-1,n-1}(W)$ for a small open neighbourhood W of a given point x in X^{an}. There are finitely many tropical charts $\{(V_j, \varphi_{U_j})\}_{j=1,\dots,m}$ in W covering $\mathrm{supp}(\alpha)$ such that $\alpha = \mathrm{trop}_{U_j}^*(\alpha_j)$ on V_j for some element $\alpha_j \in \mathrm{AZ}^{n-1,n-1}(V_j, \varphi_{U_j})$. We will use a Zariski dense very affine open subset U of $U_1 \cap \cdots \cap U_m$ which will serve as a very affine chart of integration for various forms. Now we consider the restriction of the canonical affine map $F_j : N_{U,\mathbb{R}} \to N_{U_j,\mathbb{R}}$ to $\mathrm{Trop}(U)$. Let Ω in $\mathrm{Trop}(U)$ denote the union of the preimages of the open subsets $\Omega_j := \mathrm{trop}_{U_j}(V_j)$ in $\mathrm{Trop}(U_j)$ and put $V = \mathrm{trop}_U^{-1}(\Omega)$. By Proposition 4.12 there exists a unique element $\alpha_U \in \mathrm{AZ}^{n-1,n-1}(V, \varphi_U)$ such that $\alpha|_V = \mathrm{trop}_U^*(\alpha_U)$ and such that α_U coincides for all j on the preimage of Ω_j with the pull-back of α_j. Note that Ω is an open subset of $\mathrm{Trop}(U)$ and (V, φ_U) is a tropical chart for $V := \mathrm{trop}_U^{-1}(\Omega)$. Then α_U has not necessarily compact support, but we can extend α_U by zero to an element in $\mathrm{AZ}^{n-1,n-1}(U^{\mathrm{an}}, \varphi_U)$ using that $\mathrm{supp}(\alpha) \cap U^{\mathrm{an}}$ is a closed subset of V. By abuse of notation, this extension will also be denoted by α_U. Then we have $\alpha = \mathrm{trop}_U^*(\alpha_U)$ on U^{an}. By shrinking W and using an appropriate U, we may assume that

$$-\log\|1\|_i = \phi_i \circ \mathrm{trop}_U$$

on V for a piecewise smooth function ϕ_i on $\mathrm{Trop}(U)$ and $i = 1, 2$. Since we deal with δ-metrics, we may assume that there is a piecewise smooth extension $\tilde{\phi}_i$ of ϕ_i to $N_{U,\mathbb{R}}$ and a superform γ_i on $N_{U,\mathbb{R}}$ of bidegree $(1, 1)$ such that the first Chern δ-form ω_{Y_i} is represented on V by the δ-preform $\gamma_i + \delta_{\tilde{\phi}_i \cdot N_{U,\mathbb{R}}}$ on $N_{U,\mathbb{R}}$ and such that γ_i restricts to $d'_{\mathrm{P}} d''_{\mathrm{P}} \phi_i$ on Ω (see 9.13). We have

$$\eta_{Y_i} = -\log|f_i| - \log\|1\|_i = -\log|f_i| + \phi_i \circ \mathrm{trop}_U$$

on V. Using bilinearity of $*$ and of $\wedge$, we may either assume that η_{Y_i} is equal to $-\log\|1\|_i$ or equal to $-\log|f_i|$. Hence we have to consider the following four cases:

Case 1: $s_1 = s_2 = 1$. In this case, the divisors Y_1, Y_2 are zero and $\eta_{Y_i} = -\log\|1\|_i$ for $i = 1, 2$ are piecewise smooth functions on X^{an}. Then we have

$$\langle g_{Y_1} * g_{Y_2}, \alpha\rangle = \langle \omega_{Y_1} \wedge g_{Y_2}, \alpha\rangle = \langle g_{Y_2}, \omega_{Y_1} \wedge \alpha\rangle. \tag{11.5.1}$$

Recall that g_{Y_2} is the current associated to η_{Y_2}. By 9.3, we have $\mathrm{PS}^{0,0}(W) = P^{0,0}(W)$ and hence $\eta_{Y_2}\alpha \in P_c^{n-1,n-1}(W)$. We may view it as a generalized δ-form on X^{an} given on U^{an} by $\phi_2 \alpha_U \in P^{n-1,n-1}(U^{\mathrm{an}}, \varphi_U)$. Since the first Chern δ-form ω_{Y_1} is represented on V by $\delta_{\tilde{\phi}_1 \cdot N_{U,\mathbb{R}}} + \gamma_1 \in P^{1,1}(N_{U,\mathbb{R}})$, we get

$$\langle g_{Y_1} * g_{Y_2}, \alpha\rangle = \int_{|\mathrm{Trop}(U)|} (\delta_{\phi_1 \cdot \mathrm{Trop}(U)} + d'_{\mathrm{P}} d''_{\mathrm{P}} \phi_1) \wedge \phi_2 \alpha_U. \tag{11.5.2}$$

Here, we have used that U is a very affine chart of integration for $\omega_{Y_1} \wedge \eta_{Y_2}\alpha \in P_c^{n,n}(X^{\mathrm{an}})$. Recall from 9.13 that the generalized δ-forms $c_1(L_1, \|\cdot\|_1)_{\mathrm{res}}$ and $c_1(L_1, \|\cdot\|_1)_{\mathrm{ps}}$ are represented on V by $\delta_{\tilde{\phi}_1 \cdot N_{U,\mathbb{R}}}$ and γ_1 in $P^{1,1}(N_{U,\mathbb{R}})$. We conclude that U is a very affine chart of integration for $c_1(L_1, \|\cdot\|_1)_{\mathrm{res}} \wedge \eta_{Y_2}\alpha$ and $c_1(L_1, \|\cdot\|_1)_{\mathrm{ps}} \wedge \eta_{Y_2}\alpha$ in $P_c^{n,n}(X^{\mathrm{an}})$ and hence (11.5.2) yields

$$\langle g_{Y_1} * g_{Y_2}, \alpha\rangle = \int_{|\mathrm{Trop}(U)|} \delta_{\phi_1 \cdot \mathrm{Trop}(U)} \wedge \phi_2 \alpha_U + \int_{|\mathrm{Trop}(U)|} d'_{\mathrm{P}} d''_{\mathrm{P}} \phi_1 \wedge \phi_2 \alpha_U. \tag{11.5.3}$$

Since α has compact support in W and $\operatorname{supp}(\alpha) \cap U^{\mathrm{an}} \subseteq V$, it follows from Proposition 4.21 and Corollary 5.6 that the integrands in (11.5.3) have compact support in Ω. The generalization of Corollary 5.6 to PSP-forms given in 9.5 shows that $-d''_{\mathrm{P}} \log\|1\|_1 \wedge \alpha$ has compact support contained in U^{an}. Again, we conclude that $d''_{\mathrm{P}}\phi_1 \wedge \alpha_U$ has compact support contained in Ω. Now Leibniz's rule and the theorem of Stokes (Proposition 2.7) for d'_{P} show

$$\begin{aligned}\int_{|\mathrm{Trop}(U)|} d'_{\mathrm{P}}d''_{\mathrm{P}}\phi_1 \wedge \phi_2\alpha_U &= \int_{\partial|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2\alpha_U + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge d'_{\mathrm{P}}(\phi_2\alpha_U) \\ &= \int_{\partial|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2\alpha_U + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge d'_{\mathrm{P}}\phi_2 \wedge \alpha_U \\ &\qquad + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2 d'\alpha_U. \qquad (11.5.4)\end{aligned}$$

Recall that $\phi_2\alpha_U$ is a δ-preform on $\mathrm{Trop}(U)$ and hence Remark 3.18 gives

$$\int_{|\mathrm{Trop}(U)|} \delta_{\phi_1\cdot\mathrm{Trop}(U)} \wedge \phi_2\alpha_U + \int_{\partial|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2\alpha_U = 0. \qquad (11.5.5)$$

Using (11.5.4) and (11.5.5) in (11.5.3), we get

$$\langle g_{Y_1} * g_{Y_2}, \alpha\rangle = \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge d'_{\mathrm{P}}\phi_2 \wedge \alpha_U + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge \phi_2 d'\alpha_U. \quad (11.5.6)$$

A similar computation where we replace (11.5.4) by an application of Stokes' theorem with respect to d''_{P} shows

$$\langle g_{Y_2} * g_{Y_1}, \alpha\rangle = \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}}\phi_1 \wedge d'_{\mathrm{P}}\phi_2 \wedge \alpha_U - \int_{|\mathrm{Trop}(U)|} \phi_1 d'_{\mathrm{P}}\phi_2 \wedge d''\alpha_U. \quad (11.5.7)$$

Using that U is a very affine chart of integration for $d''_{\mathrm{P}}\eta_{Y_1} \wedge \eta_{Y_2} \wedge d'\alpha \in \mathrm{PSP}^{n,n}_c(X^{\mathrm{an}})$ and for $\eta_{Y_1} \wedge d'_{\mathrm{P}}\eta_{Y_2} \wedge d''\alpha_U \in \mathrm{PSP}^{n,n}_c(X^{\mathrm{an}})$, this proves the claim in the first case.

Case 2: $s_1 = 1$ and $\|1\|_2 = 1$. In this case Y_1 is zero and $\eta_{Y_2} = -\log|f_2|$. The following computation is similar to the one in the proof of the Poincaré–Lelong formula (see Theorem 7.2) and we will use the same terminology as there. We have $g_{Y_2} * g_{Y_1} = 0$ as $\delta_{Y_1} = 0$ and $\omega_{Y_2} = c_1(O_X, \|\cdot\|_2) = 0$. It remains to show that

$$\omega_{Y_1} \wedge g_{Y_2} = -g_{Y_1} \wedge \delta_{Y_2} + d'[d''_{\mathrm{P}}\eta_{Y_1} \wedge \eta_{Y_2}] + d''[\eta_{Y_1} \wedge d'_{\mathrm{P}}\eta_{Y_2}]. \qquad (11.5.8)$$

It is enough to check the claim locally and by linearity, we may assume that f_2 is a regular function on X. By the first case, we may assume that f_2 is nonconstant. We choose a very affine open subset U, the open subset Ω of $\mathrm{Trop}(U)$ and ϕ_1 as above. We may assume that $\operatorname{supp}(\operatorname{div}(f_2)) \cap U = \varnothing$ and hence $-\log|f_2|$ is induced by an integral Γ-affine function φ_2 on $\mathrm{Trop}(U)$. We use the very affine open U to compute the term $\langle \omega_{Y_1} \wedge g_{Y_2}, \alpha \rangle$. Similarly to (11.5.1) and (11.5.2), we deduce

$$\langle \omega_{Y_1} \wedge g_{Y_2}, \alpha \rangle = \int_{|\mathrm{Trop}(U)|} \delta_{\phi_1 \cdot \mathrm{Trop}(U)} \wedge \varphi_2 \alpha_U + \int_{|\mathrm{Trop}(U)|} d'_{\mathrm{P}} d''_{\mathrm{P}} \phi_1 \wedge \varphi_2 \alpha_U. \quad (11.5.9)$$

The same computation as in (11.5.4)–(11.5.6) yields

$$\langle \omega_{Y_1} \wedge g_{Y_2}, \alpha \rangle = \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}} \phi_1 \wedge d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U + \int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}} \phi_1 \wedge \varphi_2 d' \alpha_U. \quad (11.5.10)$$

As in the first case, we have

$$\int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}} \phi_1 \wedge \varphi_2 d' \alpha_U = \langle d'[d''_{\mathrm{P}} \eta_{Y_1} \wedge \eta_{Y_2}], \alpha \rangle. \quad (11.5.11)$$

Similarly to the proof of the Poincaré–Lelong formula, we may assume that the support of α is covered by the interiors of the affinoid subdomains $W_j := \mathrm{trop}_{U_j}^{-1}(\Delta_j)$ of the tropical chart V_j for $j = 1, \ldots, m$. We set $W := \bigcup_{j=1}^m W_j$. We choose $s > 0$ sufficiently small with $\varphi_2 \leq -\log|s|$ on the compact set $\mathrm{supp}(d''_{\mathrm{P}} \phi_1 \wedge \alpha_U)$. Since W covers $\mathrm{supp}(\alpha)$, the analytic subdomain $W(s) := \{x \in W \mid |f_2(x)| \geq s\}$ of W contains $\mathrm{supp}(d''_{\mathrm{P}} \phi_1 \wedge \alpha)$ and hence we have

$$\int_{|\mathrm{Trop}(U)|} d''_{\mathrm{P}} \phi_1 \wedge d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U = \int_{\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} d''_{\mathrm{P}} \phi_1 \wedge d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U. \quad (11.5.12)$$

By the theorem of Stokes (Proposition 2.7), this is equal to

$$\int_{\partial \mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} \phi_1 d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U + \int_{\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} \phi_1 d'_{\mathrm{P}} \varphi_2 \wedge d'' \alpha_U. \quad (11.5.13)$$

By Corollary 5.6, the support of $d''\alpha$ is contained in U^{an}. We may assume that the compact set $\mathrm{supp}(d''\alpha)$ is contained in $W(s)$. Using that U is a very affine chart of integration for $\eta_{Y_1} \wedge d'_{\mathrm{P}} \eta_{Y_2} \wedge d''\alpha$, we get

$$\int_{\mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} \phi_1 d'_{\mathrm{P}} \varphi_2 \wedge d'' \alpha_U = \langle d''[\eta_{Y_1} \wedge d'_{\mathrm{P}} \eta_{Y_2}], \alpha \rangle. \quad (11.5.14)$$

Now we apply Remark 7.4 with f_2 instead of f and the generalized δ-form $\eta_{Y_1} \wedge \alpha$ instead of α and observe that φ_2 corresponds to $F^*(x_0)$ in Remark 7.4. Then Equation (7.4.1) yields

$$\int_{\partial \mathrm{trop}_U(W(s) \cap U^{\mathrm{an}})} \phi_1 d'_{\mathrm{P}} \varphi_2 \wedge \alpha_U = -\langle g_{Y_1} \wedge \delta_{Y_2}, \alpha \rangle \quad (11.5.15)$$

as W covers $\mathrm{supp}(\alpha)$. Using (11.5.11)–(11.5.15) in (11.5.10), we get (11.5.8) proving the claim in the second case.

Case 3: $\|1\|_1 = 1$ and $s_2 = 1$. The formula proved in the second case yields

$$g_{Y_2} * g_{Y_1} - g_{Y_1} * g_{Y_2} = d'[d''_{\mathrm{P}} \eta_{Y_2} \wedge \eta_{Y_1}] + d''[\eta_{Y_2} \wedge d'_{\mathrm{P}} \eta_{Y_1}]$$

The (1, 1)-current on the left-hand side is clearly symmetric. Hence the right-hand side is symmetric as well and equals

$$-d''[d'_{\mathrm{P}}\eta_{Y_2}\wedge\eta_{Y_1}]-d'[\eta_{Y_2}\wedge d''_{\mathrm{P}}\eta_{Y_1}]$$

This proves our claim in the third case.

Case 4: $\|1\|_1=1$ and $\|1\|_2=1$. In this case $\eta_{Y_1}=-\log|f_1|$ and $\eta_{Y_2}=-\log|f_2|$. We have to show that

$$g_{Y_1}\wedge\delta_{Y_2}-g_{Y_2}\wedge\delta_{Y_1}=d'[d''_{\mathrm{P}}\eta_{Y_1}\wedge\eta_{Y_2}]+d''[\eta_{Y_1}\wedge d'_{\mathrm{P}}\eta_{Y_2}].$$

Again, we may assume that f_1 and f_2 are regular functions on X. By the previous cases, we may assume that these functions are nonconstant. We use the same notation as above. Here, we choose the very affine open subset U disjoint from $\operatorname{supp}(\operatorname{div}(f_1))\cup\operatorname{supp}(\operatorname{div}(f_2))$. Then φ_1,φ_2 are integral Γ-affine functions on $\operatorname{Trop}(U)$ inducing $-\log|f_1|$, $-\log|f_2|$ on U^{an}. Going the computation in the second case backwards, we see that

$$\langle g_{Y_1}\wedge\delta_{Y_2},\alpha\rangle=-\int_{|\operatorname{Trop}(U)|}d''_{\mathrm{P}}\varphi_1\wedge d'\varphi_2\wedge\alpha_U+\langle d''[\eta_{Y_1}\wedge d'_{\mathrm{P}}\eta_{Y_2}],\alpha\rangle. \quad (11.5.16)$$

Note here that $d''_{\mathrm{P}}\varphi_1\wedge d'_{\mathrm{P}}\varphi_2\wedge\alpha_U$ has compact support in Ω. Indeed, it follows from Corollary 5.6, that $d''_{\mathrm{P}}\log|f_1|\wedge d'_{\mathrm{P}}\log|f_2|\wedge\alpha$ is a well-defined δ-form on X^{an} with compact support in U^{an} (using that the divisors intersect properly) and hence we get compactness in Ω from Proposition 4.21. Interchanging the role of Y_1, Y_2 and also of d', d'' and d'_{P}, d''_{P} in (11.5.16), we get the fourth claim. This proves the proposition. □

In the following, we denote the support of a cycle Z (resp. of a Cartier divisor D) on X by $|Z|$ (resp. $|D|$).

Corollary 11.6. *Let Z be a cycle of X of codimension p and let g_Z be any δ-current in $E^{p-1,p-1}(X)$. For $i=1,2$, let L_i be a line bundle on X with a δ-metric $\|\cdot\|_i$ and nonzero meromorphic section s_i. Let D_i denote the Cartier divisor on X defined by s_i. We assume that $|D_1|\cap|Z|$ and $|D_2|\cap|Z|$ both have codimension ≥ 1 in $|Z|$, and that $|D_1|\cap|D_2|\cap|Z|$ has codimension ≥ 2 in $|Z|$. Let $\eta_{Y_i}:=-\log\|s_i\|_i$ and let $g_{Y_i}=[\eta_{Y_i}]$ be the induced Green current for the Weil divisor Y_i of D_i. Then we have*

$$g_{Y_1}*(g_{Y_2}*g_Z)-g_{Y_2}*(g_{Y_1}*g_Z)\in d'(E^{p,p+1}(X^{\mathrm{an}}))+d''(E^{p+1,p}(X^{\mathrm{an}})).$$

Proof. This follows immediately from Proposition 11.5 applied to the analytifications of the prime components of Z. □

12. Local heights of varieties

In this section, we study the local height of a proper variety X of dimension n over K with respect to metrized line bundles endowed with δ-metrics. If the metrics are formal, then we show that these analytically defined local heights agree with the ones based on divisorial intersection theory on formal models in [Gubler 1998]. In particular, they coincide with the local heights used in Arakelov theory over number fields.

12.1. For $i = 0, \ldots, n$, let L_i be a line bundle on X endowed with a δ-metric $\|\cdot\|_i$ and a nonzero meromorphic section s_i. For the associated Cartier divisor $D_i := \operatorname{div}(s_i)$, we consider the metrized Cartier divisor $\hat{D}_i := (D_i, \|\cdot\|_i)$, i.e., a Cartier divisor D_i and a metric $\|\cdot\|_i$ on the associated line bundle $O(D_i)$. Recall from 11.2 that we obtain the Green current $g_{Y_i} := [-\log\|s_i\|_i]$ for the Weil divisor Y_i associated to D_i.

We assume that the Cartier divisors $D_0, \ldots, D_n$ intersect properly. Then we define *the local height of X with respect to $\hat{D}_0, \ldots, \hat{D}_n$* by

$$\lambda_{\hat{D}_0,\ldots,\hat{D}_n}(X) := g_{Y_0} * \cdots * g_{Y_n}(1).$$

12.2. If Z is a cycle on X of dimension t and $\hat{D}_0, \ldots, \hat{D}_t$ are δ-metrized Cartier divisors on X with $|D_0|, \ldots, |D_t|, |Z|$ intersecting properly, then 12.1 induces a local height $\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z)$ by linearity in the prime components of Z.

Remark 12.3. The problem with this definition is that it is not functorial as the pull-back of a Cartier divisor is not always well defined as a Cartier divisor. This problem is resolved by using pseudodivisors instead of Cartier divisors (see [Fulton 1984, Chapter 2]). We follow [Gubler 2003] and define a *δ-metrized pseudodivisor* as a triple $(\bar{L}, Z, s)$, where $\bar{L} = (L, \|\cdot\|)$ is a line bundle on X equipped with a δ-metric, Z is a closed subset of X, and s is a nowhere vanishing section of L over $X \setminus Z$. Using the same arguments as in [Gubler 2003], we get a local height $\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z)$ for δ-metrized pseudodivisors which is well defined under the weaker condition $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \varnothing$.

It is straightforward to show that the local height is linear in Z and multilinear in $\hat{D}_0, \ldots, \hat{D}_t$. It follows from Corollary 11.6 along the arguments in [Gubler 2003] that the local height is symmetric in $\hat{D}_0, \ldots, \hat{D}_t$.

The next result shows that the *induction formula* holds for local heights.

Proposition 12.4. *Let $\hat{D}_0, \ldots, \hat{D}_n$ be δ-metrized pseudodivisors on X with*

$$|D_0| \cap \cdots \cap |D_n| = \varnothing.$$

Then the local height $\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X)$ is equal to

$$\lambda_{\hat{D}_0,\dots,\hat{D}_{n-1}}(Y_n) - \int_{X^{\mathrm{an}}} \log\|s_n\|_n \cdot c_1(\overline{O(D_0)}) \wedge \cdots \wedge c_1(\overline{O(D_{n-1})}),$$

where we assume that D_n is a Cartier divisor with associated Weil divisor Y_n and canonical meromorphic section s_n of $O(D_n)$.

Proof. The argument is the same as for [Gubler 2003, Proposition 3.5]. □

Proposition 12.5. *Let $\varphi : X' \to X$ be a morphism of proper varieties over K and let $\hat{D}_0, \dots, \hat{D}_n$ be δ-metrized pseudodivisors on X with $|D_0| \cap \cdots \cap |D_n| = \varnothing$. Then the functoriality*

$$\deg(\varphi)\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X) = \lambda_{\varphi^*(\hat{D}_0),\dots,\varphi^*(\hat{D}_n)}(X')$$

holds.

Proof. The proof relies on the induction formula in Proposition 12.4 and the projection formula for integrals (5.9.1). We refer to [Gubler 2003] for the analogous arguments in the archimedean case. □

Proposition 12.6 (metric change formula). *Suppose that the local height $\lambda(X)$ with respect to the δ-metrized pseudodivisors $\hat{D}_0, \dots, \hat{D}_n$ is well defined. Let $\lambda'(X)$ be the local height of X obtained by replacing the metric $\|\cdot\|_0$ on $O(D_0)$ by another δ-metric $\|\cdot\|_0'$. Then $\rho := \log(\|\cdot\|_0'/\|\cdot\|_0)$ is a piecewise smooth function on X^{an} and we have*

$$\lambda(X) - \lambda'(X) = \int_{X^{\mathrm{an}}} \rho \cdot c_1(\overline{O(D_1)}) \wedge \cdots \wedge c_1(\overline{O(D_n)}).$$

Proof. This follows from linearity and symmetry of the local height in $\hat{D}_0$ and $\hat{D}_n$ and from the induction formula in Proposition 12.4. □

Remark 12.7. Now suppose that $\hat{D}_0, \dots, \hat{D}_n$ are formally metrized pseudodivisors on X with $|D_0| \cap \cdots \cap |D_n| = \varnothing$. Then the intersection theory of divisors on admissible formal K°-models given in [Gubler 1998] induces also a local height of X (see [Gubler 2003]). It also satisfies an induction formula involving Chambert-Loir's measures (see [Gubler 2003, Remark 9.5]). Since the Chambert-Loir measure agrees with the Monge–Ampère measure (see Theorem 10.5), we deduce from the induction formula in Proposition 12.4 that the local height based on intersection theory of divisors agrees with $\lambda_{\hat{D}_0,\dots,\hat{D}_n}(X)$ from Remark 12.3. In particular, this proves Theorem 0.4 stated in the introduction.

Appendix: Convex geometry

In this appendix, we gather the notions from convex geometry on a finite dimensional real vector space W coming with an *integral structure*. This means that we consider a free abelian group N of rank r with $W = N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $M := \mathrm{Hom}(N, \mathbb{Z})$ be the dual abelian group and let $V := \mathrm{Hom}(M, \mathbb{R}) = M_{\mathbb{R}}$ be the dual vector space of W. The natural duality between V and W is denoted by $\langle u, \omega \rangle$. Let Γ be a fixed subgroup of $\mathbb{R}$. In the applications, it is usually the value group of a nonarchimedean absolute value.

A.1. Let N' be another free abelian group of finite rank and let $F : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ be an affine map. Then F is called *integral* Γ*-affine* if $F = \mathbb{L}_F + \omega$ with $\omega \in N' \otimes_{\mathbb{Z}} \Gamma \subseteq N'_{\mathbb{R}}$ and with the associated linear map $\mathbb{L}_F$ induced by a homomorphism $N \to N'$.

A.2. A *polyhedron* Δ in W is defined as the intersection of finitely many half spaces $\{\omega \in W \mid \langle u_i, \omega \rangle \geq c_i\}$ with $u_i \in V$ and $c_i \in \mathbb{R}$. If we may choose all $u_i \in M$ and all $c_i \in \Gamma$, then we say that Δ is an *integral* Γ*-affine polyhedron*. A *face* of Δ is either Δ itself or the intersection of Δ with the boundary of a closed half-space containing Δ. We write $\tau \preccurlyeq \Delta$ for a face τ of Δ and we write $\tau \prec \Delta$ if additionally $\tau \neq \Delta$. The *relative interior* of Δ is defined by

$$\mathrm{relint}(\Delta) := \Delta \setminus \bigcup_{\tau \prec \Delta} \tau.$$

Note that every polyhedron is convex. A *polytope* is a bounded polyhedron.

A polyhedron Δ in W generates an affine space $\mathbb{A}_\Delta$ of the same dimension. Recall that an affine space in W is a translate of a linear subspace and $\mathbb{A}_\Delta$ is the intersection of all affine spaces in W which contain Δ. We denote the underlying vector space by $\mathbb{L}_\Delta$. If Δ is integral Γ-affine, then the integral structure of $\mathbb{A}_\Delta$ is given by the complete lattice $N_\Delta := N \cap \mathbb{L}_\Delta$ in $\mathbb{L}_\Delta$.

A.3. A *polyhedral complex* $\mathscr{C}$ in W is a finite set of polyhedra such that

(a) $\Delta \in \mathscr{C} \Rightarrow$ all closed faces of Δ are in $\mathscr{C}$;

(b) $\Delta, \sigma \in \mathscr{C} \Rightarrow \Delta \cap \sigma$ is either empty or a closed face of Δ and σ.

The polyhedral complex $\mathscr{C}$ is called *integral* Γ*-affine* if every $\Delta \in \mathscr{C}$ is integral Γ-affine. The *support* of $\mathscr{C}$ is defined as

$$|\mathscr{C}| := \bigcup_{\Delta \in \mathscr{C}} \Delta.$$

We say that a polyhedral complex $\mathscr{C}$ is *complete* if $|\mathscr{C}| = W$. A *subdivision* of $\mathscr{C}$ is a polyhedral complex $\mathscr{D}$ with $|\mathscr{D}| = |\mathscr{C}|$ and with every $\Delta \in \mathscr{D}$ contained in a polyhedron of $\mathscr{C}$. This has to be distinguished from a *subcomplex* of $\mathscr{C}$ which is a polyhedral complex $\mathscr{D}$ with $\mathscr{D} \subseteq \mathscr{C}$.

A.4. Given a polyhedral complex $\mathscr{C}$ in $N_{\mathbb{R}}$, we denote by $\mathscr{C}_n$ the subset of n-dimensional polyhedra in $\mathscr{C}$ and by $\mathscr{C}^l = \mathscr{C}_{r-l}$ the subset of polyhedra in $\mathscr{C}$ of codimension l in $N_{\mathbb{R}}$. We say that a polyhedral complex $\mathscr{C}$ is of *pure dimension n* (resp. of *pure codimension* l) if all polyhedra in $\mathscr{C}$ which are maximal with respect to $\preccurlyeq$ lie in $\mathscr{C}_n$ (resp. $\mathscr{C}^l$). Given a polyhedral complex $\mathscr{C}$ of pure dimension n and $m \leq n$, we denote by $\mathscr{C}_{\leq m}$ the polyhedral subcomplex of $\mathscr{C}$ of pure dimension m given by all $\sigma \in \mathscr{C}$ with $\dim \sigma \leq m$. We set $\mathscr{C}^{\geq l} = \mathscr{C}_{\leq r-l}$ if $r - l \leq n$. Recall here that r is the rank of N.

Definition A.5. (i) A *polyhedral set* P in $N_{\mathbb{R}}$ (of pure dimension n) is a finite union of polyhedra (of pure dimension n). Equivalently, there exists a polyhedral complex $\mathscr{D}$ (of pure dimension n) whose support is P. The polyhedral set is called *integral* Γ*-affine* if the above polyhedra can be chosen integral Γ-affine.

(ii) Let P be a polyhedral set in $N_{\mathbb{R}}$. A point $x \in P$ is called *regular* if there exists a polyhedron $\Delta \subseteq P$ such that $\mathrm{relint}(\Delta)$ is an open neighbourhood of x in P. We denote by $\mathrm{relint}(P)$ the set of regular points of the polyhedral set P.

A.6. A *cone* σ in W is characterized by $\mathbb{R}_{\geq 0} \cdot \sigma = \sigma$. A cone which is a polyhedron is called a *polyhedral cone*. An integral $\mathbb{R}$-affine polyhedral cone is simply called a *rational polyhedral cone*. A polyhedral cone is called *strictly convex* if it does not contain a line. The *local cone* $\mathrm{LC}_\omega(S)$ of $S \subseteq W$ at $\omega \in W$ is defined by

$$\mathrm{LC}_\omega(S) := \{\omega' \in W \mid \omega + [0, \varepsilon)\omega' \subseteq S \text{ for some } \varepsilon > 0\}.$$

A.7. A polyhedral complex Σ consisting of strictly convex rational polyhedral cones is called a *rational polyhedral fan*. The theory of toric varieties (see [Kempf et al. 1973; Oda 1988; Fulton 1993; Cox et al. 2011]) gives a bijective correspondence $\Sigma \mapsto Y_\Sigma$ between rational polyhedral fans on $N_{\mathbb{R}}$ and normal toric varieties over any field K with open dense torus $\mathrm{Spec}(K[M])$ (up to equivariant isomorphisms restricting to the identity on the torus). Then Σ is complete if and only if Y_Σ is a proper variety over K.

A *simplicial cone* in $N_{\mathbb{R}}$ is generated by a part of a basis. A *simplicial fan* is a fan formed by simplicial cones. A *smooth fan* in $N_{\mathbb{R}}$ is a rational polyhedral fan Σ such that every cone $\sigma \in \Sigma$ is generated by a part of a basis of N. In particular, a smooth fan is a simplicial fan. A polyhedral fan Σ is smooth if and only if Y_Σ is a smooth variety [Cox et al. 2011, Chapter 1, Theorem 3.12; Fulton 1993, 2.1 Proposition].

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