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Mateusz Michałek and Emanuele Ventura

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Finite phylogenetic complexity and combinatorics of tables

Mateusz Michałek and Emanuele Ventura

In algebraic statistics, Jukes–Cantor and Kimura models are of great importance. Sturmfels and Sullivant generalized these models by associating to any finite abelian group G a family of toric varieties $X(G, K_{1,n})$. We investigate the generators of their ideals. We show that for any finite abelian group G there exists a constant ϕ, depending only on G, such that the ideals of $X(G, K_{1,n})$ are generated in degree at most ϕ.

1. Introduction

The aim of this article is to prove the finiteness of an intriguing invariant of finite abelian groups, called *phylogenetic complexity*. The invariant was introduced in a seminal paper by Sturmfels and Sullivant [2005], where it appeared in relation to phylogenetic models. In short, to a Markov process encoded by an abelian group G on a tree T one associates a toric variety $X(G, T)$, of particular relevance in algebraic statistics [Eriksson et al. 2005; Pachter and Sturmfels 2005]. The setting above is known as a *group-based model*.

We do not describe the relations to phylogenetics in this paper, referring the interested reader to [Allman and Rhodes 2003; Casanellas 2012; Donten-Bury and Michałek 2012; Michałek 2015]. Instead, in precise, purely mathematical language we present a natural construction of *a family of lattice polytopes* $P_{G,n}$ associated to any finite abelian group G — Definition 2.1. These polytopes should be considered as the simplest combinatorial objects encoding the group action.

Associating to interesting combinatorial objects a polytope and investigating its properties is nowadays a well-developed and powerful tool on the edge of combinatorics and toric geometry [Sturmfels 1996; Ohsugi and Hibi 1998; Herzog and Hibi 2002; Sturmfels and Sullivant 2008]. However, our knowledge of properties of the polytopes $P_{G,n}$ associated to such basic objects as finite abelian groups is still very limited. This may be even more surprising, as for various groups G, these polytopes relate not only to phylogenetics, but also mathematical physics through

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conformal blocks and moduli spaces [Sturmfels and Xu 2010; Manon 2012; 2013; Kubjas and Manon 2014].

Phylogenetic complexity governs the degrees of generators of the ideal of the variety $X(G, T)$. Using the language of toric geometry, one is interested in the generators of integral relations among the vertices of $P_{G,n}$. For the introduction to toric geometry we refer the reader to [Fulton 1993; Cox et al. 2011]. The objects that encode the group action and correspond to vertices of $P_{G,n}$ are called *flows*.

Definition 1.1 [Buczyńska and Wiśniewski 2007; Michałek 2014]. Let G be a finite abelian group and $n \in \mathbb{N}$. A *flow* is a sequence of n elements of G summing up to $0 \in G$, the neutral element of G. The set of flows is equipped with a group structure via the coordinatewise action. The group of flows $\mathfrak{G}$ is (noncanonically) isomorphic to G^{n-1}.

Hence, in our article we study possible relations among n-tuples of elements of G summing up to 0. Let T_0 and T_1 be two matrices or *tables* of the same size, whose rows are flows. These two tables are *compatible* if and only if, for each $1 \leq i \leq n$, the i-th column of T_0 and the i-th column of T_1 are the same multisets — see Example 2.3. Compatible tables correspond to binomials in the ideal $I(X(G, K_{1,n}))$, where $K_{1,n}$ is a star (also called a claw-tree) — the unique tree with one inner vertex and n leaves.

Definition 1.2 [Sturmfels and Sullivant 2005]. Let T be a tree, let $K_{1,n}$ be the star with n leaves, and let $\phi(G, T)$ be the maximal degree of a generator in a minimal generating set of $I(X(G, T))$. Let $\phi(G, n) = \phi(G, K_{1,n})$. We define the *phylogenetic complexity* of G to be $\phi(G) = \sup_{n \in \mathbb{N}} \phi(G, n)$.

The main theorem of the present article is the following:

Theorem 3.12. *For any finite abelian group G, the phylogenetic complexity is finite.*

Let us briefly summarize the state of the art. We maintain the convention that G is a finite abelian group.

Prior to the present work $\phi(G)$ was shown in [Sturmfels and Sullivant 2005] to be 2 for $\mathbb{Z}_2$ and conjectured to be $\leq |G|$ for all G and exactly 4 for the biologically relevant case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ [Conjectures 29 and 30]. It was proved in [Michałek 2017] to be finite for all $\mathbb{Z}_p$, where p is prime, and equal to 3 for $\mathbb{Z}_3$.

If one considers the *projective* scheme $X_p(G, T)$ instead of $X(G, T)$, the analog of our phylogenetic complexity is ≤ 4 for $\mathbb{Z}_2 \times \mathbb{Z}_2$ (in other words, $X_p(\mathbb{Z}_2 \times \mathbb{Z}_2, T)$ can be defined by an ideal generated in degree at most 4, for any T) and is finite for all G; both results are proved in [Michałek 2013]. The case of $\mathbb{Z}_3$ was solved in [Donten-Bury 2016] with the answer 3.

Draisma and Eggermont [2015] considered a generalization of group-based models: the group G of symmetries acts on a finite alphabet that need not coincide with G. They showed that the Zariski closure of the model can be *set-theoretically* defined by polynomial equations whose degree is bounded by a constant depending only on G. Our present results can be regarded as stronger, but for a smaller class. Obtaining finiteness results on an ideal-theoretic level for equivariant models would be a major achievement, far extending the results of [Draisma and Kuttler 2014]. However, this is beyond any of the methods described in this paper, where we focus on group-based models.

Casanellas et al. [2015b; 2015a] produced a collection of explicit equations that describe the phylogenetic variety on a Zariski-open subset of interest, and showed that the corresponding degree is $\leq |G|$. (In [Michałek 2014] that degree had been shown to be 4 for $\mathbb{Z}_2 \times \mathbb{Z}_2$.)

Finiteness also plays an increasingly important role in the context of toric varieties; see [Draisma et al. 2015].

Finally, we would like to mention the reduction that we use from the very beginning, previously obtained by Sturmfels and Sullivant [2005]. Although, in general, one is interested in arbitrary trees, it is enough to consider claw-trees. This is due to the construction of toric fiber products [Sullivant 2007].

The structure of the article is as follows. In Section 2 we describe the basic notation. In particular, we recall how one encodes binomials in $I(X(G, K_{1,n}))$ as special pairs of tables with group elements. Section 3 contains the main result. First, in Section 3A, we present the sketch of the proof, without any technical details and then the complete proof in Section 3B. We hope that some of the ideas of the paper can be made effective. In particular, in future work we plan to prove [Sturmfels and Sullivant 2005, Conjecture 30].

2. Binomials, tables and moves

This section records definitions and notation needed in the rest of the paper.

Let G be a finite abelian group and let $n \in \mathbb{N}$. In Definition 1.1, we introduced the most important algebrocombinatorial objects in our setting: n-tuples of group elements summing to 0, called *flows*. From the point of view of toric geometry and phylogenetics, flows correspond to *monomials* parametrizing our variety $X(G, K_{1,n})$ [Sturmfels and Sullivant 2005; Michałek 2011]. Relations among flows — which are described by *compatible* tables — encode the binomials in $I(X(G, K_{1,n}))$. It is a standard approach in toric geometry to represent the parametrizing monomials by their exponents, as points in a lattice. The polytope, that is the convex hull of such points, captures the geometry of the parametrized variety. For the sake of completeness we present the polytopes corresponding to $X(G, K_{1,n})$.

Definition 2.1 (polytope $P_{G,n}$). Consider the lattice $M \cong \mathbb{Z}^{|G|}$ with a basis corresponding to elements of G. Consider M^n with the basis $e_{(i,g)}$ indexed by pairs

$(i, g) \in [n] \times G$. We define an injective map of sets $\mathfrak{G} \to M^n$, by

$$(g_1, \dots, g_n) \longmapsto \sum_{i=1}^{n} e_{(i,g_i)}.$$

The image of this map defines the vertices of the polytope $P_{G,n}$.

Example 2.2 [Michałek 2017]. For $G = (\mathbb{Z}_2, +)$ and $n = 3$, we have four flows:

$$(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Hence, the polytope $P_{\mathbb{Z}_2,3}$ has the following four corresponding vertices:

$$(1, 0, 1, 0, 1, 0), (1, 0, 0, 1, 0, 1), (0, 1, 1, 0, 0, 1), (0, 1, 0, 1, 1, 0) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2,$$

where $(1, 0) \in \mathbb{Z}^2$ corresponds to $0 \in \mathbb{Z}_2$ and $(0, 1) \in \mathbb{Z}^2$ corresponds to $1 \in \mathbb{Z}_2$.

A more sophisticated example is presented in [Michałek 2011, Example 4.1]. Binomials may be identified with a pair of tables of the same size T_0 and T_1 of elements of G, regarded up to row permutation. Each row of such tables has to be a flow. The identification is as follows. Every binomial is a pair of monomials; the variables in such monomials correspond to flows, given by a collection of n elements in G. Every monomial is viewed as a table, whose rows are the variables appearing in the monomial; the number of rows of the corresponding table is the degree of the monomial. Consequently, a binomial is identified with the pair of tables encoding the two monomials respectively.

A binomial belongs to $I(X(G, K_{1,n}))$ if and only if the two tables are *compatible*, i.e., for each i the i-th column of T_0 and the i-th column of T_1 are equal as multisets.

In order to generate a binomial — represented by a pair of tables T_0 and T_1 — by binomials of degree at most d we are allowed to select a subset of rows in T_0 of cardinality at most d and replace it with a compatible set of rows, repeating this procedure until both tables are equal.

Example 2.3 [Michałek 2017]. For $G = (\mathbb{Z}_2, +)$ and $n = 6$ consider the following two compatible tables:

$$T_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Note that the red subtable of T_0 is compatible with the table

$$T' = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Hence, we may *exchange* them obtaining:

$$\widetilde{T_0} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that T_0 and $\widetilde{T_0}$ are *compatible*. Now, the brown subtable of $\widetilde{T_0}$ is compatible with the table

$$T'' = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Finally, we exchange them obtaining T_1. Hence we have a sequence of tables $T_0 \rightsquigarrow \widetilde{T_0} \rightsquigarrow T_1$. More specifically, we started from a degree three binomial given by the pair T_0, T_1 and we generated it using degree two binomials, called *quadratic moves*; see also Example 2.5.

In what follows, *quadratic moves*, i.e., binomials of degree two will play a crucial role. First, let us give the precise definition and an illustrative example.

Definition 2.4 (quadratic moves). Let T be a table — whose rows are flows — of elements of G; let r_i and r_j be two rows of T. For any subsequence $\{r_{i,l_1}, \dots r_{i,l_t}\}$ of r_i, we define two rows s_i and s_j whose elements are the following:

(i) $s_{i,k} = r_{i,k}$ if $k \neq l_1, \dots, l_t$, otherwise $s_{i,k} = r_{j,k}$.

(ii) $s_{j,k} = r_{j,k}$ if $k \neq l_1, \dots, l_t$, otherwise $s_{j,k} = r_{i,k}$.

The transformation of r_i and r_j into s_i and s_j described above is a *quadratic move* if $\sum_{k=1}^{t} r_{i,l_k} = \sum_{k=1}^{t} r_{j,l_k}$; in other words, if the differences sum to $0 \in G$. We note that this condition is equivalent to the fact that s_i and s_j are flows.

To illustrate the definition of quadratic moves, we consider the following example, to be compared with Example 2.3.

Example 2.5. Let $G = (\mathbb{Z}_2, +)$. Let T be the following 2×3 table of elements in $\mathbb{Z}_2$:

$$T = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The two rows r_1 and r_2 are flows, since their elements sum up to the $0 \in \mathbb{Z}_2$. We exchange the red subsequence of elements in the first row with the blue subsequence of elements in the second row. The rows s_1 and s_2, corresponding to the chosen (red) subsequence as in Definition 2.4, are the two rows of the following table:

$$\widetilde{T} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

This is a quadratic move, since s_1 and s_2 are still flows. Hence, the table T is transformed into the table $\widetilde{T}$ by the quadratic move above. Note that quadratic moves preserve, up to permutation, each column of a table. In particular, T and $\widetilde{T}$ are two compatible tables, i.e., their columns are the same as multisets.

3. Finite phylogenetic complexity for abelian groups

The aim of this section is to use the combinatorics of tables to prove finiteness of the phylogenetic complexity of a group-based model for any finite abelian group G.

3A. ***Idea of the proof.*** Before going into technical details, let us present here the basic ideas of Theorem 3.12.

The general strategy is to prove that the function $\phi(G, n)$ is eventually constant for large n. Hence, we start with two compatible $d \times n$ tables T_0 and T_1 for large n and we want to transform T_0 to T_1. The main objective is the proof of Lemma 3.11: *One can transform T_0 and T_1, independently, using quadratic moves, in such a way that there exist two columns c_j, c_{j+1} on which both tables exactly agree.* Once this aim is achieved, the induction becomes clear — the precise argument is presented in the last paragraph of the proof of Theorem 3.12. The most involved part is to show Lemma 3.11. First, we pass to subtables. For a table T, we denoted by T' the subtable containing all rows, but only those columns where a given element $g \in G$ is one of the (possibly many) most frequent group elements. This is not a severe restriction — see Remark 3.6. Such a "reference" element g is crucial throughout the proof. Note also that, due to compatibility, the indices of columns of the subtables T_0' and T_1' are the same, as the most frequent elements of any i-th column in T_0 and T_1 coincide. In particular, T_0' and T_1' are compatible (although their rows do not have to be flows any more). In the proof, it is shown that it is easier to move elements that are *frequent* in a table than those that are *rare*; the latter ones are called *dots*. A precise definition, independent on the choice of T_0 or T_1, of frequent and rare elements is given in Definition 3.2.

Equipped with these definitions, it is enough to prove Lemma 3.10: *One can transform T_0 and T_1, independently, using quadratic moves, in such a way that there exist two columns c_j, c_{j+1} such that any row in T_0 or T_1 contains at most one dot in columns c_j and c_{j+1}.* Indeed, once the above statement is proven, as the tables are considered up to row permutations, we can make all dots in both columns in T_0 exactly equal to corresponding dots in T_1. Then Lemma 3.11 follows as the entries that are not dots can also be adjusted — details are in the proof of the lemma.

Hence, the hard part of the proof of Theorem 3.12 lies in the proof of Lemma 3.10. Here the ideas are as follows. First, (as we passed from T to a subtable T' where a given element is one of most frequent in every column) we will be passing to

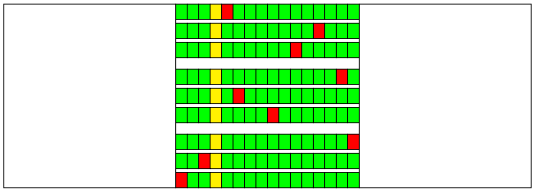


Figure 1. The subdivision algorithm.

thinner and thinner subtables. However, due to technical reasons, we must also allow their horizontal subdivisions, which motivate the following definition.

Definition 3.1 (vertical stripe). Given any table T, we define a *vertical stripe* to be

- a choice of some number of consecutive columns of T,
- a subdivision of rows into parts in the chosen columns.

Less formally, a vertical stripe is a collection of disjoint subtables in the same columns, that cover all rows of T.

Two examples of vertical stripes are presented in Figure 1. One consists of the whole colored part, where the subdivision into three subtables is given by two thick white horizontal stripes. The second stripe is the yellow one with the subdivision into nine subtables.

We would like to find a vertical stripe with (at least) two columns that has at most one dot in each row. Instead, we consider more general subtables that make vertical stripes: each subtable has k columns with at most s dots in each row. Further, we need to control how many distinct elements r of G appear as dots in the subtable. These subtables do not have to contain all rows, but appear in collections that form a vertical stripe, i.e., the collection covers all rows.

Figure 1 pictures the subdivision algorithm devised in Lemma 3.11 for T_0' and T_1'. We start with a vertical stripe — here represented by the colored part of the table. It consists of three subtables, divided by two large horizontal white stripes. In each of the subtables, we fix the same partition of columns into $t = 16$ vertical and three horizontal parts (given numbers are just examples). The new, finer horizontal subdivision is depicted with thin white stripes. In each horizontal part, we discard at most one of the subtables — these are the red squares. The yellow part drawn in the center of the picture is a vertical stripe, consisting of subtables that are not discarded in any of the horizontal parts.

The main point is that, for large k, we may decrease s or r by subdividing each subtable into $t|G|$ small subtables: columns are divided into $t \gg 0$ parts and rows into $|G|$ parts. In particular, we have t vertical parts, each consisting of $|G|$ small subtables stacked one under another. After quadratic moves, we may assume that each small subtable in almost all of the t vertical parts either has smaller number of dots in each row (decreasing s) or smaller number of distinct group elements corresponding to dots (decreasing r). As t is always much greater than the number of horizontal subdivisions (which is always some power of $|G|$) we are able to choose a whole vertical stripe (with much smaller number of columns) such that in each subtable s or r has been decreased. Further, we are able to do it in parallel in T_0' and T_1' — details are in the proof of Lemma 3.10.

We hope this discussion could shed some light on Definition 3.7. We mention here a technical remark; since we work with vertical stripes, once we focus on one subtable, we have to make sure we *do not* change the structure of other subtables. This feature is reflected in (ii) of Definition 3.7, where we restrict to quadratic moves that only modify a small part of the table. We are finally able to list the main steps towards the proof of Lemma 3.10:

(i) Bound the number of dots in each row (Lemma 3.4).

(ii) Prove that we may always subdivide a subtable, as described above, decreasing s or r (Lemma 3.9).

(iii) Show that the subdivision process can be done in parallel in T_0 and T_1 (Lemma 3.10).

3B. ***Proof.*** We start from the definition of frequent elements in a given table T with respect to a function F. Let $F(G)$ be a function of the cardinality of the group G. We assume $F(G) > |G|^2 + 3|G|$.

Definition 3.2 (F_T and dots). The set of $F(G)$-*frequent elements*, or frequent elements, in a given $d \times n$ table T is defined by

$$F_T = \{h \in G \mid \text{the number of copies of } h \text{ in } T > F(G) \cdot d\}.$$

Note that if an element is frequent, then there exists a row, where it appears at least $F(G)$ times. The elements $g \in G$ that are not in F_T are called *dots* $\bullet$.

The frequent elements have a key role in allowing quadratic moves in the table. Let us start with three basic — yet useful — lemmas.

Lemma 3.3. *Let f, f' be flows. Let I be a subset of indices and suppose $|I| \geq |G|$. There exists a* (*nonempty*) *subset $I' \subset I$ such that a quadratic move of f and f' on I' can be performed.*

Proof. Since we have $|G|$ differences, possibly repeated, of the form $f_i - f_i'$ for $i \in I$, we may find a nonempty subset I', such that $\sum_{i \in I'} (f_i - f_i') = 0 \in G$. □

Lemma 3.4. *Let T be a given table of elements of G, then we may assume that each row in T has at most $|G|(F(G)+1)$ dots.*

Proof. Note that there exists a row containing at most $|G|F(G)$ dots. Assuming the contrary, we would have at least $(|G|F(G)+1)d$ dots in T. This would imply that there would be a dot in F_T — a contradiction. Let us consider a row $r_{\max}$ with the largest number of dots. If $r_{\max}$ contains at most $|G|(F(G)+1)$ dots, this finishes the proof. Otherwise, we pick a row $r_{\min}$ with the smallest number of dots; they are at most $|G|F(G)$. Now, there exist $|G|$ dots of $r_{\max}$ in the same columns as $|G|$ elements of $r_{\min}$ which are in F_T. Exchanging a subset of them we decrease the number of rows with the largest number of dots. Repeating the process, we obtain T with all rows containing at most $|G|(F(G)+1)$ dots. □

Lemma 3.5. *Let $z \in \mathbb{N}$. For any $\epsilon > 0$, there exists $n = n(z)$ such that in any $(0, 1)$-table T of size $d \times n$, whose columns contain at least $\epsilon \cdot d$ zeros each, there exists a row with at least z zeros.*

Proof. Setting $n > z/\epsilon$ we may conclude by double counting zeros column-wise and row-wise. □

Remark 3.6. Let T be a $d \times n$ table whose entries are elements of G. In each column c_i we select the elements that appear a maximal number of times; these elements are the *most frequent elements* in c_i. Among all the columns, we select those where a *reference* element $g \in G$ appears as one of the most frequent elements.

This is not a severe restriction, as n is very large and we would restrict to a subtable with at least $n/|G|$ columns, for some $g \in G$. Such a reference element g will be important throughout the proof.

We now introduce a crucial property $S(\cdot)$ for our inductive proof.

Definition 3.7 (property $S(\cdot)$). Let $s, r, t, k \in \mathbb{N}$, let T be a $d \times n$ table whose entries are elements of G, and Q a $d' \times k$-subtable of T. Moreover, let us assume that the following hold:

(a) $g \in G$ is one of the most frequent elements in every column of T.

(b) There are at most s dots in every row of Q.

(c) There exists a subset $H \subset G$ of cardinality r, such that each dot of Q is in H.

We say that the property $S(s, r, t, k, T, Q)$ holds for the pair $Q \subset T$ if

(i) $s = 1$ and $k \geq 2$, or

(ii) $s > 1$ and we can transform T into another table $\widetilde{T}$ (transforming Q into $\widetilde{Q}$) such that we may subdivide the first $t \cdot \lfloor k/t \rfloor$ columns of $\widetilde{Q}$ into t consecutive subtables Q_i, each consisting of $\tilde{k} = \lfloor k/t \rfloor$ columns and d' rows that satisfy:

   (1) If $r = 1$ then each Q_i *except one* has the property $S(s-1, r, |G|t, \tilde{k}, T, Q_i)$.

   (2) If $r > 1$ then for every Q_i *except one* we can subdivide the rows into $|G|$ parts Q_{ij}, such that for every j either $S(s-1, r, |G|t, \tilde{k}, T, Q_{ij})$ or $S(s, r-1, |G|t, \tilde{k}, T, Q_{ij})$ holds.

Further, the transformation may only use quadratic moves that do not change dots that are in the columns of Q and in rows outside Q (i.e., it cannot move dots in the same vertical stripe, but outside Q).

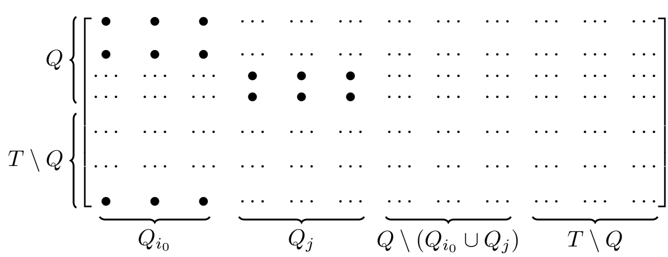
Remark 3.8. Condition (a) above is not restrictive, according to Remark 3.6, as we will be applying the definition to subtables of T_0 and T_1 for which g is one of the most frequent elements in each column.

In the next lemma, we show that one can transform and divide Q into smaller subtables decreasing either s or r, provided k is sufficiently large. This is achieved with special quadratic moves.

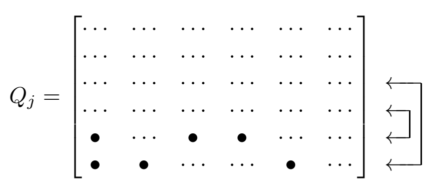
Lemma 3.9. *For every $s, r, t \in \mathbb{N}$, every k sufficiently large, and every pair $Q \subset T$ satisfying the assumptions in Definition 3.7, the property $S(s, r, t, k, T, Q)$ holds.*

Proof. The proof is by induction on s. For $s = 1$ the claim is true for $k \geq 2$ by Definition 3.7. Assume that the claim is true for s. We show the statement for $s + 1$.

If $s + 1 > |G|$, let us set $k > t \cdot \tilde{k}$, where $\tilde{k}$ is an integer such that the property $S(s, r, |G|t, \tilde{k}, \cdot, \cdot)$ holds for arbitrary pairs of tables and subtables in the last two arguments which satisfy the assumptions in Definition 3.7. Let us fix an arbitrary pair of tables $Q \subset T$ satisfying the assumptions in Definition 3.7 for $s + 1$ and r. In particular, each row of Q has at most $s+1$ dots. We fix a partition of Q into equal-sized subtables Q_j, each consisting of $\lfloor k/t \rfloor$ consecutive columns. If all the Q_j contain only rows with strictly less than $s + 1$ dots, we are done. Otherwise, we choose a subtable Q_{i_0} with a maximal number of rows containing $s+1$ dots. Every Q_j has at most as many rows with $s+1$ dots as Q_{i_0}. Hence, for any subtable Q_j different from Q_{i_0} we can pair each row of Q_j with $s+1$ dots with a row of Q_j without any dots (the latter corresponding to a row of Q_{i_0} with $s+1$ dots). The structure of T is as follows.



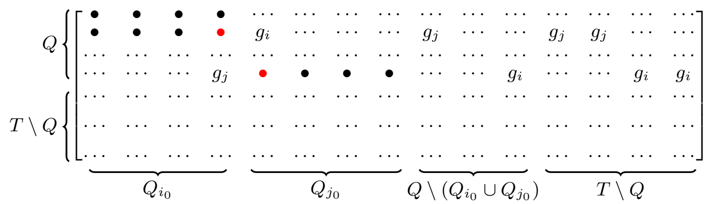
The arrows below describe the pairing between a row with $s + 1$ dots with a row without any dots in the subtable Q_j.



For each such pair we use Lemma 3.3, as $s+1 > |G|$, to make a quadratic move reducing the number of dots that a row of Q_j may have. Hence, by induction, for any $Q_j \neq Q_{i_0}$ the property $S(s, r, |G|t, \lfloor k/t \rfloor, T, Q_j)$ holds, as $k > t \cdot \tilde{k}$. Thus $S(s+1, r, t, k, T, Q)$ holds by Definition 3.7.

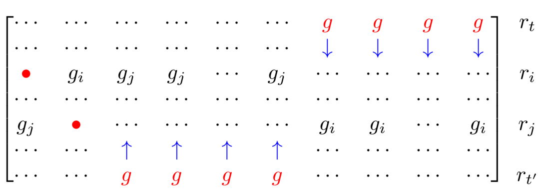
If $s+1 \leq |G|$, we proceed by induction on r.

If $r = 1$, let us set $k > t \cdot \tilde{k}$ as before. First, suppose there is only one vertical part Q_{i_0} which contains rows with $s+1$ dots. Since all the other parts Q_j have rows with at most s dots, by induction they satisfy $S(s, r, |G|t, \lfloor k/t \rfloor, T, Q_j)$, hence we may conclude this case. Otherwise, as long as there are two parts Q_{i_0} and Q_{j_0} with rows r_i and r_j respectively with $s+1$ dots, we proceed as follows. Let us fix one dot in r_i and one in r_j. Let g_i and g_j be the elements of the rows r_i and r_j in the same columns as the chosen dots. If $g_i = g_j$ we can make a quadratic move exchanging both chosen dots and the g_i. Suppose $g_i \neq g_j$.



As g_i is not a dot, there has to exist a row r_t of T with more than $F(G)$ copies of g_i. We make a quadratic move between r_j and r_t not involving the $2s+2$ columns of dots in Q_{i_0} and Q_{j_0} in the rows r_i and r_j. This procedure allows us to put at least $F(G) - 3|G|$ copies of g_i in the row r_j, without moving dots in Q — we need to subtract $|G|$ by Lemma 3.3 and $2|G| \geq 2(s+1)$ to avoid the dots. Now, we can make the same quadratic move for g_j and r_i. The result of these moves is in the table above, where the red bullets • are the chosen dots.

After performing these quadratic moves, if there is a column c_t containing g_j and g_i in rows r_i and r_j, then we make a quadratic move, exchanging the chosen dots and the elements of c_t. Otherwise, applying Lemma 3.5 for $\epsilon = 1/|G|$ to a subtable of T of columns containing g_i in the row r_j, we may find a row r_t containing at least $|G|$ copies of g, as long as $F(G) - 3|G| > |G|^2$. Then we move some copies of g to the row r_i by Lemma 3.3. Analogously for g_j, we may move some copies of g to the row r_j. Here are, depicted in red, the copies of g and, in blue, the quadratic move putting those copies of g in r_i and r_j respectively.



Applying the blue quadratic move above, we obtain a column c_i that has g in r_i and g_i in r_j. In the same way, we obtain a column c_j that has g in r_j and g_j in r_i. Now, we perform a quadratic move in the subtable below, exchanging the chosen dots:

$$\begin{bmatrix} \bullet & g_i & g_j & g \\ g_j & \bullet & g & g_i \end{bmatrix}.$$

Thus, we reduce the number of dots in both rows. This concludes the case $r = 1$.

Assume $r > 1$. Let us set $k > t \cdot \tilde{k}$, where $\tilde{k}$ is such that both of the properties $S(s, r, |G|t, k, T, \cdot)$ and $S(s+1, r-1, |G|t, k, T, \cdot)$ hold. Suppose that there is only one Q_{i_0} such that there exists a row with $s+1$ dots corresponding to r distinct group elements. Then the rows of every other part Q_j can be partitioned into at most $|G|$ parts $Q_{j,l}$ such that

(i) all the rows in $Q_{j,1}$ have at most s dots,

(ii) all the dots in $Q_{j,l}$ for $l > 1$ correspond to at most $r-1$ distinct group elements.

We conclude by induction in the case when there is only one part Q_{i_0}. We will reduce every other case to this one. Assume that there are two parts Q_{i_0} and Q_{j_0} such that there exist rows r_i and r_j with $s+1$ dots corresponding to r distinct group elements. As both rows r_i and r_j contain dots corresponding to the same r elements of the group G, we can choose one dot in each row corresponding to the same element. Now, repeating the procedure described in the case $r = 1$, we reduce the number of dots in r_i and r_j. This concludes the proof. □

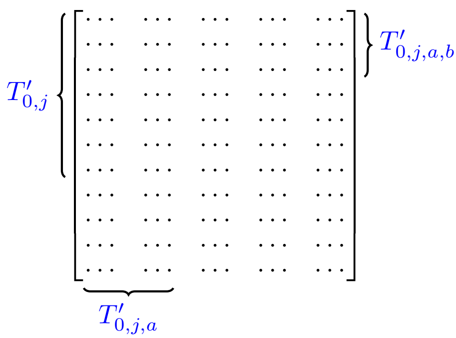
By Lemma 3.9 for any s, r, t we set $K(s, r, t)$ such that for all $k \geq K(s, r, t)$ the property $S(s, r, t, k, \cdot, \cdot)$ holds.

Lemma 3.10. *Let T_0 and T_1 be tables which are compatible and have at least $|G|K\big(|G|(F(G)+1), |G|, 3\big)$ columns. Then, we may transform them using quadratic moves into tables $\widetilde{T}_0$ and $\widetilde{T}_1$ such that the following holds: there exists j such that no row in $\widetilde{T}_0$ nor in $\widetilde{T}_1$ has a dot in both the j-th and the $(j+1)$-st columns.*

Proof. Let us restrict T_0 and T_1 to the subtables T_0' and T_1' containing all rows and those columns that have g as the most frequent element. By Lemma 3.4, we may assume that the upper bound on the number of dots in T_0' and T_1' in each row is $B = |G|(F(G)+1)$. By Remark 3.6 and the assumption on the size of T_0 and T_1, we can assume that T_0' and T_1' have at least $k_0 = K(B, |G|, 3)$ columns. Hence, in particular, the properties $S(B, |G|, 3, k_0, T_0', T_0')$ and $S(B, |G|, 3, k_0, T_1', T_1')$ hold. In the rest of the proof we transform both tables T_0' and T_1' using quadratic moves, at each step passing to a smaller vertical stripe such that each subtable in it satisfies the property $S(s, r, \cdot)$ with smaller and smaller s or r.

Starting with T_0' and T_1' we apply the following algorithm, depicted in Figure 1.

Input of step *i*: The input of the i-th step of the algorithm is two compatible tables with corresponding distinguished $k_i = \lfloor k_{i-1}/(3|G|^{i-1}) \rfloor$ consecutive columns forming a vertical stripe. The vertical stripe has at most $|G|^i$ parts (subtables). In the table T'_0 the parts are $T'_{0,j}$. For a given part $T'_{0,j}$, let $s_{0,i,j}$ be the maximal number of dots that a row may have. Let $r_{0,i,j}$ be the number of distinct group elements corresponding to dots of $T'_{0,j}$. Properties $S(s_{0,i,j}, r_{0,i,j}, 3|G|^i, k_i, T'_0, T'_{0,j})$ hold. Likewise the parts $T'_{1,j}$ of T'_1 satisfy $S(s_{1,i,j}, r_{1,i,j}, 3|G|^i, k_i, T'_1, T'_{1,j})$. Moreover, $s_{0,i,j} + r_{0,i,j} \leq B + |G| - i$ and $s_{1,i,j} + r_{1,i,j} \leq B + |G| - i$.
Output of step *i*: The output of the i-th step of the algorithm is the input of the $(i+1)$-st step.
Termination: The algorithm stops when all $s_{0,i,j}, s_{1,i,j} = 1$.
Procedure for step *i*: In the i-th step we subdivide the k_i columns into $3|G|^i$ parts, subdividing each $T'_{0,j}$ into parts $T'_{0,j,a}$, as in Definition 3.7. Now, the algorithm transforms $T'_{0,j,a}$ using Definition 3.7. Hence, we obtain a subdivision of rows of $T'_{0,j,a}$ into at most $|G|$ parts $T'_{0,j,a,b}$. Here are the parts of T'_0 highlighted in blue (the left and right brackets select horizontal parts and the bottom bracket selects vertical parts):



For each j, for every a *except one* and for every b the subtable $T'_{0,j,a,b}$ satisfies either property (1) or (2) below:

$$S(s_{0,i,j} - 1, r_{0,i,j}, 3|G|^{i+1}, k_{i+1}, T'_0, T'_{0,j,a,b}), \tag{1}$$

$$S(s_{0,i,j}, r_{0,i,j} - 1, 3|G|^{i+1}, k_{i+1}, T'_0, T'_{0,j,a,b}). \tag{2}$$

As each of the $|G|^i$ horizontal parts in T'_0 can exclude one $T'_{0,j,a}$, and each of the $|G|^i$ horizontal parts in T'_1 can exclude one $T'_{1,j,a}$ we may find an index a_0 such that the following conditions hold for every j and b:

(i) (1) or (2), with a_0 in place of a.

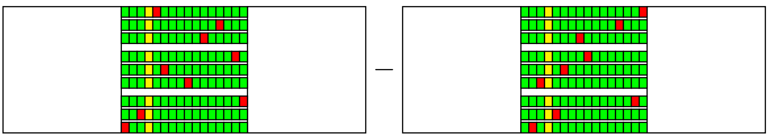
(ii) (3) or (4), given by

$$S(s_{1,i,j} - 1, r_{1,i,j}, 3|G|^{i+1}, k_{i+1}, T'_1, T'_{1,j,a_0,b}), \tag{3}$$

$$S(s_{1,i,j}, r_{1,i,j} - 1, 3|G|^{i+1}, k_{i+1}, T'_1, T'_{1,j,a_0,b}). \tag{4}$$

(Less formally, since the number of vertical stripes is much larger than the number of discarded subtables in each subdivision, we can choose two corresponding vertical stripes in both of the tables. This is pictured below.) The choice of the a_0-th vertical stripe and the subdivisions $T'_{0,j,a_0,b}$, $T'_{1,j,a_0,b}$ are the output of the i-th step of the algorithm and the input of the $(i+1)$-th step. The algorithm terminates when we reach $s = 1$. The procedure terminates in a finite number of steps as at each step either s or r decreases. Moreover, at every step of the algorithm, we have collections of subtables satisfying property $S(\cdot)$. This implies that, at the last step, $k \geq 2$. Thus the algorithm provides the desired pairs of columns.

$\tilde{T}_0 - \tilde{T}_1 =$



□

Lemma 3.11. *Let T_0 and T_1 be two compatible tables with n columns, for n sufficiently large. We can transform T_0 and T_1 using quadratic moves such that the following holds: there exists j such that the j-th and $(j+1)$-st columns in T_0 equal the j-th and $(j+1)$-st columns in T_1 respectively.*

Proof. We restrict to subtables T'_0 and T'_1 where g is the most frequent element, as in Remark 3.6. By Lemma 3.10, we may assume that in every row of T'_0 and T'_1 we have only one dot in the first two columns. Now, we can permute rows in such a way that the dots are equal in the corresponding entries. The elements in the rows which are not dots are not necessarily the same in each row. We show that given any pair of distinct elements g_i, $g_j \in F_{T'_0}$ in the first column and in the rows r_i, r_j respectively, we can exchange them.

Since g_i and g_j are in $F_{T'_0}$, we can find two rows, say r_s and r_t respectively, such that we have at least $F(G)$ copies of g_i and g_j in r_s and r_t respectively — see the table below. By Lemma 3.3, we can move at least $F(G) - |G| - 2$ copies of g_i to the row r_j and at least $F(G) - |G| - 2$ copies of g_j to r_i; here we subtract two because we are avoiding the first two columns. If there is a column c_t containing g_i and g_j in its j-th and i-th rows respectively, then we exchange them by a quadratic move on the column c_t and the first column. Otherwise, we proceed as follows. We restrict to a subtable containing columns where the row r_j has g_i as its entries. By Lemma 3.5, for $\epsilon = 1/|G|$, in this subtable we may find $|G|$ copies of g in some row r_t. Then we move some copies of g to the row r_i applying Lemma 3.3. Analogously for g_j, we may move some copies of g to the row r_j. Below are depicted in red the copies of g and in blue the quadratic moves putting those copies of g in r_i and r_j respectively.

$$T_0' = \begin{bmatrix} \bullet & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bullet & \cdots & g & g & g & g & \cdots & \cdots & \cdots & \cdots \\ \bullet & \cdots & \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \cdots & \cdots & \cdots \\ g_i & \cdots & \cdots & \cdots & \cdots & \cdots & g_j & g_j & g_j & g_j \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \bullet & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_j & \bullet & g_i & g_i & g_i & g_i & \cdots & \cdots & \cdots & \cdots \\ \cdots & \bullet & \cdots & \cdots & \cdots & \cdots & \uparrow & \uparrow & \uparrow & \uparrow \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & g & g & g & g \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Now, we perform a quadratic move exchanging g_i and g_j in a suitable subtable of T_0':

$$\begin{bmatrix} g_i & g_j & g \\ g_j & g & g_i \end{bmatrix}.$$

Such moves allow us to adjust all elements in the first two columns that are *not* dots. This concludes the proof. □

Theorem 3.12. *For any finite abelian group G, the phylogenetic complexity $\phi(G)$ of G is finite.*

Proof. Let G be a finite abelian group. Fix $N \gg |G|$. Once N is fixed, the phylogenetic complexity $\phi(G, N)$ is finite by the Hilbert basis theorem. Assume $n > N$. We will show that $\phi(G, n) \le \phi(G, n-1)$. This implies that they are equal.

Let B be a binomial in $I(X(G, K_{1,n}))$ identified with a compatible pair of $d \times n$ tables T_0 and T_1, as described in Section 2. By Lemma 3.11, we may assume there exist two columns c_j and c_{j+1} in T_0 and their corresponding columns c_j' and c_{j+1}' in T_1, for some $1 \le j \le n$, such that, for each row, c_j has the same entries as c_j', and c_{j+1} has the same entries as c_{j+1}'. Note that in Lemma 3.11 we use quadratic moves to transform two given tables T_0 and T_1 into a pair of tables such that they satisfy the condition on columns above.

Now, summing coordinatewise the columns c_j and c_{j+1} in T_0, and c_j' and c_{j+1}' in T_1, we obtain a new pair of tables $\widehat{T_0}$ and $\widehat{T_1}$ with $n-1$ columns. The new pair $\widehat{T_0}, \widehat{T_1}$ is identified with a binomial $\widehat{B} \in I(X(G, K_{1,n-1}))$. By definition, this binomial is generated by binomials of degree at most $\phi(G, n-1)$. Hence, we may transform $\widehat{T_0}$ into $\widehat{T_1}$ by exchanging in every step at most $\phi(G, n-1)$ rows. Each of these steps lifts to an exchange among at most $\phi(G, n-1)$ rows in tables T_0 and T_1. After applying all the steps, the resulting tables $\widetilde{T_0}$ and $\widetilde{T_1}$ still do not have to be equal. However, they only differ possibly on the columns c_j and c_{j+1}. Without loss of generality we may assume $j = 1$. Thus the tables $\widetilde{T_0}$ and $\widetilde{T_1}$ are as follows:

$$\widetilde{T_0} - \widetilde{T_1} = \begin{bmatrix} a_{j_1} & b_{j_1} & \cdots & \cdots & \cdots \\ a_{j_2} & b_{j_2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{j_d} & b_{j_d} & \cdots & \cdots & \cdots \end{bmatrix} - \begin{bmatrix} a_{k_1} & b_{k_1} & \cdots & \cdots & \cdots \\ a_{k_2} & b_{k_2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k_d} & b_{k_d} & \cdots & \cdots & \cdots \end{bmatrix},$$

where columns *different* from the first two are identical. Suppose there exists l such that $a_{j_l} \neq a_{k_l}$ (and $b_{j_l} \neq b_{k_l}$). Then $a_{j_l} + b_{j_l} = a_{k_l} + b_{k_l}$, since the l-th rows of $\widetilde{T}_0$ and $\widetilde{T}_1$ are identical except in the first two columns and, moreover, every row is a flow. On the other hand, there exists s such that $a_{k_l} = a_{j_s}$ and $b_{k_l} = b_{j_s}$. Thus we make a quadratic move between a_{j_l}, b_{j_l} and a_{j_s}, b_{j_s}. This concludes the proof. □

4. Open questions

In this last section, we collect some well-known open questions regarding group-based models for the convenience of the reader. We start from the central conjecture in this context.

Conjecture 4.1 [Sturmfels and Sullivant 2005, Conjecture 29]. For G, any finite abelian group, $\phi(G) \leq |G|$.

Taking into account the inductive approach presented in this article, it seems crucial to first understand the simplest tree $K_{1,3}$.

Conjecture 4.2. For G, any finite abelian group, $\phi(G, 3) \leq |G|$.

Notice that our main theorem — Theorem 3.12 — can be restated as follows: the function $\phi(G, \cdot)$ is eventually constant. The ensuing result would be a desired strengthening of ours.

Conjecture 4.3 [Michałek 2013, Conjecture 9.3]. $\phi(G, n+1) = \max(2, \phi(G, n))$.

We are grateful to Seth Sullivant for noticing that this is equivalent to $\phi(G, \cdot)$ being constant, apart from the case when $G = \mathbb{Z}_2$ and $n = 3$, when the associated variety is the whole projective space. Conjecture 4.3 also implies the following.

Conjecture 4.4 [Sturmfels and Sullivant 2005, Conjecture 30]. The phylogenetic complexity of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ is 4.

Yet another direction would be trying to find combinatorial analogs of Δ-modules presented in [Snowden 2013; Sam and Snowden 2016]. We have not pursued this approach, however we present some similarities. First, in the class of equivariant models one can apply such techniques to prove finiteness on the set-theoretic level [Draisma and Eggermont 2015]. Second, one of the properties of equivariant models — a flattening — is mimicked for group-based models (on the algebra level though, but not on the level of varieties). This is the addition of two group elements that turns a flow of length $n+1$ to a flow of length n. The latter was a crucial property that allowed us to obtain the result: generation using the "simple" equations (in our case, quadratic moves) and induced equations for smaller n. It would be very desirable to introduce a general setting for polytopes and toric varieties, which would still allow to obtain finiteness results on the ideal-theoretic level.

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mmichalek@impan.pl — *Polish Academy of Sciences, Warsaw, Poland*

emanuele.ventura@aalto.fi — *Department of Mathematics and Systems Analysis, Aalto University, FI-00076 Espoo, Finland*

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